Period integrals and other direct images of D-modules –

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ISBN 978-91-7649-182-9

Printed in Sweden by Holmbergs, Malmö, Stockholm 2015
Distributor: Department of Mathematics, Stockholm University

Abstract

This thesis consists of three papers, each touching on a different aspect of the theory of rings of differential operators and D-modules. In particular, an aim is to provide and make explicit good examples of D-module direct images, which are all but absent in the existing literature.

The first paper makes explicit the fact that *B-splines* (a particular class of piecewise polynomial functions) are solutions to D-module theoretic direct images of a class of D-modules constructed from polytopes. These modules, and their direct images, inherit all the relevant combinatorial structure from the defining polytopes, and as such are extremely well-behaved.

The second paper studies the ring of differential operator on a reduced monomial ring (aka. *Stanley-Reisner* ring), in arbitrary characteristic. The two-sided ideal structure of the ring of differential operators is described in terms of the associated abstract simplicial complex, and several quite different proofs are given.

The third paper computes the monodromy of the period integrals of Laurent polynomials about the singular point at the origin. The monodromy is describable in terms of the Newton polytope of the Laurent polynomial, in particular the combinatorial-algebraic operation of *mutation* plays an important role. Special attention is given to the class of *maximally mutable* Laurent polynomials, as these are one side of the conjectured correspondance that classifies Fano manifolds via mirror symmetry.

Sammendrag

Avhandlingen består av tre artikler, som belyser forskjellige sider av teorien om ringer av diffoperatorer og D-moduler. Spesielt har det blitt lagt vekt på å gi konkrete eksempler på direkte bilder av D-moduler, for å fylle et hull i den eksisterende literaturen.

Såkalte *B-splines*, en spesiell type stykkevis polynomielle funksjoner, er løsninger til direkte bilder av en klasse D-moduler konstruert fra polytoper. Den første artikkelen gir en eksplisitt beskrivelse av disse modulene og deres direkte bilder. Modulene bevarer all den relevante kombinatoriske strukturen til de definerende polytopene, og de oppfører seg dermed veldig pent.

Den andre artikkelen handler om ringen av diffoperatorer på en redusert monomring (også kjent som en *Stanley-Reisner*-ring), også i tilfellet der grunnkroppen har positiv karakteristikk. De tosidige idealene i ringen av diffoperatorer kan beskrives ut fra det tilhørende abstrakte simplisialkomplekset, og flere fundamentalt ulike bevis gis.

I den tredje artikkelen beregnes monodromien av periodeintegraler av Laurentpolynom om det singulære punktet i origo. Monodromien kan beskrives ut fra Newtonpolygonen til Laurentpolynomet; den kombinatorisk/algebraiske operasjonen kalt *mutasjon* spiller en viktig rolle. Spesielt omhandles klassen av *maksimalt muterbare* Laurentpolynom, som er interessant fordi disse er den ene siden av den formodede korrespondansen som klassifiserer Fanomangfoldigheter via speilsymmetri.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: **B-splines, polytopes and their characteristic D-modules**
Ketil Tveiten, *Preprint, arXiv:1306.6864*, to appear in *Communications in Algebra*

PAPER II: **Two-sided ideals in the ring of differential operators on a Stanley-Reisner ring**
Ketil Tveiten, *Preprint, arXiv:1407.1643.*

PAPER III: **Period integrals and mutation**
Ketil Tveiten, *Preprint, arXiv:1501.05095.*

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Acknowledgements

First and foremost I must thank my advisor Rikard Bøgvad, for all the usual reasons; for teaching me everything I know about D-modules, for always having somewhere new to point me when I got stuck or needed a challenge, and for being such a cheerful and supportive guy.

Next in line are the organizers and participants of the PRAGMATIC Summer School held in Catania, Sicily in 2013, in particular Alessio Corti and Al Kasprzyk of Imperial College London, and Alessandro, Andrea F, Andrea P, Diletta, Liana and Mohammad. This school was the genesis not only of the third paper in this thesis, but of a productive and continuing collaboration, and of great friendship and fun times, in Catania, London, Udine and Ferrara.

The bread and butter of my time in Stockholm has of course been the people at and connected to the Department of Mathematics. My fellow inmates in room 105 in the time I've been here — Jens Forsgård, Per Alexandersson, Elin Ottergren, Christine Jost, Stefano Marseglia, Elin Gawell and the sofa in the back — have been great company. The staff and administration of the Department have been great, in particular I want to thank Samuel Lundqvist for helping me find a place to live when I moved here. The Matlag deserves a special mention, as do Jörgen and Jaja, for livening up the place and always having interesting stuff to say. Finally, I want to thank all of the above, the climbing gang, and — in no particular order — Ivan & Ornella, Christian, Björn, Håkon & Inna, Sarah, Daniel, Qimh, Pinar, Iara, Tony, Gaultier & Claire, Rune, Felix and Christoph for being great and/or entertaining company.

Finally I must thank my non-academic friends, and my family, for all the joy they bring and support they give. In particular I want to thank Martin & Anja for all their help in my early days in Stockholm; and my parents, for everything.

1. Introduction

This thesis consists of three papers, each touching on a different aspect of the algebraic theory of differential operators, and each taking advantage of some combinatorial properties of the objects under study.

The first paper was inspired by the work of De Concini and Procesi in [14] and [13], where they apply D-module techniques to study certain particular B-splines and hyperplane arrangements. Noting that a B-spline is "essentially" a D-module theoretic direct image, and that good explicit examples of D-module direct images are few and far between, spelling out the details here would plug a gap in the existing literature. Thus, the paper gives an explicit account of a family of D-modules that models polytopes and hyperplane arrangements, and whose direct images correspond to B-splines.

In the past, much work has gone into determining conditions for when a ring of differential operators is *simple*, that is, having no nontrivial two-sided ideals (see e.g. [25]). When the ring of differential operators is not simple, it is in general difficult to describe the two-sided ideals, and this is not a very well-explored problem. The aim of Paper II is to make some headway by examining what in some sense is the simplest nontrivial case, namely reduced monomial rings. These rings, also known as Stanley-Reisner rings, are in duality to abstract simplicial complexes, so it would be natural to hope for some description of the ideal structure of the ring of differential operators in terms of the associated simplicial complex. This indeed works out: two-sided ideals in the ring of differential operators are in one-to-one correspondence with certain subcomplexes. While interesting on its own, this fact is perhaps secondary to the methods used (several proofs are given), which one might hope to generalize to less well-behaved contexts.

The third paper is somewhat different in scope; it is a part of an ongoing program to classify Fano manifolds via mirror symmetry (see [1]), in particular the dimension 2 case. One says that a Fano manifold is *mirror dual* to a Laurent polynomial if a certain differential operator constructed via quantum orbifold cohomology is equal to the differential operator that annihilates a particular period integral of the Laurent polynomial. It is known in dimension 2 that every Fano manifold is dual to a Laurent polynomial, i.e. the assignment "polynomials $\mapsto$ manifolds" is surjective (up to deformation); Paper III is concerned with gathering evidence for injectivity of this assignment, namely showing that nonequivalent Laurent polynomials produce nonequivalent differential operators. The paper computes, given a polynomial, the monodromy of the associated differential operator at the origin; this is almost enough to distinguish inequivalent polynomials, and some conjectures are made about how to find the remaining information.

1.1 Rings of differential operators and D-modules

Let us begin by giving a quick summary of the beasts we about to slay: rings of differential operators; and D-modules, which are modules over those rings. For the interested reader, further material and greater detail can be found e.g. in [5], [6], [10], [15], [17], [18], [22], and [23]. For the remainder of this section, P_n will denote the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, I will be an ideal in P_n, and $R = P_n/I$ is the quotient ring. The commutator of two elements in a ring is denoted by the bracket $[a, b] := ab - ba$.

Think first of P_n; we can differentiate polynomials by applying the partial differential operators $\partial_i := \frac{\partial}{\partial x_i}$, which act by $\partial_i(c) = 0$ for $c \in \mathbb{C}$,

$$\partial_i(x_j) = \begin{cases} 1 & i = j \\ 0 & j \neq i, \end{cases}$$

and satisfy the *Leibniz rule*: $\partial_i(f \cdot g) = f\partial_i(g) + \partial_i(f)g$, for any $f, g \in P_n$. This formula can be rewritten as

$$(\partial_i f - f\partial_i)g = (\frac{\partial f}{\partial x_i})g$$

or in other words $[\partial_i, f] = \frac{\partial f}{\partial x_i}$, thought of as operators acting on P_n. Extending P_n with variables ∂_i obeying these relations, we get the *Weyl algebra*, which is the ring of *differential operators acting on* P_n.

Definition 1.1.1. The *Weyl algebra* is the ring $\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$, where the variables commute except for the relation $[\partial_i, x_i] = 1$.

We will denote the Weyl algebra by W, or if it is important how many variables we work with, we let e.g. $X := \mathbb{C}^n, Y := \mathbb{C}^m$ and denote the Weyl algebras in n, m variables by D_X, D_Y respectively, emphasizing the connection to the geometry of affine space. Taking this a step further, if

X is a smooth variety (or complex manifold), we have associated to the sheaf of reglar functions $\mathcal{O}_X$ a sheaf of rings of differential operators $\mathcal{D}_X$, with $\mathcal{D}_X(U) = D(\mathcal{O}_X(U))$ for each affine open set U. In particular, for $U \simeq \mathbb{C}^n$, we have $\mathcal{D}_X(U) \simeq W$. We will switch back and forth between these notations as appropriate in what follows.

It is not immediately clear what a "differential operator" acting on $R = P_n/I$ should be, but there is a property of the Weyl algebra that generalizes in the right way. The Weyl algebra has a natural filtration by the order of the differential operator: ∂_i^k has order k, x_j has order zero, and this clearly respects multiplication. Now observe that the Leibniz rule implies that if δ has order k and $f \in P_n$ is any element, then $[\delta, f]$ has order $k-1$. With this in mind, Grothendieck gave the following definition:

Definition 1.1.2. Let k be an integral domain, let A be a k-algebra, let $D_k^0(A) := A$, and let $D_k^r(A) := \{\phi \in End_k(A) | \forall f \in A : [\phi, f] \in D_k^{r-1}(A)\}$. The *ring of k-linear differential operators on A* is the ring

$$D_k(A) = \bigcup_{r \geq 0} D_k^r(A).$$

In particular, for any finitely generated k-algebra R, i.e. a quotient $R = P_n/I$ of a polynomial ring, it can be shown that

$$D(R) = \{\delta \in W | \delta(I) \subset I\}/IW, \tag{1.1.1}$$

and this description does not depend on the presentation $R = P_n/I$.

The theory of D-modules is principally the theory of the modules of the Weyl algebra, so when we say "D-module" this is what we mean unless otherwise specified. The motivation originally sprung from the study of linear differential operators: $\delta \in W$ is a linear differential operator and $f_0 \in P_n$ a solution to the differential equation

$$\delta(f) = 0$$

then the W-module $W \cdot f_0$ is isomorphic to the module $W/Ann(f_0)$, where $Ann(f_0)$ is the left ideal in W of operators that annihilate f_0. In this sense, solving differential equations in P_n is equivalent to studying left ideals in W[1]. It soon turned out that D-modules were not merely

[1]The analytical situation is equivalent: replace P_n by $\mathcal{O}$, the ring of holomorphic functions in n variables, and W by $D(\mathcal{O}) = \mathcal{O}\langle \partial_1, \dots, \partial_n \rangle$, and essentially the same results are true (the proofs may be very different, however).

an algebraic tool to study analytical problems, but an interesting theory in their own right.

What follows are some of the most important results of the theory. The dimension of a D_X-module is defined via the Hilbert polynomial in the usual way: the Weyl algebra has a natural filtration by total degree, i.e. $x^a \partial^b$ has degree $|a| + |b|$. This gives an associated graded ring and to each D_X-module M an associated $gr(D_X)$-module $gr(M)$, which has a well-defined Hilbert polynomial $p(n) = dim_k(gr(M)_n)$. The *dimension* of M is the degree of the Hilbert polynomial.

Theorem 1.1.3 (Bernstein,[4]). *Let M be a nonzero D-module, then the dimension is constrained by $n \leq dim(M) \leq 2n$.*

The inequalities are sharp: $dim(W) = 2n$ and $dim(P_n) = n$, and there exist modules of every intermediate dimension. Those D-modules with $dim(M) = n$ are particularly interesting, and are called *holonomic* D-modules. In particular, the *regular holonomic*[1] D-modules are extremely well-behaved; these include "most" of the D-modules one meets in applications, e.g. modules of the form $W \cdot f$ for $f \in P_n$. In particular every holonomic D-module is cyclic, i.e. generated by a single generator. The celebrated theorem of Kashiwara tells us that regular holonomic D-modules respect the geometry of their support exactly (the analogous statement for $\mathcal{O}_X$-modules is very far from being true).

Theorem 1.1.4 (Kashiwara's Theorem,[18]). *Let X, Y be smooth schemes, and let $i : Y \hookrightarrow X$ be a closed immersion. Then the direct image functor i_+ gives an equivalence of categories between the category of regular holonomic $\mathcal{D}_Y$-modules and the category of regular holonomic $\mathcal{D}_X$-modules supported on Y.*

The keystone and central result of the theory of D-modules is the *Riemann-Hilbert correspondence*, which links D-modules, topology and representation theory.

Theorem 1.1.5 (Riemann-Hilbert correspondence). *Let X be a complex manifold. The bounded derived category of regular holonomic $\mathcal{D}_X$-modules is equivalent to the bounded derived category of perverse sheaves on X.*

The sketch proof is as follows: locally a $\mathcal{D}_X$-module has a vector space of solutions, these glue together to form a local system, and the

[1]The definition of regularity here is quite technical, so I will not bore you with it.

gluing data of a local system is equivalent to a representation of $\pi_1(X)$. The full proof fills half a book and is quite technical (see e.g. [6, chap. VI-VIII]). We need to talk about direct images, and so we will need one of these technical devices.

Definition 1.1.6. Let M be a D_X-module. The *de Rham complex* $DR_X(M)$ of M is the cochain complex

$$M \to \Omega^1_X \otimes M \to \Omega^2_X \otimes M \to \cdots \to \Omega^n_X \otimes M$$

where the differential is given by $d(\omega \otimes m) = d\omega \otimes m + \sum_i dx_i \wedge \omega \otimes \partial_i m$.

Observe that for $M = \mathcal{O}_X$, the de Rham complex $DR_X(\mathcal{O}_X)$ is the same as the usual algebraic de Rham complex, and so we may speak of de Rham cohomology of D-modules. The de Rham complex plays a central role in the theory, indeed it is the content of the Riemann-Hilbert correspondence, the general statement of which is that the functor $DR_X(-)$ (from the bounded derived category of $\mathcal{D}_X$-modules to the bounded derived category of perverse sheaves on X) is an equivalence.

To introduce direct images, we restrict to the simplest cases: inclusions of and projections to subspaces. These are the only cases we will need for explicit computations.

Definition 1.1.7. Let $X \simeq Y \times Z$, and let $i : Z \hookrightarrow X$, $\pi : X \twoheadrightarrow Y$ be the canonical inclusions and projections, respectively. Let N be a D_Z-module, and let M be a D_X-module. Then

1. the direct image i_+N is defined to be $N \otimes \mathcal{O}_Y$, and

2. the direct image π_+M is defined to be $DR_{X/Y}(M)$, where $DR_{X/Y}$ is the relative de Rham complex $\Omega^\bullet_Z \otimes M$ (with only differentials in the ∂_z's and not the ∂_y's).

Note that the direct image is not a D_Y-module, but a chain complex of D_Y-modules; in other words the direct image lives in the derived categories. While the functors f_+ are well-defined as stated above, to satisfy the properties one expects the direct image to satisfy (e.g. composition $(f \circ g)_+ = f_+ \circ g_+$, preservation of coherence and holonomicity), some assumption of properness is often necessary; either that $f : X \to Y$ is proper, or at least that $f|_{Supp(M)}$ is. See for instance Paper III, where the relevant morphism is proper, or Paper I, where it is proper when restricted to the support of the module in question.

Kashiwara's theorem makes direct images of inclusions rather trivial, so let us consider the projection $\pi : X = Y \times Z \twoheadrightarrow Y$. For $\mathcal{O}$-modules, the usual direct image $\pi_* M$ corresponds to the forgetful functor from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules (any $\mathcal{O}_X$-module is an $\mathcal{O}_Y$-module via the canonical map $\mathcal{O}_Y \to \mathcal{O}_X$), so the "functions" in the direct image on some open set are the functions defined on the inverse image of that set. For D_X-modules on the other hand, the elements of the direct image are more like the functions defined by integration over the fibers of the map; i.e. if $f(x) = f(y,z)$ is a solution to a D_X-module, then

$$(\pi_* f)(y) = \int_{\pi^{-1}(y)} f(y,z)dz \tag{1.1.2}$$

is a solution to the direct image module. I say "more like", as this is the kind of analytic construction the direct image is intended to model; precisely how well this intuition works is one of the topics of Paper I.

1.2 Distributions, B-splines, and polyhedral cell complexes

A common technique in the study of differential equations is looking at *weak solutions*, that is, instead of studying functions $f(x)$ e.g. on the real line, one studies integrals

$$\int_{-\infty}^{\infty} f(x)\phi(x)dx$$

where $\phi(x)$ is a *test function*, a smooth function with compact support. The point is that as ϕ has compact support, $\int_{-\infty}^{\infty} f(x)\partial_x\phi(x)dx = 0$, and so the integral will satisfy the same differential equations as f. Taking this idea to its logical conclusion, one shifts the viewpoint from studying *functions* to studying *linear operators* that act on the space of test functions, which lets us solve more differential equations than can be done with only smooth functions. Such operators are called *distributions*, and via the embedding $f(x) \mapsto \int f(x) \cdot - dx$ the set of distributions includes the functions we initially cared about. One often denotes the action of a distribution g on a test function $\phi(x)$ by a bracket $\langle g|\phi\rangle$. The reader who is interested in the analytic theory of distributions is referred to [16], though here the algebraic aspects are of course the important ones.

Distributions have a natural D-module structure, given by the action

$$\langle p(x)\partial^\alpha \cdot g|\phi(x)\rangle = \langle g|(-1)^{|\alpha|}\partial^\alpha(p(x)\phi(x))\rangle,$$

here $p(x)$ is some polynomial and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. In fact, it is a theorem of Kashiwara ([17]) that any regular holonomic D-module is generated by a distribution.

Example 1.2.1. Aside from smooth functions, distributions also encompass piecewise smooth functions, e.g.

$$x_+ = \begin{cases} x & x > 0 \\ 0 & x < 0 \end{cases}$$

is the distribution $x_+ : \phi(x) \mapsto \int_0^\infty x\phi(x)dx$; piecewise continous functions, e.g. the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

is the distribution $H : \phi(x) \mapsto \int_0^\infty \phi(x)dx$; the Dirac delta operator, defined by $\delta(f) = f(0)$ is a distribution. In fact, $\partial_x(x_+) = H$ and $\partial_x(H) = \delta$.

For numerical applications, an important class of functions are the *B-splines*, which are piecewise polynomial functions given by projecting a polytope in some high-dimensional space $\mathbb{R}^m$ to some lower-dimensional space $\mathbb{R}^s$, and taking the volume of the fibers over each point; see e.g. [8], [12] and [14]. Suppose $\sigma \subset \mathbb{R}^m$ is such a polytope, and $\pi : \mathbb{R}^m \to \mathbb{R}^s$ is the projections, then the associated B-spline defined on $\mathbb{R}^s$ is given by

$$\pi_*\delta_\sigma(x_1, \ldots, x_s) = vol(\pi^{-1}(x_1, \ldots, x_s) \cap \sigma)$$

or written another way

$$\pi_*\delta_\sigma(x_1, \ldots, x_s) = \int_{\sigma \cap \pi^{-1}(x)} dx_{s+1} \cdots dx_m.$$

This resembles the "solutions" to a direct image D-module, as in 1.1.2, so to answer the question of whether it actually is, Paper I begins by studying the obvious candidate: the distribution given by the characteristic function of σ. Let us introduce some notation: the affine hull of σ is denoted by H_σ, the facets of σ are $\sigma_1, \ldots, \sigma_r$ and the outward normal unit vectors of the facets are denoted by $n_1, \ldots n_r$

Definition 1.2.2. Let σ be a polytope[1] in $\mathbb{R}^m$. The *characteristic distribution* of σ is the distribution δ_σ defined by

$$\delta_\sigma(\phi) = \int_\sigma \phi dx$$

where dx is the restriction to H_σ of the standard measure on $\mathbb{R}^m$.

[1]By *polytope* I mean a closed contractible semialgebraic set defined by linear polynomials

From Stokes' theorem we have the following relations, which we call the *standard relations.*

Proposition 1.2.3 (Standard relations)**.** *(i) If* $\dim(\sigma) > 0$*, then for any directional derivative* ∂_v *where* v *is a vector tangent to* H_σ*, we have*

$$\partial_v \cdot \delta_\sigma = -\sum_i \langle v|n_i\rangle \delta_{\sigma_i}.$$

(ii) Let $I(\sigma)$ *denote the defining ideal of* H_σ*. For any* $p \in I(\sigma)$*, we have*

$$p \cdot \delta_\sigma = 0.$$

As σ *is a polyhedral body,* H_σ *is an affine space defined by* $m - \dim(\sigma)$ *equations of degree 1, and the corresponding polynomials generate* $I(\sigma)$*.*

In light of the standard relations, we see that to study δ_σ we must also study the facets δ_{σ_i}, their facets, etc; indeed the whole cell complex $\widehat{\sigma}$ made up of the faces of σ. In fact, it is convenient to consider cell complexes with polyhedral cells in general, formed by gluing together polytopes σ by their facets.

For the remainder of this section, we let X, Y and Z denote $\mathbb{R}^m, \mathbb{R}^s$ and $\mathbb{R}^{m-s}$ respectively (here $s < m$), and D_X, D_Y the Weyl algebras in m and s variables. Let $K = \bigcup \sigma$ be such a cell complex; we define the *characteristic module* of K to be the module

$$M_K := D_X \cdot \{\delta_\sigma | \sigma \subset K\}$$

generated by the characteristic distributions of all the cells in K. It is not hard to see from the standard relations that the de Rham cohomology of M_K is related to the homology of K.

Theorem 1.2.4. *The de Rham complex* $DR_X(M_K)$ *of* M_K *is quasi-isomorphic to the closed-support homology*[1] *chain complex* $C^{\mathrm{c}}_\bullet(K)$ *of* K*.*

The paper contains an algorithm to compute the D_X-annihilator ideal of δ_σ that works via the algebraic Laplace transform, and an explicit calculation shows a desirable but not obvious result:

Theorem 1.2.5. *The module* $M_{\widehat{\sigma}}$ *is generated by* δ_σ*.*

[1]Homology with closed support is also known as *Borel-Moore homology.*

The corollary is that M_K can be generated by those generators corresponding to (locally) maximally-dimensional cells.

The sequence of subcomplexes K_i made up of all the cells of dimension $\leq i$ induce a filtration $F_0 \subset F_1 \subset \cdots F_{dim(K)} = M_K$ which we call the *skeleton filtration*. The skeleton filtration is a very useful tool with which we can construct a canonical presentation (with generators and relations) of M_K as a D_X-module, prove that the standard relations generate all the relations between the generators, that M_K is regular holonomic, and that the functor $K \mapsto M_K$ also preserves the cell complex structure: it preserves the poset of subcomplexes and the gluing data of how the cells are attached to eachother.

Recall from 1.1.7 that the direct image of a projection is given by $\pi_+ M = DR_{X/Y}(M)$. We are primarily interested in the lowest-degree term of this complex, $\pi_+^0 M$, as this is where "functions" live (the higher-degree terms of the complex encode cohomological information). If we write $X = Y \times Z$ as before, with coordinates y_i, z_j on each factor, we can express this module as

$$\pi_+^0 M \simeq M / \sum_j \partial_{z_j} M.$$

Geometrically, this corresponds to "flattening" the cells in the "vertical" direction. Taking this quotient induces standard relations for the direct image module, which appear identical to the set of recurrence relations for the B-spline $\pi_* \delta_\sigma$ defined by σ, proven by De Boor and Höllig [12]. These recurrence relations are the defining relations of a D_Y-module: We denote the D_Y-module generated by the B-splines defined by the faces of σ by $S_K := D_Y \cdot \{\pi_* \delta_\tau | \tau \subset \sigma\}$. From this the main result of the paper follows, here "1-elementarily equivalent" is a slightly technical but fairly mild topological condition.

Theorem 1.2.6. *There is a canonical surjective map $\pi_+^0 M_K \twoheadrightarrow S_K$, that is an isomorphism is K is 1-elementarily equivalent to a cell complex K' with connected fibers.*

1.3 Face rings and simplicial complexes, and ideals in rings of differential operators

In Section 1.1 we discussed the properties of the Weyl algebra and the ring of differential operators on a smooth variety, which locally looks like the Weyl algebra. If the variety is singular however, the ring $D(R)$ may behave quite differently from the Weyl algebra. Notice that the

Weyl algebra is a *simple ring*, that is it has no nontrivial two-sided ideals. This is due to the Leibniz rule $[\partial_i, x_i] = 1$, which implies that whatever elements you try to generate an ideal with, you can act with commutators — the commutator $[a, b]$ is of course in both the two-sided ideals generated by a and b — until you get a unit. This may no longer hold when R is not regular, as we will see.

Let $R = P_n/I$, then recall from 1.1.1 that

$$D(R) = \{\delta \in W | \delta(I) \subset I\}/IW.$$

The important part here is that ∂_i might not preserve I, so there may be some room for two-sided ideals to live. Take e.g. $I = \langle x_1 x_2 \rangle$ in $\mathbb{C}[x_1, x_2]$; here $D(R)$ is generated by operators of the form $x_i^a \partial_i^b$ with $a \geq 1$, so the ∂_i are not in $D(R)$. By explicit computation we can show that there are three nontrivial two-sided ideals in $D(R)$: the ideals $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_1, x_2 \rangle$.

What ideals occur and how they are related can of course get very complicated, so in Paper II the idea is to study the simplest nonregular case, namely reduced monomial rings. These rings are also known as *Stanley-Reisner rings* and have powerful combinatorial structure encoded by abstract simplicial complexes.

Definition 1.3.1. Let K be an abstract simplicial complex on vertices $x_1, \ldots, x_m$. Then the *face ideal* I_K in the ring $\mathbb{C}[x_1, \ldots, x_m]$ is given by

$$I_K = \langle x_{i_1} \cdots x_{i_r} | \{x_{i_1}, \ldots, x_{i_r}\} \text{ is } \textit{not} \text{ a subset of } K \rangle.$$

The ring $R_K := \mathbb{C}[x_1, \ldots, x_m]/I_K$ is the *face ring* or *Stanley-Reisner ring* of K.

Returning to our example $I = \langle x_1 x_2 \rangle$, this is the face ideal of the simplicial complex with two isolated vertices x_1 and x_2, and we can notice that the ideals in $D(R)$ each correspond to a subcomplex. The idea of Paper II is to see what information about the ideals of $D(R_K)$ can be read off the simplicial complex K; it turns out the ideals are determined by an important class of subcomplexes called the *stars* of simplices.

Definition 1.3.2. Let $\sigma \in K$ be a simplex. The *star* of σ is the subcomplex

$$st(\sigma) := \{\tau \in K | \tau \cup \sigma \in K\}.$$

The assignment $\sigma \mapsto st(\sigma)$ is inclusion-reversing.

Example 1.3.3. Let K be the complex on vertices x, y, z with face ideal $\langle xz \rangle$. It is a union of two 1-simplices, $\{x, y\}$ and $\{y, z\}$, joined at a vertex, y. The subcomplexes that are stars are $st(x) = st(\{x, y\}) = \{x, y\}$, $st(z) = st(\{y, z\}) = \{y, z\}$, and $st(\varnothing) = st(y) = K$.

Example 1.3.4. Let K be the n-simplex Δ_n. For any simplex $\sigma \in K$ we have $st(\sigma) = K$. The face ideal of K is the zero ideal, and the face ring is the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$.

For each $\sigma \in K$, we denote $x_\sigma := \prod_{x_i \in \sigma} x_i$. We define the *support* of a monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ to be the nonvanishing locus of the set of x_i such that $a_i \neq 0$, i.e. the set of variables that appear in the monomial. It is not hard to see that the closure of the support of any monomial is the star of the simplex consisting of those x_i that appear in the monomial.

A result of Traves [26] gives an explicit description of which monomials $x^a \partial^b$ that appear in $D(R)$; in Paper I it is shown that there is an equivalent formulation in terms of stars.

Proposition 1.3.5. *Suppose $supp(x^a) = st(\sigma)$ and $supp(x^b) = st(\tau)$. Then $x^a \partial^b \in D(R)$ if and only if $st(\sigma) \subset st(\tau)$.*

Using this and some explicit computation it is possible to show that $\langle x_\sigma \rangle \subset \langle x_\tau \rangle$ if and only if $st(\sigma) \subset st(\tau)$, and a key result:

Theorem 1.3.6. *Any two-sided ideal in $D(R_K)$ is generated by monomials x_σ with $st(\sigma) \neq K$, and the ideals $\langle x_\sigma \rangle$ generate (by sums and intersections) the lattice of ideals.*

From this follows the main result of the paper:

Theorem 1.3.7. *The lattice of two-sided ideals in $D(R_K)$ is exactly the lattice of subcomplexes of K generated (by sums and intersections) by the stars of simplices in K.*

While so far all this is valid in any characteristic, in characteristic p there is another quite different proof, based on the Frobenius automorphism: let $q = p^r$, and let the subring of q'th powers be denoted by R^q. Then R is an R^q-module in a natural way, and it is a theorem of Yekutieli [27] that

$$D(R) = \bigcup_q End_{R^q}(R).$$

Here $End_{R^q}(R)$ are the R^q-linear endomorphisms of R. In our case, we can write

$$R \simeq \bigoplus_{st(\sigma) \subset K} (R^q_{st(\sigma)})^{m_\sigma}$$

where the sum is over all subcomplexes that are stars of some simplex (including the empty one), $R_{st(\sigma)}$ is the face ring of $st(\sigma)$ and m_σ are some multiplicities depending on q. From this, we get

$$D(R_K) = \bigcup_q \bigoplus_{st(\sigma), st(\tau)} Hom_{R^q}((R^q_{st(\sigma)})^{m_\sigma}, (R^q_{st(\tau)})^{m_\tau})$$

and by explicit computation prove that for each $st(\sigma) \subset K$ the ideals generated by $id : (R^q_{st(\sigma)})^{m_\sigma} \to (R^q_{st(\sigma)})^{m_\sigma}$ (for each q) are equal, and that these ideals — I call them $J(st(\sigma))$ — are the only nontrivial ones.

Theorem 1.3.8. *The ideals $J(st(\sigma))$ generate the lattice of two-sided ideals in $D(R_K)$.*

In fact, by a support argument we can see that $J(st(\sigma)) = \langle x_\sigma \rangle$.

The paper also includes a discussion of *D-stable ideals*, namely those ideals of R that are sub-$D(R)$-modules. In [26], Traves describes the D-stable ideals of R_K as those formed by sums and intersections from the minimal primes, and we obtain a new proof of this by applying the following observation.

Proposition 1.3.9. *Let $J \subset R$ be an ideal, then J is D-stable if and only if it is the restriction of a two-sided ideal in $D(R)$.*

1.4 Toric geometry, mirror symmetry, and mutations

We must introduce some basics of toric geometry. Sufficient for the purposes of Paper III is the two-dimensional case, and I will entirely skip the usual talk about torus actions as that is not relevant here; for a thorough introduction to toric geometry see e.g. [11]. For this section, we consider everything (lattices, polygons, polynomials) up to $GL_2(\mathbb{Z})$-change of variables.

Consider the lattice $\mathbb{Z}^2$, and pick two primitive elements a, b, and let $\sigma := \mathbb{R}_+\langle a, b\rangle \subset \mathbb{R}^2$ be the convex cone they span. The set of lattice points contained in this cone, $\sigma \cap \mathbb{Z}^2$, form a semigroup N_σ, generated by the lattice points contained in the triangle formed by a, b and the origin (including those on the boundary). If $\{a, b\}$ is a lattice basis, then $N_\sigma \simeq \mathbb{N}^2$, and the semigroup ring $\mathbb{C}[N_\sigma]$ is isomorphic to the polynomial ring $\mathbb{C}[x_1, x_2]$; otherwise $\mathbb{C}[N_\sigma]$ is nonregular, having a *cyclic quotient singularity.* The spectra X_σ of the rings $\mathbb{C}[N_\sigma]$ are the archetypical *affine toric varieties.* Now partition the plane $\mathbb{R}^2$ into cones of this form, so that two cones σ_1, σ_2 intersect only in the ray spanned by a common generator. Such an arrangement is called a *fan*, and we denote it by $\Sigma := \bigcup \sigma_i$. One often describes fans by the rays that span the cones, i.e. a fan is equivalent to the list of vectors that are primitive generators for the cones in the fan. The affine toric varieties X_{σ_i} can be glued together to form a variety X_Σ; this is a *projective toric variety.* The gluing goes as follows: if ρ is the common ray of σ_1 and σ_2, then the open sets where $\rho \neq 0$ in $X_{\sigma_1}, X_{\sigma_2}$ are identified.

Example 1.4.1. The fan Σ with generators $(1,0),(0,1)$ and $(-1,-1)$ gives $X_\Sigma \simeq \mathbb{P}^2$.

Most important to us is varieties constructed in this way from the *normal fan* of a convex lattice polygon. Let P be a convex lattice polygon with vertices at primitive lattice points, with the origin as a strict interior point — such a polygon is called a *Fano polygon*[1]. The inward normal vectors of the edges form a complete fan Σ_P, and so give a projective toric variety X_P. This variety contains all the information to reconstruct P, indeed there is a canonically defined divisor D_P on X_P given by

$$D_P = \sum h_i D_{P_i}$$

where the D_{P_i} are the divisors corresponding to the edges of P_i and h_i are the lattice heights of those edges. The crucial thing for us is that the global sections of D_P correspond to the Laurent polynomials with Newton polygon equal to P (see [11, 4.3.3]).

Given a Fano polygon P and a Laurent polynomial $f = \sum_{m \in P} a_m x^m$ with $Newt(f) = P$, the *classical period* of f is the integral

$$\pi_f(a,t) = \int_C \frac{1}{1 - tf(a,x)} \omega,$$

where C is the cycle $|x_1| = |x_2| = \varepsilon$, and ω is the invariant volume form on $(\mathbb{C}^*)^2$ normalized to give $\int_C \omega = 1$; we think of the coefficients a_i as parameters. The period $\pi_f(a,t)$ is a multivalued holomorphic function in a punctured disk about the origin, and we denote by $L_f \in \mathbb{C}\langle t, \nabla_t \rangle$ (here $\nabla_t = t\partial_t$) the minimal-order minimal-degree differential operator that kills it.

It is possible to explicitly compute L_f by various kinds of expensive computations (see e.g. [21]), but this is not particularly enlightening. More useful for purposes of proving things is the description of L_f as the direct image module of the $\mathcal{D}_{X_P}$-module generated by f, under the

[1]The associated variety X_P will be a Fano manifold whenever P is Fano; a manifold is *Fano* if its anticanonical bundle is ample.

birational map $\tau : X_P \dashrightarrow \mathbb{P}^1$ given by $t = \frac{1}{f}$.[1] This gives us via some general theory and the Riemann-Hilbert correspondence a nice description of the solution sheaf $Sol(L_f)$: it is the local system on $\mathbb{P}^1$ with stalk $H_1(X_t, \mathbb{Z})$ at $t \in \mathbb{P}^1$, where X_t is the fiber of τ at t. It is in general very hard to find all the singular points of τ and to find a good description of the fibers there, but at $t = 0$ we have that $X_0 \simeq D_P$, and from this it is easy to find a model for the general fiber X_t. Once we have the general fiber it is easy[2] to find the monodromy matrix at $t = 0$, which contains very much of the interesting information about L_f.

Apparently unrelated, given a Fano manifold X with cyclic quotient singularities that admits a $\mathbb{Q}$-Gorenstein degeneration (or *qG-degeneration* for short) to a toric variety, one can construct via quantum orbifold cohomology a differential operator $Q_X \in \mathbb{C}\langle t, \nabla_t \rangle$ (see [24]).

We say that X and f are *mirror dual* if $L_f = Q_X$. Both these differential operators are invariant under certain deformations; L_f is unchanged by *mutation* of f (I will define this below), while Q_X is unchanged by qG-deformation. It is thus a natural question whether the equivalence classes are in bijection, as this would reduce the problem of classifying Fano varieties (up to qG-deformation) to the much simpler problem of classifying Fano polytopes with certain Laurent polynomials on them; this is the conjectured *mirror symmetry classification of Fano manifolds* (see [1], [2], [7]). Some progress has been made on proving this conjecture:

Theorem 1.4.2 ([1])**.** *The assignment $P \mapsto X_P$ is a surjective map from the set of mutation-equivalence classes of Fano polygons to the set of qG-deformation-equivalence classes of Fano surfaces.*

Why does this result talk merely about the polygons and not the Laurent polynomials on them? There is a class of *maximally mutable* Laurent polynomials that in a precise sense capture all the relevant information about P, and it is these that are the interesting ones in this context (see Paper III and [20]).

The simplest cases of the general conjecture are proven, see [1], [9], [19], and [24]:

Theorem 1.4.3. *The set of qG-deformation classes of Fano surfaces that are smooth or have only cyclic quotient singularities of type $\frac{1}{3}(1,1)$ (respectively) is in bijection with the set of mutation-equivalence classes*

[1]Strictly speaking one resolves singularities of X_P and base points of f to get a morphism, and works with that instead.

[2]See Paper III to evaluate how big you think this understatement is.

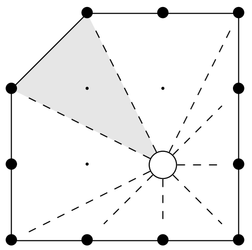

Figure 1.1: A polygon with 8 T-cones and one R-cone (grey) of type $\frac{1}{3}(1, 1)$.

of maximally mutable Laurent polynomials on Fano polygons that are smooth or have only R-cones of type $\frac{1}{3}(1, 1)$ *(respectively).*

Let P be a Fano polygon as before, and consider an edge E of lattice height h and lattice width w. We can write $w = hk + r$ for some $k \geq 0$ and $\leq r < h$, subsequently we can subdivide the cone spanned by E over the origin into k cones of width h and a cone of width r. A cone of width equal to the height is called a *T-cone*, and one with width less than the height is called an *R-cone*. From these we get a mutation-invariant measure of P called the *singularity content*, defined as the pair $(k, \mathcal{B})$, where k is the number of T-cones in P, and $\mathcal{B}$ is the set of R-cones (these concepts and terminologies are from [3]).

Example 1.4.4. The polygon pictured in Figure 1.1 has 8 T-cones, unshaded, and one R-cone of type $\frac{1}{3}(1, 1)$, shaded grey. The singularity content of the polygon is $(8, \{\frac{1}{3}(1, 1)\})$.

So, what is this *mutation* of which I speak? Suppose an edge E has a T-cone of height h. Then the mutation of P with respect to that cone is given by removing from each positive height $0 < l \leq h$ a slice of length l, and adding at each negative height $-l$ a slice of length l, as illustrated in Figure 1.2.

This amounts to removing the T-cone on E, and adding a T-cone on the opposite side. In this way one sees that the number of T-cones is preserved by mutation; the fact that mutation with respect to an R-cone does not make sense as defined here shows that the set of R-cones too is invariant. The concept of mutation originally appeared in [2].

On Laurent polynomials there is a corresponding operation, provided the coefficients satisfy some extra conditions. The mutation of P above can be described in terms of a map on the ambient lattice of the normal fan, that induces a map ϕ on the ring of rational functions $\mathbb{C}(\mathbb{N}^2)$, given

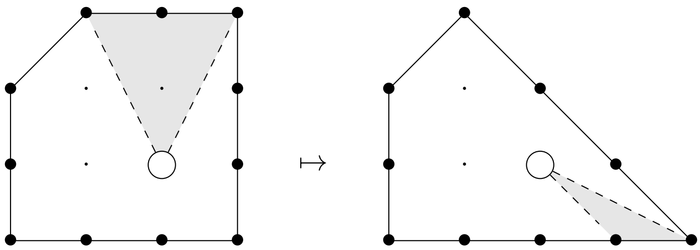

Figure 1.2: Mutation of a Fano polygon: A T-cone is contracted, and a new T-cone is inserted on the opposite side.

in suitable coordinates by $(x, y) \mapsto (x(1 + y), y)$. This map does not preserve Laurent polynomials, but we say that a Laurent polynomial f is *mutable with respect to the mutation* $P \mapsto P'$ if the image $\phi(f)$ is a Laurent polynomial. Furthermore, f is *maximally mutable* if it is mutable with respect to any mutation of P.[1]

Stated explicitly, this means that the slice f_r of monomials "at height r" must be divisible by $(1+x)^r$ at each positive height r, so the mutation can be described by

$$f_r \mapsto f_r(1 + x)^{-r},$$

i.e. at positive heights r, we divide away the factor $(1 + x)^r$, and at negative heights $-r$ we multiply by $(1 + x)^r$.

Example 1.4.5. Let $f = \frac{y^2}{x} + 2y^2 + xy^2 + \frac{1}{y}$. The Newton polygon has vertices $(-1, 2), (1, 2)$ and $(0, -1)$; the cone spanned by $(0, 0), (-1, 2)$ and $(1, 2)$ is a T-cone. Observe that we can write $f = \frac{y^2}{x}(1+x)^2+0(1+x)+\frac{1}{y}$, so f admits a mutation corresponding to this cone. The mutated polygon has vertices $(-1, 2), (1, -1)$ and $(0, -1)$, and the mutated polynomial is $\phi(f) = \frac{y^2}{x} + \frac{1}{y} + \frac{x}{y}$.

It is well known that a generic Laurent polynomial gives a curve of genus equal to the number of internal lattice points of P. Imposing the mutability condition for a T-cone of height h is equivalent to the curve defined by f having a multiple point of multiplicity h. This drops the genus by $\frac{1}{2}h(h-1)$, which is equal to the number of internal lattice points in the T-cone. Adding all this together we get an important result:

Theorem 1.4.6. *The genus of a maximally mutable Laurent polynomial with $Newt(f) = P$ is equal to the number of internal lattice points of P not lying internal to a T-cone.*

[1]The correct definition in higher dimensions is complicated, see [20].

We call this number the *mutable genus*, g_{mut}, of P; it is clearly mutation-invariant. The Cauchy-Kowalewski theorem now implies that the order (in ∇_t) of L_f and the rank of $Sol(L_f)$ both are $2g_{mut}$.

To compute the monodromy of L_f, we deform the fiber X_0 at zero to get the general fiber X_t, and find the monodromy via a number of local computations. These come in two kinds: near the intersection points between components of X_0, corresponding to the vertices of P, the special fiber looks in local coordinates like $\{x^m y^n = 0\}$ (here m, n are the multiplicities of the intersecting components); this deforms to the smooth curve $\{x^m y^n = t\}$. The other kind is deforming the components corresponding to the edges of P with R-cones on them; over such components the general fiber has positive genus, and so there are a number of ramification points to resolve. What remains is then to construct a model for an automorphism of a surface of the right genus, with the required order and number of fixpoints (and we can show that any such models are equivalent), and some power (we can find which) of this automorphism is the right one.

Paper III is a part of the effort to prove the simplest non-smooth case of the mirror symmetry correspondence, namely the case of only $\frac{1}{3}(1,1)$-singularities, i.e. a singularity content of $(k, \{n \times \frac{1}{3}(1,1)\})$.

Theorem 1.4.7. *Let P be a Fano polygon with singularity content $(k, \{n\times \frac{1}{3}(1,1)\})$, and let X_t be defined using a generic maximally mutable Laurent polynomial f with $Newt(f) = P$. Then there is a basis of cycles $\{\alpha, \beta, a_1^1, a_2^1, \ldots, a_1^n, a_2^n\}$ in $H_1(X_t, \mathbb{Z})$ such that in terms of this basis, the monodromy automorphism ω of $H_1(X_t, \mathbb{Z})$ is given by*

- $\omega(\alpha) = \alpha + (k + 2n - 12)\beta - \sum_{j=1}^{n} a_2^j$,
- $\omega(\beta) = \beta$,
- $\omega(a_1^j) = a_2^j$ *for* $1 \leq j \leq n$, *and*
- $\omega(a_2^j) = \beta - a_1^j - a_2^j$ *for* $1 \leq j \leq n$.

In fact the methods in Paper III describe the monodromy at zero for every possible singularity content, up to some minor ambiguity for R-cones of width $w \geq 2$. This ambiguity does not affect most important information:

Theorem 1.4.8. *The monodromy at $t = 0$ of L_f determines and is determined by the singularity content of $Newt(f)$.*

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