On Amoebas and Multidimensional Residues

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Abstract

This thesis consists of four papers and an introduction.

In Paper I we calculate the second order derivatives of the Ronkin function of an affine polynomial in three variables. This gives an expression for the real Monge-Ampère measure associated to the hyperplane amoeba. The measure is expressed in terms of complete elliptic integrals and hypergeometric functions.

In Paper II and III we prove that a certain semi-explicit cohomological residue associated to a Cohen-Macaulay ideal or more generally an ideal of pure dimension, respectively, is annihilated precisely by the given ideal. This is a generalization of the local duality principle for the Grothendieck residue and the cohomological residue of Passare. These results follow from residue calculus, due to Andersson and Wulcan, but the point here is that our proof is more elementary. In particular, it does not rely on the desingularization theorem of Hironaka.

In Paper IV we prove a global uniform Artin-Rees lemma for sections of ample line bundles over smooth projective varieties. We also prove an Artin-Rees lemma for the polynomial ring with uniform degree bounds. The proofs are based on multidimensional residue calculus.

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Contents

List of Papers

I: Lundqvist J., *An explicite calculation of the Ronkin function*, Preprint (2012).

II: Lundqvist J., *A local Grothendieck duality theorem for Cohen-Macaulay ideals*, Math. Scand. **111** (2012), no. 1, 42-52.

III: Lundqvist J., *A local duality principle for ideals of pure dimension*, Preprint (2012).

IV: Lundqvist J., *An effective uniform Artin-Rees lemma*, Preprint (2012).

Introduction

1 Amoebas

Amoebas are geometric objects that are connected with many different areas in mathematics such as tropical geometry, complex analysis, combinatorics, and special functions. Amoebas has a deep connection to basic classical objects like Laurent series expansions of meromorphic functions, studied by for example Hartogs in the beginning of the last century, see [Har06]. It is therefore surprising that amoebas were defined as late as 1994.

We use multiindexnotation meaning that for $\alpha \in \mathbb{Z}^n$ and $z \in \mathbb{C}^n$ we let $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $dz/z = dz_1 \cdots dz_n / z_1 \cdots z_n$.

1.1 Definition and basic observations

Let us start with a one-variable polynomial f of degree N and let us assume that the roots b_j are such that $0 < |b_1| < \cdots < |b_N|$. Then we can write

$$\frac{1}{f(z)} = \frac{A_1}{b_1 - z} + \cdots + \frac{A_n}{b_N - z},$$

where $A_1, \ldots, A_N$ are complex numbers. Using that the geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

and hence

$$\frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\sum_{k=-\infty}^{-1} z^k,$$

converge for $|z| < 1$ and for $|z| > 1$, respectively, we can expand $1/f$ as a convergent Laurent series in every annulus

$$U_j = \{z \in \mathbb{C}; b_j < |z| < b_{j+1}\}. \tag{1}$$

On the other hand, if we can write $1/f$ as a Laurent series that converges in some open set, then that series must be unique. In particular, we get a bijection between the set of various convergent Laurent series expansions of $1/f$ and the set of annuli on the form (1). If we regard the real set $\mathcal{A}_f = \{\log|b_j|; j = 1, \ldots N\}$, it follows that there is a bijection between the connected complement components of $\mathcal{A}_f$ and the various convergent

Laurent series expansions of $1/f$. This is still true when some of the roots of f have the same absolute value and it is proved in essentially the same way.

The set $\mathcal{A}_f$ is called the amoeba of f. As we saw, the amoeba of a one-variable polynomial is just a set of points but in several variables things get a bit more complicated.

Definition 1.1. *The* amoeba, $\mathcal{A}_f$, *of* f *is the image of the zero set of* f *in* $\mathbb{C}_*^n$ *under the mapping* $\mathrm{Log} : \mathbb{C}_*^n \to \mathbb{R}$ *given by*

$$\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|).$$

The concept of amoeba was introduced by Gelfand, Kapranov, and Zelevinsky in their book [GKZ94], and it is fundamental in the modern view of hypergeometric functions in several variables, introduced and studied by the same authors.

The amoeba of a polynomial f in n variables is a set in $\mathbb{R}^n$. Note that if we take a set E in the complement of $\mathcal{A}_f$ then $\mathrm{Log}(E)$ is a so-called circular domain, i.e., a domain U such that if $z = (z_1, \ldots, z_n) \in U$, then $ze^{i\theta} = (z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) \in U$ for every $\theta \in [0, 2\pi]^n$. It is well-known that if a function is holomorphic in a circular domain, then there is a Laurent series expansion of the function that converges in the domain. Since two Laurent series that both converge on an open set must coincide this means that, as in the one-variable case, there is a bijection between the connected complement components of $\mathcal{A}_f$ and the various convergent Laurent series expansions of $1/f$. Note also that if E is a connected complement component of $\mathcal{A}_f$, then E is convex. This follows from the fact that if g is holomorphic on a circular domain U, then it is holomorphic on the logarithmically convex hull of U. The observations above were made already in [GKZ94] where the authors also relates the geometry of $\mathcal{A}_f$ to the Newton polytope of f.

Definition 1.2. *Let* $f = \sum_{\alpha \in A} a_\alpha z^\alpha$, *where* $A \subset \mathbb{Z}^n$, *be a Laurent polynomial. Then the Newton polytope,* Δ_f, *of* f *is the convex hull in* $\mathbb{R}^n$ *of the set of points* α *such that* $a_\alpha \neq 0$.

It is not hard to see that the number of complement components of $\mathcal{A}_f$ is bounded from below by the number of vertices of Δ_f. It turns out that one can say a lot more than that.

Theorem 1.3 (Forsberg, Passare, Tsikh, Rullgård)**.** *Let f be a Laurent polynomial. The number of connected complement components of $\mathcal{A}_f$ is bounded from below by the number of vertices in Δ_f and from above by the number of points in $\Delta_f \cap \mathbb{Z}^n$.*

For any polytope Δ, and integer k between the lower and upper bound, there exists a Laurent polynomial g such that $\Delta_g = \Delta$ and such that the number of connected complement components of $\mathcal{A}_g$ is equal to k.

The first part of Theorem 1.3 (which is due to Forsberg, Passare, Tsikh, [FPT00]) follows since there exists an injective function from the set of connected complement components of $\mathcal{A}_f$ to $\Delta_f \cap \mathbb{Z}^n$. Such a function can be defined from a function introduced by Ronkin.

1.2 The Ronkin function and the Ronkin measure

The Ronkin function of a Laurent polynomial is a fundamental tool in order to understand the geometry of $\mathcal{A}_f$.

Definition 1.4. *Let f be a Laurent polynomial. The* Ronkin function *of f, $N_f : \mathbb{R}^n \to \mathbb{R}$, is given by*

$$N_f(x) = \frac{1}{(2\pi i)^n} \int_{|z_j|=x_j} \log|f(z)| \frac{dz}{z}. \tag{2}$$

Ronkin studied this kind of functions in [Ron74] and showed that N_f is a convex function that, in terms of amoebas, is affine linear exactly on the complement of $\mathcal{A}_f$. This means that the gradient of N_f is constant in each complement component, and it turns out that the gradient is an integer point in Δ_f, [FPT00]. In particular we get a mapping from the set of complement components of $\mathcal{A}_f$ to integer points in Δ_f and it is proved in [FPT00] that this map is injective. This imply the first part of Theorem 1.3.

Contrary to biological amoebas the mathematical ones have spines. The spine of an amoeba was first introduced in [PR04] and defined from the Ronkin function in the following way: First, the *order* of a complement component E is defined to be the image of E under the mapping $\operatorname{grad} N_f$. The complement component with order α is denoted by E_α. Let $\tilde{A}$ be the subset of $\mathbb{Z}^n \cap \Delta_f$ concisting of points α such that $\mathcal{A}_f$ has a component of order α. Then we can define the real number

$$c_\alpha = N_f(x) - \langle \alpha, x \rangle ,$$

where x is any point in E_α. Let

$$S(x) = \max_{\alpha \in \tilde{A}} (c_\alpha + \langle \alpha, x \rangle).$$

Then $S(x)$ is a convex piecewise affine linear function that agrees with N_f on the complement of the amoeba. The corner locus of $S(x)$, is called the *spine* of the amoeba of f and is denoted by $\mathcal{S}_f$.

It is proved in [PR04] that the topology of the amoeba is described by the spine. To understand that result one needs to take a closer look at the concept of duality between convex subdivisions of convex sets in $\mathbb{R}^n$.

Definition 1.5. *Let K be a convex set in $\mathbb{R}^n$ and let T be a collection of closed convex subsets of K. Then T is said to be a* convex subdivision *if it satisfies all of the following three conditions:*

1. *The union of all sets in T is equal to K.*
2. *A nonempty intersection of two sets in T belongs to T.*
3. *A subset τ of a set σ in T belongs to T if and only if τ is a face of σ.*

For two convex sets σ and τ such that $\tau \subset \sigma$ we define the convex cone $\text{cone}(\tau, \sigma)$ as

$$\text{cone}(\tau, \sigma) = \{t(x - y);\ x \in \sigma, y \in \tau, t \geq 0\}.$$

The dual cone $C^\vee$ of a convex cone C is defined to be

$$C^\vee = \{\xi \in \mathbb{R}^n;\ \langle \xi, x \rangle \leq 0, \forall x \in C\}.$$

Definition 1.6. *Let T and T' be two convex subdivisions of the sets K and K' respectively. Then T and T' are said to be dual to each other if there exists a bijective mapping from T to T', $\sigma \mapsto \sigma^*$, such that the following two conditions are satisfied for all sets $\tau, \sigma \in T$:*

1. *$\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$.*
2. *The cone $\text{cone}(\tau, \sigma)$ is dual to $\text{cone}(\sigma^*, \tau^*)$.*

Theorem 1.7 (Passare, Rullgård, [PR04])**.** *The spine $\mathcal{S}_f$ is a deformation retract of $\mathcal{A}_f$ and there exist dual subdivisions T of $\mathbb{R}^n$ and T' of $\tilde{A}$ such that $\mathcal{S}_f$ is the union of the cells in T of dimension less than n. Moreover, the cell of T dual to the point $\alpha \in \tilde{A}$ contains the complement component of order α.*

For every smooth convex function f on $\mathbb{R}^n$, the Hessian matrix of f,

$$\text{Hess}(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right),$$

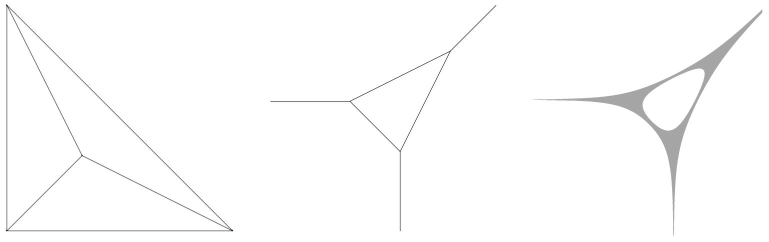

Figure 1: The amoeba of a polynomial f is a thickened graph dual to a subdivision of the Newton polytope of f. Here exemplified by the polynomial $f = 1 + z^3 + w^3 - 7zw$.

is a positive definite matrix. The determinant of the Hessian times the Lebesgue measure is called the real Monge-Ampère measure of f and is denoted by $\mathrm{M}(f)$. We can define the gradient at a point x_0 of a convex function defined in a domain Ω as

$$\operatorname{grad} f(x_0) = \{y \in \mathbb{R}^n; f(x) - f(x_0) \geq \langle y, x - x_0 \rangle, \forall x \in \Omega\}.$$

Notice that the gradient defined in this way coincide with the usual one at points where the function is differentiable. The Monge-Ampère measure can now be extended to all convex functions f by letting

$$\mathrm{M}(f)(E) = \lambda(\operatorname{grad} f(E)),$$

where λ is the Lebesgue measure and

$$\operatorname{grad} f(E) = \bigcup_{x \in E} \operatorname{grad} f(x).$$

The Monge-Ampère measure is a positive Borel measure for any convex function, see [RT77].

Since N_f is convex we may take the real Monge-Ampère measure of N_f and then we get a positive measure μ_f that we call the *Ronkin measure*. It has support on the amoeba and moreover, it has finite total mass equal to the volume of the Newton polytope of f, see [PR04].

In two variables, the Ronkin measure μ_f has some nice properties. For instance, it is proved in [PR04] that

$$\mu_f \geq \frac{\lambda}{\pi^2} \tag{3}$$

on the amoeba of f, where λ is the Lebesgue measure. From this estimate it follows immediately that

$$\text{Area}(\mathcal{A}_f) \leq \pi^2 \, \text{Area}(\Delta_f). \tag{4}$$

The inequalities (3) and (4) are sharp and it turns out that those polynomials that have amoebas with maximal areas are the ones that define the so-called Harnack curves, see [MR01]. These curves are connected to the first part of Hilbert's 16th problem. Since $f = 1 + z + w$ define a Harnack curve it follows that the Ronkin measure of f has the constant measure $(1/\pi^2)\lambda$ on the amoeba. It is also fairly easy to calculate the Ronkin measure of $f = 1 + z + w$ directly. If one consider the corresponding polynomial $f = 1 + z + w + t$ in three variables the Ronkin measure of f is much harder to calculate.

At this point, it is interesting to note that the Ronkin function is closely connected with the so-called Mahler measure from number theory. In fact the Ronkin function of f evaluated at the origin is the Mahler measure of f, and for affine linear polynomials it is easy to make the transition of the Ronkin function evaluated at a point and the Mahler measure of some other affine linear polynomial. In two variables, there are known explicit expressions for the Mahler measure of all affine linear polynomials and hence the Ronkin function itself of such a polynomial has a known explicit expression.

In three variables, very little is known about the Ronkin measure. Note that the amoebas in more than two variables have infinite volume in general, and hence there is no inequality like (4). There might still be an inequality like (3) but with $1/\pi^2$ replaced by a function. In Paper I we are interested in the Ronkin measure of the affine linear polynomial $f = 1 + z + w + t$, which should be the easiest possible three variable example. One can note that there are some known explicit formulas for the associated Mahler measure in this case but only for those polynomials that corresponds to special points lying on what is called the contour of the amoeba. In particular, this means that there is no known explicit expression for the Ronkin function to use when calculating the measure μ_f, which indicates that it might be considerably more complicated than in the two variable case. On the other hand, it seems that the Ronkin measure is easier to calculate than the Ronkin function itself. The main result of Paper I is that the Ronkin measure of an affine linear polynomial in three variables can be described in terms of complete elliptic integrals or hypergeometric functions. As an application of the calculations done there one gets new information about the regularity of the measure.

2 Multidimensional residues

We let $\mathcal{O}_0$ denote the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$. That is, φ belongs to $\mathcal{O}_0$ if there exists a small neighborhood U of 0 such that φ is holomorphic in U. We let Ω_0^k denote the $\mathcal{O}_0$-module of holomorphic germs of $(k,0)$-forms at the origin.

Let us start with the picture in one variable. Then the maximal ideal in $\mathcal{O}_0$ is generated by the coordinate function z, and every non-trivial ideal is generated by a monomial z^m for some $m \geq 1$. Given such an ideal $J = \langle z^m \rangle$ and a holomorphic germ φ we would like to answer the question whether or not φ belongs to J. Let us consider two slightly different ways of deciding that.

First, we can test whether

$$\varphi \operatorname{Res}_{z^m}(\psi) = \int_{\partial D} \frac{\varphi \psi}{z^m}$$

vanishes for every $\psi \in \Omega_0^1$. Here, D is a sufficiently (depending on ψ) small neighborhood of 0. Indeed, if $\varphi = z^m g$ for some holomorhic germ g, then the integrand is a holomorphic function in D and is consequently zero by the theorem of Cauchy. On the other hand, if the integral is zero for all $\psi \in \Omega_0^1$, then it is zero for $\psi_j := z^j dz$ for $j = 0, 1, \ldots, m-1$ and by the residue theorem in one variable this means that all negative powers in the Laurent series expansion of φ / z^m vanish, i.e., $\varphi = z^m g$ for some $g \in \mathcal{O}_0$.

Second, we may regard the principle value current $1/z^m$ which is defined by

$$\left\langle \frac{1}{z^m}, \phi \right\rangle = \lim_{\epsilon \to 0} \int_{|z^m| > \epsilon} \frac{\phi}{z^m}$$

for test forms ϕ. If we take the $\bar\partial$-operator on the principle value current in the current sense, then by Stokes' theorem we get a current defined by

$$\left\langle \bar\partial \frac{1}{z^m}, \phi \right\rangle = \lim_{\epsilon \to 0} \int_{|z^m| = \epsilon} \frac{\phi}{z^m}, \tag{5}$$

with support at the origin, and we get

$$\varphi \in (z^m) \Leftrightarrow \varphi \bar\partial \frac{1}{z^m} = 0.$$

In several variables there are generalizations of the second approach for all kind of ideals. The drawback is that it uses sophisticated and deep results such as the desingularization theorem of Hironaka. The first approach has earlier only been generalized for very special ideals. In Paper II and III we consider the first approach and generalize it to a larger set of ideals than has been considered before, in an essentially algebraic way, not using the theorem of Hironaka. Given such an ideal J, this enable us to get a semi-explicit integral condition on a germ $\varphi \in \mathcal{O}_0$ to belong to J.

2.1 The Grothendieck residue

Let $(f^1, \ldots, f^n)$ be a tuple of germs of holomorphic functions at $0 \in \mathbb{C}^n$ and assume that the common zero set consists of the single point 0. Let

$$B(z) = \frac{1}{(2\pi i)^n} \frac{\sum (-1)^{j-1} \bar{z}_j d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_j} \wedge \ldots \wedge d\bar{z}_n}{(|z_1|^2 + \ldots + |z_n|^2)^n},$$

be the so-called Bochner-Martinelli kernel. The Grothendieck residue is defined by

$$\mathrm{Res}_f(\psi) := \int_{\partial D} f^* B \wedge \psi, \quad \psi \in \Omega_0^n, \tag{6}$$

where D is a sufficiently small neighborhood (depending on ψ) of the origin. As in the one-variable case we define multiplication by a holomorphic germ φ by

$$\varphi \, \mathrm{Res}_f(\psi) = \mathrm{Res}_f(\varphi\psi).$$

The residue Res_f is said to be annihilated by a holomorphic germ φ if $\varphi \, \mathrm{Res}_f = 0$. The Grothendieck residue gives an analytic condition on a germ $\varphi \in \mathcal{O}_0$ to be in the ideal generated by the f^j:s, see [GH78].

Theorem 2.1 (The local duality theorem)**.** *If $J = \langle f^1, \ldots, f^n \rangle$ and the common zero set of the f^j:s only consists of the origin, then $\varphi \in J$ if and only if φ annihilates Res_f.*

Example 2.2. If $f^j = z_j$ and $\psi = \tilde{\psi} dz$, then $\mathrm{Res}_f(\psi) = \tilde{\psi}(0)$ by the Bochner-Martinelli formula, which is the multivariate analogue of the Cauchy formula. Thus, in this case we get that $\varphi \, \mathrm{Res}_f(\psi) = 0$ for every $\psi \in \Omega_0^n$ if and only if $\varphi(0) = 0$, and that is obviously equivalent with $\varphi \in \langle z_1, \ldots, z_n \rangle$.

2.2 Residue currents and the duality principle

One generalization of the current (5) to several variables is the Coleff-Herrera product, defined in [CH78]. Let $(f^1, \ldots, f^p)$ be a tuple of germs in $\mathcal{O}_0$ in $\mathbb{C}^n$. Then the Coleff-Herrera product is the current

$$\left\langle \bar{\partial}\frac{1}{f^p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f^1}, \phi \right\rangle := \lim_{\delta \to 0} \int_{\cap\{|f_j|=\epsilon_j(\delta)\}} \frac{\phi}{f^1 \cdots f^p}. \tag{7}$$

Here the limit is taken so that $\epsilon_j(\delta)$ tends to zero faster than any power of $\epsilon_{j-1}(\delta)$ and this ensures that the limit exists. If the zero set of the tuple $f = (f^1, \ldots, f^p)$ is nice in the sense that the dimension of the common zero set is the intuitive one, i.e., it has dimension $n-p$, then the tuple f is said to define a complete intersection. As with the Grothendieck residue a holomorphic germ φ is said to annihilate a current R if $\varphi R = 0$. The following fundamental result is proved independently by Passare in [Pas88] and Dickenstein and Sessa in [DS85].

Theorem 2.3. *If the tuple f defines a complete intersection, then φ belongs to the ideal generated by the germs in f if and only if φ annihilates the Coleff-Herrera product* (7).

All known proofs of the existence of the Coleff-Herrera product in the general case uses the desingularization theorem of Hironaka. In [TY04] the authors give a proof of the existence of the Coleff-Herrera product when the common zero set of f is a point and f defines a complete intersection that uses amoebas and avoid Hironaka's theorem. It should also be mentioned that in the complete intersection case there is a construction in [Maz10] of another current that is annihilated exactly by the given ideal. The construction in [Maz10] only use Weierstrass preparation theorem.

For a general ideal J in $\mathcal{O}_0$, Andersson and Wulcan defined a residue current R that satisfies a duality principle like in Theorem 2.3. They start with a free resolution of J:

$$0 \to \mathcal{O}_0^{\oplus r_M} \xrightarrow{f_M} \ldots \xrightarrow{f_2} \mathcal{O}_0^{\oplus r_1} \xrightarrow{f_1} \mathcal{O}_0. \tag{8}$$

Let X be a small neighborhood of 0 such that $f_j(x)$ define a complex for each $x \in X$. Moreover, let E_j be trivial vector bundles of rank r_j over X and assume that the bundles are equipped with a $\mathbb{Z}_2$-grading, a so-called super structure, so that the operators f_j and $\bar{\partial}$ on smooth $(0,q)$-sections of E_j anti-commute, see [AW07] for more details. Let $\mathcal{Z}$ be the common zero set of the tuple $f_1 = (f^1, \ldots, f^m)$ that defines J. Let σ_k be the pointwise minimal inverse of f_k on $X \setminus \mathcal{Z}$. That is,

$$\sigma_k \xi = \begin{cases} \eta, \text{ where } f_k \eta = \xi \text{ and } \eta \text{ has minimal norm, if } \xi \in \operatorname{Im} f_k, \\ 0, \text{ if } \xi \in (\operatorname{Im} f_k)^{\perp}. \end{cases}$$

It is proved in [AW07] that the E_k-valued forms

$$u_k := \sigma_k \bar{\partial}\sigma_{k-1} \cdots \bar{\partial}\sigma_1 \tag{9}$$

satisfy the equations

$$\begin{cases} f_1 u_1 = 1 \\ f_{k+1}u_{k+1} - \bar{\partial}u_k = 0, \quad 1 \le k \le M. \end{cases} \tag{10}$$

If $h_1, \dots, h_s$ are functions with $\mathcal{Z}$ as their common zero set, then the form u_k can be extended over $\mathcal{Z}$ to a current U_k defined as

$$U_k := \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon^2)u_k, \tag{11}$$

where $\chi(t)$ is a smooth function on the reals that is 0 for $t < 1$ and 1 for $t > 2$. Another and equivalent way to extend u_k is to consider the current valued function

$$\mathbb{C} \ni \lambda \mapsto u_\lambda := |h|^{2\lambda}u,$$

which is analytic for λ with a large real part. One can continue u_λ analytically to $\mathrm{Re}(\lambda) > -\epsilon$ for some small ϵ, and U is defined to be u_λ evaluated at $\lambda = 0$, [AW07].

The currents U_k satisfy the equations

$$\begin{cases} f_1 U_1 = 1 \\ f_{k+1}U_{k+1} - \bar{\partial}U_k = R_k, \end{cases} \tag{12}$$

for some currents R_k with support on $\mathcal{Z}$, cf. (10). Currents defined in such a way are called residue currents. If the tuple f defines a complete intersection, then the Koszul complex is a free resolution of J that ends at level $M = p$, where p is the codimension of $\mathcal{Z}$, and the residue R_p is the Coleff-Herrera product (7), see [AW07]. In this case $R_k = 0$ for $k \neq p$. Let R be the $E = \bigoplus E_k$-valued current $\sum R_k$. The following theorem is proved in [AW07] and is a direct generalization of Theorem 2.3.

Theorem 2.4 (Andersson, Wulcan)**.** *Let $\mathcal{J}$ be an ideal in $\mathcal{O}_0$ and let $\varphi \in \mathcal{O}_0$. Then $\varphi \in J$ if and only if φ annihilates the residue current R.*

2.3 Cohomological residues

We want to formulate and prove a result like Theorem 2.4 without using the desingularization theorem of Hironaka. Theorem 2.4 is based on a generalization of the second approach in Section 2. We now present a way to generalize the first approach and get the desired result.

If ϕ and ψ are $\bar{\partial}$-closed germs of smooth forms at the origin, then they are said to be in the same Dolbeault cohomology class if the difference $\phi - \psi$ is $\bar{\partial}$-exact, meaning that there exists a small neighborhood of the origin such that $\phi - \psi = \bar{\partial}\eta$ there for some smooth form η. We write $[\phi]_{\bar{\partial}}$ for the Dolbeault cohomology class of ϕ. We say that ϕ and ψ are in the same Dolbeault cohomology class outside of a germ of an analytic set $\mathcal{Z}$ if there exist a neighborhood U of the origin such that $\phi - \psi = \bar{\partial}\eta$ in $U \setminus \mathcal{Z}$ for some smooth form η.

Note that Stokes' theorem implies that the Grothendieck residue (6) only depends on the Dolbeault cohomology class of $\varphi f^* B$ outside of 0. It follows from Theorem 2.1 that if $[\varphi f^* B]_{\bar{\partial}} = 0$ outside of the origin, then $\varphi \in \langle f^1, \dots, f^n \rangle$. Moreover, it turns out that the reversed implication is true too and this is generalized in [Pas88] in the following way. Assume that the tuple $f = (f^1, \dots, f^p)$ defines a complete intersection and let $\mathcal{Z}$ be its zero set. Let D be a small neighborhood of the origin and denote by $\mathcal{D}_{\mathcal{Z}}$ the $\mathcal{O}$-module of smooth $(n, n-p)$-forms that are $\bar{\partial}$-closed near $\mathcal{Z}$ and with compact support in D. If D is small enough, then the residue

$$\mathrm{Res}_f^P(\psi) := \int f^* B \wedge \psi, \quad \psi \in \mathcal{D}_{\mathcal{Z}} \tag{13}$$

is defined and coincides with the Grothendieck residue when $\mathcal{Z}$ is a point. The residue Res_f^P satisfies the following result:

Theorem 2.5 (Passare). *Assume that the tuple $f = (f^1, \dots, f^p)$ defines a complete intersection and let $J = \langle f^1, \dots, f^p \rangle$. Then the following are equivalent:*

(i) $\varphi \in J$

(ii) $[\varphi f^* B]_{\bar{\partial}} = 0$ *outside of* $\mathcal{Z}$

(iii) φ *annihilates the residue* Res_f^P.

In the two-variable case when $f^1 = z$ and $f^2 = w$ it is easy to see this result directly. If a and b are holomorphic, then we see that

$$\bar{\partial} \frac{a\bar{w} - b\bar{z}}{(|z|^2 + |w|^2)} = (az + bw) B(z, w),$$

outside of the origin. This means that the Dolbeault cohomology class of φB vanishes outside of the origin if $\varphi \in \langle z, w \rangle$, the maximal ideal. If we now assume that the cohomology class of $\varphi B(z,w)$ vanishes outside of the origin for a holomorphic germ φ, then it follows that

$$0 = \int_{\partial D} \varphi B dz \wedge dw = \frac{\varphi(0)}{(n-1)!},$$

by the Bochner-Martinelli formula. This means that $\varphi \in \langle z, w \rangle$.

It is possible to generalize Theorem 2.5 to more complicated ideals. In Paper II we consider the case when J is a so-called Cohen-Macaulay ideal of codimension p. Then the resolution (8) ends at level $M = p$ and hence the form u_p defined in (9) is $\bar{\partial}$-closed by (10). We define the residue

$$\mathrm{Res}_J(\psi) := \int u_p \varphi \wedge \psi, \quad \psi \in \mathcal{D}_{\mathcal{Z}}$$

and prove that the following are equivalent:

(i) $\varphi \in J$

(ii) $[u_p \varphi]_{\bar{\partial}} = 0$ outside of $\mathcal{Z}$

(iii) φ annihilates the residue Res_J.

Note that in general φu_p is vector valued. The residue Res_J coincides with Res_f^P when J is defined by the tuple $f = (f^1, \ldots, f^m)$ and f defines a complete intersection. This generalization of Theorem 2.5 has already been proved in [AW07] but the point is that the proof in Paper II is more elementary and avoids the theorem of Hironaka used in [AW07].

Example 2.6. Let $J = \langle z_1 z_2, z_1 z_3, z_2 z_3 \rangle$. Then J is Cohen-Macaulay and has codimension 2. The complex

$$0 \to \mathcal{O}_0^2 \xrightarrow{g} \mathcal{O}_0^3 \xrightarrow{f} \mathcal{O}_0,$$

where

$$g = \begin{pmatrix} -z_3 & 0 \\ z_2 & -z_2 \\ 0 & z_1 \end{pmatrix}, \quad f = \begin{pmatrix} z_1 z_2 & z_1 z_3 & z_2 z_3 \end{pmatrix},$$

is exact outside of the common zero set of $z_1 z_2, z_1 z_3$, and $z_2 z_3$. A simple calculation shows that

$$\sigma_1 = h(z) \begin{pmatrix} \bar{z}_1 \bar{z}_2 \\ \bar{z}_1 \bar{z}_3 \\ \bar{z}_2 \bar{z}_3 \end{pmatrix}$$

and

$$\sigma_2 = h(z) \begin{pmatrix} |z_1|^2 + |z_2|^2 & |z_2|^2 \\ |z_2|^2 & |z_2|^2|z_3|^2 \end{pmatrix} \begin{pmatrix} -\bar{z}_3 & \bar{z}_2 & 0 \\ 0 & -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \qquad (14)$$

where $h(z) = 1/(|z_1 z_2|^2 + |z_1 z_3|^2 + |z_2 z_3|^2)$. We note that the matrix at the right in (14) multiplied with $\bar{\partial} h(z)$ is zero, and hence we get

$$u_2 = h^2(z) \begin{pmatrix} |z_1|^2 + |z_2|^2 & |z_2|^2 \\ |z_2|^2 & |z_2|^2 + |z_3|^2 \end{pmatrix} \begin{pmatrix} -\bar{z}_3 & \bar{z}_2 & 0 \\ 0 & -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \bar{z}_2 \\ d\bar{z}_1 \bar{z}_3 \\ d\bar{z}_2 \bar{z}_3 \end{pmatrix},$$

which is equal to

$$h^2(z) \begin{pmatrix} -\bar{z}_2\bar{z}_3|z_2|^2 d\bar{z}_1 - |z_1|^2 \bar{z}_1\bar{z}_3 d\bar{z}_2 + \bar{z}_1\bar{z}_2(|z_1|^2 + |z_2|^2) dz_3 \\ -\bar{z}_2\bar{z}_3(|z_2|^2 + |z_3|^2) d\bar{z}_1 - \bar{z}_1\bar{z}_3|z_3|^2 d\bar{z}_2 + \bar{z}_1\bar{z}_2|z_2|^2 d\bar{z}_3 \end{pmatrix}.$$

In Paper III the result is generalized to ideals of pure dimension. The difficulties in this case arise from the fact that $u_p \varphi$ does not need to be $\bar{\partial}$-closed. It turns out that this problem is circumvented if we multiply with certain holomorphic sections of the dual bundle of E_p.

3 The effective membership problem for polynomial ideals

Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$ and assume that their common zero set is empty. Then, by a version of the Nullstellensatz, we can write

$$1 = \sum Q_j F_j$$

for some polynomials Q_j. We are interested in degree estimates of $Q_j F_j$. Using methods of Hermann from [Her26] it is proved by Masser and Wüstholz in [MW83] that if $\deg F_j \leq d$, then one can choose the polynomials Q_j such that

$$\deg(Q_j F_j) \leq 2(2d)^{2^{n-1}} + d,$$

which is a double exponential bound in the degree of the F_j:s. However, this estimate can be substantially improved and the first breakthrough was done by Brownawell in [Bro87], where he proves that one can choose the polynomials Q_j such that

$$\deg(Q_j F_j) \leq \min(m,n)(nd^{\min(m,n)} + d) + d.$$

Already the year after Brownawell's result Kollár proved the sharpened estimate

$$\deg(Q_j F_j) \le d^{\min(m,n)}, \quad d \ge 3,$$

see [Kol88]. This estimate is sharp. The case when $d < 3$ was later done by Sombra in [Som99] and Jelonek in [Jel05].

If the common zero set of the F_j:s is non-empty and the polynomial Φ belongs to $J(F)$, the ideal generated by the F_j:s, then we may regard the more general equation

$$\Phi = \sum Q_j F_j. \tag{15}$$

The methods of Hermann imply that there exist polynomials Q_j such that (15) holds and where the degree of $Q_j F_j$ is double exponential. It is proved in [MM82] that this cannot be substantially improved, meaning that one can find examples where one really need something double exponential. On the other hand, with further assumptions on the zeros of Φ it may be possible to get a much better bound. For example, the theorem of Briançon-Skoda implies that if

$$|\Phi| \le C|F|^{\min(m,n)}, \tag{16}$$

then $\Phi \in J(F)$. This exponent cannot be improved. For n=2 this is seen by the example where

$$|xy| \le |x^2| + |y^2|, \quad \text{but} \quad xy \notin \langle x^2, y^2 \rangle.$$

A similar example can be made for any n. Now, if Φ belongs to $J(F)$ due to (16), then it is proved in [Hic01] that one can choose Q_j such that

$$\deg(Q_j F_j) \le \max\left(\deg \Phi + \min(m, n+1) d^{\min(m,n)}, (n+1)d - n\right). \tag{17}$$

When attacking these kind of problems it is standard first to homogenize the polynomials to $\mathbb{C}^{n+1}$. We may then regard them as sections of the line bundle $\mathcal{O}(d)$ on the projective space $\mathbb{P}^n$ and this opens up for analytic and geometric methods. In [And06] this is mixed with residue calculus to provide a general set-up to obtain these kind of membership results. With this approach and with idéas from [Hic01] and [EL99] one can get a slightly sharper estimate than (17), see [AG11].

Remark 3.1. It is possible to replace $d^{\min(m,n)}$ in (17) by d^{c_∞}, where c_∞ is a number that depends on the complexity of the common zero set of the F_j:s at infinity. In particular, $c_\infty \le \min(m,n)$. In certain nice situations $c_\infty = -\infty$ so that $d^{c_\infty} = 0$. See [EL99] and [Hic01] for more about this.

Related to these kind of theorems is the uniform Artin-Rees lemma, see [Hun92], that implies that if J is a polynomial ideal, then there exists a constant μ such that

$$I^{\mu+r} \cap J \subset I^r J$$

for every polynomial ideal I and every integer r. In view of the theorem of Briançon-Skoda we may reformulate this result:

Theorem 3.2 (Uniform Artin-Rees lemma for polynomials). *If $F_1, \dots, F_m$ are polynomials in $\mathbb{C}^n$, then there exists a constant μ such that the following holds: If $\Phi, G_1, \dots, G_\ell$ are polynomials, $r \geq 1$,*

$$\Phi \leq C|G|^{\mu+r-1},$$

and

$$\Phi \in J(F),$$

then we can write

$$\Phi = \sum_{\substack{j=1,\dots,m \\ I_1+\dots+I_\ell=r}} Q_{I,j} G_1^{I_1} \dots G_\ell^{I_\ell} F_j$$

for some polynomials $Q_{I,j}$.

In Paper IV we use a refinement of the residue calculus set-up mentioned above and an algebraic construction in [Szn11] to prove an effective version of Theorem 3.2, and we get the bound

$$\deg(Q_{I,j} G_1^{I_1} \dots G_\ell^{I_\ell} F_j) \leq \\ \max\Big((\mu+r-1)d^{\min(\ell,n)} + \deg\Phi, (n+r)d + \kappa_1, \deg\Phi + \kappa_2\Big),$$

where the constants κ_1 and κ_2 only depend on $J(F)$. In some cases the constants μ, κ_1, and κ_2 can all be explicitly calculated. For example, if $r = m = 1$ and $F_1 = 1$, then $\mu = \min(\ell, n), \kappa_1 = -n$, and $\kappa_2 = 0$. This means that we get back the effective Briançon-Skoda theorem in [AG11]. If $J(G) = (1)$, then the result of Mayr and Mayer in [MM82] imply that one in general need something double exponential. This is captured by the constant κ_2.

The effective uniform Artin-Rees lemma in Paper IV is a bit more general. There the polynomials only need to be considered on a smooth subvariety in $\mathbb{C}^n$. There is also a more abstract geometric version of our effective result, where we consider holomorphic sections of ample line bundles over smooth projective varieties. In this case the special case when $r = m = 1$ and $F_1 = 1$ corresponds to an abstract version of the theorem of Briançon-Skoda proved in [EL99].

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