Spectral estimates and Ambartsumian-type theorems for quantum graphs

Rune Suhr

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Abstract
This thesis consists of four papers and deals with the spectral theory of quantum graphs. A quantum graph is a metric graph equipped with a self-adjoint Schrödinger operator acting on functions defined on the edges of the graph subject to certain vertex conditions.

In Paper I we establish a spectral estimate implying that the distance between the eigenvalues of a Laplace and a Schrödinger operator on the same graph is bounded by a constant depending only on the graph and the integral of the potential. We use this to generalize a geometric version of Ambartsumian's Theorem to the case of Schrödinger operators with standard vertex conditions.

In Paper II we extend the results of Paper I to more general vertex conditions but also provide explicit examples of quantum graphs that show that the results are not valid for all allowed vertex conditions.

In Paper III the zero sets of almost periodic functions are investigated, and it is shown that if two functions have zeros that are asymptotically close, they must coincide. This is relevant to the spectral theory of quantum graphs as the eigenvalues of a quantum graph are given by the zeros of a trigonometric polynomial, which is almost periodic.

In Paper IV we give a proof of the result in Paper III which does not rely on the theory of almost periodic functions and apply this to show that asymptotically isospectral quantum graphs are in fact isospectral. This allows us to generalize two uniqueness results in the spectral theory of quantum graphs: we show that if the spectrum of a Schrödinger operator with standard vertex conditions on a graph is equal to the spectrum of a Laplace operator on another graph then the potential must be zero, and we show that a metric graph with rationally independent edge-lengths is uniquely determined by the spectrum of a Schrödinger operator with standard vertex conditions on the graph.

Keywords: *Spectral estimates, quantum graphs, Ambartsumian, trigonometric polynomials.*

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Stockholm University

Department of Mathematics

Stockholm University, 106 91 Stockholm

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Rune Suhr

Stockholm
University

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Sammanfattning

Denna avhandling består av fyra artiklar och behandlar spektralteorin för kvantgrafer. En kvantgraf är en metrisk graf med en tillhörande självadjungerad Schrödingeroperator som verkar på funktioner som är definierade på kanterna av grafen och uppfyller vissa hörnvillkor.

I första artikeln bevisar vi en spektraluppskattning: att avståndet mellan egenvärdena tillhörande Laplace- och Schrödingeroperatorer på samma graf är begränsad av en konstant som enbart beror på grafen och integralen av potentialen. Detta använder vi för att generalisera en geometrisk version av Ambartsumians sats till att även omfatta Schrödingeroperatorer med standardhörnvillkor.

I andra artikeln utvidgar vi resultaten i första artikeln till att omfatta mer generella hörnvillkor och vi ger explicita exempel på kvantgrafer som visar att resultaten inte kan utvidgas till alla tillåtna hörnvillkor.

I tredje artikeln undersöks nollställemängderna till nästanperiodiska funktioner och det visas att om två funktioner har nollställen som ligger asymptotiskt nära varandra så måste de sammanfalla. Detta är relevant för spektralteorin för kvantgrafer eftersom egenvärdena till en kvantgraf ges som nollställen till ett trigonometriskt polynom som är nästanperiodiskt.

I fjärde artikeln ger vi ett bevis för satsen i artikel tre som inte bygger på teorin för nästanperiodiska funktioner och använder därefter satsen för att visa att asymptotiskt isospektrala kvantgrafer är isospektrala. Med hjälp av denna sats generaliserar vi sedan två unicitetsresultat inom spektralteorin för kvantgrafer. Vi visar att om en Schrödingeroperator med standardhörnvillkor på en graf är isospektral med en Laplaceoperator på en annan graf så är potentialen noll, samt att en metrisk graf med rationellt oberoende kantlängder är entydigt bestämd av spektrumet för en Schrödingeroperator på grafen.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: **Schrödinger Operators on Graphs and Geometry II. Spectral Estimates for L_1-potentials and an Ambartsumian Theorem**
Boman, J., Kurasov, P. & Suhr, R., *Integral Equations and Operator Theory, accepted.*

PAPER II: **Schrödinger operators on graphs and geometry III. Non-standard conditions and a geometric version of the Ambartsumian theorem.**
Kurasov, P. & Suhr, R. *submitted* (2017).

PAPER III: **A note on asymptotically close zeros of almost periodic functions**
Suhr, R. *submitted* (2018).

PAPER IV: **Asymptotically isospectral quantum graphs and trigonometric polynomials**
Kurasov, P. & Suhr, R. *submitted* (2018).

Contents

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Introduction

Introduction

The subject matter of this PhD thesis is the spectral and inverse spectral theory of quantum graphs. Briefly, a quantum graph is a metric graph Γ equipped with a Schrödinger operator. More formally a quantum graph is a triple (Γ, L_q, S) with L_q a self-adjoint, edge-wise Schrödinger differential operator, and S a set of vertex conditions. We begin by giving a short explanation for the interest in spectral and inverse spectral theory and then proceed to introduce quantum graphs in more detail. We finish by describing some earlier results in the spectral theory of quantum graphs and how the results in this thesis relate to them.

For an operator A on a Hilbert space H, its spectrum $\sigma(A) \subset \mathbb{C}$ is defined as the complement of the set of $\lambda \in \mathbb{C}$ such that $A - \lambda$ is boundedly invertible and $\overline{\operatorname{ran}(A-\lambda)} = H$. Given an operator, one important problem is to determine its spectrum. On the other hand one may consider the *inverse spectral problem*, namely that of determining properties of the operator from its spectrum. Apart from being of mathematical interest, it is also a problem that arises naturally from applications. In quantum mechanics an observable — a measurable physical property — is given by a self-adjoint operator on a Hilbert space, and the values that can be measured correspond to the spectrum of the operator. Since the spectrum of the operator is the only data that is available through measurements, the inverse spectral problem assumes a central role. The first instance of an inverse spectral theorem was Ambartsumian's celebrated result [1] (Theorem 2.1 below).

1 Basics of quantum graphs

In this section we give a brief introduction to quantum graphs. For more comprehensive treatments we refer to [4] and [21].

Metric graphs

A *discrete graph* is an ordered pair (V, E) with V a non-empty set with elements that are called vertices, and E the set of edges which are just two-element subsets of V. In order to let differential operators act on functions defined on graphs, we equip the edges with a metric structure. A compact finite *metric graph* Γ is a finite collection of compact intervals glued together at some endpoints. One can construct a graph Γ by parametrising the edges $E = \{E_n\}_{n=1}^N$ of Γ by identifying them with compact intervals $[x_{2n-1}, x_{2n}]$ of separate copies of $\mathbb{R}$ (to remove any formal problems with two edges having the same parametrisation). Each vertex V_i of Γ is then identified with a collection of endpoints $V_i = \{x_{i_j}\}$, where we require that the set of vertices $\mathbf{V} = \{V_i\}_{i=1}^M$ forms a partition of the set of endpoints. The induced relation $x_i \sim x_j$ if and only if $x_i, x_j \in V_k$ for some k is then an equivalence relation on the set of endpoints, and setting $x \sim y$ if and only if $x = y$ for all other points, Γ can be seen as the quotient space

$$\bigcup_{n=1}^{N} E_n / \sim .$$

The particular choice of parametrisation of the edges will play no role in that two isometric graphs are considered to be the same graph. Thus a metric graph is determined uniquely by the ordered pair $(E, \mathbf{V})$ with E a collection of edges and $\mathbf{V}$ a partition of the set of endpoints of the intervals in E.

Differential expression

The Schrödinger operator acts as $f \mapsto -f'' + qf$ on each edge separately for some real potential function q defined on each edge. The action is in the Hilbert spaces $L^2(E_n)$ of square Lebesgue-integrable functions. We denote the operator by L_q (L_0 denotes $-d^2/dx^2$) which then acts in

$$L^2(\Gamma) = \bigoplus_{n=1}^{N} L^2(E_i).$$

Clearly $L^2(\Gamma)$ in no way reflects the connectivity of the graph: any graph Δ whose edges are of the same length as the edges of Γ will satisfy $L^2(\Delta) \simeq L^2(\Gamma)$. Instead the connectivity of Γ is encoded in the domain of L_q via the conditions on $u \in \text{dom}(L_q)$ at the vertices of Γ.

Vertex Conditions

The vertex conditions are to serve two purposes: they should reflect the topology of the graph, and they are needed to ensure that the operator given by the differential expression is self-adjoint. There are two operators in $L^2(\Gamma)$ that are naturally associated with the differential expression

$$-d^2/dx^2 = \sum_{E_n} -d^2/dx^2,$$

namely the minimal and maximal operators $L_{\min}$ and $L_{\max}$ with domains

$$\mathrm{dom}(L_{\min}(\Gamma)) = \bigoplus_{n=1}^{N} C_0^\infty(x_{2n-1}, x_{2n}) = C_0^\infty(\Gamma \setminus \mathbf{V}),$$

and

$$\mathrm{dom}(L_{\max}(\Gamma)) = \bigoplus_{n=1}^{N} W_2^2(x_{2n-1}, x_{2n}) = W_2^2(\Gamma \setminus \mathbf{V}),$$

respectively. One can show that $L_{\min} \subset L_{\max} = L_{\min}^*$, and all self-adjoint operators L associated with $-d^2/dx^2$ — i.e. self-adjoint operators satisfying $L_{\min} \subset L$ — are given as restrictions of $L_{\max}$ via vertex conditions. Functions $u \in \mathrm{dom}(L_{\max}) = W_2^2(\Gamma \setminus \mathbf{V})$ are continuous on each edge, and we let $u(x_j)$ denote the limiting values of the functions at the end points of the edges:

$$u(x_j) = \lim_{x \to x_j} u(x),$$

where the limit is taken over x inside the edge. The normal derivatives are defined similarly and are therefore independent of the chosen parametrisation of the edge:

$$\partial_n \vec{u}(x_j) = \begin{cases} u'(x_j) & x_j \text{ left end-point,} \\ -u'(x_j) & x_j \text{ right end-point.} \end{cases}$$

The vertex conditions at any vertex $V_m = \{x_{m_1}, \dots, x_{m_{d_m}}\}$ of degree d_m can be written by imposing relations between the d_m-dimensional vectors of boundary values and normal derivatives,

$$\begin{aligned} \vec{u}(V_m) &:= (u(x_{m_1}), \dots u(x_{d_m})) &\in \mathbb{C}^{d_m}, \\ \partial_n \vec{u}(V_m) &:= (\partial_n u(x_{m_1}), \dots, \partial_n u(x_{d_m})) &\in \mathbb{C}^{d_m}, \end{aligned}$$

as follows:

$$i(S_m - I)\vec{u}(V_m) = (S_m + I)\partial_n \vec{u}(V_m). \tag{1.1}$$

Here S_m is an arbitrary unitary matrix. In order to accurately reflect the topology of the graph, we shall require that each S_m is irreducible, as a reducible matrix imposes relations between the limiting values and normal derivatives of a function that more properly corresponds to a subdivision of

V_m into several vertices. We form the matrix S as the block-diagonal matrix with blocks equal to S_m (in the basis where all boundary values $u(x_j), \partial_n u(x_j)$ are arranged in accordance to the vertices they belong to).

Example 1.1. The most common vertex conditions are the *standard conditions* (also known as Kirchoff-, Neumann-, free, and natural conditions), are given at each vertex V_m by the relations

$$\begin{cases} u \text{ continuous at } V_m, \\ \sum_{x_j \in V_m} \partial_n u(x_j) = 0. \end{cases} \tag{1.2}$$

In other words, u is required to be continuous in each vertex V_m and the sum of normal derivatives must vanish. The corresponding matrix can be calculated to be :

$$(S_m^{\text{st}})_{ij} = \begin{cases} -\frac{2}{d_m}, & i \neq j, \\ 1 - \frac{2}{d_m}, & i = j. \end{cases} \tag{1.3}$$

We denote corresponding self-adjoint operator by $L_q^{\text{st}}(\Gamma)$. It is easy to check hat S_m^{st} is hermitian with -1 is an eigenvalue of multiplicity one and 1 an eigenvalue of multiplicity $d_m - 1$.

On an open edge (x_{2n-1}, x_{2n}) with $x_{2n-1} \in V_m$, the solution to $-\psi_n'' = \lambda \psi_n$ can be written in terms of incoming and outgoing waves at x_{2n-1} as $\psi = a_n e^{ik(x-x_n)} + b_n e^{-ik(x-x_n)}$. For a function ψ on Γ the vertex conditions impose a relation between the coefficients $\vec{a}_m = (a_i)_{i=1}^{d_m}$ and $\vec{b}_m = (b_i)_{i=1}^{d_m}$ at each vertex V_m which we may write as $\vec{a}_m = S_m(k)\vec{b}_m$. The matrix $S_m(k)$ is called the vertex scattering matrix for k-waves at V_m. We form the vertex scattering matrix $S_v(k)$ as the block-diagonal vector with entries $S_m(k)$. The vectors of boundary values and normal derivatives can then be written

$$\begin{aligned} \vec{\psi} &= \vec{b} + S_v(k)\vec{b}, \\ \partial\vec{\psi} &= -ik\vec{b} + ikS_v(k)\vec{b}. \end{aligned}$$

Subtituting this into the (1.1) we obtain

$$i(S_m - I)(I + S_v(k)\vec{b} = (S+I)ik(-I + S_v(k))\vec{b},$$

so provided the vertex conditions are given by (1.1) the vertex scattering matrix $S_v(k)$ is given by ([13; 14; 19])

$$S_v(k) = \frac{(k+1)S + (k-1)I}{(k-1)S + (k+1)I}. \tag{1.4}$$

Setting $k = 1$ we see that S corresponds to the vertex scattering matrix for $k = 1$, i.e. $S = S_v(1)$. Since $S_v(k)$ is unitary it may be written using the spectral representation of S, with eigenvalues $e^{i\theta_n}$ and eigenvectors $\vec{e}_n$, as [11; 12; 19]

$$\begin{aligned} S_v(k) &= \sum_{n=1}^{d} \frac{(k+1)e^{i\theta_n} + (k-1)}{(k-1)e^{i\theta_n} + (k+1)} \langle \vec{e}_n, \cdot \rangle_{\mathbb{C}^d} \vec{e}_n \\ &= \sum_{n=1}^{d} \frac{k(e^{i\theta_n}+1) + (e^{i\theta_n}-1)}{k(e^{i\theta_n}+1) - (e^{i\theta_n}-1)} \langle \vec{e}_n, \cdot \rangle_{\mathbb{C}^d} \vec{e}_n \\ &= \sum_{\theta_n = \pi} (-1) \langle \vec{e}_n, \cdot \rangle_{\mathbb{C}^d} \vec{e}_n + \sum_{\theta_n \neq \pi} \frac{k(e^{i\theta_n}+1) + (e^{i\theta_n}-1)}{k(e^{i\theta_n}+1) - (e^{i\theta_n}-1)} \langle \vec{e}_n, \cdot \rangle_{\mathbb{C}^d} \vec{e}_n. \end{aligned} \tag{1.5}$$

Thus $S_v(k)$ has the same (k-independent) eigenvectors as S but the corresponding eigenvalues are in general k-dependent. The eigenvalues ± 1 are invariant, and all other eigenvalues tend to 1 as $k \to \infty$. Thus, if S is Hermitian — so that $e^{i\theta_n} = \pm 1$ for all n — $S_v(k)$ does not depend on k. Such conditions are called *non-resonant*, and all other conditions *resonant*. Note in particular that standard conditions are non-resonant.

If S is not Hermitian we define the high energy limit of $S_v(k)$ as follows:

$$S_v(\infty) = \lim_{k \to \infty} S_v(k) = -P^{(-1)} + (I - P^{(-1)}) = I - 2P^{(-1)}, \tag{1.6}$$

where $P^{(-1)}$ is the orthogonal projection onto the eigenspace associated with -1. The high-energy limits $S_m(\infty)$ of vertex scattering matrices associated with each particular vertex are defined in an analogous way.

We can then define a new operator $L_0^{S_v(\infty)}$ that is obtained from L_0^S by letting the vertex conditions be given by $S_v(\infty)$ instead of S. In general it is not the case that $L_0^{S_v(\infty)}(\Gamma)$ is equal to $L_0^{\mathrm{st}}(\Gamma)$, even though eigenvalues of $S_v(\infty)$ can only be 1 and -1. The multiplicities may be wrong, and in fact the vertex conditions given by $S_v(\infty)$ need not even be properly connecting — the blocks $S_m(\infty)$ might be reducible and the operator therefore appropriate to a graph Γ^∞ obtained from Γ by dividing some vertices in Γ into several vertices.

Definition 1.2. We say that vertex conditions on Γ given by S are *asymptotically properly connecting* if the high energy limits $S_m(\infty)$ of all vertex scattering matrices are irreducible.

If vertex conditions are asymptotically properly connecting, then $\Gamma^\infty = \Gamma$.

Definition 1.3. We say that vertex conditions on Γ given by S are *asymptotically standard* if it is the case that $S_v(\infty) = S^{\mathrm{st}}(\Gamma^\infty)$.

Any non-resonant conditions are asymptotically properly connecting, and standard vertex conditions are asymptotically standard. See [21] for a further discussion of asymptotically standard conditions.

The operator

We may now define the Schrödinger operator on a metric graph.

Definition 1.4. Let Γ be a finite compact metric graph, q a real valued absolutely integrable potential on the graph $q \in L^1(\Gamma)$, and S_m be $d_m \times d_m$ irreducible unitary matrices. Then the operator $L_q^S(\Gamma)$ is defined on the functions from the Sobolev space $u \in W_2^1(\Gamma \setminus \mathbb{V})$ such that $-u'' + qu \in L^2(\Gamma)$ and satisfy vertex conditions (1.1).

The spectrum

The spectrum $\sigma(L_q^S(\Gamma))$ is discrete and accumulates at infinity for any finite compact Γ and $q \in L^1(\Gamma)$. The main tool for the investigation of the spectrum in this thesis is the fact that the positive spectrum $\sigma(L_0^S(\Gamma)) \setminus \{0\}$ of $L_0^S(\Gamma)$ with S non-resonant is determined by the secular equation [10; 15; 20; 22]

$$\underbrace{\det(S_v(k)S_e(k) - I)}_{=:\, p(k)} = 0, \tag{1.7}$$

where $S_e^n(k)$ is the edge-scattering matrix given by 2×2 blocks $\begin{pmatrix} 0 & e^{ik\ell_n} \\ e^{ik\ell_n} & 0 \end{pmatrix}$ on the diagonal (the diagonal form is in the basis where the endpoints are arranged in the order of the edges). To see this note that on each edge $[x_{2n-1}, x_{2n}]$ a solution to $-\psi'' = \lambda\psi$ for $\lambda > 0$ can be written both in terms of incoming or out going waves:

$$\begin{aligned} \psi(x) &= a_{2n-1}e^{i(x-x_{2n-1})} + a_{2n}e^{-i(x-x_{2n})} && (1.8)\\ &= b_{2n-1}e^{-i(x-x_{2n-1})} + b_{2n}e^{i(x-x_{2n})}. && (1.9) \end{aligned}$$

These two representations are in turn related via the edge scattering matrix $S_e^n(k)$:

$$\begin{pmatrix} b_{2n-1} \\ b_{2n} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & e^{ik\ell_n} \\ e^{ik\ell_n} & 0 \end{pmatrix}}_{=:\, S_e^n(k)} \begin{pmatrix} a_{2n-1} \\ a_{2n} \end{pmatrix}. \tag{1.10}$$

Collecting all these relations we get $\vec{B} = S_e(k)\vec{A}$, where $S_e(k)$ is block diagonal with entries $S_e^n(k)$, and the entries of $\vec{A}, \vec{B}$ are ordered as the edges of Γ. By definition the edge scattering matrix S_v must then act as $\vec{A} = S_v\vec{B}$, so $S(k)\vec{A} = S_v(k)S_e(k)\vec{A} = \vec{A}$, which has a solution if and only if $\det(S(k)-I) = 0$. For non-resonant vertex conditions S_v does not depend on k which implies $S_vS_e(k) - I$ has entries that are trigonometric polynomials and therefore its determinant is also a trigonometric polynomial. Thus we have sketched one part of the proof of the following proposition:

Proposition 1.5. *Let the vertex conditions on Γ be non-resonant, then the non-zero eigenvalues of $L_0^S(\Gamma)$ are given as zeros of a generalised trigonometric polynomial:*

- *The function $p(k)$ defined by (1.7) can be written in the form:*

$$p(k) := \det(S_v(k)S_e(k) - I) \equiv \det(SS_e(k) - I) = \sum_{j=1}^{J} a_j e^{i\omega_j k}, \qquad (1.11)$$

  with k-independent coefficients a_j[1] $\in \mathbb{C}$ *and $\omega_j \in \mathbb{R}$.*

- *A point $\lambda = k^2 > 0$ is an eigenvalue of $L_0^S(\Gamma)$ if and only if $p(k) = 0$.*

- *The multiplicity of every eigenvalue $\lambda_n(L_0^S(\Gamma)) = k_n^2$ coincides with the order of the corresponding zero of the function p.*

For the proof, see [23] and [21], see also [30].

Remark 1.6. In the proof of Proposition 1.5 one uses explicitly that $\lambda > 0$, and in general the trigonometric polynomial does not give the correct multiplicity for the eigenvalue 0. For example the circle S^1 of length π with one vertex $L_0^{\text{st}}(S^1)$ has spectrum $0, 2^2, 2^2, 4^2, 4^2, \ldots$, so that it has one eigenvalue of multiplicity 1 and all other eigenvalues are of multiplicity 2. This is not the zero set of a trigonometric polynomial (see [23]), and indeed the trigonometric polynomial associated with $L_0^{\text{st}}(S^1)$ is $p(k) = (e^{i\pi k} - 1)^2$ which has a zero of order two at $k = 0$.

This characterization of the spectrum of Laplace operators with non-resonant vertex conditions in terms of the zeros of trigonometric polynomials was used in Paper IV to prove that if the spectra of two such operators do not grow apart too quickly then the operators have to be isospectral, except that the multiplicity of the eigenvalue zero may differ. We introduced the following terminology

[1] the a_j's here are not the amplitudes appearing in (1.8)

Definition 1.7. Two semi-bounded self-adjoint operators A, B with discrete spectrum are said to be asymptotically isospectral if

$$\sqrt{\lambda_n(A)} - \sqrt{\lambda_n(B)} \to 0, \quad \text{as } n \to \infty.$$

The spectra of quantum graphs satisfy Weyl asymptotics, namely

$$\frac{\lambda_n(L_q^S(\Gamma))}{(\pi n/\mathcal{L})^2} \to 1, \quad \text{as } n \to \infty.$$

From this follows in particular that two quantum graphs are almost isospectral if the eigenvalues grow apart sub-linearly:

$$|\lambda_n(L_{q_1}^{S_1}(\Gamma_1)) - \lambda_n(L_{q_2}^{S_2}(\Gamma_2))| \le C n^{1-\epsilon}, \quad \text{for some } \epsilon > 0.$$

From an investigation of asymptotically close zeros of trigonometric polynomials in Paper III and Paper IV, it was shown that in particular

Theorem 1.8 (Paper IV)**.** *Let Γ_1, Γ_2 be finite, compact and connected. Suppose that $L_0^{\text{st}}(\Gamma_1)$ and $L_0^{\text{st}}(\Gamma_2)$ are asymptotically isospectral then $L_0^{\text{st}}(\Gamma_1)$ and $L_0^{\text{st}}(\Gamma_2)$ are isospectral.*

2 Inverse spectral theory

2.1. Classical inverse spectral theory for finite intervals Inverse spectral theory for Schrödinger operators began in 1929 — three years after Schrödinger published his equation — with Ambartsumian's classical Paper [1]. There he proved that if the spectrum of a Schrödinger operator $-\frac{d^2}{dx^2} + q$ on a finite interval with Neumann conditions at the endpoints coincided with that of $-\frac{d^2}{dx^2}$ with Neumann conditions at the endpoints, then $q \equiv 0$. Since Neumann conditions at a vertex of degree one is just standard conditions the theorem in our notation becomes

Theorem 2.1. *Let $q \in L^1([0,\ell])$. If*

$$\sigma(L_q^{\text{st}}([0,\ell])) = \sigma(L_0^{\text{st}}([0,\ell]))^1$$

then $q \equiv 0$.

For the proof we need the following standard result (see e.g. [26])

[1] $\sigma(L_0^{\text{st}}([0,\ell])) = \left\{\left(\frac{\pi}{\ell} n\right)^2 : n \in \mathbb{N}\right\}$.

Theorem 2.2. *Let $\phi(x,\lambda)$ be a solution of*

$$-\phi''_{xx} + q(x)\phi = k^2\phi,$$

with $k^2 = \lambda$, satisfying

$$\phi(0,\lambda) = 1, \quad \phi'_x(0,\lambda) = 0.$$

Then there exists a unique $K(\cdot,\cdot)$ with locally integrable first derivatives with respect to each argument, such that

$$\phi(x,\lambda) = \cos kx + \int_0^x K(x,t)\cos kt\,dt, \tag{2.1}$$

$$K(x,x) = \frac{1}{2}\int_0^x q(t)\,dt. \tag{2.2}$$

Proof. Theorem 2.1 With the representation (2.1) it is clear that $\lambda_n \in \sigma(L_q^{\mathrm{st}}[0,\ell])$ if and only if $\phi'_x(\ell,\lambda_n) = 0$, which we may then write as

$$-k_n \sin k_n\ell + K(\ell,\ell)\cos k_n\ell + \int_0^\ell K_x(\ell,t)\cos k_n t\,dt = 0. \tag{2.3}$$

For $L_q^{\mathrm{st}}([0,\pi])$ we in general have

$$k_n - \frac{\pi}{\ell}n = O(1/n).$$

Thus k_n has an asymptotic expansion

$$k_n = \frac{\pi}{\ell}n + \frac{a_0}{n} + \frac{\gamma_n}{n}$$

with γ_n second order correction terms , so in particular $\gamma_n \to 0$ as $n \to \infty$. Plugging this expansion into (2.3) we get

$$\begin{aligned} 0 = &-\left(\frac{\pi}{\ell}n + \frac{a_0}{n} + \frac{\gamma_n}{n}\right)(-1)^n\ell\left(\frac{a_0}{n} + \frac{\gamma_n}{n} + O(1/n^2)\right) \\ &+ K(\ell,\ell)(-1)^n(1 + O(1/n^2)) + \int_0^\ell K_x(\ell,t)\cos k_n t\,dt. \end{aligned} \tag{2.4}$$

Since $K_x(\cdot,t) \in L^1(0,\ell)$, by the Riemann-Lebesgue Lemma

$$\int_0^\ell K_x(\ell,t)\cos k_n t\,dt \to 0, \quad \text{as } n \to \infty.$$

Collecting terms in (2.4) we can then see that

$$a_0 = \frac{K(\ell,\ell)}{\pi} = \frac{\int_0^\ell q(t)\,dt}{2\pi},$$

so the asymptotic expansion of k_n is

$$k_n = \frac{\pi}{\ell}\left(n + \frac{1}{2}\frac{\int_0^\ell q(t)\,dt}{\ell}\frac{1}{n} + o(1/n)\right).$$

But since $k_n(L_q^{\mathrm{st}}([0,\ell])) = \frac{\pi}{\ell} n$ by assumption, we conclude that $\int_0^\ell q(t)\,dt = 0$. Plugging in the constant function $u(x) = 1/\ell$ in the quadratic form of L_q^{st} we obtain

$$Q_{L_q^{\mathrm{st}}}(u) = \int_0^\ell |u'(x)|^2\,dx + \int_0^\ell q(x)|u(x)|^2\,dx = 0,$$

so $u(x) = 1/\ell$ is an eigenfunction for the eigenvalue 0 of L_q^{st}, but then

$$-u'' + q(x)u(x) = q(x)/\ell = 0,$$

so q is zero almost everywhere. □

The same result holds for periodic boundary conditions but is false for Dirichlet boundary conditions. In general two spectra for two different boundary conditions are needed to determine q, see [5], [25] and [26].

2.2. Inverse spectral theory for quantum graphs The inverse spectral problem for a quantum graph (Γ, L_q, S) consists of determining the three elements of the triple from the spectrum $\sigma(L_q^S(\Gamma))$. This problem does not, as indicated in the previous section, admit a complete solution: already the simplest examples of quantum graphs form an obstruction as $\sigma(L_q^S([0,\ell]))$ does not determine $q \equiv 0$ uniquely, for general boundary conditions S. Furthermore the graphs themselves are not determined by the spectrum of the associated operators: examples of graphs and vertex conditions such that $\sigma(L_0^{S_1}(\Gamma_1)) = \sigma(L_0^{S_2}(\Gamma_2))$ where Γ_1 and Γ_2 are not isometric have been constructed (see for example [10], [3]).

The inverse problem may be partially solved in some cases where one compares the spectrum of an operator with the spectrum of a reference operator, where one or several elements of (Γ, L_q, S) are fixed.

If Γ is fixed and the spectrum of $L_q^{\mathrm{st}}(\Gamma)$ is compared with the spectrum of the reference operator $L_0^{\mathrm{st}}(\Gamma)$ a direct analogue of Ambartsumian's Theorem has been proven. There were several partial results in this direction where certain subclasses of graphs were considered, see [28] [29] [7] [31], [24] and the general theorem was obtained by Davies [8]:

Theorem 2.3. *Let $q \in L_\infty(\Gamma)$ and suppose that $\sigma(L_q^{\mathrm{st}}(\Gamma)) = \sigma(L_0^{\mathrm{st}}(\Gamma))$, then $q \equiv 0$.*

It turns out, however, that one doesn't actually have to fix Γ to obtain this uniqueness result for q. In Paper IV the theorem was extended to the case where $\sigma(L_q^{\text{st}}(\Gamma_1)) = \sigma(L_0^{\text{st}}(\Gamma_2))$ for two different graphs. The proof used an estimate from Paper I showing that $|\lambda_n(L_q^{\text{st}}(\Gamma)) - \lambda_n(L_0^{\text{st}}(\Gamma))| < C$ for some C and all n, together with Theorem 1.8:

Theorem 2.4 (Paper IV)**.** *Let Γ_1 be a finite compact graph, $q \in L_\infty(\Gamma_1)$ and suppose that $\sigma(L_q^{\text{st}}(\Gamma_1)) = \sigma(L_0^{\text{st}}(\Gamma_2))$ for some finite compact Γ_2. Then $q = 0$.*

If one sets $\Gamma = I$ (a finite interval), $q = 0$ and let S to be standard vertex conditions, then:

Theorem 2.5 ([27], [9] and [18])**.** *Let Γ be a finite compact metric graph with total length L. If $\sigma(L_0^{\text{st}}(\Gamma)) = \sigma(L_0^{\text{st}}([0, L]))$ then $\Gamma = I$.*

Here the reference operator is $L_0^{\text{st}}(I)$, and Theorem 2.5 can be considered as a *geometric* version of Ambartsumian's Theorem 2.1, in that the uniqueness is for the graph rather than the potential. For non-resonant vertex conditions it is crucial that the vertex conditions are standard, as was shown in Paper II. While keeping $q = 0$ in the reference operator this result was extended to $L_q^{\text{st}}(\Gamma)$ in Paper I, and $L_q^S(\Gamma)$ with S asymptotically standard conditions in Paper II:

Theorem 2.6 (Paper I)**.** *Let Γ be a finite compact metric graph and $q \in L^1(\Gamma)$. The spectrum of the standard Schrödinger operator $L_q^{\text{st}}(\Gamma)$ coincides with the spectrum of the standard Laplacian on an interval*

$$\lambda_n(L_q^{\text{st}}(\Gamma)) = \lambda_n(L_0^{\text{st}}(I)), \tag{2.5}$$

if and only if $\Gamma = I$ and $q \equiv 0$.

Spectral information about operators with two different resonant vertex conditions S_i, $i = 1, 2$, can be used to obtain spectral information about the operators with the limit vertex conditions $S_i(\infty)$ from (1.6), as it was shown in Paper II that going to the limit vertex conditions does not perturb the eigenvalues too much:

Theorem 2.7 (Paper II)**.** *Let Γ be a finite compact metric graph, $q \in L^1(\Gamma)$ and S be a unitary matrix parametrising properly connecting vertex conditions. Let $S_v(\infty)$ be the high-energy limit of the corresponding vertex scattering matrix and Γ^∞ — the corresponding metric graph so that $S_v(\infty)$ determines properly connecting vertex conditions on Γ^∞.*
Then the difference between the eigenvalues $\lambda_n(L_0^{S_v(\infty)}(\Gamma^\infty))$ and $\lambda_n(L_q^S(\Gamma))$ is bounded by a constant, i.e.

$$|\lambda_n(L_q^S(\Gamma)) - \lambda_n(L_0^{S_v(\infty)}(\Gamma^\infty))| \le C, \tag{2.6}$$

where $C = C(\Gamma, \|q\|_{L^1(\Gamma)}, S)$ is independent of n.

This together with a general version of Theorem 1.8 was used in Paper IV to show that

Theorem 2.8 (Paper IV)**.** *Suppose that* $L_{q_1}^{S_1}(\Gamma_1)$ *and* $L_{q_2}^{S_2}(\Gamma_2)$ *are asymptotically isospectral. Then* $L_0^{S_1(\infty)}(\Gamma_1^\infty)$ *and* $L_0^{S_2(\infty)}(\Gamma_2^\infty)$ *are isospectral.*

In particular for non-resonant conditions (i.e. when $S = S(\infty)$) asymptotically isospectrality implies isospectrality. For non-resonant conditions one may say that it is not possible for a potential q to move a Laplace operator $L_0^{S_1}(\Gamma_1)$ to a different isospectral class: if the spectrum of $L_q^{S_1}(\Gamma_1)$ looks (asymptotically) similar to that of $L_0^{S_2}(\Gamma_2)$, then the Laplace operators on the two graphs are isospectral.

The reconstruction of a graph from the spectrum of the Laplace operator is in general not possible as we have seen. If one is willing to restricts the class of graphs under consideration, however, it is possible to reconstruct the graph from the spectrum of the standard Laplacian $L_0^{\mathrm{st}}(\Gamma)$:

Theorem 2.9 ([10], [15] and [20])**.** *The spectrum of a Laplace operator with standard conditions on a metric graph determines the graph uniquely, provided that the graph is finite and connected, has no vertices of degree* 2*, and the edge lengths are rationally independent.*

This theorem was generalized in Paper IV in by showing that $\sigma(L_q^{\mathrm{st}}(\Gamma))$ also determines Γ if Γ satisfies the asumptions of Theorem 2.9, though an algorithm for reconstructing Γ was not given as in [10], [15] and [20]. Note however that the claim is not that no other graph Γ_2 with the same spectrum may exist, only that no other graph with rationally independent edge lengths with the same spectrum may exist.

Though this thesis contains no work on reconstructing non-zero potentials we mention some of the developments there has been in this area. Brown-Weikard [6] showed that the Dirichlet-to-Neumann map for a Schrödinger operator $L_q^{\mathrm{st}}(\Gamma)$ on a finite connected tree Γ uniquely determines q on Γ. Pivovarchik [29] showed that for star-graphs the spectrum $\sigma(L_q^{\mathrm{st}}(\Gamma))$ together with the spectra of the Dirichlet–Dirichlet problems on the edges of the graph determines q if the spectra are disjoint. An explicit procedure for recovering the potentials was also presented. Avdonin-Kurasov [2] used the boundary control method to prove that the response operator determines a quantum tree with standard conditions completely, i.e. it determines the lengths of the edges, their connectivity and the potential of each edge uniquely. Freiling-Yurko have shown that the potential may be recovered from several spectra corresponding to different vertex conditions at the boundary of so-called hedgehog-typ graphs [32].

Due to the fact that standard conditions (1.2) are usually assumed, the problem of reconstructing vertex conditions from the spectrum has received somewhat less attention than the other parts of the inverse spectral problem. The interested reader may see [16] and [17].

3 Summary of papers

Paper I

The paper generalizes the geometric version of Ambartsumian's Theorem (Theorem 2.5)to the case of $\sigma(L_q^{\mathrm{st}}(\Gamma)) = \sigma(L_0^{\mathrm{st}}([0,L]))$, with the conclusion being that also here $\Gamma = I$ and furthermore $q \equiv 0$. By Theorem 2.5 it is sufficient to show that the Laplace operators $L_0^{\mathrm{st}}(\Gamma)$ and $L_0^{\mathrm{st}}(I)$ are isospectral, for then $\Gamma = I$ and Ambartsumian's classical theorem implies $q \equiv 0$, with no extra work required. This is done by showing the uniform (in n) estimate

$$|\lambda_n(L_q^{\mathrm{st}}(\Gamma)) - \lambda_n(L_0^{\mathrm{st}}(\Gamma))| < C, \tag{3.1}$$

with the help of variational (Max-Min & Min-Max Theorems) characterizations of the spectrum of L_q. $\sqrt{\lambda_n(L_0^{\mathrm{st}}(\Gamma))}$ is given by the zeros of a trigonometric polynomial p and since $\sigma(L_q^{\mathrm{st}}(\Gamma)) = \sigma(L_0^{\mathrm{st}}([0,L]))$ it follows from (3.1) that

$$\sqrt{\lambda_n(L_0^{\mathrm{st}}(\Gamma))} - \sqrt{\lambda_n(L_0^{\mathrm{st}}(I))} \to 0.$$

Since $\sqrt{\lambda_n(L_0^{\mathrm{st}}(I))} = \pi n/|I|$ it is sufficient to show that a trigonometric polynomial which has zeros tending to $\pi n/|I|$ only have zeros at $\pi n/|I|$. This is done by using a generalization of Kronecker's Theorem and a suitable choice of subsequences of $\pi n/|I|$. Finally, the fact that $\sigma(L_0^{\mathrm{st}}(\Gamma))$ determines the Euler characteristic $\chi(\Gamma)$ of Γ is generalized by showing that $\sigma(L_q^{\mathrm{st}}(\Gamma))$ also determines $\chi(\Gamma)$.

Paper II

This paper continues the investigations of Paper I by allowing more general vertex conditions than the standard conditions that were assumed throughout Paper I. We introduce the notions of resonant and non-resonant vertex conditions S — corresponding to energy dependent and independent vertex scattering, respectively — and for resonant conditions define the high-energy limit $S(\infty)$ of the conditions. The proof of (3.1) is modified to show that also

$$|\lambda_n(L_q^S(\Gamma)) - \lambda_n(L_0^{S(\infty)}(\Gamma))| < C,$$

and we generalize the geometric version of Ambartsumian's Theorem to conditions where $S(\infty)$ coincides with standard conditions on Γ. We then present a family of graphs — intervals with an additional vertex with non-resonant non-standard conditions in the middle — with Laplacians $L_0^S(\Gamma)$ that are isospectral to the standard Laplacian on an interval of the same length. Whereas vertices of degree two with standard conditions are removable, the extra middle vertex in Γ is not removable and this shows that the geometric version of Ambartsumian's theorem can not be generalized to all vertex conditions.

Paper III

In Paper I we showed that if the zeros k_n of a trigonometric polynomial asymptotically tend to the integers, i.e. $k_n - n \to 0$ then $k_n = n$ for all n. This theorem is generalized to show that given two almost periodic functions f_1, f_2 in a horizontal strip in $\mathbb{C}$ with zeros k_n, l_n respectively, if $l_n - k_n \to 0$ then $k_n = l_n$ for all n, so if the zeros of f_1 and f_2 are asymptotically close they must coincide. The proof relies on the theory of almost periodic discrete sets and the fact that the set of zeros of an almost periodic function forms such a set.

Paper IV

The result from Paper III is generalized to deal with the case where a subsequence k_{n_k} of zeros of an almost periodic function f_1 is asymptotically close to the zeros l_n of an almost periodic function f_2, and we show that l_n must in fact me zeros of f_1 as well in this case. We give a new and more direct proof, using only basic tools from complex analysis. This result is then applied to the spectral theory of quantum graphs. Two semi-bounded operators with discrete spectra are called asymptotically isospectral if $|\sqrt{\lambda_n(A)} - \sqrt{\lambda_n(B)}| < C$, and we show that if the Laplacians with non-resonant vertex conditions on two connected graphs are asymptotically isospectral then they are in fact isospectral. We use this result to generalize Theorem 2.3 and 2.9 as described in Section 2.2 above.

Bibliography

[1] Ambartsumian, V. Über eine Frage der Eigenwerttheorie, Z. Phys., **53** 690-695 (1929) 11, 18

[2] Avdonin, S., Kurasov, P. Inverse problems for quantum trees, Inverse Problems and Imaging, **2** 1-21 (2008) 22

[3] Band, R., Parzanchevski, O., Ben-Shach, G. The isospectral fruits of representation theory: quantum graphs and drums, J. Phys. A **42**, no. 17, 175202 (2009) 20

[4] Berkolaiko, G., Kuchment, P. Introduction to Quantum Graphs. Mathematical Surveys and Monographs, Vol.186, AMS (2013) 11

[5] Borg, G. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe: Bestimmung der Differentialgleichung durch die Eigenwerte. Acta Math. **78** 1-96 (1946) 20

[6] Brown, B. M., Weikard, R. A Borg-Levinson theorem for trees. Proc. Royal Soc. A **461** 3231-3243 (2005) 22

[7] Carlson, R., Pivovarchik, V. Ambartsumian's Theorem for Trees, Elec. Jour. Diff. Eq. **142** 1-9 (2007) 20

[8] Davies, E. B. An Inverse Spectral Theorem. Jour. Op. Th.**69** 195-208 (2013) 20

[9] Friedlander, L. Extremal properties of eigenvalues for a metric graph. Ann. Inst. Fourier **55** 199-211 (2005) 21

[10] Gutkin, B., Smilansky, U. Can one hear the shape of a graph? J. Phys. A **34** 6061-6068 (2001) 16, 20, 22

[11] Harmer, M. Hermitian symplectic geometry and the factorization of the scattering matrix on graphs. J. Phys. A **33** 49, 9015-9032 (2000) 15

[12] Harmer, M. Hermitian symplectic geometry and extension theory. J. Phys. A **33** 50 9193-9203 (2000) 15

[13] Kostrykin, V., Schrader, R. Kirchhoff's rule for quantum wires. J. Phys. A **32** 4 595-630 (1999) 14

[14] Kostrykin, V., Schrader, R. Kirchhoff's rule for quantum wires. II. The inverse problem with possible applications to quantum computers. Fortschr. Phys. **48** 8 703-716 (2000) 14

[15] Kottos, T., Smilansky, U. Periodic orbit theory and spectral statistics for quantum graphs. Ann. Physics **274** 1 76-124 (1999) 16, 22

[16] Avdonin, S., Kurasov, P., Nowaczyk, M. Inverse Problems for Quantum Trees II Recovering Matching Conditions for Star Graphs. Inverse Problems and Imaging **4** 4 579-598 (2010) 23

[17] Kurasov, P. Inverse Problems for Quantum Graphs: Recent Developments and Perspectives. Acta Physica Polonica A **120** 132-141 (2011) 23

[18] Kurasov, P., Naboko, S. Rayleigh estimates for differential operators on graphs. J. Spectral Theory **4** 2211-219 (2014) 21

[19] Kurasov, P., Nowaczyk, M. Geometric properties of quantum graphs and vertex scattering matrices. Opuscula Math. **30** 3 295-309 (2010) 14, 15

[20] Kurasov, P., Nowaczyk, M. Inverse spectral problem for quantum graphs. J. Phys. A **38** 22 4901-4915 (2005) 16, 22

[21] Kurasov, P. Spectral theory of quantum graphs and inverse problems, to be published. 11, 16, 17

[22] Kurasov, P. Graph Laplacians and topology. Ark. Mat. **46** 1 95-11 (2008) 16

[23] Kurasov, P., Suhr, R. Asymptotically isospectral quantum graphs and trigonometric polynomials. Submitted. 17

[24] Law, C. K., Yanagida, E. A Solution to an Ambarsumyan Problem on Trees. Kodai Math. J. **35** 358-373 (2012) 20

[25] Levitan, B. M., Gasymov, M. G. Determination of a Differential Equation by two of its Spectra. Russ. Math. Surv. **19** 2 1-63 (1964) 20

[26] Marchenko, V. A. Sturm-Liouville Operators and Applications. Operator Theory: Advances and Applications vol. 22, Birkhäuser (1986) 18, 20

[27] Nicaise, S. Spectre Des Reseaux Topologiques Finis. Bull. Sc. math. **11** 401-413 (1987) 21

[28] Pivovarchik, V. N. Ambartsumian's Theorem for Sturm-Liouville Boundary Value Problem on a Star-Shaped Graph. Func. Anal. Appl. **39** 148-151 (2005) 20

[29] Pivovarchik, V. Inverse problem for the Sturm-Liouville equation on a star-shaped graph. Math. Nachr. **280** 1595-1619 (2007) 20, 22

[30] Schanz, H. A relation between the bond scattering matrix and the spectral counting function for quantum graphs- Quantum graphs and their applications, Contemp. Math., 415, 269–282. AMS, Providence (2006) 17

[31] Yang, C.F., Huang, Z.Y., Yang, X.P. Ambarzumyan-type theorems for the Sturm-Liouville equation on a graph.Rocky Mountain J. Math. **39** 1353-1372 (2009) 20

[32] Freiling, G., Yurko, V. Recovering Sturm-Liouville operators on hedgehog-type graphs with general matching conditions. Eurasian Math. J. **5** 4 56-69 (2014) 23

Matematiska Institutionen, Stockholms Universitet, 106 91 Stockholm
E-mail address: suhr@math.su.se