On the Extension and Wedge Product of Positive Currents

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The family comes first

To My Family

Abstract

This dissertation is concerned with extensions and wedge products of positive currents. Our study can be considered as a generalization for classical works done earlier in this field.

Paper I deals with the extension of positive currents across different types of sets. For closed complete pluripolar obstacles, we show the existence of such extensions. To do so, further Hausdorff dimension conditions are required. Moreover, we study the case when these obstacles are zero sets of strictly k-convex functions.

In Paper II, we discuss the wedge product of positive pluriharmonic (resp. plurisubharmonic) current of bidimension (p, p) with the Monge-Ampère operator of plurisubharmonic function. In the first part of the paper, we define this product when the locus points of the plurisubharmonic function are located in a $(2p-2)$-dimensional closed set (resp. $(2p-4)$-dimensional sets), in the sense of Hartogs. The second part treats the case when these locus points are contained in a compact complete pluripolar sets and $p \geq 2$ (resp. $p \geq 3$).

Paper III studies the extendability of negative S-plurisubharmonic current of bidimension (p, p) across a $(2p - 2)$-dimensional closed set. Using only the positivity of S, we show that such extensions exist in the case when these obstacles are complete pluripolar, as well as zero sets of C^2-plurisubharmoinc functions.

Acknowledgement

Contents

List of Papers.

1. ***Extension of positive currents with special properties of Monge-Ampère operators.*** To appear in Mathematica Scandinavica.

2. ***The inductive wedge product of positive currents.***

3. ***The extendability of S-plurisubharmonic currents.***

Introduction

In this introductory chapter we give a background paving the readers' way to get into the details of the papers in this thesis. In spite that many related topics to positive currents are exposed here, this introduction is not generic enough to embrace a complete account of this field. However, this chapter together with the list of references are very helpful to get your teeth into this subject.

Since our issue is local, all notions will be restricted to the case of open subsets Ω of $\mathbb{C}^n$, $n \geq 1$.

1.1 Differential Forms

Let $\mathcal{D}_{p,q}(\Omega)$ where $p, q \in \{0, ..., n\}$ be the space of C^∞ compactly supported differential forms of bidegree (p, q). Let $z_j = x_j + iy_j$ be the coordinates in $\mathbb{C}^n$, we consider the operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \text{ and } \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

If $\varphi \in \mathcal{D}_{p,q}(\Omega)$, then we define

$$\partial\varphi = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial z_j}dz_j \text{ and } \bar{\partial}\varphi = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial \bar{z}_j}d\bar{z}_j$$

We use the notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. A computation shows that

$$dd^c = 2i\partial\bar{\partial} \tag{1.1.1}$$

Recall that $\mathbb{C}^n$ is oriented. Namely, we have the (n, n)-form

$$\mathcal{V}(z) = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n \tag{1.1.2}$$

The right hand side is Lebesgue's volume form when we identify $\mathbb{C}^n$ with a real (x, y)-space. If $(w_1, ..., w_n)$ are other coordinates, we find that

$$\mathcal{V}(w) = |det(\frac{\partial w_j}{\partial z_k})|^2 \mathcal{V}(z) \tag{1.1.3}$$

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Recall that a form $\varphi \in \mathcal{D}_{n,n}(\Omega)$ is positive if there exists a non negative function γ such that $\varphi(z) = \gamma(z)\mathcal{V}(z)$. Then by (1.1.3), the positivity on $\mathcal{D}_{n,n}(\Omega)$ does not depend on the choice of the coordinates.

The Kähler form is the $(1,1)$-form defined by

$$\mathcal{K} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \tag{1.1.4}$$

One checks that n-fold exterior product of $\mathcal{K}$ is $n!$ times the volume form.

$$\frac{1}{n!}\mathcal{K}^n = \mathcal{V} \tag{1.1.5}$$

Let us remark that one has the equality

$$\beta = dd^c|z|^2 = 4\mathcal{K} \tag{1.1.6}$$

where in the literature, the left hand side is often used.

1.1.1   The Positivity on $\mathcal{D}_{p,p}(\Omega)$

Definition 1.1.1. *A form $\varphi \in \mathcal{D}_{p,p}(\Omega)$ is said to be weakly positive if for all $\alpha_1, ..., \alpha_{n-p} \in \mathcal{D}_{1,0}(\Omega)$, the (n,n)-form*

$$\varphi \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge ... \wedge i\alpha_{n-p} \wedge \overline{\alpha}_{n-p} \tag{1.1.7}$$

is positive. A form $\varphi \in \mathcal{D}_{p,p}(\Omega)$ is said to be strongly positive if φ can be written as

$$\varphi(z) = \sum_{j=1}^{N} \gamma_j(z)\, i\alpha_{1,j} \wedge \overline{\alpha}_{1,j} \wedge ... \wedge i\alpha_{p,j} \wedge \overline{\alpha}_{p,j},\ N \in \mathbb{N}$$

where $\gamma_j \geq 0$ and $\alpha_{s,j} \in \mathcal{D}_{0,1}(\Omega)$.

By this definition, we find that $\mathcal{D}_{p,p}(\Omega)$ has a basis consisting of strongly positive forms. This follows from the equality

$$4dz_j \wedge d\overline{z_k} = (dz_j + dz_k) \wedge \overline{(dz_j + dz_k)} - (dz_j - dz_k) \wedge \overline{(dz_j - dz_k)}$$
$$+ i(dz_j + idz_k) \wedge \overline{(dz_j + idz_k)} - i(dz_j - idz_k) \wedge \overline{(dz_j - idz_k)}$$

An example of strongly positive forms is the $(1,1)$-form β. The (n,n)-form β^n will play a pivotal role in the subsequent study of positive currents.

1.1.2   Pull-Back of Differential Forms

Let Ω' be an open subset of $\mathbb{C}^m$ and let f be a smooth function which maps Ω' into Ω. If $\varphi \in \mathcal{D}_{p,q}(\Omega)$, then the pull-back $f^*\varphi$ is the form on Ω' which is defined as follows.

If $\varphi = \sum_{|I|=p,|J|=q} \varphi_{I,J} dz_I \wedge d\overline{z}_J$, and if f_j are the components of f, then we have

$$f^*\varphi(z') = \sum_{|I|=p,|J|=q} \varphi_{I,J} \circ f(z') df_{i_1}(z') \wedge ... \wedge df_{i_p}(z') \wedge d\overline{f}_{j_1}(z') \wedge ... \wedge d\overline{f}_{j_q}(z') \tag{1.1.8}$$

Notice that $f^*\varphi \in \mathcal{D}_{p+q}(\Omega)$ and $\text{Supp} f^*\varphi \subset f^{-1}(\text{Supp}\varphi)$, but in general $\text{Supp} f^*\varphi$ need not to be compact. If f is analytic, then df_j is $(1,0)$-form, for all $j = 1,..,n$. Therefore, the pull-back involving analytic functions preserves both the bidegree and the positivity.

Some properties of the pull-back. Let φ and f as above, and let ψ be a differential form on Ω. Then we have the following.

1. $\partial f^*(\varphi) = f^*(\partial\varphi)$

2. $\overline{\partial} f^*(\varphi) = f^*(\overline{\partial}\varphi)$

3. $f^*(\varphi \wedge \psi) = f^*(\varphi) \wedge f^*(\psi)$

Fubini's Theorem. Let M and N be oriented differentiable manifolds of real dimensions m and n, respectively, and let f be a smooth map from M to N such that f is a submersion[I]. Take a differential form φ of degree m on M, with L^1_{loc} coefficients such that $f|_{\text{Supp}\varphi}$ is proper, i.e. $\text{Supp}\varphi \cap f^{-1}(K)$ is compact for every compact subset K of N, then for all $\psi \in \mathcal{D}_n(N)$ we have

$$\int_M \varphi \wedge f^*(\psi) = \int_{y \in N} \left(\int_{x \in f^{-1}(y)} \varphi(x) \right) \psi$$

1.2 Positive Currents

The dual space $\mathcal{D}'_{p,q}(\Omega)$ is the space of currents of bidimension (p,q) or bidegree $(n-p, n-q)$. A current $T \in \mathcal{D}'_{p,p}(\Omega)$ is said to be positive if $\langle T, \varphi \rangle \geq 0$ for all forms $\varphi \in \mathcal{D}_{p,p}(\Omega, k)$ that are strongly positive. Remember that, a famous result due to Laurent Schwartz asserts that the distribution coefficients of a positive current T are all expressed by measures. Hence we can define the mass of T over each relatively compact open subset $\Omega_1 \subset \Omega$ defined as follows

$$\|T\|_{\Omega_1} = \sup\{|T(\varphi)|, \ \varphi \in \mathcal{D}_{p,p}(\Omega_1) \text{ and } \|\varphi\| \leq 1\}$$

where $\|\varphi\|$ refers to the sum of the usual maximum norms on the continuous coefficients of φ. A fundamental result which goes back to work by Lelong asserts that there exists a constant C depending only on n and p such that

$$\frac{1}{2^p p!}(T \wedge \beta^p)(\Omega_1) \leq \|T\|_{\Omega_1} \leq C(T \wedge \beta^p)(\Omega_1)$$

In the last term, the exterior product $T \wedge \beta^p$ is a positive current of bidimension $(0,0)$, and hence a non-negative measure whose mass is evaluated on Ω_1 in the right hand side above.

1.2.1 Different Types of Currents

Let $T \in \mathcal{D}'_{p,q}(\Omega)$, we define the currents ∂T and $\overline{\partial} T$ on $\mathcal{D}_{p-1,q}(\Omega)$ and $\mathcal{D}_{p,q-1}(\Omega)$, respectively, by

$$\langle \partial T, \varphi \rangle = (-1)^{(p+q)+1} \langle T, \partial\varphi \rangle \text{ and } \langle \overline{\partial} T, \psi \rangle = (-1)^{(p+q)+1} \langle T, \overline{\partial}\psi \rangle$$

[I]Means that f is surjective and for every $x \in M$, the differential map $D_x f : T_{M,x} \longrightarrow T_{N,f(x)}$ is surjective

for all $\varphi \in \mathcal{D}_{p-1,q}(\Omega)$ and $\psi \in \mathcal{D}_{p,q-1}(\Omega)$.

Definition 1.2.1. *A current T is said to be closed if $dT = 0$. A current $T \in \mathcal{D}'_{p,p}(\Omega)$ is said to be positive plurisubharmonic if both T and $dd^c T$ are positive. In the case when T is negative and $dd^c T$ is positive we say that T is negative plurisubharmonic.*

In general, if we only assume that T and $dd^c T$ both have locally finite mass, then T is called $\mathbb{C}$-normal. Notice that the theorem by Schwartz implies that any positive (or negative) plurisubharmonic current T is $\mathbb{C}$-normal. Finally, T is said to be $\mathbb{C}$-flat if $T = F + \partial H + \overline{\partial} S + \partial\overline{\partial} R$, where F, H, S and R are currents with locally integrable coefficients.

A deep result due to Bassanelli (see [4]), asserts that every $\mathbb{C}$-normal current is $\mathbb{C}$-flat.

1.2.2   Support Theorem and Slice Formula

The support theorem (see [4]) says that for a $\mathbb{C}$-flat current T of bidimension (p,p), one has the implication

$$\mathcal{H}_{2p}(\text{Supp}T) = 0 \Rightarrow T = 0$$

Above $\mathcal{H}_{2p}$ notes $2p$-Hausdorff measure. Next, using Stokes formula one can show how currents are effected by its support. See [27] where it is proved that, when T is a positive (or negative) plurisubharmonic current with compact support, then $T = 0$.

A useful slicing formula. Let $k \leq p$ and $T \in \mathcal{D}'_{p,p}(\Omega)$ with locally integrable coefficients. Set $\pi : \mathbb{C}^n \to \mathbb{C}^k$, $\pi(z', z'') = z'$ and $i_{z'} : \mathbb{C}^{n-k} \to \mathbb{C}^n$, $i_{z'}(z'') = (z', z'')$. Then the slice $\langle T, \pi, z' \rangle$ which is defined by

$$\langle T, \pi, z' \rangle(\varphi) = \int_{z'' \in \pi^{-1}(z')} i_{z'}^* T(z'') \wedge i_{z'}^* \varphi(z''), \quad \forall \varphi \in \mathcal{D}_{p-k,p-k}(\Omega)$$

is a well defined $(p-k, p-k)$-current for a.e z', and supported in $\pi^{-1}(z')$. Notice that, the above properties of the pull-back show that

$$dd^c \langle T, \pi, z' \rangle = \langle dd^c T, \pi, z' \rangle, \ d^c \langle T, \pi, z' \rangle = \langle d^c T, \pi, z' \rangle, \ d \langle T, \pi, z' \rangle = \langle dT, \pi, z' \rangle$$

So, we deduce that for every $\mathbb{C}$-flat current T, the slice $\langle T, \pi, z' \rangle$ is well defined for a.e z'. Moreover, we have the slicing formula

$$\int_{\Omega} T \wedge \varphi \wedge \pi^* \beta'^k = \int_{z' \in \pi(\Omega)} \langle T, \pi, z' \rangle(\varphi) \beta'^k$$

This formula is helpful in many cases, and can be applied to positive (or negative) plurisubharmonic currents. For example, one can establish properties of T by testing it for the slice of T.[II]

[II]More general definition for the slice of currents can see it in [5] and [19]

1.3 <u>Currents and Plurisubharmonic Functions</u>

Definition 1.3.1. *A function u defined on Ω with values in $[-\infty, +\infty[$ is called plurisubharmonic if*

1. *u is upper semi continuous.*

2. *For arbitrary $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\xi \longmapsto u(z + \xi w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.*

From the previous definition, the $(n-1, n-1)$-current $dd^c u$ is positive. So, for positive current T and u of class C^2, the current $T \wedge dd^c u$ is positive.

The current $dd^c u$ takes its place in the study of currents. One of our main issues in this thesis is about finding the sufficient conditions on the plurisubharmonic function u and the positive current T that make $T \wedge dd^c u$ well-defined.

1.3.1 Local Potential of Closed Currents

Let $T \in \mathcal{D}'_{n-1,n-1}(\Omega)$ positive and closed. Then for all $z \in \Omega$ there exists a neighborhood V of z and $u \in Psh(V)$ such that $T = dd^c u$. For lower bidimensions, Ben Messaoud and El Mir [6] proved that, if $T \in \mathcal{D}'_{p,p}(\Omega)$ positive and closed, then locally there exist a negative current U of bidimension $(p+1, p+1)$ and a smooth form R such that $T = dd^c U + R$.

1.3.2 Pluripolar and Analytic sets

A subset A of Ω is called pluripolar if for every point $z_0 \in A$ there exists a neighborhood V of z_0 and a plurisubharmonic function u on V such that

$$V \cap A \subseteq \{z \in V, u(z) = -\infty\} \tag{1.3.1}$$

If we have equality in (1.3.1), then we call A a complete pluripolar set. If we have analytic functions $f_1, ..., f_l$ such that

$$V \cap A = \{z \in V, f_1(z) = ... = f_l(z) = 0\} \tag{1.3.2}$$

we say that A is an analytic subset. Notice that, any analytic subset A is a closed complete pluripolar set by taking $u = log(|f_1|^2 + ... + |f_l|^2)$.

In our study of extending currents, we take a current T defined outside a set A. The most general is when A is an arbitrary closed set. More specific cases occur when A is a closed complete pluripolar set, and a very special case when A is analytic. In the thesis we investigate conditions to extend T across the obstacle A to a current $\widetilde{T}$.

How to find $\widetilde{T}$? Let (χ_n) be a smooth bounded sequence which vanishes on a neighborhood of closed subset $A \subset \Omega$ and (χ_n) converges to the characteristic function $\mathbb{1}_{\Omega \backslash A}$ of $\Omega \setminus A$, and let T be a current of order zero defined on $\Omega \setminus A$. If $\chi_n T$ has a limit which does not depend on (χ_n), this limit is called the trivial extension of T by zero across A and is denoted by $\widetilde{T}$. It is clear that, $\widetilde{T}$ exists if and only if T has a locally finite mass across A.

In the case of closed complete pluripolar set, we have an appropriate choice of (χ_n). In particular, there exists an increasing sequence of smooth and plurisubharmonic functions $0 \leq u_n \leq 1$ converging uniformly to 1 on each compact subset of $\Omega \setminus A$ such that $u_n = 0$ on a neighborhood of A. The profit from using such sequence (u_n), is keeping the signs of $T \wedge dd^c u_n$ and $u_n dd^c T$. This gives us better space to deduce estimates which our whole subject is all about.

1.4   Hausdorff Measure

The announced results in section 1.2.2. show that the notion of Hausdorff measure plays a central role in this subject. Of course, not each current T can be extended over a closed obstacle A. To guarantee the existence of $\widetilde{T}$, we need to examine A and see how thick it is. Because of that, the extension of current and Hausdorff measure are often connected to each other like conjoined twins.

1.4.1   Definition and Basic Properties

Let A be a subset of $\mathbb{R}^m$, $m \geq 0$. Consider a countable covering of balls B_j for A, with radii r_j, respectively. For each $\alpha \geq 0$, we define the α-Hausdorff measure of A by

$$\mathcal{H}_\alpha(A) = \lim_{\varepsilon \to 0^+} \inf\{c_\alpha \sum_j r_j^\alpha, \ A \subset \bigcup_j B_j, \ r_j < \varepsilon\} \qquad (1.4.1)$$

For $\alpha = m$ we take the constant $c_\alpha > 0$ equal to the volume of the unit ball in $\mathbb{R}^m$; for non integers α we take it the corresponding expression with the gamma function.

$$c_\alpha = \frac{\pi^{\alpha/2}}{2^\alpha \Gamma(\alpha/2 + 1)}$$

Notice that, for $\alpha = 0$, the Hausdorff measure of A is just the number of elements of A.[III]

Let us spell out some basic properties of Hausdorff measure.

1. If $A \subset \mathbb{R}^m$ and $tA := \{tx, \ x \in A\}$ for $t > 0$, then

$$\mathcal{H}_\alpha(tA) = t^\alpha \mathcal{H}_\alpha(A)$$

2. If $\mathcal{H}_\alpha(A) < \infty$, then $\mathcal{H}_\beta(A) = 0$ for all $\beta > \alpha$. If $\mathcal{H}_\alpha(A) > 0$, then $\mathcal{H}_\gamma(A) = \infty$ for all $\gamma < \alpha$. The number $d := \inf\{\alpha, \ \mathcal{H}_\alpha(A) = 0\}$ is called the Hausdorff dimension of A.

3. If $f : X \longrightarrow X'$ is a continuous map between metric spaces that satisfies $d_{X'}(f(x), f(y)) \leq C d_X(x, y)$ for some constant C and for all $x, y \in X$, then $\mathcal{H}_\alpha(f(A)) \leq C^\alpha \mathcal{H}_\alpha(A)$ for all $A \subset X$. In particular, under the projection, Hausdorff measure does not increase.

4. If $\alpha = m$, then for all Lebesgue measurable set $A \subset \mathbb{R}^m$, we have

$$\mathcal{H}_\alpha(A) = \lambda(A)$$

[III]When A is a subset of a metric space, the definition of $\mathcal{H}_\alpha(A)$ coincides with (1.4.1), after removing the multiplicative constant c_α.

A useful result due to Bernard Shiffman ([26], Corollary 4), asserts the following

Theorem 1.4.1. *Let A be an open subset of $\mathbb{R}^m$, let $\alpha \geq 0$, and let $\pi_k : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ denote the projection onto the first k coordinates.*

1. *If $\mathcal{H}_{k+\alpha}(A) = 0$, then $\mathcal{H}_\alpha(A \cap \pi_k^{-1}(x)) = 0$ for $\mathcal{H}_k$-(a.e) $x \in \mathbb{R}^k$.*

2. *If $\mathcal{H}_{k+\alpha}(A) < \infty$, then $\mathcal{H}_\alpha(A \cap \pi_k^{-1}(x)) < \infty$ for $\mathcal{H}_k$-(a.e) $x \in \mathbb{R}^k$.*

1.4.2 Bishop's Lemma

In [7], Errett Bishop performed a spectacular work concerning the analytic sets. Theorem 1.4.2 below, was one of the main tools in his article. Actually, it stands tall as a preparation step for the results announced in section 1.5.

Theorem 1.4.2. *Let A be a closed subset of $\mathbb{C}^n$ such that $\mathcal{H}_{2s+1}(A) = 0$ for some integer $0 \leq s < n$. Then for almost all choices of unitary coordinates $(z_1, ..., z_n) = (z', z'')$, $z' = (z_1, ..., z_s)$ and $z'' = (z_{s+1}, ..., z_n)$, and almost all $B'' = B(0, r'') \subset \mathbb{C}^{n-s}$, the set $\partial B'' \times \{0\}$ does not intersect A.*

1.5 The Evolution of Currents' Extension

Once we say the word "current", a very prominent mathematician must be mentioned. The French mathematician Pierre Lelong defined the plurisubharmonic functions in his note (see [23]), and perceived the integration over analytic sets expressed via currents. His glamorous works inspired others to go further in this subject. One must also give contribution to Kiyoshi Oka who was the first to investigate plurisubharmonic functions, restricted to the case of two complex variables (see [25]).

As in every mathematical subject, the theory of currents underwent several stages. We give in this section a historical survey for the evolution of currents.

1.5.1 Integration Currents

A basic example of currents comes from the integration over analytic sets. For such currents, many papers are devoted to solve problems when singularities occur.

Lelong [24] 1957. *Let A be a pure p-dimensional analytic subset of Ω, the the current $[A]_{reg} \in \mathcal{D}'_{p,p}(\Omega \setminus A_{sing})$ has finite mass in a neighborhood of every point $z_0 \in A_{sing}$. Moreover, the current $[A]$ - the trivial extension of $[A]_{reg}$- is a closed positive current on Ω.*

Bishop [7] 1968. *Let E be an analytic subset of Ω, and let A be a pure p-dimensional analytic subset of $\Omega \setminus E$ with finite 2p-dimensional volume. Then $\overline{A} \cap \Omega$ is an analytic subset of Ω.*

1.5.2 Positive Closed Currents

Next follows two results which extend those of Lelong and Bishop.

Skoda [28] 1982. *Let A be an analytic subset of Ω and let $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ be a closed positive. Assume that T has a finite mass on a neighborhood of each point in A. Then the trivial extension $\widetilde{T}$ is a closed positive current.*

The case when A is a closed complete pluripolar set was settled by Hassine El Mir.

El Mir [16] 1982. *Let A be a closed complete pluripolar subset of Ω and let $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ be a closed positive. Assume that T has a finite mass on a neighborhood of each point in A. Then the trivial extension $\widetilde{T}$ is a closed positive current.*

The Existence Problem. El Mir and Imed Feki gave sufficient conditions that guarantee the existence of $\widetilde{T}$, without a priori assumption on local finite mass.

El Mir-Feki [18] 1998. *Let A be a closed complete pluripolar subset of an open subset Ω and T be a closed positive current of bidimension (p,p) on $\Omega \setminus A$. Assume that $\mathcal{H}_{2p}(A \cap \overline{Supp T}) = 0$. Then T extends to a closed positive current.*

The result above was inspired by a theorem due to Reese Harvey.

Harvey [20] 1974. *Let A be a closed subset of an open subset Ω and T be a closed positive current of bidimension (p,p) on $\Omega \setminus A$. If $\mathcal{H}_{2p-1}(A) = 0$, then T has a closed positive extension $\widetilde{T}$.*

1.5.3 Plurisubharmonic Currents

In the early eighties, Skoda [28] and Jean-Pierre Demailly [11] started studying a new type of currents. Sibony considered the Skoda-El Mir result for plurisubharmonic currents, and proved.

Sibony [27] 1985. *Let A be a closed complete pluripolar subset of Ω and let $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ be a positive (resp. negative) plurisubharmonic. Assume that the trivial extensions $\widetilde{T}$, $\widetilde{dT}$ and $\widetilde{dd^c T}$ exist. Then $d\widetilde{T} = \widetilde{dT}$. Moreover, the residual current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is closed positive (resp. negative) current supported in A.*

Although the current R above depends on $dd^c T$, Sibony required the existence of $\widetilde{dT}$. There remained the question whether the condition on dT can be omitted. Lucia Alessandrini and Bassanelli [2](1993) proved that the existence of $\widetilde{dT}$ is superfluous when A is an analytic set. This result was improved by El Mir [17](2001) who showed that it suffices to assume that A is a closed complete pluripolar sets.[IV]

[IV]Ivashkovich and Shiffman made some initial work to settle this problem (see [21] and [22])

The Existence Problem. Once again, this problem started surfacing. Khalifa Dabbek, Fredj Elkhadhra and El Mir kneeled this problem, and proved.

Dabbek-Elkhadhra-El Mir [10] (2003). *Let A be a closed complete pluripolar subset of an open subset Ω and T be a negative plurisubharmonic current of bidimension (p, p) on $\Omega \setminus A$. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists and is negative plurisubharmonic.*

Harvey's Extension. In the same article [10], the authors continued Harvey's studies about plurisubharmonic currents. They found a relaxed condition for a certain Hausdorff dimension which goes as follows.

Let A be a closed subset of Ω and T a negative plurisubharmonic current of bidimension (p, p) on $\Omega \setminus A$ such that $\mathcal{H}_{2p-2}(\overline{SuppT} \cap A)$ is locally finite. Then $\widetilde{T}$ exists and is plurisubharmonic. Moreover, the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is a negative current supported in A.

1.6   **The Main Motivation Behind the Dissertation**

Let $S \in \mathcal{D}'_{p-1,p-1}(\Omega)$ be a positive current. Tien-Cuong Dinh and Sibony studied the case when $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ is a negative current such that $dd^c T \geq -S$ on $\Omega \setminus A$ (such current T we call it S-plurisubharmonic). Obviously, this case is more general than plurisubharmonic currents. The authors succeeded to get the residual current R for this case.

Dinh-Sibony [15] 2007. *Let A be a closed complete pluripolar set of Ω and let T as above. Suppose that $\widetilde{T}$ exists, then $\widetilde{dd^c T}$ exists. Furthermore, there exists a negative current R supported in A such that $R = \widetilde{dd^c T} - dd^c \widetilde{T}$.*

The question marks regarding the existence problem, the case of closed obstacle, etc..., are the main motivations behind writing this thesis.

Glimpses of the Licentiate Thesis

The licentiate thesis [1] treats three main issues concerning the wedge product of currents, the extension over pluripolar sets and the continuation across zero sets of 0-convex functions. For the first issue we proved the following result.

Theorem 3.1. *Let T be a positive dd^c-negative current of bidimension (p,p) on a complex manifold X of dimension n and let A be a closed complete pluripolar subset of X such that $\mathcal{H}_{2p-1}(A) = 0$. Let S be a positive and closed current of bidimension $(n-1, n-1)$ on X and smooth on $X \setminus A$. If g is a solution of $dd^c g = S$ on an open set $U \subset X$ and g_j is a sequence of smooth plurisubharmonic functions such that $\left(g_j\right)$ converges to g in $C^2(U \setminus A)$, then the sequence $\left(dd^c g_j \wedge T\right)$ is locally bounded in mass in U.*

This result implies that, there exists a subsequence g_{j_s} such that the sequence $dd^c g_{j_s} \wedge T$ converges weakly to a current $S \wedge T$. Of course, two questions occur, immediately.

- What about the uniqueness of $S \wedge T$?[I]

- Does $g_j T$ converge ?

Paper II deals with these two questions.

The extension of currents also had its share in [1]. In fact, for Dinh-Sibony hypothesis, we showed that

Theorem 3.7. *Let A be a closed complete pluripolar subset of Ω and T be a negative current of bidimension (p,p) on $\Omega \setminus A$ such that $\underline{dd^c T} \geq -S$ on $\Omega \setminus A$ for some positive closed current S on Ω. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is closed and negative supported in A.[II]*

Our proof was basically based on [10]. Chern-Levine-Nirenberg inequality was involved in the proof, and because of this closedness of S was required.

[I]The uniqueness of $S \wedge T$ has been achieved in many different cases (see [4], [14] and [3])

[II]We should point out that Noureddine and Dabbek [9] proved the same result prior to present work.

But to what extent can this condition on S be relaxed? This is one of the major goals in paper I. However, the closedness of S was not always essential in [1]. In particular, it was neglected in the case of 0-convex functions.

Theorem 4.6.   *Let u be a positive exhaustion strictly 0-convex function on Ω and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a positive current of bidimension (p, p) on $\Omega \setminus A$ such that $dd^c T \leq S$ on $\Omega \setminus A$ for some positive current S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists. If $p \geq 2$, $dd^c S$ is of locally finite mass and $u \in C^2$, then $\widetilde{dd^c T}$ exists and $\widetilde{dd^c T} = dd^c \widetilde{T}$.*

Overview of Paper I

The aim in this paper is to relax the closedness condition on S in [1]. So, for this purpose we assume that $S \in \mathcal{D}'_{p-1,p-1}(\Omega)$ is a positive current, A is a closed subset of Ω and $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ is a negative current such that $dd^c T \geq -S$ on $\Omega \setminus A$, and the first main theorem of this paper goes as follows:

Theorem(Paper I, Theorem 3.3) *If S is plurisubharmonic and A is complete pluripolar such that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$, then $\widetilde{T}$ exists. Moreover, the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is negative and supported in A.*

In the last section of paper I, we assume that u is a positive strictly k-convex function on Ω and we set $A = \{z \in \Omega : u(z) = 0\}$. For this type of obstacles we prove our second main theorem.

Theorem(Paper I, Theorem 4.7) *If S is plurisubharmonic (or $dd^c S \leq 0$) and $p \geq k + 1$, then $\widetilde{T}$ exists. If $dd^c S \leq 0$, $p \geq k + 2$ and u is of class C^2, then $\widetilde{dd^c T}$ exists and $\widetilde{dd^c T} = dd^c \widetilde{T}$.*

We also show that in some cases the positivity of S is sufficient to get the extension of T. This is true for the following cases.

Theorem(Paper I, Theorem 4.9) *Assume that A is compact complete pluripolar set. If $p \geq 1$, then $\widetilde{T}$ exists and $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is a negative current supported in A.*

Theorem(Paper I, Theorem 4.10) *If A is closed set such that $\mathcal{H}_{2p-2}(A \cap \overline{SuppT})$ is locally finite, then $\widetilde{T}$ exists. If $dd^c S \leq 0$, then $\widetilde{dd^c T}$ exists and the residual current R is negative supported in A.*

The technical tools in this paper are essentially based on [10]. Actually, by compiling the techniques in [10] and new versions of Chern-Levine-Nirenberg we have been able to conquer the problems.

The first main theorem provides some good news concerning Monge-Ampère operators. It tells us how to control higher orders for plurisubharmonic functions. In fact, let A be a closed complete pluripolar subset of an open subset Ω and T be a closed positive current of bidimension (p, p) on $\Omega \setminus A$. Assume that $\mathcal{H}_{2p-2}(A) = 0$. Now, suppose that g is a plurisubharmonic function on Ω such that $g \in C^\infty(\Omega \setminus A)$. Of course by using ([10], Theorem 1), the extension $\widetilde{gT}$ exists. But the innovated new result is that $\widetilde{g^2 T}$ exists as well.

We should point out that this paper partially solves problems (2) and (3) in the problems' list of [1], since we still would like that the plurisubharmonicity of S to be removed in all hypotheses. This can be considered as an intersting problem to be discussed in future projects.

Overview of Paper II

This paper deals with the wedge product of positive currents. More precisely, we consider the following case.

Let A be a closed subset of Ω and $T \in \mathcal{D}'_{p,p}(\Omega)$ be a positive current. Let $g \in Psh(\Omega) \cap C^\infty(\Omega \setminus A)$ We show

Theorem (Paper II, Theorem 3.3) *Let T be a pluriharmonic current and (g_j) be a sequence of decreasing smooth plurisubharmonic functions converging pointwise to g on $\Omega \setminus A$.*

1. *If $\mathcal{H}_{2p-1}(A \cap SuppT) = 0$, then $\widetilde{dd^c g \wedge T}$ exists. Furthermore, there exists a subsequence (g_{j_s}) such that $dd^c g_{j_s} \wedge T$ converges.*

2. *In addition to the hypotheses above, if $\mathcal{H}_{2p-2}(A \cap SuppT)$ is locally finite and g_j converges to g in $C^1(\Omega \setminus A)$, then $dd^c g \wedge T$ is a well-defined current as a limit of $dd^c g_j \wedge T$.*

Theorem (Paper II, Theorem 3.13) *Let A be an analytic subset of Ω, and T be a plurisubharmonic current, $dimA < p - 1$. Then $dd^c g \wedge T$ is a well-defined current as a limit of $dd^c g_j \wedge T$.*

As a consequence of the above results, the current $S \wedge T$ is well-defined as soon as $S \in \mathcal{D}'_{n-1,n-1}(\Omega)$ is a closed positive and smooth on $\Omega \setminus A$. This is true, since we can apply the previous results for the local potential of S.

The more we improve inequalities, the more we get extensions. This is the real wealth we seek in the study of currents. Actually, the main tool to prove the existence of wedge products above is a new version of Chern-Liveine-Nirenberg inequality which asserts.

Theorem (Paper II, Lemma 3.5 and Lemma 3.12) *Let A and g as above, and let T be a pluriharmonic current such that $\mathcal{H}_{2p-1}(A \cap SuppT) = 0$. Let K and L compact sets of Ω with $L \subset K^\circ$. Then there exist a constant $C_{K,L} > 0$, and a neighborhood V of $K \cap A$ such that*

1. $\|dd^c g \wedge T\|_{L \setminus A} \leq C_{K,L} \|g\|_{\mathcal{L}^\infty(K \setminus V)} \|T\|_K$

2. *If $dd^c T \geq 0$ (or $dd^c T \leq 0$) and $\mathcal{H}_{2p-3}(A \cap SuppT) = 0$, then there exist a constant $D_{K,L} > 0$ such that*

$$\|dd^c g \wedge T\|_{L \setminus A} \leq D_{K,L} \|g\|_{\mathcal{L}^\infty(K \setminus V)} \|T\|_K$$

This part of the paper solves the first problem in [1]. Also, by induction process we give more general versions of the precedent results.

In the last part of the paper, we assume the case when A is compact complete pluripolar set. For this case we prove

Theorem (Paper II, Theorem 5.8) *Let T be a dd^c-negative current, $p \geq 3$ (resp. pluriharmonic, $p \geq 2$). Then $dd^c g \wedge T$ is well-defined as a limit of $dd^c g_j \wedge T$.*

As an application of this result, a version of Lelong number is defined in the case of dd^c-negative (resp. pluriharmonic) currents. Moreover, the study of Monge-Ampère operators on hyperconvex domains can be extended to such currents.

Overview of Paper III

In this paper we study the extendability of S-plurisubharmonic currents. More precisely, we consider the following case. Let A be a closed subset of Ω and let $T \in \mathcal{D}'_{p,p}(\Omega \setminus A)$ be a positive current such that

$$dd^c T \leq S$$

on $\Omega \setminus A$ for some positive current S on Ω. Such current T is called $dd^c(S)$-negative. Using only the positivity of S, we prove

Theorem (Paper III, Theorem 2.2) *If A is complete pluripolar and $\mathcal{H}_{2p-1}(A \cap \overline{SuppT}) = 0$, then $\widetilde{T}$ exists. Moreover, the current $R = \widetilde{dd^c T} - dd^c \widetilde{T}$ is positive and supported in A.*

To obtain the extension above, we establish a version of the Ben Messaoud-El Mir inequality which asserts.

Theorem (Paper III, Lemma 2.1) *Let A be a closed complete pluripolar subset of Ω and let v be a plurisubharmonic function of class C^2, $v \geq -1$ on Ω such that $\Omega' = \{z \in \Omega : v(z) < 0\}$ is relatively compact in Ω. Let $K \subset \Omega'$ be a compact subset and set $c_K = -\sup_{z \in K} v(z)$. Then there exists a constant $\eta \geq 0$ such that for every plurisubharmonic function u on Ω' of class C^2 satisfying that $-1 \leq u < 0$ we have,*

$$\int_{K \setminus A} T \wedge dd^c u \wedge \beta^{p-1} \leq \frac{1}{c_K} \int_{\Omega' \setminus A} T \wedge dd^c v \wedge \beta^{p-1} + \eta \|S\|_{\Omega'}$$

In the second part of the paper, we consider the case when A is the zero set of a non-negative plurisubharmonic function u on Ω of class C^2, and prove the following result.

Theorem (Paper II, Theorem 3.1) *Let L and K be compact sets of Ω such that $L \subset K^\circ$. If $\mathcal{H}_{2p-1}(A \cap \overline{SuppT}) = 0$, then there exist a constant $C_{K,L} \geq 0$ and a neighborhood V of $K \cap A$ such that*

$$\|dd^c u \wedge T\|_{L \setminus A} \leq C_{K,L} \|u\|_{\mathcal{L}^\infty(K)} (\|T\|_{K \setminus V} + \|dd^c T\|_{K \setminus V} + \|S\|_K)$$

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