Combinatorics of stable polynomials and correlation inequalities

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Abstract

This thesis contains five papers divided into two parts. In the first part, Papers I–IV, we study polynomials within the field of combinatorics. Here we study combinatorial properties as well as the zero distribution of the polynomials in question. The second part consists of Paper V, where we study correlating events in randomly oriented graphs.

In Paper I we give a new combinatorial interpretation of the stationary distribution of the partially asymmetric exclusion process in terms of colored permutations and decorated alternative trees. We also find a connection between the corresponding multivariate partition functions and the multivariate Eulerian polynomials for r-colored permutations.

In Paper II we study a multivariate refinement of P-Eulerian polynomials. We show that this refinement is stable (i.e., non-vanishing whenever the imaginary parts of its variables are all positive) for a large class of labeled posets.

In Paper III we use the technique of compatible polynomials to prove that the local h-polynomial of the rth edgewise subdivision of the $(n-1)$-dimensional simplex 2^V has only real zeros. We generalize the result and study matrices with interlacing preserving properties.

In Paper IV we introduce s-lecture hall partitions for labeled posets. We provide generating functions as well as establish a connection between statistics on wreath products and statistics on lecture hall partitions for posets.

In Paper V we prove that the events $\{s \to a\}$ (that there exists a directed path from s to a) and $\{t \to b\}$ are positively correlated in a random tournament for distinct vertices $a, s, b, t \in K_n$. We also discuss the correlation between the same events in two random graphs with random orientation.

List of papers

The following five papers are included in this thesis and will be referred to by their Roman numerals.

Contents

1. Introduction

This thesis contains five papers divided into two parts. The first part consists of Papers I–IV which were supervised by Petter Brändén, KTH. The second part consists of Paper V which was supervised by Svante Linusson, KTH.

In the first part we study polynomials of combinatorial nature and the geometry of zeros of polynomials. The second part consists of one paper, Paper V, where we study correlating events in randomly oriented graphs.

1.1 Overview of the thesis

In addition to the five research papers this thesis includes three initial chapters. First, the present one (Chapter 1), which gives an overview of the rest of the thesis and an introduction to the coming two chapters.

In Chapter 2 we start by giving an introduction to the topics relevant to the first four papers. Papers I–IV all touch on different aspects of the geometry of zeros of polynomials. We discuss real-rooted polynomials in combinatorics and properties of sequences whose generating polynomials are real-rooted. We also discuss a multivariate analog of real-rooted polynomials, stable polynomials, i.e., polynomials that are non-vanishing whenever all their variables have positive imaginary parts.

The Eulerian polynomials have been studied frequently in enumerative combinatorics, especially within the combinatorics of permutations, and are known to be real-rooted. This fact lays the foundation of the work in the first part of this thesis. We let the Eulerian polynomials act as examples through Chapter 2, as we introduce permutation statistics, stable polynomials and stability preserving operators. Linear operators preserving non-vanishing properties have been studied for centuries. We will discuss the characterization of all linear operators preserving stability by Brändén and Borcea [7], which is a generalization of the characterization for the univariate case by Pólya and Schur from 1914 [34].

At the end of Chapter 2 we give a brief overview of the Papers I–IV.

In Paper I we present a new combinatorial interpretation of the stationary distribution of the partially asymmetric exclusion process. In Paper II we study a multivariate refinement of P-Eulerian polynomials. In Paper III we answer a question addressed by Athanasiadis [3], regarding real-rootedness of polynomials in a certain family of polynomials. In Paper IV we introduce and study s-lecture hall partitions for labeled posets.

There are several surveys on the topics that will be discussed in Chapter 2. I would like to refer to Brändén [13], Brenti [10], Stanley [39] and Wagner [40].

In Chapter 3 we introduce the topics relevant for Paper V and summarize the content in that paper. Let $\{s \rightarrow a\}$ denote the event that there exists a directed path from s to a. We prove that the events $\{s \rightarrow a\}$ and $\{t \rightarrow b\}$ are positively correlated for distinct vertices $a, s, b, t \in K_n$, where every edge in K_n is oriented either way with the same probability independently of each other. We also study this correlation in two different random graph models. Paper V was inspired by a previous paper by Alm and Linusson [1], where they studied the correlation between similar events in a random tournament.

2. Real-rooted and stable polynomials in combinatorics

2.1 Real-rooted polynomials in combinatorics

The study of zeros of polynomials has a long and rich history. In combinatorics there are many examples of real-rooted polynomials. One such example is the Eulerian polynomials [23], that we will discuss in Section 2.3.

Let $A = a_0, \ldots, a_n$ be a sequence of real numbers, then

- A is *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for all $1 \leq i \leq n-1$ and

- A is *unimodal* if there exists an index $0 \leq i \leq n$ such that $a_0 \leq \cdots \leq a_i \geq \cdots \geq a_n$.

The corresponding generating polynomial

$$p_A(x) := \sum_{i=0}^n a_i x^i,$$

is said to be *real-rooted* if all its zeros are real. For convenience the zero polynomial is treated as a real-rooted polynomial.

In combinatorics many important sequences are known to be log-concave, unimodal or have a real-rooted generating polynomial [10; 13; 39]. Unimodal and log-concave sequences also arise often in algebra, analysis, computer science and statistics.

Example 1. The polynomial

$$\sum_{i=0}^n \binom{n}{i} x^i = (1+x)^n$$

is a real-rooted polynomial, in fact all its zeros are equal to -1. The corresponding sequence $\{\binom{n}{i}\}_{i=0}^n$ is log-concave and unimodal.

One of the reasons why real-rooted polynomials are of interest is the connection between the zeros and the coefficients of a polynomial. If the generating polynomial of a sequence has only real zeros it implies several properties of the sequence. For example, if p_A is a real-rooted polynomial with nonnegative coefficients and $n \geq 2$, the sequence $a_0, \ldots, a_n$ is both log-concave and unimodal. However, the converse is not true.

Example 2. The sequence $1, 1, 1$ is log-concave but the polynomial $1 + x + x^2$ does not have any real zeros.

If p_A is real-rooted, the sequence $a_0, \ldots, a_n$ also satisfies Newton's inequalities, that is,

$$a_k^2 \geq \left(\frac{k+1}{k} \right) \left(\frac{n-k+1}{n-k} \right) a_{k-1} a_{k+1}, \ 1 \leq k \leq n-1.$$

2.2 Statistics on permutations

Recall that the symmetric group, $\mathfrak{S}_n$, is the set of all permutations of the set $[n] = \{1, \ldots, n\}$. Let us use the one-line notation to denote a permutation $\pi \in \mathfrak{S}_n$, that is $\pi = \pi_1 \cdots \pi_n$. We call position i a *descent* in π if $\pi_i > \pi_{i+1}$, and an *excedance* if $\pi_i > i$. The set of descents and excedances of a permutation $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ are defined as $D(\pi) = \{i : \pi_i > \pi_{i+1}\}$ and $E(\pi) = \{i : \pi_i > i\}$, respectively.

A *permutation statistic* is a function $f : \mathfrak{S}_n \to \mathbb{N}$. We define the two permutation statistics counting descents and excedances in a permutation by

$$\mathrm{des}(\pi) = |D(\pi)|, \text{ and} \tag{2.1}$$
$$\mathrm{exc}(\pi) = |E(\pi)|. \tag{2.2}$$

Example 3. For the permutation $\pi = 4152637$ we have $\mathrm{des}(\pi) = 3$ and the descent set is $D(\pi) = \{1, 3, 5\}$.

It is well known that excedences and descents are equidistributed, also the descent and excedance statistic are both examples of so called Eulerian statistics since Eulerian polynomials may be defined as generating functions of descents (or excedances) over $\mathfrak{S}_n$, something that will be discussed further in the next section.

Another well-studied Eulerian statistic is the *ascent* statistic, it may be defined as

$$\mathrm{asc}(\pi) = |\{i : \pi_{i-1} < \pi_i\}|. \tag{2.3}$$

Statistics such as descents, ascents and excedances can also be extended to other sets of objects, such as set partitions, inversion sequences and wreath products. For example, in Paper III, we study an ascent statistic on restricted Smirnov words. Let r be a positive integer. The *wreath product* of the symmetric group of order n with the cyclic group of order r may be defined as

$$\mathbb{Z}_r \wr \mathfrak{S}_n = \{(\kappa, \sigma) : \sigma \in \mathfrak{S}_n \text{ and } \kappa : [n] \to \mathbb{Z}_r\}.$$

In Paper I we study a refined excedance statistic on wreath products and in Paper IV we establish a connection between statistics on wreath products and statistics on lecture hall partitions for posets.

The set of *inversion sequences* of length n is defined by

$$\mathrm{I}_n = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n : 0 \le e_i \le i - 1\}.$$

For an inversion sequence $e \in \mathrm{I}_n$ define an ascent statistic as

$$\mathrm{asc_I}(e) = \left| \left\{ i \in \{0, 1, \ldots, n-1\} \Big| \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\} \right|,$$

where $e_0 = 0$. The ascent statistic over inversion sequences is another example of an Eulerian statistic [37], that is

$$\sum_{e \in \mathrm{I}_n} x^{\mathrm{asc_I}(e)} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}.$$

A statistic, f, is said to be *Mahonian* if the distribution over $\mathfrak{S}_n$ is the standard q-analogue of $n!$. That is if

$$\sum_{\pi \in \mathfrak{S}_n} q^{f(\pi)} = [n]!_q,$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]!_q = [1]_q [2]_q \cdots [n]_q$. An example of a Mahonian statistic is the *Major index* defined as

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{D}(\pi)} i$$

The statistic maj was studied first by MacMahon [32], and will be discussed briefly in Paper IV.

Statistics such as descents, ascents and excedances may also be refined over permutations, for example by their values or positions.

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$, let

$$\mathrm{AB}(\pi) := \{\pi_i \in [n] : \pi_i < \pi_{i+1}\} \text{ and } \mathrm{DB}(\pi) := \{\pi_i \in [n] : \pi_{i-1} > \pi_i\},$$

where $\pi_0 = \pi_{n+1} := \infty$, denote the set of *ascent bottoms* and *descent bottoms* of π, respectively.

Example 4. If $\pi = 324156$, then $\mathrm{DB}(\pi) = \{1, 2, 3\}$ and $\mathrm{AB}(\pi) = \{1, 2, 5, 6\}$.

2.3 Eulerian polynomials

The Eulerian polynomials first appeared in Euler's work in 1736 [22], and have been studied frequently in enumerative combinatorics ever since. Eulerian polynomials are generating functions of permutation statistics such as descents and excedances, also called Eulerian statistics, over $\mathfrak{S}_n$.

The *Eulerian number*, $A_{n,k}$, counts the number of permutations in $\mathfrak{S}_n$ with exactly $k - 1$ descents. The Eulerian numbers satisfies the recurrence relation

$$A_{n+1,k} = kA_{n,k} + (n + 2 - k)A_{n,k-1}, \ 0 \le k \le n.$$

This can be proved by observing what happens to the number of descents of a permutation $\pi \in \mathfrak{S}_n$ when $n + 1$ is inserted. If $n + 1$ is inserted between π_i and π_{i+1} where i is a descent or at the end of the permutation, then the number of descents is unchanged. If $n + 1$ is inserted between π_i and π_{i+1} where i is not a descent or at the beginning, then the new permutation has one more descent than the original one.

The Eulerian polynomial, $A_n(x), n \in \mathbb{N}$, may further be defined as the generating polynomial for the descent statistic over the symmetric group. We have:

$$A_n(x) = \sum_{i=1}^{n} A_{n,i}x^i = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)+1}.$$

As stated earlier, the Eulerian polynomials are known to be real-rooted. This result has been generalized to various refinements of the Eulerian polynomials and serves as a starting point for many of the results presented in the Papers I–IV.

For a positive integer n we denote by $\mathbf{x}$ the n-tuple $(x_1, x_2, \ldots, x_n)$. We will now introduce a multivariate refinement of the Eulerian polynomials, $A_n(\mathbf{x}, \mathbf{y})$, that will be used as an example throughout this chapter. We define

$$A_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{i \in \mathrm{DB}(\pi)} x_i \prod_{j \in \mathrm{AB}(\pi)} y_j.$$

In the next section we will discuss a multivariate analog to the property of having only real zeros.

2.4 Stable polynomials in combinatorics

Choe *et al.* [16], were pioneers in realizing the importance of multivariate polynomials with zero restrictions in combinatorics. Over the past decade methods using zeros of multivariate polynomials have been used in various areas. For example, Gurvits' proof and generalization of the Van der Waerden conjecture [24], the proof of Johnson's conjectures [8], Speyer's alternative proof of Horn's problem using hyperbolic polynomials [38], the proof of the existence of infinite families of Ramanujan graphs [30], and the proof of the Kadison-Singer problem from 1959 [31], see also [6; 12; 15].

In this section we discuss the theory of stable polynomials, a multivariate generalization of real-rooted polynomials. A multivariate polynomial is said to be *stable* if it does not vanish whenever all the variables have positive imaginary parts, that is, $f \in \mathbb{C}[\mathbf{x}]$ is stable if $\text{Im}(x_i) > 0$ for all $1 \leq i \leq n$ implies $f(\mathbf{x}) \neq 0$. Observe that for a univariate polynomial with real coefficients, stability is equivalent to the property of having only real zeros.

Example 5. All monomials $\prod_{i=1}^{n} x_i$ and all linear polynomials $\sum_{i=1}^{n} a_i x_i$ with positive coefficients a_i are stable.

An important family of stable polynomials arise from determinants.

Example 6. Let $A_1, \ldots, A_m$ be positive semidefinite $n \times n$ matrices and let B be a Hermitian $n \times n$ matrix. Then the determinantal polynomial

$$\det \left(\sum_{i=1}^{m} x_i A_i + B \right)$$

is stable or identically zero [8].

There are other notions of stability for which the polynomial is required to be non-zero in another region of $\mathbb{C}^n$. The definition of Ω-stability follows. Let $n \geq 0$ and $\Omega \subset \mathbb{C}^n$. Then $f \in \mathbb{C}[\mathbf{x}]$ is said to be Ω-stable if $f(\mathbf{x}) \neq 0$ whenever $\mathbf{x} \in \Omega$. As an example, a polynomial is said to be Hurwitz stable if it is non-vanishing in the open right half-plane. A stable polynomial with real coefficients is called real stable.

Techniques to lift results concerning real-rooted univariate polynomials to concern multivariate extensions of the polynomials have been developed over the past decades [13; 15; 25]. Such refinements have many benefits. For example the stability of a multivariate polynomial implies the real-rootedness of the corresponding univariate polynomial. Another

benefit is that the proofs of the multivariate statements are often simpler. Most importantly multivariate stability implies several inequalities among the coefficients, such as refined log-concavity and unimodality [6; 40]. In Section 2.6 we also discuss the connection between stable polynomials and negative dependence.

Many multivariate refinements of the Eulerian polynomials have been studied. As stated before, the univariate Eulerian polynomials can be seen as the generating polynomial for the descent statistic over the symmetric group. When we study multivariate refinements we study e.g., descents, ascents and excedances that are refined over permutations by their values.

We will use the following multivariate notation. For a set $S \subseteq [n]$ we let $\mathbf{x}^S = \prod_{i \in S} x_i$. The cardinality of a set S will be denoted by $|S|$.

The following result was observed by Brändén in 2010.

Theorem 1. *The multivariate Eulerian polynomial*

$$A_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \mathfrak{S}_n} \mathbf{x}^{\mathrm{DB}(\pi)} \mathbf{y}^{\mathrm{AB}(\pi)}$$

is stable.

To prove Theorem 1 we need to introduce stability preserving operators. In the next section we introduce stability preserving operators and give a brief discussion of what is known. The proof of Theorem 1 will serve as an example of how this theory can be used. We will present the idea for the proof of Theorem 1 and also see how real-rootedness of the classical Eulerian polynomials follows as a corollary.

2.5 Stability preserving operators

In [34] Pólya and Schur characterized all diagonal operators preserving real-rootedness of univariate real polynomials. Borcea and Brändén [7] proved a generalization of this result and gave a characterization of all linear operators preserving stability. This characterization is of major importance since it provides a way of checking if an operator preserves stability or not. Operators preserving stability are important tools which are used to, for example, prove stability of sequences of polynomials that satisfy a linear recursion.

We call a linear transformation $T : V \to W$ *stability-preserving* if $T(f(\mathbf{x}))$ is stable, or identically zero, whenever $f(\mathbf{x})$ is stable and where V and W are $\mathbb{C}$– or $\mathbb{R}$–linear spaces of polynomials. We start with giving some simple examples of stability preserving operators.

Lemma 2 (Lemma 2.4 in [40]). *Let* $\mathbf{x} = (x_1, \ldots, x_n)$. *The following operations preserve stability of polynomials in* $\mathbb{C}[\mathbf{x}]$.

- *Permutation. For any permutation* $\sigma \in \mathfrak{S}_n$,
$$f \mapsto f(x_{\sigma(1)}, \ldots x_{\sigma(n)}).$$

- *Scaling. For* $c \in \mathbb{C}$ *and* $(a_1, \ldots, a_n) \in \mathbb{R}^n$ *with* $a_i > 0$ *for* $1 \leq i \leq n$,
$$f \mapsto cf(a_1 x_1, \ldots, a_n x_n).$$

- *Diagonalization. For* $1 \leq i < j \leq n$,
$$f \mapsto f(\mathbf{x})|_{x_i = x_j}.$$

- *Specialization. For* $a \in \{w \in \mathbb{C} : \text{Im}(w) \geq 0\}$,
$$f \mapsto f(a, x_2, \ldots, x_n).$$

- *Inversion. If* r *is the degree of* x_1 *in* f,
$$f \mapsto x_1^r f(-1/x_1, x_2, \ldots, x_n).$$

- *Differentiation or contraction. For any* $1 \leq i \leq n$,
$$f \mapsto \partial f(\mathbf{x})/\partial x_i.$$

A polynomial is said to be *multiaffine* if it has degree at most one in each variable. We denote by $\mathbb{C}[\mathbf{x}]^{ma}$ the vector space of complex polynomials with degree at most one in each variable. For a linear operator $T : \mathbb{C}[\mathbf{x}]^{ma} \to \mathbb{C}[\mathbf{x}]$ we define the *symbol* of T as

$$G_T(\mathbf{x}, \mathbf{y}) = T\left((\mathbf{x} + \mathbf{y})^{[n]}\right) = \sum_{S \subseteq [n]} T(\mathbf{x}^S) \mathbf{y}^{[n] \setminus S} \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$$

The following was proved in [7].

Proposition 3. *Let* $T : \mathbb{C}[\mathbf{x}]^{ma} \to \mathbb{C}[\mathbf{x}]$ *be a linear transformation. Then* T *preserves stability if and only if either*

- (i) $T(f) = \eta(f) \cdot g$ *for some linear functional* $\eta : \mathbb{C}[\mathbf{x}]^{ma} \to \mathbb{C}$ *and stable polynomial* $g \in \mathbb{C}[\mathbf{x}]$, *or*

- (ii) $G_T(\mathbf{x}, \mathbf{y})$ *is stable.*

We now have all the tools necessary to prove Theorem 1 and a univariate corollary.

Proof of Theorem 1. The proof is by induction on n. For the base case, clearly $A_1 = x_1 y_1$ is stable. First we observe what happens to the set of descent bottoms and ascent bottoms if 1 is inserted into a permutation $\pi^* = (\pi_1 + 1)(\pi_2 + 1) \cdots (\pi_n + 1)$ for a permutation $\pi \in \mathfrak{S}_n$. This observation yields the recursion

$$A_{n+1}(\mathbf{x}, \mathbf{y}) = (x_1 + y_1) A_n(\mathbf{x}^*, \mathbf{y}^*) + x_1 y_1 \sum_{i=2}^{n+1} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right) A_n(\mathbf{x}^*, \mathbf{y}^*),$$

where $\mathbf{x}^* = (x_2, \ldots, x_{n+1})$ and $\mathbf{y}^* = (y_2, \ldots, y_{n+1})$. Thus what remains is to prove that the linear operator

$$T = (x_1 + y_1) + x_1 y_1 \sum_{i=2}^{n+1} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right)$$

preserves stability. By a simple calculation

$$G_T = x_1 y_1 \left(\frac{1}{x_1} + \frac{1}{y_1} + \sum_{i=2}^{n+1} \left(\frac{1}{x_i + u_i} + \frac{1}{y_i + v_i} \right) \right) (\mathbf{x}^* + \mathbf{u}^*)^{[n]} (\mathbf{y}^* + \mathbf{v}^*)^{[n]}.$$

Since all terms in the large parenthesis above have negative imaginary parts when the variables have positive imaginary parts, we see that G_T is stable. The theorem follows by Proposition 3. $\square$

As a corollary to Theorem 1 we have the classical result for the univariate Eulerian polynomials, namely that they have only real zeros.

Corollary 4. *The Eulerian polynomial,*

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)+1}$$

has only real zeros.

Proof. First we use the specialization of a variable and set all $y_i = 1$. According to Lemma 2 this operation preserves stability. Applying the same lemma again, diagonalizing $\mathbf{x}$ we find that

$$\sum_{\pi \in \mathfrak{S}_n} x^{|\operatorname{DB}(\pi)|},$$

has only real zeros. This is equivalent to

$$\sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)+1}$$

having only real zeros. $\square$

2.6 Negative dependence

From random variables affecting each other in a consistent direction one can obtain inequalities that are important in statistical physics, probability theory and combinatorics. Negative dependence in probability theory models repelling particles. In [6] the class of strong Rayleigh probability measures was introduced. The geometry of zeros was used to develop a theory of negative dependence for measures on $\{0,1\}^n$ and to answer several conjectures regarding correlation inequalities in discrete probability theory. In this section we give a brief introduction to negative dependence and its connection to zeros of polynomials and refer to [6; 13; 26; 33] for wider discussions.

Let $\mathbf{1}$ denote the vector of n ones. Multiaffine polynomials with nonnegative coefficients may be viewed as discrete probability measures as follows. Let

$$f(\mathbf{x}) = \sum_{\eta \in \{0,1\}^n} a(\eta) \mathbf{x}^\eta$$

be a multiaffine polynomial in n variables with nonnegative coefficients and with $f(\mathbf{1}) = 1$. We define a discrete probability measure μ_f on $\{0,1\}^{[n]}$ by setting $\mu_f(\{\eta\}) = a(\eta)$.

Negative association is a very useful negative dependence. A discrete probability measure μ on $\{0,1\}^S$ is *negatively associated* if

$$\int fg d\mu \le \int f d\mu \int g d\mu,$$

whenever $f, g : \{0,1\}^S \to \mathbb{R}$ are increasing functions depending on disjoint sets of variables.

Strong Rayleigh is a strong form of negative dependence. If the generating polynomial of a discrete probability measure μ is stable then we say that μ is *strong Rayleigh* [6]. It was proved in [6] that if μ is strong Rayleigh then μ is negatively associated. Hence the geometry of zeros of polynomials may be used to prove strong negative dependence properties of discrete probability measures.

The symmetric exclusion process is a Markov process on $\{0,1\}^S$ that models particles jumping symmetrically on a countable set S. We think of $\eta \in \{0,1\}^S$ as a configuration of particles. This is briefly described in Section 2.7 and is discussed further in Paper I, where we also discuss negative dependence properties of the multivariate partition function satisfied by the stationary distribution with certain parameters. It was proved in [6] that the symmetric exclusion process preserves the strong Rayleigh property.

2.7   Overview of Paper I

In Paper I we study a multivariate extension of the Eulerian polynomials, namely the one that refines the excedance statistic over permutations by the set of excedances. These multivariate Eulerian polynomials appear in statistical mechanics in connection with the (partially) asymmetric exclusion process (ASEP) [18–20; 27]. The ASEP appears in various contexts, such as computation biology and traffic flow.

The ASEP model describes a system of particles hopping left and right on a set of n sites, labeled by $[n] := \{1, 2, \ldots, n\}$. Given a nonnegative matrix $Q = (q_{ij})_{i,j=1}^n$ and vectors $b = (b_i)_{i=1}^n$ and $d = (d_i)_{i=1}^n$ of nonnegative numbers, define a continuous time Markov chain on $\{0, 1\}^n$ by its transitions:

(J) a particle jumps from site i to site j at rate q_{ij},

(B) a particle is created at site i at rate b_i, and

(D) a particle is annihilated at site i at rate d_i.

A configuration of particles is represented by a vector $\eta \in \{0, 1\}^n$ where $\eta(i) = 1$ if site i is occupied, and $\eta(i) = 0$ if i is vacant.

Let μ be a discrete probability measure, then the multivariate *partition function* of μ on $\{0, 1\}^n$ is the polynomial

$$Z_\mu(\mathbf{x}) = \sum_{\eta \in \{0,1\}^n} \mu(\{\eta\}) \mathbf{x}^\eta := \sum_{\eta \in \{0,1\}^n} \mu(\{\eta\}) x_1^{\eta(1)} \cdots x_n^{\eta(n)}.$$

In a series of papers [18–20], Corteel and Williams established an important connection between permutation tableaux and a special case of the ASEP model. They also proved that the stationary distribution of the ASEP, with certain parameters, can be described in terms of multivariate Eulerian polynomials. In Paper I we generalize this result and prove that the stationary distribution, with other parameters, is a q-analog of the Eulerian polynomials for r-colored permutations. We also give a new combinatorial interpretation of the stationary distribution in terms of alternating trees and decorated alternative trees by using the matrix ansatz for the ASEP due to Liggett [27]. Towards the end of the paper we also discuss correlation inequalities and stability for the multivariate partition function.

2.8   Overview of Paper II

In Paper II we study a multivariate version of the P-Eulerian polynomials. A *labeled poset* on n elements is a poset $P = ([n], \preceq)$, where $\preceq$ is the partial order and $\leq$ is the natural order on $[n]$. We denote by $\mathscr{L}(P)$ the set of linear extensions of P, i.e., the *Jordan-Hölder set* of P:

$$\mathscr{L}(P) := \{\pi \in \mathfrak{S}_n : \text{ for all } i, j \in [n], \text{ if } \pi_i \preceq \pi_j, \text{ then } i \leq j\}.$$

The univariate P-Eulerian polynomial is defined by

$$A_P(x) := \sum_{\pi \in \mathscr{L}(P)} x^{\text{des}(\pi)+1}.$$

The Neggers-Stanley conjecture stated the following.

Conjecture 5. *For any labeled poset P, the polynomial $A_P(x)$ is real-rooted.*

The Neggers-Stanley conjecture was first proved for several classes of posets [11; 41], before it was disproved in [14]. The conjecture is still open for the class of naturally labeled graded posets, though efforts have been made to both prove and disprove it. In Paper II we prove that a multivariate P-Eulerian polynomial is stable for a large class of labeled posets, for which the univariate polynomial is known to be real-rooted.

Let $[n]' := \{i' : i \in [n]\}$ be a distinct copy of $[n]$. For a permutation $\pi \in \mathfrak{S}_n$ define a monomial in the variables $\mathbf{z} = \{z_e : e \in [n] \cup [n]'\}$:

$$A_\pi(\mathbf{z}) := \prod_{e \in \text{DB}(\pi)} z_e \prod_{e \in \text{AB}(\pi)} z_{e'} = \prod_{e \in \text{B}(\pi)} z_e,$$

where $\text{B}(\pi) = \text{DB}(\pi) \cup \{e' : e \in \text{AB}(\pi)\}$. The *multivariate P-Eulerian polynomial* is defined as

$$A_P(\mathbf{z}) := \sum_{\pi \in \mathscr{L}(P)} A_\pi(\mathbf{z}),$$

We study the effect on these multivariate Eulerian polynomials upon taking disjoint unions. The *disjoint union* of two labeled posets $P = ([n], \preceq_P)$ and $Q = ([n], \preceq_Q)$ on ground sets $[n]$ and $[m]$ is the labeled poset $P \sqcup Q = ([n+m], \preceq)$ whose set of relations is

$$\{i \preceq j : i, j \in [n] \text{ and } i \preceq_P j\} \cup \{(n+i) \preceq (n+j) : i, j \in [m] \text{ and } i \preceq_Q j\}.$$

We prove that if $A_P(\mathbf{z})$ and $A_Q(\mathbf{z})$ are stable for two labeled posets P and Q, then so is $A_{P \sqcup Q}(\mathbf{z})$, using the algebra of free quasi-symmetric

functions [21; 29]. We generalize the result and discuss applications for a multivariate peak polynomial and naturally labeled decreasing forests. In the process we identify a, to our knowledge, new algebra on Dyck paths.

2.9 Overview of Paper III

In Paper III we answer the following question posed by Athanasiadis [3], to the affirmative.

Question 1. *Is the local h-polynomial of the rth edgewise subdivision of the $(n-1)$-dimensional simplex 2^V, $l_V((2^V)^{\langle r \rangle}, x)$, real-rooted?*

The rth edgewise subdivision of the $(n-1)$-dimensional simplex 2^V can be expressed as the generating polynomial for the ascent statistic over a restricted version of the Smirnov words [3]. Due to this result we can use some already developed techniques to show that these polynomials are indeed real-rooted.

Let $\mathcal{SW}(n,r)$ be the set of words $(w_0, w_1, \ldots, w_n) \in \{0, 1, \ldots r - 1\}^{n+1}$ such that $w_i \neq w_{i+1}$ for $i = 0, 1, \ldots, n-1$, and $w_0 = w_n = 0$. The words in $\mathcal{SW}(n,r)$ are the Smirnov words with restrictions on the first and last letters [28]. We use the same definition of ascents as for permutations in the symmetric group. That is, an ascent is an index $i \in \{0, 1, \ldots, n-1\}$ such that $w_i < w_{i+1}$. Let $\mathrm{asc}(w)$ be the number of ascents in w. The polynomial $l_V((2^V)^{\langle r \rangle}, x)$ can be expressed as

$$l_V((2^V)^{\langle r \rangle}, x) = \sum_{w \in \mathcal{SW}(n,r)} x^{\mathrm{asc}(w)}.$$

We consider the polynomials

$$E_{r,n}^i = \sum_{w \in \mathcal{SW}'(n,r)} \chi(w_n = i) x^{\mathrm{asc}(w)}, \tag{2.4}$$

where $\chi(\varphi)$ is one if φ is true and zero if not, and where $\mathcal{SW}'(n,r)$ is defined as $\mathcal{SW}(n,r)$ but without the restriction of w_n to be zero. Note that $E_{r,n}^0 = l_V((2^V)^{\langle r \rangle}, x)$.

Let $f_1(x), \ldots, f_k(x)$ be polynomials over the real numbers. Then they are said to be *compatible* if, for all real and positive $c_1, \ldots, c_k$, the polynomial

$$\sum_{j=1}^{k} c_j f_j(x)$$

has only real zeros.

We use some previous results and adapted techniques by Chudnovsky-Seymour [17] and Savage-Visontai [36] on compatible polynomials to prove the following theorem.

Theorem 6. *For $i = 0, \ldots, r-1$ the polynomial $E_{r,n}^i$ has only real zeros.*

We then further generalize the tool so that it can be used to prove stability of polynomials in certain types of families of polynomials.

2.10 Overview of Paper IV

Let $s = (s_1, \ldots, s_n)$ be a sequence of positive integers. An *s-lecture hall partition* is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying

$$0 \leq \frac{\lambda_1}{s_1} \leq \cdots \leq \frac{\lambda_n}{s_n}.$$

These were studied in e.g., [35], and are generalizations of the original *lecture hall partitions*, corresponding to the case $s = (1, 2, \ldots, n)$, see [9].

Let $P = ([p], \preceq)$ be a labeled poset. A P-partition is a map $f : P \to \mathbb{R}$ such that

1. if $x \prec y$, then $f(x) \leq f(y)$, and

2. if $x \prec y$ and $x > y$, then $f(x) < f(y)$.

In Paper IV we generalize the concept of s-lecture hall partitions to labeled posets. We introduce and study a lecture hall generalization of P-partitions. Let P be a labeled poset and let $s : P \to \mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ be an arbitrary map. We define a *lecture hall (P, s)-partition* to be a map $f : P \to \mathbb{R}$ such that

1. if $x \prec y$, then $f(x)/s(x) \leq f(y)/s(y)$, and

2. if $x \prec y$ and $x > y$, then $f(x)/s(x) < f(y)/s(y)$.

In Paper IV we derive formulas for the main generating functions associated to lecture hall (P, s)-partitions. We discuss the connection between lecture hall partitions and Ehrhart theory and acquire a reciprocity theorem. We study special cases, such as when P is a naturally labeled chain or an anti-chain and find that they automatically produce results on lecture hall partitions and signed permutations, respectively. We also discuss real-rootedness and unimodality of the (P, s)-Eulerian polynomials.

3. Correlating events in randomly oriented graphs

3.1 Correlations in graphs

For a graph $G = (V, E)$, with vertex set V and edge set E, consider a model where each edge in E is oriented independently of all other edges with probability $1/2$ in each direction. This model, that will be referred to as O, has been studied frequently in combinatorics and in the context of edge percolation lately, see [1; 2; 4; 5].

Let $a, b \in V$ be two distinct vertices and denote by $\{a \to b\}$ the event that there exist a directed path from a to b. In the previous mentioned papers correlations between different events of this form have been studied. It has been proven that for three distinct vertices $a, b, s \in V$ the events $\{s \to a\}$ and $\{s \to b\}$ are positively correlated, meaning

$$P_O(\{s \to a\}) P_O(\{s \to b\}) \leq P_O(\{s \to a\} \cap \{s \to b\}).$$

In [1] the correlation between the events $\{a \to s\}$ and $\{s \to b\}$ was proven to be positive for the complete graph K_n for $n \geq 5$, negative for $n = 3$ and zero for $n = 4$. The two events were also proven to be negatively correlated if the graph, G, is a cycle or a tree.

3.2 Overview of Paper V

In Paper V we study the model O, and correlations in randomly oriented graphs. For four different vertices $a, b, s, t \in V(G)$, we study the correlation between the events $\{s \to a\}$ and $\{t \to b\}$. We prove that the events $\{s \to a\}$ and $\{t \to b\}$ are positively correlated in the complete graph.

We further investigate the two random graph models, $G(n, p)$ and $G(n, m)$ and find that the events $\{s \to a\}$ and $\{t \to b\}$ are positively correlated for fixed $p > 0$ and sufficiently large n. This situation can be compared to the events, $\{s \to a\}$ and $\{b \to s\}$, studied in [2] where a critical value of p was found and the correlation was proven to be

negative if p was below the critical value and positive if p was above the critical value.

Sammanfattning

Denna avhandling består av totalt fem artiklar uppdelade i två delar samt tre inledande kapitel. I den första delen studeras polynom av kombinatorisk natur och polynomegenskapen att ha enbart reella nollställen. Dit hör artikel I–IV. I den andra delen ingår artikel V, där vi studerar korrelerade händelser i slumpvis orienterade grafer.

Polynom med enbart reella nollställen har många intressanta egenskaper, exempelvis att koefficienterna är log-konkava och unimodala. Inom kombinatoriken finns många exempel på polynom med enbart reella nollställen. Det mest klassiska exemplet är de Eulerska polynomen, som kan beskrivas som genererande funktioner av fall över den symmetriska gruppen. Detta är ett resultat som har generaliserats och förfinats på många olika sätt och ligger till grund för en stor del av den forskning som presenteras i den här avhandlingen. I kapitel 2 introducerar vi polynom med enbart reella nollställen och diskuterar egenskaper för sekvenser vars genererande funktioner har enbart reella nollställen. Vi låter de Eulerska polynomen agera exempel genom kapitlet när vi introducerar polynomstatistiker, stabila polynom (en generalisering av polynom med enbart reella nollställen till flera variabler) och operationer som bevarar stabilitet. Vi går även kort in på negativt beroende. En introduktion och översikt till var och en av artikel I–IV avslutar kapitlet.

ASEP är en modell som beskriver partiklar som hoppar mellan olika tillstånd på ett lattice. I artikel I ger vi en ny kombinatorisk tolkning av den stationära fördelningen av ASEP med hjälp av färgade permutationer och dekorerade alternativa träd. Vi hittar ett samband mellan motsvarande partitionsfunktion i flera variabler och ett generaliserat Eulerskt polynom där överskott och deras värde behandlas över r-färgade permutationer.

I artikel II studerar vi P-Eulerska polynom i flera variabler, en utvidgning av de Eulerska polynomen till märkta pomängder. Vi visar att denna förfining är stabil för en stor klass av märkta pomängder.

I artikel III använder vi oss av kompatibla polynom för att visa att polynom i en speciell familj av polynom enbart har reella nollställen.

Vi besvarar således en öppen fråga ställd av C. Athanasiadis [3]. Vi generaliserar sedan tekniken så att den kan användas för att kontrollera om polynom i en viss typ av familjer av polynom har enbart reella nollställen.

I artikel IV introducerar och studerar vi s-hörsalspartitioner för märkta pomängder. Vi finner samband mellan statistiker på kransprodukter och statistiker på hörsalspartitioner för märkta pomängder.

I kapitel 3 diskuterar vi korrelationer mellan olika händelser i turneringar och sammanfattar resultaten i artikel V. I artikel V visar vi att om alla kanter i den kompletta grafen K_n slumpvis riktas i en av de två möjliga riktningarna oberoende av varandra så är händelserna $\{s \rightarrow a\}$ och $\{t \rightarrow b\}$ positivt korrelerade om de fyra noderna $a, s, b, t \in K_n$ är olika. Vi undersöker också korrelationen mellan dessa händelser i slumpgrafer med slumpmässig orientering.

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