Toric varieties and residues

Alexey Shchuplev

Department of Mathematics
Stockholm University
2007

Doctoral dissertation
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Typeset by LaTeX

ISBN 978-91-7155-461-1
Printed by US-AB, Stockholm, 2007

Contents

Introduction

The multidimensional residue theory as well as the theory of integral representations for holomorphic functions is a very powerful tool in complex analysis. The computation of integrals, solving algebraic or differential equations is usually reduced to some residue integral. It is a notable feature of the theory that it is based on few model differential forms called *kernels*. These are the Cauchy kernel in $\mathbb{C}^d$ whose singular set consists of all coordinate hyperplanes and the Bochner-Martinelli kernel with singularity at the origin, that is, the zero-dimensional coordinate subspace of $\mathbb{C}^d$. Observe that these two model kernels have been the source of other fundamental kernels and residue concepts by means of homological procedures.

For instance, starting with the Bochner-Martinelli formula in a domain $G \subset \mathbb{C}^d$, J. Leray was able to deduce a new integral representation formula (which he called the Cauchy-Fantappiè formula) by lifting the boundary ∂G into a complex quadric in $\mathbb{C}^{2d}$ preserving its homology class [30]. Later this idea was developed in the works of W. Koppelman [26] and others [1, 31], resulting in new representation formulas for $\overline{\partial}$-closed forms.

Another fundamental integral formula in complex analysis, namely, the Weil formula, turns out to be related to the Cauchy formula by the transformation law for the Grothendieck residue [46]. Moreover, the connection between those two formulas demonstrates how to produce various algebraic residue concepts, starting from the residue for a Laurent series as the coefficient c_{-I}. Lastly, let us point to the Sorani integral kernels [44], which are regular in $(\mathbb{C}^p)_* \times (\mathbb{C}_*)^{d-p}$. These forms appear as intermediate differential forms between the Cauchy and Bochner-Martinelli kernels in the proof of the de Rham (or Dolbeault) isomorphism using the Mayer-Vietoris cohomology principle [20].

Notice that the Cauchy and Bochner-Martinelli kernels, together with the various Sorani kernels, possess two common properties: firstly, their singular

sets are the unions of coordinate planes, and secondly, the top cohomology group of the complement to the singular set is generated by a single element.

This observation motivates the following definition. Let $\{E_\nu\}$ be a non empty finite collection of complex subspaces (of arbitrary dimensions) in $\mathbb{C}^d$.

Definition. *The family $\{E_\nu\}$ is said to be atomic if the top de Rham cohomology group $H^k(\mathbb{C}^d \setminus \bigcup_\nu E_\nu)$ is generated by one element:*

$$H^k\left(\mathbb{C}^d \setminus \bigcup_\nu E_\nu\right) \simeq \begin{cases} \mathbb{C} \ \textit{if} \ \ k = k_0, \\ 0 \ \textit{whenever} \ \ k > k_0. \end{cases}$$

A generating form of the group H^{k_0} is called a residue kernel for the atomic family $\{E_\nu\}$.

Let us illustrate the definition with some positive and negative examples.

1) The family of all coordinate hyperplanes $\mathcal{E} = \{E_\nu\}_{1\le\nu\le d}$, where

$$E_\nu := \{\zeta \in \mathbb{C}^d : \zeta_j = 0\}$$

is atomic; in this case the complement $\mathbb{C}^d \setminus \bigcup_\nu E_\nu$ is the d-dimensional complex torus $(\mathbb{C}_*)^d = \mathbb{T}^d$, which is homotopically equivalent to the real d-dimensional torus, so for this example $k_0 = d$; thus, the Cauchy form

$$\eta_C = \frac{d\zeta_1}{\zeta_1} \wedge \cdots \wedge \frac{d\zeta_d}{\zeta_d}$$

is a kernel for the family of all coordinate hyperplanes.

2) The family $\mathcal{E} = \big\{\{0\}\big\}$ consisting of the single zero dimensional coordinate plane $\{\zeta = 0\}$ is also atomic, since $\mathbb{C}^d \setminus \{0\}$ is homotopically equivalent to the $(2d-1)$-dimensional real sphere S^{2d-1}, in this example $k_0 = 2d-1$; so, the Bochner-Martinelli form

$$\eta_{BM} = \frac{\sum_{k=1}^d (-1)^{k-1} \overline{\zeta}_k d\overline{\zeta}[k] \wedge d\zeta}{||\zeta||^{2d}}$$

is a kernel for $\mathcal{E} = \big\{\{0\}\big\}$.

3) The family $\mathcal{E} = \{l_1, l_2, l_3\}$ of coordinate lines

$$l_1 := \{\zeta_2 = \zeta_3 = 0\}, l_2 := \{\zeta_1 = \zeta_3 = 0\}, l_3 := \{\zeta_1 = \zeta_2 = 0\}$$

in $\mathbb{C}^3$ is not atomic. In order to make this observation clear let us note that by de Rham's theorem it is enough to show that the top homology group $H_k(\mathbb{C}^3 \setminus \bigcup_j l_j)$ with coefficients in $\mathbb{Z}$ cannot be generated by a single element. By the Alexander-Pontryagin duality (see [2]) for any $k \in \mathbb{N}$ one has

$$H_k(\mathbb{C}^3 \setminus \bigcup_j l_j) = \widetilde{H}_{6-k-1}(\overline{l_1 \cup l_2 \cup l_3}),$$

where $\overline{l_1 \cup l_2 \cup l_3}$ denotes the closure of the lines l_j in the compactification S^6 of $\mathbb{C}^3$ and $\widetilde{H}_k$ denote the reduced homology groups; and since $\overline{l_1 \cup l_2 \cup l_3}$ consists of three 2-dimensional spheres with two common points (namely 0 and ∞), it is easy to see that

$$\widetilde{H}_{6-k-1}(\overline{l_1 \cup l_2 \cup l_3}) \simeq \begin{cases} \mathbb{Z}^2 \text{ if } k = 4, \\ 0 \text{ whenever } k > 4, \end{cases}$$

which proves $\{l_1, l_2, l_3\}$ is not an atomic family.

4) The family of 6 two-dimensional planes in $\mathbb{C}^4$

$$\mathcal{E} = \Big\{ \{\zeta_1 = \zeta_2 = 0\},\ \{\zeta_1 = \zeta_3 = 0\},\ \{\zeta_1 = \zeta_4 = 0\}, \\ \{\zeta_2 = \zeta_3 = 0\},\ \{\zeta_2 = z_4 = 0\},\ \{\zeta_3 = z_4 = 0\} \Big\}$$

is not atomic. The computations using the formula from [17, p. 238, Theorem A] show that top non-trivial homology group of the complement $H_5(\mathbb{C}^4 \setminus \bigcup_\nu E_\nu)$ is isomorphic to $\mathbb{Z}^3$.

The main objective of the thesis is to construct residue kernels for atomic families in $\mathbb{C}^d$. And although the problem of completely describing the atomic families is not yet solved, we may easily point to a rather large class of them. This class arises from the construction of toric varieties (Proposition 1.1). Notice here that the family of the fourth example above fails to be atomic, although it is related to the construction of a toric prevariety [40]. The extensively developed techniques of toric geometry applied to analytic and algebraic problems in complex analysis have already produced many explicit results. In the field of multidimensional residue theory one may recall, to mention but a few, the notion of toric residue [11], the toric residue mirror symmetry [7], the residue currents of Bochner-Martinelli type [36], the Vidras-Yger generalisation of the Jacobi residue formula [48].

In the thesis, we apply the methods of toric geometry to the following two questions of multidimensional residue theory:

- simplification of the proof of the Vidras-Yger generalisation of the Jacobi residue formula in the toric setting using the notion of a residue current of a holomorphic section of a holomorphic vector bundle;
- construction of a residue kernel associated with a toric variety and its applications in the theory of residues and integral representations.

The thesis consists of four chapters.

The first chapter of auxiliary character devoted to a brief introduction in toric geometry with focus on facts for future reference. The content of the chapter is taken mainly from [16, 33, 10]. Besides, we prove here a couple of statements which are used later on.

The atomic families appear as exceptional sets Z in the representation of toric varieties X [10] as quotients

$$(\mathbb{C}^d \setminus Z)\Big/G,$$

which generalises the well-known representation

$$\mathbb{C}^{n+1} \setminus \{0\}\Big/\sim$$

of the complex projective space $\mathbb{P}_n$. The combinatorial properties of Z and variety X itself are expressed in the terms of the associated with X fan Σ, which is a *rational polyhedral decomposition* of $\mathbb{R}^n$. In its turn, a fan defines (up to an isomorphism) a toric variety X_Σ, the exceptional set $Z(\Sigma)$ and group G in the representation of X_Σ. Furthemore, among other things, the fan of a toric variety allows explicitly compute the moment map associated with the Hamiltonian action of G on $\mathbb{C}^d \setminus Z(\Sigma)$ and the Kähler cone of X_Σ.

In **the second chapter** of illustrative character we consider the Vidras-Yger generalisation of the Jacobi residue formula. Recall that Jacobi's residue formula states if $P\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a polynomial mapping with discrete zero

set and some constraints on the growth at infinity then the sum of local residues of the meromorphic form

$$\operatorname{Res}\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]_{\mathbb{C}^n} := \sum_{w\in P^{-1}(0)} \operatorname{Res}_w\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]$$

equals zero as soon as

$$\deg Q \le \sum_{j=1}^{n} \deg P_j - n - 1.$$

Here the local (Grothendieck) residue of the above meromorphic form at a point $w \in P^{-1}(0)$ is defined by the integral

$$\frac{1}{(2\pi i)^n} \int\limits_{\Gamma_{\varepsilon,w}} \frac{Q(z)\,dz}{P_1(z)\dots P_n(z)},$$

where $\Gamma_{\varepsilon,w}$ is a connected component of the cycle

$$\{|P_1(z)| = \varepsilon, \dots, |P_n(z)| = \varepsilon\}$$

around w for sufficiently small ε. The orientation of this cycle is chosen such that $dP_1 \wedge \dots \wedge dP_n \ge 0$ on it.

This result is deeply related to the properties of P on the projective compactification of $\mathbb{C}^n$. As the projective space is a toric variety, the Jacobi formula has a toric version [24].

In 2001 A. Vidras and A. Yger managed to significantly weaken the restrictions on the mapping allowing it to have zeroes on compactifying hypersurfaces [48], in contrast to the classical case.

In this chapter we use the notion of a residue current of a holomorphic section of a holomorphic vector bundle introduced by M. Andersson [3] to give a shorter and, hopefully, clearer proof of the Vidras-Yger generalisation of the toric version of Jacobi's formula (Theorem 1). This result is going to appear in a forthcoming paper [42].

The central **third chapter** is devoted to the construction of a kernel for an atomic family coming from some toric variety X_Σ. The key role is played here by Theorem 2 stating that under some assumptions about the toric variety X_Σ it admits realisation as a "skeleton of infinity" of an ambient toric variety $X_{\tilde{\Sigma}}$, in much the same manner as the projective space $\mathbb{P}_n$ is the infinite hyperplane in the decomposition $\mathbb{P}_{n+1} = \mathbb{C}^{n+1} \sqcup \mathbb{P}_n$.

This realisation immediately leads to the construction of a kernel η for the atomic family X_Σ (Theorem 3). The model example for us here is the Bochner-Martinelli kernel, which fits the general construction as a kernel associated with the projective space. Still, the construction leaves the choice of a volume form on X_Σ essential for numerical computations, in section 3.3 we introduce the Fubini-Study volume form on X_Σ and compute the volume of the variety with respect to it. Firstly, this gives the value of the normalisation constant, and secondly, it makes it possible to interprete the integral representation with the kernel η as an averaging of the Cauchy formula with some measure over a polyhedron.

As any residue kernel, the form η produces a number of integral formulas which we prove in the final **fourth chapter**. These are the integral representation for holomorphic functions of the Bochner-Ono type (Theorem 4) and the logarithmic residue formula (Theorem 5). Moreover, in section 4.2 we exhibit a number of examples of atomic families and kernels for them.

The results of the two last chapters were partially published in [41] and [43].

The last result of the fourth chapter, which we do not prove, because it follows directly from Theorem 3 and [47, Theorem 1], is an integral realisation of the Grothendieck residue. This simple corollary opens the door for developing applications of the constructed kernels, for instance, to produce a class of residue currents associated with toric varieties. The limit cases represented by the Coleff-Herrera and the Bochner-Martinelli residue currents are well-known and already well studied. The research of the remaining ones is yet to be done.

Acknowledgements

I would like to express my cordial thanks to my supervisor Mikael Passare for his guidance and attention throughout all the stages of my work. I also feel deeply grateful to my mentor and supervisor August Tsikh for introducing me to the complex world of complex analysis, our fruitful discussions on numerous occasions, and his kind consent for including the results of our joint work in this thesis.

I am greatly indebted to all people at the department of Mathematics of Stockholm University for creating an atmosphere conducive to learning and work, especially to Tom and Marianne for their consideration, and to my colleagues David and Veronica for engaging conversations.

I really appreciate the helpful advice generously offered by Jan-Erik Björk and the valuable comments by Alain Yger who granted permission to include the results of our joint paper in this thesis.

Toric varieties

1. The definition and construction

Toric varieties is a class of algebraic varieties that generalises both projective and affine varieties. Besides those, this class includes also their products and many other varieties, for example the Hirzebruch surfaces. They are almost as simple to study but appear to be more convenient in many cases. It seems that the first definition of a toric variety is due to M. Demazure and says that an n-dimensional toric variety is a variety on which the action of the complex torus $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$ on itself by the component-wise multiplication extends to an action on the whole variety [14].

Toric varieties are characterised by the property that they admit an atlas with monomial transition functions. This fact allows us to express their algebraic properties in a purely combinatorial way and associate a toric variety with a *fan*.

Let us give basic definitions. Let N be a free $\mathbb{Z}$-module of rank n and $M = \mathrm{Hom}(N, \mathbb{T})$ be its dual. A subset σ of $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ is called a *strongly convex rational polyhedral cone* if there exists a finite number of elements $v_1, \ldots, v_s$ in the lattice N (*generators*) such that σ is generated by them, i.e.

$$\sigma = \{a_1 v_1 + \cdots + a_s v_s \colon a_i \in \mathbb{R}, a_i \geq 0\},$$

and σ does not contain any line. We say that a subset τ of σ given by some a_i being equal to zero is a *proper face* of σ and write $\tau < \sigma$. Faces of a cone

are cones also. The *dimension* of a cone σ is, by definition, the dimension of a minimal subspace of $\mathbb{R}^n$ containing σ. A cone σ is called *simplicial* if its generators can be chosen to be linearly independent. A k-dimensional simplicial cone is said to be *primitive* if its k generators form a part of a basis of the lattice N.

Definition 1.1. *A fan in $N \otimes \mathbb{R}$ is a non-empty collection Σ of strongly convex rational polyhedral cones in $N \otimes \mathbb{R}$ satisfying the following conditions:*

1. *Every face of any σ in Σ is contained in Σ.*
2. *For any σ, σ' in Σ, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.*

The set $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ is called the support of Σ.

The *dimension* of a fan is the maximal dimension of its cones. An n-dimensional fan is *simplicial* (*primitive*) if all its n-dimensional cones are simplicial (primitive). In the case $|\Sigma| = \mathbb{R}^n$, the fan is called *complete*.

Let Σ be a fan in $N \otimes \mathbb{R}$. Each k-dimensional cone $\sigma^{(k)}$ in Σ (generated by v_{i_j}) defines a finitely generated semigroup $\sigma \cap N$. The dual $(n-k)$-dimensional cone

$$\check{\sigma} = \left\{ m \in M \otimes \mathbb{R} \colon \langle m,\, v_{i_j} \rangle \geq 0 \right\}$$

is then a rational polyhedral cone in $M \otimes \mathbb{R}$ and $\check{\sigma} \cap M$ is also a finitely generated semigroup. An affine (n-dimensional) toric variety corresponding to $\sigma^{(k)}$ is the variety

$$U_\sigma := \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M].$$

If a cone τ is a face of σ then $\check{\tau} \cap M$ is a subsemigroup of $\check{\sigma} \cap M$, hence U_τ is embedded into U_σ as an open subset. The affine varieties corresponding to all cones of the fan Σ are glued together according to this rule into the toric variety X_Σ associated with Σ.

2. The homogeneous coordinates

The projective space however, being also a toric variety, is usually defined differently. The projective space $\mathbb{CP}_n$ is the set of all lines passing through the origin in $\mathbb{C}^{n+1}$. The same definition can be reformulated as follows. Consider the equivalence relation $\sim$ on the set of non-zero points $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$x \sim y \text{ iff } y = (\lambda x_1, \ldots, \lambda x_{n+1}) \text{ for } \lambda \in \mathbb{T}.$$

Then the projective space is the set of all equivalence classes.

Every point $x = (x_1, \ldots, x_{n+1}) \neq 0$ determines an element of the projective space, namely the line passing through the point x and the origin. This line is the equivalence class of all points proportional to x. As only the ratio of coordinates is then of interest, the equivalence class is commonly denoted by the $(n+1)$-tuple of *homogeneous coordinates* $(x_1 \colon \ldots \colon x_{n+1})$ which we may refer to as the Cartesian coordinates of a point in $\mathbb{C}^{n+1}$.

If one dimensional cones of Σ span $N \otimes \mathbb{R}$, in particular if the fan is complete, the corresponding toric variety X_Σ admits an analogous construction (see [10]). Let the cones of Σ be generated by d integral generators $v_1, \ldots, v_d$, $d \geq n$. Assign a variable ζ_i to each generator v_i. For every n-dimensional cone $\sigma \in \Sigma$, let $\zeta_{\hat{\sigma}}$ be the monomial

$$\zeta_{\hat{\sigma}} := \prod_{\substack{j \in \{1,\ldots,d\} \\ v_j \notin \sigma}} \zeta_j$$

and $Z(\Sigma) \subset \mathbb{C}^d$ be the zero set of the ideal generated by such monomials $\zeta_{\hat{\sigma}}$ in $\mathbb{C}[\zeta_1, ..., \zeta_d]$, i.e.

$$Z(\Sigma) = \{\zeta \in \mathbb{C}^d : \zeta_{\hat{\sigma}} = 0 \text{ for all } n\text{-dimensional cones } \sigma \text{ in } \Sigma\}. \tag{1.1}$$

The set $Z(\Sigma)$ assumes the role of the origin in the representation of the projective space as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the diagonal action of the complex torus $\mathbb{T}$. To define a group action in the analogous construction of a toric variety X_Σ, one utilises the Chow group $A_{n-1}(X_\Sigma)$ of Weil divisors modulo rational equivalence on X_Σ. The Chow group is given by the exact sequence

$$0 \longrightarrow M \xrightarrow{\nu} \mathbb{Z}^d \longrightarrow A_{n-1}(X_\Sigma) \longrightarrow 0, \tag{1.2}$$

where ν sends $m \in M$ to

$$(\langle m,\, v_1\rangle, \ldots,\, \langle m,\, v_d\rangle).$$

Then any element g of the group G which is defined to be

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma),\, \mathbb{T})$$

acts on $\zeta \in \mathbb{C}^d \setminus Z(\Sigma)$ according to

$$g \cdot \zeta = (g([\mathcal{D}_1])\zeta_1, \ldots,\, g([\mathcal{D}_d])\zeta_d) \tag{1.3}$$

with $[\mathcal{D}_i]$, $i = 1, \ldots, d$ being the classes of the basis elements of $\mathbb{Z}^d$ in $A_{n-1}(X_\Sigma)$.

Let us say some words on the structure of G. Being the quotient

$$A_{n-1}(X_\Sigma) = \mathbb{Z}^d \big/ \nu(M),$$

the Chow group is a direct sum of a free group of rank $r = d - n$ and a finite group. If any n of the integral generators span N, for example Σ contains a primitive n-dimensional cone, then $A_{n-1}(X_\Sigma) = \mathbb{Z}^r$ via mapping

$$x \mapsto (k_1(x), \ldots,\, k_r(x))$$

where $k_1, \ldots, k_r$ are linear equations defining the n-dimensional kernel of ν and G is isomorphic to $\mathbb{T}^r$. In general, the group G has a factor of finite order.

In order to describe the action of G when it has no torsion we consider the lattice of relations between generators of Σ, i.e. r linearly independent over $\mathbb{Z}$ relations between $v_1, \ldots, v_d$

$$\begin{cases} a_{11}v_1 + \cdots + a_{1d}v_d = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}v_1 + \cdots + a_{rd}v_d = 0. \end{cases} \tag{1.4}$$

These linear forms define the kernel of ν and are in fact the k_j's mentioned before. Thus by (1.3) the isomorphic to $\mathbb{T}^r$ factor of G defines an equivalence relation on $\mathbb{C}^d \setminus Z(\Sigma)$

$$\xi \sim \zeta \Leftrightarrow \exists \lambda \in \mathbb{T}^r \colon \xi = (\lambda_1^{a_{11}} \ldots \lambda_r^{a_{r1}}\zeta_1, \ldots,\, \lambda_1^{a_{1d}} \ldots \lambda_r^{a_{rd}}\zeta_d). \tag{1.5}$$

When G has no torsion, the equivalence relation on $\mathbb{C}^d \setminus Z(\Sigma)$ is given by this formula, in general the G-action is more complicated.

Let us consider an example of such a situation.

Example 1. A toric variety with the Chow group with torsion.

Let us consider in $\mathbb{R}^2$ a complete fan Σ generated by vectors $(1,\, 0)$, $(-1,\, 2)$, and $(-1,\, -2)$.

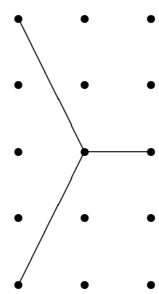


Figure 1.1: Fan Σ in $\mathbb{R}^2$.

The set $Z(\Sigma)$ here consists only of the origin, but what distinguishes this example from the case of the projective space is that the generating vectors do not span $\mathbb{Z}^2$. Therefore the orbits of G-action are not lines passing through the origin.

The image of ν in $\mathbb{Z}^3$ are precisely those points that admit the representation $(m_1,\, -m_1 + 2m_2,\, -m_1 - 2m_2)$, $(m_1,\, m_2) \in \mathbb{Z}^2$, therefore the class of $(a,\, b,\, c) \in \mathbb{Z}^3$ contains $(0,\, a+b,\, a+c)$ and either $(0,\, 0,\, 2a+b+c)$ or $(0,\, 1,\, 2a+b+c-1)$, depending on the parity of $a+b$. Thus, the Chow group $A_1(X_\Sigma)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$ via mapping

$$(a,\, b,\, c) \longrightarrow ((a+b) \bmod 2,\; 2a+b+c-((a+b) \bmod 2).$$

The group G consists then of two series of homomorphisms g and h defined as

$$\begin{aligned} &g(0,\, a) = \lambda^a,\; g(1,\, a-1) = \lambda^a \\ &h(0,\, a) = \lambda^a,\; h(1,\, a-1) = -\lambda^a, \quad a \in \mathbb{Z}. \end{aligned}$$

Notice that the only nontrivial element of second order in the Chow group, namely $(1,\, -1)$, is mapped to identity in the torus by the first series of homomorphisms and to the only nontrivial element of second order in the

torus, -1, by the second. Because the classes of the basis elements of $\mathbb{Z}^3$ in $A_1(X_\Sigma)$ are (1, 1), (1, 0), and (0, 1), respectively, the equivalence relation on $\mathbb{C}^3 \setminus \{0\}$ is given by

$$\xi \sim \zeta \Leftrightarrow \exists \lambda \in \mathbb{T}\colon \begin{array}{rcl} \xi & = & (\lambda^2\zeta_1,\, \lambda\zeta_2,\, \lambda\zeta_3) \\ & \text{or} & \\ \xi & = & (-\lambda^2\zeta_1,\, -\lambda\zeta_2,\, \lambda\zeta_3). \end{array}$$

Hence, the orbits of G-action consist of two curves.

It follows from [10] that the toric variety X_Σ can be represented as a geometric quotient

$$X_\Sigma = \big(\mathbb{C}^d \setminus Z(\Sigma)\big)\Big/G,$$

with $Z(\Sigma)$ and G defined as above. We may as well take this representation as a definition of X_Σ. The ζ_i's in this representation become the homogeneous coordinates of points of the toric variety (the G-orbits). The ring $\mathbb{C}[\zeta_1 \ldots, \zeta_d]$ is called *the homogeneous coordinate ring* of X_Σ. If z is a coordinate system in the torus then the coordinates of $z \in \mathbb{T}^n \subset X_\Sigma$ can be expressed by the monomials

$$\begin{cases} z_1 = \zeta_1^{v_1^1} \ldots \zeta_d^{v_d^1}, \\ \ldots\ldots\ldots\ldots, \\ z_n = \zeta_1^{v_1^n} \ldots \zeta_d^{v_d^n}, \end{cases} \qquad \text{or } z = \zeta^V. \tag{1.6}$$

Obviously, the set $Z(\Sigma)$ consists of coordinate planes, in general, of different dimensions. For complete simplicial fans there is an equivalent construction of the set $Z(\Sigma)$ [6]. A subset of generators $\{v_i,\, i \in I\}$ as well as the index set $I \subset \{1, \ldots, d\}$ is called a *primitive collection* if the vectors v_i do not generate any cone of Σ but every proper subset of them does so. All primitive collections I constitute a set that we shall denote by $\mathcal{P}$. Then the set $Z(\Sigma)$ coincides with the union of coordinate planes

$$Z(\Sigma) = \bigcup_{\mathcal{P}} \{\zeta_{i_1} = \cdots = \zeta_{i_k} = 0\},$$

where the union is taken over all primitive collections. This case (Σ being complete and simplicial) is particularly interesting because the set $Z(\Sigma)$ is an atomic family.

Proposition 1.1. *The collection $Z(\Sigma)$ is an atomic family.*

Proof. Since Σ is a complete fan, the set $\mathbb{C}^d \setminus Z(\Sigma)$ is a bundle over the compact n-dimensional complex simplicial toric variety X_Σ, with fibers being homeomorphic to several copies of the torus $\mathbb{T}^r$. In other words, $\mathbb{C}^d \setminus Z(\Sigma)$ is homotopically equivalent to some oriented compact real analytic cycle of real dimension $2n + r = d + n$ that one can interpret as a bundle over connected compact oriented manifold X_Σ with several disjoint copies of the real torus $\underbrace{S^1 \times \cdots \times S^1}_{r \text{ times}}$ as a fiber. □

Remark. It would be interesting to deduce Proposition 1 from the result of Goresky-MacPherson [17] (see also [13]), which could help to estabilish a general criterion for the atomicity property to hold.

Furthermore, complete fans have the following property that we use in the construction.

Proposition 1.2. *Let Σ be a complete fan in $N \otimes \mathbb{R} = \mathbb{R}^n$ with d generators. Then all coefficients a_{ij} in the lattice of relations* (1.4) *can be chosen non-negative.*

This fact seems to be well-known. Here we give its simple proof.

Proof. Let the coefficient a_{jk} in the identity $a_{j1}v_1 + \cdots + a_{jd}v_d = 0$ be negative. Consider then the vector $-v_k$. For the fan is complete, this vector lies in some cone generated by the vectors $v_{i_1}, \ldots, v_{i_m}$ and therefore can be represented as linear combination of them $-v_k = b_{i_1}v_{i_1} + \cdots + b_{i_m}v_{i_m}$ with all non-negative coefficients. The identity we started with is equivalent then to $a_{j1}v_1 + \cdots + a_{jd}v_d + |a_{ij}|(v_k + b_{i_1}v_{i_1} + \cdots + b_{i_m}v_{i_m}) = 0$ with the coefficient at v_k being equal zero and containing no new negative coefficients. Proceeding in this way we get all a_{ij} being non-negative. □

3. The T-invariant divisors

As with any set on which a group acts, X_Σ equipped with the action of torus T is a disjoint union of T-orbits. There is one such orbit for each cone of the fan. Therefore each of the one-dimensional cones of Σ determines a T-invariant irreducible Weil divisor $\mathcal{D}_i$, $i = 1, \ldots, d$ (the closure of the

corresponding orbit). The set $\mathbb{Z}^d$ with the basis consisting of $\mathcal{D}_i$'s is in fact the group of such divisors on X_Σ. The homogeneous coordinates ζ_i assigned to each one dimensional cone of Σ turn out to actually define these divisors

$$\mathcal{D}_i = \{\zeta_i = 0\}.$$

Any monomial z^α in coordinates z of the torus $\mathbb{T}^n \subset X_\Sigma$ determines a T-Cartier divisor

$$\operatorname{div}(z^\alpha) = \sum_{i=1}^{d} \langle \alpha,\, v_i \rangle\, \mathcal{D}_i.$$

In its turn, each T-Cartier divisor $D = \sum a_i \mathcal{D}_i$ on X_Σ determines a rational convex polyhedron

$$\Delta_D = \{m \in M \colon \langle m,\, v_i \rangle \geq -a_i\},$$

moreover the global sections of the line bundle $\mathcal{O}(D)$ are directly related to Δ_D

$$\Gamma(X_\Sigma,\, \mathcal{O}(D)) = \bigoplus_{m \in \Delta_D \cap M} \mathbb{C} \cdot z^m. \tag{1.7}$$

Conversely, a piecewise linear function

$$\{\varphi(v) = \langle m_\sigma,\, v \rangle \ \text{ if } v \in \sigma\} \tag{1.8}$$

on the cones of the fan Σ with the property: if $\tau < \sigma$ then for $v \in \tau$ $\langle m_\tau,\, v \rangle = \langle m_\sigma,\, v \rangle$, and all $m_\sigma \in M$, comes from a unique T-Cartier divisor

$$D = \sum_{i=1}^{d} -\varphi(v_i) \mathcal{D}_i. \tag{1.9}$$

If, in addition, φ is convex on the support $|\Sigma|$ of the fan Σ in $N \otimes \mathbb{R}$ with all maximal cones being n-dimensional, then $\mathcal{O}(D)$ is generated by its sections (the converse is also true).

This motivates the following definition

Definition 1.2. *A complete fan Σ in $N \otimes \mathbb{R}$ is compatible with a convex polyhedron Δ in $M \otimes \mathbb{R}$ if the function*

$$\varphi_\Delta(v) := \min_{m \in \Delta \cap M} \langle m,\, v \rangle$$

is linear on each cone of the fan.

Example 2. Compatible and non-compatible fans and polyhedra.

A complete fan in $\mathbb{R}^2$ generated only by vectors $(1,\,0)$, $(0,\,1)$, $(-1,\,0)$, $(0,\,-1)$ is obviously compatible with a square. Adding one more generator, for example $(1,\,1)$, does not change the situation. This shows that subdivision of fans preserves the property of being compatible with the fixed polyhedron.

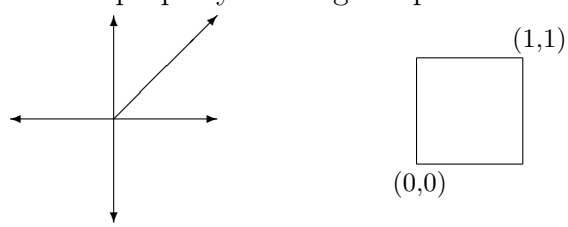


Figure 1.2: Compatible fan and polyhedron.

The fan and polyhedron in the following pair are not compatible because the function φ_Δ is not linear on the cone generated by $(-1,\,0)$ and $(0,\,-1)$. Refining this cone by adding $(-1,\,-1)$ to the list of generators we can make them compatible.

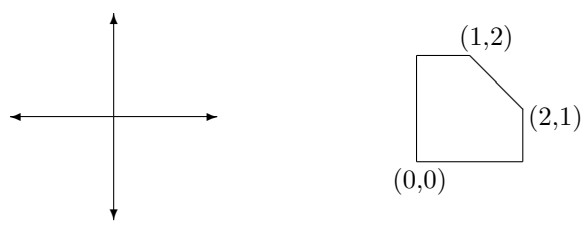


Figure 1.3: Non-compatible fan and polyhedron.

If polyhedron Δ and fan Σ are compatible then Δ defines a very ample T-Cartier divisor D_Δ (1.9) on X_Σ satisfying (1.7). This divisor is principal and given by the monomial function (written in the homogeneous coordinates)

$$\zeta^\Delta := \prod_{i=1}^{d} \zeta_i^{-\varphi_\Delta(v_i)}.$$

The line bundle $\mathcal{O}(D)$ is generated by its sections and if

$$\Delta \cap M = \{m_1, \ldots, m_N\}$$

then

$$z \mapsto (z^{m_1} : \ldots : z^{m_N}) \tag{1.10}$$

is a mapping to the projective space $\mathbb{P}_{N-1}$. This mapping is a closed embedding if for any n-dimensional cone σ the points $m_\sigma \in M$ are all different (i.e. φ_Δ is *strictly convex*) and the semigroup $\check{\sigma} \cap M$ is generated by $\{m_\sigma - m \colon m \in \Delta \cap M\}$.

In particular, this holds whenever Σ is the dual fan to an absolutely simple polyhedron Δ, i.e. the one dimensional generators of Σ are the (inward) normals to faces of Δ.

Definition 1.3. *The convex hull of a finite number of points in M is called a simple integral polytope if it is n-dimensional, and each of its vertex belongs to exactly n edges. The simple polytope is absolutely simple if, in addition, minimal integer vectors on n edges meeting at a vertex generate the lattice N.*

Notice also that if the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are generated by their sections then $\Delta_{D+E} = \Delta_D + \Delta_E$ (see [16], p. 69).

4. Projective toric varieties

Consider the case of smooth compact projective toric varieties in more details because there is even more information one can recover from the fan. The fan associated to such a variety is complete and primitive [33, Theorem 1.10] and the dual integral polyhedron is absolutely simple.

Generally, let M be a smooth complex manifold endowed with a closed nondegenerated differential form $\omega \in \bigwedge^2 \mathsf{T}^*M$, which makes (M, ω) into a symplectic manifold. The canonical example is a complex plane $\mathbb{C}$ with the form $\omega = -\frac{1}{2i} d\zeta \wedge d\bar{\zeta}$. If M is equipped with a Hermitian metric H and the associated differential form $\omega = -\mathrm{Im}(H)$ is closed then M is called *Kähler* manifold and ω *Kähler* form.

Let G be a Lie group acting on (M, ω) by diffeomorphisms

$$g \in G \colon \zeta \mapsto g \cdot \zeta.$$

A group action is called *symplectic* if every diffeomorphism $g \in G$ preserves the symplectic form ω. For every $\zeta \in M$, define a map

$$f_\zeta : G \to M,\ f_\zeta(g) = g \cdot \zeta$$

such that the image of G under this mapping $G \cdot \zeta$ is a *flow* or an *orbit* of the group actions. Its differential map at the point $1 \in G$ is a linear map

$$\mathsf{T}_1 f_\zeta : \mathfrak{g} \to \mathsf{T}_\zeta M$$

associating a tangent vector $\mathsf{T}_1 f_\zeta(X) = \underline{X}_\zeta \in \mathsf{T}_\zeta M$ to every direction $X \in \mathfrak{g}$. When ζ varies in M, we get a vector field $\underline{X}$ called the *fundamental field* associated with X.

Definition 1.4. *A vector field X on a symplectic manifold (M, ω) is called Hamiltonian if $\imath_X \omega$ is exact and locally Hamiltonian if it is closed. One writes $\mathcal{H}(M)$ and $\mathcal{H}_{loc}(M)$ for the spaces of Hamiltonian and locally Hamiltonian vector fields on M, respectively.*

Obviously, there is an exact sequence

$$0 \longrightarrow \mathcal{H}(M) \longrightarrow \mathcal{H}_{loc}(M) \longrightarrow H^1(M, \mathbb{R})$$

As ω is non-degenerate, every C^∞-function f defines a Hamiltonian vector field X_f via $\imath_{X_f} = df$. If the group action is symplectic, then all fundamental vector fields are locally Hamiltonian [5, Prop. 3.1.1.]. Combining this with the sequence, we get the following diagram

$$\begin{tikzcd}
 & C^\infty(M) \arrow[d] & \mathfrak{g} \arrow[d] & \\
0 \arrow[r] & \mathcal{H}(M) \arrow[r] & \mathcal{H}_{loc}(M) \arrow[r] & H^1(M, \mathbb{R})
\end{tikzcd}$$

Definition 1.5. *A symplectic action of G on M is Hamiltonian if there exists a linear map (morphism of Lie algebras) $\widetilde{\mu} : \mathfrak{g} \to C^\infty(M)$ making the diagram commute.*

By duality, there is an associated map μ of dual spaces

$$\mu : (C^\infty(M))^* = M \longrightarrow \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$$

defined by

$$\mu : \zeta \mapsto (X \mapsto \widetilde{\mu}_X(\zeta))$$

called the *moment map*.

In our case, the set $\mathbb{C}^d \setminus Z(\Sigma)$ endowed with the form

$$\omega = -\frac{1}{2i}\sum_{j=1}^{d} d\zeta_j \wedge d\bar{\zeta}_j$$

is a symplectic manifold. Consider the action of the maximal compact subgroup $G_{\mathbb{R}}$ of the group G defined in (1.5)

$$G_{\mathbb{R}} = \{(\lambda_1^{a_{11}} \dots \lambda_r^{a_{r1}}, \dots, \lambda_1^{a_{1d}} \dots \lambda_r^{a_{rd}})\colon \lambda_i \in S^1 \subset \mathbb{T}\}.$$

The action of $G_{\mathbb{R}}$ is clearly symplectic. The Lie algebra $\mathfrak{t}$ of $G_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^r$ as well as its dual. Denoting the columns of coefficients in (1.4) by $a^k = (a_{1k}, \dots, a_{rk})$, $k = 1, \dots, d$, we can write down the fundamental field for every $X = (x_1, \dots, x_r) \in \mathbb{R}^r$:

$$\underline{X} = -i\sum_{k=1}^{d}\langle a^k, X\rangle \left(\bar{\zeta}_k \frac{\partial}{\partial \bar{\zeta}_k} - \zeta_k \frac{\partial}{\partial \zeta_k}\right).$$

The interior product of $\underline{X}$ with the symplectic form ω is then

$$\imath_{\underline{X}}\omega = \frac{1}{2}\sum_{k=1}^{d}\langle a^k, X\rangle \left(\zeta_k d\bar{\zeta}_k + \bar{\zeta}_k d\zeta_k\right),$$

which is the full differential with respect to ζ of the function

$$\widetilde{\mu}_X(\zeta) = \frac{1}{2}\sum_{k=1}^{d}\left(\sum_{j=1}^{r} a_{jk}x_j|\zeta_k|^2\right) = \frac{1}{2}\sum_{j=1}^{r}\langle a_j, |\zeta|^2\rangle x_j,$$

where $a_j = (a_{j1}, \dots, a_{jd})$, $j = 1, \dots, r$ are the rows of the coefficients and $|\zeta|^2 = (|\zeta_1|^2, \dots, |\zeta_d|^2)$. So, for every $\zeta \in \mathbb{C}^d \setminus Z$, the image $\mu(\zeta)$ is a point $(\rho_1, \dots, \rho_r)$ in $\mathfrak{t}^* \simeq \mathbb{R}^r$ with coordinates

$$\begin{cases} a_{11}|\zeta_1|^2 + \dots + a_{1d}|\zeta_d|^2 = \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1|^2 + \dots + a_{rd}|\zeta_d|^2 = \rho_r. \end{cases} \tag{1.11}$$

For $\rho \in \mu(\mathbb{C}^d \setminus Z) \subseteq \mathfrak{t}^*$, the cycle $\mu^{-1}(\rho)$ is a smooth manifold, but the restriction of the symplectic form ω to $\mu^{-1}(\rho)$ will fail to be symplectic as it

will be degenerate. However, it is degenerate only along the orbits of action of $G_{\mathbb{R}}$, then the restriction of ω descends to the quotient $\mu^{-1}(\rho)/G_{\mathbb{R}}$ as a symplectic form. This process is called *symplectic reduction*. In such a way we get the representation of $\mathbb{P}_n$ as the quotient S^{2n+1}/S^1. It turns out that $\mu^{-1}(\rho)/G_{\mathbb{R}}$ is another representation of X_Σ.

Theorem [12]. *Let Σ be a complete primitive fan in $\mathbb{R}^n$ with d integral generators and $\rho \in \mu(\mathbb{C}^d \setminus Z)$. Then the map*

$$\mu^{-1}(\rho)/G_{\mathbb{R}} \longrightarrow (\mathbb{C}^d \setminus Z)/G = X_\Sigma$$

is a diffeomorphism.

The image of $\mathbb{C}^d \setminus Z(\Sigma)$ in $\mathfrak{t}^* \simeq A_{n-1}(X_\Sigma) \otimes \mathbb{R}$ under the moment map is usually identified with the interior of the *Kähler cone* of X_Σ. The image $\mu\left(\mathbb{C}^d \setminus Z(\Sigma)\right)$ with non-empty interior [6, Theorem 4.5] consists of strictly convex linear functions (1.8) up to linear functions. When X_Σ is smooth this cone coincides with the cone in $A_{n-1}(X_\Sigma) \otimes \mathbb{R} \simeq H^{1,1}(X_\Sigma, \mathbb{R}) \simeq \mathbb{R}^r$ of cohomology classes of closed positive (i.e. Kähler) forms on X_Σ. In the general case the analogous fact holds with the definition slightly modified [4].

There is a description of the Kähler cone of X_Σ using primitive collections of the fan. Let $\mathcal{P}_I = \{v_{i_1}, \dots, v_{i_k}\}$ be a primitive collections for the fan Σ. For the fan is complete, the sum $\sum_{i \in I} v_i$ belongs to some cone of Σ generated by $\{v_j\}$, $j \in J$, so

$$\sum_{i \in I} v_i = \sum_{j \in J} c_j v_j$$

with all c_j being positive rational numbers. Since the relations (1.4) are the basis of all relations between generators, this relation can be rewritten as

$$\sum_{i \in I} v_i - \sum_{j \in J} c_j v_j = t_1^I(a_{11}v_1 + \dots + a_{1d}v_d) + \dots + t_r^I(a_{r1}v_1 + \dots + a_{rd}v_d). \quad (1.12)$$

Then the system

$$l_I(\rho) = t_1^I \rho_1 + \dots + t_r^I \rho_r > 0 \quad (1.13)$$

for all primitive collections of Σ defines the Kähler cone of X_Σ in $\mathbb{R}^r$.

Remark. We have not used other properties of the generators except for the lattice of relations (1.4) to obtain relations (1.12). Therefore they are valid for any symbols satisfying the lattice of relations, including $|\zeta_i|^2$.

On the Vidras-Yger theorem

1. Generalisations of the Jacobi residue formula

One of the classical results of the one dimensional complex analysis is the following: for any polynomials $P(z)$ and $Q(z)$ the sum of residues of the form $Q(z)dz/P(z)$ at the zeroes of $P(z)$ is equal to the residue of this form at infinity with the opposite sign.

With the computation reduced to the computation at one point, we can make a simple observation: the residue of the form $Q(z)dz/P(z)$ at infinity is equal to zero if

$$\deg Q \leq \deg P - 2.$$

Passing on to the multidimensional situation we find that we cannot any longer uniquely define the residue at infinity because there is a wide choice of different compactifications of $\mathbb{C}^n$. But one can still generalise the observation made, so let us consider a polynomial mapping

$$P = (P_1, \ldots, P_n) \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n,$$

such that

the set of its common zeroes $V_{\mathbb{C}^n}(P)$ in $\mathbb{C}^n$ is finite and consists of the maximal possible number of points, namely $\deg P_1 \cdots \deg P_n$. (2.1a)

Then the sum of local residues of the meromorphic form $\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}$

$$\operatorname{Res}\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]_{\mathbb{C}^n} = \sum_{w\in V_{\mathbb{C}^n}(P)} \operatorname{Res}_w\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]$$

equals zero as soon as

$$\deg Q \le \sum_{j=1}^{n} \deg P_j - n - 1. \tag{2.1b}$$

This result goes back to an article by C. Jacobi [21] where he proves a number of relations for two polynomials generalising the Lagrange interpolation formula. These relations are equivalent to the formulated theorem.

This result has a toric counterpart. Let $\mathbb{T}^n = (\mathbb{C}\setminus\{0\})^n$ be a complex torus and $P = (P_1, \dots, P_n)$ be n Laurent polynomials in n variables with the Newton polyhedra Δ_j, $j = 1\dots, n$ satisfying the Bernstein condition [8]. To formulate this condition let us for $\xi \in \mathbb{R}^n\setminus\{0\}$ denote by $\widetilde{P}_{j,\xi}$ the sum of only those terms $c_{\alpha_j} z^{\alpha_j}$ of P_j whose powers satisfy

$$\langle \alpha_j, \xi\rangle = \max_{\eta\in\Delta_j}\langle \eta, \xi\rangle.$$

Then the Bernstein condition says that

for any $\xi \in \mathbb{R}^n\setminus\{0\}$ the intersection of the set $\left\{z\in\mathbb{C}^n\colon \widetilde{P}_{j,\xi}(z) = 0\right\}$ with the torus $\mathbb{T}^n$ is empty.

It implies that

the common zero set $V_{\mathbb{T}^n}(P)$ of P in $\mathbb{T}^n$ is finite with cardinality equal to $n!$ times the Minkowski mixed volume of $\Delta_1, \dots, \Delta_n$. (2.2a)

Under this hypothesis A. Khovanskii [24] noticed that the sum of toric residues

$$\operatorname{Res}\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]_{\mathbb{T}^n} := \sum_{w\in V_{\mathbb{T}^n}(P)} \operatorname{Res}_w\left[\frac{Q(z)}{P_1(z)\dots P_n(z)}\frac{dz}{z_1\dots z_n}\right]$$

equals zero for any Laurent polynomials $Q(z)$ whose Newton polyhedron Δ_Q lies in the interior of the polyhedron $\Delta_1 + \cdots + \Delta_n$:

$$\Delta_Q \subset (\Delta_1 + \cdots + \Delta_n)^\circ. \tag{2.2b}$$

Let us stress it here that both these results are corollaries of the more general theorem [19] stating that the total sum of local residues of a meromorphic $(n, 0)$-form on a compact n-dimensional manifold equals zero. The hypotheses of the mentioned theorems only ensure that when considering the integrand as being defined on some compact manifold X (such as $\mathbb{CP}_n$ in the first case or an appropriate compact toric manifold in the second case) it has no other poles apart from the hypersurfaces $P_j^{-1}(0)$, $j = 1, \ldots, n$ and the mapping P does not have any common zeroes except for $V_X(P)$.

One may now try to weaken the hypothesis which forbids the mapping to have common zeroes outside $\mathbb{C}^n$ (or $\mathbb{T}^n$ in the second case) and concentrate on controlling the numerator $Q(z)$. An important role here is played by the analytic interpretation of conditions (2.1a) and (2.2a). From this point of view condition (2.1a) is equivalent (see, for example, [48, Proposition 2.1]) to the following strong properness condition on the polynomial map $P = (P_1, \ldots, P_n)$: there exist strictly positive constants R and c such that for $||z|| \geq R$

$$\sum_{j=1}^{n} \frac{|P_j(z)|}{(1 + ||z||^2)^{\deg P_j/2}} \geq c.$$

One can weaken this condition decreasing the powers of $(1 + ||z||^2)$ allowing $V(P)$ to have components on the infinite hyperplane of $\mathbb{CP}_n$, possibly of positive dimension. In this situation A. Vidras and A. Yger, using the Bochner-Martinelli integral formula, were able to present a generalisation of the Jacobi formula [48] with applications to effectivity issues of the Nullstellensatz and results of Caley-Bacharach type.

The first theorem in their article proves the vanishing of the total sum of residues in the case of the projective space.

Theorem [48, Theorem 1.1]. *Let $P = (P_1, \ldots, P_n)$ be a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and assume that there exist positive constants c, R, and rational numbers $0 < \delta_j \leq \deg P_j$, $j = 1, \ldots, n$, such that for $||z|| \geq R$*

$$\sum_{j=1}^{n} \frac{|P_j(z)|}{(1 + ||z||^2)^{\delta_j/2}} \geq c.$$

Then for any polynomial Q which satisfies

$$\deg Q \leq \delta_1 + \cdots + \delta_n - n - 1$$

the total sum of residues vanishes

$$\operatorname{Res}\left[\frac{Q(z)\,dz}{P_1(z)\dots P_n(z)}\right]_{\mathbb{C}^n} = 0.$$

A possible analytic interpretation of the Bernstein condition was given by A. Kazarnovskii [22] (cf. [37]). Following his idea, Vidras and Yger gave a suitable modification of this interpretation [48, Proposition 2.2]: a polynomial map $P = (P_1, \dots, P_n)$ satisfies (2.2a), or equivalently, the Bernstein condition, if there exist strictly positive constants R and c such that for $z \in \mathbb{T}^n$ with $||\log|z||| \geq R$

$$\sum_{j=1}^{n} \frac{|P_j(z)|}{\mathrm{e}^{H_{\Delta_j}(\log|z|)}} \geq c,$$

where $\log|z|$ denotes the vector $(\log|z_1|, \dots, \log|z_n|)$ and H_{Δ_j} denotes the support function of the convex polyhedron defined as

$$H_{\Delta_j}(x) := \sup_{\xi \in \Delta_j} \langle x, \xi \rangle. \tag{2.3}$$

The generalisation of the toric version of the Jacobi formula has been formulated as follows:

Theorem 1. [48, Theorem 1.2] *Let $P = (P_1, \dots, P_n)$ be a system of Laurent polynomials in n variables with respective Newton polyhedra $\Delta_1, \dots, \Delta_n$. Suppose there exist constants $c > 0$, $R > 0$, and convex polyhedra $\delta_1, \dots, \delta_n$ with rational vertices such that $\delta_j \subset \Delta_j$, $j = 1, \dots, n$,*

$$\dim(\delta_1 + \cdots + \delta_n) = n, \tag{2.4}$$

and for any $z \in \mathbb{T}^n$ with $||\log|z||| \geq R$

$$\sum_{j=1}^{n} \frac{|P_j(z)|}{\mathrm{e}^{H_{\delta_j}(\log|z|)}} \geq c. \tag{2.5}$$

Then for any Laurent polynomial Q with the Newton polyhedron Δ_Q such that

$$\Delta_Q \subset (\delta_1 + \cdots + \delta_n)^\circ \tag{2.6}$$

one has

$$\operatorname{Res}\left[\frac{Q\,dz}{P_1 \dots P_n}\right]_{\mathbb{T}^n} = 0.$$

Our aim here is to give a shorter proof of this theorem using the notion of residue current of a holomorphic section of a holomorphic vector bundle introduced by M. Andersson [3].

2. Residue current of a holomorphic section

These currents appear as an obstacle when solving the division problem by means of the Koszul complex. Let $p_1, \dots, p_m$ and φ be holomorphic functions in $\mathbb{C}^n$ then one looks for holomorphic ψ_i such that

$$\sum_{i=1}^{m} p_i \psi_i = \varphi. \tag{2.7}$$

In the invariant setting we consider $p = (p_1, \dots, p_m)$ as a holomorphic section of the dual bundle E^* of a holomorphic bundle $E \to X$ of rank m over an n-dimensional manifold X. If $e_1, \dots, e_m$ is a local holomorphic frame for E and $e_1^*, \dots, e_m^*$ is the dual frame the we can write $p = \sum_j p_j e_j^*$. On the exterior algebra of E there is defined an interior multiplication by p

$$\delta_p \colon \bigwedge^{l+1} E \longrightarrow \bigwedge^{l} E.$$

For example, for section $\psi = \sum_j \psi_j e_j$ of E the result of this contraction is $\delta_p \psi = \sum \psi_j p_j$. Therefore the original problem can be reformulated as finding a holomorphic solution to

$$\delta_p \psi = \varphi. \tag{2.8}$$

One can easily find a smooth solution to this equation outside $V_X(p)$, it is

$$u_1 = \varphi \sum_{j=1}^{m} \frac{p_j}{|p|^2} e_j.$$

In order to improve this solution one must successively solve equations

$$\delta_p u_k = \bar\partial u_{k-1} \tag{2.9}$$

where u_k is a differential $(0,\, k-1)$-form with coefficient in $\bigwedge^k E$. This is always possible outside the common zero set of p because the contraction δ_p and the exterior derivation $\bar\partial$ anticommute and because the Koszul complex is exact there

$$\delta_p \bar\partial u_{k-1} = -\bar\partial \delta_p u_{k-1} = -\bar\partial\bar\partial u_{k-2} = 0.$$

The sequence $\{u_k\}$ will finally terminate (at least $\bar\partial u_m = 0$). Suppose now that all u_k have current extensions across $V_X(p)$ such that (2.9) holds. Then if X is Stein, by successively solving

$$\bar\partial v_{k-1} = u_{k-1} + \delta_p v_k$$

we arrive at the holomorphic solution to (2.8)

$$\psi = u_1 + \delta_p v_2.$$

Thus, in general all equations in (2.9) cannot hold, and the residue current associated with the section p corrects these equaton to make them hold across $V_X(p)$.

Let

$$\mathcal{L}^r(X,\, E) = \bigoplus_{k-l=r} \mathcal{D}'_{0,\,k}(X,\, \bigwedge^l E)$$

where $\mathcal{D}'_{0,\,k}(X,\, \bigwedge^l E)$ is the space of currents of bidegree $(0,\, k)$ with coefficients in $\bigwedge^l E$, then the sequence $\{u_k\}$ is an element of $\mathcal{L}^{-1}(X,\, E)$, and letting

$$\nabla_p := \delta_p - \bar\partial \colon \mathcal{L}^r \longrightarrow \mathcal{L}^{r+1}$$

we may rewrite (2.8) and (2.9) as a single equation

$$\nabla_p u = \varphi.$$

Notice that if $\nabla_p u = 1$ in $X \setminus V_X(p)$ then $\nabla_p(u\varphi) = \varphi$ for any holomorphic function φ. In order to construct a solution u, let us endow E with a Hermitian metric and let s be the section of E with the pointwise minimal norm such that $\delta_p s = |p|^2$. Outside $V_X(p)$, the Cauchy-Fantappiè-Leray form

$$u = \frac{s}{\nabla_p s} = \sum_{l=1}^{m} \frac{s \wedge (\bar\partial s)^{l-1}}{(\delta_p s)^l}$$

is well-defined and solves the equation. Note that in the case of trivial bundle $E = \mathbb{C}^m \times X$ and trivial metric, the top degree term is the Bochner-Martinelli form.

The extension U of u to X is the value at $\lambda = 0$ of the current $|p|^{2\lambda}u$ (which is smooth for $\mathrm{Re}\lambda > m$). The residue current R^p is the value at $\lambda = 0$ of the current $\bar\partial|p|^{2\lambda} \wedge u$ and is related to U by

$$\nabla_p U = 1 - R^p.$$

It follows immediately that if X is Stein then this current is the only obstacle for solving the division problem (2.7).

In our proof of the Vidras-Yger theorem we use the following properties of this current:

- R^p has its support on $V_X(p)$;
- it admits a decomposition $R^p = R^p_k + \cdots + R^p_l$ where $R^p_j \in \mathcal{D}'_{0,j}(X, \bigwedge^j E)$, $k = \operatorname{codim} V_X(p)$, and $l = \min(m, n)$;
- if h is a holomorphic function that vanishes on $V_X(p)$ and $|h| \le C||p||^j$, then $hR^p_j = 0$;
- although the current is defined without the requirement that p be a complete intersection, in this special case it has only one component R^p_m and coincides with the Coleff-Herrera residue current.

Using this notion M. Andersson gave a short and elegant proof of the generalisation of the Jacobi formula in projective case. Here we give a similiar proof in the toric case, although, in essence, our proof is close to the original one.

3. The proof of the vanishing theorem

Let us begin the proof by first considering the special case when all polyhedra $\delta_j = \Delta_j$. Let Σ be a simple refinement of the fan dual to the Minkowski sum of all polyhedra $\Delta = \Delta_1 + \cdots + \Delta_n$ and X_Σ be a smooth toric variety associated with Σ. Since every Δ_j is compatible Σ, it defines a T-Cartier divisor $D_{\Delta_j} = \sum_i a_i D_i$ where $a_i = -\min\limits_{m \in \Delta_j \cap \mathbb{Z}^n} \langle m, v_i \rangle$.

Consider the homogenization p_j of P_j as a global section of $\mathcal{O}(D_{\Delta_j})$, then the system $p = (p_1, \dots, p_n)$ is a section of $E^* = \mathcal{O}(D_{\Delta_1}) \oplus \cdots \oplus \mathcal{O}(D_{\Delta_n})$ with discrete zeroes only in $\mathbb{T}^n \subset X_\Sigma$. The residue current R^p associated with p is a $\det E^*$-valued current and coincides with the Coleff-Herrera residue current. Since all $\mathcal{O}(D_{\Delta_j})$ are very ample, $\det E^* = \mathcal{O}(D_{\Delta_1} + \cdots + D_{\Delta_n}) = \mathcal{O}(D_\Delta)$. Then the $\mathcal{O}(D_\Delta)$-homogenization of $Q(z) \wedge dz/z$ is

$$s(z) = \zeta^\Delta Q(\zeta) \frac{E(\zeta)}{\zeta_1 \dots \zeta_n}$$

where $E(\zeta)$ is the toric Euler form [11]

$$E(\zeta) = \sum_{|I|=n} \left(\prod_{j \notin I} \zeta_j \right) \det(v_I) d\zeta_I. \tag{2.10}$$

The condition on Δ_Q ensures that $s(\zeta)$ is a global $\det E$-valued holomorphic $(n, 0)$-form on X_Σ. Thus

$$\int_X q \wedge R^p = \operatorname{Res} \left[\frac{Q\, dz}{P_1 \dots P_n} \right]_{\mathbb{T}^n}$$

but this integral is equal to zero as a an integral of $\bar\partial$-exact (n, n)-form.

Let us now move on to the general case. We may assume that all polyhedra δ_j are integral. Choose an n-dimensional integral polyhedron Δ containing the origin such that $\Delta_j \subset \Delta + \delta_j$, $j = 1, \dots, n$ and write

$$\widetilde{\Delta} = n\Delta + \delta_1 + \cdots + \delta_n.$$

Then let Σ be a simple refinement of the fan dual to $\widetilde{\Delta}$ and let X_Σ be the toric variety corresponding to Σ.

Each polyhedron $\Delta + \delta_j$ defines a T-Cartier divisor $D_{\Delta+\delta_j}$ on X_Σ and a very ample bundle $\mathcal{O}(D_{\Delta+\delta_j})$. Because of the choice of Δ every $P_j(z)$ can be considered as a global section of this line bundle and can be written in the homogeneous coordinates as

$$p_j(\zeta) = \zeta^{\Delta+\delta_j} P_j(\zeta).$$

The system $p = (p_1, \dots, p_n)$ is then a section of

$$E^* = \mathcal{O}(D_{\Delta+\delta_1}) \oplus \cdots \oplus \mathcal{O}(D_{\Delta+\delta_n})$$

and defines a residue current R^p with the support on the common zero set of p. Notice that condition on the growth (2.5) implies that $V_{X_\Sigma}(p)$ is still compact in $\mathbb{T}^n$, hence discrete, and the current R^p constructed with respect to any metric on E coincides there with the Coleff-Hererra residue current. Our goal is to show that if $s(\zeta)$ is a $\det E$-homogenization of $Q(z)dz/z$ then $s(\zeta) \wedge R^p$ vanishes as a current in the neighbourhood of all $\mathcal{D}_i$, $i = 1, \dots, d$. For that, we compare the order of vanishing of both factors at $\mathcal{D}_i$.

In order to estimate the growth of $P(z)$, let us for every j, $j = 1, \dots, n$ choose $n+1$ Laurent polynomials $G_0^{(j)}, \dots, G_n^{(j)}$ with Newton polyhedron $\Delta + \delta_j$ that do not vanish simultaneously in $\mathbb{T}^n$ and such that the system $(G_1^{(j)}, \dots, G_n^{(j)})$ satisfies (2.2a) or, equivalently, the Bernstein condition. Since all $\Delta + \delta_j$ are compatible with Σ, this implies that the function

$$|G^{(j)}(z)| := \sum_{k=0}^{n} |G_k^{(j)}(z)|$$

extended to X_Σ does not vanish there. We define a metric on E^* by

$$||p||_{E^*} := \sum_{j=1}^{n} \frac{|P_j(\zeta)|}{|G^{(j)}(\zeta)|}.$$

The triangle inequality yields

$$|G^{(j)}(z)| \leq c_j e^{H_{\Delta+\delta_j}(\log|z|)},$$

indeed, by definition of the support function (2.3) the value of $e^{H_{\Delta+\delta_j}(\log|z|)}$ is the maximal value of all monomials in any of $G_k^{(j)}(z)$, and for $|\log|z|| \geq R$ we have

$$||p||_{E^*} \geq c' \sum_{j=1}^{n} \frac{|P_j(z)|}{e^{H_{\Delta+\delta_j}(\log|z|)}}$$

with $c' = \min_j \frac{1}{c_j}$.

Next we need to introduce $n+1$ Laurent polynomials $H_0, \dots, H_n$ with Newton polyhedron Δ which do not vanish simultaneously in $\mathbb{T}^n$ and such that the system $(H_1, \dots, H_n)$ satisfies (2.2a). Because Δ is n-dimensional the function $|H(z)| := |H_0(z)| + \dots + |H_n(z)|$ does not vanish in $\mathbb{T}^n$. Moreover, this function is bounded from below

$$c'' e^{H_\Delta(\log|z|)} \leq |H_1(z)| + \dots + |H_n(z)| \leq |H(z)|,$$

see [37, Lemme 2.1] or the proof of [48, Proposition 2.2]. Taking into account that $H_{\Delta+\delta_j} = H_\Delta + H_{\delta_j}$ and the hypothesis of the theorem we have

$$||p||_{E^*} \geq \frac{\widetilde{c}}{|H(z)|}$$

in the neighbourhood of the divisors $\mathcal{D}_i$. We may rewrite it in the homogenouos coordinates as

$$||p||_{E^*} \geq \frac{\widetilde{c}|\zeta^{D_\Delta}|}{|h(\zeta)|} =: \widetilde{c}||\zeta^{D_\Delta}||_{\mathcal{O}(D_\Delta)}.$$

Consider now the $\det E$-homogenization $s(\zeta)$ of $Q(z)\frac{dz}{z}$. Since

$$\det E = \mathcal{O}(D_{n\Delta+\delta_1+\cdots+\delta_n})$$

the section $s(\zeta)$ is

$$s(\zeta) = \zeta^{n\Delta+\delta_1+\cdots+\delta_n} Q(\zeta)\frac{E(\zeta)}{\zeta_1\ldots\zeta_n} = \zeta^{n\Delta} q(\zeta)\prod_{i=1}^{d}\zeta_i^{\mu_i}E(\zeta)$$

where $\mu_i = -\min\limits_{\eta\in\delta_1+\cdots+\delta_n}\langle\eta,\, v_i\rangle + \min\limits_{m\in\Delta_Q}\langle\eta,\, v_i\rangle - 1$. It remains to notice that all powers in $\zeta^{n\Delta}$ are non-negative because Δ contains the origin, therefore $\zeta^{n\Delta}$ is holomorphic at $\mathcal{D}_i$. Moreover, conditions (2.4) and (2.6) guarantee that all the numbers μ_i are non-negative. Thus, in the heighbourhood of all T-Cartier divisors $s(\zeta) = \zeta^{n\Delta}g(\zeta)$ where g is a holomorphic $(n,\, 0)$-form, and hence $s(\zeta)\wedge R^p = 0$ close to any of toric hyperplanes $\mathcal{D}_i$.

Embedding theorem and residue kernel

1. The multidimensional perspective

The Renaissance (linear or one-dimensional) perspective has given rise to the notion of projective space $\mathbb{P}_n$ as the set of all lines $\{l\}$ in an affine space $\mathbb{C}^{n+1}$ passing through the center point. Moreover $\mathbb{P}_n$ can be attached 'at the infinity' to the space $\mathbb{C}^{n+1}$ to form $\mathbb{P}_{n+1} = \mathbb{C}^{n+1} \sqcup \mathbb{P}_n$ in such a way that the closure $\bar{l}$ in $\mathbb{P}_{n+1}$ of every line intersects the attached $\mathbb{P}_n$ in a single point corresponding to l. In this section we shall prove an analogous result for toric varieties. Note that with the exception of the weighted projective spaces, all toric varieties are defined as spaces of r-dimensional orbits where $r \geq 2$. This allows us to interprete the embedding theorem as the *multidimensional perspective*.

Assume that a complete fan Σ in $\mathbb{R}^n$ contains at least one simple n-dimensional cone. With this assumption made, we prove the following

Theorem 2. *Let Σ be a simplicial complete fan in $\mathbb{R}^n$ with d generators. There exists a d-dimensional simplicial and compact toric variety*

$$X_{\widetilde{\Sigma}} = \mathbb{C}^d \sqcup (\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r)$$

with 'infinite' toric hypersurfaces $\mathcal{X}_1, \ldots, \mathcal{X}_r$ such that its 'skeleton' $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ is isomorphic to X_Σ. Moreover, for every $\zeta \in \mathbb{C}^d \setminus Z(\Sigma) \subset X_{\widetilde{\Sigma}}$

the closure $\overline{G \cdot \zeta}$ *of its orbit in* $X_{\widetilde{\Sigma}}$ *intersects 'the skeleton of infinity' in a unique point corresponding to the class of* ζ *under the isomorphism.*

To illustrate this assertion we can consider an embedding of $\mathbb{P}_1 \times \mathbb{P}_1$ into

$$\mathbb{P}_2 \times \mathbb{P}_2 = \mathbb{C}^4 \sqcup (\mathcal{X}_1 \cup \mathcal{X}_2)$$

as the intersection of two infinite hypersurfaces $\mathcal{X}_1 = \mathbb{P}_1^\infty \times \mathbb{P}_2$ and $\mathcal{X}_2 = \mathbb{P}_2 \times \mathbb{P}_1^\infty$. The fan Σ that corresponds to $\mathbb{P}_1 \times \mathbb{P}_1$has four generators (see Figure 3.1), and the set $Z(\Sigma) = E_{13} \cup E_{24}$ with $E_{13} = \{\zeta_1 = \zeta_3 = 0\}$ and $E_{24} = \{\zeta_2 = \zeta_4 = 0\}$ in $\mathbb{C}^4$. The relative arrangement of these objects is depicted on Figure 3.2 where the orbit $G \cdot \zeta$ is specified by the shadowed area and 'the skeleton of infinity' $X_\Sigma = \mathcal{X}_1 \cap \mathcal{X}_2$ attached to $\mathbb{C}^4$ is the ridge.

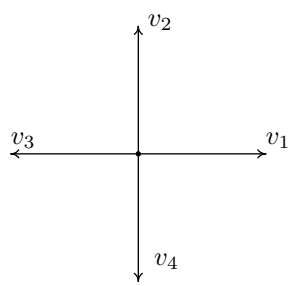


Figure 3.1.

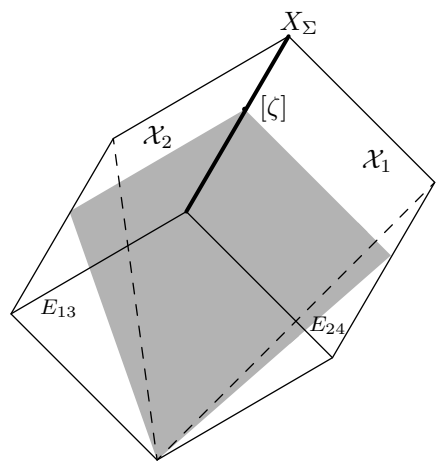


Figure 3.2

Proof. Let us construct the fan $\widetilde{\Sigma}$ in $\mathbb{R}^d$ (and by that the corresponding toric variety $X_{\widetilde{\Sigma}}$) starting from the fan Σ.

Without loss of generality, we may assume that the simple cone in Σ, mentioned in the theorem, is generated by the canonical basis $v_1 = e_1, \ldots, v_n = e_n$ of $\mathbb{Z}^n \subset \mathbb{R}^n$. Consider in $\mathbb{Z}^d = \mathbb{Z}^{n+r}$ the following $d + r$ primitive vectors:

$$\begin{aligned} \widetilde{e}_1 = (e_1,\, 0''), \quad \ldots \quad &, \widetilde{e}_n = (e_n,\, 0''), \\ \widetilde{e}_{n+1} = (0',\, e_1''), \quad \ldots \quad &, \widetilde{e}_{n+r} = (0',\, e_r''), \\ \widetilde{v}_{n+1} = (v_{n+1},\, -e_1''), \quad \ldots \quad &, \widetilde{v}_{n+r} = (v_{n+r},\, -e_r''), \end{aligned} \tag{3.1}$$

where $(e_1'', \ldots, e_r'')$ denotes the canonical basis of $\mathbb{R}^r$ and $0'$, $0''$ the neutral elements in $\mathbb{Z}^n$ and $\mathbb{Z}^r$, respectively. These $n + 2r$ distinct vectors span

$\mathbb{R}^d = \mathbb{R}^{n+r}$ as a vector space; they are going to form the set of 1-dimensional cones of our d-dimensional fan $\widetilde{\Sigma}$. We need then to describe how to organize the cones of the fan $\widetilde{\Sigma}$, starting with this collection of generators.

Let $I := \{1, ..., n\}$ and $J := \{n+1, ..., n+r\}$. The prescription rules for the organization of d-dimesional cones suggest the following three steps:

- choose any n-dimensional cone $\sigma = \langle v_{m_1}, ..., v_{m_n} \rangle$ in the fan Σ and divide the set of indices $\{m_1, ..., m_n\}$ into

$$K := \{m_1, ..., m_n\} \cap I, \quad L := \{m_1, ..., m_n\} \cap J;$$

- divide (in some arbitrary way) the complement $J \setminus L$ into two disjoint subsets Q and S, so that one gets an ordered partition $\{Q,\, S\}$ of $J \setminus L$;

- consider the d-dimensional cone $\widetilde{\sigma} = \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_L \rangle$.

Define now the fan $\widetilde{\Sigma}$ as the collection of all such d-dimensional cones $\widetilde{\sigma}$ together with all their faces. The following proposition holds.

Proposition 3.1. *The collection $\widetilde{\Sigma}$ is a complete simplicial fan in $\mathbb{R}^{n+r}$; furthermore, if Σ is simple so is $\widetilde{\Sigma}$.*

Proof of the proposition. Let us note first that the cardinals $|K|$, $|L|$, $|Q|$, $|S|$ satisfy

$$|K| + |L| = n \ , \ |L| + |Q| + |S| = r,$$

and all d generators of a cone $\widetilde{\sigma}$ are linearly independent, so every such cone is d-dimensional and simplicial.

The proof reduces to the verification of three points.

1. <u>*$\widetilde{\Sigma}$ is a fan.*</u>

To prove this, it is enough to show that if one of the generators of $\widetilde{\Sigma}$ (taken from the list (3.1)) is not an edge in some d-dimensional cone $\widetilde{\sigma}$, then it does not belong to this cone. Let us take such a cone $\widetilde{\sigma}$ and order all the index sets obtained. Write the coordinates of all the cone's generators (with respect to the basis $(\widetilde{e}_1, \ldots, \widetilde{e}_{n+r})$ into the rows of the $(n+r) \times (n+r)$-matrix

	$\overbrace{\quad\quad}^{n}$	$\overbrace{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}^{r}$			
$\widetilde{e}_K$	e_K	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$
$\widetilde{e}_L$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{matrix}$
$\widetilde{e}_Q$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$
$\widetilde{v}_S$	v_S	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} -1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -1 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$
$\widetilde{v}_L$	v_L	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}$	$\begin{matrix} -1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -1 \end{matrix}$
	I	Q	S	L

The generators that are not used as edges in the construction of the cone are:

$$\begin{array}{ll} (a) & \widetilde{e}_{I\setminus K}; \\ (b) & \widetilde{e}_S; \\ (c) & \widetilde{v}_Q. \end{array}$$

We consider cases a), b), c) separately.

a) Let $i \in I \setminus K$. If we assume that $\widetilde{e}_i \in \widetilde{\sigma}$ then in its representation as a non-negative linear combination of generators of $\widetilde{\sigma}$ the coefficients at $\widetilde{v}_S$ must be zeroes. It follows from the fact that the S-coordinates of $\widetilde{e}_i$ are zero, but at the same time in the S-column of the matrix for $\widetilde{\sigma}$ non-zero elements stand only on the corresponding diagonal of the block $\widetilde{v}_S$-row and they are negative. Hence, the vector e_i must belong to the cone $\langle e_K, v_L \rangle$ of the fan Σ that contradicts the definition of a fan and the assumption that $i \in I \setminus K$.

b) If $\widetilde{e}_s \in \widetilde{\sigma}$ then in its representation as a non-negative linear combination of generators of $\widetilde{\sigma}$ the coefficients at v_s must be -1, which is impossible.

c) This case is analogous to the previous one, it is enough to consider the Q-column.

2. *If Σ is primitive then so is $\widetilde{\Sigma}$.*

Add $\widetilde{e}_L$-rows to the $\widetilde{v}_L$-rows of the matrix for $\widetilde{\sigma}$. Then it is easy to see that the determinant

$$\det \widetilde{\sigma} = \det \begin{pmatrix} e_K \\ v_L \end{pmatrix} = \det \sigma$$

is equal to ± 1 in the case Σ is primitive.

3. *Completeness of Σ implies completeness of $\widetilde{\Sigma}$.*

In order to prove this it is enough to show that each $(d-1)$-dimensional face of any d-dimensional cone $\widetilde{\sigma}$ of $\widetilde{\Sigma}$ is a face of some other d-dimensional cone of $\widetilde{\Sigma}$.

There are five types of $(d-1)$-dimensional faces for an arbitrary d-cone $\widetilde{\sigma}$ as follows

$$\begin{aligned}
\tau_k &= \langle \tilde{e}_{K\setminus k},\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ k \in K;\\
\tau_l &= \langle \tilde{e}_K,\, \tilde{e}_{L\setminus l},\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ l \in L;\\
\tau_q &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_{Q\setminus q},\, \tilde{v}_S,\, \tilde{v}_L \rangle,\ q \in Q;\\
\tau_s &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_{S\setminus s},\, \tilde{v}_L \rangle,\ s \in S;\\
\tau_l' &= \langle \tilde{e}_K,\, \tilde{e}_L,\, \tilde{e}_Q,\, \tilde{v}_S,\, \tilde{v}_{L\setminus l} \rangle,\ l \in L.
\end{aligned}$$

Consider each of these types separately.

1) *faces of the type τ_k.* The sets of indices $K \subset I$ and $L \subset J$ are so definied that $\sigma = \langle e_K,\, v_L \rangle$ is an n-dimensional cone of Σ. The cone $\langle e_{K\setminus k},\, v_L \rangle$ is then a face of σ and since the fan Σ is complete, this cone must be a face of some other n-dimensional cone of the fan. We distinguish two cases

i) $\langle e_{K\setminus k},\, v_L \rangle \in \Sigma$ is a face of a cone of type $\langle e_{(K\setminus k)\cup i},\, v_L \rangle \in \Sigma$, $k \in K$, $i \in I$;

ii) $\langle e_{K\setminus k},\, v_L \rangle \in \Sigma$ is a face of a cone of type $\langle e_{K\setminus k},\, v_{L\cup j} \rangle \in \Sigma$, $k \in K$, $j \in J \setminus L$.

In the first case the cone τ_k is a face of the cone

$$\langle \widetilde{e}_{(K\setminus k)\cup i},\, \widetilde{e}_L,\, \widetilde{e}_Q,\, \widetilde{v}_S,\, \widetilde{v}_L \rangle.$$

In the second case it is a face of the cone

$$\langle \widetilde{e}_{K\setminus k},\, \widetilde{e}_{L\cup j},\, \widetilde{e}_{Q\setminus j},\, \widetilde{v}_S,\, \widetilde{v}_{L\cup j}\rangle \text{ if } j\in Q,$$

(this face is obtained by the elimination of $\widetilde{v}_j$ and renaming of vectors $\widetilde{e}_{L\cup j}$ and $\widetilde{e}_{Q\setminus j}$ by $\widetilde{e}_L$ and $\widetilde{e}_Q$) and of the cone

$$\langle \widetilde{e}_{K\setminus k},\, \widetilde{e}_{L\cup j},\, \widetilde{e}_{Q},\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{L\cup j}\rangle \text{ if } j\in S$$

(by the same reasoning).

Obviously, each of these three d-dimensional cones does not coincide with $\widetilde{\sigma}$ and belongs to $\widetilde{\Sigma}$, for instance, the last cone belongs to $\widetilde{\Sigma}$, because by the assumption $\langle e_{K\setminus k},\, v_{L\cup j}\rangle$ is a cone of Σ and $\{Q,\, S\setminus j\}$ is a partition of $J\setminus(L\cup j)$ (recall that $(Q,\, S)$ is a partition of $J\setminus L$ by the definition of $\widetilde{\sigma}$).

2) <u>*faces of the type τ_l.*</u> There are two cases:

iii) $\langle e_K,\, v_{L\setminus l}\rangle \in \Sigma$ is a face of a cone of type $\langle e_{K\cup i},\, v_{L\setminus l}\rangle \in \Sigma$, $l\in L$, $i\in I\setminus K$.

iv) $\langle e_K,\, v_{L\setminus l}\rangle \in \Sigma$ is a face of a cone of type $\langle e_K,\, v_{(L\setminus l)\cup j}\rangle \in \Sigma$, $l\in L$, $j\in J\setminus K$.

In case iii) the cone τ_l is a face of the cone

$$\langle \widetilde{e}_{K\cup i},\, \widetilde{e}_{L\setminus l},\, \widetilde{e}_{Q},\, \widetilde{v}_{S\cup l},\, \widetilde{v}_{L\setminus l}\rangle.$$

In the latter it is a face of the cone

$$\langle \widetilde{e}_{K},\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{Q\setminus j},\, \widetilde{v}_{S},\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j\in Q$$

and of the cone

$$\langle \widetilde{e}_{K},\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{Q},\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j\in S.$$

All these three d-dimensional cones do not coincide with $\widetilde{\sigma}$.

3) <u>*faces of the type τ_q.*</u> The cone τ_q is a face of the cone

$$\langle \widetilde{e}_{K},\, \widetilde{e}_{L},\, \widetilde{e}_{Q\setminus q},\, \widetilde{v}_{S\cup q},\, \widetilde{v}_{L}\rangle,$$

which differs from $\widetilde{\sigma}$.

4) <u>*faces of the type τ_s.*</u> The cone τ_s is a face of the cone

$$\langle \widetilde{e}_{K},\, \widetilde{e}_{L},\, \widetilde{e}_{Q\cup s},\, \widetilde{v}_{S\setminus s},\, \widetilde{v}_{L}\rangle,$$

which is different from $\widetilde{\sigma}$ as well.
5) <u>*faces of the type τ_l'.*</u> The proof is similiar to the case 2) of faces of the type τ_l. Two possibilities, iii) and iv), are conceivable. In case iii) the cone τ_l' is a face of the cone

$$\langle \widetilde{e}_{K\cup i},\, \widetilde{e}_{L\setminus l},\, \widetilde{e}_{Q\cup l},\, \widetilde{v}_S,\, \widetilde{v}_{L\setminus l}\rangle,$$

and in case iv) it is a face of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{(Q\cup l)\setminus j},\, \widetilde{v}_S,\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j\in Q$$

and of the cone

$$\langle \widetilde{e}_K,\, \widetilde{e}_{(L\setminus l)\cup j},\, \widetilde{e}_{Q\cup l},\, \widetilde{v}_{S\setminus j},\, \widetilde{v}_{(L\setminus l)\cup j}\rangle \text{ if } j\in S.$$

□

Now we can finish the proof of theorem. Compare the actions of the groups G and $\widetilde{G}$ on $\mathbb{C}^d\setminus Z(\Sigma)$ and $\mathbb{C}^{d+r}\setminus Z(\widetilde{\Sigma})$, respectively. Note that every relation

$$\mu_1 v_1+\cdots+\mu_d v_d=0$$

between the generators $v_1,\dots,\, v_d$ of the fan Σ defines the relation

$$\mu_1\widetilde{v}_1+\cdots+\mu_d\widetilde{v}_d+\mu_{n+1}\widetilde{v}_{d+1}+\cdots+\mu_{n+r}\widetilde{v}_{d+r}=0$$

between the generators

$$\widetilde{v}_i=\widetilde{e}_i,\ i=1,\dots,\, n,\ \widetilde{v}_{d+j}=\widetilde{e}_{n+j},\ j=1,\dots,\, r,\ \widetilde{v}_k,\ k=n+1,\dots,\, d$$

of the fan $\widetilde{\Sigma}$ (see (3.1)). So, if a basis of relations between $v_1,\dots,\, v_d$ consists of the vectors

$$\begin{array}{l}\mu^1=(\mu_{11},\dots,\,\mu_{1d})=p^1\oplus q^1\in\mathbb{Z}^n\oplus\mathbb{Z}^r_{\geq},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \mu^r=(\mu_{r1},\dots,\,\mu_{rd})=p^r\oplus q^r\in\mathbb{Z}^n\oplus\mathbb{Z}^r_{\geq},\end{array}$$

then the vectors

$$\begin{array}{l}\widetilde{\mu}^1=p^1\oplus q^1\oplus q^1,\\ \dots\dots\dots\dots\dots\\ \widetilde{\mu}^r=p^r\oplus q^r\oplus q^r\end{array}$$

can be chosen so that they constitute a basis of relations between $\widetilde{v}_1, \dots, \widetilde{v}_{d+r}$.

Note that $q^1, \dots, q^r$ can be chosen such that they form an identity matrix. Assign a complex variable to each of generators of the fan $\widetilde{\Sigma}$ in the following way

$$\begin{gathered}\widetilde{e}_j \longleftrightarrow \xi_j,\ j = 1, \dots, d,\\ \dots\dots\dots\dots\dots\dots\\ \widetilde{v}_{n+j} \longleftrightarrow \xi_{d+j},\ j = 1, \dots, r.\end{gathered}$$

Therefore the group actions occuring in the definitions of the varieties X_Σ and $X_{\widetilde{\Sigma}}$ are as follows:

$$G = (\lambda^p,\, \lambda^q) = (\lambda_1^{p_1^1} \dots \lambda_r^{p_1^r}, \dots,\, \lambda_1^{p_n^1} \dots \lambda_r^{p_n^r},\, \lambda_1^{q_1^1} \dots \lambda_r^{q_1^r}, \dots,\, \lambda_1^{q_r^1} \dots \lambda_r^{q_r^r}),$$

$$\widetilde{G} = (\lambda^p,\, \lambda^q,\, \lambda^q).$$

Consider now the coordinate chart $\widetilde{U} \simeq \mathbb{C}^d$ of the variety $X_{\widetilde{\Sigma}}$ corresponding to the cone $\langle \widetilde{e}_I,\, \widetilde{e}_J \rangle$ with local coordinates $\zeta = (\zeta_1, \dots, \zeta_d)$. In the homogeneous coordinates ξ of $X_{\widetilde{\Sigma}}$ this chart is defined by the condition

$$\xi_{d+1} \neq 0, \dots, \xi_{d+r} \neq 0,$$

and every class in $\widetilde{U}$ has a representative of the kind $(a,\, 1, \dots, 1)$ in $\mathbb{C}^{d+r} \setminus Z(\widetilde{\Sigma})$.

Let $a \in \mathbb{C}^d \setminus Z(\Sigma) \subset \widetilde{U} \subset X_{\widetilde{\Sigma}}$ then for every fixed $g = (\lambda^p,\, \lambda^q) \in G$ and the corresponding element $\widetilde{g} = (\lambda^p,\, \lambda^q,\, \lambda^q) \in \widetilde{G}$ one has

$$g \cdot a = [(g \cdot a,\, 1, \dots, 1)]_{\widetilde{G}} = [\widetilde{g} \cdot (a,\, \lambda^{-q})]_{\widetilde{G}} = [(a,\, \lambda^{-q})]_{\widetilde{G}},$$

where $[\cdot]$ denotes the class of an element by the action of the group in the lower index. The fact that $\lambda^{-q} = (\lambda_1^{-1}, \dots, \lambda_r^{-1})$ implies that

$$\lim_{\substack{\lambda_1 \to \infty \\ \dots \\ \lambda_r \to \infty}} g \cdot a = \lim_{\substack{\lambda_1 \to \infty \\ \dots \\ \lambda_r \to \infty}} [(a,\, \lambda^{-q})]_{\widetilde{G}} = [(a,\, 0, \dots, 0)]_{\widetilde{G}}.$$

Denoting

$$\mathcal{X}_j = \{\xi_{d+j} = 0\},\ j = 1, \dots, r$$

we conclude that the closure of the orbit $G \cdot a$ of a point $a \in \mathbb{C}^d \setminus Z(\Sigma)$ intersects

$$\mathcal{X}_1 \cap \dots \cap \mathcal{X}_r = \{\xi_{d+1} = \dots = \xi_{d+r} = 0\}$$

in a unique point $[(a,\,0,\ldots,\,0)]_{\widetilde{G}} = ([a]_G,\,0,\ldots,\,0)$ corresponding to the class $[a]_G \in X_\Sigma$. On the other hand, every point $a \in \mathbb{C}^d \setminus Z(\Sigma)$ canonically corresponds a unique point $(a,\,0,\ldots,\,0) \in \mathbb{C}^{d+r} \setminus Z(\widetilde{\Sigma})$ such that there is an isomorphism

$$X_\Sigma \simeq \mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r.$$

This completes the proof of Theorem 2. □

2. The construction of residue kernels

Let us describe residue kernels associated to complete projective toric varieties, by which we mean kernels with singularities on $Z(\Sigma)$. We assume that the fan Σ has d primitive generators and at least one simple n-dimensional cone.

The construction of kernels that we present involves volume form of the toric variety. In general, such a variety is not smooth and by a *volume form* on a compact n-dimensional toric variety X_Σ we mean a differential $(n,\,n)$-form ω that is regular in smooth points of X_Σ, positive and integrable in $\mathbb{T}^n \subset X_\Sigma$. In the homogeneous coordinates ζ of X_Σ is given by

$$\omega(\zeta) = g(\zeta,\,\overline{\zeta})\overline{E(\zeta)} \wedge E(\zeta),$$

where $E(\zeta)$ denotes the Euler form (2.10) and $g(\zeta,\,\overline{\zeta})$ is a C^∞-function in $\mathbb{C}^d \setminus Z(\Sigma)$ of appropriate homogeneous degree (in the sense of [10]) such that it makes the form ω to be G-invariant.

Denoting the set $\mathbb{C}^d \setminus \{\zeta_{\hat{\sigma}} = 0\}$ by U_σ and taking into account the structure (1.1) of Z, we have the canonical covering of the complement to $Z(\Sigma)$

$$\mathbb{C}^d \setminus Z(\Sigma) = \bigcup_\sigma U_\sigma$$

where the union is taken over all n-dimensional cones of Σ. In accordance with notation $\zeta_{\hat{\sigma}}$, for every n-dimensional cone σ we introduce the differential r-form

$$\frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} = \bigwedge_{v_j \notin \sigma} \frac{d\zeta_j}{\zeta_j}$$

regular in U_σ.

Theorem 3. *Let $\omega(\zeta)$ be a volume form of X_Σ given in homogeneous coordinates. Then the list of forms*

$$\eta_\sigma(\zeta) = \omega(\zeta) \wedge \frac{1}{\det \sigma} \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} \tag{3.2}$$

defined in U_σ for every n-dimensional cone σ of Σ constitutes a globally defined, closed and regular $(d,\, n)$-form η in $\mathbb{C}^d \setminus Z(\Sigma)$ being a kernel for $Z(\Sigma)$.

Proof. The exterior product $E(\zeta) \wedge \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}}$ consists of a single term corresponding to the index set of generators of σ:

$$E(\zeta) \wedge \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} = \det \sigma \, d\zeta.$$

Consequently, the form $\eta_\sigma(\zeta)$ in U_σ equals

$$\eta_\sigma(\zeta) = \omega(\zeta) \wedge \frac{1}{\det \sigma} \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}} = g(\zeta,\, \overline{\zeta}) \overline{E(\zeta)} d\zeta,$$

which shows that η_σ does not depend on the choice of the cone σ and the list $\{\eta_\sigma\}$, $\sigma \in \Sigma(n)$ constitutes a global form η regular in the union of all U_σ, which is $\mathbb{C}^d \setminus Z(\Sigma)$.

The differential of $\eta(\zeta)$ in some U_σ is obviously equal to $\overline{\partial}\omega(\zeta) \wedge \frac{1}{\det \sigma} \frac{d\zeta_{\hat{\sigma}}}{\zeta_{\hat{\sigma}}}$. Taking into account that $\omega(\zeta)$ comes from an $(n,\, n)$-form on an n-dimensional variety, which implies that its differential is zero, we conclude that the form $\eta(\zeta)$ is closed.

Recall that the set $\mathbb{C}^d \setminus Z(\Sigma)$ is homotopically equivalent to an oriented compact real analytic cycle γ of real dimension $d + n$ (see Theorem 2 or [10]). By Proposition 1.1, this cycle generates the top homology group of the complement to $Z(\Sigma)$. Hence, the obtained differential $(d,\, n)$-form $\eta(\zeta)$ has the right degree and is a kernel, provided that it is not cohomologically trivial.

To compute the integral

$$\int_\gamma \eta(\zeta)$$

we regard the space $\mathbb{C}^d$, where the form $\eta(\zeta)$ lives, as the chart $\widetilde{U}$ in the toric variety $X_{\widetilde{\Sigma}}$. This chart is the affine toric variety $X_{\widetilde{\sigma}}$ that corresponds to the

cone $\widetilde{\sigma} = \langle \widetilde{e}_I, \widetilde{e}_J \rangle = \mathbb{R}^d_+$ generated by the first $d = n + r$ vectors from the list (3.1). According to Theorem 2, the variety X_Σ is a complete intersection of toric hypersurfaces $\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ in $X_{\widetilde{\Sigma}}$. Let us show that this form has poles of the first order along all those hypersurfaces $\mathcal{X}_j$, $j = 1, \ldots, r$. For that we look at the form $\eta(\zeta)$ in another coordinate chart of $X_{\widetilde{\Sigma}}$ whose intersection with 'the skeleton' is dense in it. There is a number of such standard charts in the variety, we choose the one that corresponds to the simple cone $\langle \widetilde{e}_I, \widetilde{v}_J \rangle \in \widetilde{\Sigma}$. This chart $\widetilde{V}$ is homeomorphic to $\mathbb{C}^d$ and the formulas

$$\begin{gathered} w_1 = \zeta_1 \zeta_{n+1}^{v^1_{n+1}} \ldots \zeta_d^{v^1_d}, \\ \ldots\ldots\ldots\ldots\ldots \\ w_n = \zeta_n \zeta_{n+1}^{v^n_{n+1}} \ldots \zeta_d^{v^n_d}, \\ w_{n+1} = \tfrac{1}{\zeta_{n+1}}, \ldots, w_d = \tfrac{1}{\zeta_d} \end{gathered}$$

relate its local coordinates $w = (w_1, \ldots, w_d)$ to the local coordinates ζ in $\widetilde{U}$ [23, section 27.9] or [15]. Notice that the first n equalities are nothing but the relations between local and homogeneous coordinates in the variety X_Σ and easy calculation shows that in local coordinates w

$$\eta = (-1)^r \omega(w_1, \ldots, w_n) \wedge \frac{dw_{n+1}}{w_{n+1}} \wedge \cdots \wedge \frac{dw_d}{w_d}. \tag{3.3}$$

On the other hand, the relation between the local coordinates w and the homogeneous coordinates ξ of $X_{\widetilde{\Sigma}}$

$$\begin{gathered} w_1 w_{n+1}^{v^1_{n+1}} \ldots w_d^{v^1_d} = \xi_1 \xi_{d+1}^{v^1_{n+1}} \ldots \xi_{d+r}^{v^1_d}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_n w_{n+1}^{v^n_{n+1}} \ldots w_d^{v^n_d} = \xi_n \xi_{d+1}^{v^n_{n+1}} \ldots \xi_{d+r}^{v^n_d}, \\ \tfrac{1}{w_{n+1}} = \tfrac{\xi_{n+1}}{\xi_{d+1}}, \ldots, \tfrac{1}{w_d} = \tfrac{\xi_d}{\xi_{d+r}}, \end{gathered}$$

shows that a toric hypersurface $\mathcal{X}_j = \{\xi_{d+j} = 0\}$ in $\widetilde{W}$ is given by the equation $w_{n+j} = 0$. Thus, a global differential form on $X_{\widetilde{\Sigma}}$ given by $\eta(\zeta)$ in $\widetilde{U}$ has poles of the first order along 'the skeleton of infinity'.

Let us show how to choose the cycle γ. We follow the construction of the Leray coboundary for cycles in subvariety S of the complex codimension r in the variety X [30, 1]. In our case we consider the subvariety $S = \mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r$ of $X = X_{\widetilde{\Sigma}}$ as in Theorem 2. By this theorem the closure of the orbit $G \cdot \zeta$ in $X_{\widetilde{\Sigma}}$ of any $\zeta \in \mathbb{C}^d \setminus Z(\Sigma)$ intersects 'the skeleton'

$$\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_r \simeq X_\Sigma$$

in a unique point $[\zeta]_G$. Let us take in every orbit a real r-dimensional torus such that their union over all $[\zeta]$ forms a continuous family of tori in $\mathbb{C}^d \backslash Z(\Sigma)$, which we shall take as γ.

Then by (3.3) and Fubini's theorem

$$\int_\gamma \eta(\zeta) = \int_{X_\Sigma} \omega(w) \int_{S^1} \frac{dw_{n+1}}{w_{n+1}} \dots \int_{S^1} \frac{dw_d}{w_d} = (2\pi i)^r \mathrm{Vol}_\omega(X_\Sigma)$$

with the volume $\mathrm{Vol}_\omega(X_\Sigma) = \int\limits_{X_\Sigma} \omega$. □

In the next section we shall point out a natural class of volume forms on X_Σ and, correspondingly, the class of kernels η, for which the volume $\mathrm{Vol}_\omega(X_\Sigma)$ is easily computed. At the moment let us consider another class of kernels and volume forms.

Example 3.

Let Σ be a fan in $\mathbb{R}^n$. Assume that its generators are vertices of a reflexive polytope P. In this case the dual polytope

$$P^\vee = \{x \in \mathbb{R}^n \langle y,\, x\rangle \geq -1 \,\forall y \in P\}$$

is also integer.

Let $\{u_j\}$ be the vertex set of $P^\vee$. Then the differential form

$$\eta = \frac{\overline{E(\zeta)} \wedge d\zeta}{\sum_j c_j |\zeta^{\nu(u_j)+I}|^2}, \quad c_j > 0$$

is a kernel for $Z(\Sigma)$. Indeed, let u_j be dual to the cone $\sigma \in \Sigma$. According to the definition of ν in (1.2) the coordinates of $\nu(u_j)$ with indices that correspond to vectors generating σ are equal to -1 and are non-negative for other indices (since u_j is integer). Whence, taking into account the structure of the exceptional set (1.1), the form η has singularity only along $Z(\Sigma)$. The form η has degree zero, since $E(\zeta)$, $d\zeta$, and ζ^I have the same degree and the degree of $\zeta^{\nu(u_j)}$ is zero. The corresponding volume form in homogeneous coordinates is

$$\omega = \frac{\overline{E(\zeta)} \wedge E(\zeta)}{\sum_j c_j |\zeta^{\nu(u_j)+I}|^2}.$$

Analogously, for reasons of homogeneity, a vector $k = (k_1, \dots, k_d)$ of natural divisors of d defines the differential $(d, d-1)$-form

$$\eta_{[k]} = \frac{\sum_{l=1}^{d} (-1)^l k_l \zeta_l d\overline{\zeta}[l] \wedge d\zeta}{\sum\limits_{\langle s,\, k\rangle = d} c_s |\zeta|^{2s}},$$

where $c_s \geq 0$ and $c_s > 0$ for $s = (0, \dots, \frac{d}{k_j}, \dots, 0)$. This form is a kernel in $\mathbb{C}^d \setminus \{0\}$ that generalizes the Bochner-Martinelli kernel, which one obtains choosing all $k_j = 1$.

A representative γ of the only non-trivial class of the top homology group can be given by an analytic expression. This is the point where the fact that X_Σ is projective plays a key role. Indeed, in this case the results [12, Theorem 4.1] and [25, Section 7.4] suggest taking

$$\gamma = \mu^{-1}(\rho) \tag{3.4}$$

where μ is the moment map (1.11)

$$\mu \colon \mathbb{C}^d \longrightarrow \mathbb{R}^r_+$$

associated with the action of the maximal compact subgroup of G on the complement of $Z(\Sigma)$. A point ρ is taken from the interior of the Kähler cone K_Σ (1.13) of X_Σ.

3. The Fubini-Study volume form of X_Σ

There is a natural choice of the volume form in the construction of the kernel. Let Δ be an n-dimensional simple (each vertex belongs to exactly n edges) integral polytope $\Delta \subset \mathbb{R}^n$. Given Δ, there is a complete simplicial toric variety X_Σ associated to the fan Σ dual to Δ. The variety X_Σ constructed in this way admits a closed embedding into the projective space as follows.

Let $P(z) = \sum\limits_{\alpha \in \Delta \cap \mathbb{Z}^n} c_\alpha z^\alpha$ be a Laurent polynomial in the torus $\mathbb{T}^n$ with all non-negative coefficients c_α such that its Newton polytope Δ_P coincides with Δ. Put elements of $\Delta \cap \mathbb{Z}^n$ in the order $\alpha_1 \dots, \alpha_N$ and similarly to (1.10) define an embedding of the torus $f \colon \mathbb{T}^n \longrightarrow \mathbb{P}_{N-1}$ by

$$(z_1, \dots, z_n) \longmapsto (\sqrt{c_{\alpha_1}} z^{\alpha_1} \colon \dots \colon \sqrt{c_{\alpha_N}} z^{\alpha_N}).$$

The closure $\overline{f(\mathbb{T}^n)}$ is then the image of X_Σ, which may have singularities, but observe that $f(\mathbb{T}^n) \subset f(X_\Sigma)$ is always smooth.

We introduce a differential $(n,\, n)$-form ω on the torus $\mathbb{T}^n$ as the pullback image of the Fubini-Study volume form ω_{FS}^n:

$$\omega = \frac{1}{n!} f^*(\omega_{FS}^n) = \frac{1}{n!} \left(\mathrm{dd^c} \ln P(|z_1|^2, \ldots,\, |z_n|^2) \right)^n .$$

This is a well-defined positive in the torus $\mathbb{T}^n \subset X_\Sigma$ differential $(n,\, n)$-form; it may vanish or be not defined in some points of the variety, however that does not affect the value of the integral

$$\int\limits_{\mathrm{reg} X_\Sigma} \omega = \int\limits_{\mathbb{T}^n} \omega.$$

The following simple proposition gives the exact value of the volume.

Proposition 3.2.

$$\mathrm{Vol}(X_\Sigma) = \pi^n \mathrm{Vol}(\Delta).$$

Proof. The obvious change of variables in the integral gives

$$\int\limits_{\mathbb{T}^n} \omega = \frac{1}{n!} \int\limits_{f(\mathbb{T}^n)} \omega_{FS}^n.$$

The volume of this algebraic subvariety of projective space with respect to the Fubini-Study metric equals

$$\frac{\pi^n}{n!} \deg(f).$$

It only remains to compute the degree of the embedding f. Let the homogeneous linear forms $l_j(\xi)$, $j = 1, \ldots,\, n$ define a generic plane of codimension n in $\mathbb{P}_{N-1}$. Then the degree of f equals the number of solutions to the system of equations $l_j(\xi)\Big|_{f(\mathbb{T}^n)}$ with the same Newton polytope. The number of solutions to such a system is given by Kushnirenko's theorem [27] and equals $n!\mathrm{Vol}(\Delta)$, and the assertion follows. $\square$

Notice also that ω can be easily integrated with respect to angular coordinates. The form $\mathrm{dd^c} \ln P(|z_1|^2, \ldots,\, |z_n|^2)$ is equal to

$$\frac{i}{2} \sum_{l,\, m=1}^{n} \left(\frac{\sum c_{\alpha_k} \alpha_k^l \alpha_k^m |z|^{2\alpha_k}}{\sum c_{\alpha_k} |z|^{2\alpha_k}} - \frac{\left(\sum c_{\alpha_k} \alpha_k^l |z|^{2\alpha_k}\right) \left(\sum c_{\alpha_k} \alpha_k^m |z|^{2\alpha_k}\right)}{\left(\sum c_{\alpha_k} |z|^{2\alpha_k}\right)^2} \right) \frac{dz_l}{z_l} \wedge \frac{d\bar{z}_m}{\bar{z}_m},$$

and the determinant of the coefficients of this form is the coefficient at$dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ in the (n, n)-form ω. This $n \times n$-determinant is the product of

$$\left(\frac{i}{2}\right)^n \frac{1}{|z_1|^2 \dots |z_n|^2 \left(\sum c_{\alpha_k}|z|^{2\alpha_k}\right)^n}$$

and the following $(n+1) \times (n+1)$-determinant:

$$\begin{vmatrix} 1 & \sum c_{\alpha_k}\alpha_k^1|z|^{2\alpha_k} & \dots & \sum c_{\alpha_k}\alpha_k^n|z|^{2\alpha_k} \\ \dfrac{\sum c_{\alpha_k}\alpha_k^1|z|^{2\alpha_k}}{\sum c_{\alpha_k}|z|^{2\alpha_k}} & \sum c_{\alpha_k}\alpha_k^1\alpha_k^1|z|^{2\alpha_k} & \dots & \sum c_{\alpha_k}\alpha_k^1\alpha_k^n|z|^{2\alpha_k} \\ \dots & \dots & \dots & \dots \\ \dfrac{\sum c_{\alpha_k}\alpha_k^n|z|^{2\alpha_k}}{\sum c_{\alpha_k}|z|^{2\alpha_k}} & \sum c_{\alpha_k}\alpha_k^n\alpha_k^1|z|^{2\alpha_k} & \dots & \sum c_{\alpha_k}\alpha_k^n\alpha_k^n|z|^{2\alpha_k} \end{vmatrix}.$$

To check this one may eliminate all the elements of the first column but the unity and then use the Laplace expansion with respect to this column.

Factoring out the denominator in the first column, we get a determinant of a matrix which is the product of two matrices of dimensions $(n+1)\times(N+1)$ and $(N+1) \times (n+1)$ respectively:

$$\begin{pmatrix} \sqrt{c_{\alpha_0}}z^{\alpha_0} & \dots & \dots & \sqrt{c_{\alpha_N}}z^{\alpha_N} \\ \alpha_0^1\sqrt{c_{\alpha_0}}z^{\alpha_0} & \dots & \dots & \alpha_N^1\sqrt{c_{\alpha_N}}z^{\alpha_N} \\ \dots & \dots & \dots & \dots \\ \alpha_0^n\sqrt{c_{\alpha_0}}z^{\alpha_0} & \dots & \dots & \alpha_N^n\sqrt{c_{\alpha_N}}z^{\alpha_N} \end{pmatrix} \begin{pmatrix} \sqrt{c_{\alpha_0}}\bar{z}^{\alpha_0} & \dots & \alpha_0^n\sqrt{c_{\alpha_0}}\bar{z}^{\alpha_0} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \sqrt{c_{\alpha_N}}\bar{z}^{\alpha_N} & \dots & \alpha_N^n\sqrt{c_{\alpha_N}}\bar{z}^{\alpha_N} \end{pmatrix}.$$

By the Cauchy-Binet formula (see e.g. [18]) the determinant is the product of all possible $(n+1)$-minors of these two.

Thus, the form ω is $\left(\frac{i}{2}\right)^n$ multiplied by

$$\frac{\sum'_{|J|=1+n} \det^2(A_J) c_{\alpha_{j_0}} \dots c_{\alpha_{j_n}} |z|^{2\alpha_{j_0}+\dots+2\alpha_{j_n}}}{|z_1|^2 \dots |z_n|^2 P(|z|^2)^{n+1}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

where the sum is taken over all increasing subsets of the index set $0 \leqslant j_0 < \cdots < j_n \leqslant N$, and A_J denotes a matrix consisting of $n+1$ columns of A:

$$\begin{pmatrix} 1 & \dots & \dots & 1 \\ \alpha_0^1 & \dots & \dots & \alpha_N^1 \\ \dots & \dots & \dots & \dots \\ \alpha_0^n & \dots & \dots & \alpha_N^n \end{pmatrix}.$$

Polar change of coordinates $z_j = r_j e^{i\varphi_j}$ and integration with respect to φ yield

$$\mathrm{Vol}(X_\Sigma) = (2\pi)^n \int\limits_{\mathbb{R}^n_+} \frac{\sum'_{|J|=1+n} \det{}^2(A_J) c_{\alpha_{j_0}} \dots c_{\alpha_{j_n}} r^{2\alpha_{j_0}+\dots+2\alpha_{j_n}+I}}{r_1^2 \dots r_n^2 P(r^2)^{n+1}} \, dr_1 \dots dr_n.$$

The last substitution $t_j = r_j^2$ results in the formula

$$\mathrm{Vol}(\Delta) = \int\limits_{\mathbb{R}^n_+} \frac{\sum'_{|J|=1+n} \det{}^2(A_J) c_{\alpha_{j_0}} \dots c_{\alpha_{j_n}} t^{\alpha_{j_0}+\dots+\alpha_{j_n}}}{t_1 \dots t_n P(t)^{n+1}} \, dt_1 \dots dt_n$$

providing a new proof of Passare's formula, which gives the volume of a polytope as an integral of rational form over the positive orthant [35].

Example 4. The volume form of $\mathbb{P}_1 \times \mathbb{P}_1$.

The product of two copies of the Riemann sphere (projective line) is a toric variety associated with the two-dimensional complete fan Σ on Figure 3.1. Let P be a polynomial $P(z_1, z_2) = 1 + z_1 + z_2 + a z_1 z_2$ where the coefficient a is positive. Its Newton polytope N_P is the unit square in $\mathbb{R}^2$ (Figure 3.3) and is obviously dual to the fan Σ.

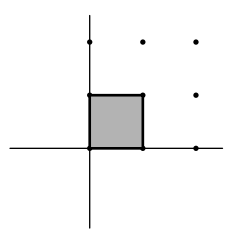


Figure 3.3: The Newton polytope of $P(z)$.

Following the construction, we define a differential form on $\mathbb{T}^2 \subset \mathbb{P}_1 \times \mathbb{P}_1$ as the pull-back of ω^2_{FS} under the mapping $f\colon (z_1, z_2) \mapsto (1\colon z_1\colon z_2\colon \sqrt{a} z_1 z_2)$

$$\omega = \frac{1}{2!} f^*(\omega^2_{FS}) = \left(\frac{i}{2}\right)^2 \frac{1 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2}{(1 + |z_1|^2 + |z_2|^2 + a|z_1|^2|z_2|^2)^3} \, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

The volume of $\mathbb{P}_1 \times \mathbb{P}_1$ with respect to this measure is equal to π^2. Note that the volume form does not coincide with the product of two volume forms on copies of $\mathbb{P}_1$ (it happens only if $a = 1$), although it gives the same volume.

Applications: some integral formulas

1. Integral representation with residue kernels

In this section we shall prove integral representation formulas with the kernels obtained. The first step in this direction is the following proposition. We keep our notation where K_Σ is the Kähler cone (see (1.13)) and η is the residue kernel (3.2) defined for the fan Σ.

Proposition 4.1. *Let $\rho \in K_\Sigma$ and U_ρ be a complete Reinhardt domain*

$$\begin{cases} a_{11}|\zeta_1|^2 + \cdots + a_{1d}|\zeta_d|^2 < \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|\zeta_1|^2 + \cdots + a_{rd}|\zeta_d|^2 < \rho_r, \end{cases} \tag{4.1}$$

with distinguished boundary $\gamma = \mu^{-1}(\rho)$. Then for every $f \in \mathcal{O}(U_\rho) \cap C(\overline{U}_\rho)$

$$f(0) = \frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_\gamma f(\zeta)\eta(\zeta). \tag{4.2}$$

Proof. Let $V \subset \mathbb{C}^d$ be a polydisc centered at the origin and relatively compact in U_ρ. The function f admits a Taylor series expansion $\sum_\beta a_\beta \zeta^\beta$ about the

origin that converges on compact subsets of V. Choose $\rho' \in K_\Sigma$ such that $\gamma' = \mu^{-1}(\rho')$ compactly lies in V. Then by Stokes' theorem the cycle γ can be replaced by γ' where the series converges absolutely and uniformly. Let us show that

$$\int\limits_{\gamma'} \zeta^\beta \eta(\zeta) = 0 \text{ if } \beta \neq 0.$$

Notice that the following change of variables

$$\begin{cases} \zeta_1 \mapsto e^{i(a_{11}t_1+\dots+a_{r1}t_r)}\zeta_1, \\ \dots\dots\dots\dots\dots\dots\dots \\ \zeta_d \mapsto e^{i(a_{1d}t_1+\dots+a_{rd}t_r)}\zeta_d \end{cases}$$

with all t_j being real, preserves the integration set and the kernel, as the latter is homogeneous with respect to this action; but the integrand gets the coefficient $e^{i((a_{11}t_1+\dots+a_{r1}t_r)\beta_1+\dots(a_{1d}t_1+\dots+a_{rd}t_r)\beta_d)}$. The rank of the matrix $A = (a_{ij})$ is r, so the image of the linear mapping given by A is $\mathbb{R}^r$. Therefore, for any $\beta \neq 0$ one can choose $t = (t_1, \dots, t_r)$ such that the coefficient is not equal to 1, so the integral must equal 0.

The statement follows now from Theorem 3. □

The integral representation formula obtained in the proposition may be seen as an averaging of the Cauchy formula. Indeed, the kernel

$$\eta(\zeta) = g(\zeta, \bar{\zeta})\overline{E(\zeta)}d\zeta = g(\zeta, \bar{\zeta}) \sum_{|I|=n} \left(\prod_{j \notin I} |\zeta_j|^2 \right) \det(v_I)\zeta_I d\overline{\zeta}_I \wedge \frac{d\zeta}{\zeta}.$$

The presence of all holomorphic differentials allows to replace $\zeta_i d\overline{\zeta}_i$ with $d|\zeta_i|^2$, therefore

$$\eta(\zeta) = g(\zeta, \bar{\zeta})E(|\zeta|^2) \wedge \frac{d\zeta}{\zeta}.$$

If the volume form of the toric variety employed in the construction of η is Fubini-Study then $g(\zeta, \bar{\zeta})$ depends only on absolute values of ζ and integral representation formula (4.2) may be rewritten as

$$\int_\gamma f(\zeta)\eta(\zeta) = \int\limits_{\Delta_\rho} g(s)E(s) \int\limits_{|\zeta|^2=s} \frac{f(\zeta)d\zeta}{\zeta}$$

where the polyhedron Δ_ρ is the intersection of the plane

$$\begin{cases} a_{11}s_1 + \cdots + a_{1d}s_d = \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}s_1 + \cdots + a_{rd}s_d = \rho_d \end{cases}$$

with the positive orthant of $\mathbb{R}^d$.

Now we extend the representation of the function at the origin (Proposition 4.1) to the representation in a domain. The formula we shall obtain combines the properties of the Bochner-Ono formula [1] (as it represents values of a function in a subdomain) as well as of the one of Sorani [44] (as one integrates over the distinguished boundary).

Recall that the Kähler cone of X_Σ is defined by the system of linear inequalities $l_I(\rho) > 0$. For a fixed ρ, define a domain D of $\mathbb{C}^d$ by the system

$$|\zeta_{i_1}|^2 + \cdots + |\zeta_{i_k}|^2 < t_1^I\rho_1 + \cdots + t_r^I\rho_r \tag{4.3}$$

for all primitive collections $I \in \mathcal{P}$ of Σ.

Proposition 4.2. *The domain D is a subdomain of U_ρ.*

Proof. Note that the rational vectors $t^I = (t_1^I, \dots, t_r^I)$ are the interior normal vectors to the faces of the Kähler cone. Therefore they generate the dual cone $b_1t^{I_1} + \cdots + b_st^{I_s}$ where $b_j \in \mathbb{R}^r$, $b_j \geq 0$. Since the Kähler cone is not empty and contained in the positive orthant $\mathbb{R}^r_+$, the dual cone is also non-empty and contains the positive orthant. This means that every basis vector e_i of $\mathbb{R}^r$ can be expressed as a linear combination of $\{t^I\}$ with non-negative rational coefficients. Hense we can sum the inequalities (4.3) multiplied by these coefficients to get ρ_i on the right side and

$$a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2 + b_1\left(\sum_{j\in J_1} c_j|\zeta_j|^2\right) + \cdots + b_s\left(\sum_{j\in J_s} c_j|\zeta_j|^2\right)$$

on the left with the same inequality sign. So, $a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2 < \rho_i$ and the proposition is proved. □

Theorem 4. *Let $f \in \mathcal{O}(U_\rho) \cap C(\overline{U}_\rho)$. Then for every $z \in D \subset U_\rho$*

$$f(z) = \frac{1}{(2i)^r\pi^d\mathrm{Vol}(\Delta)} \int_\gamma f(\zeta)\eta(\zeta - z).$$

Proof. Let z be in D. Consider the homotopy $\Gamma(t)$ of the cycle $\mu^{-1}(\rho) = \gamma$

$$\begin{cases} a_{11}|\zeta_1 - tz_1|^2 + \cdots + a_{1d}|\zeta_d - tz_d|^2 = R_1(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r1}|\zeta_1 - tz_1|^2 + \cdots + a_{rd}|\zeta_d - tz_d|^2 = R_r(t). \end{cases}$$

with the vector-function $R(t) = (R_1(t), \ldots, R_r(t))$ given by

$$R(t) = (1-t)^2\rho + t\varepsilon,\ t \in [0,\ 1],$$

where ε is a point from the Kähler cone K_Σ of X_Σ, chosen in such a way that the following two conditions are satisfied

(1) the homotopy $\Gamma(t)$ forms a $(d+n+1)$-dimensional chain in U_ρ;

(2) the cycles $\Gamma(t)$ do not intersect

$$Z(\Sigma) + z = \{\zeta \in \mathbb{C}^d \colon \zeta - z \in Z(\Sigma)\}.$$

With these conditions satisfied the Stokes formula implies that

$$\int\limits_{\gamma} f(\zeta)\,\eta(\zeta - z) = \int\limits_{\Gamma(1)} f(\zeta)\,\eta(\zeta - z)$$

and the change of variables $\zeta \mapsto \zeta + z$ in the integral gives

$$\int\limits_{\mu^{-1}(\varepsilon)} f(\zeta + z)\,\eta(\zeta).$$

Since $\varepsilon \in K_\Sigma$, it follows from (4.1) that the integral equals $(2i)^r\pi^d\mathrm{Vol}(\Delta)f(z)$. Hence, it is left to point out such ε.

Notice the fact that $\varepsilon \in K_\Sigma$ automatically implies that the whole curve $R(t)$ lies in K_Σ. Then cycles from $\{\Gamma(t)\}$, $t \in [0,\ 1]$ constitute a continuous family, i.e. a $(d+n+1)$-dimensional chain. By the triangle inequality in the standard metric of $\mathbb{R}^{2d}$

$$\begin{aligned} &\left(a_{i1}|\zeta_1 - tz_1|^2 + \cdots + a_{id}|\zeta_d - tz_d|^2\right)^{1/2} \geq \\ &\qquad \geq \left(a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2\right)^{1/2} - t\left(a_{i1}|z_1|^2 + \cdots + a_{id}|z_d|^2\right)^{1/2}; \end{aligned}$$

denoting the image $\mu(z)$ by μ we obtain

$$\left(a_{i1}|\zeta_1|^2 + \cdots + a_{id}|\zeta_d|^2\right)^{1/2} \leq \sqrt{R_i(t)} + t\sqrt{\mu_i} \text{ for all } \zeta \in \Gamma(t).$$

Therefore, to satisfy the first condition for $\Gamma(t)$ it is enough to require

$$(\sqrt{\rho_i} - t\sqrt{\mu_i})^2 - R_i(t) \geq 0 \text{ for all } i = 1, \ldots, r.$$

These inequalities hold if $\varepsilon_i < ((\sqrt{\rho_i} - \sqrt{\mu_i})^2)$, $i = 1, \ldots, r$.

For all $\zeta \in \Gamma(t)$, we have

$$l_I(R(t)) = \sum_{i \in I} |\zeta_i - tz_i|^2 - \sum_{j \in J} c_j |\zeta_j - tz_j|^2.$$

Substituting any point of $Z + z$ into corresponding identity, we get

$$-\sum_{j \in J} c_j |\zeta_j - tz_j|^2 = l_I(R(t)) - (1-t)^2 \sum_{i \in I} |z_i|^2.$$

By definition $\sum\limits_{i \in I} |z_i|^2 < l_I(\rho)$ and get the right hand side being strictly positive. This proves that $\zeta \in Z(\Sigma) + z$ does not belong to the chain $\{\Gamma(t)\}$. $\square$

2. Examples

Example 1. Integral representations associated with projective spaces

Let Δ be the standard simplex in $\mathbb{R}^n$, which is an absolutely simple polytope. Its dual fan Σ with $(n+1)$ generators is then the fan of the projective space $\mathbb{P}_n$. The volume form ω defined on it coincides with the Fubini-Study volume form ω^n_{FS}, $\mathbb{P}_n$ is embedded into $\mathbb{P}_{n+1}$ and the form η is the Bochner-Martinelli kernel η_{BM} in $\mathbb{C}^{n+1}$(see Example 1.1). The integration cycle is a sphere S^{2n+1} with the radius $\sqrt{\rho}$ (see Example 2.6). Thus, the following corollary from Theorem 6′ holds:

Let f be holomorphic in the closed ball B^{2n+2}_ρ with radius ρ and $S^{2n+1} = \partial B^{2n+2}_\rho$. Then for every $z \in B^{2n+2}_\rho$

$$f(z) = \frac{n!}{(2\pi i)^{n+1}} \int\limits_{S^{2n+1}} f(\zeta)\eta_{BM}(\zeta - z).$$

Afterwards, using analytic methods one proves this formula for any bounded domain in $\mathbb{C}^{n+1}$ with appropriate boundary.

Example 2. Integral representations associated with $\mathbb{P}_1 \times \mathbb{P}_1$

Let Δ be a unit square in $\mathbb{R}^2$ and $P(z) = 1 + z_1 + z_2 + az_1z_2$ as in example 4 on page 52. The volume form computed there produces an integral kernel

$$\eta = \frac{(\bar{\zeta}_3 d\bar{\zeta}_1 - \bar{\zeta}_1 d\bar{\zeta}_3) \wedge (\bar{\zeta}_2 d\bar{\zeta}_4 - \bar{\zeta}_4 d\bar{\zeta}_2)}{(|\zeta_1|^2|\zeta_4|^2 + |\zeta_3|^2|\zeta_4|^2 + |\zeta_2|^2|\zeta_3|^2 + a|\zeta_1|^2|\zeta_2|^2)^3} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\zeta_4$$

with singularity along $\{z_1 = z_3 = 0\} \cup \{z_2 = z_4 = 0\}$.

The domain U_ρ in this case is the product of two balls $B^4_{\rho_1} \times B^4_{\rho_2}$ in $\mathbb{C}_{z_1, z_3} \times \mathbb{C}_{z_2, z_4}$. The Kähler cone coincides with $\mathbb{R}^2_+$ so the integral represents values of a holomorphic function at every point z of U_ρ:

$$f(z) = \frac{1}{(2\pi i)^4} \int_{\partial B^4_{\rho_1} \times \partial B^4_{\rho_2}} f(\zeta)\eta(\zeta - z).$$

Example 3. Integral representation associated with the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ at the origin

Let $P(z) = 1 + z_1^2 + z_2^2 + z_1^2 z_2 + z_1 z_2^2$ with the Newton polytope depicted on Figure 4.1(a). The dual fan Σ has five integral generators and the corresponding toric variety is the blow-up of the product $\mathbb{P}_1 \times \mathbb{P}_1$ at the origin.

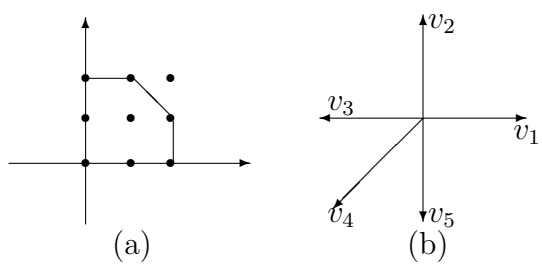


Figure 4.1: (a) the Newton polytope; (b) the fan.

Number the integral generators, and when all the calculations are done the integral kernel is

$$\eta = \frac{u(\zeta, \overline{\zeta})\overline{E(\zeta)}}{v(\zeta, \overline{\zeta})} \wedge d\zeta$$

where

$$\begin{aligned} u(\zeta, \overline{\zeta}) = {} & |\zeta_1|^8|\zeta_2|^4|\zeta_5|^4 + 4|\zeta_1|^6|\zeta_2|^4|\zeta_3|^2|\zeta_4|^2|\zeta_5|^4 + 4|\zeta_1|^6|\zeta_3|^2|\zeta_4|^6|\zeta_5|^8 + \\ & + |\zeta_1|^4|\zeta_2|^8|\zeta_3|^4 + 4|\zeta_1|^4|\zeta_2|^6|\zeta_3|^4|\zeta_4|^2|\zeta_5|^2 + 9|\zeta_1|^4|\zeta_2|^4|\zeta_3|^4|\zeta_4|^4|\zeta_5|^4 + \\ & + 16|\zeta_1|^4|\zeta_2|^2|\zeta_3|^4|\zeta_4|^6|\zeta_5|^6 + 16|\zeta_1|^2|\zeta_2|^4|\zeta_3|^6|\zeta_4|^6|\zeta_5|^4 + \\ & + 16|\zeta_1|^2|\zeta_2|^2|\zeta_3|^6|\zeta_4|^8|\zeta_5|^6 + 4|\zeta_2|^6|\zeta_3|^8|\zeta_4|^6|\zeta_5|^2, \end{aligned}$$

$$\begin{aligned} v(\zeta, \overline{\zeta}) = {} & (\ |\zeta_3|^4|\zeta_4|^6|\zeta_5|^4 + |\zeta_1|^4|\zeta_4|^2|\zeta_5|^4 + |\zeta_2|^4|\zeta_3|^4|\zeta_4|^2 + \\ & + |\zeta_1|^4|\zeta_2|^2|\zeta_5|^2 + |\zeta_1|^2|\zeta_2|^4|\zeta_3|^2\ )^3, \end{aligned}$$

and

$$\begin{aligned} E(\zeta) = {} & \zeta_3\zeta_4\zeta_5 d\zeta_1 d\zeta_2 - \zeta_2\zeta_3\zeta_5 d\zeta_1 d\zeta_4 - \zeta_2\zeta_3\zeta_4 d\zeta_1 d\zeta_5 + \\ & \zeta_1\zeta_4\zeta_5 d\zeta_2 d\zeta_3 + \zeta_1\zeta_3\zeta_5 d\zeta_2 d\zeta_4 + \zeta_1\zeta_2\zeta_5 d\zeta_3 d\zeta_4 + \\ & \zeta_1\zeta_2\zeta_4 d\zeta_3 d\zeta_5 + \zeta_1\zeta_2\zeta_3 d\zeta_4 d\zeta_5. \end{aligned}$$

The lattice of relations between the integral generators is given by

$$\begin{cases} v_1 + v_3 = 0, \\ v_2 + v_5 = 0, \\ v_1 + v_2 + v_4 = 0; \end{cases}$$

and one can easily check that the primitive collections for Σ are $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, and $\{3, 5\}$. It follows that the domain U_ρ is given by the inequalities

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ |\zeta_2|^2 + |\zeta_5|^2 < \rho_2, \\ |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2 < \rho_3, \end{cases}$$

where $\rho = (\rho_1, \rho_2, \rho_3)$ lies in the Kähler cone being given by

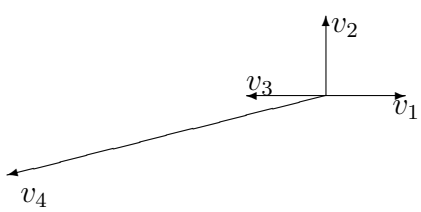
$$\begin{cases} \rho_1 > 0, \\ \rho_3 - \rho_2 > 0, \\ \rho_3 - \rho_1 > 0, \\ \rho_2 > 0, \\ \rho_1 + \rho_2 - \rho_3 > 0. \end{cases}$$

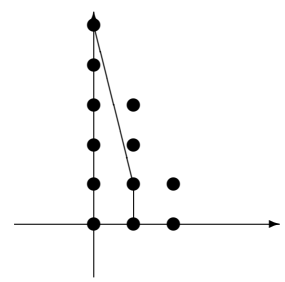
Therefore the subdomain D consists of all points z that satisfy the system

$$\begin{cases} |z_1|^2 + |z_3|^2 < \rho_1, \\ |z_1|^2 + |z_4|^2 < \rho_3 - \rho_2, \\ |z_2|^2 + |z_4|^2 < \rho_3 - \rho_1, \\ |z_2|^2 + |z_5|^2 < \rho_2, \\ |z_3|^2 + |z_5|^2 < \rho_1 + \rho_2 - \rho_3. \end{cases}$$

Example 4. Integral representation associated with a Hirzebruch surface

Let the polynomial P be equal $1 + z_1 + z_1 z_2 + z_2^5$. The dual fan to its Newton polytope corresponds to one of the Hirzebruch surfaces.





The volume form on this surface is

$$\omega = \frac{|z_1|^2 + |z_1|^2|z_2|^{10} + 25|z_2|^{10} + 25|z_2|^8}{(1 + |z_1|^2 + |z_1|^2|z_2|^2 + |z_2|^{10})^3}\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$

and associated integral kernel is

$$\eta = g(\zeta, \overline{\zeta})\overline{E(\zeta)} \wedge d\zeta$$

where

$$g(\zeta, \overline{\zeta}) = \frac{|\zeta_1|^2|\zeta_2|^{10} + 25|\zeta_2|^{10}|\zeta_3|^2|\zeta_4|^8 + 25|\zeta_2|^8|\zeta_3|^2|\zeta_4|^{10} + |\zeta_1|^2|\zeta_4|^{10}}{(|\zeta_3|^2|\zeta_4|^{10} + |\zeta_1|^2|\zeta_4|^2 + |\zeta_1|^2|\zeta_2|^2 + |\zeta_2|^{10}|\zeta_3|^2)^3}$$

and

$$E(\zeta) = \zeta_3\zeta_4\, d\zeta_1 \wedge d\zeta_2 - \zeta_2\zeta_3\, d\zeta_1 \wedge d\zeta_4 + \zeta_1\zeta_4\, d\zeta_2 \wedge d\zeta_3 + 4\zeta_1\zeta_3\, d\zeta_2 \wedge d\zeta_4 + \zeta_1\zeta_2\, d\zeta_3 \wedge d\zeta_4.$$

For functions that are holomorphic in

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ 4|\zeta_1|^2 + |\zeta_2|^2 + |\zeta_4|^2 < \rho_2, \end{cases}$$

where $\rho \in \{\rho_1 > 0,\ \rho_2 - 4\rho_1 > 0\}$, the integral represents values at the points from D given by

$$\begin{cases} |\zeta_1|^2 + |\zeta_3|^2 < \rho_1, \\ |\zeta_2|^2 + |\zeta_4|^2 < \rho_2 - 4\rho_1. \end{cases}$$

Note that the polygon with the integral generators of the fan as vertices in the case is not convex, but this is not an obstacle to construct a kernel. What really matters is the existence of the dual polytope. Similiar formulas of integral representations have been considered by A.A. Kytmanov [28] but his construction is different and does not cover this case.

Example 5. A kernel associated with a singular toric variety.

Let us fix a polynomial $P(x) = 1 + x_1^2 x_2 + x_2^2$, its Newton polyhedron Δ is a triangle in $\mathbb{R}^2$ with vertices at (0, 0), (2, 1), and (0, 2). The fan Σ dual to Δ is shown on Figure 1.1. The volume form of X_Σ defined in smooth points of the variety is equal to

$$\omega = 16 \left(\frac{i}{2}\right)^2 \frac{|z_1|^2 |z_2|^4}{(1 + |z_1|^4 |z_2|^2 + |z_2|^4)^3}\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

According to the construction of Theorem 3, in $\mathbb{C}^3 \setminus \{0\}$ we consider the differential (3, 2)-form

$$\eta(\zeta) = \omega\left(\frac{\zeta_1}{\zeta_2\zeta_3}, \frac{\zeta_2^2}{\zeta_3^2}\right) \wedge \frac{d\zeta_3}{\zeta_3},$$

which equals

$$-4|\zeta_1|^2|\zeta_2|^6|\zeta_3|^6 \frac{4\bar{\zeta}_1 d\bar{\zeta}_2 \wedge d\bar{\zeta}_3 - 2\bar{\zeta}_2 d\bar{\zeta}_1 \wedge d\bar{\zeta}_3 + 2\bar{\zeta}_3 d\bar{\zeta}_1 \wedge d\bar{\zeta}_2}{(|\zeta_1|^4 + |\zeta_2|^8 + |\zeta_3|^8)^3}\, d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3.$$

This form is obviously a kernel for the atomic family $\{0\} \subset \mathbb{C}^3$.

3. Logarithmic residue formula

For any integral formula with a closed kernel there is a related logarithmic residue formula ([1], [46]). This holds for the formula obtained in Proposition 4.1 that allows to present an integral expressing the sum of values of aholomorphic function in the zeroes of holomorphic mapping.

Let Σ be fan in $\mathbb{R}^n$ satisfying the hypothesis of Theorem 2. Further, let G be a domain in $\mathbb{C}^d$ and $f\colon G \longrightarrow \mathbb{C}^d$ be a holomorphic mapping

$$w_j = f_j(\zeta_1, \dots, \zeta_d),\ j = 1, \dots, d. \tag{4.4}$$

We assume that the mapping f is of finite type over U_ρ, i.e. $W_\rho = f^{-1}(U_\rho)$ is relatively compact in G. According to (4.1) this polyhedron is defined by the system of inequalities

$$\begin{cases} a_{11}|f_1(\zeta)|^2 + \dots + a_{1d}|f_d(\zeta)|^2 < \rho_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}|f_1(\zeta)|^2 + \dots + a_{rd}|f_d(\zeta)|^2 < \rho_r, \end{cases}$$

where ρ is taken from the Kähler cone K_Σ of X_Σ and let $\Gamma = f^{-1}(\mu^{-1}(\rho))$.

Under the assumptions made the set E of zeroes of the system $f(\zeta) = 0$ in W_ρ is finite and in this domain the Jacobian $\frac{\partial f}{\partial \zeta} \not\equiv 0$.

In the situation when the common zero set is discrete we cann assign a multiplicity to each. This follows from the following well-known fact.

Lemma [38]. *Let V_a be an open neighbourhood of a zero a of the system* (4.4) *that does not contain any other zero. Then for almost all sufficiently close to the origin points $\xi = (\xi_1, \dots, \xi_d)$ the system*

$$w_j = f_j(\zeta) - \xi_j \tag{4.5}$$

has only simple zeroes in V_a. Their number is the same for all such ξ.

The number of zeroes of (4.5) in V_a for such ξ is called a (dynamical) *multplicity* of the zero a of (4.4) and is commonly denoted as $\mu_a(f)$.

Let η be a residue kernel (3.2) associated with a toric variety X_Σ, then we prove the following result.

Theorem 5. *For any function $\varphi \in \mathcal{O}(W_\rho)$ a logarithmic residue formula*

$$\frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_\Gamma \varphi(\zeta)\eta(f(\zeta)) = \sum_{a \in E} \mu_a(f)\varphi(a).$$

holds.

Proof. The mapping f does not have zeroes on the boundary ∂W_ρ and has only finite number of isolated zeroes in W_ρ. Choose for each zero $a \in E$ an

open neighbourhood $V_a \Subset W_\rho$ so that they do not intersect. Choose now a point ρ' from the set

$$K_\Sigma \cap \left\{ \rho\colon \rho_i < \min_{a\in E} \max_{\partial V_a} (a_{i1}|f_1(\zeta)|^2 + \cdots + a_{id}|f_d(\zeta)|^2),\ i = 1, \ldots, r \right\}.$$

Note that this set is never empty as an intersection of a cone K_Σ and a parallelepiped with faces parallel to coordinate hyperplanes.

Let $Z(f)$ be the preimage of $Z(\Sigma) \subset \mathbb{C}^d$ under f, then the cycle Γ is homologous to the cycle $\Gamma' = f^{-1} \circ \mu^{-1}(\rho')$ in $W_\rho \setminus Z(f)$. Indeed, the set

$$\{\ \zeta \in G\colon a_{i1}|f_1(\zeta)|^2 + \cdots + a_{id}|f_d(\zeta)|^2 = (1-t)\rho'_i + t\rho_i, \\ t \in [0,\, 1],\ i = 1, \ldots, r\ \}$$

is relatively compact in G and is a continuous one-parametric family of cycles $f^{-1} \circ \mu^{-1}((1-t)\rho' + t\rho)$. Each of them does not intersect $Z(f)$, since the Kähler cone is convex. Because of the choice of ρ' the cycle Γ' consists of connected components $\Gamma_a = f^{-1} \circ \mu^{-1}(\rho') \cap \overline{V}_a$.

By Stokes' formula we have

$$\int_\Gamma \varphi(\zeta)\eta(f(\zeta)) = \sum_{a\in E} \int_{\Gamma_a} \varphi(\zeta)\eta(f(\zeta)).$$

If a is a simple zero then we may choose neighbourhood V_a such that the mapping f is biholomorphic there. Then

$$\int_{\Gamma_a} \varphi(\zeta)\eta(f(\zeta)) = \int_\gamma \varphi(f^{-1}\big|_{V_a}(w))\eta(w)$$

where $\gamma = \mu^{-1}(\rho')$. Then it follows from proposition 4.1, that

$$\frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_\gamma \varphi(f^{-1}\big|_{V_a}(w))\eta(w) = \varphi(f^{-1}\big|_{V_a}(0)) = \varphi(a).$$

Let now a be a multiple zero of (4.4). Then the lemma provide us with a neighbourhood of the origin in $\mathbb{C}^d$ such that for any ξ from there the system (4.5) has only simple zeroes $a^{(1)}, \ldots, a^{(k)}$ in V_a, here k is equal to the multiplicity $\mu_a(f)$ of a.

Choose a point ξ from the intersection of this neighbourhood with $D_{\rho'}$, which is also a d-dimensional neighbourhood of the origin in $\mathbb{C}^d$. The cycles Γ_a and

$$\Gamma_{a,\xi} = \{\zeta \in V_a\colon a_{i1}|f_1(\zeta) - \xi_1|^2 + \cdots + a_{id}|f_d(\zeta) - \xi_d|^2 = \varepsilon\}$$

where $\varepsilon \in K_\Sigma \cap \left\{\rho_i < \left(\sqrt{\rho_i'} - \sqrt{\mu_i(\xi)}\right)^2\right\}$ are homologous in $V_a \setminus Z(f-\xi)$ because their difference is the boundary

$$\{\zeta \in V_a \colon a_{i1}|f_1(\zeta) - t\xi_1|^2 + \cdots + a_{id}|f_d(\zeta) - t\xi_d|^2 = (1-t)^2\rho_i' + t\varepsilon_i, \\ t \in [0,\, 1],\, i = 1, \ldots,\, r\}.$$

The argument of the simple zero case yields

$$\frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_{\Gamma_{a,\,\xi}} \varphi(\zeta)\eta(f(\zeta) - \xi) = \sum_{j=1}^{k} \varphi(a^{(j)}),$$

and taking the limit as $\xi \longrightarrow 0$ we conclude that

$$\frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int_{\Gamma_a} \varphi(\zeta)\eta(f(\zeta)) = \mu_a(f)\varphi(a).$$

□

4. Integral realisation of the Grothendieck residue

Besides the results of two previous sections the pair of kernel (3.2) and cycle (3.4) produces an integral realisation of the Grothendieck residue. It follows directly from [47, Theorem 1] because the mentioned pair of kernel and cycle is a *reproducing pair* in the sense of that article. Let us recall some definitions.

Let $\operatorname{sing}\omega$ be a singular set of the form ω and assume that $0 \in \operatorname{sing}\omega$. A cycle Γ in $\mathbb{C}^d \setminus \operatorname{sing}\omega$ is called *local at* $a \in \mathbb{C}^d$ if its homology class contains representatives in any neighbourhood of a.

Definition 4.1. *A pair* $(\omega,\, \Gamma)$, ω *of a closed differential form* $\omega = \psi(w) \wedge dw$ *where* ψ *is a* $(0,\, n)$*-form and an* $(n{+}r)$*-dimensional cycle* Γ *local at* $0 \in \operatorname{sing}\omega$ *is called reproducing if for any germ* $s \in \mathcal{O}_0$

$$s(0) = \int_\Gamma s(w)\omega.$$

The kernels constructed in Theorem 3 with the corresponding cycles are clearly reproducing. Hence, we have a corollary of Proposition 4.1 and Theorem 1 of [47]

Proposition 4.3. *Let $\eta(w)$ be a kernel for an atomic family $Z(\Sigma)$ in $\mathbb{C}^d$. Then in notation $\psi(w) = \eta(w)/dw$ the local residue of a germ $h \in \mathcal{O}_a$ associated with a regular sequence $f = (f_1, \ldots, f_d)$ at $a \in f^{-1}(0)$ is represented by the integral*

$$\mathrm{Res}_f(h) = \frac{1}{(2i)^r \pi^d \mathrm{Vol}(\Delta)} \int\limits_{f^{-1}(\gamma)} h(\zeta)\psi(f(\zeta)) \wedge d\zeta$$

where $\gamma = \mu^{-1}(\rho)$ for sufficently small ρ from the Kähler cone K_Σ.

Bibliography

[1] L.A. Aizenberg, A. P. Yuzhakov *Integral Representations and Residues in Multidimensional Complex Analysis*. Translations of Mathematical Monographs, 58. AMS, 1983.

[2] P.S. Aleksandrov *Topological duality theorems. I. Closed sets*, Amer. Math. Soc. Transl. **30**, 1963, 1–102.

[3] M. Andersson *Residue currents and ideals of holomorphic functions*, Bull. Sci. Math. 128 (2004), №6, 481–512.

[4] P.S. Aspinwall, B.R. Greene, and D.R. Morrison *The monomial-divisor mirror map*. Intern. Math. Res. Notices, 1993, №12, 319 – 337.

[5] M. Audin *The topology of torus actions on symplectic manifolds*. Translated from the French by the author. Progress in Mathematics, 93. Birkhäuser Verlag, Basel, 1991.

[6] V.V. Batyrev *Quantum cohomology ring of toric manifolds*. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Astérisque №218 (1993), 9–34.

[7] V.V. Batyrev, E.N. Materov *Toric residues and mirror symmetry*, Mosc. Math. J. 2 (2002), №. 3, 435–475.

[8] D. Bernstein *The number of roots of a system of equations,* Funct. Anal. Appl. 9 (2) (1975), 183–185.

[9] P. Cartier *Arrangements d'hyperplans: un chapitre de géometrie combinatoire*, Séminaire Bourbaki (1980/1981), exp. 561.

[10] D. Cox *The homogeneous coordinate ring of a toric variety*, J. Algebraic geometry **4** (1995), 17-50.

[11] D. Cox *Toric Residues*, Ark. Mat. **34** (1996), №1, 73–96.

[12] D. Cox *Recent developments in toric geometry.* Algebraic geometry — Santa Cruz 1995, 389 – 436, Proc. Sympos. Pure Math., 62, Part 2, AMS, 1997.

[13] P. Deligne, M. Goresky, R. MacPherson *L'algèbre de cohomologie du complément, dans un espace affine, d'une famille finie de sous-espaces affines*, Michigan Math. J. **48** (2000), 121–136.

[14] M. Demazure *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup, 1970, Vol. 3, 507–588.

[15] T.O. Ermolaeva, A.K. Tsikh *Integration of rational functions over* $\mathbb{R}^n$ *by means of toric compactifications and higher-dimensional residues*, Sb. Math. **187** (1996), №9, 1301–1318.

[16] W. Fulton *Introduction to toric varieties.* Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993.

[17] M. Goresky, R. MacPherson *Stratified Morse Theory.* Ergeb. Math. Grenzgeb. 3. Folge, Bd. 14, Springer-Verlag, Berlin, 1988.

[18] F.R. Gantmacher *The theory of matrices.* Vol. 1. Chelsea Publishing Co., New York 1959

[19] Ph. Griffiths, J. Harris *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

[20] F.R. Harvey *Integral formulae connected by Dolbeault's isomorphism*, Rice Univ. Studies **56** (1970), №2, 77–97.

[21] C.G.J. Jacobi *De relationibus, quae locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum algebraicarum dati ordinis, simul cum enodatione paradoxi algebraici*, J. reine angew. Math. 15 (1836) 285–308.

[22] B.Ja. Kazarnovskii *On the zeroes of exponential sums,* Soviet Math. Doklady, **23** (1981) №6, 347–351.

[23] A. Khovanskĭ *Algebra and mixed volumes*. Addendum 3 in the book Yu.D. Burago, V.A. Zalgaller *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften, 285. Springer-Verlag, Berlin, 1988.

[24] A. Khovanskii *Newton polyhedra and the Euler-Jacobi formula*, Russian Math. Surveys **33**(1978), 237–238.

[25] F. Kirwan *Cohomology of quotients in symplectic and algebraic geometry*. Math. Notes **31**, Princeton University Press, 1984.

[26] W. Koppelman *The Cauchy integral for differential forms*, Bull. Amer. Math. Soc. **73** (1967), 554–556.

[27] A.G. Kouchnirenko *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), №1, 1–31.

[28] A.A. Kytmanov *On an analogue of the Fubini-Study form for two-dimensional toric manifolds*, Siberian Math. J. **44** (2003), №2, 286–297.

[29] A.M. Kytmanov *The Bochner-Martinelli integral and its applications*. Birkhäuser Verlag, Basel, 1995.

[30] J. Leray *Le calcul différentiel et intégral sur une variété analytique complexe. (Problème de Cauchy. III)*, Bull. Soc. Math. France **87**(1959), 81–180.

[31] I. Lieb, J. Michel *The Cauchy-Riemann complex. Integral formulae and Neumann problem*, Aspects of Mathematics, E34. Friedr. Vieweg & Sohn, Braunschweig, 2002.

[32] D. Mumford *Algebraic geometry. I. Complex projective varieties*. Grundlehren der Mathematischen Wissenschaften, №221. Springer-Verlag, Berlin-New York, 1976.

[33] T. Oda *Convex Bodies and Algebraic Geometry*. Ergeb. Math. Grenzgeb. 3. Folge, Bd. 15, Springer-Verlag, Berlin, 1988.

[34] P. Orlik, L. Solomon *Combinatorics and topology of complements of hyperplanes*, Inv. Math. 1980, Vol. 56, 167–189.

[35] M. Passare *Amoebas, convexity and the volume of integer polytopes*. Advanced Studies in Pure Mathematics, **42** (2004), 263-268.

[36] M. Passare, A. Tsikh, A. Yger *Residue currents of the Bochner-Martinelli type*, Publ. Mat. 44 (2000), №1, 85–117.

[37] T. Pellé *Identités de Bézout pour certains systèmes de sommes d'exponentielles,* Ark. Mat. **36** (1998), 131–162.

[38] J. Milnor *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, №61 Princeton University Press, Princeton, N.J.

[39] J.R. Sangwine-Yager *Mixed volumes.* Handbook of convex geometry. Vol. A. North-Holland, Amsterdam 1993, 43–71.

[40] A.V. Shchuplev *On two-dimensional toric prevarieties* (Russian), Vestnik KrasGU, Krasnoyarsk 2004, №1, 93–98.

[41] A.V. Shchuplev *On reproducing kernels in* $\mathbb{C}^d$ *and volume forms on toric varieties*, Russian Math. Surveys **60** (2005), №2, 373–375.

[42] A. Shchuplev *On the Vidras-Yger theorem*, to appear in RIMS Koukyuuroku.

[43] A. Shchuplev, A.Tsikh, A. Yger *Residual kernals with singularities on coordinate planes*, Proceedings of the Steklov Institute of Mathematics, 2006, Vol. 253, 256–274.

[44] G. Sorani *Integral representations of holomorphic functions*, Amer. J. of Math. 1966, Vol. 88, №4, 737–746.

[45] T.L. Tong *Integral representation formulae and Grotendieck residue symbol*, Amer. J. Math. 1973, V. 4, 904–917.

[46] A.K. Tsikh *Multidimensional residues and their applications.* Translations of Mathematical Monographs, 103. AMS, 1992.

[47] A.K. Tsikh, B.A. Shaimkulov *Integral realisations of the Grothendieck residue and its transformations under compositions* (Russian), Vestnik KrasGU, Krasnoyarsk 2005, №1, 151–155.

[48] A. Vidras, A. Yger *On some generalizations of Jacobi's residue formula*, Ann. Scient. Éc. Norm. Sup., 4e série, t. 34., 2001, 131–157.