Prop profiles of compatible Poisson and Nijenhuis structures

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Abstract

A prop profile of a differential geometric structure is a minimal resolution of an algebraic prop such that representations of this resolution are in one-to-one correspondence with structures of the given type. We begin this thesis with a detailed account of the algebraic tools necessary to construct prop profiles; we treat operads and props, and resolutions of these through Koszul duality.

Our main results can be summarized as follows.

Firstly, we contribute to the work of S.A. Merkulov on the prop profiles of Poisson and Nijenhuis structures. We prove that the operad of the latter prop profile is Koszul by showing that it has a PBW-basis, and we provide a geometrical interpretation of the former in terms of an L-infinity structure on the structure sheaf of a manifold.

Secondly, we construct prop profiles of compatible Poisson and Nijenhuis structures. Representations of minimal resolutions of props correspond to Maurer-Cartan elements of certain Lie algebras associated to the resolved props. Also the differential geometric structures are defined as solutions of Maurer-Cartan equations. We show the correspondence between props and differential geometry by providing explicit isomorphisms between these Lie algebras.

Thirdly, in order to construct the prop profiles of compatible Poisson and Nijenhuis structures we study operads of compatible algebraic structures. By studying Cohen-Macaulay properties of posets associated to such operads we prove the Koszulness of a large class of operads of compatible structures.

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Henrik Strohmayer, Operads of compatible structures and weighted partitions. *Journal of Pure and Applied Algebra*, 212(11):2522-2534, 2008.

II Henrik Strohmayer, Prop profile of bi-Hamiltonian structures. arXiv:0804.0596v1 [math.DG], accepted for publication in *Journal of Noncommutative Geometry*, 2009.

III Henrik Strohmayer, Operad profiles of Nijenhuis structures. arXiv:0809.2279v1 [math.DG], accepted for publication in *Differential Geometry and its applications*, 2009.

Contents

Introduction

The aim of this thesis is two-fold. To introduce the reader to the general notion of prop profiles of differential geometric structures and to construct the specific prop profiles of compatible Poisson structures and compatible Nijenhuis structures. To accomplish the first aim we will give thorough definitions of operads and props and their Koszul duality theories. Through a solid understanding of how to compute minimal resolutions of props the link to the differential geometric structures can be made very clear; the differentials of the propic resolutions correspond to the Lie brackets defining the differential geometric structures. Having made this correspondence clear through previously known examples we hope that the newly constructed prop profiles will be accessible to the reader.

Operads, dioperads, properads, and props

One perspective on operads is that they are structures which parametrize operations on a space; one forgets the space and studies properties of the operations themselves. Operads can be defined in any monoidal category but we will only consider operads in the categories of vector spaces, differential graded vector spaces and sets. The fundamental example of an operad is the endomorphism operad $\mathscr{E}nd_V$ of multilinear endomorphisms of a $\mathbb{K}$-vector space V. Explicitly, it is given by the family

$$\mathscr{E}nd_V = \left\{\mathscr{E}nd_V(n) := \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)\right\}_{n\in\mathbb{N}}.$$

The axioms of an operad are modeled after the properties of composition of multivariate functions. An associative algebra is a vector space V and an element $\mu \in \mathscr{E}nd_V(2)$ satisfying the associativity condition $\mu\circ(\mu\otimes\mathrm{Id}) = \mu\circ(\mathrm{Id}\otimes\mu)$. This condition is encoded by an operad, $\mathscr{A}s$; an operad morphism $\mathscr{A}s \to \mathscr{E}nd_V$ is equivalent to an associative algebra structure on V. Similarly there exist operads $\mathscr{C}om$ and $\mathscr{L}ie$ encoding commutative algebras and Lie algebras to name but a few.

On the other hand, operads can be considered as algebraic objects in their own right in that they constitute a generalization of associative algebras. An associative algebra consists of a vector space A and a product $\circ\colon A\otimes A\to A$; an operad $\mathscr{P}$ consists of certain families of these,

$$\{\mathscr{P}(n)\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\circ_i^{n,n'}\colon \mathscr{P}(n)\otimes\mathscr{P}(n')\to\mathscr{P}(n+n'-1)\}.$$

For an algebra there are two ways any given element $a\in A$ can be multiplied with another element $b\in A$, $b\circ a$ and $a\circ b$. For a pair of elements of an operad $p\in\mathscr{P}(n)$ and $q\in\mathscr{P}(m)$ we have the products $p\circ_1 q,\dots,p\circ_n q$, and $q\circ_1 p,\dots,q\circ_m p$. With the many ways of composing elements, a one-dimensional way of writing does not give a very clear picture of expressions involving iterated multiplications. Enter graphs. By considering graphs decorated with elements of the spaces $\mathscr{P}(n)$ we can give an illuminating and precise description of this two-dimensional multiplication. For operads the graphs used are rooted trees.

Operads are good for describing many algebraic structures but cannot model structures which have operations with both multiple inputs and multiple outputs, e.g. a Lie bialgebra structure on a vector space V consists of both a Lie bracket $[\_,\_]\colon V\otimes V\to V$ and a Lie cobracket $\Delta\colon V\to V\otimes V$. To encode such structures we need dioperads, properads, and props. They all consist of families $\{\mathscr{P}(m,n)\}_{m,n\in\mathbb{N}}$ of vector spaces but differ in according to which graphs we allow to compose elements, the restrictions being on genus, connectedness and directed loops. Our presentation of operads and generalizations will be based on the unifying approach of $\mathfrak{G}^*$-algebras [41] which are algebraic structures in which the product is modeled by classes of directed graphs. This makes it possible to define all the above structures as instances of $\mathfrak{G}^*$-algebras differing just in which class of graphs one considers.

For many applications the categories of algebraic structures such as associative or Lie algebras have proven to be too restrictive. One of the more important examples of this is the morphism between the Lie algebra of polydifferential operators of a manifold and its homology, the Lie algebra of polyvector fields, constructed in [26]. It had long been know that the two were quasi-isomorphic as vector spaces but no morphism respecting their Lie algebra structures existed. By relaxing the conditions defining a Lie algebra so that the Jacobi identity no longer needs to hold but only is required to hold up to homotopy with respect to a hierarchy of higher operations, one gets the notion of L_∞-algebra, see e.g. [27]. Similarly the notion of homomorphism is relaxed. Working in this category M. Kontsevich was able to construct an L_∞-quasi-isomorphism between the two Lie algebras which implied the settling of the deformation quantization conjecture; that there is a one-to-one correspondence between star products and Poisson structures on a manifold.

Operads and generalizations are deeply involved in this affair by giving a conceptual understanding of these relaxed structures, which have the epithets infinity or strongly homotopy; the operad governing L_∞-algebras is given by the minimal quasi-free resolution $\mathscr{L}ie_\infty$ of the operad $\mathscr{L}ie$. By computing minimal resolutions of other operads one obtains operadic descriptions of strongly homotopy structures, e.g. the minimal resolution $\mathscr{A}s_\infty$ of the operad $\mathscr{A}s$ encodes A_∞-algebras. Algebras over minimal resolutions of operads and props are in some sense more natural to consider; e.g. such structures transfer between homotopy equivalent vector spaces [28] and make it possible to define the cohomology of algebras over an operad in terms of Ext-functors [46]. As we will see, these minimal resolutions also provide a means to give concise algebraic descriptions of many differential geometric structures.

To compute the resolutions explicitly is in general a very difficult problem, but for certain operads there exists a tractable way. The theory of Koszul algebras has been generalized to operads [20], dioperads [18], properads and props [56]. For operads (and generalizations) possessing the Koszul property there exists an algorithmic way of computing its resolution. Thus proving the Koszulness of the operads involved in prop profiles will be an important task for us. Also the Koszul duality theory of the respective structures can be expressed as special cases of the Koszul duality of $\mathfrak{G}^*$-algebras.

For an introduction to and history of operads see e.g. [36] or [29]. We also refer to [28].

Compatible structures

Let $[\ \circ\ ]$ and $[\ \bullet\ ]$ be Lie brackets on a common vector space over a field $\mathbb{K}$. One can then define a new bracket $[\ ,\ ]$ by $[a,b] := \alpha[a\circ b]+\beta[a\bullet b]$, for some $\alpha,\beta\in\mathbb{K}$. Any such bracket is clearly skew symmetric and bilinear, so the only condition necessary in order for $[\ ,\ ]$ to be a Lie bracket is that the Jacobi identity $[[a,b],c]+[[b,c],a]+[[c,a],b]=0$ should hold. Direct calculation shows that this condition is equivalent to

$$[[a\circ b]\bullet c]+[[b\circ c]\bullet a]+[[c\circ a]\bullet b]+[[a\bullet b]\circ c]+[[b\bullet c]\circ a]+[[c\bullet a]\circ b]=0.$$

We call structures compatible in this way *linearly compatible*.

In [14] Khoroshkin and Dotsenko described the operad $\mathscr{L}ie^2$ which encodes pairs of compatible Lie algebras. They also considered the Koszul dual operad ${}^2\mathscr{C}om$ encoding two compatible commutative algebras. The compatibility condition in this case is quite different from the linear compatibility of Lie algebras. The commutative associative products $\circ$ and $\bullet$ are compatible in the following sense. Firstly, it should not matter in which order the products appear, i.e.

$$(a\circ b)\bullet c=(a\bullet b)\circ c,$$

and secondly, the associativity relation should be fulfilled for any order of applying the two products, i.e.

$$(a \circ b) \bullet c = a \bullet (b \circ c), \quad (a \bullet b) \circ c = a \circ (b \bullet c).$$

We call structures compatible in this way *totally compatible* since $\circ$ and $\bullet$ are totally interchangeable up to the number of each of them.

The operads of compatible Lie algebras and of compatible pre-Lie algebras will appear in the prop profile of compatible structures, thus a clear understanding of their structure will be of great use. Especially their Koszulness is of interest to us since we will compute minimal resolutions of props containing these as suboperads.

In [55] B. Vallette introduced a new method for showing the Koszulness of algebraic operads which can be obtained as the linearization of a set operad. By associating a certain poset to a set operad $\mathscr{P}$, and then studying its Cohen-Macaulay properties, one gets a concrete recipe for checking whether the algebraic operad associated to $\mathscr{P}$, and thus also its Koszul dual operad, is Koszul or not. Studying the posets of unordered and ordered pointed and multipointed partitions in [10], B. Vallette and F. Chapoton were able to prove the Koszulness of several important operads such as $\mathscr{P}erm$, $\mathscr{P}re\mathscr{L}ie$, $\mathscr{C}om\mathscr{T}rias$, $\mathscr{P}ost\mathscr{L}ie$, $\mathscr{D}ias$, $\mathscr{D}end$, $\mathscr{T}rias$ and $\mathscr{T}ri\mathscr{D}end$ over a field of any characteristic and over $\mathbb{Z}$.

To show the Koszulness of $\mathscr{L}ie^2$ and ${}^2\mathscr{C}om$, as well as several other linearly and totally compatible structures, we will use the poset method of B. Vallette. In order to handle the poset associated to an operad of two totally compatible structures we will show that it decomposes into the fiber product of two posets. The first one being the poset associated to the original structure and the other one being what we will call the poset of weighted partitions. In contrast to the posets studied by B. Vallette and F. Chapoton, these products of posets are not totally semimodular, therefore we need to refine the arguments of [10] in order to show that they are Cohen-Macaulay. In doing so we obtain:

Theorem A. *The following operads are Koszul:* ${}^2\mathscr{C}om$, $\mathscr{L}ie^2$, ${}^2\mathscr{P}erm$, $\mathscr{P}re\mathscr{L}ie^2$, ${}^2\mathscr{C}om\mathscr{T}rias$, $\mathscr{P}ost\mathscr{L}ie^2$, ${}^2\mathscr{A}s$, $\mathscr{A}s^2$, ${}^2\mathscr{D}ias$, $\mathscr{D}end^2$, ${}^2\mathscr{T}rias$ *and* $\mathscr{T}ri\mathscr{D}end^2$.

Prop profiles

In the papers [38], [39], and [40], S.A. Merkulov made the discovery that certain differential geometric structures, including Hertling-Manin, Nijenhuis, and Poisson structures, allow descriptions as the degree zero part of minimal resolutions of certain algebraic props. Merkulov called such descriptions prop profiles. Apart from the sheer beauty of these observations they provide us with new and surprising links between differential geometry, homological algebra and algebraic topology. For example, the prop profile of Hertling-Manin's weak Frobenius manifolds was shown to be given by a minimal resolution of the operad of Gerstenhaber algebras which in turn is quasi-isomorphic to the chain operad of the little 2-disc operad [11]. The prop profile of Poisson geometry, on the other hand, predicts existence of rather mysterious wheeled Poisson structures which can be deformation quantized [42, 43] in a wheeled propic way. Here by wheeled we mean that we allow graphs with oriented cycles which on the geometric side translates to traces of the involved structures. Another connection between wheeled props and differential geometry, the Batalin-Vilkovisky formalism, was studied in [22] and [44]. It is an open and interesting question whether or not the associated (wheeled) props have topological meaning as in the case of Hertling-Manin geometry.

The general philosophy of constructing prop profiles can be expressed as follows.

$$\begin{tikzcd} \text{DiffGeom} \arrow[r, "\text{(i)}"] & \text{DiffGeom} \arrow[r, "\text{(ii)}"] & \text{Props} \arrow[r, "\text{(iii)}"] & \text{Props} \arrow[lll, bend left, "\text{(iv)}"] \end{tikzcd}$$

(i) Extract the fundamental part of a differential geometric structure.

(ii) Translate this fundamental part into a prop $\mathscr{P}$, *the genome*.

(iii) Compute its minimal resolution $\mathscr{P}_\infty$, *the prop profile*.

(iv) Translate the prop profile back into a differential geometric structure.

Poisson structures

Poisson geometry plays a prominent role in Hamiltonian mechanics; the differential equations associated to a Hamiltonian system can be formulated via Poisson structures. The presence of two compatible Poisson structures makes it possible to solve a wide range of integrable Hamiltonian equations, e.g. the KdV-equations, by providing a hierarchy of integrable vector fields. This kind of geometric structure is called a Poisson pair or a bi-Hamiltonian structure. It was first considered by F. Magri in [32], Equation (3.1), for pairs of symplectic operators. The compatibility of the operators was called the coupling condition. See e.g. [2] for a treatment of Hamiltonian systems, [59] for a survey on Poisson geometry and [33] for an introduction to bi-Hamiltonian structures.

A Poisson structure on a graded manifold V is a graded Lie bracket on the structure sheaf $\mathscr{O}_V$ which acts as a derivation in each argument with respect to the multiplication on $\mathscr{O}_V$. A Poisson structure can equivalently be defined as a bivector field Γ of degree two satisfying $[\Gamma,\Gamma]_S = 0$. Here the bracket is the Schouten bracket on the polyvector fields on V. The fundamental part of Poisson structures translates into the prop $\mathscr{L}ie^1\mathscr{B}i$ of Lie 1-bialgebras, i.e. of Lie bialgebras with bracket and cobracket differing by one in degree. The prop profile of Poisson geometry, constructed in [40], is given by its minimal resolution $\mathscr{L}ie^1\mathscr{B}i_\infty$. Translating the prop profile back into differential geometry yields polyvector fields Γ of degree two, but not necessarily concentrated in $\wedge^2\mathscr{T}_V$, such that $[\Gamma,\Gamma]_S = 0$. Such a polyvector field can be interpreted as a family $\{L_n\}_{n\geq 1}$ of n-ary brackets on the structure sheaf $\mathscr{O}_V$. These families of brackets form L_∞-algebras and the brackets act as derivations in each argument with respect to the multiplication in $\mathscr{O}_V$.

One of our main results is that formal bi-Hamiltonian structures can be derived from a rather simple algebraic structure comprising a Lie bracket of degree one and two compatible Lie cobrackets of degree zero, with the further relations that each cobracket together with the Lie bracket form a Lie 1-bialgebra. We call such structures Lie_2 1-bialgebras and denote the corresponding prop by $\mathscr{L}ie^1_2\mathscr{B}i$. Using results from [18], [14], and the results of our efforts in studying operads of compatible structures, we show that its dioperadic part is Koszul, which makes it possible to compute its minimal resolution $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ and leads us to the following conclusion.

Theorem B. *There is a one-to-one correspondence between representations of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ in $\mathbb{R}^n$ and formal bi-Hamiltonian structures on $\mathbb{R}^n$ vanishing at the origin.*

In fact we prove a stronger result. When considering representations in arbitrary graded vector spaces we obtain the following result.

Theorem C. *There is a one-to-one correspondence between representations of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ in a graded vector space V and polyvector fields $\Gamma = \sum_k {}_k\Gamma\hbar^k \in \wedge^\bullet\mathscr{T}_V[[\hbar]]$ on the formal manifold associated to V which depend on the formal parameter $\hbar$ and satisfy the conditions*

- *(i)* ${}_k\Gamma \in \wedge^{\bullet\geq k+1}\mathscr{T}_V$,
- *(ii)* $|\Gamma| = 2$,
- *(iii)* $[\Gamma,\Gamma]_S = 0$,
- *(iv)* $\Gamma|_0 = 0$.

A pair of Poisson structures are called compatible if their brackets are compatible as Lie brackets. We will show how an element $\Gamma \in \wedge^\bullet\mathscr{T}_V[[\hbar]]$ with properties (i), (ii), and (iii) of Theorem C corresponds to a family $\{{}_kL_n\}_{n\geq 1, 1\leq k\leq n}$ of n-ary brackets on the structure sheaf $\mathscr{O}_V$. These brackets form an L^2_∞-algebra, the algebraic structure encoded by the minimal resolution of the operad $\mathscr{L}ie^2$, and act as derivations in each argument with respect to the multiplication in $\mathscr{O}_V$. When V is concentrated in degree zero we obtain precisely a bi-Hamiltonian structure. Property (iv) means that the structure vanishes at the distinguished point. By Remark 6.7 this is not a serious restriction.

To deformation quantize in the propic sense of Merkulov one needs a wheeled propic resolution of $\mathscr{L}ie^1_2\mathscr{B}i$. From the dioperadic resolution that we construct one obtains a propic resolution by known results. We note however that the same obstruction occurs as in the case of $\mathscr{L}ie^1\mathscr{B}i$ when trying to extend it to a resolution of wheeled props.

Nijenhuis structures

An *almost complex structure* on an even dimensional manifold M is an endomorphism J of $\mathscr{T}_M$ satisfying $J^2 = -\mathrm{Id}$. To any endomorphism of the tangent sheaf of a manifold there is associated the Nijenhuis torsion $\mathscr{N}_J$

$$\mathscr{N}_J(X,Y) := JJ[X,Y] + [JX,JY] - J[X,JY] - J[JX,Y].$$

By the Newlander-Nirenberg Theorem [47] the vanishing of the Nijenhuis torsion of an almost complex structure J is equivalent to J being a complex structure on M. This is probably the most important application of Nijenhuis structures; other examples can be found in [49].

The vector field valued differential forms $\Omega^\bullet_V \otimes \mathscr{T}_V$ of a manifold V together with the Frölicher-Nijenhuis bracket $[\_,\_]_{\text{F-N}}$ comprise a graded Lie algebra [49]. A Nijenhuis structure is an element $J \in \Omega^1_V \otimes \mathscr{T}_V$ such that $[J,J]_{\text{F-N}} = 0$. In [39] Merkulov defined a quadratic operad $\mathscr{N}ij$ such that representations of $\Omega(\mathscr{N}ij^{¡})$, the cobar construction on the Koszul dual cooperad, in a vector space V correspond to Nijenhuis structures on the formal manifold associated to V. If an operad $\mathscr{P}$ is Koszul, the cobar construction on its Koszul dual is a minimal resolution of the operad. The operad $\mathscr{N}ij$ consists of the Lie operad and the pre-Lie operad with the Lie bracket and pre-Lie product differing by one in degree and compatible in a certain sense. Until recently the available methods have not been sufficient to prove the Koszulness of $\mathscr{N}ij$; the compatibility relation of the operations does not define a distributive law and the operad does not come from a set theoretic operad, thus neither the methods of [34] nor [55] are applicable. Using the method of Poincaré-Birkhoff-Witt bases for operads, introduced by E. Hoffbeck in [23], we show that $\mathscr{N}ij$ is Koszul. Thereby we obtain the following result.

Theorem D. *There is a one-to-one correspondence between representations of $\mathscr{N}ij_\infty$ in $\mathbb{R}^n$ and formal Nijenhuis structures on $\mathbb{R}^n$ vanishing at the origin.*

When considering representations of $\mathcal{N}ij_\infty$ in a dg vector space V they correspond to arbitrary degree one vector forms J, not necessarily in $\Omega^1_V \otimes \mathcal{T}_V$, vanishing on the Frölicher-Nijenhuis bracket. Thus they correspond to a family of multivariate endomorphisms of the tangent sheaf satisfying certain quadratic relations. The geometric significance of this is an open question. Another interpretation of these structures was given by Merkulov who showed in [39] that representations of $\mathcal{N}ij_\infty$ in V correspond to contractible dg manifolds.

We say that two Nijenhuis structures J and K are compatible if their sum is again a Nijenhuis structure. To the best of the authors knowledge this kind of compatibility has not been considered before in the literature. We call such a pair a bi-Nijenhuis structure. One of our results is that bi-Nijenhuis structures can be derived from a rather simple algebraic structure. This structure consists of a Lie bracket and two pre-Lie products differing by one in degree with respect to the Lie bracket. The pre-Lie products are compatible in the sense that their sum again is a pre-Lie product and each of them is compatible with the Lie bracket in the sense that they form a $\mathcal{N}ij$ algebra. We denote the operad encoding such structures by $\mathcal{B}i\mathcal{N}ij$. Again using the PBW-basis method of Hoffbeck we show that $\mathcal{B}i\mathcal{N}ij$ is Koszul, making it possible to prove the following:

Theorem E. *There is a one-to-one correspondence between representations of $\mathcal{B}i\mathcal{N}ij_\infty$ in $\mathbb{R}^n$ and formal bi-Nijenhuis structures on $\mathbb{R}^n$ vanishing at the origin.*

Considering representations in arbitrary graded manifolds we obtain the following generalization.

Theorem F. *There is a one-to-one correspondence between representations of $\mathcal{B}i\mathcal{N}ij_\infty$ in a graded vector space V and formal power series*

$$\Gamma = \sum_k \Gamma_k \hbar^k \in (\Omega^\bullet_V \otimes \mathcal{T}_V)[[\hbar]]$$

satisfying the conditions

- *(i)* $\Gamma_k \in \Omega^{\geq k}_V \otimes \mathcal{T}_V$,
- *(ii)* $|\Gamma| = 1$,
- *(iii)* $[\Gamma,\Gamma]_{F\text{-}N} = 0$,
- *(iv)* $\Gamma|_0 = 0$.

Maurer-Cartan elements of Lie algebras

A common characteristic of all the differential geometric structures described above is that they can be defined as Maurer-Cartan elements, i.e. degree one elements γ satisfying $d(\gamma) + \frac{1}{2}[\gamma,\gamma] = 0$. On the propic side the Maurer-Cartan equations are encoded by the condition that representations of the minimal resolution of the involved props commute with differentials. In fact, this correspondence can be made clearer. To any prop, $\mathscr{P}$ and vector space V there exist an associated Lie algebra $\mathscr{L}_{\mathscr{P}}(V)$ whose elements are $\mathbb{S}$-bimodule morphisms from the Koszul dual coprop $\mathscr{P}^{¡}$ to $\mathscr{E}nd_V$. When $\mathscr{P}$ is Koszul, representations of $\mathscr{P}_\infty$ are precisely Maurer-Cartan elements of this Lie algebra. The Lie bracket of $\mathscr{L}_{\mathscr{P}}(V)$ is defined from the decomposition coproduct of $\mathscr{P}^{¡}$ as is the differential of $\mathscr{P}_\infty$. In [45] Merkulov and Vallette observed that the Lie algebras associated to $\mathscr{L}ie^1\mathscr{B}i$ and $\mathscr{N}ij$ in fact are isomorphic to the Lie algebras of polyvector fields and vector forms, respectively.

The prop profiles of bi-Hamiltonian and bi-Nijenhuis structures give rise to Lie algebras isomorphic to the following Lie algebras on the differential geometric side:

$$\mathfrak{g}_V := \{Y = \sum_{k\geq 0} {}_kY\hbar^k \in \wedge^\bullet\mathscr{T}_V[\![\hbar]\!] \mid {}_kY \in \wedge^{\geq k+1}\mathscr{T}_V\}$$

and

$$\mathfrak{h}_V := \{K = \sum_{k\geq 0} {}_kK\hbar^k \in (\Omega_V^\bullet \otimes \mathscr{T}_V)[\![\hbar]\!] \mid {}_kK \in \Omega_V^{\geq k} \otimes \mathscr{T}_V\}.$$

We summarize the prop profiles in the following table. Here the corolla $\curlyvee$ is the generator of an odd Lie algebra, the corollas $\curlywedge$, $\dot\curlywedge$, and $\dot\curlywedge$ denote generators of Lie coalgebras, and $\dot\curlyvee$, $\dot\curlyvee$, and $\dot\curlyvee$ are pre-Lie products. The notions of extended Poisson and bi-Hamiltonian structures and (bi-)Nijenhuis$_\infty$ structures are generalizations to the graded context derived from the prop profiles. See Chapters 7-10 for precise definitions. For $V = \mathbb{R}^n$ these notions coincide with the classical definitions.

Genome $\mathscr{P}$	Genes	$\mathscr{L}_{\mathscr{P}}(V)$	Maurer-Cartan elements
$\mathscr{L}ie^1$	$\curlyvee$	$\mathscr{T}_V$	Homological vector fields
$\mathscr{L}ie^1\mathscr{B}i$	$\curlyvee, \curlywedge$	$\wedge^\bullet\mathscr{T}_V$	Extended Poisson structures
$\mathscr{L}ie^1_2\mathscr{B}i$	$\curlyvee, \dot\curlywedge, \dot\curlywedge$	$\mathfrak{g}_V$	Extended bi-Hamiltonian structures
$\mathscr{N}ij$	$\curlyvee, \dot\curlyvee$	$\Omega_V^\bullet \otimes \mathscr{T}_V$	Nijenhuis$_\infty$ structures
$\mathscr{B}i\mathscr{N}ij$	$\curlyvee, \dot\curlyvee, \dot\curlyvee$	$\mathfrak{h}_V$	bi-Nijenhuis$_\infty$ structures

On the contents

Chapter 1 comprises definitions of operads, dioperads, properads and props and in Chapter 2 we give a formulation of the Koszul duality machinery that enables us to compute resolutions of these structures. These sections essentially contain no new material, rather they gather and give a detailed exposition of the relevant results for our needs.

In Chapter 3 we give definitions of operads encoding compatible structures, in particular we describe the operads $\mathscr{L}ie^2$ and ${}^2\mathscr{C}om$. We also show that there exist decompositions of the operads of compatible structures using black, white, and Hadamard products. Next, using the poset method of B. Vallette we proceed to prove the Koszulness of operads encoding a class of compatible structures. Finally we demonstrate the effectiveness of Koszul duality by computing the minimal resolution of $\mathscr{L}ie^2$. Chapter 4 recalls two other methods for proving Koszulness: W. Gan's distributive law method for dioperads [18] and E. Hoffbeck's PBW-basis method for operads [23].

In Chapter 5 we review the differential geometry necessary to define Poisson and Nijenhuis structures.

Chapter 6 is a template for the rest of the chapters. Here we explain the correspondence between L_∞-algebras and homological vector fields. The process consists of the steps extracting a prop, computing its resolution, and interpreting it geometrically. In Chapters 7-10 we iterate this procedure for Poisson, bi-Hamiltonian, Nijenhuis, and bi-Nijenhuis structures, respectively.

The correspondence between the contents of the chapters of this thesis and of the papers I, II, and III is more or less the following. Paper I corresponds to the first three sections of Chapter 3. Paper II encompasses most of Chapters 1, 2, 7, and 8 as well as the last section of Chapter I. Paper III contains the material of Chapters 9 and 10. The contents of Chapters 4, 5, and 6 are distributed between the two last papers.

Preliminaries

A few words about our notation. Given a finite set S we denote its cardinality by $|S|$. By $\mathbb{N}$ we mean the set $\{0,1,2,\dots\}$. For $n \geq 1$, we denote by $[n]$ the set $\{1,\dots,n\}$. Let $\mathbb{S}_n$ denote the symmetric group of permutations of $[n]$. By $\mathbb{1}_n$ we denote the trivial representation of $\mathbb{S}_n$ and by sgn_n the sign representation. An element $\sigma \in \mathbb{S}_{p+q}$ is called a (p,q)-*unshuffle* if $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. Let $\mathbb{S}_{(p,q)}^{\text{un-sh}} \subset \mathbb{S}_{p+q}$ denote the subset of (p,q)-unshuffles.

All vector spaces and tensor products are considered to be over a field $\mathbb{K}$ of characteristic zero unless otherwise specified. For a vector space V over $\mathbb{K}$ we denote the linear dual $\mathrm{Hom}(V,\mathbb{K})$ by V^*. Given a graded vector space

$V = \oplus_{j\in\mathbb{Z}} V^j$ we denote by $V[i]$ the vector space whose graded components are given by $V[i]^j = V^{i+j}$. The differential of a dg vector space is assumed to be of degree one.

The symmetric product of vector spaces is denoted by $\odot$. The notation $\odot^\bullet V$ stands for the direct sum

$$\bigoplus_{n\in\mathbb{N}} \underbrace{V\odot\cdots\odot V}_{n \text{ factors}}$$

and $\widehat{\odot^\bullet} V$ for its formal completion

$$\prod_{n\in\mathbb{N}} \underbrace{V\odot\cdots\odot V}_{n \text{ factors}}.$$

Given a coefficient $X^{a_1\cdots a_n}$ we denote its symmetrization by

$$X^{(a_1\cdots a_n)} := \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} X^{a_{\sigma(1)}\cdots a_{\sigma(n)}}$$

and its skew-symmetrization by

$$X^{[a_1\cdots a_n]} = \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} \mathrm{sgn}(\sigma) X^{a_{\sigma(1)}\cdots a_{\sigma(n)}}.$$

Throughout the paper we use the Einstein summation convention, i.e. we always sum over repeated upper and lower indices, $X^a\partial_a = \sum_a X^a\partial_a$.

When depicting directed graphs we generally consider the direction of an edge to be downwards.

Part I:
Operads and props

1 Operads and generalizations via graphs

In this chapter we give thorough definitions of operads, dioperads, properads, and props, as well as their co-versions, using the graph-approach of Merkulov [41].

1.1 Decorated graphs

1.1.1 $\mathbb{S}$-modules and $\mathbb{S}$-bimodules

First we define the underlying spaces of our generalized structures.

Definition. An $(\mathbb{S}_m, \mathbb{S}_n)$-*bimodule* is a vector space $\mathcal{M}$ with a right action of $\mathbb{S}_n$ and a commuting left action of $\mathbb{S}_m$. A family $\{\mathcal{M}(m,n)\}_{m,n\in\mathbb{N}}$ of $(\mathbb{S}_m, \mathbb{S}_n)$-bimodules is called an $\mathbb{S}$-*bimodule*. We say that an element of $\mathcal{M}(m,n)$ is of *arity* (m,n).

A family $\{\mathcal{M}(n)\}_{n\in\mathbb{N}}$ of right $(\mathbb{S}_n)$-modules is called an $\mathbb{S}$-*module*.

If an $\mathbb{S}$-bimodule $\mathcal{M}$ satisfies $\mathcal{M}(m,n) = 0$ whenever $m \neq 1$ we can consider it as an $\mathbb{S}$-module since the action of $\mathbb{S}_1$ is trivial. We then denote $\mathcal{M}(1,n)$ by $\mathcal{M}(n)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathbb{S}$-bimodules. An $\mathbb{S}$-*bimodule homomorphism* $\theta\colon \mathcal{M} \to \mathcal{N}$ is a family $\{\theta_{m,n}\colon \mathcal{M}(m,n) \to \mathcal{N}(m,n)\}_{m,n\in\mathbb{N}}$ of $(\mathbb{S}_m, \mathbb{S}_n)$-bimodule homomorphisms. We will often write $\theta(p)$ for $\theta_{m,n}(p)$ for $p \in \mathcal{M}(m,n)$.

1.1.2 Labeled directed graphs

Composition of elements of $\mathbb{S}$-bimodules is modeled by graphs. Intuitively we can think of these graphs as 1-dimensional regular CW-complexes with the 1-cells given an orientation. Two subsets of the 1-cells are singled out, directed towards and away from the graph, respectively, and are labeled with integers.

Definition. A *labeled directed graph* G is the data

$$(V_G, E_G, \Phi_G, E_G^{\text{in}}, E_G^{\text{out}}, \text{in}_G, \text{out}_G).$$

The elements of the set V_G are called the *vertices* of G, the elements of the set E_G the *edges*. Further $\Phi_G\colon E_G \to (V_G \times V_G) \sqcup V_G$. The edges in the preimage $\Phi_G^{-1}(V_G)$ are called *external edges* and the edges in the preimage $\Phi_G^{-1}(V_G \times V_G)$ are called *internal*. We denote the internal edges by E_G^{int}. For an edge e with $\Phi_G(e) = (u, v)$ we say that e is an edge from u to v and in this case we call the vertices u and v *adjacent*.

The set of external edges is partitioned into the sets E_G^{in} and E_G^{out} of *global input edges* and *global output edges*, respectively. We denote by n_G and m_G the cardinalities of these sets. The external edges are labeled by integers via the bijections $\mathrm{in}_G\colon [n_G] \to E_G^{\mathrm{in}}$ and $\mathrm{out}_G\colon E_G^{\mathrm{out}} \to [m_G]$.

A *labeled directed (m,n)-graph* G is a labeled directed graph with $m_G = m$ and $n_G = n$.

Note that the data $(V_G, E_G^{\mathrm{int}}, \Phi_G|_{E_G^{\mathrm{int}}})$ is an ordinary directed graph.

There exist a natural right action of $\mathbb{S}_n$ and a commuting left action of $\mathbb{S}_m$ on the class of (m,n)-graphs given by permuting the labels. For a labeled directed (m,n)-graph G the action of $\tau \in \mathbb{S}_n$ is given by $(\mathrm{in}_G \tau)(i) := \mathrm{in}_G \circ \tau(i)$. Similarly $\sigma \in \mathbb{S}_m$ acts to the left by $(\sigma\, \mathrm{out}_G)(e) := \sigma \circ \mathrm{out}_G(e)$, cf. Figure 1.1.

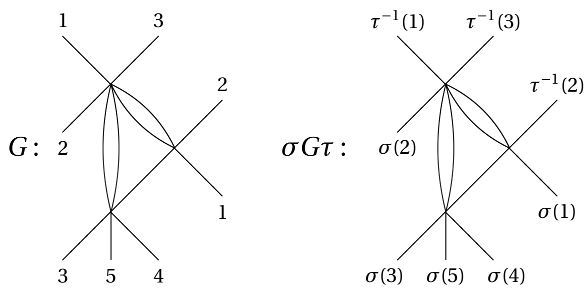


Figure 1.1: Example of action on a $(5,3)$-graph G by $\mathbb{S}_3$ from the right and by $\mathbb{S}_5$ from the left.

A *path* from a vertex u to a vertex v in a labeled directed graph is a sequence of edges $e_1, \ldots, e_r$ such that for some sequence of vertices $u = v_1, \ldots v_{r+1} = v$ either $\Phi_G(e_i) = (v_i, v_{i+1})$ or $\Phi_G(e_i) = (v_{i+1}, v_i)$. A path is called *directed* if $\Phi_G(e_i) = (v_i, v_{i+1})$ for all i and it is called *closed* if $u = v$. A closed directed path is called a *wheel*. A graph is *connected* if for each pair of vertices there is a path between them.

1.1.3 Isomorphisms of graphs

We are only interested in the structure of the graphs up to a certain level of detail: how many vertices there are, how many internal edges there are in each direction between any two vertices, how many external edges are directed towards and away from each vertex and how they are labeled. Thus we need to define isomorphisms of graphs.

Let G and G' be labeled directed graphs. An *isomorphism* of labeled directed graphs $\Psi\colon G \to G'$ is a pair (Ψ_V, Ψ_E), where $\Psi_V\colon V_G \to V_{G'}$ and $\Psi_E\colon E_G \to E_{G'}$ are bijections with the properties

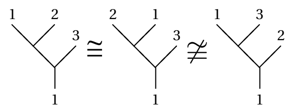
(i) $\Psi_E(E_G^{\text{in}}) = E_{G'}^{\text{in}}$ and $\Psi_E(E_G^{\text{out}}) = E_{G'}^{\text{out}}$,

(ii) $\Phi_{G'}(\Psi_E(e)) = \Psi_V \times \Psi_V(\Phi_G(e))$ for all internal edges e,

(iii) $\Phi_{G'}(\Psi_E(e)) = \Psi_V(\Phi_G(e))$ for all external edges e,

(iv) $\text{in}_{G'} = \Psi_E \circ \text{in}_G$ and $\text{out}_G = \text{out}_{G'} \circ \Psi_E$.

Example. Three graphs, of which the third not is isomorphic to the first two because of the labeling of the edges.



1.1.4 Classes of graphs

From now on we will refer to isomorphism classes of labeled directed graphs simply as graphs. We define the following classes of graphs:

(i) $\mathfrak{G}^{\circlearrowleft}$ is the class of all graphs.

(ii) $\mathfrak{G}^{\downarrow}$ is the class of graphs without wheels.

(iii) $\mathfrak{G}^{\downarrow}_c$ is the class of connected graphs without wheels.

(iv) $\mathfrak{G}^{\downarrow}_{c,0}$ is the class of connected graphs without closed paths (the class of trees).

(v) $\mathfrak{G}^{\downarrow_1}_c$ is the class of connected graphs without closed paths whose vertices have exactly one edge directed from it (the class of rooted trees).

(vi) $\mathfrak{G}^{{}^1\downarrow_1}_c$ is the class of connected graphs without directed paths whose vertices have exactly one edge directed towards it and exactly one edge directed from it (the class of ladder graphs).

We observe that $\mathfrak{G}^{{}^1\downarrow_1}_c \subset \mathfrak{G}^{\downarrow_1}_c \subset \mathfrak{G}^{\downarrow}_{c,0} \subset \mathfrak{G}^{\downarrow}_c \subset \mathfrak{G}^{\downarrow} \subset \mathfrak{G}^{\circlearrowleft}$. When depicting graphs of the classes (ii)-(vi) we think of them as having a global flow, from global input edges to global output edges, downwards.

Let $\mathfrak{G}^*$ denote an arbitrary class of the classes (i)-(vi). We denote by $\mathfrak{G}^*(m,n)$ the subclass of $\mathfrak{G}^*$ consisting of all (m,n)-graphs and by $\mathfrak{G}^*_{(i)}$ the subclass consisting of graphs with i vertices.

1.1.5 Subgraphs

In order to describe the associativity of the compositions described by graphs we need to define subgraphs and the notion of contraction of a subgraph in a graph.

Loosely speaking, a subgraph consists of some subset of the vertices of a graph, the edges attached to them, and an arbitrary global labeling.

Let G be a graph. A *subgraph* H of G is a graph satisfying

(i) $V_H \subset V_G$ and $E_H \subset E_G$,

(ii) if $\Phi_G(e) = (u, v)$ and $u, v \in V_H$, then $e \in E_H$ and $\Phi_H(e) = \Phi_G(e)$,

(iii) if $e \in E_G$, $\Phi_G(e) = (u, v)$, $u \notin V_H$ and $v \in V_H$, then $e \in E_H^{\text{in}}$ and $\Phi_H(e) = v$, similarly if $u \in V_H$, $v \notin V_H$, then $e \in E_H^{\text{out}}$ and $\Phi_H(e) = u$,

(iv) if $e \in E_G^{\text{in}}$, $\Phi(e) = v$ and $v \in V_H$, then $e \in E_H^{\text{in}}$, similarly if $e \in E_G^{\text{out}}$

Note that in_H and out_H are arbitrary labelings of the global input and output edges of H.

1.1.6 Contraction of subgraphs

The contraction of a subgraph in a graph can be thought of as replacing all vertices and internal edges of the subgraph with a single vertex.

Let H be a subgraph of a graph G. The *contraction of* H *in* G is the labeled directed graph G/H defined by the same data as G except

(i) $V_{G/H} = (V_G \setminus V_H) \sqcup \{v_H\}$, where by v_H we denote the vertex into which H is contracted,

(ii) $E_{G/H} = E_G \setminus E_H^{\text{int}}$,

(iii)

$$\Phi_{G/H}(e) = \begin{cases} \Phi_G(e) & \text{if } e \in E_G \setminus E_H \\ (u, v_H) & \text{if } \Phi_G(e) = (u, v) \text{ for some } v \in V_H \\ (v_H, u) & \text{if } \Phi_G(e) = (v, u) \text{ for some } v \in V_H \\ v_H & \text{if } \Phi_G(e) = v \text{ for some } v \in V_H. \end{cases}$$

We say that a subgraph H of a graph $G \in \mathfrak{G}^*$ is $\mathfrak{G}^*$-*admissible* if both $G/H \in \mathfrak{G}^*$ and $H \in \mathfrak{G}^*$. In Figure 1.2 we see an example of a subgraph H which is $\mathfrak{G}^{\circlearrowleft}$-admissible but not $\mathfrak{G}^{\downarrow}$-admissible.

1.1.7 Graphs decorated by $\mathbb{S}$-bimodules

Compositions of elements of $\mathbb{S}$-bimodules is described by graphs decorated with $\mathbb{S}$-bimodules. When decorating a vertex v with an element p of an $\mathbb{S}$-bimodule $\mathscr{M}$ we want to keep track of how we connect p to the internal edges attached to v.

We define the set of *local input edges* of v to be

$$E_v^{\text{in}} := \{e \in E_G^{\text{int}} \mid \Phi_G(e) = (u, v) \text{ for some } u \in V_G\} \cup \{e \in E_G^{\text{in}} \mid \Phi_G(e) = v\}$$

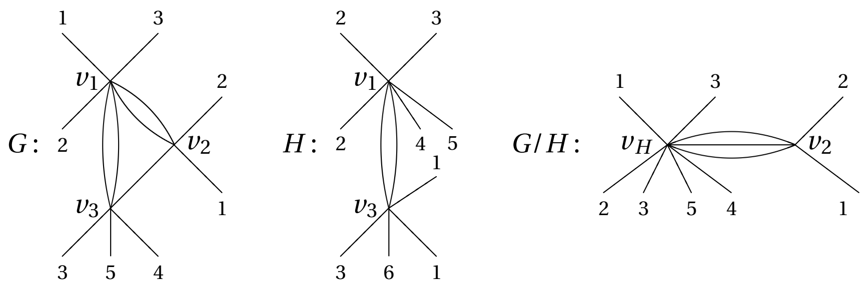


Figure 1.2: The contraction G/H of a subgraph H in a graph G.

and the set of *local output edges* of v as

$$E_v^{\text{out}} := \{e \in E_G^{\text{int}} \mid \Phi_G(e) = (v,u) \text{ for some } u \in V_G\} \cup \{e \in E_G^{\text{out}} \mid \Phi_G(e) = v\}.$$

Note that we allow edges from a vertex to itself. Such an edge will be both a local input and output edge of the same vertex.

For two finite sets I, J of the same cardinality, let $\text{Bij}(I,J)$ denote the set of bijections from I to J and let $\langle I \xrightarrow{\sim} J\rangle$ denote the vector space generated over $\mathbb{K}$ by $\text{Bij}(I,J)$. If $|I| = n$ there is a natural left action of $\mathbb{S}_n$ on $\langle I \xrightarrow{\sim} [n]\rangle$ given by $\tau g := \tau \circ g$ for $g \in \text{Bij}(I,[n])$ and $\tau \in \mathbb{S}_n$. Similarly, if $|J| = m$, $\mathbb{S}_m$ acts to the right on $\langle [m] \xrightarrow{\sim} J\rangle$ by $f\sigma := f \circ \sigma$ for $f \in \text{Bij}([m],J)$ and $\sigma \in \mathbb{S}_m$.

We define a vector space by

$$\mathcal{M}(E_v^{\text{out}}, E_v^{\text{in}}) := \langle [m] \xrightarrow{\sim} E_v^{\text{out}}\rangle \otimes_{\mathbb{S}_m} \mathcal{M}(m,n) \otimes_{\mathbb{S}_n} \langle E_v^{\text{in}} \xrightarrow{\sim} [n]\rangle,$$

where m and n are the cardinalities of E_v^{out} and E_v^{in}, respectively. Often we will denote an element $f \otimes_{\mathbb{S}_m} p \otimes_{\mathbb{S}_n} g \in \mathcal{M}(E_v^{\text{out}}, E_v^{\text{in}})$ by $\overline{p}$ or simply p.

Remark. Decorating by $\mathcal{M}(E_v^{\text{out}}, E_v^{\text{in}})$ rather than by $\mathcal{M}(m,n)$ corresponds to an additional labeling of the internal edges locally, cf. Figure 1 of [56, p4869].

We want decorated graphs to extend the notion of tensor products, but for a general graph there is no natural ordering of the vertices. Let $\{V_i\}_{i\in I}$ be a family of vector spaces indexed by some finite set I with $|I| = k$. The *unordered tensor product* of this family is defined to be

$$\bigotimes_{i\in I} V_i := \left(\bigoplus_{s \in \text{Bij}([k],I)} V_{s(1)} \otimes \cdots \otimes V_{s(k)} \right)_{\mathbb{S}_k}.$$

Here we consider the coinvariants with respect to the right action of $\mathbb{S}_k$ on $\text{Bij}([k],I)$. We denote an equivalence class in $\bigotimes_{i\in I} V_i$ by $[v_1 \otimes \cdots \otimes v_k]$, where $v_1 \otimes \cdots \otimes v_k \in V_{s(1)} \otimes \cdots \otimes V_{s(k)}$ for some $s \in \text{Bij}([k],I)$.

Definition. We define the vector space of decorations of a graph G by an $\mathbb{S}$-bimodule $\mathscr{M}$ to be $G\langle\mathscr{M}\rangle := \bigotimes_{v\in V_G} \mathscr{M}(E_v^{\text{out}}, E_v^{\text{in}})_{\text{AutG}}$, where AutG denotes the automorphism group of G.

An element $(G, [p_1 \otimes \cdots \otimes p_k])$ of $G\langle\mathscr{M}\rangle$ is called a *decorated graph.*

A *decorated subgraph* of a decorated graph $(G, [p_1 \otimes \cdots \otimes p_k])$ is a decorated graph $(H, [p_{i_1} \otimes \cdots \otimes p_{i_l}])$ such that H is a subgraph of G, $\{i_1, \ldots, i_l\} = \{i \in [k] \mid p_i \in \mathscr{M}(E_v^{\text{out}}, E_v^{\text{in}}) \text{ for some } v \in V_H\}$, and $i_1 < \cdots < i_l$.

1.1.8 $\mathbb{S}$-bimodules of decorated graphs

We define the vector space of (m, n)-graphs of $\mathfrak{G}^*$ decorated by an $\mathbb{S}$-bimodule $\mathscr{M}$ by

$$\mathfrak{G}^*\langle\mathscr{M}\rangle(m, n) := \bigoplus_{G\in\mathfrak{G}^*(m,n)} G\langle\mathscr{M}\rangle.$$

There is a natural $(\mathbb{S}_m, \mathbb{S}_n)$-bimodule structure on $\mathfrak{G}^*\langle\mathscr{M}\rangle(m, n)$ induced by the actions of $\mathbb{S}_m$ and $\mathbb{S}_n$ on G, $\sigma(G, [p_1 \otimes \cdots \otimes p_k])\tau := (\sigma G\tau, [p_1 \otimes \cdots \otimes p_k])$. Thus $\mathfrak{G}^*\langle\mathscr{M}\rangle(m, n)$ is naturally an $(\mathbb{S}_m, \mathbb{S}_n)$-bimodule. This lets us define the $\mathbb{S}$-bimodule of decorated graphs

$$\mathfrak{G}^*\langle\mathscr{M}\rangle := \{\mathfrak{G}^*\langle\mathscr{M}\rangle(m, n)\}_{m,n\in\mathbb{N}}.$$

1.2 Operads and generalizations via $\mathfrak{G}^*$-algebras

1.2.1 $\mathfrak{G}^*$-algebras

We are now ready to define the compositions in our generalized structures.

Let $\mu\colon \mathfrak{G}^*\langle\mathscr{M}\rangle \to \mathscr{M}$ be a homomorphism of $\mathbb{S}$-bimodules. We call such a morphism a *composition product* in $\mathscr{M}$. Denote by $\mu_G\colon G\langle\mathscr{M}\rangle \to \mathscr{M}$ the restriction of μ to $G\langle\mathscr{M}\rangle$. We will write $\mu_G(p_1 \otimes \cdots \otimes p_k)$ for $\mu((G, [p_1 \otimes \cdots \otimes p_k]))$.

Given an (r, s)-subgraph H of a graph G we define the morphism

$$\mu_H^G\colon G\langle\mathscr{M}\rangle \to G/H\langle\mathscr{M}\rangle$$

by

$$\mu_H^G(G, [\overline{p}_1 \otimes \cdots \otimes \overline{p}_k \otimes \overline{q}_1 \otimes \cdots \otimes \overline{q}_l]) := (G/H, [(\text{out}_H^{-1} \otimes_{\mathbb{S}_r} \mu_H(\overline{p}_1 \otimes \cdots \otimes \overline{p}_k) \otimes_{\mathbb{S}_s} \text{in}_H^{-1}) \otimes \overline{q}_1 \otimes \cdots \otimes \overline{q}_l]),$$

where $[\overline{p}_1 \otimes \cdots \otimes \overline{p}_k]$ is the decoration of H.

Definition. A $\mathfrak{G}^*$*-algebra* is an $\mathbb{S}$-bimodule $\mathscr{P}$ together with a composition product $\mu: \mathfrak{G}^*\langle\mathscr{P}\rangle \to \mathscr{P}$ satisfying the associativity condition

$$\mu_G = \mu_{G/H} \circ \mu_H^G \tag{1.1}$$

for each $G \in \mathfrak{G}^*$ and each $\mathfrak{G}^*$-admissible subgraph H of G.

Let μ_2 denote the restriction of μ to $\mathfrak{G}^*_{(2)}\langle\mathscr{M}\rangle$. We say that μ_2 is *associative* if it satisfies

$$\mu_{G/H_1} \circ \mu_{H_1}^G = \mu_{G/H_2} \circ \mu_{H_2}^G \tag{1.2}$$

for all $G \in \mathfrak{G}^*_{(3)}$ and admissible subgraphs $H_1, H_2 \in \mathfrak{G}^*_{(2)}$. We call μ_2 the *partial composition product* of $\mathscr{P}$.

Proposition 1.1. *Let* $\mathfrak{G}^*$ *be one of the subfamilies* (ii)-(vi) *of §1.1.4, then* $(\mathscr{P}, \mu)$ *is a* $\mathfrak{G}^*$*-algebra if and only if* μ_2 *is associative.*

Proof. If $(\mathscr{P}, \mu)$ is a $\mathfrak{G}^*$-algebra, then obviously μ_2 is associative. Conversely, suppose $\mathscr{P}$ is equipped with an associative morphism $\mu_2\colon \mathfrak{G}^*_{(2)}\langle\mathscr{P}\rangle \to \mathscr{P}$ then for a graph $G \in \mathfrak{G}^*_{(>2)}$ we define

$$\mu_G := \mu_{(\cdots(G/H_1)/\cdots/H_{k-1})} \circ \cdots \circ \mu_{H_2}^{G/H_1} \circ \mu_{H_1}^G,$$

where $H_1, \dots H_{k-1}$ is an arbitrary sequence of two-vertex graphs such that H_i is a $\mathfrak{G}^*$-admissible subgraph of $(\cdots(G/H_1)/\cdots/H_{i-1})$. Since μ is associative the definition of μ_G is independent of choice of sequence and the morphisms μ_G satisfy (1.1). □

1.2.2 $\mathfrak{G}^*$-coalgebras

Let $\Delta\colon \mathscr{M} \to \mathfrak{G}^*\langle\mathscr{M}\rangle$ be a homomorphism of $\mathbb{S}$-bimodules. We call such a morphism a *decomposition coproduct* in $\mathscr{M}$. Denote by ${}_G\Delta\colon \mathscr{M} \to G\langle\mathscr{M}\rangle$ the composition of Δ with the projection $\mathfrak{G}^*\langle\mathscr{M}\rangle \twoheadrightarrow G\langle\mathscr{M}\rangle$.

Given an (r,s)-subgraph H of a graph G we define the morphism

$${}^G_H\Delta\colon G/H\langle\mathscr{M}\rangle \to G\langle\mathscr{M}\rangle$$

by

$${}^G_H\Delta(G/H, [(\mathrm{out}_H^{-1} \otimes_{\mathbb{S}_r} \overline{p}_H \otimes_{\mathbb{S}_s} \mathrm{in}_H^{-1}) \otimes \overline{q}_1 \otimes \cdots \otimes \overline{q}_l]) := (G, [\overline{p}_1 \otimes \cdots \otimes \overline{p}_k \otimes \overline{q}_1 \otimes \cdots \otimes \overline{q}_l]),$$

where $(H, [\overline{p}_1 \otimes \cdots \otimes \overline{p}_k]) = \Delta_H(\overline{p}_H)$.

Definition. A $\mathfrak{G}^*$*-coalgebra* is an $\mathbb{S}$-bimodule $\mathscr{C}$ together with an $\mathbb{S}$-bimodule homomorphisms $\Delta: \mathscr{C} \to \mathfrak{G}^*\langle\mathscr{C}\rangle$ satisfying the coassociativity condition

$${}_G\Delta = {}^G_H\Delta \circ {}_{G/H}\Delta$$

for each $G \in \mathfrak{G}^*$ and $\mathfrak{G}^*$-admissible subgraph H of G.

As for $\mathfrak{G}^*$-algebras, for the subfamilies (ii)-(vi) of §1.1.4 a $\mathfrak{G}^*$-coalgebra can equivalently be defined by the coassociativity of the morphism $\Delta_2 := \pi_2 \circ \Delta$, where π_2 is the projection on $\mathfrak{G}^*_{(2)}\langle\mathscr{C}\rangle$;

$$ {}^{G}_{H_1}\Delta \circ {}_{G/H_1}\Delta = {}^{G}_{H_2}\Delta \circ {}_{G/H_2}\Delta $$

for all $G \in \mathfrak{G}^*_{(3)}$ and admissible subgraphs $H_1, H_2 \in \mathfrak{G}^*_{(2)}$. We call Δ_2 the *partial decomposition coproduct.*

1.2.3 Homomorphisms of $\mathfrak{G}^*$-(co)algebras

Let G be a graph in $\mathfrak{G}^*$ and v be a vertex of G. A homomorphism $\theta\colon \mathscr{M} \to \mathscr{N}$ of $\mathbb{S}$-bimodules canonically gives rise to a morphism $\theta_v\colon \mathscr{M}(E_v^{\text{out}}, E_v^{\text{in}}) \to \mathscr{N}(E_v^{\text{out}}, E_v^{\text{in}})$ by

$$ \theta_v(f \otimes_{\mathbb{S}_m} p \otimes_{\mathbb{S}_n} g) := f \otimes_{\mathbb{S}_m} \theta_{m,n}(p) \otimes_{\mathbb{S}_n} g. $$

We will write $\theta(p)$ for $\theta_v(f \otimes p \otimes g)$. This further extends to a morphism $\theta_G\colon G\langle\mathscr{M}\rangle \to G\langle\mathscr{N}\rangle$ by

$$ \theta_G(G, [p_1 \otimes \cdots \otimes p_k]) := (G, \theta(p_1) \otimes \cdots \otimes \theta(p_k)). $$

Finally this gives us a morphism of $\mathbb{S}$-bimodules $\theta_{\mathfrak{G}^*}\colon \mathfrak{G}^*\langle\mathscr{M}\rangle \to \mathfrak{G}^*\langle\mathscr{N}\rangle$.

Let $(\mathscr{P}, \mu_{\mathscr{P}})$ and $(\mathscr{Q}, \mu_{\mathscr{Q}})$ be $\mathfrak{G}^*$-algebras. A $\mathfrak{G}^*$*-algebra homomorphism* is a homomorphism of $\mathbb{S}$-bimodules $\theta\colon \mathscr{P} \to \mathscr{Q}$ such that for all decorated graphs $G \in \mathfrak{G}^*$ we have $\theta \circ (\mu_{\mathscr{P}})_G = (\mu_{\mathscr{Q}})_G \circ \theta_G$.

Let $(\mathscr{C}, \Delta_{\mathscr{C}})$ and $(\mathscr{D}, \Delta_{\mathscr{D}})$ be $\mathfrak{G}^*$-algebras. A $\mathfrak{G}^*$*-coalgebra homomorphism* is a homomorphism of $\mathbb{S}$-bimodules $\theta\colon \mathscr{C} \to \mathscr{D}$ such that for all decorated graphs $G \in \mathfrak{G}^*$ we have $\theta_G \circ {}_G\Delta_{\mathscr{C}} = {}_G\Delta_{\mathscr{D}} \circ \theta$.

1.2.4 $\mathfrak{G}^{\downarrow}$-(co)algebras versus $\mathfrak{G}^{\circlearrowleft}$-(co)algebras

Some notions related to $\mathfrak{G}^*$-(co)algebras allow simpler expositions when one forgets about $\mathfrak{G}^{\circlearrowleft}$-(co)algebras. Since we will only implicitly be needing $\mathfrak{G}^{\circlearrowleft}$-(co)algebras we avoid the subtleties related to them by restricting our attention to the strict subclasses of $\mathfrak{G}^{\circlearrowleft}$; from now on let $\mathfrak{G}^*$ be one of the subclasses (ii)-(vi) in §1.1.4. See e.g. [35, 42] for a treatment of $\mathfrak{G}^{\circlearrowleft}$-(co)algebras, also called wheeled props (without unit).

1.2.5 Graphs decorated by several $\mathbb{S}$-bimodules

Given a graph G with $|V_G| > 1$, we can decorate it with more than one $\mathbb{S}$-bimodule. Let $\mathscr{M}_1, \ldots, \mathscr{M}_l$ be $\mathbb{S}$-bimodules and let $V_G = V_1 \sqcup \cdots \sqcup V_l$ be a partition of the set of vertices of G. We define the vector space

$$ G\langle\mathscr{M}_1^{V_1}, \ldots \mathscr{M}_l^{V_l}\rangle := \bigotimes_{v \in V_G} \mathscr{M}_v, $$

where $\mathscr{M}_v = \mathscr{M}_i(E_v^{\text{out}}, E_v^{\text{in}})$ for $v \in V_i$.

Given morphisms of $\mathbb{S}$-bimodules $\theta_1 \colon \mathscr{M}_1 \to \mathscr{N}_1, \ldots, \theta_l \colon \mathscr{M}_l \to \mathscr{N}_l$ we define the morphism

$$(\theta_1^{V_1}, \ldots, \theta_l^{V_l}) \colon G\langle \mathscr{M}_1^{V_1}, \ldots \mathscr{M}_l^{V_l} \rangle \to G\langle \mathscr{N}_1^{V_1}, \ldots \mathscr{N}_l^{V_l} \rangle$$

by

$$(G, [p_1 \otimes \cdots \otimes p_l]) \mapsto (G, [\theta_{i_1}(p_1) \otimes \cdots \otimes \theta_{i_k}(p_k)]),$$

where $\theta_{i_j} = \theta_r$ when $p_j \in \mathscr{M}_r$.

1.2.6 Units and counits

We define the $\mathbb{S}$-bimodule I by

$$\begin{cases} I(1,1) = \mathbb{K} \\ I(m,n) = 0 \quad \text{for } (m,n) \neq (1,1) \end{cases}.$$

Let G be a (m,n)-graph satisfying $|E_u^{\text{in}}| = |E_u^{\text{out}}| = 1$ for all vertices $u \in V_G$ except for one vertex v which then satisfies $|E_v^{\text{in}}| = n$ and $|E_v^{\text{out}}| = m$. The maps in_G and out_G naturally induce maps $\widetilde{\text{in}}_G \colon [n] \to E_v^{\text{in}}$ and $\widetilde{\text{out}}_G \colon E_v^{\text{out}} \to [m]$.

Let $\mathscr{M}$ be an $\mathbb{S}$-bimodule. There exists a natural isomorphism

$$G\langle I^{V_G \setminus \{v\}}, \mathscr{M}^{\{v\}} \rangle \xrightarrow{\sim} \mathscr{M}(m,n)$$

defined by

$$(G, [\overline{c}_1 \otimes \cdots \otimes \overline{c}_{k-1} \otimes \overline{p}]) \mapsto (c_1 \cdots c_{k-1}) \sigma^{-1} p \tau^{-1},$$

where $\sigma \in \mathbb{S}_m$ and $\tau \in \mathbb{S}_n$ are permutations such that for $\overline{p} = f \otimes_{\mathbb{S}_m} p \otimes_{\mathbb{S}_n} g \in \mathscr{M}(E_v^{\text{out}}, E_v^{\text{in}})$ we have $(\tau g) \circ \widetilde{\text{in}}_G = \text{Id}_{[n]}$ and $\widetilde{\text{out}}_G \circ (f\sigma) = \text{Id}_{[m]}$.

Let μ be a composition product in $\mathscr{P}$ and let $\eta \colon I \to \mathscr{P}$ be an $\mathbb{S}$-bimodule homomorphism. We say that η is a *unit* with respect to μ if the following diagram commutes for all $m, n \in \mathbb{N}$ and $G \in \mathfrak{G}^*(m,n)$ of the above type:

$$\begin{array}{ccc} G\langle I^{V_G \setminus \{v\}}, \mathscr{P}^{\{v\}} \rangle & \xrightarrow{(\eta^{V_G \setminus \{v\}}, \text{Id}^{\{v\}}_{\mathscr{P}(m,n)})} & G\langle \mathscr{P} \rangle \\ & \searrow^{\sim} & \downarrow{\scriptstyle \mu_G} \\ & & \mathscr{P}(m,n) \end{array}.$$

We denote the element $\eta(1) \in \mathscr{P}(1,1)$ by $\mathbb{1}$. The above condition is then equivalent to that, for all G as above, the morphism μ_G satisfies

$$\mu_G(\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes (f \otimes_{\mathbb{S}_m} p \otimes_{\mathbb{S}_n} g) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) = \sigma^{-1} p \tau^{-1}. \tag{1.3}$$

On the coside, let Δ be a decomposition coproduct on an $\mathbb{S}$-bimodule $\mathscr{C}$ and let $\epsilon\colon \mathscr{C} \to I$ be an $\mathbb{S}$-bimodule homomorphism. We say that ϵ is a *counit* with respect to Δ if the following diagram commutes for all $m, n \in \mathbb{N}$ and $G \in \mathfrak{G}^*(m,n)$ of the above type:

$$\begin{array}{ccc} \mathscr{C}(m,n) & \xrightarrow{\ {}_G\Delta\ } & G\langle\mathscr{C}\rangle \\ & \searrow^{\sim} & \Big\downarrow{\scriptstyle (\epsilon^{V_G\setminus\{v\}},\mathrm{Id}^{\{v\}}_{\mathscr{C}(m,n)})} \\ & & G\langle I^{V_G\setminus\{v\}}, \mathscr{C}^{\{v\}}\rangle \end{array} \quad .$$

1.2.7 (Co)unital $\mathfrak{G}^*$-(co)algebras

Let $(\mathscr{P}, \mu)$ be a $\mathfrak{G}^*$-algebra. If there exists a morphism $\eta\colon I \to \mathscr{P}$, which is a unit with respect to μ, we call the data $(\mathscr{P}, \mu, \eta)$ a *unital* $\mathfrak{G}^*$*-algebra*.

Let $(\mathscr{C}, \Delta)$ be a $\mathfrak{G}^*$-coalgebra. If there exists a morphism $\epsilon\colon \mathscr{C} \to I$, which is a counit with respect to Δ, we call the data $(\mathscr{C}, \Delta, \epsilon)$ a *counital* $\mathfrak{G}^*$*-coalgebra*.

Definition. We have

(i) a (co)unital $\mathfrak{G}^{\downarrow}$-(co)algebra is called a *(co)prop*,

(ii) a (co)unital $\mathfrak{G}^{\downarrow}_c$-(co)algebra is called a *(co)properad*,

(iii) a (co)unital $\mathfrak{G}^{\downarrow}_{c,0}$-(co)algebra is called a *(co)dioperad*,

(iv) a (co)unital $\mathfrak{G}^{\downarrow_1}_c$-(co)algebra such that $M(m,n) = 0$ if $m \neq 1$ is called an *(co)operad*,

(v) a (co)unital $\mathfrak{G}^{\downarrow^1_1}_c$-(co)algebra such that $M(m,n) = 0$ if $m,n \neq 1$ is called a(n) *(co)associative (co)algebra*.

In Section 1.3 we show how the above definitions relate to the classical ones.

Let $(\mathscr{P}, \mu_{\mathscr{P}}, \eta_{\mathscr{P}})$ and $(\mathscr{Q}, \mu_{\mathscr{Q}}, \eta_{\mathscr{Q}})$ be $\mathfrak{G}^*$-algebras. A *homomorphism of unital* $\mathfrak{G}^*$*-algebras* is a $\mathfrak{G}^*$-algebra homomorphism $\theta\colon \mathscr{P} \to \mathscr{Q}$ such that $\eta_{\mathscr{Q}} = \theta \circ \eta_{\mathscr{P}}$.

Let $(\mathscr{C}, \Delta_{\mathscr{C}}, \epsilon_{\mathscr{C}})$ and $(\mathscr{D}, \Delta_{\mathscr{D}}, \epsilon_{\mathscr{D}})$ be $\mathfrak{G}^*$-coalgebras. A *homomorphism of counital* $\mathfrak{G}^*$*-coalgebras* is a homomorphism of $\mathbb{S}$-bimodules $\theta\colon \mathscr{C} \to \mathscr{D}$ such that $\epsilon_{\mathscr{C}} = \epsilon_{\mathscr{D}} \circ \theta$.

1.2.8 The endomorphism $\mathfrak{G}^*$-algebra

We define the endomorphism $\mathfrak{G}^*$-algebra $\mathscr{E}nd^*_V$ of a vector space V by $\mathscr{E}nd^*_V(m,n) := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$. The $(\mathbb{S}_m, \mathbb{S}_n)$ action is given by permuting the input and output. For a graph $G \in \mathfrak{G}^*$, the composition product $\mu_G\colon G\langle\mathscr{E}nd^*_V\rangle \to \mathscr{E}nd^*_V$ is defined as the composition of multivariate functions according to G. The local labelings of the vertices dictate, in an obvious way, which output is to be plugged into which input of functions decorating adjacent vertices. The global labeling plays a similar role. A unit $\eta\colon I \to \mathcal{E}nd_V^*$ is given by $\eta(1) := \mathrm{Id}_V$. We will usually suppress the $*$ from the notation.

1.2.9 Representations of $\mathfrak{G}^*$-algebras

A *representation* of a $\mathfrak{G}^*$-algebra $\mathcal{P}$ in a vector space V is a homomorphism $\rho\colon \mathcal{P} \to \mathcal{E}nd_V$ of $\mathfrak{G}^*$-algebras. We say that ρ gives V the structure of a $\mathcal{P}$-*algebra*.

We can think of a $\mathcal{P}$-algebra structure as an assignment of multilinear operations on V, possibly with several inputs and outputs, satisfying axioms encoded by structure of decorated graphs and the composition product in $\mathcal{P}$.

1.3 Relation to classical definitions

1.3.1 Operads as monoids

By endowing the category of $\mathbb{S}$-modules with a monoidal product one can define operads as monoids in the resulting monoidal category. We define the monoidal product in the category of $\mathbb{S}$-modules by:

$$\mathcal{M} \circ \mathcal{N}(n) = \bigoplus_{1\le k\le n} \left(\bigoplus_{i_1+\cdots+i_k=n} \mathcal{M}(k) \otimes (\mathcal{N}(i_1)\otimes\cdots\otimes\mathcal{N}(i_k)) \otimes_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_k},$$

where we consider the coinvariants with respect to the action of $\mathbb{S}_k$ given by $(v\otimes(w_1\otimes\cdots\otimes w_k)\otimes\sigma)\tau = (v\tau\otimes(w_{\tau(1)}\otimes\cdots\otimes w_{\tau(k)})\otimes\bar{\tau}^{-1}\sigma)$ and $\bar{\tau}$ is the induced block permutation. A unit I with respect to this product is given by the $\mathbb{S}$-module defined by

$$I_n := \begin{cases} \mathbb{1}_1 & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \end{cases}.$$

Definition. An *operad* is a monoid $(\mathcal{P}, \mu\colon \mathcal{P}\circ\mathcal{P} \to \mathcal{P}, \eta\colon I \to \mathcal{P})$ in the monoidal category ($\mathbb{S}$-modules, $\circ$, I).

For an element $(p\otimes(p_1\otimes\cdots\otimes p_k)\otimes\sigma) \in \mathcal{P}\circ\mathcal{P}$ we will suppress the permutation σ and denote $\mu(p\otimes(p_1\otimes\cdots\otimes p_k))$ by $\mu(p; p_1,\ldots,p_k)$.

Operads were originally defined by P. May in [37] where the axioms contained in the assertion that $(\mathcal{P},\mu,\eta)$ is a monoid were spelled out explicitly.

1.3.2 The monoidal product via two-level graphs

Let $\mathfrak{G}^*$ be one of the subfamilies (ii)-(vi) in §1.1.4. For a graph $G \in \mathfrak{G}^*$ the direction of the edges induces a partial order on V_G; the covering relation is defined by $v_1 \prec v_2$ if there is an edge $e \in E_G$ with $\Phi_G(e) = (v_2, v_1)$. An *n-level graph* is a graph G such that V_G is a ranked poset (i.e. all maximal chains are of the same length) of rank n. For a two-level graph, let V_u denote the vertices on the upper level and V_d the vertices on the lower level.

The monoidal product of the previous paragraph can be expressed in terms of decorated two-level graphs; we have

$$\mathcal{M} \circ \mathcal{N} = \oplus_{G \in \mathfrak{G}_c^{\downarrow 1,2}} G\langle \mathcal{M}^{\{v\}}, \mathcal{N}^{V_G \setminus \{v\}} \rangle,$$

where v is the vertex on the lower level.

1.3.3 Pseudo-operads

A pseudo-operad is the data $(\mathcal{P} = \{\mathcal{P}(n)\}_{n\in\mathbb{N}}, \{\circ_i^{n_1,n_2}\}_{1\le i\le n_1}^{n_1,n_2\in\mathbb{N}})$, where $\mathcal{P}$ is an $\mathbb{S}$-module and the maps

$$\circ_i^{n_1,n_2} : \mathcal{P}(n_1) \otimes \mathcal{P}(n_2) \to \mathcal{P}(n_1 + n_2 - 1)$$

satisfy certain associativity and $\mathbb{S}$-equivariance axioms, see e.g. [36]. The operation $\circ_i^{n_1,n_2}$ is usually denoted by $\circ_i$. It is called the *partial composition product.*

Proposition 1.2. *An $\mathbb{S}$-module $\mathcal{P}$ is a pseudo-operad if and only if it is a $\mathfrak{G}_c^{\downarrow 1}$-algebra.*

Proof. Let $(\mathcal{P}, \mu)$ be a $\mathfrak{G}_c^{\downarrow 1}$-algebra. We can give $\mathcal{P}$ a pseudo-operad structure as follows. Let $G \in \mathfrak{G}_c^{\downarrow 1}$ be the two-vertex graph depicted in Figure 1.3. We then define $p_1 \circ_i^{n_1,n_2} p_2 := \mu_G((p_1 \otimes_{\mathbb{S}} g_1) \otimes (p_2 \otimes_{\mathbb{S}} g_2))$, where p_1 and p_2 are

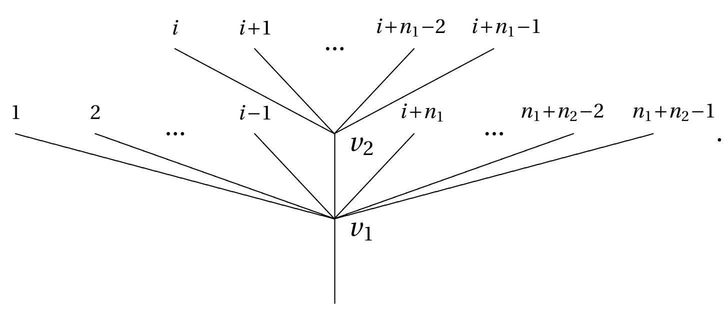


Figure 1.3: A two-vertex graph in $\mathfrak{G}_c^{\downarrow 1}$.

decorating v_1 and v_2, respectively, and g_1 and g_2 are labelings satisfying

$$\begin{cases} g_1 \circ \mathrm{in}_G(1) = 1, \dots, g_1 \circ \mathrm{in}_G(i-1) = i-1 \\ g_2 \circ \mathrm{in}_G(i) = 1, \dots, g_2 \circ \mathrm{in}_G(i+n_2-1) = n_2 \\ g_1 \circ \mathrm{in}_G(i+n_2) = i+1, \dots, g_1 \circ \mathrm{in}_G(n_1+n_2-1) = n_1. \end{cases}$$

The condition $\mu_{G/H_1} \circ \mu^G_{H_1} = \mu_{G/H_2} \circ \mu^G_{H_2}$ for all pairs of admissible two-vertex subgraphs of a three vertex graph $G \in \mathfrak{G}^{\downarrow 1}_c$ together with the $\mathbb{S}$-equivariance of μ and the structure of decorated graphs are equivalent to that the operations $\circ^{n_1,n_2}_i$ satisfy the associativity and $\mathbb{S}$-equivariance axioms of a pseudo-operad.

Conversely, if $\mathscr{P}$ has a pseudo-operad structure then we can define a $\mathfrak{G}^{\downarrow 1}_c$-algebra structure on $\mathscr{P}$ by letting μ_G, for G a two-vertex graph, be given by the appropriate $\circ^{n_1,n_2}_i$ as above. By Proposition 1.1 we are done.

□

1.3.4 Pseudo-operads and operads

Any pseudo-operad $\mathscr{P}$ is a non-unital operad with composition product

$$\mu(p; p_1, \dots, p_k) := (\cdots (p \circ_1 p_1) \circ_{n_1+1} p_2 \cdots) \circ_{n_1+\cdots n_{k-1}+1} p_k,$$

where $p_i \in \mathscr{P}(n_i)$. A unit with respect to the pseudo-operad structure is a unit with respect to the operad structure.

Conversely, any operad is a unital pseudo-operad with partial composition product

$$p_1 \circ^{n_1,n_2}_i p_2 := \mu(p_1; \eta(1), \dots, \eta(1), p_2, \eta(1), \dots, \eta(1)),$$

where p_2 is at the ith place, and with η serving as unit.

1.3.5 Dioperads, properads, and props

In [18] dioperads were defined as monoids in the category of $\mathbb{S}$-bimodules with monoidal product

$$\mathscr{P} \square_c \mathscr{Q} = \sum_{G \in \mathfrak{G}^{\downarrow,2}_{c,0}} G\langle \mathscr{P}^{V_d}, \mathscr{Q}^{V_u} \rangle.$$

They were also defined as $\mathbb{S}$-bimodules endowed with operations ${}_i\circ_j$ which can be thought of as plugging the jth input of an element into the ith input of another element. As for operads these definitions are equivalent only in the unital setting, so without unit the latter structure should be a called pseudo-dioperad.

Properads were defined in [56] as monoids with respect to the monoidal product $\boxtimes_c$ given by

$$\mathscr{P} \boxtimes_c \mathscr{Q} = \sum_{G \in \mathfrak{G}_c^{\downarrow,2}} G\langle \mathscr{P}^{V_d}, \mathscr{Q}^{V_u} \rangle.$$

Props were originally defined in [31], where they were called PROPs, in terms of two operations $\circ$ and $\otimes$. The first one corresponding to composition along two-vertex graphs with the number of output edges of the upper vertex equal to the number of input edges of the lower and all of them connected. The second operation corresponds to composition according to disconnected two-vertex graphs. Also props can be defined using a product $\boxtimes$, which is the sum of decorated two-level graphs of $\mathfrak{G}^{\downarrow}$, but they are not monoids with respect to this product, c.f. [56, 45].

2 Resolutions via Koszul duality

In this chapter we give definitions of $\mathfrak{G}^*$-algebras presented by generators and relations. To this end we describe the free $\mathfrak{G}^*$-algebra. We also set up the differential graded framework and describe two kinds of resolutions of $\mathfrak{G}^*$-algebras. One kind of resolution is based on an extension of the Koszul duality theory for associative algebras to $\mathfrak{G}^*$-algebras. As the absence of wheels in directed graphs makes a more accessible presentation possible, we restrict our attention to the strict subfamilies of $\mathfrak{G}^{\circlearrowleft}$, i.e. in this chapter $\mathfrak{G}^*$ denotes one of the subfamilies (ii)-(vi) defined in §1.1.4.

This chapter contains no new material; we merely wish to express the results we need from [20],[16],[18] and [56] in the unifying language of [41].

2.1 Differential graded framework

2.1.1 Differential graded $\mathbb{S}$-bimodules

A *graded* $\mathbb{S}$*-bimodule* is an $\mathbb{S}$-bimodule $\mathcal{M}$ which can be decomposed as $\mathcal{M}(m,n) := \bigoplus_{i\in\mathbb{Z}} \mathcal{M}(m,n)^i$. We denote by $\mathcal{M}^i$ the collection $\{\mathcal{M}(m,n)^i\}_{m,n\in\mathbb{N}}$. For an element $p \in \mathcal{M}^i$ we write $|p| = i$, and say that p is of degree i. We will refer to this degree as the *cohomological degree*.

A *homomorphism* $\theta\colon \mathcal{M} \to \mathcal{N}$ *of graded* $\mathbb{S}$*-bimodules* of degree j is a homomorphism of $\mathbb{S}$-bimodules satisfying $\theta(\mathcal{M}^i) \subset \mathcal{N}^{i+j}$.

A *differential graded* (dg) $\mathbb{S}$*-bimodule* is a pair $(\mathcal{M}, d)$, where $\mathcal{M}$ is a graded $\mathbb{S}$-bimodule and $d\colon \mathcal{M} \to \mathcal{M}$ is a homomorphism of graded $\mathbb{S}$-bimodules of degree one satisfying $d^2 = 0$.

A *homomorphism* θ *of dg* $\mathbb{S}$*-bimodules* is a degree zero homomorphism of graded $\mathbb{S}$-bimodules satisfying $d \circ \theta = \theta \circ d$.

In the differential graded framework we apply the Koszul-Quillen sign rules; whenever a symbol of degree a is moved past a symbol of degree b the sign $(-1)^{ab}$ is introduced, e.g. for graphs decorated with graded $\mathbb{S}$-bimodules we have

$$(G, [p_1 \otimes \cdots \otimes p_i \otimes p_{i+1} \otimes \cdots \otimes p_k]) = \\ (G, [(-1)^{|p_i||p_{i+1}|} p_1 \otimes \cdots \otimes p_{i+1} \otimes p_i \otimes \cdots \otimes p_k]).$$

2.1.2 Differential graded $\mathfrak{G}^*$-algebras and $\mathfrak{G}^*$-coalgebras

The differential d of a dg $\mathbb{S}$-bimodule $\mathscr{M}$ extends to a differential d^G on the vector space $G\langle\mathscr{M}\rangle$ defined by

$$d^G(G,[p_1\otimes\cdots\otimes p_k]) := (G,[\sum_{i=1}^{k}(-1)^{|p_1|+\cdots|p_{i-1}|}p_1\otimes\cdots\otimes d(p_i)\otimes\cdots\otimes p_k]).$$

The grading of $\mathscr{M}$ induces a grading on $G\langle\mathscr{M}\rangle$ given by $|(G,[p_1\otimes\cdots\otimes p_k])| = |p_1|+\cdots+|p_k|$. Together this makes $\mathfrak{G}^*\langle\mathscr{M}\rangle$ into a dg $\mathbb{S}$-bimodule.

Definition. A *dg* $\mathfrak{G}^*$*-algebra* is a triple $((\mathscr{P},d),\mu,\eta)$ where $(\mathscr{P},\mu,\eta)$ is a $\mathfrak{G}^*$-algebra, $(\mathscr{P},d)$ is a dg $\mathbb{S}$-bimodule, and μ is a morphism of dg $\mathbb{S}$-bimodules.

Explicitly, the condition that μ is a morphism of dg $\mathbb{S}$-bimodules is given by $d\mu_G = \mu_G d^G$ for all $G\in\mathfrak{G}^*$. We say that a morphism $d\colon\mathscr{P}\to\mathscr{P}$ is a $\mathfrak{G}^*$*-algebra derivation* if this condition is satisfied.

Definition. A *dg* $\mathfrak{G}^*$*-coalgebra* is a triple $((\mathscr{C},d),\Delta,\epsilon)$ where $(\mathscr{C},\Delta,\epsilon)$ is a $\mathfrak{G}^*$-coalgebra, $(\mathscr{C},d)$ is a dg $\mathbb{S}$-bimodule, and Δ is a morphism of dg $\mathbb{S}$-bimodules.

The last condition can be expressed by ${}_G\Delta d = d^G{}_G\Delta$ for all $G\in\mathfrak{G}^*$. We call an $\mathbb{S}$-bimodule homomorphism $d\colon\mathscr{C}\to\mathscr{C}$ a $\mathfrak{G}^*$*-coalgebra coderivation* if it satisfies this condition.

A *morphism of dg* $\mathfrak{G}^*$*-(co)algebras* is a morphism of $\mathfrak{G}^*$-(co)algebras which also is a morphism of dg $\mathbb{S}$-bimodules.

2.1.3 Representations in the dg framework

Let (V,d) be a dg vector space and let $\mathscr{P}$ be a dg $\mathfrak{G}^*$-algebra. The differential of V induces a differential on $\mathscr{E}nd_V$ which we also denote by d. For $f\in\mathscr{E}nd_V(m,n)$ it is defined by

$$d(f) = \sum_{i=1}^{m} d_1\circ_i f - (-1)^{|f|}\sum_{i=1}^{n} f_i\circ_1 d$$

Since the differential of a vector space usually is considered as part of the data, as in the cases of dg associative and dg Lie algebras for instance, we define a *representation* of $\mathscr{P}$ in V to be a pair (ρ,d), where d is a differential on V and $\rho\colon\mathscr{P}\to\mathscr{E}nd_V$ is a morphism of dg $\mathfrak{G}^*$-algebras.

2.1.4 Suspension and desuspension

The *suspension* $\Sigma\mathscr{M}$ of a dg $\mathbb{S}$-bimodule $\mathscr{M}$ is defined as $(\Sigma\mathscr{M})(m,n) := \mathbb{K}s\otimes\mathscr{M}(m,n)$, where s is an element of degree 1. We define the *desuspension* $\Sigma^{-1}\mathscr{M}$ by $(\Sigma^{-1}\mathscr{M})(m,n) := \mathbb{K}s^{-1}\otimes\mathscr{M}(m,n)$, where s^{-1} is an element of degree -1. Thus $(\Sigma\mathscr{M})^i = \mathscr{M}^{i-1}$ and $(\Sigma^{-1}\mathscr{M})^i = \mathscr{M}^{i+1}$.

2.1.5 Weight graded $\mathbb{S}$-bimodules and $\mathfrak{G}^*$-(co)algebras

We will need to consider an extra grading on the objects we study. We call a dg $\mathbb{S}$-bimodule $\mathscr{M}$ *weight graded* if it has a decomposition $\mathscr{M} = \bigoplus_{s\in\mathbb{N}} \mathscr{M}_{(s)}$, where each $\mathscr{M}_{(s)}$ is a dg sub-$\mathbb{S}$-bimodule. This is an extra grading which differs from the cohomological degree in that it does not effect signs, i.e. the Koszul-Quillen sign rules only apply to cohomological degree. We call $\mathscr{M}_{(s)}$ the weight s part of $\mathscr{M}$. The tensor product of weight graded $\mathbb{S}$-bimodules inherits a weight grading by $(\mathscr{M}\otimes\mathscr{N})_{(t)} = \bigoplus_{r+s=t}\mathscr{M}_{(r)}\otimes\mathscr{N}_{(s)}$.

We call a $\mathfrak{G}^*$-algebra $(\mathscr{P},\mu,\eta)$ *weight-graded* if $\mathscr{P}$ is a weight graded $\mathbb{S}$-bimodule and μ preserves the weight grading. Note that we necessarily have $\eta(I)\subset\mathscr{P}_{(0)}$.

Similarly we call a $\mathfrak{G}^*$-coalgebra $(\mathscr{C},\Delta,\epsilon)$ *weight graded* if $\mathscr{C}$ is a weight graded $\mathbb{S}$-bimodule and Δ preserves the weight grading.

2.1.6 Connected $\mathfrak{G}^*$-(co)algebras

We call an $\mathbb{S}$-bimodule *connected* if $\mathscr{M}(m,0)=0$ for all m, $\mathscr{M}(0,n)=0$ for all n, and $\mathscr{M}(1,1)=\mathbb{K}$.

A weight graded $\mathbb{S}$-bimodule $\mathscr{M}$ is *connected* if $\mathscr{M}$ is connected as an $\mathbb{S}$-bimodule, $\mathscr{M}_{(0)}(1,1)=\mathbb{K}$, and $\mathscr{M}_{(0)}(m,n)=0$ for other m,n.

We call a (weight graded) $\mathfrak{G}^*$-(co)algebra *connected* if the underlying $\mathbb{S}$-bimodule is connected.

2.1.7 (Co)ideals

An *ideal* of a $\mathfrak{G}^*$-algebra $\mathscr{P}$ is a sub-$\mathbb{S}$-bimodule $\mathscr{J}$ satisfying $\mu_G(p_1\otimes\cdots\otimes p_k)\in\mathscr{J}$ whenever at least one of the p_i is in $\mathscr{J}$. We denote the ideal generated by a subset $J\subset\mathscr{P}$ by (J).

Let $\mathscr{P}$ be a $\mathfrak{G}^*$-algebra and $\mathscr{J}$ be an ideal of $\mathscr{P}$. The *quotient* $\mathfrak{G}^*$*-algebra* $\mathscr{P}/\mathscr{J}$ is defined by $\mathscr{P}/\mathscr{J}(m,n) := \mathscr{P}(m,n)/\mathscr{J}(m,n)$. If $\mathscr{P}$ is weight graded and the ideal $\mathscr{J}$ is homogeneous with respect to this weight grading, i.e. $\mathscr{J}=\bigoplus_{s\in\mathbb{N}}\mathscr{J}_{(s)}$ and $\mathscr{J}_{(s)}=\mathscr{J}\cap\mathscr{P}_{(s)}$, then the quotient $\mathscr{P}/\mathscr{J}$ inherits a weight grading from $\mathscr{P}$.

A *coideal* of a $\mathfrak{G}^*$-coalgebra $\mathscr{C}$ is a sub-$\mathbb{S}$-bimodule $\mathscr{J}$ such that

$$\Delta_G(\mathscr{J}) \subset \bigoplus_{v\in V_G} G\langle \mathscr{C}^{V_G\setminus\{v\}}, \mathscr{J}^{\{v\}}\rangle.$$

2.1.8 (Co)augmented $\mathfrak{G}^*$-(co)algebras

We can give the $\mathbb{S}$-bimodule I, with

$$\begin{cases} I(1,1) = \mathbb{K} \\ I(m,n) = 0 \quad \text{for } (m,n) \neq (1,1), \end{cases}$$

a $\mathfrak{G}^*$-algebra structure by defining the composition product $\mu_G(c_1 \otimes \cdots \otimes c_k) := c_1 \cdots c_k$, the product of the scalars, the unit η being the identity $I \to I$.

An *augmentation* of a $\mathfrak{G}^*$-algebra $\mathscr{P}$ is a morphism of $\mathfrak{G}^*$-algebras $\epsilon \colon \mathscr{P} \to I$. We define the *augmentation ideal* of $\mathscr{P}$ by $\bar{\mathscr{P}}(m,n) := \ker(\epsilon_{m,n})$.

Note that a connected $\mathfrak{G}^*$-algebra $\mathscr{P}$ has a canonical augmentation which sends $\mathscr{P}(1,1)$ to $I(1,1)$ and is zero otherwise.

We can also give the $\mathbb{S}$-bimodule I a $\mathfrak{G}^*$-coalgebra structure by ${}_G\Delta(c) := (G, [c \cdot 1 \otimes \cdots \otimes 1])$, the counit ϵ being the identity $I \to I$.

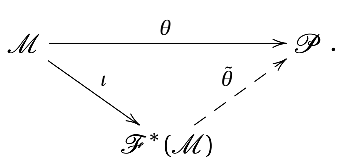
A *coaugmentation* of a $\mathfrak{G}^*$-coalgebra $\mathscr{C}$ is a morphism of $\mathfrak{G}^*$-coalgebras $\eta \colon I \to \mathscr{C}$. We define the *coaugmentation coideal* of $\mathscr{C}$ by $\bar{\mathscr{C}}(m,n) := \operatorname{coker}(\eta_{m,n})$.

Also note that a connected $\mathfrak{G}^*$-coalgebra is coaugmented; the coaugmentation maps $I(1,1)$ to $\mathscr{C}(1,1)$.

2.2 Bar and cobar constructions

2.2.1 Free $\mathfrak{G}^*$-algebras

The free $\mathfrak{G}^*$-algebra $\mathscr{F}^*(\mathscr{M})$ on an $\mathbb{S}$-bimodule $\mathscr{M}$ is characterized by the classical universal property; there exists an inclusion $\iota \colon \mathscr{M} \to \mathscr{F}^*(\mathscr{M})$ such that given any homomorphism of $\mathbb{S}$-bimodules $\theta \colon \mathscr{M} \to \mathscr{P}$ there is a unique homomorphism of $\mathfrak{G}^*$-algebras $\tilde{\theta}$ making the following diagram commute:



Here we give an explicit construction. The free non-unital $\mathfrak{G}^*$-algebra, $\mathscr{F}^*(\mathscr{M})$, on an $\mathbb{S}$-bimodule $\mathscr{M}$ has $\mathfrak{G}^*\langle \mathscr{M} \rangle$ as underlying $\mathbb{S}$-bimodule. The composition product $\mu \colon \mathfrak{G}^*\langle \mathscr{F}^*(\mathscr{M}) \rangle \to \mathscr{F}^*(\mathscr{M})$ maps a graph decorated with graphs decorated with $\mathscr{M}$ to a graph decorated with $\mathscr{M}$. Intuitively we may thing of this composition product as grafting the external edges of the decorating graphs together according to the internal edges of the graph they decorate, leaving the decoration by $\mathcal{M}$ unchanged, except for a minor modification of the internal labeling.

To be more precise, for a graph $G \in \mathfrak{G}^*$, the morphism μ_G maps

$$(G,[\overline{(G_1,[\overline{p}^1_1\otimes\cdots\otimes\overline{p}^1_{k_l}])}\otimes\cdots\otimes\overline{(G_k,[\overline{p}^k_1\otimes\cdots\otimes\overline{p}^k_{l_k}])}])\in G\langle(\mathfrak{G}^*\langle M\rangle)\rangle$$

to

$$(G(G_1,\dots,G_k),[\widetilde{p}^1_1\otimes\cdots\otimes\widetilde{p}^k_{l_k}])\in G(G_1,\dots,G_k)\langle M\rangle,$$

where $G(G_1,\dots,G_k)$ is the result of the grafting and $\widetilde{p}^a_b$ is equal to $\overline{p}^a_b$ up to a modification of the labeling to keep track of how the p^a_b connect to the grafted graph. We describe in detail the graph $G(G_1,\dots G_k)$ as well as the modification of the labeling in §A of the appendix.

Since it does not matter in which order we graft the edges, the associativity condition $\mu_G = \mu_{G/H}\circ\mu^G_H$ is immediate.

To define a unit of $\mathcal{F}^*(\mathcal{M})$ we have to add a special graph, $|$, to $\mathfrak{G}^*$ consisting of a single edge and no vertices. The space of decorations is defined as $|\langle\mathcal{M}\rangle := \mathbb{K}$, in analogy with the tensor product of zero factors. We define the grafting $G(|,\dots,|,G',|,\dots,|) := \sigma^{-1}G'\tau^{-1}$, where σ, τ, and G are defined as in (1.3) in §1.2.6 and G' is an (m,n)-graph. The unit is then defined by $\mathbb{1} := (|,[1])$.

The inclusion $\iota\colon \mathcal{M}\to\mathcal{F}^*(\mathcal{M})$ is defined as follows, for an element $p\in\mathcal{M}(m,n)$, its image $\iota(p)$ is the decorated one vertex (m,n)-graph $(G,[f\otimes_{\mathbb{S}} p\otimes_{\mathbb{S}} g])$ such that $g\circ \mathrm{in}_G = \mathrm{Id}_{[n]}$ and $\mathrm{out}_G\circ f = \mathrm{Id}_{[m]}$.

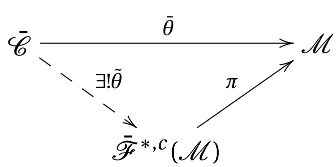
We will usually omit the $*$ and denote a free $\mathfrak{G}^*$-algebra simply by $\mathcal{F}(\mathcal{M})$ when it is clear which family of graphs we consider.

The free $\mathfrak{G}^*$-algebra has a natural weight grading by the number of vertices of a decorated graph, $\mathcal{F}^*(\mathcal{M}) = \bigoplus_{s\in\mathbb{N}}\mathcal{F}^*_{(s)}(\mathcal{M})$, where $\mathcal{F}^*_{(s)}(\mathcal{M}) := \mathfrak{G}^*_{(s)}\langle\mathcal{M}\rangle$. It is connected since $\mathcal{F}^*_{(0)}(\mathcal{M}) = |\langle\mathcal{M}\rangle$ and therefore also augmented. The augmentation ideal is given by $\bar{\mathcal{F}}^*\langle\mathcal{M}\rangle = \bigoplus_{s\geq 1}\mathfrak{G}^*_{(s)}\langle\mathcal{M}\rangle$.

2.2.2 Cofree connected $\mathfrak{G}^*$-coalgebras

A morphism $\theta\colon \mathcal{C}\to\mathcal{D}$ of connected $\mathfrak{G}^*$-coalgebras is completely determined by the restriction $\bar{\theta}$ to the coaugmentation coideal of $\mathcal{C}$.

The cofree connected $\mathfrak{G}^*$-coalgebra on an $\mathbb{S}$-bimodule $\mathcal{M}$ is characterized by the universal property obtained by reversing all arrows in the diagram characterizing free $\mathfrak{G}^*$-algebras.



Its underlying $\mathbb{S}$-bimodule is also $\mathfrak{G}^*\langle\mathcal{M}\rangle$. The decomposition product Δ is defined as follows. For a decorated graph $X = (\widetilde{G}, [p_1 \otimes \cdots \otimes p_k])$ the image of X under ${}_G\Delta$ is the sum over all decorated graphs $Y = (G, [(G_1, [p_1^1 \otimes \cdots \otimes p_{k_l}^1]) \otimes \cdots \otimes (G_k, [p_1^k \otimes \cdots \otimes p_{l_k}^k])])$ such that $\mu(Y) = X$ in the free $\mathfrak{G}^*$-algebra on M. The counit is given by $\epsilon\colon (|, [1]) \to 1$ and zero otherwise.

Remark. Note that this $\mathfrak{G}^*$-coalgebra satisfies the universal property only in the category of connected $\mathfrak{G}^*$-coalgebras. Note also that there exists a weaker notion of connected $\mathfrak{G}^*$-coalgebras, c.f. [45].

2.2.3 Derivations of free $\mathfrak{G}^*$-algebras

Let $\mathcal{F}^*(\mathcal{M})$, be the free $\mathfrak{G}^*$-algebra on an $\mathbb{S}$-bimodule $\mathcal{M}$ and let $\theta\colon \mathcal{M} \to \mathcal{F}^*(\mathcal{M})$ be an $\mathbb{S}$-bimodule homomorphism. Such a morphism θ determines a $\mathfrak{G}^*$-algebra derivation ${}_\theta d\colon \mathcal{F}^*(\mathcal{M}) \to \mathcal{F}^*(\mathcal{M})$. The morphism θ is itself determined by morphisms ${}_G\theta\colon \mathcal{M} \to G\langle\mathcal{M}\rangle$, with $|V_G| \geq 1$. For a pair of graphs $H \subset G$ we define the morphism ${}_H^G\theta\colon G/H\langle\mathcal{M}\rangle \to G\langle\mathcal{M}\rangle$ as in §1.2.2, then ${}_\theta d$ defined by

$$ {}_\theta d|_{\widetilde{G}\langle\mathcal{M}\rangle} := \sum_{G/H=\widetilde{G}} {}_H^G\theta $$

can readily be checked to satisfy the derivation property. The above sum is over all pairs H, G such that H is an admissible subgraph of G up to the global labeling of H since ${}_H^G\theta$ is not dependent on this labeling. Since ${}_G\theta$ applied to a fixed element of $\mathcal{M}$ is non-zero for only finitely many G this is true also for ${}_H^G\theta$.

Conversely a derivation d of the free $\mathfrak{G}^*$-algebra $\mathcal{F}^*(\mathcal{M})$ is determined by its restriction $d|_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{F}^*(\mathcal{M})$. Indeed,

$$\begin{aligned} &d(G, [p_1 \otimes \cdots \otimes p_k]) = \\ &\quad d\mu_G((G_1, [p_1]) \otimes \cdots \otimes (G_k, [p_k])) = \mu_G d^G((G_1, [p_1]) \otimes \cdots \otimes (G_k, [p_k])) = \\ &\sum_{i=1}^{k} (-1)^{(|p_1|+\cdots+|p_{i-1}|)|d|} \mu_G((G_1, [p_1]) \otimes \cdots \otimes (G_i, [dp_i]) \otimes \cdots \otimes (G_k, [p_k])) = \\ &\qquad\qquad \sum_{i=1}^{k} (-1)^{(|p_1|+\cdots+|p_{i-1}|)|d|} (G, [p_1 \otimes \cdots \otimes dp_i \otimes \cdots \otimes p_k]). \end{aligned}$$

Here the one-vertex graphs G_i and the local labelings of the p_i are appropriately chosen so as to satisfy the above equalities as well as $(G_i, [p_i]) \overset{\sim}{\mapsto} p_i$ under the isomorphism defined in §1.2.6.

Combining the last two observations we conclude the following (cf. Lemma 14 of [45].):

Lemma 2.1. *There is a one-to-one correspondence between $\mathfrak{G}^*$-algebra derivations of $\mathcal{F}^*(\mathcal{M})$ and $\mathbb{S}$-bimodule homomorphisms $\mathcal{M} \to \mathcal{F}^*(\mathcal{M})$.*

2.2.4 Coderivations of cofree $\mathfrak{G}^*$-coalgebras

Let $\mathscr{F}^{*,c}(\mathscr{M})$ be the free $\mathfrak{G}^*$-coalgebra on an $\mathbb{S}$-bimodule $\mathscr{M}$ and let $\theta\colon \mathscr{F}^{*,c}(\mathscr{M}) \to \mathscr{M}$ be an $\mathbb{S}$-bimodule homomorphism. Such a θ determines a $\mathfrak{G}^*$-coalgebra coderivation $d_\theta\colon \mathscr{F}^{*,c}(\mathscr{M}) \to \mathscr{F}^{*,c}(\mathscr{M})$ as follows. The morphism θ is itself determined by morphisms $\theta_G\colon G\langle\mathscr{M}\rangle \to \mathscr{M}$. For a pair of graphs $H \subset G$, we define the morphism $\theta_H^G\colon G\langle\mathscr{M}\rangle \to G/H\langle\mathscr{M}\rangle$ as in §1.2.1, then d_θ defined by

$$d_\theta|_{G\langle\mathscr{M}\rangle} := \sum_{H\subset G} \theta_H^G,$$

can readily be checked to satisfy the coderivation property. Here the sum is over, up to the labeling of H, all $\mathfrak{G}^*$-admissible subgraphs H of G.

Conversely a coderivation d of the cofree connected $\mathfrak{G}^*$-coalgebra $\mathscr{F}^{*,c}(\mathscr{M})$ is uniquely determined by the projection $\pi_{\mathscr{M}} d\colon \mathscr{F}^{*,c}(\mathscr{M}) \to \mathscr{M}$. First we observe that

$$d(\mathscr{F}^{*,c}_{(s)}(\mathscr{M})) \subset \bigoplus_{r\le s} \mathscr{F}^{*,c}_{(r)}(\mathscr{M}).$$

This claim is verified by induction on the number of vertices. Now suppose $d|_{\mathscr{F}^{*,c}_{(r)}(\mathscr{M})}$ is known for all $r < s$ and consider $X = (G, [p_1 \otimes \cdots \otimes p_s])$. First we note that $d(X)$ is a sum of decorated graphs with at most s vertices. Next for $G' \in \mathfrak{G}^*_{(2)}$ we observe that ${}_{G'}\Delta d(X)$, if non-zero, consists of terms where either one of the vertices is decorated with $(|, [1])$ or both vertices are decorated with graphs with at most $s-1$ vertices. It is clear that in order to determine the part of $d(X)$ which consists of graphs with more than one vertex, it is enough to know the part of ${}_{G'}\Delta d(X)$ without trivially decorated vertices, for all $G' \in \mathfrak{G}^*_{(2)}$. Thus if we consider only such terms in the equality ${}_{G'}\Delta d(X) = d^{G'}\,{}_{G'}\Delta(X)$, then in the right hand side d is applied only to graphs with less than s vertices and is therefore known by the induction assumption. Hence $d(X)$ is fully known if we only know the projection of d to $\mathscr{F}^{*,c}_{(1)}(\mathscr{M}) \cong \mathscr{M}$. We have proved the following (cf. Lemma 15 of [45]):

Lemma 2.2. *There is a one-to-one correspondence between $\mathfrak{G}^*$-coalgebra coderivations of $\mathscr{F}^{*,c}(\mathscr{M})$ and $\mathbb{S}$-bimodule homomorphisms $\mathscr{F}^*(\mathscr{M}) \to \mathscr{M}$.*

2.2.5 Quasi-(co)free dg $\mathfrak{G}^*$-(co)algebras

A free $\mathfrak{G}^*$-algebra on a dg $\mathbb{S}$-bimodule $(\mathscr{M}, d)$ has a natural differential induced by d, as defined in §2.1.2. We will consider also free $\mathfrak{G}^*$-algebras where the differential differs from the differential freely generated by d. We call a free $\mathfrak{G}^*$-algebra $\mathscr{F}^*(\mathscr{M})$ with a differential ${}_\theta\delta = d + {}_\theta d$, where ${}_\theta d$ is a derivation determined by a morphism $\theta\colon \mathscr{M} \to \mathscr{F}^*(\mathscr{M})$ (cf. §2.2.3), a *quasi-free* $\mathfrak{G}^*$-algebra.

Similarly we call a cofree $\mathfrak{G}^*$-coalgebra $\mathscr{F}^{*,c}(\mathscr{M})$ *quasi-cofree* if its codifferential is a sum $\delta_\theta = d + d_\theta$ of the codifferential induced by the one on $\mathscr{M}$ and a coderivation d_θ determined by a morphism $\theta \colon \mathscr{F}^*(\mathscr{M}) \to \mathscr{M}$ (cf. §2.2.4).

2.2.6 Bar and cobar constructions

For the rest of the section let $\mathfrak{G}^*$ be one of $\mathfrak{G}_c^{\downarrow_1^1}$, $\mathfrak{G}_c^{\downarrow_1}$, $\mathfrak{G}_{c,0}^{\downarrow}$, and $\mathfrak{G}_c^{\downarrow}$. Let $\mathscr{P}$ be a dg $\mathfrak{G}^*$-algebra. Consider the cofree $\mathfrak{G}^*$-coalgebra $\mathscr{F}^{*,c}(\Sigma^{-1}\bar{\mathscr{P}})$. It comes equipped with the codifferential d induced by the differential of $\mathscr{P}$, cf. §2.1.2. The partial composition product μ_2 of $\mathscr{P}$ induces a degree one morphism $\tilde{\mu} \colon \mathfrak{G}^*_{(2)}\langle\Sigma^{-1}\bar{\mathscr{P}}\rangle \to \Sigma^{-1}\bar{\mathscr{P}}$. By §2.2.4, the morphism $\tilde{\mu}$ determines a coderivation ${}_{\tilde{\mu}}d$ of $\mathscr{F}^{*,c}(\Sigma^{-1}\bar{\mathscr{P}})$. The associativity of μ_2 implies ${}_{\tilde{\mu}}d^2 = 0$ and that μ_2 is a morphism of dg $\mathbb{S}$-bimodules implies $d({}_{\tilde{\mu}}d) + ({}_{\tilde{\mu}}d)d = 0$. Hence $\delta := d + {}_{\tilde{\mu}}d$ satisfies $\delta^2 = 0$. We define the *bar construction* of $\mathscr{P}$ to be the quasi-cofree $\mathfrak{G}^*$-coalgebra $\mathrm{B}^*(\mathscr{P}) := (\mathscr{F}^{*,c}(\Sigma^{-1}\bar{\mathscr{P}}), \delta)$.

Now let $\mathscr{C}$ be a dg $\mathfrak{G}^*$-coalgebra. We define the *cobar construction* of $\mathscr{C}$ to be the quasi-free $\mathfrak{G}^*$-algebra $\Omega^*(\mathscr{C}) := (\mathscr{F}^*(\Sigma\bar{\mathscr{C}}), \delta)$ where the differential $\delta := d + d_{\tilde{\Delta}}$ is defined as follows. The $\mathfrak{G}^*$-algebra $\mathscr{F}^*(\Sigma\bar{\mathscr{C}})$ has a differential d induced by the codifferential of $\mathscr{C}$, cf. §2.1.2. The partial decomposition coproduct Δ_2 of $\mathscr{C}$ induces a degree one morphism $\tilde{\Delta} \colon \Sigma\bar{\mathscr{C}} \to \mathfrak{G}^*_{(2)}\langle\Sigma\bar{\mathscr{C}}\rangle$. By §2.2.3, $\tilde{\Delta}$ determines a derivation $d_{\tilde{\Delta}}$ of $\mathscr{F}^*(\Sigma\bar{\mathscr{C}})$. The coassociativity of Δ_2 implies $d_{\tilde{\Delta}}^2 = 0$. That Δ_2 is a morphism of dg $\mathbb{S}$-bimodules implies $d(d_{\tilde{\Delta}}) + (d_{\tilde{\Delta}})d = 0$. From this we conclude that $\delta^2 = 0$.

When we do not want to emphasize which family of graphs we are considering we will usually omit the $*$ from the notation of the bar and cobar constructions.

2.2.7 Quasi-free resolutions

A dg $\mathfrak{G}^*$-algebra $\mathscr{P}$ together with a quasi-isomorphism $\phi \colon \mathscr{P} \to \mathscr{Q}$ of dg $\mathfrak{G}^*$-algebras is called a *resolution* of $\mathscr{Q}$. We call the resolution *quasi-free* if $\mathscr{P} = (\mathscr{F}^*(\mathscr{M}), {}_\theta\delta)$ is a quasi-free $\mathfrak{G}^*$-algebra. If ${}_\theta\delta$ satisfies ${}_\theta\delta(\mathscr{M}) \subset \bigoplus_{i\geq 2}\mathscr{F}^*_{(i)}(\mathscr{M})$ we call the resolution *minimal.*

2.2.8 Bar-cobar resolutions

Applying first the bar and then the cobar construction to a $\mathfrak{G}^*$-algebra $\mathscr{P}$ yields a quasi-free resolution of $\mathscr{P}$.

Theorem. *Let $\mathscr{P}$ be a connected dg $\mathfrak{G}^*$-algebra, where $\mathfrak{G}^*$ is one of $\mathfrak{G}_c^{\downarrow_1^1}$, $\mathfrak{G}_c^{\downarrow_1}$, $\mathfrak{G}_{c,0}^{\downarrow}$, and $\mathfrak{G}_c^{\downarrow}$. In this case the morphism*

$$\mathscr{F}^*(\Sigma\bar{\mathscr{F}}^{*,c}(\Sigma^{-1}\bar{\mathscr{P}})) \to \mathscr{P}$$

induced by the projection

$$\Sigma\bar{\mathscr{F}}^{*,c}(\Sigma^{-1}\bar{\mathscr{P}}) \to \Sigma\bar{\mathscr{F}}^{*,c}_{(1)}(\Sigma^{-1}\bar{\mathscr{P}}) \cong \bar{\mathscr{P}} \subset \mathscr{P}$$

induces a quasi-isomorphism of dg $\mathfrak{G}^*$*-algebras*

$$\Omega(\mathrm{B}(\mathscr{P})) \xrightarrow{\sim} \mathscr{P}.$$

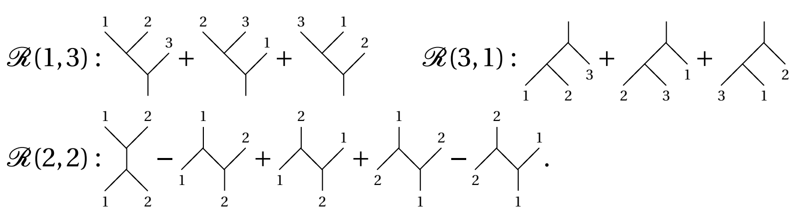
This was proved for operads in [20], for dioperads in [18], and for properads in [56]. The problem with the bar-cobar resolution is that it can be very difficult to compute explicitly. Fortunately there exists a large class of $\mathfrak{G}^*$-algebras for which there is a more easily computable resolution.

2.3 Koszul duality

2.3.1 Quadratic $\mathfrak{G}^*$-algebras

Definition. A *quadratic* $\mathfrak{G}^*$*-algebra* is a $\mathfrak{G}^*$-algebra $\mathscr{P} = \mathscr{F}^*(\mathscr{M})/(\mathscr{R})$, where $\mathscr{R} \subset \mathscr{F}^*_{(2)}(\mathscr{M})$. A quadratic $\mathfrak{G}^*$-algebra is called *binary* if $\mathscr{M}(m,n) =$ for $(m,n) \neq (1,2),(2,1)$.

Example 2.3 (Dioperad of Lie bialgebras)**.** Let $\mathscr{M}$ be the $\mathbb{S}$-bimodule given by $\mathscr{M}(1,2) = \mathbb{1}_1 \otimes \mathrm{sgn}_2$, $\mathscr{M}(2,1) = \mathrm{sgn}_2 \otimes \mathbb{1}_1$, and $\mathscr{M}(m,n) = 0$ for other m,n. We denote a graph decorated with the natural basis element of $\mathscr{M}(1,2)$ by Y and a graph decorated with the basis element of $\mathscr{M}(2,1)$ by ⅄. Consider the binary quadratic dioperad $\mathscr{L}ie\mathscr{B}i = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ where $\mathscr{R} = \mathscr{R}(1,3) \sqcup \mathscr{R}(3,1) \sqcup \mathscr{R}(2,2)$, with $\mathscr{R}(i,j) \subset \mathscr{F}(\mathscr{M})(i,j)$, is the following set of relations:



(2.1)

(2.2)

A representation $\rho\colon \mathscr{L}ie\mathscr{B}i \to \mathscr{E}nd_V$ in a vector space V makes V into a Lie bialgebra. The Lie bracket is given by ρ(Y): $V^{\otimes 2} \to V$ and the Lie co-bracket by ρ(⅄): $V \to V^{\otimes 2}$. That ρ is map of dioperads ensures that the Jacobi and co-Jacobi identities (2.1) are satisfied as well as the compatibility of the brackets (2.2).

See e.g. [20] for a treatment of quadratic operads, [18] for quadratic dioperads and [56] for quadratic properads and props.

2.3.2 Koszul $\mathfrak{G}^*$-algebras

In addition to the weight grading given by the number of vertices, the cofree $\mathfrak{G}^*$-coalgebra on a weight graded $\mathbb{S}$-bimodule $\mathscr{M}$ inherits another weight grading, the total weight,

$$\mathscr{F}^{*,c}(\mathscr{M})_{(s)} := \bigoplus_{\substack{G\in\mathfrak{G}^* \\ \{v_1,\dots,v_k\}=V_G \\ s_1+\dots+s_k=s}} G\langle(\mathscr{M}_{(s_1)})^{v_1},\dots,(\mathscr{M}_{(s_k)})^{v_k}\rangle.$$

For a weight graded $\mathbb{S}$-bimodule $\mathscr{M}$ concentrated in positive weight we observe that

$$\begin{cases} \mathscr{F}^{*,c}_{(s)}(\mathscr{M})_{(s)} = \mathscr{F}^{*,c}_{(s)}(\mathscr{M}_{(1)}) & \\ \mathscr{F}^{*,c}_{(r)}(\mathscr{M})_{(s)} = 0 & \text{for } r > s. \end{cases}$$

Now consider the bar construction $\mathrm{B}(\mathscr{P})$ on a connected weight graded $\mathfrak{G}^*$-algebra $\mathscr{P}$. By the above observations we see that $\mathrm{B}(\mathscr{P})$ is bi-graded by the number of vertices and the total weight. We also observe that $\Sigma^{-1}\bar{\mathscr{P}}$ is concentrated in positive weight since $\mathscr{P}$ is connected. By construction we see that ${}_{\theta}d(\mathrm{B}_{(r)}(\mathscr{P})_{(s)}) \subset \mathrm{B}_{(r-1)}(\mathscr{P})_{(s)}$. The compatibility of ${}_{\theta}d$ and d yields a complex of dg $\mathbb{S}$-bimodules

$$0 \to \mathrm{B}_{(s)}(\mathscr{P})_{(s)} \to \mathrm{B}_{(s-1)}(\mathscr{P})_{(s)} \to \dots$$

One can show that the weight graded sub-$\mathbb{S}$-bimodule given by

$$(\mathscr{P}^{\text{i}})_{(s)} := \mathrm{H}_s(\mathrm{B}_{(\bullet)}(\mathscr{P})_{(s)}, {}_{\theta}d) = \ker({}_{\theta}d\colon\ \mathrm{B}_{(s)}(\mathscr{P})_{(s)} \to \mathrm{B}_{(s-1)}(\mathscr{P})_{(s)})$$

is a weight graded sub-$\mathfrak{G}^*$-coalgebra of $\mathrm{B}(\mathscr{P})$. We call $\mathscr{P}^{\text{i}}$ the *Koszul dual $\mathfrak{G}^*$-coalgebra* of $\mathscr{P}$ and we say that $\mathscr{P}$ is *Koszul* if the inclusion $\mathscr{P}^{\text{i}} \hookrightarrow \mathrm{B}(\mathscr{P})$ is a quasi-isomorphism. It is shown in the above mentioned references that a Koszul $\mathfrak{G}^*$-algebra is necessarily quadratic.

Remark. Note that the Koszul dual $\mathfrak{G}^*$-coalgebra is defined as the *homology* of $(\mathrm{B}(\mathscr{P}), {}_{\theta}d)$ with respect to the weight grading. The codifferential raises the cohomological degree by one but lowers the weight by one.

2.3.3 Koszul resolutions

For Koszul $\mathfrak{G}^*$-algebras we have the following well-known result.

Theorem 2.4 ([50, 20, 18, 56])**.** *Let $\mathscr{P}$ be a dg $\mathfrak{G}^*$-algebra, where $\mathfrak{G}^*$ is one of $\mathfrak{G}^{\downarrow^1_1}_c$, $\mathfrak{G}^{\downarrow_1}_c$, $\mathfrak{G}^{\downarrow}_{c,0}$, and $\mathfrak{G}^{\downarrow}_c$. In this case $\mathscr{P}$ is Koszul if and only if the morphism of the bar-cobar resolution induces a quasi-free resolution*

$$\Omega(\mathscr{P}^{i}) \xrightarrow{\sim} \mathscr{P}.$$

We denote this resolution by $\mathscr{P}_\infty$. Representations of $\mathscr{P}_\infty$ yield strongly homotopy, also called infinity, versions of the algebras corresponding to $\mathscr{P}$; e.g. algebras over the operad $\mathscr{L}ie_\infty$ are called strongly homotopy Lie algebras or L_∞-algebras.

If $\mathscr{P}$ is a Koszul $\mathfrak{G}^*$-algebra with zero differential, then all we need to know in order to compute the differential of $\Omega(\mathscr{P}^{¡})$ is the structure of the decomposition coproduct of $\mathscr{P}^{¡}$. Next we will consider a shortcut to determining this coproduct.

2.3.4 Koszul dual $\mathfrak{G}^*$-algebras

To a quadratic $\mathfrak{G}^*$-algebra there is an associated dual $\mathfrak{G}^*$-algebra defined as follows.

Let $\mathscr{M}$ be an $\mathbb{S}$-bimodule. The *Czech dual* $\mathbb{S}$-bimodule $\mathscr{M}^\vee$ of $\mathscr{M}$ is defined by $\mathscr{M}^\vee(m,n) := \mathrm{sgn}_m \otimes \mathscr{M}(m,n)^* \otimes \mathrm{sgn}_n$. Now consider the free $\mathfrak{G}^*$-algebra on a connected $\mathbb{S}$-bimodule $\mathscr{M}$ satisfying in addition that $\mathscr{M}(1,1) = 0$ and that $\mathscr{M}(m,n)$ is finite dimensional for all (m,n). The components $\mathscr{F}^*_{(s)}(\mathscr{M})(m,n)$ are then all finite dimensional and the linear dual $(\mathscr{F}^*(\mathscr{M}))^*$ is naturally isomorphic to $\mathscr{F}^{*,c}(\mathscr{M}^*)$ as $\mathfrak{G}^*$-coalgebras. This isomorphism induces a pairing

$$\langle \_,\_ \rangle : \mathscr{F}^*_{(2)}(\mathscr{M}^\vee) \otimes \mathscr{F}^*_{(2)}(\mathscr{M}) \to \mathbb{K}$$

defined by

$$(G, [e_a^* \otimes e_b^*]) \otimes (G', [e_c \otimes e_d]) \mapsto \delta_{G,G'}\delta_{a,c}\delta_{b,d},$$

where the δ:s are Kronecker deltas, $\{e_i\}$ is a basis of $\mathscr{M}$, $\{e_i^*\}$ the dual basis, and we assume in the case $G = G'$ that e_a decorates the same vertex as e_c.

Now let $\mathscr{P} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ be a quadratic $\mathfrak{G}^*$-algebra such that $\mathscr{M}$ satisfies the above conditions. Let $\mathscr{R}^\perp$ be a subset of $\mathscr{F}^*_{(2)}(\mathscr{M}^\vee)$ satisfying that $(\mathscr{R}^\perp)_{(2)}$ is the orthogonal complement to $(\mathscr{R})_{(2)}$ with respect to the pairing $\langle \_,\_ \rangle$. We define the *Koszul dual* $\mathfrak{G}^*$ *-algebra* of $\mathscr{P}$ to be $\mathscr{P}^! := \mathscr{F}(\mathscr{M}^\vee)/(\mathscr{R}^\perp)$.

Remark. For a quadratic $\mathfrak{G}^*$-algebra $\mathscr{P} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ such that $\mathscr{M}(m,n)$ is finite dimensional for all $m,n \in \mathbb{N}$ we have that $(\mathscr{P}^!)^! = \mathscr{P}$.

The property of being Koszul is preserved under Koszul duality.

Proposition 2.5 ([20],[56])**.** *Let $\mathscr{P}$ be a quadratic $\mathfrak{G}^*$-algebra. Then $\mathscr{P}$ is Koszul if and only if $\mathscr{P}^!$ is Koszul.*

The Koszul dual $\mathfrak{G}^*$-algebra $\mathscr{P}^!$ of a quadratic $\mathfrak{G}^*$-algebra $\mathscr{P}$ relates to the Koszul dual $\mathscr{P}^{¡}$ in the following way.

$$(\mathscr{P}^{¡})_{(s)}(m,n) \cong \Sigma^{-s}((\mathscr{P}^!)_{(s)}(m,n))^\vee,$$

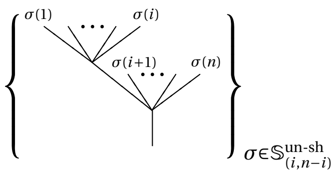
where the isomorphism is of $\mathfrak{G}^*$-coalgebras. Thus, computing the Koszul dual $\mathfrak{G}^*$-algebra and its composition product gives us an accessible way of determining the differential of the cobar construction on $\mathscr{P}^{¡}$.

2.4 $\mathscr{P}_\infty$-algebras as Maurer-Cartan elements

2.4.1 The Lie algebra associated to a $\mathfrak{G}^*$-algebra

The total space of a dg $\mathfrak{G}^*$-algebra $(\mathscr{P}, \mu, d_{\mathscr{P}})$ is defined by $\underline{\mathscr{P}} := \prod_{m,n} \mathscr{P}(m,n)$. There exists a binary product $\star$ on $\underline{\mathscr{P}}$; for p and q of homogeneous arities the product $p \star q$ is defined as the sum of all possible compositions of p and q along two-vertex graphs, in case the graph is connected with p decorating the lower vertex and q the upper vertex.

For operads this product defines a pre-Lie algebra structure on $\underline{\mathscr{P}}$ [45]. The subset of two-vertex graphs in $(\mathfrak{G}_c^{\downarrow_1})(n)$ is explicitly given by



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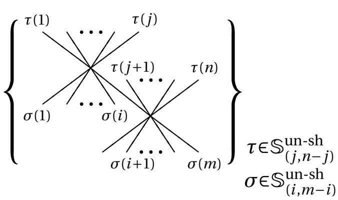
Remark. Note that for operads $p \star q$ often denotes the operation $p \star q = \sum_{i=1}^n p \circ_i q$, for $p \in \mathscr{P}(n)$, which also is a pre-Lie product [24]. We will always refer to the former product.

In the case of properads the product $\star$ is a Lie admissible product and for props it is an associative product [45].

For dioperads the subset of two-vertex graphs of $(\mathfrak{G}_{c,0}^{\downarrow})(n)$ is given by



.

In this case the product is of yet another type.

Let V be a vector space and let $\star\colon V \otimes V \to V$ be a bilinear operation on V. We define the associator of $\star$ to be $A_\star(u,v,w) := (u \star v) \star w - u \star (v \star w)$, for $u, v, w \in V$.

Definition. A vector space V endowed with a product $\star$ is called a *left-symmetric algebra* if the product satisfies $A_\star(u,v,w) = A_\star(v,u,w)$ for all

$u, v, w \in V$. It is called a *right-symmetric algebra* if $\star$ satisfies $A_\star(u, v, w) = A_\star(u, w, v)$ for all $u, v, w \in V$, and it is called a *bi-symmetric algebra* [3] if it is both a left-symmetric and a right-symmetric algebra.

Remark. Right-symmetric and left-symmetric algebras are often called pre-Lie algebras and left-symmetric algebras are known also as Vinberg algebras. Bi-symmetric algebras can equivalently be defined by $A_\star(v_1, v_2, v_3) = A_\star(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$ for all $v_1, v_2, v_3 \in V$ and all $\sigma \in \mathbb{S}_3$. They are also called assosymmetric algebras [25].

It is not hard to show the following:

Proposition 2.6. *For a dioperad $\mathscr{P}$ the product $\star$ is bi-symmetric.*

In all the above cases, for $\mathscr{P}$ a $\mathfrak{G}^*$-algebra the commutator of $\star$ together with the differential of $\mathscr{P}$ makes $\underline{\mathscr{P}}$ into a dg Lie algebra. See [28] and [45].

2.4.2 Convolution $\mathfrak{G}^*$-algebras

Let $(\mathscr{C}, \Delta, \delta_\mathscr{C})$ be a dg $\mathfrak{G}^*$-coalgebra and $(\mathscr{P}, \mu, \delta_\mathscr{P})$ a dg $\mathfrak{G}^*$-algebra. The collection $\mathscr{P}^\mathscr{C} = \mathrm{Hom}_\mathbb{K}(\mathscr{C}, \mathscr{P})$ of all homomorphisms of graded $\mathbb{K}$-modules is an $\mathbb{S}$-bimodule with components $\mathrm{Hom}_\mathbb{K}(\mathscr{C}, \mathscr{P})(m, n) = \mathrm{Hom}_\mathbb{K}(\mathscr{C}(m, n), \mathscr{P}(m, n))$ and the $\mathbb{S}$-action given by $(\sigma f \tau)(x) = \sigma f(\sigma^{-1} x \tau^{-1})\tau$. The invariants $(\mathscr{P}^\mathscr{C})^\mathbb{S}$ of this action are the $\mathbb{S}$-equivariant maps. The $\mathbb{S}$-module $(\mathscr{P}^\mathscr{C})^\mathbb{S}$ has a $\mathfrak{G}^*$-algebra structure, see [4] and [45] for the cases of operads and props, respectively, defined as follows. For a graph G and an element $(G, [f_1 \otimes \cdots \otimes f_k] \in G\langle(\mathscr{P}^\mathscr{C})^\mathbb{S}\rangle$, let $G(f_1, \ldots, f_k)\colon G\langle\mathscr{C}\rangle \to G\langle\mathscr{P}\rangle$ denote the morphism which applies f_i to the decoration of the corresponding vertex. The composition product of $(\mathscr{P}^\mathscr{C})^\mathbb{S}$ is given by

$$\mu_G(f_1 \otimes \cdots \otimes f_k) := \mu_G \circ G(f_1, \ldots, f_k) \circ {}_G\Delta.$$

The differential $\delta_\mathscr{P}$ and codifferential $\delta_\mathscr{C}$ induce a differential ∂ given by $\partial(f) = \delta_\mathscr{P} \circ f - (-1)^{|f|} f \circ \delta_\mathscr{C}$. Together this gives $(\mathscr{P}^\mathscr{C})^\mathbb{S}$ a structure of dg $\mathfrak{G}^*$-algebra called the *convolution $\mathfrak{G}^*$-algebra.*

2.4.3 Maurer-Cartan elements of convolution Lie algebras

Definition. A *Maurer-Cartan element* in a dg Lie algebra $(\mathfrak{g}, [\_,\_], d)$, is a degree one element γ satisfying the Maurer-Cartan equation

$$d(\gamma) + \frac{1}{2}[\gamma, \gamma] = 0.$$

Let $\mathscr{P}$ be a dg $\mathfrak{G}^*$-algebra and let $\mathscr{C}$ denote the dg $\mathfrak{G}^*$-coalgebra $\overline{\mathscr{P}^{¡}}$. Further let (V, d) be a dg vector space and let $\mathscr{E}$ denote the endomorphism

$\mathfrak{G}^*$-algebra of V. Since a $\mathfrak{G}^*$-algebra morphism from a free $\mathfrak{G}^*$-algebra is determined by its restriction to the space of generators, the dg $\mathfrak{G}^*$-algebra morphisms $\Omega(\mathscr{P}^{\mathsf{i}}) = (\mathscr{F}(\Sigma\mathscr{C}), \delta = d_{\Sigma\mathscr{C}} + d_{\tilde{\Delta}}) \to (\mathscr{E}, d_{\mathscr{E}})$ correspond to the degree zero $\mathbb{S}$-invariants $\rho \in \mathscr{E}^{\Sigma\mathscr{C}}$ which satisfy

$$\rho \circ \delta = d_{\mathscr{E}} \circ \rho \Longleftrightarrow \rho \circ d_{\tilde{\Delta}} = d_{\mathscr{E}} \circ \rho - \rho \circ d_{\Sigma\mathscr{C}}. \tag{2.3}$$

Let $\mathscr{L}_{\mathscr{P}}(V) := (\underline{((\mathscr{E}^{\mathscr{C}})^{\mathbb{S}})}, [\_,\_], -\partial)$ denote the dg Lie algebra associated to the convolution $\mathfrak{G}^*$-algebra $(\mathscr{E}^{\mathscr{C}})^{\mathbb{S}}$. We call it the *convolution Lie algebra*. A degree r $\mathbb{S}$-bimodule morphism $\rho\colon \Sigma\mathscr{C} \to \mathscr{E}$ induces a degree $r+1$ element $\tilde{\rho} \in \underline{((\mathscr{E}^{\mathscr{C}})^{\mathbb{S}})}$. That ρ satisfies (2.3) is equivalent to that $\tilde{\rho}$ satisfies

$$\sum_{G \in \mathfrak{G}_{(2)}} \mu_G \circ G\langle\tilde{\rho}\rangle \circ {}_G\Delta = d_{\mathscr{E}} \circ \tilde{\rho} - \tilde{\rho} \circ d_{\mathscr{C}} \Longleftrightarrow -\partial(\tilde{\rho}) + \frac{1}{2}[\tilde{\rho}, \tilde{\rho}] = 0.$$

Thus we obtain the following result.

Theorem 2.7 (Theorem 62 (ii) in [45])**.** *Let $\mathscr{P}$ be a Koszul operad. Then there is a one-to-one correspondence between representations of $\mathscr{P}_\infty = \Omega(\mathscr{P}^{\mathsf{i}})$ in (V, d) and Maurer-Cartan elements in $\mathscr{L}_{\mathscr{P}}(V)$.*

3 Operads of compatible structures

In this chapter we describe operads encoding two different kinds of compatibility of algebraic structures. We show that there exist decompositions of these in terms of black and white products and we prove that they are Koszul for a large class of algebraic structures by using the poset method of B. Vallette. In particular we show that this is true for the operads of compatible Lie, associative and pre-Lie algebras.

3.1 Compatibility of structures encoded by operads

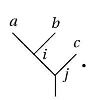
3.1.1 Linearly compatible structures

Let $\mathcal{O} = \mathcal{F}(\mathcal{M})/(\mathcal{R})$ be a binary quadratic operad with a $\mathbb{K}$-basis $e_1, \ldots, e_s$ of $\mathcal{M}$. A representation ρ of $\mathcal{O}$ in a vector space U can be thought of as the data $(U, \{\rho(e_1), \ldots, \rho(e_s)\})$, where $\rho(e_i)$ are binary operations on U subject to axioms encoded by the relations $\mathcal{R}$ and the $\mathbb{S}$-module structure of $\mathcal{M}$.

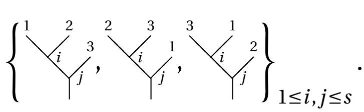
Definition. Let $\mathcal{O}$ be a binary quadratic operad and U a vector space over $\mathbb{K}$. Let $A = (U, \mu_1 \ldots, \mu_k)$ and $B = (U, \nu_1 \ldots, \nu_k)$ be $\mathcal{O}$-algebra structures on U. Define new operations by $\eta_i := \alpha\mu_i + \beta\nu_i$ for some $\alpha, \beta \in \mathbb{K}$. We say that A and B are *linearly compatible* if $C = (U, \eta_1, \ldots, \eta_k)$ is an $\mathcal{O}$-algebra for any choice of α and β. Note that this is equivalent to requiring C to be an $\mathcal{O}$-algebra for $\alpha = \beta = 1$.

3.1.2 Koszul dual operad of a binary operad

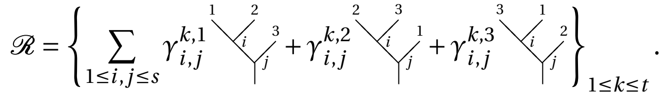
For an $\mathbb{S}$-module $\mathcal{M}$ concentrated in $\mathcal{M}(2)$ we have that $\mathcal{F}_{(2)}(\mathcal{M}) = \mathcal{F}_{(2)}(\mathcal{M})(3)$. Given such a module $\mathcal{M}$ with $\mathbb{K}$-basis $\{\curlyvee^1, \ldots, \curlyvee^s\}$ we denote a labeled tree in $\mathcal{F}_{(2)}(\mathcal{M})$ decorated with $\curlyvee^i$ above $\curlyvee^j$ by



The space $\mathcal{F}_{(2)}(\mathcal{M})$ is then spanned by the trees



and is $3s^2$-dimensional. Thus for a binary quadratic operad $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ we have that $\mathscr{R}$ consists of $t \le 3s^2$ linearly independent relations

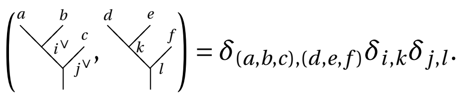


(3.1)

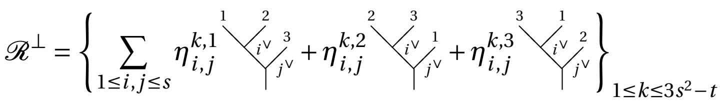
Recall from §2.3.4 that the Koszul dual operad of $\mathscr{O}$ is given by $\mathscr{O}^! = \mathscr{F}(\mathscr{M}^\vee)/(\mathscr{R}^\perp)$, where $\mathscr{M}^\vee$ is defined by $\mathscr{M}^\vee(n) = \mathscr{M}(n)^* \otimes \mathrm{sgn}_n$ and $\mathscr{R}^\perp$ is a maximal set of relations orthogonal to $\mathscr{R}$ with respect to the natural pairing

$$\langle \_,\_ \rangle \colon \mathscr{F}_{(2)}(\mathscr{M}^\vee) \otimes \mathscr{F}_{(2)}(\mathscr{M}) \to \mathbb{K}.$$

Explicitly, for a binary quadratic operad this pairing is given by



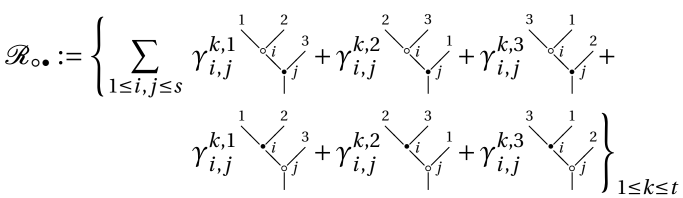
Given relations $\mathscr{R}$ as in (3.1), there are $3s^2 - t$ linearly independent orthogonal relations



(3.2)

3.1.3 Operads of linearly compatible structures

Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ be a binary quadratic operad. Consider two operads $\mathscr{O}_\circ = \mathscr{F}(\mathscr{M}_\circ)/(\mathscr{R}_\circ)$ and $\mathscr{O}_\bullet = \mathscr{F}(\mathscr{M}_\bullet)/(\mathscr{R}_\bullet)$ both isomorphic to $\mathscr{O}$. We choose $\mathbb{K}$-bases $\curlyvee^{\circ}_{1},\dots,\curlyvee^{\circ}_{s}$ of $\mathscr{M}_\circ$ and $\curlyvee^{\bullet}_{1},\dots,\curlyvee^{\bullet}_{s}$ of $\mathscr{M}_\bullet$ so that there exists an isomorphism $\phi\colon \mathscr{O}_\circ \to \mathscr{O}_\bullet$ with $\phi(\curlyvee^{\circ}_{i}) = \curlyvee^{\bullet}_{i}$. The relations $\mathscr{R}_\circ$ and $\mathscr{R}_\bullet$ can then be given by the same coefficients $\gamma^{k,l}_{i,j}$, cf. (3.1). By embedding $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ into $\mathscr{F}(\mathscr{M}_\circ \oplus \mathscr{M}_\bullet)/(\mathscr{R}_\circ \cup \mathscr{R}_\bullet)$ we obtain an operad whose representations are pairs of $\mathscr{O}$-algebras which not necessarily are compatible in any way. In order to encode linear compatibility we define the following relations:



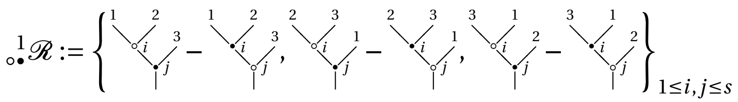
Definition. Given $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ and $\mathscr{R}_{\circ\bullet}$ as above we define a new binary quadratic operad by $\mathscr{O}^2 := \mathscr{F}(\mathscr{M}_\circ \oplus \mathscr{M}_\bullet)/(\mathscr{R}_\circ \cup \mathscr{R}_\bullet \cup \mathscr{R}_{\circ\bullet})$.

Proposition 3.1. *A representation of $\mathscr{O}^2$ is a pair of linearly compatible $\mathscr{O}$-algebras.*

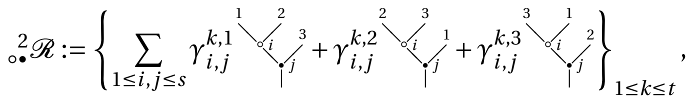
Proof. By direct calculation. □

3.1.4 Totally compatible structures

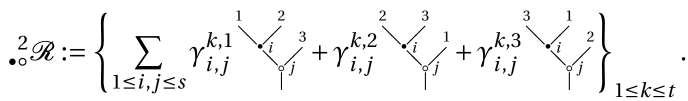
We now turn our attention to the other kind of compatibility which should generalize the compatibility of ${}^2\mathscr{C}om$. Given a binary quadratic operad $\mathscr{O}$ and isomorphic operads $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ as in the previous paragraph, we define



These relations encode that the order in which we apply operations of $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ is irrelevant. Next we define



which encodes that the relations of the original operad are satisfied for combinations of the operations of $\mathscr{O}_\circ$ and $\mathscr{O}_\bullet$ if we first apply an operation of $\mathscr{O}_\circ$ and then an operation of $\mathscr{O}_\bullet$. Note that a consequence of ${}_{\circ\bullet}^{\;1}\mathscr{R}$ and ${}_{\circ\bullet}^{\;2}\mathscr{R}$ is



We define ${}_{\circ\bullet}\mathscr{R} := {}_{\circ\bullet}^{\;1}\mathscr{R} \cup {}_{\circ\bullet}^{\;2}\mathscr{R}$.

Definition. Given $\mathscr{O}_\circ$, $\mathscr{O}_\bullet$ and ${}_{\circ\bullet}\mathscr{R}$ as above we define a new binary quadratic operad by ${}^2\mathscr{O} := \mathscr{F}(\mathscr{M}_\circ \oplus \mathscr{M}_\bullet)/(\mathscr{R}_\circ \cup \mathscr{R}_\bullet \cup {}_{\circ\bullet}\mathscr{R})$. A representation of ${}^2\mathscr{O}$ is a pair of $\mathscr{O}$-algebras with the compatibility given by ${}_{\circ\bullet}\mathscr{R}$. We call structures compatible in this way *totally compatible.*

In the next paragraph we will see that ${}^2\mathscr{C}om$ is an operad of this form, but first we note how linear and total compatibility relate under Koszul duality.

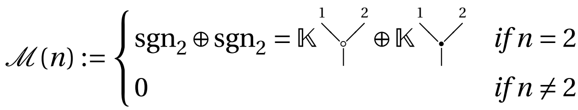
Proposition 3.2. *Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ be a binary quadratic operad such that $\mathscr{M}(n)$ is finite dimensional for all $n \in \mathbb{N}$. We have $(\mathscr{O}^2)^! = {}^2(\mathscr{O}^!)$ and $({}^2\mathscr{O})^! = (\mathscr{O}^!)^2$.*

Proof. By direct calculation. □

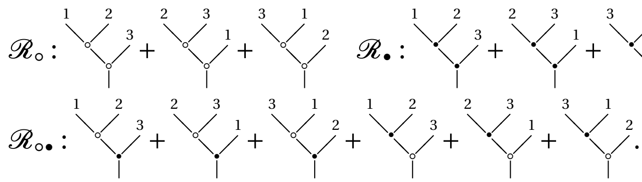
3.1.5 The operads $\mathscr{L}ie^2$ and ${}^2\mathscr{C}om$

Here we give descriptions of the operads encoding linearly compatible Lie algebras and totally compatible commutative algebras using the definitions and notation of the previous two paragraphs. Thereby we show that the notation for compatible structures agrees with the notation already given to these operads. Both results follow from straightforward verifications.

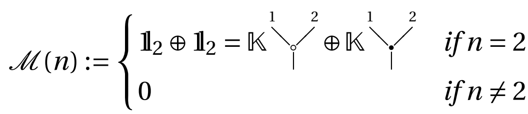
Proposition 3.3. *The operad $\mathscr{L}ie^2$, encoding pairs of linearly compatible Lie algebras, is the quadratic operad $\mathscr{F}(\mathscr{M})/(\mathscr{R})$ where the $\mathbb{S}$-module $\mathscr{M}$ is given by*



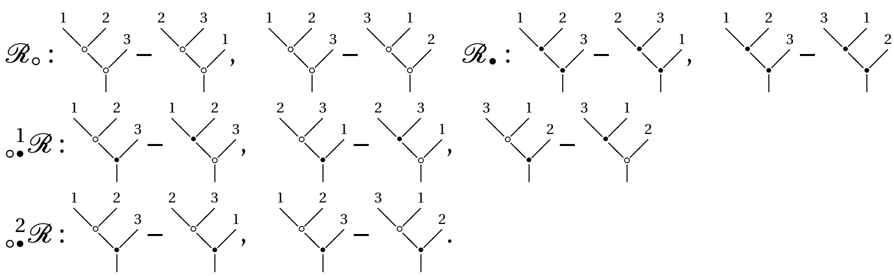
and the relations $\mathscr{R} = \mathscr{R}_\circ \cup \mathscr{R}_\bullet \cup \mathscr{R}_{\circ\bullet}$ are as follows



Proposition 3.4. *The operad ${}^2\mathscr{C}om$, encoding pairs of totally compatible commutative algebras, is the quadratic operad $\mathscr{F}(\mathscr{M})/(\mathscr{R})$, where the $\mathbb{S}$-module $\mathscr{M}$ is given by*



and the relations $\mathscr{R} = \mathscr{R}_\circ \cup \mathscr{R}_\bullet \cup {}^{\,1}_{\circ\bullet}\mathscr{R} \cup {}^{\,2}_{\circ\bullet}\mathscr{R}$ are given by



Note that by Proposition 3.2 $\mathscr{L}ie^2$ is Koszul dual to ${}^2\mathscr{C}om$.

Remark. The operads $\mathscr{L}ie^2$ and ${}^2\mathscr{C}om$ were denoted by $\mathscr{L}ie_2$ and $\mathscr{C}om_2$, respectively, in [14]. We denote them with upper index so that the notation does not interfere with that of set operads, see §3.2.1, and we have the indices to the right and left respectively to emphasize that these are two different kinds of compatibility.

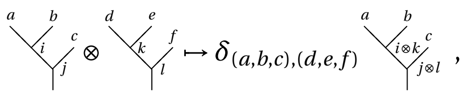
3.1.6 Black product and white product

In [20, 21] Ginzburg and Kapranov generalized the notions of black and white products for algebras to binary quadratic operads. B. Vallette generalized the notion further to arbitrary operads given by generators and relations and to properads in [54]. He also established some results about black and white products for operads. For more details we refer the reader to these papers.

The definition of the black product for binary quadratic operads is given in terms of a certain map Ψ. Note that for an $\mathbb{S}$-module $\mathscr{M}$ concentrated in $\mathscr{M}(2)$ we have that $\mathscr{F}(\mathscr{M})(3)$ is equal to $\mathscr{F}_{(2)}(\mathscr{M})$ and that $\mathscr{F}_{(2)}(\mathscr{M})$ is spanned by three types of decorated trees, corresponding to the possible labelings of the leaves, see §3.1.2. Given two binary quadratic operads $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ and $\mathscr{Q} = \mathscr{F}(\mathscr{N})/(\mathscr{S})$ the map

$$\Psi\colon \mathscr{F}(\mathscr{M})(3) \otimes \mathscr{F}(\mathscr{N})(3) \otimes \mathrm{sgn}_3 \to \mathscr{F}(\mathscr{M} \otimes \mathscr{N} \otimes \mathrm{sgn}_2)$$

is defined by



where by abuse of notation $i \otimes k$ denotes the tensor product of the elements decorating the trees.

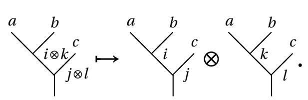
Definition. Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ and $\mathscr{Q} = \mathscr{F}(\mathscr{N})/(\mathscr{S})$ be binary quadratic operads whose $\mathbb{S}$-modules of generators $\mathscr{M}$ and $\mathscr{N}$ are finite dimensional. We define their *black product* by

$$\mathscr{O} \bullet \mathscr{Q} := \mathscr{F}(\mathscr{M} \otimes \mathscr{N} \otimes \mathrm{sgn}_2)/(\Psi(\mathscr{R} \otimes \mathscr{S})).$$

The white product is defined through another map

$$\Phi\colon \mathscr{F}(\mathscr{M} \otimes \mathscr{N})(3) \to \mathscr{F}(\mathscr{M})(3) \otimes \mathscr{F}(\mathscr{N})(3)$$

which is given by



Definition. Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ and $\mathscr{Q} = \mathscr{F}(\mathscr{N})/(\mathscr{S})$ be binary quadratic operads. We define their *white product* by

$$\mathscr{O} \circ \mathscr{Q} := \mathscr{F}(\mathscr{M} \otimes \mathscr{N})/(\Phi^{-1}(\mathscr{R} \otimes \mathscr{F}(\mathscr{N})(3) + \mathscr{F}(\mathscr{M})(3) \otimes \mathscr{S})).$$

We have the following relation between black and white products which was stated in [20, 21] and explicitly proven in [54].

Proposition 3.5 (Theorem 2.2.6 in [20])**.** *Let $\mathscr{O}$ and $\mathscr{Q}$ be binary quadratic operads generated by finite dimensional $\mathbb{S}$-modules, then $(\mathscr{O} \circ \mathscr{Q})^! = \mathscr{O}^! \bullet \mathscr{Q}^!$.*

3.1.7 Decomposition of operads of compatible structures

We now reach the highlight of this section with the following theorem.

Theorem 3.6. *Let $\mathscr{O}$ be a binary quadratic operad. We have $\mathscr{O}^2 = \mathscr{O} \bullet \mathscr{L}ie^2$.*

Proof. Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ be generated by the $\mathbb{S}$-module $\mathscr{M} = \mathbb{K}\curlyvee_1 \oplus \cdots \oplus \mathbb{K}\curlyvee_s$ and with relations

$$\mathscr{R} = \left\{ \sum_{1\le i,j\le s} \gamma^{k,1}_{i,j} \begin{smallmatrix}1\ \ 2\ \ \\ \curlyvee_i\ \ 3\\ \curlyvee_j\end{smallmatrix} + \gamma^{k,2}_{i,j} \begin{smallmatrix}2\ \ 3\ \ \\ \curlyvee_i\ \ 1\\ \curlyvee_j\end{smallmatrix} + \gamma^{k,3}_{i,j} \begin{smallmatrix}3\ \ 1\ \ \\ \curlyvee_i\ \ 2\\ \curlyvee_j\end{smallmatrix} \right\}_{1\le k\le t}.$$

Further denote the generators of $\mathscr{L}ie^2$ as in §3.1.5 by $\curlyvee^{\circ}$ and $\curlyvee^{\bullet}$ and the relations by $\mathscr{S} = \mathscr{S}_\circ \cup \mathscr{S}_\bullet \cup \mathscr{S}_{\circ\bullet}$.

By the definition of the black product we see that $\mathscr{O} \bullet \mathscr{L}ie^2$ is generated by $(\mathbb{K}\curlyvee_1 \oplus\cdots\oplus \mathbb{K}\curlyvee_s)\otimes(\mathbb{K}\curlyvee^{\circ} \oplus \mathbb{K}\curlyvee^{\bullet})\otimes \mathrm{sgn}_2$. We denote the generator of $\mathbb{K}\curlyvee_i \otimes \mathbb{K}\curlyvee^{\circ} \otimes \mathrm{sgn}_2$ by $\curlyvee^{\circ}_i$ and since $\mathbb{K}\curlyvee^{\circ} \otimes \mathrm{sgn}_2 = \mathrm{sgn}_2 \otimes \mathrm{sgn}_2 \cong \mathbb{1}_2$ we have that $\mathscr{M}_\circ = \mathbb{K}\curlyvee^{\circ}_1 \oplus \cdots \oplus \mathbb{K}\curlyvee^{\circ}_s$ is isomorphic to $\mathscr{M}$ as an $\mathbb{S}$-module. Of course the same is true for $\mathscr{M}_\bullet = \mathbb{K}\curlyvee^{\bullet}_1 \oplus\cdots\oplus \mathbb{K}\curlyvee^{\bullet}_s$, with the obvious meaning of $\curlyvee^{\bullet}_i$.

Next we see that

$$\mathscr{R}_\circ = \Psi(\mathscr{R}\otimes\mathscr{S}_\circ) = \left\{ \sum_{1\le i,j\le s} \gamma^{k,1}_{i,j} \begin{smallmatrix}1\ \ 2\ \ \\ \curlyvee^{\circ}_i\ \ 3\\ \curlyvee^{\circ}_j\end{smallmatrix} + \gamma^{k,2}_{i,j} \begin{smallmatrix}2\ \ 3\ \ \\ \curlyvee^{\circ}_i\ \ 1\\ \curlyvee^{\circ}_j\end{smallmatrix} + \gamma^{k,3}_{i,j} \begin{smallmatrix}3\ \ 1\ \ \\ \curlyvee^{\circ}_i\ \ 2\\ \curlyvee^{\circ}_j\end{smallmatrix} \right\}_{1\le k\le t}.$$

and similarly for $\mathscr{R}_\bullet = \Psi(\mathscr{R}\otimes\mathscr{S}_\bullet)$. Finally for $\mathscr{R}_{\circ\bullet} = \Psi(\mathscr{R}\otimes\mathscr{S}_{\circ\bullet})$ we have

$$\begin{aligned}\mathscr{R}_{\circ\bullet} = \Bigg\{ \sum_{1\le i,j\le s} & \gamma^{k,1}_{i,j} \begin{smallmatrix}1\ \ 2\ \ \\ \curlyvee^{\circ}_i\ \ 3\\ \curlyvee^{\bullet}_j\end{smallmatrix} + \gamma^{k,2}_{i,j} \begin{smallmatrix}2\ \ 3\ \ \\ \curlyvee^{\circ}_i\ \ 1\\ \curlyvee^{\bullet}_j\end{smallmatrix} + \gamma^{k,3}_{i,j} \begin{smallmatrix}3\ \ 1\ \ \\ \curlyvee^{\circ}_i\ \ 2\\ \curlyvee^{\bullet}_j\end{smallmatrix} + \\ & \gamma^{k,1}_{i,j} \begin{smallmatrix}1\ \ 2\ \ \\ \curlyvee^{\bullet}_i\ \ 3\\ \curlyvee^{\circ}_j\end{smallmatrix} + \gamma^{k,2}_{i,j} \begin{smallmatrix}2\ \ 3\ \ \\ \curlyvee^{\bullet}_i\ \ 1\\ \curlyvee^{\circ}_j\end{smallmatrix} + \gamma^{k,3}_{i,j} \begin{smallmatrix}3\ \ 1\ \ \\ \curlyvee^{\bullet}_i\ \ 2\\ \curlyvee^{\circ}_j\end{smallmatrix} \Bigg\}_{1\le k\le t}.\end{aligned}$$

where $1 \le i \le n$. Thus we see that $\mathscr{O} \bullet \mathscr{L}ie^2 = \mathscr{F}(\mathscr{M}_\circ \oplus \mathscr{M}_\bullet)/(\mathscr{R}_\circ \cup \mathscr{R}_\bullet \cup \mathscr{R}_{\circ\bullet}) = \mathscr{O}^2$. □

Corollary 3.7. *Let $\mathscr{O}$ be a binary quadratic operad. We have ${}^2\mathscr{O} = \mathscr{O} \circ {}^2\mathscr{C}om$.*

Proof. By Proposition 3.2 we have that $((\mathscr{O}^!)^2)^! = {}^2\mathscr{O}$ and by Theorem 3.6 that $(\mathscr{O}^!)^2 = \mathscr{O}^! \bullet \mathscr{L}ie^2$. By Proposition 3.5 we know that $(\mathscr{O}^! \bullet \mathscr{L}ie^2)^! = \mathscr{O} \circ {}^2\mathscr{C}om$. Putting this together we conclude that

$${}^2\mathscr{O} = ((\mathscr{O}^!)^2)^! = (\mathscr{O}^! \bullet \mathscr{L}ie^2)^! = \mathscr{O} \circ {}^2\mathscr{C}om.$$

□

3.1.8 White product and Hadamard product

In practice the white product can be difficult to compute explicitly. In [54] a useful result was proven relating the white product and Hadamard product for certain operads.

Definition. The *Hadamard product* $\mathscr{O} \otimes_H \mathscr{Q}$ of two operads $\mathscr{O}$ and $\mathscr{Q}$ is defined as $(\mathscr{O} \otimes_H \mathscr{Q})(n) := \mathscr{O}(n) \otimes \mathscr{Q}(n)$. The composition μ is given by

$$\mu(e \otimes q; e_1 \otimes q_1, \dots, e_k \otimes q_k) := \mu(e; e_1, , \dots, e_k) \otimes \mu(q; q_1, \dots, q_k).$$

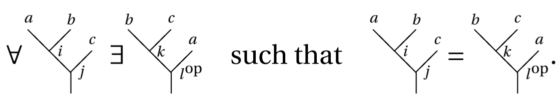
For a quadratic operad $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ let $\pi_{\mathscr{O}} \colon \mathscr{F}(\mathscr{M}) \to \mathscr{O}$ be the natural projection. Denote by T a labeled binary tree with $n-1$ internal vertices. We order the internal vertices linearly in an arbitrary way and let $\mathscr{L}_T^{\mathscr{M}}$ denote the induced decoration morphism $\mathscr{L}_T^{\mathscr{M}} \colon \mathscr{M}^{\otimes(n-1)} \to \mathscr{F}(\mathscr{M})$ which decorates the internal vertices of T with elements of $\mathscr{M}$.

Proposition 3.8 (Proposition 15 in [54])**.** *Let $\mathscr{O}$ be a binary quadratic operad such that for every $n \geq 3$ and every labeled binary tree T with $n-1$ vertices the composite map $\pi_{\mathscr{O}} \circ \mathscr{L}_T^{\mathscr{M}} \colon \mathscr{M}^{\otimes(n-1)} \to \mathscr{O}(n)$ is surjective. For every binary quadratic operad, $\mathscr{Q}$, the white product $\mathscr{O} \circ \mathscr{Q}$ is equal to the Hadamard product $\mathscr{O} \otimes_H \mathscr{Q}$.*

3.1.9 Weakly associative operads

Since we will use the condition in Proposition 3.8 later we extract it into a definition.

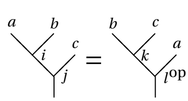
Definition. Let $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ be a binary quadratic operad with a $\mathbb{K}$-basis $\{Y^1, \dots, Y^s\}$ of $\mathscr{M}$. Denote the element $Y^i(12)$ by $Y^{i\,\mathrm{op}}$. We call $\mathscr{O}$ *weakly associative* if



Note that an operation Y^i is associative in the usual sense precisely when the above condition is satisfied for $i = j = k = l$.

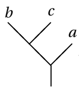
Proposition 3.9. *Let $\mathscr{O}$ be binary quadratic operad, then $\mathscr{O}$ is weakly associative if and only if it has the property of Proposition 3.8.*

Proof. Assume that $\mathscr{O} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ is weakly associative. Let T be any labeled binary tree. By repeatedly using the identity

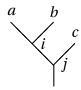


any decorated labeled binary tree T' is equivalent to a decorated tree with the same shape and labeling as T. Hence the map $\pi_{\mathscr{O}} \circ \mathscr{L}_T^{\mathscr{M}}$ is surjective.

Now assume instead that $\pi_{\mathscr{O}} \circ \mathscr{L}_T^{\mathscr{M}}$ is surjective for any labeled binary tree T. Let

$T =$  .

Then since $\pi_{\mathscr{O}} \circ \mathscr{L}_T^{\mathscr{M}}$ is surjective, any decorated tree



is equivalent to a decorated tree of the same shape and labeling as T, which is exactly the condition for being weakly associative. □

Corollary 3.10. *For every binary quadratic operad $\mathscr{O}$ we have $\mathscr{O} \circ {}^2\mathscr{C}om = \mathscr{O} \otimes_H {}^2\mathscr{C}om$.*

Proof. Clearly ${}^2\mathscr{C}om$ is weakly associative, thus by Proposition 3.9 it satisfies the condition of Proposition 3.8 whence we obtain the desired result. □

3.2 Operadic partition posets of set operads

3.2.1 Set operads

An $\mathbb{S}$-set is a collection of sets, $\mathscr{S} = (\mathscr{S}_n)_{n\in\mathbb{N}}$, equipped with a right action of the symmetric group $\mathbb{S}_n$ on $\mathscr{S}_n$. Define a monoidal product in the category of $\mathbb{S}$-sets by:

$$\mathscr{S} \circ \mathscr{T}_n = \bigsqcup_{1\le k\le n} \left(\bigsqcup_{i_1+\cdots+i_k=n} \mathscr{S}_k \times (\mathscr{T}_{i_1} \times \cdots \times \mathscr{T}_{i_k}) \times_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \mathbb{S}_n \right)_{\mathbb{S}_k},$$

where we consider the coinvariants with respect to the action of $\mathbb{S}_k$ given by $(s,(t_1,\ldots,t_k),\sigma)\tau = (s\tau,(t_{\tau(1)},\ldots,t_{\tau(k)}),\bar{\tau}^{-1}\sigma)$ and $\bar{\tau}$ is the induced block permutation. A unit I with respect to this product is given by the $\mathbb{S}$-module defined by

$$I_n := \begin{cases} [1] & \text{if } n = 1 \\ \emptyset & \text{if } n \neq 1. \end{cases}$$

Definition. A *set operad* is a monoid $(\mathscr{P},\mu\colon \mathscr{P}\circ\mathscr{P} \to \mathscr{P},\varepsilon\colon I \to \mathscr{P})$ in the monoidal category $(\mathbb{S}\text{-sets},\circ,I)$. For an element $(p,(p_1,\ldots,p_k),\sigma) \in \mathscr{P}\circ\mathscr{P}$ we will suppress the sigma and denote $\mu(p,(p_1,\ldots,p_k))$ by $\mu(p;p_1,\ldots,p_k)$.

To any set operad $\mathscr{P}$ one can associate an algebraic operad $\widetilde{\mathscr{P}}$ by considering formal linear combinations of the elements. We define $\widetilde{\mathscr{P}}(n) := \mathbb{K}\mathscr{P}_n$, the $\mathbb{K}$-vector space spanned by $\mathscr{P}_n$. We call $\widetilde{\mathscr{P}}$ the *linearization* of $\mathscr{P}$. Often we will use the same notation for a set operad as for its linearization. It should be clear from the context which of the two is referred to.

To an element $(p_1, \ldots, p_k) \in \mathscr{P}_{i_1} \times \cdots \times \mathscr{P}_{i_k}$ one can associate a map

$$\mu_{p_1,\ldots,p_k} : \mathscr{P}_k \to \mathscr{P}_{i_1+\cdots+i_k}$$

defined as $\mu_{p_1,\ldots,p_k}(p) := \mu(p; p_1, \ldots, p_k)$. The following definition was introduced in [55] since it is a crucial property for set operads in order to use the poset method.

Definition 3.11. A set operad $\mathscr{P}$ is called a *basic-set operad* if the map $\mu_{p_1,\ldots,p_k}$ is injective for all $(p_1, \ldots, p_k) \in \mathscr{P}_{i_1} \times \cdots \times \mathscr{P}_{i_k}$.

Proposition 3.12. *The operad ${}^2\mathscr{C}om$ is the linearization of a basic-set operad.*

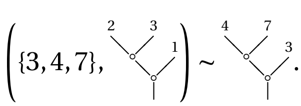
Proof. The operad ${}^2\mathscr{C}om$ is the linearization of $\mathscr{P}$, where $\mathscr{P}_n = \{D_i^n\}$ and D_i^n are as in Proposition 3.21. It is immediate from the formula for the composition product that $\mathscr{P}$ is basic-set. □

3.2.2 Operadic partition posets

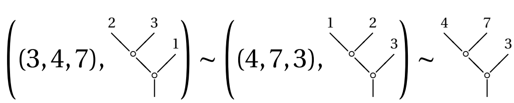
For definitions of the various notions related to posets see [5, 55].

Definition. Let $\mathscr{P}$ be a set operad. A *$\mathscr{P}$-partition* of $[n]$ is the data $\{(B_1, p_1), \ldots, (B_s, p_s)\}$, where $\{B_1, \ldots, B_s\}$ is a partition of $[n]$ and $p_i \in \mathscr{P}_{|B_i|}$. We let $\Pi_{\mathscr{P}}(n)$ denote the set of all $\mathscr{P}$-partitions of $[n]$ and let $\Pi_{\mathscr{P}}$ denote the collection $\{\Pi_{\mathscr{P}}(n)\}_{n\in\mathbb{N}}$. For an algebraic operad $\mathscr{O}$ which is the linearization of a set operad $\mathscr{P}$, i.e. $\mathscr{O} = \widetilde{\mathscr{P}}$, we will sometimes write $\Pi_{\mathscr{O}}$ for $\Pi_{\mathscr{P}}$.

Remark 3.13. One can think of this as enriching a partition with elements of an operad or, shifting the perspective, as labeling the input of the operation that an element $p_i \in \mathscr{P}_{|B_i|}$ describes with the elements of the block B_i instead of with $[|B_i|]$. E.g. one can identify



The definition in [55] uses ordered sequences of elements of the blocks instead of unordered blocks and then considers equivalence classes of pairs (S_B, p), where S_B is an ordered sequence of the elements of a block B where each element appears exactly once and $p \in \mathscr{P}_{|S_B|}$. E.g.



Our definition corresponds to choosing the representative of a class with the elements of the sequence in ascending order. In the following we will assume that, given a partition $\alpha = \{(A_1, p_1), \ldots, (A_r, p_r)\}$, the elements of a block $A_i = \{a^i_1, \ldots, a^i_{m_i}\}$ are indexed in ascending order, i.e. $a^i_j < a^i_{j+1}$.

Next we define a partial order on $\Pi_{\mathscr{P}}(n)$.

Definition 3.14. Let $\alpha = \{(A_1, p_1), \ldots, (A_r, p_r)\}$ and $\beta = \{(B_1, q_1), \ldots, (B_s, q_s)\}$ be two $\mathscr{P}$-partitions of $[n]$. We define $\alpha \leq \beta$ if

(i) $\{A_1, \ldots, A_r\}$ is a refinement of $\{B_1, \ldots, B_s\}$, i.e. each B_j is the union of one or more A_i.

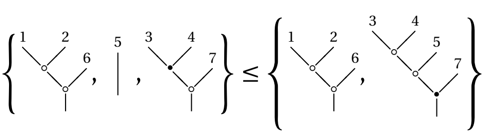
(ii) when $B_j = A_{i_1} \cup \cdots \cup A_{i_t}$ then there exists a $p \in \mathscr{P}_t$ such that $q_j = \mu(p; p_{i_1}, \ldots, p_{i_t})\sigma^{-1}$, where $\sigma \in \mathbb{S}_{|B_j|}$ is the obvious permutation associated to

$$\begin{pmatrix} b^j_1 \ldots b^j_{|B_j|} \\ a^{i_1}_1 \ldots a^{i_t}_{m_{i_t}} \end{pmatrix}.$$

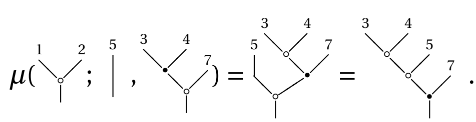
We call $\Pi_{\mathscr{P}}$ together with this partial order the *operadic partition poset* of $\mathscr{P}$.

Remark. We define the order in the opposite way to the one in [55] to make it correspond to the way it is defined in [10]. Note that with this in mind our definition leads to the same ordering of the corresponding equivalence classes.

Example. Using the identification in Remark 3.13 we see that in $\Pi_{2\mathscr{C}om}(7)$



since



In [55], Vallette studied homological properties of the order complex associated to the partition poset of an operad. The following is the main result.

Theorem 3.15 (Theorem 9 of [55])**.** *Let $\mathscr{P}$ be a basic-set quadratic operad. The operad $\widetilde{\mathscr{P}}$ is Koszul if and only if each subposet $[\hat{0}, \gamma]$ of each $\Pi_{\mathscr{P}}(n)$ is Cohen-Macaulay, where γ is a maximal element of $\Pi_{\mathscr{P}}(n)$.*

3.2.3 Fiber product of operadic partition posets

In [6] a product of posets was introduced under the name Segre product and a particular case was studied. We prefer to call it the fiber product because it corresponds to this categorical construction.

Definition. Let P,Q and S be posets. Given poset maps $f\colon P \to S$ and $g\colon Q \to S$ we define $P \times_{f,g} Q$, the *fiber product* of P and Q over f, g, to be the subset of $P \times Q$ consisting of pairs (p,q) such that $f(p) = g(q)$. The order on $P \times_{f,g} Q$ is induced by the order on $P \times Q$ which is given by $(p,q) \leq (p',q')$ if $p \leq p'$ and $q \leq q'$.

Let Π_n denote the poset of partitions of $[n]$ and let Π denote the collection $\{\Pi_n\}_{n\in\mathbb{N}}$. Further, given operadic partition posets $\Pi_{\mathscr{P}}$ and $\Pi_{\mathscr{Q}}$, let $f\colon \Pi_{\mathscr{P}} \to \Pi$ and $g\colon \Pi_{\mathscr{Q}} \to \Pi$ be the natural projections which send an element $\alpha = \{(A_1,p_1),\ldots,\ (A_m,p_m)\} \in \Pi_{\mathscr{P}}$ to the underlying partition $\{A_1,\ldots,A_m\}$ and similarly for g. Then $\Pi_{\mathscr{P}} \times_{f,g} \Pi_{\mathscr{Q}}$ consists of pairs (α,β), where $\alpha = \{(A_1,p_1),\ldots,(A_m,p_m)\}$ and $\beta = \{(A_1,q_1),\ldots,(A_m,q_m)\}$. This poset is isomorphic to the poset consisting of elements $\alpha = \{(A_1,p_1,q_1),\ldots,(A_m,p_m,q_m)\}$, where $p_i \in \mathscr{P}_{|A_i|}$ and $q_i \in \mathscr{Q}_{|A_i|}$, with the order given by $\alpha \leq \alpha'$ if

(i) $\{A_1,\ldots,A_r\}$ is a refinement of $\{A'_1,\ldots,A'_s\}$.

(ii) when $A'_j = A_{i_1}\cup\cdots\cup A_{i_t}$ then there exist a $p \in \mathscr{P}_t$ and a $q \in \mathscr{Q}_t$ such that $p'_j = \mu(p;p_{i_1},\ldots,p_{i_t})\sigma^{-1}$ and $q'_j = \mu(q;q_{i_1},\ldots,q_{i_t})\sigma^{-1}$, where $\sigma \in \mathbb{S}_{|A'_j|}$ is the permutation given in Definition 3.14.

We will denote this fiber product by $\Pi_{\mathscr{P}} \times_\Pi \Pi_{\mathscr{Q}}$. Note that $\Pi_{\mathscr{C}om} = \Pi$ whence $\Pi_{\mathscr{P}} \times_\Pi \Pi_{\mathscr{C}om} = \Pi_{\mathscr{P}}$, for any $\mathscr{P}$.

Definition. The *Hadamard product* $\mathscr{P} \times_H \mathscr{Q}$ of two set operads $\mathscr{P}$ and $\mathscr{Q}$ is defined as $(\mathscr{P} \times_H \mathscr{Q})_n = \mathscr{P}_n \times \mathscr{Q}_n$, where $\times$ denotes the Cartesian product. The composition μ is given by

$$\mu((p,q);(p_1,q_1),\ldots,(p_k,q_k)) := (\mu(p,;p_1,,\ldots,p_k),\mu(q;q_1,\ldots,q_k)).$$

Proposition 3.16. *For any set operads $\mathscr{P}$, $\mathscr{Q}$ the following equalities hold.*

(i) $\Pi_{\mathscr{P}\times_H\mathscr{Q}} = \Pi_{\mathscr{P}} \times_\Pi \Pi_{\mathscr{Q}}$

(ii) $\widetilde{\mathscr{P} \times_H \mathscr{Q}} = \widetilde{\mathscr{P}} \otimes_H \widetilde{\mathscr{Q}}$

Proof. Immediate from the definitions involved. □

Next we describe the operadic partition poset associated to an operad encoding totally compatible structures.

Corollary 3.17. *Let $\mathscr{O}$ be an algebraic operad which is the linearization of a set operad $\mathscr{P}$. Then*

(i) ${}^2\mathscr{O} = \widetilde{\mathscr{P} \times_H {}^2\mathscr{C}}om$ *and*

(ii) $\Pi_{{}^2\mathscr{O}} = \Pi_{\mathscr{P}} \times_\Pi \Pi_{{}^2\mathscr{C}om}$.

Proof. Using Corollary 3.7 and Propositions 3.10 and 3.16 (ii) we have that

$$^2\mathscr{O} = \mathscr{O} \circ {}^2\mathscr{C}om = \mathscr{O} \otimes_H {}^2\mathscr{C}om = \widetilde{\mathscr{P} \times_H {}^2\mathscr{C}om}.$$

Thus by Proposition 3.16 (i) we have $\Pi_{^2\mathscr{O}} = \Pi_{\mathscr{P}} \times_H \Pi_{^2\mathscr{C}om}$. □

Let $\mathscr{P}$ be a set operad. We define $^2\mathscr{P} := \mathscr{P} \times_H {}^2\mathscr{C}om$ and observe that $\widetilde{^2\mathscr{P}} = {}^2\widetilde{\mathscr{P}}$.

Proposition 3.18. *Let $\mathscr{P}$ and $\mathscr{Q}$ be set operads. If $\mathscr{P}$ and $\mathscr{Q}$ are basic-set, then so is $\mathscr{P} \times_H \mathscr{Q}$.*

Proof. We want to show that the map

$$\mu_{(\nu_1,\eta_1),\dots,(\nu_k,\eta_k)} : \mathscr{P}_k \times \mathscr{Q}_k \to \mathscr{P}_{i_1+\dots+i_k} \times \mathscr{Q}_{i_1+\dots+i_k}$$

given by

$$\mu_{(\nu_1,\eta_1),\dots,(\nu_k,\eta_k)}(\alpha,\beta) = (\mu(\alpha;\nu_1,\dots,\nu_k), \mu(\beta;\eta_1,\dots,\eta_k))$$

is injective for all $((\nu_1,\eta_1),\dots,(\nu_k,\eta_k)) \in (\mathscr{P}_{i_1} \times \mathscr{Q}_{i_1}) \times \dots \times (\mathscr{P}_{i_k} \times \mathscr{Q}_{i_k})$. Now let $(\alpha,\beta),(\alpha',\beta') \in \mathscr{P}_k \times \mathscr{Q}_k$ be such that $(\alpha,\beta) \neq (\alpha',\beta')$. Then $\alpha \neq \alpha'$ or $\beta \neq \beta'$ and thus, since $\mathscr{P}$ and $\mathscr{Q}$ are basic-set, either $\mu(\alpha;\nu_1,\dots,\nu_k) \neq \mu(\alpha';\nu_1,\dots,\nu_k)$ or $\mu(\beta;\eta_1,\dots,\eta_k)) \neq \mu(\beta';\eta_1,\dots,\eta_k))$. □

Corollary 3.19. *Let $\mathscr{P}$ be a basic-set operad, then so is $^2\mathscr{P}$.*

Proof. By Proposition 3.12 we know that $^2\mathscr{C}om$ is basic-set. Thus we can apply Proposition 3.18 to $^2\mathscr{P} = \mathscr{P} \times_H {}^2\mathscr{C}om$. □

3.3 Koszulness of a class of compatible structures

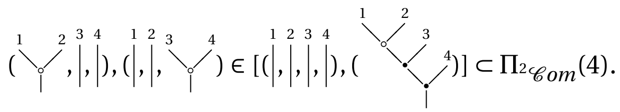
3.3.1 $^2\mathscr{C}om$ and weighted partitions

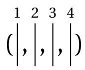
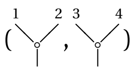
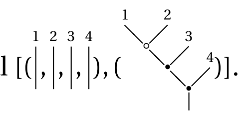
Theorem 3.15 was used in [54] and [10] to show the Koszulness of several operads. There it was shown that for the associated posets all maximal intervals $[\hat{0},\gamma]$ were totally semimodular. Hence by Corollary 5.2 of [5] they are CL-shellable and by Proposition 2.3 of the same paper shellable whence it follows that they are Cohen-Macaulay by Theorem 4.2 of [19]. The chain of implications is

$$\text{totally semimodular} \implies \text{CL-shellable} \implies \text{shellable} \implies \text{Cohen-Macaulay}. \tag{3.3}$$

Definition. A finite poset P is called *semimodular* if it is bounded, i.e. has a least and a greatest element, and for any distinct $\kappa, \lambda \in P$ covering a $\nu \in P$ there exists a $\omega \in P$ covering both κ and λ. The poset P is said to be *totally semimodular* if it is bounded and all intervals $[\zeta, \xi]$ are semimodular.

Remark. Contrary to the claims in [14], the maximal chains of $\Pi_{^2\mathscr{C}om}$ are not necessarily totally semimodular. For example, consider the elements



They both cover  but the only element covering both of them is  which does not belong to the interval .

By the chain of implications (3.3) we see that to show Cohen-Macaulayness of $\Pi_{^2\mathscr{C}om}$ and thus Koszulness of $^2\mathscr{C}om$, it is in fact sufficient to show that the maximal intervals of $\Pi_{^2\mathscr{C}om}$ are CL-shellable. A poset is CL-shellable if a certain kind of labeling of the maximal chains is possible, see [5]. By Theorem 3.2 of [5], showing CL-shellability of a poset is equivalent to showing that it admits a recursive atom ordering. Recall that the atoms of a poset are the elements covering $\hat{0}$.

Definition. A graded poset P admits a *recursive atom ordering* if the length of the poset is 1 or if the length is greater than 1 and there is an ordering $\alpha_1, \ldots, \alpha_m$ of the atoms of P satisfying

(i) For all $j \in [m]$, $[\alpha_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[\alpha_j, \hat{1}]$ that come first in the ordering are those that cover some α_i, where $i < j$.

(ii) For all $i < j$, if $\alpha_i, \alpha_j < \lambda$ then there is a $k < j$, not necessarily distinct from i, and an element $\kappa \leq \lambda$ such that κ covers both α_j and α_k

We will soon see that $\Pi_{^2\mathscr{C}om}$ admits a recursive atom ordering, but first we make the structure of $\Pi_{^2\mathscr{C}om}$ explicit by the following partition poset.

Definition. Given a partition $\beta = \{B_1, \ldots, B_s\}$ of $[n]$, we assign a weight $w_i \in \mathbb{N}$ to each block $B_i = \{b^i_1, \ldots, b^i_{k_i}\}$, with $0 \leq w_i \leq k_i - 1$. The weight of the block is denoted by $\mathrm{w}(B_i) := w_i$. The weight of a partition β is $\mathrm{w}(\beta) := \mathrm{w}(B_1) + \cdots + \mathrm{w}(B_s)$. We call a partition with this extra structure a *weighted partition* and we denote the set of weighted partitions of $[n]$ by Π^{w}_n. The collection $\{\Pi^{\mathrm{w}}_n\}_{n \in \mathbb{N}}$ is denoted by Π^{w}.

Let $\mathrm{n}(\beta)$ be the number of blocks of β. Then we can define a partial order on Π^{w}_n by letting $\alpha \leq \beta$ if

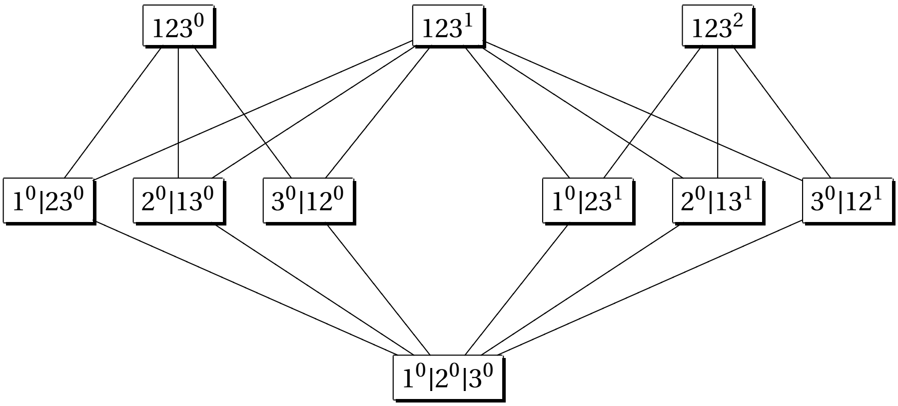


Figure 3.1: The poset Π_3^{w}

(i) the partition of α is a refinement of the partition of β and

(ii) $\mathrm{w}(\beta) - \mathrm{w}(\alpha) \leq \mathrm{n}(\alpha) - \mathrm{n}(\beta)$.

We call Π^{w} together with this partial order the *poset of weighted partitions.*

Remark 3.20. We see that the covering relation $\lessdot$ of the above partial order is given by $\alpha \lessdot \beta$ if

(i) the partition of α is a refinement of that of β obtained by splitting exactly one block of β into two and

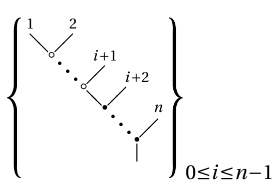
(ii) $0 \leq \mathrm{w}(\beta) - \mathrm{w}(\alpha) \leq 1$.

Any element α of Π_n^{w} can be described by $\alpha = \{(A_1, w_1), \ldots, (A_m, w_m)\}$ where $\{A_1, \ldots A_r\}$ is a partition of $\{1, \ldots, n\}$ and $w_i = \mathrm{w}(A_i)$. We observe that Π_n^{w} is a pure poset, i.e. all maximal chains have the same length.

Remark. In Figure 3.1. the weight w of a block $B = \{b_1, \ldots, b_k\}$ is indicated by $b_1 \cdots b_k^w$. For example, the block $\{1,2\}$ with weight 1 is denoted by 12^1.

To show the relation between the posets Π_n^{w} and $\Pi_{^2\mathscr{C}om}(n)$ we present a basis of $^2\mathscr{C}om$ and the composition product with respect to this basis.

Proposition 3.21. *We have that* $^2\mathscr{C}om(n) = \mathbb{1}_n \oplus \cdots \oplus \mathbb{1}_n$, *where the sum consists of* n *terms. In terms of labeled trees decorated with* $\mathscr{M}^\vee$, *a* $\mathbb{K}$*-basis for* $^2\mathscr{C}om(n)$ *is given by*



.

Denote by D_i^n the basis element in ${}^2\mathscr{C}om(n)$ corresponding to i white products. The composition product in ${}^2\mathscr{C}om$ is then given by

$$\mu(D_i^n; D_{i_1}^{m_1}, \ldots, D_{i_n}^{m_n}) = D_{i+i_1+\cdots+i_n}^{m_1+\cdots+m_n}$$

Proof. This follows from the relations of ${}^2\mathscr{C}om$ being homogeneous in the number of black and white products. Thus any element of ${}^2\mathscr{C}om(n)$ is determined by the number of white products, which can be at most $n-1$. □

Proposition 3.22. *The poset $\Pi_{{}^2\mathscr{C}om}(n)$ is isomorphic to Π_n^w.*

Proof. There is an obvious bijection between the elements of $\Pi_{{}^2\mathscr{C}om}(n)$ and Π_n^w where a block B enriched with an element $D_i^{|B|}$ with i white product(s) corresponds to the same block B with weight i in Π_n^w.

Now let $\alpha = \{(A_1, p_1), \ldots, (A_m, p_m)\}$ be a ${}^2\mathscr{C}om$-partition, then β covers α if and only if

$$\beta = \{(A_j \cup A_k, \mu(\curlyvee_{\bullet}; p_j, p_k)), (A_1, p_1), \ldots, \widehat{(A_j, p_j)}, \ldots, \widehat{(A_k, p_k)}, \ldots, (A_m, p_m)\}$$

or

$$\beta = \{(A_j \cup A_k, \mu(\curlyvee_{\circ}; p_j, p_k)), (A_1, p_1), \ldots, \widehat{(A_j, p_j)}, \ldots, \widehat{(A_k, p_k)}, \ldots, (A_m, p_m)\}.$$

The first case corresponds to increasing the weight by one when merging two blocks of a weighted partition and the second case to keeping it constant, which precisely is the covering relation of Π_n^w. □

3.3.2 Proof of Koszulness

Lemma 3.23. *Let $\mathscr{P}$ be a weakly associative binary quadratic set operad such that the maximal intervals of $\Pi_{\mathscr{P}}$ are totally semimodular. Then the maximal intervals of $\Pi_{{}^2\mathscr{P}}$ are CL-shellable.*

Proof. By Propositions 3.17 and 3.22 we have that $\Pi_{{}^2\mathscr{P}} = \Pi_{\mathscr{P}} \times_\Pi \Pi_{{}^2\mathscr{C}om} = \Pi_{\mathscr{P}} \times_\Pi \Pi^w$. By Theorem 3.2 of [5] CL-shellable is equivalent to admitting a recursive atom ordering. We aim to show that $\Pi_{\mathscr{P}} \times_\Pi \Pi^w$ admits such an ordering.

When denoting decorated partitions we will suppress the blocks only containing one element e.g.

$$\{(\{i,j\}, p), (\{k,l\}, p')\} = \{(\{i,j\}, p), (\{k,l\}, p'), (\{1\}, 1), \ldots, \widehat{(\{i\}, 1)}, \ldots, \widehat{(\{j\}, 1)}, \ldots, \widehat{(\{k\}, 1)}, \ldots, \widehat{(\{l\}, 1)}, \ldots, (\{n\}, 1)\}.$$

Denote the maximal elements $\{([n], p, w)\}$ of $\Pi_{\mathscr{P}}(n) \times_{\Pi_n} \Pi_n^w$ by $\mu_{p,w}$. Similarly denote the maximal elements $\{([n], p)\}$ of $\Pi_{\mathscr{P}}(n)$ by μ_p. Assume that the length of a maximal interval $[\hat{0}, \mu_{p,w}]$ is greater than 1, otherwise we are done. We may also assume that the weight w satisfies $0 < w < n-1$. Otherwise $[\hat{0}, \mu_{p,w}]$ is isomorphic to $[\hat{0}, \mu_p] \in \Pi_{\mathscr{P}}(n)$ which is totally semimodular by assumption. Thus by Corollary 5.2 of [5] it is CL-shellable.

Denote the atom $\{(\{i,j\},p)\} \in \Pi_{\mathscr{P}}(n)$ by $\alpha_{i,j}^{p}$. Similarly denote the atoms of $\Pi_{\mathscr{P}}(n) \times_{\Pi_n} \Pi_n^{w}$ by $\alpha_{i,j}^{p,w}$.

For a maximal interval $[\hat{0}, \mu_{p,w}]$ with $w > 0$ we claim that any ordering of the form

$$\alpha_{i_1,j_1}^{p_1,0} \dashv \alpha_{i_1,j_1}^{p_1,1} \dashv \cdots \dashv \alpha_{i_1,j_1}^{p_r,0} \dashv \alpha_{i_1,j_1}^{p_r,1} \dashv \cdots \\ \dashv \alpha_{i_m,j_m}^{p_1,0} \dashv \alpha_{i_m,j_m}^{p_1,1} \dashv \cdots \dashv \alpha_{i_m,j_m}^{p_r,0} \dashv \alpha_{i_m,j_m}^{p_r,1} \tag{3.4}$$

satisfies the second criterion of being a recursive atom ordering, where $\{(\{i_k,j_k\},p_1),\ldots,(\{i_k,j_k\},p_r)\}$ is some indexing of the atoms in $[\hat{0},\mu_p]$ and $\alpha \dashv \beta$ means that α is less than β in the atom ordering. Note that given an atom $\alpha_{i_t,j_t}^{p_s,v} \in [\hat{0}, \mu_{p,w}]$, the atom $\alpha_{i_1,j_1}^{p_r,\tilde{v}}$ will also be in the interval since we assume $0 < w < n-1$. Here $\tilde{v}$ denotes the element of $\{0,1\} \setminus \{v\}$.

Let $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ be distinct atoms with $\alpha_{i,j}^{p_1,w_1} \dashv \alpha_{k,l}^{p_2,w_2}$ and suppose $\alpha_{i,j}^{p_1,w_1}$,$\alpha_{k,l}^{p_2,w_2} \le \gamma$, for some $\gamma = \{(C_1,q_1,v_1),\ldots,(C_s,q_s,v_s)\} \in [\hat{0}, \{([n],p,w)\}]$. We want to show that there is a $\delta \le \gamma$ and an $\alpha_{i',j'}^{p',w'} \dashv \alpha_{k,l}^{p_2,w_2}$ such that $\alpha_{i',j'}^{p',w'}, \alpha_{k,l}^{p_2,w_2} \prec \delta$. Let $\gamma' = \{(C_1,q_1),\ldots,(C_s,q_s)\}$. We have three main cases to consider:

(I) $\{i,j\}$=$\{k,l\}$. Since the length of $[\hat{0}, \{([n],p,w)\}]$ is greater than 1 we have that $n \ge 3$. We get two further subcases:

- (i) $p_1 = p_2$: Since $\alpha_{i,j}^{p_1,w_1}$,$\alpha_{i,j}^{p_1,w_2} \le \gamma$ and $w_1 \ne w_2$ there must be at least one decorated block (C_r,q,u) of γ such that $\alpha_{i,j}^{p_1} \le \{(C_r,q)\}$ for some $q \in \mathscr{P}_{|C_r|}$ and $|C_r| \ge 3$. Further, since $\mathscr{P}$ is weakly associative there exist $q' \in \mathscr{P}_3$ and $m \in C_r \setminus \{i,j\}$, for some r, such that $\delta' = \{(\{i,j,m\},q')\} \succ \alpha_{i,j}^{p_1}$ and $\delta' \le \gamma'$. Then $\delta = \{(\{i,j,m\},q',\max(w_1,w_2))\} \le \gamma$ and covers $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$.
- (ii) $p_1 \ne p_2$: Since $\Pi_{\mathscr{P}}(n)$ is totally semimodular there exists a $\delta' = \{(\{i,j,m\},p)\} \in [\hat{0},\gamma']$ covering both $\alpha_{i,j}^{p_1}$ and $\alpha_{k,l}^{p_2}$. Then $\delta = \{(\{i,j,\ m\},q,v)\} \le \gamma$ covers both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$, where

$$v = \begin{cases} \max(w_1,w_2) & \text{if } w_1 \ne w_2 \\ w & \text{if } w_1 = w_2 = w \\ w_1 + 1 & \text{if } w_1 = w_2 \le w \end{cases}$$

(II) $\{i,j\} \cap \{k,l\} = \{m\}$, for some $m \in \{i,j\}$. Let m' be the element of $\{k,l\} \setminus \{m\}$. Since both atoms are less then γ we must have that $\{i,j,m'\}$ is a subset of a block C_r in γ. Since $\Pi_{\mathscr{P}}(n)$ is totally semimodular there exists a $\delta' = \{(\{i,j,m'\},q)\} \in [\hat{0},\gamma']$ covering both $\alpha_{i,j}^{p_1}$ and $\alpha_{k,l}^{p_2}$. Then $\delta = \{(\{i,j,m'\},q,v)\}$ is an element covering both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ and which is less than γ, where v is as in case (ii).

(III) $\{i,j\} \cap \{k,l\} = \emptyset$. Here we have two subcases:

(i) $w_1 \neq w_2$: $\delta = \{(\{i,j\},p_1,w_1),(\{k,l\},p_2,w_2)\}$ covers both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ and will always be less than or equal to any γ greater than both atoms.

(ii) $w_1 = w_2$: By the ordering of the atoms $\alpha_{i,j}^{p_1,w_1} \dashv \alpha_{k,l}^{p_2,w_2}$ implies $\alpha_{i,j}^{p_1,\tilde{w}_1} \dashv \alpha_{k,l}^{p_2,w_2}$, where $\tilde{w}_1$ is the element in $\{0,1\} \setminus \{w_1\}$. Now since $\alpha_{i,j}^{p_1,w_1}, \alpha_{k,l}^{p_2,w_2} \leq \gamma$ either $\delta = \{(\{i,j\},p_1,w_1),(\{k,l\},p_2,w_2)\} \leq \gamma$ or $\tilde{\delta} = \{(\{i,j\},p_1,\tilde{w}_1),(\{k,l\},p_2,w_2)\} \leq \gamma$, where δ covers $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ whereas $\tilde{\delta}$ covers $\alpha_{i,j}^{p_1,\tilde{w}_1}$ and $\alpha_{k,l}^{p_2,w_2}$.

We also need to show that, given an ordering of the form (3.4), any interval $[\alpha_{i,j}^{q,v},\mu_{p,w}]$ satisfies the first criterion of being a recursive atom ordering. Note that $\alpha_{i,j}^{q} \leq \mu_p$ implies that there exists a $p' \in \mathscr{P}_{n-1}$ such that $p = \mu(p';q,1,\ldots,1)$ whence we observe that $[\alpha_{i,j}^{q,v},\mu_{p,w}] \cong [\hat{0},\mu_{p',w-v}] \subset \Pi_{\mathscr{P}}(n-1) \times_{\Pi_{n-1}} \Pi_{n-1}^{\mathrm{w}}$.

Thus checking the above step is readily done if we may order the atoms of $[\alpha_{i,j}^{q,v},\mu_{p,w}]$ in the same way as above. We only need to show that some way of ordering the atoms in pairs as above satisfies that the first atoms are the ones covering some atom $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$. After that we can proceed by induction.

We may assume that the length of $[\alpha_{i,j}^{q,v},\mu_{p,w}]$ is greater than 1, since otherwise we are done. We may also assume that $0 < w-v < n-2$, since otherwise the interval $[\alpha_{i,j}^{q,v},\mu_{p,w}]$ is isomorphic to $[\alpha_{i,j}^{q},\mu_p] \subset \mathscr{P}_n$ which is totally semimodular by assumption. Thus in the case that $w-v=0$ or $w-v=n-2$ we have by Theorem 5.1 of [5] that any ordering of the atoms is a recursive atom ordering. We would therefore be able to freely order the atoms of $[\alpha_{i,j}^{q,v},\mu_{p,w}]$ so that the atoms that come first are those that cover some atom less than $\alpha_{i,j}^{q,v}$ in the ordering (3.4).

Now the atoms are either of the form $\{(\{i,j\},q,v),(\{k,l\},t,u)\}$ which we denote by $\beta_{k,l}^{t,u}$ or of the form $\{(\{i,j,k\},t,v+u)\}$ which we denote by $\beta_{k}^{t,u}$, where $u \in \{0,1\}$. Let $\tilde{u}$ be the element of $\{0,1\} \setminus \{u\}$.

We have that $\beta_{k,l}^{t,u}$ covers some $\alpha_{i',j'}^{q',v'} \dashv \alpha_{i,j}^{q,v}$, namely $\alpha_{i',j'}^{q',v'} = \alpha_{k,l}^{t,u}$, if and only if $\alpha_{k,l}^{t,u} \dashv \alpha_{i,j}^{p,w}$. Since by the atom ordering of $[\hat{0},\mu_{p,w}]$ we have that $\alpha_{k,l}^{t,u} \dashv$

$\alpha^{q,v}_{i,j}$ if and only if $\alpha^{t,\tilde{u}}_{k,l} \dashv \alpha^{q,v}_{i,j}$, we have that $\beta^{t,u}_{k,l}$ covers some $\alpha^{q',v'}_{i',j'} \dashv \alpha^{q,v}_{i,j}$ if and only if $\beta^{t,\tilde{u}}_{k,l}$ covers some $\alpha^{q',v'}_{i',j'} \dashv \alpha^{q,v}_{i,j}$.

Similarly we have that $\beta^{t,u}_{k}$ may cover some $\alpha^{q',v'}_{i',j'} \dashv \alpha^{q,v}_{i,j}$, where $\{i',j'\} \subset \{i,j,k\}$ and q' is an appropriate element of $\mathscr{P}_2$. Again $\alpha^{q',v'}_{i',j'} \dashv \alpha^{p,w}_{i,j}$ if and only if $\alpha^{q',\tilde{v}'}_{i',j'} \dashv \alpha^{q,v}_{i,j}$. Hence $\beta^{t,u}_{k}$ covers some $\alpha^{q',v'}_{i',j'} \dashv \alpha^{q,v}_{i,j}$ if and only if $\beta^{t,\tilde{u}}_{k}$ does.

Thus we may order the atoms of $[\alpha^{q,v}_{i,j}, \mu_{p,w}]$ by first putting all pairs of atoms, differing only in weight, covering some atom less than $\alpha^{q,v}_{i,j}$ followed by all pairs of atoms not covering any atom less than $\alpha^{q,v}_{i,j}$. Using the aforementioned identification $[\alpha^{q,u}_{i,j}, \mu_{p,w}] \cong [\hat{0}, \mu_{p',w-u}]$, we proceed by induction.

□

Remark. Note that the assumption that $\mathscr{P}$ is weakly associative and the assumption that the maximal intervals of its associated poset are totally semimodular are both necessary for the proof to go through. Both are used, in subcase (I)(i) and e.g. subcase (I)(ii), respectively, and neither of the two properties implies the other.

Theorem 3.24. *Let $\mathscr{P}$ be a weakly associative binary quadratic basic-set operad such that the maximal intervals of $\Pi_{\mathscr{P}}$ are totally semimodular, then ${}^2\widetilde{\mathscr{P}}$ and $(\widetilde{\mathscr{P}}^!)^2$ are Koszul.*

Proof. By Lemma 3.23 and the chain of implications (3.3) we obtain that the maximal intervals of $\Pi_{{}^2\mathscr{P}}$ are Cohen-Macaulay and by Corollary 3.19 we see that ${}^2\mathscr{P}$ is basic-set. Thus we can apply Theorem 3.15 and conclude that ${}^2\widetilde{\mathscr{P}}$ is Koszul. Since ${}^2\widetilde{\mathscr{P}}$ and $(\widetilde{\mathscr{P}}^!)^2$ are Koszul dual to each other we are done. □

We get the following immediate corollary.

Corollary 3.25. *The following operads are Koszul: ${}^2\mathscr{C}om$, $\mathscr{L}ie^2$, ${}^2\mathscr{P}erm$, $\mathscr{P}re\mathscr{L}ie^2$, ${}^2\mathscr{C}om\mathscr{T}rias$, $\mathscr{P}ost\mathscr{L}ie^2$, ${}^2\mathscr{A}s$, $\mathscr{A}s^2$, ${}^2\mathscr{D}ias$, $\mathscr{D}end^2$, ${}^2\mathscr{T}rias$ and $\mathscr{T}ri\mathscr{D}end^2$.*

Proof. The operads $\mathscr{C}om$, $\mathscr{P}erm$, $\mathscr{C}om\mathscr{T}rias$, $\mathscr{A}s$, $\mathscr{D}ias$, and $\mathscr{T}rias$ are all algebraic operads which are linearizations of weakly associative basic-set operads whose associated posets have totally semimodular maximal intervals. The other operads are their Koszul dual operads. See [55, 10] for these results and definitions of the operads. □

3.4 The minimal resolution of $\mathscr{L}ie^2$

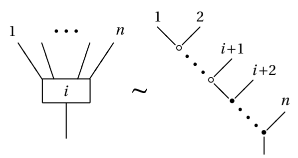
3.4.1 The resolution

The minimal resolution of the operad $\mathscr{L}ie^2$ of pairs of compatible Lie algebras will play an important role in the interpretation of bi-Hamiltonian structures on formal graded manifolds as algebraic structures on the structure sheaf. As a first application of the techniques presented in Chapter 2 we compute this resolution.

Theorem 3.26. *The minimal resolution $(\mathscr{L}ie^2)_\infty$ of the operad $\mathscr{L}ie^2$ is the quasi-free operad on the $\mathbb{S}$-module $\mathscr{E} = \{\mathscr{E}(n)\}_{n\geq 2}$ where*

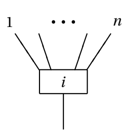
$$\mathscr{E}(n) = \begin{cases} \underbrace{\mathrm{sgn}_n \oplus \cdots \oplus \mathrm{sgn}_n}_{n \text{ copies}}[n-2] & \text{if } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

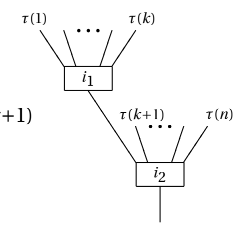
Denote the natural basis element of $\mathscr{E}(n)$ corresponding to the element of ${}^2\mathscr{C}om$ with i white products by



$$0 \leq i \leq n-1.$$

The differential of $(\mathscr{L}ie^2)_\infty$ is then given by

$$\delta :$$ 

$$\mapsto \sum_{\substack{2\leq k\leq n-1 \\ i_1+i_2=i \\ \tau\in\mathbb{S}^{\text{un-sh}}_{(k,n-k)}}} (-1)^{\mathrm{sgn}(\tau)+(k-1)(n-k+1)}$$ 

.

Proof. From the Koszulness of $\mathscr{L}ie^2$ it follows by Theorem 2.4 that $(\mathscr{L}ie^2)_\infty = \Omega((\mathscr{L}ie^2)^{¡})$ is a quasi-free resolution of $\mathscr{L}ie^2$. Since $\mathscr{L}ie^2$ has no differential, whence $(\mathscr{L}ie^2)^{¡}$ has no codifferential, it follows from the definition of the cobar construction that the resolution is minimal. The underlying $\mathbb{S}$-bimodule of the cobar construction is given by $\Omega((\mathscr{L}ie^2)^{¡}) = \mathscr{F}(\Sigma\overline{(\mathscr{L}ie^2)^{¡}})$. We observed in §2.3.4 that for an operad $\mathscr{P}$ we have $(\mathscr{P}^{¡})_{(s)}(n) \cong \Sigma^{-s}((\mathscr{P}^!)_{(s)}(n))^\vee$. The weight-grading of $\mathscr{C}om^1$ is given by the number of vertices of a decorated graph. Thus $\mathscr{C}om^1(n)$ is concentrated in weight $n-1$ and it follows from Proposition 8.3 that

$$(\mathscr{L}ie^2)^{¡}(n) = \underbrace{\mathrm{sgn}_n \oplus \cdots \oplus \mathrm{sgn}_n}_{n \text{ copies}}[n-1].$$

Setting $\mathscr{E} = \overline{\Sigma(\mathscr{L}ie^2)^{¡}}$ the first assertion of the theorem follows.

A derivation of a free operad is completely determined by its restriction to the $\mathbb{S}$-module of generators. Since $(\mathscr{L}ie^2)^{¡}$ has no codifferential it follows that the differential δ of $\Omega((\mathscr{L}ie^2)^{¡})$ is fully determined by the decomposition coproduct of $(\mathscr{L}ie^2)^{¡}$. Through straightforward graph calculations one can determine the composition product of $(\mathscr{L}ie^2)^!$. Considering the linear dual of this product yields a coproduct on $((\mathscr{L}ie^2)^!)^*$ which induces the coproduct of $(\mathscr{L}ie^2)^{¡}$.

□

We denote the operad of the above theorem by $\mathscr{L}ie^2_\infty$.

3.4.2 Algebras over the resolved operad

Algebras over the operad $\mathscr{L}ie^2_\infty$ are defined as follows:

Definition. A dg vector space V together with a family $\{{}_k l_n\}_{n\geq 1, 1\leq i\leq n}$ of maps ${}_i l_n\colon \wedge^n V \to V$ of degree $2-n$ is called an L^2_∞*-algebra* if the following condition is satisfied for all $n, k \geq 1$ with $2 \leq k \leq n+1$

$$\sum_{\substack{r+s=n+1\\ i+j=k\\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(s,r-1)}}} \epsilon(\sigma)\,\mathrm{sgn}(\sigma)(-1)^{r(s-1)}\,{}_i l_r({}_j l_s(v_{\sigma(1)},\dots v_{\sigma(s)}), v_{\sigma(s+1)},\dots,v_{\sigma(n)}).$$

Here the sign $\epsilon(\sigma)$ is the sign appearing from the Koszul-Quillen sign rule.

Remark. Note that the subfamilies $\{{}_1 l_n\}_{n\geq 1}$ and $\{{}_n l_n\}_{n\geq 1}$ both are L_∞-algebras (see §6.1.2) sharing the same differential ${}_1 l_1$. The rest of the brackets model the higher homotopies of the compatibility of the brackets ${}_1 l_2$ and ${}_2 l_2$. If these two are the only non-zero brackets, then an L^2_∞-algebra is a pair of compatible Lie algebras.

Remark. As pointed out by the referee of Paper II, a family of morphisms $\{{}_k l_n\}_{n\in\mathbb{N}\geq 1, 1\leq k\leq n}$ is an L^2_∞-algebra if and only if for all $\lambda\in\mathbb{K}$ the family of morphisms $\{L_n\}_{n\in\mathbb{N}\geq 1}$, where $L_n = \sum_{k=1}^n \lambda^{k-1}\,{}_k l_n$, is an L_∞-algebra, i.e. an L^2_∞-algebra is a non-linear pencil of L_∞-algebras.

4 Two methods for showing Koszulness

To compute prop profiles of differential geometric structures we need to establish that the involved $\mathfrak{G}^*$-algebras are Koszul. In this chapter we review two methods to this end that will come in handy: W. Gan's distributive law method for dioperads [18] and E. Hoffbeck's PBW-basis method for operads [23].

4.1 Distributive laws for dioperads

4.1.1 The operads of a dioperad

To a dioperad $\mathscr{P}$ one can associate its *opposite dioperad* defined by $\mathscr{P}^{\mathrm{op}}(m,n) := \mathscr{P}(n,m)$. The composition product μ^{op} is obtained from μ by reversing the direction of all graphs. Further, from a quadratic dioperad $\mathscr{P}$ we can extract two operads $\mathscr{P}_u$ and $\mathscr{P}_d$ defined by $\mathscr{P}_u(n) := \mathscr{P}(1,n)$ and $\mathscr{P}_d(n) := \mathscr{P}^{\mathrm{op}}(1,n)$. Explicitly, for a binary quadratic dioperad $\mathscr{P} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ we have

$$\mathscr{P}_u = \mathscr{F}(\mathscr{M}(1,2))/(\mathscr{R}(1,3)), \qquad \mathscr{P}_d = \mathscr{F}(\mathscr{M}(2,1)^{\mathrm{op}})/(\mathscr{R}(3,1)^{\mathrm{op}}),$$

where $\mathscr{R}(1,3)$ is the part of $\mathscr{R}$ in $\mathscr{F}_{(2)}(\mathscr{M})(1,3)$, $\mathscr{M}(2,1)^{\mathrm{op}}$ is the $\mathbb{S}$-module given by $\mathscr{M}(2,1)^{\mathrm{op}}(2) = \mathscr{M}(2,1)$ and zero otherwise, and $\mathscr{R}(3,1)^{\mathrm{op}}$ are the relations in $\mathscr{F}_{(2)}(\mathscr{M}(2,1)^{\mathrm{op}})$ obtained from $\mathscr{R}(3,1) \subset \mathscr{F}_{(2)}(\mathscr{M})(3,1)$ by reversing the direction of the decorated graphs.

We also note that to any operad $\mathscr{P}$ one can associate a dioperad $\widetilde{\mathscr{P}}$ defined by $\widetilde{\mathscr{P}}(1,n) := \mathscr{P}(n)$ and $\widetilde{\mathscr{P}}(m,n) = 0$ for $m \neq 1$.

4.1.2 Distributive laws

The inclusions $(\widetilde{\mathscr{P}_d})^{\mathrm{op}} \hookrightarrow \mathscr{P}$ and $\mathscr{P}_u \hookrightarrow \mathscr{P}$ induce $\mathbb{S}$-bimodule morphisms $\widetilde{\mathscr{P}_u}\Box(\widetilde{\mathscr{P}_d})^{\mathrm{op}} \to \mathscr{P}$ and $(\widetilde{\mathscr{P}_d})^{\mathrm{op}}\Box\widetilde{\mathscr{P}_u}$, see §1.3.5 for the definition of $\Box$. We say that a quadratic dioperad $\mathscr{P}$ is given by a *distributive law* if $\widetilde{\mathscr{P}_u}\Box(\widetilde{\mathscr{P}_d})^{\mathrm{op}} = \mathscr{P}$ or if $(\widetilde{\mathscr{P}_d})^{\mathrm{op}}\Box\widetilde{\mathscr{P}_u} = \mathscr{P}$.

The following theorem was proved by W. Gan, generalizing results in [34] and [13].

Theorem 4.1 (Theorem 5.9 of [18]). *Let $\mathscr{P}$ be a binary quadratic dioperad. If $\mathscr{P}_u$ and $\mathscr{P}_d$ are Koszul operads and $\widetilde{\mathscr{P}_u}\square(\widetilde{\mathscr{P}_d})^{\mathrm{op}}(i,j) = \mathscr{P}(i,j)$ (respectively, if $(\widetilde{\mathscr{P}_d})^{\mathrm{op}}\square\widetilde{\mathscr{P}_u}(i,j) = \mathscr{P}(i,j)$) for $(i,j) = (2,2),(2,3),(3,2)$, then $\widetilde{\mathscr{P}_u}\square(\widetilde{\mathscr{P}_d})^{\mathrm{op}} = \mathscr{P}$ (respectively $(\widetilde{\mathscr{P}_d})^{\mathrm{op}}\square\widetilde{\mathscr{P}_u} = \mathscr{P}$) and $\mathscr{P}$ is Koszul.*

Example 4.2. It was shown in [18] that the dioperad $\mathscr{L}ie\mathscr{B}i$ of Example 2.3 is given by a distributive law.

4.1.3 Extending dioperadic resolutions

To compute the minimal resolution of a quadratic prop it is often easier to work in a sub-category if possible, e.g. when the quadratic relations are dioperadic, and then to extend the resolution to a resolution of props.

There exists a forgetful functor from the category of properads to the category of dioperads which is denoted by $\mathscr{U}^{\text{dioperad}}_{\text{properad}}$. It keeps the same underlying $\mathbb{S}$-bimodule but only allows composition along graphs of genus zero. The functor $\mathscr{U}^{\text{dioperad}}_{\text{properad}}$ has a left adjoint which is denoted by $\mathscr{F}^{\text{properad}}_{\text{dioperad}}$. For a quadratic dioperad $\mathscr{P} = \mathscr{F}^{\downarrow}_{c,0}(\mathscr{M})/(\mathscr{R})$ we have $\mathscr{F}^{\text{properad}}_{\text{dioperad}}(\mathscr{P}) = \mathscr{F}^{\downarrow}_{c}(\mathscr{M})/(\mathscr{R})$, where in the latter case $(\mathscr{R})$ is the properadic ideal generated by $\mathscr{R}$. The functor $\mathscr{F}^{\text{properad}}_{\text{dioperad}}$ is not exact, Theorem 47 of [45], however in the same paper it is proved, Proposition 50, that if a dioperad is given by a distributive law then a quasi-free resolution of the dioperad is still a resolution when this functor is applied.

The step from properads to props is less troublesome. There exists a similar pair of functors $\mathscr{U}^{\text{properad}}_{\text{prop}}$ and $\mathscr{F}^{\text{prop}}_{\text{properad}}$. Also here it is true that for a quadratic properad $\mathscr{P} = \mathscr{F}^{\downarrow}_{c}(\mathscr{M})/(\mathscr{R})$ we have $\mathscr{F}^{\text{prop}}_{\text{properad}}(\mathscr{P}) = \mathscr{F}^{\downarrow}(\mathscr{M})/(\mathscr{R})$, where $(\mathscr{R})$ is the propic ideal generated by $\mathscr{R}$. By §7.4 of [56] the functor $\mathscr{F}^{\text{prop}}_{\text{properad}}$ is exact.

Let $\mathscr{F}^{\text{prop}}_{\text{dioperad}}$ denote the composition $\mathscr{F}^{\text{prop}}_{\text{properad}} \circ \mathscr{F}^{\text{properad}}_{\text{dioperad}}$. For this composition of functors we observe:

Proposition 4.3. *Let $\mathscr{P} = \mathscr{F}^{\downarrow}_{c,0}(M)(R)$ be a quadratic dioperad given by a distributive law and let $\mathscr{Q} = (\mathscr{F}^{\downarrow}_{c,0}(E),\delta)$ be a quasi-free resolution of $\mathscr{P}$, then we have*

$$\mathscr{F}^{\text{prop}}_{\text{dioperad}}(\mathscr{P}) = \mathscr{F}^{\downarrow}(M)(R) \quad \textit{and} \quad \mathscr{F}^{\text{prop}}_{\text{dioperad}}(\mathscr{Q}) = (\mathscr{F}^{\downarrow}(E),\delta),$$

moreover, the latter is a quasi-free resolution of the former.

4.2 PBW-bases for operads

In [23] E. Hoffbeck generalized the notion of PBW-bases from quadratic algebras to quadratic operads. Our interest in PBW-bases lies in that if an operad has a PBW-basis then it is Koszul.

4.2.1 A planar representation of trees

By a *tree* we mean an element of $\mathfrak{G}_c^{\downarrow 1}$, i.e. a directed, rooted, labeled tree. We refer to the external output edge as the *root* and the external input edges simply as external edges. A tree is called *reduced* if all vertices have at least one input edge. If an $\mathbb{S}$-module satisfies $\mathcal{M}(0) = 0$, then all non-zero decorated trees in $\mathcal{F}_c^{\downarrow 1}(\mathcal{M})$ are reduced. An operad is called reduced if it is spanned by reduced trees. A reduced tree has a natural planar representation.

(i) To every $e \in E_v^{\text{in}}$ we associate the minimum of the labels of the external edges of the tree which are linked to e through a directed path (we consider an external input edge to be linked to itself).

(ii) We place the edges of E_v^{in} (and thus the vertices directly above) from left to right above v in ascending order.

4.2.2 An ordering of decorated trees

Let $\mathcal{M}$ be an $\mathbb{S}$-module, let $B^{\mathcal{M}}$ be a $\mathbb{K}$-basis of $\mathcal{M}$, and let $B^{\mathcal{F}(\mathcal{M})}$ denote all trees decorated with elements of $B^{\mathcal{M}}$. The set $B^{\mathcal{F}(\mathcal{M})}$ is a $\mathbb{K}$-basis of $\mathcal{F}(\mathcal{M})$. For a tree τ, we denote by $B_\tau^{\mathcal{F}(\mathcal{M})}$ the subset of $B^{\mathcal{F}(\mathcal{M})}$ consisting of the elements whose underlying tree is τ.

Given an order of the elements of $B^{\mathcal{M}}$, and using the above defined planar representations of trees, we define an order on $B^{\mathcal{F}(\mathcal{M})(n)}$. To each decorated tree in $\mathcal{F}(\mathcal{M})(n)$ we associate a sequence of n words in the alphabet $B^{\mathcal{M}}$ as follows. There is a unique path of vertices from the root of the tree to the external edge labeled by $i \in [n]$. Let a_i be the word consisting (from left to right) of the labels of these vertices (from bottom to top). Thus we obtain the sequence $\bar{a} = (a_1, \ldots, a_n)$. The words are ordered by the length lexicographical order; we first compare two words a and b by their length ($a < b$ if $l(a) < l(b)$, where l denotes the length) and if the lengths are equal we compare them lexicographically using the order on $B^{\mathcal{M}}$. We compare two sequences $\bar{a}$ and $\bar{b}$ associated to $\alpha, \beta \in B^{\mathcal{F}(\mathcal{M})}(n)$ by first comparing a_1 with b_1, next a_2 with b_2, and so forth. This order is compatible with the operad structure of $\mathcal{F}(\mathcal{M})$, see [23] for details. Another compatible order is obtained by considering the reverse length lexicographic order.

4.2.3 Definition of PBW-bases

To each internal edge e of a tree τ we define the restricted tree τ_e as follows. The vertices of τ_e are the two vertices v_1, v_2 adjacent to e. The edges of τ_e are all edges adjacent to v_1 and v_2. The external input edges of τ_e are given labels according to which labels are directly linked to them; the external edge of τ_e linked to the external edge of τ labeled by 1 is given this label, the external edge of τ_e linked to the external edge of τ which has the least of the labels not linked to the previous edge is given the label 2, and so forth.

Definition. Let $B^{\mathscr{M}}$ be an ordered $\mathbb{K}$-basis of $\mathscr{M}$. A *PBW-basis* for a quadratic operad $\mathscr{P} = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ with respect to the order on $B^{\mathscr{M}}$ is a $\mathbb{K}$-basis $B^{\mathscr{P}} \subset B^{\mathscr{F}(\mathscr{M})}$ of $\mathscr{P}$ containing $\mathbb{1}$ and satisfying the conditions:

(i) for $\alpha \in B^{\mathscr{F}(\mathscr{M})}$, either $\alpha \in B^{\mathscr{P}}$ or the elements of the basis $\gamma_i \in B^{\mathscr{P}}$ which appear in the unique decomposition $\alpha \equiv \sum_i c^i \gamma_i$, satisfy $\gamma_i > \alpha$,

(ii) a decorated tree $\alpha \in B_\tau^{\mathscr{F}(\mathscr{M})}$ is in $B_\tau^{\mathscr{P}}$ if and only if for every internal edge e of τ, the restricted decorated tree $\alpha|_{\tau_e}$ is in $B_{\tau_e}^{\mathscr{P}}$.

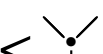
The following result makes it easier to verify that a given basis is a PBW-basis.

Proposition 4.4 (Proposition 3.9 of [23])**.** *Let $\mathscr{M}$ be finitely generated. If condition* (i) *is verified when the underlying tree of α has two vertices and condition* (ii) *is satisfied, then condition* (i) *is satisfied for all α.*

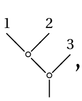
Using a filtration induced by the ordering on $\mathscr{F}(\mathscr{M})$ Hoffbeck was able to prove the following theorem.

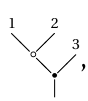
Theorem 4.5 (Theorem 3.10 of [23])**.** *A reduced operad which has a PBW-basis is Koszul.*

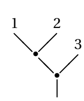
In fact we already have come across an example of a PBW-basis of an operad.

Proposition 4.6. *The basis of ${}^2\mathscr{C}om$ presented in Proposition 3.21 is a PBW-basis with respect to the length lexicographic order defined by*  $<$ .

Proof. Condition (ii) is easily verified. Using Proposition 4.4 it is not hard to see that



,


, and


are maximal among the decorated graphs equal to them. □

Part II:

Differential geometric structures and prop profiles

5 Poisson and Nijenhuis geometry

In this chapter we recall the definitions of Poisson and Nijenhuis structures for ordinary and graded manifolds. We also present definitions of compatible Poisson structures, called bi-Hamiltonian structures, and introduce a definition of compatible Nijenhuis structures.

5.1 Poisson geometry

In this section we recall basic facts concerning Poisson structures.

5.1.1 Poisson structures

Let M be a manifold and denote by $\mathscr{O}_M$ the *structure sheaf* of M, i.e. the sheaf of commutative $\mathbb{K}$-algebras of smooth functions on M. A *Poisson bracket* on M is an operation $\{\_,\_\}\colon \mathscr{O}_M \otimes \mathscr{O}_M \to \mathscr{O}_M$ satisfying the following properties:

(i) $\{f,g\} = -\{g,f\}$ (skew-symmetry)

(ii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)

(iii) $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (Leibniz property of $\{f,\_\}$).

Thus a Poisson bracket is a Lie bracket on $\mathscr{O}_M$ which in each argument acts as a derivation with respect to the multiplication of smooth functions on $\mathscr{O}_M$. A Poisson bracket is often called a *Poisson structure.*

5.1.2 Polyvector fields and the Schouten bracket

To a manifold M there is associated the *tangent sheaf* $\mathscr{T}_M$ of derivations of $\mathscr{O}_M$. Global sections of the tangent sheaf are called *vector fields* and there is a Lie bracket on $\mathscr{T}_M$ given by

$$[X,Y] := X \circ Y - Y \circ X.$$

Consider now the exterior algebra $\wedge^\bullet \mathscr{T}_M$ of *polyvector fields,* where the exterior algebra is considered over $\mathscr{O}_M$. It has a natural grading given by the tensor length, i.e. $\wedge^i \mathscr{T}_M$ are precisely the elements of degree i. The bracket of $\mathscr{T}_M$ can be extended to a degree -1 Lie bracket on $\wedge^\bullet \mathscr{T}_M$ through a standard construction. The extended bracket

$$[\_,\_]_{\tilde{S}} \colon \wedge^k \mathcal{T}_M \wedge_{\mathbb{K}} \wedge^l \mathcal{T}_M \to \wedge^{k+l-1} \mathcal{T}_M, \tag{5.1}$$

which we call the *odd Schouten bracket*, is defined by

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l]_{\tilde{S}} := \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_l,$$

for $k \geq 1$, $l = 0$, i.e $Y_0 \in \mathcal{O}_M$, by

$$[X_1 \wedge \cdots \wedge X_k, Y_0]_{\tilde{S}} := \sum_i (-1)^{i+k} X_i(Y_0) \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k,$$

for $k = 0$, $l \geq 1$ by

$$[X_0, Y_1 \wedge \cdots \wedge Y_l]_{\tilde{S}} := \sum_j (-1)^j Y_j(X_0) \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_l,$$

and for $k = l = 0$ by

$$[X_0, Y_0]_{\tilde{S}} := 0.$$

Note that with the degree shift $\wedge^\bullet \mathcal{T}_M[1]$, the odd Schouten induces the ordinary degree zero *Schouten bracket* $[\_,\_]_S$, cf. §6.1.4. We will usually suppress this degree shift from the notation, but keep the distinction in the notation of the brackets.

5.1.3 Poisson structures as Maurer-Cartan elements

The *cotangent sheaf* of a manifold M is defined by $\Omega^1_M := \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{T}_M, \mathcal{O}_M)$ and the *de Rham algebra* by $\Omega^\bullet_M := \wedge^\bullet \Omega^1_M$, with multiplication given by the wedge product. Note that there is a natural isomorphism $\Omega^i_M \cong \mathrm{Hom}_{\mathcal{O}_M}(\wedge^i \mathcal{T}_M, \mathcal{O}_M)$. The elements of Ω^i_M are called *i-forms*. The morphism $d \colon \mathcal{O}_M \to \Omega^1_M$, defined by $df(X) := X(f)$ for a smooth function f and a vector field X, extends to a differential on $\Omega^\bullet_M$ called the *de Rham differential*.

From a *bivector field* P, i.e. an element of $\wedge^2 \mathcal{T}_M$, one obtains an operation

$$\{\_,\_\}_P \colon \mathcal{O}_M \otimes \mathcal{O}_M \to \mathcal{O}_M$$

defined by

$$\{f, g\}_P := df \wedge dg(P).$$

This operation satisfies properties (i) and (iii) of §5.1.1 since P is a bivector field. Conversely, any bilinear operation $\mathcal{O}_M \wedge \mathcal{O}_M \to \mathcal{O}_M$ satisfying the properties (i) and (iii) can be described by a bivector field in this way. The condition that $\{\_,\_\}_P$ satisfies the Jacobi identity is equivalent to $[P,P]_{\tilde{S}} = 0$. Thus the following definition is equivalent to the one given in §5.1.1.

Definition 5.1. A *Poisson structure* on a manifold M is a polyvector field $P \in \wedge^2 \mathcal{T}_M$ satisfying $[P, P]_{\tilde{S}} = 0$.

Under the degree shift $(\wedge^\bullet \mathcal{T}_M)[1]$ the elements of $\wedge^2 \mathcal{T}_M$ are of degree one. Since the Schouten algebra of polyvector fields has no differential, a Poisson-structure is precisely a Maurer-Cartan element (§2.4.3) in $((\wedge^\bullet \mathcal{T}_M)[1], [\_,\_]_S)$.

5.1.4 Generalized Poisson structures

In fact one does not need to consider only the solutions of $[P, P]_S = 0$ which are of degree two. One generalization of Poisson geometry is to n-ary Poisson brackets. For n even, a polyvector field $P \in \wedge^n \mathcal{T}_M$ defines a *generalized Poisson structure* if $[P, P]_S = 0$. The associated n-ary Poisson bracket is defined analogously to the case $n = 2$; for a polyvector field $P \in \wedge^n \mathcal{T}_M$ it is given by

$$\{f_1, \ldots, f_n\} := df_1 \wedge \cdots \wedge df_n(P).$$

The condition $[P, P]_S = 0$ translates into a generalized Jacobi identity. Notice that for polyvector fields of a non-graded manifold the expression $[P, P]_S$ identically vanishes for n odd. It is possible to define a Poisson bracket with properties mimicking the classical case also for n odd, but then the generalized Jacobi identity can not be expressed through the Schouten bracket. See e.g. [12] and [57] for more on n-ary Poisson brackets. In Section 7.3 we study a generalization of Poisson structures in the setting of graded manifolds.

5.1.5 Bi-Hamiltonian structures

Let M be a manifold equipped with a pair of Poisson brackets $\{\_,\_\}_\circ$ and $\{\_,\_\}_\bullet$. Consider the bracket defined by their sum

$$\{\_,\_\} := \{\_,\_\}_\circ + \{\_,\_\}_\bullet.$$

This bracket is obviously skew symmetric and it satisfies the Leibniz property, but it does not always satisfy the Jacobi identity. The Poisson brackets $\{\_,\_\}_\circ$ and $\{\_,\_\}_\bullet$ are called *compatible* if their sum satisfies the Jacobi identity (i.e. if they are linearly compatible as Lie brackets, cf. §3.1.1) and thus itself is a Poisson bracket.

Definition. A pair of compatible Poisson structures is called a *bi-Hamiltonian structure* or a *Poisson pair*.

Let $P_\circ$ and $P_\bullet$ be bivector fields corresponding to a pair of Poisson brackets, thus they satisfy $[P_\circ, P_\circ]_S = 0$ and $[P_\bullet, P_\bullet]_S = 0$, respectively. The compatibility of the Poisson brackets is equivalent to $[P_\circ + P_\bullet, P_\circ + P_\bullet]_S = 0$ which in turn, if $P_\circ$ and $P_\bullet$ are Poisson structures, is equivalent to $[P_\circ, P_\bullet]_S = 0$. By introducing a formal parameter $\hbar$, the above conditions are together equivalent to

$$[P_\circ + P_\bullet \hbar, P_\circ + P_\bullet \hbar]_{S_\hbar} = 0.$$

Here the bracket is the linearization in $\hbar$ of the Schouten bracket.

5.2 Nijenhuis geometry

Here we review basic definitions concerning Nijenhuis structures and define a notion of compatibility.

5.2.1 Nijenhuis structures

To a morphism $J\colon \mathcal{T}_M \to \mathcal{T}_M$ one can associate a morphism $\mathcal{N}_J\colon \wedge^2 \mathcal{T}_M \to \mathcal{T}_M$, called the *Nijenhuis torsion*, defined by

$$\mathcal{N}_J(X,Y) := JJ[X,Y] + [JX,JY] - J[X,JY] - J[JX,Y].$$

Definition. We call an endomorphism J of $\mathcal{T}_M$ a *Nijenhuis structure* if it satisfies $\mathcal{N}_J = 0$.

5.2.2 Vector forms and the Frölicher-Nijenhuis bracket

A *vector field valued differential form*, or *vector form* for short, is a tensor field in $\Omega^\bullet_M \otimes \mathcal{T}_M$, where the tensor product is over $\mathcal{O}_M$. One can identify vector i-forms with homomorphisms $\mathrm{Hom}_{\mathcal{O}_M}(\wedge^i \mathcal{T}_M, \mathcal{T}_M)$; for $K = \omega \otimes X \in \Omega^i_M \otimes \mathcal{T}_M$ the associated morphism is defined by $K(Y) := \omega(Y)X$, where $Y \in \wedge^i \mathcal{T}_M$.

In [17] Frölicher and Nijenhuis studied derivations of $\Omega^\bullet_M$. They showed that there exist two types of derivations, derivations of type i_* which vanish on $\mathcal{O}_M$ and of type d_* which commute with the de Rham differential, and that any derivation can be uniquely decomposed as a sum of two derivations, one of each type.

To any vector form $K \in \Omega^r_M \otimes \mathcal{T}_M$ one can associate two derivations: a derivation $i_K\colon \Omega^\bullet_M \to \Omega^{\bullet + r-1}_M$ of type i_* defined by

$$i_K(\omega)(Y_1 \wedge \cdots \wedge Y_{r+s-1}) := \sum_{\sigma \in \mathbb{S}^{\text{un-sh}}_{(r,s-1)}} \omega(K(Y_{\sigma(1)}, \dots Y_{\sigma(r)}), Y_{\sigma(r+1)}, \dots, Y_{\sigma(r+s-1)}),$$

for $\omega \in \Omega^s_M$ and $Y_1 \wedge \cdots \wedge Y_{r+s-1} \in \wedge^{r+s-1}\mathcal{T}_M$, and a derivation $d_K\colon \Omega^\bullet_M \to \Omega^{\bullet+r}_M$ of type d_* defined by $d_K := i_K \circ d - (-1)^{r-1} d \circ i_K$. Conversely, any derivation of respective type uniquely determines a vector form.

The commutator of two derivations is again a derivation and the subsets of derivations of type i_* and of type d_* are both closed under the formation of commutators. Thus, to a pair of vector forms K and L there exists an associated vector form $[K,L]_{\text{F-N}}$ uniquely determined by the condition

$$d_{[K,L]_{\text{F-N}}} = [d_K, d_L].$$

This was shown to be a Lie bracket by A. Nijenhuis [49] and is called the *Frölicher-Nijenhuis bracket*. A formula for this bracket is

$$\begin{aligned}[K,L]_{\text{F-N}} = \phi\wedge\psi\otimes[X,Y] + \phi\wedge d_X(\psi)\otimes Y - d_Y(\phi)\wedge\psi\otimes X \\ + (-1)^r\left(d\phi\wedge i_X(\psi)\otimes Y + i_Y\phi\wedge d(\psi)\otimes X\right),\end{aligned}$$

where $K=\phi\otimes X\in\Omega^r_M\otimes\mathcal{T}_M$ and $L=\psi\otimes Y\in\Omega^s_M\otimes\mathcal{T}_M$. It generalizes the ordinary Lie bracket of vector fields in the sense that when K and L are vector fields, then $[K,L]_{\text{F-N}}=[K,L]$.

5.2.3 Nijenhuis structures as Maurer-Cartan elements

Given two endomorphisms J and K of $\mathcal{T}_V$ one can consider the morphism $\mathcal{N}_{J,K}\colon \mathcal{T}_V\otimes\mathcal{T}_V\to\mathcal{T}_V$ defined by

$$\mathcal{N}_{J,K}(X,Y) = JK[X,Y] + [JX,KY] - J[X,KY] - K[JX,Y].$$

When J and K are *commuting*, i.e. $J\circ K = K\circ J$, we have that $\mathcal{N}_{J,K}$ corresponds to a (1,2) tensor field which in general is not alternating. It was introduced in [48] in the study of the problem of when eigenvectors of a tangent bundle endomorphism form an integrable distribution. It was further considered in [7] where an explicit description was given of the relation between $\mathcal{N}_J$ and $\mathcal{N}_{A,B}$ when A and B are polynomials in J.

For arbitrary $J,K\in\Omega^1_V\otimes\mathcal{T}_V$ Nijenhuis showed in [49] that the following identity holds:

$$\mathcal{N}_{J,K} + \mathcal{N}_{K,J} = [J,K]_{\text{F-N}}.$$

Thus the following definition is equivalent to the one given in §5.2.1.

Definition. A *Nijenhuis structure* on a manifold M is a vector form $J\in\Omega^1_M\otimes\mathcal{T}_M$ satisfying $[J,J]_{\text{F-N}}=0$.

In other words, a Nijenhuis structure is a Maurer-Cartan element in the Lie algebra $(\Omega^\bullet_M\otimes\mathcal{T}_M, [\_,\_]_{\text{F-N}})$.

5.2.4 Compatible Nijenhuis structures

We call two Nijenhuis structures J and K *compatible* if their sum again is a Nijenhuis structure and we call such a pair a *bi-Nijenhuis structure*. Note that this is equivalent to that $\alpha J + \beta K$ is a Nijenhuis structure for any $\alpha, \beta \in \mathbb{K}$. For the Nijenhuis torsion of the sum of two Nijenhuis structures J and K we have

$$\mathcal{N}_{J+K} = \mathcal{N}_J + \mathcal{N}_{J,K} + \mathcal{N}_{K,J} + \mathcal{N}_K = \mathcal{N}_{J,K} + \mathcal{N}_{K,J} = [J,K]_{\text{F-N}}.$$

The compatibility of J and K is thus equivalent to $[J,K]_{\text{F-N}} = 0$. Introducing a formal parameter $\hbar$ and considering the linearization in $\hbar$ of the Frölicher-Nijenhuis bracket, the pair J and K is a bi-Nijenhuis structure precisely when

$$[J + \hbar K, J + \hbar K]_{\text{F-N}} = 0.$$

Remark. The notion of compatible Nijenhuis structures has been defined differently elsewhere, e.g. in [30] it is defined to be what we call commuting Nijenhuis structures.

5.3 Structures on formal graded manifolds

5.3.1 Formal graded manifolds

A *formal manifold* is a manifold consisting of a formal neighborhood of a single point. Let (V, d) be a dg vector space with a homogeneous basis $\{e_a\}$ and associated dual basis $\{t^a\}$. We may view V as a formal graded manifold by considering a formal neighborhood of the origin. For the structure sheaf we have $\mathcal{O}_V \cong \mathbb{K}[\![t^a]\!]$ and the tangent sheaf $\mathcal{T}_V$ is generated as an $\mathcal{O}_V$-module by $\{\partial_a\}$, where we write ∂_a for $\frac{\partial}{\partial t^a}$. The cotangent sheaf Ω^1_V in turn, is generated by $\{dt^a\}$ with $dt^a(\partial_b) = \delta_{a,b}$.

5.3.2 Graded Poisson structures

A *graded Poisson bracket* on a formal graded manifold V is a degree zero bilinear operation $\{\_,\_\}\colon \mathcal{O}_V \wedge \mathcal{O}_V \to \mathcal{O}_V$ satisfying the properties

(i) $\{f,g\} + (-1)^{|f||g|}\{g,f\}$ (graded skew-symmetry)

(ii) $(-1)^{|f||h|}\{f,\{g,h\}\} + (-1)^{|g||f|}\{g,\{h,f\}\} + (-1)^{|h||g|}\{h,\{f,g\}\}$ (graded Jacobi identity)

(iii) $\{f, gh\} = \{f,g\}h + (-1)^{|f||g|} g\{f,h\}$ (Leibniz property of $\{f,\_\}$).

We see that a graded Poisson structure is a graded Lie algebra on $\mathcal{O}_V$ with the extra property that the Lie bracket is a graded derivation in each argument with respect to the graded commutative multiplication on $\mathcal{O}_V$.

5.3.3 Graded Poisson structures as bivector fields

The sheaf of polyvector fields is defined as

$$\wedge^\bullet \mathcal{T}_V := \odot^\bullet(\mathcal{T}_V[-1]),$$

where the symmetric product is considered over $\mathcal{O}_V$.

We denote the generators $s\partial a$ by v_a, where s is a formal symbol of degree one. Thus $|v_a| = -|t^a| + 1$. With this notation we have $\wedge^\bullet \mathcal{T}_V \cong \mathbb{K}[\![t, v]\!]$. The degree of a homogeneous polyvector field

$$P = P^{a_1 \dots a_i}_{b_1 \dots b_j} t^{b_1} \dots t^{b_j} v_{a_1} \cdots v_{a_i}$$

is given by

$$|P| = |t^{b_1}| + \cdots + |t^{b_j}| + |v_{a_1}| \cdots |v_{a_i}|.$$

Note that $\wedge^\bullet \mathcal{T}_V$ also has the grading described in §5.1.2; we will refer to this grading as the *weight* and to the former as the *cohomological degree* or simply as the degree. When V is concentrated in degree zero these gradings coincide.

In local coordinates the odd Schouten bracket is given by

$$[X, Y]_{\tilde{S}} := X \bullet Y + (-1)^{|X||Y|+|X|+|Y|} Y \bullet X$$

where we use the notation

$$X \bullet Y := \frac{\partial X}{\partial v_a} \frac{\partial Y}{\partial t^a}.$$

Note that the with the above grading the Schouten bracket is a degree -1 (cohomological as well as weight) Lie bracket and if V is concentrated in degree zero, then this definition coincides with (5.1). An interpretation of graded Poisson structures in terms of bivector fields vanishing on the Schouten bracket, analogous to that of §5.1.3 can be found in [8].

Definition. A *graded Poisson structure* on a formal graded manifold V is an element $P \in \wedge^2 \mathcal{T}_V$ of degree two satisfying $[P, P]_{\tilde{S}} = 0$.

Remark. Working with $\mathbb{Z}$-graded manifolds instead of with $\mathbb{Z}_2$-graded also called super manifolds introduces a technical detail on the grading of $\wedge^\bullet \mathcal{T}_V$ and $\Omega^\bullet_V$. For i-forms to induce morphisms $\wedge^i \mathcal{T}_V \to \mathcal{T}_V$ one should define the de Rham algebra as $\Omega^\bullet_V := \widehat{\odot}^\bullet(\Omega^1_V[1])$, where the symmetric product is considered over $\mathcal{O}_V$, causing i-forms of a manifold concentrated in degree zero to be of degree $-i$, but of *weight* i. We will work also with the definition $\Omega^\bullet_V := \widehat{\odot}^\bullet(\Omega^1_V[-1])$ (§5.3.1) as the link to the classical non-graded definition is clearer with this grading; weight and degree then coincide for ordinary manifolds.

5.3.4 Graded bi-Hamiltonian structures

A graded Bi-Hamiltonian structure on a formal manifold V is defined analogously to the non-graded case; it is a pair $P_\circ$ and $P_\bullet$ of graded Poisson structures such that their sum $P_\circ + P_\bullet$ again is a graded Poisson structure. In particular this implies that the associated Poisson brackets are a pair of compatible graded Lie brackets.

5.3.5 Nijenhuis structures on graded manifolds

A *graded Nijenhuis structure* is a degree zero morphism $J\colon \mathscr{T}_V \to \mathscr{T}_V$ such that the Nijenhuis torsion $\mathscr{N}_J$ vanishes.

Let us use the gradings $\Omega_V^\bullet := \widehat{\odot^\bullet}(\Omega_V^1[-1])$ and $\wedge^\bullet \mathscr{T}_V := \odot^\bullet(\mathscr{T}_V[1])$, cf. §5.3.3. Denoting sdt^a by γ^a, where s is a formal symbol of degree one, the de Rham algebra is given by $\Omega_V^\bullet \cong \mathbb{K}[\![t^a,\gamma^b]\!]$ and the de Rham differential by $d = \gamma^a\partial_a$. Using the interpretation $\Omega_V^i \otimes \mathscr{T}_V = \mathrm{Hom}_{\mathscr{O}_V}(\wedge^i\mathscr{T}_V, \mathscr{T}_V)$ a graded Nijenhuis structure is equivalent to a degree one element $J \in \Omega_V^1 \otimes \mathscr{T}_V$ such that $[J,J]_{\text{F-N}} = 0$.

Let

$$K = K^i_{[a_1\cdots a_p]}(t)\gamma^{a_1}\cdots\gamma^{a_p}\partial_i$$

and

$$L = L^j_{[c_1\cdots c_q]}(t)\gamma^{c_1}\cdots\gamma^{c_q}\partial_j$$

be vector forms, $p,q \geq 0$. In local coordinates the Frölicher-Nijenhuis bracket is defined by

$$\begin{aligned}[K,L]_{\text{F-N}} = \Big(&K^i_{[a_1\cdots a_p|}\partial_i L^j_{|a_{p+1}\cdots a_{p+q}]} - pK^j_{[a_1\cdots a_{p-1}|i|}\partial_{a_p} L^i_{a_{p+1}\cdots a_{p+q}]}\\ &(-1)^{pq}(L^i_{[a_1\cdots a_q|}\partial_i K^j_{|a_{q+1}\cdots a_{p+q}]} - qL^j_{[a_1\cdots a_{q-1}|i|}\partial_{a_q} K^i_{a_{q+1}\cdots a_{p+q}]})\Big)\\ &\gamma^{a_1}\cdots\gamma^{a_{p+q}}\partial_j.\end{aligned}$$

A pair of graded Nijenhuis structures are called *compatible* if their sum is a graded Nijenhuis structure.

5.3.6 Pointed manifolds and structures

A manifold V with a distinguished point p is called *pointed*. A differential geometric structure, e.g. a vector form or a polyvector field, is called *pointed* if it vanishes at the distinguished point.

Note that a formal graded manifold is naturally pointed with the origin as the distinguished (and only) point. We denote the origin by 0.

6 Operad profile of homological vector fields

A common property of the prop profiles computed by S.A. Merkulov [38, 39, 40, 42, 43, 44] is that they all contain the minimal resolution $\mathscr{L}ie^1_\infty$ of the operad $\mathscr{L}ie^1$ of odd Lie algebras as a suboperad. On the differential geometric side this corresponds to that the geometric structures all contain homological vector fields. In this chapter we will explain the correspondence between homological vector fields and representations of $\mathscr{L}ie^1_\infty$.

6.1 From vector fields to operads

6.1.1 Homological vector fields

A *homological vector field* on a formal graded manifold V is a degree one vector field Q such that $[Q,Q] = 0$, i.e. it is a Maurer-Cartan element in the dg Lie algebra $(\mathscr{T}_V, [\_,\_], 0)$. Homological vector fields were introduced by V. Šander in the study of differential equations on supermanifolds [51]. They are also useful in that various mathematical objects can be described in terms of homological vector fields, see e.g. [52, 53].

6.1.2 L_∞-algebras

Definition. A graded vector space V together with a family $\{l_n\}_{n\geq 1}$ of maps $l_n\colon \wedge^n V \to V$ of degree $2-n$ is called an L_∞*-algebra* if the following condition is satisfied for all $n \in \mathbb{N} \geq 1$

$$\sum_{\substack{r+s=n+1\\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(s,r-1)}}} \epsilon(\sigma)\,\mathrm{sgn}(\sigma)(-1)^{r(s-1)}\, l_r(l_s(v_{\sigma(1)},\dots v_{\sigma(s)}), v_{\sigma(s+1)},\dots, v_{\sigma(n)}). \qquad (6.1)$$

Here the sign $\epsilon(\sigma)$ is the sign appearing from the Koszul-Quillen sign rule.

It was observed by M. Kontsevich [26] that a homological vector field on a formal manifold V is equivalent to an L_∞-algebra structure on $V[-1]$. We will take a detour through the world of operads before arriving at this conclusion in order to set the stage for the operad and prop profiles to be studied in the following chapters.

6.1.3 Extracting an operad

To a degree one vector field

$$Q = \sum_{i\geq 1} Q^b_{(c_1\cdots c_i)} t^{c_1}\cdots t^{c_i}\partial_b$$

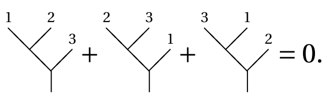
on a formal graded manifold V we can associate a family of degree one maps $\{q_i\colon \odot^i V \to V\}$ by

$$q_i(e_{c_1}\odot\cdots\odot e_{c_i}) = Q^b_{(c_1\cdots c_i)} e_b.$$

Of the operations of an L_∞-algebra the binary operation is fundamental while the higher operations are homotopies of its relations, see e.g. [27]. Encoding just the properties of this fundamental part in an operad captures the essence of the structure. The rest of the structure is the result of plugging the operad into the Koszul duality machinery. We depict the operation q_2 by the corolla $\curlyvee$. That q_2 is symmetric implies $\curlyvee(12) = \curlyvee$. Let us denote the part of Q corresponding to q_2 by $\hat{Q}$. The condition

$$[\hat{Q},\hat{Q}] = 0 \tag{6.2}$$

translates to



(6.3)

Definition. An *odd dg Lie algebra* is a dg vector space (V, d) together with a bilinear symmetric degree one map $[\_\bullet\_]$ satisfying the odd Jacobi identity

$$(-1)^{|u||w|+|u|+|w|}[u\bullet[v\bullet w]] + (-1)^{|v||u|+|v|+|u|}[v\bullet[w\bullet u]]+ \\ (-1)^{|w||v|+|w|+|v|}[w\bullet[u\bullet v]] = 0.$$

Next we define the operad governing odd Lie algebras.

Definition. The operad $\mathscr{L}ie^1$ is the quadratic operad $\mathscr{F}(\mathscr{M})/(\mathscr{R})$ where $\mathscr{M}$ is the $\mathbb{S}$-module given by $\mathscr{M}(2) = \mathbb{K}\curlyvee = \mathbb{1}_2[-1]$ and zero for other n, and the relations $\mathscr{R}$ are given by (6.3).

6.1.4 Odd versus ordinary Lie algebras

An odd Lie algebra on a vector space V naturally induces an ordinary Lie algebra structure on $V[-1]$ as follows. The vector space $V[-1]$ can be considered as $\mathbb{K}s\otimes V$, where s is a formal symbol of degree one. We will write sv for $s\otimes v$. Let $s\colon V\to V[-1]$ be the degree one morphism $v\mapsto sv$ and let $(V,[\_\bullet\_])$ be an odd Lie algebra. It is not hard to show that

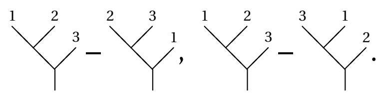
$$[\_,\_] := s\circ[\_\bullet\_]\circ s^{-1}\otimes s^{-1}\colon V[-1]\otimes V[-1]\to V[-1]$$

is a Lie bracket. Conversely, given an ordinary Lie bracket $[\_,\_]$ on $V[-1]$, the construction $[\_ \bullet \_] := s^{-1} \circ [\_,\_] \circ s \otimes s \colon V \otimes V \to V$ defines an odd Lie bracket on V. Rephrasing this correspondence we observe that a representation of the operad $\mathscr{L}ie^1$ in V is equivalent to a representation of $\mathscr{L}ie$ in $V[-1]$.

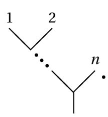
6.2 An application of Koszul duality

6.2.1 Koszul dual of $\mathscr{L}ie^1$

Proposition 6.1. *The Koszul dual operad of $\mathscr{L}ie^1$ is the quadratic operad $\mathscr{C}om^1 := \mathscr{F}(\mathscr{N})/(\mathscr{S})$, where $\mathscr{N}$ is the $\mathbb{S}$-module defined by $\mathscr{N}(2) = \mathbb{K}\curlyvee = \mathrm{sgn}_2[1]$ and $\mathscr{N}(n) = 0$ for $n \neq 2$, and the relations $\mathscr{S}$ are given by*



The underlying $\mathbb{S}$-bimodule of $\mathscr{C}om^1$ is given by $\mathscr{C}om^1(n) = \mathrm{sgn}_n[n-1]$. In terms of the operations $\curlyvee$, a $\mathbb{K}$-basis for $\mathscr{C}om^1(n)$ is given by the element



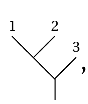
Proof. The first claim is a straightforward verification using the definition of Koszul dual operads §2.3.4. The second claim follows from graph computations using the relations $\mathscr{S}$. □

6.2.2 Koszulness of $\mathscr{L}ie^1$

To calculate a minimal resolution of $\mathscr{L}ie^1$ we first need to establish Koszulness of $\mathscr{L}ie^1$.

Proposition 6.2. *The operad $\mathscr{C}om^1$ is Koszul.*

Proof. Using the notation of §4.2.2, let $B^{\mathscr{N}} = \curlyvee$. We claim that the basis in Proposition 6.1 is a PBW-basis of $\mathscr{C}om^1$. To the basis element of $\mathscr{C}om^1(n)$ we associate the sequence of words $(w_{n-1}, w_{n-1}, w_{n-2} \ldots, w_1)$, where w_i is the word which contains the letter $\curlyvee$ i times. Clearly this is the largest sequence of words associated to a decorated tree in $B^{\mathscr{F}(\mathscr{N})(n)}$, whence condition (i) is satisfied. Since the basis element of $\mathscr{C}om^1(n)$ is the unique decorated tree of $B^{\mathscr{F}(\mathscr{N})(n)}$ for which the restricted decorated tree of any internal edge is



also condition (ii) is satisfied. □

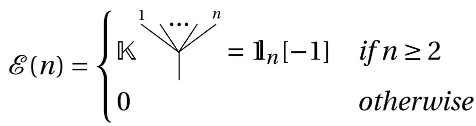
By Theorem 2.5 we obtain the following immediate corollary.

Corollary 6.3. *The operad $\mathscr{L}ie^1$ is Koszul.*

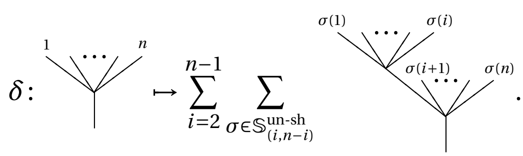
6.2.3 The minimal resolution of $\mathscr{L}ie^1$

The Koszulness of $\mathscr{L}ie^1$ enables us to prove the following:

Theorem 6.4. *The minimal resolution $\mathscr{L}ie^1_\infty$ of $\mathscr{L}ie^1$ is the quasi-free operad $(\mathscr{F}(\mathscr{E}),\delta)$, where*



and the differential δ is given by

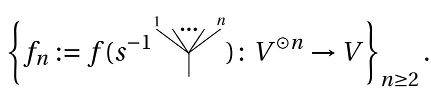


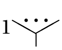
Proof. The proof is completely analogous to that of Theorem 3.26. □

6.3 From operads to vector fields

6.3.1 An isomorphism of Lie algebras

Recall from §2.4.3 that to any dg operad $\mathscr{P}$ and dg vector space (V,d) there is an associated dg Lie algebra $\mathscr{L}_{\mathscr{P}}(V)$. In the case when $\mathscr{P}$ is Koszul, representations of $\mathscr{P}_\infty$ in V are Maurer-Cartan elements in this Lie algebra. An arbitrary element $f \in \mathscr{L}_{\mathscr{L}ie^1}(V)$ is determined by the image of the generators of $\overline{(\mathscr{L}ie^1)^{¡}}$ and can thus be considered as a family of linear maps



We note that s^{-1} is of degree zero and from now on omit s^{-1} from the notation.

From f we construct a vector field $X_f \in \mathscr{T}_V \cong \mathbb{K}[\![t^b]\!] \otimes \mathbb{K}\partial_a$ on the formal graded manifold associated to V as follows. For a vector field X we denote by X_n the part of X of polynomial degree n in the variables $t^b q$. We define X_f by

$$(X_f)_n := \frac{1}{n!} X^a_{(b_1 \cdots b_n)} t^{b_1} \cdots t^{b_n} \partial_a,$$

where the coefficients $X^a_{(b_1 \cdots b_n)}$ are given by $f_n(e_{b_1} \odot \cdots \odot e_{b_n}) = X^a_{(b_1 \cdots b_n)} e_a$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{K}[\![t^b]\!]$. We define a subset of vector fields

$$\widetilde{\mathcal{T}_V} := \{X = X^a(t)\partial_a \in \mathcal{T}_V \ \mid \ X^a(t) \in \mathfrak{m}^2\},$$

further, we note that $\widetilde{\mathcal{T}_V}$ is a Lie subalgebra of $\mathcal{T}_V$ and that

$$\widetilde{\mathcal{T}_V} = \{X_f \ \mid \ f \in \mathcal{L}_{\mathcal{L}ie^1}(V)\}.$$

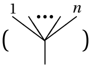
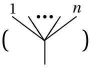
In the same manner the differential d of V corresponds to a degree one vector field D, linear in t; let $D = D^a_b t^b \partial_a$, where $D^a_b \in \mathbb{K}$ are defined by $d(e_a) = D^b_a e_b$. That d is a differential, i.e. $d^2 = 0$, is equivalent to $[D, D] = 0$. From this it follows that $\delta_D := [D, \_]$ defines a differential on $\mathcal{T}_V$ and we note that it is compatible with the Lie bracket and that $\delta_D(\widetilde{\mathcal{T}_V}) \subset \widetilde{\mathcal{T}_V}$. Thus $(\widetilde{\mathcal{T}_V}, [\_,\_], \delta_D)$ is a dg Lie algebra. We let $(\widetilde{\mathcal{T}}^{\mathrm{op}}_V, [\_,\_]^{\mathrm{op}}, \delta^{\mathrm{op}}_D)$ denote the opposite Lie algebra defined by $[X, Y]^{\mathrm{op}} := [Y, X]$.

Theorem 6.5. *The morphism*

$$\Phi \colon \mathcal{L}_{\mathcal{L}ie^1}(V) \to \widetilde{\mathcal{T}}^{\mathrm{op}}_V, \ \ f \mapsto X_f$$

is an isomorphism of dg Lie algebras.

Proof. Let $f \in \mathcal{L}_{\mathcal{L}ie^1}(V)$ be a homogeneous element of degree r. Explicitly, this means that if a structure coefficient $X^a_{(b_1 \cdots b_n)}$ of f is non-zero, then $|e_a| - |e_{b_1}| - \cdots - |e_{b_n}| = r$. Since $|\partial_a| = |e_a|$ and $|t^b| = -|e_b|$ it follows that X_f is of degree r. That the maps f_n are graded symmetric is equivalent to that X_f is graded commutative in the variables t^b. Applying the differential δ to f we obtain a new family of maps $\{\delta(f)_n\}_{n \geq 2}$ where

$\delta(f)_n := \delta(f)($  $) = d \circ f($  $) = d(f_n) =$

$$d \circ_1 f_n - (-1)^r \sum_{i=1}^{n} f_n \circ_i d = \sum d \circ_1 f_n - (-1)^r n f_n \circ_1 d.$$

From this we get that

$$\Phi(\delta(f))_n = \left(\frac{1}{n!} X^c_{b_1 \cdots b_n} D^a_c - (-1)^r \frac{1}{(n-1)!} D^c_{b_1} X^a_{c b_2 \cdots b_n} \right) t^{b_1} \cdots t^{b_n} \partial_a =$$

$$([X_f, D])_n = \delta^{\mathrm{op}}_D (\Phi(f))_n.$$

Thus Φ defines an isomorphism of dg vector spaces.

Now let f and g be homogeneous elements of $\mathscr{L}_{\mathscr{L}ie^1}(V)$ of degrees r and s, respectively, and let $Y^a_{b_1\cdots b_n}$ denote the structure coefficients of the morphisms $g(\overset{\cdots}{\curlyvee})\colon V^{\odot n} \to V$. By definition

$$
\begin{aligned}
&[f,g]\Big(\overset{1\,\cdots\,n}{\curlyvee}\Big) = \\
&\qquad \sum_{G\in\mathfrak{G}_{(2)}} \mu_G \circ \Big((G,[f^d\otimes g^u]) - (-1)^{rs}(G,[g^d\otimes f^u])\Big)\circ_G \Delta\Big(\overset{1\,\cdots\,n}{\curlyvee}\Big) = \\
&\qquad\qquad \sum_{\substack{i+j=n+1\\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(j,i-1)}}} (f_i\circ_1 g_j - (-1)^{rs} g_j \circ_1 f_i)\circ\phi_\sigma.
\end{aligned}
$$

Here $\phi_\sigma\colon V^{\otimes n} \to V^{\otimes n}$ denotes the morphism defined by

$$
\phi_\sigma\colon (-1)^\epsilon v_1\otimes\cdots\otimes v_n \mapsto v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)},
$$

where ϵ is determined by the Koszul-Quillen sign-rule. Note that the cardinality of $\mathbb{S}^{\text{un-sh}}_{(i,j)}$ is $(i+j)!/(i!j!)$. Now it follows that

$$
\begin{aligned}
&\Phi([f,g])_n = \\
&\quad \sum_{i=1}^{n-1} \frac{1}{i!(n-i)!}\Big(Y^c_{b_1\cdots b_i} X^a_{cb_{i+1}\cdots b_n} - (-1)^{rs} X^c_{b_1\cdots b_i} Y^a_{cb_{i+1}\cdots b_n}\Big) t^{b_1}\cdots t^{b_n}\partial_a = \\
&\qquad\qquad [X_g, X_f]_n = [\Phi(f),\Phi(g)]^{\text{op}}_n.
\end{aligned}
$$

□

6.3.2 Representations of $\mathscr{L}ie^1_\infty$ and homological vector fields

For an element $X_f \in \widetilde{\mathscr{T}}_V$ we have

$$
\begin{aligned}
[D+X_f, D+X_f] = 0 &\Longleftrightarrow \delta_D(X_f) + \frac{1}{2}[X_f,X_f] = 0 \\
&\Longleftrightarrow \delta^{\text{op}}_D(X_f) + \frac{1}{2}[X_f,X_f]^{\text{op}} = 0,
\end{aligned}
$$

i.e. X_f is a Maurer-Cartan element in $(\widetilde{\mathscr{T}}_V, [\_,\_], \delta_D)$ if and only if $D+X_f$ is a pointed Maurer-Cartan element in $(\mathscr{T}_V, [\_,\_])$.

Recall that we consider the differential of a vector space to be part of the representation of an operad in V, see §2.1.3. From the isomorphism between $\mathscr{L}_{\mathscr{L}ie^1}(V)$ and $\widetilde{\mathscr{T}}_V$ we thus obtain the following corollary.

Corollary 6.6. *There is a one-to-one correspondence between representations of $\mathscr{L}ie^1_\infty$ in a vector space V and pointed homological vector fields on the formal manifold associated to V.*

Remark 6.7. That the homological vector field considered in Corollary 6.6 are pointed is no serious restriction. Given an arbitrary non-pointed homological vector field on a formal graded manifold V, i.e. an element $Q \in \mathscr{T}_V$ such that $[Q,Q]=0$ and $Q|_0 \neq 0$, it can be obtained from $\mathscr{L}ie^1_\infty$ by considering representations in $V \oplus \mathbb{K}$. For a formal variable x, viewed as a coordinate on $\mathbb{K}$, we have that $xQ \in \mathscr{T}_{V\oplus\mathbb{K}}$ vanishes at the distinguished point of $V \oplus \mathbb{K}$ and since xQ still satisfies $[xQ, xQ] = 0$, it corresponds to a representation of $\mathscr{L}ie^1_\infty$.

Using the isomorphism $\wedge^n(V[-1]) \cong (\odot^n V)[-n]$, see e.g. [1], we arrive at the conclusion that a homological vector field is equivalent to an L_∞-algebra structure on $V[-1]$

7 Prop profile of Poisson structures

In this chapter we first review the prop profile of Poisson structures originally constructed in [40], following the outline of the previous chapter, whereafter we give an interpretation of the prop profile as a family of brackets comprising an L_∞-algebra on the structure sheaf of the manifold.

7.1 Extracting the prop

7.1.1 The main idea

Consider the formal graded manifold associated to a vector space V. Recall that a Poisson structure on V is a degree two bivector field $P \in \wedge^\bullet \mathcal{T}_V$ satisfying $[P,P]_{\tilde{S}} = 0$. To be precise we consider a pointed Poisson structure, i.e. $P|_0 = 0$. With the notation of Section 5.3 we have

$$P = \sum_{n\geq 1} P^{[a_1 a_2]}_{(b_1\cdots b_n)} t^{b_1}\cdots t^{b_n} v_{a_1} v_{a_2}.$$

We can interpret this as a collection of degree zero maps

$$p_n \colon \odot^n V \to \wedge^2 V$$

defined by

$$p_n(e_{b_1}\odot\cdots\odot e_{b_n}) \to P^{a_1 a_2}_{b_1\cdots b_n} e_{a_1}\wedge e_{a_2}.$$

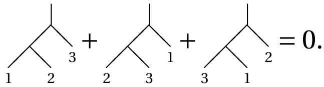
The condition $[P,P]_{\tilde{S}} = 0$ then translates into a sequence of quadratic relations of these maps. Merkulov's idea [40] was that this algebraic structure corresponds to just the degree zero part of the resolution of a prop. This means that a certain part of the structure is fundamental and the rest of the maps are higher homotopies, many of which may not be visible in degree zero.

7.1.2 The visible part

Of the operations p_i the fundamental one is p_1. We denote the part of P corresponding to p_1 by $\hat{P}$. We will see that for Poisson structures on graded manifolds there exist a hierarchy of higher homotopies of p_1 which for degree reasons do not appear in a classical Poisson structure. We depict the map p_1 with the corolla ⅄. The condition

$$[\hat{P},\hat{P}]_{\tilde{S}} = 0 \tag{7.1}$$

is then equivalent to the relation



(7.2)

Definition. A *Lie coalgebra* is a vector space V together with a linear map $\Delta\colon V \to V \wedge V$ which satisfies the co-Jacobi identity

$$(\Delta \otimes \mathrm{Id}) \circ \Delta + \sigma(\Delta \otimes \mathrm{Id}) \circ \Delta + \sigma^2(\Delta \otimes \mathrm{Id}) \circ \Delta = 0,$$

where σ is the cyclic permutation $(1,2,3) \mapsto (2,3,1)$.

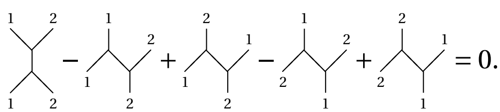
Thus the fundamental part of a Poisson structure gives V a Lie coalgebra structure.

7.1.3 Adding a homological vector field

To obtain also the maps p_i with $i \geq 2$ we need to add a homological vector field Q. In Chapter 6 we saw that the properties of a homological vector field is encoded in the condition $[\hat{Q},\hat{Q}] = 0$. To obtain the relations of the prop profile of Poisson structures we use that the Schouten bracket is a generalization of the Lie bracket of vector fields and consider the condition

$$[\hat{P},\hat{Q}]_{\tilde{S}} = 0, \tag{7.3}$$

which translates to



(7.4)

The compatibility condition (7.4) has an interpretation in terms of Chevalley-Eilenberg cohomology. Let V be an odd Lie algebra and consider the associated Chevalley-Eilenberg cochain complex with values in $V \wedge V$. Then $\Delta\colon V \to V \wedge V$ is a 1-cocycle if and only if condition (7.4) is satisfied. See e.g. [15] for a treatment of Lie bialgebras.

Note that a vector field of degree r is of degree $r+1$ when considered as an element of $(\wedge^\bullet \mathcal{T}_V, [\_,\_]_{\tilde{S}})$. Thus $\hat{P}$ and $\hat{Q}$ are both of degree two.

7.1.4 The genome

Since $[\hat{Q},\hat{Q}]_{\tilde{S}} \in \wedge^1\mathcal{T}_V$, $[\hat{Q},\hat{P}]_{\tilde{S}} \in \wedge^2\mathcal{T}_V$, and $[\hat{P},\hat{P}]_{\tilde{S}} \in \wedge^3\mathcal{T}_V$, we can simultaneously express the conditions (6.2), (7.1), and (7.3) which we want to encode by

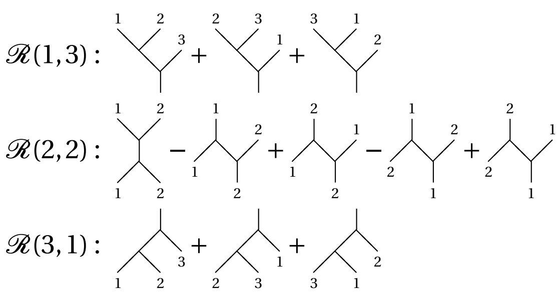
$$[\hat{P}+\hat{Q},\hat{P}+\hat{Q}]_{\tilde{S}} = 0.$$

To describe Poisson geometry as a minimal resolution of an algebraic object we need to go beyond operads; since p_1 has multiple outputs and q_2 multiple inputs operads, are to restrictive. As the relations (7.4) are given by genus zero graphs and constitute what is called a distributive law, it suffices to encode the fundamental part of the geometric structure as a dioperad. Its resolution is then straightforwardly extended to a resolution of the corresponding prop, see §4.1.3.

Definition. The dioperad $\mathscr{L}ie^1\mathscr{B}i$ is the quadratic dioperad $\mathscr{F}(\mathscr{M})/(\mathscr{R})$ where $\mathscr{M}$ is the $\mathbb{S}$-bimodule given by

$$\mathscr{M}(m,n) = \begin{cases} \mathbb{K}\curlyvee = 1\!\!1_1 \otimes 1\!\!1_2[-1] & \text{if } (m,n) = (1,2) \\ \mathbb{K}\curlywedge = \mathrm{sgn}_2 \otimes 1\!\!1_1 & \text{if } (m,n) = (2,1) \\ 0 & \text{otherwise} \end{cases}$$

The relations $\mathscr{R}$ are given by (6.3), (7.2), and (7.4):



Remark. This dioperad is similar to the dioperad $\mathscr{L}ie\mathscr{B}i$ of Example 2.3 with the difference being that the bracket and cobracket lie in degrees differing by one, explaining the 1 in the notation.

Merkulov called the generators and relations of $\mathscr{L}ie^1\mathscr{B}i$ the *genes* and *engineering rules* of Poisson geometry, together constituting its *genome*.

7.2 Computing the resolution

7.2.1 Koszulness of $\mathscr{L}ie^1\mathscr{B}i$

Since $\mathscr{L}ie$ [20] and $\mathscr{L}ie^1$ (§6.2.2) are Koszul and the relations of $\mathscr{L}ie^1\mathscr{B}i$ are given by a distributive law we obtain:

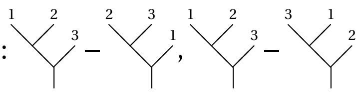
Proposition 7.1 (Proposition 3.1 of [40])**.** *The dioperad $\mathscr{L}ie^1\mathscr{B}i$ is Koszul.*

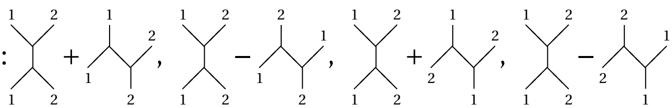
7.2.2 The Koszul dual dioperad of $\mathscr{L}ie^1\mathscr{B}i$

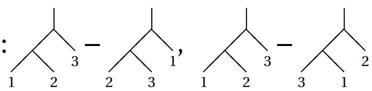
Proposition 7.2. *The Koszul dual dioperad of $\mathscr{L}ie^1\mathscr{B}i$ is the quadratic dioperad $\mathscr{F}(\mathscr{N})/(\mathscr{S})$, where $\mathscr{N}$ is the $\mathbb{S}$-bimodule given by*

$$\mathscr{N}(m,n) = \begin{cases} 1\!\!1_1 \otimes \mathrm{sgn}_2[1] = \mathbb{K}\curlyvee & \text{if } (m,n) = (1,2) \\ 1\!\!1_2 \otimes 1\!\!1_1 = \mathbb{K}\curlywedge & \text{if } (m,n) = (2,1) \\ 0 & \text{otherwise,} \end{cases}$$

and the relations $\mathscr{S}$ are given by

$\mathscr{S}(1,3)$:  (7.5)

$\mathscr{S}(2,2)$:  (7.6)

$\mathscr{S}(3,1)$:  (7.7)

Proof. For $\mathscr{L}ie^1\mathscr{B}i = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ we first observe that $\mathscr{N} = \mathscr{M}^\vee$. Recalling the pairing described in 2.3.4 we notice that $(\mathscr{S})_{(2)}$ is the orthogonal complement to $(\mathscr{R})_{(2)}$ with respect to this pairing. □

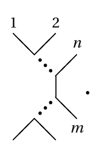
We see that $\mathscr{L}ie^1\mathscr{B}i^!$ is constructed from the operads $(\mathscr{L}ie^1\mathscr{B}i^!)_U = \mathscr{C}om^1$ and $(\mathscr{L}ie^1\mathscr{B}i^!)_D = \mathscr{L}ie$. The relations (7.6) are orthogonal to the compatibility relations (7.4) of $\mathscr{L}ie^1\mathscr{B}i$ and are related to the dioperad of Frobenius algebras; the dioperad of Frobenius algebras is Koszul dual to the dioperad of Lie bialgebras, see e.g. [18].

By straightforward graph calculations we obtain the following result.

Proposition 7.3 ([40])**.** *The dioperad $\mathscr{L}ie^1\mathscr{B}i^!$ has as underlying $\mathbb{S}$-bimodule*

$$\mathscr{L}ie^1\mathscr{B}i^!(m,n) = \begin{cases} 1\!\!1_m \otimes \mathrm{sgn}_n[n-1] & \text{if } m+n \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, in terms of the operations $\curlyvee$ and $\curlywedge$, the basis-element of $\mathscr{L}ie^1\mathscr{B}i^!(m,n)$ is given by the

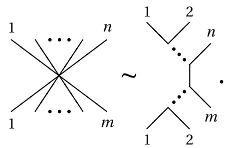


7.2.3 A minimal resolution of $\mathscr{L}ie^1\mathscr{B}i$

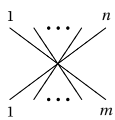
Theorem 7.4 (Theorem 3.2 of [40])**.** *The Koszul resolution $\mathscr{L}ie^1\mathscr{B}i_\infty$ of the dioperad $\mathscr{L}ie^1\mathscr{B}i$ is the quasi-free dioperad $(\mathscr{F}(\mathscr{E}),\delta)$ where*

$$\mathscr{E}(m,n) = \begin{cases} \mathrm{sgn}_m \otimes \mathbb{1}_n[m-2] & \text{if } m+n \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

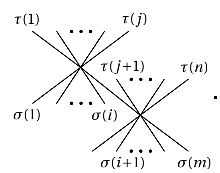
We denote the element of $\mathscr{E}(m,n)$ corresponding to the basis element of $\mathscr{L}ie^1\mathscr{B}i^{¡}(m,n)$ by



The differential of $\mathscr{L}ie^1\mathscr{B}i_\infty$ is then given by

$\delta:$ 

$$\mapsto \sum_{\substack{\tau \in \mathbb{S}^{\text{un-sh}}_{(j,n-j)} \\ \sigma \in \mathbb{S}^{\text{un-sh}}_{(i,m-i)}}} (-1)^{\mathrm{sgn}(\sigma)+i(m-i)}$$



Proof. The proof is completely analogous to that of Theorem 3.26. We just make two remarks: $\mathscr{L}ie^1\mathscr{B}i^{¡}(m,n)$ is concentrated in weight $m+n-2$ and the graph calculations though still straightforward are somewhat more tedious. □

7.2.4 From dioperads to props

From the results reviewed in §4.1.3 and the fact that the relations define a distributive law we obtain the following corollary.

Corollary 7.5. *With the notation*

$$\mathscr{L}ie^1\mathscr{B}i = \mathscr{F}^{\downarrow}_{c,0}(\mathscr{M})(\mathscr{R}) \quad \textit{and} \quad \mathscr{L}ie^1\mathscr{B}i_\infty = (\mathscr{F}^{\downarrow}_{c,0}(\mathscr{E}),\delta)$$

we have

$$\mathscr{F}^{\text{prop}}_{\text{dioperad}}(\mathscr{L}ie^1\mathscr{B}i) = \mathscr{F}^{\downarrow}(\mathscr{M})(\mathscr{R}) \quad \textit{and} \quad \mathscr{F}^{\text{prop}}_{\text{dioperad}}(\mathscr{L}ie^1\mathscr{B}i_\infty) = (\mathscr{F}^{\downarrow}(\mathscr{E}),\delta),$$

moreover, the latter is a quasi-free resolution of the former.

We will use the same notation for $\mathscr{L}ie^1\mathscr{B}i$ when considering it as prop.

7.2.5 From props to wheeled props

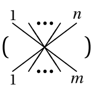
There exist a pair of adjoint functors $\mathscr{U}^{\text{prop}}_{\text{wheeledprop}}$ and $\mathscr{F}^{\text{wheeledprop}}_{\text{prop}}$ between props and wheeled props. Unfortunately the latter functor is not exact; it has been shown that when applying $\mathscr{F}^{\text{wheeledprop}}_{\text{prop}}$ to the propic resolution of $\mathscr{L}ie^1\mathscr{B}i$ new cohomology classes arise, Remark 4.2.4 of [42]. In the same paper it was shown though, that a minimal quasi free wheeled propic resolution exists, Theorem 4.5.1, but neither the differential nor the $\mathbb{S}$-bimodule by which it is generated need necessarily be directly obtained from the propic resolution. The explicit calculation of the wheeled resolution is a highly non-trivial problem and has not yet been accomplished.

7.3 Geometrical interpretation

7.3.1 An isomorphism of Lie algebras

We are going to follow the paradigm of Chapter 6, this time constructing a morphism from $\mathscr{L}_{\mathscr{L}ie^1\mathscr{B}i}(V)$ to a Lie subalgebra of $\wedge^\bullet\mathscr{T}_V[1]$. Let us use the same notation for the basis of $\mathscr{L}ie^1\mathscr{B}i^{\circlearrowleft}$ as for the basis of the $\mathbb{S}$-bimodule $\mathscr{E}$ of Theorem 7.4. A degree r element $f \in \mathscr{L}_{\mathscr{L}ie^1\mathscr{B}i}(V)$ is equivalent to a family of degree r linear maps

$$\left\{ f^n_m := f(\quad) : \odot^n V \to \wedge^m V[1-m] \right\}_{\substack{m,n\geq 1\\ m+n\geq 3}}.$$



From f we construct a polyvector field $X_f \in \wedge^\bullet\mathscr{T}_V[1] \cong \mathbb{K}[\![t^b, v_a]\!][1]$ on the formal graded manifold associated to V as follows. We denote the part of a polyvector field X of polynomial degree n in the variables t and m in v by X^n_m. We define X_f by

$$(X_f)^n_m := \frac{1}{m!n!} X^{[a_1\cdots a_m]}_{(b_1\cdots b_n)} t^{b_1}\cdots t^{b_n} v_{a_1}\cdots v_{a_m},$$

where the coefficients $X^{[a_1\cdots a_m]}_{(b_1\cdots b_n)}$ are given by

$$f^n_m(e_{b_1}\odot\cdots\odot e_{b_n}) = X^{[a_1\cdots a_m]}_{(b_1\cdots b_n)} e_{a_1}\wedge\cdots\wedge e_{a_m}.$$

Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{K}[\![t^b]\!]$. We define a subset of polyvector fields

$$\widetilde{\wedge^\bullet\mathscr{T}_V} := \{X = X^a(t)v_a \in \wedge^1\mathscr{T}_V[1] \mid X^a(t) \in \mathfrak{m}^2\} \cup \{X \in \wedge^{\geq 2}\mathscr{T}_V[1] \mid X|_0 = 0\},$$

further, we note that $\widetilde{\wedge^\bullet\mathscr{T}_V}$ is a Lie subalgebra of $\wedge^\bullet\mathscr{T}_V[1]$ and that $X_f \in \widetilde{\wedge^\bullet\mathscr{T}_V}$.

The differential d of V corresponds to a degree one vector field D, as noted in §6.3.1, and $\delta_D := [D,\_]_s$ defines a differential on $\wedge^\bullet \mathcal{T}_V[1]$. Thus $(\widetilde{\wedge^\bullet \mathcal{T}_V}, [\_,\_]_s, \delta_D)$ is a dg Lie algebra. We let $(\widetilde{\wedge^\bullet \mathcal{T}_V}^{\mathrm{op}}, [\_,\_]_s^{\mathrm{op}}, \delta_D^{\mathrm{op}})$ denote the opposite Lie algebra.

The following result was stated in [45]:

Theorem 7.6. *Consider $\mathcal{L}ie^1\mathcal{B}i$ with dioperad structure, then the morphism*

$$\tilde{\Phi}\colon \mathcal{L}_{\mathcal{L}ie^1\mathcal{B}i}(V) \to \widetilde{\wedge^\bullet \mathcal{T}_V}^{\mathrm{op}}, \quad f \mapsto X_f$$

is an isomorphism of dg Lie algebras.

Proof. Let $f \in \mathcal{L}_{\mathcal{L}ie^1}(V)$ be a homogeneous element of degree r. Explicitly, this means that if a structure coefficient $X^{[a_1\cdots a_m]}_{(b_1\cdots b_n)}$ of f is non-zero, then $|e_{a_1}| + \cdots + |e_{a_m}| - |e_{b_1}| - \cdots - |e_{b_n}| = r + 1 - m$. Since $|v_a| = |e_a| + 1$ and $|t^b| = -|e_b|$ it follows that X_f is of degree r. Applying the differential δ to f we obtain a new family of maps $\{\delta(f)^n_m\}_{m,n\geq 1, m+n\geq 3}$ where

$$\delta(f)^n_m := \delta(f)\Big(\underset{1\,\cdots\,m}{\overset{1\,\cdots\,n}{\times}}\Big) = d \circ f\Big(\underset{1\,\cdots\,m}{\overset{1\,\cdots\,n}{\times}}\Big) = d(f^n_m) =$$
$$\sum_{i=1}^{m} d_1 \circ_i f^n_m - (-1)^r \sum_{i=1}^{n} f^n_m{}_i\circ_1 d.$$

From this we get that

$$(X_{\delta(f)})^n_m = \Bigg(\frac{1}{(m-1)!n!} X^{a_1\cdots a_{m-1}c}_{b_1\cdots b_n} D^{a_m}_c$$
$$-(-1)^r \frac{1}{m!(n-1)!} D^c_{b_1} X^{a_1\cdots a_m}_{cb_2\cdots b_n}\Bigg) t^{b_1}\cdots t^{b_n} v_{a_1}\cdots v_{a_m} =$$
$$([X_f, D]_s)^n_m = (\delta^{\mathrm{op}}_D(X_f))^n_m.$$

Thus $\tilde{\Phi}$ defines an isomorphism of dg vector spaces.

Now let f and g be homogeneous elements of $\mathcal{L}_{\mathcal{L}ie^1\mathcal{B}i}(V)$ of degrees r and s, respectively, and let $Y^{a_1\cdots a_m}_{b_1\cdots b_n}$ denote the structure coefficients of the vector space endomorphisms corresponding to g. By definition

$$[f,g]\Big(\underset{1\,\cdots\,m}{\overset{1\,\cdots\,n}{\times}}\Big) =$$
$$\sum_{G\in\mathfrak{G}_{(2)}} \mu_G \circ \Big((G,[f^d\otimes g^u]) - (-1)^{rs}(G,[g^d\otimes f^u])\Big) \circ_G \Delta\Big(\underset{1\,\cdots\,m}{\overset{1\,\cdots\,n}{\times}}\Big) =$$
$$\sum_{\substack{i+k=n+1\\ \tau\in\mathbb{S}^{\text{un-sh}}_{(k,i-1)}\\ j+l=m+1\\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(j-1,l)}}} \operatorname{sgn}(\sigma)\phi_\sigma \circ (f^i_j{}_1\circ_l g^k_l - (-1)^{rs} g^i_j{}_1\circ_l f^k_l) \circ \phi_\tau.$$

Here $\phi_\sigma\colon V^{\otimes n} \to V^{\otimes n}$ denotes the morphism defined by

$$\phi_\sigma\colon (-1)^\epsilon v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

where $\sigma \in \mathbb{S}_n$ and ϵ is determined by the Koszul-Quillen sign-rule. Note that the cardinality of $\mathbb{S}^{\text{un-sh}}_{(i,j)}$ is $(i+j)!/(i!j!)$. Now it follows that

$$\Phi([f,g])^n_m = \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} \frac{1}{i!(n-i)!}\frac{1}{(j-1)!(m-j+1)!}$$
$$\left(Y^{[a_1\cdots a_{j-1}c]}_{(b_1\cdots b_i)} X^{[a_j\cdots a_m]}_{(cb_{i+1}\cdots b_n)} - (-1)^{rs} X^{[a_1\cdots a_{j-1}c]}_{(b_1\cdots b_i)} Y^{[a_j\cdots a_m]}_{(cb_{i+1}\cdots b_n)}\right) t^{b_1}\cdots t^{b_n} \nu_{a_1}\cdots\nu_{a_m} =$$
$$([X_g, X_f]_s)^n_m = ([\Phi(f),\Phi(g)]^{\text{op}}_s)^n_m.$$

□

7.3.2 Extended Poisson structures

Any pointed polyvector field $Y \in \wedge^{\geq 1}\mathcal{T}_V[1]$ can be uniquely decomposed as a sum $Y = D + Y_f$ for some pair (f,d), where $f \in \mathcal{L}_{\mathcal{L}ie^1\mathcal{B}i}(V)$ and D is the vector field corresponding to a differential d of V. Since

$$[D+Y_f, D+Y_f]_s = 0 \Longleftrightarrow \delta_D(Y_f) + \frac{1}{2}[Y_f, Y_f]_s = 0$$
$$\Longleftrightarrow \delta^{\text{op}}_D(Y_f) + \frac{1}{2}[Y_f, Y_f]^{\text{op}}_s = 0,$$

we obtain the following corollary of Theorem 7.6.

Corollary 7.7 (Proposition 1.5.1 of [40])**.** *There is a one-to-one correspondence between representations of $\mathcal{L}ie^1\mathcal{B}i_\infty$ in a dg vector space V and pointed Maurer-Cartan elements of $\wedge^{\geq 1}\mathcal{T}_V[1]$.*

We propose the following definition.

Definition. An *extended Poisson structure* on a formal graded manifold V is a Maurer-Cartan element in the odd Lie algebra $(\wedge^{\geq 1}\mathcal{T}_V, [\_,\_]_{\tilde{S}})$, i.e. a degree two element $P \in \wedge^{\geq 1}\mathcal{T}_V$ satisfying $[P,P]_{\tilde{S}} = 0$.

Note that if V is concentrated in degree zero, then an extended Poisson structure is an ordinary Poisson structure on V, i.e. in this case $P \in \wedge^2\mathcal{T}_V$.

By Remark 6.7 the prop profile $\mathcal{L}_{\mathcal{L}ie^1\mathcal{B}i}(V)$ essentially describes all extended Poisson structures on V.

7.3.3 The family of brackets of an extended Poisson structure

To a polyvector field $P = \sum_{n\geq 1} P_n$, with $P_n := P^{[a_1\ldots a_n]}(t)\nu_{a_1}\cdots\nu_{a_n}$, we associate a family of brackets

$$\left\{L_n\colon \otimes^n \mathcal{O}_V \to \mathcal{O}_V\right\}$$

where

$$\begin{aligned} L_n(f_1,\dots,f_n) &:= P_n(df_1\wedge\cdots\wedge df_n) \\ &= (-1)^{\epsilon} P^{[a_1\dots a_n]}(t)(\partial_{a_1} f_1)\cdots(\partial_{a_n} f_n) \end{aligned}$$

and the sign $(-1)^{\epsilon}$ is given by $\epsilon =$

$$|\partial_{a_n}|(|f_1|+\cdots+|f_{n-1}|+n-1)+|\partial_{a_{n-1}}|(|f_1|+\cdots+|f_{n-2}|+n-2)+\cdots+|\partial_{a_2}|(|f_1|+1).$$

Proposition 7.8. *The brackets L_n associated to a polyvector field $P \in \wedge^{\bullet\geq 1}\mathcal{T}_V$ as above are graded skew commutative and have the graded Leibniz property in each argument, i.e. for all $n \geq 1$ and all $1 \leq j \leq n$*

$$\begin{aligned} &L_n(f_1,\dots,f_{j-1},gh,f_{j+1},\dots f_n) = \\ &(-1)^{\epsilon_1} gL_n(f_1,\dots,f_{j-1},h,f_{j+1},\dots f_n) + (-1)^{\epsilon_2} L_n(f_1,\dots,f_{j-1},g,f_{j+1},\dots f_n)h, \end{aligned}$$

where $\epsilon_1 = |g|(|f_1|+\cdots+|f_{j-1}|+2-n)$ and $\epsilon_2 = |h|(|f_{j+1}|+\cdots+|f_n|)$. Moreover, the family of brackets $\{L_n\}_{n\geq 1}$ gives $\mathcal{O}_V$ the structure of L_∞-algebra if and only if P is an extended Poisson structure.

Proof. That the brackets L_n are graded skew symmetric is immediate from the definition. The Leibniz property is satisfied since $L_n(f_1,\dots,f_{j-1},\_,f_{j+1},\dots f_n)$ is a vector field. We notice that

$$|L_n| = |P^{a_1\dots a_n}(t)| + (|\partial_{a_1}|+\cdots+|\partial_{a_n}|) = |P_n| - n = 2-n.$$

Thus P is of degree two if and only if L_n is of degree $2-n$. For the Poisson bracket associated to a bivector field Y the condition $[Y,Y]_{\tilde{S}} = 0$ is equivalent to the Poisson bracket satisfying the Jacobi identity. That the L_i satisfy the L_∞-conditions is proven much in the same way. It is a tedious but straightforward computation to verify that the brackets L_n associated to a polyvector field P of degree two satisfy equation (6.1) if and only if $[P,P]_{\tilde{S}} = 0$. □

This leads to another definition of extended Poisson structures on formal graded manifolds, which by the preceding proposition is equivalent to the one we gave in §7.3.2.

Definition. An *extended Poisson structure* on a formal graded manifold V is an L_∞-algebra $\{L_n\}_{n\geq 1}$ on $\mathcal{O}_V$ such that the brackets L_n have the Leibniz property in each argument.

Remark. That a polyvector field gives an L_∞-algebra structure on the structure sheaf was observed in [58] using the notion of higher derived brackets.

8 Prop profile of bi-Hamiltonian structures

In this chapter we define a prop such that the degree zero part of representations of its minimal resolution in a vector space V are in a one-to-one correspondence with bi-Hamiltonian structures on the formal manifold associated to V. In the general case representations correspond to Maurer-Cartan elements in a certain Lie subalgebra of $\wedge^\bullet \mathcal{T}_V[\![\hbar]\!]$. We call such elements extended bi-Hamiltonian structures. We also give an interpretation of extended bi-Hamiltonian structures as a family of brackets which gives $\mathcal{O}_V$ the structure of L^2_∞-algebra.

8.1 Extracting the prop

8.1.1 The compatibility relation

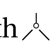
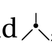
A bi-Hamiltonian structure on the formal manifold associated to a vector space V is a pair of bivector fields P_1 and P_2 satisfying $[P_1, P_1]_{\tilde{S}} = 0$, $[P_2, P_2]_{\tilde{S}} = 0$, and $[P_1, P_2]_{\tilde{S}} = 0$. We want again to extract a prop encoding the fundamental part of this structure. As in §7.1.2 we let $\hat{P}_1$ and $\hat{P}_2$ denote the parts of P_1 and P_2 corresponding to maps $V \to \wedge^2 V$. The conditions

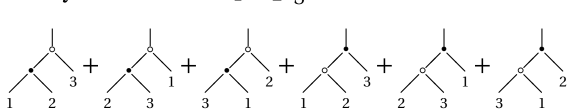
$$[\hat{P}_1, \hat{P}_1]_{\tilde{S}} = 0 \quad \text{and} \quad [\hat{P}_2, \hat{P}_2]_{\tilde{S}} = 0 \tag{8.1}$$

are equivalent to that the maps corresponding to $\hat{P}_1$ and $\hat{P}_2$ each give V the structure of Lie coalgebra.

Definition. Let V be a vector space and let Δ_1 and Δ_2 be Lie cobrackets on V. We say that the cobrackets are *compatible* if their sum $\Delta_1 + \Delta_2$ again is a Lie cobracket. We denote the quadratic dioperad encoding this structure by $\mathcal{C}o\mathcal{L}ie^2$.

We note that $\mathcal{C}o\mathcal{L}ie^2{}_D = \mathcal{L}ie^2$, cf §4.1.1.

We depict the maps corresponding to $\hat{P}_1$ and $\hat{P}_2$ with  and , respectively. The compatibility condition $[\hat{P}_1, \hat{P}_2]_{\tilde{S}} = 0$ can then be illustrated by



= 0,

i.e. the pair $(\hat{P}_1, \hat{P}_2)$ gives V the structure of compatible Lie coalgebras.

8.1.2 Adding a homological vector field

From the experience of constructing the prop profile of Poisson structures we expect a homological vector field Q compatible with both P_1 and P_2 to be present, i.e. satisfying $[P_1, Q]_{\tilde{S}} = 0$ and $[P_2, Q]_{\tilde{S}} = 0$. The compatibility of the fundamental part $\hat{Q}$ with $\hat{P}_1$ and $\hat{P}_2$ means that the maps corresponding to the pairs $(\hat{P}_1, \hat{Q})$ and $(\hat{P}_2, \hat{Q})$ both give V the structure of $\mathcal{L}ie^1\mathcal{B}i$ algebra.

8.1.3 The genome

To express all the above conditions with a single equation we introduce a formal parameter $\hbar$. The conditions

$$[\hat{Q}, \hat{Q}]_{\tilde{S}} = 0, \quad [\hat{P}_1, \hat{Q}]_{\tilde{S}} = 0, \quad [\hat{P}_1, \hat{P}_1]_{\tilde{S}} = 0, \quad [\hat{P}_2, \hat{P}_2]_{\tilde{S}} = 0, \quad \text{and} \quad [\hat{P}_1, \hat{P}_2]_{\tilde{S}} = 0$$

are then all subsumed by

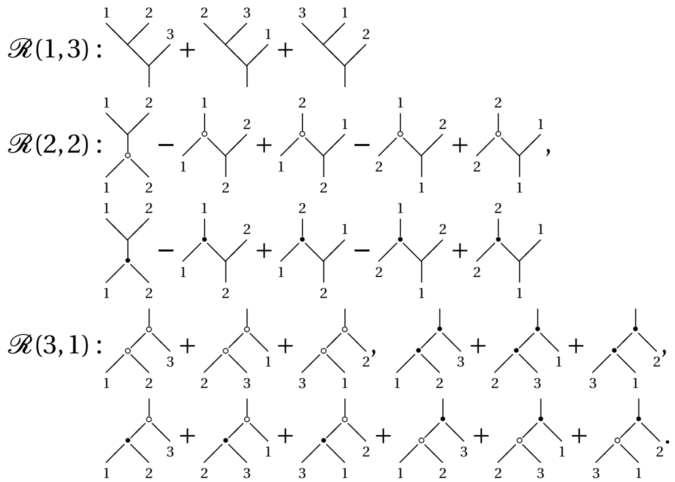
$$[\hat{Q} + \hat{P}_1 + \hat{P}_2\hbar, \hat{Q} + \hat{P}_1 + \hat{P}_2\hbar]_{S_\hbar} = 0. \tag{8.2}$$

Here the bracket is the linearization in $\hbar$ of the Schouten bracket. As in the case of Poisson structures, the relations (8.2) are dioperadic and in order to make the computation of the resolution easier we extract the dioperad encoding these relations.

Definition. The dioperad $\mathcal{L}ie^1_2\mathcal{B}i$ is the quadratic dioperad $\mathcal{F}(\mathcal{M})/(\mathcal{R})$, where $\mathcal{M}$ is the $\mathbb{S}$-bimodule defined by

$$\mathcal{M}(m,n) = \begin{cases} \mathbb{1}_1 \otimes (\mathbb{1}_2[-1]) = \mathbb{K}\curlyvee & \text{if } (m,n) = (1,2) \\ (\mathrm{sgn}_2 \oplus \mathrm{sgn}_2) \otimes \mathbb{1}_1 = \mathbb{K}\curlywedge \oplus \mathbb{K}\curlywedge & \text{if } (m,n) = (2,1)\,, \\ 0 & \text{otherwise} \end{cases}$$

and the relations $\mathcal{R} = \mathcal{R}(1,3) \sqcup \mathcal{R}(2,2) \sqcup \mathcal{R}(3,1)$ consist of the following subsets $\mathcal{R}(i,j) \subset \mathcal{F}_{(2)}(M)(i,j)$:



(8.3)
(8.4)
(8.5)
(8.6)
(8.7)

By this we have obtained the genome of bi-Hamiltonian structures. We are now ready to plug it into the machinery of Koszul resolutions.

8.2 Computing the resolution

8.2.1 Koszulness of $\mathscr{L}ie^1_2\mathscr{B}i$

We begin by showing the following result.

Proposition 8.1. *The dioperad $\mathscr{L}ie^1_2\mathscr{B}i$ is Koszul.*

Proof. We observe that $\mathscr{L}ie^1_2\mathscr{B}i_U = \mathscr{L}ie^1$ and $\mathscr{L}ie^1_2\mathscr{B}i_D = \mathscr{L}ie^2$. We showed in §3.3.2 that the operad $\mathscr{L}ie^2$ is Koszul and in §6.2.2 that $\mathscr{L}ie^1$ is Koszul. It is straightforward to see that $\mathscr{L}ie^1 \square (\mathscr{L}ie^2)^{\mathrm{op}}(i,j) = \mathscr{L}ie^1_2\mathscr{B}i(i,j)$ for $(i,j) = (2,2),(2,3),(3,2)$; thus by Theorem 4.1 we obtain that $\mathscr{L}ie^1_2\mathscr{B}i$ is Koszul. $\square$

8.2.2 The Koszul dual dioperad of $\mathscr{L}ie^1_2\mathscr{B}i$

Proposition 8.2. *The Koszul dual dioperad of $\mathscr{L}ie^1_2\mathscr{B}i$ is the quadratic dioperad $\mathscr{F}(\mathscr{N})/(\mathscr{S})$, where*

$$\mathscr{N}(m,n) = \begin{cases} \mathbb{1}_1 \otimes \mathrm{sgn}_2[1] = \curlyvee & \textit{if } (m,n) = (1,2) \\ (\mathbb{1}_2 \oplus \mathbb{1}_2) \otimes \mathbb{1}_1 = \mathbb{K}\curlyvee \oplus \mathbb{K}\curlyvee & \textit{if } (m,n) = (2,1) \\ 0 & \textit{otherwise} \end{cases}$$

and the relations $S = S(1,3) \sqcup S(3,1) \sqcup S(2,2)$ are given by

$S(1,3):$ (8.8)

$S(2,2):$ (8.9)

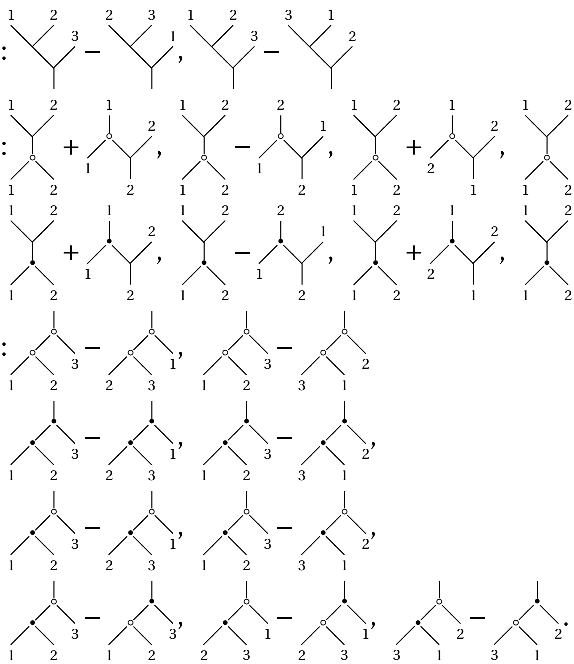
(8.10)

$S(3,1):$ (8.11)

(8.12)

(8.13)

(8.14)



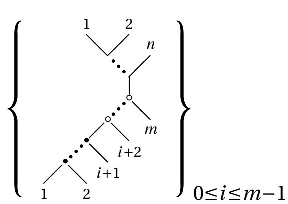
Proof. For $\mathscr{L}ie^1_2\mathscr{B}i = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ we first observe that $\mathscr{N} = \mathscr{M}^\vee$. Recalling the pairing described in 2.3.4 we notice that $(\mathscr{S})_{(2)}$ is the orthogonal complement to $(\mathscr{R})_{(2)}$ with respect to this pairing. □

Like $\mathscr{L}ie^1_2\mathscr{B}i$, its Koszul dual dioperad $\mathscr{L}ie^1_2\mathscr{B}i^!$ is constructed from two operads: $(\mathscr{L}ie^1_2\mathscr{B}i^!)_U = \mathscr{C}om^1$ and $(\mathscr{L}ie^1_2\mathscr{B}i^!)_D = {}^2\mathscr{C}om$, see Sections 6.2 and 3.1, respectively. By straightforward graph calculations we obtain the following result.

Proposition 8.3. *The dioperad $\mathscr{L}ie^1_2\mathscr{B}i^!$ has as underlying $\mathbb{S}$-bimodule*

$$\mathscr{L}ie^1_2\mathscr{B}i^!(m,n) = \begin{cases} \underbrace{(\mathbb{1}_m \oplus \cdots \oplus \mathbb{1}_m)}_{m \text{ terms}} \otimes \mathrm{sgn}_n[n-1] & \text{if } m+n \geq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, a $\mathbb{K}$-basis for $\mathscr{L}ie^1_2\mathscr{B}i^!(m,n)$ is given by



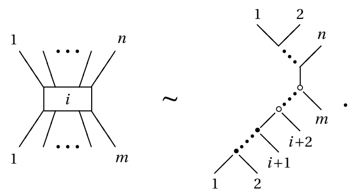
8.2.3 A minimal resolution of $\mathscr{L}ie^1_2\mathscr{B}i$

We now have everything we need to describe a minimal resolution of $\mathscr{L}ie^1_2\mathscr{B}i$ explicitly.

Theorem 8.4. *The Koszul resolution $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ of the dioperad $\mathscr{L}ie^1_2\mathscr{B}i$ is the quasi-free dioperad $(\mathscr{F}(\mathscr{E}),\delta)$, where*

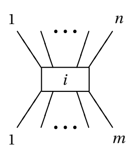
$$\mathscr{E}(m,n)=\begin{cases}\underbrace{(\mathrm{sgn}_m\oplus\cdots\oplus\mathrm{sgn}_m)}_{m\ \text{terms}}\otimes\mathbb{1}_n[m-2] & \text{if } m+n\geq 3\\ 0 & \text{otherwise.}\end{cases}$$

We denote the element of $\mathscr{E}$ corresponding to the basis element of $\mathscr{L}ie^1_2\mathscr{B}i^{!}(m,n)$ with i black operations by

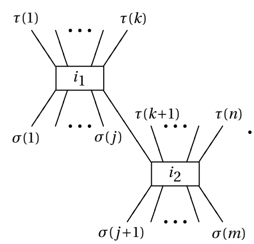


The differential of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ is then given by

$$\delta:$$



$$\longmapsto \sum_{\substack{1\leq k\leq n\\ 0\leq j\leq m-1\\ 2\leq j+k\leq m+n-2\\ i_1+i_2=i\\ \tau\in\mathbb{S}^{\text{un-sh}}_{(k,n-k)}\\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(j,m-j)}}}(-1)^{\mathrm{sgn}(\sigma)+j(m-j)}$$



Proof. The proof is completely analogous to that of Theorems 3.26 and 7.4.

□

We will use the same notation for $\mathscr{L}ie^1_2\mathscr{B}i$ when considering it as prop.

8.2.4 From dioperads to props to wheeled props

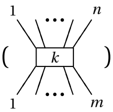
Since the relations of $\mathscr{L}ie^1_2\mathscr{B}i$ constitute a distributive law the dioperadic resolution extends to a propic resolution as in §7.2.4. Because $\mathscr{L}ie^1\mathscr{B}i_\infty$ is present in $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ as a subcomplex, consider e.g. all generators with only white operations, at least the same difficulties arise when trying to extend the propic resolution of $\mathscr{L}ie^1_2\mathscr{B}i$, cf. §7.2.5.

8.3 Geometrical interpretation

8.3.1 An isomorphism of Lie algebras

Now we are going to construct a morphism from $\mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V)$ to a Lie subalgebra of $\wedge^\bullet\mathscr{T}_V[\![\hbar]\!][1]$, where $\hbar$ denotes a formal variable of degree zero. For the rest of this paragraph we will suppress the degree shift. Let us use the same notation for the basis of $\mathscr{L}ie^1_2\mathscr{B}i^{\text{¡}}$ as for the basis of the $\mathbb{S}$-bimodule $\mathscr{E}$ of Theorem 7.4. A degree r element $f \in \mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V)$ is equivalent to a family of degree r linear maps

$$\Big\{ {}_k f^n_m := f\big($$



$$\big) : \odot^n V \to \wedge^m V[1-m] \Big\}_{\substack{m,n\geq 1\\ m+n\geq 3\\ 0\leq k\leq m-1}}.$$

From f we construct a formal power series $Y_f \in \wedge^\bullet\mathscr{T}_V[\![\hbar]\!] \cong \mathbb{K}[\![t^b, \nu_a, \hbar]\!]$ as follows. For a power series Y we denote by ${}_kY^n_m$ the part of Y of polynomial degree n in the variables t^b, m in ν^a, and k in $\hbar$. Further, we denote by ${}_kY$ the polyvector field coefficient of $\hbar^k$. We define Y_f by

$${}_k(Y_f)^n_m := \frac{1}{m!n!}\,{}_kY^{[a_1\cdots a_m]}_{(b_1\cdots b_n)}\, t^{b_1}\cdots t^{b_n}\nu_{a_1}\cdots\nu_{a_m},$$

where the coefficients ${}_kY^{[a_1\cdots a_m]}_{(b_1\cdots b_n)}$ are given by

$${}_kf^n_m(e_{b_1}\odot\cdots\odot e_{b_n}) = {}_kY^{[a_1\cdots a_m]}_{(b_1\cdots b_n)}\, e_{a_1}\wedge\cdots\wedge e_{a_m}.$$

The role of the formal parameter $\hbar$ is to distinguish polyvector fields of the same weight from each other. We let $[\_,\_]_{S_\hbar}$ denote the linearization in $\hbar$ of the Schouten bracket.

We define a subset of $\wedge^\bullet\mathscr{T}_V[\![\hbar]\!]$ by

$$\tilde{\mathfrak{g}}_V := \{Y = \sum_{k\geq 0} {}_kY\hbar^k \in \wedge^\bullet\mathscr{T}_V[\![\hbar]\!] \;\mid\; {}_kY \in \wedge^{\geq k+1}\mathscr{T}_V,\;\; Y|_0 = 0,\; \text{and}\; {}_0Y^1_1 = 0\}.$$

The following is immediate from the definition of $\tilde{\mathfrak{g}}_V$.

Lemma 8.5. *The subset $\tilde{\mathfrak{g}}_V$ is a Lie subalgebra of $\wedge^\bullet\mathscr{T}_V[\![\hbar]\!]$ and*

$$\tilde{\mathfrak{g}}_V = \{Y_f \;\mid\; f \in \mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V)\}.$$

As noted in the preceding chapters, the differential d of V corresponds to a degree one vector field D and $\delta_D := [D,\_]_{S_\hbar}$ defines a differential on $\wedge^\bullet\mathscr{T}_V[\![\hbar]\!]$. Thus $(\tilde{\mathfrak{g}}_V, [\_,\_]_{S_\hbar}, \delta_D)$ is a dg Lie algebra.

Theorem 8.6. *The morphism*

$$\Phi\colon \mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V) \to \tilde{\mathfrak{g}}_V^{\mathrm{op}},\quad f \mapsto Y_f$$

is an isomorphism of dg Lie algebras.

Proof. The proof is completely analogous to the proof of Theorem 7.6. □

8.3.2 Extended bi-Hamiltonian structures

We define a Lie subalgebra of the odd Lie algebra $\wedge^\bullet \mathscr{T}_V[\![\hbar]\!]$ by

$$\mathfrak{g}_V := \{Y = \sum_{k\geq 0} {}_kY\hbar^k \in \wedge^\bullet \mathscr{T}_V[\![\hbar]\!] \mid {}_kY \in \wedge^{\geq k+1}\mathscr{T}_V\}.$$

Any pointed element $Y \in \mathfrak{g}_V$ can be uniquely decomposed as a sum $Y = D + Y_f$ for some pair (f, d), where $f \in \mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V)$ and D is the vector field corresponding to a differential d of V. Since

$$[D + Y_f, D + Y_f]_{\tilde{S}_\hbar} = 0 \Longleftrightarrow \delta_D^{\mathrm{op}}(Y_f) + \frac{1}{2}[Y_f, Y_f]^{\mathrm{op}}_{S_\hbar} = 0,$$

we obtain the following corollary to Theorem 8.6.

Corollary 8.7. *There is a one-to-one correspondence between representations of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ in a dg vector space V and pointed Maurer-Cartan elements of $\mathfrak{g}_V$.*

This is just another formulation of Theorem C in the introduction. The preceding corollary suggests the following definition.

Definition. An *extended bi-Hamiltonian structure* on a formal graded manifold V is a Maurer-Cartan element of $\mathfrak{g}_V$, i.e. a degree two element $P \in \mathfrak{g}_V$ satisfying $[P, P]_{\tilde{S}_\hbar} = 0$.

8.3.3 Representations of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ in non-graded vector spaces

If V is a vector space concentrated in degree zero, then the maps ${}_kf^n_m$ corresponding to an element $f \in \mathscr{L}_{\mathscr{L}ie^1_2\mathscr{B}i}(V)$ vanish unless $m = 2$. Thus $Y_f = P_\circ + P_\bullet\hbar$, where $P_\circ$ and $P_\bullet$ are pointed bivector fields. The condition $[Y_f, Y_f]_{\tilde{S}_\hbar} = 0$ is therefore equivalent to

$$[P_\circ, P_\circ]_{\tilde{S}} + ([P_\circ, P_\bullet]_{\tilde{S}} + [P_\bullet, P_\circ]_{\tilde{S}})\hbar + [P_\bullet, P_\bullet]_{\tilde{S}}\hbar^2 = 0$$

and we observe that representations of $\mathscr{L}ie^1_2\mathscr{B}i_\infty$ in V are in one-to-one correspondence with pointed bi-Hamiltonian structures on the formal manifold associated to V. In particular this proves Theorem B.

8.3.4 The family of brackets of an extended bi-Hamiltonian structure

To an element $P = \sum_{k\geq 0} {}_kP\hbar^k \in \mathfrak{g}_V$ with ${}_kP = \sum_{i\geq k+1} {}_kP_i$ and

$${}_kP_i := {}_kP^{a_1\ldots a_i}(t)\nu_{a_1}\cdots\nu_{a_i}$$

we associate a family of brackets as follows. For $1 \leq k \leq n$ we define an n-ary bracket ${}_kL_n \colon \otimes^n \mathscr{O}_V \to \mathscr{O}_V$ by

$$\begin{aligned} {}_kL_n(f_1, \ldots, f_n)_i :&= {}_{k-1}P_n df_1 \wedge \cdots \wedge df_n \\ &= (-1)^\epsilon {}_{k-1}P^{a_1\ldots a_n}(t)(\partial_{a_1} f_1)\cdots(\partial_{a_n} f_n). \end{aligned}$$

Here the sign $(-1)^\epsilon$ is given by

$$\epsilon = |\partial_{a_n}|(|f_1| + \cdots + |f_{n-1}| + n - 1) + \\ |\partial_{a_{n-1}}|(|f_1| + \cdots + |f_{n-2}| + n - 2) + \cdots + |\partial_{a_2}|(|f_1| + 1).$$

Theorem 8.8. *The brackets ${}_kL_n$ associated to a power series $P \in \mathfrak{g}_V$ as above satisfy the Leibniz property in each argument, i.e.*

$$ {}_kL_n(f_1, \dots, f_{j-1}, gh, f_{j+1}, \dots f_n) = \\ (-1)^{\epsilon_1} g\, {}_kL_n(f_1, \dots, f_{j-1}, h, f_{j+1}, \dots f_n) + \\ (-1)^{\epsilon_2}\, {}_kL_n(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots f_n) h.$$

where $\epsilon_1 = |g|(|f_1| + \cdots + |f_{j-1}| + 2 - n)$ *and* $\epsilon_2 = |h|(|f_{j+1}| + \cdots + |f_n|)$. *Moreover, the family of brackets* $\{{}_kL_n\}_{n\geq 1, 1\leq k\leq n}$ *gives* $\mathscr{O}_V$ *the structure of* L^2_∞*-algebra if and only if* P *is of degree two and satisfies* $[P, P]_{\tilde{S}_\hbar} = 0$.

Proof. The proof is completely analogous to that of Proposition 7.8. □

This leads to another definition of extended bi-Hamiltonian structures on formal graded manifolds, which by the preceding theorem is equivalent to the one we gave in §8.3.1.

Definition. An *extended bi-Hamiltonian structure* on a formal graded manifold V is an L^2_∞-algebra $\{{}_kL_n\}_{n\geq 1, 1\leq k\leq n}$ on $\mathscr{O}_V$ such that the brackets ${}_kL_n$ have the Leibniz property in each argument.

9 Operad profile of Nijenhuis structures

In [39] S.A. Merkulov described Nijenhuis structures as corresponding to representations of the cobar construction on the Koszul dual of a certain quadratic operad. In this chapter we first review the construction of this operad. Thereafter we prove, using the PBW-basis method of E. Hoffbeck [23], that the operad governing Nijenhuis structures is Koszul, thereby showing that Nijenhuis structures correspond to representations of the minimal resolution of this operad. Finally, we consider the geometric meaning of the operad profile.

9.1 Extracting the operad

9.1.1 The visible part

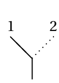
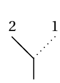
Let V be a dg vector space considered as a formal graded manifold. Recall from §5.3.5 that a pointed graded Nijenhuis structure on V is a degree one element $J \in \Omega^1_V \otimes \mathcal{T}_V$ such that $[J,J]_{\text{F-N}} = 0$ and $J|_0 = 0$. Using the notation of §5.3.1 we have

$$J = \sum_{i\geq 1} J^b_{(c_1\cdots c_i)a} t^{c_1} \cdots t^{c_i} \gamma^a \partial_b,$$

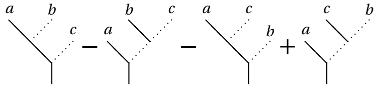
where $J^b_{(c_1\cdots c_i)a} \in \mathbb{K}$. Here we consider the grading $\Omega^\bullet_V = \widehat{\odot^\bullet}(\Omega^1_V[-1)])$. The vector form J defines a family of degree zero maps $\{j_i\colon \odot^i V \otimes V \to V\}$ by

$$j_i(e_{c_1} \odot \cdots \odot e_{c_i} \otimes e_a) = J^b_{c_1\cdots c_i a} e_b.$$

Let $\hat{J}$ denote the part of J corresponding to j_1. It was observed in [39] that j_1 gives V a pre-Lie algebra structure if and only if $[\hat{J},\hat{J}]_{\text{F-N}} = 0$. We want to encode this in the language of operads. To j_1 we associate the corolla . Denoting the nontrivial element of $\mathbb{S}_2$ by (12) we depict the elements  and (12) by the planar corollas

 and ,

respectively. The condition $[\hat{J},\hat{J}]_{\text{F-N}} = 0$ then translates to

$\mathscr{R}^{\text{PL}}_{a,b,c} :=$  $= 0,$

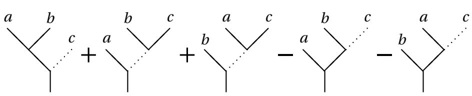
for a, b, c being the cyclic permutations of $1, 2, 3$. Let $\mathscr{M}_{\text{PL}} = \mathbb{K}\,\curlyvee \oplus \mathbb{K}\,\curlyvee(12) = \mathbb{K}[\mathbb{S}_2]$ and $\mathscr{R}_{\text{PL}} = \mathscr{R}^{\text{PL}}_{1,2,3} \cup \mathscr{R}^{\text{PL}}_{2,3,1} \cup \mathscr{R}^{\text{PL}}_{3,1,2}$, then the operad of pre-Lie algebras $\mathscr{P}re\mathscr{L}ie$ is given by $\mathscr{F}(\mathscr{M}_{\text{PL}})/(\mathscr{R}_{\text{PL}})$. This operad was shown to be Koszul in [9] and thus its minimal resolution $\mathscr{P}re\mathscr{L}ie_\infty$ can be computed explicitly. Representations of $\mathscr{P}re\mathscr{L}ie_\infty$ correspond to Nijenhuis structures which are linear in t. To obtain arbitrary pointed Nijenhuis structures we need to add a homological vector field.

9.1.2 Adding a homological vector field

In §6.1.3 we translated the properties of the fundamental part $\hat{Q}$ of a homological vector field to operadic relations. Let us denote the relations of (6.3) by $\mathscr{R}_{\text{L}}$. The last step in encoding the operad profile of Nijenhuis structures is achieved by considering the interplay between $\hat{J}$ and $\hat{Q}$. Here we use that the Frölicher-Nijenhuis bracket is an extension of the Lie bracket of vector fields. The compatibility is given by

$$[\hat{J},\hat{Q}]_{\text{F-N}} = 0.$$

This translates to

$\mathscr{R}^{\text{C}}_{a,b,c} :=$  $= 0,$

for a, b, c being the cyclic permutations of $1, 2, 3$. Let $\mathscr{R}_{\text{C}} = \mathscr{R}^{C}_{1,2,3} \cup \mathscr{R}^{C}_{2,3,1} \cup \mathscr{R}^{C}_{3,1,2}$.

9.1.3 The genome

Since $[\hat{Q},\hat{Q}]_{\text{F-N}} \in \Omega^0_V \otimes \mathscr{T}_V$, $[\hat{J},\hat{Q}]_{\text{F-N}} \in \Omega^1_V \otimes \mathscr{T}_V$, and $[\hat{J},\hat{J}]_{\text{F-N}} \in \Omega^2_V \otimes \mathscr{T}_V$, the relations $\mathscr{R}_{\text{L}}$, $\mathscr{R}_{\text{PL}}$, and $\mathscr{R}_{\text{C}}$ can all be simultaneously expressed by the single condition

$$[\hat{Q}+\hat{J},\hat{Q}+\hat{J}]_{\text{F-N}} = 0.$$

We encode this in an operad.

Definition (Merkulov)**.** The operad $\mathscr{N}ij$ is the quadratic operad

$$\mathscr{F}(\mathscr{M}_{\text{PL}} \oplus \mathscr{M}_{\text{L}})/(\mathscr{R}_{\text{L}} \cup \mathscr{R}_{\text{PL}} \cup \mathscr{R}_{\text{C}}).$$

The operad $\mathscr{N}ij$ thus contains the Lie operad and the pre-Lie operad with the operations differing by one in degree and compatible in the sense of $\mathscr{R}_C$. See [39] for interpretations of this compatibility.

Remark. The operad $\mathscr{N}ij$ was denoted by *pre-$\mathscr{L}ie^2$* in [39]. We renamed it to avoid confusion with the operad of compatible pre-Lie algebras.

9.2 Computing the resolution

9.2.1 The Koszul dual operad of $\mathscr{N}ij$

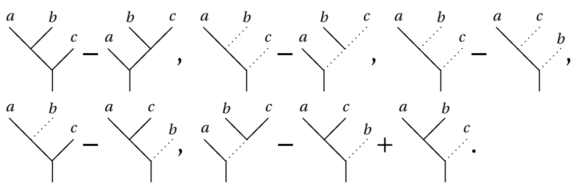
Proposition 9.1 ([39])**.** *The Koszul dual dioperad of $\mathscr{N}ij$ is*

$$\mathscr{N}ij^{!} = \mathscr{F}(\mathscr{N})/(\mathscr{S})$$

where $\mathscr{N}$ is the $\mathbb{S}$-module defined by

$$\mathscr{N}(n) = \begin{cases} \mathrm{sgn}_2[1] \oplus \mathbb{K}[\mathbb{S}_2] \otimes \mathrm{sgn}_2 = \mathbb{K}\curlyvee \oplus \mathbb{K}\curlyvee^{\cdot\cdot} \oplus \mathbb{K}\curlyvee^{\cdot\cdot}(12) & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The relations $\mathscr{S}$ are given by



(9.1)

(9.2)

Proof. For $\mathscr{N}ij = \mathscr{F}(\mathscr{M})/(\mathscr{R})$ we first observe that $\mathscr{N} = \mathscr{M}^{\vee}$. Recalling the pairing described in 2.3.4 we notice that $(\mathscr{S})_{(2)}$ is the orthogonal complement to $(\mathscr{R})_{(2)}$ with respect to this pairing. □

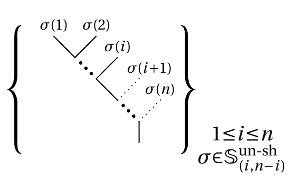
We note that $\mathscr{N}ij^{!}$ has the operads $\mathscr{P}erm$, of permutative algebras (see e.g. [9]), and $\mathscr{C}om^1$ as suboperads.

For H a subgroup of G and M an H-module we define $\mathrm{Ind}_H^G M := \mathbb{K}[G] \otimes_{\mathbb{K}[H]} M$. Using the relations $\mathscr{S}$ of the previous proposition one can show the following:

Proposition 9.2. *The operad $\mathscr{N}ij^{!}$ has as underlying $\mathbb{S}$-bimodule*

$$\mathscr{N}ij^{!}(n) = \begin{cases} \bigoplus_{i=1}^{n} \mathrm{Ind}_{\mathbb{S}_i \times \mathbb{S}_{n-i}}^{\mathbb{S}_n} \mathrm{sgn}_i \otimes \mathbb{1}_{n-i}[i-1] & \text{if } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, in terms of the operations $\curlyvee$ and $\curlyvee^{\cdot\cdot}$, a $\mathbb{K}$-basis of $\mathscr{N}ij^{!}(n)$ is given by



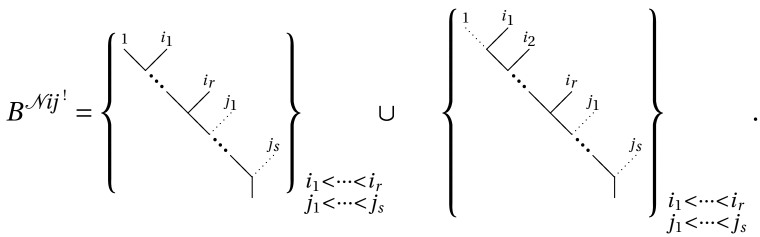
$$\left\{ \quad \right\}_{\substack{1 \leq i \leq n \\ \sigma \in \mathbb{S}_{(i,n-i)}^{\text{un-sh}}}} .$$

9.2.2 Koszulness of $\mathcal{N}ij$

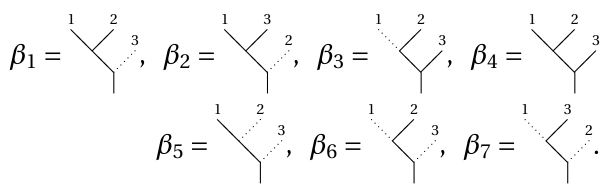
In the following theorem we use the notation

in order to write the trees using the planar representation of §4.2.1.

Proposition 9.3. *Let $B^{\mathcal{M}} = \{\ , \ , \ \}$. The following is a PBW-basis of $\mathcal{N}ij^!$ with respect to the ordering $\ < \ < \ $:*



Proof. Through straightforward graph calculations one can verify that $B^{\mathcal{N}ij^!}$ is a basis of $\mathcal{N}ij^!$, cf. §3.6 of [39] and note that the dotted edges here correspond to the wavy edges in [39]. Condition (ii) is easily verified. We denote the elements of $B^{\mathcal{N}ij^!(3)}$ by



To show that (i) is satisfied, it is sufficient, using Proposition 4.4, to observe that for any decorated two-vertex graph $\alpha \in B^{\mathcal{F}(\mathcal{M})} \setminus B^{\mathcal{N}ij^!}$ with $\alpha = \sum c_i \beta_i$, $c_i \in \mathbb{K}$, we have $c_i \neq 0 \implies \beta_i > \alpha$. Here $\mathcal{M} = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. □

Together with Theorems 4.5 and 2.5 we obtain:

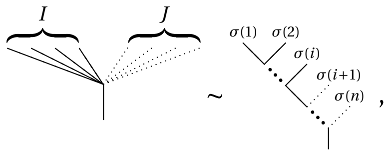
Corollary 9.4. *The operads $\mathcal{N}ij^!$ and $\mathcal{N}ij$ are Koszul.*

9.2.3 The minimal resolution of $\mathcal{N}ij$

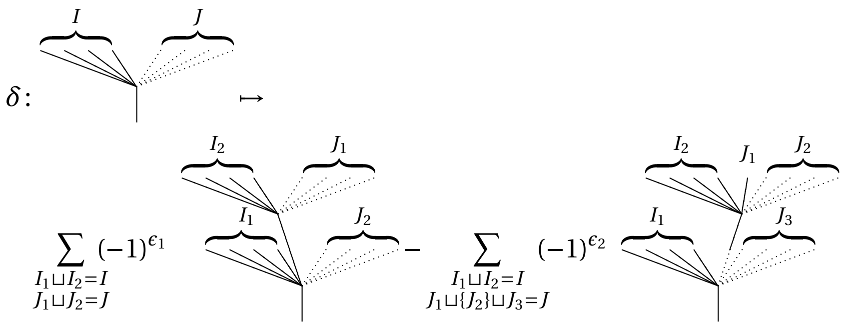
Theorem 9.5. *The minimal resolution $\mathcal{N}ij_\infty$ of the operad $\mathcal{N}ij$ is the quasi-free operad $(\mathcal{F}(\mathcal{E}), \delta)$ where the $\mathbb{S}$-module $\mathcal{E}$ is given by*

$$\mathcal{E}(n) = \begin{cases} \bigoplus_{j=0}^{n-1} \mathrm{Ind}_{\mathbb{S}_{n-j}\times\mathbb{S}_j}^{\mathbb{S}_n} \mathbb{1}_{n-j} \otimes \mathrm{sgn}_j[j-1]) & \text{if } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

We denote the basis elements of $\mathscr{E}(n)$ corresponding to the basis elements of $\mathscr{N}ij^{\text{!`}}(n)$ by



where $I = \{\sigma(1), \ldots, \sigma(i)\}$ and $J = \{\sigma(i+1), \ldots, \sigma(n)\}$. Note that they are symmetric in the input edges labeled by I, skew-symmetric in the ones labeled by J, and of degree $1 - |J|$. The differential of $\mathscr{N}ij_\infty$ is then given by



Here $\epsilon_1 = |J_2| + \pi(J_1 \sqcup J_2)$ and $\epsilon_2 = |J_2| + |J_3| + \pi(J_1 \sqcup J_2 \sqcup J_3)$, and $\pi(J_1 \sqcup J_2)$ and $\pi(J_1 \sqcup J_2 \sqcup J_3)$ denote the parities of the permutations $J \mapsto J_1 \sqcup J_2$ and $J \mapsto J_1 \sqcup J_2 \sqcup J_3$, where we assume the elements of each disjoint set to be ordered ascendingly.

Proof. The proof is completely analogous to that of Theorem 3.26. The graph calculations are more tedious though.

□

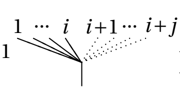
9.3 Geometrical interpretation

9.3.1 An isomorphism of Lie algebras

We are going to follow the pattern of Chapter 6 once more, this time constructing a morphism from $\mathscr{L}_{\mathscr{N}ij}(V)$ to a Lie subalgebra of $\Omega^\bullet_V \otimes \mathscr{T}_V$.

A degree r element $f \in \mathscr{L}_{\mathscr{N}ij}(V)$ is equivalent to a family of degree r linear maps

$$\left\{ f_{(i,j)} := f(s^{-1}$$



$$) \colon \odot^i V \otimes \wedge^j V \to V[-j] \right\}_{\substack{i \geq 1 \\ j \geq 0 \\ i+j \geq 2}} .$$

That we only need to consider the image of corollas labeled as above is a consequence of the $\mathbb{S}$-equivariance of f.

From f we construct a vector form $K_f \in \Omega^\bullet_V \otimes \mathcal{T}_V \cong \mathbb{K}[\![t^a, \gamma^b]\!] \otimes \mathbb{K}\partial_c$ as follows. Let

$$(K_f)_{(i,j)} := \frac{1}{i!j!} K^c_{(a_1\cdots a_i)[b_1\cdots b_j]} t^{a_1} \cdots t^{a_i} \gamma^{b_1} \cdots \gamma^{b_j} \partial_c,$$

where the elements $K^c_{(a_1\cdots a_i)[b_1\cdots b_j]} \in \mathbb{K}$ are given by

$$f_{(i,j)}(e_{a_1} \odot \cdots \odot e_{a_i} \otimes e_{b_1} \wedge \cdots \wedge e_{b_j}) = K^c_{(a_1\cdots a_i)[b_1\cdots b_j]} e_c. \tag{9.3}$$

We define

$$K_f := \sum_{\substack{i\geq 1, j\geq 0\\ i+j\geq 2}} (K_f)_{(i,j)}.$$

Further, we define a subset of vector forms

$$\widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V} := \{K \in \Omega^{\geq 1}_V \otimes \mathcal{T}_V \mid K|_0 = 0\} \cup \widetilde{\mathcal{T}_V},$$

where $\widetilde{\mathcal{T}_V}$ was defined in §6.3.1. We note that $\widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V}$ is a Lie subalgebra of $\Omega^\bullet_V \otimes \mathcal{T}_V$ and that $K_f \in \widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V}$.

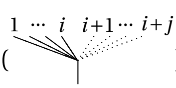
As noted in §6.3.1 the differential d of V corresponds to a degree one vector field D and $\delta_D := [D, \_]_{\text{F-N}}$ defines a differential on $\Omega^\bullet_V \otimes \mathcal{T}_V$. Thus $(\widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V}, [\_,\_]_{\text{F-N}}, \delta_D)$ is a dg Lie algebra. We let $(\widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V}^{\text{op}}, [\_,\_]^{\text{op}}_{\text{F-N}}, \delta^{\text{op}}_D)$ denote the opposite Lie algebra.

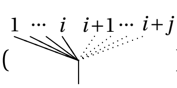
Theorem 9.6. *The morphism*

$$\Phi\colon \mathcal{L}_{\mathcal{N}ij}(V) \to \widetilde{\Omega^\bullet_V \otimes \mathcal{T}_V}^{\text{op}}, \quad f \mapsto K_f$$

is an isomorphism of dg Lie algebras.

Proof. Let $f \in \mathcal{L}_{\mathcal{N}ij}(V)$ be a homogeneous element of degree r. Explicitly, this means that if a structure coefficient $K^c_{(a_1\cdots a_i)[b_1\cdots b_j]}$ of f is non-zero, then $|e_c| - |e_{a_1}| - \cdots - |e_{a_i}| - |e_{b_1}| - \cdots - |e_{b_j}| = r - j$. Since $|\partial_c| = |e_c|$, $|t^a| = -|e_a|$, and $|\gamma^b| = -|e_b| + 1$, it follows that K_f is of degree r. Applying the differential δ to f we obtain a new family of maps $\{\delta(f)_{(i,j)}\}$ determined by the morphisms

$\delta(f)_{(i,j)} = \delta(f)($ 

$) = d \circ f($ 

$) =$

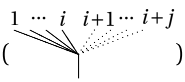
$$d \circ_1 f_{(i,j)} - (-1)^r \sum_{k=1}^{i+j} f_{(i,j)} \circ_k d.$$

From this we get that

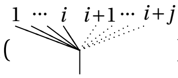
$$\Phi(\delta(f))_{(i,j)} = \left(\frac{1}{i!j!} K^e_{(a_1\cdots a_i)[b_1\cdots b_n]} D^c_e - (-1)^r \frac{1}{(i-1)!j!} D^e_{a_i} K^a_{(a_1\cdots a_{i-1}e)[b_1\cdots b_j]} \right.$$
$$\left. -(-1)^r \frac{1}{(i!(j-1)!} D^e_{b_1} K^c_{(a_1\cdots a_i)[eb_2\cdots b_j]} \right) t^{a_1}\cdots t^{a_i} \gamma^{b_1}\cdots\gamma^{b_j}\partial_c =$$
$$([\Phi(f), D]_{\text{F-N}})_{(i,j)} = (\delta^{\text{op}}_D(\Phi(f)))_{(i,j)}.$$

Thus Φ defines an isomorphism of dg vector spaces.

Now let f and g be homogeneous elements of $\mathscr{L}_{\mathscr{L}ie^1}(V)$ of degrees r and s, respectively, and let $L^c_{(a_1\cdots a_i)[b_1\cdots b_j]}$ denote the structure coefficients of the morphisms $g(\ ) \colon \odot^i V \otimes \wedge^j V \to V$. By definition

$$[f,g]($$


$$) =$$

$$\sum_{G\in\mathfrak{G}_{(2)}} \mu_G \circ \Big((G,[f^d,g^u] - (-1)^{rs}(G,[g^d,f^u]) \Big) \circ_G \Delta($$


$$) =$$

$$\sum_{\substack{i_1+i_2=i+1\\ j_1+j_2=j\\ \sigma\in\mathbb{S}'_{i+j}}} \widetilde{\text{sgn}}(\sigma) \left(f_{(i_1,j_1)} \circ_{i_1} g_{(i_2,j_2)} - (-1)^{rs} g_{(i_1,j_1)} \circ_{i_1} f_{(i_2,j_2)}\right) \circ \phi_\sigma$$
$$+ \sum_{\substack{i_1+i_2=i\\ j_1+j_2=j+1\\ \sigma\in\mathbb{S}''_{i+j}}} \widetilde{\text{sgn}}(\sigma) \left((f_{(i_1,j_1)} \circ_{i_1+1} g_{(i_2,j_2)} - (-1)^{rs} g_{(i_1,j_1)} \circ_{i_1+1} f_{(i_2,j_2)})\right) \circ \phi_\sigma$$

Here $\mathbb{S}'_{i+j}$ and $\mathbb{S}''_{i+j}$ denote the subset of permutations of $\mathbb{S}_{i+j}$ defined in Theorem 9.5 and $\widetilde{\text{sgn}}(\sigma)$ denotes the sign of the permutation of the inputs $i+1,\ldots,i+j$. It follows that

$$\Phi([f,g])_{(i,j)} = ([\Phi(f),\Phi(g)]^{\text{op}}_{\text{F-N}})_{(i,j)}.$$

□

9.3.2 Representations of $\mathscr{N}ij_\infty$ and vector forms

We note that any pointed vector form $K \in \Omega^\bullet_V \otimes \mathscr{T}_V$ can be decomposed as the sum $K = D + K_f$ where $f \in \mathscr{L}_{\mathscr{N}ij}(V)$ and D is a vector field derived from a differential of V. Thus Theorem 9.6 yields the following:

Corollary 9.7. *There is a one-to-one correspondence between representations of $\mathscr{N}ij_\infty$ in V and pointed Maurer-Cartan elements of $\Omega^\bullet_V \otimes \mathscr{T}_V$.*

Remark. Using the above correspondence between $\mathbb{S}$-module morphisms $\mathscr{N}ij^{¡} \to \mathscr{E}nd_V$ and vector forms Merkulov showed in [39] that there is a one-to-one correspondence between representations of $\Omega(\mathscr{N}ij^{¡})$ in V and pointed Maurer-Cartan elements of $\Omega^\bullet_V \otimes \mathscr{T}_V$. This result together with Corollary 9.4 immediately implies Corollary 9.7. We note also that the differential given in §3.6 of [39] really is the differential of the minimal resolution of $\mathscr{N}ij$.

When V is concentrated in degree zero the Maurer-Cartan elements of $\Omega^\bullet_V \otimes \mathscr{T}_V$ lie in $\Omega^1_V \otimes \mathscr{T}_V$. Thus representations of $\mathscr{N}ij_\infty$ correspond precisely to pointed classical Nijenhuis structures.

9.3.3 Nijenhuis$_\infty$ structures

The Maurer-Cartan elements of $\Omega^\bullet_V \otimes \mathscr{T}_V$ were studied in [39].

Definition (Merkulov)**.** A *Nijenhuis$_\infty$ structure* on a graded manifold V is a Maurer-Cartan element of $\Omega^\bullet_V \otimes \mathscr{T}_V$.

One of the results obtained by Merkulov is that Nijenhuis$_\infty$ structures correspond to contractible dg manifolds, see Section 5 of [39] for details.

A Nijenhuis$_\infty$ structure can be interpreted as a family of maps $\{J_i\colon \wedge^i \mathscr{T}_V \to \mathscr{T}_V\}_{i\in\mathbb{N}}$ (or as a map $J\colon \wedge^\bullet \mathscr{T}_V \to \mathscr{T}_V$) satisfying certain quadratic relations. The geometrical (or algebraic) significance of these maps is an interesting and open question.

10 Operad profile of compatible Nijenhuis structures

In this chapter we define the operad encoding bi-Nijenhuis structures. Again we use the PBW-method to show that the operad $\mathscr{B}i\mathscr{N}ij$ of compatible Nijenhuis structures is Koszul which enables us to calculate its minimal resolution. Finally we give a geometrical interpretation of the operad $\mathscr{B}i\mathscr{N}ij_\infty$.

10.1 Extracting the operad

10.1.1 The fundamental parts

Recall that a bi-Nijenhuis structure is pair (J, K) of Nijenhuis structures such that their sum is a Nijenhuis structure. This is equivalent to that the following conditions are satisfied:

$$[J,J]_{\text{F-N}} = 0, \quad [J,K]_{\text{F-N}} = 0, \quad \text{and} \quad [K,K]_{\text{F-N}} = 0,$$

or equivalently

$$[J + \hbar K, J + \hbar K]_{\text{F-N}} = 0.$$

We want to define an operad capturing the fundamental part of this structure, analogously to how a Nijenhuis structure is encoded.

Let $\hat{J}$ and $\hat{K}$ denote the fundamental parts of J and K, respectively, and let $\hat{Q}$ again denote the fundamental part of a homological vector field. That J and K are Nijenhuis structures is encoded by translating

$$[\hat{Q},\hat{Q}]_{\text{F-N}} = 0, \quad [\hat{Q},\hat{J}]_{\text{F-N}} = 0, \quad [\hat{J},\hat{J}]_{\text{F-N}} = 0, \quad [\hat{Q},\hat{K}]_{\text{F-N}} = 0, \quad \text{and} \quad [\hat{K},\hat{K}]_{\text{F-N}} = 0$$

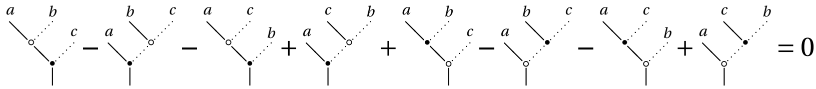
to corresponding operadic relations. In doing this we denote the corollas encoding $\hat{J}$ and $\hat{K}$ by  and , respectively.

10.1.2 Linearly compatible pre-Lie algebras

The compatibility of J and K is captured by

$$[\hat{J},\hat{K}]_{\text{F-N}} = 0,$$

which translates to



for a, b, c being the cyclic permutations of $1, 2, 3$. We make the following definition.

Definition. Let $\circ$ and $\bullet$ be pre-Lie products on a vector space V. We call them *compatible* if their sum $\circ + \bullet$, defined by $a(\circ + \bullet)b := a \circ b + a \bullet b$ is a pre-Lie product. Equivalently, the products are compatible if they satisfy

$$(a \circ b) \bullet c - a \bullet (b \circ c) - (a \circ c) \bullet b + a \bullet (c \circ b) + (a \bullet b) \circ c - a \circ (b \bullet c) - (a \bullet c) \circ b + a \circ (c \bullet b) = 0.$$

Thus the compatibility of $\hat{J}$ and $\hat{K}$ means that the corresponding maps j_1 and k_1 give V the structure of compatible pre-Lie algebras.

10.1.3 The genome

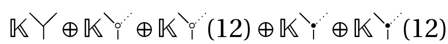
We can encode all the above conditions on $\hat{J}$, $\hat{K}$, and $\hat{Q}$ by the single equation

$$[\hat{Q} + \hat{J} + \hbar\hat{K}, \hat{Q} + \hat{J} + \hbar\hat{K}]_{\text{F-N}} = 0.$$

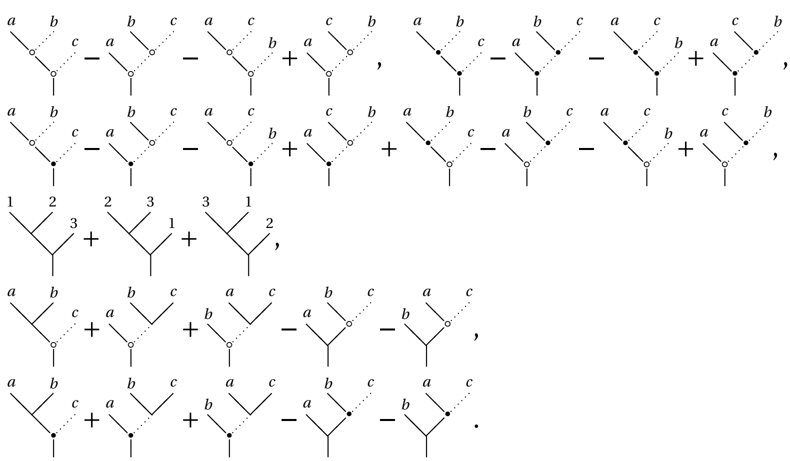
Translated to the language of operads we obtain the following:

Definition. The operad $\mathscr{B}i\mathscr{N}ij$ is the quadratic operad $\mathscr{F}(\mathscr{M})/(\mathscr{R})$, where $\mathscr{M}$ is the $\mathbb{S}$-module given by

$$\mathscr{M}(n) = \begin{cases} \mathbb{1}_2[-1] \oplus \mathbb{K}[\mathbb{S}_2] \oplus \mathbb{K}[\mathbb{S}_2] = \\ \text{(see diagram)} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$



and the relations $\mathscr{R}$ are given by



To sum up this definition, a $\mathscr{B}i\mathscr{N}ij$-algebra is a pair of $\mathscr{N}ij$-algebras sharing the same Lie bracket and such that the pre-Lie products are compatible.

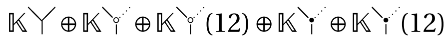
10.2 Computing the resolution

10.2.1 The Koszul dual operad of $\mathscr{B}i\mathscr{N}ij$

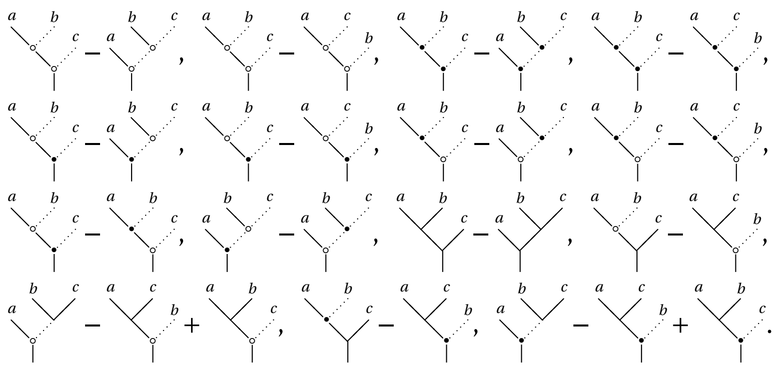
From the definition of Koszul dual operads we immediately obtain the following:

Proposition 10.1. *The Koszul dual operad of $\mathscr{B}i\mathscr{N}ij$ is the quadratic operad $\mathscr{F}(\mathscr{N})/(\mathscr{S})$, where $\mathscr{N}$ is the $\mathbb{S}$-module given by*

$$\mathscr{N}(n) = \begin{cases} \operatorname{sgn}_2[1] \oplus \mathbb{K}[\mathbb{S}_2] \otimes \operatorname{sgn}_2 \oplus \mathbb{K}[\mathbb{S}_2] \otimes \operatorname{sgn}_2 & \\ \mathbb{K}\,[\text{tree}] \oplus \mathbb{K}\,[\text{tree}] \oplus \mathbb{K}\,[\text{tree}](12) \oplus \mathbb{K}\,[\text{tree}] \oplus \mathbb{K}\,[\text{tree}](12) & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$



and the relations $\mathscr{S}$ are given by



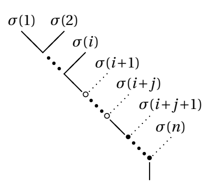
Straightforward graph computations yield the following:

Proposition 10.2. *The underlying $\mathbb{S}$-module of $\mathscr{B}i\mathscr{N}ij^{¡}$ is given by*

$$\mathscr{B}i\mathscr{N}ij^{¡}(n) = \begin{cases} \bigoplus_{0\le p\le n-1} (\operatorname{Ind}_{\mathbb{S}_{n-p}\times\mathbb{S}_p}^{\mathbb{S}_n} \operatorname{sgn}_{n-p} \otimes (\underbrace{\mathbb{1}_p \oplus \cdots \oplus \mathbb{1}_p}_{p+1 \text{ terms}})[n-p-1]) & \text{if } n \ge 2 \\ 0 & \text{otherwise.} \end{cases}$$

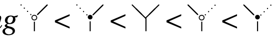
Explicitly, a $\mathbb{K}$-basis for $\mathscr{B}i\mathscr{N}ij^{¡}(n)$ is given by

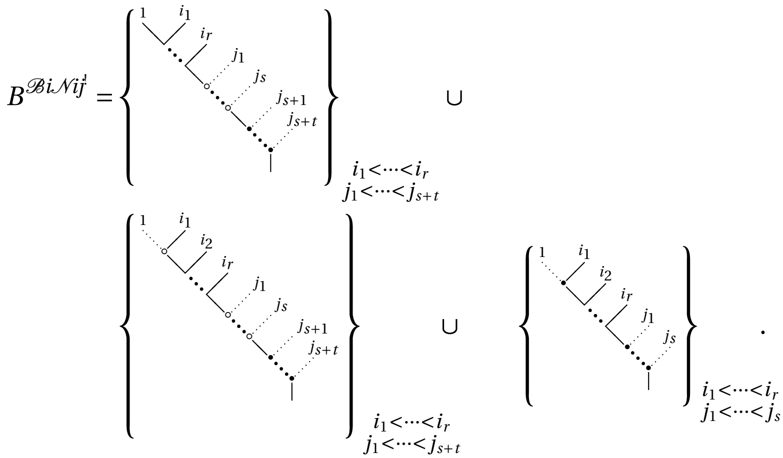
$$\left\{ \quad [\text{tree}] \quad \right\}_{\substack{i\ge1,\, j\ge0 \\ \sigma\in\mathbb{S}^{\text{un-sh}}_{(i,n-i)}}} . \qquad (10.1)$$



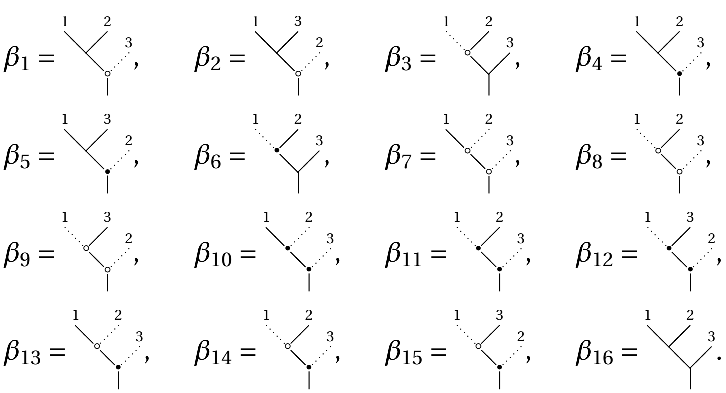
10.2.2 Koszulness of $\mathscr{B}i\mathscr{N}ij$

Through a slight modification of the basis (10.1) we obtain a PBW-basis for $\mathscr{B}i\mathscr{N}ij^{!}$.

Proposition 10.3. *The following is a PBW-basis of $\mathscr{B}i\mathscr{N}ij^{!}$ with respect to the ordering* 



Proof. Through straightforward graph calculations one can verify that $B^{\mathscr{B}i\mathscr{N}ij^{!}}$ is a basis of $\mathscr{B}i\mathscr{N}ij^{!}$, cf. the basis (10.1). Condition (ii) is easily verified. We denote the elements of $B^{\mathscr{B}i\mathscr{N}ij^{!}(3)}$ by



To show that (i) is satisfied it is sufficient, using Proposition 4.4, to observe that for any decorated two-vertex graph $\alpha \in B^{\mathscr{F}(\mathscr{N})} \setminus B^{\mathscr{B}i\mathscr{N}ij^{!}}$ with $\alpha = \sum c_i \beta_i$, $c_i \in \mathbb{K}$, we have $c_i \neq 0 \implies \beta_i > \alpha$. □

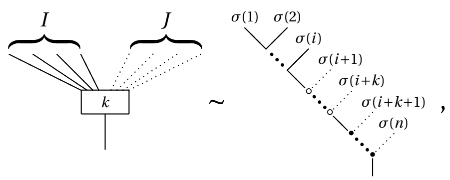
Corollary 10.4. *The operad $\mathscr{B}i\mathscr{N}ij^{!}$, and thus also $\mathscr{B}i\mathscr{N}ij$, is Koszul.*

10.2.3 The minimal resolution of $\mathscr{B}i\mathscr{N}ij$

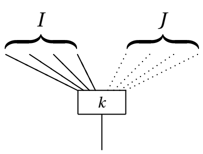
Theorem 10.5. *The minimal resolution $\mathscr{B}i\mathscr{N}ij_\infty$ of the operad $\mathscr{B}i\mathscr{N}ij$ is the quasi-free operad $(\mathscr{F}(E),\delta)$ where $E=\{E(n)\}_{n\geq 0}$ is the $\mathbb{S}$-module*

$$E(n)=\begin{cases}\bigoplus_{0\leq j\leq n-1}(\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{n-j}\times\mathbb{S}_j}\mathbb{1}_{n-j}\otimes(\underbrace{\mathrm{sgn}_j\oplus\cdots\oplus\mathrm{sgn}_j}_{j+1\ \text{terms}})[j-1]) & \text{if } n\geq 2\\ 0 & \text{otherwise.}\end{cases}$$

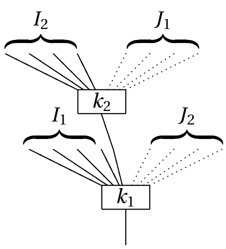
We denote the basis elements of $E(n)$ corresponding to the basis elements of $\mathscr{B}i\mathscr{N}ij^{¡}(n)$ with k white operations by

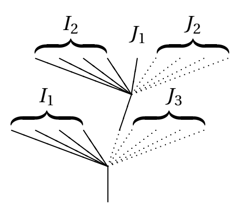


where $I=\{\sigma(1),\ldots,\sigma(i)\}$ and $J=\{\sigma(i+1),\ldots,\sigma(n)\}$. Note that they are symmetric in the input edges labeled by I, skew-symmetric in the ones labeled by J, and of degree $1-|J|$. The differential of $\mathscr{B}i\mathscr{N}ij_\infty$ is then given by

$\delta:$ 

$\mapsto$

$\sum_{\substack{k_1+k_2=k\\ I_1\sqcup I_2=I\\ J_1\sqcup J_2=J}}(-1)^{\epsilon_1}$ 

$-\sum_{\substack{k_1+k_2=k\\ I_1\sqcup I_2=I\\ \{J_1\}\sqcup J_2\sqcup J_3=J}}(-1)^{\epsilon_2}$ 

.

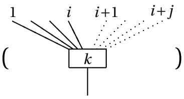
Here $\epsilon_1=|J_2|+\pi(J_1\sqcup J_2)$ and $\epsilon_2=|J_2|+|J_3|+\pi(J_1\sqcup J_2\sqcup J_3)$, and $\pi(J_1\sqcup J_2)$ and $\pi(J_1\sqcup J_2\sqcup J_3)$ denote the parities of the permutations $J\mapsto J_1\sqcup J_2$ and $J\mapsto J_1\sqcup J_2\sqcup J_3$, where we assume the elements of each disjoint set to be ordered ascendingly.

Proof. The proof is completely analogous to that of Theorem 3.26 and 9.5. □

10.3 Geometrical interpretation

10.3.1 An isomorphism of Lie algebras

We will construct a morphism Φ from $\mathscr{L}_{\mathscr{L}ie^{1}_{2}\mathscr{B}i}(V)$ to a Lie subalgebra of $(\Omega^{\bullet}_{V} \otimes \mathscr{T}_{V})[\![\hbar]\!]$. A degree r element $f \in \mathscr{L}_{\mathscr{L}ie^{1}_{2}\mathscr{B}i}(V)$ is equivalent to a family of degree zero linear maps

$\Big\{ {}_k f_{i,j} := f($  $) \colon \odot^{i} V \otimes \wedge^{j} V \to V[1-j] \Big\}_{\substack{j\geq 0, i\geq 1\\ i+j\geq 2\\ 0\leq k\leq j}}.$

From such a family of maps we construct a family of vector forms as follows. Let

$${}_k K_{i,j} = \frac{1}{i!j!}\, {}_k K^{c}_{(a_1\cdots a_i)[b_1\cdots b_j]}\, t^{a_1}\cdots t^{a_i}\gamma^{b_1}\cdots\gamma^{b_j}\partial_c,$$

where the numbers ${}_k K^{c}_{(a_1\cdots a_i)[b_1\cdots b_j]} \in \mathbb{K}$ are given by

$${}_k f_{i,j}(e_{a_1}\odot\cdots\odot e_{a_i}\otimes e_{b_1}\wedge\cdots\wedge e_{b_j}) = {}_k K^{c}_{(a_1\cdots a_i)[b_1\cdots b_j]} e_c$$

We assemble these vector forms into a formal power series in the formal parameter $\hbar$ with vector form coefficients:

$$K_f = \sum_{k\geq 0} {}_k K \hbar^{k}, \qquad \text{where } {}_k K = \sum_{\substack{i\geq 1\\ j\geq k}} {}_k K_{i,j}.$$

We introduce $\hbar$ to distinguish vector forms of the same weight but corresponding to different maps. We write $[\_,\_]_{\text{F-N}_{\hbar}}$ for the linearization in $\hbar$ of the Frölicher-Nijenhuis bracket.

$$\tilde{\mathfrak{h}}_V := \{K \in (\Omega^{\geq 1}_{V}\otimes\mathscr{T}_V)[\![\hbar]\!] \mid {}_k K \in (\Omega^{\geq k}_{V}\otimes\mathscr{T}_V)[\![\hbar]\!] \text{ and } K|_0 = 0\} \cup \widetilde{\mathscr{T}_V},$$

where $\widetilde{\mathscr{T}_V}$ is defined as in §6.3.1.

Before we proceed to the main theorem we make an observation.

Lemma 10.6.

Theorem 10.7. *The morphism*

$$\Phi\colon \mathscr{L}_{\mathscr{N}ij}(V) \to \tilde{\mathfrak{h}}^{\text{op}}_{V}, \quad f \mapsto K_f$$

is an isomorphism of dg Lie algebras.

Proof. The proof is completely analogous to the proof of Theorem 9.6. □

10.3.2 Bi-Nijenhuis$_\infty$ structures

We define another Lie subalgebra of $\wedge^\bullet \mathscr{T}_V[\![\hbar]\!]$ by

$$\mathfrak{h}_V := \{K = \sum_{k\geq 0} {}_kK\hbar^k \in (\Omega_V^\bullet \otimes \mathscr{T}_V)[\![\hbar]\!] \;\mid\; {}_kK \in \Omega_V^{\geq k} \otimes \mathscr{T}_V\}.$$

Any pointed element $K \in \mathfrak{g}_V$ can be uniquely decomposed as a sum $K = D + K_f$ for some pair (f, d), where $f \in \mathscr{L}_{\mathscr{B}i\mathscr{N}ij}(V)$ and D is the vector field corresponding to a differential d of V. The equivalence

$$[D + K_f, D + K_f]_{\text{F-N}_\hbar} = 0 \Longleftrightarrow \delta_D^{\text{op}}(K_f) + \frac{1}{2}[K_f, K_f]^{\text{op}}_{\text{F-N}_\hbar} = 0,$$

yields the following corollary to Theorem 10.7.

Corollary 10.8. *There is a one-to-one correspondence between representations of $\mathscr{B}i\mathscr{N}ij_\infty$ in a dg vector space V and pointed Maurer-Cartan elements of $\mathfrak{h}_V$.*

Note that Theorem F in the introduction is just another formulation of the preceding theorem. In analogy with Nijenhuis$_\infty$ structures we make the following definition.

Definition. A bi-Nijenhuis$_\infty$ structure on a manifold V is a Maurer-Cartan element in $\mathfrak{g}_V$.

10.3.3 Representations of $\mathscr{B}i\mathscr{N}ij_\infty$ in non-graded vector spaces

Let V be a vector space concentrated in degree zero, then a morphism $f \in \mathscr{L}_{\mathscr{B}i\mathscr{N}ij}(V)$ corresponds to an element $J_f = J_1 + J_2\hbar \in \mathfrak{h}_V$ such that J_1 and J_2 are pointed elements of $\Omega_V^1 \otimes \mathscr{T}_V$. That f is a representation of $\mathscr{B}i\mathscr{N}ij_\infty$ in V is thus equivalent to that J_f is a pointed bi-Nijenhuis structure on the formal manifold associated to V. In particular this proves Theorem E.

A Details on $\mathfrak{G}^*$-algebras

Composition product of free $\mathfrak{G}^*$-algebras

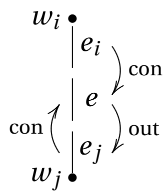
We keep the notation of §2.2.1. When describing the grafting of graphs we will denote $G(G_1, \dots G_k)$ by $\widetilde{G}$.

The vertices of the graph $\widetilde{G}$ are given by $V_{\widetilde{G}} := V_{G_1} \sqcup \cdots \sqcup V_{G_k}$, the internal edges by $E^{\text{int}}_{\widetilde{G}} := E^{\text{int}}_{G_1} \sqcup \cdots \sqcup E^{\text{int}}_{G_k} \sqcup E^{\text{int}}_{G}$, and the external edges by $E^{\text{in}}_{\widetilde{G}} := E^{\text{in}}_{G}$ and $E^{\text{out}}_{\widetilde{G}} := E^{\text{out}}_{G}$. Defining the incidence morphism $\Phi_{\widetilde{G}}$ is more complicated.

For an edge $e \in E^{\text{int}}_{G_i}$ we define $\Phi_{\widetilde{G}}(e) := \Phi_{G_i}(e)$. Let $e \in E^{\text{int}}_{G}$ be an edge with $\Phi_G(e) = (v_i, v_j)$ and let the vertices v_i and v_j of G be decorated with $f_i \otimes_{\mathbb{S}} G_i \otimes_{\mathbb{S}} g_i$ and $f_j \otimes_{\mathbb{S}} G_j \otimes_{\mathbb{S}} g_j$, respectively. Via the local labeling of G and the global labelings of G_i and G_j, this edge connects two vertices, $w_i \in V_{G_i}$ and $w_j \in V_{G_j}$ of $\widetilde{G}$, as follows. Let e_i be the edge in $E^{\text{out}}_{G_i}$ with $f_i \circ \text{out}_{G_i}(e_i) = e$. Note that this composition is well defined since another representative, $f'_i \otimes_{\mathbb{S}} G'_i \otimes_{\mathbb{S}} g'_i$ of the decoration of v_i, will satisfy $\text{out}_{G'_i} = \sigma\, \text{out}_{G_i}$ and $f'_i = f_i \sigma^{-1}$ for some permutation σ, implying $f'_i \circ \text{out}_{G'_i} = f_i \circ \sigma^{-1} \circ \sigma \circ \text{out}_{G_i} = f_i \circ \text{out}_{G_i}$. By composing further with $\text{in}_{G_j} \circ g_j$, which by a similar argument also is well defined, we obtain an edge $e_j = \text{in}_{G_j} \circ g_j \circ f_i \circ \text{out}_{G_i}(e_i) \in E^{\text{in}}_{G_j}$. Let $w_i = \Phi_{G_i}(e_i)$ and $w_j = \Phi_{G_j}(e_j)$, then we set $\Phi_{\widetilde{G}}(e) := (w_i, w_j)$.

For an external edge $e \in E^{\text{in}}_{\widetilde{G}}$ with $\Phi_G(e) = v_i$ let $e_i = \text{in}_{G_i} \circ g_i(e) \in E^{\text{in}}_{G_i}$ and $w_i = \Phi_{G_i}(e_i)$. We define $\Phi_{\widetilde{G}}(e) := w_i$. Similarly for an external edge $e \in E^{\text{out}}_{\widetilde{G}}$ with $\Phi_G(e) = v_i$ let $e_i = f_i \circ \text{out}_{G_i}(e) \in E_{G_i}$ and $w_i = \Phi_{G_i}(e_i)$. We define $\Phi_{\widetilde{G}}(e) := w_i$. By the same arguments as above this is well defined. The global labeling of the external edges is directly induced by the one of G, $\text{in}_{\widetilde{G}} := \text{in}_G$ and $\text{out}_{\widetilde{G}} := \text{out}_G$.

For three edges e, e_i, e_j connected as above we will use the notation $e_{\text{in}} := e_i$, $e_{\text{out}} := e_j$, and $(e_i)_{\text{con}} = (e_j)_{\text{con}} := e$. We will use the same notation for two connected edges.



The elements $\widetilde{p}_b^a$ are defined as follows. If $\overline{p}_b^a = f_b^a \otimes_{\mathbb{S}} p_b^a \otimes_{\mathbb{S}} g_b^a$ is an element decorating a vertex $w \in V_{G_i}$ with $|E_w^{\text{out}}| = m$ and $|E_w^{\text{in}}| = n$, then $\widetilde{p}_b^a = \widetilde{f}_b^a \otimes_{\mathbb{S}} p_b^a \otimes_{\mathbb{S}} \widetilde{g}_b^a$, where the bijections $\widetilde{f}_b^a \colon [m] \to E_w^{\text{out}}$ and $\widetilde{g}_b^a \colon E_w^{\text{in}} \to [n]$ are given by

$$\widetilde{f}_b^a(i) = \begin{cases} f_b^a(j) & \text{if } f_b^a(j) \in E_{G_i}^{\text{int}} \\ f_b^a(j)_{\text{con}} & \text{if } f_b^a(j) \in E_{G_i}^{\text{out}} \end{cases}$$

and

$$\widetilde{g}_b^a(e) = \begin{cases} g_b^a(e) & \text{if } e \in E_{G_i}^{\text{int}} \\ g_b^a(e_{\text{out}}) & \text{if } e \in E_G \cap (E_w^{\text{in}})_{\text{con}}. \end{cases}$$

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