Digital Geometry, Combinatorics,

and Discrete Optimization

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The cover image shows a Persian chessboard with a king. Henri-Auguste Delannoy (1833–1915) published in 1895 an article with the title “Emploi de l’échiquier pour la résolution de certains problèmes de probabilités” (Use of the chessboard for the solution of certain problems in probability), *Association Française de Bordeaux* **24**, 70–90, where he introduced the numbers which are now known as the Delannoy array (see Paper III) and which describe the possible movements of the king.

Photo by Ayaz (Pedram) Razmjooei.

Abstract

This thesis consists of two parts: digital geometry and discrete optimization. In the first part we study the structure of digital straight line segments. We also study digital curves from a combinatorial point of view.

In Paper I we study the straightness in the 8-connected plane and in the Khalimsky plane by considering vertical distances and unions of two segments. We show that we can investigate the straightness of Khalimsky arcs by using our knowledge from the 8-connected plane.

In Paper II we determine the number of Khalimsky-continuous functions with 2, 3 and 4 points in their codomain. These enumerations yield examples of known sequences as well as new ones. We also study the asymptotic behavior of each of them.

In Paper III we study the number of Khalimsky-continuous functions with codomain $\mathbb{Z}$ and $\mathbb{N}$. This gives us examples of Schröder and Delannoy numbers. As a byproduct we get some relations between these numbers.

In Paper IV we study the number of Khalimsky-continuous functions between two points in a rectangle. Using a generating function we get a recurrence formula yielding this numbers.

In the second part we study an analogue of discrete convexity, namely lateral convexity.

In Paper V we define by means of difference operators the class of lateral convexity. The functions have $+\infty$ in their codomain. For the real-valued functions we need to check the difference operators for a smaller number of points. We study the relation between this class and integral convexity.

In Paper VI we study the marginal function of real-valued functions in this class and its generalization. We show that for two points with a certain distance we have a Lipschitz property for the points where the infimum is attained. We show that if a function is in this class, the marginal function is also in the same class.

To the peaceful future of all children,
especially mine: Aein and Artina

List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I	Samieinia, Shiva. 2010. Chord properties of digital straight line segments. *Math. Scand.* **106**, 169–195
II	Samieinia, Shiva. 2010. The number of Khalimsky-continuous functions on intervals. *Rocky Mountain J. Math.* **40**(5), 1667–1687.
III	Samieinia, Shiva. 2010. The number of continuous curves in digital geometry. *Portugaliae Mathematica* **67**, Issue 1, 75–89.
IV	Samieinia, Shiva. 2010. The number of Khalimsky-continuous functions between two points. Submitted.
V	Samieinia, Shiva. 2010. Discrete convexity built on differences. Manuscript.
VI	Kiselman, Christer O. and Samieinia, Shiva. 2010. Convexity of marginal functions in the discrete case. Manuscript.

Contents

Introduction

This thesis deals with two domains: Digital Geometry and Discrete Optimization. The first one was also studied from a combinatorial point of view. We shall make a brief description of both subjects as well as of the results which were obtained.

1. Digital Geometry

Drawing an object on paper is an approximate illustration of a real object. Euclidean geometry played a leading role in the study of these objects. If we instead draw them by computer, the things that we get are collections of small elements, namely pixels. Our eyes with the help of the brain put these small elements together in order to see the object. People made digital objects for thousands of years before computers. We refer to mosaics and different kind of carpets which are thousands of years old.

After the invention of computers and initiation of research in the fields of computer graphics and digital image analysis, the unsuitability of Euclidean geometry and the necessity of defining a new kind of geometry became evident. Digital geometry as an application-oriented field is being built up to do so. It deals with grid points or cells with different kinds of connections based on topological or non-topological structure. Digital geometry can be considered as a subdiscipline of discrete geometry with mathematical roots in graph theory and digital topology (see Klette [27]). As a brief description for this new kind of geometry we may refer to it as the geometry of the computer screen (Kiselman [21]).

Digital geometry was considered as a theory of n-dimensional digital spaces (cellular or grid point spaces) oriented toward the understanding of geometric objects (Klette [27]).

In a book chapter, Johnston and Rosenfeld [14] studied the geometric operations on digital images by considering an image as a finite subset of grid points as well as squares (cells). In the digital geometry chapter in [46], Rosenfeld considered a digital picture as a finite set of grid points and provided a theoretical basis for some picture analysis algorithms. These are just two pioneering works in which digital geometry was studied as cellular or grid point spaces. This holds for current work as well. In some cases one of these models may be more convenient than the other.

It is understood that in this new kind of geometry—digital geometry—all fundamental concepts of Euclidean geometry should be redefined in the discrete case. The way of doing this is not always unique but based on the problems we face; they can vary. Digital geometry can be considered partly as digitized Euclidean geometry because in the analysis of pictures the ideas of Euclidean geometry are frequently used but adapted to the discrete setting.

Digital geometry started in the 1960s, but it has grown increasingly so that hundreds of journal papers has been published so far. Citations of important research in that field are provided in the book by Klette and Rosenfeld [28]. The proceedings of the DGCI conferences (Discrete Geometry for Computer Imagery) show current research of this field. In the lecture notes by Kiselman [21] we can find more about the mathematical knowledge required for digital geometry.

Following the citations in digital geometry, we would like to mention a person who played a leading role in research in this field and established a pioneering theoretical framework in nearly every fundamental area of that field. Azriel Rosenfeld (1931–2004) was working in the field of computer image analysis and wrote the first textbook on this field in 1969 (see [43]) with a chapter on digital geometry. The foundation of current research in digital geometry was built during the 1960s and 1970s by Rosenfeld's research on digital image analysis.

1.1 Background

We shall make a brief description of the mathematical framework which is applicable to the study of digital objects and used also in this thesis.

Digital geometry can be defined as a theory of n-dimensional digital spaces (grid points or cell spaces) oriented toward the understanding of geometric objects. In this thesis we consider the grid plane both with 8-adjacency and with a topological adjacency. We follow the definition of digital space by Herman [12].

We define a digital space as a pair (V, π), where V is a set of points and π is a symmetric, reflexive and binary relation on V. Two distinct points x, y of V are called *adjacent* if $(x, y) \in \pi$. The space is called *connected* if for any two distinct points $x, y \in V$, there is a finite sequence $x_1, \ldots, x_n$ of points in V with the property that x_i is adjacent to x_{i+1} for all $1 \leq i \leq n-1$, and $x = x_1$, $y = x_n$.

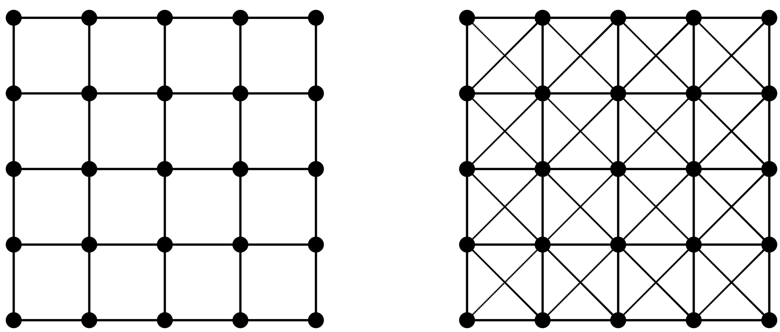
If V is finite, then V is just an undirected graph, but we allow V to be infinite. In many applications V is supposed to be $\mathbb{Z}^2$ or $\mathbb{Z}^3$.

Digital geometry often deals with real-valued functions from a set of points. A digital picture is such a function defined on a (finite) subset G of $V = \mathbb{Z}^n$. As we mentioned the value of n is most often 2 or 3, and we refer to 2D or 3D digital pictures. The set G is called a grid. (It is sometimes called a set of lattice points which is actually not as in lattice theory. In order not to be confusing the theories in which lattices are studied as partially ordered sets, we prefer to use the term *grid*.) An element of a two-dimensional G is called a *pixel*, which is the short term used for 'picture element'. The analogous term in a three-dimensional grid G is *voxel*, which is short for 'volume element'.

Following the definition of digital spaces, we are free to choose the adjacency relations. The most common adjacency relations used in $\mathbb{Z}^2$ are the 4- and 8-adjacency. The 4-adjacency is defined by the l^1 metric. Two grid points $p, q \in \mathbb{Z}^2$ are called 4-adjacent if $d_1(p, q) = |p_1 - q_1| + |p_2 - q_2| \leq 1$. By this adjacency relation each isolated grid point p will be connected to the four neighboring points $(p_1 \pm 1, p_2)$ and $(p_1, p_2 \pm 1)$.

The 8-adjacency is defined by the l^∞ metric. In this case, two grid points $p, q \in \mathbb{Z}^2$ are called 8-adjacent if $d_\infty(p,q) = \max\{|p_1 - q_1|, |p_2 - q_2|\} \leq 1$. Through this adjacency relation the point p is connected to the eight neighboring points $(p_1 \pm 1, p_2)$, $(p_1, p_2 \pm 1)$, $(p_1 \pm 1, p_2 + 1)$ and $(p_1 \pm 1, p_2 - 1)$.

The concepts of 4- and 8-adjacency were introduced in picture analysis in 1966 during the work on sequential local operations on neighborhoods by Rosenfeld and Pfaltz [47]. (However, the prefixes "4-" and "8-" were not used in this work. The earliest uses of these prefixes seem to have been made a few years later, in 1970 in [44] and [46].)



(a) 4-adjacency (b) 8-adjacency

Figure 1.1: (a) and (b) show the digital plane $\mathbb{Z}^2$ equipped with 4- and 8-adjacency, respectively.

1.2 Digital topology

One of the powerful fields in mathematics is topology. A topology on a set X is a collection $\tau = U(X)$ of subsets of X which is closed under the formation of arbitrary unions and finite intersections. The elements of $U(X)$ are called *open sets*. The empty set $\emptyset$ and the whole space X are always open. Using the topology of $\mathbb{R}^n$ is not appropriate in image processing, since in this kind of topology every pixel (discrete point) is an open set. Therefore an image will be a set of disjoint pixels, which does not carry any information about connections and neighboring points. Therefore, even though using topology in image processing is desirable, it should be constructed according to the discrete nature of pixels and the problems which we are expected to solve.

In a topological space it is not always true that the intersection of any family of open sets is open. If this is true, it implies that every point in the space can possess a smallest neighborhood containing it. A topological space with this property was introduced by Aleksandrov [1] and developed in [2]. It is called an *Aleksandrov space*, a *P. S. Aleksandrov space* or a *smallest-neighborhood space*. (The name Aleksandrov space is used also in differential geometry so we prefer to use the name smallest-neighborhood space.)

In a topological space X, let $N(B)$ be the intersection of all open sets containing B. This is in general not open. In a smallest-neighborhood space this set is open. The set $N(\{x\}) = N(x)$ is in that case the smallest neighborhood of x.

According to what we wrote until here, it is not far-fetched to say that digital geometry is the geometry of $\mathbb{Z}^n$. In this thesis we deal with the digital plane $\mathbb{Z}^2$. In higher dimensions, most applications are in $\mathbb{Z}^3$. In image processing the digital plane is a mathematical model of digitized black and white images. The set $S \subset \mathbb{Z}^2$ of black points and its complement, which is the set of white points, represents a digital image. In this thesis we deal with such black and white images. As already mentioned, there are two adjacency relations, 4- and 8-adjacency, which are of interest in the study of digital images.

Since the points in $\mathbb{Z}^n$ are isolated, connectedness of the set is usually defined by an adjacency relation. It is of interest to know which kind of connectedness based on the adjacency relation can be defined using a topological connectedness.

The 4-adjacency relation is such an adjacency relation defined also by a topological basis (see Rosenfeld [46]:9). Kong [30] went on to show that under some restrictions on the notion of adjacency, for any positive integer n there are only finitely many topological adjacency relations on $\mathbb{Z}^n$.

Khalimsky topology

There is a topology on $\mathbb{Z}^n$ by which we can equip the space with two adjacency relations: $(3^n - 1)$- and $2n$-adjacency. The topology called Khalimsky topology was defined in the 1960s by Efim Khalimsky [15] and [16]. He studied the topology of ordered segments and products of ordered segments. At that time the uses of this topology in image processing were not known. Some years later, Khalimsky [17], Khalimsky et al. [18], and Kopperman [31] studied applications of the Khalimsky topology in digital geometry.

We define the Khalimsky topology on $\mathbb{Z}$ (in a different way than the original one) by declaring that for every even integer $2n$, the set $\{2n-1, 2n, 2n+1\}$ is open, thus $N(2n) = \{2n-1, 2n, 2n+1\}$ and for every odd integer $2n+1$ the singleton $\{2n+1\}$ is open, thus $N(2n+1) = \{2n+1\}$. The complement of an even point $2n$ is the union of all smallest neighborhoods $N(x)$, $x \neq 2n$, which is an open set. Thus the even points are closed. Through this construction we present the Khalimsky topology by a topological basis given by

$$\big\{\{2n+1\}, \{2n-1, 2n, 2n+1\}; n \in \mathbb{Z}\big\}.$$

Thus a subset A of the Khalimsky line $\mathbb{Z}$ is open if and only if, for each even number $2n \in A$, we have $2n \pm 1 \in A$. Figure 1.2 illustrates the Khalimsky line.

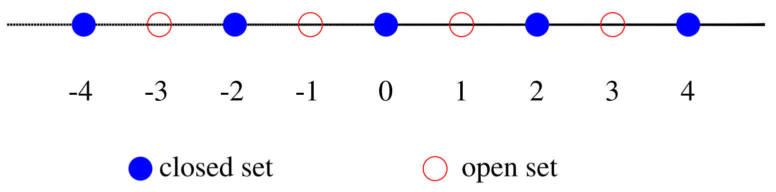


Figure 1.2: The Khalimsky line.

A Khalimsky interval is an interval $[a,b]_{\mathbb{Z}} = [a,b]_{\mathbb{R}} \cap \mathbb{Z}$ equipped with the Khalimsky topology on $\mathbb{Z}$. The Khalimsky plane is the Cartesian product of two Khalimsky lines, and, more generally, Khalimsky space is the Cartesian product of n copies of $\mathbb{Z}$. Equivalently, we can define Khalimsky topology on $\mathbb{Z}^n$ by declaring $\{x \in \mathbb{Z}^n; \|x-c\|_\infty \leq 1\}$ to be open for any point $c \in (2\mathbb{Z})^n$ and then taking all intersections of such sets as open sets, then all unions of such intersections.

In the Khalimsky plane a point with both coordinates odd is open. If both coordinates are even, the point is closed. These types of points are called *pure points*. The points with one even and one odd coordinate are neither open nor closed; these are called *mixed points*. Note that the mixed points are only connected to their four neighbors, whereas the pure points are connected to all eight neighbors.

A subset A of the Khalimsky plane is open if and only if, for every pair of even numbers $x = (2m, 2n)$, all pairs $y \in \mathbb{Z}^2$ with $\|y-x\|_\infty \leq 1$ belong to A, for every pair $(2m, 2n+1) \in A$ also $(2m \pm 1, 2n+1) \in A$, and, finally, for every pair $(2m+1, 2n) \in A$, also $(2m+1, 2n \pm 1) \in A$.

A topological space X is said to be *connected* if the only sets that are both open and closed are the empty set and the whole space X. The 4- and 8-adjacency were defined as graph connections. A connected path was also defined according to this. In a topological space we can define an adjacency relation between two distinct points x, y by declaring the set $\{x, y\}$ to be connected in the adjacency sense if it is so in the topological sense. Therefore in a smallest-neighborhood space we can say that x and y are adjacent if and only if $y \in N(x)$ or $x \in N(y)$. The adjacency relation in the Khalimsky plane is a mixture of 4- and 8-adjacency. A mixed point is adjacent to its four neighboring points, whereas a pure point is adjacent to its eight neighboring points. More precisely, at each mixed point (p,q) we have 4-adjacency and it is connected to its four neighbors $(p \pm 1, q)$ and $(p, q \pm 1)$, and on the other hand at each pure point (p,q) we have 8-adjacency and it is connected to its eight neighbors which consists of the four neighbors already mentioned and the points $(p \pm 1, q+1)$ and $(p \pm 1, q-1)$.

We mention that in general topology it might be that the concept of path connectedness is equivalent to the connectivity based on the topological structure. This also happens in digital topology as we wrote for the 4-connected plane, and for the Khalimsky plane. More information on the Khalimsky plane and the Khalimsky topology can be found in Khalimsky et al. [18], Kiselman [21] and Melin [34]. The Khalimsky plane is illustrated in Figure 1.3.

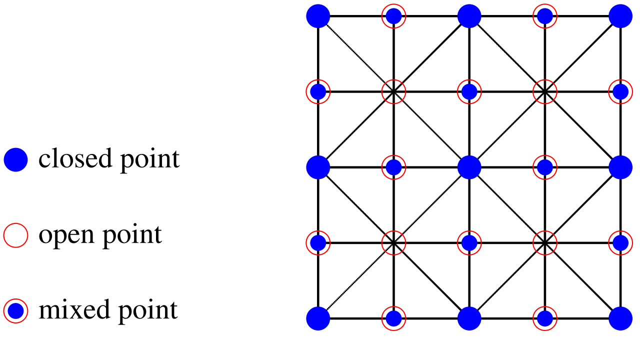


Figure 1.3: The Khalimsky plane.

Khalimsky-continuous functions

The concept of continuity is one of the main subjects in general topology. It would be useful for many applications to have such a concept in discrete spaces. When we equip $\mathbb{Z}$ with the Khalimsky topology, we can speak of continuous functions $\mathbb{Z} \to \mathbb{Z}$, i.e., functions for which the inverse image of an open set is open.

A function $f\colon X \to Y$ from one smallest-neighborhood space into another is continuous at a point x if and only if the direct image of $N_X(x)$ is contained in $N_Y(f(x))$, or, equivalently, the inverse image of $N_Y(f(x))$ contains $N_X(x)$:

$$f(N_X(x)) \subset N_Y(f(x)), \text{ equivalently } N_X(x) \subset f^{-1}(N_Y(f(x))). \tag{1.2.1}$$

Here $N_X(x)$ and $N_Y(y)$ denote the smallest neighborhoods of $x \in X$ and $y \in Y$, respectively.

If we let $X = Y = \mathbb{Z}$ equipped with the Khalimsky topology, this means that:
1. For odd x, property (1.2.1) always holds;
2. For even x, if $f(x)$ is odd, we shall have $f(x \pm 1) = f(x)$;
3. For even x, if $f(x)$ is even, we shall have $|f(x \pm 1) - f(x)| \leq 1$.

In particular, a continuous function is Lip-1, but sometimes it must be constant in some intervals meaning that when it takes an odd value at an even point, or conversely an even value at an odd point.

We also observe that the following functions are continuous:
(1) $\mathbb{Z} \ni x \to a \in \mathbb{Z}$, where a is constant;

(2) $\mathbb{Z} \ni x \to \pm x + c \in \mathbb{Z}$, where c is an even constant;
(3) $\max(f,g)$ and $\min(f,g)$ if f and g are continuous.

Actually every continuous function on a bounded Khalimsky interval can be obtained by a finite succession of the rules (1), (2), (3). For more information see Kiselman [21]:74.

1.3 Digitization

In $\mathbb{R}^n$ we do continuum (non-digital) geometry but in $\mathbb{Z}^n$ we do digital geometry. Digitization is a way in which we can go from $\mathbb{R}^n$ to $\mathbb{Z}^n$.

Digitization might be understood as finding a good approximation in a digital space of real objects. As two pioneering methods which are related to this we mention Gauss digitization which goes back to the time that Gauss (1777–1855) studied the measurement of the area of a planar set $S \subset \mathbb{R}^2$ by counting the grid points $(i, j) \in \mathbb{Z}^2$ contained in S, and Jordan digitization which goes back to Jordan (1838–1922) when he used grids to estimate the volumes of subsets of $\mathbb{R}^3$ (see Klette and Rosenfeld [28]:56 and 58).

A kind of digitization which is commonly used for arcs and curves is grid-intersection digitization. It is defined as the closest grid points (i, j) to the intersection of curve or arc with the grid lines. The distance measures by Euclidean distance and this digitization works for planar curves and arcs.

A conflict appears when the intersection points have the same distance to the two different grid points. In this case we have to limit the digitization process by choosing for example the left or the down point, respectively.

The result of this digitization is an ordered sequence of grid points. Each two successive pairs of grid points are at distance 1 along a grid line, or $\sqrt{2}$ along a diagonal.

We have eight possible directions to go from one pixel to the next adjacent pixel. Freeman [7] proposed a technique in which directions can be represented by codes $0, 1, \ldots, 7$. The corresponding Freeman chain code of lines with slope $0 \le \alpha \le 1$ can consist only 0 or 1. In the right side of Figure 1.4 we can see these directions and their related codes which are in a counterclockwise order beginning from 0.

Hence each digitized arc can be represented geometrically as a polygonal arc with vertices at grid points, and also by chain codes which are given by the direction. Using chain codes yields simple algorithms for image processing operations (see Freeman [7] and [8]). It is also an appropriate way to investigate the problems from a combinatorial point of view.

Kiselman [23] put the definition of digitization in a mathematical form as follows:
Let P be a subset of $\mathbb{R}^n$. We may define the P-digitization of a set $M \subset \mathbb{R}^n$ as the set

$$dig_P(M) = (M + P) \cap \mathbb{Z}^n, \qquad M \subset \mathbb{R}^n.$$

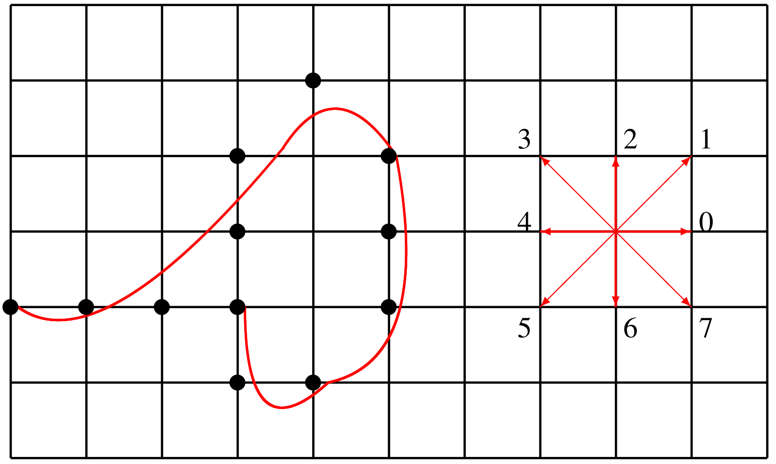


Figure 1.4: The left part shows the digitization of a curve by using grid intersection. In this digitization we choose the intersection points to the left or down. The right part shows the directional codes proposed by Freeman. The related Freeman chain code to the curve on the left side is 00121766542.

Here P may be a pixel or voxel or any subset of $\mathbb{R}^n$. Note that $M \mapsto M + P$ is a dilation whose role is to fatten the set M before intersecting with the gird points.

If we let $P = \big(\{0\} \times]-\frac{1}{2}, \frac{1}{2}]\big) \cup \big(]-\frac{1}{2}, \frac{1}{2}] \times \{0\}\big) \subset \mathbb{R}^2$, we get the digitization which Rosenfeld [45] used for digitization of lines.

The Rosenfeld digitization does not work well in the Khalimsky plane. In general the Rosenfeld digitization of a straight line segment is not connected for this topology. Figure 1.5 shows the Rosenfeld digitization of a line segment which is not connected in the Khalimsky plane.

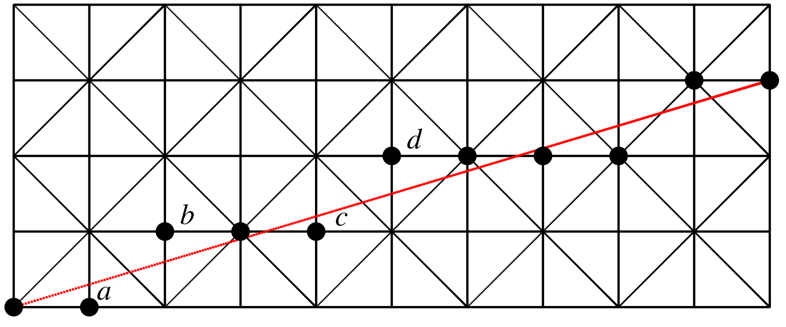


Figure 1.5: The Rosenfeld digitization of line segment in the Khalimsky plane. The picture shows that there is no connection between the points a and b or between the points c and d.

To introduce a digitization for the Khalimsky plane which makes the digital line segments connected, Melin [33] introduced a Khalimsky-continuous digitization. This digitization maps each line segments in $\mathbb{R}^2$ to a Khalimsky arc in $\mathbb{Z}^2$.

The main idea of his work is to first define a digitization for the pure points and then add the mixed points to obtain connectedness.

Let

$$D(0) = \{(t,t) \in \mathbb{R}^2;\ -1/2 < t \leq 1/2\} \cup \{(t,-t) \in \mathbb{R}^2;\ -1/2 < t \leq 1/2\}.$$

For each pure point $p \in \mathbb{Z}^2$, define $D(p) = D(0) + p$. Note that $D(p)$ is a cross, rotated 45°, with center at p.

Melin [33] defined the *pure digitization* $D_P(A)$ of a subset A of $\mathbb{R}^2$ as

$$D_P(A) = \{p \in \mathbb{Z}^2;\ p \text{ is pure and } D(p) \cap A \neq \emptyset\}.$$

By using the pure digitization he defined for a Khalimsky-connected set of points L, the continuous digitization $D(L)$ as follows:
If L is horizontal or vertical, $D(L)$ is the Rosenfeld digitization. Otherwise define $D_M(L)$ as

$$D_M(L) = \{p \in \mathbb{Z}^2;\ (p_1 \pm 1, p_2) \in D_P(L)\} \cup \{p \in \mathbb{Z}^2;\ (p_1, p_2 \pm 1) \in D_P(L)\}$$

and let $D(L) = D_P(L) \cup D_M(L)$.

1.4 Digital straight line segments

A straight line in a real plane is the set $\{(1-t)a + tb; t \in \mathbb{R}\}$, where a and b are two distinct points in the plane. A straight line segment is a connected subset of a straight line (perhaps the whole line).

In this thesis we work with two different digital planes, one with 8-adjacency and the other with the adjacency defined by the Khalimsky topology.

For the plane with 8-adjacency we follow the Rosenfeld digitization. For a set D of the points of digital plane $\mathbb{Z}^2$ we say that D is a digital straight line segment if and only if there exists a real line segment the Rosenfeld digitization of which is equal to this set.

For the slopes $-45^\circ < \alpha < 45^\circ$, the digitization process depends on its crossing of vertical grid lines, and the horizontal grid lines do not give any extra digitization points. For the other slopes it may happen that we get points from vertical as well as horizontal grid lines. In this case we get a fat digitization set of points which is not of our interest. Since the slope zero is not a complicated case (this is just a horizontal line) and the symmetry which we have between first and fourth quadrant, we consider lines and straight line segments with slope strictly between 0 and 45° in the 8-connected case as well as in the Khalimsky plane.

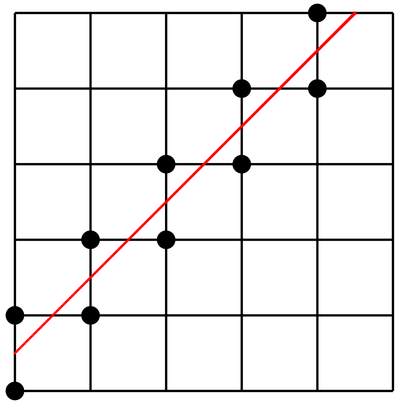


Figure 1.6: The Rosenfeld digitization of the line segment $y = x + \frac{1}{2}$ which is a fat digitization. For this line both horizontal and vertical grid lines yield the digitization points.

Digital straight line segments have been studied extensively from the beginning of the field digital geometry. Freeman [7] introduced the directional chain code as a technique to represent 8-connected arcs and lines. He used this technique in [9] to encode the boundary lines. Rosenfeld [45] characterized it by the chord property which we shall discuss in subsection 1.5. Hung and Kasvand [13] gave a necessary and sufficient condition for a digital arc to have the chord property. Rosenfeld [45] stated the definition of runs and some of its properties. Studies of digital lines based on the concept of runs were done by Smeulders and Dorst [48], Stephenson [49], and Uscka-Wehlou [51].

Digital straightness was also investigated using digital convexity by Kim [19]. Reveillès [41] did it arithmetically by introducing the concept of naive digital line by double Diophantine inequalities. As a generalization of this definition we easily get naive digital hyperplanes. Kiselman [20, 23, 26] generalized Reveillès' definition of a digital hyperplane by allowing more freely strict and non-strict inequalities. He represented a digital hyperplane as a graph of a function which is both convex and concave.

Bruckstein [3] presented some transformations on sequences composed of two symbols, 0 and 1. Kiselman [23, 25] characterized straightness by using the calculus of difference operators. Klette and Rosenfeld [29] presented a review of the various concepts of digital lines. In the book [28] we can see citations of important research on this concept in two and three dimensions.

Digital straight line segments in the Khalimsky plane were studied by Melin [33]. He provided a continuous digitization and chord measure for this concept.

1.5 Summary of results in Digital Geometry

The digital geometry part of this thesis deals with the study of straightness in both the 8-connected plane and the Khalimsky plane. Khalimsky-countinuous functions were also studied from a combinatorial point of view.

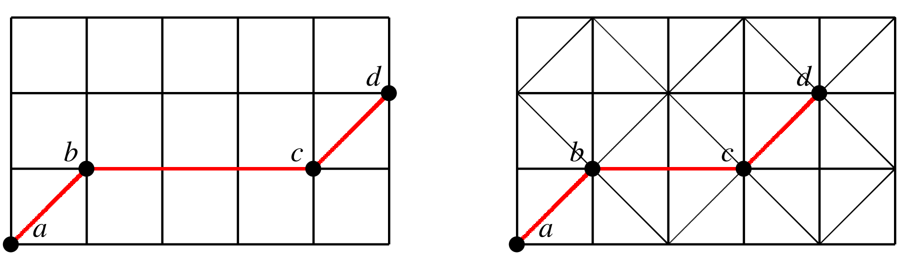
Straightness in the 8-connected plane and the Khalimsky plane

An important problem related to straightness is how to recognize a set of pixels or codes representing a digital straight line segment. Given any subset P of $\mathbb{R}^2$ we define its *chord set* $\mathsf{chord}(P)$ as the union of all chords, i.e., all segments with endpoints in P, as $\mathsf{chord}(P) = \bigcup_{x,y \in P}[x,y] \subseteq \mathbb{R}^2$. Rosenfeld [45] characterized straightness by the chord property. A subset $P \subseteq \mathbb{R}^2$ has the *chord property* if $\mathsf{chord}(P)$ is contained in $P + B^{\infty}_{<}(0,1)$. This means that the chord between two points of set is not far from the set itself. Ronse [42] presented a simpler proof for Rosenfeld's theorem.

Melin [33] generalized the chord property by chord measure in order to characterize the straightness in both 8- and Khalimsky-connectedness. This measure is qualitative (a smaller measure implies a better approximation).

In Paper I, we define a geometric object called boomerang. It is the union of two segments in the digital curve: horizontal and diagonal.

Definition 1.5.1. When a graph P is given, we shall say that a digital curve consisting of $m+1$ points, $B = (b^i)_{i=0}^{m}$, $m \geq 2$, is a *boomerang in P* if it consists of a horizontal segment $[b^0, b^k]$, where $0 < k < m$, followed by a diagonal segment $[b^k, b^m]$, or conversely, and if B is maximal with this property. We shall call the horizontal and diagonal segments, $\mathsf{Con}(B)$ and $\mathsf{Inc}(B)$, respectively.



(a) (b)

Figure 1.7: (a) and (b) show the boomerangs in the 8- and the Khalimsky connected plane. In both planes, the union of two segments ab and bc shows a concave boomerang, and bc and cd a convex one. The point b is the vertex of concave boomerang, and the point c is the vertex of convex boomerang.

The cardinalities of the horizontal and diagonal segments are denoted by $|\mathsf{Con}(B)|$ and $|\mathsf{Inc}(B)|$, and they are equal to the number of zeros and ones in the related chain code, respectively. The cardinality of a boomerang $|B|$ is defined as the sum of $|\mathsf{Con}(B)|$ and $|\mathsf{Inc}(B)|$. The boomerangs need not be disjoint and the last segment of a boomerang may be a starting segment of the next boomerang, so the number of boomerangs is equal to the number of vertices.

We modified the chord property in both planes by using vertical distances at the vertices of boomerangs. These modification of chord property made the straightness easier to check.

Theorem 1.5.2. *Let $P = (p^i)_{i=0}^n$ be an 8-connected sequence of points which is the graph of a function and has b boomerangs. Let $V = (v^i)_{i=1,\ldots,b}$ be the sequence of all vertices of its boomerangs. Then $P \in DSLS_8$ if and only if for all $i = 1,\ldots,b$ and all real points $a \in \mathsf{chord}(P)$ such that $a_1 = v_1^i$ we have $d_\mathrm{v}(v^i,a) = |v_2^i - a_2| < 1$.*

This result is similar to the characterization of digital lines by Reveillès [41]. He introduced the concept of naive digital line by double Diophantine inequalities. His definition characterizes the straightness arithmetically, and this is a useful tools in the analysis of medical images (see Figueiredo [5]). In our work we have shown that we need just to check the inequalities for certain points, namely vertices of boomerangs.

In the Khalimsky plane the vertices do not play the same role as in the 8-connected plane. In the Khalimsky plane, the vertical distances at mixed points are important for characterizing the straightness.

Theorem 1.5.3. *Suppose that $P = (p^i)_{i=0}^n$ is a Khalimsky-connected sequence with pure endpoints and let b be the number of its boomerangs. Let M be the set of all mixed points in P. Then $P \in DSLS_{\mathrm{Kh}}$ if and only if for all $m \in M$ and all $a \in \mathsf{chord}(P)$ with $a_1 = m_1$ we have $d_\mathrm{v}(m,a) = |m_2 - a_2| < 1$.*

According to two propositions in Paper I, we consider two classes when we study digital straightness, dominant increasing if $|\mathsf{Inc}(B_i)| = 1$, and dominant constant if $|\mathsf{Con}(B_i)| = 1$ in the 8-connected case and $|\mathsf{Con}(B_i)| = 2$ in the Khalimsky plane.

Rosenfeld [45] stated that an 8-connected digitized line can only contain runs of two different lengths and their lengths must be consecutive integers. This fact plays a leading role in many works afterwards.

We mention some works in which the notion of runs was developed. To decompose the digital curves, Smeulders and Dorst [48] worked with runs as the sequences of successive elements in the Freeman chain code with the same values, and go to the higher order by classifying runs with the same number of elements. They showed that there are two different values for the runs of nonfinal orders. Stephenson [49] studied digital lines with rational slopes by considering runs. Uscka-Wehlou [51] did it for irrational slopes as well.

For the cardinality of boomerangs in the 8-connected plane, we have the same result as for runs in [45], but not exactly the same in the Khalimsky plane. For a digital straight line segments we have at most two possible values for the cardinality of the boomerangs in both planes. In the 8-connected plane they can differ by at most one. However in the Khalimsky plane, the cardinality of boomerangs for the dominant increasing differs by at most 1, and for the dominant constant by at most 2. These conditions are necessary but not sufficient for straightness.

In Paper I, we found a sufficient condition for straightness. We have just two possibilities for the values of $|B_i|$. Thus we map the boomerangs with greater cardinality to 1 and the other boomerangs to 0. This gives a transformation from the set $\{0,1\}^{\mathbb{N}}$ into the same set. The graph of this function is an 8-connected set. Hence, to investigate the straightness in the Khalimsky plane we can refer to the previous works on the 8-connected case.

Theorem 1.5.4. *We define a function f on a subset of the set $\{0,1\}^{\mathbb{N}}$ of sequences of zeros and ones and with values in the same set: $f(C)$ is defined for those chain codes that represent dominant increasing or dominant constant sequences which arise from sets of boomerangs of at most two different lengths. We define $f(C)$ as the sequence obtained by replacing the chain code of a long concave boomerang by* 1 *and that of a short concave boomerang by* 0*. Then*
(I) C is the chain code of an element of $DSLS_8$ if and only if $f(C) \in DSLS_8$, and
(II) C is the chain code of an element of $DSLS_{\mathrm{Kh}}$ if and only if C is the chain code of a Khalimsky-connected set and $f(C) \in DSLS_8$.

Digital curves from a combinatorial point of view

In Papers II, III and IV, we studied the Khalimsky-continuous functions from a combinatorial point of view.

In Paper II, we determined the number of Khalimsky-continuous functions with two, three, and four points in their codomain.

For the case of two points in the codomain the number of such functions is given by the Fibonacci numbers.

Theorem 1.5.5. *Let a_n be the number of Khalimsky-continuous functions $[0,n-1]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$. Then $a_n = F_{n+2}$, where $(F_n)_0^\infty$ is the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.*

The case of three points in the codomain gives an example of the Tribonacci and Jacobsthal sequences for odd and even indices.

Theorem 1.5.6. *Let b_n be the number of Khalimsky-continuous functions $[0,n-1]_{\mathbb{Z}} \to [0,2]_{\mathbb{Z}}$. Then $b_1 = 3$, $b_2 = 5$, and*

$$\begin{aligned} b_{2k} &= b_{2k-1} + b_{2k-2} + b_{2k-3} = 2b_{2k-2} + 3b_{2k-3}, && k \geq 2, \\ b_{2k-1} &= b_{2k-2} + 2b_{2k-3}, && k \geq 2. \end{aligned} \tag{1.5.1}$$

Considering four points in the codomain led us to the following theorem.

Theorem 1.5.7. *Let c_n be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \to [0,3]_{\mathbb{Z}}$ and let c_n^i be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \to [0,3]_{\mathbb{Z}}$ such that $f(n-1) = i$ for $i = 0,1,2,3$. Then $c_1^1 = c_1^2 = 1$, $c_2 = 7$, $c_3 = 15$ and*

$$c_n = c_{n-1} + 2c_{n-2} + c_{n-3}^1 + c_{n-3}^2. \tag{1.5.2}$$

Formula (1.5.2) together with two other formulas in the proof determine the value of c_n.

In all three cases we also studied the asymptotic behavior.

In Paper III, we determined the number of Khalimsky-continuous functions with codomain $\mathbb{Z}$ and $\mathbb{N}$.

Enumeration of Khalimsky-continuous functions with codomain $\mathbb{Z}$ gave us the 30th example of the Delannoy numbers. The 29 previous examples are listed in Sulanke [50].

Theorem 1.5.8. *Let f_n^s, $|s| \le n$, be the number of Khalimsky-continuous functions $f\colon [0,n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$ and $f(n) = s$, and $d_{i,j}$ be the Delannoy numbers. Then we have that $f_n^s = d_{i,j}$ for $i = \frac{1}{2}(n+s)$ and $j = \frac{1}{2}(n-s)$ where $n+s \in 2\mathbb{Z}$, and $f_n^s = f_{n-1}^s$ for $n+s$ odd.*

We also went on to show that these numbers has the same recursion relation as the Pell numbers, but with different initial values.

Theorem 1.5.9. *Let f_n be the number of Khalimsky-continuous functions $f\colon [0,n]_{\mathbb{Z}} \mapsto \mathbb{Z}$ such that $f(0) = 0$. Then*

$$f_n = 2f_{n-1} + f_{n-2} \quad \text{for } n \ge 2. \tag{1.5.3}$$

When we consider the codomain $\mathbb{N}$ we get an example of Schröder numbers as follows.

Theorem 1.5.10. *Let $g_n^s = \operatorname{card}\{g\colon [0,n] \to \mathbb{N}; g(0) = 0, g(n) = s\}$ for $s \in \mathbb{N}$ and $s \le n$, and $r_{i,j}$ be the Schröder numbers. Then we have $g_n^s = r_{i,j}$ for $i = \frac{1}{2}(n+s)$ and $j = \frac{1}{2}(n-s)$, where $n+s \in 2\mathbb{N}$.*

As a byproduct we showed some relations between the Delannoy and Schröder arrays.

Theorem 1.5.11. *Let g_n^s be the number of Khalimsky-continuous functions $g\colon [0,n] \to \mathbb{N}$ such that $g(0) = 0$ and $g(n) = s$ for $s \in \mathbb{N}$ and $s \le n$. Let*

$$p_{t,n} = \sum_{i=0}^{\frac{n-t}{2}} g_n^{t+2i} \text{ where } 0 \le t \le n \text{ and } n+t \in 2\mathbb{N}.$$

Then $p_{t,n} = d_{i,j}$ where $i = \frac{n+t}{2}$, $j = \frac{n-t}{2}$ and the $d_{i,j}$ are Delannoy numbers.

Corollary 1.5.12. *Let $r_{i,j}$ and $d_{i,j}$ be the Schröder numbers and Delannoy numbers, respectively. Then $d_{i,j} = \sum_{l=0}^{j} r_{i+l,j-l}$ for $i \geq j$.*

We also went on to show that the number of Khalimsky-continuous functions with codomain $\mathbb{N}$ can be obtained by summing two consecutive numbers of other sequences.

Theorem 1.5.13. *Let g_n be the number of Khalimsky-continuous functions $g\colon [0,n] \to \mathbb{N}$ such that $g(0) = 0$. Let $p_n = p_{0,n}$ for n even and $p_n = p_{1,n}$ for n odd, where $p_{t,n}$ are the numbers defined in Theorem 1.5.11. Then*

$$g_n = p_n + p_{n-1}. \tag{1.5.4}$$

What we perceive from Euclidean geometry is that there is just one line segment between two points. On the other hand, there are infinitely many line segments with endpoints in a given rectangle. However the discrete nature of lines in $\mathbb{Z}^2$ leads to a countable number of these. Considering digital objects (digital straight line segments or curves) exhibits great differences between Euclidean and digital geometry. Another striking contrast is shown by the fact that the number of Khalimsky-continuous functions between two given points is finite.

In Paper IV, we defined a generating function by which we introduced a recurrence formula to enumerate the number of Khalimsky-continuous functions not exiting from a rectangle and with two endpoints. To be precise, we proved the following result.

Theorem 1.5.14. *Let f_n^s be the number of Khalimsky-continuous functions $f\colon [0,n]_{\mathbb{Z}} \to [0,s]_{\mathbb{Z}}$ such that $f(0) = 0$ and $f(n) = s$. Then*

$$f_{2k+1}^1 = (-1)^k \binom{n-1}{k} f_n^n + (-1)^{k-1} \binom{n-1}{k-1} f_{n+2}^n + \cdots + (-1)\binom{n-1}{1} f_{n+2(k-1)}^n + f_{n+2k}^n$$

for all natural numbers $n,k \geq 1$. The formula serves to define f_{n+2k}^n in terms of f_j^n for $j < n+2k$ and f_{2k+1}^1.

2. Discrete Optimization

An important concept in optimization theory of real variables is convexity of sets and functions. For a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ we have the following results from convex analysis of real-valued functions.
(I). Local minima: A local minimum of a convex function is global;
(II). Separating hyperplane: There exists a separating hyperplane between two disjoint convex sets;
(III). The marginal function of a convex function is convex.

The aim of much work in discrete convex analysis is to provide a theoretical framework for discrete convexity by which these three properties and some others are valid as for convexity of functions of real variables.

A natural way to define the concept of convexity for functions of integer variables is to define it as a function which admits a convex extension defined in the space $\mathbb{R}^n$ of real variables. There are some examples in Murota [38] and Kiselman [22, 26] which show that this is not a good discrete analogue of convexity.

To improve the concept of discrete convexity in order to have the theoretical framework comparable with convexity in real variables, several types of discrete convexity have been studied. Let us mention some classes of functions, like the discretely convex functions introduced by Miller [35], integrally convex functions defined by Favati and Tardella [4], M-convexity introduced by Murota [36] and L-convexity by Murota [37].

In this thesis we shall take up the first and third problems mentioned above and point to convenient solutions.

2.1 Background

Any good discrete analogue of convexity should cover the fundamental properties of convexity in real variables as local minima being global; the existence of separating hyperplanes; convexity of marginal functions, as well as some other properties. We mention here some pioneering work on discrete convexity.

Miller [35] found a condition called discrete convexity, which is a sufficient condition for a local minimum to be global. This class of functions is not closed under addition and its members are in general not convex extensible (see Murota and Shioura [40]:161).

Favati and Tardella [4] introduced the notion of integral convexity by means of ordinary convexity.

Definition 2.1.1. Given a function $f\colon \mathbb{Z}^n \to [-\infty, +\infty]$, we define its *canonical extension* of f by $\mathbf{can}(f)\colon \mathbb{R}^n \to \mathbb{R}$ as the convex envelope of f on each unit hypercube $a + [0,1]^n$, $a \in \mathbb{Z}^n$.

Definition 2.1.2. A function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is called *integrally convex* if its canonical extension $\mathbf{can}(f)$ is convex.

The elements of this class are convex extensible by definition. The property of local minima being global was proved in [4]. We mention that this class is not closed under addition when $n \geq 3$ (see Murota and Shioura [40]:161).

The concept of M-convexity was introduced by Murota [36] in terms of an exchange axiom. An M$^\natural$-convex function can be obtained by a restriction of an M-convex function to a certain $(n-1)$-dimensional plane. For the first time it was introduced by Murota and Shioura [39]. It is also possible to characterize it by an exchange property.

The concept of L-convex function was introduced by Murota [37]. The concept of L$^\natural$-convexity is defined as a variant of L-convexity defined by restriction to an $(n-1)$-dimensional plane by Fujishige and Murota [11].

The class of L- and M-convexity are conjugate to each other through a discrete version of the Legendre–Fenchel transformation. We note that the letter L stands for lattice, and M for matroid.

We recall the definition of submodularity and supermodularity as well as modularity.
A function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is *submodular* if we have

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \ (\forall x, y \in \mathbb{R}^n),$$

where $(x \vee y)_i = \max(x_i, y_i)$ and $(x \wedge y)_i = \min(x_i, y_i)$, for $i = 1, \ldots, n$.
The function is *supermodular* if we have the inequality on the other direction; it is *modular* if equality holds.

Similarly we may define a submodular function on the lattice of all subsets of a set with n elements. There is a relation between this submodularity and convexity given by László Lovász [32]: A set function is submodular if and only if its Lovász extension is convex. It should be noted, however, that in this extension, the empty set must correspond to the origin, and the full set to $(1, 1, \ldots, 1)$, the opposite point in the hypercube—no translation or reflection is allowed.

The study of submodular and supermodular functions by Frank [6] and the min-max duality theorem by Fujishige [10] together with other works on submodularity in the 1980s led to a similarity between submodularity and convexity. Thus it has became of interest afterwards to compare submodularity with convexity and not to concavity, and study of duality for submodularity-supermodularity in the discrete sense. However, it should be noted that submodularity is not a good analogue of convexity in all cases. Among other things, we would like our class to be closed under simple coordinate changes like $(x_1,\dots,x_n) \mapsto (x_1,\dots,-x_n)$. Hence finding a new class of functions and a new framework in order to compare the convexity in real variables with discrete analogues of this is still of interest.

The concept of integral convexity was studied in detail in the two-dimensional case by Kiselman [22, 26]. He presented several equivalent definitions for this class in the two-dimensional case. One of these definitions is based on difference operators.

The structure of canonical extension plays a leading role in his work. We can see that the canonical extension $\mathbf{can}(f)$ over a unit square contains two affine planes which intersect on the line segment $[(0,0),(1,1)]$ if the function is submodular on the unit square $\{0,1\}^2$, and if on the other hand the function is supermodular on this unit square then the two affine plane intersects on $[(1,0),(0,1)]$. We notice that it will be just one affine plane if we have modularity.

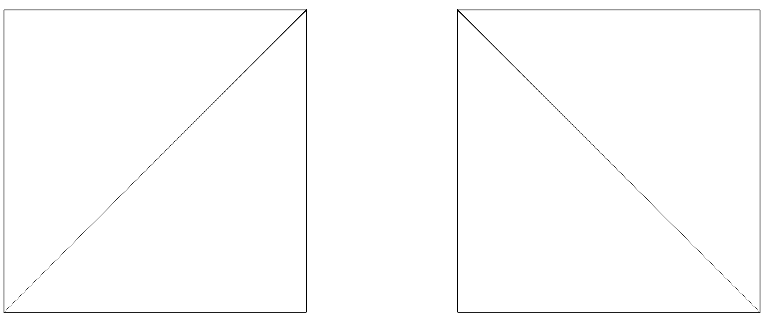


Figure 2.1: The left and right squares show the structure of canonical extension for a submodular and supermodular function, respectively.

The structure of canonical extension in dimension more than three is not as easy as the two-dimensional case. We refer to the works by Kiselman [24].

The canonical extension is not in general linear. We have $\mathbf{can}(f)+\mathbf{can}(g) \le \mathbf{can}(f+g)$. The equality holds if at least one of the functions f and g is linear (Favati and Tardella [4]). For the two-dimensional case, Kiselman [26] defined a function as a difference between two sides of the inequality and described it exactly. He went on to show that equality holds if and only if both functions are submodular or supermodular. In the case of local minimum he found a suitable neighborhood depending on the function to have the property of local minimum being global.

2.2 Summary of results in Discrete Optimization

The discrete optimization part of this thesis deals with the study of a new class of discrete convexity, namely lateral convexity. We study two important properties of convexity, local minima and marginal functions.

Lateral convexity and local minima

Two-dimensional integrally convex functions were defined using difference operators by Kiselman [22]. In Paper V, using difference operators, we generalized his definition to the n-dimensional case. We show also that this new class is a subclass of the integrally convex functions but not equal to it in general.

We use difference operators which were introduced by Kiselman [26]. For a given $a \in \mathbb{Z}^n$, a difference operator D_a on functions $f\colon \mathbb{Z}^n \to \mathbb{R}$ is defined by

$$(D_a f)(x) = f(x+a) - f(x), \quad x \in \mathbb{Z}^n.$$

D_a is a difference operator from $\mathbb{R}^{\mathbb{Z}^n}$ to $\mathbb{R}^{\mathbb{Z}^n}$.

The second-order operators $D_b D_a$ will be as follows:

$$(D_b D_a f)(x) = f(x+a+b) - f(x+a) - f(x+b) + f(x).$$

Using an extension of operation of addition, Kiselman [26] extended D_a to $(D_a)_!\colon (\mathbb{R}_!)^{\mathbb{Z}^n} \to (\mathbb{R}_!)^{\mathbb{Z}^n}$ for functions $f\colon \mathbb{Z}^n \to \mathbb{R}_!$ by

$$(D_a)_! f(x) = f(x+a) \dot{+} (-f(x)), \qquad x, a \in \mathbb{Z}^n.$$

We also define

$$(D_a D_b)_! f(x) = f(x+a+b) \dot{+} (-f(x+a)) \dot{+} (-f(x+b)) \dot{+} f(x).$$

A-lateral convexity can be defined as follows:

Definition 2.2.1 (Kiselman [26])**.** Given any subset A of $\mathbb{Z}^n \times \mathbb{Z}^n$ we shall say that a function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is A-laterally convex if $D_a D_b f \geq 0$ for all $(a,b) \in A$. For a function f with infinite values, $f\colon \mathbb{Z}^n \to \mathbb{R}_!$, we shall say that it is A-laterally convex if $(D_a D_b)_! f \geq 0$ for all $(a,b) \in A$.

From the definition it is obvious that the class of real-valued A-laterally convex functions is closed under addition and multiplication by a nonnegative scalar. The set

$$-A = \{(-a,-b); (a,b) \in A\}$$

defines the same class as A. The same is true for

$$A^{\smallsmile} = \{(b,a); (a,b) \in A\}.$$

We define

$$A^{\mathrm{sym}} = A \cup (-A) \cup A\check{~} \cup (-A)\check{~},$$

which may have up to four times as many elements as A but still defines the same class.

We have also that

$$(D_b D_a f)(x) + (D_c D_a f)(x+b) = (D_{b+c} D_a f)(x). \tag{2.2.1}$$

Thus if $D_b D_a f \geq 0$ and $D_c D_a f \geq 0$, then we also have $D_{b+c} D_a f \geq 0$. This means that the set of pairs $\{(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n\}$ such that the inequality holds is closed under partial addition:

$$(a,b) +_2 (a,c) = (a, b+c),$$

i.e., if the first elements agree, we may add the second elements. For sets we define

$$B +_2 C = \{(a, b+c); (a,b) \in B, (a,c) \in C\}.$$

Similarly we can define of course $(a,b) +_1 (c,b) = (a+c, b)$ and

$$B +_1 C = \{(a+c, b); (a,b) \in B, (c,b) \in C\}$$

when the two second elements are the same.

By repeated use of these formulas we see that A-lateral convexity is equivalent to B-lateral convexity, where B is any class such that $A \subset B \subset \tilde{A}$, denoting by $\tilde{A}$ the smallest set which contains

$$(\mathbb{Z}^n \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times \mathbb{Z}^n) \cup A^{\mathrm{sym}}$$

and is closed under both partial additions

$$(B,C) \mapsto B +_1 C \text{ and } (B,C) \mapsto B +_2 C.$$

Thus $\tilde{A}$ contains sets such as $A^{\mathrm{sym}} +_1 A^{\mathrm{sym}}$, $A^{\mathrm{sym}} +_2 (A^{\mathrm{sym}} +_1 A^{\mathrm{sym}})$ and so on.

In view of some examples in Paper VI in this thesis, it is normally required that

$$(e^{(j)}, e^{(j)}) \in A, \qquad j = 1, \ldots, n. \tag{2.2.2}$$

and

$$(a,b) \in A \text{ implies } a_j b_j \geq 0, \qquad j = 1, \ldots, n. \tag{2.2.3}$$

Property (2.2.2) shows that all A-laterally convex functions are separately $\{(1,1)\}$-laterally convex.

In paper V, we considered

$$B = \left\{ \left(\textstyle\sum_{j \in V} e_j, \sum_{j \in V} e_j + \sum_{i \notin V} a_i \right); V \subset \{1, \ldots, n\} \text{ and } a_i \in [-1,1]_{\mathbb{Z}} \right\},$$

which serves as a generalization of two-dimensional set of points in Kiselman [26]. An example in [26] shows that the set B cannot be modified by a smaller number of points for function with infinite values to have the convexity of the canonical extension. On the other hand when the range consists of real numbers only, we showed that it is not necessary to consider all possible directions and it is enough to check B-lateral convexity using a smaller number of points. The result is as follows.

Theorem 2.2.2. *Let*

$$B = \left\{(\textstyle\sum_{j\in V} e_j, \sum_{j\in V} e_j + \sum_{i\notin V} a_i);\, V \subset \{1,\dots,n\} \text{ and } a_i \in [-1,1]_{\mathbb{Z}}\right\},$$

and

$$C = \left\{(e_j, e_j + \textstyle\sum_{i\neq j} a_i);\, j \in \{1,\dots,n\} \text{ and } a_i \in [-1,1]_{\mathbb{Z}}\right\}. \tag{2.2.4}$$

A function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is B-laterally convex if and only if it is C-laterally convex.

The set C cannot be reduced; the same example as we mentioned for the set B can show this fact.

We defined a metric $\delta_{k,l}$ for $k = 1,\dots,n$ and $l \in \mathbb{R}$, $l > 0$, by

$$\delta_{k,l}(x,y) = \max\left(\tfrac{1}{l}|x_k - y_k|, \max_{i\neq k, 1\le i\le n} |x_i - y_i|\right), \qquad x,y \in \mathbb{Z}^n. \tag{2.2.5}$$

Using this metric, we introduced an equivalent property for lateral convexity which stated as follows.

Theorem 2.2.3. *A function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is C-laterally convex, where C is defined by* (2.2.4)*, if and only if we have the following property for all $1 \le k \le n$;*

$$f(x) + f(y) \ge f(x+e_k) + f(y-e_k), \tag{2.2.6}$$

for each two points x and y in $\mathbb{Z}^n$ where $y_k - x_k = 2$ and $\delta_{k,2}(x,y) = 1$.

In Paper V, we showed that the class of C-laterally convex functions is a subclass of the class of integrally convex functions.

Theorem 2.2.4. *A C-laterally convex function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is integrally convex.*

By an example, we showed that the converse is not true in general. We also prove the following result.

Theorem 2.2.5. *Consider a unit hypercube $a + \{0,1\}^n$, $a \in \mathbb{Z}^n$. The sides of this hypercube is the set*

$$a + \sum_{i\in J} e_i,$$

where $I = \{1,\dots,n\}$ and J is a subset of I of cardinality m, $1 \le m \le n-1$. If an integrally convex function $f\colon \mathbb{Z}^n \to \mathbb{R}$ is modular on the sides of each unit hypercube $a + \{0,1\}^n$, $a \in \mathbb{Z}^n$, then it is C-laterally convex function.

Since a laterally convex function is integrally convex, we have indeed the property of local minimum.

Lateral convexity and marginal functions

In Paper VI, we studied the marginal functions of laterally convex functions.

For the two-dimensional case, Kiselman [22] proved the convexity of marginal functions for integrally (laterally) convex functions. He also represented some equivalent conditions for integral convexity by using marginal functions.

We studied the relation between A-lateral convexity and the interval (possibly empty) where the infimum defining the marginal function is attained.

Theorem 2.2.6. *Let us define, for any function $f\colon \mathbb{Z}^n \to \mathbb{R}$,*

$$M_f(x_1,\dots,x_{n-1}) = \{b \in \mathbb{Z}; f(x_1,\dots,x_{n-1},b) = \inf_{s\in\mathbb{Z}} f(x_1,\dots,x_{n-1},s)\},$$

where $(x_1,\dots,x_{n-1}) \in \mathbb{Z}^{n-1}$. We also define

$$f_\beta(x) = f(x) - \beta x_n, \qquad x = (x_1,\dots,x_n) \in \mathbb{Z}^n, \quad \beta \in \mathbb{R}.$$

Now let $a = (a', a_n) \in \mathbb{Z}^n$, where $a' = (a_1,\dots,a_{n-1})$ and $a_n \geq 0$, and define

$$A = \{(e^{(n)}, e^{(n)}), (e^{(n)}, (a', a_n)), (e^{(n)}, (-a', a_n))\}.$$

Then f is A-laterally convex if and only if $s \mapsto f(x,s)$ is convex extensible for every x and

$$M_{f_\beta}(x+a') \subset M_{f_\beta}(x) + [-a_n, a_n]_{\mathbb{Z}}, \qquad x \in \mathbb{Z}^{n-1}, \quad \beta \in \mathbb{R}.$$

By permuting the variables, we concluded that

Corollary 2.2.7. *Given a function $f\colon \mathbb{Z}^n \to \mathbb{R}$, we define, for $1 \leq j \leq n$ and $(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbb{Z}^{n-1}$,*

$$M_{j,f}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n-1}) = \left\{b \in \mathbb{Z}; f(x_1,\dots,x_{j-1},b,x_{j+1},\dots,x_{n-1},x_n) = \inf_{x_j\in\mathbb{Z}} f(x)\right\}.$$

We also define

$$f_{j,\beta}(x) = f(x) - \beta x_j, \quad x = (x_1,\dots,x_n) \in \mathbb{Z}^n, \quad j = 1,\dots,n, \quad \beta \in \mathbb{R}.$$

Fix a set A which contains $(a, e^{(j)})$ and $(\bar{a}, e^{(j)})$, where

$$\bar{a} = 2a_j e^{(j)} - a = (-a_1,\dots,a_j,\dots,-a_n),$$

and satisfies (2.2.2) *and* (2.2.3). *If f is A-laterally convex, then f is convex extensible in each variable separately and we have*

$$M_{j,f_{j,\beta}}(x+a') \subset M_{j,f_{j,\beta}}(x) + [-a_j, a_j]_{\mathbb{Z}}, \qquad x \in \mathbb{Z}^{n-1},$$

where now $a' = (a_1,\dots,a_{j-1},a_{j+1},\dots,a_n)$.

Kiselman [26] using the marginal functions $f_{1,\beta}$ and $f_{2,\beta}$, $\beta \in \mathbb{R}$, found a necessary and sufficient condition for a function of two integer variables to be integrally convex function.

Theorem 2.2.8. *Let $A \subset \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$ and $B \subset \mathbb{Z}^n \times \mathbb{Z}^n$ be given. We assume that* (2.2.2) *and* (2.2.3) *hold both for A and B. Assume that*

$$\textit{If } (a,b) \in A \textit{ and } s \in [-1,1]_{\mathbb{Z}}, \textit{then } ((a,s),(b,0)) \textit{ belongs to } \tilde{B}; \tag{2.2.7}$$

that

$$\textit{If there exists } c \in \mathbb{Z}^{n-1} \textit{ such that } (a,c) \in A, \textit{ then } ((a,1),e^{(n)}) \in \tilde{B}; \tag{2.2.8}$$

and finally that

$$\textit{If } ((a,1),e^{(n)}) \in B, \textit{ then } ((-a,1),e^{(n)}) \in \tilde{B}. \tag{2.2.9}$$

If $f\colon \mathbb{Z}^n \to \mathbb{R}$ is B-laterally convex, then its marginal function

$$h(x) = \inf_{s \in \mathbb{Z}} f(x,s), \qquad x \in \mathbb{Z}^{n-1},$$

is A-laterally convex, provided that it does not take the value $-\infty$.

By iteration we concluded that

Corollary 2.2.9. *Let us define $B^{(0)} = \{(0,0)\}$, $B^{(1)} = \{(1,1)\}$, and generally $B^{(n)} \subset \mathbb{Z}^n \times \mathbb{Z}^n$ such that $B^{(n)}$ and $B^{(n-1)}$ satisfy the conditions for B and A in Theorem 2.2.8, $n = 2,3,\ldots$ If $f\colon \mathbb{Z}^n \to \mathbb{R}$ is a given $B^{(n)}$-laterally convex function, then the successive marginal functions $h_n = f$,*

$$h_m(x) = \inf_{s \in \mathbb{Z}} h_{m+1}(x,s), \qquad x = (x_1,\ldots,x_m) \in \mathbb{Z}^m, \quad m = n-1,\ldots,0,$$

are $B^{(m)}$-laterally convex, provided that the constant h_0 is not $-\infty$. In particular, the marginal function h_1 of one variable is $\{(1,1)\}$-laterally convex, equivalently convex extensible.

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