# Free loop spaces, Koszul duality and A-infinity algebras


## Kaj Börjeson


Academic dissertation for the Degree of Doctor of Philosophy in Mathematics at Stockholm University to be publicly defended on Friday 22 September 2017 at 13.00 in sal 14, hus 5, Kräftriket, Roslagsvägen 101.


### Abstract

This thesis consists of four papers on the topics of free loop spaces, Koszul duality and $A_\infty$-algebras.

In Paper I we consider a definition of differential operators for noncommutative algebras. This definition is inspired by the connections between differential operators of commutative algebras, $L_\infty$-algebras and BV-algebras. We show that the definition is reasonable by establishing results that are analoguous to results in the commutative case. As a by-product of this definition we also obtain definitions for noncommutative versions of Gerstenhaber and BV-algebras.

In Paper II we calculate the free loop space homology of (n-1)-connected manifolds of dimension of at least 3n-2. The Chas-Sullivan loop product and the loop bracket are calculated. Over a field of characteristic zero the BV-operator is determined as well. Explicit expressions for the Betti numbers are also established, showing that they grow exponentially.

In Paper III we restrict our coefficients to a field of characteristic 2. We study the Dyer-Lashof operations that exist on free loop space homology in this case. Explicit calculations are carried out for manifolds that are connected sums of products of spheres.

In Paper IV we extend the Koszul duality methods used in Paper II by incorporating $A_\infty$-algebras and $A_\infty$-coalgebras. This extension of Koszul duality enables us to compute free loop space homology of manifolds that are not necessarily formal and coformal. As an example we carry out the computations for a non-formal simply connected 7-manifold.



**Keywords:** *Koszul duality, free loop spaces, a-infinity algebras, BV-algebras, string topology.*

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# Sammanfattning


Denna avhandling består av fyra artiklar inom ämnena fria öglerum, Koszuldualitet och $A_\infty$-algebror.

I Artikel I behandlar vi en definition av differentialoperatorer för ickekommutativa algebror. Denna definition är inspirerad av kopplingar mellan differentialoperatorer för kommutativa algebror, $L_\infty$-algebror och BV-algebror. Vi visar att definitionen är rimlig genom att etablera resultat som är analoga med resultat i det kommutativa fallet. Som en biprodukt får vi också definitioner för ickekommutativa varianter av Gerstenhaber och BV-algebror.

I Artikel II beräknar vi den fria öglerumshomologin av $(n-1)$-sammanhängande mångfalder av dimension minst $3n-2$. Chas-Sullivans ögleprodukt och öglehake beräknas. Över en kropp av karakteristik noll beräknas även BV-operatorn. Explicita uttryck för Bettitalen fastställs också, vilka visar att de växer exponentiellt.

I Artikel III begränsar vi koefficienterna till en kropp av karakteristik 2. Vi studerar Dyer-Lashofoperationer som existerar på den fria öglerumshomologin i detta fall. Explicita beräkningar görs för mångfalder som är sammanhängande summor av produkter av sfärer.

I Artikel IV utvidgar vi Koszuldualitetmetoden som används i Artikel II genom att inkorporera $A_\infty$-algebror och $A_\infty$-koalgebror. Denna utvidgning av Koszuldualitet gör det möjligt att beräkna fri öglerumshomologi för mångfalder som inte nödvändigtvis är formella och koformella. Som ett exempel utför vi beräkningar för en ickeformell enkelt sammanhängande 7-mångfald.

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Finally, I am deeply thankful to my parents, Lennart and Thelma and my girlfriend Helena.

# List of papers

The following four papers are included in this thesis and will be referred to by their Roman numerals.

PAPER I: **$A_\infty$-algebras derived from associative algebras with a non-derivation differential**
K. Börjeson
*J. Generalized Lie Theory and Applications*, 9:214 (2015)

PAPER II: **Free loop space homology of highly connected manifolds**
A. Berglund, K. Börjeson
*Forum Mathematicum*, **29**, page 201–228 (2017)

PAPER III: **Restricted Gerstenhaber algebra structure on the free loop homology of $(S^n \times S^n)^{\#m}$**
K. Börjeson

PAPER IV: **Koszul $A_\infty$-algebras and free loop space homology**
A. Berglund, K. Börjeson

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# Contents

## Introduction

This thesis contains four papers, the first paper studies differential operators of noncommutative algebras and the remaining three study free loop space homology of manifolds.

A notion of a differential operator on a noncommutative algebra is introduced in Paper I. It is an analogue of A. Grothendieck's notion of differential operator for any commutative algebra. The idea came from considering the following problem in homotopical algebra: If a graded vector space is both a graded commutative algebra and a chain complex there is an $L_\infty$- algebra structure that measures the incompatibility of the two structures [Akm97]. The operations of the structure can be straightforwardly interpreted as being zero if the differential is a differential operator of a certain order. In [Ber07] it is asked if this result could be generalized to noncommutative algebras. There it is also proved that such an $L_\infty$-structure cannot be found in general in this case. In Paper I, we instead define an $A_\infty$-structure that satisfies analogous properties. This structure has many properties similar to the commutative case, among other things a notion of differential operator for noncommutive algebras comes for free. It also leads to noncommutative analogues of Gerstenhaber and BV-algebras.

The article [Bör15] was uploaded to the arXiv e-print server in April 2013. To my pleasant surprise, several articles have since built upon the results. M. Markl generalized the construction which he calls Börjeson's braces in [Mar15a] and put it in the context of formal noncommutative geometry in [Mar15b]. V. Dotsenko, S. Shadrin and B. Vallette puts what they call Börjeson products in a geometric and operadic context with strong analogies to the algebraic geometry of the moduli spaces of curves in [DSV15].

The topic of Papers II to IV is free loop space homology. Free loop space homology is important for applications to the study of geodesics on Riemannian manifolds [GM69]. For example, one can establish bounds on the number of geodesics from the Betti numbers. When the underlying space is a manifold the free loop space homology carries extra algebraic structure, the study of which is the subject of string topology [CS99]. Unfortunately, free loop space homology is notoriously hard to compute in general so there are not many fully calculated examples. We calculate free loop space homology using Koszul duality and Hochschild cohomology as our main tools. In Paper II we study the free loop space homology of spaces that are formal and coformal, in particular we study highly connected manifolds. In Paper III we study Dyer-Lashof operations in the free loop space homology of connected sums of products of spheres. In Paper IV we extend the methods of Paper II to spaces that are either formal or coformal. This involves developing Koszul duality for $A_\infty$-(co)algebras.

This thesis has several topics and concepts that appear multiple times. Here we give a brief and informal introduction to some of the often occurring concepts.

**Gerstenhaber and BV-Algebras.** A *Gerstenhaber algebra* is a graded commutative algebra together with a Lie bracket of degree +1 such that

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c].$$

In many examples Gerstenhaber algebras come with an extra operation. A *BV-algebra* is a Gerstenhaber algebra together with a square-zero operator $\Delta$ satisfying the identity

$$[a, b] = \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b).$$

Alternatively, a BV-algebra is a graded commutative algebra together with a square-zero operator $\Delta$ satisfying

$$\Delta(abc) - \Delta(ab)c - (-1)^{|a|}a\Delta(bc) - (-1)^{(|a|+1)|b|}b\Delta(ac)+$$
$$\Delta(a)bc + (-1)^{|a|}a\Delta(b)c + (-1)^{|a|+|b|}ab\Delta(c) = 0.$$

The second definition has the advantage of never mentioning a Lie bracket. However, by letting

$$[a, b] := \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b),$$

we can recover the Lie bracket. Three of the papers in this thesis deal with Gerstenhaber algebras in some way. Paper II and III calculate the Gerstenhaber algebra structure on free loop space homology and Paper I defines alternative versions of Gerstenhaber and BV-algebras that do not require commutativity in the definition. This requires changing the definition considerably but gives analoguous results.

**Loop spaces and string topology.** The last three papers all deal with loop spaces of manifolds. Fix a manifold $M$ and consider the set of all closed loops in $M$. That is, we consider all continuous functions from the circle $S^1$ into our manifold $M$. Since $M$ is a topological space, this set can be made into a topological space as well. This is called the *free loop space* of $M$ and we denote it $LM$. The homology $H_*(LM)$ of this space is a topological invariant of $M$. In [CS99] M. Chas and D. Sullivan realized that one could use the fact that $M$ is a manifold and not just any topological space to equip $H_*(LM)$ with extra algebraic structure. In particular, this turns $H_*(LM)$ into a BV-algebra. This structure and related algebraic structures on $H_*(LM)$ go under the name of *string topology.*

An important subspace of $LM$ is the *based loop space* $\Omega M$. To the define $\Omega M$ we need to fix a base point in $M$. Then, $\Omega M$ is the space of all loops that begin and end at the base point. The choice of base point is often suppressed from the notation since all choices give homeomorphic results in the case when $M$ is connected. The homology of $\Omega M$ is usually much simpler to calculate than the homology of the free loop space. Since the loops begin and end at the same point we can concatenate them to obtain new loops. If one is careful with how one parametrizes loops, this gives $\Omega M$ a strictly associative multiplication. Taking the singular chains gives us the strictly associative differential graded algebra $C_*(\Omega M)$ and taking homology gives us the algebra $H_*(\Omega M)$. These are both important ingredients in our calculations of $H_*(LM)$.

**Hochschild cohomology.** Hochschild cohomology is a cohomology theory for algebras introduced in [Hoc45] by G. Hochschild. Gerstenhaber algebras were introduced in [Ger63] by M. Gerstenhaber with Hochschild cohomology as the original example motivated by the study of deformations of algebras. The Hochschild cochain complex of a graded algebra $A$ is the cochain complex

$$C^*(A,A): \quad \mathrm{Hom}(sA,A) \xrightarrow{d} \mathrm{Hom}((sA)^{\otimes 2},A) \xrightarrow{d} \mathrm{Hom}((sA)^{\otimes 3},A) \to \cdots,$$

where $s$ denotes a degree shift and the coboundary operator is given by

$$\begin{aligned} d(f)(sx_1,\dots,sx_{k+1}) =& x_1 f(sx_2,\dots,sx_{k+1}) \\ &+ \sum_{i=1}^{k} \pm f(sx_1,\dots,sx_i x_{i+1},\dots,sx_{k+1}) \\ &\pm f(sx_1,\dots,sx_k)x_{k+1}. \end{aligned}$$

Both the free loop space homology $H_*(LM)$ and Hochschild cohomology of an algebra have the structure of a Gerstenhaber algebra. It is in fact possible to compute $H_*(LM)$ as the Hochschild cohomology of the algebra $C_*(\Omega M)$, which is something that we use a lot. However, this gives us a complex that is in general very large, which makes it hard to calculate anything with. A large part of this thesis is devoted to the idea that we can make this complex much smaller for sufficiently nice manifolds.

**$A_\infty$-algebras.** A differential graded algebra $A$ is a chain complex that is at the same time an associative algebra in a compatible way. For example, consider the algebra $C_*(\Omega M)$ mentioned above. The compatibility means that the multiplication in $A$ induces a multiplication in $H_*(A)$ by multiplying representatives. Unfortunately, it is not in general possible to recover the original algebra $A$ from $H_*(A)$ with only this multiplication, not even up to some appropriate notion of homotopy. However, we can treat $H_*(A)$ as an $A_\infty$-algebra instead. This gives $H_*(A)$ the extra structure needed to recover $A$ up to homotopy.

**Definition.** An $A_\infty$-algebra structure on a graded vector space $A$ consists of a collection of maps $m_k\colon A^{\otimes k} \to A$ of degree $k-2$ such that the following identity is satisfied for every $k \geq 1$:

$$\sum (-1)^{r+st} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0.$$

The sum is over all $r, s, t$ such that $r + s + t = k$.

In the above definition, if the maps $m_k$ are all zero for $k \geq 3$ it defines a differential graded algebra where $m_1$ is the differential and $m_2$ is the multiplication. Thus $A_\infty$-algebras generalize differential graded algebras. There are explicit formulas for the $A_\infty$-structure on $H_*(A)$. The $A_\infty$-algebra $H_*(A)$ is often much smaller than $A$ and for many purposes we can use $H_*(A)$ instead of $A$. In some nice cases the maps $m_k$ vanish for all $k \geq 3$ in the $A_\infty$-structure on $H_*(A)$. In this case the structure is actually the same as the induced algebra structure on $H_*(A)$. This is a very important type of differential graded algebra called *formal*.

Many other algebraic structures can be generalized in a similar way, for example, there are $L_\infty$-algebras that generalize Lie algebras.

**Koszul duality.** Koszul duality was originally introduced as a duality theory of quadratic algebras with particularly nice homological properties in [Pri70]. For our purposes it is convenient to formulate it as a duality theory of graded quadratic algebras and coalgebras. A quadratic algebra $A$ is obtained by specifying a graded vector space of generators $V$ and a space of relations $R \subseteq V \otimes V$ and defining

$$A = T(V)/\langle R \rangle,$$

the tensor algebra of $V$ modulo the relations generated by $R$. From the same $V$ and $R$ we can also look at

$$C = \Bbbk \oplus sV \oplus \cdots \oplus \left( \bigcap_{i+2+j=k} (sV)^{\otimes i} \otimes s^2 R \otimes (sV)^{\otimes j} \right) \oplus \ldots,$$

where $s$ denotes a shift in degree. Now, $C$ sits inside the bar construction $BA$ as a sub-coalgebra. In case $BA$ is quasi-isomorphic to $C$ we say that $A$ and $C$ are *Koszul* (co)algebras and that they are *Koszul dual* to each other. This definition is symmetric in the sense that we could equivalently ask that the cobar construction of $C$ is quasi-isomorphic to $A$. Koszul duality is very helpful in many situations since $C$ is much smaller than $BA$ but have the same homological properties. Topological spaces are great sources of Koszul (co)algebras. A space $X$ is *formal* if $C_*(X)$ is a formal coalgebra, that is, if it is quasi-isomorphic to $H_*(X)$. Similarly, $X$ is *coformal* if $C_*(\Omega X)$ is a formal algebra, that is, it is quasi-isomorphic to $H_*(\Omega X)$. If a space is both formal and coformal, $H_*(X)$ and $H_*(\Omega X)$ are Koszul (co)algebras and they are Koszul dual to each other.

## Paper I: $A_\infty$-algebras derived from associative algebras with a non-derivation differential

This paper has already been published in Generalized Lie theory and Applications and also appears in the author's Licentiate thesis.

This paper deals with the question of defining differential operators for algebras that are not commutative. The motivation is by analogy to the commutative case and the study of generalized BV-algebras.

Since the paper appeared on the arXiv in 2013 there has been progress following up on this [BV14, DSV15, Mar15a, Mar15b]. We will describe some background motivation for the paper as well as describe how these newer results relate to the article.

In [Akm97] the definition of a BV-algebra is generalized as follows.

**Definition.** A *generalized BV algebra* is a graded commutative algebra together with a square-zero operator $\Delta$ of degree $+1$.

In a non-generalized BV-algebra we require the identity

$$\Delta(abc) - \Delta(ab)c - (-1)^{|a|}a\Delta(bc) - (-1)^{(|a|+1)|b|}b\Delta(ac)$$
$$+\Delta(a)bc + (-1)^{|a|}a\Delta(b)c + (-1)^{|a|+|b|}ab\Delta(c) = 0$$

to hold. Without this identity the expression $\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)$ is not necessarily a degree $+1$ Lie bracket anymore. However, it turns out to have an $L_\infty$-structure.

**Theorem.** [Akm97] *Any generalized BV-algebra is also equipped with a degree +1 $L_\infty$-structure given by*

$$\ell_n(a_1, \dots, a_n) = [[[\Delta, L_{a_1}], \dots], L_{a_n}](1)$$

*where $L_{a_i}$ is left multiplication with elements $a_i$, $[\cdot,\cdot]$ is the graded commutator of operators and we let $\Delta(1) = 0$.*

Differential graded algebras and ordinary BV-algebras are a subset of generalized BV-algebras. They are characterized by the following proposition.

**Proposition.** *The pair $(A, \Delta)$ is a dg algebra, that is, $\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = 0$, if and only if the maps $\ell_n$ vanish for all $n \geq 2$. Similarly, the maps $\ell_n$ vanish for all $n \geq 3$ exactly when the generalized BV-algebra is an ordinary BV-algebra.*

The higher maps $\ell_n$ have a similar interpretation as measuring how much $\Delta$ fails to be a differential operator of a certain order.

Now one may ask if there are analogous constructions for other types of algebras $A$ together with a square-zero degree $+1$ operator.

The first natural thing to consider is to drop the commutativity assumption and consider a graded associative algebra that is not necessarily commutative with a square-zero operator $\Delta$. If one tries to define an $L_\infty$-structure with the same operations it fails immediately since the definition relies on being able to permute elements and only change the result by a sign. Even if one tries a more symmetrical definition it does not work as smoothly, there are results saying that obtaining a reasonable $L_\infty$-structure in this way is impossible (see [Ber07] for a precise statement and [Man15] for more discussion). With a view toward Koszul duality of operads, one can guess that a natural thing to aim for would be an $A_\infty$-structure instead of an $L_\infty$-structure (since the commutative operad is dual to the Lie operad and the associative operad is self-dual). This turns out to work and the main results of the paper can be summarized as follows.

**Theorem.** [Bör15] *Consider a graded associative algebra equipped with a square-zero degree* $+1$ *operator* $\Delta$. *Let*

$$m_1(a) = \Delta(a),\ m_2(a,b) = \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)$$

*and for* $n \geq 3$ :

$$m_n(a_1,\ldots,m_n) = \Delta(a_1a_2\ldots a_{n-1}a_n) - \Delta(a_1a_2\ldots a_{n-1})a_n$$
$$-(-1)^{|a_1|}a_1\Delta(a_2\ldots a_{n-1}a_n) + (-1)^{|a_1|}a_1\Delta(a_2\ldots a_{n-1})a_n.$$

*These maps define a degree +1* $A_\infty$*-structure, which shares the following properties with the degree +1* $L_\infty$*-structure constructed for a generalized BV-algebra.*

(1) *It is natural, that is, built using only multiplication, composition and the operator* $\Delta$.

(2) *It is defined over the integers.*

(3) *It is hereditary, that is,* $m_i = 0$ *implies* $m_j = 0$ *if* $j > i$.

(4) *The operation* $m_2$ *measures the deviation of* $\Delta$ *from being a derivation, i.e.*

$$m_2(a,b) = \Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b).$$

(5) *The coefficient in front of* $\Delta(a_1\ldots a_n)$ *in* $m_n(a_1,\ldots,a_n)$ *is* $\pm 1$.

This theorem leaves open the question whether this structure is unique. This was answered affirmatively in the paper [Mar15a]. This structure is unique up to strict isomorphism. Markl also provides an alternative construction of the $A_\infty$-structure using twistings of $A_\infty$-structures.

**Definition.** Let $T^c(A)$ denote the coalgebra $\bigoplus_{n\geq 1} A^{\otimes n}$ with comultiplication given by deconcatenation. Let $\phi : T^c(A) \to T^c(A)$ be an automorphism and denote the components $\phi_k : A^{\otimes n} \to A$. Let $(A, m_1, \dots)$ be a degree $+1$ $A_\infty$-algebra. Then the degree $+1$ $A_\infty$-algebra given by the maps

$$m_n^\phi = \sum_{\substack{s,t,j \\ i_1+\dots i_{s+t-1}=n}} (\phi^{-1})_s \circ (id^j \otimes m_t \otimes id^{s-j-1}) \circ (\phi_{i_1}, \dots, \phi_{i_{s+t-1}})$$

is called *twisted* by $\phi$.

**Theorem.** [Mar15a] *The $A_\infty$-structure defined in* [Bör15] *can be constructed by taking the $A_\infty$-structure given by $m_1 = \Delta, m_{\geq 2} = 0$ and then twisting it with the automorphism where $\phi_k : A^k \to A$ is given by k-fold multiplication.*

Noncommutative $BV$-algebras and noncommutative Gerstenhaber algebras come up naturally, as shown by the following examples.

**Example.** The bar construction of an associative algebra is always a differential graded coalgebra. It also has an algebra structure since it is isomorphic to the tensor algebra. However, with this algebra structure the differential is not a derivation, but it gives us a noncommutative BV-algebra in the above sense. Alternatively, this can be seen as saying that the bar differential is a noncommutative differential operator of order 2. Similarly, the bar construction of an $A_\infty$-algebra gets an $A_\infty$-structure. One may look at this as having the structure of a non-commutative analogue of $BV_\infty$-algebras (where $BV_\infty$-algebras are a homotopical weakening of the notion of BV-algebras). One may ask if this construction embeds the category of $A_\infty$-algebras in the category of noncommutative $BV_\infty$-algebras. In [BV14] the analogous question for $L_\infty$-algebras and $BV_\infty$-algebras is explored. They prove that $L_\infty$-algebras gives a subcategory of $BV_\infty$-algebras such that the commutative structure is free. One may ask whether the analogous result holds for $A_\infty$-algebras considered as noncommutative $BV_\infty$-algebras or more generally for any homotopy algebra over any Koszul operad.

**Example.** In [GS12] the authors develop a noncommutative deformation theory of associative algebras. They consider a deformation complex given by the Hochschild complex of $A$ with coefficients in the tensor algebra $T(A)$. There is the Hochschild differential $b$ and a cup product

$$\smile: C^p(A, A^{\otimes k}) \otimes C^q(A, A^{\otimes m})) \to C^{p+q}(A, A^{\otimes(k+m-1)})$$

induced by the multiplication in the tensor product. There's also a composition operation

$$\vdash: C^p(A, A^{\otimes k}) \otimes C^q(A, A^{\otimes m})) \to C^{p+q-1}(A, A^{\otimes(k+m-1)})$$

where

$$(f \vdash g)(a_1, \dots, a_\ell) = \sum \pm a_1 \otimes \cdots \otimes f(a_i, \dots, a_{i+p}) \otimes \cdots \otimes a_{p+q-1}.$$

With the above operations $(C^*(A, T(A))$ is a noncommutative BV-algebra, where $b$ takes the role of the noncommutative BV-operator.

In [DSV15], V. Dotsenko, S. Shadrin and B. Vallette considers these parallel constructions in an algebraic geometric context. Classically, there is a strong connection between BV-algebras and the moduli space of genus zero algebraic curves. The homology of the operad of framed little discs describe BV-algebras and a homotopy quotient of the same operad gives a compactification of the moduli space of genus zero curves. With inspiration from [Bör15] and [GS12] they generalize this picture to a noncommutative setting with the noncommutative BV-algebras instead of the classical BV-algebras. To this end they define a new noncommutative replacement for the moduli space of curves. Both the classical and the noncommutative settings features operads constructed via de Concini-Procesi wonderful models of hyperplane arrangements.

## Paper II: Free Loop Space Homology of Highly Connected Manifolds

The second paper is joint with A. Berglund. It has already been published in Forum Mathematicum and also appears in the author's Licentiate thesis.

We study the free loop space homology of $(n-1)$-connected manifolds of dimension at most $3n-2$. We compute the Gerstenhaber algebra structure and for coefficients in a field of characteristic zero we also describe the BV-algebra structure. It turns out that the structure can be completely described in terms of the intersection form of the manifold.

The free loop space homology of highly connected closed 4-manifolds has been studied in [BB13], but the methods used there do not extend to higher dimensions and thus cannot be used in our case. The free loop space homology of $(n-1)$-connected $2n$-dimensional manifolds has also been studied in [BS12] using different methods, but the calculations there are not complete.

Our approach to compute the free loop space homology is to compute the Hochschild cohomology for the dg algebra $C_*(\Omega M;\Bbbk)$, where $\Omega M$ is the loop space of $M$. For coefficients in a field of characteristic zero this Hochschild cohomology is a $BV$-algebra that is isomorphic to the Chas-Sullivan BV-algebra [FT08, Tra08]. For fields of characteristic $p$, one needs to be more careful in the light of [Men09], so in this case we only determine the Gerstenhaber algebra structure, where we rely on [FMT05, Mal10] for the isomorphism between Hochschild cohomology and string topology.

Our method for computing Hochschild cohomology relies on the fact that the manifolds $M$ we consider are both *formal* and *coformal* over the coefficient field $\Bbbk$, that is, $C_*(M;\Bbbk)$ is weakly equivalent to $H_*(M;\Bbbk)$ and

$C_*(\Omega M;\Bbbk)$ is weakly equivalent to $H_*(\Omega M;\Bbbk)$. This means that by the results in [Ber14] we can connect this to the study of Koszul algebras. For a formal and coformal manifold the coalgebra $H_*(M;\Bbbk)$ a Koszul coalgebra and the algebra $H_*(\Omega M;\Bbbk)$ is its Koszul dual. In addition to highly connected manifolds, over $\mathbb{Q}$ many spaces are formal and coformal. They include for example suspensions, loop spaces and (ordered) configuration spaces of points in $\mathbb{R}^n$.

For any such formal and coformal space we can write down a small complex that computes free loop space homology. We need the notion of twisted convolution algebra to formulate the result.

**Definition.** Let $C$ be a differential graded coalgebra and let $A$ be a differential graded algebra. The *convolution algebra* $Hom(C, A)$ is a differential graded algebra where the underlying module consists of the graded module homomorphisms $Hom(C, A)$. The multiplication is given by

$$f \star g = \mu_A \circ (f \otimes g) \circ \Delta_C,$$

where $\Delta_C$ is the comultiplication of $C$ and $\mu_A$ is the multiplication of $A$. The differential is given by

$$\partial(f) = d_A \circ f - (-1)^f f \circ d_C.$$

A *twisting morphism* is an element $\tau \in Hom(C, A)$ of degree $-1$ such that

$$\partial(\tau) + \tau \star \tau = 0.$$

Given such a twisting morphism $\tau$, the *twisted convolution algebra* is the differential graded algebra

$$\mathrm{Hom}^\tau(C, A) = \big(\mathrm{Hom}(C, A), \star, \partial^\tau\big),$$

with differential $\partial^\tau = \partial + [\tau, -]$, where

$$\partial(f) = d_A \circ f - (-1)^f f \circ d_C, \quad [\tau, f] = \tau \star f - (-1)^{|f|} f \star \tau.$$

The main calculational tool can then be formulated as follows.

**Theorem.** *Let $\Bbbk$ be a field and let $X$ be a simply connected space of finite $\Bbbk$-type such that $X$ is both formal and coformal over $\Bbbk$. Then $H_*(X;\Bbbk)$ and $H_*(\Omega X;\Bbbk)$ are Koszul dual and the twisted convolution algebra*

$$Hom^\tau(H_*(X;\Bbbk), H_*(\Omega X;\Bbbk))$$

*is quasi-isomorphic to the Hochschild cochains of $C_*(\Omega X;\Bbbk)$ as a differential graded algebra.*

By working with this smaller complex we can make explicit calculations for the ranks of the homology groups $H_{*+d}(LM;\Bbbk)$. The Betti numbers $H_{*+d}(LM;\Bbbk)$ for different fields $\Bbbk$ is of geometrical interest since there is a close relationship between the ranks of these homology groups and the number of closed geodesics on a manifold with a generic Riemannian metric (technically called a bumpy metric). The relation is established using an approach by D. Gromoll and W. Meyer [GM69]. For example one can prove the following.

**Proposition.** [BZ82] *Suppose that $g$ is a bumpy Riemannian metric on a simply connected closed manifold $M$. Then there exists $\alpha > 0$ and $\beta > 0$ such that the number of geometrically distinct closed geodesics of length less than $T$ is greater than*

$$\alpha \max_{n \leq \beta T} rank(H_n(LM; R))$$

*for any principal ideal domain $R$ and for $T$ large enough.*

M. Gromov have conjectured that exponential growth of the number of closed geodesics holds for 'most' manifolds [Gro78]. For the case of rational coefficients, exponential growth of Betti numbers has been proved for all rationally hyperbolic coformal manifolds in [Lam01b]. Together with the results in [NM78] this implies exponential growth of the number of closed geodesics for $(n-1)$-connected manifolds of dimension at most $3n-2$ (with the extra condition that $\dim H^*(M; \mathbb{Q}) > 4$). For an arbitrary coefficient field, exponential growth of Betti numbers has been proved for all manifolds that can be written as a connected sum $M_1 \# M_2$ where $H^*(M_1; \Bbbk)$ is not generated by a single element and $H^*(M_2; \Bbbk)$ does not have the same homology as a sphere (see [Lam01a]).

With our methods we prove the following theorem.

**Theorem.** *Let $\Bbbk$ be any field and let $M$ be an $(n-1)$-connected closed manifold of dimension at most $3n-2$ ($n \geq 2$) such that $\dim H^*(M; \Bbbk) > 4$. Then the sequence $\dim(H_n(LM; \Bbbk))$ grows exponentially.*

Comparing with the results of Lambrechts, this gives new information for prime manifolds in positive characteristic.

*Remark.* After publication of the paper, we found out that one of the proofs were incomplete. An erratum is included after the paper to correct this fact.

## Paper III: Restricted Gerstenhaber algebra structure on the free loop homology of $(S^n \times S^n)^{\#m}$

In Paper II, we studied $(n-1)$-connected manifolds of dimension at most $3n-2$. We calculated the Gerstenhaber algebra structure on the Hochschild cohomology $HH^*(C_*(\Omega M; \Bbbk))$ by writing down a smaller complex given by $Hom^{\tau}(H_*(X; \Bbbk), H_*(\Omega X; \Bbbk))$. However, there is more structure on the Hochschild cohomology complex than what is needed to define the Gerstenhaber algebra structure. According to the solution of Deligne's conjecture (see [MS02]), the Hochschild cohomology complex is an $E_2$-algebra, that is, it is an algebra over the chains of the little disc operad. For coefficient fields of finite characteristic, Hochschild cohomology has extra operations called Dyer-Lashof operations. In particular, this means that the Gerstenhaber algebra structure on Hochschild cohomology is enhanced with an extra operation $Q$, called the restriction.

**Definition 1.1.** Let $G$ be a graded vector space over a field $\Bbbk$ of characteristic 2. A restricted Gerstenhaber algebra structure on $G$ consists of

(1) A graded linear map of degree 0: $\smile: G \otimes G \to G$.

(2) A graded linear map of degree -1: $[-,-] : G \otimes G \to G$.

(3) A function $Q : G_k \to G_{2k-1}$

satisfying

(1) $a \smile b = b \smile a$

(2) $(a \smile b) \smile c = a \smile (b \smile c)$

(3) $[a, b] = [b, a]$

(4) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

(5) $[ab, c] = [a, c] + [b, c]$

(6) $[Q(a), b] = [a, [a, b]]$

(7) $Q(a + b) = Q(a) + [a, b] + Q(b)$.

This structure has been computed in the context of string topology for spheres and projective spaces in [Wes05]. Our goal is to compute the restricted Gerstenhaber structure where the underlying manifold has more than one algebra generator in cohomology. We choose $(S^k \times S^k)^{\#m}$, that is, connected sums of products of spheres as our examples. These are examples of highly connected manifolds with a simple cohomology algebra structure. However, the method used in [BB16] cannot be applied directly to compute the restricted Gerstenhaber algebra structure since there is no explicit $E_2$-algebra structure on $Hom^\tau(H_*(X;\Bbbk), H_*(\Omega X;\Bbbk))$. Instead we use a different method. The idea is to make the retraction

$$h \circlearrowright C^*(H_*(\Omega M), H_*(\Omega M)) \underset{g}{\overset{f}{\rightleftarrows}} (Hom^\tau(H_*(M), H_*(\Omega M))$$

explicit so that we can lift any cycle to the Hochschild cohomology complex. There, $[-,-]$ and $Q$ are defined explicitly in terms of Gerstenhaber's pre-Lie structure. To build this explicit retract we use a PBW-basis for $H_*(\Omega X)$ and repeated applications of the perturbation lemma. These computations show that it is feasible to work with these retracts explicitly by choosing good bases.

## Paper IV: Koszul $A_\infty$-algebras and free loop space homology

In Paper II and III, we computed the free loop space homology of manifolds by writing down the complex

$$Hom^\tau(H_*(X), H_*(\Omega X)).$$

The crucial property of $X$ that makes this work is that it is both formal and coformal over the coefficient ring. In this case $H_*(X)$ and $H_*(\Omega X)$ are Koszul dual (co)algebras.

The goal of this paper is to introduce Koszul duality for $A_\infty$-algebras in such a way that we can generalize the above method to cases where $X$ is not necessarily formal and coformal.

In our definition, a Koszul $A_\infty$-algebra is an $A_\infty$-algebra that can be equipped with a negative weight grading such that the homology of the bar construction is concentrated in weight 0.

**Definition.** A *weight grading* on an $A_\infty$-algebra $A$ is a decomposition of $A$ as a direct sum of graded $\Bbbk$-modules,

$$A = \bigoplus_{k\in\mathbb{Z}} A(k),$$

such that $m_n\colon A^{\otimes n} \to A$ is homogeneous of weight $n-2$.

A weight grading on $A$ is called *negative* if $A(k) = 0$ for $k \geq 0$.

(1) A *Koszul weight grading* on an $A_\infty$-algebra $A$ is a negative weight grading such that the homology of the bar construction $BA$ is concentrated in weight 0.

(2) An $A_\infty$-algebra is called *Koszul* if it admits a Koszul weight grading.

The above definition is inspired by the following well-known characterization of Koszul algebras: a quadratic algebra $A$ is Koszul if and only if the grading induced by the (negative) wordlength in the generators is a Koszul weight grading, see e.g. [LV12, Theorem 3.4.4]. In this case, the weight grading on $BA$ corresponds to the 'syzygy degree' of [LV12, §3.3.1]. It is possible to define the dual notion of Koszul $A_\infty$-coalgebra as well, in that case we replace the bar construction with the cobar construction. With this notion of Koszul $A_\infty$-algebra we can apply the theory to topological spaces that are *either* formal or coformal. The main technical results of the paper are summarized as follows.

**Theorem.** *Let $M$ be a simply connected topological space and let $\Bbbk$ be a field.*

(1) *The space $M$ is formal over $\Bbbk$ if and only if the homology of the based loop space $H_*(\Omega M;\Bbbk)$ admits a minimal Koszul $A_\infty$-algebra structure making it quasi-isomorphic to $C_*(\Omega M;\Bbbk)$. In this situation, the*

*homology of $M$ is isomorphic to the Koszul dual coalgebra,*

$$H_*(M;\Bbbk) \cong H_*(\Omega M;\Bbbk)^{\text{¡}}.$$

(2) *The space $M$ is coformal over $\Bbbk$ if and only if its homology $H_*(M;\Bbbk)$ admits a minimal Koszul $A_\infty$-coalgebra structure making it quasi-isomorphic to $C_*(M;\Bbbk)$. In this situation, the homology of the based loop space is isomorphic to the Koszul dual algebra,*

$$H_*(\Omega M;\Bbbk) \cong H_*(M;\Bbbk)^!.$$

*In either situation, there is a twisting morphism*

$$\kappa\colon H_*(M;\Bbbk) \to H_*(\Omega M;\Bbbk)$$

*such that the twisted convolution $A_\infty$-algebra*

$$\mathrm{Hom}^\kappa(H_*(M;\Bbbk), H_*(\Omega M;\Bbbk))$$

*is quasi-isomorphic, as an $A_\infty$-algebra, to the Hochschild cochain complex of $C_*(\Omega M;\Bbbk)$.*

*In particular, if $M$ is a $d$-dimensional manifold that is formal or coformal over $\Bbbk$, then there is an isomorphism of graded algebras*

$$H_{*+d}(LM;\Bbbk) \cong H_* \mathrm{Hom}^\kappa(H_*(M;\Bbbk), H_*(\Omega M;\Bbbk)),$$

*where the left hand side carries the Chas-Sullivan loop product.*

This generalizes the results of Paper II. As an illustration that this is more powerful, we offer two case studies where the methods of Paper II do not apply, but where the new methods do apply. The first is an example of a formal but non-coformal manifold, $\mathbb{C}\mathrm{P}^n$. The Chas-Sullivan algebra of $\mathbb{C}\mathrm{P}^n$ was computed in [CJY04], but the methods here give a quick streamlined computation. The second example is a certain coformal but non-formal 7-manifold $M$, obtained by pulling back the Hopf fibration $\eta\colon S^7 \to S^4$ along the collapse map $S^2 \times S^2 \to S^4$. We show that this manifold is coformal but not formal over $\mathbb{Z}$ and compute $H_{*+7}(LM;\mathbb{Z})$.

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