# Chordal and Complete Structures in <br> Combinatorics and Commutative Algebra 

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### 0.1 Abstract

This thesis is divided into two parts. The first part is concerned with the commutative algebra of certain combinatorial structures arising from uniform hypergraphs. The main focus lies on two particular classes of hypergraphs called chordal hypergraphs and complete hypergraphs, respectively. Both these classes arise naturally as generalizations of the corresponding well known classes of simple graphs. The classes of chordal and complete hypergraphs are introduced and studied in Chapter 2 and Chapter 3 respectively. Chapter 4, that is the content of [14], answers a question posed at the P.R.A.G.MAT.I.C. summer school held in Catania, Italy, in 2008. In Chapter 5 we study hypergraph analogues of line graphs and cycle graphs. Chapter 6 is concerned with a connectedness notion for hypergraphs and in Chapter 7 we study a weak version of shellability.

The second part is concerned with affine monoids and their monoid rings. Chapter 8 provide a combinatorial study of a class of positive affine monoids that behaves in some sense like numerical monoids. Chapter 9 is devoted to the class of numerical monoids of maximal embedding dimension. A combinatorial description of the graded Betti numbers of the corresponding monoid rings in terms of the minimal generators of the monoids is provided. Chapter 10 is concerned with monomial subrings generated by edge sets of complete hypergraphs.

### 0.2 List of Papers

The thesis is based on the following papers.
[23] E. Emtander. Betti numbers of hypergraphs. Comm. Algebra, 37(5):15451571, 2009.
[24] E. Emtander. A class of hypergraphs that generalizes chordal graphs. Math. Scand., 106(1):50-66, 2010.
[14] V. Crispin Quinonez and E. Emtander. Componentswise linearity of ideals arising from graphs. Matematiche (Catania), 63(2):185-189, 2008.
[27] E. Emtander, F. Mohammadi, and S. Moradi. Some algebraic properties of hypergraphs. To appear in Czechoslovak Mathematical Journal.
[26] E. Emtander, R. Fröberg, F. Mohammadi, and S. Moradi. Poincaré series of some hypergraph algebras. To appear in Mathematica Scandinavica.
[25] E. Emtander. Betti numbers of some semigroup rings. J. Commut. Algebra, 2(3), Fall 2010.
[22] E. Emtander. On positive affine monoids. Submitted.

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## Chapter 1

## Introduction and preliminaries

### 1.1 Introduction

Combinatorial commutative algebra lies, as the name suggests, in the intersection of combinatorics and commutative algebra and was created by Hochster and Stanley in the mid-seventies. A wealth of mathematics has, and still do, come out of the subject and new connections between combinatorics and commutative algebra and other fields of mathematics and related sciences are being discovered.

Edge ideals of simple graphs were introduced by Villarreal in the nineties, [67], and have since then been extensively studied. See for example [17, 28, 29, $30,32,39,46,45,50,49,54,55,60,61,63,70,71]$. Among all simple graphs chordal graphs and various complete graphs have been in particular focus and many nice results concerning their algebraic properties have been presented. An interesting question is then of course if there are natural hypergraph analogous of chordal and complete graphs that provides similar results. This thesis is mainly devoted to studying this question.

A milestone and a starting point for many algebraic investigations of graph algebras is a famous theorem by Fröberg that classifies all simple graphs for which the corresponding graph algebra has linear resolution:

Theorem 1.1.1 (Fröberg, [39]). Let $\mathcal{G}$ be a simple graph on $n \in \mathbb{N}$ vertices. Then the graph algebra $k\left[x_{1}, \ldots, x_{n}\right] / I(\mathcal{G})$ has linear resolution precisely when $\mathcal{G}^{c}$ is chordal.

Inspired by this beautiful theorem researchers have subsequently been attacking the much harder problem of giving classes of (often square-free) monomial ideals generated in some degree $d \geq 3$ that have linear resolutions. By Alexander duality this is the same thing as providing classes of Cohen-Macaulay
rings. Thus such tasks are interesting but of course one should not hope to classify all monomial ideals with linear resolutions.

The key in investigating hypergraph algebras is often to find a suitable way of translating the combinatorial information of the hypergraphs or of the corresponding simplicial complexes to the algebras themselves. There are two celebrated results by Hochster that often do the trick. One of them links the local cohomology of Stanley-Reisner rings with the simplicial homology of certain simplicial complexes and the other, called Hochster's formula, provides a combinatorial interpretation of the Betti numbers of Stanley-Reisner rings. The later result is well suited when considering linear resolutions and is the key ingredient of Fröberg's proof of the above theorem.

Shellable simplicial complexes were introduced by Björner and Wachs, see $[8,9]$. Many natural combinatorial objects gives rise to shellable simplicial complexes. It is well known that shellability implies Cohen-Macaulayness and that the Alexander dual notion of shellability is that of linear quotients. Besides [ 8,9$]$ the interested reader should consider also [47, 48, 55, 56, 65, 70]. In Chapter 7 we study a weak version of shellability that combinatorially resembles shellability but homologically behaves different.

The notion of being connected is natural for simple graphs and simplicial complexes. It can be stated in terms of the homology of the corresponding chain complexes. In Chapter 6 we consider a connectedness property for uniform hypergraphs that is inspired by this homological interpretation of the connectedness notion for graphs. It turns out that our connectedness notion is related to the depths of the corresponding hypergraph algebras. However, due to its complexity it is hard to understand how a connected uniform hypergraph "should" look and behave, and other authors have considered other notions of connected hypergraphs, see [46].

The polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ is naturally $\mathbb{N}^{n}$-graded. If one instead considers a grading of $k\left[x_{1}, \ldots, x_{n}\right]$ by an arbitrary positive affine monoid $S$ one stumbles into the land of toric varieties and polyhedral cones. For positive affine monoids $S$ much information about the monoid rings $k[S]$, for example their dimensions and descriptions of their monomial prime ideals, is contained in the monoid itself and also in its geometric manifestation $\mathbb{R}_{\geq 0} S$. The geometry of the cone $\mathbb{R}_{\geq 0} S$ is intimately connected to the local cohomology of $k[S]$. Details may be found in the books [11, 59].

If one considers positive affine monoids whose sets of minimal generators satisfy some strong combinatorial condition, one usually obtains strong results for the corresponding monoid rings $k[S]$. There are many nice results concerning monoid rings corresponding in this way to various kinds of matroids. The fundamental underlying combinatorial property here is the exchange property of the set of bases of a matroid. We recommend the papers [3, 15, 48, 56] and the books [66] and [68, Chapters 7, 8 and 9].

### 1.2 Preliminaries and notation

In this section we briefly recall relevant background material and establish our notation. General background on commutative algebra may be found in the books [2, 57]. A reference on hypergraphs in general is Berge's book [7]. [11, 19, 59] are excellent resources on combinatorial commutative algebra. Background in topology and homological algebra, respectively, may be found in [58] and [69]. The book [10] covers monoids and monoid rings and the related geometry of polyhedra. For background on numerical monoids we recommend [43].

### 1.2.1 Rings and ideals

Let $k$ be a field and denote by $R$ the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ in the indeterminates $\left\{x_{1}, \ldots, x_{n}\right\}$. By $\mathbb{N}$ we mean the set $\{0,1,2,3, \ldots\}$ of non-zero integers. A monomial $m$ in $R$ is a product $m=x_{1}^{a_{1}} \cdots x_{n}^{a_{n}}$, where $a_{i} \in \mathbb{N}$ for all $i \in\{1,2, \ldots, n\}$. If $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{N}^{n}$, we also use the notation $m=\mathbf{x}^{\mathbf{a}}$ for $m$. We frequently identify a subset $W \subseteq\{1,2, \ldots, n\}$ with its characteristic vector $v(W) \in \mathbb{N}^{n}$ whose $i$ th component is 1 if $i \in W$ and zero otherwise. Employing this we also write $\mathbf{x}^{W}$ for the monomial $\mathbf{x}^{v(W)}$.

The ring $R$ is naturally both $\mathbb{N}$ - and $\mathbb{N}^{n}$-graded where in the $\mathbb{N}$-grading $\operatorname{deg} x_{i}=1$ for all $i \in\{1, \ldots, n\}$ whereas $\operatorname{deg} x_{i}=e_{i}$ in the $\mathbb{N}^{n}$-grading. Here $e_{i}$ is the $i$ th unit vector $\mathbf{x}^{v(\{i\})}$. With the $\mathbb{N}$-grading $R$ is a standard graded $k$-algebra.

If $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{N}^{n}$ we call $\operatorname{supp}(\mathbf{a})=\left\{i \in\{1, \ldots, n\} ; a_{i} \neq 0\right\}$ the support of $\mathbf{a}$.

A vector $\mathbf{a} \in \mathbb{N}^{\mathbf{n}}$ is called square-free if its non-zero components are 1 and a monomial $\mathbf{x}^{\mathbf{a}} \in R$ is called square-free it its exponent vector $\mathbf{a}$ is square-free.

A monomial ideal $I \subseteq R$ is an ideal generated by monomials and $I$ is called a square-free monomial ideal if it is generated by square-free monomials. By Hilbert's basis theorem and the fact that a monomial ideal $I$ is a $\mathbb{N}^{n}$-graded $R$-submodule of $R$, there is a unique minimal set of monomial generators of $I$. We denote this set $\mathcal{G}(I)$. If $J \subseteq R$ is an ideal generated by the elements $\left\{g_{1}, \ldots, g_{r}\right\}$, we denote this by $J=\left(g_{1}, \ldots, g_{r}\right)$.

The depth of a finitely generated $R$-module is

$$
\operatorname{depth} M=\min \left\{i ; \operatorname{Ext}_{R}^{i}(R / \mathfrak{m}, M) \neq(0)\right\}
$$

where (0) denotes the zero $R$-module. The depth of an $R$-module $M$ equals the length of the longest possible $M$-sequence in $R$. The dimension $\operatorname{dim} M$ of a $R$-module $M$, is the Krull dimension of $R / \operatorname{Ann} M$. We always have depth $M \leq$ $\operatorname{dim} M$ and in case of equality one says that $M$ is a Cohen-Macaulay $R$-module.

Let $I \subseteq R$ be a monomial ideal minimally generated by the set $\left\{g_{1}, \ldots, g_{t}\right\}$. $I$ is said to have linear quotients if there exists an ordering $g_{1}<\cdots<g_{t}$ of the minimal generators of $I$ such that for each $s \in\{1, \ldots, t\}$, the colon ideal $\left(g_{1}, \ldots, g_{s-1}\right): g_{s}$ is generated by a subset of the variables $\left\{x_{1}, \ldots, x_{n}\right\}$.

### 1.2.2 Hypergraphs

Let $\mathcal{V}$ be a finite set and $\mathcal{E}=\left\{E_{1}, \ldots, E_{s}\right\}$ a finite collection of non-empty distinct subsets of $\mathcal{V}$. The pair $\mathcal{H}=(\mathcal{V}, \mathcal{E})$ is called a hypergraph. The elements of $\mathcal{V}$ and $\mathcal{E}$, respectively, are called the vertices and the edges, respectively, of the hypergraph. If we want to specify what hypergraph we consider, we write $\mathcal{V}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ for the vertices and edges respectively. If $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$ we usually identify $\mathcal{V}$ and the generic set of $n$ objects $[n]=\{1,2, \ldots, n\}$.

Let $\mathcal{H}$ be a hypergraph. A sub-hypergraph $\mathcal{K}$ of $\mathcal{H}$ is a hypergraph such that $\mathcal{V}(\mathcal{K}) \subseteq \mathcal{V}(\mathcal{H})$, and $\mathcal{E}(\mathcal{K}) \subseteq \mathcal{E}(\mathcal{H})$. If $\mathcal{W} \subseteq \mathcal{V}$, the induced hypergraph on $\mathcal{W}$, $\mathcal{H}_{\mathcal{W}}$, is the sub-hypergraph with $\mathcal{V}\left(\mathcal{H}_{\mathcal{W}}\right)=\mathcal{W}$ and with $\mathcal{E}\left(\mathcal{H}_{\mathcal{W}}\right)$ consisting of the edges of $\mathcal{H}$ that lie entirely in $\mathcal{W}$. A hypergraph $\mathcal{H}$ is said to be $d$-uniform if $\left|E_{i}\right|=d$ for every edge $E_{i} \in \mathcal{E}(\mathcal{H})$. We always assume $d \geq 2$. By a uniform hypergraph we mean a hypergraph that is $d$-uniform for some $d \in \mathbb{N}$. Note that a 2 -uniform hypergraph is an ordinary simple graph. A free vertex $v$ of a hypergraph is a vertex $v$ that lies in at most one edge. The complementary hypergraph, $\mathcal{H}^{c}$, of a $d$-uniform hypergraph $\mathcal{H}$ is defined as the hypergraph with $\mathcal{V}(\mathcal{H})^{c}=\mathcal{V}(\mathcal{H})$ and with edge set

$$
\mathcal{E}\left(\mathcal{H}^{c}\right)=\{F \subseteq \mathcal{V}(\mathcal{H}) ;|F|=d, F \notin \mathcal{E}(\mathcal{H})\} .
$$

### 1.2.3 Simplicial complexes

An (abstract) simplicial complex on vertex set $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$ is a collection, $\Delta$, of distinct subsets of $\mathcal{V}$ with the property that $G \subseteq F, F \in \Delta \Rightarrow G \in \Delta$. As for hypergraphs, we usually identify $\mathcal{V}$ and $[n]$. The elements of $\Delta$ are called the faces of the complex and the maximal (under inclusion) faces are called facets. The set of facets of $\Delta$ we denote by $\mathcal{F}(\Delta)$. The dimension, $\operatorname{dim} F$, of a face $F$ in $\Delta$ is defined as $\operatorname{dim} F=|F|-1$. The dimension of $\Delta$ is $\operatorname{dim} \Delta=\max \{\operatorname{dim} F ; F \in \Delta\}$. The $r$-skeleton of a simplicial complex $\Delta$ is the collection of faces of $\Delta$ of dimension at most $r$. The empty set $\emptyset$ is the unique -1 dimensional face of every complex that is not the void complex $\}$ which has no faces. The dimension of the void complex we define as $-\infty$. If $\mathcal{W} \subseteq \mathcal{V}$ we denote by $\Delta_{W}$ the simplicial complex

$$
\Delta_{W}=\{F \subseteq V ; F \in \Delta, F \subseteq W\}
$$

Given a simplicial complex $\Delta, \tilde{\mathcal{C}} .(\Delta ; k)$ denotes its reduced chain complex with coefficients in the field $k$ and $\tilde{H}_{n}(\Delta ; k)=Z_{n}(\Delta ; k) / B_{n}(\Delta ; k)$ the $n$th reduced homology group of this complex. For convenience, we define the homology of the void complex to be zero.

Let $\Delta$ be an arbitrary simplicial complex on vertex set $\mathcal{V}$. We then define the Alexander dual simplicial complex of $\Delta$ as

$$
\Delta^{*}=\{F \subseteq \mathcal{V} ; \mathcal{V} \backslash F \notin \Delta\}
$$

Note that $\left(\Delta^{*}\right)^{*}=\Delta$.

If $X$ and $Y$ are two sets, we denote their disjoint union by $X \sqcup Y$. Let $\Delta$ and $\Gamma$ be two non-empty simplicial complexes on disjoint vertex sets $\left\{x_{1}, \ldots, x_{n}\right\}$ and $\left\{y_{1}, \ldots, y_{m}\right\}$ respectively. The join $\Delta * \Gamma$ of $\Delta$ and $\Gamma$ is the simplicial complex on vertex set $\left\{x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right\}$ having faces $\left\{x_{i_{1}}, \ldots, x_{i_{r}}, y_{j_{1}}, \ldots, y_{j_{s}}\right\}$, where $\left\{x_{i_{1}}, \ldots, x_{i_{r}}\right\}$ and $\left\{y_{j_{1}}, \ldots, y_{j_{s}}\right\}$ are faces of $\Delta$ and $\Gamma$ respectively.

Given a finite collection $\left\{F_{1}, \ldots, F_{t}\right\}$ of non-empty subsets of $\mathcal{V}$ we denote by $\left\langle F_{1}, \ldots, F_{t}\right\rangle$ the simplicial complex with $\mathcal{F}(\Delta)=\left\{F_{1}, \ldots, F_{t}\right\}$.

We recall the definition of the classes of pure shellable and non-pure shellable simplicial complexes:

Definition 1.2.1. Let $\Delta$ be a simplicial complex with $\mathcal{F}(\Delta)=\left\{F_{1}, \ldots, F_{t}\right\}$. $\Delta$ is called pure shellable if
(i) $\left|F_{i}\right|=\left|F_{j}\right|$ for every pair of indices $1 \leq i<j \leq t$.
(ii) There exists an ordering $F_{1}<\cdots<F_{t}$ of the facets such that $\left\langle F_{j}\right\rangle \cap$ $\left\langle F_{1}, \ldots F_{j-1}\right\rangle$ is generated by a non-empty set of proper maximal faces of $\left\langle F_{j}\right\rangle$ for every $j \in\{2, \ldots, t\}$.

A simplicial complex $\Delta$ is called non-pure shellable if (ii) but not ( $i$ ) holds. If $\Delta$ is a shellable (pure of non-pure) simplicial complex a linear order $F_{1}<\cdots<F_{t}$ of $\mathcal{F}(\Delta)$ as in $(i i)$ is called a shelling of $\Delta$.

For other equivalent descriptions of shellable simplicial complexes, see Theorem 7.1.8.

### 1.2.4 Stanley-Reisner rings and ideals

Let $\Delta$ be a simplicial complex on vertex set $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$. The StanleyReisner ring $k[\Delta]$ of $\Delta$ is the quotient of the ring $R=k\left[x_{1}, \ldots, x_{n}\right]$ by the Stanley-Reisner ideal

$$
I_{\Delta}=\left(\mathbf{x}^{F} ; F \notin \Delta\right)
$$

In this way a vertex $v_{i}$ of $\Delta$ is identified with a variable $x_{i}$ in the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$. Conversely, to every square-free monomial ideal $I$ one may associate a simplicial complex $\Delta_{I}$ in such way that its Stanley-Reisner ideal is precisely $I$.

The above correspondence between simplicial complexes and square-free monomial ideals yields the following duality: If $I=I_{\Delta}$ is the Stanley-Reisner ideal of a simplicial complex $\Delta$ we may associate to it its Alexander dual ideal $\left(I_{\Delta}\right)^{*}=I_{\Delta^{*}}$. One verifies that $\left(\left(I_{\Delta}\right)^{*}\right)^{*}=I_{\Delta}$. A simplicial complex $\Delta$ is called Cohen-Macaulay if the Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay.

It is well known that a Stanley-Reisner ideal $I_{\Delta}$ has linear quotients if and only if the Alexander dual simplicial complex $\Delta^{*}$ is shellable. See also Theorem 7.1.7.

### 1.2.5 Edge ideals and independence complexes

Let $\mathcal{H}$ be a $d$-uniform hypergraph on vertex set $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$ and consider the simplicial complex

$$
\Delta(\mathcal{H})=\left\{F \subseteq \mathcal{V} ; E_{i} \nsubseteq F, E_{i} \in \mathcal{E}(\mathcal{H})\right\}
$$

This is called the independence complex of $\mathcal{H}$. Note that the edges of $\mathcal{H}$ are precisely the minimal non-faces in $\Delta(\mathcal{H})$. Thus the Stanley-Reisner ideal of $\Delta(\mathcal{H})$ is

$$
I(\mathcal{H})=\left(\mathbf{x}^{E_{i}} ; E_{i} \in \mathcal{H}\right) .
$$

This ideal is called the edge ideal of $\mathcal{H}$. The Stanley-Reisner ring $k[\Delta(\mathcal{H})]$ is called the also called hypergraph algebra of $\mathcal{H}$. Another simplicial complex we associate to $\mathcal{H}$ is its clique complex:

$$
\Delta_{\mathcal{H}}=\left\{F \subseteq \mathcal{V} ;\binom{F}{d} \subseteq \mathcal{E}(\mathcal{H})\right\}
$$

where $\binom{F}{d}$ denotes the set of all cardinality $d$ subsets of the set $F$. Observe that all sets $F \subseteq \mathcal{V}$ with $|F|<d$ are faces of $\Delta_{\mathcal{H}}$. If $\mathcal{H}$ is uniform, we have $\Delta_{\mathcal{H}}=\Delta\left(\mathcal{H}^{c}\right)$.

### 1.2.6 Resolutions and Betti numbers

To every finitely generated $\mathbb{N}$-graded module $M$ over the polynomial ring $R=$ $k\left[x_{1}, \ldots, x_{n}\right]$ we may, in an essentially unique way, associate a minimal graded free resolution

$$
0 \rightarrow \bigoplus_{j} R(-j)^{\beta_{l, j}(M)} \rightarrow \cdots \rightarrow \bigoplus_{j} R(-j)^{\beta_{0, j}(M)} \rightarrow M \rightarrow 0
$$

where $l \leq n$ and $R(-j)$ is the $R$-module obtained from $R$ by shifting the degrees by $j$. Thus, $R(-j)$ is the graded $R$-module in which the grade $i$ component $(R(-j))_{i}$ is $R_{i-j}$.

The number $\beta_{i, j}(M)$ is called the $i$ th $\mathbb{N}$-graded Betti number of $M$ in degree $j$. The total $i$ 'th Betti number is $\beta_{i}(M)=\sum_{j} \beta_{i, j}$. Observe that we may equally well consider the $\mathbb{N}^{n}$-graded minimal free resolution and Betti numbers of $M$. Then we instead use shifts $R(-\mathbf{j}), \mathbf{j} \in \mathbb{N}^{n}$.

We are interested in the Betti numbers of quotient rings $R / I$. Hence, since in this situation $\beta_{0}(R / I)=1$, the interesting parts of the resolutions are the homological degrees greater than zero. In the various formulas for Betti numbers we give we therefore sometimes omit homological degree 0 .

The Betti numbers of a finitely generated graded $R$-module $M$ occur as the $k$-dimensions of certain $R$-modules: Let $\mathfrak{m}=\left(x_{1}, \ldots, x_{n}\right)$ be the unique graded maximal ideal of $R$ and put $k=R / \mathfrak{m}$. We have

$$
\beta_{i, \mathbf{j}}(M)=\operatorname{dim}_{k} \operatorname{Tor}_{i}^{R}(M, k)_{\mathbf{j}}
$$

Thus the Betti numbers depend on the field $k$. However, for convenience, we omit $k$ in the notation for the Betti numbers.

The projective dimension $\operatorname{pd} M$ of $M$ over $R$ is

$$
\operatorname{pd} M=\max \left\{i ; \exists \beta_{i, j}(M) \neq 0\right\} .
$$

A minimal free resolution of $M$ is said to be linear if for $i>0, \beta_{i, j}(M)=0$ whenever $j \neq i+d-1$ for some fixed natural number $d \geq 1$. In connection to this we mention the Eagon-Reiner theorem:

Theorem 1.2.2 ([18], Theorem 3). Let $\Delta$ be a simplicial complex and $\Delta^{*}$ its Alexander dual simplicial complex. Then $k[\Delta]$ is Cohen-Macaulay if and only if $k\left[\Delta^{*}\right]$ has linear minimal free resolution.

A common way of displaying the Betti numbers of a $R$-module $M$ is giving the Betti diagram of $M$ :

|  | 0 | $\cdots$ | $s$ | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | $\beta_{0,0}(M)$ | $\cdots$ | $\beta_{s, s}(M)$ | $\cdots$ |
| $\ldots$ | $\ldots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $i$ | $\beta_{0, i}(M)$ | $\cdots$ | $\beta_{s, s+i}(M)$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Observe that if $M$ has linear resolution there is at most one $i>0$ for which there are non-zero entries in row $i$ in the Betti diagram. More about Betti diagrams may be found in for example [20].

One further result which we will use is the Auslander-Buchsbaum formula ([19], Exercise 19.8):

Theorem 1.2.3 (The Auslander-Buchsbaum formula). Let $R$ be a finitely generated graded $k$-algebra for some field $k$ and $M \neq 0$ a finitely generated graded $R$-module with $\operatorname{pd} M<\infty$, then

$$
\operatorname{pd} M+\operatorname{depth} M=\operatorname{depth} R
$$

### 1.2.7 Hochster's formula

In topology one defines Betti numbers in a somewhat different manner. Hochster's formula provides a link between these and the Betti numbers defined above.

Theorem 1.2.4 (Hochster's formula, [11], Theorem 5.5.1.). Let $k[\Delta]$ be the Stanley-Reisner ring of a simplicial complex $\Delta$. The non-zero Betti numbers of $k[\Delta]$ are only in squarefree degrees $\mathbf{j}$ and may be expressed as

$$
\beta_{i, \mathbf{j}}(k[\Delta])=\operatorname{dim}_{k} \tilde{H}_{|\mathbf{j}|-i-1}\left(\Delta_{\operatorname{supp}(\mathbf{j})} ; k\right)
$$

Hence the total ith Betti number may be expressed as

$$
\beta_{i}(k[\Delta])=\sum_{V \subseteq[n]} \operatorname{dim} \tilde{H}_{|V|-i-1}\left(\Delta_{V} ; k\right) .
$$

If one has $\mathbb{N}^{n}$-graded Betti numbers, it is easy to obtain the $\mathbb{N}$-graded ones since

$$
\beta_{i, j}(k[\Delta])=\sum_{\substack{\mathbf{j}^{\prime} \in \mathbb{N}^{n} \\\left|\mathbf{j}^{\prime}\right|=j}} \beta_{i, \mathbf{j}^{\prime}}(k[\Delta]) .
$$

Thus

$$
\beta_{i, j}(k[\Delta])=\sum_{\substack{V \subseteq[n] \\|V|=j}} \operatorname{dim} \tilde{H}_{|V|-i-1}\left(\Delta_{V} ; k\right) .
$$

### 1.2.8 The Mayer-Vietoris sequence

If $R$ is a commutative ring and we have an exact sequence of complexes of $R$-modules

$$
\mathbf{0} \rightarrow \mathbf{L} \rightarrow \mathbf{M} \rightarrow \mathbf{N} \rightarrow \mathbf{0}
$$

there is a long exact homology sequence associated to it

$$
\cdots \rightarrow \tilde{H}_{r}(N) \rightarrow \tilde{H}_{r-1}(L) \rightarrow \tilde{H}_{r-1}(M) \rightarrow \tilde{H}_{r-1}(N) \rightarrow \cdots
$$

Suppose we have a simplicial complex $N$ and two sub-complexes $L$ and $M$, such that $N=L \cup M$. This gives us an exact sequence of reduced chain complexes

$$
\begin{equation*}
0 \rightarrow \tilde{\mathcal{C}} .(L \cap M ; k) \rightarrow \tilde{\mathcal{C}} .(L ; k) \oplus \tilde{\mathcal{C}} .(M ; k) \rightarrow \tilde{\mathcal{C}} .(N ; k) \rightarrow 0 \tag{1.1}
\end{equation*}
$$

where the non trivial maps are defined by $x \mapsto(x,-x)$ and $(x, y) \mapsto x+y$. The long exact homology sequence associated to (1.1) is called the Mayer-Vietoris sequence. More about the Mayer-Vietoris sequence can be found in [58, Section 4.4].

### 1.2.9 Künneth's theorem for simplicial homology

Let $\tilde{\mathcal{C}} .(\Delta ; k)$ and $\tilde{\mathcal{C}} .(\Gamma ; k)$ be the reduced chain complexes, with coefficients in a field $k$, of the simplicial complexes $\Delta$ and $\Gamma$ respectively. Künneth's tensor formula ([58, Theorem 10.1], or [69, Theorem 3.6.3]) then says

$$
\begin{equation*}
\tilde{H}_{n}(\tilde{\mathcal{C}} .(\Delta ; k) \otimes \tilde{\mathcal{C}} .(\Gamma ; k))=\bigoplus_{\substack{r+s=n \\ r, s \geq 0}} \tilde{H}_{r}(\Delta ; k) \otimes \tilde{H}_{s}(\Gamma ; k) \tag{1.2}
\end{equation*}
$$

We will use of this formula in connection to the join operation on simplicial complexes: It is easy to verify that the chain complex $\mathcal{C} .(\Delta * \Gamma ; k)$ of the join of two simplicial complexes $\Delta$ and $\Gamma$ is isomorphic to $(\tilde{\mathcal{C}} .(\Delta ; k) \otimes \tilde{\mathcal{C}} .(\Gamma ; k))(-1)$. This is the same as the complex $(\tilde{\mathcal{C}} .(\Delta ; k) \otimes \tilde{\mathcal{C}} .(\Gamma ; k))$ if we shift the homological degrees by 1 .

### 1.2.10 Affine monoids and monoid rings

An affine monoid $S=\left\langle s_{1}, \ldots, s_{n}\right\rangle$ is a finitely generated sub-monoid of $\mathbb{Z}^{r}$ for some $r \in \mathbb{N}, r \geq 1$. We denote by gp $S$ the group inside $\mathbb{Z}^{r}$ generated by $S$. Observe that every element $x \in \operatorname{gp} S$ can be written as $x=s-s^{\prime}$ for some elements $s$ and $s^{\prime}$ in $S$ and that gp $S$ is free of rank at most $r$. The rank of $S$, $\operatorname{rank} S$, is by definition the rank of gp $S$. We assume all affine monoids $S$ are embedded in $\mathbb{Z}^{d}$ where $d=\operatorname{rank} S$.

Our main concern will be positive affine monoids: An affine monoid is called positive if zero is the only element whose inverse in gp $S$ also lies in $S$. A positive affine monoid $S=\left\langle s_{1}, \ldots, s_{n}\right\rangle$ of rank $d$ is isomorphic to an affine monoid $T$ inside $\mathbb{N}^{d}$. Thus in the sequel all positive affine monoids $S$ will be considered to be inside $\mathbb{N}^{d}$ where $d=\operatorname{rank} S$.

Assume $S=\left\langle s_{1}, \ldots, s_{n}\right\rangle$ is a positive affine monoid of rank one such that $\operatorname{gcd}\left(s_{1}, \ldots, s_{n}\right)=1$. Then $S$ is called a numerical monoid.

Any affine (resp. positive affine) monoid $S=\left\langle s_{1}, \ldots, s_{n}\right\rangle$ gives rise to a cone (resp. pointed cone) $\mathbb{R}_{\geq 0} S=\mathbb{R}_{\geq 0}\left\{s_{1}, \ldots, s_{n}\right\}$, whose dimension $\operatorname{dim} \mathbb{R}_{\geq 0} S$ equals rank $S$. Recall that such a cone is the intersection of finitely many halfspaces in $\mathbb{R}^{d}$, where $d=\operatorname{rank} S$. Let $H^{+}$be a half-space in $\mathbb{R}_{\geq 0} S$ with bounding hyperplane $H$. Assume $H$ intersects $\mathbb{R}_{\geq 0} S$ and that $\mathbb{R}_{\geq 0} S$ lies entirely inside $H^{+}$. Then $F=H \cap \mathbb{R}_{\geq 0} S$ is called a face of the cone $\mathbb{R}_{\geq 0}$. The dimension of a face is by definition the dimension of its affine hull. A face of dimension $d-1$, where $d=\operatorname{dim} \mathbb{R}_{\geq 0} S$, is called a facet. The faces form a lattice under inclusion. See [10] for details.

If $S$ is a positive affine monoid the set $S_{+}=S \backslash\{0\}$ is called the maximal ideal of $S$.

The normalization of an affine monoid $S$, denoted $\bar{S}$, is the monoid

$$
\bar{S}=\{x \in \operatorname{gp} S ; m x \in S \text { for some } m \in \mathbb{N}, m>1\}
$$

We have $\bar{S}=\mathbb{R}_{\geq 0} S \cap \operatorname{gp} S$ and $\bar{S}$ is affine (resp. positive affine) when $S$ is affine (resp. positive affine). The normalization of $S$ is (by construction) a normal monoid. That is, a monoid $T$ such that if $m x \in T$ for some $m \in \mathbb{N}, m>1$, and $x \in \operatorname{gp} T$, then $x \in T$. An affine monoid is normal precisely when $S=\bar{S}$, see [10, Proposition 2.22].

As for numerical monoids we define the set of gaps of an affine monoid $S$ as $H(S)=\bar{S} \backslash S$. Also, we define a set $T(S)$ by

$$
T(S)=\left\{x \in \operatorname{gp} S ; x \notin S, x+S_{+} \subseteq S_{+}\right\}
$$

Remark 1.2.5. For numerical monoids the cardinality of the set $T(S)$ is called the type of $S$, denoted type $S$, and agrees with the Cohen-Macaulay type of the corresponding monoid ring.

Assume $S$ is an affine monoid. Then, by considering the elements $s \in S$ that lie in some bounding hyperplane of the cone $\mathbb{R}_{\geq 0} S$ and the affine form defining that hyperplane, we see that $T(S) \subseteq \mathbb{R}_{\geq 0} S$ and, in fact, $T(S) \subseteq \bar{S}$. Thus for affine monoids $S$ we have $T(S) \subseteq H(S)$.

We associate to an affine monoid $S=\left\langle s_{1}, \ldots, s_{n}\right\rangle$ its monoid ring $k[S], k$ being a field. This is the $k$-algebra $k\left[\mathbf{t}^{s} ; s \in S\right] \subseteq k\left[t_{1}, \ldots, t_{d}\right], d=\operatorname{rank} S$, with multiplication

$$
\mathbf{t}^{s} \cdot \mathbf{t}^{s^{\prime}}=\mathbf{t}^{s+s^{\prime}}, s \in S, s^{\prime} \in S
$$

The dimension of $k[S]$ coincides with the rank of $S$. The ring $k[S]$ is naturally $S$-graded. That is, it is a graded ring with non-zero components only in degrees $s \in S$.

Two important classes of numerical monoids are the classes of symmetric and quasi-symmetric numerical monoids. These classes of numerical monoids are characterized by the fact that $T(S)$ consists of one, respectively two, elements.

If $R$ is the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ we can define a homomorphism

$$
R \xrightarrow{\phi} k[S]
$$

by $x_{i} \mapsto \mathbf{t}^{s_{i}}$. Since $\phi$ is surjective we have a kernel $\operatorname{ker} \phi=\mathfrak{p}$, and consequently an isomorphism $k[S] \cong R / \mathfrak{p}$. The ideal $\mathfrak{p}$ is a prime ideal generated by binomials.

Definition 1.2.6. A monoid ring $k[S]=k\left[\mathbf{t}^{s_{1}}, \ldots, \mathbf{t}^{s_{n}}\right]$ corresponding to a positive affine monoid is called homogeneous if there is a vector $v \in \mathbb{Q}^{d}$ with

$$
s_{i} \cdot v=1
$$

for all $i \in\{1, \ldots, n\}$.
It is well known that $k[S]$ is homogeneous if and only if it is standard graded with respect to the grading

$$
\begin{equation*}
k[S]_{i}=\sum_{|b|=i} k\left\{\left(\mathbf{t}^{s_{1}}\right)^{b_{1}} \cdots\left(\mathbf{t}^{s_{n}}\right)^{b_{n}}\right\} \tag{1.3}
\end{equation*}
$$

where $|b|=b_{1}+\cdots+b_{n}$ for any vector $b \in \mathbb{N}^{n}$. For details, see [68], Proposition 7.2.39.

Lastly, let $S$ be an affine monoid. Then it is known that the integral closure $\overline{k[S]}$ of $k[S]$ in its field of fractions is $k[\bar{S}]$. For details, see for example [59, Proposition 7.25].

## Part I

## Hypergraph algebras

## Chapter 2

## Various complete hypergraphs

The complete graph $K_{n}$ is a fundamental and important graph. For $d$-uniform hypergraphs there is a natural analogue to $K_{n}$ that we call the $d$-complete hypergraph. If one instead tries to construct a hypergraph analogue to the complete multipartite graph $K_{n_{1}, \ldots, n_{t}}$ one realizes rather quickly that this may be done in several equally natural ways. In this chapter we present gradually inclusive classes of complete hypergraphs. We compute the Betti numbers of the hypergraph algebras of two of these classes and show that the hypergraph algebras corresponding to all classes have linear resolutions. The last fact here has subsequently been improved: In [60], Mohammadi, Moradi, and Kiani show that the corresponding edge ideals even have linear quotients. They do this by proving that the edge sets of all our complete hypergraph are weakly polymatroidal sets. The notion of weakly polymatroidal sets was introduced by Kokubo and Hibi. Among other things they show that weakly polymatroidal sets do provide linear quotients, see [56] for details.

### 2.1 Some results on induced hypergraphs

Let $\mathcal{H}$ be a $d$-uniform hypergraph. We say that two edges $E$ and $E^{\prime}$ are disjoint if $E \cap E^{\prime}=\emptyset$. By considering the Taylor resolution (see [6]) of $k[\Delta(\mathcal{H})]$, one can prove the following results, which are essentially due to Jacques, see [54].
Proposition 2.1.1. Let $\mathcal{H}$ be a d-uniform hypergraph. Then $\beta_{i, i d}(k[\Delta(\mathcal{H})])$ equals the number of induced hypergraphs that consist of $i$ disjoint edges.
Proof. For $d=2$ this is [54, Theorem 3.3.5]. The proof given there holds also for $d>2$.

Proposition 2.1.2. Let $\mathcal{H}$ be a hypergraph and $\mathcal{K}$ an induced hypergraph. Then

$$
\beta_{i, j}(k[\Delta(\mathcal{K})]) \leq \beta_{i, j}(k[\Delta(\mathcal{H})]) .
$$

Proof. Since $\mathcal{K}=\mathcal{H}_{W}$ for some $W \subseteq \mathcal{V}(\mathcal{H})$, we have

$$
\begin{gathered}
\beta_{i, j}(k[\Delta(\mathcal{H})])=\sum_{\substack{V \subseteq \mathcal{V}(\mathcal{H}) \\
|V|=j}} \operatorname{dim}_{k} \tilde{H}_{|V|-i-1}\left(\Delta(\mathcal{H})_{V} ; k\right) \geq \\
\sum_{\substack{V \subseteq W \\
|V|=j}} \operatorname{dim}_{k} \tilde{H}_{|V|-i-1}\left(\Delta(\mathcal{K})_{V} ; k\right)=\beta_{i, j}(k[\Delta(\mathcal{K})]) .
\end{gathered}
$$

Corollary 2.1.3. Let $\mathcal{H}$ be a hypergraph and $\mathcal{K}$ an induced hypergraph. Then

$$
\begin{aligned}
\beta_{i}(k[\Delta(\mathcal{K})]) & \leq \beta_{i}(k[\Delta(\mathcal{H})]) \\
\operatorname{pd} k[\Delta(\mathcal{K})] & \leq \operatorname{pd} k[\Delta(\mathcal{H})] .
\end{aligned}
$$

### 2.2 The $d$-complete hypergraph

We start by defining the $d$-complete hypergraph.
Definition 2.2.1. The $d$-complete hypergraph $K_{n}^{d}$ on a vertex set $\mathcal{V}$ of cardinality $n$ is defined by

$$
\mathcal{E}\left(K_{n}^{d}\right)=\binom{\mathcal{V}}{d}
$$

If $n<d$, we interpret $K_{n}^{d}$ as $n$ isolated points.
Remark 2.2.2. Observe that the combinatorial configuration $\mathcal{E}\left(K_{n}^{d}\right)$ occur elsewhere: It is used in a hypergraph setting by Berge, [7]. Also, it is clear that the set $\mathcal{E}\left(K_{n}^{d}\right)$ is the set of bases of a matroid. Hence it occurs also in such contexts. One may also find this configuration in purely algebraic contexts since the edges correspond to the generators of the square-free Veronese subring, see Remark 10.2.3.

We now compute the Betti numbers of $k\left[\Delta\left(K_{n}^{d}\right)\right]$.
Theorem 2.2.3. The ring $k\left[\Delta\left(K_{n}^{d}\right)\right]$ has linear minimal free resolution and the $\mathbb{N}$-graded Betti numbers may be written as

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n}^{d}\right)\right]\right)= \begin{cases}\binom{n}{j}\binom{j-1}{d-1}, & j=i+(d-1) \\ 0, & j \neq i+(d-1)\end{cases}
$$

In particular the Betti numbers are independent of the characteristic of the field $k$.

Proof. The fact that the minimal free resolution is linear will follow from the more general Theorem 2.6.3. Let $\mathcal{V},|\mathcal{V}|=n$, be the vertex set of $K_{n}^{d}$. Recall that Hochster's formula says

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n}^{d}\right)\right]\right)=\sum_{\substack{V \subseteq \mathcal{V} \\|V|=j}} \operatorname{dim}_{k} \tilde{H}_{|V|-i-1}\left(\Delta\left(K_{n}^{d}\right)_{V} ; k\right) .
$$

Consider $\Delta\left(K_{n}^{d}\right)_{V}$ for some $V \subseteq \mathcal{V}$. It is clear that every cycle in $Z_{d-2}\left(\Delta\left(K_{n}^{d}\right)_{V} ; k\right)$ is a linear combination of "elementary cycles", by which we mean the derivatives of $(d-1)$-simplices in the simplex $\langle\mathcal{V}\rangle$. Denote this generating set by $\mathcal{G}_{V}$.

We note that we may extract a smaller generating set out of $\mathcal{G}_{V}$. Namely, we claim that it is enough to consider the elements that contain a fixed vertex $x \in V$ (by containing $x$ we mean that some term in the cycle contains $x$ ). Denote this set by $\mathcal{G}_{V}(x)$ and consider an element $\partial\left(\left\{x_{1}, \ldots, x_{d}\right\}\right)$ in $\mathcal{G}_{V}$, that do not contain $x$. This cycle is a linear combination of elements in $\mathcal{G}_{V}(x)$, which may be seen by first forming the cone (see [11] p. 230) $x *\left\{x_{1}, \ldots, x_{d}\right\}$, and then taking the derivative of the $(d-1)$-skeleton of this cone. This proves our claim.

Furthermore the images $\bar{\sigma}$ in the homology group $\tilde{H}_{d-2}\left(\Delta\left(K_{n}^{d}\right) ; k\right)$ of the elements $\sigma \in \mathcal{G}_{V}(x)$ are linearly independent: Assume that $\sum_{i=1}^{t} a_{i} \bar{\sigma}_{i}=0$, $a_{i} \in k=k\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1}, \ldots, x_{n}\right)$ and $\sigma_{i} \in \mathcal{G}_{V}(x)$. Every $\sigma_{i}$ contains a unique term which does not contain $x$. This is because $\sigma_{i}=\partial\left(\Sigma_{i}\right)$, where $\Sigma_{i}$ is a $(d-1)$ simplex. Hence $a_{i}=0$ for every $i \in\{1, \ldots, t\}$.

Now we are done, since if $|V|=j$, the cardinality of $\mathcal{G}_{V}(x)$ clearly is $\binom{j-1}{d-1}$, and the number of $j$-sets $V \subseteq \mathcal{V}$ are $\binom{n}{j}$.

Due to Corollary 2.2.4, the above result also follows from [51, Theorem 1]. Corollary 2.2 .4 seems to be well known, but we did not find a published proof.

Since $\Delta\left(K_{n}^{d}\right)$ has a specially nice structure, it is easy to determine its Alexander dual. As the minimal non-faces of $\Delta\left(K_{n}^{d}\right)$ are all $\left\{x_{i_{1}}, \ldots, x_{i_{d}}\right\}, x_{i_{j}} \in \mathcal{V}$, the facets of $\Delta\left(K_{n}^{d}\right)^{*}$ are all $\left\{x_{i_{1}}, \ldots, x_{i_{n-d}}\right\}, x_{i_{j}} \in \mathcal{V}$. Whence $\Delta\left(K_{n}^{d}\right)^{*} \cong$ $\Delta\left(K_{n}^{n-d+1}\right)$.
Corollary 2.2.4. The ring $k\left[\Delta\left(K_{n}^{d}\right)\right]$ is Cohen-Macaulay and we have

$$
\begin{aligned}
\beta_{i}\left(k\left[\Delta\left(K_{n}^{d}\right)\right]\right) & =\binom{n}{j}\binom{j-1}{d-1} \\
\operatorname{pd} k\left[\Delta\left(K_{n}^{d}\right)\right] & =n-(d-1)
\end{aligned}
$$

where $j=i+(d-1)$.
Proof. The last two claims follows directly from the theorem. We know, by the Eagon-Reiner theorem, that a Stanley-Reisner ring $k[\Delta]$ of a simplicial complex $\Delta$ has a linear resolution precisely when the Stanley-Reisner ring $k\left[\Delta^{*}\right]$ of the Alexander dual complex is Cohen-Macaulay. Since $\Delta\left(K_{n}^{d}\right)^{*} \cong \Delta\left(K_{n}^{n-d+1}\right)$ we are done.

One should note that $\Delta\left(K_{n}^{d}\right)$ is in fact shellable. A shelling is easy to construct using the lexicographic order on $n$-tuples.

Corollary 2.2.5. The ring $k\left[\Delta\left(K_{n}^{d}\right)^{*}\right]$ is Cohen-Macaulay and we have

$$
\begin{aligned}
\operatorname{dim} \Delta\left(K_{n}^{d}\right)^{*} & =n-d-1 \\
\operatorname{dim} k\left[\Delta\left(K_{n}^{d}\right)^{*}\right] & =n-d \\
\operatorname{pd} k\left[\Delta\left(K_{n}^{d}\right)^{*}\right] & =d
\end{aligned}
$$

Proof. The Cohen-Macaulayness is now clear and the first equation follows from the definitions. The second equation follows from the first one since $\operatorname{dim} k[\Delta]=$ $\operatorname{dim} \Delta+1$ for any simplicial complex $\Delta$ (see [11, Theorem 5.1.4]). The second equation and the Cohen-Macaulayness together with the Auslander-Buchsbaum formula imply the third equation.

In [54] Jacques studies the graph algebra of $K_{n}$, which we denote $K_{n}^{2}$, and obtains the formula

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n}\right)\right]\right)= \begin{cases}\binom{n}{j} i, & j=i+1 \\ 0, & j \neq i+1\end{cases}
$$

Note that this is a special case of our formula for $\beta_{i, j}\left(k\left[\Delta\left(K_{n}^{d}\right)\right]\right)$.

### 2.3 The $d$-complete multipartite hypergraph

We start by defining the $d$-complete multipartite hypergraph $K_{n_{1}, \ldots, n_{t}}^{d}$.
Definition 2.3.1. Let $\mathcal{V}=\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$ be a disjoint union of sets $\mathcal{V}_{i}$ of cardinality $n_{i}, i \in\{1,2, \ldots, t\}$, respectively. The $d$-complete multipartite hypergraph $K_{n_{1}, \ldots, n_{t}}^{d}$ on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph whose edge set $\mathcal{E}\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)$ consists of all sets $E \subseteq \mathcal{V}$ of cardinality $d$ except those of the form $\left\{x_{i_{1}}, \ldots, x_{i_{d}}\right\}$ where $x_{i_{j}} \in \mathcal{V}_{i}$ for all $j \in\{1, \ldots, d\}$.

Lemma 2.3.2. The ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]$ has linear minimal free resolution. That is, if $\beta_{i, j}\left(k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]\right) \neq 0$, then $j=i+d-1$.

Remark 2.3.3. The lemma will follow from Theorem 2.6.3.
Theorem 2.3.4. The $\mathbb{N}$-graded Betti numbers $\beta_{i, j}=\beta_{i, j}\left(k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]\right.$ of the ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]$ are independent of the characteristic of the field $k$ and may be written as
$\beta_{i, j}= \begin{cases}\binom{N}{j}\binom{j-1}{d-1}-\sum_{\substack{\left.j_{1}, \ldots, j_{t}\right) \in \mathbb{N}^{t} \\ j_{1}+\ldots+j_{t}=j}}\left[\prod_{s=1}^{t}\binom{n_{s}}{j_{s}}\right] \cdot \sum_{s=1}^{t}\binom{j_{s}-1}{d-1}, & j=i+(d-1) \\ 0, & j \neq i+(d-1)\end{cases}$
where $N=\sum_{s=1}^{t} n_{s}$.

Proof. In order to get the notations as clear as possible, we prove here only the case where $t=2$. It will be obvious that the same proof holds also when $t>2$.

For $t=2$ the formula for $\beta_{i, j}=\beta_{i, j}\left(k\left[\Delta\left(K_{n, m}^{d}\right)\right)\right]$ has the form

$$
\beta_{i, j}=\binom{n+m}{j}\binom{j-1}{d-1}-\sum_{j_{1}=0}^{j}\binom{n}{j_{1}}\binom{m}{j-j_{1}}\left[\binom{j_{1}-1}{d-1}+\binom{j-j_{1}-1}{d-1}\right]
$$

Our idea is to compare $\tilde{H}_{|V|-i-1}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)$ and $\tilde{H}_{|V|-i-1}\left(\Delta\left(K_{n+m}^{d}\right)_{V} ; k\right)$.
We realize that

$$
\operatorname{dim}_{k} \tilde{H}_{|V|-i-1}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right) \leq \operatorname{dim}_{k} \tilde{H}_{|V|-i-1}\left(\Delta\left(K_{n+m}^{d}\right)_{V} ; k\right)
$$

for every set $V \subseteq \mathcal{V}_{1} \sqcup \mathcal{V}_{2}$. The possible difference lies in the fact that there might very well be faces $F \in \Delta\left(K_{n, m}^{d}\right)$ such that $|F| \geq d$. This would result in a non-zero boundary group $B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right) ; k\right)$ in the chain complex of $\Delta\left(K_{n, m}^{d}\right)$.

It is an elementary fact that $\operatorname{dim}_{k} \tilde{H}_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)$ equals

$$
\operatorname{dim}_{k} Z_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)-\operatorname{dim}_{k} B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)
$$

Since the cycle groups $Z_{d-2}\left(\Delta\left(K_{n+m}^{d}\right)_{V} ; k\right)$ and $Z_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)$ clearly coincide and since $B_{d-2}\left(\Delta\left(K_{n+m}^{d}\right)_{V} ; k\right)=0$, we only have to compute the dimension over $k$ of $B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)$. If we write $V=V_{1} \sqcup V_{2}$, where $V_{1} \subseteq \mathcal{V}_{1}$ and $V_{2} \subseteq \mathcal{V}_{2}$, it is clear that

$$
B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V} ; k\right)=B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V_{1}} ; k\right) \oplus B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V_{2}} ; k\right)
$$

This is because the potential $(d-1)$-faces of $\Delta\left(K_{n, m}^{d}\right)$ lies either in $\Delta\left(K_{n, m}^{d}\right) \mathcal{V}_{1}$ or in $\Delta\left(K_{n, m}^{d}\right) \mathcal{V}_{2}$, which are disjoint.

Now, we have already computed (this was done in the proof of Theorem 2.2.3) $\operatorname{dim}_{k} B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V_{\nu}} ; k\right), \nu=1,2$. Thus,

$$
\begin{aligned}
& \operatorname{dim}_{k} B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V_{1}} ; k\right)=\binom{\left|V_{1}\right|-1}{d-1} \\
& \operatorname{dim}_{k} B_{d-2}\left(\Delta\left(K_{n, m}^{d}\right)_{V_{2}} ; k\right)=\binom{\left|V_{2}\right|-1}{d-1} .
\end{aligned}
$$

If we put $\left|V_{1}\right|=j_{1}$ the theorem follows as we sum over all possible $V \subseteq \mathcal{V}_{1} \sqcup$ $\mathcal{V}_{2}$.

Corollary 2.3.5. Given $K_{n_{1}, \ldots, n_{t}}^{d}$ we have

$$
\begin{aligned}
& \qquad \beta_{i}\left(k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]\right)=\binom{N}{j}\binom{j-1}{d-1}-\sum_{\substack{\left(j_{1}, \ldots, j_{t}\right) \in \mathbb{N}^{t} \\
j_{1}+\ldots+j_{t}=j}}\left[\prod_{s=1}^{t}\binom{n_{s}}{j_{s}}\right] \cdot \sum_{s=1}^{t}\binom{j_{s}-1}{d-1}, \\
& \left.\operatorname{pd} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right)\right]=N-(d-1) \\
& \text { where } N=\sum_{s=1}^{t} n_{s} \geq d \text { and } j=i+(d-1) .
\end{aligned}
$$

Proof. The fact that $\left.\operatorname{pd} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right)\right] \leq N-(d-1)$ follows from directly from the expression for the Betti number. By putting $j=N$ we get

$$
\begin{equation*}
\binom{N-1}{d-1}-\sum_{s=1}^{t}\binom{n_{s}-1}{d-1} . \tag{2.1}
\end{equation*}
$$

This expression is strictly greater than 0 , which we may prove as follows: Consider the vertex set $\mathcal{V}=\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$. Pick an arbitrary element $x_{s}$ from each one of the sets $\mathcal{V}_{s}, s \in\{1, \ldots, t\}$. The first term in the above display counts the number of ways of choosing $d-1$ elements from $\mathcal{V} \backslash\left\{x_{1}\right\}$ whereas $\sum_{\mathcal{V}=1}^{t}\binom{n_{s}-1}{d-1}$ is the number of $(d-1)$ element subsets of $\mathcal{V}$ that do lie in some $\mathcal{V}_{s} \backslash x_{s}$.

Example 1. Denote the vertex set of $K_{2,3}^{3}$ by $\{a, b\} \sqcup\{A, B, C\}$. Then we have

$$
\mathcal{E}\left(K_{2,3}^{3}\right)=\{a b A, a b B, a b C, a A B, b A B, a A C, b A C, a B C, b B C\} .
$$

The Betti numbers are $\beta_{0}\left(k\left[\Delta\left(K_{2,3}^{3}\right)\right]\right)=1, \beta_{1}\left(k\left[\Delta\left(K_{2,3}^{3}\right)\right]\right)=9, \beta_{2}\left(k\left[\Delta\left(K_{2,3}^{3}\right)\right]\right)=$ $13, \beta_{3}\left(k\left[\Delta\left(K_{2,3}^{3}\right)\right]\right)=5$.

By construction, the edges in a hypergraph $\mathcal{H}$ are the minimal non faces in $\Delta(\mathcal{H})$. This makes it easy to determine the facets of $\Delta(\mathcal{H})^{*}$ once $\mathcal{E}(\mathcal{H})$ is known: they are the complements of the edges.

Corollary 2.3.6. The ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*}\right]$ is Cohen-Macaulay and we have

$$
\begin{aligned}
\operatorname{dim} \Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*} & =N-d-1 \\
\operatorname{dim} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*}\right] & =N-d \\
\operatorname{pd} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*}\right] & =d .
\end{aligned}
$$

Proof. The Cohen-Macaulayness follows from Lemma 2.3.2 and the EagonReiner theorem. The first equation is clear and the second follows from the fact that $\operatorname{dim} k[\Delta]=\operatorname{dim} \Delta+1$ for any simplicial complex. The third equation follows since $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*}\right]$ is Cohen-Macaulay.

Also in this case we have generalized a formula given by Jacques in [54]. By studying the graph algebra of $K_{n, m}$ he obtains the formula

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n, m}\right)\right]\right)= \begin{cases}\sum_{j_{1}=1}^{j-1}\binom{n}{j_{1}}\binom{m}{j-j_{1}}, & j=i+1 \\ 0, & j \neq i+1\end{cases}
$$

A priori this looks quite different from our result. But, if one put $d=2$ and use that $\binom{n}{d}$ is defined as 0 if $n<d$, our formula simplifies to this one.

Contrary to when we considered $\Delta\left(K_{n}^{d}\right)$, the structure of the Alexander dual $\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)^{*}$ is not transparent. A natural question is: When do $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]$ both have linear resolution and the Cohen-Macaulay property? We answer this below.

Lemma 2.3.7. Let $N=\sum_{s=1}^{t} n_{s}$ and $n_{s} \leq d-1$ for $s \in\{1, \ldots, t\}$. Then $K_{N}^{d}=K_{n_{1}, \ldots, n_{t}}^{d}$.
Proof. $\mathcal{E}\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)=\mathcal{E}\left(K_{N}^{d}\right)$ and $\mathcal{V}\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)=\mathcal{V}\left(K_{N}^{d}\right)$.
Proposition 2.3.8. The Stanley-Reisner ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]$ of a d-complete multipartite hypergraph on vertex set $\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$ is Cohen-Macaulay precisely when $n_{s} \leq d-1$ for all $s \in\{1, \ldots, t\}$.

Proof. The Auslander-Buchsbaum formula tells us that

$$
\operatorname{pd} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]+\operatorname{depth} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]=N
$$

where $N=\sum_{s=1}^{t} n_{s}$. Since we already have computed the projective dimension, the above formula says

$$
\operatorname{depth} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]=d-1
$$

and it is clear that

$$
\operatorname{dim} \Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)=\max \left\{n_{i}-1, d-2 ; i \in\{1, \ldots, t\}\right\}
$$

Thus since depth $M \leq \operatorname{dim} M$ holds for every finitely generated $R$-module $M$ we see that $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)\right]$ is Cohen-Macaulay precisely when $n_{s} \leq d-1$ for all $s \in\{1, \ldots, t\}$. Furthermore, according to the lemma, in this case we have $K_{n_{1}, \ldots, n_{t}}^{d}=K_{N}^{d}$.

### 2.4 The Alexander dual of a join

Let $\Delta$ be a simplicial complex on vertex set $\mathcal{V},|\mathcal{V}|=n$, with Stanley-Reisner ideal $I_{\Delta}=\left(\mathbf{x}^{F} ; F \notin \Delta\right)$. It is well known (follows from [59, Theorem 1.7]) that the Stanley-Reisner ideal of the Alexander dual simplicial complex $\Delta^{*}$ then may be written

$$
\begin{equation*}
I_{\Delta^{*}}=\bigcap_{\mathcal{V} \backslash F \notin \Delta} \mathfrak{p}_{F}, \tag{2.2}
\end{equation*}
$$

where $\mathfrak{p}_{F}$ is the prime ideal generated by the variables $x_{i}, i \in \mathcal{V} \backslash F$.
Let $\Delta$ and $\Gamma$ be simplicial complexes on disjoint vertex sets $\mathcal{V}$ and $\mathcal{W}$, respectively. Denote the minimal non-faces of $\Delta$ and $\Gamma$ by $F_{i}, i \in\{1, \ldots, r\}$, and $G_{j}, j \in\{1, \ldots, s\}$, respectively.

If we consider $I_{\Delta} \subseteq k\left[x_{1}, \ldots, x_{n}\right]$ and $I_{\Gamma} \subseteq k\left[y_{1}, \ldots, y_{m}\right]$ as ideals in the ring $k\left[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right]$, it follows that

$$
I_{\Delta * \Gamma}=I_{\Delta}+I_{\Gamma}=\left(\mathbf{x}^{F_{i}}, \mathbf{x}^{G_{j}} ; i \in\{1, \ldots, r\}, j \in\{1, \ldots, s\}\right)
$$

Hence, by the equation (2.2) the Stanley-Reisner ideal of $(\Delta * \Gamma)^{*}$ is

$$
\begin{equation*}
I_{(\Delta * \Gamma)^{*}}=I_{\Delta^{*}} \cap I_{\Gamma^{*}}=I_{\Delta^{*}} I_{\Gamma^{*}} \tag{2.3}
\end{equation*}
$$

Suppose hypergraphs $\mathcal{H}=(\mathcal{V}, \mathcal{E}(\mathcal{H}))$ and $\mathcal{K}=(\mathcal{W}, \mathcal{E}(\mathcal{K}))$ are given. We define the product $\mathcal{H} \cdot \mathcal{K}$ of $\mathcal{H}$ and $\mathcal{K}$ to be the hypergraph on vertex set $\mathcal{V} \sqcup \mathcal{W}$ and with edges $\left\{x_{1}, \ldots, x_{r}, y_{1}, \ldots, y_{s}\right\}$, where $\left\{x_{1}, \ldots, x_{r}\right\}$ is an edge in $\mathcal{H}$ and $\left\{y_{1}, \ldots, y_{s}\right\}$ is an edge in $\mathcal{K}$. Thus $\mathcal{E}(\mathcal{H} \cdot \mathcal{K})$ may be identified with the cartesian product $\mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{K})$.

Lemma 2.4.1. Let $\mathcal{H}=(\mathcal{V}, \mathcal{E}(\mathcal{H}))$ and $\mathcal{K}=(\mathcal{W}, \mathcal{E}(\mathcal{K}))$ be $d$ - and $d^{\prime}$-uniform hypergraphs respectively. Then $\mathcal{H} \cdot \mathcal{K}$ is a $\left(d+d^{\prime}\right)$-uniform hypergraph, and $k[\Delta(\mathcal{H} \cdot \mathcal{K})]$ has linear resolution if and only if both $k[\Delta(\mathcal{H})]$ and $k[\Delta(\mathcal{K})]$ have linear resolutions.

Proof. The fact that $\mathcal{H} \cdot \mathcal{K}$ is $\left(d+d^{\prime}\right)$-uniform is clear from the definition. If we put $\Delta=\Delta(\mathcal{H})^{*}$ and $\Gamma=\Delta(\mathcal{K})^{*}$ equation (2.3) yields

$$
\Delta(\mathcal{H} \cdot \mathcal{K})=\left(\Delta(\mathcal{H})^{*} * \Delta(\mathcal{K})^{*}\right)^{*}
$$

It is known, and proved in for example [38], that the join $\Delta * \Gamma$ of two simplicial complexes $\Delta$ and $\Gamma$ is Cohen-Macaulay precisely when both $\Delta$ and $\Gamma$ are CohenMacaulay. Thus by the Eagon-Reiner theorem the Stanley-Reisner ring of $\left(\Delta(\mathcal{H})^{*} * \Delta(\mathcal{K})^{*}\right)^{*}$ has linear resolution precisely when both $\Delta(\mathcal{H})^{*}$ and $\Delta(\mathcal{K})^{*}$ are Cohen-Macaulay. This is, again by the Eagon-Reiner theorem, the same thing as saying both $k[\Delta(\mathcal{H})]$ and $k[\Delta(\mathcal{K})]$ have linear resolutions.

Let $V=V_{1} \sqcup V_{2} \subseteq \mathcal{V} \sqcup \mathcal{W}$ where $V_{1} \subseteq \mathcal{V}$ and $V_{2} \subseteq \mathcal{W}$. Since

$$
\Delta(\mathcal{H} \cdot \mathcal{K})=(\Delta(\mathcal{H}) *\langle W\rangle) \cup(\langle V\rangle * \Delta(\mathcal{K}))
$$

we have an exact sequence

$$
\begin{align*}
0 & \rightarrow \tilde{\mathcal{C}} .\left((\Delta(\mathcal{H}) * \Delta(\mathcal{K}))_{V} ; k\right) \rightarrow \\
\tilde{\mathcal{C}} .\left((\Delta(\mathcal{H}) *\langle\mathcal{W}\rangle)_{V} ; k\right) & \oplus \tilde{\mathcal{C}} .\left((\langle\mathcal{V}\rangle * \Delta(\mathcal{K}))_{V} ; k\right) \rightarrow \tilde{\mathcal{C}} .\left(\Delta(\mathcal{H} \cdot \mathcal{K})_{V} ; k\right) \rightarrow 0 . \tag{2.4}
\end{align*}
$$

Our aim is to use Hochster's formula and thus we interested in the homology of $\Delta(\mathcal{H} \cdot \mathcal{K})_{V}$. If $V_{1}$ or $V_{2}$ is empty then $\Delta(\mathcal{H} \cdot \mathcal{K})_{V}$ has zero homology. Therefore assume $V_{1}$ and $V_{2}$ are non-empty. In this case the complexes $(\Delta(\mathcal{H}) *\langle\mathcal{W}\rangle)_{V}$ and $(\langle\mathcal{V}\rangle * \Delta(\mathcal{K}))_{\mathcal{V}}$ are cones, so if we consider the Mayer-Vietoris sequence obtained from (2.4) we get that the following equation holds for every $V=V_{1} \sqcup V_{2} \subseteq$ $\mathcal{V} \sqcup \mathcal{W}, V_{1} \neq \emptyset, V_{2} \neq \emptyset$, and $r \geq-1$ :

$$
\tilde{H}_{r}\left(\Delta(\mathcal{H} \cdot \mathcal{K})_{V} ; k\right) \cong \tilde{H}_{r-1}\left((\Delta(\mathcal{H}) * \Delta(\mathcal{K}))_{V} ; k\right)
$$

From equation (1.2) in Section 1.2.9 it follows that

$$
\tilde{H}_{r}\left(\Delta(\mathcal{H} \cdot \mathcal{K})_{V} ; k\right) \cong \bigoplus_{\substack{r_{1}+r_{2}=r-2 \\ r_{1}, r_{2} \geq 0}} \tilde{H}_{r_{1}}\left(\Delta(\mathcal{H})_{V_{1}} ; k\right) \otimes \tilde{H}_{r_{2}}\left(\Delta(\mathcal{K})_{V_{2}} ; k\right)
$$

Thus by Hochster's formula we get

$$
\beta_{i, j}(k[\Delta(\mathcal{H} \cdot \mathcal{K})])=\sum_{\substack{r_{i} \geq 0 \\|V|=j \\ V=V_{1} \cup V_{2} \\ r_{1}+r_{2}=j-i-3}} \operatorname{dim}_{k} \tilde{H}_{r_{1}}\left(\Delta(\mathcal{H})_{V_{1}} ; k\right) \cdot \operatorname{dim}_{k} \tilde{H}_{r_{2}}\left(\Delta(\mathcal{K})_{V_{2}} ; k\right)
$$

We extend this to products of more than two hypergraphs inductively.
Remark 2.4.2. In the below theorem and corollary $V_{i}, i \in\{1, \ldots, t+1\}$, denotes a subset of $\mathcal{V}_{i}$.

Theorem 2.4.3. Assume hypergraphs $\mathcal{H}_{i}, i \in\{1, \ldots, t+1\}$, on disjoint vertex sets $\mathcal{V}_{i}$ respectively are given. The $\mathbb{N}$-graded Betti numbers of $k\left[\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t+1}\right)\right]$ are given by

$$
\beta_{i, j}\left(k\left[\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t+1}\right)\right)\right]=\sum_{\substack{r_{i} \geq 0 \\|V|=j \\ V=V_{1} \cup \cdots \cup V_{t+1}}}\left[\prod_{l=1}^{t+1} \operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(\mathcal{H}_{l}\right)_{V_{l}} ; k\right)\right] .
$$

Proof. We have seen that the formula holds for $t=1$. It follows by induction that

$$
\begin{equation*}
\operatorname{dim}_{k} \tilde{H}_{r}\left(\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t}\right)_{V} ; k\right)=\sum_{\substack{ \\r_{1}+\cdots+r_{t}=r-2(t-1) \\ r_{i} \geq 0}}\left[\prod_{l=1}^{t} \operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(\mathcal{H}_{l}\right)_{V_{l}} ; k\right)\right] \tag{2.5}
\end{equation*}
$$

holds for every $r \geq-1$ and $V_{i} \subseteq \mathcal{V}_{i}, i \in\{1, \ldots, t\}, V_{i} \neq \emptyset$. By the case $t=1$, $\beta_{i, j}\left(k\left[\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t+1}\right)\right]\right)$ equals

$$
\sum_{\begin{array}{r}
r \geq 0, r_{t+1} \geq 0  \tag{2.6}\\
|V|=j \\
V=V_{1} \sqcup \cdots \sqcup V_{t+1} \\
r+r_{t+1}=j-i-3
\end{array}} \operatorname{dim}_{k} \tilde{H}_{r}\left(\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t}\right)_{V_{1} \sqcup \cdots \sqcup V_{t}} ; k\right) \cdot \operatorname{dim}_{k} \tilde{H}_{r_{t+1}}\left(\Delta\left(\mathcal{H}_{t+1}\right)_{V_{t+1}} ; k\right) .
$$

By putting the expression for $\operatorname{dim}_{k} \tilde{H}_{r}\left(\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t}\right)_{V} ; k\right)$ from (2.5) into (2.6) we get

$$
\sum_{\substack{r \geq 0, r_{i} \geq 0 \\|V|=j \\ V=V_{1} \leq \cdots \cup V_{t+1} \\ r+r_{t+1}=j-i-3 \\ r_{1}+\cdots+r_{t}=r-2(t-1)}}\left[\prod_{l=1}^{t} \operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(\mathcal{H}_{l}\right)_{V_{l}} ; k\right)\right] \cdot \operatorname{dim}_{k} \tilde{H}_{r_{t+1}}\left(\Delta\left(\mathcal{H}_{t+1}\right)_{V_{t+1}} ; k\right)
$$

which after cleaning of the summation symbol is the asserted formula.
The above formula for the Betti numbers becomes much nicer if we know that each $\mathcal{H}_{i}$ has linear resolution. This is because then know that for each $l$, $\operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(\mathcal{H}_{l}\right)_{V_{l}} ; k\right)$ can be non-zero only in one specific degree $r_{l}$.

Corollary 2.4.4. Let hypergraphs $\mathcal{H}_{i}, i \in\{1, \ldots, t+1\}$, on disjoint vertex sets $\mathcal{V}_{i}$ respectively be given. Assume for $i \in\{1, \ldots, t+1\}$ that $\mathcal{H}_{i}$ is $a_{i}$-uniform and that $k\left[\Delta\left(\mathcal{H}_{i}\right)\right]$ has linear minimal free resolution. Then the ith $\mathbb{N}$-graded Betti number in degree $j$ of $k\left[\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t+1}\right)\right]$ is given by the following expression.

$$
\beta_{i, j}\left(k\left[\Delta\left(\mathcal{H}_{1} \cdots \mathcal{H}_{t+1}\right)\right]\right)=\sum_{\substack{|V|=j \\ V=V_{1} \sqcup \cdots \sqcup V_{t+1}}}\left[\prod_{l=1}^{t+1} \operatorname{dim}_{k} \tilde{H}_{a_{l}-2}\left(\Delta\left(\mathcal{H}_{l}\right)_{V_{l}} ; k\right)\right] .
$$

### 2.5 The $d\left(a_{1}, \ldots, a_{t}\right)$-complete multipartite hypergraph

Consider the $d$-complete multipartite hypergraph $K_{n_{1}, \ldots, n_{t}}^{d}$ on vertex set $\mathcal{V}=$ $\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$. Recall that the edge set $\mathcal{E}\left(K_{n_{1}, \ldots, n_{t}}^{d}\right)$ consists of all cardinality $d$ subsets of $\mathcal{V}$ except those of the form $\left\{x_{i_{1}}, \ldots, x_{i_{d}}\right\}, x_{i_{j}} \in \mathcal{V}_{s}$ for some $s \in$ $\{1, \ldots, t\}$. In the case of the ordinary graph $K_{n, m}$ this just tells us that we have all edges between the disjoint sets $\mathcal{V}_{1}$ and $\mathcal{V}_{2}$ of vertices. This one may think of as an edge being a choice of two vertices, prescribing a certain number of vertices in each one of the sets $\mathcal{V}_{1}$ and $\mathcal{V}_{2}$, namely one in each. This is the idea behind the following definition.

Definition 2.5.1. Let $\mathcal{V}=\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$ be a disjoint union of sets $\mathcal{V}_{i}$ of cardinality $n_{i}, i \in\{1,2, \ldots, t\}$, respectively. The $d\left(a_{1}, \ldots, a_{t}\right)$-complete multipartite hypergraph $K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}$ on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph whose edge set $\mathcal{E}\left(K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}\right)$ consists of all sets $E \subseteq \mathcal{V}$ of cardinality $d$ such that precisely $a_{s}$ elements of $E$ comes from the set $\mathcal{V}_{s}, a_{s} \in \mathbb{N}, a_{s} \geq 1, \sum_{s=1}^{t} a_{s}=d$.

To simplify the notation, for the rest of the section $a_{1}, \ldots, a_{t}=\mathbf{a}, n_{1}, \ldots, n_{t}=$ $\mathbf{n}$ and $d=\sum_{s=1}^{t} a_{s}$. Thus, $d\left(a_{1}, \ldots, a_{t}\right)=d(\mathbf{a})$ and $K_{\mathbf{n}}^{d(\mathbf{a})}=K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}$.

Lemma 2.5.2. The ring $k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]$ has linear minimal free resolution. That is, if $\beta_{i, j}\left(k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]\right) \neq 0$, then $j=i+(d-1)$.

Remark 2.5.3. The lemma will follow from Theorem 2.6.3. We give a proof here anyway since it is very short and also provides a description of the Alexander dual simplicial complex $\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*}$.

Proof. By considering the facets of the Alexander dual complex, we realize that

$$
\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*}=\Delta\left(K_{n_{1}}^{n_{1}-a_{1}+1}\right) * \cdots * \Delta\left(K_{n_{t}}^{n_{t}-a_{t}+1}\right)
$$

Thus $\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*}$ is Cohen-Macaulay since we know that each $\Delta\left(K_{n_{s}}^{n_{s}-a_{s}+1}\right)$ is Cohen-Macaulay. The lemma now follows from the Eagon-Reiner theorem.

We now compute the Betti numbers of $K_{\mathbf{n}}^{d(\mathbf{a})}$. It follows directly from the definitions that

$$
K_{\mathbf{n}}^{d(\mathbf{a})}=\prod_{s=1}^{t} K_{n_{s}}^{a_{s}} .
$$

Thus we may apply the results from the previous section.
Theorem 2.5.4. The $\mathbb{N}$-graded Betti numbers $\beta_{i, j}=\beta_{i, j}\left(k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]\right)$ are independent of the characteristic of the field $k$ and may be written as

$$
\beta_{i, j}= \begin{cases}\sum_{r_{1}+\cdots+r_{t}=i+t-1}\left[\prod_{l=1}^{t} \beta_{r_{l}, r_{l}+a_{l}-1}\left(K_{n_{l}}^{a_{l}}\right)\right] & \text { if } \quad j=i+(d-1) \\ 0 & \text { if } \quad j \neq i+(d-1)\end{cases}
$$

Proof. Using the notation of Section 2.4 we know that $\operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(K_{n_{l}}^{a_{l}}\right)_{V_{l}} ; k\right) \neq$ 0 only when $r_{l}=a_{l}-2$, and in this case we have

$$
\operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(K_{n_{l}}^{a_{l}}\right)_{V_{l}} ; k\right)=\binom{j_{l}-1}{a_{l}-1}
$$

where $j_{l}=\left|V_{l}\right|$. Thus the expression

$$
\sum_{\substack{r_{i} \geq 0 \\|V|=j \\ V=V_{1} \cup \cdots \cup V_{t} \\ r_{1}+\cdots+r_{t}=j-i-(2 t-1)}}\left[\prod_{l=1}^{t} \operatorname{dim}_{k} \tilde{H}_{r_{l}}\left(\Delta\left(K_{n_{l}}^{a_{l}}\right)_{V_{l}} ; k\right)\right]
$$

obtained from Theorem 2.4.3 simplifies, via the formula in Corollary 2.4.4, to

$$
\sum_{\substack{|V|=j \\ V=V_{1} \cup \cdots \sqcup V_{t}}} \prod_{l=1}^{t}\binom{j_{l}-1}{a_{l}-1} .
$$

This may in turn be written as

$$
\sum_{\substack{j_{l} \geq 0 \\ j=j_{1}+\cdots+j_{t}}} \prod_{l=1}^{t}\binom{n_{l}}{j_{l}}\binom{j_{l}-1}{a_{l}-1} .
$$

Now, if some $j_{l} \leq a_{l}-1$ the corresponding term is zero. So, we may write $j_{l}=r_{l}+a_{l}-1$ where $r_{l} \geq 1$ for $l \in\{1, \ldots, t\}$. The above expression then becomes

$$
\begin{equation*}
\sum_{\substack{r_{l} \geq 1 \\ r_{1}+\cdots+r_{t}=j-d+t}} \prod_{l=1}^{t}\binom{n_{l}}{r_{l}+a_{l}-1}\binom{r_{l}+a_{l}-2}{a_{l}-1} \tag{2.7}
\end{equation*}
$$

We know that the resolution is linear so for non-zero Betti numbers $\beta_{i, j}$ we have $j=i+(d-1)$. Using this in (2.7) we get the formula in the theorem.

Corollary 2.5.5. The $\mathbb{N}$-graded Betti numbers of the $d(a, b)$-complete bipartite hypergraph $K_{n, m}^{d(a, b)}$ may be written as
$\beta_{i, j}\left(k\left[\Delta\left(K_{n, m}^{d(a, b)}\right)\right]\right)=\sum_{\substack{r+s=i+1 \\ r, s \geq 1}}\binom{n}{r+a-1}\binom{r+a-2}{a-1}\binom{m}{s+b-1}\binom{s+b-2}{b-1}$.
Note that by putting $a=b=1$ we get

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n, m}^{d(1,1)}\right)\right]\right)=\sum_{\substack{p+q=j \\ p, q \geq 1}}\binom{n}{p}\binom{m}{q} .
$$

Now $K_{n, m}^{d(1,1)}=K_{n, m}$, so we have another proof of Jacques' formula for the Betti numbers $\beta_{i, j}\left(k\left[\Delta\left(K_{n, m}\right)\right]\right)$.
Corollary 2.5.6. Given $K_{\mathbf{n}}^{d(\mathbf{a})}$ we have

$$
\begin{aligned}
& \beta_{i}\left(k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]\right)=\sum_{\substack{r_{1}+\cdots+r_{t}=i+t-1 \\
r_{i} \geq 1}}\left[\prod_{l=1}^{t} \beta_{r_{l}, r_{l}+a_{l}-1}\left(K_{n_{l}}^{a_{l}}\right)\right] \\
& \operatorname{pd} k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right] \quad=\quad N-(d-1)
\end{aligned}
$$

where $j=i+(d-1)$.
Proof. The first assertion is clear. If we put $i=N-(d-1)$ in the formula we get

$$
\beta_{N-(d-1)}\left(k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]\right)=\prod_{l=1}^{t} \beta_{n_{l}-\left(a_{l}-1\right)}\left(k\left[\Delta\left(K_{n_{l}}^{a_{l}}\right)\right]\right)
$$

which is non-zero. At the same time we see that if $i>N-(d-1)$ every term in the sum is zero because some factor in every term is zero.
Example 2. Consider $\mathcal{H}=K_{3,3}^{4(2,2)}$. If we denote the set of vertices of this hypergraph by $\{a, b, c\} \sqcup\{x, y, z\}$, we get

$$
I(\mathcal{H})=(a b x y, a b x z, a b y x, a c x y, a c x z, a c y z, b c x y, b c x z, b c y z)
$$

The Betti diagram of $k[\Delta(\mathcal{H})]$ is

|  | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | - | - | - |
| 1 | - | - | - | - |
| 2 | - | - | - | - |
| 3 | - | 9 | 12 | 4 |

Considering Theorem 2.5.4 this diagram may be constructed from the diagram

|  | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 1 | - | - |
| 1 | - | 3 | 2 |

of $k\left[\Delta\left(K_{3}^{2}\right)\right]$.

Corollary 2.5.7. The ring $k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*}\right]$ is Cohen-Macaulay and we have

$$
\begin{aligned}
\operatorname{dim} \Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*} & =N-d-1 \\
\operatorname{dim} k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)^{*}\right] & =N-d \\
\operatorname{pd} k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a}}\right)^{*}\right] & =d .
\end{aligned}
$$

Proof. The Cohen-Macaulayness follows from Lemma 2.5.2 and the EagonReiner theorem and imply the third equation. By considering the description of the Alexander dual given in Lemma 2.5.2 the first and second equation are clear.
Proposition 2.5.8. The ring $k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]$ is Cohen-Macaulay precisely when $a_{s}=n_{s}$ for all $s \in\{1, \ldots, t\}$ but possibly one. This single $a_{i}$ is such that it maximizes the expression $a_{i}+\sum_{j \neq i, j=1}^{t} n_{j}$.

Proof. For $s \in\{1, \ldots, t\}$ let $I_{s} \subseteq \mathcal{V}_{s}$. It is necessary and sufficient that at least one set $I_{i}$ satisfy $\left|I_{i}\right|<a_{i}$ for $I_{1} \sqcup \cdots \sqcup I_{t}$ to be a face of $\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)$. Thus the dimension of $\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)$ is

$$
\max \left\{a_{i}-2+\sum_{j \neq i, j=1}^{t} n_{j} ; i \in\{1, \ldots, t\}\right\},
$$

so

$$
\operatorname{dim} k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]=\max \left\{a_{i}-1+\sum_{j \neq i, j=1}^{t} n_{j}\right\}
$$

We know that $\operatorname{pd} k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]=N-(d-1)$, so depth $k\left[\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)\right]=d-1$. Now, since by construction $d=\sum_{s=1}^{t} a_{s}$ we are done.

In [7], Berge defines what he calls the $d$-partite complete hypergraph. In our language this is $K_{n_{1}, \ldots, n_{d}}^{d(1, \ldots, 1)}$. Its $i$ th Betti number in degree $j$ is

$$
\beta_{i, j}\left(k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d(1, \ldots, 1)}\right)\right]\right)=\sum_{\substack{r_{1}+\cdots+r_{t}=i+t-1 \\ r_{s} \geq 1}} \prod_{l=1}^{t}\binom{n_{l}}{r_{l}} .
$$

### 2.6 The $d\left(I_{1}, \ldots, I_{t}\right)$-complete multipartite hypergraph

Definition 2.6.1. Let $\mathcal{V}=\mathcal{V}_{1} \sqcup \mathcal{V}_{2} \sqcup \cdots \sqcup \mathcal{V}_{t}$ be a disjoint union of sets $\mathcal{V}_{i}$ of cardinality $n_{i}, i \in\{1,2, \ldots, t\}$, respectively. For each $s \in\{1, \ldots, t\}$ let $I_{s}=$ $\left[\alpha_{s}, \beta_{s}\right]$ be an interval in $\left\{0, \ldots, n_{s}\right\}$. The $d\left(I_{1}, \ldots, I_{t}\right)$-complete multipartite hypergraph on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph whose edge set consists of all sets $I_{1}\left(a_{1}\right) \sqcup \cdots \sqcup I_{t}\left(a_{t}\right)$ of cardinality $d$, were $I_{s}\left(a_{s}\right)$ is a subset of $\mathcal{V}_{s}$ of cardinality $a_{s} \in I_{s}$ and $d=\sum_{s=1}^{t} a_{s}$.

We immediately see why this generalizes previously considered hypergraphs:

- If $I_{s}=\left\{0, \ldots, n_{s}\right\}$ for all $s \in\{1, \ldots, t\}$ we have the $d$-complete hypergraph $K_{N}^{d}, N=\sum_{i=1}^{t} n_{i}$.
- If $I_{s}=\left\{0, \ldots, \min \left\{n_{s}, d-1\right\}\right\}$ we obtain the $d$-complete multipartite hypergraph $K_{n_{1}, \ldots, n_{s}}^{d}$.
- By letting $I_{s}$ consist of only one non zero element for all $s$, we obtain the $d\left(a_{1}, \ldots, a_{t}\right)$-complete hypergraph $K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}$.

One easily realizes that two different sets of intervals $I_{1}, \ldots, I_{t}$ and $J_{1}, \ldots, J_{t}$ say, may yield the same hypergraph. Just consider the case where $I_{s}=\left\{a_{s}\right\}$ for all $s, \sum_{s=1}^{t} a_{s}=d$, and $J_{s}=\left\{a_{s}\right\}$ for all $s \neq 1, J_{1}=\left[a_{1}, a_{1}+1\right]$.

Without loss of generality we assume that the sequence of intervals $I_{1}, \ldots, I_{t}$ corresponding to a hypergraph $K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}$ satisfies the following property: If $I_{s}=\left[\alpha_{s}, \beta_{s}\right], s \in\{1, \ldots, t\}$, then

$$
\alpha_{s}+\sum_{j \neq s} \beta_{j} \geq d
$$

and

$$
\beta_{s}+\sum_{j \neq s} \alpha_{j} \leq d
$$

holds for all $s \in\{1, \ldots, t\}$. This guarantees that there is no redundancy in the intervals.

Remark 2.6.2. It is clear that a set of intervals $I_{1}, \ldots, I_{t}$ corresponding to a hypergraph $K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}$ can be constructed from a sequence $a_{1}, \ldots, a_{t}, d=$ $\sum_{s=1}^{t} a_{s}, a_{j} \in I_{j}$ for all $j \in\{1, \ldots, t\}$, by successively extending one or two of the intervals in such a way that the inequalities above remain true in each step. The following example will clarify this idea.

Example 3. Suppose $a_{1}+a_{2}+a_{3}=d$ and consider consider the hypergraph $K_{n_{1}, n_{2}, n_{3}}^{d\left(I_{1}, I_{2}, I_{3}\right)}$ with $I_{1}=\left[a_{1}-1, a_{1}\right], I_{2}=\left[a_{2}-1, a_{2}+1\right], I_{3}=\left[a_{3}, a_{3}+1\right]$. The intervals can be constructed in the following steps:

$$
\begin{aligned}
& \left\{a_{1}\right\},\left\{a_{2}\right\},\left\{a_{3}\right\} \rightsquigarrow\left[a_{1}-1, a_{1}\right],\left[a_{2}, a_{2}+1\right],\left\{a_{3}\right\} \\
& \\
& \rightsquigarrow\left[a_{1}-1, a_{1}\right],\left[a_{2}-1, a_{2}+1\right],\left[a_{3}, a_{3}+1\right] .
\end{aligned}
$$

Theorem 2.6.3. The ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)\right]$ has linear minimal free resolution. That is, if $\beta_{i, j}\left(k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)\right]\right) \neq 0$, then $j=i+(d-1)$.

Proof. The theorem follows from [60, Theorem 3.3], where the authors show that the edge ideal $I\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)$ is weakly polymatroidal (see [56] for details about weakly polymatroidal ideals). It is known that all weakly polymatroidal ideals have linear quotients.

Remark 2.6.4. A straight forward proof using induction may be found in [23].
Example 4. Consider $\mathcal{H}=K_{3,3,3}^{5\left(I_{1}, I_{2}, I_{3}\right)}$ where $I_{1}=[1,2], I_{2}=\{1\}, I_{3}=[2,3]$. There are 365 -edges in this hypergraph and the Betti numbers are $\beta_{0}(\mathcal{H})=$ $1, \beta_{1}(\mathcal{H})=36, \beta_{2}(\mathcal{H})=90, \beta_{3}(\mathcal{H})=87, \beta_{4}(\mathcal{H})=39, \beta_{5}(\mathcal{H})=7$.

By considering the edges in $K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}$ and the description of the Alexander dual of $\Delta\left(K_{\mathbf{n}}^{d(\mathbf{a})}\right)$, we obtain the following description of $\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*}$.

$$
\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*}=\bigcup_{\substack{a_{1}+\cdots+a_{t}=d \\ a_{s} \in I_{s}}} \Delta\left(K_{n_{1}}^{n_{1}-a_{1}+1}\right) * \cdots * \Delta\left(K_{n_{t}}^{n_{t}-a_{t}+1}\right)
$$

We immediately get the following
Corollary 2.6.5. The ring $k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*}\right]$ is Cohen-Macualay and we have

$$
\begin{aligned}
\operatorname{dim} \Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*} & =N-d-1 \\
\operatorname{dim} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*}\right] & =N-d \\
\operatorname{pd} k\left[\Delta\left(K_{n_{1}, \ldots, n_{t}}^{d\left(I_{1}, \ldots, I_{t}\right)}\right)^{*}\right] & =d .
\end{aligned}
$$

## Chapter 3

## Chordal hypergraphs

The class of chordal graphs is a well studied class of graphs and indeed turns out to have many nice properties, graph-theoretical as well as algebraic. A corner stone in the algebraic investigations of chordal graphs is the following theorem by Fröberg:

Theorem 3.0.6 (Fröberg, [39]). Let $\mathcal{G}$ be a simple graph on $n \in \mathbb{N}$ vertices. Then the graph algebra $k\left[x_{1}, \ldots, x_{n}\right] / I(\mathcal{G})$ has linear resolution precisely when $\mathcal{G}^{c}$ is chordal.

### 3.1 The classes of chordal and triangulated hypergraphs

Definition 3.1.1. Two distinct vertices $x, y$ of a hypergraph $\mathcal{H}$ are neighbors if there is an edge $E \in \mathcal{E}(\mathcal{H})$, such that $x, y \in E$. For any vertex $x \in \mathcal{V}(\mathcal{H})$, the neighborhood of $x$, denoted $N(x)$, is the set

$$
N(x)=\{y \in \mathcal{V}(\mathcal{H}) ; y \text { is a neighbor of } x\} .
$$

If $N(x)=\emptyset, x$ is called isolated. Furthermore, we let $N[x]=N(x) \cup\{x\}$ denote the closed neighborhood of $x$.

Remark 3.1.2. Let $\mathcal{H}$ be a hypergraph and $V \subseteq \mathcal{V}(\mathcal{H})$. Denote by $N_{V}[x]$ the closed neighborhood of $x$ in the induced hypergraph $\mathcal{H}_{V}$. For ordinary graphs it is clear that $N_{V}[x]=N[x] \cap V$. This is not always the case for hypergraphs, as is shown in the example below. Note that the notation $N_{V}[x]$ will only occur in this remark and the example below.
Example 5. Consider the hypergraph $\mathcal{H}$ on vertex set $\mathcal{V}(\mathcal{H})=\{a, b, c, d, e\}$ and edge set $\mathcal{E}(\mathcal{H})=\{\{a, b, c\},\{a, d, e\},\{b, c, d\}\}$. Let $V=\{a, b, c, d\}$. Then $N_{V}[a]=\{a, b, c\}$ but $N[a] \cap V=\{a, b, c, d\}$.

Definition 3.1.3. A $d$-uniform hypergraph $\mathcal{H}$ is called triangulated if for every non-empty subset $V \subseteq \mathcal{V}(\mathcal{H})$, either there exists a vertex $x \in V$ such that the induced hypergraph $\mathcal{H}_{N[x] \cap V}$ is isomorphic to a $d$-complete hypergraph $K_{n}^{d}$, $n \geq d$, or else the edge set of $\mathcal{H}_{V}$ is empty.

This definition is basically due to Hà and Van Tuyl, see [46, Definition 5.5]. However, in [46] the property of being triangulated is defined on a special class of hypergraphs called properly-connected. For a further discussion see Subsection 3.2 below.

Recall that $\Delta_{\mathcal{H}}$ denotes the clique complex of a uniform hypergraph $\mathcal{H}$.
Definition 3.1.4. A $d$-uniform hypergraph $\mathcal{H}$ is called triangulated* if for every non-empty subset $V \subseteq \mathcal{V}(\mathcal{H})$, either there exists a vertex $x \in V$ such that $N[x] \cap V$ is a facet of $\left(\Delta_{\mathcal{H}}\right)_{V}$ of dimension greater than or equal to $d-1$, or else the edge set of $\mathcal{H}_{V}$ is empty.

We will show (Theorem-definition 3.1.12) that the above two definitions, and also the following two definitions are equivalent.

Definition 3.1.5. A chordal hypergraph is a $d$-uniform hypergraph, obtained inductively as follows:

- $K_{n}^{d}$ is a chordal hypergraph, $\{n, d\} \subseteq \mathbb{N}$.
- If $\mathcal{G}$ is chordal, then so is $\mathcal{H}=\mathcal{G} \bigcup_{K_{j}^{d}} K_{i}^{d}$, for $0 \leq j<i$. (This we think of as glueing $K_{i}^{d}$ to $\mathcal{G}$ by identifying some edges, or parts of some edges, of $K_{i}^{d}$ with the corresponding part, $K_{j}^{d}$, of $\mathcal{G}$.)

Remark 3.1.6. For $d=2$ this specializes precisely to the class of generalized trees, i.e. generalized $n$-trees for some $n$, as defined in [39].
Remark 3.1.7. Recall that a simple graph is called chordal if every induced cycle of length $>3$, has a chord. It follows from [16, Theorem 1, Theorem 2], that the chordal graphs are precisely the generalized trees.

Another characterization of chordal graphs may be found in [42]. There it is shown that a simple graph is chordal precisely when it has a perfect elimination order. Recall that a perfect elimination order of a graph $\mathcal{G}=(\mathcal{V}, \mathcal{E})$ is an ordering of its vertices, $x_{1}<x_{2}<\cdots<x_{n}$, such that for each $i, \mathcal{G}_{N\left[x_{i}\right] \cap\left\{x_{i}, x_{i+1}, \ldots, x_{n}\right\}}$ is a complete graph. The concept of perfect elimination order is well suited for generalizations. We make the following

Definition 3.1.8. A $d$-uniform hypergraph $\mathcal{H}$ is said to have a perfect elimination order if its vertices can be ordered $x_{1}<x_{2}<\cdots<x_{n}$, such that for each $i$, either $\mathcal{H}_{N\left[x_{i}\right] \cap\left\{x_{i}, x_{i+1}, \ldots, x_{n}\right\}}$ is isomorphic to a $d$-complete hypergraph $K_{n}^{d}, n \geq d$, or else $x_{i}$ is isolated in $\mathcal{H}_{\left\{x_{i}, x_{i+1}, \ldots, x_{n}\right\}}$

This specializes precisely to the definition of perfect elimination order for simple graphs if we put $d=2$.

Lemma 3.1.9. Let $\mathcal{H}$ be a d-uniform hypergraph and $x \in V \subseteq \mathcal{V}(\mathcal{H})$ a vertex such that $\mathcal{H}_{N[x]} \cong K_{m}^{d}, m \geq d$. Then $\mathcal{H}_{N[x] \cap V}$ either is isomorphic to a dcomplete hypergraph $K_{m^{\prime}}^{d}, m^{\prime} \geq d$, or else $x$ is isolated in $V$.

Proof. Either $|N[x] \cap V| \geq d$ or else $|N[x] \cap V|<d$.
Lemma 3.1.10. If a d-uniform hypergraph $\mathcal{H}$ with $\mathcal{E}(\mathcal{H}) \neq \emptyset$ has a perfect elimination order, then it has a perfect elimination order $x_{1}<x_{2}<\cdots<x_{n}$ in which $x_{1}$ is not isolated.

Proof. Let $x_{1}<x_{2}<\cdots<x_{n}$ be a perfect elimination order of $\mathcal{H}$, and put

$$
t=\min \left\{i ; x_{i} \text { is not isolated }\right\}
$$

We claim that $x_{t}<\cdots<x_{n}<x_{1}<\cdots<x_{t-1}$ also is a perfect elimination order of $\mathcal{H}$. Since $x_{1}, \ldots, x_{t-1}$ are isolated, we need only verify that $\mathcal{H}_{N\left[x_{i}\right] \cap\left\{x_{i}, x_{i+1}, \ldots, x_{n}, x_{1}, \ldots, x_{t-1}\right\}} \cong K_{m_{i}}^{d}$ for some $m_{i} \geq d, i \in\{t, \ldots, n\}$. However, this is clear since $\mathcal{H}_{N\left[x_{i}\right] \cap\left\{x_{i}, x_{i+1}, \ldots, x_{n}, x_{1}, \ldots, x_{t-1}\right\}}=\mathcal{H}_{N\left[x_{i}\right] \cap\left\{x_{i}, x_{i+1}, \ldots, x_{n}\right\}}$.
Lemma 3.1.11. If a d-uniform hypergraph $\mathcal{H}$ is triangulated (triangulated*, chordal), or, has a perfect elimination order, the same holds for $\mathcal{H}_{V}$ for every $V \subseteq \mathcal{V}(\mathcal{H})$.

Proof. Let $V \subseteq \mathcal{V}(\mathcal{H})$. If $\mathcal{E}\left(\mathcal{H}_{V}\right)=\emptyset, \mathcal{H}_{V}$ clearly is triangulated and triangulated*. It is also chordal since we can add one vertex at a time until we have the desired discrete hypergraph, and any ordering of $V$ yields a perfect elimination order. Thus we may assume that $\mathcal{E}\left(\mathcal{H}_{V}\right) \neq \emptyset$.

The lemma is clear for the classes of triangulated and triangulated* hypergraphs, since if $W \subseteq V$, we have that $\left(\mathcal{H}_{V}\right)_{W}=\mathcal{H}_{W}$. Now, let $\mathcal{H}=\mathcal{G} \bigcup_{K_{j}^{d}} K_{i}^{d}$, $0 \leq j<i$, be chordal. If $V \subseteq \mathcal{V}(\mathcal{G})$, or if $V \subseteq \mathcal{V}\left(K_{i}^{d}\right)$, we are done by induction. If this is not the case, it is easy to realize that $\mathcal{H}_{V}=\mathcal{G}_{V} \bigcup_{\left(K_{j}^{d}\right)_{V}}\left(K_{i}^{d}\right)_{V}$. Since $\mathcal{G}_{V}$ is chordal by induction, the result follows. Finally, assume $\mathcal{H}$ has a perfect elimination order $x_{1}<x_{2}<\cdots<x_{n}$. Then $V$ inherits an ordering $x_{i_{1}}<x_{i_{2}}<\cdots<x_{i_{|V|}}$. The fact that this is a perfect elimination order of $\mathcal{H}_{V}$ follows from Lemma 3.1.9.

Theorem-definition 3.1.12. Let $\mathcal{H}=(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be a d-uniform hypergraph. Then the following are equivalent.
(i) $\mathcal{H}$ is triangulated.
(ii) $\mathcal{H}$ is triangulated*.
(iii) $\mathcal{H}$ is chordal.
(iv) $\mathcal{H}$ has a perfect elimination order.

Proof. Due to Lemma 3.1.11, we need only consider the full set $\mathcal{V}(\mathcal{H})$ of vertices in our arguments, and we may assume that $\mathcal{E}(\mathcal{H}) \neq \emptyset$.
$(i) \Rightarrow(i i)$. Since we assume $\mathcal{E}(\mathcal{H}) \neq \emptyset$ and consider only the case where $V=\mathcal{V}(\mathcal{H})$, there is a vertex $x$ such that $\mathcal{H}_{N[x]} \cong K_{n}^{d}, n \geq d$. Then, $N[x]$ clearly is a face in $\Delta_{\mathcal{H}}$ of dimension at least $d-1$. Furthermore it has to be a facet, since if there were a $y \in \mathcal{V}(\mathcal{H}), y \neq x$, such that $N[x] \cup\{y\} \in \Delta_{\mathcal{H}}$, then there would exist an edge $E$ with $x, y \in E$. Hence, $y \in N[x]$.
$(i i) \Rightarrow(i)$. By assumption, there is a vertex $x$ such that $N[x]$ is a facet in $\Delta_{\mathcal{H}}$ of dimension greater than or equal to $d-1$, whence it is clear (from the definition of $\Delta_{\mathcal{H}}$ ) that $\mathcal{H}_{N[x]} \cong K_{n}^{d}$ for some $n \geq d$.
$(i) \Rightarrow($ iii $)$. By assumption there is a vertex $x \in \mathcal{V}(\mathcal{H})$ such that $\mathcal{H}_{N[x]} \cong$ $K_{n}^{d}$, for some $n \geq d$. Let $\mathcal{G}$ be the induced hypergraph on $\mathcal{V}(\mathcal{H}) \backslash\{x\}$. Then $\mathcal{E}(\mathcal{G})$ consists of all edges of $\mathcal{H}$, except those that contain $x$. This yields $\mathcal{H}=\mathcal{G} \cup_{\mathcal{K}} K_{n}^{d}$, where $\mathcal{K}=K_{|N(x)|}^{d}$ on vertex set $N(x)$, and by induction we are done.
$($ iii $) \Rightarrow(i)$. Assume $\mathcal{H}=\mathcal{G} \cup_{K_{j}^{d}} K_{i}^{d}, 0 \leq j<i$, is chordal, where $\mathcal{G}$ is chordal by construction. If $i \geq d$, any vertex $x \in \mathcal{V}\left(K_{i}^{d}\right) \backslash \mathcal{V}(\mathcal{G})$ will do, since $\mathcal{H}_{N[x]} \cong K_{i}^{d}$ for such $x$. If $i<d$, we find, by induction, a vertex $x \in \mathcal{V}(\mathcal{G})$ with the property that $\mathcal{H}_{N[x]}=\mathcal{G}_{N[x]} \cong K_{n}^{d}$ for some $n \geq d$, since otherwise the edge set of $\mathcal{H}$ would be empty, contrary to our assumptions.
$(i) \Rightarrow(i v) . \quad$ By assumption we find a vertex $x=x_{1}$ such that $\mathcal{H}_{N\left[x_{1}\right]} \cong K_{n}^{d}$, $n \geq d$. Since the induced hypergraph on $\mathcal{V}(\mathcal{H}) \backslash\left\{x_{1}\right\}$ is triangulated, by induction it has a perfect elimination order $x_{2}<\cdots<x_{n}$. If we put $x_{1}<x_{2}$ we are done.
$(i v) \Rightarrow(i) . \quad$ By Lemma 2.2 there is a perfect elimination order $x_{1}<\cdots<$ $x_{n}$, such that $\mathcal{H}_{N\left[x_{1}\right] \cap V} \cong K_{m}^{d}$ for some $m \geq d$.

We use hypergraphs from Chapter 2 to create some examples.
Example 6. Consider the complement $\mathcal{H}=\left(K_{n, m}^{d}\right)^{c}$ of the complete hypergraph $K_{n, m}^{d}$ on vertex set $\mathcal{V} \sqcup \mathcal{W},|\mathcal{V}|=n,|\mathcal{W}|=m$. We claim that $\mathcal{H}$ is chordal. It is easy to see, considering the the Stanley-Reisner ring, that

$$
\Delta_{\mathcal{H}}=\Delta\left(K_{n+m}^{d}\right) \cup\langle\mathcal{V}, \mathcal{W}\rangle .
$$

From this one may conclude that $\mathcal{H}$ is the disjoint union two $d$-complete hypergraphs,

$$
\mathcal{H}=K_{n}^{d} \cup_{K_{0}^{d}} K_{m}^{d}
$$

so $\mathcal{H}$ is chordal.
The analogous case of the $d$-complete multipartite hypergraph, $K_{n_{1}, \ldots, n_{t}}^{d}$ goes through similarly.

Example 7. Let $K_{n, m}^{d}$ be as in the previous example and consider the complex $\Delta_{K_{n, m}^{d}}$. If $n<d$ and $m<d$ we have an isomorphism $K_{n, m}^{d} \cong K_{n+m}^{d}$, so in this case $K_{n, m}^{d}$ is chordal. If $n$ or $m$ is greater than or equal to $d, K_{n, m}^{d}$ is not chordal. This is because no matter which vertex $x$ we choose, the induced
hypergraph on $N[x]$ cannot be $d$-complete, since it would then contain an edge lying entirely in either $\mathcal{V}$ or $\mathcal{W}$, which is impossible.

The analogous case of the $d$-complete multipartite hypergraph, $K_{n_{1}, \ldots, n_{t}}^{d}$ goes through similarly.

Example 8. In Chapter 5 we will consider a class of hypergraphs that we call line hypergraphs. A line hypergraph $L_{n}^{d, \alpha}$ may be written

$$
L_{n}^{d, \alpha}=K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} \cdots \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d}
$$

and is thus chordal.

### 3.2 Various concepts of chordality

In recent years several authors have generalized the properties of chordal graphs and since such generalizations may be made in many different directions, no particular standard concerning the use of the word chordal has been established. Thus there is the risk of different, but often somehow related, concepts getting similar names. We comment here on a couple of interesting papers in which the concept of chordality/triangulability has been studied.

As mentioned after Definition 3.1.3, the concept of triangulated hypergraphs also occurs in [46]. There the authors (among other things) aim for a generalization of Theorem 3.0.6. However, the triangulated property is used on the complementary hypergraphs compared with how Fröberg and we use it. The class of triangulated hypergraphs in the sense of [46], is properly included in the class of triangulated (chordal) hypergraphs considered here.

In [13] the authors (indirectly via matriods) define two classes of uniform hypergraphs, called $D$-perfect and triangulable respectively. It is then shown that a $D$-perfect hypergraph $\mathcal{H}$ is also triangulable. These hypergraphs and our chordal hypergraphs are related, but the relationship is not completely transparent.

In [62] the authors note that chordal graphs may be characterized as follows: A graph $\mathcal{G}$ is chordal if and only if its vertices can be labelled by numbers in $[n]=\{1,2, \ldots, n\}$ so that $\mathcal{G}$ has no induced subgraph $\mathcal{G}_{\{i<j<k\}}$ with edges $(i, j),(i, k)$ but without the edge $(j, k)$. The authors call a graph with this property perfectly labelled. This description of chordal graphs is then used to show that (see [62, Definition 6.1, Example 6.2, Definition 9.2, and Theorem 9.4] for background) a certain kind of building sets, called graphical building sets, are chordal if and only if the underlying graph is chordal. It seems possible that chordal hypergraphs (or some variant thereof) may be connected to chordal building sets, but it is not clear to us how.

Recall that a clutter is a (not necessarily uniform) hypergraph. Recently Woodroofe, [70], introduced a class of clutters that he calls chordal. The classes of chordal clutters and chordal hypergraphs do have non-empty intersection, but are in general quite different. Below we show that if $\mathcal{H}$ is a generalized chordal hypergraph, then $I_{\Delta_{\mathcal{H}}}$ has linear resolution. In the case of chordal hypergraphs
one can even show that $I_{\Delta_{\mathcal{H}}}$ has linear quotients. This was first proved in [27, Theorem 4.3] and also follows from Woodroofe's stronger result [70, Proposition 6.13]: Woodroofe proves that $\left(\Delta_{\mathcal{H}}\right)^{*}$ is vertex decomposable, which implies shellability.

### 3.3 Generalized chordal hypergraphs

It is easy to find an example of a uniform hypergraph $\mathcal{H}$ that is not chordal, but such that the Stanley-Reisner ring of $\Delta_{\mathcal{H}}$ has linear resolution.

Example 9. Let $\mathcal{H}$ be the 3-uniform hypergraph with $\mathcal{V}(\mathcal{H})=\{a, b, c, d\}$, and edge set

$$
\mathcal{E}(\mathcal{H})=\{\{a, b, c\},\{a, c, d\},\{a, b, d\}\} .
$$

The following simple picture lets us visualize $\mathcal{H}$.

$k\left[\Delta_{\mathcal{H}}\right]$ has linear resolution, but $\mathcal{H}$ is not chordal.
If $\Delta$ is a simplicial complex on $\mathcal{V}$ and $E$ is a finite set, we denote by $\Delta \cup E$ the simplicial complex on vertex set $\mathcal{V} \cup E$ whose set of facets, $\mathcal{F}(\Delta \cup E)$, is $\mathcal{F}(\Delta) \cup\{E\}$. Similarly, if $\mathcal{H}$ is a (not necessarily uniform) hypergraph and $E$ a finite set, we denote by $\mathcal{H} \cup E$ the hypergraph on $\mathcal{V}(\mathcal{H}) \cup E$ whose edge set is $\mathcal{E}(\mathcal{H} \cup E)=\mathcal{E}(\mathcal{H}) \cup\{E\}$.

Definition 3.3.1. A generalized chordal hypergraph is a $d$-uniform hypergraph, obtained inductively as follows:

- $K_{n}^{d}$ is a generalized chordal hypergraph, $\{n, d\} \subseteq \mathbb{N}$.
- If $\mathcal{G}$ is generalized chordal, then so is $\mathcal{H}=\mathcal{G} \bigcup_{K_{j}^{d}} K_{i}^{d}$, for $0 \leq j<i$.
- If $\mathcal{G}$ is generalized chordal and $E \subseteq \mathcal{V}(\mathcal{G}),|E|=d$, is such that $E$ has non-empty intersection with some edge of $\mathcal{G}$ and at least one element of $\binom{E}{d-1}$ is not a subset of any edge of $\mathcal{G}$, then $\mathcal{G} \cup E$ is generalized chordal.

Remark 3.3.2. The hypergraph in Example 9 is generalized chordal.
Remark 3.3.3. It is clear that every chordal hypergraph is also a generalized chordal hypergraph. Furthermore, for $d=2$ chordal graphs and generalized chordal graphs are the same.

Theorem 3.3.4. Let $\mathcal{H}$ be a generalized chordal hypergraph and $k$ a field of arbitrary characteristic. Then the Stanley-Reisner ring $k\left[\Delta_{\mathcal{H}}\right]$ has linear resolution.

Proof. We consider the three instances of Definition 4.1 one at a time. If $\mathcal{H} \cong K_{n}^{d}$ we are done, since if $n \geq d$ we have a simplex so the situation is trivial, and if $n<d$ the claim is proved for example in Theorem 2.6.3. So, we may assume $\mathcal{H} \not \neq K_{n}^{d}$. Let $\mathcal{H}=\mathcal{G} \cup_{K_{j}^{d}} K_{i}^{d}, 0 \leq j<i$, where $\mathcal{G}$ is generalized chordal. Let $C$ and $B$ be the simplices determined by $K_{j}^{d}$ and $K_{i}^{d}$, respectively, and consider the complex $\Delta_{\mathcal{H}}^{\prime}=\Delta_{\mathcal{G}} \cup B$. Note that $\Delta_{\mathcal{G}} \cap B=C, B \neq C$. We first show that $\Delta_{\mathcal{H}}^{\prime}$ has linear resolution. For every $V \subseteq \mathcal{V}(\mathcal{H})$, we have an exact sequence of chain complexes

$$
0 \rightarrow \tilde{\mathcal{C}} .\left(C_{V} ; k\right) \rightarrow \tilde{\mathcal{C}} .\left(\left(\Delta_{\mathcal{G}}\right)_{V} ; k\right) \oplus \tilde{\mathcal{C}} .\left(B_{V} ; k\right) \rightarrow \tilde{\mathcal{C}} .\left(\left(\Delta_{\mathcal{H}}^{\prime}\right)_{V} ; k\right) \rightarrow 0
$$

By induction, via Hochster's formula, we know that $\left(\Delta_{\mathcal{G}}\right)_{V}$ can have non-zero homology only in degree $d-2$. But then, since both $B_{V}$ and $C_{V}$ are simplices and accordingly have no homology at all, by considering the Mayer-Vietoris sequence we conclude that the only possible non-zero homologies of $\left(\Delta_{\mathcal{H}}^{\prime}\right)_{V}$ lies in degree $d-2$.

Note that it is not in general true that $\Delta_{\mathcal{H}}=\Delta_{\mathcal{H}}^{\prime}$. In fact, this holds only when $d=2$. However, the difference between the two complexes is easy to understand, and we may use the somewhat easier looking $\Delta_{\mathcal{H}}^{\prime}$ to show that $\Delta_{\mathcal{H}}$ has linear resolution as well.

To this end, let $\Gamma$ be the $(d-2)$-skeleton of the full simplex on vertex set $\mathcal{V}(\mathcal{H})$. Then one sees that

$$
\Delta_{\mathcal{H}}=\Delta_{\mathcal{H}}^{\prime} \cup \Gamma
$$

The $(d-2)$-faces that we add to $\Delta_{\mathcal{H}}^{\prime}$ to obtain $\Delta_{\mathcal{H}}$ can certainly not cause any homology in degrees greater than $d-2$, that did not already exist in $\Delta_{\mathcal{H}}^{\prime}$. Indeed, suppose $\sum_{i} a_{i} \sigma_{i}$ is a cycle in a degree $r>d-2$, where $a_{i} \in k$ and the $\sigma_{i}$ 's are faces of $\Delta_{\mathcal{H}}$ of dimension $r$. Since every face $\sigma_{i}$ actually lies in $\Delta_{\mathcal{H}}^{\prime}$, it follows that $\sum_{i} a_{i} \sigma_{i}$ is a cycle also in $\Delta_{\mathcal{H}}^{\prime}$. Thus, if $\Delta_{\mathcal{H}}^{\prime}$ has linear resolution, so does $\Delta_{\mathcal{H}}$.

Finally, let $\mathcal{H}=\mathcal{G} \cup E$. Let $\left\{F_{1}, \ldots, F_{t}\right\}$ be the set of elements of $\binom{E}{d-1}$ that are not subsets of any edge of $\mathcal{G}$. Note that $\Delta_{\mathcal{H}}=\Delta_{\mathcal{G}} \cup E$. Take $V \subseteq \mathcal{V}(\mathcal{H})$. If $E \nsubseteq V$, then $\left(\Delta_{\mathcal{H}}\right)_{V}=\left(\Delta_{\mathcal{G}}\right)_{V}$, so, by induction we conclude that the only possible non-zero homologies of $\left(\Delta_{\mathcal{H}}\right)_{V}$ lies in degree $d-2$. Hence we may assume that $E \subseteq V$. Then we have an exact sequence

$$
0 \rightarrow \tilde{\mathcal{C}} .\left(\left(\Delta_{\mathcal{G}} \cap E\right)_{V} ; k\right) \rightarrow \tilde{\mathcal{C}} .\left(\left(\Delta_{\mathcal{G}}\right)_{V} ; k\right) \oplus \tilde{\mathcal{C}} .\left(E_{V} ; k\right) \rightarrow \tilde{\mathcal{C}} .\left(\left(\Delta_{\mathcal{H}}\right)_{V} ; k\right) \rightarrow 0 .
$$

Note that $E_{V}$ is a simplex so it has zero homology, and, by induction, we know that $k\left[\Delta_{\mathcal{G}}\right]$ has linear resolution. Using Hochster's formula, we may conclude that $\tilde{H}_{d-1}\left(\left(\Delta_{\mathcal{G}}\right)_{V} ; k\right)=0$. Hence, the Mayer-Vietoris sequence obtained from the above exact sequence looks as follows:

$$
0 \rightarrow \tilde{H}_{d-1}\left(\left(\Delta_{\mathcal{H}}\right)_{V} ; k\right) \rightarrow
$$

$$
\tilde{H}_{d-2}\left(\left(\Delta_{\mathcal{G}} \cap E\right)_{V} ; k\right) \rightarrow \tilde{H}_{d-2}\left(\left(\Delta_{\mathcal{G}}\right)_{V} ; k\right) \rightarrow \tilde{H}_{d-2}\left(\left(\Delta_{\mathcal{H}}\right)_{V} ; k\right) \rightarrow 0
$$

Let $z=\sum_{j} a_{j} \sigma_{j}$ be an element in $Z_{d-1}\left(\left(\Delta_{\mathcal{H}}\right)_{V} ; k\right)$, where $\sigma_{1}=E$. Consider the expression for the derivative of this cycle

$$
0=d(z)=\cdots+\sum_{i=1}^{t} \pm a_{1} F_{i}+\cdots
$$

Since $\sum_{i=1}^{t} \pm a_{1} F_{i}$ only can come from $d(E)$, we conclude that $a_{1}=0$. Hence $z \in Z_{d-1}\left(\left(\Delta_{\mathcal{G}}\right)_{V} ; k\right)$, and, using Hochster's formula, we may conclude that the Stanley-Reisner ring of $\Delta_{\mathcal{H}}$ has linear resolution.

Corollary 3.3.5. Let $\mathcal{H}$ be a generalized chordal hypergraph and $k$ a field of arbitrary characteristic. Then the Stanley-Reisner ring $k\left[\Delta_{\mathcal{H}}^{*}\right]$ of the Alexander dual complex $\Delta_{\mathcal{H}}^{*}$ is Cohen-Macaulay.

Proof. This follows by the Eagon-Reiner theorem.
Corollary 3.3.6. Theorem 3.3.4 and Corollary 3.3.5 in particular applies to triangulated and triangulated* hypergraphs, and also to hypergraphs that have perfect elimination orders.

As mentioned in Section 3.2 we have
Theorem 3.3.7. If $\mathcal{H}$ is a chordal hypergraph, then $I_{\Delta_{\mathcal{H}}}$ has linear quotients.
Proof. This is proved in [27, Theorem 4.3] and also follows from Woodroofe's stronger result [70, Proposition 6.13].

Remark 3.3.8. Observe that this improves Theorem 3.3.4 in the case of chordal hypergraphs.

Corollary 3.3.9. A graph $\mathcal{G}$ is chordal if and only if $I_{\Delta_{\mathcal{G}}}$ has linear quotients.
Proof. The fact that the ideal $I_{\Delta_{\mathcal{G}}}$ has linear quotients follows from the theorem. Assume $I_{\Delta_{\mathcal{G}}}$ has linear quotients. Then it has linear resolution and thus by Fröberg's theorem, Theorem 3.0.6, $\mathcal{G}$ is chordal.

In [39] Fröberg considers a class of chordal graphs called $n$-trees.
Definition 3.3.10. An $n$-tree is a chordal graph defined inductively as follows:

- $K_{n+1}$ is a $n$-tree.
- If $\mathcal{G}$ is a $n$-tree, then so is $\mathcal{H}=\mathcal{G} \bigcup_{K_{n}} K_{n+1}$. (We attach $K_{n+1}$ to $\mathcal{G}$ in a common (under identification) $K_{n}$ )

Now, consider the corresponding subclass $\mathcal{T}_{d}$ of the class of chordal hypergraphs. That is, $\mathcal{T}_{d}$ is the class of chordal hypergraphs described as follows:

- $K_{n+1}^{d}$ belongs to $\mathcal{T}_{d}$.
- If $\mathcal{G}$ belongs to $\mathcal{T}_{d}$, then so does $\mathcal{H}=\mathcal{G} \bigcup_{K_{n}^{d}} K_{n+1}^{d}$. (We attach $K_{n+1}^{d}$ to $\mathcal{G}$ in a common (under identification) $K_{n}^{d}$.)
We get the following results:
Theorem 3.3.11. For any hypergraph $\mathcal{H}$ in $\mathcal{T}_{d}$, the clique complex $\Delta_{\mathcal{H}}$ is pure shellable and hence Cohen-Macaulay.
Proof. The proof is by induction. If $\mathcal{H}=K_{n+1}^{d}$, then $\Delta_{\mathcal{H}}$ is a simplex and pure shellable. Let $\mathcal{H}=\mathcal{G} \cup_{K_{n}^{d}} K_{n+1}^{d}$ and $F_{1}<\cdots<F_{r}$ be a shelling for $\Delta_{\mathcal{G}}$. Then $\Delta_{\mathcal{H}}=\left\langle F_{1}, \ldots, F_{r}, F_{r+1}\right\rangle$, where $F_{r+1}=\mathcal{V}\left(K_{n+1}^{d}\right)$. Let $\mathcal{V}\left(K_{n}^{d}\right)=L$. Then $L \subseteq F_{i}$ for some $1 \leq i \leq r$. We claim that $F_{1}<\cdots<F_{r}<F_{r+1}$ is a shelling for $\Delta_{\mathcal{H}}$. Let $F_{r+1}=L \cup\{v\}$. Then for any $j \leq r$, one has $v \in F_{r+1} \backslash F_{j}$ and $F_{r+1} \backslash F_{i}=\{v\}$.
Corollary 3.3.12. For any d-tree $\mathcal{G}$, the clique complex $\Delta_{\mathcal{G}}$ is pure shellable and hence Cohen-Macaulay.

In the proof of the following proposition, we will use the fact that the complexes $\Delta\left(K_{n}^{d}\right)$ are shellable.
Proposition 3.3.13. Let $\mathcal{H}=K_{m}^{d} \cup_{K_{j}^{d}} K_{i}^{d}, m \geq d$. Then $I(\mathcal{H})$ has linear quotients precisely when
(i) $i<d$, or
(ii) $i \geq d$, and $j=m-1$ (or symmetrically $j=i-1$ ).

Proof. Put $A=\mathcal{V}(\mathcal{H}) \backslash \mathcal{V}\left(K_{m}^{d}\right)$ and $B=\mathcal{V}(\mathcal{H}) \backslash \mathcal{V}\left(K_{i}^{d}\right)$. We show that the Alexander dual complex $\Delta(\mathcal{H})^{*}$ is shellable precisely in the cases mentioned above. Assume that $F_{1}^{\prime}<\cdots<F_{t}^{\prime}$ is a shelling of $\Delta\left(K_{m}^{d}\right)^{*}$. For $i \in\{1, \ldots, t\}$, set $F_{i}=F_{i}^{\prime} \cup A$. In case $(i)$ the sequence $F_{1}<\cdots<F_{t}$ is a shelling of $\Delta(\mathcal{H})^{*}$.

In case (ii) we give a shelling when $i \geq d$ and $j=m-1$. The case $j=i-1$ is similar. Construct sets $F_{i}$ as above, put $\mathcal{V}\left(K_{m}^{d}\right) \backslash \mathcal{V}\left(K_{j}^{d}\right)=\{v\}$, and let $G_{1}^{\prime}<\cdots<G_{s}^{\prime}$ be a shelling of $\Delta\left(K_{i}^{d}\right)^{*}$. Set $G_{i}=G_{i}^{\prime} \cup B$ for $i \in\{1, \ldots, s\}$. It is easy to see that the set of facets of $\Delta(\mathcal{H})^{*}$ is $\left\{F_{i}\right\}_{i=1}^{t} \cup\left\{G_{j}\right\}_{j=1}^{s}$. We claim that the ordering $G_{1}<\cdots<G_{s}<F_{1}<\cdots<F_{t}$ is a shelling of $\Delta(\mathcal{H})^{*}$. Let $G_{i}<F_{j}$, where $G_{i}=\mathcal{V}(\mathcal{H}) \backslash E_{1}$ and $F_{j}=\mathcal{V}(\mathcal{H}) \backslash E_{2}$ for some edges $E_{1}$ of $K_{i}^{d}$ and $E_{2}$ of $K_{m}^{d}$. Let $E_{1}=\left\{w_{1}, \ldots, w_{d}\right\}$ and $E_{2}=\left\{v, v_{1}, \ldots, v_{d-1}\right\}$, where $\left\{v_{1}, \ldots, v_{d-1}\right\} \subseteq \mathcal{V}\left(K_{j}^{d}\right)$. Then there exists $1 \leq l \leq d$ such that $w_{l} \in F_{j} \backslash G_{i}$. Set $E_{3}=\left\{w_{l}, v_{1}, \ldots, v_{d-1}\right\}$, then $\mathcal{V}(\mathcal{H}) \backslash E_{3}=G_{k}$ for some $k$ and $F_{j} \backslash G_{k}=\left\{w_{l}\right\}$.

Now, assume ( $i$ ) and (ii) do not hold. Then $i \geq d, m-j \geq 2$ and $i-j \geq 2$. We first claim that if $j \leq d-2$, there is no shelling: Consider the intersection $G_{j} \cap F_{i}$ (with the same notation as above). The two facets here correspond to two edges in $\mathcal{H}$, one from $K_{m}^{d}$ and one from $K_{i}^{d}$. These two edges can at most have $j$ elements in common. Hence, by considering the set complements of these edges we realize that the two facets can at most have $|\mathcal{V}(\mathcal{H})|-d-2$ vertices in common. This shows that no ordering of the $F_{i}$ 's and the $G_{j}$ 's can be a shelling, since $\operatorname{dim} F_{i}=\operatorname{dim} G_{j}=\mathcal{V}(\mathcal{H})-d-1$ for every $i \in\{1, \ldots, t\}, j \in\{1, \ldots, s\}$.

So, we assume $j \geq d-2$. Let $\left\{v_{1}, v_{2}\right\} \subseteq \mathcal{V}\left(K_{m}^{d}\right) \backslash \mathcal{V}\left(K_{j}^{d}\right),\left\{w_{1}, w_{2}\right\} \subseteq \mathcal{V}\left(K_{i}^{d}\right) \backslash$ $\mathcal{V}\left(K_{j}^{d}\right)$ and $\left\{u_{1}, \ldots, u_{d-2}\right\} \subseteq \mathcal{V}\left(K_{j}^{d}\right)$. To finish the proof by contradiction, we assume $\Delta(\mathcal{H})^{*}$ is shellable. Consider the edges $E_{1}=\left\{u_{1}, \ldots, u_{d-2}, v_{1}, v_{2}\right\}$ and $E_{2}=\left\{u_{1}, \ldots, u_{d-2}, w_{1}, w_{2}\right\}$. Then $H_{i}=\mathcal{V}(\mathcal{H}) \backslash E_{i}$ for $i \in\{1,2\}$ are facets of $\Delta(\mathcal{H})^{*}$. Without loss of generality may assume $H_{1}<H_{2}$. Hence there exists vertex $v \in H_{2} \backslash H_{1}$ and facet $H_{3}$ such that $H_{2} \backslash H_{3}=\{v\}$. Let $H_{3}=\mathcal{V}(\mathcal{H}) \backslash E_{3}$ for some edge $E_{3}$. Since $H_{2} \backslash H_{1}=\left\{v_{1}, v_{2}\right\}$, we have $v=v_{1}$ or $v=v_{2}$. Therefore $v \in E_{3}$ so $E_{3} \subseteq \mathcal{V}\left(K_{m}^{d}\right)$. Thus $\left\{w_{1}, w_{2}\right\} \subseteq H_{3}$ which is a contradiction since $\left|H_{3}\right|=\left|H_{2}\right|$.

We end this section with a result on the diameter of the complement of a chordal graph. Recall that the diameter of a connected graph $\mathcal{G}$ is defined as

$$
\operatorname{diam}(\mathcal{G})=\max \{\operatorname{dist}(u, v) ; u, v \in \mathcal{V}(\mathcal{G})\}
$$

where $\operatorname{dist}(u, v)$ is the number of edges in a shortest path between $u$ to $v$. If $\mathcal{G}$ is not connected we set the diameter to be $\infty$.

Proposition 3.3.14. Let $\mathcal{G}$ be a connected chordal graph. Then the diameter of the complementary graph $\mathcal{G}^{c}$ is at most 3.

Proof. If $\operatorname{diam}\left(\mathcal{G}^{c}\right)>3$ we find vertices $u$ and $v$ with $\operatorname{dist}(u, v)=4$. Hence there are vertices $\left\{v_{1}, v_{2}, v_{3}\right\}$ such that the induced graph of $\mathcal{G}^{c}$ on $\left\{u, v_{1}, v_{2}, v_{3}, v\right\}$ is the path $u v_{1}, v_{1} v_{2}, v_{2} v_{3}, v_{3} v$. The graph complement of $\mathcal{G}_{\left\{u, v_{1}, v_{2}, v_{3}, v\right\}}^{c}$ contains a 4 -cycle without any chord. This contradiction gives our result.

## $3.4 d$-uniform hypergraphs and Quasi-Forests

It is known that a certain class of simplicial complexes, called quasi-trees (see definition below), and chordal graphs, in a sense contain the same information.

The following lemma is basically [49, Lemma 3.1].
Lemma 3.4.1. Let $\Delta$ be a simplicial complex. Then $\Delta$ is a quasi-forest precisely when $\Delta=\Delta_{\mathcal{G}}$ for some chordal graph $\mathcal{G}$.

In this section we will see that there is a close connection also between quasi-trees and the class of chordal hypergraphs.

Definition 3.4.2 (Faridi, [29], Zheng, [71]). Let $\Delta$ be a simplicial complex. A sub-collection $\Gamma$, of $\Delta$, is a sub-complex of $\Delta$ such that $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$. A facet $F$ of $\Delta$ is called a leaf if either $F$ is the only facet of $\Delta$, or there exists a facet $G$ in $\Delta, G \neq F$, such that $F \cap H \subseteq F \cap G$ for any facet $H$ in $\Delta, H \neq F$.
Assume $\Delta$ is connected. Then $\Delta$ is called a tree if every sub-collection of $\Delta$ has a leaf, and $\Delta$ is called a quasi-tree if there exists an order $F_{1}, \ldots, F_{t}$ of the facets of $\Delta$ such that for each $i \in\{1, \ldots, t\}, F_{i}$ is a leaf of the simplicial complex $\left\langle F_{1}, \ldots, F_{i}\right\rangle$. The order $F_{1}, \ldots, F_{t}$ is called a leaf order. A simplicial complex with the property that every connected component is a (quasi-)tree is called a (quasi-)forest.

Remark 3.4.3. A tree is clearly a quasi-tree, but the converse need not hold. Consider the quasi-tree

$$
\Delta=\langle\{a, b, c\},\{b, c, d\},\{c, d, e\},\{b, d, f\}\rangle
$$

One sees that the sub-complex $\langle\{a, b, c\},\{c, d, e\},\{b, d, f\}\rangle$ has no leaf.
Let $\Delta$ be a simplicial complex. Denote by $\mathcal{R}_{d}(\Delta)$ the simplicial complex obtained from $\Delta$ by removing every facet $F$ with $1 \leq \operatorname{dim} F \leq d-2$, and all faces $G \subseteq F$, with $1 \leq \operatorname{dim} G \leq \operatorname{dim} F$, that are not faces of some facet of dimension greater than $d-2$. Conversely, denote by $\mathcal{A}_{d}(\Delta)$ the simplicial complex obtained from $\Delta$ by adding, as a facet, every face of dimension $d-2$ that is not already in the complex.

Lemma 3.4.4. Let $\Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{G}}$ be the clique complexes of a d-uniform hypergraph $\mathcal{H}$, and a graph $\mathcal{G}$, respectively. Then the following holds:

- $\mathcal{A}_{d}\left(\mathcal{R}_{d}\left(\Delta_{\mathcal{H}}\right)\right)=\Delta_{\mathcal{H}}$
- $\mathcal{R}_{d^{\prime}}\left(\mathcal{A}_{d^{\prime}}\left(\Delta_{\mathcal{G}}\right)\right)=\Delta_{\mathcal{G}}$ for all $d^{\prime}<\min \left\{\operatorname{dim} F ; F\right.$ facet in $\left.\Delta_{\mathcal{G}}\right\}$.

Proof. This follows immediately from the definition of clique complex.
Lemma 3.4.5. Let $\mathcal{H}=\mathcal{G} \cup_{K_{j}^{d}} K_{i}^{d}$ be a chordal hypergraph. If $i<d$ (that is $K_{i}^{d}$ is consists of $i$ isolated vertices), we may exchange the attaching of $K_{i}^{d}$ to $K_{j}^{d}$, with $i-j$ attachings of the form

$$
\mathcal{H}^{\prime}=\mathcal{G}^{\prime} \cup_{K_{0}^{d}} K_{1}^{d}
$$

Proof. This is clear, since either way, we are just adding a number of isolated vertices.

Proposition 3.4.6. Let $\mathcal{H}$ be a d-uniform chordal hypergraph, and let $\mathcal{G}$ be a chordal graph. Then the following holds:
(i) $\mathcal{R}_{d}\left(\Delta_{\mathcal{H}}\right)$ is the clique complex of a chordal graph.
(ii) $\mathcal{A}_{d^{\prime}}\left(\Delta_{\mathcal{G}}\right)$ is, for any $d^{\prime}$, the clique complex of a $d^{\prime}$-uniform chordal hypergraph.

Proof. A chordal hypergraph $\mathcal{H}$ may, according to its inductive construction, be represented by a sequence of pairs of $d$-complete hypergraphs

$$
\left(K_{0}^{d}, K_{i_{1}}^{d}\right), \ldots,\left(K_{j_{t}}^{d}, K_{i_{t}}^{d}\right)
$$

where in each step of the construction of $\mathcal{H}, K_{i_{s}}^{d}$ is attached to $K_{j_{s}}^{d}$. We assume that in the construction of $\mathcal{H}$, Lemma 3.4.5 has been used if necessary. Then every $d$-complete hypergraph in the sequence $\left(K_{0}^{d}, K_{i_{1}}^{d}\right), \ldots,\left(K_{j_{t}}^{d}, K_{i_{t}}^{d}\right)$ yields a complete graph, and, by considering the facets, it is clear that $\mathcal{R}_{d}\left(\Delta_{\mathcal{H}}\right)$ is the complex of the chordal graph that is represented by the sequence of pairs $\left(K_{0}, K_{i_{1}}\right), \ldots,\left(K_{j_{t}}, K_{i_{t}}\right)$. This proves $(i)$.

Now let $\left(K_{0}, K_{j_{1}}\right), \ldots,\left(K_{j_{t}}, K_{i_{t}}\right)$ denote a chordal graph $\mathcal{G}$. If $d^{\prime}-2 \geq$ $\operatorname{dim} \Delta_{\mathcal{G}}$, then the claim (ii) is trivial, so we assume $d^{\prime}-2<\operatorname{dim} \Delta_{\mathcal{G}}$. It is obvious that $\mathcal{A}_{d^{\prime}}\left(\Delta_{\mathcal{G}}\right)$ will be the complex of a $d^{\prime}$-uniform hypergraph $\mathcal{H}$, since every minimal non-face has dimension $d^{\prime}-1$. We now show that $\mathcal{H}$ is chordal. We do this by constructing a sequence of pairs $\left(K_{0}^{d}, K_{i_{1}}^{d}\right), \ldots,\left(K_{j_{r}}^{d}, K_{i_{r}}^{d}\right), r \geq t$, from the sequence $\left(K_{0}, K_{i_{1}}\right), \ldots,\left(K_{j_{t}}, K_{i_{t}}\right)$, and showing that this sequence actually defines $\mathcal{H}$.

First note that if $i_{s} \geq d^{\prime}$, a complete graph $K_{i_{s}}$ immediately yields a $d^{\prime}$ complete hypergraph $K_{i_{s}}^{d^{\prime}}$. For such $i_{s}$, we get a pair $\left(K_{j_{s}}^{d^{\prime}}, K_{i_{s}}^{d^{\prime}}\right)$, corresponding to the pair $\left(K_{j_{s}}, K_{i_{s}}\right)$ in the sequence representing $\mathcal{G}$. If $i_{s}<d^{\prime}$, we may instead associate to the pair $\left(K_{j_{s}}, K_{i_{s}}\right)$ a sequence of "trivial pairs", as in Lemma 3.4.5. Continuing in this way, we obtain a sequence $\left(K_{j_{1}}^{d}, K_{i_{i}}^{d}\right), \ldots,\left(K_{j_{r}}^{d}, K_{i_{r}}^{d}\right)$, representing a $d^{\prime}$-uniform chordal hypergraph $\mathcal{H}^{\prime}$.

The $d^{\prime}$-uniform chordal hypergraph that correspond to the constructed sequence yields the same complex as $\mathcal{H}$, and hence we conclude that they must be the same.

Corollary 3.4.7. To every chordal hypergraph $\mathcal{H}$ we may associate a quasiforest $\Delta$, and vice versa.

Proof. If $\Delta$ is a quasi-forest, then $\Delta=\Delta_{\mathcal{G}}$ for some chordal graph ([49, Lemma 3.1]). Then, according to the proposition, we may associate to $\Delta$ the chordal hypergraph $\mathcal{H}$ whose clique complex is the complex $\mathcal{A}_{d^{\prime}}\left(\Delta_{\mathcal{G}}\right)$ in the proposition. Conversely, given a chordal hypergraph $\mathcal{H}$ we may associate to it the quasi-forest $\mathcal{R}_{d}\left(\Delta_{\mathcal{H}}\right)$ from the proposition.

## Chapter 4

## Componentwise linearity of ideals arising from graphs

Let $\mathcal{G}$ be a simple graph on $n \in \mathbb{N}$ vertices. Francisco and Van Tuyl have shown that if $\mathcal{G}$ is chordal, then $I(\mathcal{G})^{*}=\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G )}}\left(x_{i}, x_{j}\right)$ is componentwise linear. It is natural to ask for which $t_{i j}>1$ the ideal $\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G})}\left(x_{i}, x_{j}\right)^{t_{i j}}$ is componentwise linear, if $\mathcal{G}$ is chordal. We show that $\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G})}\left(x_{i}, x_{j}\right)^{t}$ is componentwise linear for all $n \geq 3$ and positive $t$, if $\mathcal{G}$ is a complete graph. We give also an example where $\mathcal{G}$ is chordal, but the intersection ideal is not componentwise linear for any $t>1$.

### 4.1 Intersections for complete graphs

For a graded ideal $I$ in a polynomial ring we denote by $I_{\langle d\rangle}$ the ideal generated by the elements of degree $d$ that belong to $I$. In [47] Herzog and Hibi defined $I$ to be componentwise linear if $I_{\langle d\rangle}$ has a linear resolution for all $d$.

Here we examine componentwise linearity of ideals arising from complete graphs and of the form

$$
\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G})}\left(x_{i}, x_{j}\right)^{t}
$$

Let $K_{n}$ be a complete graph on $n$ vertices. We write

$$
K_{n}^{(t)}=\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}\left(K_{n}\right)}\left(x_{i}, x_{j}\right)^{t}
$$

We will show that the ideal $K_{n}^{(t)}$ is componentwise linear for all $n \geq 3$ and $t \geq 1$. Recall that a vertex cover of a graph $\mathcal{G}$ is a subset $A \subset \mathcal{V}(\mathcal{G})$ such that every edge of $\mathcal{G}$ is incident to at least one vertex of $A$. It is not hard to show that $I(\mathcal{G})^{*}=\left(x_{i_{1}} \cdots x_{i_{k}} \mid\left\{x_{i_{1}}, \ldots, x_{i_{k}}\right\}\right.$ a vertex cover of $\mathcal{G}$ ). A $t$-vertex cover (or a vertex cover of order $t)$ of $\mathcal{G}$ is a vector $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with $a_{i} \in \mathbb{N}$ such that $a_{i}+a_{j} \geq t$ for all $\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G})$.

In the proof of the theorem below, we use the following proposition.
Proposition 4.1.1 (Proposition 2.6 in [31] and Lemma 4.1 in [12]). If $I$ is a homogeneous ideal with linear quotients, then $I$ is componentwise linear.

Theorem 4.1.2. The ideal $K_{n}^{(t)}$ is componentwise linear for all $n \geq 3$ and $t \geq 1$.
Proof. For calculating an explicit generating system of $K_{n}^{(t)}$ we will use $t$-vertex covers. Pick any monomial $m$ in the generating set of $K_{n}^{(t)}$ and, for some $k$ and $l$, consider the greatest exponents $t_{k}$ and $t_{l}$ such that $x_{k}^{t_{k}} x_{l}^{t_{l}}$ is a factor in $m$. As $m$ is contained in $\left(x_{k}, x_{l}\right)^{t}$ we must have $t_{k}+t_{l} \geq t$. Hence, $K_{n}^{(t)}$ is generated by the monomials of the form $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a}$ is a $t$-cover of $K_{n}$. That is, the sum of the two lowest exponents in every (monomial) generator of $K_{n}^{(t)}$ is at least $t$.

First we assume that $t=2 m+1$ is odd. Considering the minimal monomial generators ( $t$-covers) we get

$$
\begin{array}{r}
K_{n}^{(t)}=K_{n}^{(2 m+1)}=\begin{array}{ccc}
\left(x_{1}^{m} \prod_{i \neq 1} x_{i}^{m+1},\right. & \ldots & , x_{n}^{m} \prod_{i \neq n} x_{i}^{m+1} \\
x_{1}^{m-1} \prod_{i \neq 1} x_{i}^{m+2}, & \ldots & , x_{n}^{m-1} \prod_{i \neq 1} x_{i}^{m+2} \\
\vdots & \\
\prod_{i \neq 1} x_{i}^{2 m+1}, & \ldots & \left., \prod_{i \neq n} x_{i}^{2 m+1}\right)
\end{array} .
\end{array}
$$

The generators of the ideal are ordered, increasing from left to right, using the degree lexicographic ordering with $x_{1} \prec x_{2} \prec \cdots \prec x_{n}$. This ordering yields linear quotients and hence $K_{n}^{(t)}$ is componentwise linear by Proposition 4.1.1.

If $t=2 m$ is even we get instead

$$
\begin{array}{r}
K_{n}^{(t)}=K_{n}^{(2 m)}=\left(\prod_{i=1}^{2 m} x_{i}^{m}, \quad x_{1}^{m-1} \prod_{i \neq 1} x_{i}^{m+1},\right. \\
\ldots
\end{array}, \quad, x_{n}^{m-1} \prod_{i \neq n} x_{i}^{m+1},
$$

which also yields linear quotients.

## Example 10.

$$
K_{12}^{(5)}=\left(\left\{x_{j}^{2} \prod_{i \neq j} x_{i}^{3}\right\}_{1 \leq j \leq 12},\left\{x_{j} \prod_{i \neq j} x_{i}^{4}\right\}_{1 \leq j \leq 12},\left\{\prod_{i \neq j} x_{i}^{5}\right\}_{1 \leq j \leq 12}\right)
$$

and

$$
K_{5}^{(6)}=\left(\prod_{i=1}^{5} x_{i}^{3},\left\{x_{j}^{2} \prod_{i \neq j} x_{i}^{4}\right\}_{1 \leq j \leq 5},\left\{x_{j} \prod_{i \neq j} x_{i}^{5}\right\}_{1 \leq j \leq 5},\left\{\prod_{i \neq j} x_{i}^{6}\right\}_{1 \leq j \leq 5}\right)
$$

Remark 4.1.3. A monomial ideal is called polymatroidal if it is generated in one degree and its minimal generators satisfy a certain "exchange condition". In [52] Herzog and Takayama show that polymatroidal ideals have linear resolutions. Later Francisco and Van Tuyl proved, [31, Theorem 3.1], that some families of ideals $I$ are componentwise linear by showing that $I_{\langle d\rangle}$ is polymatroidal for all $d$.

The ideals $K_{n}^{(t)}$ are also polymatroidal, but proving this using the same techniques as in the proof of [31, Theorem 3.1] is rather tedious.

### 4.2 A Counterexample

Claim 1. There exists a chordal graph $\mathcal{G}$ such that $\bigcap_{\left\{x_{i}, x_{j}\right\} \in \mathcal{E}(\mathcal{G})}\left(x_{i}, x_{j}\right)^{t}$ is not componentwise linear for any $t>1$.

Proof. Let $\mathcal{G}$ be the chordal graph

and denote the intersection $(a, b)^{t} \cap(a, c)^{t} \cap(b, c)^{t} \cap(b, d)^{t} \cap(c, d)^{t}$ by $I^{(t)}$. We have

$$
I^{(1)}=(b c)+(a b d, a c d)
$$

and

$$
I^{(2)}=\left(b^{2} c^{2}, a b c d\right)+\left(a^{2} b^{2} d^{2}, a^{2} c^{2} d^{2}\right)
$$

Considering the $t$-covers in the same way as we did for $K_{n}^{(t)}$ we see the following:

If $t_{a} \leq\left\lfloor\frac{t}{2}\right\rfloor$ then $t_{b}=t-t_{a}=t_{c}$ (the sum $t_{b}+t_{c} \geq t$ automatically) and $t_{d}=t-t_{b}=t-t_{c}=t_{a}$. Thus, we get the set of minimal generators of degree $2 t$ :

$$
\left\{a^{i}(b c)^{t-i} d^{i}\right\}_{0 \leq i \leq\left\lfloor\frac{t}{2}\right\rfloor}
$$

If $t_{a}>\left\lfloor\frac{t}{2}\right\rfloor$, then either $t_{b}=t-t_{a}$ and $t_{c}=t-t_{b}=t_{a}$, or $t_{c}=t-t_{a}$ and $t_{b}=t_{a}$. Further $t_{d}=t_{a}$. The set of minimal generators we get in this way is equal to

$$
\left\{(a c d)^{i} b^{t-i}\right\}_{\left\lfloor\frac{t}{2}\right\rfloor<i \leq t} \cup\left\{(a c d)^{i} b^{t-i}\right\}_{\left\lfloor\frac{t}{2}\right\rfloor<i \leq t} .
$$

The generators in this set are of degree at least $(2 t+1)$ for odd $t$ and of degree at least $(2 t+2)$ for even $t$.

Now consider the minimal free resolution $\mathcal{F}$. of $\left(I^{(t)}\right)_{\langle 2 t\rangle}$. Since $\mathcal{F}$. is contained in any free resolution $\mathcal{G}$. of $\left(I^{(t)}\right)_{\langle 2 t\rangle}$ we have that if $F_{1}$ (the component
of $\mathcal{F}$. in homological degree 1) has a non-zero component in a certain degree, then so do $G_{1}$. Let $\mathcal{G}$. be the Taylor resolution of $\left(I^{(t)}\right)_{\langle 2 t\rangle}$. The degrees in which $G_{1}$ has non-zero components come from least common mutliples of pairs of minimal generators of $\left(I^{(t)}\right)_{\langle 2 t\rangle}$. By considering the above description of the minimal generators in degree $2 t$, one sees that $G_{1}$ has non-zero components only in degrees strictly larger than $2 t+1$. Thus $\mathcal{F}$. cannot be a linear resolution and hence, $I^{(t)}$ is not componentwise linear.

## Chapter 5

## Hypercycles and Line Hypergraphs

In [54] Jacques computes the Betti numbers of the graph algebras for graphs that are lines and cycles. In this chapter we define line hypergraphs $L_{n}^{d, \alpha}$ and hypercycles $C_{n}^{d, \alpha}$ and compute the Betti numbers of their hypergraph algebras.

### 5.1 Hypercycles and Line Hypergraphs

Definition 5.1.1. Let $\mathcal{V}$ be a finite set and $\alpha$ and $d$ positive integers such that $d \geq 2 \alpha$. The line hypergraph $L_{n}^{d, \alpha}$ on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph with edge set $\mathcal{E}\left(L_{n}^{d, \alpha}\right)=\left\{E_{1}, \ldots, E_{n}\right\}$ such that
(i) $\mathcal{V}\left(L_{n}^{d, \alpha}\right)=\bigcup_{i=1}^{n} E_{i}$.
(ii) For any $0 \leq i<j \leq n, E_{i} \bigcap E_{j} \neq \emptyset$ if and only if $j=i+1$.
(iii) $\left|E_{i} \bigcap E_{i+1}\right|=\alpha$ for all $i \in\{1, \ldots, n-1\}$.

The length of a line hypergraph is defined as the number of edges.
Example 11. Every line hypergraph $L_{n}^{d, \alpha}$ is a chordal hypergraph since it may be written as

$$
L_{n}^{d, \alpha}=K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} \cdots \cup_{K_{\alpha}^{d}} K_{d}^{d} \cup_{K_{\alpha}^{d}} K_{d}^{d}
$$

Definition 5.1.2. Let $\mathcal{V}$ be a finite set and $\alpha$ and $d$ positive integers such that $d \geq 2 \alpha$. The hypercycle $C_{n}^{d, \alpha}$ on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph with edge set $\mathcal{E}\left(C_{n}^{d, \alpha}\right)=\left\{E_{1}, \ldots, E_{n}\right\}$ such that
(i) $\mathcal{V}\left(C_{n}^{d, \alpha}\right)=\bigcup_{i=1}^{n} E_{i}$.
(ii) For any $i \neq j$ we have $E_{i} \bigcap E_{j} \neq \emptyset$ if and only if $|j-i| \equiv 1 \bmod n$.
(iii) $\left|E_{i} \bigcap E_{i+1}\right|=\alpha$ for all $i \in\{1, \ldots, n-1\}$ and $\left|E_{1} \bigcap E_{n}\right|=\alpha$.

The length of a hypercycle is defined as the number of edges.
Remark 5.1.3. Hypercycles are generalized chordal hypergraphs but not chordal hypergraphs. Thus a chordal hypergraph can not contain any induced hypercycle $C_{n}^{d, \alpha}$.
Remark 5.1.4. For $d=2$ Definition 5.1.1 and Definition 5.1.2 give the usual concepts of a line graph and a cycle graph.

### 5.1.1 Betti numbers of Line Hypergraphs

Recall that a free vertex of a hypergraph is a vertex that lies in at most one edge.

If each edge of a hypergraph $\mathcal{H}$ contains a free vertex it is particularly easy to compute the Betti numbers of the corresponding hypergraph algebra.

Lemma 5.1.5. Let $\mathcal{H}$ be a hypergraph such that each edge of $\mathcal{H}$ contains a free vertex. Then $\beta_{i}(k[\Delta(\mathcal{H})])=\binom{n}{i}$, where $n$ is the number of edges of $\mathcal{H}$.

Proof. Let $\mathcal{E}(\mathcal{H})=\left\{E_{1}, \ldots, E_{n}\right\}$ and $v_{i} \in E_{i}, 1 \leq i \leq n$, be a free vertex. Then, for any $j, 1 \leq j \leq n$,

$$
x^{E_{j}} \nmid \operatorname{lcm}\left(x^{E_{1}}, \ldots, x^{E_{j-1}}, x^{E_{j+1}}, \ldots, x^{E_{n}}\right)
$$

since

$$
v_{j} \notin \bigcup_{\substack{i \neq j \\ E_{i} \in \mathcal{E}(\mathcal{H})}} E_{i} .
$$

Therefore the Taylor resolution (see [6]) of $k[\Delta(H)]$ is minimal and $\beta_{i}(k[\Delta(H)])=$ $\binom{n}{i}$.

It is now easy to give a combinatorial interpretation of the graded Betti numbers:

Theorem 5.1.6. Let $\mathcal{H}$ be a hypergraph with $\mathcal{E}(\mathcal{H})=\left\{E_{1}, \ldots, E_{n}\right\}$ such that each edge has a free vertex. Then

$$
\beta_{i, j}(k[\Delta(\mathcal{H})])=\left|\left\{F \subseteq[n] ;|F|=i,\left|\bigcup_{k \in F} E_{k}\right|=j\right\}\right| .
$$

Proof. Since each edge of $\mathcal{H}$ has a free vertex, it is enough to find the number of basis elements $e_{k_{1}, \ldots, k_{i}}$ of degree $j$ in the Taylor resolution of $k[\Delta(\mathcal{H})]$ that sits in homological degree $i$. We have $\operatorname{deg}\left(e_{k_{1}, \ldots, k_{i}}\right)=\operatorname{deg}\left(\operatorname{lcm}\left(x^{E_{k_{1}}}, \ldots, x^{E_{k_{i}}}\right)\right)$. Since all $x^{E_{k_{l}}}$ 's are square-free, $\operatorname{deg}\left(\operatorname{lcm}\left(x^{E_{k_{1}}}, \ldots, x^{E_{k_{i}}}\right)\right)=\left|\bigcup_{l=1}^{i} E_{k_{l}}\right|$, which completes the proof.

Corollary 5.1.7. Let $L_{n}^{d, \alpha}$ be a line hypergraph such that $d>2 \alpha$. Then

$$
\beta_{n, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)= \begin{cases}0 & \text { if } j \neq n(d-\alpha)+\alpha \\ 1 & \text { if } j=n(d-\alpha)+\alpha\end{cases}
$$

Proof. Since $d>2 \alpha$, each edge has a free vertex. Thus by Theorem 5.1.6,

$$
\beta_{n, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\left|\left\{F \subseteq[n] ;|F|=n,\left|\bigcup_{k \in F} E_{k}\right|=j\right\}\right| .
$$

Therefore it equals 1 if $j=\left|\bigcup_{i=1}^{n} E_{i}\right|$ and 0 if $j \neq\left|\bigcup_{i=1}^{n} E_{i}\right|$. Since $\left|\bigcup_{i=1}^{n} E_{i}\right|=$ $n(d-\alpha)+\alpha$, the assertion holds.

We now continue towards a formula for all Betti numbers in the case when $d>2 \alpha$.

For $i \in\{1, \ldots, r\}$ let $s_{i}$ be positive integers and let $L_{n}^{d, \alpha}$ be a line hypergraph of length $n$. Set $E\left(s_{1}, \ldots, s_{r}, n\right)=\left\{\mathcal{H} ; \mathcal{H}\right.$ is a sub-hypergraph of $L_{n}^{d, \alpha}$, which is comprised of $r$ disjoint line hypergraphs of lengths $\left.s_{1}, \ldots, s_{r}\right\}$. Thus if $\mathcal{H} \in$ $E\left(s_{1}, \ldots, s_{r}, n\right) \mathcal{H}$ have no isolated vertices.

Lemma 5.1.8. Let $s_{1}, \ldots, s_{r}$ be positive integers such that $1 \leq s_{1}=\cdots=s_{l_{1}}<$ $s_{l_{1}+1}=\cdots=s_{l_{1}+l_{2}}<s_{l_{1}+l_{2}+1}=\cdots=s_{l_{1}+l_{2}+l_{3}}<\cdots<s_{l_{1}+\cdots+l_{t-1}+1}=\cdots=$ $s_{l_{1}+\cdots+l_{t}}=s_{r}$ and $s_{1}+\cdots+s_{r}=i$, then

$$
\left|E\left(s_{1}, \ldots, s_{r}, n\right)\right|=\frac{r!}{l_{1}!\cdots l_{t}!}\binom{n-i+1}{r}
$$

Proof. Let $L_{n}^{d, \alpha}$ be a line hypergraph of length $n$ and $S$ be the set of hypergraphs $\mathcal{H} \in E\left(s_{1}, \ldots, s_{r}, n\right)$ such that $\mathcal{H}$ is comprised of line hypergraphs $Q_{1}, \ldots, Q_{r}$ such that the length of $Q_{i}$ is $s_{i}$ and for any $1 \leq i<j \leq r$, if $v_{\lambda} \in \mathcal{V}\left(Q_{i}\right)$ and $v_{\gamma} \in \mathcal{V}\left(Q_{j}\right)$, then $\lambda<\gamma$. By basic counting arguments one sees that

$$
\begin{equation*}
\left|E\left(s_{1}, \ldots, s_{r}, n\right)\right|=\frac{r!}{l_{1}!\cdots l_{t}!}|S| . \tag{5.1}
\end{equation*}
$$

We claim that there is a bijection between $S$ and the set of integer solutions $\left(t_{1}, \ldots, t_{r+1}\right)$ to the equation

$$
\sum_{l=1}^{r+1} t_{l}=n-i
$$

subject to the conditions $t_{1} \geq 0, t_{r+1} \geq 0$ and $t_{i} \geq 1$ for $i \in\{2, \ldots, r\}$. For any $\mathcal{H} \in S$, let $\mathcal{H}^{\prime}$ be the sub-hypergraph of $L_{n}^{d, \alpha}$ with edge set $\mathcal{E}\left(L_{n}^{d, \alpha}\right) \backslash \mathcal{E}(\mathcal{H})$ considered as a line hypergraph. Assume $\mathcal{H}^{\prime}$ is comprised of $l$ line hypergraphs $Q_{1}^{\prime}, \ldots, Q_{l}^{\prime}$ such that for any $1 \leq i<j \leq l$, if $v_{\lambda} \in \mathcal{V}\left(Q_{i}^{\prime}\right)$ and $v_{\gamma} \in \mathcal{V}\left(Q_{j}^{\prime}\right)$, then $\lambda<\gamma$. We have $r-1 \leq l \leq r+1$. Set $v_{\mathcal{H}}=\left(t_{1}, \ldots, t_{r+1}\right)$ where the $t_{i}$, $1 \leq i \leq r+1$, are as follows:
(i) If $l=r-1$, set $t_{1}, t_{r+1}=0$ and $t_{i}=\left|\mathcal{E}\left(Q_{i-1}^{\prime}\right)\right|$ for any $2 \leq i \leq r$.
(ii) If $l=r$ and $v_{0} \in \mathcal{V}\left(Q_{1}\right)$, set $t_{1}=0$ and $t_{i}=\left|\mathcal{E}\left(Q_{i}^{\prime}\right)\right|$ for any $2 \leq i \leq r+1$.
(iii) If $l=r$ and $v_{0} \in \mathcal{V}\left(Q_{1}^{\prime}\right)$, set $t_{r+1}=0$ and $t_{i}=\left|\mathcal{E}\left(Q_{i}^{\prime}\right)\right|$ for any $1 \leq i \leq r$.
(iv) If $l=r+1$, set $t_{i}=\left|\mathcal{E}\left(Q_{i}^{\prime}\right)\right|$ for any $1 \leq i \leq r+1$.

Observe that $\sum_{k=1}^{r+1} t_{k}=\sum_{k=1}^{l}\left|\mathcal{E}\left(Q_{k}^{\prime}\right)\right|=n-i$. We define a function

$$
\phi: S \longrightarrow\left\{\left(t_{1}, \ldots, t_{r+1}\right), t_{1}, t_{r+1} \geq 0, t_{2}, \ldots, t_{r} \geq 1, \sum_{l=1}^{r+1} t_{l}=n-i\right\}
$$

by $\phi(\mathcal{H})=v_{\mathcal{H}}$. It is easy to see that $\phi$ is a bijection and hence

$$
|S|=\binom{n-i+1}{r}
$$

Thus by equation (5.1) $\left|E\left(s_{1}, \ldots, s_{r}, n\right)\right|=\frac{r!}{l_{1}!\cdots l_{t}!}\binom{n-i+1}{r}$.
Theorem 5.1.9. Let $i<n$ be a positive integer and let $L_{n}^{d, \alpha}$ be a line hypergraph such that $d>2 \alpha$. Then

$$
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\binom{i-1}{r-1}\binom{n-i+1}{r}
$$

for any $r \in\{1, \ldots, i\}$ and $\beta_{i, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=0$ for all other degrees $j$.
Proof. Let $i<n$ and $j$ be integers such that $\beta_{i, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right) \neq 0$. Since $d>2 \alpha$, each edge has a free vertex. Thus as was shown in Theorem 5.1.6,

$$
\beta_{i, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\left|\left\{F \subseteq[n] ;|F|=i,\left|\bigcup_{k \in F} E_{k}\right|=j\right\}\right| .
$$

Let $E_{l_{1}}, \ldots, E_{l_{i}}$ be some edges of $L_{n}^{d, \alpha}$ such that $\left|\bigcup_{t=1}^{i} E_{l_{t}}\right|=j$ and let $\mathcal{H}$ be the line sub-hypergraph of $L_{n}^{d, \alpha}$ with edge set $\left\{E_{l_{1}}, \ldots, E_{l_{i}}\right\}$ and assume that $\mathcal{H}$ is comprised of $r$ line hypergraphs which are of lengths $s_{1}, \ldots, s_{r}$. Observe that $s_{1}+\cdots+s_{r}=i$ Let $L_{t}^{d, \alpha}, t \in\{1, \ldots, r\}$, be the corresponding line hypergraph of length $s_{t}$, so that $\left|\mathcal{V}\left(L_{t}^{d, \alpha}\right)\right|=s_{t} d-\alpha\left(s_{t}-1\right)$. Therefore

$$
|\mathcal{V}(\mathcal{H})|=\sum_{t=1}^{r}\left(s_{t} d-\alpha\left(s_{t}-1\right)\right)=i d-\alpha(i-r)
$$

and so $j=i d-\alpha(i-r)$ for some $r \in\{1, \ldots, i\}$. Hence

$$
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\sum_{\substack{1 \leq s_{1} \leq \cdots \leq s_{r} \\ s_{1}+\cdots+s_{r}=i}}\left|E\left(s_{1}, \ldots, s_{r}, n\right)\right|
$$

and $\beta_{i, j}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=0$ for those $j$ that can not be written in the form $i d-$ $\alpha(i-r)$ for some $r$. Let $l_{1}, \ldots, l_{m}$ be positive integers and $P_{\left(l_{1}, \ldots, l_{m}\right)}$ be the number of integer solutions to the equation

$$
x_{1}+\cdots+x_{r}=i
$$

such that

- $x_{j} \geq 1$ for all $j \in\{1, \ldots, r\}$, and
- $l_{i}$ components of $\left(x_{1}, \ldots, x_{r}\right)$ are equal for all $i \in\{1, \ldots, m\}$.

Then by basic counting arguments

$$
\sum_{m \geq 1, l_{i} \geq 1} P_{\left(l_{1}, \ldots, l_{m}\right)}=\binom{i-1}{r-1} .
$$

Also the number of solutions of $x_{1}+\cdots+x_{r}=i$ such that $1 \leq x_{1}=\cdots=x_{l_{1}}<$ $x_{l_{1}+1}=\cdots=x_{l_{1}+l_{2}}<x_{l_{1}+l_{2}+1}=\cdots=x_{l_{1}+l_{2}+l_{3}}<\cdots<x_{l_{1}+\cdots+l_{m-1}+1}=$ $\cdots=x_{l_{1}+\cdots+l_{m}}=x_{r}$ is equal to $P_{\left(l_{1}, \ldots, l_{m}\right)} \frac{l_{1}!\cdots l_{m}!}{r!}$. Thus using Lemma 5.1.8 we see that

$$
\begin{gathered}
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\sum_{m \geq 1, l_{i} \geq 1} P_{\left(l_{1}, \ldots, l_{m}\right)} \frac{l_{1}!\cdots l_{m}!}{r!} \frac{r!}{l_{1}!\cdots l_{m}!}\binom{n-i+1}{r} \\
=\sum_{m \geq 1, l_{i} \geq 1} P_{\left(l_{1}, \ldots, l_{m}\right)}\binom{n-i+1}{r}=\binom{i-1}{r-1}\binom{n-i+1}{r} .
\end{gathered}
$$

The proof is complete.
The remaining case when $d=2 \alpha$ reduces to the graph case found in [54].
Theorem 5.1.10. Let $L_{n}^{d, \alpha}$ be a line hypergraph such that $d=2 \alpha$. Then the non-zero Betti numbers of $k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]$ are in degrees $j \alpha$, where $2 j \geq i$, and are as follows:
$\beta_{i, j \alpha}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\binom{j-i}{2 i-j}\binom{n+1-2 j+2 i}{j-i}+\binom{j-i-1}{2 i-j}\binom{n+1-2 j+2 i}{j-i-1}$.
Proof. Let $\mathcal{E}\left(L_{n}^{d, \alpha}\right)=\left\{E_{1}, \ldots, E_{n}\right\}$, where $E_{i}=\left\{x_{1, i}, \ldots, x_{d, i}\right\}$. Set $X_{i}=$ $x_{1, i} \cdots x_{\alpha, i}$ and $X_{i+1}=x_{\alpha+1, i} \cdots x_{d, i}$ for any $i, 1 \leq i \leq n$. Here $\left\{x_{\alpha+1, i} \cdots x_{d, i}\right\}$ are, for every $i \in\{1, \ldots, n-1\}$, the vertices in the intersection $E_{i} \cap E_{i+1}$. Then $I\left(L_{n}^{d, \alpha}\right)=\left(X_{1} X_{2}, \ldots, X_{n} X_{n+1}\right)$. Since the $X_{i}$ 's are independent variables and $\operatorname{deg}\left(X_{i}\right)=\alpha$, we have

$$
\beta_{i, j \alpha}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\beta_{i, j}\left(R /\left(X_{1} X_{2}, \ldots, X_{n} X_{n+1}\right)\right),
$$

where $R$ is the polynomial ring over $k$ in the variables $\left\{X_{1}, \ldots, X_{n+1}\right\}$. The result now follows, using Theorem 7.7.34 of [54].

### 5.1.2 Betti numbers of Hypercycles

We start with the following observation similar to Corollary 5.1.7.
Corollary 5.1.11. Let $C_{n}^{d, \alpha}$ be a hypercycle such that each edge has a free vertex. Then

$$
\beta_{n, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)= \begin{cases}0 & \text { if } j \neq n(d-\alpha) \\ 1 & \text { if } j=n(d-\alpha) .\end{cases}
$$

Proof. By Theorem 5.1.6,

$$
\beta_{n, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\left|\left\{F \subseteq[n] ;|F|=n,\left|\bigcup_{k \in F} E_{k}\right|=j\right\}\right| .
$$

Therefore it is equal to 1 if $j=\left|\bigcup_{t=1}^{n} E_{t}\right|$ and 0 if $j \neq\left|\bigcup_{t=1}^{n} E_{t}\right|$. Since $\left|\bigcup_{t=1}^{n} E_{t}\right|=n(d-\alpha)$, the assertion holds.

We compute the Betti numbers of the hypergraph algebra of $C_{n}^{d, \alpha}$ in the same manner as we did for $L_{n}^{d, \alpha}$ in the previous section. That is, we consider the two cases $d>2 \alpha$ and $d=2 \alpha$ separately.

For $i \in\{1, \ldots, r\}$ let $s_{i}$ be positive integers and let $C_{n}^{d, \alpha}$ be a cycle of length $n$. Set $F\left(s_{1}, \ldots, s_{r}, n\right)=\left\{\mathcal{H} ; \mathcal{H}\right.$ is a sub-hypergraph of $C_{n}^{d, \alpha}$, which is comprised of $r$ disjoint line hypergraphs of lengths $\left.s_{1}, \ldots, s_{r}\right\}$.

Lemma 5.1.12. Let $s_{1}, \ldots, s_{r}$ be positive integers such that $s_{1}=\cdots=s_{l_{1}}<$ $s_{l_{1}+1}=\cdots=s_{l_{1}+l_{2}}<s_{l_{1}+l_{2}+1}=\cdots=s_{l_{1}+l_{2}+l_{3}}<\cdots<s_{l_{1}+\cdots+l_{t-1}+1}=\cdots=$ $s_{l_{1}+\cdots+l_{t}}=s_{r}$ and $s_{1}+\cdots+s_{r}=i$, then

$$
\left|F\left(s_{1}, \ldots, s_{r}, n\right)\right|=\frac{n(r-1)!}{l_{1}!\cdots\left(l_{t}-1\right)!}\binom{n-i-1}{r-1}
$$

Proof. Let $\left\{E_{k_{1}}, \ldots, E_{k_{s_{r}}}\right\} \subseteq \mathcal{E}\left(C_{n}^{d, \alpha}\right)$. The number of sub-hypergraphs of $C_{n}^{d, \alpha}$ that are comprised of $r$ line hypergraphs $Q_{1}, \ldots, Q_{r}$ of lengths $s_{1}, \ldots, s_{r}$ with $\mathcal{E}\left(Q_{r}\right)=\left\{E_{k_{1}}, \ldots, E_{k_{s_{r}}}\right\}$ is equal to the number of sub-hypergraphs of $L_{n-s_{r}-2}^{d, \alpha}$, which are comprised of $r-1$ line hypergraphs of lengths $s_{1}, \ldots, s_{r-1}$. Therefore

$$
\left|F\left(s_{1}, \ldots, s_{r}, n\right)\right|=n\left|E\left(s_{1}, \ldots, s_{r-1}, n-s_{r}-2\right)\right|
$$

since the line hypergraph $Q_{r}$ can start from any of the $n$ edges. By Lemma 5.1.8

$$
\left|E\left(s_{1}, \ldots, s_{r-1}, n-s_{r}-2\right)\right|=\frac{(r-1)!}{l_{1}!\cdots\left(l_{t}-1\right)!}\binom{n-i-1}{r-1}
$$

which completes the proof.
Theorem 5.1.13. Let $i<n$ be a positive integer and let $C_{n}^{d, \alpha}$ be a hypercycle such that $d>2 \alpha$. Then

$$
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}
$$

for any $r \in\{1, \ldots, i\}$ and $\beta_{i, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=0$ for all other degrees $j$.
Proof. Let $i<n$ and $j$ be integers such that $\beta_{i, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right) \neq 0$. Since $d>2 \alpha$, each edge has a free vertex. Thus as was shown in Theorem 5.1.6,

$$
\beta_{i, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\left|\left\{F \subseteq[n] ;|F|=i,\left|\bigcup_{k \in F} E_{k}\right|=j\right\}\right|
$$

Let $E_{l_{1}}, \ldots, E_{l_{i}}$ be some edges of $C_{n}^{d, \alpha}$ such that $\left|\bigcup_{t=1}^{i} E_{l_{t}}\right|=j$ and let $\mathcal{H}$ be the line sub-hypergraph of $C_{n}^{d, \alpha}$ with edge set $\left\{E_{l_{1}}, \ldots, E_{l_{i}}\right\}$ and assume that $\mathcal{H}$ is comprised of $r$ line hypergraphs which are of lengths $s_{1}, \ldots, s_{r}$. Observe that $s_{1}+\cdots+s_{r}=i$. Let $L_{t}^{d, \alpha}, t \in\{1, \ldots, r\}$, be the corresponding line hypergraph of length $s_{t}$ so that $\left|\mathcal{V}\left(L_{t}^{d, \alpha}\right)\right|=s_{t} d-\alpha\left(s_{t}-1\right)$. Therefore

$$
|\mathcal{V}(\mathcal{H})|=\sum_{t=1}^{r}\left(s_{t} d-\alpha\left(s_{t}-1\right)\right)=i d-\alpha(i-r)
$$

and so $j=i d-\alpha(i-r)$ for some $r \in\{1, \ldots, i\}$. Hence

$$
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\sum_{\substack{1 \leq s_{1} \leq \cdots \leq s_{r} \\ s_{1}+\cdots+s_{r}=i}}\left|F\left(s_{1}, \ldots, s_{r}, n\right)\right|
$$

and $\beta_{i, j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=0$ for all $j$ that can not be written in the form $i d-\alpha(i-r)$ for some $r \in\{1, \ldots, i\}$.

Construct the numbers $P_{\left(l_{1}, \ldots, l_{m}\right)}$ and $P_{\left(l_{1}, \ldots, l_{m}\right)} \frac{l_{1}!\cdots l_{m}!}{r!}$ in the same way as in the proof of Theorem 5.1.9. Thus using Lemma 5.1.12 we see that

$$
\begin{aligned}
\beta_{i, i d-\alpha(i-r)}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)= & \sum_{m \geq 1, l_{i} \geq 1} P_{\left(l_{1}, \ldots, l_{m}\right)} \frac{l_{1}!\cdots l_{m}!}{r!} \frac{n(r-1)!}{l_{1}!\cdots l_{m}!}\binom{n-i-1}{r-1}= \\
& =\frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}
\end{aligned}
$$

and the proof is complete.
Now consider the case where $d=2 \alpha$. Also this reduces to the graph case in [54].
Theorem 5.1.14. Let $C_{n}^{d, \alpha}$ be a hypercycle such that $d=2 \alpha$. Then the nonzero Betti numbers of $k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]$ are in degrees $j \alpha$, where $j \leq n$, and are as follows:
(i) If $j<n$ and $2 i \geq j$, then

$$
\beta_{i, \alpha j}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\frac{n}{n-2(j-i)}\binom{j-i}{2 i-j}\binom{n-2(j-i)}{j-i} .
$$

(ii) If $n \equiv 1 \bmod 3$, then

$$
\beta_{\frac{2 n+1}{3}, \alpha n}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=1 .
$$

(iii) If $n \equiv 2 \bmod 3$, then

$$
\beta_{\frac{2 n-1}{3}, \alpha n}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=1 .
$$

(iv) If $n \equiv 0 \bmod 3$, then

$$
\beta_{\frac{2 n}{3}, \alpha n}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=2 .
$$

Proof. Let $\mathcal{E}\left(C_{n}^{d, \alpha}\right)=\left\{E_{1}, \ldots, E_{n}\right\}$, where $E_{i}=\left\{x_{1, i}, \ldots, x_{d, i}\right\}$. Set $X_{i}=$ $x_{1, i} \cdots x_{\alpha, i}$ and $X_{i+1}=x_{\alpha+1, i} \cdots x_{d, i}$ for any $i, 1 \leq i \leq n-1$, where $X_{i}$ and $X_{i+1}$ denote the same things as they do in the proof of Theorem 5.1.10. Then

$$
I\left(C_{n}^{d, \alpha}\right)=\left(X_{1} X_{2}, \ldots, X_{n-1} X_{n}, X_{n} X_{1}\right) .
$$

Since $X_{i}$ are independent variables and $\operatorname{deg}\left(X_{i}\right)=\alpha$, we have

$$
\beta_{i, j \alpha}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\beta_{i, j}\left(R /\left(X_{1} X_{2}, \ldots, X_{n-1} X_{n}, X_{n} X_{1}\right)\right)
$$

where $R$ is the polynomial ring over $k$ in the variables $\left\{X_{1}, \ldots, X_{n}\right\}$. Using Theorem 7.6.28 of [54], the result follows.

We recall from [54] that a star graph is a complete bipartite graph $K_{1, n}$ for some $n$. One generalization of this graph is the $d$-complete bipartite hypergraph $K_{1, n}^{d}$. Another way of generalizing the star graph is to focus on its appearance. The following is a picture of $K_{1,4}$, here on vertex set $\{y\} \sqcup\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ :


Definition 5.1.15. Let $\mathcal{V}$ be a finite set and $\alpha$ and $d$ positive integers such that $d \geq \alpha \geq 1$. The star hypergraph $S_{n}^{d, \alpha}$ on vertex set $\mathcal{V}$ is the $d$-uniform hypergraph with edge set $\mathcal{E}\left(S_{n}^{d, \alpha}\right)=\left\{E_{1}, \ldots, E_{n}\right\}$ such that
(i) $\mathcal{V}\left(S_{n}^{d, \alpha}\right)=\bigcup_{i=1}^{n} E_{i}$.
(ii) For any $i \neq j,\left|E_{i} \cap E_{j}\right|=\left|\bigcap_{j=1}^{n} E_{j}\right|=\alpha$.

Remark 5.1.16. If $d>\alpha$, so that the situation is non-trivial, then each edge contains a free vertex.

Theorem 5.1.17. Let $\mathcal{V}$ be a finite set and let $S_{n}^{d, \alpha}, d>\alpha$, be a star hypergraph on vertex set $\mathcal{V}$. Then $\beta_{i, j}\left(k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]\right) \neq 0$ if and only if $j=i d-\alpha(i-1)$ and

$$
\beta_{i, i d-\alpha(i-1)}\left(k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]\right)=\binom{n}{i} .
$$

Proof. For any $i$ and any edges $E_{l_{1}}, \ldots, E_{l_{i}}$, we have $\left|\bigcup_{t=1}^{i} E_{l_{t}}\right|=i d-\alpha(i-1)$. The number of different unions $\bigcup_{t=1}^{i} E_{l_{t}}$ with $\left|\bigcup_{t=1}^{i} E_{l_{t}}\right|=i d-\alpha(i-1)$ is $\binom{n}{i}$. Therefore using Theorem 5.1.6, we have $\beta_{i, j}\left(k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]\right) \neq 0$ if and only if $j=i d-\alpha(i-1)$ and $\beta_{i, i d-\alpha(i-1)}\left(k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]\right)=\binom{n}{i}$.

### 5.2 Poincaré series of Hypercycles and Line Hy pergraphs

We determine the Poincaré series $P_{k[\Delta(\mathcal{H})]}(t)$ for line hypergraphs, hypercycles, and stars hypergraphs. Recall that the Poincaré series of a graded $k$-algebra $R=k\left[x_{1}, \ldots, x_{n}\right] / I$ is $P_{R}(t)=\sum_{i=1}^{\infty} \operatorname{dim}_{k} \operatorname{Tor}_{i}^{R}(k, k) t^{i}$. [44] is an excellent source for results on Poincaré series.

### 5.2.1 The case $d=2 \alpha$

We start with the case $d=2 \alpha$. Let $E_{i}=\left\{x_{i 1}, \ldots, x_{i \alpha}, x_{i 1}^{\prime}, \ldots, x_{i \alpha}^{\prime}\right\}$, where $\left\{x_{i j}^{\prime}\right\} \in E_{i+1}$ and denote by $M$ the set of all binomials $x_{i k}-x_{i l}$ and $x_{i k}^{\prime}-x_{i l}^{\prime}$. The elements of $M$ form a linear regular sequence (this is proved by Fröberg in [36]) of length $(n+1)(\alpha-1)$ for the line hypergraph and of length $n(\alpha-1)$ for the hypercycle. Factor out by these sequences in $k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]$ and $k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]$, respectively, and denote the results by

$$
k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]^{\prime}=k\left[x_{1}, \ldots, x_{n+1}\right] /\left(x_{1}^{\alpha} x_{2}^{\alpha}, x_{2}^{\alpha} x_{3}^{\alpha}, \ldots, x_{n}^{\alpha} x_{n+1}^{\alpha}\right)
$$

and

$$
k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]^{\prime}=k\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1}^{\alpha} x_{2}^{\alpha}, x_{2}^{\alpha} x_{3}^{\alpha}, \ldots, x_{n-1}^{\alpha} x_{n}^{\alpha}, x_{n}^{\alpha} x_{1}^{\alpha}\right)
$$

respectively. Then by [44, Theorem 3.4.2 (ii)] we have

$$
P_{k\left[\Delta\left(L_{n}^{2 \alpha, \alpha}\right)\right]}(t)=(1+t)^{(n+1)(\alpha-1)} P_{k\left[\Delta\left(L_{n}^{2 \alpha, \alpha}\right)\right]^{\prime}}(t)
$$

and

$$
P_{k\left[\Delta\left(C_{n}^{2 \alpha, \alpha}\right)\right]}(t)=(1+t)^{n(\alpha-1)} P_{k\left[\Delta\left(C_{n}^{2 \alpha, \alpha}\right)\right]^{\prime}}(t)
$$

Now $k\left[\Delta\left(L_{n}^{2 \alpha, \alpha}\right)\right]^{\prime}$ and $k\left[\Delta\left(C_{n}^{2 \alpha, \alpha}\right)\right]^{\prime}$ obviously have the same (ungraded) Poincaré series as the graph algebras

$$
k\left[\Delta\left(L_{n}\right)\right]=k\left[\Delta\left(L_{n}^{2,1}\right)\right]=k\left[x_{1}, \ldots, x_{n+1}\right] /\left(x_{1} x_{2}, x_{2} x_{3}, \ldots, x_{n} x_{n+1}\right)
$$

and

$$
k\left[\Delta\left(C_{n}\right)\right]=k\left[\Delta\left(C_{n}^{2,1}\right)\right]=k\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1} x_{2}, x_{2} x_{3}, \ldots, x_{n-1} x_{n}, x_{n} x_{1}\right)
$$

respectively.
For a graded $k$-algebra $R=\oplus_{i=0}^{\infty} R_{i}$ the Hilbert series of $R$ is defined as $H_{R}(t)=\sum_{i=0}^{\infty} \operatorname{dim}_{k}\left(R_{i}\right) t^{i}$. The exact sequence (of $k$-algebras)

$$
0 \longrightarrow\left(x_{n+1}\right) \longrightarrow k\left[\Delta\left(L_{n}\right)\right] \longrightarrow k\left[\Delta\left(L_{n}\right)\right] /\left(x_{n+1}\right) \longrightarrow 0
$$

and $k\left[\Delta\left(L_{n}\right)\right] /\left(x_{n+1}\right) \cong k\left[\Delta\left(L_{n-1}\right)\right]$ and $\left(x_{n+1}\right) \cong k\left[\Delta\left(L_{n-2}\right)\right] \otimes x_{n+1} \cdot k\left[x_{n+1}\right]$ gives

$$
\begin{equation*}
H_{k\left[\Delta\left(L_{n}\right)\right]}(t)=H_{k\left[\Delta\left(L_{n-1}\right)\right]}(t)+\frac{t}{1-t} H_{k\left[\Delta\left(L_{n-2}\right)\right]}(t) . \tag{5.2}
\end{equation*}
$$

The exact sequences

$$
0 \longrightarrow\left(x_{1}, x_{n-1}\right)(-1) \longrightarrow k\left[\Delta\left(C_{n}\right)\right](-1) \xrightarrow{x_{n}} k\left[\Delta\left(C_{n}\right)\right] \longrightarrow k\left[\Delta\left(L_{n-2}\right)\right] \longrightarrow 0
$$

and

$$
0 \longrightarrow\left(x_{1}, x_{n-1}\right) \longrightarrow k\left[\Delta\left(C_{n}\right)\right] \longrightarrow k\left[\Delta\left(C_{n}\right)\right] /\left(x_{1}, x_{n-1}\right) \longrightarrow 0
$$

and $k\left[\Delta\left(C_{n}\right)\right] /\left(x_{1}, x_{n-1}\right) \cong k\left[\Delta\left(L_{n-4}\right)\right] \otimes k\left[x_{n}\right]$ gives

$$
\begin{equation*}
H_{k\left[\Delta\left(C_{n}\right)\right]}(t)=H_{k\left[\Delta\left(L_{n-2}\right)\right]}(t)+\frac{t}{1-t} H_{k\left[\Delta\left(L_{n-4}\right)\right]}(t) \tag{5.3}
\end{equation*}
$$

Now $k\left[\Delta\left(C_{n}\right)\right]$ and $k\left[\Delta\left(L_{n}\right)\right]$ are (as all graph algebras) Koszul algebras [34, Corollary 2], so

$$
P_{k\left[\Delta\left(C_{n}\right)\right]}(t)=1 / H_{k\left[\Delta\left(C_{n}\right)\right]}(-t)
$$

and

$$
P_{k\left[\Delta\left(L_{n}\right)\right]}(t)=1 / H_{k\left[\Delta\left(L_{n}\right)\right]}(-t) .
$$

Since $k\left[\Delta\left(L_{0}\right)\right]=k\left[x_{1}\right]$ and $k\left[\Delta\left(L_{1}\right)\right]=k\left[x_{1}, x_{2}\right] /\left(x_{1} x_{2}\right)$, we have

$$
H_{k\left[\Delta\left(L_{0}\right)\right]}(t)=1 /(1-t)
$$

and

$$
H_{k\left[\Delta\left(L_{1}\right)\right]}(t)=(1+t) /(1-t)
$$

If we put $k\left[\Delta\left(L_{-1}\right)\right]=1$ equations (5.2) and (5.3) give the first Hilbert series:

- $H_{k\left[\Delta\left(L_{2}\right)\right]}(t)=\left(1+t-t^{2}\right) /(1-t)^{2}$
- $H_{k\left[\Delta\left(L_{3}\right)\right]}(t)=(1+2 t) /(1-t)^{2}$
- $H_{k\left[\Delta\left(L_{4}\right)\right]}(t)=\left(1+2 t-t^{2}-t^{3}\right) /(1-t)^{3}$
- $H_{k\left[\Delta\left(L_{5}\right)\right]}(t)=\left(1+3 t+t^{2}-t^{3}\right) /(1-t)^{3}$
- $H_{k\left[\Delta\left(C_{3}\right)\right]}(t)=(1+2 t) /(1-t)$
- $H_{k\left[\Delta\left(C_{4}\right)\right]}(t)=\left(1+2 t-t^{2}\right) /(1-t)^{2}$
- $H_{k\left[\Delta\left(C_{5}\right)\right]}(t)=\left(1+3 t+t^{2}\right) /(1-t)^{3}$
- $H_{k\left[\Delta\left(C_{6}\right)\right]}(t)=\left(1+3 t-2 t^{3}\right) /(1-t)^{3}$.

Thus we get

- $P_{k\left[\Delta\left(L_{2}\right)\right]}(t)=(1+t)^{2} /\left(1-t-t^{2}\right)$
- $P_{k\left[\Delta\left(L_{3}\right)\right]}(t)=(1+t)^{2} /(1-2 t)$
- $P_{k\left[\Delta\left(L_{4}\right)\right]}(t)=(1+t)^{3} /\left(1-2 t-t^{2}+t^{3}\right)$
- $P_{k\left[\Delta\left(L_{5}\right)\right]}(t)=(1+t)^{3} /\left(1-3 t+t^{2}+t^{3}\right)$
- $P_{k\left[\Delta\left(C_{3}\right)\right]}(t)=(1+t) /(1-2 t)$
- $P_{k\left[\Delta\left(C_{4}\right)\right]}(t)=(1+t)^{2} /\left(1-2 t-t^{2}\right)$
- $P_{k\left[\Delta\left(C_{5}\right)\right]}(t)=(1+t)^{2} /\left(1-3 t+t^{2}\right)$
- $P_{k\left[\Delta\left(C_{6}\right)\right]}(t)=(1+t)^{3} /\left(1-3 t+2 t^{3}\right)$.

We collect the results in
Theorem 5.2.1. The Poincaré series of $k\left[\Delta\left(L_{n}\right)\right]$ and $k\left[\Delta\left(C_{n}\right)\right]$ satisfy the recursion formulas

$$
P_{k\left[\Delta\left(L_{n}\right)\right]}(t)=\frac{(1+t) P_{k\left[\Delta\left(L_{n-1}\right)\right]}(t) P_{k\left[\Delta\left(L_{n-2}\right)\right]}(t)}{(1+t) P_{k\left[\Delta\left(L_{n-2}\right)\right]}(t)-t P_{k\left[\Delta\left(L_{n-1}\right)\right]}(t)}
$$

where $P_{k\left[\Delta\left(L_{0}\right)\right]}(t)=1+t$ and $P_{k\left[\Delta\left(L_{1}\right)\right]}(t)=(1+t) /(1-t)$ and

$$
P_{k\left[\Delta\left(C_{n}\right)\right]}(t)=\frac{(1+t) P_{k\left[\Delta\left(L_{n-2}\right)\right]}(t) P_{k\left[\Delta\left(L_{n-4}\right)\right]}(t)}{P_{k\left[\Delta\left(L_{n-2}\right)\right]}(t)+(1+t) P_{k\left[\Delta\left(L_{n-4}\right)\right]}(t)}
$$

Furthermore

$$
P_{k\left[\Delta\left(L_{n}^{2 \alpha, \alpha}\right)\right]}(t)=(1+t)^{(n+1)(\alpha-1)} P_{k\left[\Delta\left(L_{n}\right)\right]}(t)
$$

and

$$
P_{k\left[\Delta\left(C_{n}^{2 \alpha, \alpha}\right)\right]}(t)=(1+t)^{n(\alpha-1)} P_{k\left[\Delta\left(C_{n}\right)\right]}(t)
$$

### 5.2.2 The case $2 \alpha<d$

Next we turn to the case $2 \alpha<d$. Now each edge has a free vertex so by Lemma 5.1.5 the Taylor resolution of the respective hypergraph algebra is minimal. In that case there is a formula for the Poincaré series in terms of the graded homology of the Koszul complex [35, Corollary to Proposition 2]: Let $R$ be a monomial ring for which the Taylor resolution is minimal and denote by $K_{R}$ the Koszul complex. Then the homology $H\left(K_{R}\right)$ is of the form $H\left(K_{R}\right)=$ $k\left[u_{1}, \ldots, u_{N}\right] / I$, where $I$ is generated by a set of monomials of degree 2. Define a bi-grading induced by $\operatorname{deg}\left(u_{i}\right)=\left(1,\left|u_{i}\right|\right)$, where $\left|u_{i}\right|$ is the homological degree. Then $P_{R}(t)=(1+t)^{e} / H_{R}(-t, t)$, where $e$ is the embedding dimension and $H_{R}(x, y)$ is the bi-graded Hilbert series of $H\left(K_{R}\right)$, see [35].

We begin with the hypercycle. The homology of the Koszul complex (which computes the Betti numbers) is generated by $\left\{z_{I}\right\}$, where $I=\{i, i+1, \ldots, j\}$ corresponds to a path $\left\{E_{i}, E_{i+1}, \ldots, E_{j}\right\}$ in $C_{n}^{d, \alpha}$ (indices counted modulo $n$ ). Thus there are $n$ generators in all homological degrees $<n$ and one generator in homological degree $n$. We have $z_{I} z_{J}=0$ if $I \cap J \neq \emptyset$. Thus the surviving monomials are of the form $m=z_{I_{1}} \cdots z_{I_{r}}$, where $I_{i} \cap I_{j}=\emptyset$ if $i \neq j$. Let $\sum_{j=1}^{r}\left|I_{j}\right|=i$. If the bi-degree of $m$ is $\left(r, \sum_{j=1}^{r}\left|I_{j}\right|\right)$ then $m$ lies in $H\left(K_{R}\right)_{i, d i-\alpha(i-r)}$. The graded Betti numbers are determined in Section 5.1.2. The non-zero Betti numbers are

$$
\beta_{i, d i-\alpha(i-r)}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=\frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}, \quad 1 \leq r \leq i<n
$$

and

$$
\beta_{n, n(d-\alpha)}\left(k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]\right)=1 .
$$

This gives the Poincaré series.
Next we consider the line hypergraph. The homology of the Koszul complex is generated by $\left\{z_{I}\right\}$, where $I=\{i, i+1, \ldots, j\}$ corresponds to a path $\left\{E_{i}, E_{i+1}, \ldots, E_{j}\right\}$ in $L_{n}^{d, \alpha}$. Thus there are $n+1-i$ generators of homological degree $i$. We have $z_{I} z_{J}=0$ if $I \cap J \neq \emptyset$. The graded Betti numbers are determined in Section 5.1.1. The non-zero Betti numbers are

$$
\beta_{i, d i-(i-r) \alpha}\left(k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]\right)=\binom{i-1}{r-1}\binom{n-i+1}{r}, \quad 1 \leq r \leq i \leq n
$$

The same reasoning as above gives the Poincaré series. We state the results in a theorem.

Theorem 5.2.2. If $2 \alpha<d$, then

$$
P_{k\left[\Delta\left(C_{n}^{d, \alpha}\right)\right]}(t)=\frac{(1+t)^{n(d-\alpha)}}{1+\sum_{1 \leq r \leq i<n}(-1)^{r} \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1} t^{i+r}-t^{n+1}},
$$

and

$$
P_{k\left[\Delta\left(L_{n}^{d, \alpha}\right)\right]}(t)=\frac{(1+t)^{n(d-\alpha)+\alpha}}{1+\sum_{1 \leq r \leq i \leq n}(-1)^{r}\binom{i-1}{r-1}\binom{n-i+1}{r} t^{i+r}} .
$$

### 5.3 The Hyperstar

The edge ideal of the star hypergraph is of the form $m\left(m_{1}, \ldots, m_{n}\right)$, where $m$ is a monomial of degree $\alpha$. Then $k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]$ is Golod [44, Theorem 4.3.2]. This means that

Theorem 5.3.1.
$P_{k\left[\Delta\left(S_{n}^{d, \alpha}\right)\right]}(t)=(1+t)^{|V|} /\left(1-\sum \beta_{i} t^{i+1}\right)=(1+t)^{n(d-\alpha)+\alpha} /\left(1-\sum\binom{n}{i} t^{i+1}\right)$.

## Chapter 6

## Connectivity for uniform hypergraphs

For graphs and simplicial complexes there is a natural notion of being connected. This property may be described purely in terms of 0 -homologies of certain chain complexes. As we have mentioned, the notion of a connected hypergraph may be defined in several ways. In this chapter we introduce in a homological fashion a concept of connected hypergraph.

### 6.1 Connectivity and depth

Definition 6.1.1. Let $\mathcal{H}$ be a $d$-uniform hypergraph and $k$ be a field. The connectivity of $\mathcal{H}$ over $k, \operatorname{con}(\mathcal{H})$, is defined as

$$
\operatorname{con}(\mathcal{H})=\min \left\{|V| ; V \subseteq[n], \operatorname{dim} \tilde{H}_{d-2}\left(\left(\Delta_{\mathcal{H}_{[n] \backslash V}} ; k\right) \neq 0\right\} .\right.
$$

Definition 6.1.2. Let $k$ be a field. If $\mathcal{H}$ is a $d$-uniform hypergraph with nonzero connectivity over $k$, we say that $\mathcal{H}$ is homologically connected over $k$. If $\mathcal{H}$ is homologically connected over every field, we say that $\mathcal{H}$ is homologically connected.

Note that in the case of graphs, this is the usual notion of connectedness. Also, in terms of homological connectedness, the connectivity of a $d$-uniform hypergraph $\mathcal{H}$, is the cardinality of a minimal disconnecting set of vertices.

Proposition 6.1.3. If $\mathcal{H}$ is homologically connected over $\mathbb{Q}$, it is homologically connected over every field $k$.

Proof. By the Universal Coefficient Theorem we have

$$
\tilde{H}_{i}\left(\Delta_{\mathcal{H}} ; k\right) \cong \tilde{H}_{i}\left(\Delta_{\mathcal{H}} ; \mathbb{Q}\right) \otimes k \oplus \operatorname{Tor}_{1}^{\mathbb{Z}}\left(\tilde{H}_{i-1}\left(\Delta_{\mathcal{H}}\right), k\right)
$$

Now if we consider the clique complex $\Delta_{\mathcal{H}}$ of a non-empty $d$-uniform hypergraph, $\tilde{H}_{l}\left(\Delta_{\mathcal{H}} ; k\right)=0$ for every $l \leq d-3$ over every field $k$.

Proposition 6.1.4. If $\mathcal{G}$ is an induced hypergraph of a d-uniform hypergraph $\mathcal{H}$, such that $\mathcal{G}$ is not homologically connected over $k$, then

$$
\beta_{|\mathcal{V}(\mathcal{G})|-d+1}\left(k\left[\Delta_{\mathcal{H}}\right]\right) \neq 0
$$

Proof. We will use the fact that $\left(\Delta_{\mathcal{H}}\right)_{V}=\Delta_{\mathcal{H}_{V}}$. Consider Hochster's formula with $i=|\mathcal{V}(\mathcal{G})|-d+1$;

$$
\begin{gathered}
\beta_{|\mathcal{V}(\mathcal{G})|-d+1}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=\sum_{V \subseteq \mathcal{V}(\mathcal{H})} \operatorname{dim}_{k} \tilde{H}_{|V|-|\mathcal{V}(\mathcal{G})|+d-2}\left(\Delta_{\mathcal{H}_{V}} ; k\right) \geq \\
\operatorname{dim}_{k} \tilde{H}_{d-2}\left(\Delta_{\mathcal{V}(\mathcal{G})} ; k\right)>0 .
\end{gathered}
$$

Corollary 6.1.5. If $\mathcal{G}$ is an induced hypergraph of a d-uniform hypergraph $\mathcal{H}$, such that $\mathcal{G}$ is not homologically connected over $k$, then

$$
\begin{gathered}
|\mathcal{V}(\mathcal{G})|-d+1 \leq \operatorname{pd} k\left[\Delta_{\mathcal{H}}\right] \leq n \\
0 \leq \operatorname{depth} k\left[\Delta_{\mathcal{H}}\right] \leq n-|\mathcal{V}(\mathcal{G})|+d-1
\end{gathered}
$$

where $n=|\mathcal{V}(\mathcal{H})|$.
Proof. It is well know that (Hilbert's syzygy theorem) $n \geq \operatorname{pd} R / I_{\Delta_{\mathcal{H}}}$. Furthermore, according to the lemma, $\beta_{|\mathcal{V}(\mathcal{G})|-d+1}(\mathcal{H})>0$. This gives the first assertion. The second follows from the first using the Auslander-Buchsbaum formula.

Corollary 6.1.6. If $\mathcal{H}$ is a d-uniform hypergraph that is not homologically connected over $k$, then

$$
\begin{gather*}
n-d+1 \leq \operatorname{pd} k\left[\Delta_{\mathcal{H}}\right] \leq n,  \tag{6.1}\\
0 \leq \operatorname{depth} k\left[\Delta_{\mathcal{H}}\right] \leq d-1 \tag{6.2}
\end{gather*}
$$

where $n=|\mathcal{V}(\mathcal{H})|$.
If $\mathcal{H}$ is a $d$-uniform hypergraph that is not homologically connected we will see (Theorem 6.1.10 below) that the left inequality in (6.1) and the right inequality in (6.2) are in fact both equalities. First, we prove the following theorem, which connects the depth of the Stanley-Reisner ring $k\left[\Delta_{\mathcal{H}}\right]$, with the connectivity of $\mathcal{H}$.

Theorem 6.1.7. Let $\Delta_{\mathcal{H}}$ be the clique complex of a d-uniform hypergraph $\mathcal{H}$ with $|\mathcal{V}(\mathcal{H})|=n$ and put $g=\operatorname{depth} k\left[\Delta_{\mathcal{H}}\right]$. Then,

$$
\operatorname{con}(\mathcal{H})=g-d+r+1
$$

where $r$ is the minimal number such that $\beta_{n-g-r, n-g-r+d-1}\left(k\left[\Delta_{\mathcal{H}}\right]\right) \neq 0$. That is, $r$ is the minimal number such that there exists a $V \subseteq \mathcal{V}(\mathcal{H}),|V|=n-(g-$ $d+r+1)$ with $\tilde{H}_{d-2}\left(\Delta_{\mathcal{H}_{V}} ; k\right) \neq 0$

Remark 6.1.8. If $\mathcal{H}$ is a $d$-uniform hypergraph, recall that the linear strand of a resolution of $k\left[\Delta_{\mathcal{H}}\right]$ is the part of the resolution that is of degrees $(i, i+d-1)$. Note that $r=\operatorname{pd} k\left[\Delta_{\mathcal{H}}\right]-\max \left\{i ; \beta_{i, i+d-1}\left(k\left[\Delta_{\mathcal{H}}\right]\right) \neq 0\right\}$.

Proof. We know that $\operatorname{Tor}_{n-g}^{R}\left(k\left[\Delta_{\mathcal{H}}\right], k\right) \neq 0$, but $\operatorname{Tor}_{n-i}^{R}\left(k\left[\Delta_{\mathcal{H}}\right], k\right)=0$ for every $i<g$. In particular, $\operatorname{Tor}_{n-i}^{R}\left(k\left[\Delta_{\mathcal{H}}\right], k\right)_{j}=0$ in every degree $j$ if $i<g$. This gives, via Hochster's formula, that $\tilde{H}_{|V|-(n-i+1)}\left(\Delta_{\mathcal{H}_{V}} ; k\right)=0$ for every $V \subseteq \mathcal{V}(\mathcal{H})$, $i<g$, and that there exists a $V \subseteq \mathcal{V}(\mathcal{H})$ such that $\tilde{H}_{|V|-(n-g+1)}\left(\Delta_{\mathcal{H}_{V}} ; k\right) \neq 0$. Let $r \geq 0$ be the minimal number such that $\operatorname{Tor}_{n-(g+r)}^{R}\left(k\left[\Delta_{\mathcal{H}}\right], k\right)_{j} \neq 0$ for $j=n-(g-d+r+1)$. This is the same thing as saying that there exists a $V \subseteq \mathcal{V}(\mathcal{H}),|V|=n-(g-d+r+1)$, such that $\tilde{H}_{d-2}\left(\Delta_{\mathcal{H}_{V}} ; k\right) \neq 0$ but at the same time, for any $V \subseteq \mathcal{V}(\mathcal{H}),|V|>n-(g-d+r+1)$, the homology of $\Delta_{\mathcal{H}_{V}}$ in degree $d-2$ is zero. This means precisely that $\operatorname{con}(\mathcal{H})=g-d+r+1$.

If in the following corollary $\mathcal{H}$ is 2 -uniform we get as a special case Lemma 3 in [39].

Corollary 6.1.9. Let $\mathcal{H}$ be a d-uniform hypergraph and suppose the length of the linear strand of $k\left[\Delta_{\mathcal{H}}\right]$ is maximal. Then

$$
\operatorname{depth} k\left[\Delta_{\mathcal{H}}\right]=\operatorname{con}(\mathcal{H})+d-1
$$

Theorem 6.1.10. Let $\mathcal{H}$ be a d-uniform hypergraph with $|\mathcal{V}(\mathcal{H})|=n$. Then $\mathcal{H}$ is not homologically connected over $k$ precisely when

$$
\begin{aligned}
& \operatorname{pd} k\left[\Delta_{\mathcal{H}}\right]=n-d+1 \\
& \operatorname{depth} k\left[\Delta_{\mathcal{H}}\right]=d-1
\end{aligned}
$$

and the length of the linear strand of $k\left[\Delta_{\mathcal{H}}\right]$ is maximal.
Proof. We know that $n-d+1 \leq \operatorname{pd} k\left[\Delta_{\mathcal{H}}\right] \leq n$. Put pd $k\left[\Delta_{\mathcal{H}}\right]=n-r$, $0 \leq r \leq d-1$. Hochster's formula gives

$$
\beta_{n-r}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=\sum_{V \subseteq \mathcal{V}(\mathcal{H})} \operatorname{dim}_{k} \tilde{H}_{|V|-(n-r)-1}\left(\Delta_{\mathcal{H}_{V}} ; k\right) .
$$

If $r \leq d-2$ then $|V|-(n-r)-1 \leq|V|-n+d-3 \leq d-3$. But $\tilde{H}_{l}\left(\Delta_{\mathcal{H}_{V}} ; k\right)=0$ for all $l \leq d-3$ and for all $V \subseteq \mathcal{V}(\mathcal{H})$.

The last claim follows from Theorem 6.1.7 since not being homologically connected is the same thing as having connectivity 0 .

Example 12. Since homologically connected and connected are the same things for an ordinary simple graph $\mathcal{G}$, we have $\operatorname{pd} k\left[\Delta_{\mathcal{G}}\right]=n-1$ and depth $k\left[\Delta_{\mathcal{G}}\right]=$ 1 for any simple graph $\mathcal{G}$ that is not connected. Furthermore, the length of the linear strand of $k\left[\Delta_{\mathcal{G}}\right]$ is maximal. This special case of Theorem 6.1.10 is Theorem 4.2.6 in [54].

Corollary 6.1.11. Let $\mathcal{H}$ be a d-uniform hypergraph. If $k\left[\Delta_{\mathcal{H}}\right]$ is CohenMacaulay of dimension at least $d$, then $\mathcal{H}$ has non-zero connectivity. Put another way, the only d-uniform hypergraph $\mathcal{H}$ with connectivity 0 such that $k\left[\Delta_{\mathcal{H}}\right]$ is Cohen-Macaulay, is the discrete hypergraph.

Proposition 6.1.12. Let $\mathcal{H}$ be a d-uniform hypergraph with $|\mathcal{V}(\mathcal{H})|=n$. Then the Betti number $\beta_{n-d+1}\left(k\left[\Delta_{\mathcal{H}}\right]\right)$ can be non-zero only in degree $n$. Furthermore, it determines whether $\mathcal{H}$ has non-zero connectivity or not.

Proof. This follows from Hochster's formula and the fact that $\beta_{i, j}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=0$ if $j<i+d-1$.

Remark 6.1.13. In the case of ordinary simple graphs it follows from the above proposition that the number $\beta_{n-1}\left(k\left[\Delta_{\mathcal{H}}\right]\right)+1$ is the number of connected components of $\mathcal{G}$.
Example 13. Let $\mathcal{H}$ be the 3-uniform hypergraph on vertex set $\{a, b, c, d\}$ and with edge set $\mathcal{E}(\mathcal{H})=\{\{a, b, c\},\{b, c, d\}\}$. We may visualize $\mathcal{H}$ as follows:


By computing the Betti numbers of $k\left[\Delta_{\mathcal{H}}\right]$ one sees that $\beta_{2}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=1$. If we add to the edge set the edge $\{a, b, d\}$, the resulting hypergraph has non-zero connectivity.

Example 14. Let $\mathcal{H}$ be the 3 -uniform hypergraph on vertex set $\{a, b, c, d, e\}$ and with edge set $\mathcal{E}(\mathcal{H})=\{\{a, b, c\},\{c, d, e\}\}$. $\mathcal{H}$ is illustrated below:


The Betti number $\beta_{3}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=4$ shows that $\mathcal{H}$ has 0 connectivity. If we create new hypergraphs by, in turn, adding to the edge sets the edges $\{a, c, d\},\{b, c, e\}$ and $\{a, b, d\}$, we do not obtain hypergraphs with non-zero connectivity. If we finally add the edge $\{a, b, e\}$ we arrive at a hypergraph with edge set

$$
\{\{a, b, c\},\{c, d, e\},\{a, c, d\},\{b, c, e\},\{a, b, d\},\{a, b, e\}\}
$$

that has non-zero connectivity.

Example 15. Let $\mathcal{H}=\left(K_{n}^{d}\right)^{c}$. Then we know that (Theorem 2.2.3)

$$
\beta_{i, j}\left(k\left[\Delta_{\mathcal{H}}\right]\right)=\binom{n}{j}\binom{j-1}{d-1} .
$$

Hence $\mathcal{H}$ has connectivity 0 . This is quite natural since it generalizes the fact that the discrete graph on $n$ vertices has $n=\binom{n-1}{2-1}+1$ connected components.

Example 16. We have seen that $k\left[\Delta_{K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}}\right]$ has linear minimal free resolution and projective dimension $n-d+1$, where $n=\sum_{i=1}^{t} n_{i}$. Hence $\left(K_{n_{1}, \ldots, n_{t}}^{d\left(a_{1}, \ldots, a_{t}\right)}\right)^{c}$ has connectivity 0 .

Lemma 6.1.14 ([37], Lemma 7). Let $k[\Delta]$ be a Stanley-Reisner ring with $\operatorname{dim} k[\Delta]=e$ and with embedding dimension $n$. Then $k[\Delta]$ is Cohen-Macaulay if and only if $\tilde{H}_{i}\left(\Delta_{V} ; k\right)=0$ for every $i$ and $V \subseteq \mathcal{V}(\mathcal{H})$ such that $|V|=n-e+i+2$.

Assume $\mathcal{H}$ is a $d$-uniform hypergraph such that $k\left[\Delta_{\mathcal{H}}\right]$ has linear resolution. Since induced complexes $\Delta_{\mathcal{H}_{V}}$ can only have homology in degree $d-2$, one gets:

Corollary 6.1.15. Let $\mathcal{H}$ be a d-uniform hypergraph with $|\mathcal{V}(\mathcal{H})|=n$. Assume $\operatorname{dim} k\left[\Delta_{\mathcal{H}}\right]=e$ and that $k\left[\Delta_{\mathcal{H}}\right]$ has linear resolution. Then $k\left[\Delta_{\mathcal{H}}\right]$ is also CohenMacaulay if and only if $\tilde{H}_{d-2}\left(\Delta_{\mathcal{H}_{V}} ; k\right)=0$ for every $V \subseteq \mathcal{V}(\mathcal{H})$ with $|V|=$ $n-(e-d)$. Furthermore, in this case we have that

$$
e=\operatorname{con}(\mathcal{H})+d-1
$$

Proof. The if and only if statement follows from Lemma 6.1.14 and the last claim follows from Corollary 6.1.9.

## Chapter 7

## A weak version of shellability

Pure shellable simplicial complexes is somewhat of a cornerstone of combinatorial commutative algebra. This is perhaps mostly since they provide a rather non-technical way of showing that a complex $\Delta$ is Cohen-Macaulay. Also, shellability may be used together with Alexander duality to show that certain rings have linear quotients.

## $7.1 d$-shellability

We start by recalling the definition of pure and non-pure shellability.
Definition 7.1.1. Let $\Delta$ be a simplicial complex with $\mathcal{F}(\Delta)=\left\{F_{1}, \ldots, F_{t}\right\}$. $\Delta$ is called pure shellable if
(i) $\left|F_{i}\right|=\left|F_{j}\right|$ for every pair of indices $1 \leq i<j \leq t$.
(ii) There exists an ordering $F_{1},<\cdots<, F_{t}$ of the facets such that $\left\langle F_{j}\right\rangle \cap$ $\left\langle F_{1}, \ldots F_{j-1}\right\rangle$ is generated by a non-empty set of proper maximal faces of $\left\langle F_{j}\right\rangle$ for every $j \in\{2, \ldots, t\}$.

A simplicial complex $\Delta$ is called non-pure shellable if (ii) but not (i) holds. If $\Delta$ is a shellable (pure or non-pure) simplicial complex a linear order $F_{1}<\cdots<F_{t}$ of $\mathcal{F}(\Delta)$ as in $(i i)$ is called a shelling of $\Delta$.

Remark 7.1.2. It is well known, see for example [11, Theorem 5.1.13], that pure shellability implies Cohen-Macaulayness. This also follows from Corollary 7.1.15 below and the Eagon-Reiner Theorem.

In the following two definitions we introduce the concepts of $d$-shellability and $d$-quotients.

Definition 7.1.3. Let $\Delta$ be a simplicial complex with $\mathcal{F}(\Delta)=\left\{F_{1}, \ldots, F_{t}\right\}$. $\Delta$ is called $d$-shellable if its facets can be ordered $F_{1}<\cdots<F_{t}$, such that $\left\langle F_{j}\right\rangle \cap\left\langle F_{1}, \ldots, F_{j-1}\right\rangle$ is generated by a non-empty set of proper faces of $\left\langle F_{j}\right\rangle$ of dimension $\left|F_{j}\right|-d-1$ for every $j \in\{2, \ldots, t\}$.

Remark 7.1.4. The concepts of being 1-shellable and shellable coincides. If $\Delta$ is a simplicial complex, a linear ordering of $\mathcal{F}(\Delta)$ satisfying the conditions of Definition 7.1.3 is called a $d$-shelling of $\Delta$.

Definition 7.1.5. Let $I$ be a monomial ideal. We say that $I$ has $d$-quotients if there exists an ordering $x^{m_{1}} \leq \cdots \leq x^{m_{t}}$ of the minimal generators of $I$ such that if for every $s \in\{1, \ldots, t\}$ we put $I_{s}=\left(x^{m_{1}}, \ldots, x^{m_{s}}\right)$, then there are monomials $x^{b_{s_{i}}}, i \in\left\{1, \ldots, r_{s}\right\}, \operatorname{deg} x^{b_{s_{i}}}=d$ for all $i$, such that

$$
I_{s-1}: x^{m_{s}}=\left(x^{b_{s_{1}}}, \ldots, x^{b_{s_{r_{s}}}}\right)
$$

The motivation behind these definitions is the following well known theorem, which we generalize below.

Theorem 7.1.6. Let $I_{\Delta}=\left(x^{m_{1}}, \ldots, x^{m_{t}}\right)$ be a square-free monomial ideal. Then $I_{\Delta}$ has linear quotients (that is, 1-quotients) precisely when the simplicial complex $\Delta^{*}$ is shellable.

Example 17. The simplicial complexes $\Delta_{L_{n}^{d, \alpha}}$ and $\Delta_{C_{n}^{d, \alpha}}$ are both $(d-\alpha)$ shellable.

Theorem 7.1.7. Let $I_{\Delta}$ be a square-free monomial ideal. Then $I_{\Delta}$ has $d$ quotients precisely when the simplicial complex $\Delta^{*}$ is $d$-shellable.

Proof. Assume the vertex set of $\Delta$ is $\mathcal{V}$. Let $I_{\Delta}=\left(x^{m_{1}}, \ldots, x^{m_{t}}\right)$, where the $x^{m_{i}}$ are the minimal generators. The set of facets of $\Delta^{*}$ is $\mathcal{F}\left(\Delta^{*}\right)=\left\{F_{1}, \ldots, F_{t}\right\}$, where $F_{i}=\mathcal{V} \backslash m_{i}$ for $i \in\{1, \ldots, t\}$.

Assume $I_{\Delta}$ has $d$-quotients. If for every $1 \leq i<j \leq t, x^{a_{j_{i}}}$ denotes the minimal generator of $\left(x^{m_{i}}\right): x^{m_{j}}$, then $I_{j-1}: x^{m_{j}}$ is minimally generated by the set $\left\{x^{a_{j_{1}}}, \ldots, x^{a_{j_{r}}}\right\}$, for some $r \leq j-1$. This is equivalent to saying that the sets $a_{j_{\alpha}}, \alpha \in\left\{1, \ldots, j_{r}\right\}$, that all have cardinality $d$ by assumption, are precisely the minimal subsets of $\mathcal{V}$ such that $F_{j} \backslash a_{j_{\alpha}} \subseteq F_{i}$ for some $1 \leq i<j$ and that $\left\langle F_{j}\right\rangle \cap\left\langle F_{1}, \ldots, F_{j-1}\right\rangle$ is pure of dimension $\left|F_{j}\right|-d-1$ and equals $\left\langle F_{j} \backslash a_{j_{1}}, \ldots, F_{j} \backslash a_{j_{r}}\right\rangle$.

The converse is proved by a similar argument: Assume $\Delta^{*}$ is $d$-shellable, and let $\mathcal{F}\left(\Delta^{*}\right)=\left\{F_{1}, \ldots, F_{t}\right\}$. Put $m_{i}=\mathcal{V} \backslash F_{i}$. Then the ideal $I_{\Delta}$ is minimally generated by the monomials $x^{m_{i}}, i \in\{1, \ldots, t\}$. For every $j \in\{2, \ldots, t\}$, we let $a_{j_{\alpha}}, \alpha \in\left\{1, \ldots, j_{r}\right\}$ denote the minimal subsets of $F_{j}$ that one has to remove in order for $F_{j} \backslash a_{j_{\alpha}}$ to be a generator of $\left\langle F_{j}\right\rangle \cap\left\langle F_{1}, \ldots, F_{j-1}\right\rangle$. Then the monomials $x^{a_{j_{\alpha}}}$ are precisely the minimal generators of $I_{j-1}: x^{m_{j}}$.

The following theorem occurs frequently in the literature. It shows that simplicial complexes that are 1-shellable may be defined in (at least) three equivalent ways:

Theorem 7.1.8. Let $\Delta$ be a simplicial complex on vertex set $\mathcal{V}$ with $\mathcal{F}(\Delta)=$ $\left\{F_{1}, \ldots, F_{t}\right\}$. Then the following conditions are equivalent:
(i) $\Delta$ is shellable and $F_{1}<\cdots<F_{t}$ is a shelling.
(ii) For all $i, j, 1 \leq i<j \leq t$, there exist $a$ vertex $v$ and an integer $k$ with $1 \leq k<j$, such that $v \in F_{j} \backslash F_{i}$ and $F_{j} \backslash F_{k}=\{v\}$.
(iii) The set $\left\{F \in \mathcal{V} \mid F \in\left\langle F_{1}, \ldots, F_{j}\right\rangle, F \notin\left\langle F_{1}, \ldots, F_{j-1}\right\rangle\right\}$ has a unique minimal element for all $2 \leq i \leq t$.

Two of these statements, slightly modified, remain equivalent in the case of $d$-shellable complexes also for $d>1$.

Theorem 7.1.9. Let $\Delta$ be a simplicial complex on vertex set $\mathcal{V}$ with $\mathcal{F}(\Delta)=$ $\left\{F_{1}, \ldots, F_{t}\right\}$. Then the following conditions are equivalent:
(i) $\Delta$ is d-shellable and $F_{1}<\cdots<F_{t}$ a d-shelling.
(ii) For all $i, j, 1 \leq i<j \leq t$, there exist some set $a_{j} \subseteq \mathcal{V},\left|a_{j}\right|=d$, and a $k$ with $1 \leq k<j$, such that $a_{j} \subseteq F_{j}, a_{j} \cap F_{i}=\emptyset$ and $F_{j} \backslash F_{k}=a_{j}$.

Proof. The implication $(i) \Rightarrow$ (ii) follows by considering the proof of Theorem 7.1.7. For the converse let $F$ be a face of $\left\langle F_{j}\right\rangle \cap\left\langle F_{1}, \ldots, F_{j-1}\right\rangle$. Then $F$ lies in some $\left\langle F_{i}\right\rangle, i<j$. Let $a_{j}$ be a set that fits the description in (ii). Then $F$ is also a face of $\left\langle F_{j} \backslash a_{j}\right\rangle$ so $\left\langle F_{j}\right\rangle \cap\left\langle F_{1}, \ldots, F_{j-1}\right\rangle$ is pure of dimension $\left|F_{j}\right|-d-1$.

Remark 7.1.10. Let $\Delta$ be a simplicial complex on vertex set $\mathcal{V}$ and $F$ a face of $\Delta$. The link of $F, \mathrm{lk}_{\Delta} F$, is by definition the simplicial complex with faces

$$
\{G \subseteq \mathcal{V} ; G \cap F \neq \emptyset, G \cup F \in \Delta\}
$$

As for shellable complexes, links of faces of $d$-shellable complexes stay $d$ shellable:

Proposition 7.1.11. Let $\Delta$ be a d-shellable complex and $F$ a face of $\Delta$. Then $\mathrm{lk}_{\Delta}(F)$ is again d-shellable.

Proof. Assume $F_{1}<\cdots<F_{t}$ is a $d$-shelling of $\Delta$ and that the face $F$ lies in the facets $F_{i_{1}}, \ldots, F_{i_{r}}$, where $i_{1}<\cdots<i_{r}$. Put $G_{i_{j}}=F_{i_{j}} \backslash F$. Then $\mathrm{lk}_{\Delta}(F)=\left\{G_{i_{1}}, \ldots, G_{i_{r}}\right\}$. If $j \leq r$ and $G$ is a face of $G_{i_{j}} \cap\left\langle G_{i_{1}}, \ldots, G_{i_{j-1}}\right\rangle$, then $F \cup G$ is a face of $F_{i_{j}} \cap\left\langle F_{1}, \ldots, F_{i_{j-1}}\right\rangle$. Hence, if $G$ is maximal we see that $|G|=\left|F_{j_{i}}\right|-|F|-d$, which is our result.

Splittable monomial ideals were introduced by Eliahou and Kervaire in [21] and has been studied in for example [33, 46, 45]. This class of ideals is well behaved in the sense that their Betti numbers satisfy the Eliahou-Kervaire formula, see [21] Proposition 3.1. The following definition (that is Definition 1.1 in [33]), captures the content of the Eliahou-Kervaire formula in an axiomatic way.

Definition 7.1.12. Let $I, J$ and $K$ be monomial ideals such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I=J+K$ is a Betti splitting of $I$ if

$$
\beta_{i, j}(I)=\beta_{i . j}(J)+\beta_{i, j}(K)+\beta_{i-1, j}(J \cap K)
$$

for all $i \in \mathbb{N}$ and (multi)degrees $j$.
It follows from [33, Proposition 2.1] that $I=J+K$ is a Betti splitting of $I$ if and only if the mapping cone of the lifting of the map $J \cap K \rightarrow J \oplus K$ to the corresponding minimal free resolutions is a minimal free resolution of $I$.

Theorem 7.1.13. Denote by $R$ the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ and let $I=$ $\left(x^{m_{1}}, \ldots, x^{m_{t}}\right) \subseteq R, \operatorname{deg} x^{m_{i}}=d^{\prime}$ for every $i \in\{1, \ldots, t\}$, be a square-free monomial ideal with $d$-quotients, $d \leq d^{\prime}$. For each $s \in\{1, \ldots, t\}$ put $I_{s}=$ $\left(x^{m_{1}}, \ldots, x^{m_{s}}\right)$. Then
(i) $I_{s}=I_{s-1}+\left(x^{m_{s}}\right)$ is a Betti splitting.
(ii) $\beta_{i, j}\left(R /\left(I_{s-1}: x^{m_{s}}\right)\left(-d^{\prime}\right)\right)$ and $\beta_{i, j}\left(R / I_{s-1}\right)$ are not non-zero in any common degree $j$ for any $i \geq 2, s \in\{2, \ldots, t\}$.
(iii) For all $i, 2 \leq i \leq \operatorname{pd}_{R}(R / I)$, we have

$$
\beta_{i}(R / I)=\sum_{s=2}^{t} \beta_{i-1}\left(R /\left(I_{s-1}: x^{m_{s}}\right)\left(-d^{\prime}\right)\right)
$$

Proof. By [33, Proposition 2.1] (i) and (ii) are equivalent. (iii) is a consequence of $(i)$ (and ( $i i$ )) since if we assume that (i) holds, then for every $s \in\{2, \ldots, t\}$ we have an exact sequence

$$
0 \rightarrow R /\left(I_{s-1}: x^{m_{s}}\right)\left(-d^{\prime}\right) \xrightarrow{x^{m_{s}}} R / I_{s-1} \rightarrow R / I_{s} \rightarrow 0
$$

where the second map is multiplication by $x^{m_{s}}$. It follows from the long exact Tor-sequence that $\beta_{i}\left(R / I_{s}\right)=\beta_{i}\left(R / I_{s-1}\right)+\beta_{i-1}\left(R /\left(I_{s-1}: x^{m_{t}}\right)\left(-d^{\prime}\right)\right)$. Noting that $\beta_{2}\left(R / I_{1}\right)=0,(i i)$ follows by induction.

To prove (i), let $2 \leq r \leq t$ and consider the following exact sequence

$$
\begin{equation*}
0 \rightarrow I_{r-1} \cap\left(x^{m_{r}}\right) \rightarrow I_{r-1} \oplus\left(x^{m_{r}}\right) \rightarrow I_{r} \rightarrow 0 \tag{7.1}
\end{equation*}
$$

The non trivial maps are $x \mapsto(x,-x)$ and $(x, y) \mapsto x+y$. Let $F$. and $G$. be the minimal free resolutions of $I_{r-1} \cap\left(x^{m_{r}}\right)$ and $I_{r-1} \oplus\left(x^{m_{r}}\right)$ respectively. We show that the mapping cone, cone $(\alpha)$, of the lifting $\alpha: F . \rightarrow G$. of the left map in the above exact sequence is the minimal free resolution of $I_{r}$. By looking at the generators of the ideals $I_{r-1}: x^{m_{r}}$ and $I_{r-1} \cap\left(x^{m_{r}}\right)$, it is clear that we have an homogeneous $R$-module isomorphism

$$
\left(I_{r-1}: x^{m_{r}}\right)\left(-d^{\prime}\right) \rightarrow I_{r-1} \cap\left(x^{m_{r}}\right)
$$

the map being multiplication by $x^{m_{r}}$. Hence we wish to show that the mapping cone of the lifting of the injection

$$
\left(I_{r-1}: x^{m_{r}}\right)\left(-d^{\prime}\right) \rightarrow I_{r-1} \oplus\left(x^{m_{r}}\right)
$$

gives a minimal free resolution of $I_{r}$. Note that the minimal free resolutions of $I_{r-1}$ and $I_{r-1} \oplus\left(x^{m_{r}}\right)$ only differ in a simple way at the bottom degrees. Hence the mapping cone of the lifting of the map $\left(I_{r-1}: x^{m_{r}}\right)\left(-d^{\prime}\right) \rightarrow I_{r-1} \oplus\left(x^{m_{r}}\right)$ is essentially the same (except possibly in the bottom degrees) as the mapping cone of the lifting of the injection $\left(I_{r-1}: x^{m_{r}}\right)\left(-d^{\prime}\right) \rightarrow I_{r-1}$.

Let $F^{\prime}$. and $G^{\prime}$. be the minimal free resolutions of $R /\left(I_{r-1}: x^{m_{r}}\right)\left(-d^{\prime}\right)$ and $R / I_{r-1}$ respectively, and $\alpha^{\prime}: F^{\prime} . \rightarrow G^{\prime}$. a lifting of the map $R /\left(I_{r-1}\right.$ : $\left.x^{m_{r}}\right)\left(-d^{\prime}\right) \rightarrow R / I_{r-1}$. Hence cone $(\alpha)$ and cone $\left(\alpha^{\prime}\right)$ are the same except possibly in the bottom degrees after shifting the homological degrees one step. It is known that, see Exercise A3.30 in [19], if we have an ideal $J / I$ in a quotient ring $R / I$, the mapping cone of the lifting of the inclusion $J / I \rightarrow R / I$ provide a free resolution of $R / J$. Since $R /\left(I_{r-1}: x^{m_{r}}\right) \cong I_{r} / I_{r-1}$, the result follows after separately considering the bottom degrees.

Corollary 7.1.14. Denote by $R$ the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ and let $I=$ $\left(x^{m_{1}}, \ldots, x^{m_{t}}\right) \subseteq R, \operatorname{deg} x^{m_{i}}=d^{\prime}$ for every $i \in\{1, \ldots, t\}$, be a square-free monomial ideal with $d$-quotients, $d \leq d^{\prime}$. Assume the minimal generators of $I_{s-1}: x^{m_{s}}$ forms an $R$-sequence for every $s \in\{1, \ldots, t\}$. Then $\beta_{i, j}(R / I)$ is non-zero only for $j=i+d^{\prime}-1+(i-1)(d-1)$ and for all $i, 2 \leq i \leq \operatorname{pd}(R / I)$, we have

$$
\beta_{i, j}(R / I)=\sum_{s=2}^{t}\binom{r_{s}}{i-1}
$$

where $r_{s}$ is the cardinality of the minimal generating set of $I_{s-1}: x^{m_{s}}$.
Proof. Let $I_{s-1}: x^{m_{s}}=\left(x^{b_{s_{1}}}, \ldots, x^{b_{s r_{s}}}\right)$. It is then easy to see that $\beta_{i}\left(R /\left(I_{s-1}\right.\right.$ : $\left.\left.x^{m_{s}}\right)\left(-d^{\prime}\right)\right)=\binom{r_{s}}{i}$ in degree $j=i d+d^{\prime}$ and zero in all other degrees. By induction, $\beta_{i}\left(R / I_{s-1}\right)$ is non-zero only in degree $j=i+d^{\prime}-1+(i-1)(d-1)=$ $d^{\prime}+i d-d$. This shows that $\beta_{i}\left(R / I_{s}\right)$ may be non-zero only in degree $j=i+d^{\prime}-$ $1+(i-1)(d-1)$ and that $\beta_{i}\left(R / I_{s}\right)=\beta_{i}\left(R / I_{s-1}\right)+\beta_{i-1}\left(R /\left(I_{s-1}: x^{m_{s}}\right)\left(-d^{\prime}\right)\right)$. The result now follows by induction and Theorem 7.1.13.

Corollary 7.1.15. Denote by $R$ the polynomial ring $k\left[x_{1}, \ldots, x_{n}\right]$ and let $I=$ $\left(x^{m_{1}}, \ldots, x^{m_{t}}\right) \subseteq R, \operatorname{deg} x^{m_{i}}=d^{\prime}$ for every $i \in\{1, \ldots, t\}$, be a square-free monomial ideal with linear quotients. If for each $s \in\{2, \ldots, t\}, I_{s-1}: x^{m_{s}}=$ $\left(x_{s_{1}}, \ldots, x_{s_{r_{s}}}\right)$, then for all $2 \leq i \leq \operatorname{pd} R / I, \beta_{i, j}(R / I)$ is non-zero only in degree $j=i+d^{\prime}-1$ and we have

$$
\beta_{i, j}(R / I)=\sum_{s=2}^{t}\binom{r_{s}}{i-1}
$$

## Part II

## Positive affine monoids

## Chapter 8

## Positive affine monoids

For numerical monoids $S$ the Krull dimension of the ring $k[\bar{S}] / k[S]$ is always zero. This is of course because the set of gaps $H(S)$ is finite, a property that does not hold in general for positive affine monoids of higher rank. We examine here in a combinatorial fashion positive affine monoids $S$ with $H(S)$ finite, or equivalently, positive affine monoids with $\operatorname{dim} k[\bar{S}] / k[S]=0$. This class of monoids turns out to behave in some respects like numerical monoids. In particular we describe the maximal elements in certain posets whose elements are positive affine monoids. This description provides natural higher dimensional versions of familiar classes of numerical monoids such as the class of symmetric numerical monoids.

### 8.1 Iterated Frobenius numbers

Recall from the introduction that we assume all affine monoids $S$ are embedded in $\mathbb{Z}^{d}$ where $d=\operatorname{rank} S$ is the rank of gp $S$. Thus in this chapter whenever $\mathbb{N}^{d}$ or $\mathbb{Z}^{d}$ occur without further explanation, $d$ is the rank of $S$. Also recall that $d$ equals the dimension of the cone $\mathbb{R}_{\geq 0} S$ generated by $S$. Positive affine monoids $S$ are assumed to be embedded in $\overline{\mathbb{N}}^{d}, d=\operatorname{rank} S$.

In the sequel we will use the following partial ordering:

- If $S$ is an affine monoid and $a$ and $b$ are two elements in gp $S$ we say that $a \leq_{S} b$ if and only if $a+s=b$ for some element $s \in S$.

Since $\mathbb{N}^{d}$ is a monoid we obtain as a special case the familiar ordering $\leq_{\mathbb{N}^{d}}$, where for any two elements $a=\left(a_{1}, \ldots, a_{d}\right)$ and $b=\left(b_{1}, \ldots, b_{d}\right)$ we have $a \leq_{\mathbb{N}^{d}} b$ if and only if $a_{i} \leq b_{i}$ holds for every $i \in\{1, \ldots, d\}$. Note that if $a$ and $b$ are elements of a positive affine monoid $S$ and $a \leq_{S} b$, then $a \leq_{\mathbb{N}^{d}} b$.

Given an affine monoid $S$ we defined in the introduction a set $T(S)$ by

$$
T(S)=\left\{x \in \operatorname{gp} S ; x \notin S, x+S_{+} \subseteq S_{+}\right\}
$$

We may deduce that $T(S)$ is finite.

Lemma 8.1.1. Let $S=\left\langle s_{1}, \ldots s_{n}\right\rangle$ be an affine monoid. Then $|T(S)|<\infty$.
Proof. Let $s$ be any non-zero element of $S$ and consider the $S$-graded ideal $\left(\mathbf{t}^{s+u} ; u \in T(S)\right)$ of $k[S] /\left(\mathbf{t}^{s}\right)$. The fact that $T(S)$ is finite follows since $k[S] /\left(\mathbf{t}^{s}\right)$ is Noetherian and the minimal generators of the ideal $\left(\mathbf{t}^{s+u} ; u \in T(S)\right.$ ) are in 1-1 correspondence with the elements in $T(S)$.

Remark 8.1.2. In the case of numerical monoids the above lemma yields the fact (see Remark 1.2.5) that $|T(S)|$ equals the Cohen-Macaulay type of $k[S]$. See [40] for details.

Example 18. We note that $S$ being finitely generated need not imply $H(S)$ finite. Let $S$ be the sub-monoid of $\mathbb{N}^{2}$ generated by the elements

$$
\{(0,2),(1,0),(1,1)\}
$$

Here $T(S)$ is the empty set but $H(S)$ consists of all points $(0,2 k+1), k \in \mathbb{N}$.
Remark 8.1.3. For any subsets $A$ and $B$ of $\mathbb{Z}^{d}$, we denote by $A-B$ the set of differences $\{a-b ; a \in A, b \in B\}$. If the set $A$ consists of only one element, $a$ say, we write $a-B$ instead of $A-B$.

Let $S$ be an affine monoid and denote by $T_{0}(S)=\left\{h_{0,1}, \ldots, h_{0, r_{0}}\right\}$ the set of maximal elements in $T(S)$ with respect to the partial order $\leq_{\mathbb{N}^{d}}$. We define sets $T_{i}(S)$ recursively as follows: Assuming we have defined already $T_{j}(S), j \in$ $\{0,1, \ldots, i-1\}$ we define $T_{i}(S)$ to consist of the elements $x \in \mathbb{R}_{\geq 0} S \cap \operatorname{gp} S$ that are maximal relative $\leq_{\mathbb{N}^{d}}$ with the properties

- $x \notin S$.
- $x \notin T_{j}(S)-S, j \in\{0, \ldots, i-1\}$.

Remark 8.1.4. If $T_{i}(S)=\emptyset$ for some number $i$, then $T_{j}(S)=\emptyset$ for all $j>i$ as well. This follows readily from the definition of the sets $T_{i}(S)$.
Remark 8.1.5. Let $S$ be an affine monoid of rank $d$. A finite subset $T_{0} \subseteq \mathbb{N}^{d}$ can satisfy $T_{0}=T_{0}(S)$ only if $T_{0}$ is an anti-chain in the poset $\left(\mathbb{Z}^{d}, \leq_{\mathbb{N}^{d}}\right)$.
Example 19. Let $S$ be the sub-monoid of $\mathbb{N}^{2}$ generated by the elements

$$
\{(1, k) ; k \in \mathbb{N}\}
$$

Then $T(S)$ consists of all integer points on the $y$-axis so $|T(S)|=\infty$ but $T_{0}(S)=$ $\emptyset$. According to Remark 8.1.5, $T_{i}(S)=\emptyset$ for all $i \geq 0$.

Definition 8.1.6. The elements in the set $\bigcup_{i \geq 0} T_{i}(S)$ are called the iterated Frobenius numbers of $S$.

We now display an important property of the iterated Frobenius numbers.
Proposition 8.1.7. Let $S$ be a positive affine monoid with $H(S)$ finite. Then

$$
T(S)=\bigcup_{j \geq 0} T_{j}(S)
$$

Proof. The elements of $T_{0}(S)$ belong to $T(S)$ by definition. Assume all elements in $\bigcup_{j=0}^{i-1} T_{j}(S)$ belong to $T(S)$ and consider an arbitrary iterated Frobenius number $h_{i} \in T_{i}(S)$. If $h_{i} \notin T(S)$ there is an element $s \in S_{+}$such that $h_{i}+s \notin S_{+}$. By considering the maximal property defining $h_{i}$ we conclude that $h_{i}+s \in T_{\alpha}(S)-S$ for some $\alpha \in\{0, \ldots, i-1\}$. This however yields a contradiction since if $h_{\alpha} \in T_{\alpha}(S)$ and $s^{\prime} \in S$ we have

$$
\begin{gathered}
h_{i}+s=h_{\alpha}-s^{\prime} \\
\Longleftrightarrow \\
h_{i}=h_{\alpha}-\left(s^{\prime}+s\right) \in h_{\alpha}-S .
\end{gathered}
$$

Thus $\bigcup_{i \geq 0} T_{i}(S) \subseteq T(S)$.
By the first part of the proof $T_{i}(S)$ can be non-empty only for a finite number of integers $i$. Assume $T_{j}(S)=\emptyset$ for $j>i$. If $T(S) \neq \bigcup_{j=0}^{i} T_{j}(S)$ the finite set

$$
\left\{x \in \mathbb{R}_{\geq 0} S \cap \operatorname{gp} S ; x \notin S, x \notin \bigcup_{j \geq 0} T_{j}(S)-S\right\}
$$

is non-empty. This however would imply $T_{i+1}(S) \neq \emptyset$ which is a contradiction. Thus $T(S) \subseteq \bigcup_{j \geq 0} T_{j}(S)$ and we are done.

Corollary 8.1.8. Let $S$ be a positive affine monoid and assume $h_{i} \in T_{i}(S)$ and $h_{k} \in T_{k}(S)$. Then either $h_{i}+h_{k} \in S$ or $h_{i}+h_{k} \in T_{r}(S)$ where $0 \leq r<\min \{i, k\}$.

Proof. Assume $h_{i}+h_{k} \notin S$ and let $s \in S_{+}$. Then

$$
h_{i}+h_{k}+s=h_{i}+\left(h_{k}+s\right)=h_{i}+s^{\prime}=s^{\prime \prime}
$$

where $s^{\prime}$ and $s^{\prime \prime}$ belong to $S$. Hence $h_{i}+h_{k} \in T(S)$ so $h_{i}+h_{k} \in T_{r}(S)$ for some $r \geq 0$. It follows from the definition of the iterated Frobenius numbers that $0 \leq r<\min \{i, k\}$.

Remark 8.1.9. The property in Corollary 8.1 .8 of the iterated Frobenius numbers will be used many times in the sequel.
Remark 8.1.10. For numerical monoids $S$ the elements of the set $T(S)$ are known as Pseudo-Frobenius numbers, see [43]. However, due to the above proposition the name iterated Frobenius numbers is motivated.

The following lemma describes for positive affine monoids $T(S)$ as a subset of $H(S)$. For numerical monoids the lemma is part of Proposition 1.19 in [43].

Lemma 8.1.11. Let $S$ be a positive affine monoid. Then we have the following:
(i) $T(S)$ consists of the elements of $H(S)$ that are maximal with respect to the partial order $\leq_{S}$.
(ii) $T_{0}(S)$ consists of the elements of $H(S)$ that are maximal with respect to the partial order $\leq_{\mathbb{N}^{d}}$.

Proof. The elements of $H(S)$ that are maximal with respect to the partial order $\leq_{S}$ are precisely the elements that are characterized by the fact that $x+s \in S$ for any $s \in S_{+}$. This proves $(i)$. The second assertion follows from the definition of $T_{0}(S)$ and the fact that being maximal with respect to $\leq_{\mathbb{N}^{d}}$ implies being maximal with respect to $\leq_{S}$.

Proposition 8.1.12. Let $S$ be a positive affine monoid. Then the following are equivalent:
(i) $H(S)$ is finite.
(ii) $H(S)=(T(S)-S) \cap \mathbb{N}^{d}$.
(iii) If $x \in H(S)$ there is an element $s \in S$ such that $x+s \in T(S)$.

Proof. The fact that (i) and (iii) are equivalent follows from Lemma 8.1.11 and Proposition 8.1.7. (iii) clearly implies (ii) and (ii) implies (i) since in this case $H(S)$ lies in a bounded region of $\mathbb{N}^{d}$.

Consider a positive affine monoid $S$ with $H(S)$ finite and let $x$ be an arbitrary non-zero element in $-\bar{S}=\left\{x \in \mathbb{Z}^{d} ;-x \in \bar{S}\right\}$. Since $x \in \operatorname{gp} S$ we have $x=s-s^{\prime}$ for some elements $s$ and $s^{\prime}$ in $S$. Then

$$
\begin{equation*}
0 \leq_{\mathbb{N}^{d}} h_{0, i} \leq_{\mathbb{N}^{d}} h_{0, i}-x=h_{0, i}-\left(s-s^{\prime}\right)=h_{0, i}+s^{\prime}-s=s^{\prime \prime}-s \in \operatorname{gp} S \tag{8.1}
\end{equation*}
$$

Now, since $H(S)$ is finite we have

$$
\begin{equation*}
h_{0, i} \in T_{0}(S), h_{0, i}<_{\mathbb{N}^{d}} y, y \in \bar{S} \Longrightarrow y \in S \tag{8.2}
\end{equation*}
$$

and so $h_{0, i}-x \in S$ by (8.1). This proves
Corollary 8.1.13. Let $S$ be a positive affine monoid such that $H(S)$ is finite. Then

$$
\begin{equation*}
-\bar{S} \subseteq \bigcap_{1 \leq i \leq r_{0}}\left(h_{0, i}-S\right) \tag{8.3}
\end{equation*}
$$

Remark 8.1.14. The corollary provides a generalization of the fact that all negative integers are in $g-S$ when $S$ is a numerical monoid with Frobenius number $g$. Also, we may view (8.2) as generalizing the fact that every integer greater than the Frobenius number lies in $S$ when $S$ is a numerical monoid.

It is well known that numerical monoids have Cohen-Macaulay monoid rings, a property that does not hold in general for positive affine monoids, in particular not if $H(S)$ is finite non-empty and $\operatorname{rank} S \geq 2$. Indeed, Hoa and Trung have characterized the positive affine monoids that have Cohen-Macaulay monoid rings, see Theorem 8.1.15 below. We review the notions that are used in that theorem:

Let $S$ be a positive affine monoid and denote by $F_{i}, i \in\{1, \ldots, m\}$, the set of facets of the cone $\mathbb{R}_{\geq 0} S$. Put

$$
S_{i}=\left\{x \in \operatorname{gp} S ; x+s \in S, \text { for some } s \in S \cap F_{i}\right\}
$$

and $S^{\prime}=\bigcap_{i=1}^{m} S_{i}$. Furthermore, for every subset $J \subseteq\{1, \ldots, m\}$ we put

$$
G_{J}=\bigcap_{i \notin J} S_{i} \backslash \bigcup_{j \in J} S_{j}
$$

Finally, we let $\pi_{J}$ be the abstract simplicial complex consisting of the nonempty subsets $I \subseteq J$ for which

$$
\bigcap_{i \in I} S \cap F_{i} \neq\{0\}
$$

Theorem 8.1.15 (Hoa and Trung [53]). The monoid ring of a positive affine monoid $S$ is Cohen-Macaulay if and only if

- $S=S^{\prime}$, and
- for every non-empty subset $J \subseteq\{1, \ldots, m\}$, either $G_{J}$ is empty or else the chain complex of $\pi_{J}$ has zero reduced homology, that is, $\pi_{J}$ is acyclic.

Remark 8.1.16. Let $S$ be a numerical monoid. Then $\mathbb{R}_{\geq 0} S=\mathbb{N}$, so the only facet is $\{0\}$. Then $S^{\prime}=S_{1}=S$ so the first condition is satisfied. The second condition is trivially satisfied since there are no non-empty proper subsets of $\{1\}$. Thus follows the well known fact that all numerical monoids have CohenMacaulay monoid rings.
Remark 8.1.17. Assume $S$ is a positive affine monoid and $k[S]$ is not CohenMacaulay. Then rank $S \geq 2$ and, as one easily sees, $T(S) \subseteq S^{\prime}$.

The following results, Proposition 8.1.19, Corollary 8.1.21, and Corollary 8.1.22, are easy to come by in a purely algebraic way since $\operatorname{dim} k[\bar{S}] / k[S]=0$ if $H(S)$ is finite. However, we prove them here using our combinatorial tools and the following lemma.
Lemma 8.1.18. Let $S$ be a positive affine monoids with $\operatorname{rank} S \geq 2$ and let $F$ be a facet of $\mathbb{R}_{\geq 0} S$. Then there is a non-zero element $s \in S \cap F$.

Proof. Assume $S$ is minimally generated by $\left\{s_{1}, \ldots, s_{n}\right\}$ and let $R_{i}, i \in\{1, \ldots, t\}$ be the set of one dimensional faces of $\mathbb{R}_{\geq 0} S$. Let $x_{i}$ be any non-zero element in $R_{i}$. By [10, Proposition 1.20] the finite set of elements $\left\{x_{i}\right\}_{i=1}^{t}$ is, up to scalar multiples, the unique set of minimal generators of $\mathbb{R}_{\geq 0} S$. Now, since $\left\{s_{1}, \ldots, s_{n}\right\}$ generate $\mathbb{R}_{\geq 0} S$ we conclude that the elements $x_{i}$ can be chosen from $S$.

Example 20. The condition that $S$ is finitely generated in the lemma is crucial. Consider the monoid

$$
S=\langle(n, m) ; n \geq 1, m \geq 1\rangle \subseteq \mathbb{N}^{2}
$$

The facets of $\mathbb{R}_{\geq 0} S$ are the coordinate axes. Hence no non-zero element of $S$ can lie in a facet.

Lemma 8.1.18 lets us prove the following

Proposition 8.1.19. Let $S$ be a positive affine monoid with $\operatorname{rank} S \geq 2$ and assume $H(S)$ is finite. Then $S^{\prime}=\bar{S}$.

Proof. By considering the affine forms defining the facets of $\mathbb{R}_{\geq 0} S$, one may conclude that an element $x \in S_{i}$ must lie in $\bar{S}$, so $S^{\prime} \subseteq \bar{S}$. Consider an element $x \in H(S)$. By Proposition 8.1.12 $x=h-s$ for some elements $h \in T(S)$ and $s \in S$. Let $s^{\prime}$ be an element in $S \cap F_{k}$ for some facet $F_{k}$. Such an element $s^{\prime}$ exists by the lemma. Now, $h-s+s^{\prime}=\left(h+s^{\prime}\right)-s=s^{\prime \prime}-s \in \operatorname{gp} S$. If $s^{\prime \prime}-s$ does not already lie in $S$ we substitute $s^{\prime}$ with $n s^{\prime}, n \in \mathbb{N}$ being a large integer. Then, since $H(S)$ is finite, we conclude that $x=h-s \in S_{k}$. Since we only have a finite number of facets, it follows that $H(S) \subseteq S^{\prime}$ so $S^{\prime}=\bar{S}$.

Remark 8.1.20. Let $S$ be an affine monoid and assume $\left\{F_{1}, \ldots, F_{m}\right\}$ is the set of facets of $\mathbb{R}_{\geq 0} S$. The sets $S_{i}$ are the localizations $S-F_{i}$ of $S$ along the facets $F_{i}$, that is, $S_{i}$ is generated as a monoid by $S$ and the inverses of the elements of $S$ that lie in $F_{i}$. Thus $S^{\prime}=\bigcap_{i=1}^{m}\left(S-F_{i}\right)$. This implies that $k\left[S^{\prime}\right]=\bigcap_{i=1}^{m} k[S]_{\mathfrak{p}_{F_{i}}}$, where $\mathfrak{p}_{F_{i}}$ is the monomial prime ideal corresponding to the facet $F_{i}$. See [59, Section 7.2] for details.

Corollary 8.1.21. Let $S$ be a positive affine monoid with $\operatorname{rank} S \geq 2$ such that $k[S]$ is Cohen-Macaulay. Then $T(S)=\emptyset$. If in addition $H(S)$ is finite, then $H(S)=\emptyset$.

Proof. $S=S^{\prime}$ if $k[S]$ is Cohen-Macaulay so $T(S)$ must be empty since $T(S) \subseteq$ $S^{\prime}$. The last claim follows from Lemma 8.1.11.

In particular, if $S$ is as in Corollary 8.1.21, $\operatorname{rank} S=d$, and $H(S)$ is infinite, then there are no $\leq_{\mathbb{N}^{d}}$-maximal elements in $H(S)$.

Hochster proved ([10, Theorem 6.10]) that normal affine monoids have CohenMacaulay monoid rings. Using this we get
Corollary 8.1.22. Let $S$ be a positive affine monoid with $\operatorname{rank} S \geq 2$ such that $k[S]$ is Cohen-Macaulay but not normal. Then $H(S)$ is infinite.

Let $S$ be a positive affine monoid. If $H(S)$ is not finite we would like to construct a positive affine monoid $\tilde{S}, S \subseteq \tilde{S} \subseteq \bar{S}$, such that $H(\tilde{S})$ is finite and $T(\tilde{S})=T(S)$. It is however not clear how to proceed to obtain this. The last couple of results in this section, Proposition 8.1.25 and Proposition 8.1.26, provide a "partial answer" to this problem.

Lemma 8.1.23. Let $S$ be a positive affine monoid with $H(S)$ infinite. Then there exists a positive affine monoid $\tilde{S}$ with $H(\tilde{S})$ finite such that $S \subseteq \tilde{S}$.

Proof. For any element $x=\left(x_{1}, \ldots, x_{d}\right) \in \bar{S}$, let $|x|=x_{1}+\cdots+x_{d}$. Put

$$
a=\left\{\begin{array}{lll}
1+\max \left\{\left|h_{i, j}\right| ; h_{i, j} \in T(S)\right\} & \text { if } T(S) \neq \emptyset \\
1 & \text { if } T(S)=\emptyset
\end{array}\right.
$$

and denote by $H^{+}$the positive half-space

$$
H^{+}=\left\{x \in \mathbb{R}^{d} ; x_{1}+\cdots+x_{d} \geq a\right\}
$$

Then $P=\mathbb{R}_{\geq 0} S \cap H^{+}$is a polyhedron. Also, by construction, the intersection

$$
Q=\mathbb{R}_{\geq 0} S \cap H
$$

of the cone $\mathbb{R}_{\geq 0} S$ and the bounding hyperplane $H$ of $H^{+}$, is a convex polytope. By [10, Proposition 1.28], $P$ is the Minkowski sum

$$
\begin{equation*}
P=Q+\mathbb{R}_{\geq 0} S \tag{8.4}
\end{equation*}
$$

We claim that $P \cap \operatorname{gp} S$ is finitely generated, by which we mean that all $x \in$ $P \cap \operatorname{gp} S$ are positive integer combinations of a finite set of vectors. To prove this, assume $Q=\operatorname{conv}\left\{b_{1}, \ldots, b_{t}\right\}$. By (8.4), an element $x \in P \cap \operatorname{gp} S$ may be written as

$$
\begin{equation*}
x=\alpha_{1} b_{1}+\cdots+\alpha_{t} b_{t}+\beta_{1} s_{1}+\cdots+\beta_{n} s_{n} \tag{8.5}
\end{equation*}
$$

where $\sum_{0 \leq \alpha_{i} \leq 1} \alpha_{i}=1$ and $0 \leq \beta_{j}$ for all $i \in\{1, \ldots, t\}$ and $j \in\{1, \ldots, n\}$. Put $\tilde{S}=\langle S, B\rangle$ where $B$ is the set of elements $x \in P \cap \operatorname{gp} S$ as in (8.5) with $0 \leq \beta_{j}<1$ for all $j \in\{1, \ldots, n\}$. Clearly $B$ is a bounded set and thus $B$ is finite. Hence $\tilde{S}$ is positive affine.

Lemma 8.1.24. Let $S$ be a positive affine monoid and let $x \in H(S)$. Then $S \cup\{x\}$ is a positive affine monoid if and only if $2 x \in S$ and $x \in T(S)$.
Proof. If $2 x \in S$ and $x \in T(S)$ clearly $S \cup\{x\}=\langle S, x\rangle$ so in this case $S \cup\{x\}$ is a positive affine monoid. On the other hand if $S \cup\{x\}$ is a positive affine monoid then, since $2 x \neq x, 2 x$ must belong to $S$ and $x$ must belong to $T(S)$.

Proposition 8.1.25. Let $S$ be a positive affine monoid with $H(S)$ infinite. Then there is a positive affine monoid $\tilde{S}$ such that
(i) $S \subseteq \tilde{S}$.
(ii) $H(\tilde{S})$ is finite.
(iii) $T(S) \subseteq T(\tilde{S})$.
(iv) $T_{0}(S)=T_{0}(\tilde{S})$.

Proof. We define a set $S_{0}$ by

$$
S_{0}=S \cup\left\{x \in \bar{S} ; x \not Z_{\mathbb{N}^{d}} h_{0} \in T_{0}(S)\right\} .
$$

Adding any two elements from $S_{0}$ yields a new element in $S_{0}$, so $S_{0}$ is a submonoid of $\mathbb{N}^{d}$ containing $S$. Also, $S_{0}$ may differ from the positive affine monoid $\tilde{S}$ constructed in Lemma 8.1.23 only by a finite number of elements. Thus $S_{0}$ is positive affine with $H\left(S_{0}\right)$ finite. It is easy to see that $T(S) \subseteq T\left(S_{0}\right)$. If there is an element $h \in T_{0}\left(S_{0}\right) \backslash T(S)$ we put $S_{1}=S_{0} \cup\{h\}$. Since $2 h \in S_{0}$ this is again a positive affine monoid with $H\left(S_{1}\right)$ finite and $T(S) \subseteq T\left(S_{1}\right)$. In this way we obtain in a finite number of steps a positive affine monoid $S_{k}$ with $H\left(S_{k}\right)$ finite and $T(S) \subseteq T\left(S_{k}\right)$ and $T_{0}\left(S_{k}\right) \subseteq T(S)$. In fact, since the elements in $T_{0}\left(S_{k}\right)$ are $\mathbb{N}^{d}$-maximal in $H\left(S_{k}\right)$, we see that $T_{0}\left(S_{k}\right) \subseteq T_{0}(S)$. Then $T_{0}(S) \subseteq T_{0}\left(S_{k}\right)$ must hold so $T_{0}\left(S_{k}\right)=T_{0}(S)$. Put $\tilde{S}=S_{k}$ and we are done.

Proposition 8.1.26. Let $S$ be a positive affine monoid with $H(S)$ finite and let $M \subseteq T_{i}(S)$ for some $i \geq 0$. Then there exists a positive affine monoid $\tilde{S}$ such that $S \subseteq \tilde{S}$ and $T_{0}(\tilde{S})=M$.

Proof. By Lemma 8.1.24 and Corollary 8.1.8 we see that

$$
S_{0}=\left(\bigcup_{j<i} T_{j}(S)\right) \cup S \cup\left(T_{i}(S) \backslash M\right)
$$

is a positive affine monoid and $S \subseteq S_{0}$. By Corollary 8.1.8 it follows that $M \subseteq T\left(S_{0}\right)$. Put $B_{0}=T_{0}\left(S_{0}\right) \backslash M$ and $S_{1}=S_{0} \cup B_{0}$. Again $S_{1}$ is a positive affine monoid and $M \subseteq T\left(S_{1}\right)$. Clearly $S \subseteq S_{0} \subseteq S_{1}$. Since $H(S)$ is finite we obtain in a finite number of steps a positive affine monoid $S_{k}$ such that $S \subseteq S_{k}$ and $M=T_{0}\left(S_{k}\right)$. Put $\tilde{S}=S_{k}$ and we are done.

Question 1. Let $S$ be a positive affine monoid and assume $H(S)$ is not finite. Is there is a positive affine monoid $\tilde{S}, S \subseteq \tilde{S} \subseteq \bar{S}$, such that $H(\tilde{S})$ is finite and $T(\tilde{S})=T(S)$.

Remark 8.1.27. Using the notation introduced in the next section, given a positive affine monoid $S$ with $H(S)$ infinite, we ask for a positive affine monoid $\tilde{S} \in \mathcal{S}_{T_{0}(S)}$ with $T(\tilde{S})=T(S)$.

### 8.2 Maximal objects in the poset $\mathcal{S}_{T_{0}}$

For symmetric (resp. quasi-symmetric, depending on the parity of $g$ ) numerical monoids one has that for all $x \in \mathbb{Z}$ either $x \in S$ or else $g-x \in S$ (resp. $x \in S$ or, $g-x \in S$, or $x=\frac{g}{2}$ ), where $g$ is the Frobenius number of $S$. These two particular classes of numerical monoids are also characterized by the following fact (see [4, 41]): A numerical monoid $S$ is maximal (with respect to inclusion) among the numerical monoids with fixed Frobenius number $g=g(S)$ if and only if it is symmetric (resp. quasi-symmetric). The following lemma lets us prove a similar result, Theorem 8.2.3, for positive affine monoids.

Lemma 8.2.1. Let $S$ be a positive affine monoid and assume $H(S)$ is finite. For any integer $a>1$ and any $h_{i} \in T_{i}(S)$ we have

$$
a^{i} h_{i} \in S \Longleftrightarrow a^{i} h_{i} \notin T_{0}(S)
$$

Proof. The "only if" part is clear and the result holds for $i=0$. Assume the result holds for $j \in\{1, \ldots, i-1\}$ and consider an element $h_{i} \in T_{i}(S)$. If $a^{i} h_{i} \notin T_{0}(S)$ and $a^{i} h_{i} \notin S$, then $a h_{i} \notin S$. But $h_{i}<_{\mathbb{N}^{d}} a h_{i}$ so $a h_{i} \in h_{k}-S$ for some iterated Frobenius number $h_{k}$ with $k<i$. But $a h_{i}=h_{k}-s$ implies that $s=0$, so $a h_{i}=h_{k}$. Now, by the induction hypothesis, either $a^{k} h_{k} \in S$ or else $a^{k} h_{k} \in T_{0}(S)$. In either case, the equation

$$
a^{i} h_{i}=a^{i-k-1} a^{k} h_{k}
$$

yields a contradiction.

Note in particular that it follows from the lemma that $2^{i+1} h_{i} \in S$ for all $i$ and all $h_{i} \in T_{i}(S)$.

Lemma 8.2.2. Let $S$ be a positive affine monoid and assume $h_{i} \in T_{i}(S), i>0$, is such that $2^{i} h_{i} \in T_{0}(S)$. Then $i=1$.

Proof. By Corollary 8.1 .8 we see that all multiples $k h_{i}, 2 \leq k \leq 2^{i}-1$, are iterated Frobenius numbers. Thus we must have $i+1=2^{i}$ which can only hold for $i=0$ and $i=1$.

Let $T_{0}$ be a finite non-empty set of vector in $\mathbb{N}^{d}$ and denote by $\mathcal{S}_{T_{0}}$ the set of positive affine monoids $S$ of rank $d$ with $H(S)$ finite and $T_{0}(S)=T_{0} . \mathcal{S}_{T_{0}}$ is a partially ordered set with respect to inclusion. Let $S$ be a non-maximal element in $\mathcal{S}_{T_{0}}$. According to the proof of the following theorem, a monoid $\hat{S}$, $S \subset \hat{S} \subseteq \bar{S}$, with $T_{0}(S)=T_{0}=T_{0}(\hat{S})$ can be constructed.

Theorem 8.2.3. Let $S$ be a positive affine monoid. Assume $H(S)$ is finite and put $T_{0}(S)=T_{0}$. Then $S$ is maximal in $\mathcal{S}_{T_{0}}$ if and only if $2^{i} h_{i} \in T_{0}$ for all $i$ and all $h_{i} \in T_{i}(S)$. In particular, if $S$ is maximal in $\mathcal{S}_{T_{0}}$ then $T(S)=T_{0}(S) \cup T_{1}(S)$.

Proof. Assume $S$ is such that $2^{i} h_{i} \in T_{0}$ for all $i$ and all $h_{i} \in T_{i}(S)$, and pick an element $b \notin S, b \in \operatorname{gp} S$. Then by Proposition 8.1.12 $b=h_{i}-s$ for some element $s \in S$ and some iterated Frobenius number $h_{i}$, and thus $h_{i} \in\langle S, b\rangle$. But, then $2^{i} h_{i} \in\langle S, b\rangle$ so $T_{0}(S) \nsubseteq T_{0}(\langle S, b\rangle)$ so $S$ is maximal in $\mathcal{S}_{T_{0}}$.

Now assume $S$ does not have the property that $2^{i} h_{i} \in T_{0}$ for all $i$ and all $h_{i} \in T_{i}(S)$. Then, by Lemma 8.2.1 (with $a=2$ ), there is an element $h_{i} \in T(S)$ with $2^{i} h_{i} \in S$. Assume $i=\min \left\{k \in \mathbb{N} ; \exists h_{k} \in T_{k}(S), 2^{k} h_{k} \in S\right\}$. Note that this implies $i>0$. Also, put $b=\min \left\{k \in \mathbb{N} ; k h_{i} \in S\right\}$. If $h_{0} \in\left\langle S, h_{i}\right\rangle$ for some $h_{0} \in T_{0}(S)$ we have $h_{0}=s+n h_{i}, n \in \mathbb{N}$. Then clearly $s=0$ so $h_{0}=n h_{i}$. Here we must have $n>1$ and thus $2 \leq n<b \leq 2^{i}$. Observe that this implies $3 \leq b$ and $2 \leq i$. Since $(b-1) h_{i} \notin S$, by Lemma 8.1 .8 we have $(b-1) h_{i} \in T(S)$, so $(b-1) h_{i}=h_{r} \in T_{r}(S)$ where $0 \leq r<i$. From the equation

$$
b h_{i}<_{\mathbb{N}^{d}} 2(b-1) h_{i}=2 h_{r}
$$

we conclude that $2 h_{r} \in S$. This implies $r=0$ since $r \geq 1$ would contradict the minimality of $i$. Also, we may conclude that $b=3$ : Considering Corollary 8.1.8, $b>3$ implies that $(b-2) h_{i}=h_{s} \in T_{s}(S)$ for some $s \geq 1$ and that $b-1<2(b-2)$. But, then

$$
(b-1) h_{i}=h_{0}<_{\mathbb{N}^{d}} 2(b-2) h_{i}=2 h_{s} \in S
$$

which contradicts the minimality of $i$.
In summary we know that $i \geq 2,2 h_{i}=h_{0} \in T_{0}(S)$, and that $3 h_{i} \in S$. In particular this implies $h_{i} \notin S$ and $h_{i} \notin T_{0}(S)-S$. Since $i \geq 2, h_{i}$ is not $\mathbb{N}^{d}$-maximal with the properties $h_{i} \notin S$ and $h_{i} \notin T_{0}(S)-S$. Thus there is a non-zero element $y \in \mathbb{N}^{d}$ such that $h_{i}+y=h_{1} \in T_{1}(S)$. Then $2\left(h_{i}+y\right)=2 h_{1}$, but $2\left(h_{i}+y\right)=h_{0}+2 y \in S$ so $2 h_{1} \in S$. This again contradicts the minimality of $i$.

We conclude that $S \subset\left\langle S, h_{i}\right\rangle$ and $T_{0}(S) \subseteq T_{0}\left(\left\langle S, h_{i}\right\rangle\right)$. If $T_{0}(S) \neq T_{0}\left(\left\langle S, h_{i}\right\rangle\right)$ we may use the same procedure as in Proposition 8.1.26 and produce a positive affine monoid $\tilde{S}$ with $S \subset \tilde{S}$ and $T_{0}(S)=T_{0}(\tilde{S})$. Thus $S$ is not maximal in $\mathcal{S}_{T_{0}}$. The fact that $T(S)=T_{0}(S) \cup T_{1}(S)$ if $S$ is maximal in $\mathcal{S}_{T_{0}}$ now follows from Lemma 8.2.2.

Remark 8.2.4. The theorem should be compared to the situation for numerical monoids: Let $S$ be a numerical monoid. If $g=g(S)$ is odd, then $S$ is maximal in $\mathcal{S}_{g}$ if and only if $T(S)=\{g\}$. If $g$ is even then $S$ is maximal in $\mathcal{S}_{g}$ if and only if $T(S)=\left\{\frac{g}{2}, g\right\}$.

Corollary 8.2.5. Assume $S$ is a positive affine monoids such that $H(S)$ is finite and $T(S)=T_{0}(S)$. Then $S$ is maximal in $\mathcal{S}_{T_{0}}$ and provide a generalization of a symmetric numerical monoid.

Corollary 8.2.6. Let $S$ be a positive affine monoid. Assume $H(S)$ is finite and that for all $h_{i} \in T_{i}(S)$ there exist positive integers $a_{i}$ such that $a_{i} h_{i} \in T_{0}(S)$. Then $S$ is maximal in $\mathcal{S}_{T_{0}}$.
Proof. Pick an element $b \in H(S)$. Then $b \in h_{i}-S$ for some iterated Frobenius number $h_{i}$. Thus $b=h_{i}-s$ for some $s \in S$ and it follows that $h_{i} \in\langle S, b\rangle$. Hence, since $a_{i} h_{i}=h_{0} \in T_{0}(S), T_{0}(S) \nsubseteq T_{0}(\langle S, b\rangle)$ so $S$ is maximal in $\mathcal{S}_{T_{0}}$.

Corollary 8.2.7. Let $S$ be a positive affine monoid and assume $H(S)$ is finite. If for all $h_{0} \in T_{0}(S)$ the coordinates of the vector $h_{0}$ have no common divisor that is even, then either $S$ is not maximal in $\mathcal{S}_{T_{0}(S)}$ or else $T(S)=T_{0}(S)$.

Example 21. Let $S_{1}$ and $S_{2}$ be two numerical monoids. Put $S_{1}$ on the positive $x$-axis and $S_{2}$ on the positive $y$-axis in $\mathbb{N}^{2}$ and fill in all integer points in the interior of $\mathbb{N}^{2}$. This gives a positive affine monoid $S$ with $T_{0}(S)=\left\{g\left(S_{1}\right), g\left(S_{2}\right)\right\}$.

Proposition 8.2.8. Let $S$ be an positive affine monoid. Then the following conditions on $S$ are equivalent.
(i) $\left.S \cup\left(\left(T_{0}(S)-S\right)\right) \cap \mathbb{N}^{d}\right)=\bar{S} \backslash \bigcup_{q \geq 1} T_{q}(S)$.
(ii) For all $x \in \bar{S} \cup-\bar{S}$ it holds that $x \in S$ or $x \in h_{0}-S$ for some $h_{0} \in T_{0}(S)$ or $x \in T(S) \backslash T_{0}(S)$.

Proof. Taking into account Corollary 8.1.13 the two statements are merely reformulations of each other.

Definition 8.2.9. A positive affine monoid as in Proposition 8.2.8 is called almost symmetric.

Note in particular that $H(S)$ is finite if $S$ is almost symmetric.
Proposition 8.2.10. Let $S$ be a positive affine monoid and assume $H(S)$ is finite. Put $T_{1}^{\prime}(S)=\left\{x \in T_{1}(S) ; 2 x \in T_{0}(S)\right\}$. Then $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$ if and only if for all $x \in \bar{S} \cup-\bar{S}$ it holds that either $x \in S$, or $x \in T_{0}(S)-S$, or $x \in T_{1}^{\prime}(S)$. In particular, if $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$ then $S$ is almost symmetric.

Proof. Assume $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$. In order to prove that for all $x \in \bar{S} \cup-\bar{S}$ it holds that either $x \in S$, or $x \in T_{0}(S)-S$, or $x \in T_{1}^{\prime}(S)$, by Proposition 8.1.12, Corollary 8.1.13 and Theorem 8.2.3 it is sufficient to show that every element $h_{1}-s, h_{1} \in T_{1}(S), s \in S_{+}$, can be written as $h_{0}-s^{\prime}$, where $h_{0} \in T_{0}(S)$ and $s^{\prime} \in S$. Let $h_{1}$ be an arbitrary element in $T_{1}(S)$. Since $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$ we know that $2 h_{1}=h_{0}$ for some $h_{0} \in T_{0}(S)$, so $h_{1}=h_{0}-h_{1}$. Pick any element $s \in S_{+}$. Then $h_{1}-s=h_{0}-\left(h_{1}+s\right)=h_{0}-s^{\prime}, s^{\prime} \in S$.

Now assume $S$ is a positive affine monoid such that $H(S)$ is finite and assume that for all $x \in \bar{S} \cup-\bar{S}$ it holds that either $x \in S$, or $x \in T_{0}(S)-S$, or $x \in T_{1}^{\prime}(S)$. Then

$$
\left.\left.S \cup\left(T_{0}(S)-S\right)\right) \cap \mathbb{N}^{d}\right)=\bar{S} \backslash T_{1}^{\prime}(S)
$$

and clearly $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$.
Remark 8.2.11. By Proposition 8.2.8 and Proposition 8.2.10 the class of almost symmetric monoids as defined above, naturally generalizes the class of almost symmetric numerical monoids, see [5] for details about almost symmetric numerical monoids.

Corollary 8.2.12. Let $S$ be an almost symmetric monoid. Then $S$ is maximal in $\mathcal{S}_{T_{0}(S)}$ in the sense that for any monoid $S^{\prime} \in \mathcal{S}_{T_{0}(S)}$ strictly containing $S$, we have $\left|T\left(S^{\prime}\right)\right|<|T(S)|$.

Corollary 8.2.13. Let $S$ be an almost symmetric monoid with $T_{0}(S)=\left\{h_{0}\right\}$. Then an iterated Frobenius number $h_{i}$ lies in $T(S) \backslash T_{0}(S)$ if and only if $h_{0}-h_{i} \in$ $T(S) \backslash T_{0}(S)$.

Proof. We know that

$$
S \cup\left(\left(h_{0}-S\right) \cap \mathbb{N}^{d}\right)=\bar{S} \backslash \bigcup_{q \geq 1} T_{q}(S) .
$$

Pick an element $x \in \bigcup_{q \geq 1} T_{q}(S)$. We see that $h_{0}-x$ cannot belong to neither $S$ nor to $h_{0}-S$. Thus $h_{0}-x \in \bigcup_{q \geq 1} T_{q}(S)$.

Remark 8.2.14. If $S$ is as in Corollary 8.2.13 then the elements of $\bigcup_{i \geq 1} T_{i}(S)$ occur in pairs. This fact is in the case of numerical monoids observed already in [41].

Let $S \in \mathcal{S}_{T_{0}}$ be an almost symmetric monoid. We thus have

$$
S \cup\left(\left(T_{0}(S)-S\right) \cap \mathbb{N}^{d}\right)=\bar{S} \backslash \bigcup_{q \geq 1} T_{q}(S)
$$

Assume $S$ is not maximal in $\mathcal{S}_{T_{0}}$. Then there exists an element $h_{i} \in T_{i}(S)$ such that $\left\langle S, h_{i}\right\rangle \in \mathcal{S}_{T_{0}}$. It is natural to ask if $\left\langle S, h_{i}\right\rangle \in \mathcal{S}_{T_{0}}$ is almost symmetric. Since $\bar{S}=\overline{\left\langle S, h_{i}\right\rangle}$ we clearly have

$$
\begin{equation*}
\left.\left\langle S, h_{i}\right\rangle \cup\left(T_{0}(S)-\left\langle S, h_{i}\right\rangle\right) \cap \mathbb{N}^{d}\right)=\overline{\left\langle S, h_{i}\right\rangle} \backslash\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}}\right) \tag{8.6}
\end{equation*}
$$

where $\left.T_{h_{i}}=\left\langle S, h_{i}\right\rangle \cup\left(T_{0}(S)-\left\langle S, h_{i}\right\rangle\right) \cap \mathbb{N}^{d}\right) \cap\left(\bigcup_{q \geq 1} T_{q}\right)$. It is not hard to see that

$$
T_{h_{i}}=\left\{h_{l}-m h_{i}, m h_{i}\right\}_{h_{l}, m} \cap\left(\bigcup_{q \geq 1} T_{q}(S)\right)
$$

where $h_{l} \in T_{0}(S)$ and $1 \leq m \leq m_{i}=\max \left\{k \in \mathbb{N} ; k h_{i} \notin S\right\}$.
Proposition 8.2.15. Let $S \in \mathcal{S}_{T_{0}}$ be an almost symmetric monoid that is not maximal in $\mathcal{S}_{T_{0}}$ and assume $h_{i} \in T_{i}(S)$ is such that $\left\langle S, h_{i}\right\rangle \in \mathcal{S}_{T_{0}}$. Then $\left\langle S, h_{i}\right\rangle$ is almost symmetric precisely when

$$
\left(\bigcup_{\alpha<i} T_{\alpha}(S)-m h_{i}\right) \cap\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}}\right)=\emptyset
$$

where $1 \leq m \leq m_{i}=\max \left\{k \in \mathbb{N} ; k h_{i} \notin S\right\}$.
Proof. It follows from equation (8.6) that

$$
\begin{equation*}
T\left(\left\langle S, h_{i}\right\rangle\right) \backslash T_{0}\left(\left\langle S, h_{i}\right\rangle\right) \subseteq \bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}} \subseteq H\left(\left\langle S, h_{i}\right\rangle\right) \tag{8.7}
\end{equation*}
$$

and $\left\langle S, h_{i}\right\rangle$ is almost symmetric precisely when the left inclusion is an equality. If the left inclusion is strict there is an element $h_{k} \in T_{k}(S)$ with

$$
h_{k} \in\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}}\right) \backslash T\left(\left\langle S, h_{i}\right\rangle\right)
$$

Since $h_{k} \in H\left(\left\langle S, h_{i}\right\rangle\right) \backslash T\left(\left\langle S, h_{i}\right\rangle\right)$ by Proposition 8.1.12 there is a non-zero element $s+m h_{i} \in\left\langle S, h_{i}\right\rangle$ such that $h_{k}+\left(s+m h_{i}\right)=h_{\alpha} \in T\left(\left\langle S, h_{i}\right\rangle\right)$. Clearly we must have $s=0$ and $1 \leq m \leq m_{i}$ and $h_{k}+m h_{i}$ cannot lie in $S$. Thus, by Corollary 8.1.8, $h_{\alpha} \in T_{\alpha}(S)$ where $\alpha<\min \{i, k\}$. Hence $h_{k}=h_{\alpha}-m h_{i}$ and

$$
\left(\bigcup_{\alpha<i} T_{\alpha}(S)-m h_{i}\right) \cap\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}}\right) \neq \emptyset
$$

To prove the converse assume

$$
h_{k} \in\left(\bigcup_{\substack{\alpha<i \\ 1 \leq m \leq m_{i}}} T_{\alpha}(S)-m h_{i}\right) \cap\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{i}}\right) .
$$

Hence $h_{k}$ has the form $h_{k}=h_{\alpha}-m h_{i}$. If $h_{k} \in T\left(\left\langle S, h_{i}\right\rangle\right)$ in particular we have

$$
\begin{equation*}
h_{k}+m h_{i}=h_{\alpha}=s+k h_{i} \tag{8.8}
\end{equation*}
$$

for some $s \in S$ and $k \in \mathbb{N}$. If $s=0$ then $h_{k}=(k-m) h_{i}$. This yields a contradiction since then

$$
(k-m) h_{i} \in S \quad \text { or } \quad(k-m) h_{i} \in T_{h_{i}} \quad \text { or } \quad(k-m) h_{i} \leq_{\mathbb{N}^{d}} 0
$$

On the other hand if $s \neq 0$ it follows from (8.8) that $h_{\alpha}=s+k h_{i} \in S$ which is impossible since $h_{\alpha} \in T(S)$. Thus $h_{k} \notin T\left(\left\langle S, h_{i}\right\rangle\right)$ and the left inclusion in (8.7) is strict.

Corollary 8.2.16. Let $S \in \mathcal{S}_{T_{0}}$ be an almost symmetric monoid that is not maximal in $\mathcal{S}_{T_{0}}$. If there is an element $h_{1} \in T_{1}(S)$ such that $\left\langle S, h_{1}\right\rangle \in \mathcal{S}_{T_{0}}$, then $\left\langle S, h_{i}\right\rangle$ is almost symmetric.

Proof. This follows since any element

$$
h_{k} \in\left(\bigcup_{q \geq 1} T_{q}(S) \backslash T_{h_{1}}\right) \backslash T\left(\left\langle S, h_{1}\right\rangle\right)
$$

would by the proof of Proposition 8.2.15 lie in $T_{h_{1}}$ which is a contradiction.
Corollary 8.2.17. Let $S$ be an almost symmetric numerical monoid that is not maximal in $\mathcal{S}_{g(S)}$. Then $\left\langle S, h_{1}\right\rangle$ is almost symmetric.
Proof. The fact that $S$ is not maximal in $\mathcal{S}_{g(S)}$ implies $\frac{g}{2}<h_{1}$. Thus $2 h_{1} \in S$ so $\left\langle S, h_{1}\right\rangle \in \mathcal{S}_{g(S)}$ and is almost symmetric by Corollary 8.2.16.

### 8.3 Apéry sets

As for numerical monoids one may define the Apéry set of a positive affine monoid $S$ with respect to any non-zero element $m \in S$ :

$$
\operatorname{Ap}(S, m)=\{x \in S ; x-m \notin S\} .
$$

For numerical monoids the following is Proposition 7 in [41].
Proposition 8.3.1. Let $S$ be a positive affine monoid and $m$ a non-zero element of $S$. Then the following conditions on an element $t \in \mathbb{Z}^{d}$ are equivalent.
(i) $t-m \in T(S)$.
(ii) $t$ is maximal in $\mathrm{Ap}(S, m)$ with respect to the partial order $\leq_{S}$.

Proof. Assuming (i) we have $t=h_{i, j}+m$ for some iterated Frobenius number $h_{i, j}$ so, clearly, $t \in \operatorname{Ap}(S, m)$. Let $s \in S_{+}$and consider the element $t+s-m=$ $\left(h_{i, j}+m\right)+s-m$. Since this element belong to $S$ we see that $t$ is maximal in $\operatorname{Ap}(S, m)$. If (ii) holds then $t+s-m \in S$ for every $s \in S_{+}$, that is, $t-m \in T(S)$.

Corollary 8.3.2. Let $S$ be a positive affine monoid and $m \in S_{+}$. Then there is a 1-1 correspondence between the elements of $T(S)$ and the elements of $\operatorname{Ap}(S, m)$ that are maximal relative $\leq_{S}$. In particular, $\operatorname{Ap}(S, m)$ is finite.

Proof. This follows from Proposition 8.3.1.
Just as for numerical monoids, the set $T_{0}(S)$ can be described using $\operatorname{Ap}(S, m)$ for any non-zero element $m \in S$.

Proposition 8.3.3. Let $S$ be a positive affine monoid. For any non-zero element $m \in S$ we have

$$
T_{0}(S)=\max _{\mathrm{S}_{\mathrm{N}}}\{x \in \operatorname{Ap}(S, m)\}-m .
$$

Proof. We know that $T(S) \subseteq \operatorname{Ap}(S, m)-m$. On the other hand all elements in the set $\operatorname{Ap}(S, m)-m$ belong to $H(S)$, so the result follows from Lemma 8.1.11.

### 8.4 Special case of numerical monoids

In this section we confine ourselves to numerical monoids. We present here separately versions of a few results seen in the previous sections since in case of numerical monoids one may say a bit more. Recall from the introduction that in case of numerical monoids $S$ the cardinality of the set $T(S)$ is called the type of $S$ and is denoted by type $S$. Also, type $S$ equals the Cohen-Macaulay type of $k[S]$.

For numerical monoids $S$ all negative integers belong to $g(S)-S$. The definition of the iterated Frobenius numbers thus takes the following form:

Definition 8.4.1. Let $S$ be a numerical monoid with Frobenius number $g(S)$. Put $h_{0}(S)=g(S)$ and define the iterated Frobenius numbers $h_{i}(S)$ by

$$
h_{i}(S)=\max \left\{x \in \mathbb{Z} ; x \notin S, x \notin h_{j}(S)-S, j \in\{0, \ldots, i-1\}\right\} .
$$

Remark 8.4.2. The number $h_{1}(S)$ is the the number $h(S)$ explored already in [41].

Since a numerical monoid is an affine monoid, we have
Proposition 8.4.3. Let $S$ be a numerical monoid and assume there are $r+1$ iterated Frobenius numbers $\left\{h_{r}, \ldots, h_{1}, g\right\}$. Then type $S=r+1$ and

$$
T(S)=\left\{h_{r}, \ldots, h_{1}, g\right\} .
$$

Remark 8.4.4. In particular we see that $h_{i}$ is the $(i+1)$ st largest element in $T(S)$.

Observe that for numerical monoids $\operatorname{gn} S=\mathbb{Z}$. Using this we recall from Proposition 8.2.8 that a numerical monoid satisfying the equivalent conditions in Proposition 8.4.5 below, is called an almost symmetric numerical monoid.

Proposition 8.4.5. Let $S=\left\langle s_{0}, \ldots, s_{t}\right\rangle$ be a numerical monoid with $T(S)=$ $\left\{h_{r}, \ldots, h_{1}, g\right\}$. Then the following conditions on $S$ are equivalent:
(i) $S \cup(g-S)=\mathbb{Z} \backslash\left\{h_{r}, \ldots, h_{1}\right\}$.
(ii) For all $x \in \mathbb{Z}$ it holds that $x \in S$ or $x \in g-S$ or $x \in\left\{h_{r}, \ldots, h_{1}\right\}$.
(iii) If $x+y=g$, then either $x \in S$ or $y \in S$, or both $x$ and $y$ belong to $\left\{h_{r}, \ldots, h_{1}\right\}$.
(iv) There are equally many elements in the set

$$
\{0,1, \ldots, g\} \backslash\left\{h_{r}, \ldots, h_{1}\right\}
$$

from $S$ as there are from outside $S$.
(v) $h_{i}-S_{+} \subseteq g-S_{+}$for all $0 \leq i \leq r$.

Proof. We first consider the case when $g$ is odd. (i) and (ii) are equivalent by Proposition 8.2.8. (ii) implies (iii) since if neither $x$ nor $y$ lies in $S$, then $x+y=g$ implies that neither $x$ nor $y$ lies in $g-S$ and hence both $x$ and $y$ lie in $\left\{h_{r}, \ldots, h_{1}\right\}$. (iii) implies (i) since for any integer $x$, we have that if neither $x$ nor $g-x$ lies in $S$, they both belong to the set $\left\{h_{r}, \ldots, h_{1}\right\}$. The fact that (iii) and (iv) are equivalent is easily verified considering the equation $x+y=g$ and Corollary 8.2.13. Finally, by considering the equation

$$
S \cup(g-S) \cup \cdots \cup\left(h_{r}-S\right)=\mathbb{Z}
$$

we see that $(i)$ and $(v)$ are equivalent since $h_{i} \notin g-S$ for any $i$ other than 0 .
The case $g$ even follows in the exact same way considering Remark 8.4.7 and Lemma 8.4.8 below.

Remark 8.4.6. Proposition 8.4.5 partially generalizes Lemma 1 and Lemma 3 in [41].

Remark 8.4.7. In order for the proof of Proposition 8.4.5 to go through also in the case $g$ even, we need to know that $\frac{g}{2}$ belongs to $T(S)$ in case $S$ satisfies the equivalent conditions $(i)$ and (ii). Lemma 8.4.8 below shows that this is the case. However observe that $\frac{g}{2}$ does not always belong $T(S)$ if $g(S)$ is even. The numerical monoid $\langle 3,11,13\rangle$ is an example of this since $g(\langle 3,11,13\rangle)=10$ but $5+3=8 \notin S$.

Lemma 8.4.8. Let $S$ be an almost symmetric numerical monoid. Then
(i) type $S$ is odd whenever $g(S)$ is odd.
(ii) type $S$ is even whenever $g(S)$ is even and in particular, in this case $\frac{g}{2}$ always belongs to $T(S)$.

Proof. Clearly we may assume $S$ is not maximal in $\mathcal{S}_{g}$. By Corollary 8.2.17 we know that $\left\langle S, h_{1}\right\rangle$ is almost symmetric and Proposition 8.2 .15 gives that $T\left(\left\langle S, h_{1}\right\rangle\right)=T(S) \backslash\left\{g-h_{1}, h_{1}\right\}$. The result follows by induction on $r$.

Example 22. Consider the numerical monoid $S=\langle 8,12,14,15,17,18,21,27\rangle$. We have $g(S)=19$ and $T(S)=\{6,9,10,13,19\}$ so $S$ is not symmetric. However, $\mathbb{N} \cap(g-S)=\{1,2,3,4,5,7,11\}$ so we see that $S$ is almost symmetric.

Example 23. Consider the numerical monoid $S=\langle 4,17,18,23\rangle$. We have $g(S)=19, \mathbb{N} \cap(g-S)=\{1,2,3,7,11,15\}$ and $T(S)=\{13,14,19\}$. Since $5 \notin S$, $5 \notin g-S$ and $5 \notin T(S)$ we conclude that $S$ is not almost symmetric.

Proposition 8.4.9. Let $S=\left\langle s_{0}, \ldots, s_{t}\right\rangle$ be an almost symmetric numerical monoid with $g$ odd (resp. even) and assume type $S=2 r+1$ (resp. 2( $r+1$ )). Then there exists a strict sequence of almost symmetric numerical monoids

$$
S \subset S_{1} \subset \cdots \subset S_{r}
$$

such that $g(S)=g\left(S_{i}\right)$ for all $i \in\{1, \ldots, r\}$ and with $S_{r}$ symmetric (resp. quasi-symmetric).

Proof. We may assume that $S$ is not maximal in $\mathcal{S}_{g}$. As in Lemma 8.4.8 we have $T\left(\left\langle S, h_{1}\right\rangle\right)=T(S) \backslash\left\{g-h_{1}, h_{1}\right\}$ and $\left\langle S, h_{1}\right\rangle$ is almost symmetric. The result follows by induction on $r$.

## Chapter 9

## Numerical monoids of maximal embedding dimension

We compute the Betti numbers of all monoid rings corresponding to numerical monoids of maximal embedding dimension. A description, in terms of the generators of $S$, precisely in which degrees the non-zero graded Betti numbers occur is given. We show that for arithmetic numerical monoids of maximal embedding dimension, the graded Betti numbers occur symmetrically in two respects.

### 9.1 Betti numbers

Let $S$ be a numerical monoid with minimal generators $\left\{s_{0}, \ldots, s_{n}\right\}$. Recall that the Betti numbers of $k[S]$ over $R=k\left[x_{0}, \ldots, x_{n}\right]$ are the invariants $\beta_{i}(k[S])=$ $\operatorname{dim} \operatorname{Tor}_{i}^{R}(k[S], k)$. The Betti numbers inherit a grading from the chosen grading of $R$ and we denote by $\beta_{i, j}(k[S])=\operatorname{dim} \operatorname{Tor}_{i}^{R}(k[S], k)_{j}$ the $i$ th Betti number in degree $j$.

Assume $S=\left\langle s_{0}, s_{1}, \ldots, s_{n}\right\rangle$ where $s_{0}<\cdots<s_{n}$ and recall that the number $e(S)=n+1$ is called the embedding dimension of $S$. The number $s_{0}$ is called the multiplicity of $S$ and is denoted by $m(S)$. We always have $e(S) \leq m(S)$ and if $e(S)=m(S)$ we say that $S$ has maximal embedding dimension.

Our results all rely on the following lemma.
Lemma 9.1.1. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be a numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Put $\bar{R}=R /\left(x_{0}\right)$ and let $\mathfrak{m}=\left(x_{1}, \ldots, x_{n}\right)$. Then

$$
\beta_{i, j}(k[S])=\beta_{i, j}\left(\bar{R} / \mathfrak{m}^{2}\right)
$$

Proof. Let

$$
\mathcal{G} . \quad \cdots \rightarrow \oplus_{i} R\left(-b_{2 i}\right) \rightarrow \oplus_{i} R\left(-b_{1 i}\right) \rightarrow R \rightarrow R / I \rightarrow 0
$$

be a minimal $R$-free resolution of the monoid ring $R / I \cong k[S]$. Since $x_{0}$ is not a zero divisor on $R$ or on $R / I$, by [11, Proposition 1.1.5], the tensored complex $\mathcal{G} . \otimes R /\left(x_{0}\right)$ is an $R$-free resolution of $R / I \otimes R /\left(x_{0}\right)$, that is in fact also minimal. Recall that $s_{0}<\cdots<s_{n}$ and that $x_{0}$ correspond to $t^{s_{0}}$ in the isomorphism $k[S] \cong R / I$. Since $S$ has maximal embedding dimension $\left\{s_{0}, \ldots, s_{n}\right\}$ represents a full system of residue classes modulo $s_{0}$. This yields

$$
\begin{equation*}
R / I \otimes R /\left(x_{0}\right) \cong R /\left(I+\left(x_{0}\right)\right) \cong \bar{R} / \mathfrak{m}^{2} \tag{9.1}
\end{equation*}
$$

Remark 9.1.2. Because of Lemma 9.1.1, in the results below we will write $\beta_{i, j}(k[S])$ for the Betti numbers, even if the computations will be explicitely made for $k[S] /\left(t^{s_{0}}\right) \cong \bar{R} / \mathfrak{m}^{2}$.

In the standard grading of $R$ it is plain that the Hilbert series is of $\bar{R} / \mathfrak{m}^{2}$ is

$$
\begin{equation*}
H\left(\bar{R} / \mathfrak{m}^{2} ; z\right)=1+n z=\frac{(1+n z)(1-z)^{n}}{(1-z)^{n}} \tag{9.2}
\end{equation*}
$$

It is well known ([11, Theorem 4.1.13]) that the polynomial $(1+n z)(1-z)^{n}$ here may be written in the form

$$
\begin{equation*}
\sum_{i, j}(-1)^{i} \beta_{i, j}\left(\bar{R} / \mathfrak{m}^{2}\right) z^{j} \tag{9.3}
\end{equation*}
$$

Since $\bar{R} / \mathfrak{m}^{2}$ clearly has 2 -linear resolution over $R$ (that is, $\beta_{i, j}\left(\bar{R} / \mathfrak{m}^{2}\right) \neq 0$ only for $j=i+1$ ) when using the standard grading, we only have to identify the coefficients from the denominator of (9.2) with those from (9.3).

Proposition 9.1.3. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be a numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Then

$$
\beta_{i}(k[S])=\left\{\begin{array}{ccc}
1 & \text { if } & i=0 \\
i\binom{n+1}{i+1} & \text { if } & i \geq 1
\end{array}\right.
$$

Proof. From (9.1), (9.2), (9.3) and the equation

$$
(1+n z)(1-z)^{n}=1+\sum_{k=1}^{n}\left[(-1)^{k}\binom{n}{k}+(-1)^{k-1} n\binom{n}{k-1}\right] z^{k}+(-1)^{n} n z^{n+1}
$$

it follows that

$$
\beta_{i}(k[S])=\left\{\begin{array}{ccc}
1 & \text { if } & i=0 \\
n\binom{n}{i}-\binom{n}{i+1} & \text { if } & i \geq 1
\end{array}\right.
$$

However, it is easily seen that $i\binom{n+1}{i+1}=n\binom{n}{i}-\binom{n}{i+1}$ for all $i \geq 1$.

Remark 9.1.4. The Betti numbers in Proposition 9.1.3 occur elsewhere, for example as the Betti numbers of certain graph algebras: One may view (via polarization, see [36] for details) the ideal $\mathfrak{m}^{2}$ as the edge ideal $I(\mathcal{G})$ of the simple graph $\mathcal{G}$ on $2 n$ vertices $\left\{x_{1}, \ldots, x_{n}\right\} \cup\left\{y_{1}, \ldots, y_{n}\right\}$ whose edge set consists of all edges on the variables $x_{i}$, and all edges of the form $\left\{x_{i}, y_{i}\right\}, 1 \leq i \leq n$. Also, consider the edge ideal $I\left(K_{n+1}\right)$ of the complete graph $K_{n+1}$ on $n+1$ vertices $\left\{z_{1}, \ldots, z_{n}, w\right\}$. Define an onto $\operatorname{map} S_{1} / I(\mathcal{G}) \xrightarrow{\varphi} S_{2} / I\left(K_{n+1}\right)$ by

$$
x_{i} \mapsto z_{i}, \quad y_{i} \mapsto w
$$

for all $1 \leq i \leq n$. Here $S_{1}$ and $S_{2}$ are polynomial rings over $k$ in $2 n$ and $n+1$ variables, respectively. Clearly the ideal generated by the elements $y_{1}-y_{j}$, $j>1$, lies in the kernel which in turn must lie inside the ideal generated by the $y_{i}, 1 \leq i \leq n$. Hence

$$
\operatorname{ker} \varphi=\left(y_{1}-y_{j} ; j>1\right)
$$

It is not hard to see that the elements $y_{1}-y_{j}, j>1$, form a regular sequence on $S_{1} / I(\mathcal{G})$. Thus, using the same kind of arguments as in the proof of Lemma 9.1.1, we see that $S_{1} / I(\mathcal{G})$ and $S_{2} / I\left(K_{n+1}\right)$ have the same graded Betti numbers. The proposition now follows from Theorem 2.2.3.

As mentioned before Proposition 9.1.3, the standard grading of $R$ yields a 2-linear resolution of $k[S] \cong \bar{R} / \mathfrak{m}^{2}$. This is not the case if one use the grading given by $\operatorname{deg}\left(x_{i}\right)=s_{i}$ instead. This fact is illustrated in the following two examples.
Example 24. Consider the monoid $S=\langle 5,9,13,17,21\rangle$. Below we see the Betti diagram of $\bar{R} / \mathfrak{m}^{2}$ considering the standard grading of $R$.

|  | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | - | - | - | - |
| 1 | - | 10 | 20 | 15 | 4 |

In the sequel we will collect the Betti numbers in tables of the following form instead of using the Betti diagrams. The numbers in the column to the right are $j\left(\beta_{i, j}(k[S])\right)$.

| $i$ | $\beta_{i}$ | $j\left(\beta_{i, j}\right)$ |
| :---: | :---: | :---: |
| 1 | 10 | $2(10)$ |
| 2 | 20 | $3(20)$ |
| 3 | 15 | $4(15)$ |
| 4 | 4 | $5(4)$ |

Example 25. We consider the same monoid as in the previous example but instead with the grading of $R$ defined by $\operatorname{deg}\left(x_{i}\right)=s_{i}$. We get the following table:

| $i$ | $\beta_{i}$ | $j\left(\beta_{i, j}\right)$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 10 | $18(1)$ | $22(1)$ | $26(2)$ | $30(2)$ | $34(2)$ | $38(1)$ | $42(1)$ |  |
| 2 | 20 | $31(1)$ | $35(2)$ | $39(3)$ | $43(4)$ | $47(4)$ | $51(3)$ | $55(2)$ | $59(1)$ |
| 3 | 15 | $48(1)$ | $52(2)$ | $56(3)$ | $60(3)$ | $64(3)$ | $68(2)$ | $72(1)$ |  |
| 4 | 4 | $69(1)$ | $73(1)$ | $77(1)$ | $81(1)$ |  |  |  |  |

We now determine, in general, the degrees $j$ for which $\beta_{i, j}(k[S])$ is nonzero. Since $\mathfrak{m}^{2}$ is a stable ideal the Eliahou-Kervaire resolution, see [21] for details, provides a minimal $R$-free resolution of $\bar{R} / \mathfrak{m}^{2}$. The minimal generators of $L_{i}$, the $i$ th component of the Eliahou-Kervaire resolution, are the symbols $e(\sigma, u)$ where $\sigma=\left(q_{1}, \ldots, q_{i}\right)$ is a sequence of integers satisfying

$$
\begin{equation*}
1 \leq q_{1}<\cdots<q_{i}<\max u \tag{9.4}
\end{equation*}
$$

and $u$ a minimal generator of $\mathfrak{m}^{2}$. Here $\max (u)$ denotes the maximal index of a variable $x_{i}$ that divides $u$. For $i=0$ the condition (9.4) is considered as void, so that the symbols of $L_{0}$ are in 1-1 correspondence with the minimal generators of $\mathfrak{m}^{2}$. The Eliahou-Kervaire resolution is in fact graded and in the standard grading of $R$ the degree of a symbol $e(\sigma, u) \in L_{i}$ is by definition $\operatorname{deg}(u)+i$. Thus in our case, in the standard grading of $R$, the degree of $e(\sigma, u) \in L_{i}$ is $2+i$.
Remark 9.1.5. Note that the Eliahou-Kervaire resolution resolves the ideal $\mathfrak{m}^{2}$. Thus, in the formulas below the homological degrees are shifted one step since we resolve $\bar{R} / \mathfrak{m}^{2}$.

Lemma 9.1.6. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be a numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Then $\beta_{i+1, j}(k[S])$ is non-zero precisely in the degrees $j$ that may be written

$$
\begin{equation*}
j=s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}} \tag{9.5}
\end{equation*}
$$

for some $1 \leq k \leq l \leq n$ and $1 \leq q_{1}<\cdots<q_{i}<l$, and equals the number of different ways in which this can be done.

Proof. Recall that if $u=x_{k} x_{l}$, in the standard grading the degree of a symbol $e(\sigma, u) \in L_{i}$ is $\operatorname{deg}\left(x_{k} x_{l}\right)+i$. If we translate this via the isomorphism $k[S] \cong R / I$ to the corresponding $R$-free resolution of $k[S]$ in the grading given by $\operatorname{deg}\left(x_{i}\right)=$ $s_{i}$, we see that the degree of the symbol $e(\sigma, u)$ becomes $s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}}$.

Let $j \in \mathbb{N}$. Recall that a partition of $j$ with $i$ parts on a set $I \subseteq\{1,2, \ldots, j\}$, is an expression

$$
j=x_{1}+\cdots+x_{i}
$$

where $1 \leq x_{1} \leq \cdots \leq x_{i}$ and $x_{k} \in I$ for all $k \in\{1, \ldots, i\}$. We denote the number of partitions of $j$ with $i$ parts on the set $\{1,2, \ldots, n\}$ by $p(j, i, n)$. Motivated by Lemma 9.1.6, we define an Eliahou-Kervaire partition of an integer $j$ with $i+2$ parts on the set $\{1,2, \ldots, n\}$ to be a partition $j=k+l+q_{1}+\cdots+q_{i}$ where
(i) $1 \leq k \leq l \leq n$
(ii) $1 \leq q_{1}<\cdots<q_{i}<l$.

Also, let $\operatorname{EKP}(j, i, n)$ denote the number of Eliahou-Kervaire partitions of $j$ with $i+2$ parts on $\{1,2, \ldots, n\}$.

If $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ is a numerical monoid of maximal embedding dimension with $s_{0}<\cdots<s_{n}$, then $\left\{s_{0}, \ldots, s_{n}\right\}$ represents a full system of congruence classes modulo $s_{0}$. Thus we may re-index so that $s_{i} \equiv i \bmod s_{0}$. We assume this is done for the rest of this section.

If $\beta_{i+1, j}(k[S]) \neq 0$ (so that there is a partition $j=s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}}$ ) we see that there is a number $m_{j}$ such that $j=\left(k+l+q_{1}+\cdots+q_{i}\right)+m_{j} s_{0}$. This is clear since for every $i \in\{0,1, \ldots, n\}, s_{i}$ may be written as $s_{i}=i+n_{i} s_{0}$. For given $i$ and $j$ denote by $M_{i, j}$ the set of all such numbers $m_{j}$.
Proposition 9.1.7. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be a numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Then

$$
\begin{equation*}
\beta_{i+1, j}(k[S])=\sum_{m_{j} \in M_{i, j}} \operatorname{EKP}\left(j-m_{j} s_{0}, i, n\right) . \tag{9.6}
\end{equation*}
$$

Proof. Assume $\beta_{i+1, j}(R)$ is nonzero and that $j=s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}}=$ $\left(k+l+q_{1}+\cdots+q_{i}\right)+m_{j} s_{0}$. Clearly every Eliahou-Kervaire partition of $j-m_{j} s_{0}$ with $i+2$ parts on $\{1,2, \ldots, n\}$ will contribute by 1 to the number $\beta_{i+1, j}(k[S])$. Hence, taking the sum over all elements in $M_{i, j}$ yields $\beta_{i+1, j}(k[S])$.

Example 26. Let $S=\langle 3,5,7\rangle$. We may use Lemma 9.1.6 directly to find the degrees where the relations generating the ideal $I$ lies. The Betti numbers that keep track of this information lies in homological degree 1 , so we have $i=0$. Thus the degrees $j$ are

$$
\left\{\begin{array}{l}
j=5+5=(2+2)+2 \cdot 3 \\
j=5+7=(2+1)+3 \cdot 3 \\
j=7+7=(1+1)+4 \cdot 3
\end{array}\right.
$$

From the expressions to the right we see that the corresponding Eliahou-Kervaire partitions are $4=2+2,3=2+1$ and $2=1+1$.

Assume we have an Eliahou-Kervaire partition $k+l+q_{1}+\cdots+q_{i}$ of $j$. By subtracting $h$ from $q_{h+1}$ for every $h \in\{1,2, \ldots, i-1\}$ we obtain a partition $q_{1}^{\prime}+\cdots+q_{i}^{\prime}$ of $j-k-l-\binom{i}{2}$ with $i$ parts on $\{1,2, \ldots, n-i\}$. The number $k$ still satisfies $1 \leq k \leq l$. This yields:
Lemma 9.1.8. For the number $\operatorname{EKP}(j, i, n)$ we have the equality

$$
\operatorname{EKP}(j, i, n)=\sum_{\substack{i+1 \leq l \leq n \\ 1 \leq k \leq l}} p\left(j-k-l-\binom{i}{2}, i, n-i\right)
$$

By inserting this into (9.6) we get
Theorem 9.1.9. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be a numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Then

$$
\beta_{i+1, j}(k[S])=\sum_{\substack{i+1 \leq l \leq n \\ 1 \leq k \leq l \\ m_{j} \in M_{i, j}}} p\left(j-m_{j} s_{0}-k-l-\binom{i}{2}, i, n-i\right) .
$$

To make this theorem more explicit we give, in Proposition 9.1.12 below, a generating series for the numbers $\operatorname{EKP}(j, i, n)$ for fixed $i$ and $n$. We will use the following two lemmas.

The following lemma describes, for any integer $l$, the bivariate generating series

$$
G(x, y)=\sum_{\substack{n \geq 0 \\ k \geq 0}} p(n, k, l) y^{k} x^{n}
$$

Lemma 9.1.10. Given a positive integer $l$, the bivariate generating series $G(x, y)$ of the numbers $p(n, k, l)$ is given by

$$
\begin{equation*}
G(x, y)=\prod_{j=1}^{l} \frac{1}{1-y x^{j}} \tag{9.7}
\end{equation*}
$$

If we consider the above generating series for fixed $k$, we have the vertical generating series

$$
G_{k}(x)=\sum_{n \geq k} p(n, k, l) x^{n}
$$

We may use the vertical generating series to rewrite $G(x, y)$ as

$$
\begin{equation*}
G(x, y)=\sum_{k \geq 0} G_{k}(x) y^{k} \tag{9.8}
\end{equation*}
$$

Lemma 9.1.11. With $k$ fixed and the notations as above, we have

$$
G_{k}(x)=x^{k} \prod_{j=1}^{k} \frac{1-x^{j+l-1}}{1-x^{k}}
$$

Proof. It is easy to verify that $G(x, x y)\left(1-y x^{l+1}\right)=G(x, y)(1-y x)$. From this and equation (9.8) we get that

$$
G_{k}(x)=\frac{x\left(1-x^{k+l-1}\right)}{1-x^{k}} \cdot G_{k-1}(x)
$$

which proves the assertion by induction.
In light of Proposition 9.1.7 and Lemma 9.1.8 our interest lies in the coefficients of $G_{i}(x)=x^{i} \prod_{j=1}^{i} \frac{1-x^{j+n-i-1}}{1-x^{i}}$ (we use $l=n-i$ ) that stand before powers of $x$ with exponents of the form $j-k-l-\binom{i}{2}$, where $1 \leq k \leq l \leq n$. Considering the double sum in Lemma 9.1.8 we see that the maximal value the expression $j-k-l-\binom{i}{2}$ takes there, is $j-i-\binom{i}{2}-2$. For each pair $k, l$ of summation indices, we let $\alpha_{k, l}$ denote the number $k+l-i-2=\left(j-i-\binom{i}{2}-2\right)-\left(j-k-l-\binom{i}{2}\right)$. Observe that $\alpha_{k, l}$ does not depend on $j$. Using this we obtain the generating series for the numbers $\operatorname{EKP}(r, i, n)$ :

Proposition 9.1.12. The generating series for the numbers $\operatorname{EKP}(j, i, n)$, $i$ and $n$ fixed, is given by

$$
\sum_{j \geq 0} \operatorname{EKP}(j, i, n) x^{r}=x^{2 i+\binom{i}{2}+2} \cdot \sum_{\substack{i+1 \leq l \leq n \\ 1 \leq k \leq l}}\left[x^{\alpha_{k, l}} \prod_{r=1}^{i} \frac{1-x^{r+n-i-1}}{1-x^{i}}\right]
$$

Proof. We have added generating series of the summands that occur in Lemma 9.1.8 giving them an additional "weight", $x^{\alpha_{k, l}+2 i+\binom{i}{2}+2}$, so that the coefficient before $x^{j}$ in the sum counts precisely $\operatorname{EKP}(j, i, n)$.

Example 27. Consider the numerical monoid $S=\langle 3,5,7\rangle$ from Example 3. We showed there that $\beta_{1,10}(k[S])=\beta_{1,12}(k[S])=\beta_{1,14}(k[S])=1$. If we do the same kind of computations but with $i=1$ instead we get

$$
\left\{\begin{array}{l}
j=5+5+5=(2+2+2)+3 \cdot 3 \\
j=5+5+7=(2+2+1)+4 \cdot 3 \\
j=5+7+7=(2+1+1)+5 \cdot 3 \\
J=7+7+7=(1+1+1)+6 \cdot 3
\end{array}\right.
$$

Considering the right hand side expressions we see that only the middle two degrees $j$ are in fact Betti degrees. We have $\beta_{2,17}(k[S])=\beta_{2,19}(k[S])=1$. Computing the generating series from the above proposition (with $i=1$ and $n=2$ ) gives

$$
\sum_{j \geq 0} \operatorname{EKP}(j, 1,2) x^{j}=x^{4}+x^{5}
$$

The exponents 4 and 5 here correspond to the Eliahou-Kerviare partitions $4=$ $2+1+1$ and $5=2+2+1$.

We may also confirm the fact that $\operatorname{pd} k[S]=2$ : Consider the following computations

$$
\left\{\begin{array}{l}
j=5+5+5+5=(2+2+2+2)+4 \cdot 3 \\
j=5+5+5+7=(2+2+2+1)+5 \cdot 3 \\
j=5+5+7+7=(2+2+1+1)+6 \cdot 3 \\
j=5+7+7+7=(2+1+1+1)+7 \cdot 3 \\
j=7+7+7+7=(1+1+1+1)+8 \cdot 3
\end{array}\right.
$$

None of the expressions on the right hand side give Eliahou-Kervaire partitions, so $\beta_{3}(k[S])=0$.

### 9.2 Arithmetic numerical monoids

We do no longer assume that the minimal generators of $S$ satisfy $s_{i} \equiv i \bmod s_{0}$. Consider the sets $M_{i, j}$ from Proposition 9.1.7. If $\left|M_{i, j}\right|=1$ for every $i$ and $j$, the description of the Betti numbers can be made more explicit. We call a numerical monoid $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ arithmetic if $s_{i}=s_{0}+i d$ for all $i \in\{0, \ldots, n\}$, where $d$ is some integer $1 \leq d<s_{0}$ with $\operatorname{gcd}\left(d, s_{0}\right)=1$.

Example 28. $S=\langle 5,9,13,17,21\rangle$ is an example of an arithmetic numerical monoid with $d=4$. Another example is $S=\langle 5,6,7,8,9\rangle$ where $d=1$.

Remark 9.2.1. For $n=3$, the following results are included in more general results by Sengupta, [64]. In [64] minimal resolutions for all monomial curves in $\mathbb{A}^{4}$ defined by an arithmetic sequence are given. Hence the information about the Betti numbers below can be obtained from these resolutions in the case where $n=3$.

Proposition 9.2.2. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be an arithmetic numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. Assume $s_{i}=s_{0}+i d$ for every $i \in\{0,1, \ldots, n\}$. Then the following holds.
(i) The non-zero Betti numbers $\beta_{i+1, j}(k[S])$ lie in degrees $j$ that are of the form $(i+2) s_{0}+m_{j} d$. The integer $m_{j}$ is uniquely determined by $j$ and $m_{j}$ has an Eliahou-Kervaire partition with $i+2$ parts on $\{1,2, \ldots, n\}$.
(ii) $\beta_{i+1, j}(k[S])$ equals the number of Eliahou-Kervaire partitions of $m_{j}$ with $i+2$ parts on $\{1,2, \ldots, n\}$.
(iii) The minimal and maximal degrees, $j_{\min }$ and $j_{\max }$ respectively, for which $\beta_{i+1, j}(k[S])$ is nonzero are

$$
\begin{gathered}
j_{\min }=(i+2) s_{0}+\left(1+\binom{i+2}{2}\right) d \\
j_{\max }=(i+2) s_{0}+\left((i+2) n-\binom{i+2}{2}\right) d
\end{gathered}
$$

(iv) $\beta_{i+1, j}(k[S])$ is nonzero in every degree $j=(i+2) s_{0}+m_{j} d$ for which

$$
1+\binom{i+2}{2} \leq m_{j} \leq(i+2) n-\binom{i+2}{2}
$$

Remark 9.2.3. Part (iv) of the proposition says that there is a certain kind of symmetry in the Betti numbers. Namely, if $s_{i}=s_{0}+i d$ for every $i \in$ $\{0,1, \ldots, n\}$, then $\beta_{i+1, j}(k[S])$ is non-zero in every $d$ th degree $j$ between two specific degrees $j_{\text {min }}$ and $j_{\text {max }}$.

Proof. (i), (iii) and (iv) follows directly by considering (9.5), so let us prove the second assertion. Let $m_{j}$ be the unique number for which $j=(i+2) s_{0}+m_{j} d$. By mapping the partition $j=s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}}$ to the Eliahou-Kervaire partition $k+l+q_{1}+\cdots+q_{i}$ of $m_{j}$, we obtain not only an injection, but in fact a bijection between the set $B_{i+1, j}$ consisting of partitions $j=s_{k}+s_{l}+s_{q_{1}}+\cdots+s_{q_{i}}$ of $j$ and the set of Eliahou-Kervaire partitions of integers with $i+2$ parts on $\{1,2, \ldots, n\}$.

Proposition 9.2.4. Let $S=\left\langle s_{0}, \ldots, s_{n}\right\rangle$ be an arithmetic numerical monoid of maximal embedding dimension and $k[S]$ the corresponding monoid ring. If $j_{1}, \ldots, j_{r}$ are the degrees in which $\beta_{i+1, j}(k[S])$ is non-zero, then

$$
\begin{equation*}
\beta_{i+1, j_{k}}(k[S])=\beta_{i+1, j_{r-k+1}}(k[S]), \tag{9.9}
\end{equation*}
$$

for every $0 \leq k \leq r$.
Proof. Define a map $\phi$ on the set of generators of $S$ by $s_{i} \mapsto s_{n-i+1}, 0 \leq i \leq n$. If $e(\sigma, u)$ is a generator of $L_{i}$ corresponding to the sequence $\sigma=\left(q_{1}, \ldots, q_{i}\right)$ and the minimal generator $u=x_{k} x_{l}$ of $\mathfrak{m}^{2}, 1 \leq k \leq l \leq n$, we consider the set

$$
N_{\sigma, u}=\left\{s_{k}, s_{l}, s_{q_{1}}, \ldots, s_{q_{i}}\right\} .
$$

Here all but possibly two elements are distinct, and $\phi$ maps $N_{\sigma, u}$ to a set $\phi\left(N_{\sigma, u}\right)=\left\{\phi\left(s_{k}\right), \phi\left(s_{l}\right), \phi\left(s_{q_{1}}\right), \ldots, \phi\left(s_{q_{i}}\right)\right\}$, that in turn correspond to some other generator of $L_{i}$. In light of Proposition 9.2.2 and the fact that $\phi$ preserves, but reverses, all relations $s_{t} \leq s_{u}, s_{t}, s_{u} \in N_{\sigma, u}$, the proposition follows.

Thus, if $S$ is arithmetic of maximal embedding dimension equation (9.9) tells us that the contributions to $\beta_{i}(k[S])$ from its graded components $\beta_{i+1, j}(k[S])$ are symmetric relative to the non-zero degrees $j$.

Note that for arithmetic numerical monoids of maximal embedding dimension, there are two kinds of symmetries in the Betti numbers. One described just above, and one in Remark 9.2.3. It is natural to ask for other numerical monoids for which both these symmetries hold.

Example 29. Let $S=\langle 5,7,9\rangle$. Below are the Betti numbers of $k[S]$ using the grading given by $\operatorname{deg}\left(x_{i}\right)=s_{i}$. Clearly the Betti numbers are symmetric in the sense of Proposition 9.2.4, but not in the sense of (iv) in Proposition 9.2.2.

| $i$ | $\beta_{i}$ | $j\left(\beta_{i, j}\right)$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | $14(1)$ | $25(1)$ | $27(1)$ |
| 2 | 2 | $28(1)$ | $30(1)$ |  |

Example 30. The symmetry in the Betti numbers in Proposition 9.2.4 does not hold in general. Consider for example the monoid $S=\langle 5,11,17,18,19\rangle$. Below are the Betti numbers of $k[S]$.

| $i$ | $\beta_{i}$ | $j\left(\beta_{i, j}\right)$ |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 10 | $22(1)$ | $28(1)$ | $29(1)$ | $30(1)$ | $34(1)$ | $35(1)$ | $36(1)$ | $37(1)$ | $38(1)$ |
| 2 | 20 | $39(1)$ | $40(1)$ | $41(1)$ | $45(1)$ | $46(2)$ | $47(3)$ | $48(2)$ | $4(1)$ | $52(1)$ |
| 3 |  | $53(2)$ | $54(2)$ | $55(2)$ | $56(1)$ | 4 |  |  |  |  |
|  | 15 | $57(1)$ | $58(1)$ | $59(1)$ | $63(1)$ | $64(2)$ | $65(3)$ | $66(2)$ | $67(1)$ | $71(1)$ |
| 4 | 4 | $76(1)$ | $73(1)$ |  | $82(1)$ | $83(1)$ | $84(1)$ |  |  |  |
|  | $76(1)$ | 8 |  |  |  |  |  |  |  |  |

## Chapter 10

## Monomial subrings of complete hypergraphs

Let $a_{i}, i \in\{1,2, \ldots, n\}$, and $d$ be non-negative integers. Recall that for any $\mathbf{u}=\left(u_{1}, \ldots, u_{n}\right) \in \mathbb{N}^{n}$ we denote by $|\mathbf{u}|=u_{1}+\cdots+u_{n}$. Consider the set

$$
A=\left\{\mathbf{u} \in \mathbb{N}^{n} ; 0 \leq u_{i} \leq a_{i}, i \in\{1,2, \ldots, n\},|\mathbf{u}|=d\right\}
$$

The monomial subring $k[A] \subseteq k\left[t_{1}, \ldots, t_{n}\right]$ generated by the set of monomials $\left\{\mathbf{t}^{\mathbf{u}} ; \mathbf{u} \in A\right\}$ is said to be of Veronese type and is well studied, see for example $[48,66]$. Since the set $A$ generates a positive affine monoid the ring $k[A]$ is isomorphic to a quotient $k\left[x_{\mathbf{u}} ; \mathbf{u} \in A\right] / \mathfrak{p}_{A}$ via the map $x_{u} \mapsto \mathbf{t}^{\mathbf{u}}$. Here $k\left[x_{\mathbf{u}} ; \mathbf{u} \in\right.$ $A]$ is a polynomial ring with one variable for each element in $A$.

Sturmfels showed, [66, Theorem 14.2], that there exists a monomial order on $k\left[x_{u} ; u \in A\right]$ such that the standard monomials modulo $\mathfrak{p}_{A}$ are the so called sorted monomials and that $\mathfrak{p}_{A}$ in this monomial order has a Gröbner basis consisting of square-free quadratic binomials called sorting relations. Later De Negri, [15], showed that these properties are inherited by certain sub-algebras of $k[A]$. The monomial algebras considered by De Negri are generated by so called sortable subsets of $A$.

We use the results of De Negri and Sturmfels to study monomial subrings $k[\mathcal{H}]$ of $k\left[t_{1}, \ldots, t_{n}\right]$ that are generated by the edge sets of complete hypergraphs $\mathcal{H}$.

### 10.1 Preliminaries

Recall that an integral domain $D$ is called normal if it is integrally closed in its field of fractions. The following result is part of [66, Proposition 13.15].
Proposition 10.1.1. If $S$ is a positive affine monoid and $k[S]$ is a homogeneous monomial subring whose toric ideal has a square-free initial ideal with respect to some term order, then $k[S]$ is normal.

Let $M_{d} \subseteq \mathbb{N}^{n}$ be the set of integral vectors $\mathbf{u}=\left(u_{1}, \ldots, u_{n}\right)$ such that $|\mathbf{u}|=u_{1}+\cdots+u_{n}=d$ and assume $B \subseteq M_{d}$. Assume further $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$ belong to $B$. Write $\mathbf{t}^{\mathbf{u}_{1}} \mathbf{t}^{\mathbf{u}_{\mathbf{2}}}=t_{i_{1}} \cdots t_{i_{2 d}}$ where $i_{1} \leq \cdots \leq i_{2 d}$. We set $\mathbf{t}^{\mathbf{u}_{1}^{\prime}}=\prod_{j=1}^{d} t_{i_{2 j-1}}$ and $\mathbf{t}^{\mathbf{u}_{2}^{\prime}}=\prod_{j=1}^{d} t_{i_{2 j}}$. This defines a map

$$
\text { sort: } B \times B \rightarrow M_{d} \times M_{d}, \quad \operatorname{sort}\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right)=\left(\mathbf{u}_{1}^{\prime}, \mathbf{u}_{2}^{\prime}\right)
$$

called the sorting operator.
A subset $B \subseteq M_{d}$ is called sorted if $\operatorname{sort}(B \times B) \subseteq B \times B$, and a pair $(\mathbf{u}, \mathbf{v}) \in B \times B$ is called sorted if $\operatorname{sort}(\mathbf{u}, \mathbf{v})=(\mathbf{u}, \mathbf{v})$.
Remark 10.1.2. The sorting operator was introduced by Sturmfels in [66]. It is also used by Herzog and Hibi in [48].

Let $M$ be any subset of $M_{d}$ and consider the monomial subring $k[M] \cong$ $k\left[x_{\mathbf{u}} ; \mathbf{u} \in M\right] / \mathfrak{p}_{M}$. Binomials $x_{\mathbf{u}} x_{\mathbf{v}}-x_{\mathbf{u}^{\prime}} x_{\mathbf{v}^{\prime}} \in \mathfrak{p}_{M}$, where $\left(\mathbf{u}^{\prime}, \mathbf{v}^{\prime}\right)=\operatorname{sort}(\mathbf{u}, \mathbf{v})$, are called sorting relations. Thus the before mentioned result by Sturmfels, [66, Theorem 14.2], says that if $A$ is of Veronese type, then

$$
\mathfrak{p}_{A}=\left(x_{\mathbf{u}} x_{\mathbf{v}}-x_{\mathbf{u}^{\prime}} x_{\mathbf{v}^{\prime}} ;\left(\mathbf{u}^{\prime}, \mathbf{v}^{\prime}\right)=\operatorname{sort}(\mathbf{u}, \mathbf{v})\right)
$$

and there exists a monomial order on $k\left[x_{\mathbf{u}} ; \mathbf{u} \in A\right]$ such that the standard monomials modulo $\mathfrak{p}_{A}$ are the sorted monomials.

### 10.2 Monomial subrings of complete hypergraphs

Proposition 10.2.1. Let $\mathcal{H}$ be any $d\left(I_{1}, \ldots, I_{t}\right)$-complete hypergraph. Then the toric ideal $\mathfrak{p}_{\mathcal{H}}$ of $\mathcal{H}$ has a Gröbner basis consisting of square-free binomials of degree two and $k[\mathcal{H}]$ is normal.

Proof. The normality follows either directly from the fact that the edge set is the set of bases of a discrete polymatroid (see [48]), or else will follow from Proposition 10.1.1. We show that the set $\mathcal{E}(\mathcal{H})$ is sortable. Pick two edges $E_{i}$ and $E_{j}$ and denote by $a_{i}$ and $b_{i}$ respectively the number of vertices from $\mathcal{V}_{i}$ in $E_{i}$ and $E_{j}$ respectively. If $I_{i}=\left[\alpha_{i}, \beta_{i}\right]$, by definition $\alpha_{i} \leq a_{i} \leq \beta_{i}$ and $\alpha_{i} \leq b_{i} \leq \beta_{i}$. Consider the image $\operatorname{sort}\left(E_{i}, E_{j}\right)=(M, N)$ and let $a_{i}^{\prime}$ and $b_{i}^{\prime}$ denote the number of vertices from $\mathcal{V}_{i}$ in $M$ and $N$ respectively. We claim $a_{i}^{\prime} \in I_{i}, b_{i}^{\prime} \in I_{i}$. Clearly

$$
2 \alpha_{i} \leq a_{i}+b_{i} \leq 2 \beta_{i}
$$

If $a_{i}+b_{i}=2 k, k \in \mathbb{N}$, then $\alpha_{i} \leq k \leq \beta_{i}$ and $k=a_{i}^{\prime}=b_{i}^{\prime}$. Hence in this case $a_{i}^{\prime} \in I_{i}$ and $b_{i}^{\prime} \in I_{i}$. Assume $a_{i}+b_{i}=2 k+1, k \in \mathbb{N}$. Then $\alpha_{i} \leq k+\frac{1}{2} \leq \beta_{i}$ and either $a_{i}^{\prime}=k+1, b_{i}^{\prime}=k$ or $a_{i}^{\prime}=k, b_{i}^{\prime}=k+1$. Hence also in this case $a_{i}^{\prime} \in I_{i}$ and $b_{i}^{\prime} \in I_{i}$. Since this holds for every $i \in\{1, \ldots, t\}$ and since $M$ and $N$ both have cardinality $d, M$ and $N$ are in $\mathcal{E}(\mathcal{H})$ and hence $\mathcal{E}(\mathcal{H})$ is sortable. The result now follows from [66, Theorem 14.2] and [15, Proposition 2.1].

Corollary 10.2.2. If $\mathcal{H}$ is a $d\left(I_{1}, \ldots, I_{t}\right)$-complete hypergraph, then $k[\mathcal{H}]$ is Koszul.

Proof. This follows since there is a square-free quadratic Gröbner basis, see [1].

Remark 10.2.3. The case $\mathcal{H}=K_{n}^{d}$ is the square-free Veronese and is treated by Sturmfels in [66].

Example 31. Consider the complete hypergraph $\mathcal{H}=K_{3,2}^{3(2,1)}$. Let the vertex set be $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{2}\right\}$. The edge set $\mathcal{E}(\mathcal{H})$ then is

$$
\left\{\left\{v_{1}, v_{2}, w_{1}\right\},\left\{v_{1}, v_{2}, w_{2}\right\},\left\{v_{1}, v_{3}, w_{1}\right\},\left\{v_{1}, v_{3}, w_{2}\right\},\left\{v_{2}, v_{3}, w_{1}\right\},\left\{v_{2}, v_{3}, w_{2}\right\}\right\}
$$

The ideal $\mathfrak{p}_{\mathcal{H}}$ is the kernel of the map

$$
k\left[x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}\right] \rightarrow k\left[t_{1}, t_{2}, t_{3}, t_{4}, t_{5}\right]
$$

defined by

$$
x_{1} \mapsto t_{1} t_{2} t_{4}, x_{2} \mapsto t_{1} t_{2} t_{5}, x_{3} \mapsto t_{1} t_{3} t_{4}, x_{4} \mapsto t_{1} t_{3} t_{5}, x_{5} \mapsto t_{2} t_{3} t_{4}, x_{6} \mapsto t_{2} t_{3} t_{5}
$$

Considering all sorting relations we get

$$
\mathfrak{p}_{\mathcal{H}}=\left(x_{2} x_{3}-x_{1} x_{4}, x_{2} x_{5}-x_{1} x_{6}, x_{4} x_{5}-x_{3} x_{6}\right)
$$

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