Low-dimensional cohomology of current Lie algebras

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Abstract

We deal with low-dimensional homology and cohomology of current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is an associative commutative algebra.

We derive, in two different ways, a general formula expressing the second cohomology of current Lie algebra with coefficients in the trivial module through cohomology of L, cyclic cohomology of A, and other invariants of L and A. The first proof is achieved by using the Hopf formula expressing the second homology of a Lie algebra in terms of its presentation. The second proof employs a certain linear-algebraic technique, ideologically similar to "separation of variables" of differential equations.

We also obtain formulas for the first and, in some particular cases, for the second cohomology of the current Lie algebra with coefficients in the "current" module, and the second cohomology with coefficients in the adjoint module in the case where L is the modular Zassenhaus algebra.

Applications of these results include: description of modular semi-simple Lie algebras with a solvable maximal subalgebra; computations of structure functions for manifolds of loops in compact Hermitian symmetric spaces; a unified treatment of periodizations of semi-simple Lie algebras, derivation algebras (with prescribed semi-simple part) of nilpotent Lie algebras, and presentations of affine Kac-Moody algebras.

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Contents

Preface

Current Lie algebras are curious objects. Algebraically, they are defined in a very simple way: given a Lie algebra L and associative commutative algebra A, the *current Lie algebra* is defined as the vector space $L \otimes A$ equipped with the Lie bracket:

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab$$

for $x,y \in L$, $a,b \in A$. Yet, this simple construction leads to a large body of interesting mathematics.

Current Lie algebras are ubiquitous in various branches of mathematics and physics. Let us mention only two examples. According to the classical Block's theorem about the structure of modular semisimple Lie algebras, the latter are described in terms of the direct sums of algebras of the type

$$S \otimes K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) \ltimes 1 \otimes D, \tag{1}$$

where $A \ltimes B$ denotes the semidirect sum of modules (algebras) with A a submodule (ideal), S is a simple Lie algebra, and D is an algebra of derivations of the reduced polynomial algebra $K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ (p being the characteristic of the ground field). Thus, current Lie algebras of this special type, extended by a "tail" of derivations, provide a typical model of modular semisimple Lie algebras.

Essentially the same structural picture holds for the celebrated affine Kac-Moody algebras: they are nothing but "extended" current Lie algebras of the form

$$\mathbb{C}z \ltimes \left(\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \ltimes \mathbb{C}t\frac{d}{dt} \right) \simeq \left(\mathbb{C}z \ltimes \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \right) \ltimes \mathbb{C}t\frac{d}{dt}, \tag{2}$$

or twisted versions thereof. Here $\mathfrak{g}$ is a complex finite-dimensional simple Lie algebra, $\mathbb{C}[t,t^{-1}]$ is the algebra of Laurent polynomials, and z is the central term. Their generalizations – toroidal and Krichever-Novikov algebras (Lie algebras of functions or vector fields on tori or compact Riemann surfaces) – also follow this scheme.

Kac-Moody algebras, Krichever-Novikov algebras and variations thereof appear as algebras of symmetries in various physical theories (in fact, the term "current algebra" came from physics). It is known that physicists are (somewhat justly) obsessed with central extensions, and, luckily for them, the current Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ has a well-known beautiful central extension

(2) turning it to a Kac-Moody algebra. Our cohomological computations show how this central extension fits a broader picture.

Besides of being an amusement to physicists, low-dimensional cohomology is important in the structural theory of Lie algebras itself: to describe generators and relations, derivations and deformations. As special types of current algebras and algebras close to them pop up in various situations in the modular Lie algebra theory, it is only natural to try to consider these invariants for current Lie algebras in general.

So the problem arises to "compute" the cohomology of the current Lie algebra $L \otimes A$ – that is, to describe it in terms of some structures associated with L and A. In general, this problem seems to be hopeless. But in some special low-dimensional situations this seems to be possible. Moreover, the additive K-theory of Loday-Quillen and Feigin-Tsygan, linking the cohomology of Lie algebras of infinite size matrices over an associative algebra A (which, in the case of A commutative, are, again, nothing but current Lie algebras), with the cyclic cohomology of A, gives us a hint that the cyclic cohomology is involved. And this is indeed so: our low-dimensional computations, though more pedestrian than general additive K-theoretic ones, but encompassing a broader class of algebras and modules, provide a further insight how various cohomological theories – Hochschild cohomology, cyclic cohomology, Harrison cohomology – are intertwined via the current Lie algebras cohomology.

Each chapter of this thesis is either an edited version of a published paper, or paper accepted for publication.

In the first chapter, "The second homology group of current Lie algebras", published in Astérisque **226** (1994), 435–452, we derive a general formula for the second homology of the current algebra with coefficients in the trivial module, using the Hopf formula expressing the second homology of an algebra in terms of its presentation.

In the second chapter, "Deformations of $W_1(n) \otimes A$ and modular semisimple Lie algebras with a solvable maximal subalgebra", published in J. Algebra **268** (2003), 603–635, we compute the second cohomology of the current Lie algebra of the form $W_1(n) \otimes A$, where $W_1(n)$ is the modular Zassenhaus algebra. This has an interesting application in the structure theory of modular Lie algebras: according to a result of Weisfeiler, semisimple Lie algebras with a solvable maximal subalgebra are filtered deformations of algebras of the form (1) with $S = W_1(n)$. The result we obtain about the second cohomology of $W_1(n) \otimes A$ with coefficients in the adjoint module, allows us to describe these filtered deformations, and thus complete Weisfeiler's description of such algebras. In the proof of the central Theorem 3.1 of that paper, I use some earlier results from another of my paper, "Central extensions of current algebras", Trans. Amer. Math. Soc. **334** (1992), 143–152 (not included in this thesis).

In the third chapter, "Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds", published in Lin. Algebra Appl. **407** (2005), 71–104, we obtain further results about the low-dimensional cohomology of arbitrary current Lie algebra with coefficients in the "current" module. Though the complexity of even these partial results demonstrates that in general, it seems to be impossible to give an adequate (and rather graphic) description of (co)homology of $L \otimes A$ in terms of invariants of L and A, in some particular situations such description is possible. One of such situations is the case of the second cohomology and abelian L, yielding a description of structure functions on some manifolds of loops, via the known link to the Spencer cohomology and Cartan prolongations.

In the fourth chapter, "Invariants of Lie algebras extended over commutative algebras without unit", to be published in J. Nonlin. Math. Phys., we generalize some of the results of preceding chapters to the situation when the commutative algebra A is not necessary unital. As an application, this provides a unified treatment of seemingly separate subjects, like generators and relations of periodizations of semisimple Lie algebras and of Kac-Moody algebras, and existence of derivation algebras of nilpotent Lie algebras, with prescribed semisimple part.

1. The second homology group of current Lie algebras

Astérisque **226** (1994), 435–452

We derive a general formula expressing the second homology of a Lie algebra of the form $L \otimes A$ with coefficients in the trivial module through homology of L, cyclic homology of A, and other invariants of L and A. This is achieved by using the Hopf formula expressing the second homology of a Lie algebra in terms of its presentation. We also derive a similar formula for the associated Lie algebra of the tensor product of two associative algebras.

0 Introduction

It is a well known fact that the current Lie algebra $\mathscr{G} \otimes \mathbb{C}[t,t^{-1}]$ associated to a simple finite-dimensional Lie $\mathbb{C}$-algebra $\mathscr{G}$ has a central extension leading to the affine non-twisted Kac-Moody algebra $\mathbb{C}z \ltimes \mathscr{G} \otimes \mathbb{C}[t,t^{-1}]$ with bracket (z is the central element)

$$\{x \otimes f, y \otimes g\} = [x,y] \otimes fg + (x,y) Res(\frac{df}{dt} g)\, z$$

where $(\cdot\,,\cdot)$ is the Killing form on $\mathscr{G}$ (cf. [Kac]).

In view of the known relationship between central extensions and the second (co)homology group with coefficients in the trivial module, one of the main results of this paper can be considered as a generalization of this fact for the case of general current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is associative commutative algebra, equipped with bracket

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab.$$

Theorem 0.1. *Let L be an arbitrary Lie algebra over a field K of characteristic $p \neq 2$ and A an associative commutative algebra with unit over K. Then there is an isomorphism of K-vector spaces:*

$$H_2(L \otimes A) \simeq H_2(L) \otimes A \bigoplus B(L) \otimes HC_1(A) \bigoplus \wedge^2(L/[L,L]) \otimes Ker(S^2(A) \to A) \bigoplus S^2(L/[L,L]) \otimes T(A) \quad (0.1)$$

where the mapping $S^2(A) \to A$ is induced by the multiplication in A,

$$T(A) = \langle ab \wedge c + ca \wedge b + bc \wedge a \,|\, a,b,c \in A \rangle,$$

$B(L)$ is the space of coinvariants of the L-action on $S^2(L)$, $HC_1(A)$ is the first cyclic homology of A, and $\wedge^2$ and S^2 denote the skew and symmetric products, respectively.

Notice that in the case $L=[L,L]$, the third and fourth terms in the right-hand side of (0.1) vanish.

Many particular cases of this theorem were proved previously by different authors. An exhaustive description of previous works on this theme may be found in [H] and [S].

For the first time, a cohomology formula of the type (0.1) has appeared in [S], where Theorem 0.1 was proved assuming that L is 1-generated over an augmentation ideal of its enveloping algebra. A. Haddi [H] obtained a result similar to Theorem 0.1 in the case where K is a field of characteristic zero (however, it seems that his arguments work over any field of characteristic $p \neq 2,3$).

Our method of proof differs from previous ones known to us, and is based on the Hopf formula expressing $H_2(L)$ in terms of a presentation $0 \to I \to \mathscr{L} \to L \to 0$, where $\mathscr{L} = \mathscr{L}(X)$ is the free Lie algebra over K freely generated by the set X:

$$H_2(L) \simeq ([\mathscr{L},\mathscr{L}] \cap I)/[\mathscr{L},I] \tag{0.2}$$

(see, for example, [KS]).

The contents of the paper are as follows. §1 is devoted to some technical preliminary results. In §2 we determine the presentation of a current Lie algebra $L \otimes A$. In §3 Theorem 0.1 is proved. As it corollary we get in §4 a description of the space $B(L \otimes A)$. In §5 a "noncommutative version" of Theorem 0.1 is proved (Theorem 5.1). Namely, we derive the formula for the second homology group of the Lie algebra $(A \otimes B)^{(-)}$, where A, B are associative (noncommutative) algebras with unit, and (–) in superscript denotes passing to the associated Lie algebra. The technique used here is no longer based on the Hopf formula, but on more or less direct computations in some quotients of the spaces of cycles. However, arguments used in proof, resemble, to a great extent, the previous ones. Getting a particular case $B = M_n(K)$, we recover, after a slight modification, an isomorphism $H_2(sl_n(A)) \simeq HC_1(A)$ obtained in [KL].

The following notational convention will be used: the letters $a,b,c,\dots$, possibly with sub- and superscripts, denote elements of algebra A, while letters $u,v,w,\dots$ denote elements of the free Lie algebra $\mathscr{L}(X)$ with the set of generators $X = \{x_i\}$, unless stated otherwise. $\mathscr{L}^n(X)$ denotes the nth term in the derived series of $\mathscr{L}(X)$. The arrows $\rightarrowtail$ and $\twoheadrightarrow$ denote injection and surjection, respectively.

All other undefined notions and notation are standard, and may be found, for example, in [F] for Lie algebra (co)homology, and in [LQ] for cyclic homology. In some places we use diagram chasing and 3×3-Lemma without explicitly mentioning it.

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1 Preliminaries

Looking at formula (0.1), one can distinguish between the first two "principal" terms and other two "non-principal" ones. In order to simplify calculations, we will obtain a version of the Hopf formula leading to the appearance of "principal" terms only, and then the general case will be derived.

Each nonperfect Lie algebra L, i.e., not coinciding with its commutant $[L,L]$, possesses "trivial" homology classes of 2-cycles with coefficients in the module K, namely, classes whose representatives do not lie in $L \wedge [L,L]$. More precisely, consider a natural homomorphism

$$\psi : H_2(L) \to H_2(L/[L,L]) \simeq \wedge^2(L/[L,L])$$

and denote $H_2^{ess}(L) = Ker\psi$, the homology classes of "essential" cycles.

Lemma 1.1. *One has an exact sequence*

$$0 \to H_2^{ess}(L) \to H_2(L) \xrightarrow{\psi} \wedge^2(L/[L,L]) \xrightarrow{\pi} [L,L]/[[L,L],L] \to 0$$

where π is induced by multiplication in L.

Proof. This is just an obvious consequence of a 5-term exact sequence derived from the Hochschild-Serre spectral sequence $H_n(L/[L,L], H_m([L,L])) \Rightarrow H_{n+m}(L)$. □

Further, we need a version of Hopf formula for $H_2^{ess}(L)$.

Lemma 1.2. *Given a presentation* $0 \to I \to \mathscr{L} \to L \to 0$ *of a Lie algebra L, one has*

$$H_2^{ess}(L) \simeq \frac{\mathscr{L}^3 \cap I}{\mathscr{L}^3 \cap [\mathscr{L}, I]}. \tag{1.1}$$

Proof. Since $L/[L,L] \simeq \mathscr{L}/(\mathscr{L}^2 + I)$, the Hopf formula (0.2) being applied to the algebra $L/[L,L]$ gives $H_2(L/[L,L]) \simeq \mathscr{L}^2/[\mathscr{L}, \mathscr{L}^2 + I]$, and

$$Ker\psi = Ker\left(\frac{\mathscr{L}^2 \cap I}{[\mathscr{L}, I]} \to \frac{\mathscr{L}^2}{[\mathscr{L}, \mathscr{L}^2 + I]}\right) \simeq \frac{\mathscr{L}^2 \cap I \cap [\mathscr{L}, \mathscr{L}^2 + I]}{[\mathscr{L}, I]} \simeq \frac{\mathscr{L}^3 \cap I}{\mathscr{L}^3 \cap [\mathscr{L}, I]}.$$

□

Now consider an action of a Lie algebra L on $S^2(L) = L \vee L$ via

$$[z, x \vee y] = [z, x] \vee y + x \vee [z, y].$$

Let $B(L) = S^2(L)/[L, S^2(L)]$ be the space of coinvariants of this action. The dual $B(L)^*$ is the space of symmetric bilinear invariant forms on L.

Let I, J be ideals of L. Define $B(I, J)$ to be the space of coinvariants of the L-action on $I \vee J$. One has a natural embedding $B(I, J) \to B(L)$. The natural map

$$L \vee J \to (L/I) \vee ((I + J)/I)$$

defines a surjection $B(L, J) \to B(L/I, (I + J)/I)$.

Lemma 1.3. *The short sequence*

$$0 \to B(L, I \cap J) + B(I, J) \to B(L, J) \to B(L/I, (I + J)/I) \to 0 \tag{1.2}$$

is exact.

Proof. Since $Ker(L \vee J \to L/I \vee (I + J)/I) = L \vee (I \cap J) + I \vee J$, the factorization through $[L, S^2(L)]$ yields

$$Ker(B(L, J) \to B(L/I, (I + J)/I)) = (L \vee (I \cap J) + I \vee J + [L, S^2(L)])/[L, S^2(L)] \simeq B(L, I \cap J) + B(I, J).$$

□

Remark. Actually we need the following two cases of this Lemma:

(1) $J = [L,L]$. Since $I \vee [L,L]$ and $[I,L] \vee L$ are congruent modulo $[L,S^2(L)]$ and $[I,L] \subseteq I \cap [L,L]$, then $B(I,[L,L]) \subseteq B(L, I \cap [L,L])$ and we get a short exact sequence

$$0 \to B(L, I \cap [L,L]) \to B(L,[L,L]) \to B(L/I,[L/I,L/I]) \to 0. \tag{1.3}$$

(2) $I = [L,L]$ and $J = L$. Then taking into account that for an abelian Lie algebra M, $B(M) \simeq S^2(M)$, the short exact sequence (1.2) becomes

$$0 \to B(L,[L,L]) \to B(L) \to S^2(L/[L,L]) \to 0. \tag{1.4}$$

2 Presentation of $L \otimes A$

In this section starting from a presentation of L we construct a presentation of $L \otimes A$.

Let $0 \to I \to \mathscr{L}(X) \xrightarrow{p} L \to 0$ be a presentation of the Lie algebra L. Tensoring by A, we get a short exact sequence

$$0 \to I \otimes A \to \mathscr{L}(X) \otimes A \xrightarrow{p \otimes 1} L \otimes A \to 0. \tag{2.1}$$

Let $X(A)$ be a set of symbols $x(a)$, where $x \in X, a \in A$. Define a homomorphism $\phi : \mathscr{L}(X(A)) \to \mathscr{L}(X) \otimes A$ by

$$\phi : u(x_1(a_1), \ldots, x_n(a_n)) \mapsto u(x_1, \ldots, x_n) \otimes a_1 \ldots a_n.$$

Obviously this mapping is surjective, and taking (2.1) into account, gives rise to the following exact sequence:

$$0 \to \phi^{-1}(I \otimes A) \to \mathscr{L}(X(A)) \xrightarrow{(p \otimes 1) \circ \phi} L \otimes A \to 0 \tag{2.2}$$

which gives a presentation of $L \otimes A$.

In order to determine the structure of $\phi^{-1}(I \otimes A)$, let us introduce one notation. For each homogeneous element $u = u(x_1, \ldots, x_n)$ of $\mathscr{L}(X)$, define $u(a)$ to be $u(x_1(a), x_2(1), \ldots, x_n(1))$. Now having an arbitrary element $u \in \mathscr{L}(X)$, define $u(a)$ as $u_1(a) + \cdots + u_n(a)$, where $u = u_1 + \cdots + u_n$ is decomposition of u into the sum of homogeneous components, with respect to the standard grading on $\mathscr{L}(X)$ induced by word lengths.

Lemma 2.1. *(i) $Ker\phi$ is spanned by elements of the form*

$$\sum_j u(x_{i_1}(a_1^j), \ldots, x_{i_n}(a_n^j)), \tag{2.3}$$

where $u(x_{i_1}, \ldots, x_{i_n})$ is a homogeneous element of $\mathscr{L}(X)$ and $\sum_j a_1^j \ldots a_n^j = 0$.

(ii) $\phi^{-1}(I \otimes A)$ *is spanned modulo* $Ker\phi$ *by elements of the form* $u(a)$, *where* $u \in I$.

Proof. (1) Evidently each element of $\mathscr{L}(X(A))$ may be expressed as a sum of elements of the form $u(a)$ and elements of the form (2.3), the latter lying in $Ker\phi$. To prove that they exhaust all $Ker\phi$, take a nonzero element $\sum_i \sum_j u_i(a_{ij})$ belonging to $Ker\phi$, where u_i's are linearly independent, and obtain $\sum_i \sum_j u_i \otimes a_{ij} = 0$, which implies $\sum_j a_{ij} = 0$ for each i.

(2) The quotient space $\phi^{-1}(I \otimes A)/Ker\phi$, consisting of cosets $u(a) + Ker\phi$, maps onto $I \otimes A$, whence the conclusion. □

We also need the following technical result.

Lemma 2.2. *For any* $u, v, w \in \mathscr{L}(X)$ *and* $a, b, c \in A$, *the elements*

$$[[w,u](a), v(b)] - [[w,u](b), v(a)] + [[w,v](a), u(b)) - [[w,v](b), u(a)]$$

and

$$\begin{aligned} &[[u,v](ab), w(c)] - [[u,v)(c), w(ab)] \\ +&[[u,v](ca), w(b)] - [[u,v](b), w(ca)] \\ +&[[u,v](bc), w(a)] - [[u,v](a), w(bc)] \end{aligned}$$

belong to $[\mathscr{L}(X(A)), Ker\phi]$.

Proof. Consider the first case only, the second one is analogous. We have modulo $[\mathscr{L}(X(A)), Ker\phi]$:

$$\begin{aligned} &[[w,u](a), v(b)] - [[w,u](b), v(a)] + [[w,v)(a), u(b)] - [[w,v](b), u(a)] \\ &\qquad \equiv [[w(1), u(a)], v(b)] + [[v(b), w(1)], u(a)] \\ &\qquad + [[w(1), v(a)], u(b)] + [[u(b), w(1)], v(a)] \\ &\qquad\qquad \equiv -[[u(a), v(b)], w(1)] + [[u(b), v(a)], w(1)] \equiv 0 \end{aligned}$$

□

3 The second homology of $L \otimes A$

The aim of this section is to prove Theorem 0.1.

Consider the following commutative diagram with exact rows and columns, where ϕ^{-1} stands for $\phi^{-1}(I \otimes A)$ (we will use this notation in some places further):

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}]\cap Ker\phi & \to & \mathscr{L}^3(X(A))\cap Ker\phi & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}] & \to & \mathscr{L}^3(X(A))\cap\phi^{-1} & \to H_2^{ess}(L\otimes A) & \to 0\\
 & & \phi\downarrow & & \phi\downarrow & & \\
0 & \to & (\mathscr{L}^3(X)\otimes A)\cap[\mathscr{L}(X)\otimes A, I\otimes A] & \to & (\mathscr{L}^3(X)\otimes A)\cap(I\otimes A) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The middle row follows from the Lemma 1.2 applied to the presentation (2.2).

Completing this diagram to the third column, we get a short exact sequence

$$0 \to \frac{\mathscr{L}^3(X(A))\cap Ker\phi}{\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)]\cap Ker\phi} \to H_2^{ess}(L\otimes A) \\ \to \frac{\mathscr{L}^3(X)\cap I}{\mathscr{L}^3(X)\cap[\mathscr{L}(X),I]}\otimes A \to 0. \quad (3.1)$$

According to Lemma 1.2, the right term here is nothing but $H_2^{ess}(L)\otimes A$. Let us compute the left term.

Let $\mathscr{F}(Y)$ be a free skewcommutative algebra on an alphabet Y with nonassociative product denoted by $[\cdot,\cdot]$. Define a mapping $\alpha : \mathscr{F}^2(X(A)) \to S^2(\mathscr{F}(X))\otimes(A\wedge A)$ by

$$\alpha : [u(x_1(a_1),\dots,x_n(a_n)), v(x_1(b_1),\dots,x_m(b_m))] \mapsto \\ (u(x_1,\dots,x_n)\vee v(x_1,\dots,x_m))\otimes(a_1\dots a_n\wedge b_1\dots b_m). \quad (3.2)$$

(recall that $\mathscr{F}^2(Y)$ is just $[\mathscr{F}(Y),\mathscr{F}(Y)]$).

It is easy to see that this mapping is well defined and surjective.

Let $J(Y)$ be an ideal of $\mathscr{F}(Y)$ generated by elements of the form

$$[[u,v],w]+[[w,u],v]+[[v,w],u], \text{ where } u,v,w\in\mathscr{F}(Y),$$

so $\mathscr{F}(Y)/J(Y)\simeq\mathscr{L}(Y)$.

Lemma 3.1.

$$\alpha(J(X(A))) = (J(X)\vee\mathscr{F}(X)+[\mathscr{F}(X),S^2(\mathscr{F}(X))])\otimes(A\wedge A) \\ +(\mathscr{F}^2(X)V\vee\mathscr{F}(X))\otimes T(A).$$

Proof. Writing the generic element in $J(X(A))$, it is easy to see, by considering graded degree, that every element in $\alpha(J(X(A)))$ can be written as a sum of an element lying in $(J(X)\vee\mathscr{F}(X))\otimes(A\wedge A)$ and an element of the form

$$([u,v]\vee w)\otimes(ab\wedge c)+([w,u]\vee v)\otimes(ca\wedge b)+([v,w]\vee u)\otimes(bc\wedge a) \quad (3.3)$$

for certain $u,v,w\in\mathscr{F}(X)$ and $a,b,c\in A$.

Substituting $b=c=1$ in (3.3), we get an element

$$([u,v]\vee w+[w,u]\vee v-[v,w]\vee u)\otimes(1\wedge a).$$

Now permuting the letters u,v in the last expression, one easily get

$$(\mathscr{F}^2(X)\vee\mathscr{F}(X))\otimes(1\wedge A)\subset\alpha(J(A(X))).$$

Substituting $c=1$ in (3.3) and taking into account the last relation, we get

$$([w,u]\vee v+u\vee[w,v])\otimes(A\wedge A)\subset\alpha(J(A(X))). \quad (3.4)$$

Any element in (3.3) is congruent modulo (3.4) to an element of the form

$$(\mathscr{F}^2(X)\vee\mathscr{F}(X))\otimes(ab\wedge c+ca\wedge b+bc\wedge a)$$

proving the Lemma. □

Now factorizing the surjection α through $J(X(A))$ and using Lemma 3.1, we get a map

$$\overline{\alpha}:\mathscr{L}^2(X(A))\longrightarrow B(\mathscr{L}(X))\otimes HC_1(A)+(KX\vee KX)\otimes(A\wedge A),$$

(KX denotes the space of linear terms in $\mathscr{F}(X)$, so $\mathscr{F}(X)=KX+\mathscr{F}^2(X)$), which being restricted to $\mathscr{L}^3(X(A))$, gives rise to the surjection

$$\overline{\alpha}:\mathscr{L}^3(X(A))\longrightarrow B(\mathscr{L}(X),\mathscr{L}^2(X))\otimes HC_1(A),$$

where $HC_1(A)=(A\wedge A)/T(A)$ (for the definition of $T(A)$, see Theorem 0.1) is a first cyclic homology of A.

Further, the restriction of the mapping ϕ defined in §2 to $\mathscr{L}^3(X(A))$ leads to a surjection $\phi:\mathscr{L}^3(X(A))\to\mathscr{L}^3(X)\otimes A$.

Lemma 3.2. $\overline{\alpha}(\mathscr{L}^3(X(A))\cap Ker\phi)=\overline{\alpha}(\mathscr{L}^3(X(A)))$.

Proof. The Lemma follows immediately from Lemma 2.1 and equality

$$\overline{\alpha}[u(a),v(b)]=\frac{1}{2}\overline{\alpha}([u(a),v(b)]-[u(b),v(a)]),$$

where the argument in the right-hand side lies in $Ker\phi$. □

Lemma 3.3.

$$\overline{\alpha}(\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)]) = B(\mathscr{L}(X),I\cap\mathscr{L}^2(X))\otimes HC_1(A)$$

Proof. According to Lemma 2.1, $\overline{\alpha}(\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)])$ consists from the linear span of elements of the following two types: first,

$$\overline{u\vee v}\otimes\overline{a\wedge b}$$

where either $u\in\mathscr{L}^2(X), v\in I$ or $u\in\mathscr{L}(X), v\in I\cap\mathscr{L}^2(X)$, and second,

$$\sum_j \overline{u\vee v}\otimes\overline{a\wedge b_j}$$

where $\sum_j b_j = 0$. The elements of the second type obviously vanish.

Modulo $[\mathscr{L}(X),S^2(\mathscr{L}(X))]$ we have:

$$\mathscr{L}^2(X)\vee I \equiv \mathscr{L}(X)\vee[I,\mathscr{L}(X)]\subseteq\mathscr{L}(X)\vee(I\cap\mathscr{L}^2(X)),$$

which implies the assertion of Lemma. □

Lemmas 3.2 and 3.3 imply that the mapping α, being restricted to $\mathscr{L}^3(X(A))\cap\phi^{-1}(I\otimes A)$ and factorized through

$$\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)]$$

gives rise to a surjection

$$\beta : \frac{\mathscr{L}^3(X(A))\cap\phi^{-1}(I\otimes A)}{\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)]}\to\frac{B(\mathscr{L}(X),\mathscr{L}^2(X))}{B(\mathscr{L}(X),I\cap\mathscr{L}^2(X))}\otimes HC_1(A). \tag{3.5}$$

The right-hand side here is by (1.3) isomorphic to $B(L,[L,L])\otimes HC_1(A)$.

Further, according to (3.1) and Lemma 3.2, β can be restricted to a surjection

$$\beta : \frac{\mathscr{L}^3(X(A))\cap Ker\phi}{\mathscr{L}^3(X(A))\cap[\mathscr{L}(X(A)),\phi^{-1}(I\otimes A)]\cap Ker\phi}\to B(L,[L,L])\otimes HC_1(A). \tag{3.6}$$

Lemma 3.4. *β in (3.6) is injective.*

Proof. Denoting the source of the map (3.5) as *Frac*, consider the following diagram:

$$\begin{array}{ccccc}
Ker([L,L]\otimes A\wedge L\otimes A\to L\otimes A) & \xrightarrow{h} & H_2^{ess}(L\otimes A) & \xrightarrow{j} & Frac \\
{\scriptstyle i}\big\uparrow & & & & \big\downarrow{\scriptstyle\beta} \\
L\vee[L,L]\otimes A\wedge A & & \xrightarrow{n} & & B(L,[L,L])\otimes HC_1(A)
\end{array}$$

where h is the obvious factorization, j is the isomorphism following from Lemma 1.2 applied to presentation (2.2), $n = l \otimes s$, where $l : L \vee [L,L] \to B(L,[L,L])$ and $s : A \wedge A \to HC_1(A)$ are obvious factorizations, and i is defined to be

$$i : (x \vee y) \otimes (a \wedge b) \mapsto \frac{1}{2}(x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a) \tag{3.7}$$

for any $x \in [L,L], y \in L$.

The following calculation verifies the commutativity of this diagram:

$$\begin{aligned}
&\beta \circ j \circ h \circ i((x \vee y) \otimes (a \wedge b)) = \frac{1}{2}\beta \circ j \circ h(x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a) \\
&= \frac{1}{2}\beta \circ j(\overline{x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a}) \\
&= \frac{1}{2}\beta \circ j(\overline{(u(a) + \phi^{-1}) \wedge (v(b) + \phi^{-1}) - (u(b) + \phi^{-1}) \wedge (v(a) + \phi^{-1})}) \\
&= \frac{1}{2}\beta(\overline{[u(a), v(b)] - [u(b), v(a)]}) = \frac{1}{2}(\overline{(x \vee y) \otimes (a \wedge b) - (x \vee y) \otimes (b \wedge a)}) \\
&= \overline{x \vee y} \otimes \overline{a \wedge b} \\
&= n((x \vee y) \otimes (a \wedge b))
\end{aligned}$$

where the overlined elements denote cosets in the corresponding quotient spaces, and $x = u + I, y = v + I$.

It is also clear from the calculation just performed, and from Lemmas 2.1 and 3.2, that the image of $j \circ h \circ i$ coincides with the left-hand side of (3.6).

Thus the kernel of the mapping (3.6) can be evaluated as

$$
\begin{aligned}
Ker\phi &= j\circ h\circ i(Ker\,n)\\
&= j\circ h\circ i(\langle [z,x]\vee y+[z,y]\vee x\rangle \otimes \langle a\wedge b\rangle\\
&+\langle [x,y]\vee z\rangle \otimes \langle ab\wedge c+ca\wedge b+bc\wedge a\rangle)\\
&= j(\langle \overline{[z,x]\otimes a\wedge y\otimes b-[z,x]\otimes b\wedge y\otimes a}\\
&+\overline{[z,y]\otimes a\wedge x\otimes b-[z,y]\otimes b\wedge x\otimes a}\rangle\\
&+\langle \overline{[x,y]\otimes ab\wedge z\otimes c-[x,y]\otimes c\wedge z\otimes ab}\\
&+\overline{[x,y]\otimes ca\wedge z\otimes b-[x,y]\otimes b\wedge z\otimes ca}\\
&+\overline{[x,y]\otimes bc\wedge z\otimes a-[x,y]\otimes a\wedge z\otimes bc}\rangle)\\
&= \langle \overline{[[w,u](a),v(b)]-[[w,u](b),v(a)]}\\
&+\overline{[[w,v](a),u(b)]-[[w,v](b),u(a)]}\rangle\\
&+\langle \overline{[[u,v](ab),w(c)]-[[u,v](c),w(ab)]}\\
&+\overline{[[u,v](ca),w(b)]-[[u,v](b),w(ca)]}\\
&+\overline{[[u,v](bc),w(a)]-[[u,v](a),w(bc)]}\rangle
\end{aligned}
$$

(here $u=x+I$, $v=y+I$, $w=z+I$). The last expression vanishes thanks to Lemma 2.2. □

Putting together (3.1), (3.6) and Lemma 3.4, we get

Proposition 3.5. $H_2^{ess}(L\otimes A)\simeq H_2^{ess}(L)\otimes A\,\oplus\,B(L,[L,L])\otimes HC_1(A)$.

By Lemma 1.1 we have an exact sequence

$$
0\to H_2^{ess}(L\otimes A)\to H_2(L\otimes A)\to \wedge^2(L/[L,L]\otimes A)\xrightarrow{\pi_A}[L,L]/[[L,L],L]\otimes A \to 0. \quad (3.8)
$$

Lemma 3.6.

$$
\begin{aligned}
Ker\pi_A \simeq\ & Ker(\wedge^2(L/[L,L])\xrightarrow{\pi}[L,L]/[[L,L],L])\otimes A\\
&\oplus \wedge^2(L/[L,L])\otimes Ker(S^2(A)\to A)\,\oplus\, S^2(L/[L,L])\otimes\wedge^2(A).
\end{aligned}
$$

Proof. The following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & S^2(L/[L,L])\otimes\wedge^2(A) & \xrightarrow{\equiv} & S^2(L/[L,L])\otimes\wedge^2(A) & \xrightarrow{\pi} & 0 \\
 & & & & i\downarrow & & \downarrow \\
0 \to & & Ker\pi_A & \to & \wedge^2(L/[L,L]\otimes A) & \xrightarrow{\pi_A} & [L,L]/[[L,L],L]\otimes A \to 0 \\
 & & & & k\downarrow & & \| \\
0 \to & & Ker(\pi\otimes m) & \to & \wedge^2(L/[L,L])\otimes S^2(A) & \xrightarrow{\pi\otimes m} & [L,L]/[[L,L],L]\otimes A \to 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
$$

where i is defined in (3.7), and

$$
\begin{aligned}
k: x\otimes a\wedge y\otimes b &\mapsto (x\wedge y)\otimes(a\vee b)\\
m: a\vee b &\mapsto ab
\end{aligned}
$$

for any $x,y\in L/[L,L]$ and $a,b\in A$, implies

$$Ker\pi_A \simeq Ker(\pi\otimes m)\oplus S^2(L/[L,L])\otimes\wedge^2(A). \tag{3.9}$$

Considering the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & \wedge^2(L/[L,L])\otimes Ker m & \xrightarrow{\equiv} & \wedge^2(L/[L,L])\otimes Ker m & \xrightarrow{\pi} & 0 \\
 & & & & \downarrow & & \downarrow \\
0 \to & & Ker(\pi\otimes m) & \to & \wedge^2(L/[L,L])\otimes S^2(A) & \xrightarrow{\pi\otimes m} & [L,L]/[[L,L],L]\otimes A \to 0 \\
 & & & & 1\otimes m\downarrow & & \| \\
0 \to & & Ker(\pi\otimes 1) & \to & \wedge^2(L/[L,L])\otimes A & \xrightarrow{\pi\otimes 1} & [L,L]/[[L,L],L]\otimes A \to 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
$$

we get

$$
\begin{aligned}
Ker(\pi\otimes m) \simeq \wedge^2(L/[L,L])\otimes Ker(S^2(A)\to A)&\\
\oplus\ Ker(\wedge^2(L/[L,L]) \xrightarrow{\pi} [L,L]/[[L,L],L])\otimes A.&
\end{aligned} \tag{3.10}
$$

Putting (3.9) and (3.10) together proves the Lemma. □

Combining Proposition 3.5, (3.8) and Lemma 3.6, we get

$$
\begin{aligned}
H_2(L\otimes A) \simeq H_2^{ess}(L)\otimes A \,\oplus\, Ker(\wedge^2(L/[L,L]) \to [L,L]/[L,[L,L]])\otimes A \\
\oplus\, B(L,[L,L])\otimes HC_1(A) \,\oplus\, S^2(L/[L,L])\otimes \wedge^2(A) \\
\oplus\, \wedge^2(L/[L,L])\otimes Ker(S^2(A)\to A).
\end{aligned}
$$

By Lemma 1.1 the first two terms here give $H_2(L)\otimes A$. Using a (noncanonical) splitting $\wedge^2(A) = HC_1(A)\oplus T(A)$ and the exact sequence (1.4), the third and fourth terms give

$$B(L)\otimes HC_1(A)\oplus S^2(L/[L,L])\otimes T(A).$$

Combining these identifications gives Theorem 0.1.

Remark. It is interesting to compare Theorem 0.1 with a particular (for $n=2$) case of the homological operation

$$H_n(L\otimes A) \to \bigoplus_{i+j=n-1} HC_i(U(L))\otimes HC_j(A)$$

defined in [FT] ($U(L)$ is the universal enveloping algebra of L and the ground field assumed to be of characteristic zero). Taking $n=2$, we obtain a mapping

$$H_2(L\otimes A) \to HC_1(U(L))\otimes HC_0(A) \,\oplus\, HC_0(U(L))\otimes HC_1(A). \qquad (3.11)$$

Cyclic homology of universal enveloping algebras was studied in [FT] and [Kas2]. Using their results, we observe that if $S(L)$ denotes the symmetric algebra over L, then

$$HC_0(U(L)) = H_0(L,S(L)) = S(L)/[L,S(L)]$$

and $HC_1(U(L))$ is a certain quotient space of $H_1(L,S(L))$ containing $H_2(L)$. This implies that in general (3.11) is neither injection, nor surjection. However, if $L=[L,L]$, then (3.11) is an injection.

4 Computation of $B(L\otimes A)$

Theorem 0.1 allows us to compute $B(L\otimes A)$ in terms of L and A (of course, an alternative but longer proof may be given by means of direct computations).

Theorem 4.1. $B(L\otimes A)\simeq B(L,[L,L])\otimes A \,\oplus\, S^2(L/[L,L])\otimes A.$

Proof. It is more convenient to use Proposition 3.5 rather then Theorem 0.1 to obtain an expression for $B(L\otimes A,[L,L]\otimes A)$ and then to derive the general case.

Take any commutative unital algebra A' with $HC_1(A') \simeq K$. According to Proposition 3.5,

$$\begin{aligned} H_2^{ess}(L\otimes A\otimes A') &\simeq H_2^{ess}(L\otimes A)\otimes A' \oplus B(L\otimes A, [L,L]\otimes A) \\ &\simeq H_2^{ess}(L)\otimes A\otimes A' \oplus B(L,[L,L])\otimes HC_1(A)\otimes A' \oplus B(L\otimes A, [L,L]\otimes A). \end{aligned} \tag{4.1}$$

On the other hand,

$$\begin{aligned} H_2^{ess}(L\otimes A\otimes A') &\simeq H_2^{ess}(L)\otimes A\otimes A' \oplus B(L,[L,L])\otimes HC_1(A\otimes A') \\ &\simeq H_2^{ess}(L)\otimes A\otimes A' \oplus B(L,[L,L])\otimes HC_1(A)\otimes A' \oplus B(L,[L,L])\otimes A. \end{aligned} \tag{4.2}$$

The last isomorphism follows from the partial first-order commutative case of the Künneth formula for cyclic homology (cf. [Kas1]):

$$HC_1(A\otimes A') \simeq HC_1(A)\otimes A' + A\otimes HC_1(A').$$

Comparing (4.1) and (4.2), and using the naturalness condition guaranteeing compatibility, one has

$$B(L\otimes A, [L,L]\otimes A) \simeq B(L,[L,L])\otimes A.$$

Now the assertion of Theorem easily follows from the last isomorphism and the short exact sequence (1.4) applied to the algebra $L\otimes A$. □

5 The second homology of $A\otimes B$

Let characteristic of the ground field K be different from 2. Recall that given an associative algebra A, we may consider the Lie algebra $A^{(-)}$ with the same underlying space A and the bracket $[a,b] = ab - ba$, as well as the Jordan algebra $A^{(+)}$ with multiplication $a\circ b = \frac{1}{2}(ab+ba)$.

Recall that

$$T(A) = \langle ab\wedge c + ca\wedge b + bc\wedge a \mid a,b,c\in A\rangle.$$

For the sake of convenience we will also use the following notation:

$$T(A,[A,A]) = \frac{T(A)+[A,A]\wedge A}{[A,A]\wedge A}$$

$$HC_1(A,[A,A]) = \frac{A\wedge A}{[A,A]\wedge A + T(A)} \simeq \frac{\wedge^2(A/[A,A])}{T(A,[A,A])}$$

(the second one is an analog of $H_2^{ess}(L)$ for cyclic homology).

The aim of this section is to prove the following

Theorem 5.1. *Let A, B be associative algebras with unit over a field K of characteristic $p \neq 2$. Let $F(A,B)$ denote the direct sum of the following four vector spaces:*

(1) $A[A,A]/[A,A] \otimes HC_1(B)$
(2) $A/A[A,A] \otimes H_2(B^{(-)})$
(3) $(Ker(S^2(A) \to A/[A,A]))/[A,S^2(A)] \otimes HC_1(B,[B,B])$
(4) $Ker(S^2(A/[A,A]) \to A/A[A,A]) \otimes T(B,[B,B])$

where arrows in (3) and (4) are induced by (associative or Jordan) multiplication in A.

Then $H_2((A \otimes B)^{(-)}) \simeq F(A,B) \oplus F(B,A)$.

The proof is divided into several steps.

We employ the following short exact sequence:

$$0 \to \wedge^2 A \otimes S^2 B \xrightarrow{i} \wedge^2 (A \otimes B) \xrightarrow{p} S^2 A \otimes \wedge^2 B \to 0$$

where the middle term is identified with the direct sum of its neighbors via

$$a_1 \otimes b_1 \wedge a_2 \otimes b_2 \leftrightarrow a_1 \wedge a_2 \otimes b_1 \vee b_2 + a_1 \vee a_2 \otimes b_1 \wedge b_2,$$

and i and p are obvious, by an imbedding and projection, respectively. In what follows, this will be used without explicitly mentioning it.

The arguments are quite analogous to the ones at the beginning of §3. Here they are applied to $H_2((A \otimes B)^{(-)}) \simeq Ker\, d/Im\, d$ (d is the differential in the standard homology complex of $(A \otimes B)^{(-)}$). The mapping p gives rise to the following short exact sequence:

$$0 \to \frac{Ker\, p \cap Ker\, d}{Ker\, p \cap Im\, d} \to H_2((A \otimes B)^{(-)}) \to \frac{p(Ker\, d)}{p(Im\, d)} \to 0. \qquad (5.1)$$

The Lie bracket on $A \otimes B$ may be expanded as a sum

$$[a_1 \otimes b_1, a_2 \otimes b_2] = [a_1, a_2] \otimes b_1 \circ b_2 + a_1 \circ a_2 \otimes [b_1, b_2].$$

The proof of the following statement is quite analogous to the proof of (3.10).

Lemma 5.1. *We have:*

$$p(Ker\, d) \simeq Ker(A \vee A \to A/[A,A]) \otimes B \wedge B + A \vee A \otimes Ker(B \wedge B \to B)$$

where the first arrow is induced by (associative or Jordan) multiplication in algebra A and the second one is the Lie multiplication in $B^{(-)}$.

Lemma 5.2. *The space $p(Im\, d)$ is a linear span of the following elements:*

(1) $[A, S^2(A)] \otimes B \wedge B$
(2) $[A,A] \vee A \otimes T(B)$

(3) $(a_1 \vee a_2 - 1 \vee a_1 \circ a_2) \otimes [B,B] \wedge B, a_i \in A$
(4) $A \vee A \otimes ([b_1,b_2] \wedge b_3 + [b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1), b_i \in B.$

Proof. The notation $x \equiv 0$ reflects the fact that certain element $x \in (A \vee A) \otimes (B \wedge B)$ lies in $p(Im\, d)$. The generic relation defining the quotient by $p(Im\, d)$ is

$$\begin{aligned}
&[a_1,a_2] \vee a_3 \otimes (b_1 \circ b_2) \wedge b_3 + (a_1 \circ a_2) \vee a_3 \otimes [b_1,b_2] \wedge b_3 \\
&+[a_3,a_1] \vee a_2 \otimes (b_3 \circ b_1) \wedge b_2 + (a_3 \circ a_1) \vee a_2 \otimes [b_3,b_1] \wedge b_2 \\
&+[a_2,a_3] \vee a_1 \otimes (b_2 \circ b_3) \wedge b_1 + (a_2 \circ a_3) \vee a_1 \otimes [b_2,b_3] \wedge b_1 \equiv 0.
\end{aligned} \tag{5.2}$$

Symmetrizing this relation with respect to a_1, a_2, we get:

$$\begin{aligned}
&2(a_1 \circ a_2) \vee a_3 \otimes [b_1,b_2] \wedge b_3 \\
&\quad + ([a_3,a_1] \vee a_2 - [a_2,a_3] \vee a_1) \otimes ((b_3 \circ b_1) \wedge b_2 - (b_2 \circ b_3) \wedge b_1) \\
&+ ((a_3 \circ a_1) \vee a_2 + (a_2 \circ a_3) \vee a_1) \otimes ([b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1) \equiv 0.
\end{aligned} \tag{5.3}$$

Cyclic permutations of a_1, a_2, a_3 in the last relation yield:

$$\begin{aligned}
&((a_1 \circ a_2) \vee a_3 + (a_3 \circ a_1) \vee a_2 + (a_2 \circ a_3) \vee a_1) \\
&\qquad \otimes ([b_1,b_2] \wedge b_3 + [b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1) \equiv 0.
\end{aligned}$$

This relation, in its turn, evidently implies

$$A \wedge A \otimes ([b_1,b_2] \wedge b_3 + [b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1) \equiv 0. \tag{5.4}$$

Now rewriting (5.3) modulo (5.4) and substituting $a_3 = 1$ and $b_2 = 1$, we get, respectively:

$$(a_1 \vee a_2 - 1 \vee a_1 \circ a_2) \otimes [B,B] \wedge B \equiv 0 \tag{5.5}$$

and

$$([a_3,a_1] \vee a_2 - [a_2,a_3] \vee a_1) \otimes (b_1 \wedge b_3 + 1 \wedge b_1 \circ b_3) \equiv 0.$$

Symmetrizing the last relation with respect to b_1, b_3, one gets:

$$([a_3,a_1] \vee a_2 - [a_2,a_3] \vee a_1) \otimes B \wedge B \equiv 0. \tag{5.6}$$

Particularly, taking in (5.6) $a_2 = 1$, one gets

$$(1 \vee [A,A]) \otimes (B \wedge B) \equiv 0. \tag{5.7}$$

Now, (5.2) is equivalent modulo (5.4)–(5.6) to

$$
\begin{aligned}
&[a_1,a_2]\vee a_3\otimes(b_1b_2\wedge b_3+b_3b_1\wedge b_2+b_2b_3\wedge b_1)\\
&+1\vee((a_1\circ a_2)\circ a_3-(a_2\circ a_3)\circ a_1)\otimes[b_1,b_2]\wedge b_3\\
&+1\vee((a_3\circ a_1)\circ a_2-(a_2\circ a_3)\circ a_1)\otimes[b_3,b_1]\wedge b_2\equiv 0.
\end{aligned}
\tag{5.8}
$$

Taking into account the identity

$$
(a\circ b)\circ c-(a\circ c)\circ b=\frac{1}{4}[a,[b,c]]
$$

(cf. [J], p.37), and (5.7), the relation (5.8), in its turn, is equivalent to

$$
[A,A]\vee A\otimes T(B)\equiv 0. \tag{5.9}
$$

Putting together (5.4)–(5.6) and (5.9), we get exactly the statement of the Lemma. □

Lemma 5.3. *We have:*
(1) $p(Ker\,d)/p(Im\,d)\simeq F(A,B)$.
(2) $(Ker\,p\cap Ker\,d)/(Ker\,p\cap Im\,d)\simeq F(B,A)$.

Proof. (1) is derived from Lemmas 5.1 and 5.2 after a number of routine transformations.

(2) Define a projection $p':\wedge^2(A\otimes B)\to\wedge^2A\otimes S^2B$. Due to an obvious fact that p' is the identity map on $Ker\,p=(A\wedge A)\otimes(B\vee B)$, we have an isomorphism

$$
\frac{Ker\,d\cap Ker\,p}{Ker\,p\cap Im\,d}\simeq\frac{p'(Ker\,d\cap Ker\,p)}{p'(Ker\,p\cap Im\,d)}=\frac{p'(Ker\,d)}{p'(Im\,d)}
$$

But the right-hand term here is computed as in the part (1), up to permutation of A and B. □

Now Theorem 5.1 follows immediately from (5.1) and Lemma 5.3.

Remark. Taking $B=M_n(K)$ in Theorem 5.1, we get, after a series of elementary transformations, an isomorphism

$$
H_2(gl_n(A))\simeq HC_1(A)\oplus\wedge^2(A/[A,A]).
$$

We have the following central extension of Lie algebras:

$$
0\to sl_n(A)\to gl_n(A)\to A/[A,A]\to 0.
$$

Using the Hochschild-Serre spectral sequence of $gl_n(A)$ with respect to the subalgebra $sl_n(A)$, we derive

$$
H_2(sl_n(A))\simeq HC_1(A)
$$

which is a result of C. Kassel and J.-L. Loday [KL].

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2. Deformations of $W_1(n) \otimes A$ and modular semisimple Lie algebras with a solvable maximal subalgebra

J. Algebra **268** (2003), 603–635

In one of his last papers, Boris Weisfeiler proved that if a modular semisimple Lie algebra possesses a solvable maximal subalgebra which defines a long filtration, then the associated graded algebra is isomorphic to the one constructed from the Zassenhaus algebra tensored by the divided powers algebra. We completely determine the class of such algebras, calculating in the process low-dimensional cohomology of the Zassenhaus algebra tensored by any associative commutative algebra.

Introduction

The ultimate goal of this paper is to describe semisimple finite-dimensional Lie algebras over an algebraically closed field of characteristic $p > 5$, having a solvable maximal subalgebra (that is, a maximal subalgebra which is solvable) which determines a "long filtration", as defined below. We hope, however, that some intermediate results contained here are of independent interest.

Simple Lie algebras with a solvable maximal subalgebra were described by B. Weisfeiler [W]. We heavily depend on his results and to a certain extent this paper may be considered as a continuation of Weisfeiler's paper. Since its appearance the classification of modular simple Lie algebras has been completed (announced in [SW] and elaborated in a series of papers among which [Str] is the last one), and recently the approach to the classification problem has been reworked in a series of papers among which [PS] is the latest, including the cases of low characteristic.

Though the classification provides a powerful tool for solutions of many problems in the modular Lie algebra theory, the question considered here remains non-trivial even modulo this classification. Moreover, we hope that the result we obtain here in particular, and the cohomological technique we use to prove it in general, may simplify to certain degree the classification itself.

Let us recall the contents of Weisfeiler's paper. He considers the semisimple modular Lie algebra $\mathfrak{L}$ with a solvable maximal subalgebra $\mathfrak{L}_0$ which defines a filtration in $\mathfrak{L}$ via

$$\mathfrak{L}_{i+1} = \{x \in \mathfrak{L}_i \,|\, [x, \mathfrak{L}] = \mathfrak{L}_i\}$$

(though in general the filtration can be prolonged also to the negative side, in the case under consideration we can let $\mathfrak{L}_{-1} = \mathfrak{L}$).

When the term $\mathfrak{L}_1$ of this filtration does not vanish, the filtration is called *long*, otherwise it is called *short*. Weisfeiler proved that when the filtration is long, the associated graded algebra is isomorphic to $S \otimes O_m + 1 \otimes \mathfrak{D}$, where S coincides either with $sl(2)$, the three-dimensional simple algebra, or with $W_1(n)$, the Zassenhaus algebra, O_m is the ring of truncated polynomials in m indeterminates, and $\mathfrak{D}$ is a subalgebra of the derivation algebra of O_m. The grading is "thick" in the sense that it is completely determined by the standard grading (defined below) of $W_1(n)$ or $sl(2)$, containing, therefore, the whole tensor factor O_m in each component. In the case of a short filtration, Weisfeiler proved that the initial algebra $\mathfrak{L}$ possesses a $\mathbb{Z}_p$-grading with very restrictive conditions. Then, considering the case of simple $\mathfrak{L}$, he derived that in the case of a long filtration, $\mathfrak{L}$ is isomorphic either to $sl(2)$ or to $W_1(n)$ (in fact, this follows immediately from the results of Kuznetsov [K], which are also important for us here), and the case of a short filtration does not occur.

Here we study the case of a long filtration. We determine all filtered algebras whose associated graded algebra is $W_1(n) \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ or $sl(2) \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ with the above-mentioned "thick" grading (recall that $A \ltimes B$ denotes the semidirect sum of modules or algebras with A a submodule or ideal). This is done in the framework of the deformation theory due to Gerstenhaber. In this theory the second cohomology group of a Lie algebra with coefficients in the adjoint module plays a significant role (for an excellent account of this subject, see [GS]). As it turns out that the "tail" $1 \otimes \mathfrak{D}$ is not important in these considerations, one needs to compute $H^2(W_1(n) \otimes O_m, W_1(n) \otimes O_m)$. In a (slightly) more general setting, we compute $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$ for an arbitrary associative commutative algebra A with unit. (Cohomology $H^2(sl(2) \otimes A, sl(2) \otimes A)$ was earlier computed by Cathelineau [C].) This calculation seems sufficiently interesting for its own sake, as a nontrivial example of the low-dimensional cohomology of the current Lie algebras $L \otimes A$. This may also be considered as a complement to Cathelineau's computation of the second cohomology of the current Lie algebra $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a classical simple Lie algebra, as well as generalization of the Dzhumadil'daev-Kostrikin computations of $H^2(W_1(n), W_1(n))$ [DK].

The knowledge of $H^2(W_1(n) \otimes O_m, W_1(n) \otimes O_m)$ allows one to solve the problem of determining all filtered algebras associated with the graded structure mentioned above. The answer is not very surprising – all such algebras have a socle† isomorphic to $W_1(k) \otimes O_l$ for some k and l.

The contents of this paper is as follows. §1 contains some preliminary material, the most significant of which is a representation of $W_1(n)$ as a deformation of $W_1(1) \otimes O_{n-1}$, due to Kuznetsov [K]. It turns out that it is much easier to perform cohomological calculations using this

† The *socle* of a Lie algebra is the (necessarily direct) sum of its minimal ideals.

representation. Following Kuznetsov, we define a class of Lie algebras $\mathfrak{L}(A,D)$ which are certain deformations of $W_1(1) \otimes A$ defined by means of a derivation D of A. Then we compute $H^2(\mathfrak{L}(A,D), \mathfrak{L}(A,D))$ in two steps: in the first step (in §2), we compute $H^2(W_1(1) \otimes A, W_1(1) \otimes A)$, and then, in §3, we determine $H^2(\mathfrak{L}(A,D), \mathfrak{L}(A,D))$, using a spectral sequence abutting to $H^*(\mathfrak{L}(A,D), \mathfrak{L}(A,D))$ with the E_1-term isomorphic to $H^*(W_1(1) \otimes A, W_1(1) \otimes A)$.

In parallel to the results for the second cohomology, we state similar results for the first cohomology, as well as for the second cohomology with trivial coefficients, which are useful later, in §5.

In §4, using Kuznetsov's isomorphism, we transform the results on $H^2(\mathfrak{L}(A,D), \mathfrak{L}(A,D))$ into those on $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$. This section contains also all necessary computations related to the rings of truncated polynomials, particularly, of their Harrison cohomology. After that, in §5 we formulate a theorem about filtered deformations of $W_1(n) \otimes A \ltimes 1 \otimes \mathfrak{D}$ and of $sl(2) \otimes A \ltimes 1 \otimes \mathfrak{D}$ and almost immediately derive it from preceding results. It turns out that each such deformation strictly related to the class $\mathfrak{L}(A,D)$ (for different A's), so in §6 we determine all, up to an isomorphism, semisimple algebras in this class, completing, therefore, the consideration of the case of a long filtration (Theorem 6.4).

Since the present paper is overloaded with different kinds of computations, we omit some of them which are similar to those already presented, or just too tedious. We believe that this will not cause inconvenience to the reader.

1 Preliminaries

In this section we recall all necessary notions, notation, definitions, results and theories, as well as define a class of algebras $\mathfrak{L}(A,D)$ important for further considerations.

The ground field K is assumed to be of characteristic $p > 3$, unless otherwise is explicitly stated. When appealing to Weisfeiler's results, we have to assume the ground field is algebraically closed of characteristic $p > 5$.

As we deal with modular Lie algebras, it is not surprising that the divided powers algebra $O_1(n)$ plays a significant role in our considerations. Recall that $O_1(n)$ is the commutative associative algebra with basis $\{x^i \mid 0 \le i < p^n\}$ and multiplication $x^i x^j = \binom{i+j}{j} x^{i+j}$. It is isomorphic to the reduced polynomial ring $O_n = K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$, the isomorphism is given by

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \mapsto \alpha_1! \alpha_2! \dots \alpha_n! x^{\alpha_1 + p\alpha_2 + \dots + p^{n-1}\alpha_n} \tag{1.1}$$

The subalgebra $\{x^i \mid 1 \le i < p^n\}$, denoted as $O_1(n)^+$ (or O_n^+) is the single maximal ideal of $O_1(n)$. The invertible elements of $O_1(n)$ are exactly those not lying in $O_1(n)^+$.

The Zassenhaus algebra $W_1(n)$ is the Lie algebra of derivations of $O_1(n)$ of the kind $u\partial$, where $u \in O_1(n)$ and $\partial(x^j) = x^{j-1}$. It possesses a basis $\{e_i = x^{i+1}\partial \mid -1 \le i \le p^n - 2\}$ with the bracket

$$[e_i, e_j] = N_{ij}e_{i+j}, \qquad \text{where } N_{ij} = \binom{i+j+1}{j} - \binom{i+j+1}{i}.$$

The grading $W_1(n) = \bigoplus_{i=-1}^{p^n-2} Ke_i$ is called *standard*. In the case $n = 1$ it coincides with the root space decomposition relative to the action of the semisimple element e_0.

Notice the following properties of the coefficients N_{ij}:

(1) $N_{ij} = 0$ if $-1 \le i, j \le p-2, i+j \ge p-2$;

(2) $N_{ij} = N_{i-1,j} + N_{i,j-1}$;

(3) $N_{i,j-p} = N_{ij}$ if $-1 \le i \le p-2 \le j$.

The first two are obvious, the third may be found, for example, in [DK]. Notice also that if $0 \le i \le j \le p$, then $\binom{p-i}{p-j} = (-1)^{j-i}\binom{j-1}{i-1}$ and if $i = \sum_{n \ge 0} i_n p^n$, $j = \sum_{n \ge 0} j_n p^n$ are p-adic decompositions, then

$$\binom{i}{j} = \prod_{n \ge 0} \binom{i_n}{j_n}$$

(the latter equality is known as *Lucas' theorem*). Here and below, we will deal occasionally with rational fractions whose denominators are integers not divisible by p. Their reduction modulo p is well defined.

The derivation algebra $Der(W_1(n))$ is generated (linearly) by inner derivations of $W_1(n)$ and derivations $(ad\, e_{-1})^{p^t}, 1 \le t \le n-1$ (so the latter constitute a basis of $H^1(W_1(n), W_1(n))$ (cf. [B1] or [D2]).

The whole derivation algebra of $O_1(n) \simeq O_n$, known as a general Lie algebra of Cartan type W_n, is freely generated, as $O_1(n)$-module, by $\{\partial^{p^i} \mid 0 \le i \le n-1\}$ (or, in terms of O_n, by $\{\frac{\partial}{\partial x_i} \mid 1 \le i \le n\}$) (cf. [BIO], [K], or [W]).

Let L be a Lie algebra and A an associative commutative algebra with unit. Define a Lie structure on the tensor product $L \otimes A$ by setting

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \tag{1.1a}$$

for any $x, y \in L$, $a, b \in A$. If $\mathfrak{D}$ is a subalgebra of $Der(A)$, then $L \otimes A \ltimes 1 \otimes \mathfrak{D}$ is defined as a semidirect product, where $1 \otimes \mathfrak{D}$ acts on $L \otimes A$ by $[x \otimes a, 1 \otimes d] = x \otimes d(a)$. For any $a \in A$, the symbol R_a stands for the multiplication by a in A.

We will need the following elementary results.

Proposition 1.1.

(i) $Z(L\otimes A) = Z(L)\otimes A$ *(here* $Z(L)$ *is the center of* L*);*

(ii) $(1\otimes Der(A))\cap ad(L\otimes A) = 0$.

Proof. (i) Obviously $Z(L)\otimes A \subseteq Z(L\otimes A)$. Let $\sum z_i\otimes a_i \in Z(L\otimes A)$. We may assume that all a_i are linearly independent. Then

$$[\sum z_i\otimes a_i, x\otimes 1] = \sum [z_i,x]\otimes a_i = 0$$

for every $x\in L$, which together with our assumption implies $z_i\in Z(L)$ for all i.

(ii) Let $1\otimes d = \sum ad\, x_i\otimes a_i \in (1\otimes Der(A))\cap ad(L\otimes A)$. Applying it to $y\otimes 1$, we get $\sum [y_i,x_i]\otimes a_i = 0$ whence $x_i\in Z(L)$ for all i, and $d=0$. □

Definition. Let $D\in Der(A)$. Define $\mathfrak{L}(A,D)$ to be a Lie algebra with the underlying vector space $W_1(1)\otimes A$ and Lie bracket $\{x,y\} = [x,y] + \Phi_D(x,y)$, where $[\cdot,\cdot]$ is the bracket on $W_1(1)\otimes A$ given by (1.1a), and

$$\Phi_D(e_i\otimes a, e_j\otimes b) = \begin{cases} e_{p-2}\otimes (aD(b) - bD(a)), & i=j=-1 \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

for any $a,b\in A$.

Remark. (Referee). In [Re], Ree considered a class of Lie algebras which are subalgebras of $Der(B)$ for a commutative associative algebra B (freely) generated over B. The algebras $\mathfrak{L}(A,D)$ belong to this class. Indeed, let $D\in Der(A)$ and consider

$$d = \partial\otimes 1 + x^{p-1}\otimes D \in Der(O_1\otimes A).$$

One can easily see, by identifying O_1 and $W_1(1)$ as vector spaces via

$$x^i \mapsto e_{i-1}, \quad 0\le i < p,$$

that $\mathfrak{L}(A,D)$ is the Lie algebra of derivations of $O_1\otimes A$ of the form $\{bd \mid b\in O_1\otimes A\}$ (i.e., freely generated, as a module over $O_1\otimes A$, by a single derivation d).

The following is crucial for our considerations.

Proposition 1.2 (Kuznetsov). $W_1(n)\otimes A \simeq \mathfrak{L}(O_1(n-1)\otimes A, \partial\otimes 1)$.

Proof. This obviously follows from the isomorphism $W_1(n)\simeq \mathfrak{L}(O_1(n-1),\partial)$, noticed in [K]. A direct calculation shows that the mapping

$$e_{pk+i}\mapsto e_i\otimes x^k, \qquad -1\le i\le p-2, \quad 0\le k < p^{n-1}$$

provides the isomorphism desired. □

The reason why we prefer to deal with such a realization of $W_1(n)$ lies in the fact that $e_0 \otimes 1$ is a semisimple element in $\mathfrak{L}(A,D)$ with root spaces $e_i \otimes A$. So we obtain the grading of length p, and not of length p^n as in the case of $W_1(n) \otimes A$. The significance of the "good"(=short) root space decomposition follows from the well-known theorem about the invariance of the Lie algebra cohomology under the torus action.

Introduce a filtration

$$\mathfrak{L}(A,D) = \mathfrak{L}_{-1} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \cdots \supset \mathfrak{L}_{p-2} \tag{1.3}$$

by setting $\mathfrak{L}_i = \bigoplus_{j \geq i} e_j \otimes A$.

In general, for a given decreasing filtration $\{\mathfrak{L}_i\}$ of the Lie algebra $\mathfrak{L}$, let $gr\mathfrak{L} = \bigoplus_i \mathfrak{L}_i/\mathfrak{L}_{i+1}$ denote its associated graded algebra.

The following is evident.

Proposition 1.3. *The graded Lie algebra $gr\mathfrak{L}(A,D)$ associated with filtration (1.3), is isomorphic to $W_1(1) \otimes A = \bigoplus_{i=-1}^{p-2} e_i \otimes A$.*

This is the place where deformation theory enters the game. It is known that each filtered algebra can be considered as a deformation of its associated graded algebra $L = \bigoplus L_i$ (for this fact as well as for all necessary background in the deformation theory we refer to [GS]). One calls such deformations *filtered deformations* (or $\{L_i\}$-deformations in the terminology of [DK]). As the space of infinitesimal deformations coincides with the cohomology group $H^2(L,L)$, the space of infinitesimal filtered deformations coincides with its subgroup

$$H^2_+(L,L) = \{\overline{\phi} \in H^2(L,L) \mid \phi(L_i,L_j) \subset \bigoplus_{k \geq 1} L_{i+j+k}\}.$$

To describe all filtered deformations, one needs to investigate prolongations of infinitesimal ones, obstructions to which are described by Massey products $[\phi,\psi] \in H^3(L,L)$ defined as follows, where the symbol $\curvearrowright$ after an expression refers to the sum of all cyclic permutations of letters and indices occurring in that expression:

$$[\phi,\psi](x,y,z) = \phi(\psi(x,y),z) + \psi(\phi(x,y),z) + \curvearrowright .$$

(This product arises from the graded Lie (super)algebra structure on $H^*(L,L)$.)

We formulate just a small part of this broad subject needed for our purposes.

Proposition 1.4 (cf. [GS] or direct verification). *Let L be a finitely graded Lie algebra such that the Massey product of any two elements of $Z^2_+(L,L)$ is zero. Then any filtered Lie algebra $\mathfrak{L}$ such that $gr\,\mathfrak{L} \simeq L$ (as graded algebras), is isomorphic to a Lie algebra with underlying vector space L and Lie bracket $\{\cdot,\cdot\} = [\cdot,\cdot] + \Phi$ for some $\Phi \in Z^2_+(L,L)$.*

Note in this connection that $[\Phi_D, \Phi_D] = 0$. We will see later that this holds also for other 2-cocycles on $W_1(1) \otimes A$ (and more generally, on $W_1(n) \otimes A$), which preserve the standard filtration, so Proposition 1.4 will be applicable in our situation.

Now we formulate Weisfeiler's main result [W]:

Theorem 1.5 (Weisfeiler). *(The ground field K is algebraically closed of characteristic $p > 5$).*

Let $\mathfrak{L}$ be a semisimple Lie algebra with a solvable maximal subalgebra $\mathfrak{L}_0$. Suppose that $\mathfrak{L}_0$ defines a long filtration in $\mathfrak{L}$. Then $\mathfrak{L}$ is a filtered deformation of a graded Lie algebra $L = S \otimes O_m \ltimes 1 \otimes \mathfrak{D}$, where $S = sl(2)$ or $W_1(n)$ equipped with the standard grading $\bigoplus_i Ke_i$, $\mathfrak{D} \subset Der(O_m)$, and the homogeneous components are:

$$L_i = \begin{cases} e_0 \otimes O_m \ltimes 1 \otimes \mathfrak{D}, & i = 0 \\ e_i \otimes O_m, & i \neq 0. \end{cases}$$

Further, the Harrison cohomology $Har^*(A,A)$ with coefficients in the adjoint module A plays an important role in our considerations. Note that $Har^1(A,A) = Der(A)$ and Harrison 2-cocycles, denoted by $\mathscr{Z}^2(A,A)$, are just symmetrized Hochschild 2-cocycles (cf. [Ha] where this cohomology was introduced and [GS] for a more modern treatment). δ refers to the Harrison (=Hochschild) coboundary operator, i.e.

$$\delta G(a,b) = aG(b) + bG(a) - G(ab)$$
$$\delta F(a,b,c) = aF(b,c) - F(ab,c) + F(a,bc) - F(a,b)c$$

for $G \in Hom(A,A)$ and $F \in Hom(A \otimes A, A)$. The action of $Der(A)$ on $Har^2(A,A)$ is defined via

$$D \star F(a,b) = F(D(a),b) + F(a,D(b)) - D(F(a,b)).$$

The same formula defines the action of $Der(L)$ on the cohomology $H^2(L,L)$ of the Lie algebra L.

Considering the L-action on $H^*(L,L)$, the well-known theorem says that if T is an abelian subalgebra relative to which L decomposes into a sum of eigenspaces $L = \bigoplus L_\alpha$, then one can decompose the cochain complex into the sum of subcomplexes

$$C^n_\alpha = \{\phi \in C^n(L,L) \,|\, \phi(L_{\alpha_1}, \ldots, L_{\alpha_n}) \subseteq L_{\alpha_1 + \cdots + \alpha_n + \alpha}\}$$

and, moreover, $H^*(C_\alpha) = 0$ for $\alpha \neq 0$ (cf. [F], Theorem 1.5.2).

Similarly, any $\mathbb{Z}$-grading $L = \bigoplus L_i$ induces a $\mathbb{Z}$-grading on the cohomology group $H^*(L,L)$, as the initial complex $C^*(L,L)$ splits into the sum of subcomplexes $C^*_i(L,L)$, where

$$C^n_i(L,L) = \{\phi \in C^n(L,L) \,|\, \phi(L_{i_1}, \ldots, L_{i_n}) \subseteq L_{i_1 + \cdots + i_n + i}\}. \qquad (1.4)$$

The cocycles, coboundaries and cohomology of these subcomplexes form the modules denoted by $Z_i^n(L,L)$, $B_i^n(L,L)$ and $H_i^n(L,L)$ respectively. If there is an element $e \in L$ whose action on L_i is multiplication by i, then $H^*(C_i) = 0$ for any $i \neq 0 \, mod \, p$.

2 Low-dimensional cohomology of $W_1(1) \otimes A$

The aim of this section is to establish the following isomorphisms.

Proposition 2.1.
(i) $H^1(W_1(1) \otimes A, W_1(1) \otimes A) \simeq Der(A)$
(ii) $H^2(W_1(1) \otimes A, W_1(1) \otimes A)$
$$\simeq H^2(W_1(1), W_1(1)) \otimes A \bigoplus Der(A) \bigoplus Der(A) \bigoplus Har^2(A,A).$$

Before beginning the proof, let us make several remarks.

Part (i) follows from [B2], Theorem 7.1 (formulated in terms of derivation algebras). Alternatively, one may prove it in a similar (and much easier) way as (ii). Perhaps it should only be remarked that the basic 1-cocycles on $W_1(1) \otimes A$ can be given as $1 \otimes D$ for $D \in Der(A)$.

So we will concentrate our attention on (ii).

The cohomology $H^2(W_1(n), W_1(n))$ was computed in [DK]. Namely, $dim H^2(W_1(1), W_1(1)) = 1$ and the single basic cocycle can be chosen as:

$$\phi(e_i, e_j) = \begin{cases} (N_{ij}/p) \cdot e_{i+j-p}, & i+j \geq p-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where (N_{ij}/p) denotes a (well defined) element of the field K, which is obtained from N_{ij} by division by p and further reduction modulo p.

The appearance of the first and last summands in (ii) is evident: the corresponding parts of the cohomology are spanned by the classes of cocycles

$$\Theta_{\phi,u} : x \otimes a \wedge y \otimes b \mapsto \phi(x,y) \otimes abu \tag{2.2}$$
$$\Upsilon_F : x \otimes a \wedge y \otimes b \mapsto [x,y] \otimes F(a,b) \tag{2.3}$$

respectively, where $\phi \in Hom(L \otimes L, L)$, $u \in A$, and $F \in Hom(A \otimes A, A)$. We will denote the cochains of type (2.2) with $u = 1$ by Θ_ϕ (so, actually, $\Theta_{\phi,u} = (1 \otimes R_u) \circ \Theta_\phi$).

We have the following simple proposition.

Proposition 2.2. *Let L be a Lie algebra which is not 2-step nilpotent. Then*
(i) $(1 \otimes R_u) \circ \Theta_\phi \in Z^2(L \otimes A, L \otimes A)$ *if and only if either* $\phi \in Z^2(L,L)$ *or* $u = 0$
(ii) $\Upsilon_F \in Z^2(L \otimes A, L \otimes A)$ *if and only if* $F \in \mathscr{Z}^2(A,A)$.

Proof. We will prove the second part only, the first one is similar. The cocycle equation for Υ_F together with Jacobi identity gives

$$[[x,y],z] \otimes \delta F(a,c,b) + [[z,x],y] \otimes \delta F(a,b,c) = 0. \tag{2.4}$$

Since $[[L,L],L] \neq 0$ and $p \neq 3$, one may choose $x,y \in L$ such that $[[y,x],x] \neq 0$. Setting $z = x$, one gets $F \in \mathscr{Z}^2(A,A)$. Conversely, the last condition implies (2.4). □

It is possible to prove also that for any Lie algebra L these cocycles are cohomologically independent, whence $H^2(L\otimes A, L\otimes A)$ must contain $H^2(L,L)\otimes A$ and $Har^2(A,A)$ as direct summands.

Let us define now explicitly the remaining classes of basic cocycles: Φ_D is already defined by (1.2), and

$$\Psi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes \left(\binom{i+j+1}{j} bD(a) - \binom{i+j+1}{i} aD(b)\right), & -2 < i+j < p-1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Lemma 2.3. *For any $D \in Der(A)$, we have $\Psi_D, \Phi_D \in Z^2(W_1(1)\otimes A, W_1(1)\otimes A)$.*

Proof. We perform necessary calculations for Ψ_D, leaving the easier case of Φ_D to the reader (the fact that Φ_D is a 2-cocycle on $W_1(1)\otimes A$ follows from the Jacobi identity in $\mathfrak{L}(A,D)$).

Isolating the coefficient of $e_{i+j+k}\otimes abD(c)$ in the cocycle equation for Ψ_D, we get

$$-N_{ij}\binom{i+j+k+1}{i+j} + N_{jk}\binom{i+j+k+1}{i} + N_{ki}\binom{i+j+k+1}{j} + N_{i,j+k}\binom{j+k+1}{j} - N_{j,k+i}\binom{k+i+1}{i} = 0.$$

The last relation can be verified immediately. □

The element $e_0 \otimes 1$ acts semisimply on $W_1(1)\otimes A$, as well as on $\mathfrak{L}(A,D)$. The roots of $ad(e_0\otimes 1)$-action lie in the prime subfield of the ground field K, and the root spaces are:

$$L_{[i]} = e_i \otimes A, \quad [i] \in \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}, \quad -1 \leq i \leq p-2. \tag{2.6}$$

This action induces a corresponding action on the cohomology, and any cocycle in $Z^2(W_1(1)\otimes A, W_1(1)\otimes A)$ is cohomologous to a cocycle in

$$Z^2_{[0]}(W_1(1)\otimes A, W_1(1)\otimes A) = Z^2_{-p} \oplus Z^2_0 \oplus Z^2_p \tag{2.7}$$

as noted above.

Lemma 2.4. *Let u_i be linearly independent elements of A, D_i linearly independent derivations of A, and F_i cohomologically independent cocycles in $\mathscr{Z}^2(A,A)$.*

Then the cocycles $(1\otimes R_{u_i})\circ\Theta_\phi$, Ψ_{D_i}, Υ_{F_i}, Φ_{D_i} defined in (2.2), (2.5), (2.3) and (1.2), respectively, are cohomologically independent.

Proof. As the cocycles $(1 \otimes R_{u_i}) \circ \Theta_\phi$ belong to the $(-p)$th component of $Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$, the cocycles Ψ_{D_i} and Υ_{F_i} – to the zero component, the cocycles Φ_{D_i} – to the pth component, and the degree of any coboundary lies between $1-p$ and $p-1$, one needs only to show the cohomological independence of cocycles of the form Ψ_{D_i} and Υ_{F_i}.

Suppose there is a linear combination of the above-mentioned cocycles equal to a coboundary $d\omega$. Clearly this condition can be written as

$$\Psi_D + \Upsilon_F = d\omega \tag{2.8}$$

where D, F are some linear combinations of D_i's and F_i's respectively.

Due to the $e_0 \otimes 1$-action on $\mathfrak{L}(A,D)$ and its cohomology, we may assume that ω preserves the root space decomposition (2.6), i.e.

$$\omega(e_i \otimes a) = e_i \otimes X_i(a)$$

for some $X_i \in Hom(A,A)$.

Evaluating the left and right sides of (2.8) for the pair $e_0 \otimes a$, $e_0 \otimes 1$, one gets $D = 0$. Then (2.8) reduces to

$$F(a,b) = aX_j(b) + bX_i(a) - X_{i+j}(ab) \tag{2.9}$$

for all i, j such that $N_{ij} \neq 0$.

Substituting $j = 0$ in (2.9) and using the symmetry of F, we get $F = \delta X_0$. Since F is a linear combination of cohomologically independent Harrison cocycles, $F = 0$. We see that all elements entering (2.8) vanish, whence all coefficients in the initial linear combinations of cocycles are equal to zero. □

Now, to prove Proposition 2.1(ii), one merely needs to show that each cocycle $\phi \in Z^2_{[0]}(W_1(1) \otimes A, W_1(1) \otimes A)$ is cohomologous to the sum of previous cocycles.

Let

$$\phi = \phi_{-p} + \phi_0 + \phi_p, \quad \phi_k(e_i \otimes A, e_j \otimes A) \subseteq e_{i+j+k} \otimes A, \quad k = -p, 0, p \tag{2.10}$$

be a decomposition corresponding to (2.7). It is immediate that

$$d\phi = 0 \iff d\phi_{-p} = d\phi_0 = d\phi_p = 0.$$

The next three lemmas elucidate the form of cocycles $\phi_{-p}, \phi_0, \phi_p$, respectively. Two of the lemmas are formulated in a slightly more general setting which will be used later, in §3.

Lemma 2.5. *$\phi_{-p} = (1 \otimes R_u) \circ \Theta_\phi$ for some $u \in A$.*

Proof. Write

$$\phi_{-p}(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j-p} \otimes X_{ij}(a,b), & i+j \geq p-1 \\ 0, & \text{otherwise} \end{cases}$$

for certain $X_{ij} \in Hom(A \otimes A, A)$. From the cocycle equation for triples $e_i \otimes a$, $e_j \otimes b$, $e_{-1} \otimes 1$ and $e_i \otimes a$, $e_j \otimes 1$, $e_0 \otimes c$, one obtains, respectively:

$$X_{ij}(a,b) = X_{i-1,j}(a,b) + X_{i,j-1}(a,b), \quad i+j > p-1 \tag{2.11}$$

$$X_{ij}(ac) = \frac{i+j}{j} cX_{ij}(a,1) - \frac{i}{j} X_{ij}(ac,1), \quad i+j \geq p-1 \tag{2.12}$$

The last equality in the case $i+j=p$ entails

$$X_{ij}(a,c) = X_{ij}(ac,1), \quad i+j = p.$$

Now the cocycle equation for the triple $e_i \otimes a$, $e_j \otimes b$, $e_1 \otimes 1$, where $i+j = p-1, i, j \neq 1$, implies

$$X_{ij}(a,b) = -N_{1,i}(X_{i+1,j}(a,b) + X_{i,j+1}(a,b)), \quad i+j = p-1.$$

Substitution of the penultimate equality into the last one yields

$$X_{ij}(a,b) = Y_{ij}(ab), \quad i+j = p-1, \quad i,j \neq 1 \tag{2.13}$$

where $Y_{ij} = -N_{1,i}(X_{i+1,j}(a,1) + X_{i,j+1}(a,1))$. Substituting this, in its turn, in (2.12) (with $i+j=p-1$), one gets

$$Y_{ij}(ac) = cY_{ij}(a)$$

which implies $Y_{ij}(a) = au_{ij}$ for some $u_{ij} \in A$. Hence

$$X_{ij}(a,b) = abu_{ij}, \quad i+j = p-1, \quad i,j \neq 1.$$

Evaluating the cocycle equation at the triple $e_1 \otimes a$, $e_1 \otimes 1$, $e_{p-2} \otimes 1$, one obtains

$$X_{1,p-2}(a,1) = aX_{1,p-2}(1,1).$$

Substituting this in (2.12) in the particular case $i=1, j=p-2$, one deduces (2.13) also in this case, with $u_{1,p-2} = X_{1,p-2}(1,1)$. Then writing the cocycle equation for the triple $e_i \otimes 1$, $e_j \otimes 1$, $e_1 \otimes 1$, where $i+j=p-2$ and $i,j \neq 0$, and taking into account (2.13), one obtains

$$N_{1,i}u_{i+1,j} + N_{1,j}u_{i,j+1} - N_{ij}u_{1,p-2} = 0, \quad i+j = p-2, \quad i,j \neq 0.$$

The last relation for $i = 2,3,\dots,p-4$ ($i=1$ and $p-3$ give trivial relations) together with the equality $u_{\frac{p-1}{2},\frac{p-1}{2}} = 0$ (which follows from (2.13)) gives $p-5$ equations for $p-5$ unknowns $u_{2,p-3},\dots,u_{p-3,2}$. One easily checks that

$$u_{ij} = (N_{ij}/p) \cdot u, \quad i+j = p-1$$

for a certain $u \in A$ (actually, $u = -\frac{2}{3}u_{1,p-2}$), provides a unique solution.

With the aid of (2.11) this equality can be extended to all i,j such that $i+j \geq p-1$. □

Lemma 2.6. Let $d\phi_0 = \xi$, where $\phi_0 \in C_0^2(W_1(1) \otimes A, W_1(1) \otimes A)$ and $\xi \in C_0^3(W_1(1) \otimes A, W_1(1) \otimes A)$ such that $\xi(e_i \otimes a, e_j \otimes b, e_k \otimes c)$ is (possibly) nonzero only when one of the indices i, j, k is equal to -1 and the sum of the other two indices is equal to $p-1$.

Then $\xi = 0$ and ϕ_0 is a cocycle cohomologous to $\Upsilon_F + \Psi_D$ for some $F \in \mathscr{Z}^2(A,A)$ and $D \in Der(A)$.

Proof. Write

$$\phi_0(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes X_{ij}(a,b), & -2 < i+j < p-1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $\omega \in C^1(W_1(1) \otimes A, W_1(1) \otimes A)$ as follows:

$$\omega(e_{-1} \otimes a) = 0, \quad \omega(e_i \otimes a) = e_i \otimes \sum_{j=0}^{i} X_{-1,j}(1,a), \quad i \geq 0.$$

Then $d\omega(e_{-1} \otimes 1, e_i \otimes a) = e_{i-1} \otimes X_{-1,i}(1,a) = \phi_0(e_{-1} \otimes 1, e_i \otimes a)$ and replacing ϕ_0 by $\phi_0 - d\omega$ (without changing the notation), one can assume that

$$X_{-1,j}(1,a) = 0. \tag{2.14}$$

Writing the equation $d\phi_0 = \xi$ for the triple $e_{-1} \otimes a$, $e_{-1} \otimes b$, $e_i \otimes c$, one obtains

$$X_{-1,i-1}(b,ac) - X_{-1,i-1}(a,bc) = aX_{-1,i}(b,c) - bX_{-1,i}(a,c). \tag{2.15}$$

Setting here $b = 1$ and using (2.14), one gets $X_{-1,i-1}(b,c) = X_{-1,i}(b,c)$ which implies

$$X_{-1,i}(a,b) = X_{-1,0}(a,b). \tag{2.16}$$

Together with (2.15) this gives

$$X_{-1,0}(b,ac) - X_{-1,0}(a,bc) - aX_{-1,0}(b,c) + bX_{-1,0}(a,c) = 0. \tag{2.17}$$

Writing the equation $d\phi_0 = \xi$ for the triple $e_i \otimes a$, $e_j \otimes b$, $e_{-1} \otimes c$, where $i + j \leq p-2$, one obtains

$$\begin{aligned} &N_{ij}X_{-1,i+j}(c,ab) - X_{i-1,j}(ac,b) - X_{i,j-1}(a,bc) + cX_{ij}(a,b) \\ &\quad - N_{i,j-1}aX_{-1,j}(c,b) - N_{i-1,j}bX_{-1,i}(c,a) = 0, \quad i+j \leq p-2, \quad i,j \geq 0. \end{aligned} \tag{2.18}$$

Setting $c = 1$ in the last equality, one gets

$$X_{ij}(a,b) = X_{i-1,j}(a,b) + X_{i,j-1}(a,b).$$

The last relation together with (2.16) permits to prove, by induction on $i+j$, the following equality:

$$X_{ij}(a,b)=\binom{i+j+1}{j}X_{-1,0}(a,b)-\binom{i+j+1}{i}X_{-1,0}(b,a) \tag{2.19}$$

Setting $i=j=0$ in (2.18) and since $X_{00}(a,b)=X_{-1,0}(a,b)-X_{-1,0}(b,a)$ (which follows from (2.19)), one obtains

$$\begin{aligned}X_{-1,0}(bc,a)-X_{-1,0}(ac,b)-bX_{-1,0}(c,a)-cX_{-1,0}(b,a)\\ +cX_{-1,0}(a,b)+aX_{-1,0}(c,b)=0.\end{aligned} \tag{2.20}$$

Set

$$\begin{aligned}F(a,b)&=\frac{1}{2}(X_{-1,0}(a,b)+X_{-1,0}(b,a)-X_{-1,0}(ab,1))\\ &=X_{-1,0}(b,a)-aX_{-1,0}(b,1);\\ D(a)&=X_{-1,0}(a,1).\end{aligned}$$

Using (2.17) and (2.20) it is easy to see that $F\in\mathscr{Z}^2(A,A)$ and $D\in Der(A)$, and hence (2.19) implies

$$X_{ij}(a,b)=N_{ij}F(a,b)+\binom{i+j+1}{j}bD(a)-\binom{i+j+1}{i}aD(b).$$

Thus, ϕ_0 is a cocycle, whence $\xi=0$. □

Lemma 2.7. *Let $d\phi_p=\xi$, where $\phi_p\in C^2_p(W_1(1)\otimes A,W_1(1)\otimes A)$ and $\xi\in C^3_p(W_1(1)\otimes A,W_1(1)\otimes A)$ such that the only possibly nonzero values of ξ are given by*

$$\xi(e_{-1}\otimes a,e_{-1}\otimes b,e_0\otimes c)=e_{p-2}\otimes(aG(b,c)-bG(a,c))$$

for some $G\in Hom(A\otimes A,A)$.

Then G is a Harrison 2-coboundary and $\phi_p=\Phi_D$ for some $D\in End(A)$. If $G=0$, then $D\in Der(A)$.

Proof. Write

$$\phi_p(e_i\otimes a,e_j\otimes b)=\begin{cases}e_{p-2}\otimes X(a,b), & i=j=-1\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, X is skew-symmetric. Writing the equation $d\phi_p=\xi$ for the triples $e_{-1}\otimes a,e_{-1}\otimes b,e_{-1}\otimes 1$ and $e_{-1}\otimes a,e_{-1}\otimes 1,e_0\otimes b$, one gets, respectively:

$$X(a,b)=aX(b,1)-bX(a,1) \tag{2.21}$$

and

$$-X(ab,1)+X(b,a)+2bX(a,1)=aG(1,b)-G(a,b). \tag{2.22}$$

Setting $D(a)=X(1,a)$, we obtain $\phi_p=\Phi_D$. Substitution of (2.21) into (2.22) gives

$$G(a,b)-aG(1,b)=\delta D(a,b).$$

Symmetrizing the last equality, one gets

$$G(a,b)=\delta D(a,b)+abG(1,1)=\delta(D+R_{G(1,1)})(a,b).$$

If $G=0$, then $\delta D=0$, i.e., $D\in Der(A)$. □

This completes the proof of Proposition 2.1(ii).

Similar but more elementary computations can be used to prove

Proposition 2.8.
(i) $H^1(sl(2)\otimes A, sl(2)\otimes A)\simeq Der(A)$
(ii) $H^2(sl(2)\otimes A, sl(2)\otimes A)\simeq Har^2(A,A)$.

Proof. We refer to the paper of Cathelineau [C]. Though formally it contains a slightly different result – namely, the computation of $H_2(\mathfrak{g}\otimes A,\mathfrak{g}\otimes A)$ for any finite-dimensional simple Lie algebra $\mathfrak{g}$ over a field of characteristic zero, the methods employed there can be easily adapted to our case. Alternatively, one may go along the lines of our proof for the case $W_1(1)\otimes A$. All basic cocycles turn out to be of the type (2.3). □

3 Low-dimensional cohomology of $\mathfrak{L}(A,D)$

Let super- and subscripts denote the kernel and cokernel respectively of the corresponding action of D (which is, for (i), given by $Df(a)\mapsto f(D(a))$ for any $f\in A^*$, and for (ii) and (iii), is the standard action on Harrison (=Hochschild) cocycles described in §1).

Theorem 3.1.
(i) $H^2(\mathfrak{L}(A,D),K)\simeq (A^*)^D$
(ii) $H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D))\simeq Der(A)^D$
(iii) $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))\simeq A^D\oplus Der(A)_D\oplus Der(A)^D\oplus Har^2(A,A)^D$.

Part (i) is borrowed from [Z], where it is proved along the lines of the present paper (though the computations are easier).

We give also an explicit basis of $H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ and $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$.

There are at least three ways to compute the cohomology of the deformed algebra knowing the cohomology of an initial one. The first way is the Coffee-Gerstenhaber lifting theory (cf. [GS]) which tells how to determine obstructions to lifting cocycles from a Lie algebra L to its deform $\mathfrak{L}$.

The second way is applicable when $\mathfrak{L}$ is a filtered deform of L, i.e., $\mathfrak{L}$ is a filtered Lie algebra with descending filtration $\{\mathfrak{L}_i\}$ and $L = gr\mathfrak{L}$. One can define a descending filtration in the Chevalley-Eilenberg complex $C^*(\mathfrak{L},\mathfrak{L})$:

$$C_i^n(\mathfrak{L},\mathfrak{L}) = \{\phi \in C^n(\mathfrak{L},\mathfrak{L}) \mid \phi(\mathfrak{L}_{i_1},\dots,\mathfrak{L}_{i_n}) \subseteq \mathfrak{L}_{i_1+\dots i_n+i}\}.$$

Then the associated graded complex will be $C^*(L,L)$ with the grading defined by (1.4), and the general theory about filtered complexes says that there is a spectral sequence abutting to $H^*(\mathfrak{L},\mathfrak{L})$ whose E_1-term is $H^*(L,L)$.

The third way is applicable in a special situation when $\mathfrak{L}$ is a deform of L, linear with respect to the deformation parameter, i.e., the multiplication in $\mathfrak{L}$ is given by

$$\{x,y\} = [x,y] + \phi(x,y)t,$$

where $[\cdot,\cdot]$ is a multiplication in L and $\phi \in Z^2(L,L)$. Then we have three complexes defined on the underlying module $C^*(L,L)$: the first one, responsible for the cohomology of L with differential d, the second one, with differential $b = [\cdot,\phi]$ (Massey bracket), and the third one is responsible for the cohomology of $\mathfrak{L}$ with differential $b+d$. Moreover, the Jacobi identity for $\{\cdot,\cdot\}$ implies $bd+db=0$. In this situation it is possible to define a double complex on $C^*(L,L)$ whose horizontal arrows are given by d and the vertical ones by b. The total complex $\mathscr{T}$ of this double complex is closely related to the Chevalley–Eilenberg complex $\mathscr{C} = C^*(\mathfrak{L},\mathfrak{L})$ responsible for the cohomology of $\mathfrak{L}$. Namely, there is a surjection

$$\mathscr{T}^n = \bigoplus_{i=1}^{n} C^n(L,L) \to C^n(\mathfrak{L},\mathfrak{L})$$

defined by the summation of all coordinates, whose kernel $\mathscr{K}$ is closely related to the shifted complex $\mathscr{T}[-1]$. So one can determine the cohomology $H^n(\mathfrak{L},\mathfrak{L})$ from the long exact sequence associated with the short exact sequence of complexes $0 \to \mathscr{K} \to \mathscr{T} \to \mathscr{C} \to 0$.

However, in our even more specific situation, we will use the fourth method employing a special $\mathbb{Z}$-grading. Its advantage is that we will be able not only to determine $H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ and $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ as modules, but also to find explicit expressions for cocycles.

As noted in §1, when considering the cohomology both of $W_1(n)\otimes A$ and $\mathfrak{L}(A,D)$, we may restrict our attention to a subcomplex preserving the $\mathbb{Z}_p$-grading of $W_1(1)\otimes A$:

$$C_{[0]}^n(W_1(1)\otimes A, W_1(1)\otimes A) = \bigoplus_{i\in\mathbb{Z}} C_{ip}^n(W_1(1)\otimes A, W_1(1)\otimes A).$$

Let d and d_D be the differentials in the Chevalley-Eilenberg complexes $C^*(W_1(1)\otimes A, W_1(1)\otimes A)$ and $C^*(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ respectively. We obviously have

$$d_D = d + [\cdot,\Phi_D] \tag{3.1}$$

where $[\cdot,\cdot]$ denotes the graded Lie (super)algebra structure (given by the Massey bracket) on $H^*(W_1(1)\otimes A, W_1(1)\otimes A)$.

Since $\Phi_D \in C^2_p(W_1(1)\otimes A, W_1(1)\otimes A)$, the bracket $b = [\cdot,\Phi_D]$ acts as a differential of bidegree $(1,p)$ on the bigraded module $C^*_*(W_1(1)\otimes A, W_1(1)\otimes A)$ (the first grading is the usual cohomology grading, the second one comes from the $\mathbb{Z}$-grading on $W_1(1)\otimes A$). Denoting for convenience the module $C^n_{ip}(W_1(1)\otimes A, W_1(1)\otimes A)$ by $\widehat{C}^n_i$, we have a double complex

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\widehat{C}^0_1 & \xrightarrow{d} & \widehat{C}^1_1 & \xrightarrow{d} & \widehat{C}^2_1 & \xrightarrow{d} & \widehat{C}^3_1 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
 & & \widehat{C}^0_0 & \xrightarrow{d} & \widehat{C}^1_0 & \xrightarrow{d} & \widehat{C}^2_0 & \longrightarrow & \cdots \\
 & & & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
 & & & & \widehat{C}^0_{-1} & \xrightarrow{d} & \widehat{C}^1_{-1} & \longrightarrow & \cdots \\
 & & & & & & \uparrow & & \\
 & & & & & & \cdots & &
\end{array}
$$

In view of (3.1), the total complex associated with this double complex is exactly the Chevalley-Eilenberg complex computing the cohomology $H^*(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$. Therefore the first spectral sequence $\{E^{st}_r\}$ associated with the double complex, has the E_1-term

$$E^{st}_1 \simeq H^{s+t}_{ps}(W_1(1)\otimes A, W_1(1)\otimes A).$$

A necessary condition for $\widehat{C}^n_i \neq 0$ is that there exists a solution to inequality $-1 \le i_1+\cdots+i_n+ip \le p-2$ for $-1\le i_k \le p-2$. This implies the inequalities $-n+\frac{2n-1}{p} \le i \le 1+\frac{n-2}{p}$ (so $i=0$ for $n=1$, $-1\le i\le 1$ for $n=2$, and $-2\le i \le 1$ for $n=3$). Thus, in each degree, there is finite number of non-vanishing components and the spectral sequence converges to $H^{s+t}(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$.

The only possibly nonzero terms responsible for the 1st and 2nd cohomology are:

$$E^{01}_r, E^{-1,3}_r, E^{02}_r, E^{11}_r, E^{-2,5}_r, E^{-1,4}_r, E^{03}_r, E^{12}_r.$$

Hence the only possibly nonzero differentials affecting the values of $H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ and $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ are:

$$
\begin{gathered}
d^{01}_1 : E^{01}_1 \to E^{11}_1 \\
d^{-1,3}_1 : E^{-1,3}_1 \to E^{03}_1 \\
d^{02}_1 : E^{02}_1 \to E^{12}_1 \\
d^{-1,3}_2 : E^{-1,3}_2 \to E^{12}_2.
\end{gathered}
$$

Consequently,

$$\begin{aligned}
&E_\infty^{01} = E_1^{01} = Ker\, d_1^{01}\\
&E_\infty^{-1,3} = E_3^{-1,3} = Ker\, d_2^{-1,3}; \quad E_2^{-1,3} = Ker\, d_1^{-1,3}\\
&E_\infty^{02} = E_2^{02} = Ker\, d_1^{02}\\
&E_\infty^{11} = E_2^{11} = E_1^{11}/Im\, d_1^{01}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D)) \simeq E_\infty^{01}\\
&H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D)) \simeq E_\infty^{-1,3} \oplus E_\infty^{02} \oplus E_\infty^{11}.
\end{aligned}$$

Proposition 2.1 (strictly speaking, the explicit basic cocycles provided in its proof) yields

$$\begin{aligned}
E_1^{01} &\simeq H_0^1(W_1(1)\otimes A, W_1(1)\otimes A) \simeq Der(A)\\
E_1^{-1,3} &\simeq H_{-p}^2(W_1(1)\otimes A, W_1(1)\otimes A) \simeq H^2(W_1(1), W_1(1))\otimes A \simeq A\\
E_1^{02} &\simeq H_0^2(W_1(1)\otimes A, W_1(1)\otimes A) \simeq Der(A) \oplus Har^2(A,A)\\
E_1^{11} &\simeq H_p^2(W_1(1)\otimes A, W_1(1)\otimes A) \simeq Der(A).
\end{aligned}$$

In the next lemmas we will determine all necessary kernels and images in (3.2).

Lemma 3.2. $E_\infty^{-1,3} \simeq A^D$.

Proof. In order to determine $Ker\, d_1^{-1,3}$, one needs to consider the equation

$$[(1\otimes R_u)\circ\Theta_\phi, \Phi_D] = d\Lambda_u \tag{3.3}$$

for some $\Lambda_u \in C_0^2(W_1(1)\otimes A, W_1(1)\otimes A)$. The elements of $E_2^{-1,3}$ will be of the form

$$(1\otimes R_u)\circ\Theta_\phi - \Lambda_u$$

for appropriate solutions of (3.3).

Direct computations show that

$$[(1\otimes R_u)\circ\Theta_\phi, \Phi_D] = (1\otimes R_u)\circ[\Theta_\phi, \Phi_D] + (1\otimes R_{D(u)})\circ\Gamma \tag{3.4}$$

where $\Gamma \in C^3(W_1(1)\otimes A, W_1(1)\otimes A)$ is defined as (assuming $i \le j \le k$)

$$\Gamma(e_i\otimes a, e_j\otimes b, e_k\otimes c) = \begin{cases} e_{p-2}\otimes(N_{jk}/p)\cdot abc, & i=-1, j+k=p-1\\ 0, & \text{otherwise}\end{cases}$$

and

$$[\Theta_\phi, \Phi_D](e_i\otimes a, e_j\otimes b, e_k\otimes c) = \begin{cases} e_{p-2}\otimes(N_{jk}/p)\cdot(bcD(a) - aD(bc)), & i=-1, j+k=p-1\\ 0, & \text{otherwise.}\end{cases} \tag{3.5}$$

(Notice that $N_{jk}/p = (-1)^k \frac{2k+1}{k(k+1)}$ if $j+k = p-1$, see p. 36, under "Lucas' theorem".)

Define

$$\Theta'(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes (\lambda_{ij} aD(b) - \lambda_{ji} bD(a)), & -2 < i+j < p-1 \\ 0, & \text{otherwise} \end{cases}$$

for some $\lambda_{ij} \in K$.

Let us verify first that there are λ_{ij} such that $\Theta_\phi + \Theta' \in E_2^{-1,3}$. Writing the equation (3.3) under conditions $u = 1$ and $\Lambda_u = -\Theta'$ for the triple $e_{-1} \otimes a$, $e_j \otimes b$, $e_k \otimes c$, where $j+k = p-1$ and for all remaining possible triples, we obtain respectively:

$$\lambda_{j-1,k} + \lambda_{j,k-1} = (-1)^k \frac{2k+1}{k(k+1)} \tag{3.6}$$

$$\lambda_{j,k-1} - \lambda_{k,j-1} = 2(-1)^k \lambda_{j,-1} + 2(-1)^{k+1} \lambda_{k,-1} + (-1)^{k+1} \frac{2k+1}{k(k+1)} \tag{3.7}$$

where $j+k = p-1$, and

$$N_{ij}\lambda_{i+j,k} - N_{jk}\lambda_{i,j+k} + N_{ik}\lambda_{j,i+k} + N_{j+k,i}\lambda_{jk} - N_{i+k,j}\lambda_{ik} = 0, \quad i,j,k \geq 0, \quad i+j+k < p-1. \tag{3.8}$$

The left-hand side in the latter equality is a basic expression for the coefficient of $abD(c)$ in $d\Theta'$.

Lemma 3.3. *The element*

$$\lambda_{ij} = \sum_{k=1}^{i} \binom{i+j+1-k}{j+1} \frac{k+2}{k(k+1)}, \quad -1 \leq i,j \leq p-2$$

provides solution for equations (3.6)–(3.8).

Proof. Note the following properties of the just defined coefficients λ_{ij}:

(i) $\lambda_{-1,j} = \lambda_{0j} = 0; \quad \lambda_{1,j} = \frac{3}{2}$

(ii) $\lambda_{i,-1} = \sum_{k=1}^{i} \frac{k+2}{k(k+1)}$

(iii) $\lambda_{ij} = \lambda_{i-1,j} + \lambda_{i,j-1}$.

Now, (3.6) may be reformulated as

$$\lambda_{ij} = (-1)^j \frac{2j+1}{j(j+1)}, \quad i+j = p-1, \quad i,j \geq 1$$

which can be proved with the help of simple transformations of binomial coefficients in the spirit of the first few pages of [Ri].

Equation (3.7) is proved by induction on j, using equation (3.6) in the induction step.

Finally, equation (3.8) is proved by induction on $i+j+k$. The induction step is:

$$\begin{aligned}
&N_{ij}\lambda_{i+j,k} - N_{jk}\lambda_{i,j+k} + N_{ik}\lambda_{j,i+k} + N_{j+k,i}\lambda_{jk} - N_{i+k,j}\lambda_{ik} \\
&\qquad = N_{ij}\lambda_{i+j-1,k} + N_{ij}\lambda_{i+j,k-1} - N_{jk}\lambda_{i-1,j+k} - N_{jk}\lambda_{i,j+k-1} + N_{ik}\lambda_{j-1,i+k} \\
&\qquad + N_{ik}\lambda_{j,i+k-1} + N_{j+k,i-1}\lambda_{jk} + N_{j+k-1,i}\lambda_{jk} - N_{i+k,j-1}\lambda_{ik} - N_{i+k-1,j}\lambda_{ik} \\
&= (N_{i-1,j}\lambda_{i+j-1,k} - N_{jk}\lambda_{i-1,j+k} + N_{i-1,k}\lambda_{j,i+k-1} + N_{j+k,i-1}\lambda_{jk} - N_{i+k-1,j}\lambda_{i-1,k}) \\
&+ (N_{i,j-1}\lambda_{i+j-1,k} - N_{j-1,k}\lambda_{i,j+k-1} + N_{ik}\lambda_{j-1,i+k} + N_{j+k-1,i}\lambda_{j-1,k} - N_{i+k,j-1}\lambda_{ik}) \\
&+ (N_{ij}\lambda_{i+j,k-1} - N_{j,k-1}\lambda_{i,j+k-1} + N_{i,k-1}\lambda_{j,i+k-1} + N_{j+k-1,i}\lambda_{j,k-1} - N_{i+k-1,j}\lambda_{i,k-1}) \\
&= 0
\end{aligned}$$

where the first equality follows from recurrent relations for λ_{ij}, the second one from those for N_{ij}, and the third one from the induction assumption for the triple $(i-1,j,k)$, $(i,j-1,k)$ and $(i,j,k-1)$. □

Continuation of the proof of Lemma 3.2. Now consider a general solution of (3.3). Taking equation (3.4) into account, the partial solution $[\Theta_\phi, \Phi_D] = -d\Theta'$, and the commutativity of operators d and R_u, equation (3.3) can be rewritten as

$$d(\Lambda_u + (1 \otimes R_u) \circ \Theta') = (1 \otimes R_{D(u)}) \circ \Gamma.$$

By Lemma 2.6, $D(u) = 0$ and

$$\Lambda_u = -(1 \otimes R_u) \circ \Theta' \mod Z_0^2(W_1(1) \otimes A, W_1(1) \otimes A).$$

Hence $E_2^{-1,3}$ consists of elements of the form

$$\overline{\Theta}_u = (1 \otimes R_u) \circ (\Theta_\phi + \Theta'), \quad u \in A^D$$

and $E_2^{-1,3} \simeq A^D$.

To compute $Ker\, d_2^{-1,3}$, take a look at $Im\, d_2^{-1,3}$, i.e. at the space of elements of the form $[(1 \otimes R_u) \circ (\Theta_\phi + \Theta'), \Phi_D]$, where $u \in A^D$. This expression is equal to $(1 \otimes R_u) \circ [\Theta', \Phi_D]$ modulo $B^3(W_1(1) \otimes A, W_1(1) \otimes A)$. Direct computations show that

$$\begin{aligned}
&[\Theta', \Phi_D](e_i \otimes a, e_j \otimes b, e_k \otimes c) \\
&\qquad = \begin{cases} e_{p-2} \otimes \lambda_{p-2,0}(aD(b) - bD(a))D(c), & i = j = -1, k = 0 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

But

$$\lambda_{p-2,0} = -\sum_{k=1}^{p-2}(1 + \frac{2}{k}) = -(p-2) - 2\sum_{k=1}^{p-1} k + \frac{2}{p-1} = 0.$$

Consequently, $d_2^{-1,3} = 0$ and $E_\infty^{-1,3} = E_3^{-1,3} = E_2^{-1,3} \simeq A^D$. □

Lemma 3.4. $E_\infty^{02} \simeq Der(A)^D \oplus Har^2(A,A)^D$.

Proof. To determine $Ker\, d_p^{02}$, one needs to solve two equations

$$[\Psi_E, \Phi_D] = d\Lambda_E \tag{3.9}$$
$$[\Upsilon_F, \Phi_D] = d\Lambda_F \tag{3.10}$$

for $\Lambda_E, \Lambda_F \in C_p^2(W_1(1)\otimes A, W_1(1)\otimes A)$. Let

$$\Psi'_E(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (D(a)E(b) - E(a)D(b)), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

By means of direct computations one gets

$$[\Psi_E, \Phi_D] = d\Psi'_E + \Gamma_E$$

where the only possibly nonzero values of Γ_E are given by

$$\Gamma_E(e_{-1}\otimes a, e_{-1}\otimes b, e_{-1}\otimes c) = e_{p-2} \otimes (a[E,D](b) - b[E,D](a))c.$$

But equation (3.9) implies that $\Gamma_E = d(\Lambda_E - \Psi'_E)$. One easily checks that each coboundary vanishes on the triple $e_{-1}\otimes a$, $e_{-1}\otimes 1$, $e_0 \otimes 1$. This implies $[E,D] = 0$ and $\Gamma_E = 0$. Consequently,

$$\Lambda_E = \Psi'_E \mod Z_p^2(W_1(1)\otimes A, W_1(1)\otimes A),$$

and the set of elements $\{\Psi_E - \Psi'_E \mid E \in Der(A)^D\}$ embeds into E_2^{02}.

To solve equation (3.10), define

$$\Upsilon'_F(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (F(D(a),b) - F(a, D(b))), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

By means of direct computations one gets

$$[\Upsilon_F, \Phi_D] = d\Upsilon'_F + \Gamma_F \tag{3.11}$$

where the only possibly nonzero values of Γ_F are given by (for the definition of the $\star$ see Introduction to this chapter)

$$\Gamma_F(e_{-1}\otimes a, e_{-1}\otimes b, e_0 \otimes c) = e_{p-2} \otimes (aD\star F(b,c) - bD\star F(a,c)).$$

By (3.11), $\Gamma_F = d(\Lambda_F - \Upsilon'_F)$. According to Lemma 2.7, $\Lambda_F - \Upsilon'_F = \Phi_H$ for some $H \in End(A)$ and moreover, $D\star F = \delta H$ (we may suppose that $D\star F(1,1) = 0$).

Conversely, if $D\star F = \delta H$, then $\Gamma_F = -d\Phi_H$, which in view of (3.11) leads to the solution $\Lambda_F = \Upsilon'_F + \Phi_H$ of equation (3.10).

So, $E_\infty^{02} = E_2^{02}$ is the direct sum of two subspaces consisting of elements of the form $\Psi_E - \Psi'_E$ and $\Upsilon_F - \Upsilon'_F - \Phi_H$ for appropriate E, F and H. These subspaces are isomorphic to $Der(A)^D$ and $Har^2(A,A)^D$, respectively. □

Lemma 3.5.
(i) $E_\infty^{01} \simeq Der(A)^D$
(ii) $E_\infty^{11} \simeq Der(A)_D$.

Proof. d_1^{01} acts on the space $E_1^{01} \simeq Der(A)$ as

$$1 \otimes E \mapsto [1 \otimes E, \Phi_D] = \Phi_{[D,E]},$$

where $E \in Der(A)$. Hence $Ker\, d_1^{01} \simeq Der(A)^D$, proving (i).

$Im\, d_1^{01} \simeq [D, Der(A)]$, $E_\infty^{11} = E_2^{11} \simeq Der(A)_D$ and E_∞^{11} consists of elements Φ_E for $E \in Der(A)$. These elements are independent modulo $[D, Der(A)]$; this proves (ii). □

Putting all these calculations together, we get statements (ii) and (iii) of Theorem 3.1. (Lemma 3.5(i) is used to get a formula for cohomology of degree 1, while all the rest is used to get a formula for cohomology of degree 2.)

For convenience we summarize here the cocycles whose cohomology classes constitute a basis of $H^1(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$ and $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$.

Basic cocycles of degree 1 are just mappings of the form $1 \otimes E$, where $E \in Der(A)^D$.

All cocycles of degree 2 constructed here have their counterparts in $Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$ (in fact, they are liftings, in Gerstenhaber's terminology [GS], of 2-cocycles on $W_1(1) \otimes A$). Each class of cocycles denoted by overlined capital Greek letter is lifted from the corresponding class of §2 denoted by the same letter.

So, let $\overline{\Theta}_u$, $\overline{\Upsilon}_{F,H}$, $\overline{\Psi}_E$ and $\overline{\Phi}_E$ be 2-cochains on $\mathfrak{L}(A,D)$ defined by the following formulas, where the top line comes from the appropriate cocycle of §2 (the "regular" components), and the second line represent a new component coming from the deformation:

$$\overline{\Theta}_u(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j-p} \otimes (N_{ij}/p) \cdot abu, & i+j \geq p-1 \\ e_{i+j} \otimes (\lambda_{ij} aD(b) - \lambda_{ji} bD(a))u, & -2 < i+j < p-1 \\ 0, & \text{otherwise} \end{cases}$$

where $u \in A^D$ and the coefficients λ_{ij} defined as in Lemma 3.3 (the regular component is given by equation (2.2)),

$$\overline{\Upsilon}_{F,H}(e_i \otimes a, e_j \otimes b) \\ = \begin{cases} e_{i+j} \otimes N_{ij} F(a,b), & -2 < i+j < p-1 \\ e_{p-2} \otimes (bH(a) - aH(b) - F(D(a),b) + F(a,D(b))), & i=j=-1 \\ 0, & i+j \geq p-1 \end{cases}$$

where $F \in \mathscr{Z}^2(A,A)^D$ and $H \in End(A)$ such that $D \star F = \delta H$ (the regular component is given by equation (2.3)),

$$\overline{\Psi}_E(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (E(a)D(b) - E(b)D(a)), & i = j = -1 \\ e_{i+j} \otimes \left(\binom{i+j+1}{j} bE(a) - \binom{i+j+1}{i} aE(b) \right), & -2 < i+j < p-1 \\ 0, & i+j \geq p-1 \end{cases}$$

where $E \in Der(A)^D$ (the regular component is given by equation (2.5)), and, finally, $\overline{\Phi}_E = \Phi_E$ (the regular component is given by equation (1.2); there is no deformation component).

Lemma 2.4 (stating cohomological independence of the initial cocycles on $W_1(1) \otimes A$) together with the spectral sequence construction assure the cohomological independence of the corresponding cocycles on $\mathfrak{L}(A,D)$. More precisely, the following is true:

Proposition 3.6. *Let u_i be linearly independent elements of A, F_i cohomologically independent cocycles in $\mathscr{Z}^2(A,A)$, H_i elements in $Hom(A,A)$ linearly independent modulo $Der(A)$ and such that $D \star F_i = \delta H_i$, E_i linearly independent elements in $Z_D(Der(A))$, and E'_i derivations of A linearly independent modulo $[D, Der(A)]$.*

Then the cocycles $\overline{\Theta}_{u_i}$, $\overline{\Upsilon}_{F_i,H_i}$, $\overline{\Psi}_{E_i}$, $\overline{\Phi}_{E'_i}$ are cohomologically independent.

4 Low-dimensional cohomology of $W_1(n) \otimes A$

Now our objective is to transform the results obtained so far for $\mathfrak{L}(A,D)$ into those for $W_1(n) \otimes A$.

For this, take $A = O_1(n-1) \otimes B$ and $D = \partial \otimes 1$. By Proposition 1.2, $\mathfrak{L}(A,D)$ in this case is isomorphic to $W_1(n) \otimes B$, and Theorem 3.1 entails

$$\begin{aligned} & H^2(W_1(n) \otimes B, W_1(n) \otimes B) \\ & \simeq (O_1(n-1) \otimes B)^{\partial \otimes 1} \oplus Der(O_1(n-1) \otimes B)_{\partial \otimes 1} \oplus Der(O_1(n-1) \otimes B)^{\partial \otimes 1} \\ & \qquad \oplus Har^2(O_1(n-1) \otimes B, O_1(n-1) \otimes B)^{\partial \otimes 1}. \end{aligned} \tag{4.1}$$

The next lemmas collect all necessary information for evaluating the four summands appearing on the right side of this isomorphism. Just for notational convenience, we put $m = n-1$.

Lemma 4.1.

(i) $(O_1(m) \otimes B)^{\partial \otimes 1} = 1 \otimes B$

(ii) $Der(O_1(m) \otimes B)_{\partial \otimes 1} \simeq \langle x^{p^m-1} \partial^{p^k} \mid 0 \leq k \leq m-1 \rangle \otimes B \bigoplus x^{p^m-1} \otimes Der(B)$

(iii) $Der(O_1(m) \otimes B)^{\partial \otimes 1} \simeq \langle \partial^{p^k} \mid 0 \leq k \leq m-1 \rangle \otimes B \bigoplus 1 \otimes Der(B)$.

Proof. (i) Obvious, as $Ker_{O_1(m)}\partial = K1$.

(ii) Since

$$Der(O_1(m)\otimes B) \simeq Der(O_1(m))\otimes B + O_1(m)\otimes Der(B)$$

and $Der(O_1(m))$ is a free $O_1(m)$-module with basis $\{\partial^{p^k} \mid 0 \le k \le m-1\}$, it follows that

$$\begin{aligned} &[\partial\otimes 1, Der(O_1(m)\otimes B)] \simeq [\partial, Der(O_1(m))]\otimes B \oplus \partial(O_1(m))\otimes Der(B) \\ &= \langle x^i\partial^{p^k} \mid 0 \le i < p^m-1, 0 \le k \le m-1\rangle \otimes B \oplus \langle x^i \mid 0 \le i < p^m-1\rangle \otimes Der(B). \end{aligned}$$

As $\langle x^{p^m-1}\partial^{p^k} \mid 0 \le k < m-1\rangle$ is a complement in $Der(O_1(m))$ to the tensor factor in the first summand, and $\langle x^{p^m-1}\rangle$ is a complement in $O_1(m)$ to those in the second summand, we get the isomorphism desired.

(iii) Analogous to (ii). □

Further, according to [Ha], Theorem 5,

$$\begin{aligned} &Har^2(O_1(m)\otimes B, O_1(m)\otimes B)^{\partial\otimes 1} \\ &\quad \simeq Har^2(O_1(m), O_1(m))^{\partial}\otimes B \oplus O_1(m)^{\partial}\otimes Har^2(B,B) \quad (4.2) \end{aligned}$$

(as $O_1(m)^\partial \simeq K$, the second summand is actually just $Har^2(B,B)$).

So we need to compute the second Harrison cohomology of the divided powers algebra $O_1(m)$. First we determine its Hochschild cohomology. It is more convenient to work with the ring of truncated polynomials O_m.

Note that O_m is a quotient of a polynomial algebra as well as the group algebra of an elementary abelian group, and for both class of algebras all sort of cohomological computations have been done. Instead of digging the result we need out of the literature (which will require some additional computations anyway, see, e.g., [L], §7.4 and [Ho] and references therein), we give a direct simple proof suited for our needs.

We use multi-index notations: $\digamma_m = \{\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{Z}^m \mid 0 \le \alpha_i < p\}$, $x^\alpha = x_1^{\alpha_1}\dots x_m^{\alpha_m}$, and $\varepsilon_i = (0,\dots,0,1,0,\dots,0)$ (1 in the ith position).

Proposition 4.2. *$H^i(O_m,O_m)$ is a free O_m-module of rank $\binom{i+m-1}{i}$.*

Proof. Consider first the case $m=1$, i.e. an algebra $O_1 = K[x]/(x^p)$. We use a very simple (and nice) free O_1^e-resolution of O_1 presented in [RS] (as O_1 is commutative, $O_1^e \simeq O_1\otimes O_1$):

$$\dots \xrightarrow{d_1} O_1\otimes O_1 \xrightarrow{d_0} O_1\otimes O_1 \xrightarrow{m} O_1 \longrightarrow 0$$

where m is the multiplication in O_1 and

$$d_i(a\otimes b) = \begin{cases} a\otimes xb - ax\otimes b, & i \text{ even} \\ \sum_{k=0}^{p-1} ax^k\otimes x^{p-1-k}b, & i \text{ odd}. \end{cases}$$

Applying the functor $Hom_{O_1^e}(-, O_1)$, we get a (deleted) complex whose all but first differentials are zero:

$$0 \longrightarrow O_1 \xrightarrow{id} O_1 \xrightarrow{0} O_1 \xrightarrow{0} \dots$$

Therefore, $H^i(O_1, O_1) \simeq O_1$ for each i.

Now the general case is proved by induction on m by applying the Künneth formula to the decomposition $O_m \simeq O_{m-1} \otimes O_1$. □

Lemma 4.3.

(i) *$Har^2(O_m, O_m)$ is a free O_m-module of rank m. The basic cocycles (over O_m) can be chosen as*

$$F_i(x^\alpha, x^\beta) = \begin{cases} x^{\alpha+\beta-p\varepsilon_i}, & \alpha_i + \beta_i \geq p \\ 0, & \alpha_i + \beta_i < p \end{cases}$$

for $1 \leq i \leq m$.

(ii) *$rkHar^2(O_1(m), O_1(m))^\partial = m$. The basic ∂-invariant cocycles can be chosen as*

$$F_i(x^\alpha, x^\beta) = \begin{cases} \left(\binom{\alpha+\beta}{\beta}/p\right) \cdot x^{\alpha+\beta-p^i}, & \alpha_i + \beta_i \geq p \\ 0, & \alpha_i + \beta_i < p \end{cases} \tag{4.3}$$

where $\alpha = \sum_{i \geq 1} \alpha_i p^{i-1}, \beta = \sum_{i \geq 1} \beta_i p^{i-1}$ are p-adic decompositions.

Proof. (i) By Proposition 4.2, $H^2(O_m, O_m)$ is a free O_m-module of rank $\frac{m(m+1)}{2}$. We assert that the two classes of cocycles, F_i, where $1 \leq i \leq m$, and $\frac{\partial}{\partial x_i} \cup \frac{\partial}{\partial x_j}$, where $1 \leq i < j \leq m$ and $\cup$ denotes the usual cohomological cup product, form a basis of this module. Indeed, the cocycle condition is verified immediately. As we have $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ cocycles, it remains to check their independence. Since the F_i are symmetric and $\frac{\partial}{\partial x_i} \cup \frac{\partial}{\partial x_j}$ are skew, one suffices to perform this check only for F_i (remember that 2-coboundaries are symmetric). Suppose

$$\sum_{i=1}^{m} u_i F_i = \delta G \tag{4.4}$$

for some $G \in Hom(O_m, O_m)$ and $u_i \in O_m$.

Then $\delta G(x^\alpha, x^\beta) = 0$ if $\alpha_i + \beta_i < p$ for each i. This implies that G acts as derivation on products $x^\alpha x^\beta$ if $\alpha_i + \beta_i < p$ for all i, hence

$$G(x^\alpha) = \sum_{i=1}^{m} \alpha_i x^{\alpha-\varepsilon_i} G(x^{\varepsilon_i})$$

what in its turn entails that G is a derivation, and thus $\delta G(x^\alpha, x^\beta) = 0$ for all α, β. Then evaluating the left side of equation (4.4) for all pairs (x^α, x^β) such that $\alpha_j+\beta_j=\delta_{ji}p$ for each j and a fixed i, we get $u_i=0$. This shows that cocycles F_i are cohomologically independent.

Now picking from the basic cocycles of $H^2(O_m,O_m)$ those which are symmetric, we obtain a basis $\{F_i\,|\,1\leq i\leq m\}$ of $Har^2(O_m,O_m)$ (as a module over O_m). The freeness of $Har^2(O_m,O_m)$ follows either from the previous reasonings or from the fact that the Harrison cohomology is a direct summand of the Hochschild one (cf. [GS]).

(ii) Using the isomorphism (1.1), the cocycles of part (i) may be rewritten as (4.3). An easy direct check shows that the cocycles F_i are ∂-invariant (in fact, $\partial\star F_i=0$). The identity $\partial\star(uF)=u\star\partial F-(\partial u)F$ shows that the equality $\partial\star(u_1F_1+\cdots+u_kF_k)=0$ implies

$$(\partial u_1)F_1+\cdots+(\partial u_k)F_k=0$$

which due to the freeness of $Har^2(O_1(m),O_1(m))$ over $O_1(m)$ entails that all $u_i\in K1$, and the assertion desired follows. □

Now, collecting (4.1), (4.2) and Lemmas 4.1 and 4.3(ii), we get an isomorphism

$$H^2(W_1(n)\otimes B,W_1(n)\otimes B)\simeq\mathscr{H}\otimes B\oplus Der(B)\oplus Der(B)\oplus Har^2(B,B) \quad (4.5)$$

where $\mathscr{H}$ is a vector space with basis $\{1,x^{p^{n-1}-1}\partial^{p^k},\partial^{p^k},F_{k+1}\,|\,0\leq k\leq n-2\}$.

To obtain an explicit basis of this cohomology, regroup the basis of $H^2(\mathfrak{L}(A,D),\mathfrak{L}(A,D))$, exhibited in §3, according to the direct summands in equation (4.5) as follows.

The classes of $3n-2$ cocycles

$$\overline{\Theta}_{1\otimes u},\overline{\Upsilon}_{F_{i+1}\otimes R_u,0},\overline{\Psi}_{\partial^{p^i}\otimes R_u},\overline{\Phi}_{x^{p^{n-1}-1}\partial^{p^i}\otimes R_u},\quad 0\leq i\leq n-2,\quad u\in B$$

form a module denoted in (4.5) by $\mathscr{H}\otimes B$. It is easy to see that all these cocycles are of the form $(1\otimes R_u)\circ\Theta_\phi$ for an appropriate $\phi\in Z^2(W_1(n),W_1(n))$. As by Proposition 3.6 all these cocycles are independent, the corresponding $3n-2$ cocycles on $W_1(n)$ are also independent. But according to [DK], $dim\,H^2(W_1(n),W_1(n))=3n-2$, whence $\mathscr{H}\simeq H^2(W_1(n),W_1(n))$. It should be noted that the 2-cocycles derived here constitute a basis of $H^2(W_1(n),W_1(n))$, different from the one presented in [DK].

The classes of cocycles $\overline{\Psi}_{1\otimes D}$ and $\overline{\Phi}_{x^{p^{n-1}-1}\otimes D}$, where $D\in Der(B)$, denoted from now for the convenience as ψ_D and ϕ_D respectively, form two modules isomorphic to $Der(B)$. They are just obvious generalizations of cocycles Ψ_D and Φ_D to arbitrary n:

$$\psi_D(e_i\otimes a,e_j\otimes b)=e_{i+j}\otimes\left(\binom{i+j+1}{j}bD(a)-\binom{i+j+1}{i}aD(b)\right),\quad -1\leq i,j\leq p^n-2 \quad (4.6)$$

$$\phi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p^n-2} \otimes (aD(b) - bD(a)), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases} \quad (4.7)$$

And finally, the classes of cocycles $\overline{\Upsilon}_{1\otimes F,0}$ where $F \in \mathscr{Z}^2(B,B)$, generate a module isomorphic to $Har^2(B,B)$. These cocycles are of the form Υ_F (cf. Proposition 2.2).

Thus, we get a generalization of Proposition 2.1:

Theorem 4.4. *For an arbitrary associative commutative unital algebra B,*

$$H^2(W_1(n) \otimes B, W_1(n) \otimes B) \\ \simeq H^2(W_1(n), W_1(n)) \otimes B \bigoplus Der(B) \bigoplus Der(B) \bigoplus Har^2(B,B).$$

The basic cocycles can be chosen among $(1 \otimes R_u) \circ \Theta_\phi$ *for* $\phi \in Z^2(W_1(n), W_1(n))$, ψ_D, ϕ_D *for* $D \in Der(B)$, *and* Υ_F *for* $F \in \mathscr{Z}^2(B,B)$, *given by formulas (2.2), (4.6), (4.7) and (2.3), respectively.*

We conclude this section with formulation of all necessary results needed for our further purposes, which are obtained in a similar (and much simpler) way as Theorem 4.4 and/or can be found elsewhere (cf. [B2], [C], [Z]):

$$H^1(W_1(n) \otimes B, W_1(n) \otimes B) \simeq H^1(W_1(n), W_1(n)) \otimes B \bigoplus Der(B) \quad (4.8)$$

$$H^2(W_1(n) \otimes B, K) \simeq H^2(W_1(n), K) \otimes B^* \quad (4.9)$$

$$H^2(sl(2) \otimes A, K) \simeq HC^1(A). \quad (4.10)$$

5 Filtered deformations of the Lie algebra $W_1(n) \otimes A \ltimes 1 \otimes \mathfrak{D}$

As we explained in §1, we are interested in filtered Lie algebras whose associated graded algebra is $S \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ for $S = W_1(n)$ or $sl(2)$, where $\mathfrak{D}$ is a subalgebra of $Der(B)$. First, using Theorem 4.4, we compute the second cohomology of such algebras.

Lemma 5.1. *Let L be a Lie algebra which can be written as the semidirect product* $L = I \rtimes Q$, *where I is a centerless perfect ideal of L, Q is a subalgebra, and* $Q \cap ad(I) = 0$ *(in the last equality, Q and* $ad(I)$ *are considered as subspaces of* $End(I)$*). Then the terms relevant to the cohomology group* $H^2(L,L)$ *in the Hochschild-Serre spectral sequence of L with respect to I with the general* E_2*-term* $E_2^{pq} = H^p(Q, H^q(I,L))$, *are the following ones:*

$$E_\infty^{20} = 0$$
$$E_\infty^{11} = E_2^{11} = H^1(Q, H^1(I,I)/Q)$$
$$E_2^{02} = H^2(I,I)^Q \oplus (Ker\, F)^Q$$
$$E_\infty^{02} = E_3^{02} = Ker\, d_2^{02}$$

where $F : H^2(I) \otimes Q \to H^3(I,I)$ *is induced by the mapping*

$$
\begin{aligned}
C^2(I,Q) &\to C^3(I,I) \\
&\quad \phi \mapsto (x \wedge y \wedge z \mapsto [x, \phi(y,z)] + \curvearrowright).
\end{aligned}
$$

Proof. One has $E_2^{p0} = H^p(Q, L^I)$. The condition $Q \cap ad(I) = 0$ entails $L^I = Z(I) = 0$, so $E_2^{p0} = 0$. Thus $E_\infty^{20} = 0$ and $E_\infty^{02} = E_3^{02}$ follow from standard considerations. As d_2 maps E_2^{11} to $E_2^{30} = 0$, it follows that $E_\infty^{11} = E_2^{11}$. We also have

$$
\begin{aligned}
E_2^{02} &= H^0(Q, H^2(I,L)) = H^2(I,L)^Q \\
E_2^{11} &= H^1(Q, H^1(I,L)) \\
E_2^{21} &= H^2(Q, H^1(I,L)).
\end{aligned}
$$

Consider a piece of the long exact cohomology sequence associated with the short exact sequence $0 \to I \to L \to Q \to 0$ of I-modules (Q is considered as the trivial I-module):

$$
\begin{aligned}
H^0(I,L) \to H^0(I,Q) \to H^1(I,I) \to H^1(I,L) \to H^1(I,Q) \to H^2(I,I) \\
\to H^2(I,L) \to H^2(I,Q) \xrightarrow{F} H^3(I,I) \quad (5.1)
\end{aligned}
$$

(F is the connecting homomorphism).

We obviously have: $H^0(I,L) = L^I = 0$, $H^0(I,Q) = Q^I = Q$, $H^1(I,Q) = H^1(I) \otimes Q = 0$, $H^2(I,Q) = H^2(I) \otimes Q$. Hence

$$
\begin{aligned}
H^1(I,L) &\simeq H^1(I,I)/Q \\
H^2(I,L) &\simeq H^2(I,I) \oplus Ker F
\end{aligned}
$$

(note that since $Q \cap ad(I) = 0$, Q consists of outer derivations of I, and therefore embeds in $H^1(I,I)$).

As $L = I \oplus Q$ as Q-modules and the differential commutes with $ad\,x$ for any $x \in Q$, the Q-action commutes with inclusion and projection arrows in (5.1) (but not necessarily with connecting homomorphism), and we get

$$
H^2(I,L)^Q \simeq H^2(I,I)^Q \oplus (Ker F)^Q.
$$

Putting all this together, we obtain the equalities asserted. □

Passing to our specific case, define the grading on $L = S \otimes B \ltimes 1 \otimes \mathfrak{D}$ as in Theorem 1.5, i.e.,

$$
L_i = \begin{cases} e_0 \otimes B \oplus 1 \otimes \mathfrak{D}, & i = 0 \\ e_i \otimes B, & i \neq 0 \end{cases} \quad (5.2)
$$

and consider the induced grading on $H^2(L,L)$. Let $H^2_+(L,L)$ denote the positive part of that induced grading.

Proposition 5.2. *Let $L = S\otimes B \ltimes 1\otimes\mathfrak{D}$, where $\mathfrak{D}\subseteq Der(B)$. Then:*
(i) *if $S = W_1(n)$, then $H^2_+(L,L) \simeq H^2_+(S,S)\otimes B^{\mathfrak{D}} \oplus Der(B)^{\mathfrak{D}}$*
(ii) *if $S = sl(2)$, then $H^2_+(L,L) = 0$.*

Proof. (i) Proposition 1.1 ensures that Lemma 5.1 is applicable here if we put $I = S\otimes B$ and $Q = 1\otimes\mathfrak{D}$. Using Theorem 4.4, (4.8) and (4.9) and considering the action of $1\otimes\mathfrak{D}$ on appropriate cohomology on the level of explicit cocycles, one gets

$$E_2^{11} \simeq H^1(S,S)\otimes H^1(\mathfrak{D},B) \oplus H^1(\mathfrak{D},Der(B)/\mathfrak{D})$$
$$E_2^{02} \simeq H^2(S,S)\otimes B^{\mathfrak{D}} \oplus Der(B)^{\mathfrak{D}} \oplus Har^2(B,B)^{\mathfrak{D}} \oplus (Ker F)^{\mathfrak{D}}.$$

The grading (5.2) induces a $\mathbb{Z}$-grading on each term of the spectral sequence (cf. [F]).

The knowledge of $H^1(W_1(n),W_1(n))$, $H^2(W_1(n),K)$ (cf. [B1] or [D2]) and $H^2(W_1(n),W_1(n))$ (cf. [DK]), allows one to write down all nonzero homogeneous components of $E_2 = E_2^{11}\oplus E_2^{02}$ with respect to grading (5.2):

$$(E_2)_{-p^n} \simeq H^2_{-p^n}(W_1(n),W_1(n))\otimes B^{\mathfrak{D}} \oplus (Ker F)^{\mathfrak{D}}$$
$$(E_2)_{-p^t} \simeq H^2_{-p^t}(W_1(n),W_1(n))\otimes B^{\mathfrak{D}} \oplus H^1_{-p^t}(W_1(n),W_1(n))\otimes H^1(\mathfrak{D},B)$$
$$(E_2)_0 \simeq Der(B)^{\mathfrak{D}} \oplus Har^2(B,B)^{\mathfrak{D}} \oplus H^1(\mathfrak{D},Der(B)/\mathfrak{D})$$
$$(E_2)_{p^n-p^t} \simeq H^2_{p^n-p^t}(W_1(n),W_1(n))\otimes B^{\mathfrak{D}}$$
$$(E_2)_{p^n} \simeq Der(B)^{\mathfrak{D}}$$

where $1\le t\le n-1$.

The last two classes constitute $(E_2)_+$ and are generated by cocycles $(1\otimes R_u)\circ\Theta_{\psi_t}$, where $u\in B^{\mathfrak{D}}$,

$$\psi_t(e_i,e_j) = \begin{cases} e_{p^n-2}, & i=-1, j=p^t-1 \\ 0, & \text{otherwise} \end{cases}$$

and ϕ_D, where $D\in Der(B)^{\mathfrak{D}}$, respectively. Strictly speaking, these cocycles are extended from the corresponding cocycles from $Z^2(W_1(n)\otimes B, W_1(n)\otimes B)$ by letting them vanish on $W_1(n)\otimes B\wedge 1\otimes\mathfrak{D}$ and $1\otimes\mathfrak{D}\wedge 1\otimes\mathfrak{D}$.

According to Lemma 5.1, the corresponding E_3-term is

$$(E_3^{02})_+ = Ker((E_2^{02})_+ \xrightarrow{(d_2^{02})_+} (E_3^{02})_+).$$

Theorem 4.4 shows that all $\mathfrak{D}$-invariant cohomology classes in

$$(E_2^{02})_+ \simeq H^2_+(S,S)\otimes B^{\mathfrak{D}} \oplus (Der(B))^{\mathfrak{D}}$$

can be represented by $\mathfrak{D}$-invariant cocycles. This implies $(d_2^{02})_+ = 0$ and the positive part of the spectral sequence stabilizes in the relevant range.

(ii) Quite analogous (and simpler). □

Remark. For $G = S\otimes B \ltimes 1\otimes\mathfrak{D}$, one can compute the whole cohomology $H^2(G,G)$ by the following scheme: First, it is possible to evaluate $(Ker\,F)^{\mathfrak{D}}$ in the spirit of §2 or §3, and, in particular, to show that in this case, the Q-action commutes with F, which, in its turn, implies

$$E_2^{02} = H^2(S\otimes B, S\otimes B)^{\mathfrak{D}} \oplus (Ker\,F \cap (H^2(S\otimes B)\otimes\mathfrak{D})^{\mathfrak{D}}).$$

Then the same reasoning as at the end of the proof of Proposition 5.2 shows that $d_2^{02} = 0$ in general.

Since all the cocycles constituting the basis of $H^2_+(L,L)$ are of the type $(1\otimes R_u)\circ\Theta_{\psi_t}$ and ϕ_D that do not possibly vanish only at the $(-1)st$, $1st$ and $(p^t-1)st$ $(1\le t\le n-1)$ components of L and take values in the $(p^n-2)th$ component, it follows that each Massey product of two such cocycles is obviously zero, and by Proposition 1.4 we have

Theorem 5.3. *Let L be as in Proposition 5.2 with grading defined by (5.2). Let $\mathfrak{L}$ be a filtered algebra whose associated graded algebra is isomorphic (as graded algebra) to L. We have:*
(i) *if $S = W_1(n)$, then the bracket in $\mathfrak{L}$ is of the form*

$$\{\cdot,\cdot\} = [\cdot,\cdot] + \sum_{t=1}^{n-1}(1\otimes R_{u_t})\circ\Theta_{\psi_t} + \phi_D$$

for some $u_t\in B^{\mathfrak{D}}$ and $D\in Der(B)^{\mathfrak{D}}$
(ii) *if $S = sl(2)$, then $\mathfrak{L}\simeq L$.*

Note that the basic cocycles in $H^2_+(L,L)$ are of the form Φ_d for an appropriate $d\in Der(O_1(n-1)\otimes B)$: indeed, $\phi_D = \Phi_{x^{p^{n-1}-1}\otimes D}$ and

$$(1\otimes R_u)\circ\Theta_{\psi_t} = \Phi_{x^{p^{n-1}-1}\partial^{p^t-1}\otimes R_u},$$

where $1\le t\le n-1$. Therefore the algebras appearing in part (i) of Theorem 5.3 are of the kind

$$\mathfrak{L}(A,D)\ltimes 1\otimes\mathfrak{D}$$

for $A = O_1(n-1)\otimes B$ and $\mathfrak{D}\subseteq Der(A)$. Note that the Jacobi identity implies that $[D,\mathfrak{D}] = 0$. Combining this fact (in the particular case $B = O_m$) with Theorem 1.5, we conclude:

Corollary 5.4. *The ground field is algebraically closed of characteristic $p>5$.*
Let $\mathfrak{L}$ be a semisimple Lie algebra with a solvable maximal subalgebra defining in $\mathfrak{L}$ a long filtration. Then either $\mathfrak{L}\simeq sl(2)\otimes O_m\ltimes 1\otimes\mathfrak{D}$, or $\mathfrak{L}\simeq\mathfrak{L}(O_m,D)\ltimes 1\otimes\mathfrak{D}$ for some $m\in\mathbb{N}$, $D\in Der(O_m)$ and a solvable subalgebra $\mathfrak{D}$ in $Der(O_m)$ such that $[D,\mathfrak{D}] = 0$ and O_m has no $\langle\mathfrak{D},D\rangle$-invariant ideals.

Remarks. (i) Close inspection of Weisfeiler's results shows that if in Theorem 1.5 $\mathfrak{L}_0$ is a solvable maximal subalgebra, then after passing to the associated graded algebra, $\mathfrak{L}_0$ goes to $\langle e_0, e_1\rangle \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ (in the case $S = sl(2)$) or to $W_1(n)_0 \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ (in the case $S = W_1(n)$). Our computations of filtered deformations show that actually $\mathfrak{L}_0$ coincides with these algebras (as they do not change under deformations).

(ii) Since $[D, \mathfrak{D}] = 0$, the algebra $\langle \mathfrak{D}, D\rangle \subseteq Der(O_m)$ either coincides with $\mathfrak{D}$ (if $D \in \mathfrak{D}$), or is an 1-dimensional abelian extension of $\mathfrak{D}$ (if $D \notin \mathfrak{D}$).

So, to classify semisimple Lie algebras with a solvable maximal subalgebra occurring in Theorem 1.5, it remains to describe algebras appearing in Corollary 5.4 up to an isomorphism and to identify them with the known semisimple Lie algebras. We accomplish this task in the next section.

6 Classification of the semisimple Lie algebras $\mathfrak{L}(A,D) \ltimes 1 \otimes \mathfrak{D}$

The object of this section is the class of Lie algebras $\mathfrak{L}(A,D) \ltimes 1 \otimes \mathfrak{D}$, where $1 \otimes \mathfrak{D}$ acts on $\mathfrak{L}(A,D)$ as on $W_1(1) \otimes A$ and $[D, \mathfrak{D}] = 0$. The case $A = O_m$ is of particular importance.

Throughout this section, all algebras are assumed to be finite-dimensional.

Note that dimension considerations immediately imply that no algebra of the form $\mathfrak{L}(O_n, D)$ is isomorphic to some $sl(2) \otimes O_m$.

Lemma 6.1. *Each ideal of $\mathfrak{L}(A,D) \ltimes 1 \otimes \mathfrak{D}$ is of the form $\mathfrak{L}(I,D) \ltimes 1 \otimes \mathfrak{E}$, where I is a $\langle \mathfrak{D}, D\rangle$-invariant ideal of A, $\mathfrak{E}$ is an ideal of $\mathfrak{D}$, and $\mathfrak{E}(A) \subseteq I$.*

Proof. Let $\mathfrak{I}$ be an ideal of $\mathfrak{L}(A,D) \ltimes 1 \otimes \mathfrak{D}$, then $\mathfrak{I} \cap \mathfrak{L}(A,D)$ is an ideal of $\mathfrak{L}(A,D)$. Passing to the associated graded algebra (as in Proposition 1.3), we see that $gr(\mathfrak{I} \cap \mathfrak{L}(A,D))$ is an ideal of $W_1(1) \otimes A$. Either a direct calculation in $W_1(1)$, or the general result of [Ste], yields that

$$gr(\mathfrak{I} \cap \mathfrak{L}(A,D)) = W_1(1) \otimes I \tag{6.1}$$

for some ideal I of A. In particular, $e_{p-2} \otimes I \subset \mathfrak{I} \cap \mathfrak{L}(A,D)$. Multiplying elements from $e_{p-2} \otimes I$ a necessary number of times by $e_{-1} \otimes 1$, one gets $e_i \otimes I \subset \mathfrak{I} \cap \mathfrak{L}(A,D)$ for each $-1 \le i \le p-2$, that is, $W_1(1) \otimes I \subseteq \mathfrak{I} \cap \mathfrak{L}(A,D)$. Due to equation (6.1) this inclusion is actually an equality (of vector spaces): $\mathfrak{I} \cap \mathfrak{L}(A,D) = W_1(1) \otimes I$. In particular, $W_1(1) \otimes I$ is closed under the bracket $\{\cdot,\cdot\}$ (cf. Definition and equation (1.2) in §1), which is equivalent to $D(I) \subseteq I$.

Now, taking an arbitrary element $\sum\limits_{i=-1}^{p-2} e_i \otimes a_i + 1 \otimes d \in \mathfrak{I}$, and multiplying it by $e_0 \otimes 1$ and $e_{-1} \otimes 1$, we get, respectively, $\sum\limits_i ie_i \otimes a_i \in \mathfrak{I} \cap \mathfrak{L}(A,D)$ and $\sum\limits_i e_{i-1} \otimes a_i \in \mathfrak{I} \cap \mathfrak{L}(A,D)$ showing therefore that $a_i \in I$ for all i. This proves that $\mathfrak{I} = \mathfrak{L}(I,D) + 1 \otimes \mathfrak{E}$ for some subalgebra $\mathfrak{E} \subseteq \mathfrak{D}$. The rest of the conditions in the assertion follow immediately. □

Lemma 6.2. *Let a Lie algebra $\mathfrak{L} = \mathfrak{L}(A,D) + 1 \otimes \mathfrak{D}$ be semisimple. Then the following hold:*

(i) $A \simeq \bigoplus_i O_{n_i}$ *for some $n_i \in \mathbb{N}$, each O_{n_i} has no $\langle \mathfrak{D}, D \rangle$-invariant ideals;*

(ii) $\mathfrak{L}(A,D) \simeq \bigoplus_i S_i \otimes O_{m_i}$ *for some $m_i \in \mathbb{N}$ and simple Lie algebras S_i;*

(iii) $S_i \simeq \mathfrak{L}(O_{k_i}, d_i)$ *for some $k_i \in \mathbb{N}$ and $d_i \in Der(O_{k_i})$, O_{k_i} has no d_i-invariant ideals.*

Proof. The proof merely consists of multiple applications of the classical Block's results [B2].

By Lemma 6.1, A has no $\langle \mathfrak{D}, D \rangle$-invariant nilpotent ideals, i.e. A is $\langle \mathfrak{D}, D \rangle$-semisimple in the terminology of Block [B2]. According to [B2], Main Theorem and Corollary 8.3, A is isomorphic to the direct sum $\bigoplus_i O_{n_i}$ of rings of truncated polynomials having no $\langle \mathfrak{D}, D \rangle$-invariant ideals. Hence $D = \sum_i D_i$, where each D_i acts as derivation on O_{n_i} and by zero on O_{n_j}, if $j \neq i$. Obviously

$$\mathfrak{L}(\bigoplus_i O_{n_i}, \sum_i D_i) \simeq \bigoplus_i \mathfrak{L}(O_{n_i}, D_i)$$

and by Lemma 6.1 each minimal ideal of $\mathfrak{L}$ coincides with one of the $\mathfrak{L}(O_{n_i}, D_i)$. Thus by [B2], Theorem 1.3, $\mathfrak{L}(O_{n_i}, D_i) \simeq S_i \otimes O_{m_i}$ for some simple Lie algebra S_i and $m_i \in \mathbb{N}$.

Applying Lemma 6.1 again, we see that each ideal of $\mathfrak{L}(O_{n_i}, D_i)$ is of the form $\mathfrak{L}(I, D_i)$ for some D_i-invariant ideal I of O_{n_i}, and by [Ste] each ideal of $S_i \otimes O_{m_i}$ is of the form $S_i \otimes J$ for some ideal J of O_{m_i}. But $O_{m_i}^{+}$ is the greatest (it contains all other ideals) ideal of O_{m_i} whence there is a greatest D_i-invariant ideal I_i of O_{n_i}, $\mathfrak{L}(I_i, D_i) \simeq S \otimes O_{m_i}^{+}$, and

$$\mathfrak{L}(O_{n_i}, D_i)/\mathfrak{L}(I_i, D_i) \simeq (S_i \otimes O_{m_i})/(S_i \otimes O_{m_i}^{+}) \simeq S_i.$$

It is easy to see that the left side here is isomorphic to $\mathfrak{L}(O_{n_i}/I_i, d_i)$, $d_i \in Der(O_{n_i}/I_i)$ being induced from D_i. Since S_i is simple, O_{n_i}/I_i has no d_i-invariant ideals and again by Block's theorem, $O_{n_i}/I_i \simeq O_{k_i}$ for some $k_i \in \mathbb{N}$. □

Now we determine simple Lie algebras in the class $\mathfrak{L}(A,D)$.

Lemma 6.3. *The ground field is perfect of characteristic $p > 3$.*
$\mathfrak{L} = \mathfrak{L}(A,D)$ is simple if and only if $\mathfrak{L} \simeq W_1(n)$ for some $n \in \mathbb{N}$.

Proof. The "if" part is contained in Proposition 1.2. So suppose $\mathfrak{L}(A,D)$ is simple. According to Lemmas 6.1 and 6.2, $A \simeq O_n$ for some $n \in \mathbb{N}$. Hence $\mathfrak{L}$ has a subalgebra $\mathfrak{L}_0 = (e_{-1} \otimes O_n^{+}) \oplus (\langle e_0, \dots, e_{p-2} \rangle \otimes O_n)$ of codimension 1. Then by [D1], $\mathfrak{L}$ is isomorphic to either $sl(2)$ or $W_1(n)$, the first case is impossible by dimension consideration. □

Remarks. (i) If the ground field is algebraically closed, one may deduce the assertion of Lemma 6.3 from many other results in the literature, e.g., [Re] (by utilizing the fact that algebras under consideration are Ree's algebras, see remark after definition of $\mathfrak{L}(A,D)$ in §1), or [K] or [W] (by noting that that $\mathfrak{L}_0$ is solvable).

(ii) Combining Theorem 5.3(i) (with remark after it) and Lemma 6.3, we recover the fact that each filtered deform (with respect to the standard grading) of $W_1(n)$ is isomorphic to $W_1(n)$. This fact is important in considering of some classes of Lie algebras with given properties of subalgebras or elements and was proved by Benkart, Isaacs and Osborn in [BIO], §3 and Dzhumadil'daev in [D1].

Now summarizing all our results, we obtain the final classification of the long filtration case.

Theorem 6.4. *The ground field is algebraically closed of characteristic $p > 5$.*
$\mathfrak{L}$ is a semisimple Lie algebra with a solvable maximal subalgebra defining in it a long filtration, if and only if either $\mathfrak{L} \simeq sl(2) \otimes O_m \ltimes 1 \otimes \mathfrak{D}$, or

$$W_1(n) \otimes O_m \subset \mathfrak{L} \subset Der(W_1(n)) \otimes O_m \ltimes 1 \otimes W_m,$$

where $\mathfrak{D}$ in the first case, and $pr_{W_m}\mathfrak{L}$ in the second one, are solvable subalgebras of W_m such that O_m has no $\mathfrak{D}$- or $pr_{W_m}\mathfrak{L}$-invariant ideals.

Proof. "only if" part. Summarizing results of Lemmas 6.2 and 6.3, we get that the semisimple Lie algebras of the form $\mathfrak{L}(A,D) \ltimes 1 \otimes \mathfrak{D}$ are exactly those whose socle is a direct sum of algebras $W_1(n) \otimes O_m$ for some $n,m \in \mathbb{N}$. By Corollary 5.4, these algebras (with solvable $\mathfrak{D}$), along with $sl(2) \otimes O_m \ltimes 1 \otimes \mathfrak{D}$, exhaust all possible semisimple Lie algebras with a solvable maximal subalgebra defining a long filtration. Obviously a socle of such an algebra should consist of only one minimal ideal, and the assertion desired follows.

"if" part. In the $sl(2)$ case, it is evident that $\mathfrak{L}_0 = \langle e_0, e_1 \rangle \otimes O_m \ltimes 1 \otimes \mathfrak{D}$ is a solvable maximal subalgebra.

In the $W_1(n)$ case, we have

$$W_1(n) \otimes O_m \subset \mathfrak{L} \subset Der(W_1(n) \otimes O_m) \simeq Der(W_1(n)) \otimes O_m \ltimes 1 \otimes Der(O_m).$$

By Proposition 1.2, we identify $W_1(n) \otimes O_m$ with $\mathfrak{L}(O_1(n-1) \otimes O_m, \partial \otimes 1)$. By Theorem 3.1(ii), $\mathfrak{L} = \mathfrak{L}(O_1(n-1) \otimes O_m, \partial \otimes 1) \ltimes 1 \otimes \mathfrak{D}$ for some solvable subalgebra $\mathfrak{D} \subset Der(O_1(n-1) \otimes O_m)$ (a further elucidation of the structure of $\mathfrak{D}$ is possible due to conditions imposed on $pr_{Der(O_m)}\mathfrak{L}$ and Theorem 4.1(iii), but we don't need it here).

Consider a maximal subalgebra $\mathfrak{L}_0$ containing a subalgebra

$$\langle e_0, e_1, \dots, e_{p-2} \rangle \otimes O_1(n-1) \otimes O_m \ltimes 1 \otimes \mathfrak{D}.$$

Obviously

$$\mathfrak{L}_0 = (e_{-1} \otimes I \oplus \langle e_0, e_1, \dots, e_{p-2} \rangle \otimes O_1(n-1) \otimes O_m) \ltimes 1 \otimes \mathfrak{D}$$

for some $I \lhd O_1(n-1) \otimes O_m$. Since $O_1(n-1) \otimes O_m$ is a ring of truncated polynomials itself, each its ideal is nilpotent, whence $\mathfrak{L}_0$ is solvable. □

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3. Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds

Lin. Algebra Appl. **407** (2005), 71–104

We obtain formulas for the first and second cohomology of the general current Lie algebra with coefficients in the "current" module, and apply them to compute structure functions for manifolds of loops with values in compact Hermitian symmetric spaces.

Introduction

We deal with the low-dimensional cohomology of current Lie algebras with coefficients in the "current module". Namely, let L be a Lie algebra, M an L-module, A an associative commutative algebra with unit, V a symmetric unital A-module. Then the Lie algebra structure on $L \otimes A$ and the $L \otimes A$-module structure on $M \otimes V$ are defined via obvious formulas:

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab,$$
$$(x \otimes a) \bullet (m \otimes v) = (x \bullet m) \otimes (a \bullet v)$$

for any $x,y \in L$, $m \in M$, $a,b \in A$, $v \in V$, where $\bullet$ denotes, by abuse of notation, a respective module action.

The aim of this paper is twofold. First, we want to demonstrate that the problem of description of such cohomology in terms of the tensor factors L and A probably does not have an adequate general solution, as even a partial answer for the two-dimensional cohomology seems to be overwhelmingly complex. Second, we want to demonstrate, nevertheless, computability of this cohomology in some cases and its application to some differential geometric questions.

In §1 we establish an elementary result from linear algebra which will be useful in the course of subsequent algebraic manipulations. In §2 we get a formula for the first cohomology group. In §3 we compute the second cohomology group in the two cases – where L is abelian and where L acts trivially on the whole cohomology group $H^2(L \otimes A, M \otimes V)$. At the end of this section, we present the list of 13 types of 2-cocycles (so-called cocycles of rank 1, generated by decomposable elements in the tensor product) in the general case. However, this list is a priori not complete. In §4 a certain spectral sequence is sketched, which may provide a more conceptual framework for computations in preceding sections. However, we do not go into details and other sections are not dependent on this one. The last §5 is devoted to an application. We show how to derive from previous computations obstructions to integrability (structure functions) of certain canonical connections on the manifolds of loops with values in compact Hermitian symmetric spaces.

One should note that the result about the first cohomology group (in particular, about derivations of the current Lie algebra) can be found (in different forms) in the literature and is a sort of folklore, and partial results on the second cohomology were obtained by Cathelineau [C], Haddi [Had], Lecomte and Roger [R] and the author [Z]. However, all these results do not provide the whole generality we need, as various restrictions, notably the zero characteristic of the ground field and perfectness of the Lie algebra L were imposed. Moreover, as we see in §5, the case in a sense opposite to the case of perfect L, namely, the case of abelian L, does lead to some interesting application (first considered by Poletaeva).

The technique used is highly computational and linear-algebraic in nature and based on applying various symmetrization operators to the cocycle equation.

Notation

The ground field K is assumed to be arbitrary field of characteristic $\neq 2,3$ in §1–4, and $\mathbb{C}$ in §5.

$H^n(L,M)$, $C^n(L,M)$, $Z^n(L,M)$, $B^n(L,M)$ stand, respectively, for the spaces of cohomology, cochains, cocycles and coboundaries of a Lie algebra L with coefficients in a module M.

$M^L = \{m \in M \,|\, x \bullet m = 0 \text{ for any } x \in L\}$ is a submodule of L-invariants.

If M,N are two L-modules, $Hom(M,N)$ bears a standard L-module structure via

$$(x \bullet \varphi)(m) = \varphi(x \bullet m) - x \bullet \varphi(m)$$

for any $x \in L, m \in M$. $Hom_L(M,N)$ is another notation for $Hom(M,N)^L$.

$S^n(A,V)$ stands for the space of symmetric n-linear maps $A \times \cdots \times A \to V$.

$\wedge^n(V)$ and $T^n(V)$ stand, respectively, for the spaces of n-fold skew and tensor products of a module V.

$Har^n(A,V)$ and $\mathcal{Z}^n(A,V)$ stand, respectively, for the spaces of Harrison cohomology and Harrison cocycles of an associative commutative algebra A with coefficients in a module V (for $n = 2$, these are just symmetric Hochschild cocycles; see [Har], where this cohomology was introduced, and [GeSc] for a more modern treatment).

$Der(A)$ denotes the derivation algebra of an algebra A. More generally, $Der(A,V)$ denotes the space of derivations of A with values in a A-module V.

All other (nonstandard and inavoidably numerous) notations for different spaces of multilinear maps and modules are defined as they introduced in the text.

The symbol $\curvearrowright$ after an expression refers to the sum of all cyclic permutations (under S_3 of letters and indices occurring in that expression).

1 A lemma from linear algebra

If either both L and M or both A and V are finite-dimensional, then each cocycle $\Phi \in Z^n(L \otimes A, M \otimes V)$ can be represented as an element of $Hom(L^{\otimes n}, M) \otimes Hom(A^{\otimes n}, V)$:

$$\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i \tag{1.1}$$

where φ_i, α_i are n-linear maps $L \times \cdots \times L \to M$ and $A \times \cdots \times A \to V$ respectively. We restrict our considerations to this case. The minimal possible number $|I|$ such that the cocycle Φ can be written in the form (1.1) will be called the *rank of cocycle*.

Representing $H^n(L \otimes A, M \otimes V)$ in terms of pairs (L, M) and (A, V), we encounter conditions such as

$$\sum_{i \in I} S\varphi_i \otimes T\alpha_i = 0, \tag{1.2}$$

where S and T are some linear operators defined on the spaces of n-linear maps $L \times \cdots \times L \to M$ and $A \times \cdots \times A \to V$, respectively.

For example, the substitution $a_1 = \cdots = a_{n+1} = 1$ in the cocycle equation

$$d\Phi(x_1 \otimes a_1, \dots, x_{n+1} \otimes a_{n+1}) = 0,$$

where Φ is as in (1.1), yields

$$\sum_{i \in I} d\varphi_i(x_1, \dots, x_{n+1}) \otimes a_i(1, \dots, 1) = 0.$$

Another example: applying the symmetrization operator Y with respect to the letters $x_1, \dots, x_{n+1}$, to the cocycle equation, we get:

$$\sum_{i \in I} \left(Y(x_1 \bullet \varphi_i(x_2, \dots, x_{n+1})) \otimes \sum_{j=1}^{n+1} (-1)^j a_j \bullet \alpha_i(a_1, \dots, \widehat{a_j}, \dots, a_{n+1}) \right) = 0.$$

So, suppose that a condition of type (1.2) holds. Since

$$Ker(S \otimes T) = Hom(A^{\otimes n}, V) \otimes Ker\, T + Ker\, S \otimes Hom(L^{\otimes n}, M),$$

it follows that replacing α_i's and φ_i's by appropriate linear combinations, one can find a decomposition of the set of indices $I = I_1 \cup I_2$ such that

$$S\varphi_i = 0, \quad i \in I_1 \qquad \text{and} \qquad T\alpha_i = 0, \quad i \in I_2. \tag{1.3}$$

Suppose that another equality of type (1.2) holds:

$$\sum_{i\in I} S'\varphi_i \otimes T'\alpha_i = 0. \tag{1.2$'$}$$

Then it determines a new decomposition $I = I_1' \cup I_2'$ such that $S'\varphi_i = 0$ if $i \in I_1'$ and $T'\alpha_i = 0$ if $i \in I_2'$. It turns out that it is possible to replace φ_i's and α_i's by their linear combinations so that both decompositions will hold simultaneously.

Lemma 1.1. *Let U,W be two vector spaces, $S,S' \in Hom(U,\cdot)$, $T,T' \in Hom(W,\cdot)$. Then*

$$\begin{aligned} &Ker(S\otimes T)\cap Ker(S'\otimes T') \\ &\qquad \simeq (Ker S \cap Ker S')\otimes W + Ker S \otimes Ker T' + Ker S' \otimes Ker T \\ &\qquad\qquad + U \otimes (Ker T \cap Ker T'). \end{aligned}$$

Proof. Since $Ker(S\otimes T) = Ker S \otimes W + U \otimes Ker T$ and analogously for $Ker(S'\otimes T')$, the equality to prove is a particular case of

$$\begin{aligned} &(U_1\otimes W + U\otimes W_1)\cap(U_2\otimes W + U\otimes W_2) \\ &\qquad = (U_1\cap U_2)\otimes W + U_1\otimes W_2 + U_2\otimes W_1 + U\otimes(W_1\cap W_2) \end{aligned} \tag{1.4}$$

provided U_1,U_2 and W_1,W_2 are subspaces of U and W, respectively.

Assume for the moment that $U_1\cap U_2 = W_1\cap W_2 = 0$. Then expressing $U = U_1\oplus U_2\oplus U'$ and $W = W_1\oplus W_2\oplus W'$ for some subspaces U',W' and substituting this in the left side of (1.4), we get:

$$\begin{aligned} &(U_1\otimes W \oplus U_2\otimes W_1 \oplus U'\otimes W_1)\cap(U_1\otimes W_2 \oplus U_2\otimes W \oplus U'\otimes W_2) \\ &\qquad = U_1\otimes W_2 \oplus U_2\otimes W_1. \end{aligned}$$

To prove (1.4) in the general case, pass to quotient modulo $(U_1\cap U_2)\otimes W + U\otimes(W_1\cap W_2)$ and obtain $U_1\otimes W_2 + U_2\otimes W_1$ by the just proved. □

Below, in numerous applications of Lemma 1.1, we will, by abuse of language, say *"by (1.2) and (1.2)$'$, one gets a decomposition $I = I_1\cup I_2\cup I_3\cup I_4$ such that $S\varphi_i = S'\varphi_i = 0$ for $i\in I_1$, $S\varphi_i = T'\alpha_i = 0$ for $i\in I_2$, $S'\varphi_i = T\alpha_i = 0$ for $i\in I_3$, and $T\alpha = T'\alpha_i = 0$ for $i\in I_4$"*. This means that one can find a new expression $\Phi = \sum_{i\in I}\varphi_i\otimes\alpha_i$ with indicated properties (where the new φ_i's and α_i's are linear combinations of the old ones).

Unfortunately, for the "triple intersection"

$$Ker(S\otimes T)\cap Ker(S'\otimes T')\cap Ker(S''\otimes T'')$$

the analogous decomposition is no longer true. That is why dealing with the second cohomology in §3, we are unable to obtain a general result and restrict our considerations with cocycles of rank 1 or with some special cases. For the first cohomology, however, Lemma 1.1 suffices to consider the general case, but at the end of the proof it turns out that it is possible to choose a basis consisting of cocycles of rank 1.

2 The first cohomology

From now on (in this and subsequent sections), either both L and M or both A and V are finite-dimensional.

Theorem 2.1.

$$H^1(L\otimes A, M\otimes V) \simeq H^1(L,M)\otimes V$$
$$\oplus Hom_L(L,M)\otimes Der(A,V)\oplus Hom(L/[L,L],M^L)\otimes \frac{Hom(A,V)}{V+Der(A,V)}. \quad (2.1)$$

Each cocycle in $Z^1(L\otimes A, M\otimes V)$ is a linear combination of cocycles of the three following types (which correspond to the summands in (2.1)):
(i) $x\otimes a \mapsto \varphi(x)\otimes(a\bullet v)$ *for some* $\varphi\in Z^1(L,M), v\in V$
(ii) $x\otimes a\mapsto \varphi(x)\otimes\alpha(a)$ *for some* $\varphi\in Hom_L(L,M), a\in Der(A,V)$
(iii) *as in (ii) with* $\varphi(L)\subseteq M^L, \varphi([L,L])=0, \alpha\in Hom(A,V)$.

Remark. Theorem 2.1 was obtained earlier by Santharoubane [Sa] in the particular case where $M=L^*$, $V=A^*$ and L is 1-generated as an $U(L)^+$-module, where $U(L)^+$ is the augmentation ideal of the universal enveloping algebra $U(L)$, and by Haddi [Had] (in homological form) in the case of characteristic zero and L perfect.

Proof. Let $\Phi=\sum\limits_{i\in I}\varphi_i\otimes\alpha_i$ be a cocycle of

$$Z^1(L\otimes A, M\otimes V)\subset Hom(L,M)\otimes Hom(A,V).$$

The cocycle equation $d\Phi=0$ reads

$$\sum_{i\in I}(x\bullet\varphi_i(y)\otimes a\bullet\alpha_i(b) - y\bullet\varphi_i(x)\otimes b\bullet\alpha_i(a) - \varphi_i([x,y])\otimes\alpha_i(ab)) = 0. \quad (2.2)$$

Symmetrizing this equation with respect to x,y, we get:

$$\sum_{i\in I}(x\bullet\varphi_i(y)+y\bullet\varphi_i(x))\otimes(a\bullet\alpha_i(b)-b\bullet\alpha_i(a)) = 0.$$

Substitute $a=b=1$ in (2.2):

$$\sum_{i\in I} d\varphi_i(x,y)\otimes\alpha_i(1) = 0.$$

Applying Lemma 1.1 to the last two equations, we get a decomposition $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

$$\begin{array}{lll} d\varphi_i = 0, & x \bullet \varphi_i(y) + y \bullet \varphi_i(x) = 0 & \text{for any } i \in I_1, \\ d\varphi_i = 0, & a \bullet \alpha_i(b) = b \bullet \alpha_i(a) & \text{for any } i \in I_2, \\ x \bullet \varphi_i(y) + y \bullet \varphi_i(x) = 0, & \alpha_i(1) = 0 & \text{for any } i \in I_3, \\ a \bullet \alpha_i(b) = b \bullet \alpha_i(a), & \alpha_i(1) = 0 & \text{for any } i \in I_4. \end{array}$$

It is easy to see that $\alpha_i(a) = a \bullet \alpha_i(1)$ for each $i \in I_2$, and the maps $x \otimes a \mapsto \varphi_i(x) \otimes \alpha_i(a)$ are cocycles of type (i) from the statement of the Theorem 2.1, and that $\alpha_i = 0$ for each $i \in I_4$.

Substitute $b = 1$ in the cocycle equation (2.2):

$$\sum_{i \in I_1 \cup I_3} (x \bullet \varphi_i(y) - \varphi_i([x,y])) \otimes (\alpha_i(a) - a \bullet \alpha_i(1)) = 0.$$

Now apply Lemma 1.1 again. For elements φ_i, where $i \in I_1$, we see that if $x \bullet \varphi_i(y) - \varphi_i([x,y]) = 0$, then $\varphi_i([L,L]) = 0$ and $\varphi_i(L) \subseteq M^L$, what gives rise to cocycles of type (iii), and if $\alpha_i(a) - a \bullet \alpha_i(1) = 0$, then cocycles of type (i) appear, an already considered case. We have $x \bullet \varphi_i(y) = \varphi_i([x,y])$ for all (remaining) $i \in I_3$.

Hence (2.2) can be rewritten as

$$\sum_{i \in I_3} \varphi_i([x,y]) \otimes (a \bullet \alpha_i(b) + b \bullet \alpha_i(a) - \alpha_i(ab)) = 0. \tag{2.2a}$$

The vanishing of the first and second tensor factors in each summand in (2.2a) gives rise to cocycles of type (iii) and (ii), respectively.

Hence we have

$$\begin{aligned} Z^1(L \otimes A, M \otimes V) &= Z^1(L,M) \otimes V \\ &\quad + Hom_L(L,M) \otimes Der(A,V) + Hom(L/[L,L], M^L) \otimes Hom(A,V) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} Z^1(L \otimes A, M \otimes V) &= Z^1(L,M) \otimes V \\ &\oplus Hom_L(L,M) \otimes Der(A,V) \oplus Hom(L/[L,L], M^L) \otimes \frac{Hom(A,V)}{V + Der(A,V)}. \end{aligned}$$

From the considerations above we easily deduce:

$$B^1(L \otimes A, M \otimes V) = B^1(L,M) \otimes V$$

and (2.1) now follows. □

Corollary 2.2. The derivation algebra of the current Lie algebra $L \otimes A$ is isomorphic to

$$Der(L) \otimes A \,\oplus\, Hom_L(L,L) \otimes Der(A) \,\oplus\, Hom(L/[L,L], Z(L)) \otimes \frac{End(A)}{A + Der(A)}.$$

This overlaps with [B, Theorem 7.1] and [BM, Theorem 1.1].

Note that $Hom_L(L,L)$ is nothing but a *centroid* of an algebra L (the set of all linear transformations in $End(L)$ commuting with algebra multiplications).

Specializing to particular cases of L and A, we get on this way (largely known) results about derivations of some particular classes of Lie algebras. So, letting $L = \mathfrak{g}$, a finite-dimensional simple Lie algebra over $\mathbb{C}$, and $A = \mathbb{C}[t, t^{-1}]$, the Laurent polynomial ring, we get an expression for the derivation algebra of the loop algebra:

$$Der(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \simeq \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \ltimes 1 \otimes W,$$

where $W = Der(\mathbb{C}[t, t^{-1}])$ is the famous Witt algebra.

More generally, replacing the Laurent polynomial ring by an algebra of functions meromorphic on a compact Riemann surface and holomorphic outside the fixed finite set of punctures on the surface, we get a similar formula for the derivation algebra of any Krichever-Novikov algebra of affine type, where the Witt algebra is replaced by a Krichever-Novikov algebra of Witt type.

3 The second cohomology

In this section we obtain some particular results on the second cohomology $H^2(L \otimes A, M \otimes V)$. The computations go along the same scheme as for H^1 but are more complicated.

As we want to express H^2 in terms of the tensor products of modules depending on (L, M) and (A, V), it is natural to do so for underlying modules of the Chevalley–Eilenberg complex. We have (under the same finiteness assumptions as before):

$$\begin{aligned} &C^1(L \otimes A, M \otimes V) \simeq C^1(L, M) \otimes C^1(A, V) \\ &C^2(L \otimes A, M \otimes V) \simeq C^2(L, M) \otimes S^2(A, V) \bigoplus S^2(L, M) \otimes C^2(A, V). \end{aligned} \tag{3.1}$$

To obtain a similar decomposition in the third degree, let us denote (by abuse of language) the Young symmetrizer corresponding to tableau λ by the same symbol λ. We have the following decomposition of the unit element in the group algebra $K[S_3]$:

$$e = \frac{1}{6}\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \frac{1}{3}\left(\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2\\\hline 3 \\\cline{1-1}\end{array}\right) + \frac{1}{6}\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline\end{array}.$$

Then, using the natural isomorphism $i: T^3(L\otimes A) \simeq T^3(L)\otimes T^3(A)$ and the projection $p: T^3(L\otimes A) \to \wedge^3(L\otimes A)$, one can decompose the third exterior power of the tensor product as follows:

$$\begin{aligned}\wedge^3(L\otimes A) &= p\circ(e\times e)\circ i(T^3(L\otimes A))\\ &\simeq \wedge^3(L)\otimes S^3(A) \bigoplus \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(L)\otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(A) + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(L)\otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(A)\right)\\ &\qquad\qquad \bigoplus S^3(L)\otimes\wedge^3(A)\end{aligned}$$

(all other components appearing in $T^3(L)\otimes T^3(A)$ vanish under the projection). One directly verifies that

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(u) = 0 \quad \text{if and only if} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(u) = 0$$

for each $u \in \wedge^3(L\otimes A)$.

Hence we get a (noncanonical) isomorphism:

$$\wedge^3(L\otimes A) \simeq \wedge^3(L)\otimes S^3(A) \bigoplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(L)\otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(A) \bigoplus S^3(L)\otimes\wedge^3(A).$$

Passing to $Hom(\,\cdot\,, M\otimes V) \simeq Hom(\,\cdot\,, M)\otimes Hom(\,\cdot\,, V)$, one gets:

$$\begin{aligned}&C^3(L\otimes A, M\otimes V)\\ &\simeq C^3(L,M)\otimes S^3(A,V) \bigoplus Y^3(L,M)\otimes \widetilde{Y}^3(A,V) \bigoplus S^3(L,M)\otimes C^3(A,V),\end{aligned} \tag{3.2}$$

where $Y^3(L,M) = Hom\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(L), M\right)$, $\widetilde{Y}^3(A,V) = Hom\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}(A), V\right)$.

According to (3.1)–(3.2) one can decompose H^2 as

$$H^2(L\otimes A, M\otimes V) = (H^2)' \oplus (H^2)'' \tag{3.3}$$

where $(H^2)'$ is the space of classes of cocycles lying in $C^2(L,M)\otimes S^2(A,V)$ and $(H^2)''$ is the space of classes of cocycles of the form $\Phi+\Psi$, where $\Phi \in S^2(L,M)\otimes C^2(A,V)$, $\Psi \in C^2(L,M)\otimes S^2(A,V)$ such that $\Phi \neq 0$. We will compute $(H^2)'$ and obtain some particular results on $(H^2)''$ (actually, $(H^2)'$ and $(H^2)''$ are limit terms of a certain spectral sequence; see §4).

The differentials of the low degree in the piece

$$C^1(L\otimes A, M\otimes V) \xrightarrow{d^1} C^2(L\otimes A, M\otimes V) \xrightarrow{d^2} C^3(L\otimes A, M\otimes V)$$

of the standard Chevalley-Eilenberg complex can be decomposed as follows:

$$\begin{aligned}d^1 &= d_1 + d_2\\ d^2 &= \sum_{\substack{1\le i\le 2\\ 1\le j\le 3}} d_{ij},\end{aligned}$$

where

$$\begin{aligned}
d_1 &: C^1(L,M)\otimes C^1(A,V) \to C^2(L,M)\otimes S^2(A,V),\\
d_2 &: C^1(L,M)\otimes C^1(A,V) \to S^2(L,M)\otimes C^2(A,V),\\
d_{11} &: C^2(L,M)\otimes S^2(A,V) \to C^3(L,M)\otimes S^3(A,V),\\
d_{12} &: C^2(L,M)\otimes S^2(A,V) \to Y^3(L,M)\otimes \widetilde{Y}^3(A,V),\\
d_{13} &: C^2(L,M)\otimes S^2(A,V) \to S^3(L,M)\otimes C^3(A,V),\\
d_{21} &: S^2(L,M)\otimes C^2(A,V) \to C^3(L,M)\otimes S^3(A,V),\\
d_{22} &: S^2(L,M)\otimes C^2(A,V) \to Y^3(L,M)\otimes \widetilde{Y}^3(A,V),\\
d_{23} &: S^2(L,M)\otimes C^2(A,V) \to S^3(L,M)\otimes C^3(A,V).
\end{aligned}$$

Direct computations show:

$$\begin{aligned}
&d_1(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2) =\\
&\qquad \frac{1}{2}(x_2\bullet\varphi(x_1) - x_1\bullet\varphi(x_2))\otimes(a_1\bullet\alpha(a_2)+a_2\bullet\alpha(a_1))\\
&\qquad\qquad - \varphi([x_1,x_2])\otimes\alpha(a_1a_2);
\end{aligned}$$

$$\begin{aligned}
&d_2(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2) =\\
&\qquad \frac{1}{2}(x_1\bullet\varphi(x_2) + x_2\bullet\varphi(x_1))\otimes(a_2\bullet\alpha(a_1)-a_1\bullet\alpha(a_2));
\end{aligned}$$

$$\begin{aligned}
&d_{11}(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2, x_3\otimes a_3) =\\
&\qquad \frac{1}{3}(\varphi([x_1,x_2],x_3)+\curvearrowright)\otimes(\alpha(a_1a_2,a_3)+\curvearrowright)\\
&\qquad -\frac{1}{3}(x_1\bullet\varphi(x_2,x_3)+\curvearrowright)\otimes(a_1\bullet\alpha(a_2,a_3)+\curvearrowright);
\end{aligned}$$

$$\begin{aligned}
&d_{12}(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2, x_3\otimes a_3) =\\
&(2\varphi([x_1,x_2],x_3)+\varphi([x_1,x_3],x_2)-\varphi([x_2,x_3],x_1))\otimes(\alpha(a_1a_2,a_3)-\alpha(a_2a_3,a_1))\\
&\qquad +(-x_1\bullet\varphi(x_2,x_3)+x_2\bullet\varphi(x_1,x_3)+2x_3\bullet\varphi(x_1,x_2))\\
&\qquad\qquad \otimes(a_1\bullet\alpha(a_2,a_3)-a_3\bullet\alpha(a_1,a_2));
\end{aligned}$$

$$d_{13}(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2, x_3\otimes a_3) = 0;$$

$$\begin{aligned}
&d_{22}(\varphi\otimes\alpha)(x_1\otimes a_1, x_2\otimes a_2, x_3\otimes a_3) =\\
&(2\varphi([x_1,x_2],x_3)+\varphi([x_1,x_3],x_2)-\varphi([x_2,x_3],x_1))\otimes(\alpha(a_1a_2,a_3)-\alpha(a_2a_3,a_1))\\
&\qquad +(-x_1\bullet\varphi(x_2,x_3)+x_2\bullet\varphi(x_1,x_3))\\
&\qquad\qquad \otimes(a_1\bullet\alpha(a_2,a_3)+a_3\bullet\alpha(a_1,a_2)+2a_2\bullet\alpha(a_1,a_3));
\end{aligned}$$

$$d_{23}(\varphi\otimes\alpha)(x_1\otimes a_1,x_2\otimes a_2,x_3\otimes a_3)=$$
$$\frac{1}{3}(x_1\bullet\varphi(x_2,x_3)+\curvearrowright)\otimes(a_1\bullet\alpha(a_2,a_3)+\curvearrowright)$$

(the absence of d_{21} in this list is merely a technical matter: at a relevant stage of computations, it will be convenient to use the entire differential d rather than d_{21}).

Now the reader should be prepared for a bunch of tedious and cumbersome definitions. Our excuse is that all this stuff provides building blocks for $H^2(L\otimes A,M\otimes V)$ and one can hardly imagine that it may be defined in a simpler way. Taking a glance at the expressions below, one can believe that the general formula for $H^n(L\otimes A,M\otimes V)$ hardly exists – if it does, one should give correct n-dimensional generalizations of definitions below (in a few cases this is evident – like Harrison or cyclic cohomology, but in most cases it is not).

Definitions.

(i) Define $d^{[]},d^{\bullet}:Hom(L^{\otimes 2},M)\to Hom(L^{\otimes 3},M)$ as follows:

$$d^{[]}\varphi(x,y,z)=\varphi([x,y],z)+\curvearrowright$$
$$d^{\bullet}\varphi(x,y,z)=x\bullet\varphi(y,z)+\curvearrowright.$$

(ii) Define $\wp,D:Hom(A^{\otimes 2},V)\to Hom(A^{\otimes 3},V)$ as follows:

$$\wp\alpha(a,b,c)=\alpha(ab,c)+\curvearrowright$$
$$D\alpha(a,b,c)=a\bullet\alpha(b,c)+\curvearrowright.$$

(iii) $\mathcal{B}(L,M)=\{\varphi\in C^2(L,M)\mid\varphi([x,y],z)+z\bullet\varphi(x,y)=0;d^{[]}\varphi(x,y,z)=0\}$.

(iv) $Q^2(L,M)=\{d\psi\mid\psi\in Hom(L,M);x\bullet\psi(y)=y\bullet\psi(x)\}$;
$H^2_M(L)=(Z^2(L,M^L)+Q^2(L,M))/Q^2(L,M)$.

(v) $\mathcal{K}(L,M)=\{\varphi\in C^2(L,M)\mid d^{[]}\varphi(x,y,z)=2x\bullet\varphi(y,z)\}$;
$\mathcal{J}(L,M)=\{\varphi\in C^2(L,M)\mid\varphi(x,y)=\psi([x,y])-\frac{1}{2}x\bullet\psi(y)+\frac{1}{2}y\bullet\psi(x)$ for $\psi\in Hom(L,M)\}$;
$\mathcal{H}(L,M)=(\mathcal{K}(L,M)+\mathcal{J}(L,M))/\mathcal{J}(L,M)$.

(vi) $\mathcal{X}(L,M)=\{\varphi\in C^2(L,M)\mid 2\varphi([x,y],z)=z\bullet\varphi(x,y);$
$\varphi([x,y],z)=\varphi([z,x],y)\}$.

(vii) $\mathcal{T}(L,M)=\{\varphi\in C^2(L,M)\mid 3\varphi([x,y],z)=2z\bullet\varphi(x,y);$
$\varphi([x,y],z)=\varphi([z,x],y)\}$.

(viii) $Poor_-(L,M)=\{\varphi\in C^2(L,M^L)\mid\varphi([L,L],L)=0\}$;
$Poor_+(L,M)=\{\varphi\in S^2(L,M^L)\mid\varphi([L,L],L)=0\}$.

(ix) $Sym^2(L,M)=\{\varphi\in S^2(L,M)\mid x\bullet\varphi(y,z)=y\bullet\varphi(x,z)\}$;
$SB^2(L,M)=\{\varphi\in S^2(L,M)\mid\varphi(x,y)=x\bullet\psi(y)+y\bullet\psi(x)$ for $\psi\in Hom(L,M)\}$;
$SH^2(L,M)=(Sym^2(L,M)+SB^2(L,M))/SB^2(L,M)$.

(x) Define an action of L on $Hom(L^{\otimes 2}, M)$ via

$$z \circ \varphi(x,y) = z \bullet \varphi(x,y) + \varphi([x,z],y) + \varphi(x,[y,z]).$$

$\mathcal{S}^2(L,M) = \{\varphi \in S^2(L,M)^L \mid \varphi([x,y],z) + \curvearrowright = 0\}$.

(xi) $D(A,V) = \{\beta \in Hom(A,V) \mid \beta(abc) = a \bullet \beta(bc) - bc \bullet \beta(a) + \curvearrowright\}$.

(xii) $HC^1(A,V) = \{\alpha \in C^2(A,V) \mid \wp\alpha = 0\}$.

(xiii) $\mathcal{C}^2(A,V) = \{\alpha \in C^2(A,V) \mid \alpha(ac,b) - \alpha(bc,a) + a \bullet \alpha(b,c) - b \bullet \alpha(a,c) + 2c \bullet \alpha(a,b) = 0\}$.

(xiv) $\mathcal{P}_-(A,V) = \{\alpha \in C^2(A,V) \mid \alpha(ab,c) = a \bullet \alpha(b,c) + b \bullet \alpha(a,c)\}$;
$\mathcal{P}_+(A,V) = \{\alpha \in S^2(A,V) \mid \alpha(ab,c) = a \bullet \alpha(b,c) + b \bullet \alpha(a,c)\}$.

(xv) $\mathcal{A}(A,V) = \{\alpha \in S^2(A,V) \mid 2D\alpha = \wp\alpha\}$.

The spaces defined in (xi), (xv) are relevant in computation of $Ker\, d_{11}$ (Lemma 3.2), the spaces defined in (iii)–(viii), (xiv) are relevant in computation of $Ker\, d_{11} \cap Ker\, d_{12}$ (see (3.6)), the spaces defined in (ix) are relevant in computation for the particular case where L is abelian (Proposition 3.5), and the spaces defined in (x), (xii)–(xiii) are relevant in computation of the relative cohomology group $H^2(L \otimes A; L, M \otimes V)$ (Proposition 3.8).

Remarks.

(i) d (the Chevalley-Eilenberg differential) $= d^{[]} + d^{\bullet}$.

(ii) As $B^2(L, M^L) \subseteq Q^2(L,M)$, there is a surjection

$$H^2(L) \otimes M^L \to H^2_M(L).$$

(iii) If $V = K$, then $HC^1(A,V)$ is just the first-order cyclic cohomology $HC^1(A)$.

(iv) The following relations hold:

$$\begin{gathered} Poor_-(L,M) \subseteq \mathcal{B}(L,M) \subseteq Z^2(L,M), \\ \mathcal{B}(L,M) \cap Z^2(L,M^L) = Poor_-(L,M), \\ \mathcal{S}^2(L,M) \cap S^2(L,M^L)^L = Poor_+(L,M), \\ \mathcal{C}^2(A,V) \cap HC^1(A,V) = \mathcal{P}_-(A,V), \\ \mathcal{Z}^2(A,V) \cap \mathcal{A}(A,V) = \mathcal{P}_+(A,V), \\ Der(A,V) \subseteq D(A,V). \end{gathered}$$

Proposition 3.1.

$$(H^2)' \simeq H^2(L,M)\otimes V \oplus H^2_M(L)\otimes \frac{Hom(A,V)}{V\oplus Der(A,V)} \oplus \mathcal{H}(L,M)\otimes Der(A,V)$$
$$\oplus\, \mathcal{B}(L,M)\otimes \frac{Har^2(A,V)}{\mathcal{P}_+(A,V)} \oplus C^2(L,M)^L\otimes \mathcal{P}_+(A,V)$$
$$\oplus\, \mathcal{X}(L,M)\otimes \frac{\mathcal{A}(A,V)}{\mathcal{P}_+(A,V)} \oplus \mathcal{T}(L,M)\otimes \frac{D(A,V)}{Der(A,V)}$$
$$\oplus\, Poor_-(L,M)\otimes \frac{S^2(A,V)}{Hom(A,V)+D(A,V)+Har^2(A,V)+\mathcal{A}(A,V)}.$$

Each cocycle which lies in $C^2(L,M)\otimes S^2(A,V)$ is a linear combination of cocycles of the eight following types (which correspond to the respective direct summands in the isomorphism):

(i) *$x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes ab\bullet v$, where $\varphi\in Z^2(L,M)$ and $v\in V$;*
(ii) *$x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes \beta(ab)$, where $\varphi\in Z^2(L,M^L)$ and $\beta\in Hom(A,V)$;*
(iii) *as in (ii) with $\varphi\in \mathcal{K}(L,M)$ and $\beta\in Der(A,V)$;*
(iv) *$x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes \alpha(a,b)$, where $\varphi\in \mathcal{B}(L,M)$ and $\alpha\in \mathcal{Z}^2(A,V)$;*
(v) *as in (iv) with $\varphi\in C^2(L,M)^L$ and $\alpha\in \mathcal{P}_+(A,V)$;*
(vi) *as in (iv) with $\varphi\in \mathcal{X}(L,M)$ and $\alpha\in \mathcal{A}(A,V)$;*
(vii) *$x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes(3a\bullet\beta(b)+3b\bullet\beta(a)-2\beta(ab))$, where $\varphi\in \mathcal{H}(L,M)$ and $\beta\in D(A,V)$;*
(viii) *as in (iv) with $\varphi\in Poor_-(L,M)$ and $\alpha\in S^2(A,V)$.*

Proof. We have
$$(H^2)' = \frac{Ker\, d_{11}\cap Ker\, d_{12}}{Im\, d_1}. \tag{3.4}$$

We compute the relevant spaces in the subsequent series of lemmas.

Lemma 3.2.

$$\begin{aligned}Ker\, d_{11} = {} & Z^2(L,M)\otimes V\\ & + \{\varphi\in C^2(L,M)\,|\,2d^{[]}\varphi + d^\bullet\varphi = 0\}\otimes \mathcal{A}(A,V)\\ & + \{\varphi\in C^2(L,M)\,|\,3d^{[]}\varphi + 2d^\bullet\varphi = 0\}\otimes D(A,V)\\ & + \{\varphi\in C^2(L,M)\,|\,d^{[]} = d^\bullet\varphi = 0\}\otimes S^2(A,V).\end{aligned}$$

Proof. Substituting $a_1=a_2=a_3=1$ into the equation $d_{11}\Phi=0$ (as usual, $\Phi=\sum_{i\in I}\varphi_i\otimes\alpha_i$), one derives the equality

$$\sum_{i\in I} d\varphi_i(x_1,x_2,x_3)\otimes\alpha_i(1,1)=0 \tag{3.5}$$

and a decomposition $I=I_1\cup I_2$ with $d\varphi_i=0$ for $i\in I_1$ and $\alpha_i(1,1)=0$ for $i\in I_2$.

Substituting then $a_2 = a_3 = 1$ into the same equation, one gets

$$\sum_{i\in I}(3d^{[]}\varphi_i + 2d^{\bullet}\varphi_i)\otimes(\alpha_i(1,a_1) - a_1\bullet\alpha_i(1,1)) = 0$$

and by Lemma 1.1 there is a decomposition $I = I_{11}\cup I_{12}\cup I_{21}\cup I_{22}$ with

$$\begin{array}{lll} d\varphi_i = 0, & 3d^{[]}\varphi_i + 2d^{\bullet}\varphi_i = 0 & \text{for any } i\in I_{11} \\ d\varphi_i = 0, & \alpha_i(1,a) = \alpha_i(1,1) & \text{for any } i\in I_{12} \\ 3d^{[]}\varphi_i + 2d^{\bullet}\varphi_i = 0, & \alpha_i(1,1) = 0 & \text{for any } i\in I_{21} \\ \alpha_i(1,1) = 0, & \alpha_i(1,a) = a\bullet\alpha_i(1,1) & \text{for any } i\in I_{22}. \end{array}$$

Obviously $d\varphi_i = d^{\bullet}\varphi_i = 0$ for any $i\in I_{11}$, so components with $i\in I_{11}$ lie in $Ker\, d_{11}$, and $\alpha_i(1,a) = 0$ for any $i\in I_{22}$.

Further, substituting $a_3 = 1$ in our equation, we get

$$\sum_{i\in I}(2d^{[]}\varphi_i + d^{\bullet}\varphi_i)\otimes(\alpha_i(a_1,a_2) - 3a_1\bullet\alpha_i(1,a_2) - 3a_2\bullet\alpha_i(1,a_1) + 2\alpha_i(1,a_1a_2) + 3a_1a_2\bullet\alpha_i(1,1)) = 0.$$

In order to apply Lemma 1.1 again, we unite the sets I_{12} and I_{22} (with the common defining condition $\alpha_i(1,a) = a\bullet\alpha_i(1,1)$) and obtain a decomposition $I = I'_1\cup I'_2\cup I'_3\cup I'_4$ such that

$$\begin{array}{ll} 2d^{[]}\varphi_i + d^{\bullet}\varphi_i = 0,\ \alpha_i(1,a) = a\bullet\alpha_i(1,1) & \text{for any } i\in I'_1 \\ \alpha_i(a,b) = ab\bullet\alpha_i(1,1) & \text{for any } i\in I'_2 \\ d^{[]}\varphi_i = d^{\bullet}\varphi_i = 0,\ \alpha_i(1,1) = 0 & \text{for any } i\in I'_3 \\ 3d^{[]}\varphi_i + 2d^{\bullet}\varphi_i = 0,\ \alpha_i(a,b) = 3a\bullet\alpha_i(1,b) + 3b\bullet\alpha_i(1,a) - 2\alpha_i(1,ab) & \text{for any } i\in I'_4. \end{array}$$

Note that components $\varphi_i\otimes\alpha_i$ with $i\in I'_3$ are among those with $i\in I_{11}$ (and lie in $Ker\, d_{11}$).

Now, since the contribution of terms with $i\in I'_4$ to the left side of (3.5) vanishes, we may apply Lemma 1.1 again, and obtain a decomposition

$$I'_1\cup I'_2 = I'_{11}\cup I'_{12}\cup I'_{21}\cup I'_{22}$$

such that

$$\begin{array}{lll} 2d^{[]}\varphi_i + d^{\bullet}\varphi_i = 0, & \alpha_i(1,a) = 0 & \text{for any } i\in I'_{12} \\ d\varphi_i = 0, & \alpha_i(a,b) = ab\bullet\alpha_i(1,1) & \text{for any } i\in I'_{21}, \end{array}$$

and the two remaining types of components do not contribute to the whole picture: those with indices from I'_{11} satisfy $d^{[]}\varphi_i = d^{\bullet}\varphi_i = 0$, the case covered by previous cases, and those with indices from I'_{22} vanish, as

$\alpha_i(a,b) = ab \bullet \alpha_i(1,1) = 0$. Moreover, the components with indices from I'_{21} lie in $Ker\, d_{11}$.

The remaining part of the equation $d_{11}\Phi = 0$ now reads:

$$\sum_{i\in I'_{12}\cup I'_4} d^{[]}\varphi_i(x_1,x_2,x_3)\otimes(\wp\alpha(a_1,a_2,a_3) - 2D\alpha_i(a_2,a_3) + 3a_1a_2 \bullet \alpha_i(1,a_3) - a_3 \bullet \alpha_i(1,a_1a_2) + \curvearrowright) = 0.$$

Applying Lemma 1.1 again, and noting that the vanishing of the first tensor factor in each summand above yields the already considered case $d^{[]}\varphi_i = d^{\bullet}\varphi_i = 0$, we see that the second tensor factor vanishes for all $i \in I'_{12} \cup I'_4$.

Consequently, we obtain two types of components $\varphi_i \otimes \alpha_i$ lying in $Ker\, d_{11}$:

$$2d^{[]}\varphi_i + d^{\bullet}\varphi_i = 0;\ \wp\alpha_i = 2D\alpha_i$$

and

$$3d^{[]}\varphi_i + 2d^{\bullet}\varphi_i = 0;\ \wp\alpha_i = \frac{3}{2}D\alpha_i;\ \alpha_i \text{ satisfies the defining condition for } i \in I'_4.$$

The last two conditions imposed on α_i imply $\alpha_i(1,\cdot) \in D(A,V)$.

Summarizing all this, we obtain the statement of the Lemma. □

Lemma 3.3.

$$\begin{aligned} &Ker\, d_{12} \\ &= C^2(L,M)\otimes V + \{\varphi \in C^2(L,M) \mid x\bullet\varphi(y,z) = z\bullet\varphi(x,y)\}\otimes Hom(A,V) \\ &+ \{\varphi \in C^2(L,M) \mid \varphi([x,y],z) - \varphi([y,z],x) - x\bullet\varphi(y,z) + z\bullet\varphi(x,y) = 0\} \\ &\otimes \mathcal{Z}^2(A,V) \\ &+ \{\varphi \in C^2(L,M) \mid x\bullet\varphi(y,z) = z\bullet\varphi(x,y); \varphi([x,y],z) = \varphi([y,z],x)\} \\ &\otimes S^2(A,V). \end{aligned}$$

Proof. Let $\Phi = \sum_{i\in I} \varphi_i \otimes \alpha_i \in Ker\, d_{12}$. Substituting $a_2 = 1$ in the equation $d_{12}\Phi = 0$, one gets:

$$\sum_{i\in I}(-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi(x_1,x_3) + 2x_3 \bullet \varphi_i(x_1,x_2)) \otimes (a_1 \bullet \alpha_i(1,a_3) - a_3 \bullet \alpha_i(1,a_1)) = 0.$$

Hence we have a decomposition $I = I_1 \cup I_2$ such that, for $i \in I_1$, the first tensor factor in each summand above vanishes, and, for $i \in I_2$, the second one vanishes. Elementary transformations show that

$$\begin{aligned} x\bullet\varphi_i(y,z) &= z\bullet\varphi_i(x,y) \quad \text{for any } i \in I_1 \\ \alpha_i(1,a) &= a\bullet\alpha_i(1,1) \quad \text{for any } i \in I_2. \end{aligned}$$

Then substituting $a_3 = 1$ into the same initial equation $d_{12}\Phi = 0$, one gets

$$\sum_{i\in I}(2\varphi_i([x_1,x_2],x_3) + \varphi_i([x_1,x_3],x_2) - \varphi_i([x_2,x_3],x_1) \\ - x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3) + 2x_3 \bullet \varphi_i(x_1,x_2)) \\ \otimes (\alpha_i(1,a_1a_2) - \alpha_i(a_1,a_2)) = 0.$$

Applying Lemma 1.1 and the fact that the vanishing of the first tensor factor here is equivalent to the condition

$$\varphi_i([x,y],z) - \varphi_i([y,z],x) - x \bullet \varphi_i(y,z) + z \bullet \varphi_i(x,y) = 0,$$

we get a decomposition $I = I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$ such that

$$\begin{array}{lll} x \bullet \varphi_i(y,z) = z \bullet \varphi_i(x,y), & \varphi_i([x,y],z) = \varphi_i([y,z],x) & \text{for any } i \in I_{11} \\ x \bullet \varphi_i(y,z) = z \bullet \varphi_i(x,y), & \alpha_i(a,b) = \alpha_i(1,ab) & \text{for any } i \in I_{12} \\ \multicolumn{2}{l}{\varphi_i([x,y],z) - \varphi_i([y,z],x) - x \bullet \varphi_i(y,z) + z \bullet \varphi_i(x,y) = 0,} & \\ & \alpha_i(1,a) = a \bullet \alpha_i(1,1) & \text{for any } i \in I_{21} \\ \alpha_i(a,b) = ab \bullet \alpha_i(1,1) & & \text{for any } i \in I_{22}. \end{array}$$

It is easy to see that the components $\varphi_i \otimes \alpha_i$ with indices belonging to I_{11}, I_{12} and I_{22}, already lie in $Ker\, d_{12}$.

The remaining part of the equation $d_{12}\Phi = 0$ becomes:

$$\sum_{i\in I_{21}} (2\varphi_i([x_1,x_2],x_3) + \varphi_i([x_1,x_3],x_2) - \varphi_i([x_2,x_3],x_1)) \otimes \delta\alpha_i(a_1,a_2,a_3) = 0,$$

where δ is Harrison(=Hochschild) differential. Thus there is a decomposition $I_{21} = I'_1 \cup I'_2$, where φ_i for $i \in I'_1$ satisfies the same relations as for $i \in I_{11}$, and $\alpha_i \in \mathcal{Z}^2(A,V)$ for any $i \in I'_2$.

Putting all these computations together yields the formula desired (the four summands there correspond to the defining conditions for I_{22}, I_{12}, I'_2 and I_{11}, respectively; the sum, in general, is not direct). □

Elementary but tedious transformations of expressions entering in defining conditions of summands of $Ker\, d_{11}$ and $Ker\, d_{12}$, allow us to write their intersection as the following direct sum:

$$\begin{aligned}
Ker\,d_{11} \cap Ker\,d_{12} \simeq Z^2(L,M) \otimes V \,\oplus\, Z^2(L,M^L) \otimes \frac{Hom(A,V)}{V \oplus Der(A,V)} \\
\oplus\, \mathcal{K}(L,M) \otimes Der(A,V) \,\oplus\, \mathcal{B}(L,M) \otimes \frac{\mathcal{Z}^2(A,V)}{\mathcal{P}_+(A,V)} \\
\oplus\, C^2(L,M)^L \otimes \mathcal{P}_+(A,V) \,\oplus\, \mathcal{X}(L,M) \otimes \frac{\mathcal{A}(A,V)}{\mathcal{P}_+(A,V)} \\
\oplus\, \mathcal{T}(L,M) \otimes \frac{D(A,V)}{Der(A,V)} \\
\oplus\, Poor_-(L,M) \otimes \frac{S^2(A,V)}{Hom(A,V) + D(A,V) + \mathcal{Z}^2(A,V) + \mathcal{A}(A,V)}.
\end{aligned} \tag{3.6}$$

According to (3.4), to compute $(H^2)'$, we must consider the equation

$$\Phi = d_1 \Psi, \quad \text{where } \Phi \in Ker\,d_{11} \cap Ker\,d_{12} \text{ and } \Psi \in Hom(L \otimes A, M \otimes V),$$

which is equivalent to elucidation of all possible cohomological dependencies between the obtained classes of cocycles.

Lemma 3.4. *Let:*
$\{\varphi_i\}$ be cohomologically independent cocycles in $Z^2(L,M)$,
$\{\theta_i\}$ be cocycles in $Z^2(L,M^L)$ independent modulo $Q^2(L,M)$,
$\{\kappa_i\}$ be elements of $\mathcal{K}(L,M)$ independent modulo $\mathcal{T}(L,M)$,
$\{\varepsilon_i\}$ be linearly independent cocycles in $\mathcal{B}(L,M)$,
$\{\rho_i\}$ be linearly independent elements in $C^2(L,M)^L$,
$\{\chi_i\}$ be linearly independent elements in $\mathcal{X}(L,M)$,
$\{\tau_i\}$ be linearly independent elements in $\mathcal{T}(L,M)$,
$\{\xi_i\}$ be linearly independent cocycles in $Poor_-(L,M)$,
$\{v_j\}$ be linearly independent elements in V,
$\{\delta_j\}$ be linearly independent derivations in $Der(A,V)$,
$\{\beta_j\}$ be maps in $D(A,V)$ independent modulo $Der(A,V)$,
$\{\gamma_j\}$ be maps in $Hom(A,V)$ independent both modulo $Der(A,V)$ and modulo maps $a \mapsto a \bullet v$ for all $v \in V$,
$\{F_j\}$ be cocycles in $\mathcal{Z}^2(A,V)$ independent both cohomologically and modulo $\mathcal{P}_+(A,V)$,
$\{P_j\}$ be linearly independent elements in $\mathcal{P}_+(A,V)$,
$\{A_j\}$ be elements in $\mathcal{A}(A,V)$ independent modulo $\mathcal{P}_+(A,V)$,
$\{G_j\}$ be maps in $S^2(A,V)$ independent simultaneously modulo:
maps $a \wedge b \mapsto \gamma(ab)$ for all $\gamma \in Hom(A,V)$,
maps $a \wedge b \mapsto 3a \bullet \beta(b) + 3b \bullet \beta(a) - 2\beta(ab)$ for all $\beta \in D(A,V)$, and $\mathcal{Z}^2(A,V) + \mathcal{A}(A,V)$.

Then the elements of $Ker\,d_{11} \cap Ker\,d_{12}$:

$$\varphi_i \otimes (R_{v_j} \circ m),\ \theta_i \otimes (\gamma_j \circ m),\ \kappa_i \otimes (\delta_j \circ m),\ \varepsilon_i \otimes F_j,\ \rho_i \otimes P_j,\ \chi_i \otimes A_j,$$
$$\tau_i \otimes (3\delta\beta_j - \beta_j \circ m),\ \xi_i \otimes G_j$$

(m stands for multiplication in A and R_v is an element in $Hom(A,V)$ defined by $a \mapsto a \bullet v$), are independent modulo $Im\,d_1$.

Proof. We must prove that if

$$\begin{aligned}
&\sum \varphi_i(x,y) \otimes ab \bullet v_j + \sum \theta_i(x,y) \otimes \gamma_j(ab) + \sum \kappa_i(x,y) \otimes \delta_j(ab) \\
&+ \sum \varepsilon_i(x,y) \otimes F_j(a,b) + \sum \rho_i(x,y) \otimes P_j(a,b) + \sum \chi_i(x,y) \otimes A_j(a,b) \\
&+ \sum \tau_i(x,y) \otimes (3a \bullet \beta_j(b) + 3b \bullet \beta_j(a) - 2\beta_j(ab)) + \sum \xi_i(x,y) \otimes G_j(a,b) \\
&= \sum_{i \in I} \left(\psi_i([x,y]) \otimes \alpha_i(ab) + \frac{1}{2}(-x \bullet \psi_i(y) + y \bullet \psi_i(x)) \otimes (a \bullet \alpha_i(b) + b \bullet \alpha_i(a)) \right)
\end{aligned} \tag{3.7}$$

for some $\sum\limits_{i \in I} \psi_i \otimes \alpha_i \in Hom(L,M) \otimes Hom(A,V)$ (the right side here is the generic element in $Im\,d_1$), then all terms in the left side vanish.

One has $\delta_j(1) = \beta_j(1) = P_j(1,a) = A_j(1,a) = 0$ and one may assume that $\gamma_j(1) = F_j(1,a) = G_j(1,a) = 0$. Substitute $a = b = 1$ in (3.7):

$$\sum \varphi_i(x,y) \otimes v_j = \sum_{i \in I} d\psi_i(x,y) \otimes \alpha_i(1).$$

As φ_i's are cohomologically independent and v_j's are linearly independent, the last equality implies that all summands $\varphi_i(x,y) \otimes v_j$ vanish and there is a decomposition $I = I_1 \cup I_2$ with $d\psi_i = 0$ for $i \in I_1$ and $\alpha_i(1) = 0$ for $i \in I_2$.

Now substitute $b = 1$ in (3.7):

$$\begin{aligned}
&\sum \theta_i(x,y) \otimes \gamma_j(a) + \sum \kappa_i(x,y) \otimes \delta_j(a) + \sum \tau_i(x,y) \otimes \beta_j(a) \\
&\quad = \sum_{i \in I} (\psi_i([x,y]) + \frac{1}{2}(-x \bullet \psi_i(y) + y \bullet \psi_i(x))) \otimes (\alpha_i(a) - a \bullet \alpha_i(1)).
\end{aligned} \tag{3.8}$$

Substituting (3.8) in (3.7), one gets:

$$\begin{aligned}
&\sum \varepsilon_i(x,y) \otimes F_j(a,b) + \sum \rho_i(x,y) \otimes P_j(a,b) + \sum \chi_i(x,y) \otimes A_j(a,b) \\
&\qquad + 3\sum \tau_i(x,y) \otimes \delta\beta_j(a,b) + \sum \xi_i(x,y) \otimes G_j(a,b) \\
&\qquad\qquad = \frac{1}{2} \sum (-x \bullet \psi_i(y) + y \bullet \psi_i(x)) \otimes (\delta\alpha_i(a,b) - ab \bullet \alpha_i(1)).
\end{aligned}$$

The independence conditions of Lemma imply that all summands in the left side vanish and, due to Lemma 1.1, for $\sum\limits_{i \in I} \psi_i \otimes \beta_i$, there exists a decomposition

$I = I_{11} \cup I_{12} \cup I_{21} \cup I_{22}$ with

$$\begin{array}{lll}
d\psi_i = 0, & x \bullet \psi_i(y) = y \bullet \psi_i(x) & \text{for any } i \in I_{11} \\
d\psi_i = 0, & \delta\alpha_i(a,b) = ab \bullet \alpha_i(1) & \text{for any } i \in I_{12} \\
x \bullet \psi_i(y) = y \bullet \psi_i(x), & \alpha_i(1) = 0 & \text{for any } i \in I_{21} \\
\alpha_i(1) = 0, & \delta\alpha_i(a,b) = ab \bullet \alpha_i(1) & \text{for any } i \in I_{22}.
\end{array}$$

Denoting $\alpha_i'(a) = \alpha_i(a) - a \bullet \alpha_i(1)$ for $i \in I_{12}$, we get $\alpha_i' \in Der(A,V)$.

Substituting all this information back into (3.8), one finally obtains

$$\begin{aligned}
&\sum \theta_i(x,y) \otimes \gamma_j(a) + \sum \kappa_i(x,y) \otimes \delta_j(a) \\
&\quad = \frac{1}{2} \sum_{i \in I_{12}} \psi_i([x,y]) \otimes (\alpha_i(a) - a \bullet \alpha_i(1)) + \sum_{i \in I_{21}} \psi_i([x,y]) \otimes \alpha_i(a) \\
&\qquad + \sum_{i \in I_{22}} \left(\psi_i([x,y]) + \frac{1}{2}(-x \bullet \psi_i(y) + y \bullet \psi_i(x)) \right) \otimes \alpha_i(a).
\end{aligned}$$

The independence conditions of Lemma imply that all terms appearing in the last equality vanish, and the desired assertion follows. □

Conclusion of the proof of Proposition 3.1.

Lemma 3.3 implies that

$$\begin{aligned}
Im\, d_1 \simeq\; & B^2(L,M) \otimes V \,\oplus\, (Q^2(L,M) \cap Z^2(L,M^L)) \otimes \frac{Hom(A,V)}{V \oplus Der(A,V)} \\
& \oplus\, (\mathcal{T}(L,M) \cap \mathcal{K}(L,M)) \otimes Der(A,V) \,\oplus\, \mathcal{B}(L,M) \otimes Der(A,V)
\end{aligned}$$

which together with (3.6) entails the asserted isomorphism. □

Now we turn to computation of the second summand in (3.3), $(H^2)''$.

We are unable to compute it in general (and are in doubt about the existence of a closed general formula for $(H^2)''$) and confine ourselves to two particular cases (in both of them, it turns out that $(H^2)''$ coincides with the space of classes of cocycles lying in $S^2(L,M) \otimes C^2(A,V)$).

Proposition 3.5. *Suppose L is abelian. Then*

$$\begin{aligned}
(H^2)'' \simeq\; & S^2(L,M^L) \otimes \frac{C^2(A,V)}{\{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in Hom(A,V)\}} \\
& \oplus SH^2(L,M) \otimes \{a \bullet \beta(b) - b \bullet \beta(a) \mid \beta \in Hom(A,V)\}.
\end{aligned}$$

First, we establish a lemma valid in the general situation (where L is not necessarily abelian).

Lemma 3.6.

(i) $Ker\, d_{23} = Ker\, d^{\bullet} \otimes C^2(A,V)$
$+ S^2(L,M) \otimes \{\alpha \in C^2(A,V) \mid \alpha(a,b) = a \bullet \beta(b) - b \bullet \beta(a)\}$;
(ii) $Im\, d_2 = \{\varphi \in S^2(L,M) \mid \varphi(x,y) = x \bullet \psi(y) + y \bullet \psi(x)\}$
$\otimes \{\alpha \in C^2(A,V) \mid \alpha(a,b) = a \bullet \beta(b) - b \bullet \beta(a)\}$.

Proof. The only thing which perhaps needs a proof here is the equality

$$Ker\, D = \{\alpha \in C^2(A,V) \mid \alpha(a,b) = a \bullet \beta(b) - b \bullet \beta(a)\}.$$

The validity of it is verified by appropriate substitution of 1's. □

Proof of Proposition 3.5. Let $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i \in Ker\, d_{23}$, with a decomposition on the set of indices $I = I_1 \cup I_2$ such that

$$\begin{aligned} d^{\bullet} \varphi_i(x_1,x_2,x_3) &= 0 \quad \text{for any } i \in I_1 \\ \alpha_i(a,b) &= -a \bullet \beta_i(b) + b \bullet \beta_i(a) \quad \text{for any } i \in I_2. \end{aligned}$$

By Lemma 3.6(i), we may also assume that elements α_i, where $i \in I_1$, are independent modulo $\{a \bullet \beta(b) - b \bullet \beta(a)\}$, and hence $\alpha_i(1,a) = 0$ for each $i \in I_1$.

Suppose there is

$$\Psi = \sum_{i \in I'} \varphi_i' \otimes \alpha_i' \in C^2(L,M) \otimes S^2(A,V) \tag{3.9}$$

such that the class of $\Phi - \Psi$ belongs to $(H^2)''$. This, in particular, means that $d_{22}\Phi = d_{12}\Psi$:

$$\begin{aligned} &\sum_{i \in I} (-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3)) \\ &\qquad \otimes (a_1 \bullet \alpha_i(a_2,a_3) + a_3 \bullet \alpha_i(a_1,a_2) + 2a_2 \bullet \alpha_i(a_1,a_3)) \\ &= \sum_{i \in I'} (-x_1 \bullet \varphi_i'(x_2,x_3) + x_2 \bullet \varphi_i'(x_1,x_3) + 2x_3 \bullet \varphi_i'(x_1,x_2)) \\ &\qquad \otimes (a_1 \bullet \alpha_i'(a_2,a_3) - a_3 \bullet \alpha_i'(a_1,a_2)). \end{aligned}$$

Substituting here $a_2 = 1$, one gets

$$\begin{aligned} &2 \sum_{i \in I_1} (-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3)) \otimes \alpha_i(a_1,a_3) \\ &\qquad + 3 \sum_{i \in I_1} (-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3)) \otimes \alpha_i(a_1,a_3) \\ &= \sum_{i \in I'} (-x_1 \bullet \varphi_i'(x_2,x_3) + x_2 \bullet \varphi_i'(x_1,x_3) + 2x_3 \bullet \varphi_i'(x_1,x_2)) \\ &\qquad \otimes (a_1 \bullet \alpha_i'(1,a_3) - a_3 \bullet \alpha_i'(1,a_1)). \end{aligned}$$

Hence, due to the independence condition imposed on α_i for $i \in I_1$,

$$-x_1 \bullet \varphi_i(x_2,x_3) + x_2 \bullet \varphi_i(x_1,x_3) = 0, \quad i \in I_1. \tag{3.10}$$

This, together with condition $\varphi_i \in Ker d^\bullet$, evidently implies $\varphi_i(L,L) \subseteq M^L$ for each $i \in I_1$. Note that the terms from $S^2(L,M^L) \otimes C^2(A,V)$ lie in $Z^2(L\otimes A, M\otimes V)$.

Now, for elements of $S^2(L,M) \otimes \{a \bullet \beta(b) - b \bullet \beta(a)\}$, write the cocycle equation:

$$\sum_{i\in I_2} (x_1 \bullet \varphi_i(x_2,x_3) \otimes (a_1a_2 \bullet \beta_i(a_3) - a_1a_3 \bullet \beta_i(a_2)) \\ + x_2 \bullet \varphi_i(x_1,x_3) \otimes (-a_1a_2 \bullet \beta_i(a_3) + a_2a_3 \bullet \beta_i(a_1)) \\ + x_3 \bullet \varphi_i(x_1,x_2) \otimes (a_1a_3 \bullet \beta_i(a_2) - a_2a_3 \bullet \beta_i(a_1))) = 0.$$

Substituting $a_2 = a_3 = 1$, we get

$$\sum_{i\in I_2} (x_2 \bullet \varphi_i(x_1,x_3) - x_3 \bullet \varphi_i(x_1,x_2)) \otimes (\beta_i(a_1) - a_1 \bullet \beta_i(1)) = 0.$$

As the vanishing of the second tensor factor $\beta_i(a_1) - a_1 \bullet \beta_i(1)$ in the l;ast equality leads to the vanishing of the whole α_i, we see that the condition (3.10) holds also in this case, i.e., for all $i \in I_2$. Conversely, if (3.10) holds, then the cocycle equation is satisfied. Thus the space of cocycles in $Z^2(L\otimes A, M\otimes V)$ whose cohomology classes lie in $(H^2)''$, coincides with

$$S^2(L,M^L) \otimes \frac{C^2(A,V)}{\{a\bullet\beta(b) - b\bullet\beta(a) \mid \beta \in Hom(A,V)\}} \\ \oplus \frac{Sym^2(L,M) + SB^2(L,M)}{SB^2(L,M)} \otimes \{a\bullet\beta(b) - b\bullet\beta(a) \mid \beta \in Hom(A,V)\}$$

(note that we can always take $\Psi = 0$ in (3.9)).

To conclude the proof, one can observe that all these cocycles are cohomologically independent. This is proved in a pretty standard way, as in Lemma 3.4. □

Summarizing Proposition 3.1 (for the case where L is abelian) and Proposition 3.5, we obtain

Theorem 3.7. *Let L be an abelian Lie algebra. Then*

$$H^2(L\otimes A, M\otimes V) \simeq H^2(L,M)\otimes V \oplus \mathcal{H}(L,M)\otimes Der(A,V) \\ \oplus C^2(L,M^L) \otimes \frac{S^2(A,V)}{V \oplus Der(A,V)} \\ \oplus S^2(L,M^L) \otimes \frac{C^2(A,V)}{\{a\bullet\beta(b) - b\bullet\beta(a) \mid \beta \in Hom(A,V)\}} \\ \oplus SH^2(L,M) \otimes \{a\bullet\beta(b) - b\bullet\beta(a) \mid \beta \in Hom(A,V)\}. \tag{3.10a}$$

Each cocycle in $Z^2(L\otimes A, M\otimes V)$ is a linear combination of cocycles of the following four types (which correspond, respectively, to the first, the sum of the second and the third, the fourth and the fifth summands in the isomorphism (3.10a)):

(i) $x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes ab\bullet v$ *for some* $\varphi\in Z^2(L,M)$ *and* $v\in V$*;*
(ii) $x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes\alpha(a,b)$ *for some* $\varphi\in C^2(L,M^L)$ *and* $\alpha\in S^2(A,V)$*;*
(iii) *as in (ii) with* $\varphi\in S^2(L,M^L)$ *and* $\alpha\in C^2(A,V)$*;*
(iv) $x\otimes a\wedge y\otimes b\mapsto \varphi(x,y)\otimes(a\bullet\beta(b)-b\bullet\beta(a))$ *for some* $\varphi\in Sym^2(L,M)$ *and* $\beta\in Hom(A,V)$.

Remark. Let $\mathcal{H}(L,M)$ consists of classes of cocycles taking values in M^L. It is easy to see that if L is abelian, then there is inclusion $\mathcal{H}(L,M)\subseteq H^2(L,M)$. Hence, singling out appropriate terms from the first three direct summands in the isomorphism (3.10a), we obtain $\mathcal{H}(L,M)\otimes S^2(A,V)$ as a direct summand of $H^2(L\otimes A,M\otimes V)$.

Now we want to perform another particular computation, namely, compute the relative cohomology $H^2(L\otimes A;L,M\otimes V)$.

We easily see that all constructions can be restricted to the relative complex

$$Hom(\wedge^\star(L\otimes A/K1),M\otimes V)$$

with a single (but greatly simplifying the matter) difference that all maps from $C^3(A,V)$, $Y^3(A,V)$ and $S^3(A,V)$ vanish whenever one of their arguments is 1.

Let $(H^2_L)'$ and $(H^2_L)''$ denote the corresponding components of $H^2(L\otimes A;L,M\otimes V)$.

Proposition 3.8.

$$(H^2_L)''\simeq S^2(L,M^L)^L\otimes HC^1(A,V)\,\oplus\,\mathcal{S}^2(L,M)\otimes\frac{\mathcal{C}^2(L,M)}{\mathcal{P}_-(A,V)}$$
$$\oplus\,\frac{S^2(L,M)^L}{S^2(L,M^L)^L}\otimes\mathcal{P}_-(A,V)\,\oplus\,Poor_+(L,M)\otimes\frac{C^2(A,V)}{HC^1(A,V)+\mathcal{C}^2(A,V)}$$

Proof. The proof goes along the same scheme as of Proposition 3.1. By Lemma 3.6(i), $Ker\,d_{23}=Ker\,d^\bullet\otimes C^2(A,V)$ (as the second tensor factor in the second component there vanishes in this case).

For $\Phi=\sum\varphi_i\otimes\alpha_i\in Ker\,d_{23}$ and $\Psi=\sum\varphi_i'\otimes\alpha_i'\in C^2(L,M)\otimes S^2(L,M)$, the condition $d_{22}\Phi=d_{12}\Psi$ reads:

$$\begin{aligned}
&\sum_{i\in I}(2\varphi_i([x_1,x_2],x_3)+\varphi_i([x_1,x_3],x_2)-\varphi_i([x_2,x_3],x_1))\\
&\quad\otimes(\alpha_i(a_1a_2,a_3)-\alpha_i(a_2a_3,a_1))+(-x_1\bullet\varphi_i(x_2,x_3)+x_2\bullet\varphi_i(x_1,x_3))\\
&\qquad\otimes(a_1\bullet\alpha_i(a_2,a_3)+a_3\bullet\alpha_i(a_1,a_2)+2a_2\bullet\alpha_i(a_1,a_3))\\
&\qquad=\sum_{i\in I'}(2\varphi_i'([x_1,x_2],x_3)+\varphi_i'([x_1,x_3],x_2)-\varphi_i'([x_2,x_3],x_1))\\
&\qquad\qquad\otimes(\alpha_i'(a_1a_2,a_3)-\alpha_i'(a_2a_3,a_1))\\
&\qquad+(-x_1\bullet\varphi_i'(x_2,x_3)+x_2\bullet\varphi_i'(x_1,x_3)+2x_3\bullet\varphi_i'(x_1,x_2))\\
&\qquad\qquad\otimes(a_1\bullet\alpha_i'(a_2,a_3)-a_2\bullet\alpha_i'(a_1,a_2)).
\end{aligned}$$

Substituting here $a_2 = 1$, we obtain (remember about vanishing of all α's if one of arguments is 1):

$$\sum_{i\in I}(2\varphi_i([x_1,x_2],x_3)+\varphi_i([x_1,x_3],x_2)-\varphi_i([x_2,x_3],x_1)$$
$$-x_1\bullet\varphi_i(x_2,x_3)+x_2\bullet\varphi_i(x_1,x_3))\otimes\alpha_i(a_1,a_3)=0.$$

This implies

$$2\varphi_i([x_1,x_2],x_3)+\varphi_i([x_1,x_3],x_2)-\varphi_i([x_2,x_3],x_1)$$
$$-x_1\bullet\varphi(x_2,x_3)+x_2\bullet\varphi_i(x_1,x_3)=0,\quad i\in I. \quad (3.11)$$

Since $\varphi \in Ker\, d^{\bullet}$,

$$x_1\bullet\varphi_i(x_2,x_3)+x_3\bullet\varphi_i(x_1,x_3)+x_3\bullet\varphi_i(x_1,x_2)=0,\quad i\in I. \quad (3.12)$$

With the help of elementary transformations, (3.11) and (3.12) yield

$$\varphi_i([x_1,x_3],x_2)+\varphi_i([x_2,x_3],x_1)+x_3\bullet\varphi_i(x_1,x_2)=0$$

or, in other words, $\varphi_i \in S^2(L,M)^L$ for each $i \in I$.

Now, writing the cocycle equation for $\sum_{i\in I}\varphi_i\otimes\alpha_i \in S^2(L,M)^L\otimes C^2(A,V)$, one gets

$$\sum_{i\in I}(\varphi_i([x_1,x_2],x_3)\otimes(\alpha_i(a_1a_2,a_3)-a_1\bullet\alpha_i(a_2,a_3)-a_2\bullet\alpha_i(a_1,a_3))$$
$$+\varphi_i([x_1,x_3],x_2)\otimes(-\alpha_i(a_1a_3,a_2)-a_1\bullet\alpha_i(a_2,a_3)+a_3\bullet\alpha_i(a_1,a_2))$$
$$+\varphi_i([x_2,x_3],x_1)\otimes(\alpha_i(a_2a_3,a_1)+a_2\bullet\alpha_i(a_1,a_3)+a_3\bullet\alpha_i(a_1,a_2)))=0.$$

Antisymmetrize this expression with respect to a_1, a_2:

$$\sum_{i\in I}(\varphi_i([x_1,x_3],x_2)+\varphi_i([x_2,x_3],x_1))\otimes(-\alpha_i(a_1a_3,a_2)+\alpha_i(a_2a_3,a_1)$$
$$-a_1\bullet\alpha_i(a_2,a_3)+a_2\bullet\alpha_i(a_1,a_3)+2a_3\bullet\alpha_i(a_1,a_2))=0.$$

Consequently, we have a decomposition $I = I_1 \cup I_2$ with

$$\varphi_i([x_1,x_3],x_2)+\varphi_i([x_2,x_3],x_1)=0,\quad i\in I_1 \quad (3.13)$$
$$\alpha_i\in\mathcal{C}^2(A,V),\quad i\in I_2. \quad (3.14)$$

Note that (3.13) together with condition $\varphi_i \in S^2(L,M)^L$ implies $\varphi_i(L,L) \subseteq M^L$ for any $i \in I_1$. Applying the symmetrizer $e-(13)+(123)$ to the condition (3.14), we get

$$a_1\bullet\alpha_i(a_2,a_3)+a_2\bullet\alpha_i(a_1,a_3)$$
$$=\frac{1}{3}(2\alpha_i(a_1a_2,a_3)-\alpha_i(a_2a_3,a_1)-\alpha_i(a_1a_3,a_2)),\quad i\in I_2. \quad (3.15)$$

Taking into account (3.13)–(3.15), the cocycle equation can be rewritten as

$$\sum_{i\in I}(\varphi_i([x_1,x_2],x_3)-\varphi_i([x_1,x_3],x_2)+\varphi_i([x_2,x_3],x_1)) \\ \otimes(\alpha_i(a_1a_2,a_3)+\alpha_i(a_1a_3,a_2)+\alpha_i(a_2a_3,a_1))=0.$$

By Lemma 1.1, there is a decomposition $I=I_{11}\cup I_{12}\cup I_{21}\cup I_{22}$ such that

$$\begin{array}{lll} \varphi_i([x,y],z)=\varphi_i(x,[y,z]), & \varphi_i([x,y],z)+\curvearrowright=0 & \text{for any } i\in I_{11} \\ \varphi_i([x,y],z)=\varphi_i(x,[y,z]), & \alpha_i\in HC^1(A,V) & \text{for any } i\in I_{12} \\ \varphi_i([x,y],z)+\curvearrowright=0, & \alpha_i\in \mathcal{C}^2(A,V) & \text{for any } i\in I_{21} \\ \alpha_i\in\mathcal{C}^2(A,V)\cap HC^1(A,V) & & \text{for any } i\in I_{22}. \end{array}$$

Evidently, $\varphi_i([L,L],L)=0$ for any $i\in I_{11}$ and $\alpha_i\in\mathcal{P}_-(A,V)$ for any $i\in I_{22}$. All these four types of components are cocycles in $Z^2(L\otimes A,M\otimes V)$.

Therefore, the space of cocycles whose cohomology classes lie in $(H^2_L)''$ is as follows:

$$^LZ^{02}\simeq S^2(L,M^L)^L\otimes HC^1(A,V)+\mathcal{S}^2(L,M)\otimes\mathcal{C}^2(A,V) \\ +S^2(L,M)^L\otimes\mathcal{P}_-(A,V)+Poor_+(L,M)\otimes C^2(A,V).$$

(the four summands here correspond to the components indexed by I_{12},I_{21},I_{22} and I_{11} respectively; note that, in this case, we may let $\Psi=0$ again).

Rewriting this as a direct sum, we get:

$$S^2(L,M^L)^L\otimes HC^1(A,V)\oplus\mathcal{S}^2(L,M)\otimes\frac{\mathcal{C}^2(A,V)}{\mathcal{P}_-(A,V)} \\ \oplus\frac{S^2(L,M)^L}{S^2(L,M^L)^L}\otimes\mathcal{P}_-(A,V)\oplus Poor_+(L,M)\otimes\frac{C^2(A,V)}{HC^1(A,V)+\mathcal{C}^2(A,V)}.$$

And finally, one may show in the same fashion as previously, that all these cocycles are cohomologically independent, and the assertion of the Proposition follows. □

Summarizing Propositions 3.1 and 3.8, we obtain:

$$H^2(L\otimes A;L,M\otimes V)\simeq(H^2_L)'\oplus(H^2_L)''$$

where

$$(H^2_L)'\simeq\mathcal{B}(L,M)\otimes\frac{Har^2(A,V)}{\mathcal{P}_+(A,V)}\bigoplus C^2(L,M)^L\otimes\mathcal{P}_+(A,V) \\ \bigoplus\mathcal{X}(L,M)\otimes\frac{\mathcal{A}(A,V)}{\mathcal{P}_+(A,V)} \\ \bigoplus Poor_-(L,M)\otimes\frac{S^2(A,V)}{Hom(A,V)+D(A,V)+Har^2(A,V)+\mathcal{A}(A,V)}$$

and $(H^2_L)''$ is described by Proposition 3.8.

We conclude this section with enumeration (for the case of generic L) of all possible cocycles of rank 1, i.e. those which can be written in the form

$$\varphi \otimes \alpha \in Hom(L^{\otimes 2}, M) \otimes Hom(A^{\otimes 2}, V).$$

In view of (3.3), Propositions 3.1 and 3.8, it suffices to consider cocycles of rank 1 whose cohomology classes lie in $(H^2)''$ and which are independent modulo $(H^2_L)''$. Let us denote this space of cocycles by Z''.

Proposition 3.9. *Each element of Z'' is cohomologic to the sum of cocycles of the following two types:*

(i) $x \otimes a \wedge y \otimes b \mapsto \varphi(x,y) \otimes (a \bullet \beta(b) - b \bullet \beta(a))$*, where* $\varphi \in Sym^2(L,M)$ *is such that* $\varphi([L,L],L) = 0$*, and* $\beta \in Hom(A,V)$*;*

(ii) *as in (i) with* $\varphi \in S^2(L,M)$*, where* $2\varphi([x,y],z) = x \bullet \varphi(y,z) - y \bullet \varphi(x,z)$*, and* $\beta \in Der(A,V)$*.*

Proof. Mainly repetition of arguments of Proposition 3.1, Lemma 3.2, Lemma 3.3, Proposition 3.5, Propositiion 3.8. □

Therefore, there are, in general, 13 types of cohomologically independent cocycles of rank 1 (7 coming from Proposition 3.1 + 4 coming from Proposition 3.8 + 2 coming from Proposition 3.9). Of course, in particular cases some of these cocycles may vanish.

We see that, for $H^2(L \otimes A; L, M \otimes V)$ and for $H^2(L \otimes A, M \otimes V)$, L abelian, it is possible (in both cases) to choose a basis consisting of rank 1 cocycles. In general this is, however, not true. The case of $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$, where $W_1(n)$ is the Zassenhaus algebra of positive characteristic, treated in [Z], shows that there are cocycles of rank 2 not cohomologic to (any sum of) cocycles of rank 1.

4 A sketch of a spectral sequence

The computations performed in preceding sections can be described (and generalized) in terms of a certain spectral sequence. Let us indicate briefly the main idea (hopefully, the full treatment with further applications will appear elsewhere).

One has a *Cauchy formula*

$$\wedge^n (L \otimes A) \simeq \bigoplus_{\lambda \vdash n} Y_\lambda(L) \otimes Y_{\lambda^\sim}(A),$$

where Y_λ is the Schur functor associated with the Young diagram λ, and $\lambda^\sim$ is the Young diagram obtained from λ by interchanging its rows and columns (see, e.g., [F, p. 121]).

Applying the functor $Hom(\cdot, M\otimes V) \simeq Hom(\cdot, M) \otimes Hom(\cdot, V)$ to both sides of this isomorphism one gets a decomposition of the underlying modules in the Chevalley–Eilenberg complex:

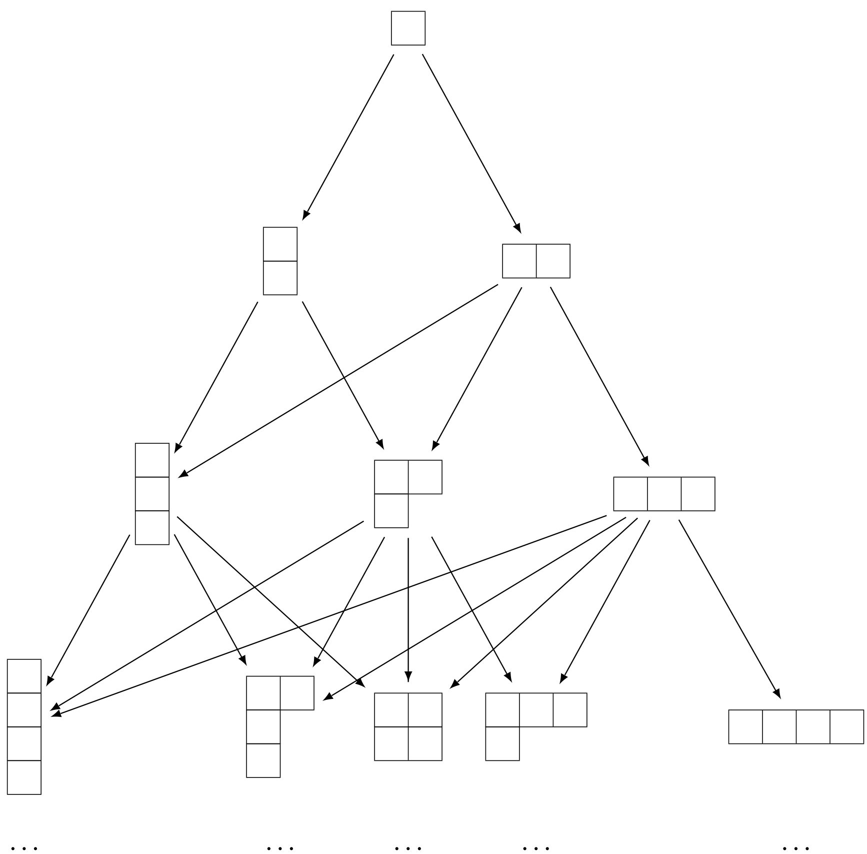
$$C^n(L\otimes A, M\otimes V) \simeq \sum_{\lambda\vdash n} C_\lambda(L,M)\otimes C_{\lambda^\sim}(A,V), \tag{4.1}$$

where $C_\lambda(U,W) = Hom(Y_\lambda(U),W)$. The two extreme terms here are $C^n(L,M)\otimes S^n(A,V)$ and $S^n(L,M)\otimes C^n(A,V)$.

So each differential $d : C^n(L\otimes A, M\otimes V) \to C^{n+1}(L\otimes A, M\otimes V)$ in the Chevalley-Eilenberg complex decomposes according to (4.1) into components

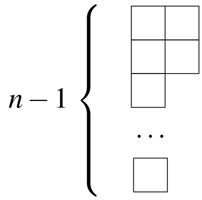
$$d_\lambda^{\lambda'} : C_{\lambda'}(L,M)\otimes C_{\lambda'^\sim}(A,V) \to C_\lambda(L,M)\otimes C_{\lambda^\sim}(A,V)$$

for each pair $\lambda' \vdash n$ and $\lambda \vdash (n+1)$. Therefore the following graph of all Young diagrams



may be interpreted in the following way: each Young diagram λ of size n designates a module $C_\lambda(L,M)\otimes C_{\lambda^\sim}(A,V)$ and the arrow from λ' to λ represents $d_\lambda^{\lambda'}$.

One can prove that nonzero arrows $d_{\lambda}^{\lambda'}$ are exactly the following: all arrows going "from right to left" and those going "from left to right" for which either λ' is a column of height n and λ is a diagram of size $n+1$ and of the following shape:



or λ' is included in λ.

Using this, we can define a decreasing nonnegative filtration $F^k C^\star$ on the complex $(C^\star(L \otimes A, M \otimes V), d)$ as the sum of all terms $C_\lambda(L,M) \otimes C_{\lambda^\sim}(A,V)$ with λ belonging to a "closure" under nonzero arrows of a single column of height $k+1$.

Now we may consider a (first quadrant) spectral sequence $\{E_r^{\star\star}, d_r\}$ associated with this filtration. Since the filtration is finite in each degree, the spectral sequence converges to the desired cohomology group $H^\star(L \otimes A, M \otimes V)$.

Then $E_\infty^{20} = 0$ and $(H^2)'$ and $(H^2)''$ from §3 are just E_∞^{11} and E_∞^{02}, respectively.

5 Structure functions

In this section we show how the result from §3 may be applied to the geometric problem of calculation of structure functions on manifolds of loops with values in compact Hermitian symmetric spaces.

Recall that the base field in this section is $\mathbb{C}$, what is stipulated by a geometric nature of the question considered. However, all algebraic considerations remain true over any field of characteristic 0.

Let us briefly recall the necessary notions and results. Let M be a complex manifold endowed with a G-structure (so G is a complex Lie group). *Structure functions* are sections of certain vector bundles over M. Their importance stems from the fact that they constitute the complete set of obstructions to integrability (= possibility of local flattening) of a given G-structure. In the case $G = O(n)$ structure functions are known under the, perhaps, more common name *Riemann tensors* (and constitute one of the main objects of study in the Riemannian geometry).

A remarkable fact is that structure functions admit a purely algebraic description. Starting with $\mathfrak{g}_{-1} = T_m(M)$, the tangent space at a point $m \in M$, and $\mathfrak{g}_0 = Lie(G)$, one may construct, via apparatus of Cartan prolongations, a graded Lie algebra $\mathfrak{g} = \bigoplus\limits_{i \geq -1} \mathfrak{g}_i$. Namely, for $i > 0$, we have:

$$\mathfrak{g}_i = \{X \in Hom(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) \,|\, (X(v))(w) = (X(w))(v) \text{ for all } v, w \in \mathfrak{g}_{-1}\}. \quad (5.1)$$

For any such graded Lie algebra, one may define the *Spencer cohomology groups* $H^{pq}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$. Then the space of structure functions of order k, i.e. obstructions to identification of the kth infinitesimal neighborhood of a point $m \in M$ with that of a point of the manifold with a flat G-structure, is isomorphic to the group $H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$. Note that since $H^2(\mathfrak{g}_{-1}, \mathfrak{g}) = \bigoplus_{k\geq 1} H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, to compute structure functions for a given G-structure on a manifold, one merely needs to evaluate the usual Chevalley-Eilenberg cohomology group $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ of an abelian Lie algebra $\mathfrak{g}_{-1}$ with coefficients in the whole $\mathfrak{g}$ and to identify structure functions of order k with the graded component

$$\{\overline{\varphi} \in H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \,|\, Im\,\varphi \subseteq \mathfrak{g}_{k-2}\}, \text{ where } k \geq 1.$$

We refer for the classical text [St, Chapter VII] for details.

One of the nice examples of manifolds endowed with a G-structure are (irreducible) compact Hermitian symmetric spaces (CHSS). There are two naturally distinguishable cases: $rank M = 1$ and $rank M > 1$.

If $rank M = 1$, then $X = \mathbb{C}P^n$, a complex projective space. In this case $\mathfrak{g}$ turns out to be a general (infinite-dimensional) Lie algebra of Cartan type $W(n)$ with a standard grading of depth 1; we recall that $W(n)$ can be defined as a Lie algebra of derivations of the polynomial ring in n indeterminates, and consists of differential operators of the form

$$\sum f_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \text{ where } f_i(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n].$$

The result of Serre on cohomology of involutive Lie algebras of vector fields (see [GuSt] for the original Serre's letter and [LPS], Theorem 1 or [P], p. 9 for a more explicit formulation) implies that structure functions in this case vanish. We will refer for this case as a *rank one case*.

If $rank M > 1$, $\mathfrak{g}$ turns out to be a classical simple Lie algebra with a grading of depth 1 and length 1: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In particular, Cartan prolongations of order > 1 vanish, so we might only have structure functions of orders 1, 2 and 3 only (see [G1], Proposition 4 or [G2], Proposition 4.2). Corresponding structure functions were determined by Goncharov ([G1], Theorem 1 or [G2], Theorem 4.5). We will refer for this case as a *general case*.

Remark. In the sequel we will need the following well-known fact: for any rank,

$$\{x \in \mathfrak{g}_i \,|\, [x, \mathfrak{g}_{-1}] = 0\} = 0, \quad i = 0, 1 \tag{5.2}$$

(this condition sometimes is referred as *transitivity* of the corresponding graded Lie algebra; see, e.g., [D] and references therein). In particular, $\mathfrak{g}_{-1}$ is a faithful $\mathfrak{g}_0$-module.

During the last decade, there was a big amount of activity by Grozman, Leites, Poletaeva, Serganova and Shchepochkina in determining structure functions of various classes of (super)manifolds and G-structures on them (see, e.g., [GLS], [LPS] and [P] with a transitive closure of references therein).

Here we describe structure functions of manifolds M^{S^1} of loops with values in a (finite-dimensional) CHSS M. The group G here is formally no longer a Lie group, but its infinite-dimensional analogue, the group of loops, and the corresponding Lie algebra is a loop Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ with a grading inherited from $\mathfrak{g}$:

$$\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] = \bigoplus_{i\geq -1}\mathfrak{g}_i\otimes\mathbb{C}[t,t^{-1}].$$

The last statement follows from the next simple but handy observation:

Proposition 5.1. *Let* $\bigoplus\limits_{i\geq -1}\mathfrak{g}_i$ *be the Cartan prolongation of a pair* $(\mathfrak{g}_{-1},\mathfrak{g}_0)$, *where* $\mathfrak{g}_{-1},\mathfrak{g}_0$ *are finite-dimensional. Then* $\bigoplus\limits_{i\geq -1}(\mathfrak{g}_i\otimes A)$ *is the Cartan prolongation of the pair* $(\mathfrak{g}_{-1}\otimes A,\mathfrak{g}_0\otimes A)$.

Proof. Induction on i. As all $\mathfrak{g}_i$ are finite-dimensional, in the inductive definition (5.1) of Cartan prolongation, any element $X\in Hom(\mathfrak{g}_{-1}\otimes A,\mathfrak{g}_{i-1}\otimes A)$ may be expressed in the form $\sum\limits_{i\in I}\varphi_i\otimes\alpha_i$, where $\varphi_i\in Hom(\mathfrak{g}_{-1},\mathfrak{g}_{i-1})$, $\alpha_i\in End(A)$. The rest goes as in the proof of Theorem 2.1. □

Thus, we shall obtain, so to speak, a “loopization” of Serre’s and Goncharov’s results.

In November 1993, Dimitry Leites showed to author a handwritten note by Elena Poletaeva containing computations of structure functions of manifolds of loops corresponding to the following two cases: the (rank one) case $\mathfrak{g} = W(1)$ and the (general) case $\mathfrak{g} = sl(4)$ with graded components $\mathfrak{g}_{-1} = V\otimes V^{\star}$, $\mathfrak{g}_0 = sl(2)\oplus gl(2)$, $\mathfrak{g}_1 = V^{\star}\otimes V$, where V is the identity 2-dimensional $gl(2)$-module. Unfortunately, this note has never been published and seems to be lost, and more than 10 years later nobody from the involved parties cannot recollect the details. Though formally the main results of this section are generalizations of those Poletaeva’s forgotten results, it should be noted that Poletaeva considered already the typical representatives in both – rank one and general – cases and observed all the main components and phenomena occurring in cohomology under consideration.

Definitions. (i) The structure functions (identified with elements of the second cohomology group) generated by cocycles of the form

$$(x\otimes a)\wedge(y\otimes b)\mapsto\varphi(x,y)\otimes abu,\quad x,y\in\mathfrak{g}_{-1},\, a,b\in\mathbb{C}[t,t^{-1}],$$

where φ is a structure function of CHSS and $u\in\mathbb{C}[t,t^{-1}]$, will be called *induced*.

(ii) The structure functions generated by cocycles of the form

$$(x\otimes a)\wedge(y\otimes b)\mapsto\varphi(x,y)\otimes\alpha(a,b),\quad x,y\in\mathfrak{g}_{-1},\, a,b\in\mathbb{C}[t,t^{-1}],$$

where $\varphi \in C^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ and $\alpha \in S^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])$, will be called *almost induced*.

(iii) Define a symmetric analogue of $H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, denoted as $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, to be the quotient space

$$\frac{S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})}{\{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \mid \varphi(x,y) = [x, \psi(y)] + [y, \psi(x)] \text{ for some } \psi \in Hom(\mathfrak{g}_{-1}, \mathfrak{g}_0)\}}.$$

Clearly, induced and almost induced structure functions arise, respectively, from the direct summands $H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes \mathbb{C}[t,t^{-1}]$ and $\mathcal{H}(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes S^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])$ of the cohomology $H^2(\mathfrak{g}_{-1} \otimes \mathbb{C}[t,t^{-1}], \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])$ (see Remark after Theorem 3.7 and compare with the paragraph after the proof of Proposition 2.2 in [Z]).

Theorem 5.2. *For the manifold M^{S^1} of loops with values in a CHSS M, the following hold: (i) Structure functions can be only of order 1, 2 or 3.*

(ii) The space of structure functions of order 1 modulo almost induced structure functions is isomorphic to

$$B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{S^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])}{(\mathbb{C}1 \oplus \mathbb{C}\frac{d}{dt}) \otimes \mathbb{C}[t,t^{-1}]} \oplus S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \otimes \frac{C^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])}{End\,(\mathbb{C}[t,t^{-1}])} \oplus SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{End(\mathbb{C}[t,t^{-1}])}{\mathbb{C}[t,t^{-1}]}.$$

(iii) If $\operatorname{rank} M = 1$*, the third direct summand in the last expression vanish.*

If $\operatorname{rank} M = 1$*, almost induced structure functions of order 1 and all structure functions of order 2 and 3 vanish.*

If $\operatorname{rank} M > 1$*, all structure functions of order 2 and 3 are induced.*

Remarks. (i) $B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ is the space of corresponding Spencer coboundaries, i.e., the space of maps $\varphi \in C^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ of the form

$$\varphi(x,y) = [x, \psi(y)] - [y, \psi(x)] \text{ for some } \psi \in Hom\,(\mathfrak{g}_{-1}, \mathfrak{g}_0).$$

(ii) Theorem 3.7 suggests the way in which denominator is embedded into numerator in the three quotient spaces involving $\mathbb{C}[t,t^{-1}]$ in (ii). In the first quotient space, the element $(\lambda 1 + \mu \frac{d}{dt})t^n \in (\mathbb{C}1 \oplus \mathbb{C}\frac{d}{dt}) \otimes \mathbb{C}[t,t^{-1}]$ corresponds to the map $\alpha \in S^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])$ defined by

$$\alpha(t^i, t^j) = \lambda t^{i+j+n} + \mu(i+j)t^{i+j+n-1}.$$

In the second one, the map $\beta(t^i) = \sum_n \lambda_{in} t^n \in End(\mathbb{C}[t,t^{-1}])$ corresponds to the map $\alpha \in C^2(\mathbb{C}[t,t^{-1}], \mathbb{C}[t,t^{-1}])$ defined by

$$\alpha(t^i, t^j) = \sum_n (\lambda_{j,n-i} - \lambda_{i,n-j})t^n.$$

In the third one, the element $t^n \in \mathbb{C}[t,t^{-1}]$ corresponds to the map $\beta \in End(C[t,t^{-1}])$ which is multiplication by t^n:

$$\beta(t^i) = t^{i+n}.$$

Proof. Our task is to compute $H^2(\mathfrak{g}_{-1} \otimes \mathbb{C}[t,t^{-1}], \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])$ for an appropriate $\mathfrak{g}$. It turns out that the concrete structure of the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$ is not important in our approach, and for notational convenience we replace it by an arbitrary (associative commutative unital) algebra A.

Substitute our specific data into the equation of Theorem 3.7:

$$\begin{aligned} H^2(\mathfrak{g}_{-1} \otimes A, \mathfrak{g} \otimes A) \simeq H^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes A \,\oplus\, &\mathcal{H}(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes Der(A) \\ &\oplus C^2(\mathfrak{g}_{-1}, \mathfrak{g}^{\mathfrak{g}_{-1}}) \otimes \frac{S^2(A,A)}{A \oplus Der(A)} \\ &\oplus S^2(\mathfrak{g}_{-1}, \mathfrak{g}^{\mathfrak{g}_{-1}}) \otimes \frac{C^2(A,A)}{\{a\beta(b) - b\beta(a) \mid \beta \in End(A)\}} \\ &\oplus SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) \otimes \{a\beta(b) - b\beta(a) \mid \beta \in End(A)\}. \end{aligned} \tag{5.3}$$

The next technical lemma determines components appearing in this isomorphism.

Lemma 5.3.
(i) $\mathfrak{g}^{\mathfrak{g}_{-1}} = \mathfrak{g}_{-1}$
(ii) $\mathcal{H}(\mathfrak{g}_{-1}, \mathfrak{g}) = H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$
(iii) $SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) = SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$.

Proof. (i) Evident in view of (5.2).

(ii) Follows from definitions of appropriate spaces, (5.2) and part (i).

(iii) Any grading of $\mathfrak{g}$ induces a grading of $SH^2(\mathfrak{g}_{-1}, \mathfrak{g})$:

$$SH^2(\mathfrak{g}_{-1}, \mathfrak{g}) = \bigoplus_{i \geq -1} SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}),$$

where

$$\begin{aligned} SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}) &= (Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) + SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i))/SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i), \\ Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) &= \{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \mid [x, \varphi(y,z)] = [y, \varphi(x,z)] \text{ for all } x,y,z \in \mathfrak{g}_{-1}\}, \\ SB^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) &= \{\varphi \in S^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \mid \varphi(x,y) = [x, \psi(y)] + [y, \psi(x)] \\ &\qquad \text{for some } \psi \in Hom(\mathfrak{g}_{-1}, \mathfrak{g}_{i+1})\}. \end{aligned}$$

We immediately see that $Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) = S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$, and $\varphi(\cdot, y)$ belongs to the $(i+1)$st Cartan prolongation of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ for each $\varphi \in Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$, where $i \geq 0$, and $y \in \mathfrak{g}_{-1}$. Hence, each $\varphi \in Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$ can be written in the form

$$\varphi(x,y) = [x, F(\varphi, y)], \quad \text{for all } x,y \in \mathfrak{g}_{-1}, \tag{5.4}$$

for a certain bilinear map $F : Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+1}$.

But the symmetry of φ implies that $F(\varphi, \cdot) \in Hom(\mathfrak{g}_{-1}, \mathfrak{g}_{i+1})$ belongs to the $(i+2)$nd Cartan prolongation of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. Hence

$$F(\varphi, y) = [y, G(F, \varphi)], \quad \text{for all } \varphi \in Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i), y \in \mathfrak{g}_{-1}, \tag{5.5}$$

for a certain bilinear map $G : Hom(Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1}, \mathfrak{g}_{i+1}) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+2}$.

Combining (5.4) and (5.5) together, one gets $\varphi(x,y) = [x, [y, H(\varphi, y)]]$ for a certain bilinear map $H : Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i) \times \mathfrak{g}_{-1} \to \mathfrak{g}_{i+2}$. Applying again symmetry of φ, we see that H is constant in the second argument, and hence each element $\varphi \in Sym^2(\mathfrak{g}_{-1}, \mathfrak{g}_i)$ can be written in the form $\varphi(x,y) = [x, [y, h]]$ for an appropriate $h = H(\varphi, \cdot) \in \mathfrak{g}_{i+2}$.

But then $\varphi(x,y) = [x, \psi(y)] + [y, \psi(x)]$ for $\psi = -\frac{ad(h)}{2}$, and $SH^2_i(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$ for $i \geq 0$.

Therefore, $SH^2(\mathfrak{g}_{-1}, \mathfrak{g})$ does not vanish only in the (-1)st graded component, and the desired equality follows. □

Continuation of the proof of Theorem 5.2. Substituting the results of Lemma 5.3 into (5.3), decomposing the Chevalley-Eilenberg cohomology $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ into the direct sum of corresponding Spencer cohomologies, and rearranging the summands as indicated in Remark after Theorem 3.7, we obtain:

$$\begin{aligned} H^2(\mathfrak{g}_{-1} \otimes A, \mathfrak{g} \otimes A) \simeq\ & H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes S^2(A,A) \\ & \oplus \Big(\bigoplus_{k>1} H^{k2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})\Big) \otimes A \\ & \oplus B^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \frac{S^2(A,A)}{A \oplus Der(A)} \\ & \oplus S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \otimes \frac{C^2(A,A)}{\{a\beta(b) - b\beta(a) \mid \beta \in End(A)\}} \\ & \oplus SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \otimes \{a\beta(b) - b\beta(a) \mid \beta \in End(A)\}. \end{aligned}$$

The first tensor product here consists of almost induced structure functions of order 1 and the second one consists of induced structure functions of order > 1. This implies (ii).

As was noted earlier, in the rank one case the first and second tensor product vanish (this follows from Serre's theorem); this implies (iv). In the general case, the second tensor product reduces to structure functions of order 2 and 3, i.e., to $(H^{2,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) \oplus H^{3,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})) \otimes A$. This proves (i) and (v) (well, after the final substitution $A = \mathbb{C}[t, t^{-1}]$).

Part (iii) follows from

Lemma 5.4. *For $\mathfrak{g} = W(n)$ with the standard grading, $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1}) = 0$.*

Proof. Denoting $\mathfrak{g}_{-1}$ as V, we have $\mathfrak{g}_0 = gl(V)$, and the statement reduces to the following: for any $\varphi \in S^2(V,V)$, there is a $\psi \in Hom(V, gl(V))$ such that

$$\varphi(x,y) = \psi(x)(y) + \psi(y)(x).$$

But this is obvious: take $\psi(x)(y) = \frac{1}{2}\varphi(x,y)$. □

Remark. In fact, this trivial reasoning shows that *any* linear map $V \times V \to V$, not necessarily symmetric one, may be represented in the form $\varphi(x,y) = \psi(x)(y) + \psi(y)(x)$ for a certain $\psi \in Hom(V, gl(V))$. In particular, it shows that the second Spencer cohomology $H^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ vanishes for $\mathfrak{g} = W(n)$, which is a particular case of Serre's theorem.

This completes the proof of the Theorem 5.2. □

Theorem 5.2 tells how to describe structure functions of manifolds of loops with values in CHSS in terms of structure functions of underlying CHSS (Spencer cohomology groups), and the space $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$, which is a sort of a symmetric analogue of the Spencer cohomology group.

The thorough treatment of the latter symmetric analogue, including its calculation for various $\mathfrak{g}$'s, as well as related construction of a symmetric analogue of Cartan prolongation and some questions pertained to Jordan algebras and Leibniz cohomology, will, hopefully, appear elsewhere. Here we only briefly outline how $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ can be determined in the general case (i.e., for any finite-dimensional simple Lie algebra $\mathfrak{g}$) in terms of the corresponding root system.

All gradings of length 1 and depth 1 of any finite-dimensional simple Lie algebra may be obtained in the following way (see, e.g., [D]). Let R be a root system of $\mathfrak{g}$ corresponding to a Cartan subalgebra $\mathfrak{h}$, B a basis of R, $\{h_\beta, e_\alpha \mid \beta \in B, \alpha \in R\}$ a Chevalley basis of $\mathfrak{g}$. Let $N_{\alpha,\alpha'}$ be structure constants in this basis: $[e_\alpha, e_{\alpha'}] = N_{\alpha,\alpha'} e_{\alpha+\alpha'}$ if $\alpha + \alpha' \in R$. Fix a root $\beta \in B$ such that β enters in decomposition of each root only with coefficients $-1, 0, 1$ (the existence of such root implies that R is not of type G_2, F_4 or E_8). Denote by R_i, where $i = -1, 0, 1$, the set of roots in which β enters with coefficient i. Then

$$\mathfrak{g}_{-1} = \bigoplus_{\alpha \in R_{-1}} \mathbb{C} e_\alpha, \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_0} \mathbb{C} e_\alpha, \quad \mathfrak{g}_1 = \bigoplus_{\alpha \in R_1} \mathbb{C} e_\alpha.$$

Now, consider the map

$$\begin{aligned} T : Hom(\mathfrak{g}_{-1}, \mathfrak{g}_0) &\to S^2(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \\ \psi(x) &\mapsto (T\psi)(x,y) = [x, \psi(y)] + [y, \psi(x)]. \end{aligned}$$

The question of determining $SH^{1,2}_{\mathfrak{g}_0}(\mathfrak{g}_{-1})$ evidently reduces to evaluation of $Ker T$.

Writing

$$\psi(e_r) = \sum_{\alpha \in B} \lambda^r_\alpha h_\alpha + \sum_{\alpha \in R_0} \mu^r_\alpha e_\alpha$$

for $r \in R_{-1}$ and parameters $\lambda^r_\alpha, \mu^r_\alpha \in \mathbb{C}$, we see that the equation

$$[x, \psi(y)] + [y, \psi(x)] = 0$$

is equivalent to the following three conditions:

$$\sum_{\alpha\in B}\lambda^s_\alpha r(h_\alpha)=\mu^r_{r-s}N_{s,r-s}\quad \text{for all } r,s\in R_{-1} \text{ such that } r-s\in R_0\,;$$
$$\sum_{\alpha\in B}\lambda^s_\alpha r(h_\alpha)=0\quad \text{for all } r,s\in R_{-1} \text{ such that } r-s\notin R\,;$$
$$\mu^r_\alpha N_{s,\alpha}=0\quad \text{for all } r,s\in R_{-1},\alpha\in R_0 \text{ such that } r-s\neq\alpha$$

which serve as (linear) defining relations for the space $Ker\,T$ and may be computed in each particular case.

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4. Invariants of Lie algebras extended over commutative algebras without unit

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We establish results about the second cohomology with coefficients in the trivial module, symmetric invariant bilinear forms, and derivations of a Lie algebra extended over a commutative associative algebra without unit. These results provide a simple unified approach to a number of questions treated earlier in completely separated ways: periodization of semisimple Lie algebras (Anna Larsson), derivation algebras, with prescribed semisimple part, of nilpotent Lie algebras (Benoist), and presentations of affine Kac-Moody algebras.

Introduction

In this paper we consider current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra, A is a commutative associative algebra, and the multiplication in $L \otimes A$ being defined by the formula

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab$$

for any $x, y \in L$, $a, b \in A$.

We are interested in the second cohomology of $L \otimes A$ with trivial coefficients, the space of symmetric invariant bilinear forms on $L \otimes A$, and the algebras of derivations of $L \otimes A$. These invariants were determined for numerous particular cases of current Lie algebras (see, for example, [S]), the general formulae for the second homology with trivial coefficients in terms of invariants of L and A were obtained in [Ha], [Z1] and [NW], and the similar formulae for the space of symmetric invariant bilinear forms and derivation algebras were obtained in [Z1] and [Z2], respectively.

So why return to these settled questions? In all considerations until now, the algebra A was supposed to have a unit. However, there are many interesting examples of current algebras were A is not unital. For example, in [La], the so-called periodization of semisimple Lie algebras $\mathfrak{g}$ was considered, which is nothing but $\mathfrak{g} \otimes t\mathbb{C}[t]$. It is known that the second homology of any nilpotent Lie algebra with trivial coefficients has interpretation in terms of presentation of the algebra, so allowing A to be nilpotent allows us to obtain presentation of $L \otimes A$ irrespective of the properties of L.

It turns out that elementary arguments similar to those in [Z2] allow us to extend the above mentioned results to the case of non-unital A. In particular, concerning the second cohomology and symmetric invariant bilinear forms, we provide another proof, considerably shorter than all the previous ones even in the case of unital A.

The contents of this paper are as follows. In §§1–3 we establish the general formulae for 2-cocycles, symmetric invariant bilinear forms, and get partial results about derivations of the current Lie algebras, respectively. This is followed by applications: in §4 we reprove the result from [La] about presentations of periodizations of the semisimple Lie algebras. In passing, we also mention how to derive from our results the theorem from [Be] about semisimple components of the derivation algebras of certain current Lie algebras, and the Serre defining relations between Chevalley generators of the non-twisted affine Kac-Moody algebra.

In all these cases, the absence of unit in A is essential. All these proofs are significantly shorter than the original ones, and reveal various almost trivial, but so far unnoticed or unpublished, links between different concepts and results. These links are, perhaps, the main virtue of this paper.

It seems that everything considered here can be extended in a straightforward way to twisted, Leibniz and super settings, but we will not venture into this, at least for now.

Notation and conventions

All algebras and vector spaces are defined over the base field K of characteristic different from 2 and 3 unless stated otherwise (some of the results are valid in characteristic 3, but we will not go into this).

In what follows, L denotes a Lie algebra, A is an associative commutative algebra.

Given an L-module M, let $B^n(L,M)$, $Z^n(L,M)$, $C^n(L,M)$ and $H^n(L,M)$ denote the space of nth degree coboundaries, cocycles, cochains, and cohomology of L with coefficients in M respectively (we will be mainly interested in the particular cases of degree 2 and the trivial module K, or degree 1 and the adjoint module or its dual). Note that $C^2(L,K)$ is the space of all skew-symmetric bilinear forms on L. The space of all symmetric bilinear forms on L will be denoted as $S^2(L,K)$.

Let $\mathscr{Z}(L)$, $[L,L]$ and $Der(L)$ denote the center, the commutant (the derived algebra), and the Lie algebra of derivations of L, respectively. Similarly, let $Ann(A) = \{a \in A \,|\, Aa = 0\}$ and AA denote the annulator and the square of A, respectively, and let $HC^*(A)$ denotes its cyclic cohomology.

A bilinear form $\varphi : L \times L \to K$ is said to be *cyclic* if

$$\varphi([x,y],z) = \varphi([z,x],y)$$

for any $x,y,z \in L$. Note that if φ is symmetric, this condition is equivalent to the *invariance* of the form φ:

$$\varphi([x,y],z) + \varphi(y,[x,z]) = 0,$$

while if φ is skew-symmetric, the notions of cyclic and invariant forms differ.

Let $\mathscr{B}(L)$ denote the space of all symmetric bilinear invariant (=cyclic) forms on L. Similarly, a bilinear form $\alpha : A \times A \to K$ is said to be *cyclic* if

$$\alpha(ab,c) = \alpha(ca,b)$$

for any $a,b,c \in A$. If the form α is symmetric, it is cyclic if and only if it is *invariant* on A.

1 The second cohomology

Theorem 1.1. *Let L be a Lie algebra, A an associative commutative algebra, and at least one of L and A be finite-dimensional. Then each cocycle in $Z^2(L \otimes A, K)$ can be represented as the sum of decomposable cocycles $\varphi \otimes \alpha$, where $\varphi : L \times L \to K$ and $\alpha : A \times A \to K$ are of one of the following* 8 *types:*

(i) $\varphi([x,y],z) + \varphi([z,x],y) + \varphi([y,z],x) = 0$ *and α is cyclic,*
(ii) *φ is cyclic and* $\alpha(ab,c) + \alpha(ca,b) + \alpha(bc,a) = 0$*,*
(iii) $\varphi([L,L],L) = 0$*,*
(iv) $\alpha(AA,A) = 0$*,*

where each of these 4 *types splits into two subtypes: with φ skew-symmetric and α symmetric, and with φ symmetric and α skew-symmetric.*

Proof. Each cocycle $\Phi \in Z^2(L \otimes A, K)$, being an element of

$$End(L \otimes A \otimes L \otimes A, K) \simeq End(L \otimes L, K) \otimes End(A \otimes A, K),$$

can be expressed in the form $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$ are bilinear maps. Using this representation, write the cocycle equation for an arbitrary triple $x \otimes a$, $y \otimes b$, $z \otimes c$, where $x,y,z \in L$ and $a,b,c \in A$:

$$\sum_{i \in I} (\varphi_i([x,y],z) \otimes \alpha_i(ab,c) + \varphi_i([z,x],y) \otimes \alpha_i(ca,b) + \varphi_i([y,z],x) \otimes \alpha_i(bc,a)) = 0. \tag{4.1}$$

Symmetrizing this equality with respect to x,y, we get:

$$\sum_{i \in I} \Big(\varphi_i([x,z],y) + \varphi_i([y,z],x)\Big) \otimes \Big(\alpha_i(bc,a) - \alpha_i(ca,b)\Big) = 0.$$

On the other hand, cyclically permuting x,y,z in (4.1) and summing up the 3 equalities obtained, we get:

$$\sum_{i \in I} \Big(\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x)\Big) \otimes \Big(\alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b)\Big) = 0.$$

Applying Lemma 1.1 from [Z2] to the last two equalities, we get a partition of the index set $I = I_1 \cup I_2 \cup I_3 \cup I_4$ such that

$$\begin{gathered}
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0,\ \ \varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0 \\
\text{for } i \in I_1 \\
\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0,\ \ \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 \\
\text{for } i \in I_2 \\
\varphi_i([x,y],z) + \varphi_i([z,x],y) + \varphi_i([y,z],x) = 0,\ \ \alpha_i(bc,a) - \alpha_i(ca,b) = 0 \\
\text{for } i \in I_3 \\
\alpha_i(bc,a) - \alpha_i(ca,b) = 0,\ \ \alpha_i(ab,c) + \alpha_i(bc,a) + \alpha_i(ca,b) = 0 \\
\text{for } i \in I_4.
\end{gathered}$$

It is obvious that if the characteristic of K is different from 3, then $\varphi_i([L,L],L) = 0$ for $i \in I_1$, and $\alpha_i(AA,A) = 0$ for $i \in I_4$. It is obvious also that $\varphi_i \otimes \alpha_i$ satisfies the cocycle equation (4.1) for each $i \in I_1, I_2, I_3, I_4$.

Now write the condition of skew-symmetry of Φ:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) + \varphi_i(y,x) \otimes \alpha_i(b,a) = 0 \tag{4.2}$$

and symmetrize it with respect to x, y:

$$\begin{aligned}
&\sum_{i \in I} (\varphi_i(x,y) - \varphi_i(y,x)) \otimes (\alpha_i(a,b) - \alpha_i(b,a)) = 0 \\
&\sum_{i \in I} (\varphi_i(x,y) + \varphi_i(y,x)) \otimes (\alpha_i(a,b) + \alpha_i(b,a)) = 0.
\end{aligned}$$

From the last two equalities, using again Lemma 1.1 from [Z2], we see that each set I_1, I_2, I_3, I_4 can be split further into two subsets, one having skew-symmetric φ_i and symmetric α_i, and the other one having symmetric φ_i and skew-symmetric α_i. □

Remark. As all our bilinear maps are K-valued, the cocycles of the form $\varphi \otimes \alpha$ are, of course, just products of bilinear maps $\varphi\alpha$. However, we have retained the symbol $\otimes$, to make it easier to track dependence on the more general situation of [Z2].

Remark. The second subdivision in the statement of Theorem 1.1, which follows from equality (4.2), is merely a manifestation of the vector space isomorphism

$$C^2(L \otimes A, K) \simeq \left(S^2(L,K) \otimes C^2(A,K)\right) \oplus \left(C^2(L,K) \otimes S^2(A,K)\right).$$

Let $d\Omega$ be the 2-coboundary defined by a given linear map $\Omega : L \otimes A \to K$. The latter can be written in the form $\Omega = \sum_{i \in I} \omega_i \otimes \beta_i$ for some linear maps $\omega_i : L \to K$

and $\beta_i : A \to K$. Then

$$d\Omega(x \otimes a, y \otimes b) = \sum_{i \in I} \omega_i([x,y]) \otimes \beta_i(ab),$$

i.e., coboundaries always lie in the direct summand $C^2(L,K) \otimes S^2(A,K)$. Consequently, nonzero cocycles from different direct summands can never be cohomologically dependent, and the cocycles from the direct summand $S^2(L,K) \otimes C^2(A,K)$ are cohomologically independent if and only if they are linearly independent.

One may try to formulate Theorem 1.1 as a statement about $H^2(L \otimes A, K)$, but in full generality this will lead only to cumbersome complications. In each case of interest, one can easily obtain an information about the cohomology. For example, assuming A contains a unit, one immediately gets that cocycles of type (i) necessarily have φ skew-symmetric and α symmetric, cocycles of type (ii) necessarily have φ symmetric and α skew-symmetric, and cocycles of type (iv) vanish. This leads to a known formula for $H^2(L \otimes A, K)$, where the cocycles of type (i) contribute to the term $H^2(L,K) \otimes A^*$, the cocycles of type (ii) contribute to the term $\mathscr{B}(L) \otimes HC^1(A)$, and the cocycles of type (iii) are non-essential (in terminology of [Z1]).

Another, more concrete, application is given in §4.

2 Symmetric invariant bilinear forms

Theorem 2.1. *Let L be a Lie algebra, A an associative commutative algebra, and at least one of L and A be finite-dimensional. Then each symmetric invariant bilinear form on $L \otimes A$ can be represented as a sum of decomposable forms $\varphi \otimes \alpha$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$ of one of the* 6 *following types:*

(i) *both φ and α are cyclic,*
(ii) $\varphi([L,L],L) = 0$,
(iii) $\alpha(AA,A) = 0$,

where each of these 3 *types splits into two subtypes: with both φ and α symmetric, and with both φ and α skew-symmetric.*

Proof. The proof is absolutely similar to that of Theorem 1.1. As in the proof of Theorem 1.1, we may write a symmetric invariant bilinear form Φ on $L \otimes A$ as $\sum_{i \in I} \varphi_i \otimes \alpha_i$ for suitable bilinear maps $\varphi_i : L \times L \to K$ and $\alpha_i : A \times A \to K$. The invariance condition, written for a given triple $x \otimes a$, $y \otimes b$, $z \otimes c$, reads:

$$\sum_{i \in I} \varphi_i([x,y],z) \otimes \alpha_i(ab,c) + \varphi_i([x,z],y) \otimes \alpha_i(ca,b) = 0. \tag{4.3}$$

Symmetrizing this with respect to x, y, we get:

$$\sum_{i \in I} \Big(\varphi_i([x,z],y) + \varphi_i([y,z],x) \Big) \otimes \alpha_i(ca,b) = 0.$$

Hence the index set can be partitioned $I = I_1 \cup I_2$ in such a way that

$$\varphi_i([x,z],y) + \varphi_i([y,z],x) = 0$$

for any $i \in I_1$, and $\alpha_i(AA,A) = 0$ for any $i \in I_2$. Then (4.3) can be rewritten as

$$\sum_{i \in I_1} \varphi_i([x,y],z) \otimes \Big(\alpha_i(ab,c) - \alpha_i(ca,b)\Big) = 0.$$

Hence there is a partition $I_1 = I_{11} \cup I_{12}$ such that $\varphi_i([L,L],L) = 0$ for any $i \in I_{11}$, and $\alpha_i(ab,c) = \alpha_i(ac,b)$ for any $i \in I_{12}$.

The condition of symmetry of Φ:

$$\sum_{i \in I} \varphi_i(x,y) \otimes \alpha_i(a,b) - \varphi_i(y,x) \otimes \alpha_i(b,a) = 0,$$

being symmetrized with respect to x, y, allows us to partition further each of the sets I_{11}, I_{12}, I_2 into two subsets, one having both φ_i and α_i symmetric, and the other one having both φ_i and α_i skew-symmetric. □

This generalizes [Z1, Theorem 4.1], where a similar statement is proved for unital A.

3 Derivations

Naturally, one may try to apply the same approach to description of the derivations of a given current algebra $L \otimes A$ (for unital A, see [Z2, Corollary 2.2]). Indeed, each derivation D of $L \otimes A$, being an element of

$$End(L \otimes A, L \otimes A) \simeq End(L,L) \otimes End(A,A),$$

can be expressed in the form $D = \sum\limits_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \to L$ and $\alpha_i : A \to A$ are bilinear maps. The condition that D is a derivation, written for an arbitrary pair $x \otimes a$ and $y \otimes b$, where $x, y \in L$ and $a, b \in A$, reads:

$$\sum_{i \in I} \varphi_i([x,y]) \otimes \alpha_i(ab) - [\varphi_i(x),y] \otimes \alpha_i(a)b - [x,\varphi_i(y)] \otimes a\alpha_i(b) = 0.$$

Symmetrizing this equality with respect to a, b (this is equivalent to symmetrization with respect to x, y):

$$\sum_{i \in I} \Big([\varphi_i(x),y] - [x,\varphi_i(y)]\Big) \otimes \Big(a\alpha_i(b) - b\alpha_i(a)\Big) = 0,$$

we get a partition of the index set I into two subsets satisfying, respectively, conditions $[\varphi_i(x),y] = [x,\varphi_i(y)]$ and $a\alpha_i(b) = b\alpha_i(a)$. But as there are only two variables in each of L and A, no other symmetrization is possible, so the last equality is all what we can get in this way.

The failure of this method can be also explained by looking at the simple example of the Lie algebra $sl(2) \otimes tK[t]$. In $sl(2)$, fix a basis $\{e_-, h, e_+\}$ with multiplication

$$[h, e_-] = -e_-, \quad [h, e_+] = e_+, \quad [e_-, e_+] = h. \tag{4.4}$$

It is easy to check that the map defined, for any $n \in \mathbb{N}$, by the formula

$$\begin{aligned} e_- \otimes t^n &\mapsto e_- \otimes (nt^n - t^{n+1}) \\ e_+ \otimes t^n &\mapsto e_+ \otimes (nt^n + t^{n+1}) \\ h \otimes t^n &\mapsto h \ \otimes nt^n \end{aligned}$$

is a derivation of $sl(2) \otimes tK[t]$. It is obvious that this map is not a decomposable one, i.e., of the form $\varphi \otimes \alpha$ for some $\varphi : sl(2) \to sl(2)$ and $\alpha : tK[t] \to tK[t]$. But for this approach to succeed, all the maps in question should be representable by the end of the day in such a way.

However, under additional assumption on $L \otimes A$, we can derive information about $Der(L \otimes A)$ from the results of the preceding sections, using the relationship between $H^1(L, L^*)$, $H^2(L, K)$, and $\mathscr{B}(L)$. In the literature, this relationship was noted many times in a slightly different form, and goes back to the classical works of Koszul and Hochschild–Serre [HS]. Namely, there is an exact sequence

$$0 \to H^2(L, K) \xrightarrow{u} H^1(L, L^*) \xrightarrow{v} \mathscr{B}(L) \xrightarrow{w} H^3(L, K) \tag{4.5}$$

where for the representative $\varphi \in Z^2(L, K)$ of a given cohomology class, we have to take the class of $u(\varphi)$, the latter being given by

$$(u(\varphi)(x))(y) = \varphi(x, y)$$

for any $x, y \in L$, v is sending the class of a given cocycle $D \in Z^1(L, L^*)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x, y) = D(x)(y) + D(y)(x),$$

and w is sending a given symmetric bilinear invariant form $\varphi : L \times L \to K$ to the class of the cocycle $\omega \in Z^3(L, K)$ defined by

$$\omega(x, y, z) = \varphi([x, y], z)$$

(see, for example, [D1], where a certain long exact sequence is obtained, of which this one is the beginning, and references therein for many earlier particular variations; this exact sequence was also established in [NW, Proposition 7.2] with two additional terms on the right).

In the case where $L \simeq L^*$ as L-modules, this sequence provides a way to evaluate $H^1(L, L)$ given $H^2(L, K)$ and $\mathscr{B}(L)$. The L-module isomorphism $L \simeq L^*$

implies the existence of a symmetric invariant non-degenerate invariant form $\langle \cdot, \cdot \rangle$ on L. In terms of this form, u is sending the class of a given cocycle $\varphi \in Z^2(L,K)$ to the class of the cocycle $u(\varphi) \in Z^1(L,L)$ defined by

$$\langle (u(\varphi))(x), y \rangle = \varphi(x,y),$$

and v is sending the class of a given cocycle $D \in Z^1(L,L)$ to the bilinear form $v(D) : L \times L \to K$ defined by the formula

$$v(D)(x,y) = \langle D(x), y \rangle + \langle x, D(y) \rangle.$$

Turning to current Lie algebras, we will make even stronger assumption: that $L \simeq L^*$ and $A \simeq A^*$. Then, utilizing the results of preceding sections about $H^2(L \otimes A, K)$ and $\mathscr{B}(L \otimes A)$, we will derive results about $Der(L \otimes A)$.

In the literature, given $H^1(L, L^*)$, the space $H^2(L,K)$ was computed for various Lie algebras L (see, for example, [S], [D1] and references therein). Here we utilize this connection in the other direction.

Theorem 3.1. *Let L be a nonabelian Lie algebra, A an associative commutative algebra, both L and A finite-dimensional and with symmetric invariant non-degenerate bilinear form. Then each derivation of $L \otimes A$ can be represented as the sum of decomposable linear maps $d \otimes \beta$, where $d : L \to L$ and $\beta : A \to A$ are of one of the following types:*

(i) $d([x,y]) = \lambda([d(x),y] + [x,d(y)])$, $\beta(ab) = \mu\beta(a)b$ *for certain* $\lambda, \mu \in K$ *such that* $\lambda\mu = 1$,

(ii) $d([x,y]) = \lambda[d(x),y]$, $\beta(ab) = \mu(\beta(a)b + a\beta(b))$ *for certain* $\lambda, \mu \in K$ *such that* $\lambda\mu = 1$,

(iii) $[d(x),y] + [x,d(y)] = 0$, $\beta(AA) = 0$, $\beta(a)b = a\beta(b)$,

(iv) $d([L,L]) = 0$, $[d(x),y] + [x,d(y)] = 0$, $\beta(a)b = a\beta(b)$,

(v) $d([L,L]) = 0$, $[d(x),x] = 0$, $\beta(a)b + a\beta(b) = 0$,

(vi) $[d(x),x] = 0$, $\beta(AA) = 0$, $\beta(a)b + a\beta(b) = 0$,

(vii) $d([L,L]) = 0$, $d(L) \subseteq \mathscr{Z}(L)$,

(viii) $d([L,L]) = 0$, $\beta(A) \subseteq Ann(A)$,

(ix) $d(L) \subseteq \mathscr{Z}(L)$, $\beta(AA) = 0$,

(x) $\beta(AA) = 0$, $\beta(A) \subseteq Ann(A)$.

Proof. By abuse of notation, let $\langle \cdot, \cdot \rangle$ denote a symmetric invariant non-degenerate bilinear form both on L and A. Obviously, the tensor product of these forms defines a symmetric invariant non-degenerate bilinear form on $L \otimes A$, for which by even bigger abuse of notation we will use the same symbol:

$$\langle x \otimes a, y \otimes b \rangle = \langle x, y \rangle \langle a, b \rangle.$$

We have $L^* \simeq L$ as L-modules, $A^* \simeq A$ as A-modules, and $(L \otimes A)^* \simeq L \otimes A$ as $L \otimes A$-modules.

As a vector space, $H^1(L\otimes A, L\otimes A)$ can be represented as the direct sum of $Ker\,v$ and $Im\,v$, and the exact sequence (4.5) tells that $Ker\,v = Im\,u$ and $Im\,v = Ker\,w$.

By Theorem 1.1, $H^2(L\otimes A, K)$ is spanned by cohomology classes which can be represented by decomposable cocycles $\varphi\otimes\alpha$ for appropriate $\varphi : L\times L \to L$ and $\alpha : A\times A \to K$. For each such pair φ and α, there are unique linear maps $d : L \to L$ and $\beta : A \to A$ such that

$$\langle d(x), y\rangle = \varphi(x,y) \tag{4.6}$$

for any $x, y \in L$, and

$$\langle \beta(a), b\rangle = \alpha(a,b) \tag{4.7}$$

for any $a, b \in A$. Hence the decomposable linear map $d\otimes\beta : L\otimes A \to L\otimes A$ satisfies

$$\langle (d\otimes\beta)(x\otimes a), y\otimes b\rangle = (\varphi\otimes\alpha)(x\otimes a, y\otimes b),$$

i.e., coincides with $u(\varphi\otimes\alpha)$. Thus, $Im\,u$ is spanned by cohomology classes whose representatives are decomposable derivations.

Similarly, by Theorem 2.1, $\mathscr{B}(L\otimes A)$ is spanned by decomposable elements $\varphi\otimes\alpha$, and $Ker\,w$ is spanned by such elements of types (ii) and (iii), i.e., either $\varphi([L,L],L) = 0$ or $\alpha(AA,A) = 0$. Again, for each such element we can find $d : L \to L$ and $\beta : A \to A$ satisfying (4.6) and (4.7) respectively. Furthermore, we may assume that for each such $\varphi\otimes\alpha$, the maps φ and α are either both symmetric, or both skew-symmetric, and hence both d and β are either self-adjoint or skew-self-adjoint, respectively, with respect to $\langle\cdot,\cdot\rangle$. In both cases we have:

$$\begin{aligned}\langle (d\otimes\beta)(x\otimes a), y\otimes b\rangle &+ \langle x\otimes a, (d\otimes\beta)(y\otimes b)\rangle \\ &= \langle d(x), y\rangle\langle\beta(a), b\rangle + \langle x, d(y)\rangle\langle a, \beta(b)\rangle = 2\langle d(x), y\rangle\langle \beta(a), b\rangle \\ &= 2\varphi(x,y)\otimes\alpha(a,b)\end{aligned} \tag{4.8}$$

for any $x, y \in L$ and $a, b \in A$.

The condition $\varphi([L,L],L) = 0$ ensures that $d([L,L]) = 0$, and an equivalent condition $\varphi(L,[L,L]) = 0$ ensures that

$$\langle [d(x), z], y\rangle = -\langle d(x), [y,z]\rangle = 0,$$

implying $d(L) \subseteq \mathscr{Z}(L)$, and hence $d\otimes\beta$ is a derivation of $L\otimes A$. Quite analogously, the condition $\alpha(AA,A) = \alpha(A,AA) = 0$ implies also that $d\otimes\beta$ is a derivation of $L\otimes A$. The equality (4.8) ensures that v maps the cohomology class of this derivation to $2\varphi\otimes\alpha$. Thus $Im\,v$ is spanned by images of cohomology classes whose representatives are decomposable derivations.

Putting all this together, we see that $H^1(L\otimes A, L\otimes A)$ is spanned by the cohomology classes whose representatives are decomposable derivations. As inner derivations of $L\otimes A$ are, obviously, also spanned by decomposable inner derivations $ad\,(x \otimes a) = ad\,x \otimes R_a$, where R_a is the multiplication on $a \in A$, any derivation of $L \otimes A$ is representable as the sum of decomposable derivations.

The rest is easy. The condition that $d \otimes \beta$ is a derivation, reads:

$$d([x,y]) \otimes \beta(ab) - [d(x),y] \otimes \beta(a)b - [x,d(y)] \otimes a\beta(b) = 0 \tag{4.9}$$

for any $x,y \in L$, $a,b \in A$. Symmetrizing (4.9) as in the beginning of this section, we see that either $[d(x),y] = [x,d(y)]$ for any $x,y \in L$, or $\beta(a)b = a\beta(b)$ for any $a,b \in A$. The equation (4.9) is equivalent to

$$d([x,y]) \otimes \beta(ab) - [d(x),y] \otimes (\beta(a)b + a\beta(b)) = 0$$

in the first case, and to

$$d([x,y]) \otimes \beta(ab) - ([d(x),y] + [x,d(y)]) \otimes \beta(a)b = 0.$$

in the second case. Now trivial case-by-case considerations involving vanishing and linear dependence of the linear operators occurring as tensor product factors in these two equalities, produce the final list of derivations. □

Theorem 3.1 can be applied, for example, to the Lie algebra $\mathfrak{g} \otimes tK[t]/(t^n)$, where $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra over any field of characteristic 0, to obtain a very short proof of the result of Benoist [Be] about realization of any semisimple Lie algebra as semisimple part of the Lie algebra of derivations of a nilpotent Lie algebra (another short proof with direct calculation of $Der(\mathfrak{g} \otimes tK[t]/(t^3))$ follows from [LL, Proposition 3.5]). Indeed, as noted, for example, in [BB, Lemma 2.2], $tK[t]/(t^n)$ possesses a symmetric nondegenerate invariant bilinear form B, hence $\mathfrak{g} \otimes tK[t]/(t^n)$ possesses such a form (being the product of the Killing form on $\mathfrak{g}$ and B), so Theorem 3.1 is applicable. As $\mathfrak{g}$ is perfect and centerless, the derivations of types (iv), (v), (vii), (viii), (ix) vanish. The remaining types can be handled, for example, by appealing to results of [Ho], [F] or [LL], which imply that in the case $\mathfrak{g} \not\simeq sl(2)$, the corresponding mappings d vanish also for types (ii) and (iii), and for the rest of the types are either inner derivations of $\mathfrak{g}$, or multiplications by scalar. Then, performing elementary calculations with conditions imposed on β's in the remaining types, and rearranging the obtained spaces of derivations, we get the following isomorphism of vector spaces:

$$Der(\mathfrak{g} \otimes tK[t]/(t^n)) \simeq (\mathfrak{g} \otimes K[t]/(t^n)) \oplus (End(\mathfrak{g})/\mathfrak{g}) \oplus K. \tag{4.10}$$

Elements of the first summand are assembled from types (i) and (x), and act on $\mathfrak{g} \otimes tK[t]/(t^n)$ as the Lie multiplication by an element of $\mathfrak{g}$ and the associative commutative multiplication by an element of $K[t]/(t^n)$. Elements of the second summand are assembled from types (i), (vi) and (x), and act by the rule

$$\begin{aligned} x \otimes t &\mapsto F(x) \otimes t^{n-1} \\ x \otimes t^k &\mapsto 0 \qquad \text{if } k \geq 2, \end{aligned}$$

where $x \in \mathfrak{g}$, $F \in End(\mathfrak{g})$, and elements of $\mathfrak{g}$ are assumed to be embedded into $End(\mathfrak{g})$ as inner derivations. Elements of the third, one-dimensional, summand are assembled from types (i) and (vi), and are proportional to the following derivation:

$$
\begin{aligned}
x \otimes t &\mapsto x \otimes t^{n-2} \\
x \otimes t^2 &\mapsto 2x \otimes t^{n-1} \\
x \otimes t^k &\mapsto 0 \qquad \text{if } k \geq 3,
\end{aligned}
$$

where $x \in \mathfrak{g}$.

All this implies that, as a Lie algebra, $Der(\mathfrak{g} \otimes tK[t]/(t^n))$ splits into the semidirect sum of the semisimple part isomorphic to $\mathfrak{g}$ (identified with the part $\mathfrak{g} \otimes 1$ of the first summand in (4.10)), and the nilpotent radical consisting of $\mathfrak{g} \otimes tK[t]/(t^n)$ from the first summand, and the whole second and third summands.

The case $\mathfrak{g} = sl(2)$ can be treated separately and easily.

4 Periodization of semisimple Lie algebras

For a given Lie algebra L, its *periodization* is an $\mathbb{N}$-graded Lie algebra, with component in each degree isomorphic to L. In other words, the periodization of L is $L \otimes tK[t]$.

In [La], Anna Larsson studied periodization of semisimple finite-dimensional Lie algebras $\mathfrak{g}$ over any field K of characteristic 0. She proved that, unless $\mathfrak{g}$ contains direct summands isomorphic to $sl(2)$, its periodization possesses a presentation with only quadratic relations. Since generators and relations of (generalized graded) nilpotent Lie algebras can be interpreted as the first and second homology, Larsson's statement can be formulated in homological terms.

Interest in periodizations, and whether they admit generators subject to quadratic relations, arose from an earlier work of Löfwall and Roos about some amazing Hopf algebras (for further details, see [La]).

As noted in [La], the whole space $H^*(\mathfrak{g} \otimes t\mathbb{C}[t], \mathbb{C})$ was studied by much more sophisticated methods in the celebrated paper by Garland and Lepowsky [GL] (actually, a particular case interesting for us here was already sketched in [G]; the case of $\mathfrak{g} = sl(2)$ was also treated in [FF, p. 233]). They determined the eigenvalues of the Laplacian on the corresponding Chevalley-Eilenberg (co)chain complex. However, to extract from [GL] exact results about (co)homology of interest requires nontrivial combinatorics with the Weyl group, as demonstrated in [HW], and case-by-case analysis for each series of the simple Lie algebras. Here we derive results for the second cohomology in a uniform way from the results of §1 using elementary methods, what provides an alternative short proof of Larsson's result. Also, our approach clearly shows why the case of $sl(2)$ is exceptional.

Theorem 4.1 (Larsson). *Let* $\mathfrak{g}$ *be a finite-dimensional semisimple Lie algebra over an algebraically closed field* K *of characteristic* 0. *Then*

$$H^2(\mathfrak{g}\otimes tK[t],K) \simeq C^2(\mathfrak{g},K)/B^2(\mathfrak{g},K) \bigoplus \left(\bigoplus S_\alpha\right),$$

where the second direct sum is taken over all simple direct summands of $\mathfrak{g}$ *isomorphic to* $sl(2)$, *and each* S_α *is a certain* 5*-dimensional space of symmetric bilinear forms on the corresponding direct summand. The basic cocycles can be chosen among cocycles of the form*

$$\Phi(x\otimes t^i, y\otimes t^j) = \begin{cases} \varphi(x,y) & \textit{if } i=j=1,\\ 0 & \textit{otherwise}, \end{cases} \tag{4.11}$$

where $x,y\in\mathfrak{g}$ *and* φ *is a skew-symmetric bilinear form on* $\mathfrak{g}$, *and the cocycles* Ψ *whose only non-vanishing values on all pairs of the simple direct summands of* $\mathfrak{g}$ *are determined by the formula*

$$\Psi(x\otimes t^i, y\otimes t^j) = \begin{cases} \psi(x,y) & \textit{if } i=1, j=2,\\ -\psi(x,y) & \textit{if } i=2, j=1,\\ 0 & \textit{otherwise}, \end{cases} \tag{4.12}$$

where x,y *belong to the corresponding* $sl(2)$*-direct summand, and* ψ *is a symmetric bilinear form on this direct summand satisfying*

$$\psi(e_-,e_+) = \frac{1}{2}\psi(h,h) \tag{4.13}$$

in the standard $sl(2)$*-basis* (4.4).

Proof. Consider the cocycles appearing in Theorem 1.1 for $L=\mathfrak{g}$ and $A=tK[t]$ case-by-case.

Cocycles of type (i). Writing the cycle condition of α for the triple t^{i-1},t^j,t, we get

$$\alpha(t^{i+j-1},t) = \alpha(t^i,t^j) \tag{4.14}$$

for any $i\ge 2, j\ge 1$.

(ia). φ is skew-symmetric, thus $\varphi\in Z^2(\mathfrak{g},K)$, and α is symmetric. Since $H^2(\mathfrak{g},K)=0$, it follows that $\varphi = d\omega$ for some linear map $\omega:\mathfrak{g}\to K$. Define a linear map $\Omega:\mathfrak{g}\otimes tK[t]\to K$ by setting, for any $x\in\mathfrak{g}$,

$$\Omega(x\otimes t^i) = \begin{cases} 0 & \text{if } i=1,\\ \omega(x)\alpha(t^{i-1},t) & \text{if } i\ge 2. \end{cases}$$

Then, taking (4.14) into account, we get $\varphi\otimes\alpha = d\Omega$, i.e., cocycles of this type are trivial.

(ib). φ is symmetric, α is skew-symmetric.

If $\mathfrak{g} = sl(2)$, direct calculation shows that the space of corresponding φ's coincides with the space of all symmetric bilinear forms on $sl(2)$ satisfying the condition (4.13), and hence is 5-dimensional (an equivalent calculation is contained in [D2, Theorem 6.5]). This case is exceptional, as shows the following

Lemma 4.2 (Dzhumadil'daev–Bakirova). *Let $\mathfrak{g} \not\simeq sl(2)$ be simple. Then any symmetric bilinear form φ on $\mathfrak{g}$ satisfying*

$$\varphi([x,y],z) + \varphi([z,x],y) + \varphi([y,z],x) = 0 \tag{4.15}$$

for any $x,y,z \in \mathfrak{g}$, vanishes.

This Lemma is proved in [DB] by considering the Chevalley basis of $\mathfrak{g}$ and performing computations with the corresponding root system. In [D2] and [DB], symmetric bilinear forms satisfying the condition (4.15) are called *commutative* 2*-cocycles* and arise naturally in connection with classification of algebras satisfying skew-symmetric identities. We will give a different proof which stresses the connection with yet another notions and results.

Proof. Consider a map from the space of bilinear forms on a Lie algebra L to the space of linear maps from L to L^*, by sending a bilinear form φ to the linear map $D : L \to L^*$ such that $D(x)(y) = \varphi(x,y)$. It is easy to see that a symmetric bilinear form φ satisfying (4.15) maps to a linear map D satisfying

$$D([x,y]) = -y \bullet D(x) + x \bullet D(y),$$

where $\bullet$ denotes the standard L-action on the dual module L^* (this is completely analogous to the embedding of $H^2(L,K)$ into $H^1(L,L^*)$ mentioned in §3).

For any finite dimensional simple Lie algebra, there is an isomorphism of $\mathfrak{g}$-modules $\mathfrak{g} \simeq \mathfrak{g}^*$, and we have an embedding of the space of bilinear forms in question into the space of linear maps $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the condition

$$D([x,y]) = -[D(x),y] - [x,D(y)].$$

Such maps are called *antiderivations* and were studied in several papers. In particular, in [Ho, Theorem 5.1] it is proved that central simple classical Lie algebras of dimension > 3 do not have nonzero antiderivations (generalizations of this result in different directions obtained further in the series of papers by Filippov, of which [F] is the last, and in [LL]). Hence, these Lie algebras do not have nonzero symmetric bilinear forms satisfying (4.15) either. □

This can be also compared with the fact that Leibniz cohomology (and, in particular, the second Leibniz cohomology with trivial coefficients) of $\mathfrak{g}$ vanishes (for homological version, see [P, Proposition 2.1] or [N]). The condition for a symmetric bilinear form φ to be a Leibniz 2-cocycle can be expressed as

$$\varphi([x,y],z)+\varphi([z,x],y)-\varphi([y,z],x)=0$$

which differs with (4.15) in the sign of the last term.

In the general case, where $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_n$, it is easy to see that $\varphi(\mathfrak{g}_i,\mathfrak{g}_j)=0$ for $i\neq j$, and hence the space of symmetric bilinear forms on $\mathfrak{g}$ satisfying (4.15) decomposes into the direct sum of appropriate spaces on each of $\mathfrak{g}_i$. The latter are determined by (4.13) if $\mathfrak{g}_i\simeq sl(2)$ and vanishes otherwise.

Now, turning to α's, and permuting in (4.14) i and j, we get $\alpha(t^i,t^j)=0$ for all $i,j\geq 2$ and $\alpha(t^k,t)=0$ for all $k\geq 3$. Conversely, it is easy to see that a skew-symmetric map α satisfying these conditions is cyclic. Hence, the space of skew-symmetric cyclic maps on $tK[t]$ is 1-dimensional and each cocycle of this type can be written in the form (4.12).

Cocycles of type (ii).

(iia). φ is skew-symmetric, α is symmetric.

Lemma 4.3. *Any skew-symmetric cyclic form on* $\mathfrak{g}$ *vanishes.*

Proof. The proof is almost identical to the proof of the well-known fact that any skew-symmetric invariant form on $\mathfrak{g}$ vanishes (see, for example, [Bo, Chapter 1, §6, Exercises 7(b) and 18(a,b)]). Namely, let φ be a skew-symmetric cyclic form on $\mathfrak{g}$. First, let $\mathfrak{g}$ be simple. There is a linear map $\sigma:\mathfrak{g}\to\mathfrak{g}$ such that $\varphi(x,y)=\langle\sigma(x),y\rangle$, where $\langle\cdot,\cdot\rangle$ is the Killing form on $\mathfrak{g}$. Then, for any $x,y,z\in\mathfrak{g}$:

$$\langle\sigma([x,y]),z\rangle=\varphi([x,y],z)=-\varphi(y,[z,x])=-\langle\sigma(y),[z,x]\rangle=-\langle[x,\sigma(y)],z\rangle.$$

Hence σ anticommutes with each $ad\,x$ and, in particular, $[\sigma(x),x]=0$ for any $x\in\mathfrak{g}$. But then by [Be, Lemme 2] (or by more general results from [LL]), σ belongs to the centroid of $\mathfrak{g}$, and hence is a scalar. Consequently, φ is proportional to the Killing form, and vanishes due to its skew-symmetry.

In the general case where $\mathfrak{g}$ is the direct sum of simple ideals $\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_n$, it is easy to see that $\varphi(\mathfrak{g}_i,\mathfrak{g}_j)=0$ for any $i\neq j$. But by just proved φ vanishes also on each $\mathfrak{g}_i$, and hence vanishes on the whole of $\mathfrak{g}$. □

(iib). φ is symmetric and α is skew-symmetric, so $\alpha\in HC^1(tK[t])$. There is an isomorphism of graded algebras

$$HC^*(K[t])\simeq HC^*(K)\oplus HC^*(tK[t])$$

(for a general relationship between cyclic homology of augmented algebra and its augmentation ideal, see [LQ, §4]; the cohomological version can be obtained in exactly the same way). On the other hand,

$$HC^*(K[t])\simeq HC^*(K)\oplus(\text{terms concentrated in degree 0})$$

(see, for example, [Lo, §3.1.7]). Hence $HC^1(tK[t]) = 0$. Of course, the vanishing of $HC^1(tK[t])$ can be established also by direct easy calculations.

Cocycles of type (iii) vanish.

Cocycles of type (iv). Obviously $\alpha(t^i, t^j) = 0$ for $(i, j) \neq (1, 1)$. Hence α is symmetric, and each cocycle of this type has the form (4.11).

To summarize: cocycles of type (ia), (ii) and (iii) either are trivial or vanish, and cocycles of type (ib) are given by formula (4.12) and vanish if $\mathfrak{g}$ does not contain direct summands isomorphic to $sl(2)$. So in the latter case, all nontrivial cocycles are of type (iv) and given by formula (4.11). Considering the natural grading in $\mathfrak{g} \otimes tK[t]$ by degrees of t, observing that the second cohomology is finite-dimensional and hence is dual to the second homology, and turning to interpretation of the 2-cycles as relations between generators, we get the assertion proved in [La] – that $\mathfrak{g} \otimes tK[t]$ admits a presentation with quadratic relations, provided $\mathfrak{g}$ does not contain direct summands isomorphic to $sl(2)$.

Let us decide now when cocycles given by (4.11) and (4.12) are cohomologically independent. According to Remark 1 in §1, any cohomological dependency beyond linear dependency can occur for cocycles of type (4.11) only. Writing, as in Remark 1, $\Phi = d\Omega$ for $\Omega = \sum_{i \in I} \omega_i \otimes \beta_i$, where $\omega_i : \mathfrak{g} \to \mathfrak{g}$ and $\beta_i : tK[t] \to tK[t]$ are some linear maps, we get, for $k \geq 3$:

$$\sum_{i \in I} \omega_i([x,y]) \beta_i(t^2) = \varphi(x,y) \quad \text{and} \quad \sum_{i \in I} \omega_i([x,y]) \beta_i(t^k) = 0.$$

Hence it is clear that cocycles of type (4.11) are cohomologically independent if and only if the corresponding skew-symmetric bilinear forms are independent modulo 2-coboundaries.

□

In [La], the author speculates about the possibility to derive Theorem 4.1 from the standard presentation of the affine Kac-Moody algebra. Let us indicate briefly how one can do the opposite: namely, how Theorem 4.1 allows us to recover the Serre relations between Chevalley generators of non-twisted affine Kac-Moody algebras.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over an algebraically closed field K of characteristic 0, and $\widehat{L} = \mathfrak{g} \otimes K[t, t^{-1}]$ be a "centerless, derivation-free" part of affine non-twisted Kac-Moody algebra. It is well-known that it admits a triangular decomposition which, with slight rearrangements of terms, can be written in the form

$$\widehat{L} = \Big((\mathfrak{g} \otimes t^{-1} K[t^{-1}]) \oplus (\mathfrak{n}_- \otimes 1) \Big) \oplus \Big(\mathfrak{h} \otimes 1 \Big) \oplus \Big((\mathfrak{g} \otimes tK[t]) \oplus (\mathfrak{n}_+ \otimes 1) \Big),$$

where $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is the triangular decomposition of the simple Lie algebra $\mathfrak{g}$ ([K, §7.6]; all direct sums are direct sums of vector spaces).

Consider the Hochschild–Serre spectral sequence abutting to $H^*((\mathfrak{g} \otimes tK[t]) \oplus (\mathfrak{n}_+ \otimes 1), K)$ with respect to the ideal $\mathfrak{g} \otimes tK[t]$. The general result contained implicitly in [HS] and explicitly, for example, in [R, Lemma 1], tells that this spectral sequence degenerates at E_2 term. For the sake of simplicity, let us exclude the case where $\mathfrak{g} = sl(2)$. Then the E_2 terms affecting the second cohomology in question are:

$$
\begin{aligned}
E_\infty^{02} &= E_2^{02} = H^2(\mathfrak{n}_+, H^0(\mathfrak{g}\otimes tK[t], K)) \simeq H^2(\mathfrak{n}_+, K),\\
E_\infty^{11} &= E_2^{11} = H^1(\mathfrak{n}_+, H^1(\mathfrak{g}\otimes tK[t], K)) \simeq H^1(\mathfrak{n}_+, \mathfrak{g}),\\
E_\infty^{20} &= E_2^{20} = H^0(\mathfrak{n}_+, H^2(\mathfrak{g}\otimes tK[t], K)) \simeq (C^2(\mathfrak{g}, K)/B^2(\mathfrak{g}, K))^{\mathfrak{n}_+}.
\end{aligned}
$$

The first and second isomorphisms here are obvious, the third one follows from Theorem 4.1. Here the first term corresponds to relations between elements of $\mathfrak{n}_+ \otimes 1$, which are (classical) Serre relations for the finite-dimensional Lie algebra $\mathfrak{g}$, the second term corresponds to relations between elements of $\mathfrak{g}\otimes tK[t]$ and $\mathfrak{n}_+ \otimes 1$, and the third one corresponds to relations between elements of $\mathfrak{g}\otimes tK[t]$. Writing them as Chevalley generators in terms of the corresponding Cartan matrix, we get the corresponding part of the Serre relations.

Repeating the similar reasonings for the "minus" part, and completing relations in an obvious manner between the "plus" and "minus" parts and the "Cartan subalgebra" $\mathfrak{h}\otimes 1$, we get the complete set of the Serre relations for $\widehat{L}$. The whole affine non-twisted Kac-Moody algebra is obtained from $\widehat{L}$ by the well-known construction which adds central extension and derivation, and its presentation readily follows from the presentation of $\widehat{L}$.

This approach is by all means not new. For example, we find in [LP, Remark in §2]: "Similar calculations by induction on the rank for simple finite-dimensional and loop algebras give the shortest known to us proof of completeness of the Serre defining relations". "Induction on the rank" means induction with repetitive application of the Hochschild–Serre spectral sequence relative to a Kac-Moody algebra build upon the simple finite-dimensional subalgebra of $\mathfrak{g}$. Here we use the Hochschild-Serre spectral sequence in a different way, and only once, thus getting even shorter proof.

It is mentioned in [K, §9.16] that "a simple cohomological proof" of the completeness of the Serre defining relations "was found by O. Mathieu (unpublished)". We presume that the approach outlined here is similar to that unpublished proof.

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