Around power ideals

From Fröberg's conjecture to zonotopal algebra

Gleb Nenashev

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Abstract

In this thesis we study power algebras, which are quotient of polynomial rings by power ideals. We will study Hilbert series of such ideals and their other properties. We consider two important special cases, namely, zonotopal ideals and generic ideals. Such ideals have a lot combinatorial properties.

In the first chapter we study zonotopal ideals, which were defined and used in several earlier publications. The most important works are by F.Ardila and A.Postnikov and by O.Holtz and A.Ron. These papers originate from different sources, the first source is homology theory, the second one is the theory of box splines. We study quotient algebras by these ideals; these algebras have a nice interpretation for their Hilbert series, as specializations of their Tutte polynomials. There are two important subclasses of these algebras, called unimodular and graphical. The graphical algebras were defined by A.Postnikov and B.Shapiro. In particular, the external algebra of a complete graph is exactly the algebra generated by the Bott-Chern forms of the corresponding complete flag variety. One of the main results of the thesis is a characterization of external algebras. In fact, for the case of graphical and unimodular algebras we prove that external algebras are in one-to-one correspondence with graphical and regular matroids, respectively.

In the second chapter we study Hilbert series of generic ideals. By a generic ideal we mean an ideal generated by forms from some class, whose coefficients belong to a Zariski-open set. There are two main classes to consider: the first class is when we fix the degrees of generators; the famous Fröberg's conjecture gives the expected Hilbert series of such ideals; the second class is when an ideal is generated by powers of generic linear forms. There are a few partial results on Fröberg's conjecture, namely, when the number of variables is at most three. In both classes the Hilbert series is known in the case when the number of generators is at most (n+1). In both cases we construct a lot of examples when the degree of generators are the same and the Hilbert series is the expected one.

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Preface

This PhD thesis was written under the supervision of Prof. Boris Shapiro. I met him for the first time in the fall 2013, when he suggested me a problem about invariants of plane curves. He also introduced me to the graphical algebras defined in [34]. This project was my main reason to move to Stockholm University and work under his supervision. I have completed 5 papers on this topic **[A1-A5]** and it is the explanation of the second half of the subtitle *From Fröberg's conjecture to zonotopal algebra*. In the fall 2014, the *problem solving seminar in commutative algebras* organized by B. Shapiro and R. Fröberg started at Stockholm University, where we discussed maximal rank properties, Waring's type problems and others. It was my first term in Stockholm and I started to work on these problems. I wrote 4 papers on this topic; three of them **[B1-B3]** are included in this thesis.

The structure of the PhD thesis is as follows: All definitions and results from [**A1-A5, B1-B3]** are formulated in two chapters, which are written using common notations. You need to read the actual papers if you want to undestand the proofs of the formulated results. Papers **[A1]** and **[A2]** were included in the licentiate thesis [28].

List of papers

The following papers are included in this thesis.

[A1] Postnikov-Shapiro Algebras, Graphical Matroids and their generalizations
Gleb Nenashev, *Preprint: arXiv:1509.08736v4*

[A2] "K-theoretic" analog of Postnikov-Shapiro algebra distinguishes graphs
Gleb Nenashev, Boris Shapiro, *Journal of Combinatorial Theory, Series A, 148 2017, pp. 316–332*

[A3] On Q-deformations of Postnikov-Shapiro algebras
Anatol N. Kirillov, Gleb Nenashev, *Séminaire Lotharingien de Combinatoire, 78B.55, FPSAC 2017, 12 pp.*

[A4] Unimodular zonotopal algebra
Anatol N. Kirillov, Gleb Nenashev

[A5] Classification of external zonotopal algebras
Gleb Nenashev, *Preprint: arXiv:1803.09966v1*

[B1] A note on Fröberg's conjecture for forms of equal degrees
Gleb Nenashev, *Comptes Rendus Mathematique, 355(3) 2017, pp. 272–276*

[B2] New proof of Fröberg's conjecture for three variables
Gleb Nenashev

[B3] On ideals generated by two generic quadratic forms in the exterior algebra
Veronica Crispin Quiñonez, Samuel Lundqvist, Gleb Nenashev, *Preprint: arXiv:1803.08918v1*

Contents

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I am very grateful to all the participants of the *problem solving seminar in commutative algebras* at SU organized by my advisor and by Ralf Fröberg, as well as all participants of the *combinatorial seminar* at KTH, especially to its organizers Anders Björner, Petter Brändén and Svante Linusson. I also would like to thank all the people I met during my visits to MIT, in the spring 2017, and to MSRI, in the fall 2017, especially to Alex Postnikov. In addition, I would like to thank all the people whose courses I attended.

My sincere gratefulness goes to people from the three main mathematical places in Saint-Petersburg: PhML №239, PDMI and SPbSU. Especially I am very grateful to Inga Andreeva and Nadezhda Kushpel, who taught me at PhML №239. I would also like to thank Fedor Petrov, who coached me at all these three places, as well as Nick Gravin, who suggested me to contact Boris.

I would like to thank all my other collaborators during these years, especially to those that contributed to this thesis: Veronica Crispin Quiñonez, Anatol N. Kirillov and Samuel Lundqvist. I am also very grateful to all people who have read my texts and gave me comments.

Last, I would like to thank my parents for bringing me to this world, and my brother for suggesting my name. I am also grateful for all their support during my life; without it, much would not have been possible.

Abstract

In this thesis we study power algebras, which are quotient of polynomial rings by power ideals. By definition a power ideal is an ideal in a polynomial ring generated by powers of linear forms, i.e., ideal $\mathcal{I} = \langle \ell_i^{d_i},\ i \in [k] \rangle \subset \mathbb{K}[x_1, \dots, x_n]$, where $\ell_i,\ i \in [k]$ are linear forms. We will study Hilbert series of such ideals and their other properties. We consider two important special cases, namely, zonotopal ideals (see § 1) and generic ideals (see § 2). Such ideals have a lot combinatorial properties.

In chapter § 1 we study zonotopal ideals, which were defined and used in several earlier publications. The most important works are [4] by F. Ardila and A. Postnikov and [20] by O. Holtz and A. Ron. These papers originate from different sources, the first source is homology theory, the second one is the theory of box splines. We study quotient algebras by these ideals; these algebras have a nice interpretation for their Hilbert series, as specializations of their Tutte polynomials. There are two important subclasses of these algebras, called unimodular and graphical (see § 1.2). The graphical algebras were defined by A.Postnikov and B.Shapiro in [34]. In particular, the external algebra of a complete graph is exactly the algebra generated by the Bott-Chern forms of the corresponding complete flag variety (see [35]). In my opinion the most important result of the thesis is a characterization of external algebras. In fact, for the case of graphical and unimodular algebras we prove that external algebras are in one-to-one correspondence with graphical and regular matroids, respectively.

In chapter § 2 we study Hilbert series of generic ideals. By a generic ideal we mean an ideal generated by forms from some class, whose coefficients belong to a Zariski-open set. There are two main classes to consider: the first class is when we fix the degrees of generators; the famous Fröberg's conjecture gives the expected Hilbert series of such ideals; the second class is when an ideal is generated by powers of generic linear forms. There are a few partial results on Fröberg's conjecture, namely, when the number of variables is at most 3 (see [3]). In both classes the Hilbert series is known in the case when the number of generators is at most $n+1$ (see [39]). In both cases we construct a lot of examples when the degree of generators are the same and the Hilbert series is the expected one.

Sammanfattning

In denna avhandling studerar vi potensalgebror, dvs kvotringar av polynomringar modulo potensideal. Per definition är ett potensideal ett ideal i en polynomring som genereras av potenser av linjära former, dvs ideal $\mathcal{I} = \langle \ell_i^{d_i},\ i \in [k] \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$, där $\ell_i,\ i \in [k]$ är linjära former. Vi studerar Hilbertserierna av dessa ideal samt en del andra egenskaper. Vi betrakta tvåviktiga speciella fall, nämligen zonotopideal (se § 1) och generiska ideal (se § 2). Sådana ideal har många intressanta kombinatoriska egenskaper.

I kapitel § 1 studerar vi zonotopideal som definierats och använts i en rad tidigare publikationer. De mest relevanta arbetena är [4] av F. Ardila och A. Postnikov samt [20] av O. Holtz och A. Ron. Dessa artiklar är motiverade av olika källor; den första källan är homologiteorin och den andra är teorin för box splines. Vi betraktar kvotalgebror modulo dessa ideal; det finns en trevlig tolkning av dessa algebrors Hilberserier i termer av specialiseringar av deras Tuttepolynom. Det finns tvånaturliga delklasser av dessa algebror som kallas unimodala och grafiska (se § 1.2). De grafiska algebrorna definierades av A. Postnikov och B. Shapiro i [34]. Speciellt sammanfaller den externa algebran av den fullständiga grafen med algebran som genereras av Bott-Chern-former påmångfalden av fullständiga flaggor (se [35]). I min mening är det mest intressanta resultatet av denna avhandling en karakterisering av externa algebror. Speciellt kommer vi att bevisa att i fallet av grafiska och unimodala algebror finns det en 1-1-korrespondens mellan de respektive externa algebrorna och de grafiska och reguljära matroiderna.

I kapitel § 2 studerar vi Hilbertserierna av generiska ideal. Med ett generiskt ideal menar vi ett ideal genererat av former tillhörande en viss klass vars koefficienter ligger in en viss Zariskiöppen mängd. Det finns tvåviktiga delklasser att betrakta. I första fallet fixerar vi grader av alla generatorer; i detta fall ger den kända Fröbergförmodan den förväntade Hilbertserien av sådana ideal. Den andra klassen består av ideal genererade av potenser av generiska linjära former. Det finns enstaka partiella resultat angående Fröbergsförmodan som gäller dåantalet variabler är högst 3 (se [3]). Utöver detta är Hilbertserien känd för bägge klasserna om antalet generatorer är högst $n+1$ (se [39]). I bägge dessa situationer konstruerar vi många exempel dåalla generatorer har samma grad och den riktiga Hilbertserier sammanfaller med den förväntade.

1. Zonotopal algebras

In this chapter we discuss zonotopal algebras. There are three types of these algebras: external, central, and internal. These algebras are defined either for matrices or for graphs. Their Hilbert series are specializations of the corresponding Tutte polynomials, see Theorem 1.1.

This topic of study has 2 different sources:

(i) The starting point are papers [36] and [35]. The first paper was written by B. Shapiro and M. Shapiro, the second paper jointly with A. Postnikov. It comes from homology theory and was motivated by the following problem posed by V. I. Arnold, [6]:

Describe algebra $\mathcal{C}_n$ generated by the curvature forms of the tautological Hermitian linear bundles over the type A complete flag variety $\mathcal{F}l_n$. Surprisingly enough, it was observed and conjectured in [36] and proved in [35], that $\dim \mathcal{C}_n$ is equal to the number of spanning forests of the complete graph K_n on n labeled vertices.

Latter, in paper [34], A. Postnikov and B. Shapiro generalized this construction to all graphs. They introduced two types of graphical algebras, whose total dimensions are the number of spanning trees and forests respectively. (see definition in § 1.2). The algebra counting spanning trees is related to the parking ideal of a graph, they have complementing Hilbert series. This parking ideal is generated by the stable configurations of the chip-firing game (other names for the same object are the abelian sandpile model, the critical group of the graph and etc.). In the papers [22; 34; 37] and **[A1-A3]** the study of graphical algebras was continued

The culminating paper in this direction was [4] by F. Ardila and A. Postnikov. They considered quotient algebras by zonotopal ideals. In fact, they introduced a little bit more, but we will focus only on three types of ideals, namely external, central, and internal zonotopal ideals. In [4], the authors found their Hilbert series, which are specializations of the Tutte polynomials. (See also papers [8; 9].)

(ii) The culminating paper in this story was [20] by O. Holtz and A. Ron, where they defined all three types of zonotopal algebras. They studied pairs of dual spaces: an ideal and its kernel. The motivation comes from the box-splines theory (see [2; 15]) and from the Dahmen-Micchelli spaces (see [13]). Unlike the first story, the central case is more natural here, it was considered in [16] by N. Dyn and A. Ron. Box-splines of a zonotope correspond to the bases of its matroid, therefore, zonotopal algebras are related to matroids (see [29]). The other relations of zonotopal algebras is Ehrhart theory, in fact the dimensions of unimodular zonotopal algebras count lattice points of the corresponding zonotope, see Theorem 1.5. (See also papers [14; 21; 30; 31; 40].)

We work only with a field $\mathbb{K}$ of zero characteristic; it is fixed across the whole thesis. The reader can assume that $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Below we study the quotient algebras by zonotopal ideals. Almost all Hilbert series will be specializations of the Tutte polynomials, see, for example, Theorem 1.1. The total dimensions are values of the Tutte polynomials of certain points. In particular, for the graphical case, the total dimensions of the external and central zonotopal algebras are the number of forests and trees, see Theorem 1.2.

The structure of this chapter is the following. In section § 1.1 we present the definition of zonotopal ideals from [4; 20] and in section § 1.2 we present the definition of zonotopal ideals for graphs and totally unimodular matrices. § 1.3 covers the original definition of external algebras from [35; 36], where we generalize a result of [34] to all three types of unimodular zonotopal algebras. Sections § 1.4 and §,1.5 describe two natural ways of generalizations of our square-free definitions. In section § 1.6 we classify all external algebras; furthermore, in section § 1.7 we considered "K-theoretic" filtration which contains the information about the whole graph. In section § 1.8 we consider definitions of trees and forests of hypergraphs, which can be used for the hypergraphical matroid. Sections § 1.4, § 1.7, and § 1.8 were included in the lincentiate thesis [28].

1.1 Definition of zonotopal algebras

Let $A \in \mathbb{K}^{n\times m}$ be a matrix of rank n. Denote by $y_1,\ldots,y_m \in \mathbb{K}^n$ its columns. For a matrix A, we define its zonotope

$$Z_A := \bigoplus_{i\in[m]} [0,y_i] \subset \mathbb{K}^n$$

as the Minkovskii sum of the intervals $[0,y_i],\ i \in [m]$. By $\mathcal{F}(A)$ we denote the set of all facets of Z_A. For any facet $H \in \mathcal{F}(A)$, we define $m(H)$ as the number of non-zero coordinates of vector $\eta_H A \in \mathbb{K}^m$, where $\eta_H \in \mathbb{K}^n$ is a normal for H.

Let $\mathcal{C}_A^{(k)}$ be the quotient algebra

$$\mathcal{C}_A^{(k)} := \mathbb{K}[x_1,\ldots,x_n]/\mathcal{I}_A^{(k)},$$

where $\mathcal{I}_A^{(k)}$ is the *zonotopal ideal* generated by polynomials

$$p_H^{(k)} = (\eta_h \cdot (x_1,\ldots,x_n))^{m(H)+k},\ \ H \in \mathcal{F}(A).$$

There are 3 main special cases, where $k = \pm 1$ and 0; they were considered in [4; 20].

- $k = 1:\ \ \mathcal{C}_A^{\mathcal{E}x} := \mathcal{C}_A^{(1)}$ is called the *external zonotopal algebra* for A;
- $k = 0:\ \ \mathcal{C}_A^{C} := \mathcal{C}_A^{(0)}$ is called the *central zonotopal algebra* for A;
- $k = -1:\ \ \mathcal{C}_A^{\mathcal{I}n} := \mathcal{C}_A^{(-1)}$ is called the *internal zonotopal algebra* for A.

Remark 1.1 *The case $k > 1$ is not "zonotopal", because the ideal $\hat{\mathcal{I}}_A^{(k)}$ generated by*

$$p_h = (h \cdot (x_1,\ldots,x_n))^{m(h)+k},\ \ h \in \mathbb{K}^n$$

is different from $\mathcal{I}^{(k)}$. They coincide only when $k \le 1$.

In the case $k < -4$, Hilbert series is not a specialization of the corresponding Tutte polynomial, see [4].

Theorem 1.1 (cf. [4; 8; 20; 30], External [35], Central for graphs [34]) *For a matrix $A \in \mathbb{K}^{n\times m}$ of rank n, the Hilbert series of its zonotopal algebras are given by*

- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{E}x}}(t) = t^{m-n} T_A(1+t, \frac{1}{t})$;
- $\mathcal{H}_{\mathcal{C}_A^{C}}(t) = t^{m-n} T_A(1, \frac{1}{t})$;

- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{I}n}}(t) = t^{m-n}T_A(0, \frac{1}{t})$,

where T_A is the Tutte polynomial of the vector configuration of columns of A (i.e., vectors $y_1, \dots, y_m$).

Let us recall the definitions of a matroid and of its Tutte polynomial. A *matroid* M is a pair (E, I), where E is a ground set and I is a family of independent subsets of E. Here a ground set is finite and independent subsets have the following properties:

- The empty set is independent;
- Every subset of an independent set is independent;
- If A and B are two independent sets and $|A| > |B|$, then there exists an element e in A such that $B \cup \{e\}$ is also independent.

For a graph G, the ground set of its graphical matroid is the set of all edges of G; and spanning forests of G are independent subsets. We will mostly work with vector matroids, where a ground set consists of vectors in $\mathbb{K}^n$ and independent subsets are just linearly independent subsets of these vectors. (See more information about matroids and their Tutte polynomials in e.g., [32], [11].) Inside the thesis, for a matrix $\mathbb{K}^{n\times m}$, we define its matroid M_A as the vector matroid, whose elements are the columns of A.

To describe their Hilbert series, we need to recall the definition of the Tutte polynomial of a matroid M. There are a lot of different definitions of the Tutte polynomial; below we present the one using the Whitney rank generating function. (See more information about this polynomial in any classic graph theory book, for example, in Tutte's book [41] or in [11].) For a matroid M, define the *Tutte polynomial* of M as

$$T_M(x,y) := \sum_{B\subset E}(x-1)^{rk(E)-rk(B)}(y-1)^{|B|-rk(B)},$$

where $rk(B)$ is the size of a maximal independent subset of B.

Example 1.1 *Consider the following matrix*

$$A := \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}.$$

In fact this matrix corresponds to the graphical case; the corresponding graph and its zonotope Z_A are shown in fig. 1.1.

Matroid M_A is the vector matroid with elements $y_1 = (-1,0)$, $y_2 = (0,-1)$, $y_3 = y_4 = (1,-1)$. Its Tutte polynomial is given by

$$T_A(x,y) = x + y + x^2 + xy + y^2.$$

The zonotope Z_A has 6 facets. We need the set of its normals (note that parallel facets have the same normal up to a factor). There are 3 different normals:

- $\eta_1 = (1,0)$;
- $\eta_2 = (0,1)$;
- $\eta_3 = (1,1)$.

It is easy to check that $m(\eta_1) = m(\eta_2) = 3$ and $m(\eta_3) = 2$. Hence,

$$\mathcal{I}_A^{(k)} = \langle x_1^{3+k}, x_2^{3+k}, (x_1+x_2)^{2+k} \rangle.$$

Then

- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{E}x}}(t) = 1 + 2t + 3t^2 + 3t^3 + t^4$ *and* $dim(\mathcal{C}_A^{\mathcal{E}x}) = 10$*;*
- $\mathcal{H}_{\mathcal{C}_A^{C}}(t) = 1 + 2t + 2t^2$ *and* $dim(\mathcal{C}_A^{C}) = 5$*;*
- $\mathcal{H}_{\mathcal{C}_A^{\mathcal{I}n}}(t) = 1 + t$ *and* $dim(\mathcal{C}_A^{\mathcal{I}n}) = 2$.

1.2 Graphical and unimodular zonotopal algebras

Both are special cases of a general definition. However, in both these cases we present their square-free definitions see section § 1.3.

1.2.1 Graphical zonotopal algebras

In this subsection we will work with undirected graphs. A. Postnikov and B. Shapiro [34] constructed algebras counting spanning forests (external) and counting spanning trees (central) of a graph.

For a graph G on a vertex set $V(G) = \{0\} \cup [n]$ and a subset of vertices $I \subseteq [n]$, we denote by d_I the number of edges of G connecting I and $V(G) \setminus I$.

Let $\mathcal{J}_G^{(k)}$ be the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p_I^{(k)} = \left(\sum_{i \in I} x_i \right)^{d_I + k}, \quad I \subseteq [n].$$

Define $\mathcal{C}_G^{(k)}$ as the quotient algebra

$$\mathcal{C}_G^{(k)} := \mathbb{K}[x_1, \ldots, x_n] / \mathcal{J}_G^{(k)}.$$

Remark 1.2 *This algebra is independent of the choice of the root (i.e., which vertex has label* 0*).*

Again we have three main cases (in paper [34] only the first two were defined).

- $k = 1$: $\mathcal{C}_G^{\mathcal{E}x} = \mathcal{C}_G^{(1)}$ is the *external Zonotopal algebra* for G or the *Postnikov-Shapiro algebra counting forests* of G;
- $k = 0$: $\mathcal{C}_G^{C} = \mathcal{C}_G^{(0)}$ is the *central Zonotopal algebra* for G or the *Postnikov-Shapiro algebra counting spanning trees* of G;
- $k = -1$: $\mathcal{C}_G^{\mathcal{I}n} = \mathcal{C}_G^{(-1)}$ is the *internal Zonotopal algebra* for G.

To describe their Hilbert series in terms of graphs, we need to define external activities of forests. Fix a linear order of edges of G. For a forest $F \subset E(G)$, by the *external activity* $act_G(F)$ denote the number of all externally active edges of F, i.e., the number of edges $e \in E(G) \setminus F$ such that: *(i)* the subgraph $F + e$ has a cycle; *(ii)* e is the minimal edge in this cycle in the above linear order. (Note that the external activity and external algebra are different notions, they are not related to each other.)

Theorem 1.2 (cf. [34]) *Given a graph G, the total dimension of algebra* $\mathcal{C}_G^{\mathcal{E}x}$ *(*$\mathcal{C}_G^{\mathcal{I}n}$*) is equal to the number of forests (trees) of G. The dimension of the k-th graded component of the algebra equals the number of forests (trees)* $F \subset E(G)$ *in G with external activity* $|E(G)| - |F| - k$.

Note that external algebras are in one-to-one correspondence with graphical matroids, see Theorem 1.26, § 1.6.

These algebras are particular cases of zonotopal algebras. For a graph G, we can form the following matrix. Consider any reference orientation $\mathcal{G}$ of edges of G. We assume that

$$c_{i,e} = \begin{cases} 1, \text{ if } \vec{e} = (i,j); \\ -1, \text{ if } \vec{e} = (j,i); \\ 0, \text{ otherwise.} \end{cases} \tag{1.1}$$

We define the matrix $A_0 = \{c_{i,e},\ i \in [n],\ e \in E(G)\}$, and delete several rows to get A, whose rank remains the same (i.e., $rk(A_0) = rk(A)$) and it is equal to number of rows of A.

Proposition 1.3 *Given a graph G and its reference orientation* $\mathcal{G}$*, their algebras* $\mathcal{C}_G^{\mathcal{E}x}$ *and* $\mathcal{C}_A^{\mathcal{E}x}$ *are isomorphic. If G is a connected graph, then algebras* $\mathcal{C}_G^{C}$ *and* $\mathcal{C}_A^{C}$ *are isomorphic and algebras* $\mathcal{C}_G^{\mathcal{I}n}$ *and* $\mathcal{C}_A^{\mathcal{I}n}$ *are isomorphic.*

Remark 1.3 *If G is disconnected, then by definition from [34], the algebra* $\mathcal{C}_G^{C}$ *should be trivial, i.e., equal to* 0*. But for the zonotopal definition, it is non-trivial. So for all graphical algebras in the central and internal cases, we always assume that G is connected.*

In fact these algebras dependent only on the blocks of G. The induced subgraph on a vertex set $V' \subset V(G)$ is called a *block*, if it is connected and remains connected after deleting of any vertex $v \in V'$.

Proposition 1.4 *Given a graph G, the algebra* $\mathcal{C}_G^{(k)}$, $k \in \{\pm 1, 0\}$ *is the Cartesian product of algebras corresponding to all blocks of G.*

Example 1.2 *Let G be the graph on the vertex set* $\{0,1,2\}$ *with* 4 *edges*

$$(0,1),(0,2),(1,2), \textit{ and } (1,2),$$

see fig. 1.1 (left).

Let us orient all edges of G according to the increase of the corresponding number. Consider the incidence matrix

$$A_0 := \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}.$$

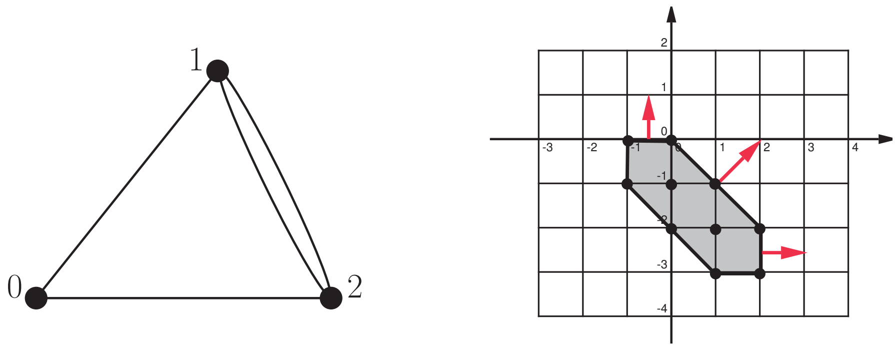

Figure 1.1: A concrete graph and the corresponding zonotope.

After forgetting its 0*-th row, we get exactly the matrix from Example 1.1. It is easy to check that dimensions* 10 *and* 5 *are exactly the number of forests and trees of* G*.*

1.2.2 Unimodular zonotopal algebras

A matrix $A \in \mathbb{K}^{n\times m}$ is called *totally unimodular* if all its minors are equal to ± 1 or 0. If A is totally unimodular, we call algebras $\mathcal{C}_A^{(k)}$ *unimodular zonotopal algebras*. All graphical algebras are particular cases of unimodular zonotopal algebras. Similar to the the graphical case, O.Holtz and A.Ron [20] founded combinatorial interpretation of their total dimensions.

Theorem 1.5 (cf. [20]) *Given a totally unimodular matrix* $A \in \mathbb{K}^{n\times m}$ *of rank* n*, we get that*

- *the total dimension of* $\mathcal{C}_A^{\mathcal{E}x}$ *is equal to the number of lattice points of* Z_A*;*
- *the total dimension of* $\mathcal{C}_A^{C}$ *is equal to the volume of* Z_A*;*
- *the total dimension of* $\mathcal{C}_A^{\mathcal{I}n}$ *is equal to the number of interior lattice points of* Z_A*.*

Example 1.3 *We observe that Example 1.1 present a graphical algebra. Thus the matrix*

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

is totally unimodular. The total dimensions $10,\ 5,$ *and* 2 *are exactly the number of lattice points, the area, and the number of interior lattice points of* Z_A *resp., see fig. 1.1 (right).*

Furthermore similarly to the graphical case, we can define the algebras without Z_A. In the graphical case we have at most 2^n relations and in the unimodular case we have at most 3^n relations.

Theorem 1.6 *Given a totally unimodular matrix $A \in \mathbb{K}^{n\times m}$ of rank n, the zonotopal ideals $\mathcal{I}_A^{(k)}$, $k \in \{\pm 1, 0\}$ are generated by polynomials*

$$p_s^{(k)} = \left(\sum_{j:\ s_i=1} x_i - \sum_{j:\ s_j=1} x_j \right)^{|supp(s\cdot A)|+1}, \quad s \in \{\pm 1, 0\}^n.$$

Furthermore, it is enough to consider only facets, for which $(s \cdot A) \in \{\pm 1, 0\}^m$.

1.3 Square-free definitions

(*based on* **[A3]** *and* **[A4]**)

The square-free definitions of the previous algebras come from homology theory and these definitions were the original ones. The algebra $\mathcal{C}^{\mathcal{E}x}_{K_n}$ is generated by the curvature forms of tautological Hermitian linear bundles over the type A complete flag variety $\mathcal{F}l_n$. In paper [35] A.Postnikov, B.Shapiro, and M.Shapiro generalized this construction to all vector configurations, which gives a definition of external zonotopal algebras. In this definition external zonotopal algebra was presented as a subalgebra of a square-free algebra.

In paper [34] they generalized to the case of central graphical algebras, in papers **[A3]** and **[A4]** the author with A. N. Kirillov generalized it for all three types of unimodular algebras.

In § 1.3.1 we present a definition from [35] for all external zonotopal algebras. In § 1.3.2 we present definitions for all three cases of unimodular zonotopal algberas.

1.3.1 Square-free definitions of external zonotopal algebras.

Let Φ_m be the square-free commutative algebra generated by $\phi_i,\ i \in [m]$, i.e., satisfying the relations

$$\phi_i\phi_j = \phi_j\phi_i,\ i,j \in [m] \quad \text{and} \quad \phi_i^2 = 0,\ i \in [m].$$

Theorem 1.7 (cf. [35]) *Given a matrix $A \in \mathbb{K}^{n\times m}$ of rank n, the external algebra $\mathcal{C}^{\mathcal{E}x}_A$ is isomorphic to the subalgebra of $\Phi^{\mathcal{E}x}_A := \Phi_m$ generated by*

$$X_i := t_i \cdot (\phi_1, \ldots, \phi_m),\ i \in [n],$$

where $t_i \in \mathbb{K}^m$ is the i-th row of A.

The proof of this theorem is clear in one direction. Namely, the constructed subalgebra satisfies all the relations from $\mathcal{C}^{\mathcal{E}x}_A$, because by the definition $m(H)$ is exactly the number of $\phi_j,\ j \in m$ with non-zero coefficients in the sum $\eta_H \cdot (X_1, \ldots, X_n)$. Hence, to prove the converse, it is enough to show that they have the same total dimension, see [35].

Example 1.4 *Consider again the matrix from Example 1.1*

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}.$$

We have

$$X_1 = -\phi_1 + \phi_3 + \phi_4;$$

$$X_2 = -\phi_2 - \phi_3 - \phi_4$$

implying the relations

$$X_1^4 = X_2^4 = (X_1 + X_2)^3 = 0.$$

Let us calculate the total dimension and the Hilbert series.

- *The* 0*-th graded component is* $\mathbb{K}$*; its dimension is* 1.
- *The* 1*-st graded component is span*$\langle X_1, X_2 \rangle$*; its dimension is* 2.
- *The* 2*-nd graded component is span*$\langle X_1^2, X_2^2, X_1X_2 \rangle$*; its dimension is* 3.
- *The* 3*-rd graded component is the span of*

$$X_1^3 = -6\phi_1\phi_3\phi_4;$$

$$X_2^3 = -6\phi_2\phi_3\phi_4;$$

$$X_1^2X_2 = 4\phi_1\phi_3\phi_4 - 2\phi_2\phi_3\phi_4 + 2\phi_1\phi_2(\phi_3 + \phi_4);$$

$$X_1X_2^2 = -2\phi_1\phi_3\phi_4 + 4\phi_2\phi_3\phi_4 - 2\phi_1\phi_2(\phi_3 + \phi_4);$$

  its dimension is 3.

- *The* 4*-th graded component is* $\mathbb{K}(\phi_1\phi_2\phi_3\phi_4)$; *its dimension is* 1.

Thus the Hilbert series and the total dimension are the same as in Example 1.1.

1.3.2 Square-free definitions of unimodular zonotopal algebras.

Let $A \in \mathbb{K}^{n\times m}$ be a totally unimodular matrix of rank n. We say that $s \in \{\pm 1, 0\}^n$ is a *cut-vector* if $s \cdot A \in \{\pm 1, 0\}^m$. We call a subset $C \subseteq [m]$ a *cut* if and only if there is a cut-vector s such that $\operatorname{supp}(s \cdot A) = C$. They are related to the usual cuts of a graph, where a set $C \subseteq E(G)$ is a *simple cut* of G if $G \setminus C$ is disconnected and C is a minimal.

Proposition 1.8 *Let G be a graph and A be a matrix for G. Then $C \subseteq [m]$ is a cut of A if and only if there is a representation $C = C_1 \sqcup C_2 \sqcup \ldots \sqcup C_k$, where C_k are simple cuts of G.*

Define the algebra Φ_A^C as quotient algebra

$$\Phi_A^C := \Phi_A^{\mathcal{E}x} / \langle \prod_{e \in C} \phi_e,\ C \text{ is a cut} \rangle.$$

Theorem 1.9 *Given a totally unimodular matrix $A \in \mathbb{K}^{n \times m}$ of rank n, the central algebra $\mathcal{C}_A^C$ is isomorphic to the subalgebra of Φ_A^C generated by*

$$X_i := t_i \cdot (\phi_1, \ldots, \phi_m),\ i \in [n],$$

where $t_i \in \mathbb{K}^m$ is the i-th row of A.

In the internal case we can define a derivative of a cut. Let s be a cut-vector; define its *derivative* by the formula

$$\delta(s) := (s \cdot A \cdot (\phi_1, \ldots, \phi_m)^t)^{|\mathrm{supp}(s \cdot A)| - 1}.$$

Define the algebra $\Phi_A^{\mathcal{I}n}$ as the quotient algebra

$$\Phi_A^{\mathcal{I}n} := \Phi_A^{\mathcal{E}x} / \langle \delta(s),\ s \text{ is a cut-vector} \rangle.$$

Note that

$$\delta(s) \cdot (a \cdot A \cdot (\phi_1, \ldots, \phi_m)) = (s \cdot A \cdot (\phi_1, \ldots, \phi_m)^t)^{|\mathrm{supp}(s \cdot A)|} =$$
$$= |supp(s \cdot A)|! \prod_{i \in supp(s \cdot A)} \phi_i,$$

which means that Φ_A^C is a bigger algebra than $\Phi_A^{\mathcal{I}n}$.

Theorem 1.10 *Given a totally unimodular matrix $A \in \mathbb{K}^{n \times m}$ of rank n, the internal algebra $\mathcal{C}_G^{\mathcal{I}n}$ is isomorphic to the subalgebra $\Phi_A^{\mathcal{I}n}$ generated by*

$$X_i := t_i \cdot (\phi_1, \ldots, \phi_m),\ i \in [n],$$

where $t_i \in \mathbb{K}^m$ is the i-th row of A.

Example 1.5 *Continuing our example*

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix},$$

we see that it has 3 *cuts*

$$\phi_1\phi_3\phi_4,\ \phi_2\phi_3\phi_4,\ \phi_1\phi_2.$$

It is easy to check that these are exactly the cuts of the corresponding graph, see fig. 1.1. Let us obtain the total dimension and the Hilbert series of the subalgebra of Φ_A^C.

- *The* 0*-th graded component is* $\mathbb{K}$*; its dimension is* 1.
- *The* 1*-st graded component is span*$\langle X_1, X_2\rangle$*; its dimension is* 2.
- *The* 2*-nd graded component is spanned by*

$$X_1^2 = -\phi_1(\phi_3+\phi_4)+\phi_3\phi_4;$$
$$X_2^2 = \phi_2(\phi_3+\phi_4)+\phi_3\phi_4;$$
$$(X_1+X_2)^2 = 2\phi_1\phi_2 = 0;$$

  its dimension is 2.

Then the Hilbert series and the total dimension are the same as in Example 1.1. Since we know that all relations from $\mathcal{C}_A^C$ *are satisfied, then this definition gives the same algebra.*

Now we will check a similar claim for $\Phi_A^{\mathcal{I}n}$*. It has* 6 *cut vectors, but we need only half of them (the last three are minus the first three)*

$$s_1 = (1,0),\ \ s_2 = (0,1),\ \text{and}\ s_3 = (1,1).$$

Their derivatives are given by

$$\delta(s_1) = \phi_3\phi_4 - \phi_1(\phi_3+\phi_4);$$
$$\delta(s_2) = \phi_3\phi_4 + \phi_2(\phi_3+\phi_4);$$
$$\delta(s_3) = -\phi_1 - \phi_2.$$

Let us calculate the total dimension and the Hilbert series of the subalgebra of $\Phi_A^{\mathcal{I}n}$.

- *The* 0*-th graded component is* $\mathbb{K}$*; its dimension is* 1.
- *The* 1*-st graded component is span*$\langle X_1, X_2\rangle$*, but*

$$X_1 + X_2 = \delta(s_3) = 0,$$

  then its dimension is 1.
- *The* 2*-nd graded component is empty because* $X_1+X_2 = \delta(s_3)$ *and* $X_1^2 = \delta(s_1)$.

Notice that the Hilbert series and the total dimension are the same as in Example 1.1. Since we know that all relations from $\mathcal{C}_A^{\mathcal{I}n}$ *are satisfied, then this definition gives the same algebra.*

1.4 t-labelled algebras

(*based on §3 of* **[A1]**)

In this section we substitute the square-free algebra $\Phi_G^{\mathcal{E}x}$ by the $(t+1)$-free algebra $\Phi_G^{\mathcal{E}x_t}$. In the case of trees we additionally change relations corresponding to the cuts. Subalgebras of $\Phi_G^{\mathcal{E}x_t}$ and $\Phi_G^{C_t}$ have properties similar to $\mathcal{C}_G^{(k)}$. They enumerate the so-called t-labelled forests and trees resp.

Consider a finite labelling set $\{1,2,\dots,t\}$ containing t different labels; each label being a number from 1 to t. A forest/tree of G with a label from $\{1,2,\dots,t\}$ on each edge is called a *t-labelled forest/tree*. The *weight* of a t-labelled forest F, denoted by $\omega(F)$, is the sum of the labels of all its edges.

The structure of this section is as follows: subsection § 1.4.1 discusses the algebra $\mathcal{C}_G^{\mathcal{E}x_t}$ counting t-labelled forests and subsection § 1.4.2 discusses the algebra $\mathcal{C}_G^{C_t}$ counting t-labelled spanning trees.

We can also define algebras $\mathcal{C}_G^{\mathcal{E}x_t}$ and $\mathcal{C}_G^{C_t}$ as quotient algebras, which is a particular case of algebras introduced in paper [4]. In fact, the authors of [4] consider algebras where each edge e is replaced by its a_e clones and their Hilbert series are specializations of the multivariate Tutte polynomial of a graph (see definition in [38]). Also the Hilbert series of $\mathcal{C}_G^{\mathcal{E}x_t}$ and $\mathcal{C}_G^{T_t}$ were calculated in [4; 30].

1.4.1 Algebras $\mathcal{C}_G^{\mathcal{E}x_t}$, $t \geq 1$

Let G be a graph without loops on the vertex set $\{0,...,n\}$ with fixed reference orientation $\mathcal{G}$. Let $t > 0$ be a positive integer.

Let $\Phi_G^{\mathcal{E}x_t}$ be the commutative algebra over $\mathbb{K}$ generated by $\{\phi_e : e \in E(G)\}$ satisfying the relations

$$(\phi_e)^{t+1} = 0, \quad \text{for any edge } e \in E(G).$$

Define $\mathcal{C}_G^{\mathcal{E}x_t}$ as the subalgebra of $\Phi_G^{\mathcal{E}x_t}$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1,...,n$, where $c_{i,e}$ are as in 1.1, § 1.2.1.

Similarly to the case of $\mathcal{C}_G^{\mathcal{E}x}$, we have the following property.

Proposition 1.11 *Given a graph G, the algebra $\mathcal{C}_G^{\mathcal{E}x_t}$ is the Cartesian product of algebras corresponding to all blocks of G.*

We can again present $\mathcal{C}_G^{\mathcal{E}x_t}$ as a quotient algebra of a polynomial ring. Let $\mathcal{J}_G^{\mathcal{E}x_t}$ be the ideal in the ring $\mathbb{K}[x_1,\cdots,x_n]$ generated by

$$p_I^{\mathcal{E}x_t} = \left(\sum_{i\in I} x_i\right)^{t\cdot d_I+1},$$

where $I \subset [n]$ and d_I is the total number of edges between vertices in I and in the complementary set $\bar{I} = [n] \cup \{0\} \setminus I$.

Theorem 1.12 *For any graph G and a positive integer t, the algebra $\mathcal{C}_G^{\mathcal{E}x_t}$ is isomorphic to*

$$\mathbb{K}[x_1,\ldots,x_n]/\mathcal{J}_G^{\mathcal{E}x_t}.$$

Its total dimension over $\mathbb{K}$ is equal to the number of t-labelled forests in G.

In fact, these algebras are not completely new objects. Indeed, algebra $\mathcal{C}_G^{\mathcal{E}x_t}$ is isomorphic to algebra $\mathcal{C}_{\widehat{G}}^{\mathcal{E}x}$, where $\widehat{G}$ is the *t-strength* copy of G, i.e., the graph $\widehat{G}$ is constructed from G by replacing every edge by its t clones.

Theorem 1.13 *For any graph G and a positive integer t, algebras $\mathcal{C}_G^{\mathcal{E}x_t}$ and $\mathcal{C}_{\widehat{G}}^{\mathcal{E}x}$ are isomorphic, where $\widehat{G}$ is the t-strength copy of G.*

Graphical matroids of G and of $\widehat{G}$ uniquely restore each other. Since by Theorem 1.25, we get that, for any positive $t > 0$, algebras $\mathcal{C}_G^{\mathcal{E}x_t}$ and $\mathcal{C}_G^{\mathcal{E}x}$ contain the same information about G.

Corollary 1.14 *Given two graphs G_1, G_2 and positive integer $t > 0$, algebras $\mathcal{C}_{G_1}^{\mathcal{E}x_t}$ and $\mathcal{C}_{G_2}^{\mathcal{E}x_t}$ are isomorphic if and only if their graphical matroids are isomorphic.*

Given two graphs G_1, G_2 and positive integers $t_1, t_2 > 0$, algebras $\mathcal{C}_{G_1}^{\mathcal{E}x_{t_1}}$ and $\mathcal{C}_{G_2}^{\mathcal{E}x_{t_1}}$ are isomorphic if and only if $\mathcal{C}_{G_1}^{\mathcal{E}x_{t_2}}$ and $\mathcal{C}_{G_2}^{\mathcal{E}x_{t_2}}$ are isomorphic.

(The algebraic isomorphism can be thought of either as graded or as non-graded; the statement holds in both cases.)

Also we can get that the Hilbert series of $\mathcal{C}_G^{\mathcal{E}x_t}$ is a specialization of the Tutte polynomial of G.

Corollary 1.15 *Given a graph G, the Hilbert series of algebra $\mathcal{C}_G^{\mathcal{E}x_t}$ is given by*

$$\mathcal{H}_{\mathcal{C}_G^{\mathcal{E}x_t}}(z) = T_G\left(\frac{z^{t+1}-1}{z^{t+1}-z}, \frac{1}{z^t}\right) \cdot z^{t(|E(G)|-n-1+c(G))} \cdot \left(\frac{1-z^t}{1-z}\right)^{n+1-c(G)},$$

where $c(G)$ is the number of connected components of G.

Furthermore, in this case we have one more interesting property. Namely, it is possible to reconstruct the Tutte polynomial of G from the Hilbert series of $\mathcal{C}_G^{\mathcal{E}x_t}$ for sufficiently large t; for an algorithm see the proof of the following theorem in paper **[A1]**.

Theorem 1.16 *For any positive integer $t \geq n$, it is possible to restore the Tutte polynomial of any connected graph G on n vertices knowing only the dimensions of each graded component of the algebra $\mathcal{C}_G^{\mathcal{E}x_t}$.*

1.4.2 Algebras $\mathcal{C}_G^{C_t}$, $t \geq 1$

Let G be a connected graph without loops on the vertex set $\{0, \ldots, n\}$ and $t > 0$ be a positive integer. Let $\Phi_G^{C_t}$ be the commutative algebra over $\mathbb{K}$ generated by $\{\phi_e : e \in E(G)\}$ satisfying the relations:

$$(\phi_e)^{t+1} = 0, \quad \text{for any edge } e \in G;$$

$$\left(\prod_{e \in C} \phi_e \right)^t = 0, \quad \text{for any cut } C \subset E(G).$$

Define $\mathcal{C}_G^{C_t}$ as the subalgebra of $\Phi_G^{C_t}$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1, \ldots, n$, where $c_{i,e}$ are as in 1.1, § 1.2.1.

Similarly to the previous case, we have the following proposition.

Proposition 1.17 *Given a connected graph G and positive $t > 0$, the algebra $\mathcal{C}_G^{C_t}$ is the Cartesian product of algebras corresponding to all blocks of G.*

We can also present $\mathcal{C}_G^{C_t}$ as a quotient algebra of a polynomial ring. Let $\mathcal{J}_G^{C_t}$ be the ideal in the ring $\mathbb{K}[x_1, \cdots, x_n]$ generated by

$$p_I^{C_t} = \left(\sum_{i \in I} x_i \right)^{t \cdot d_I},$$

where I ranges over all nonempty subsets of vertices and d_I is the total number of edges between vertices in I and in the complementary set $\bar{I}$.

Similarly to the above,

Theorem 1.18 *For any graph G and a positive integer t, algebras $\mathcal{C}_G^{C_t}$ and $\mathcal{C}_{\widehat{G}}^{C}$ are isomorphic, where $\widehat{G}$ is the t-strength copy of G. Furthermore, they are isomorphic to $\mathbb{K}[x_1,\ldots,x_n]/\mathcal{J}_G^{C_t}$. Their total dimension over $\mathbb{K}$ is equal to the number of t-labelled spanning trees in G.*

As a consequence we can get that the Hilbert series of $\mathcal{C}_G^{C_t}$ is a specialization of the Tutte polynomial of G.

Corollary 1.19 *Given a connected graph G, the Hilbert series of the algebra $\mathcal{C}_G^{C_t}$ is given by*

$$\mathcal{H}_{\mathcal{C}_G^{C_t}}(z) = T_G\left(\frac{z-1}{z^{t+1}-z}, \frac{1}{z^t}\right) \cdot z^{t(|E(G)|-n)} \cdot \left(\frac{1-z^t}{1-z}\right)^n.$$

However, in this case it is impossible to restore the Tutte polynomial from the Hilbert series of $\mathcal{H}_{\mathcal{C}_G^{C_t}}$. The reason is that all spanning trees have the same size and therefore $\mathcal{H}_{C_G^{C_t}}$ has the same information that $\mathcal{H}_{\mathcal{C}_G^{C}}$.

For $t > 1$, the graph $\widehat{G}$ has no bridges, thus its bridge-free matroid and its graphical matroid coincide. Then these algebras have the following property.

Proposition 1.20 *Given two connected graphs G_1, G_2 with isomorphic graphical matroid and $t > 1$, algebras $\mathcal{C}_{G_1}^{C_t}$ and $\mathcal{C}_{G_2}^{C_t}$ are isomorphic as graded algebra.*

In fact, it is possible to prove the converse of Proposition 1.20 by using the same idea as in the proof of the graphical case of Theorem 1.26 (see **[A1]**) with one additional change. For the element X_i corresponding to the vertex i, $\frac{d(X_i)}{t}$ is not the degree of the vertex i. However, for vertices i and j, $\left\lceil \frac{d(X_i+aX_j)-d(X_i+X_j)+1}{t} \right\rceil$ is the number of edges between i and j. We do not present the proof of the converse statement here or in paper **[A1]**, because we think that the most interesting problem is Conjecture 1.4, and the converse of Proposition 1.20 is just its corollary.

1.5 Hecke deformations of algebras

(*based on* **[A4]** *and* **[A3]**)

In this case we consider deformations of the square-free definitions. It is based on joint with A. N. Kirillov papers **[A4]** and **[A3]**. For this deformations it is easier to find their total dimensions, than in the case of the original algebras. As a corollary of these results we get a new definition of unimodular zonotopal algebras, see section § 1.3.

1.5.1 External zonotopal algebras

Let $Q = (q_i \in \mathbb{K} : \ i \in [m])$ be the set of parameters. Consider the commutative algebra $\Phi^{\mathcal{E}x}_{A,Q}$ generated by $u_i, \ i \in [m]$ with relations

$$u_i^2 = q_i u_i.$$

Let $\Psi^{\mathcal{E}x}_{A,Q}$ be the filtered subalgebra generated by

$$X_i = t_i \cdot (u_1, \ldots, u_m), \ i \in [n]$$

with the filtered structure from $X_i, \ i \in [n]$.

Example 1.6 (i) *Let G be a graph with two vertices and two multiple edges a, b. Consider the Hecke deformation of its $\mathcal{C}_G$, i.e., satisfying $q_a = q_b = q$.*

The generators are $X_0 = -(a+b)$, $X_1 = a+b = -X_0$. One can easily check that the filtered structure is given by

- $F_0 = <1>$;
- $F_1 = <1, \ a+b>$;
- $F_2 = <1, \ a+b, \ ab>$.

The Hilbert polynomial $\mathcal{H}(t)$ of $\Psi^{\mathcal{E}x}_{G,Q}$ is given by

$$\mathcal{H}(t) = 1 + t + t^2.$$

The defining relation for X_1 is given by

$$X_1(X_1 - q)(X_1 - 2q) = 0.$$

(ii) *For the same graph as before, consider the case when $Q = \{q_a, q_b\}$, $q_a^2 \neq q_b^2$.*

The generators are the same: $X_0 = -(a+b)$, $X_1 = a+b = -X_0$. *Since*

$$X_1^3 = q_a^2 a + q_b^2 b + 3(q_a+q_b)ab = \frac{3(q_a+q_b)}{2}X_1^2 - \frac{q_a^2+3q_b^2}{2}a - \frac{3q_a^2+q_b^2}{2}b$$
$$= \frac{3(q_a+q_b)}{2}X_1^2 - \frac{3q_a^2+q_b^2}{2}X_1 + (q_a^2-q_b^2)a,$$

we have

- $F_0 = <1>$;
- $F_1 = <1,\ a+b>$;
- $F_2 = <1,\ a+b,\ q_a a + q_b b + 2ab>$;
- $F_3 = <1,\ a,\ b,\ ab>$.

The Hilbert polynomial $\mathcal{H}(t)$ *of* $\Psi^{\mathcal{E}x}_{G,Q}$ *is given by*

$$\mathcal{H}(t) = 1 + t + t^2 + t^3.$$

Observe that in this case the algebra $\Psi^{\mathcal{E}x}_{G,Q}$ *coincides with the whole* $\Phi^{\mathcal{E}x}_{G,Q}$ *as a linear space, but has a different filtration. The defining relation for* X_1 *is given by*

$$X_1(X_1-q_a)(X_1-q_b)(X_1-q_a-q_b) = 0.$$

If all coefficients are nonvanishing, then it is very easy to calculate the dimension of such an algebra.

Theorem 1.21 *Given a matrix* $A \in \mathbb{K}^{n\times m}$ *of rank* n *and a set of non-zero parameters* $Q \in (\mathbb{K}\setminus\{0\})^m$, *the dimension of* $\Psi^{\mathcal{E}x}_{A,Q}$ *is equal to the number of different sums of* $q_i y_i$, *i.e.,*

$$dim\left(\Psi^{\mathcal{E}x}_{A,Q}\right) = \#\{A\cdot((q_1,\dots,q_m)\circ\chi(E))^t \in \mathbb{K}^n :\ E \subseteq [m]\},$$

where $\circ$ *is the Hadamard product and* $\chi(E) \in \mathbb{K}^m$ *is the characteristic vector of* $E \subseteq [m]$.

The above theorem gives a nice interpretation of the total dimension of $\Psi^{\mathcal{E}x}_{A,Q}$, but at the moment we can not find its Hilbert series.

Problem 1.1 *Describe the Hilbert series of* $\Psi^{\mathcal{E}x}_{A,Q}$.

Problem 1.2 *Describe all relations of* $\Psi^{\mathcal{E}x}_{A,Q}$.

Below we present solutions to these problems for the so-called Hecke deformations of the unimodular zonotopal algebras.

1.5.2 Unimodular zonotopal algebras

In this subsection we consider the special case, when all parameters coincide. Given $q = q_i,\ i \in [m]$, we simply denote algebras by $\Psi^{(k)}_{A,q}$ and $\Psi^{(k)}_{A,q}$ resp. The algebra $\Psi^{(k)}_{A,q},\ k \in \{\pm 1, 0\}$ is called *Hecke-deformations of zonotopal algebras*. In case of totally unimodular matrix, we can define Hecke deformations of the central and internal zonotopal algebras and, furthermore, present all relations between their generators.

Let $A \in \mathbb{K}^{n\times m}$ be a totally unimodular matrix. We want to find the relations for $\Phi^{\mathcal{C}}_{A,q}$ and $\Phi^{\mathcal{I}n}_{A,q}$. In both cases we will know relations for the cut-vectors, see definition in section § 1.3. Let s be a cut-vector, define its relations as

$$\sigma(s,q) := \left(\prod_{i\in supp^+(s\cdot A)} u_i\right)\left(\prod_{i\in supp^-(s\cdot A)} (u_i - q)\right) \in \Phi^{\mathcal{E}x}_{A,q},$$

where $supp^+(s\cdot A) \subseteq supp(s\cdot A)$ is the set of positive coordinates of $s\cdot A$ (i.e., those equal to $+1$) and $supp^-(s\cdot A) \subseteq supp(s\cdot A)$ is the set of negative coordinates of $s\cdot A$ (i.e., those equal to -1). Note that $\sigma(s,q) \neq \sigma(-s,q)$.

Define the algebras $\Phi^{\mathcal{C}}_{A,q}$ and $\Phi^{\mathcal{I}n}_{A,q}$ as quotient algebras

$$\Phi^{\mathcal{C}}_{A,q} := \Phi^{\mathcal{E}x}_{A,q}/\{\sigma(s),\ s \text{ is a cut-vector s.t. } (s\cdot A)(2^m, 2^{m-1}, \ldots, 1) > 0\};$$

$$\Phi^{\mathcal{I}n}_{A,q} := \Phi^{\mathcal{E}x}_{A,q}/\{\sigma(s),\ s \text{ is a cut-vector}\}.$$

Define Hecke-deformations $\Psi^{\mathcal{C}}_{A,q}$ and $\Psi^{\mathcal{I}n}_{A,q}$ as the filtered subalgebras of $\Phi^{\mathcal{C}}_{A,q}$ and $\Phi^{\mathcal{I}n}_{A,q}$ resp. generated by

$$X_i = t_i \cdot (u_1, \ldots, u_m),\ i \in [n]$$

with the filtered structures coming from $X_i,\ i \in [n]$.

To describe relations, we need the following two numbers associated to a vector $s \in \mathbb{K}^n$

$$d_s^+ := \max(s\cdot A\cdot \chi_E^{\mathfrak{t}} :\ E \subseteq [m]),$$

and

$$d_s^- := \min(s\cdot A\cdot \chi_E^{\mathfrak{t}} :\ E \subseteq [m]).$$

Theorem 1.22 *Given a totally unimodular matrix $A \in \mathbb{K}^{n\times m}$ of rank n and $0 \neq q \in \mathbb{K}$, the Hecke-deformations $\Psi^{\mathcal{E}x}_{A,q}$, $\Psi^{\mathcal{C}}_{A,q}$, and $\Psi^{\mathcal{I}n}_{A,q}$ are isomorphic to the quotient algebra $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}^{(k)}_{A,q}$, $k = \{1, 0, -1\}$ resp. Here*

- *the ideal $\mathcal{I}^{\mathcal{E}x}_{A,q}$ is generated by*

$$p^{\mathcal{E}x}_{s,q} = \prod_{i=d_s^-}^{d_s^+} (s\cdot(x_1,\dots,x_n) - qi), \ \ s \in \{0,\pm 1\}^n;$$

- *the ideal $\mathcal{I}^{C}_{A,q}$ is generated by*

$$p^{C}_{s,q} = \prod_{i=d_s^- - sign(s)+1}^{d_s^+ - sign(s)} (s\cdot(x_1,\dots,x_n) - qi), \ \ s \in \{0,\pm 1\}^n,$$

  where $sign(s)$ is equal to 1 if scalar product $(s\cdot A)(2^m, 2^{m-1},\dots,1)$ is positive and to 0 otherwise;

- *the ideal $\mathcal{I}^{\mathcal{I}n}_{A,q}$ is generated by*

$$p^{\mathcal{I}n}_{s,q} = \prod_{i=d_s^-+1}^{d_s^+-1} (s\cdot(x_1,\dots,x_n) - qi), \ \ s \in \{0,\pm 1\}^n.$$

Furthermore, it is enough to only consider the relations corresponding to the cut-vectors.

In graphical the case we can reformulate the above theorem. Namely, for a subset $I \subset V(G)$, we denote by $in(I)$ (resp. $out(I)$) the number of incoming (resp. outcoming) edges of I.

Theorem 1.23 *Given a connected graph G on vertex set $[n]\cup\{0\}$ and its reference orientation $\mathcal{G}$, the Hecke-deformations $\Psi^{\mathcal{E}x}_{G,q}$, $\Psi^{C}_{G,q}$, and $\Psi^{\mathcal{I}n}_{G,q}$ are isomorphic to $\mathbb{K}[x_1,\dots,x_n]/\mathcal{I}^{(k)}_{G,q}$, $k = \{1,0,-1\}$ resp. Here*

- *the ideal $\mathcal{I}^{\mathcal{E}x}_{G,q}$ is generated by*

$$p^{\mathcal{E}x}_{I,q} = \prod_{i=-out(I)}^{in(I)} (s\cdot(x_1,\dots,x_n) - qi), \ \ I \subseteq [n];$$

- *the ideal $\mathcal{I}^{C}_{G,q}$ is generated by*

$$p^{C}_{I,q} = \prod_{i=-out(I)+1}^{in(I)} (s\cdot(x_1,\dots,x_n) - qi), \ \ I \subseteq [n];$$

- *the ideal $\mathcal{I}^{\mathcal{I}n}_{G,q}$ is generated by*

$$p^{\mathcal{I}n}_{I,q} = \prod_{i=-out(I)+1}^{in(I)-1} (s\cdot(x_1,\ldots,x_n) - qi), \;\; I \subseteq [n].$$

Their Hilbert series are independent on q (including the case $q=0$).

Proposition 1.24 *Given a totally unimodular matrix $A \in \mathbb{K}^{n\times m}$ of rank n, for $k = \{\pm 1, 0\}$, the filtrations of its Hecke deformation $\Psi^{(k)}_{A,q}$ induced by X_i and induced from the algebra $\Phi^{(k)}_{A,q}$ coincide.*

Furthermore, the Hilbert series of the Hecke deformation $\Psi^{(k)}_{A,q}$, $k = \{\pm 1, 0\}$ are given by

- $\mathcal{H}_{\Psi^{\mathcal{E}x}_{A,q}}(t) = t^{m-n}T_A(1+t, \frac{1}{t})$;
- $\mathcal{H}_{\Psi^{C}_{A,q}}(t) = t^{m-n}T_A(1, \frac{1}{t})$;
- $\mathcal{H}_{\Psi^{\mathcal{I}n}_{A,q}}(t) = t^{m-n}T_A(0, \frac{1}{t})$.

Example 1.7 *Continuing our example*

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix},$$

we get that it has 3 cut-vectors, namely

$$(1,0),\; (0,1),\; \textit{and}\; (1,1).$$

(By the definition it has also minus these as cut-vectors, but they give the same relations.) For all these vectors $(s\cdot A)(8,4,2,1)$ are negative and the Hecke-deformations of the zonotopal ideals are given by

$$\mathcal{I}^{(1)}_A = \langle \prod_{i=-1}^{2}(x_1 - qi), \prod_{i=-3}^{0}(x_2 - qi), \prod_{i=-2}^{0}(x_1 + x_2 - qi)\rangle;$$

$$\mathcal{I}^{(0)}_A = \langle \prod_{i=0}^{2}(x_1 - qi), \prod_{i=-2}^{0}(x_2 - qi), \prod_{i=-1}^{0}(x_1 + x_2 - qi)\rangle;$$

$$\mathcal{I}^{(-1)}_A = \langle \prod_{i=0}^{1}(x_1 - qi), \prod_{i=-2}^{-1}(x_2 - qi), \prod_{i=-1}^{-1}(x_1 + x_2 - qi)\rangle.$$

1.6 Classification of external algebras

(*based on §2 of* **[A1]** *and* **[A5]**)

In this section we classify all external zonotopal algebras up to an isomorphism. In paper **[A1]** we did this for the graphical case, in paper **[A5]** we did it for all external zonotopal algebras. For the unimodular zonotopal algebras, it depends only on their matroids, see Theorem 1.26. For other matrices we should define a zonotopal equivalency. At the end of section we formulate some conjectures for the central case.

Definition 1.1 *Given two linear spaces $V_1 \subset \mathbb{K}^{m_1}$ and $V_2 \subset \mathbb{K}^{m_2}$, they are called z-equivalent if $m_1 = m_2 = m$ and there is an invertible diagonal matrix $D \in \mathbb{K}^{m\times m}$ and a permutation $\pi \in S_m$ such that*

$$V_1 = V_2(\pi D).$$

Matrices $A_1 \in \mathbb{K}^{n_1\times m_1}$ of rank n_1 and $A_2 \in \mathbb{K}^{n_2\times m_2}$ of rank n_2 are called z-equivalent if the span of the rows of A_1 is z-equivalent to the span of the rows of A_2.

Remark 1.4 *It is easy to see that z-equivalence is an equivalence relation.*

In the case when A_1 does not have proportional columns and A_2 does not have proportional columns, we can say that A_1 is equivalent to A_2 if and only if their zonotopes are equivalent (because in this case we can reconstruct a "matrix" from its zonotope).

It is easy to check that $\mathcal{C}^{\mathcal{E}x}_{A_1}$ and $\mathcal{C}^{\mathcal{E}x}_{A_2}$ are isomorphic if A_1 and A_2 are z-equivalent. The converse also holds.

Theorem 1.25 *Let $A_1 \in \mathbb{K}^{n_1\times m_1}$ and $A_2 \in \mathbb{K}^{n_2\times m_2}$ be two matrices of ranks n_1 and n_2 resp., then the following statements are equivalent:*

- *$\mathcal{C}^{\mathcal{E}x}_{A_1}$ and $\mathcal{C}^{\mathcal{E}x}_{A_2}$ are isomorphic as non-graded algebras;*
- *$\mathcal{C}^{\mathcal{E}x}_{A_1}$ and $\mathcal{C}^{\mathcal{E}x}_{A_2}$ are isomorphic as graded algebras;*
- *A_1 and A_2 are z-equivalent.*

The following theorem shows that unimodular external zonotopal algebras are in one-to-one correspondence with regular matroids.

Theorem 1.26 *Let $A_1 \in \mathbb{K}^{n_1\times m_1}$ and $A_2 \in \mathbb{K}^{n_2\times m_2}$ be two unimodular matrices of ranks n_1 and n_2 resp., then the following statements are equivalent*

- $\mathcal{C}^{\mathcal{E}x}_{A_1}$ *and* $\mathcal{C}^{\mathcal{E}x}_{A_2}$ *are isomorphic as non-graded algebras;*
- $\mathcal{C}^{\mathcal{E}x}_{A_1}$ *and* $\mathcal{C}^{\mathcal{E}x}_{A_2}$ *are isomorphic as graded algebras;*
- A_1 *and* A_2 *are z-equivalent.*
- *the matroids* M_{A_1} *and* M_{A_2} *are isomorphic.*

Since, for graphs, we have that their matrices are totally unimodular, all graphical matroids are regular, the converse is almost true. Every regular matroid may be constructed by combining graphic matroids, co-graphic matroids, and a certain ten-element matroid R_{10}, see [33] or book [32].

In the graphical case, Theorem 1.26 means that algebra $\mathcal{C}^{\mathcal{E}x}_{G}$ contains a lot of information about G. For example, if G is 3-connected (i.e. it remains connected after deletion of any 2 vertices), then the algebra remembers the whole graph. However if G is not 3-connected this is not always true. In fact, it is true up to 2-isomorphisms of a graph.

Theorem 1.27 (Whitney's 2-isomorphism theorem, see [42], [32]) *Let* G_1 *and* G_2 *be two graphs. Then their graphical matroids are isomorphic if and only if* G_1 *can be transformed to a graph, which is isomorphic to* G_2 *by a sequence of operations of vertex identification, cleaving and twisting.*

These three operations are defined below.
1a) *Identification*: Let v and v' be vertices from different connected components of the graph. We modify the graph by identifying v and v' as a new vertex v''.
1b) *Cleaving* (the inverse of identification): A graph can only be cleft at a cut-vertex.
2) *Twisting*: Suppose that the graph G is obtained from two disjoint graphs G_1 and G_2 by identifying vertices u_1 of G_1 and u_2 of G_2 as the vertex u of G and additionally identifying vertices v_1 of G_1 and v_2 of G_2 as the vertex v of G. In the twisting of G about the vertex pair $\{u,v\}$, we identify u_1 with v_2 and u_2 with v_1 to get a new graph G'.

Example 1.8 *Consider the graphs* G_1 *and* G_2*, see Fig. 1.2. It is clear that they have isomorphic matroids. It means that algebras* $\mathcal{C}^{\mathcal{E}x}_{G_1}$ *and* $\mathcal{C}^{\mathcal{E}x}_{G_2}$ *should be isomorphic by Theorem 1.26. Let us check it.*

Let $\mathbb{K}[x_1,x_2,x_3]$ *be the polynomial ring for* G_1*. Then the ideal* $\mathcal{J}^{\mathcal{E}x}_{G_1}$ *is given by*

$$\mathcal{J}^{\mathcal{E}x}_{G_1} = \langle x_1^4,\ x_2^4,\ x_3^4,\ (x_1+x_2)^5,\ (x_2+x_3)^3,\ (x_1+x_3)^7,\ (x_1+x_2+x_3)^4\rangle =$$

$$\langle x_1^4,\ x_2^4,\ x_3^4,\ (x_1+x_2)^5,\ (x_2+x_3)^3,\ (x_1+x_2+x_3)^4\rangle,$$

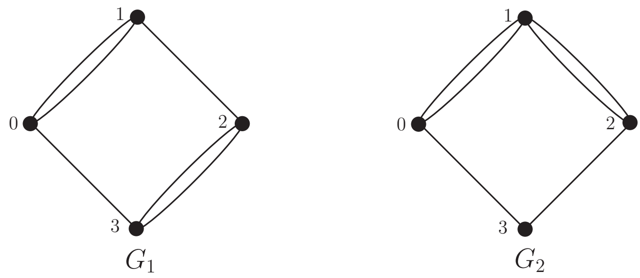

Figure 1.2: Non-isomorphic graphs with isomorphic graphical matroids

because $(x_1+x_3)^7 \in \langle x_1^4, x_2^4 \rangle$.

Let $\mathbb{K}[\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3]$ *be the polynomial ring for* G_2. *Then the ideal* $\mathcal{J}_{G_2}^{\mathcal{E}x}$ *is given by*

$$\mathcal{J}_{G_2}^{\mathcal{E}x} = \langle \widetilde{x}_1^5,\ \widetilde{x}_2^4,\ \widetilde{x}_3^3,\ (\widetilde{x}_1+\widetilde{x}_2)^4,\ (\widetilde{x}_2+\widetilde{x}_3)^4,\ (\widetilde{x}_1+\widetilde{x}_3)^7,\ (\widetilde{x}_1+\widetilde{x}_2+\widetilde{x}_3)^4 \rangle =$$

$$= \langle \widetilde{x}_1^5,\ \widetilde{x}_2^4,\ \widetilde{x}_3^3,\ (\widetilde{x}_1+\widetilde{x}_2)^4,\ (\widetilde{x}_2+\widetilde{x}_3)^4,\ (\widetilde{x}_1+\widetilde{x}_2+\widetilde{x}_3)^4 \rangle,$$

because $(\widetilde{x}_1+\widetilde{x}_3)^7 \in \langle \widetilde{x}_1^5, \widetilde{x}_2^3 \rangle$.

Consider the ring isomorphism $\psi:\ \mathbb{K}[x_1,x_2,x_3] \to \mathbb{K}[\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3]$, *defined by:*

- $\psi(x_1) = -\widetilde{x}_1 - \widetilde{x}_2$;
- $\psi(x_2) = \widetilde{x}_2$;
- $\psi(x_3) = -\widetilde{x}_2 - \widetilde{x}_3$.

The isomorphism ψ *acts on the ideal* $\mathcal{J}_{G_1}^{\mathcal{E}x}$ *as follows:*

$$\psi(\mathcal{J}_{G_1}^{\mathcal{E}x}) = \langle (-\widetilde{x}_1-\widetilde{x}_2)^4,\ \widetilde{x}_2^4,\ (-\widetilde{x}_2-\widetilde{x}_3)^4,\ (-\widetilde{x}_1)^5,\ (-\widetilde{x}_3)^3,\ (-\widetilde{x}_1-\widetilde{x}_3)^4 \rangle = \mathcal{J}_{G_2}^{\mathcal{E}x}.$$

Then we get

$$\psi(\mathcal{C}_{G_1}^{\mathcal{E}x}) = \psi\left(\mathbb{K}[x_1,x_2,x_3]/\mathcal{J}_{G_1}^{\mathcal{E}x}\right) = \mathbb{K}[\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3]/\mathcal{J}_{G_2}^{\mathcal{E}x} = \mathcal{C}_{G_2}^{\mathcal{E}x},$$

hence, algebras $\mathcal{C}_{G_1}^{\mathcal{E}x}$ *and* $\mathcal{C}_{G_2}^{\mathcal{E}x}$ *are isomorphic.*

We conjecture that a similar statement holds in the central case. For a matrix A, we say that its column is a *bridge-column* if after deleting it the rank decreases.

Conjecture 1.3 *Let* $A_1 \in \mathbb{K}^{n_1 \times m_1}$ *and* $A_2 \in \mathbb{K}^{n_2 \times m_2}$ *be two matrices of ranks* n_1 *and* n_2 *resp. Then the following* 3 *statements are equivalent:*

- $\mathcal{C}^C_{A_1}$ *and* $\mathcal{C}^C_{A_2}$ *are isomorphic as non-graded algebras;*
- $\mathcal{C}^C_{A_1}$ *and* $\mathcal{C}^C_{A_2}$ *are isomorphic as graded algebras;*
- A'_1 *and* A'_2 *are z-equivalent, where* $A'_i \in \mathbb{K}^{(n_i-k_i)\times(n_i-k_i)}$ *is a submatrix of* A_i *obtained after deleting of all* k_i *bridge-columns and some* k_i *rows such that* $rk(A'_i) = n_i - k_i$.

Conjecture 1.4 *Let* $A_1 \in \mathbb{K}^{n_1\times m_1}$ *and* $A_2 \in \mathbb{K}^{n_2\times m_2}$ *be two unimodular matrices of ranks* n_1 *and* n_2 *resp. Then the following statements are equivalent:*

- $\mathcal{C}^C_{A_1}$ *and* $\mathcal{C}^C_{A_2}$ *are isomorphic as non-graded algebras;*
- $\mathcal{C}^C_{A_1}$ *and* $\mathcal{C}^C_{A_2}$ *are isomorphic as graded algebras;*
- A'_1 *and* A'_2 *are z-equivalent, where* $A'_i \in \mathbb{K}^{(n_i-k_i)\times(n_i-k_i)}$ *is a submatrix of* A_i *obtained after deleting of all* k_i *bridge-columns and some* k_i *rows such that* $rk(A'_i) = n_i - k_i$;
- *the bridge-free matroids* $M_{A'_1}$ *and* $M_{A'_2}$ *are isomorphic.*

1.7 “K-theoretic” filtration

(*based on paper* **[A2]**)

This section deals with "K-theoretic" filtered algebras $\mathcal{K}_G^{(k)}$, which was suggested by A. N. Kirillov and B. Shapiro. The section is based on paper **[A2]**, which was written by the author jointly with B. Shapiro. Here we study a "K-theoretical" filtration of algebras $\mathcal{C}_G^{\mathcal{E}x}$ and $\mathcal{C}_G^{C}$.

Generators of the algebra $\mathcal{K}_G^{(k)}$ are non-homogeneous. We can consider its filtration $\mathbb{K} = F_0 \subset F_1 \subset \ldots \mathcal{K}_G^{(k)}$, where F_k is the span of F_{k-1} and of all products of k-tuples of generators (not necessarily distinct). The Hilbert series of the filtered algebra is just the Hilbert series of its associated graded algebra. In other words,

$$\mathcal{H}_{\mathcal{K}_G^{(k)}} = 1 + (dim(F_1) - 1)t + (dim(F_2) - dim(F_1))t^2 + \ldots.$$

The algebra $\mathcal{K}_G^{\mathcal{E}x}$ contains the information about the whole graph G, see Theorem 1.31; its total dimension is the number of forests of G. However, at the moment we do not know the combinatorial meaning of its Hilbert series. In this case the Hilbert series is not a specialization of the Tutte polynomial of G, see Example 1.10. There is a similar problem for $\mathcal{K}_G^{C}$. Besides that we introduce other filtered algebras, see their definitions and properties in paper **[A2]**.

The structure of this section is as follows: Subsection § 1.7.1 deals with the algebra $\mathcal{K}_G^{\mathcal{E}x}$ counting spanning forests and subsection § 1.7.2 deals with the algebra $\mathcal{K}_G^{C}$ counting spanning trees.

1.7.1 Algebra $\mathcal{K}_G^{\mathcal{E}x}$

In notation of subsection § 1.3.2, our next object of consideration is the filtered subalgebra $\mathcal{K}_G^{\mathcal{E}x} \subset \Phi_G^{\mathcal{E}x}$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G}(1 + c_{i,e}\phi_e),\ i = 0, \ldots, n.$$

Notice that the set of generators $Y_i - 1$, $i \in \{0, 1, \ldots, n\}$ gives the same filtered structure of Y_i and it is easier to work with them, because they are nilpotent. We have one more generator than for $\mathcal{C}_G^{\mathcal{E}x}$, since the sum $(Y_0 - 1) + \ldots + (Y_n - 1)$ is not always zero.

Furthermore, because $Y_i - 1$ is a nilpotent and $\ln(1 + (Y_i - 1)) = X_i$, then algebras $\mathcal{K}_G^{\mathcal{E}x}$ and $\mathcal{C}_G^{\mathcal{E}x}$ coincide as non-filtered subalgebras of $\Phi_G^{\mathcal{E}x}$, however we interested in the filtered structure of $\mathcal{K}_G^{\mathcal{E}x}$. Since Y_i is obtained by exponentiation of X_i, we call $\mathcal{K}_G^{\mathcal{E}x}$ the “K-*theoretic" filtration* of $\mathcal{C}_G^{\mathcal{E}x}$.

Theorem 1.28 *For any graph G, algebras $\mathcal{C}_G^{\mathcal{E}x}$ and $\mathcal{K}_G^{\mathcal{E}x}$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of forests of G.*

Example 1.9 *Consider the complete graph K_3 on vertices $\{0,1,2\}$ and edges $\{a,b,c\}$, see Fig 1.3. The generators of $\mathcal{K}_\Delta^{\mathcal{E}x}$ are*

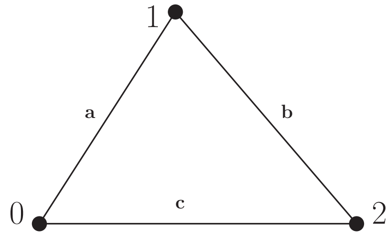

Figure 1.3: graph K_3

$$Y_0 - 1 = e^{X_0} - 1 = \phi_a + \phi_c + \phi_a\phi_c;$$

$$Y_1 - 1 = e^{X_1} - 1 = -\phi_a + \phi_b - \phi_a\phi_b;$$

$$Y_2 - 1 = e^{X_2} - 1 = -\phi_b - \phi_c + \phi_b\phi_c.$$

Then the filtered structure is

- $F_0 = span\{1\}$, $dim = 1$;
- $F_1 = span\{1,\ Y_0,\ Y_1,\ Y_2\}$, $dim(F_1) - dim(F_0) = 3$;
- $F_2 = span\{1,\ Y_0,\ Y_1,\ Y_2,\ Y_0^2,\ Y_1^2,\ Y_0Y_1,\ldots\}$, $dim(F_2) - dim(F_1) = 3$.

It is clear that $X_i = (Y_i - 1) - \frac{(Y_i-1)^2}{2}$ in this example, implying that $\mathcal{K}_\Delta^{\mathcal{E}x}$ and $\mathcal{C}_\Delta^{\mathcal{E}x}$ coincide as subalgebras of $\Phi_G^{\mathcal{E}x}$.

Similarly to the algebra $\mathcal{C}_G^{\mathcal{E}x}$, this filtered algebra is the Cartesian product of filtered algebras corresponding to the connected components of G; however, a stronger statement does not hold.

Proposition 1.29 *Given a graph G, the filtered algebra $\mathcal{K}_G^{\mathcal{E}x}$ is the Cartesian product of the filtered algebras corresponding to all connected components of G.*

Similarly to the case of algebra $\mathcal{C}_G^{\mathcal{E}x}$, we can present $\mathcal{K}_G^{\mathcal{E}x}$ as a quotient algebra.

Theorem 1.30 *For any graph G, the algebra $\mathcal{K}_G^{\mathcal{E}x}$ is isomorphic as a filtered algebra to the quotient*

$$\mathcal{K}_G^{\mathcal{E}x} \cong \mathbb{K}[y_0, \dots, y_n]/\mathcal{J}_G^{\mathcal{E}x},$$

where the ideal $\mathcal{J}_G^{\mathcal{E}x}$ in $\mathbb{K}[y_0, y_1, \dots, y_n]$ is generated by the polynomials

$$q_I^{\mathcal{E}x} = \left(\prod_{i \in I} y_i - 1\right)^{d_I+1}, \quad I \subseteq V(G). \tag{1.2}$$

The algebra $\mathcal{K}_G^{\mathcal{E}x}$ considered as a non-filtered algebra remembers the graphical matroid of G and only it. However, as a filtered algebra it contains the complete information about G.

Theorem 1.31 *Given two graphs G_1 and G_2 without isolated vertices, $\mathcal{K}_{G_1}^{\mathcal{E}x}$ and $\mathcal{K}_{G_2}^{\mathcal{E}x}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic.*

Example 1.10 *Consider two graphs G_1 and G_2 presented in fig. 1.2. We know that algebras $\mathcal{C}_{G_1}^{\mathcal{E}x}$ and $\mathcal{C}_{G_2}^{\mathcal{E}x}$ are isomorphic as graded algebras. However, by Theorem 1.31, filtered algebras $\mathcal{K}_{G_1}^{\mathcal{E}x}$ and $\mathcal{K}_{G_2}^{\mathcal{E}x}$ should distinguish graphs. Furthermore, in this case algebras $\mathcal{K}_{G_1}^{\mathcal{E}x}$ and $\mathcal{K}_{G_2}^{\mathcal{E}x}$ have different Hilbert series, namely,*

$$\mathcal{H}_{\mathcal{K}_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,$$
$$\mathcal{H}_{\mathcal{K}_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.$$

This means that they are not isomorphic as filtered algebras. It also means that, in general, it is impossible to calculate the Hilbert series of $\mathcal{K}_G^{\mathcal{E}x}$ from the Tutte polynomials of G, because in the above case G_1 and G_2 have the same Tutte polynomial.

1.7.2 Algebra $\mathcal{K}_G^C$

In notation of subsection § 1.3.2, our next object is the filtered subalgebra $\mathcal{K}_G^C \subset \Phi_G^C$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e}\phi_e), \; i = 0, \dots, n.$$

Here we usually have one more generator than in the case of the algebra $\mathcal{C}_G^C$.

As above in subsection § 1.7.1, $\mathcal{K}_G^C$ and $\mathcal{C}_G^C$ coincide as subalgebras of Φ_G^C, however, we are interested in the filtered structure of $\mathcal{K}_G^C$.

Theorem 1.32 *For any graph G, algebras $\mathcal{C}_G^C$ and $\mathcal{K}_G^C$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning trees of G.*

Example 1.11 *Consider the complete graph K_3 on vertices $\{0,1,2\}$ and with edges $\{a,b,c\}$, see fig 1.3. Generators of $\mathcal{K}_\Delta^C$ are*

$$Y_0 - 1 = e^{X_0} - 1 = \phi_a + \phi_c;$$

$$Y_1 - 1 = e^{X_1} - 1 = -\phi_a + \phi_b;$$

$$Y_2 - 1 = e^{X_2} - 1 = -\phi_b - \phi_c.$$

Then the filtered structure is

- $F_0 = span\{1\}$, $dim = 1$
- $F_1 = span\{1,\ Y_0,\ Y_1,\ Y_2\}$, $dim(F_1) - dim(F_0) = 2$

Then for K_3, we have that $\mathcal{C}_\Delta^C$ and $\mathcal{K}_\Delta^C$ are isomorphic as filtered algebras (filtered structure of $\mathcal{C}_G^C$ coincides with its graded structure). In general $\mathcal{C}_G^C$ and $\mathcal{K}_G^C$ are not isomorphic as filtered algebras.

The next statement is similar to Proposition 1.4.

Proposition 1.33 *Given a connected graph G, the filtered algebra $\mathcal{K}_G^C$ is the Cartesian product of the filtered algebras corresponding to all 2-edge connected components of G.*

As it happens for algebras $\mathcal{C}_G^{\mathcal{E}x}$ and $\mathcal{C}_G^C$, we can present this algebra as a quotient algebra.

Theorem 1.34 *For any graph G, the algebra $\mathcal{K}_G^C$ is isomorphic as a filtered algebra to the quotient*

$$\mathcal{K}_G^C \cong \mathbb{K}[y_0, \dots, y_n]/\mathcal{J}_G^C,$$

where the ideal $\mathcal{J}_G^C$ in $\mathbb{K}[y_0, y_1, \dots, y_n]$ is generated by the polynomials

$$q_I^C = \left(\prod_{i \in I} y_i - 1\right)^{d_I}, \quad I \subseteq V(G). \tag{1.3}$$

Similarly to the case of $\mathcal{C}_G^C$, we get the following implication and the problem. Define the Δ-subgraph $\widehat{G} \subset G$ as the subgraph obtained from G after removal of all its bridges and produced isolated vertices.

Proposition 1.35 *Given connected graphs G_1 and G_2 with isomorphic Δ-subgraphs $\widehat{G}_1 \cong \widehat{G}_2$, algebras $\mathcal{K}^C_{G_1}$ and $\mathcal{K}^C_{G_2}$ are isomorphic as filtered algebras.*

Problem 1.5 *Is it true that if $\mathcal{K}^C_{G_1}$ and $\mathcal{K}^C_{G_2}$ are isomorphic, then their Δ-subgraphs are also isomorphic? If not, then what is a criterium?*

Example 1.12 *Consider two graphs G_1 and G_2 presented in fig. 1.2. We know that algebras $\mathcal{C}^C_{G_1}$ and $\mathcal{C}^C_{G_2}$ are isomorphic as graded algebras; however, in this case filtered algebras $\mathcal{K}^C_{G_1}$ and $\mathcal{K}^C_{G_2}$ distinguish graphs. Furthermore, algebras $\mathcal{K}^C_{G_1}$ and $\mathcal{K}^C_{G_2}$ have different Hilbert series, namely,*

$$\mathcal{H}_{\mathcal{K}^C_{G_1}}(t) = 1 + 5t + 3t^2,$$

$$\mathcal{H}_{\mathcal{K}^C_{G_2}}(t) = 1 + 6t + 2t^2,$$

which means that they are not isomorphic as filtered algebras. This also means that it is impossible to calculate the corresponding Hilbert series from their Tutte polynomials, because they have the same Tutte polynomial.

1.8 Hypergraphical matroid

(*based on §4 of* **[A1]**)

There are many definitions of spanning trees of a hypergraph, for example: a spanning cactus in [1]; a hypertree in [10] (also known as an arboreal hypergraph in [7]). However, all these definitions allow trees to have different number of edges, whence spanning trees of a usual graph should have the same number of edges. We define spanning trees such that this property and also other natural properties hold. Also we define the hypergraphical matroid and the corresponding Tutte polynomial, whose points $T(2,1)$ and $T(1,1)$ calculate the numbers of forests and of spanning trees, resp. A similar definition of spanning trees and forests was presented in [24]; for that definition there exists a corresponding Tutte polynomial for a hypergraph, however, there is no matroid.

Let H be a hypergraph. Define $C = \{c_{i,e} \in \mathbb{K} : i \in [n],\ e \in H\}$ as a *set of parameters of* H s.t.

- $c_{i,e} = 0$, for any edge $e \in H$ and vertex $i \notin e$;
- $\sum_{i=1}^{n} c_{i,e} = 0$.

Our original motivation was to consider algebras $\mathcal{C}_C^{\mathcal{E}x}$, which constitute a family $\hat{\mathcal{C}}_H^{\mathcal{E}x}$ of algebras determined by H. Note, that for a usual graph G almost all algebras from $\hat{\mathcal{C}}_G^{\mathcal{E}x}$ are isomorphic to $\mathcal{C}_G^{\mathcal{E}x}$.

We define a hypergraphical matroid using the following definition of an independent set of edges of a hypergraph.

Let H be a hypergraph on n vertices. A set F of its edges is called *independent* if there is a set of parameters C of H, such that vectors corresponding to edges from F are linearly independent. In other words, F is independent if, for a generic set of parameters of H, vectors are linearly independent. Define the *hypergraphical matroid* of H as the matroid with the ground set $E(H)$. The *Tutte polynomial* $T_H(x,y)$ of H is the Tutte polynomial of the corresponding matroid. See Example 1.13.

There is a combinatorial definition of an independent set of edges. First we need to define a cycle of H.

A subset of edges $C \subset E$ is called a *cycle* if

- $|C| = |\cup_{e\in C}\, e|$
- There is no subset $|C' \subset C|$, such that the first property holds for C'.

Theorem 1.36 *A subset of edges $X \subset E$ is dependent if and only if there is a cycle $C \subset X$.*

Now we can define trees and forests of a hypergraph.

A set of edges F is called a *forest* if F has no cycles. In other words, F is a forest if and only if F is an independent set (by Theorem 1.36). A set of edges $T \subset H$ is called a *spanning tree* if it is a forest and T has exactly $v(H) - 1$ edges.

A hypergraph H is called *strongly connected* if it has at least one spanning tree.

Proposition 1.37 *Maximal forests of a hypergraph have the same number of edges. In fact, if $H = (V, E)$ is a strongly connected hypergraph, then for any forest $F \subset E$ there is a spanning tree T which contains F (i.e., $F \subset T \subset E$).*

Clearly we have

- $T_H(2,1)$ is the number of forests of H;
- $T_H(1,1)$ is the number of maximal forests of H, which is the number of trees for a strongly connected hypergraph.

Example 1.13 *Let H be the hypergraph with the vertex set $V = \{v_1, v_2, v_3\}$ and edges $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_1, v_2, v_3)\}$, see Fig. 1.4.*

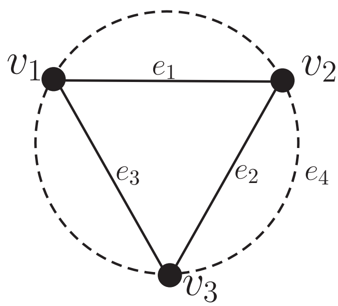

Figure 1.4

Vectors corresponding to edges are

- $e_1: \ (x, -x, 0) = x \cdot (1, -1, 0);$
- $e_2: \ (0, y, -y) = y \cdot (0, 1, -1);$
- $e_3: \ (-z, 0, z) = z \cdot (-1, 0, 1);$
- $e_4: \ (p, q, -p-q) = p \cdot (1, 0, -1) + q \cdot (0, 1, -1),$

where x, y, z, p and q are generic real numbers. It is easy to see that any three edges are linearly dependent and any two are independent.

It is clear that any three edges form a cycle of H, i.e., they are linearly dependent. So H has $\binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 11$ forests and $\binom{4}{2} = 6$ of them are trees.

The Tutte polynomial of the corresponding hypergraphical matroid is $x^2 + y^2 + 2x + 2y$. We get $T_H(1,1) = 6$ and $T_H(2,1) = 11$. Thus $T_H(1,1)$ and $T_H(2,1)$ are the number of trees and forests of H resp.

Proposition 1.38 *Given a hypergraph H, let C be a generic set of parameters for H. Then the Hilbert series are given by:*

- $\mathcal{H}_{\mathcal{C}_C^{\mathcal{E}x}}(t) = t^{m-n} T_H(1+t, \frac{1}{t})$;
- $\mathcal{H}_{\mathcal{C}_C^{C}}(t) = t^{m-n} T_H(1, \frac{1}{t})$;
- $\mathcal{H}_{\mathcal{C}_C^{\mathcal{I}n}}(t) = q^{m-n} T_H(0, \frac{1}{t})$.

There is still another characterization of our forests/trees of H, which again shows that our notion is a generalization of forests/trees of a usual graph.

Theorem 1.39 *A subset of edges $X \subset E$ is a forest (tree) if and only if there is a map from the edges to pairs: $e_k \to (i, j)$, where $v_i, v_j \in e_k$, such that these pairs form a forest (tree) in the complete graph K_n.*

By the induced subgraph on vertices $V' \subset V$, we mean a hypergraph (V', E'), where E' are all edges of E, which have vertices only from V' (i.e. $e \in E'$ if $e \subset V'$). This definition works well with colorings of hypergraphs, because if we want to color a hypergraph in such a way that there are no monochromatic edges, then it is the same as splitting vertices into sets with empty induced subgraphs. Also this definition works well with the standard notion of connectivity.

Proposition 1.40 *Let V_1 and V_2 be subsets of vertices such that the induced subgraphs of H on V_j are strongly connected and $V_1 \cap V_2 \neq \emptyset$. Then the induced subgraph of H on $V_1 \cup V_2$ is also strongly connected.*

2. Generic ideals

In this chapter we study Hilbert series of generic ideals. Let $S := \mathbb{K}[x_1,\dots,x_n]$ be the polynomial ring in n variables, where $\mathbb{K}$ is a filed of zero characteristic. Denote by S_d the d-th graded component of S, i.e., the linear space of all homogeneous polynomials of degree d in n variables. According to R.Stanley's result [39], the Hilbert series of generic ideals is known when the number of generators is small.

Theorem 2.1 (cf. [39]) *Given $z \le n+1$ and positive integer numbers $a_1,\dots,a_z$, let $\ell_1,\dots,\ell_z$ be generic linear forms and $I = \langle \ell_1^{a_1},\dots,\ell_z^{a_z}\rangle$. The Hilbert series of S/I is given by:*

$$\mathcal{H}_{S/I}(t) = \left[\frac{\prod_{i=1}^{z}(1-t^{a_i})}{(1-t)^n}\right],$$

where $[..]$ means that we truncate a real formal power series at its first negative term.

In [17] R. Fröberg formulated the following general conjecture.

Conjecture 2.1 (Fröberg's Conjecture) *Let $f_1,\dots,f_z$ be generic forms of degrees $a_1,\dots,a_z$ respectively. Set $I = \langle f_1,\dots,f_z\rangle$. The Hilbert series of S/I is given by:*

$$\mathcal{H}_{S/I}(t) = \left[\frac{\prod_{i=1}^{z}(1-t^{a_i})}{(1-t)^n}\right],$$

where $[..]$ means that we truncate a real formal power series at its first negative term.

R. Fröberg proved Conjecture 2.1 for 2 variables and noticed that the left-hand side is bigger than or equal to the right-hand side in the lexicographic sense. Theorem 2.1 implies that Fröberg's Conjecture holds for $z \le n+1$. Later in [3] D. J. Anick proved Conjecture 2.1 for 3 variables.

Theorem 2.2 (cf. [3]) *For the number of variables $n \le 3$, let $f_1,\dots,f_z$ be generic forms of degrees $a_1,\dots,a_z$ respectively. Set $I = \langle f_1,\dots,f_z\rangle$. The Hilbert series of S/I is given by:*

$$\mathcal{H}_{S/I}(t) = \left[\frac{\prod_{i=1}^{z}(1-t^{a_i})}{(1-t)^n}\right].$$

Anick's proof is hard to read, below in section § 2.2 we present an outline of our proof (which works in the case of 3 variables), where we work with a stronger conjecture from [27].

There are some extra results in the case where all degrees $a_1 = \ldots = a_z = d$ are the same. In [19] M. Hochster and D. Laksov showed that the dimension of $d+1$-th graded component is the expected one. In section § 2.1 we show that Fröberg's Conjecture is true in many different cases; the percentage of these cases compared to the total number of cases tends to 1 as $d \to +\infty$. Furthermore the same is true when f_i are linear forms raised to power d. (Notice that in general Fröberg's conjecture is not true for such type of ideals, see Remark 2.2.)

There are other generalizations of Fröberg's Conjecture. The most famous is Fröberg-Iarrobino's conjecture [12; 23]. See more problems in the recent paper [18] of participants of the *Problem solving seminar in commutative algebra* in Stockholm.

In the last section§ 2.3 we consider the first case, which is not covered by Theorem 2.1, namely when all a_i's are equal to 2 and their number is $z = n+2$. We prove the upper bound for the dimension of a quotient algebra, which is very unusual, since, for generic algebras, it is easy to prove lower bounds. Furthermore, this problem is related to generic ideals in exterior algebras.

2.1 Infinite series of examples

(*based on* **[B1]**)

Let $\mathcal{D}_d$ be any nonempty class of forms of degree d closed under the linear changes of coordinates. For example: $\mathcal{D}_d = S_d$ or $\mathcal{D}_d = \{\ell^d : \ \ell \in S_1\}$, i.e., the set of all d-th powers of linear forms.

We will work with the Hilbert function of an ideal; it is easy to convert it to the Hilbert function of the quotient algebra, because the sum of dimensions of m-th graded components of S/I and of I is the dimension of S_m. For $\mathcal{D}_d$ and z, denote by $\mathcal{HF}_{(\mathcal{D}_d,z)}(m)$ the dimension of the m-th graded component of an ideal generated by z generic forms from $\mathcal{D}_d$; denote by $\mathcal{H}_{(\mathcal{D}_d,z)}(t) = \sum \mathcal{HF}_{(\mathcal{D}_d,z)}(m)t^m$ the Hilbert series of this ideal. In [19] M. Hochster and D. Laksov found the values of $\mathcal{HF}_{(S_d,z)}(d+1)$ for any d and z. Below we generalize their result for the $(d+k)$-th graded component, but we avoid $2\dot{d}im(S_k)$ possible values of z.

Theorem 2.3 *Let d and k be positive integers. Then*

- *for* $z \leq \frac{dim(S_{d+k})}{dim(S_k)} - dim(S_k)$, $\quad \mathcal{HF}_{(\mathcal{D}_d,z)}(d+k) = z\dot{d}im(S_k)$*;*
- *for* $z \geq \frac{dim(S_{d+k})}{dim(S_k)} + dim(S_k)$, $\quad \mathcal{HF}_{(\mathcal{D}_d,z)}(d+k) = dim(S_{d+k})$.

Remark 2.1 *The condition about zero characteristic of the ground field is important here. For example, if $\mathbb{K}$ is a field of characteristic 2, $n = 3$, $d = 2$ and $\mathcal{D}_2$ is the set of squares of linear forms, then $\mathcal{HF}_{(\mathcal{D}_2,z)}(3) \leq dim(S_3) - 1$, because the form $x_1x_2x_3$ does not belong to the 3rd graded component for any z.*

In [5] M. Aubry obtained a result of the first type, but his result covers only a very thin set of cases (d is larger than some complicated function of k and z). In [25] J. Migliore and R. M. Mirós-Roig also formulated a similar result as a consequence of Anick's work [3]. However, their result holds only for small z.

As a consequence of Theorem 2.3 we get the following statement.

Proposition 2.4 *Let d and z be positive integers. If there exists r such that*

$$\frac{dim(S_{d+r+1})}{dim(S_{r+1})} + dim(S_{r+1}) \leq z \leq \frac{dim(S_{d+r})}{dim(S_r)} - dim(S_r),$$

then the Hilbert series of the ideal generated by z generic forms from $\mathcal{D}_d$ is given by

$$\mathcal{H}_{(\mathcal{D}_d,z)} = \sum_{k=0}^{\infty} min(z \cdot dim(S_k), dim(S_{d+k}))t^{d+k} = \frac{1}{(1-t)^n} - \left[\frac{(1-t^d)^z}{(1-t)^n}\right].$$

Of course, all interesting cases correspond to $z \leq dim(S_d)$; otherwise $\mathcal{HF}_{(\mathcal{D}_d,z)}(m) = dim(S_m)$ for $m \geq d$. Denote by

$$p_d := \frac{\#\{z \leq dim(S_d) \text{ satisfying Proposition 2.4}\}}{dim(S_d)}$$

the "probability" that a given $z \leq dim(S_d)$ is covered by Proposition 2.4.

Example 2.1 *For $n = 5$ and $d = 10$, $dim(S_d) = 1001$;*
$dim(S_1) = 5$ *and* $\frac{dim(S_{d+1})}{dim(S_1)} = 273$;
$dim(S_2) = 15$ *and* $\frac{dim(S_{d+2})}{dim(S_2)} = \frac{364}{3} = 121\frac{1}{3}$;
$dim(S_3) = 35$ *and* $\frac{dim(S_{d+3})}{dim(S_3)} = 68$.
Then the Hilbert series is given by Fröberg's conjecture at least if the number of generators z belongs to one of the following intervals:

- $z \geq 278$;
- $268 \geq z \geq 137$;
- $106 \geq z \geq 103$.

In other words, the Hilbert series is the standard one except possibly for $141 = 9 + 30 + 102$ *cases. Thus*

$$p_{10} = 1 - \frac{141}{1001} = 0,859..$$

For larger d, we get $p_{15} = 0,927..$; $p_{25} = 0,968..$; $p_{40} = 0,986..$

Proposition 2.5 *For any fixed number of variables, the probability p_d tends to 1 as $d \to +\infty$.*

Proposition 2.5 means that Proposition 2.4 gives the criterion, which covers a huge number of nontrivial cases for large d. As a consequence, we get that Fröberg's conjecture is true for many previously unknown cases for large d when the degrees of all forms are the same.

Remark 2.2 *It is also gives the same result for power ideals, when $\mathcal{D}_d = \{\ell^d : \ell \in S_1\}$, but we can not formulate the same conjecture for power algebras. Consider the case $(n,d,z) = (5,2,7)$. Then*

$$\mathcal{H}(S_5/\langle \ell_i^2,\ i \in [7]\rangle) = 1 + 5t + 8t^2 + t^3,$$

where ℓ_i are generic linear forms, but

$$\mathcal{H}(S_5/\langle f_i,\ i \in [7]\rangle) = 1 + 5t + 8t^2,$$

where f_i are generic quadratic forms.

2.2 Stronger conjecture

(*based on* **[B2]**)

In this section we discuss Fröberg's conjecture and present a stronger conjecture and an idea of its proof which works in the case of 3 variables. At first we should define a total order of monomials.

Let B be the set of monomials in $X = \{x_1, \dots, x_n\}$ and $B_k \subset B$ be the set of monomials of degree k. We consider the reverse lexicographic order on each B_k:

$$x_1^k < x_1^{k-1}x_2 < \dots < x_1^{k-1}x_n < x_1^{k-2}x_2^2 < \dots < x_{n-1}^k < x_{n-1}^{k-1}x_n < \dots < x_n^k.$$

Also we need the majorization partial order: for $\alpha, \beta \in B_k$, $\alpha \preceq \beta$ if and only if, for any $1 \le j \le n$, $\sum_{i=j}^n deg(\alpha)_{x_i} \le \sum_{i=j}^n deg(\beta)_{x_i}$. Note that this partial order agrees with the total order, i.e.

$$\alpha \prec \beta \Rightarrow \alpha < \beta.$$

For a linear space $C \subset S_k$, we define *leading monomials* $lm(C)$ of C as the minimal subset of B_k, such that, for any $g \in C$, the leading term of g belongs to C. In other words, leading monomials of C are

$$\mathrm{lm}(C) := \{\mathrm{lm}(g) : \ g \in C\},$$

where $\mathrm{lm}(g)$ is the leading monomial of form g.

It is clear that $dim(C) = |\mathrm{lm}(C)|$, which implies that it is enough to study leading monomials of graded components of ideals.

We fix the sequence $a_1, a_2, \dots, a_z, \dots$ of degrees of generators. Let $m_{k,z}$ be the k-th coefficient of $\frac{1}{(1-t)^n} - \left[\frac{\prod_{i=1}^z (1-t^{a_i})}{(1-t)^n}\right]$, in other words, $m_{k,z}$ is the expected dimension of the k-th graded component of an ideal I_z generated by generic forms $f_1, f_2, \dots, f_z$ of degrees $d_1, d_2, \dots, d_z$ resp.

Now we construct the set $M_{k,z} \subseteq B_k$ of monomials. We construct it by induction

- $M_{-1,z} = M_{-2,z} = \dots = \emptyset$,
- $M_{k,z} = X \cdot M_{k-1,z} \cup \tilde{M}_{k,z}$, where $\tilde{M}_{k,z}$ is the set of the first $(m_{k,z} - |X \cdot M_{k-1,z}|)$ reverse lexicographic maximal monomials from $B_k \setminus (X \cdot M_{k-1,z})$.

We think that it is exactly the set of leading monomials and formulate the stronger conjecture.

Conjecture 2.2 *Let $f_1,\ldots,f_z$ be generic forms of degrees $a_1,\ldots,a_z$ respectively. Set $I_{k,z} = \langle f_1,\ldots,f_z\rangle_k$. Then the set $lm(I_{k,z})$ of leading monomials coincides with $M_{k,z}$.*

Since $|M_{k,z}| = m_{k,z}$ (which is the expected dimension), then Fröberg's Conjecture is a consequence of Conjecture 2.2.

We suggest to prove Conjecture 2.2 and Fröberg's Conjecture by induction on a pair (k,z). We prove that our approach works for 2 and 3 variables. For the general case, it is still possible to make half of the induction step.

Proposition 2.6 *Assume that Conjecture 2.2 holds for all pairs smaller than (k,z). Then Fröberg's Conjecture holds for (k,z).*

In fact, we can also prove the following important lemma, where $dim(A : D), A \subseteq S_k,\ D \subseteq B_k$ is the dimension of A in $S_k/span(B_k \setminus D)$.

Lemma 2.1 *Assume that Conjecture 2.2 holds for all pairs smaller than (k,z). Set $C_{k,z} := span\{f_z \cdot \beta :\ \beta \in (B_{k-a_z} \setminus M_{k-a_z,z-1})\}$. If $dim(C_{k,z} : M_{k,z} \setminus M_{k,z-1}) = m_{k,z} - m_{k,z-1}$, then $lm(I_{k,z}) = M_{k,z}$.*

We can not prove the second condition of Lemma 2.1 in general. However, we can prove it in the case of three variables.

Proposition 2.7 *Assume that the number of variables n is 2 or 3. Let f_z be generic form of degree a_z, then $dim(C_{k,z} : M_{k,z} \setminus M_{k,z-1}) = m_{k,z} - m_{k,z-1}$, where $C_{k,z} := span\{f_z \cdot \beta :\ \beta \in (B_{k-a_z} \setminus M_{k-a_z,z-1})\}$.*

From Proposition 2.7 and Lemma 2.1 we get the following theorem.

Theorem 2.8 *Conjecture 2.2 holds for 2 and 3 variables.*

All proofs except Proposition 2.7 are either simple or technical (but without serious ideas). To prove Proposition 2.7 we should construct a unique chain (i.e., there is no other chain with the same product) of length $m_{k,z} - m_{k,z-1}$ in the matrix $\mathcal{M}$, where the rows of $\mathcal{M}$ correspond to the elements of $B_{k-a_z} \setminus M_{k-a_z,z-1}$, columns of $\mathcal{M}$ to $M_{k,z} \setminus M_{k,z-1}$, and an element (β,γ) is equal to the coefficient of γ in $f_z \cdot \beta$.

We construct this chain using the following algorithm:

- $A_{k,z} := M_{k,z-1}$ `and` $C := B_{k-a_z} \setminus M_{k-a_z,z-1}$.
- `repeat while` $C \neq \emptyset$:

1. `Choose maximal` $\beta \in C$ `and delete it from` C `(i.e.,` $C := C - \beta$`),`
2. `Choose maximal (if possible)` $\psi_{k,z}(\beta) := \alpha \in B_{a_z}$ `such that` $\alpha\beta \notin A_{k,z}$ `and add this product to` $A_{k,z}$ `(i.e.,` $A_{k,z} := A_{k,z} + \alpha\beta$`).`

In the case of three variables, this algorithm has two good properties.

Theorem 2.9 *For any k and z, we have*

- $A_{k,z} = M_{k,z}$;
- *if* $\beta_1 < \beta_2 \in B_{k-a_z} \setminus M_{k-a_z,z-1}$ *and* $\beta_1 | (\psi_{k,z}(\beta_2)\beta_2)$, *then* $\psi_{k,z}(\beta_1) \geq \psi_{k,z}(\beta_2)$.

The first property shows that $\mathcal{M}$ has a chain, and the second one shows that there is no chain with the same product of elements. Since the coefficients of f_z are generic, we get that the determinant of $\mathcal{M}$ does not vanish which finishes the proof of Proposition 2.7.

2.3 Ideals generated by two quadratic forms in the external and square-free algebras

(*based on* **[B3]**)

In this section we study the Hilbert series of two classes of algebras. (It is a joint work with V. Crispin and S. Lundqvist.)

The first class is $E_n/\langle f,g\rangle$, where E_n is the exterior algebra and f,g are two quadratic forms. The second class is $\Phi_n/\langle \ell_1^2,\ell_2^2\rangle$, where Φ_n is the square-free algebra, i.e., with relations

$$\phi_i\phi_j=\phi_j\phi_i,\ i,j\in[n] \quad \text{and} \quad \phi_i^2=0,\ i\in[n],$$

and ℓ_1 and ℓ_2 are two linear forms in Φ_n.

Proposition 2.10 (cf. [26]) *Given the exterior algebra E_n, let f be a generic form of degree $2d$, then the Hilbert series of $E_n/\langle f\rangle$ is given by*

$$\mathcal{H}_{E_n/\langle f\rangle}=[(1+t)^n(1-t^{2d})].$$

The following proposition is a particular case of Theorem 2.1.

Proposition 2.11 (cf. [39]) *Given the square-free algebra Φ_n and a positive integer d, let ℓ be a generic linear form. Then the Hilbert series of $\Phi_n/\langle \ell^d\rangle$ is given by*

$$\mathcal{H}_{\Phi_n/\langle \ell^d\rangle}=[(1+t)^n(1-t^{d})].$$

Let us return to our case. We have a conjecture about the Hilbert series. Let $a(n,k)$ be the number of lattice paths inside the rectangle $(n+2-2k)\times(n+2)$ from the bottom-left corner to the top-right corner with moves of two types: $(x,y)\to(x+1,y+1)$ or $(x-1,y+1)$, see Figure 2.1.

Conjecture 2.3 *Let f and g be two generic quadratic forms in the exterior algebra E_n. Then the Hilbert series of $E_n/\langle f,g\rangle$ is equal to*

$$\mathcal{H}_{E_n/\langle f,g\rangle}=1+a(n,1)t+a(n,2)t^2+\cdots+\cdots+a(n,\lceil\frac{n}{2}\rceil)t^{\lceil\frac{n}{2}\rceil}.$$

Conjecture 2.4 *Let ℓ_1 and ℓ_2 be generic linear forms in the square-free algebra Φ_n. Then the Hilbert series of $\Phi_n/\langle \ell_1^2,\ell_2^2\rangle$ is equal to*

$$\mathcal{H}_{\Phi_n/\langle \ell_1^2,\ell_2^2\rangle}=1+a(n,1)t+a(n,2)t^2+\cdots+\cdots+a(n,\lceil\frac{n}{2}\rceil)t^{\lceil\frac{n}{2}\rceil}.$$

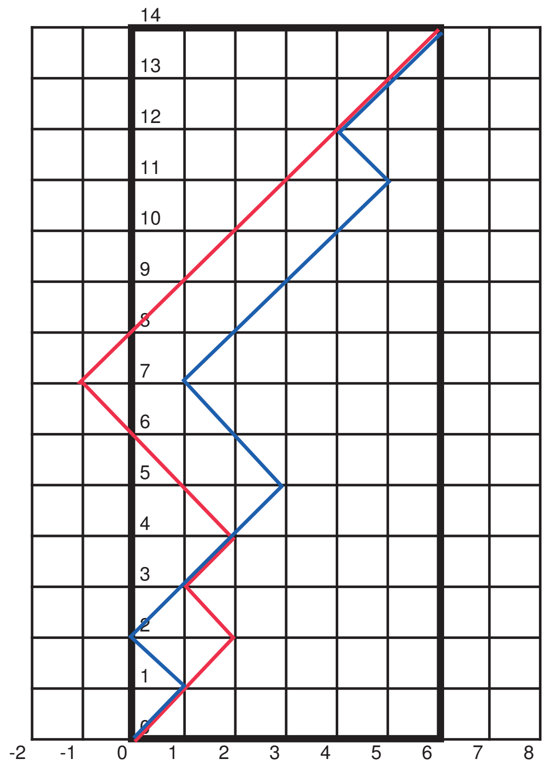

Figure 2.1: Case $(n,k) = (12,4)$. The red path is not inside the rectangle while the blue path is inside the rectangle.

Coefficients $a(n,k)$ are presented in Table 2.1, we checked the first conjecture for random linear forms and $n \leq 20$ and proved it for odd $n \leq 19$. Furthermore, we proved the upper bound.

Theorem 2.12 *In notations of Conjectures 2.3 and 2.4, the dimension of k-graded of* $E_n/\langle f,g\rangle$ *and of* $\Phi_n/\langle \ell_1^2, \ell_2^2\rangle$ *are at most* $a(n,k)$.

Although the above conjectures give the same Hilbert series for different and non-isomorphic algebras, Conjecture 2.4 is weaker than Conjecture 2.3.

Theorem 2.13 *The following statements are equivalent:*

- *Conjecture 2.3 holds for any even n;*
- *Conjecture 2.4 holds for any n.*

Theorem 2.14 *In notation of Conjectures 2.3 and 2.4 and given integer numbers n and* $k \leq \lfloor \frac{n+1}{3} \rfloor$, *the dimensions of k-graded components of* $E_n/\langle f,g\rangle$ *and of* $\Phi_n/\langle \ell_1^2, \ell_2^2\rangle$ *are equal to* $a(n,s)$.

For the exterior, algebra we also have the equality for the last components.

Theorem 2.15 *In notation of Conjecture 2.3, the dimension of* $\lceil \frac{n}{2} \rceil$*-graded of* $E_n/\langle f,g\rangle$ *is equal to* $a(n, \lceil \frac{n}{2} \rceil)$*, which is* 1 *for odd n and* $2^{\frac{n}{2}}$ *for even n.*

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0
2	1	2	0	0	0	0	0	0	0	0	0
3	1	3	1	0	0	0	0	0	0	0	0
4	1	4	4	0	0	0	0	0	0	0	0
5	1	5	8	1	0	0	0	0	0	0	0
6	1	6	13	8	0	0	0	0	0	0	0
7	1	7	19	21	1	0	0	0	0	0	0
8	1	8	26	40	16	0	0	0	0	0	0
9	1	9	34	66	55	1	0	0	0	0	0
10	1	10	43	100	121	32	0	0	0	0	0
11	1	11	53	143	221	144	1	0	0	0	0
12	1	12	64	196	364	364	64	0	0	0	0
13	1	13	76	260	560	728	377	1	0	0	0
14	1	14	89	336	820	1288	1093	128	0	0	0
15	1	15	103	425	1156	2108	2380	987	1	0	0
16	1	16	118	528	1581	3264	4488	3280	256	0	0
17	1	17	134	646	2109	4845	7752	7753	2584	1	0
18	1	18	151	780	2755	6954	12597	15504	9841	512	0
19	1	19	169	931	3535	9709	19551	28101	25213	6765	1
20	1	20	188	1100	4466	13244	29260	47652	53296	29524	1024

Table 2.1: The values of $a(n,k)$ and of the Hilbert function of $E_n/\langle f,g\rangle$ with random quadratic forms f and g.

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