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# Quark and Gluon Form Factors At Three Loops In Perturbative QCD

Nehir Ikizlerli

A Thesis presented for the degree of  
Doctor of Philosophy



IPPP  
Department of Physics  
University of Durham  
England

September 2009

*Dedicated to*  
To My Father's Memory

# Quark And Gluon Form Factors At Three Loops in Perturbative QCD

Nehir Ikizlerli

Submitted for the degree of Doctor of Philosophy  
September 2009

## Abstract

We compute the quark and gluon form factors at up to three-loop order within massless perturbative Quantum Chromodynamics by studying the photon-quark-anti-quark vertex and the effective vertex of a Higgs boson and two gluons. We use Feynman diagram methods to derive expressions for the form-factors in terms of tensor loop integrals in  $D = 4 - 2\epsilon$  dimensions. We review various methods for relating tensor integrals to a basis set of master integrals and utilize the FIRE package based on Integration-By-Parts to perform the reduction, thereby enabling the form factors to be expressed (in  $D$ -dimensions) as a sum of master integrals. We assemble the known results for master integrals and use them to provide a Laurent expansion in  $\epsilon$  through to  $\mathcal{O}(\epsilon^0)$ . The results for the three-loop form factors may provide the building blocks for many third-order cross section calculations.

# Declaration

The work in this thesis is based on research carried out at the Institute For Particle Physics Phenomenology, Department of Physics, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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# Chapter 1

## Introduction

In this chapter we give a brief introduction to the theory of the strong interactions Quantum ChromoDynamics (QCD), emphasizing only the aspects needed for the rest of this thesis. More details can be found in the relevant textbooks [3–10].

### 1.1 Quark Model

The particles which interact strongly are called hadrons. They are observed either in fermionic (baryons) or bosonic (mesons) states. According to the quark model, the baryons are bound states of three quarks ( $qqq$ ) while the mesons are bound states of a quark and an anti-quark ( $q\bar{q}$ ). Six species (flavors) of quarks have been observed. The dynamics of the elementary quarks is described by Quantum Chromodynamics ( $QCD$ ). Quarks are considered to be point-like entities, as demonstrated by the scaling behavior observed in deep inelastic experiments, carrying color charge. The quarks also possess some properties depending of the type (flavour) of the quark. The corresponding anti-quarks are denoted by a negative sign.

As yet, no free quarks have been observed in nature and no other fractionally charged particles found. In order not to contradict the fundamental assumption of quantum mechanics, the wave function of any state must be anti-symmetric under the exchange of any identical spin 1/2 fermions, another property is attributed to the wave function called color. The wave function describing the quarks is therefore composed of spatial, spin and color parts. The quarks have color states denoted by

{**r**-red,**g**-green,**b**-blue}. According to the color confinement hypothesis, the hadrons can only occur in color singlets that have zero values of color charges and this explains why hadrons have integer electromagnetic charges. The existence of three color states was confirmed by the experimental observations of the ratio

$$R \equiv \frac{\sigma(e^+e^- \longrightarrow \text{hadrons})}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)}$$

as well as several other observables.

In QCD the quarks interact via gauge bosons called gluons which both undergo local  $SU(3)$  (Special Unitary) transformations. The properties of this transformation will be detailed in the next section. A three-color model of quarks has a local  $SU(3)$  symmetry, with currents reflecting its group structure. This suggested a local non-abelian gauge theory of the type originally introduced by Yang and Mills many years before.

As seen in the table below (taken from the Review of Particle Physics. by the Particle Data Group [11]), the masses of the quarks are light compared to the top quark.

Property/Quark	d	u	s	c	b	t
Q-electric charge	-1/3	+2/3	-1/3	+2/3	-1/3	+2/3
I-isospin	+1/2	+1/2	0	0	0	0
$I_z$ -isospin	-1/2	+1/2	0	0	0	0
S-strangeness	0	0	-1	0	0	0
C-charm	0	0	0	+1	0	0
B-bottomness	0	0	0	0	-1	0
T-topness	0	0	0	0	0	+1
Mass	$5.04^{+0.96}_{-1.54}$ Mev	$2.55^{0.75}_{-1.05}$ Mev	$105^{+25}_{-35}$ Mev	$1.27^{+0.07}_{-0.11}$ Gev	$4.20^{+0.17}_{-0.07}$ Gev	$171.2 \pm 2.1$ Gev

Table 1.1: Additive quantum numbers and masses of quarks. (The u,d,s quark masses are estimates of the "current quark masses", c and b quarks masses are "running" masses in  $\overline{MS}$  scheme.)



## 1.2 QCD Lagrangian

The strong interaction is based on the Lagrangian density

$$\mathcal{L}_{QCD} = \mathcal{L}_{classical} + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost} \quad (1.1)$$

which is a non-Abelian gauge theory based on a  $SU(3)$  group.

The Classical Lagrangian of QCD is given by:

$$\mathcal{L}_{classical} = \sum_f \bar{\psi}_{f,i} (i\not{D} - m_f \delta_{ij}) \psi_{f,j} - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \quad (1.2)$$

where the index  $f$  runs over all quark flavours. The quark fields  $\psi_{f,i}$  live in the fundamental representation with colour index  $i = 1, \dots, N$  whereas the gluon fields  $A_\mu^a$  are in the adjoint representation with  $a = 1, \dots, N^2 - 1$ . Here  $N$  represents the number of colours,  $N = 3$ . Conventional notation for  $\not{D} = \gamma_\mu D^\mu$  where the gamma matrices satisfy the Clifford algebra,

$$\{\gamma^\nu, \gamma^\mu\} \equiv \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 2g^{\nu\mu}. \quad (1.3)$$

The covariant derivative in the non-Abelian gauge theory is defined as:

$$D_{ij}^\mu = \partial_{ij}^\mu - ig A_\mu^a T_{ij}^a \quad (1.4)$$

where  $g$  is the strong coupling constant. The generators of the fundamental representation group satisfy the commutation relation,

$$[T^a, T^b] = if_{abc} T^c \quad (1.5)$$

where  $f_{abc}$  is known as the structure constant. The Pauli matrices relevant for the  $SU(2)$  gauge theory describing weak interactions have a simple generalization to  $SU(3)$ . For  $SU(3)$  there are  $3^2 - 1 = 8$  generators which are  $3 \times 3$  hermitian, traceless Gell-Mann matrices and  $f_{abc}$  is totally antisymmetric. The commutator of two covariant derivatives gives the field strength tensor of the gluon fields which is related to the kinetic energy term in the classical Lagrangian part of QCD,

$$[D_\mu, D_\nu] = ig T^a F_{\mu\nu}^a \quad (1.6)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c. \quad (1.7)$$

Unlike the abelian theory Quantum Electrodynamics (QED), gluons undergo self interactions. The last term of eq. (1.2) exhibits the non-abelian character of QCD and produces interactions amongst gluons (three and four gluon vertices) in the theory. As required,  $\mathcal{L}$  is invariant under local  $SU(3)$  gauge transformations. Under these transformations the quark fields transform in the fundamental representation, and gluon fields transform in the adjoint representation of  $SU(3)$  so that,

$$\psi_f \longrightarrow U(x)\psi_f, \quad (1.8)$$

$$T^a A_\mu^a \longrightarrow U(x) \left( T^a A_\mu^a - \frac{i}{g} U^{-1}(x) \partial_\mu U(x) \right) U^{-1}(x), \quad (1.9)$$

$$U(x) = \exp(-iT^a \theta^a(x)), \quad (1.10)$$

and where  $\theta^a(x)$  is an arbitrary spacetime dependent function.

If one tries to canonically quantize the theory, a major problem occurs. The canonical momentum of the gluon fields  $A_a^\mu$  vanishes. This is because of the freedom to make gauge transformations of the gluon fields. A spin-1 massless particle has two physical degrees of freedom (polarization), therefore one has to put a constraint on the gluon field  $A^\mu$  in order to avoid unphysical states. This is achieved by the *Lorentz condition*  $\partial^\mu A_\mu = 0$  which leads to an additional gauge-fixing term inserted in the Lagrangian:

$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2\xi} (\partial^\mu A_\mu^a)^2. \quad (1.11)$$

The  $\xi$  is called the gauge parameter and we are free to choose any value for  $\xi$ . The physical predictions arising from it, are independent of  $\xi$ . Some common choices are:

$$\underbrace{\xi = 0}_{\text{Landau-gauge}}, \quad \underbrace{\xi = 1}_{\text{Feynman-gauge}}, \quad \underbrace{\xi \rightarrow \infty}_{\text{Unitarity-gauge}}. \quad (1.12)$$

In this thesis we work in axial gauge for the external gluons,  $\eta_\mu \cdot A^\mu = 0$ ,  $\eta^2 \neq 0$  and Feynman gauge for the internal gluons due to simplify the problem at hand. In the axial gauge there is no need to account for the ghosts. In QCD the longitudinal part of the gluon field can interact with the transverse (physical) component of  $A_\mu^a$ . Those unphysical longitudinal components contribute to gluon loops and have to be subtracted. In order to do this we introduce a new fictitious field called the

Faddeev-Popov ghost. They behave like a scalar field but are quantized as a fermion field:

$$\mathcal{L}_{ghost} = (\partial_\mu \eta^{a*})(\partial^\mu \delta_{ab} + g f_{abc} A_c^\mu) \eta^b. \quad (1.13)$$

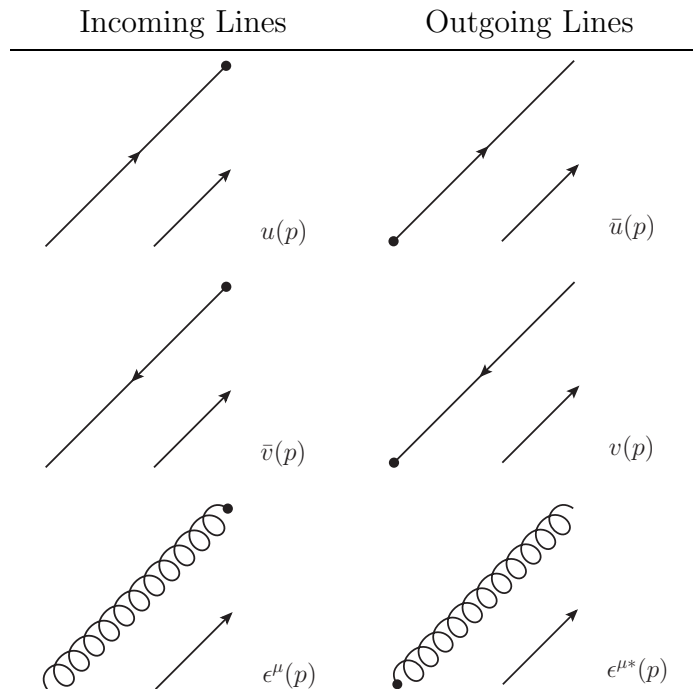
The final form of the Lagrangian is,

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f (i \not{D} - m_f \mathbf{1}) \psi_f - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 + (\partial_\mu \eta^{a*}) D_{ab}^\mu \eta^b. \quad (1.14)$$

### 1.3 Feynman Rules

Feynman devised a pictorial method to calculate the terms in the perturbative expansion called Feynman rules. This method was directly derived from the Lagrangian. With the help of these rules one can calculate the perturbative predictions of the physical observables such as the decay width ( $\Gamma$ ) and cross-section ( $\sigma$ ) of the related process. According to the established conventions, quarks are represented as solid lines, gluons as curly lines and ghosts are dashed lines.

For the external legs for quarks and gluons we have.



In order to calculate the matrix elements we need some identities for the fermion

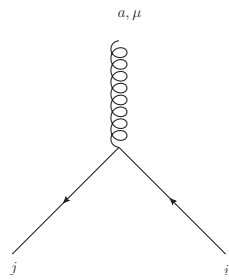
spins and gluon polarizations:

$$\begin{aligned} \sum_{spins} \bar{u}(p)u(p) &= \not{p} + m, \\ \sum_{spins} \bar{v}(p)v(p) &= \not{p} - m, \\ \sum_{pols} ((\epsilon^\mu(p))^*)\epsilon^\nu(p) &= \left[ -g^{\mu\nu} + \frac{p^\mu\eta^\nu + p^\nu\eta^\mu}{p\cdot\eta} - \frac{\eta^2 p^\mu p^\nu}{(p\cdot\eta)^2} \right]. \end{aligned}$$

As mentioned in eq. (1.12), in the Feynman gauge  $\xi \rightarrow 1$ , Landau gauge  $\xi \rightarrow 0$ . The propagators for quarks, gluons and ghosts are:

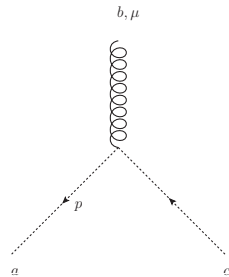
$$\begin{aligned} & \begin{array}{c} p, m, i \qquad \qquad p, m, j \\ \longrightarrow \\ \longrightarrow \end{array} \qquad \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \delta_{ij} \\ \\ & \begin{array}{c} p, a, \mu \qquad \qquad p, b, \nu \\ \text{oooooooooooooooooooo} \\ \longrightarrow \end{array} \qquad \frac{-i}{p^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{p^\mu\eta^\nu + p^\nu\eta^\mu}{p\cdot\eta} + \frac{\eta^2 p_\mu p_\nu}{(p\cdot\eta)^2} \right] \delta^{ab} \\ \\ & \begin{array}{c} p, a \qquad \qquad p, b \\ \text{.....} \\ \longrightarrow \end{array} \qquad \frac{-i}{p^2 + i\epsilon} \delta^{ab} \end{aligned}$$

The Lorentz indices are denoted by  $\{\mu, \nu, \dots\}$ . The color indices for gluon and ghost are denoted by  $\{a, b, \dots\}$  and for quarks  $\{i, j, \dots\}$ . We assigned a positive imaginary part  $+i\epsilon$  to the denominator to ensure the propagation from past to future. The interaction vertices for quark-gluon, gluon-ghost and gluon self interactions are as follows:



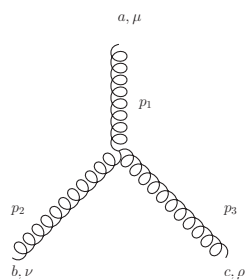
$$-ig\gamma^\mu(T^a)_{ij}.$$

Figure 1.1: The quark-gluon vertex



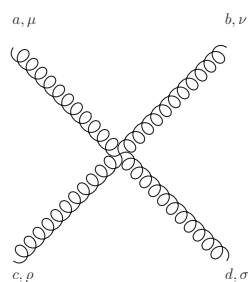
$$gf^{abc}p^\mu$$

Figure 1.2: The ghost-gluon vertex



$$\begin{aligned} & -gf^{abc}[(p_1 - p_2)^\rho g^{\mu\nu} \\ & \quad + (p_2 - p_3)^\mu g^{\nu\rho} \\ & \quad + (p_3 - p_1)^\nu g^{\mu\rho}] \end{aligned}$$

Figure 1.3: The three gluon vertex



$$\begin{aligned} & -ig^2 f^{abe} f^{cde} (g^{\nu\sigma} g^{\mu\rho} - g^{\mu\rho} g^{\nu\sigma}) \\ & -ig^2 f^{ace} f^{bde} (g^{\rho\sigma} g^{\mu\nu} - g^{\mu\sigma} g^{\nu\rho}) \\ & -ig^2 f^{ade} f^{cbe} (g^{\nu\rho} g^{\mu\sigma} - g^{\rho\sigma} g^{\mu\nu}) \end{aligned}$$

Figure 1.4: The four gluon vertex

The amplitude for a particular process is constructed by first writing down all distinct Feynman diagrams of the required order in the coupling constant using above rules. To compute the amplitude one must follow the prescription given below:

- Multiply by -1 for each fermion or ghost loop.
- Impose momentum conservation at each vertex.
- Integrate over any unconstrained momenta appearing in closed loops with the measure  $\int \frac{d^4 p}{(2\pi)^4}$ .
- Multiply by a symmetry factor to allow for permutation of the fields.

## 1.4 Regularization

Having briefly described the main ingredients of the theory in the last section, we are now able to do some calculations at leading order (LO) in the strong coupling  $\alpha_s$ . The interaction of any set of quarks and gluons is described by the invariant matrix element  $\mathcal{M}_{i \rightarrow f}$ . The squared matrix element is always proportional to an even power of the coupling  $g$ , therefore it is usual to make the perturbative expansion in powers of  $\alpha_s$ , where

$$\alpha_s = \frac{g^2}{4\pi}. \quad (1.15)$$

In a tree-level process one does not deal with divergences. But if precision results are to be required, one must go beyond leading order to next-to-leading order (NLO) and beyond. As soon as higher orders in  $\alpha_s$  are considered, divergences emerge in the intermediate steps of the calculation, although the final result describing a physically measurable quantity has to be finite. This does not mean that the Lagrangian is incapable of defining the theory. At this stage it is better to sketch some Feynman diagrams as to grasp what we mean by LO and NLO.

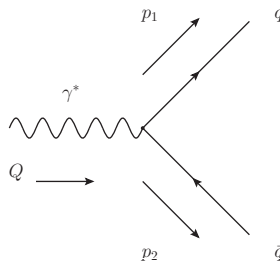


Figure 1.5: Leading Order Feynman diagram for  $\gamma^* \rightarrow q\bar{q}$

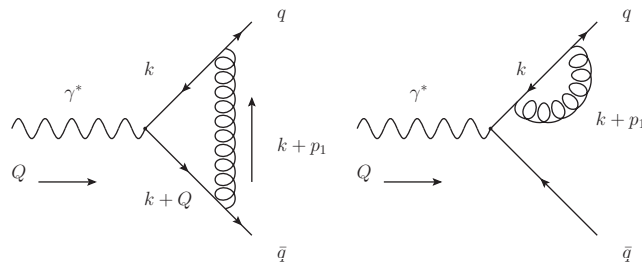


Figure 1.6: Next-to-Leading Order Feynman diagrams for the virtual corrections to  $\gamma^* \rightarrow q\bar{q}$

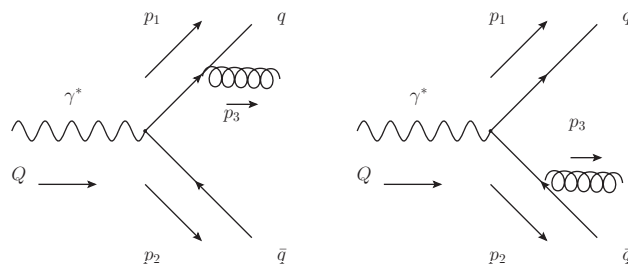


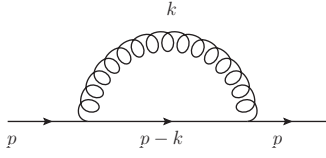
Figure 1.7: Leading Order Feynman diagrams for the real corrections to  $\gamma^* \rightarrow q\bar{q}$ .

The divergent pieces are generated by the behavior of the integrand at high and low virtual momenta. There are basically two sources of singularities:

- Ultraviolet divergences (UV) emerge when the loop momentum is not bounded and can take arbitrarily large values.

The below integral  $\mathcal{I}$  diverges as  $k \rightarrow \infty$ .

- Infrared divergences (IR) emerge when ;
  - a) the momentum of an emitted parton approaches zero.
  - b) a parton is emitted collinearly and propagates in the same direction as the



$$\mathcal{I} = \int \frac{d^4k}{(2\pi)^4} \frac{f(k)}{k^2(k-p)^2}$$

Figure 1.8: The one-loop quark self-energy diagram

parent.

In order to isolate and mathematically manipulate the divergent behavior of these integrals a technique called *regularization* has to be applied. A proper regularization procedure should respect Lorentz invariance and unitarity and preserve the gauge symmetry of the theory. In the literature some regularisation methods can be found such as Pauli-Villars [12], but in this thesis we will be using a well tried and tested one called Dimensional Regularization (DR). This method is based on extending the space-time dimension from four to  $D = 4 - 2\epsilon$  with a small parameter  $\epsilon$ . Contributions which are simultaneously softly and collinearly divergent and simple IR singularities cancel in the sum of all contributions in a suitably defined observable. After a careful cancellation of the poles the limit  $\epsilon \rightarrow 0$  can be taken for a physical result.

### 1.4.1 Dimensional Regularization

In Dimensional Regularization (DR) the Feynman integral is regarded as an analytical continuation of space-time dimension to  $4 - 2\epsilon$  dimensions where as  $\epsilon$  being a small parameter. The actual calculation of partonic matrix elements in the framework of dimensional regularization requires an extension of the Dirac algebra to  $D$  dimensions. The details of the method can be found in ref. [13]. Here we only give a summary of prescription to be applied using the scheme:

- The Clifford algebra becomes  $D$  dimensional. Instead of having four dimensions now we have  $D$  dimensions and the matrices obey:

$$g^{\mu\nu}g_{\mu\nu} = D, \quad \gamma^\mu\gamma^\nu\gamma_\mu = -(D-2)\gamma^\nu, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho$$



- The loop integral measure in the Feynman rules changes

$$\int \frac{d^4 k}{(2\pi)^4} \rightarrow \int \frac{d^D k}{(2\pi)^D}$$

- The measure of the phase-space integration changes

$$\int \frac{d^3 p}{2E(2\pi)^3} \dots (2\pi)^4 \delta^4(p_i - p_f) \rightarrow \int \frac{d^{D-1} p}{2E(2\pi)^{D-1}} \dots (2\pi)^D \delta^D(p_i - p_f)$$

- The action has to be modified as to keep it dimensionless

$$\mathcal{S} = \int d^D x \mathcal{L}.$$

The mass dimensions of the fields are:

$$\begin{aligned} m \bar{\psi}_f \psi_f &\implies [\psi_f] = \frac{D-1}{2} \\ \partial_\mu A_\nu^a \partial_\nu A_\mu^a &\implies [A_\mu^a] = \frac{D-2}{2} \end{aligned}$$

Considering the interaction part of the Lagrangian  $g \bar{\psi} A \psi$ , the coupling  $g$  acquires a dimension in  $D$  dimensions.

$$g \rightarrow \mu^\epsilon g, \quad \epsilon = 2 - \frac{D}{2}$$

After the application of DR to regularize the theory we naturally have introduced a new scale  $\mu$  to the theory. This scale is not fixed a priori and leads to an unphysical scale dependence.

There is still one piece missing after having fixed the dimensionality of the action. We are free to choose the polarizations of the internal/external quark and gluon fields. In this thesis we use *Conventional Dimensional Regularization (CDR)* scheme. In *CDR* there is no distinction between real and virtual partons. There are two helicity states for quarks and there are  $(D-2)$  polarizations for the gauge fields. There are several other *DR* schemes that undertake different helicity/polarization configurations available in the literature. [13,14]

## 1.5 Renormalization

The technique of regularization gives us a way to ‘parameterize the infinities’. Now we must develop a way to get rid of the infinities appearing in the loop integrals. This is the step of renormalization. The main idea of renormalization is to rewrite the original Lagrangian of a quantum field theory by a set of new terms, labelled by the Feynman graphs that encode the perturbative expansion of the theory. In the procedure of multiplicative renormalization, we redefine the fields and coupling with a multiplicative factor. The multiplicative constants absorb all UV divergences to all orders in perturbative QCD and we get UV-finite renormalized Green’s functions. One of the problems with any renormalization procedure is a systematic treatment of nested/overlapping divergences in multiloop diagrams. In practice we write down the Lagrangian with a set of new quantities:

$$\begin{aligned}
 \psi_f^i &\rightarrow Z^{1/2}\psi_{f,R}^i \\
 A_\mu^a &\rightarrow Z_A^{1/2}A_{\mu,R}^a \\
 \eta^a &\rightarrow Z_\eta^{1/2}\eta_R^a \\
 g &\rightarrow Z_g g_R \\
 m &\rightarrow Z_m m_R \\
 \xi &\rightarrow Z_A \xi_R
 \end{aligned}
 \tag{1.16}$$

The subscript  $R$  stands for *renormalized* quantities. By changing the bare quantities to the renormalized ones we would expect no change in the action  $S = \int d^D x \mathcal{L}$ . The ultimate goal of this procedure, however, is to obtain a priori unrenormalized so-called bare Green’s functions from the rewritten  $\mathcal{L}_{QCD}$  in a form that all UV singularities can be reshuffled into the multiplicative renormalization constants  $Z_i$ . By this procedure we can make physical predictions of the observables of the theory such as cross-sections and decay rates. The cancellation of the UV divergences works at all orders for all Green’s functions by readjusting the multiplicative factors  $Z$  at each order. The proof of this leads to so called Slavnov-Taylor identities [15]. One can deduce that, with the help of the aforementioned ideas, QCD is a renormalizable theory. So far it has been proven that renormalization procedure worked in all orders in perturbative QCD.

There is a certain amount of arbitrariness in the renormalization procedure. In order to remove the divergences from the Green's functions, one has to choose a subtraction scheme. In this thesis *Modified Minimal Subtraction* ( $\overline{MS}$ ) which removes the  $UV$  pole, plus a fixed finite contribution, which corresponds to the replacement

$$\frac{1}{\epsilon} \rightarrow \frac{1}{\bar{\epsilon}} \equiv (4\pi)^\epsilon \exp(-\epsilon\gamma) \frac{1}{\epsilon}$$

where  $\gamma = 0.5772\dots$  is the Euler-Mascheroni constant.

The predictions for the observables may vary depending on the chosen value for the scale  $\mu$ . This variation should not be perceived as an inconsistency in the theory, but is a due to the truncation of the perturbative series. Nevertheless, the requirement that the physical observable is independent of the choice of  $\mu$  leads to certain restrictions on the renormalized fields and couplings. *The renormalization group equations* are differential equations which are generated by requiring that the physical observables are independent of the scale.

### 1.5.1 Running Coupling $\alpha_s$

From eq. (1.16) the renormalized strong coupling can be written as:

$$\alpha_0 = Z_g^2(\mu^2)^\epsilon \alpha_R, \quad \alpha_0 = \frac{g_0^2}{4\pi}$$

where ' $0$ ' and ' $R$ ' stands for the bare and renormalized coupling. The definition of the 4-dimensional  $\beta$  - *function* is:

$$\frac{\partial \alpha_s}{\partial \ln \mu^2} = \beta(\alpha_s) = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 - \beta_3 \alpha_s^5 + \mathcal{O}(\alpha_s^6)$$

where  $\alpha_s = \alpha_s/4\pi = g^2/16\pi^2$ ,  $g = g(\mu^2)$ .

The recent analytical four-loop calculation of the QCD  $\beta$ -function in the  $\overline{MS}$ -scheme is [16]:

$$\begin{aligned}
\beta_3 = & C_A C_F T_F^2 n_F^2 \left( \frac{17152}{243} + \frac{448}{9} \zeta_3 \right) + C_A C_F^2 T_F n_F \left( -\frac{4204}{27} + \frac{352}{9} \zeta_3 \right) \\
& + \frac{424}{243} C_A T_F^3 n_F^3 + C_A^2 C_F T_F n_F \left( \frac{7073}{243} - \frac{656}{9} \zeta_3 \right) \\
& + C_A^2 T_F^2 n_F^2 \left( \frac{7930}{81} + \frac{224}{9} \zeta_3 \right) + \frac{1232}{243} C_F T_F^3 n_F^3 \\
& + C_A^3 T_F n_F \left( -\frac{39143}{81} + \frac{136}{3} \zeta_3 \right) + C_A^4 \left( \frac{150653}{486} - \frac{44}{9} \zeta_3 \right) \\
& + C_F^2 T_F^2 n_F^2 \left( \frac{1352}{27} - \frac{704}{9} \zeta_3 \right) + 46 C_F^3 T_F n_F \\
& + n_F \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left( \frac{512}{9} - \frac{1664}{3} \zeta_3 \right) + n_F^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left( -\frac{704}{9} + \frac{512}{3} \zeta_3 \right) \\
& + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{80}{9} + \frac{704}{3} \zeta_3 \right) \tag{1.17}
\end{aligned}$$

$$\begin{aligned}
\beta_2 = & \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_F + \frac{158}{27} C_A T_F^2 n_F^2 + \frac{44}{9} C_F T_F^2 n_F^2 \\
& - \frac{205}{9} C_F C_A T_F n_F + 2 C_F^2 T_F n_F, \\
\beta_1 = & \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_F - 4 C_F T_F n_F, \\
\beta_0 = & \frac{11}{3} C_A - \frac{4}{3} T_F n_F. \tag{1.18}
\end{aligned}$$

where  $N_F$  and  $N_A$  are the dimensions of the fundamental and adjoint representations.

The specific values for the  $SU(N)$  group are:

$$T_F = \frac{1}{2}, \quad C_F = \frac{N^2 - 1}{2N}, \quad C_A = N, \quad \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N^4 - 6N^2 + 18}{96N^2}, \tag{1.19}$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N(N^2 + 6)}{48}, \quad \frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N^2(N^2 + 36)}{24}, \quad N_A = N^2 - 1, \tag{1.20}$$

The Riemann Zeta function  $\zeta_n$  is defined by:

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

At this point, it is useful to write down some of the numerical values of some Zeta functions that will be relevant for the following chapters.

$$\zeta_2 = \frac{\pi^2}{6} = 1.64493 \quad \zeta_3 = 1.20206 \quad \zeta_4 = \frac{\pi^4}{90} = 1.08232 \quad \zeta_5 = 1.03693.$$

The solution of the  $\beta$ -function with respect to the  $\mu$  gives us the strong coupling with the scale  $\mu$ .

$$\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\alpha}{\beta(\alpha)} = \log\left(\frac{\mu^2}{\mu_0^2}\right)$$

To first order,

$$\alpha_s(\mu^2) = \frac{\alpha_s\mu_0^2}{1 + \alpha_s\beta_0 \log\left(\frac{\mu^2}{\mu_0^2}\right)}.$$

This equation points a significant property of QCD known as "*asymptotic freedom*". As  $\mu$  increases  $\alpha_s$  decreases. Roughly speaking at shorter and shorter distances, the coupling decreases in size, so that the theory acts like a free theory. Conversely at larger and larger distances, the coupling increases, so that at a certain point the perturbation calculations can no longer be valid. The quarks bind more tightly together, giving rise to confinement. This is called "*infrared slavery*". The measured values of  $\alpha_s(Q)$ , covering energy scales from  $Q = M_\tau = 1.78$  GeV to 209 GeV, exactly follow the energy dependence predicted by QCD and therefore significantly test the concept of Asymptotic Freedom. The recent world average value of the strong coupling at Z-pole is [1]:

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007.$$

This value is obtained from  $\tau$  decays, the proton structure function  $F_2$ , hadronic event shapes and jet production in  $e^+e^-$  annihilation, jet production in deep inelastic scattering and from  $\Upsilon$  decays and heavy quarkonia based on unquenched QCD lattice calculations. This constitutes a striking test of asymptotic freedom in QCD.

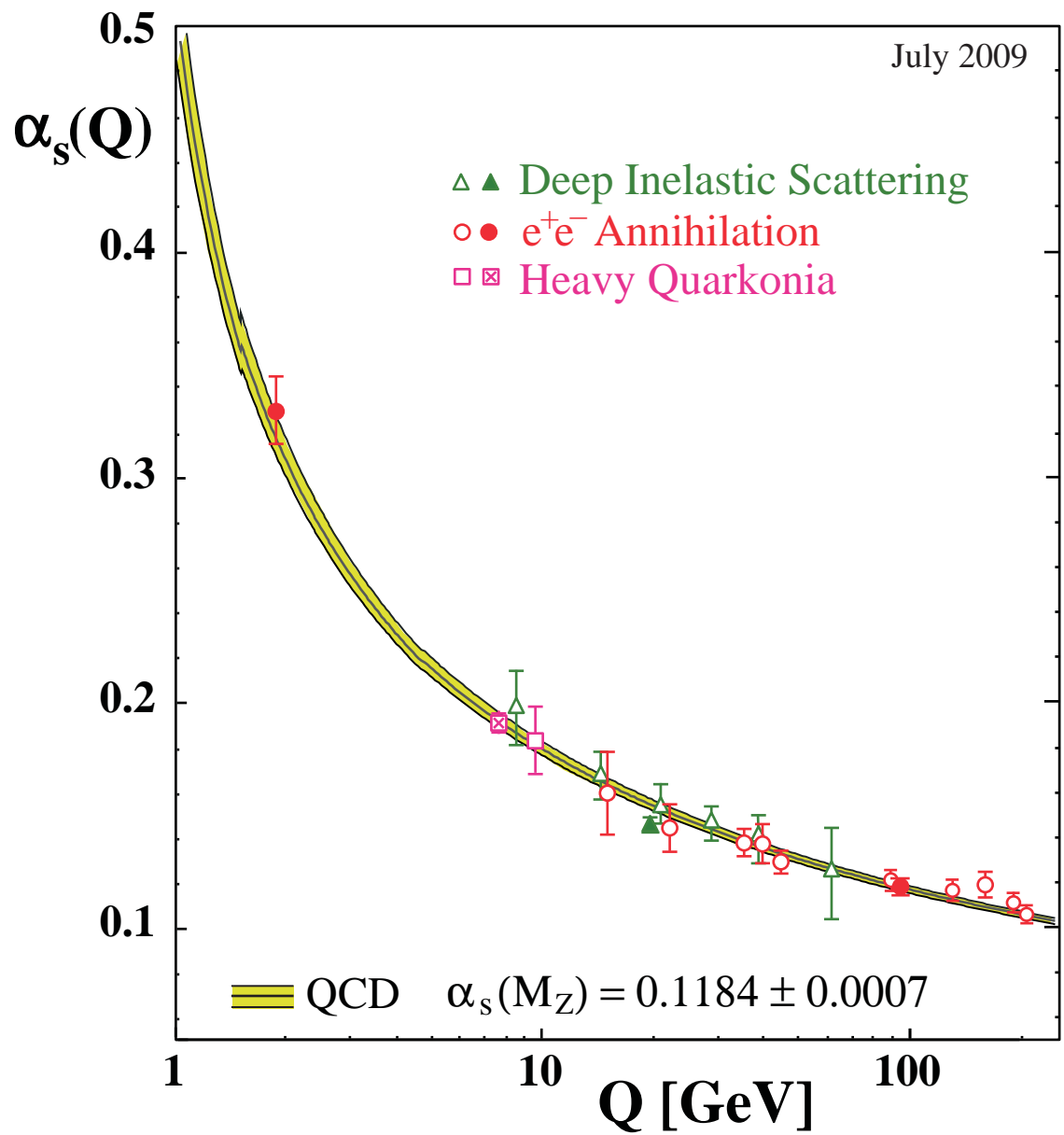


Figure 1.9: World average of the strong coupling constant  $\alpha_s$  as a function of the scale  $Q$ . [1]

# Chapter 2

## Loop Integral Methods

In this chapter we will explain some of the methods to calculate the divergent integrals which appear when applying the Feynman rules to a process. In any perturbative *QCD* calculation at some point, one has to deal with these types of integrals which is not a trivial task. There are many methods to evaluate the loop integrals which can be put into three categories. Pure numerical, pure analytical and semi-numerical. In our study we will only deal with the analytical methods and try to demonstrate them as clearly as possible. In order to ease the reasoning we need a notation to carry out in all steps. We first put the problem in its general form and then establish general methods for simplifying it.

The generic integral with  $L$ -loops and  $N$ -propagators raised to arbitrary powers  $\nu_i$  in  $D$  dimensions is denoted by:

$$J^D(\{\nu_i\}; \{Q_i^2\})[1; k_1^\mu; k_2^\mu; k_1^\mu k_1^\nu; k_1^\mu k_2^\nu; \dots] = \int \frac{d^D k_1}{i\pi^{D/2}} \dots \int \frac{d^D k_L}{i\pi^{D/2}} \frac{[1; k_1^\mu; k_2^\mu; k_1^\mu k_1^\nu; k_1^\mu k_2^\nu; \dots]}{A_1^{\nu_1} \dots A_N^{\nu_N}} \quad (2.1)$$

where  $\{Q_i^2\}$  represents the external momentum scales present in the problem. In the case of a scalar integral, the numerator  $J^D(\{\nu_i\}; \{Q_i^2\})$  is unity. Tensor integrals have powers of the loop momenta in the numerator, and are much harder to compute than scalar integrals. The propagators in the loop have the form:

$$\frac{1}{A_i} = \frac{1}{(\sum_j \xi_{ij} k_j + q_i)^2 - M_i^2 + i\epsilon} \quad (2.2)$$

where  $M_i$  is the mass of the particle associated with propagator  $i$ ,  $\xi_{ij} = 0, 1, -1$  for  $j = 1 \dots L$ , and  $q_i$  is the linear combination of the external momenta.

Loop diagrams are classified according to their topological structure. There are two types of diagrams. If the diagram doesn't have crossings between its propagators it is called *planar* otherwise it is a *non-planar* graph. Generally non-planar diagrams are harder to compute than planar ones. The loop integrals become more difficult to evaluate for increasing numbers of loops, increasing numbers of propagators, higher rank tensors and with increasing numbers of external legs and scales.

In the next sections we will give the calculational details to evaluate a certain type of topology for several methods. In doing so we begin with a simple topology assuming that all propagators are massless and that all external particles are on shell.

## 2.1 Feynman Parameterization

This very popular method was devised by Richard Feynman during the sixties. The idea is to arrange the denominators of the propagators into a single quadratic form in the loop-momenta with the help of the  $\delta$ -function.

Explicitly, the identity reads:

$$\frac{1}{A_1^{\nu_1} \dots A_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots dx_N \delta(1 - \sum x_j) \frac{x_1^{\nu_1-1} \dots x_N^{\nu_N-1}}{[\sum x_j A_j]^{\sum \nu_j}} \quad (2.3)$$

and defining  $A_j = (B_j^2 - M_j^2)$ , we can insert (2.2) into eq. (2.1) and obtain the generic integral:

$$J^D(\{\nu_j\}; \{Q_j^2\})[1] = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \overrightarrow{dx} \delta(1 - \sum_{j=1}^N x_j) \int \frac{\prod_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \frac{1}{\left[ \sum_{j=1}^N x_j B_j^2 - \sum_{j=1}^N x_j m_j^2 \right]^{N_\nu}} \quad (2.4)$$

where  $\overrightarrow{dx} = dx_1 \dots dx_N \prod_{j=1}^N x_j^{\nu_j-1}$  and  $N_\nu = (\nu_1 + \dots + \nu_N)$ .

We expand and reorder the sum  $\sum_{j=1}^N x_j B_j^2$  such that

$$\sum_{j=1}^N x_j B_j^2 = \sum_{i=1}^L \sum_{j=1}^L q_i A_{ij} q_j - 2 \sum_{i=1}^L k_i q_i + J \quad (2.5)$$



or equivalently in the matrix form,

$$\sum_{j=1}^N x_j B_j^2 = \mathbf{q}^t \mathbf{A} \mathbf{q} - 2\mathbf{k}^t \mathbf{q} + \mathbf{J} \quad (2.6)$$

where:

$\mathbf{A} \implies$  Symmetric matrix of dimension  $L \times L$ , whose elements are functions of the Feynman parameters  $\underline{x}$  only:  $\mathbf{A} = \mathbf{A}(\underline{x})$ .

$\mathbf{q} \implies$  dimension  $L$ -vector, whose components are the loop momenta:  $\mathbf{q}^t = [q_1 \dots q_L]$ .

$\mathbf{k} \implies$   $L$ -vector, whose components are linear combinations of external momenta, with coefficients that are functions of the Feynman parameters  $\underline{x}$  only, so  $\mathbf{k} = \mathbf{k}(\underline{x}, \underline{p})$ .

$J \implies$  Scalar term, which is a linear combination of scalar products of external momenta, with coefficients that depend on the Feynman parameters  $\underline{x}$  only,  $J = J(\underline{x}, \underline{p})$ .

The precise form of these quantities depend on the topology of the integral. Putting these definitions into eq. (2.4) we find the parametric representation a general integral:

$$J^D = \frac{\Gamma(N_\nu)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^N x_j) \int \frac{\sum_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \frac{1}{\left[ q^t \mathbf{A} q - 2\mathbf{k}^t q + J - \sum_{j=1}^N x_j M_j^2 \right]^{N_\nu}} \quad (2.7)$$

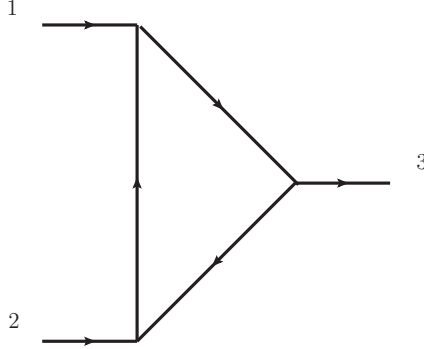
After evaluating the  $D$ -dimensional integral we obtain:

$$J^D = \frac{(-1)^{N_\nu} \Gamma(N_\nu - \frac{LD}{2})}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^N x_j) \frac{[\det \mathbf{A}]^{N_\nu - (L+1)\frac{D}{2}}}{\left[ \det \mathbf{A} \left( \sum_{j=1}^N x_j M_j^2 - J + \mathbf{k}^t \mathbf{A}^{-1} \mathbf{k} \right) \right]^{N_\nu - \frac{LD}{2}}} \quad (2.8)$$

Only the integrations over the Feynman parameters remain to be evaluated.

### 2.1.1 Example: One-Loop Triangle

We can now use (2.8) to evaluate the one-loop triangle with two legs on-shell  $p_1^2 = p_2^2 = 0$  and one leg off-shell  $p_3^2 \neq 0$ ,



Unless stated otherwise we will assume all propagators to be massless throughout our examples. The loop integral is given by

$$I_3^D(\nu_1, \nu_2, \nu_3, s^2) = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3}} \quad (2.9)$$

$$A_1 = k_1^2 + i0, \quad (2.10)$$

$$A_2 = (k_1 + p_1)^2 + i0, \quad (2.11)$$

$$A_3 = (k_1 + p_1 + p_2)^2 + i0. \quad (2.12)$$

We will frequently use the shorthand notation,

$$\nu_{ijk} = \nu_1 + \nu_2 + \nu_3, \quad \nu_{ij} = \nu_i + \nu_j, \quad \text{etc.}$$

Applying the Feynman parameterization (2.3) we find,

$$I_3^D = \frac{\Gamma(\nu_{123})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^3 x_j) \int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{\Omega^3} \quad (2.13)$$

where  $\Omega$  is given by:

$$\Omega = (x_1 + x_2 + x_3)k_1^2 + 2[(x_2 + x_3)p_1 + x_3 p_2]k_1 + (x_2 + x_3)p_1^2 + 2x_3 p_1 \cdot p_2 + x_3 p_2^2 \quad (2.14)$$

Here,  $L = 1$  and

$$\mathbf{A} = (x_1 + x_2 + x_3) \quad (2.15)$$

$$\mathbf{k} = -(x_2 + x_3)p_1 - x_3 p_2 \quad (2.16)$$

$$\mathbf{J} = (x_2 + x_3)p_1^2 + 2x_3 p_1 \cdot p_2 + x_3 p_2^2 \quad (2.17)$$

According to eq. (2.8) the scalar integral is given by:

$$I_3^D = (-1)^{D/2} \frac{\Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int d\vec{x} \delta(1 - \sum_{j=1}^3 x_j) \frac{[\det \mathbf{A}]^{\nu_{123}-D}}{[\det \mathbf{A} (-J + \mathbf{k}^t \mathbf{A}^{-1} \mathbf{k})]^{\nu_{123}-\frac{D}{2}}}. \quad (2.18)$$

evaluating each term in the expression:

$$\begin{aligned} \det \mathbf{A} &= x_1 + x_2 + x_3 \\ \mathbf{A}^{-1} &= \frac{1}{x_1 + x_2 + x_3} \\ \mathbf{k}^t \mathbf{A}^{-1} \mathbf{k} &= \frac{[(x_2 + x_3)p_1 + x_3 p_2]^2}{x_1 + x_2 + x_3} \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \det \mathbf{A} (-J + \mathbf{k}^t \mathbf{A}^{-1} \mathbf{k}) &= -(x_1 x_2 + x_1 x_3) p_1^2 - 2x_1 x_3 p_1 \cdot p_2 - (x_1 x_3 + x_2 x_3) p_2^2 \\ &= -x_1 x_2 p_1^2 - x_2 x_3 p_2^2 - x_1 x_3 (p_1 + p_2)^2 \\ &= -x_1 x_2 p_1^2 - x_2 x_3 p_2^2 - x_1 x_3 p_3^2. \end{aligned} \quad (2.20)$$

Putting everything together we find:

$$I_3^D = (-1)^{D/2} \frac{\Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^3 x_j) \frac{[x_1 + x_2 + x_3]^{\nu_{123}-D}}{[-(x_1 x_2) p_1^2 - (x_2 x_3) p_2^2 - (x_1 x_3) p_3^2]^{\nu_{123}-D/2}}, \quad (2.21)$$

with  $d\vec{x} = dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1}$ .

In our case, we have

$$p_1^2 = 0, \quad p_2^2 = 0, \quad \text{and} \quad (p_1 + p_2)^2 = p_3^2 = s_{12} \quad (2.22)$$

and the  $\delta$  function sets  $x_1 + x_2 + x_3 = 1$ . Finally we obtain:

$$I_3^D = (-1)^{D/2} \frac{\Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^3 x_j) \frac{1}{[-x_1 x_3 s_{12}]^{\nu_{123}-D/2}}. \quad (2.23)$$

With the change of variables:

$$x_1 = \chi \quad (2.24)$$

$$x_2 = (1 - \chi)\rho \quad (2.25)$$

$$x_3 = (1 - \chi)(1 - \rho), \quad (2.26)$$

we find,

$$I_3^D(\nu_1, \nu_2, \nu_3; s_{12}) = (-1)^{D/2} \frac{\Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} (s_{12})^{D/2 - \nu_{123}} \quad (2.27)$$

$$\times \int_0^1 d\chi \chi^{\nu_1 - 1} (1 - \chi)^{D - \nu_{123} - 1} \quad (2.28)$$

$$\times \int_0^1 d\rho \rho^{D/2 - \nu_{13} - 1} (1 - \rho)^{D/2 - \nu_{12} - 1}. \quad (2.29)$$

The  $\chi$  and  $\rho$  integrations can be solved with the  $\beta$ -function identity:

$$\int_0^1 dx x^{A-1} (1-x)^{B-1} = \frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)} \quad (2.30)$$

Finally we have the answer:

$$I_3^D(\nu_1, \nu_2, \nu_3, s_{12}) = (-1)^{D/2} (s_{12})^{D/2 - \nu_{123}} \frac{\Gamma(\nu_{123} - D/2)\Gamma(D/2 - \nu_{12})\Gamma(D/2 - \nu_{13})}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(D - \nu_{123})} \quad (2.31)$$

This is the analytic result for arbitrary dimension and arbitrary powers of the propagators. We can also evaluate sub-topologies of triangle such as the bubble integral for two massless propagators and external momentum  $p_3$  by pinching one of the propagators, i.e. setting  $\nu_1 \rightarrow 0$ .

## 2.2 Schwinger Parameterization

The fundamental identity for the Schwinger parameterization is :

$$\frac{1}{A_j^{\nu_j}} = \int_0^\infty dx_j x_j^{\nu_j - 1} \exp(-x_j A_j), \quad (2.32)$$

As in the last section  $A_j^2 = B_j^2 - M_j^2$ . Inserting (2.32) into our generic integral (2.1) we find:

$$J^D = \frac{1}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^\infty d\vec{x} \exp\left(\sum_{j=1}^N x_j m_j^2\right) \int \frac{\prod_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \exp\left(\sum_{j=1}^N x_j B_j^2\right). \quad (2.33)$$

Using (2.6), we find;

$$J^D = \frac{1}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^\infty d\vec{x} \exp\left(\sum_{j=1}^N x_j M_j^2 - J\right) \int \frac{\prod_{j=1}^L d^D q_j}{(i\pi^{D/2})^L} \exp(-\mathbf{q}^t \mathbf{A} \mathbf{q} + 2\mathbf{k}^t \mathbf{q}). \quad (2.34)$$

In an analogous way, after integration over internal momenta we obtain the Schwinger parameterization of  $J$ :

$$J^D = \frac{(-1)^{\frac{LD}{2}}}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^\infty d\vec{x} [\det \mathbf{A}]^{-\frac{D}{2}} \exp\left(\sum_{j=1}^N x_j m_j^2 - J + \mathbf{k}^t \mathbf{A}^{-1} \mathbf{k}\right). \quad (2.35)$$

### 2.2.1 Example: One-Loop Triangle

We will evaluate the same example in the last section with Schwinger representation. The steps from eq. (2.9) to eq. (2.20) are the same for both. If we put our findings in eq. (2.35) we find:

$$I_3^D = \frac{(-1)^{D/2}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^\infty d\vec{x} (x_1 + x_2 + x_3)^{-D/2} \exp(-x_1 x_3 s_{12}) \quad (2.36)$$

where  $d\vec{x} = dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1}$ .

Introducing new variables

$$x_1 = \xi_1 \eta \quad (2.37)$$

$$x_2 = \xi_2 \eta \quad (2.38)$$

$$x_3 = (1 - \xi_1 - \xi_2) \eta \quad (2.39)$$

The next step is the integration over  $\eta$  variable. Performing the Gaussian integral over  $\eta$ , we end up with the resulting intermediate expression:

$$I_3^D = \frac{(-1)^{\nu_{123}} \Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \frac{\xi_1^{\nu_1-1} \xi_2^{\nu_2-1} (1 - \xi_1 - \xi_2)^{\nu_3-1}}{(\xi_1 \xi_2 s_{12})^{\nu_{123}-D/2}} \quad (2.40)$$

Note that the boundaries of the integrals change after the new variables. After some straightforward manipulations we get the same result in the last section for the one loop triangle.

$$I_3^D(\nu_1, \nu_2, \nu_3, s_{12}) = (-1)^{D/2} (s_{12})^{D/2-\nu_{123}} \frac{\Gamma(\nu_{123} - D/2) \Gamma(D/2 - \nu_{12}) \Gamma(D/2 - \nu_{13})}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_{123})} \quad (2.41)$$

## 2.3 Mellin-Barnes Technique

Although Mellin-Barnes representation is known quite a long time it was not used till the work by M.C.Bergère and Y.M.P.Lam in 1974 [17, 18]. They realized Mellin

transforms and Mellin integrals can be used as a tool for Feynman diagrams. Since then it has become a very popular and powerful method in the calculation of Feynman loop integrals.

The method of Mellin-Barnes relies on a special identity to fully factorize the sums in the numerator and denominator after the Feynman parameterization is used. So it's a two step process. First Feynman parameterization is applied and then we continue with the Mellin-Barnes prescription.

The basic tool of the Mellin-Barnes method is given by the identity:

$$\frac{1}{(A_1 + A_2)^N} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\xi A_1^\xi A_2^{-N-\xi} \frac{\Gamma(-\xi)\Gamma(N+\xi)}{\Gamma(N)} \quad (2.42)$$

The identity is derived by using the transformation formulae whence the method got its name: Mellin transformations,

$$\mathcal{F}(s) = \int_0^\infty dx f(x) x^{s-1} \quad (2.43)$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \mathcal{F}(s) x^{-s} \quad (2.44)$$

By the iteration of the formulae (2.42) we can generalize for many terms in the denominator.

$$\begin{aligned} \frac{1}{(A_1 + \dots + A_m)^N} &= \frac{1}{(2\pi i)^{m-1}} \int_{-i\infty}^{+i\infty} d\xi_1 \dots d\xi_{m-1} A_1^{\xi_1} \dots A_{m-1}^{\xi_{m-1}} A_m^{-N-\xi_1-\dots-\xi_{m-1}} \\ &\times \frac{\Gamma(-\xi_1) \dots \Gamma(-\xi_{m-1}) \Gamma(N + \xi_1 + \dots + \xi_{m-1})}{\Gamma(N)} \end{aligned} \quad (2.45)$$

After the introduction of the Mellin-Barnes identities we are now to use contour integration to evaluate the integral. The transformed expressions have  $\Gamma$  functions which have poles. This causes difficulty and may result in unfavorable situations. In order to perform the integrations we need to choose straight lines parallel to the imaginary axis and close the contour to the right or left whichever we choose. With the employment of the Cauchy Residue theorem we can evaluate the expression usually in terms of  $\Gamma$  functions. Sometimes it becomes very hard to separate the overlapping poles. A number of algorithms and codes have been developed to overcome this issue in a safe way. [19–21]

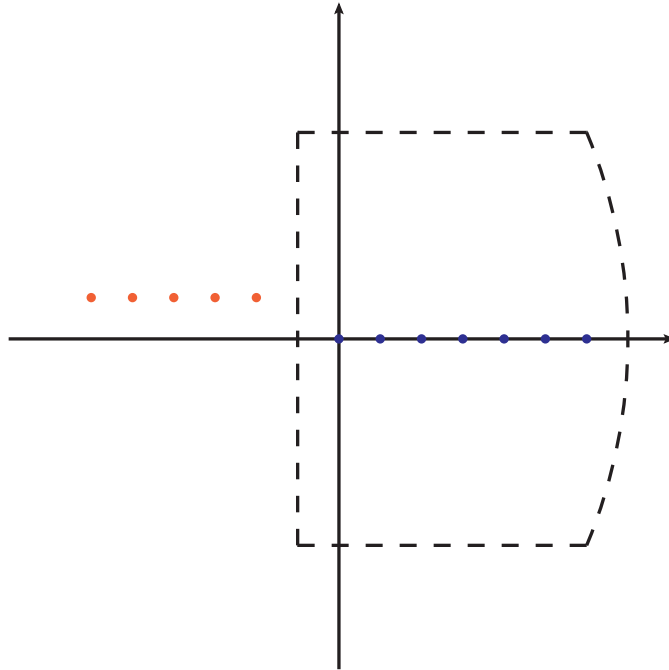


Figure 2.1: The separation of the poles for Mellin-Barnes integration

From (2.45), poles produced by  $\Gamma(\cdots + \xi)$  are to the left of the contour and the poles from  $\Gamma(\cdots - \xi)$  lie to the right of the contour. The value of  $c$  is chosen so that it separates the poles correctly. One further condition is that though the numbers may be complex they must satisfy  $|\arg(A_1) - \arg(A_2)| < \pi$ .

If we use the Residue theorem on the contour enclosing the poles to the right;

$$\oint d\xi f(\xi) = 2\pi i \sum_i \text{Res}\{f(\xi_i)\} \quad (2.46)$$

we see that the right-hand side of the equation is equal to the Taylor expansion of the rearranged form of the left-hand side.

$$\frac{1}{A_1^\alpha (1 + A_2/A_1)^\alpha} = \frac{1}{A_1^\alpha} \sum_{m=0}^{\infty} \left(\frac{A_2}{A_1}\right)^m \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)m!} \quad (2.47)$$

The residue of the  $\Gamma$  function at  $n = 0, 1, 2, \dots$

$$\text{Res}\{\Gamma(\xi)\}_{\xi=-n} = \text{Res}\{\Gamma(y-n)\}_{y=0} = \text{Res}\left\{\frac{\Gamma(1+y)}{y(y-1)\dots(y-n)}\right\}_{y=0} = \frac{(-1)^n}{n!} \quad (2.48)$$

### 2.3.1 Example: One Loop Triangle with Two Off-Shell Legs

We begin our calculation with eq. (2.21):

$$I_3^D = (-1)^{D/2} \frac{\Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \times \int_0^1 d\vec{x} \delta(1 - \sum_{j=1}^3 x_j) \frac{[x_1 + x_2 + x_3]^{\nu_{123} - D}}{[-(x_1 x_2) p_1^2 - (x_2 x_3) p_2^2 - (x_1 x_3) p_3^2]^{\nu_{123} - D/2}}, \quad (2.49)$$

with  $d\vec{x} = dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1}$ . We set  $p_3^2 = 0$ , and keep  $p_2^2$  and  $p_1^2$  off-shell.

Generally one introduces the shorthand:

$$\begin{aligned} \mathcal{Q} &= x_2 x_3 p_1^2 + x_1 x_2 p_2^2 \\ \mathcal{P} &= x_1 + x_2 + x_3 = 1. \end{aligned} \quad (2.50)$$

Performing the integral over  $x_3$  we get:

$$I_3^D = (-1)^{D/2} \frac{\Gamma(\nu_{123} - \frac{D}{2})}{\Gamma(\nu_1) \dots \Gamma(\nu_3)} \times \int_0^1 dx_1 \int_0^{1-x_1} dx_2 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} (1 - x_1 - x_2)^{\nu_3 - 1} \mathcal{Q}^{D/2 - \nu_{123}} \quad (2.51)$$

with  $\mathcal{Q} = x_2(1 - x_1 - x_2)p_1^2 + x_1 x_2 p_2^2$ .

The boundaries of the integrals  $[0, 1]$ ,  $[0, 1 - x_1]$  can be arranged for the later use of the  $\beta$ -function with the following substitution:

$$\begin{aligned} x_1 &= \xi_1 \\ x_2 &= \xi_2(1 - \xi_1) \end{aligned} \quad (2.52)$$

It can be easily seen that the lower bounds of the both integrals become 0 and the upper bounds turn to 1 so that:

$$\begin{aligned} I_3^D &= (-1)^{D/2} \frac{\Gamma(\nu_{123} - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\xi_1 \int_0^1 d\xi_2 (1 - \xi_1) \xi_1^{\nu_1 - 1} \xi_2^{\nu_2 - 1} (1 - \xi_1)^{\nu_2 + \nu_3 - 2} \\ &\times (1 - \xi_2)^{\nu_3 - 1} [\xi_2(1 - \xi_1)^2(1 - \xi_2)p_1^2 + \xi_1 \xi_2(1 - \xi_1)p_2^2]^{D/2 - \nu_{123}} \\ &= (-1)^{D/2} \frac{\Gamma(\nu_{123} - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \xi_1^{\nu_1 - 1} \xi_2^{D/2 - \nu_1 + \nu_2 - 1} (1 - \xi_2)^{\nu_3 - 1} \\ &\times (1 - \xi_1)^{D/2 - \nu_{123} + \nu_2 + \nu_3 - 1} [(1 - \xi_1)(1 - \xi_2)p_1^2 + \xi_1 p_2^2]^{D/2 - \nu_{123}} \end{aligned} \quad (2.53)$$



In the integrand we have;

$$A_1 \equiv \xi_1 p_2^2 \quad (2.54)$$

$$A_2 \equiv (1 - \xi_1)(1 - \xi_2)p_1^2 \quad (2.55)$$

Keeping in mind the general tool of the Mellin-Barnes method,

$$\frac{1}{(A_1 + A_2)^{\nu_{123}}} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\xi A_1^\xi A_2^{-\nu_{123}-\xi} \frac{\Gamma(-\xi)\Gamma(\nu_{123} + \xi)}{\Gamma(N)} \quad (2.56)$$

eq. (2.53) takes the form.

$$\begin{aligned} I_3^D &= (-1)^{D/2} \frac{\Gamma(\nu_{123} - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_{-\infty}^{+\infty} d\omega \frac{\Gamma(\nu_{123} - D/2 + \omega)}{\Gamma(\nu_{123} - D/2)} (p_1^2)^2 (p_2^2)^{D/2 - \nu_{123} - \omega} \\ &\times \int_0^1 d\xi_1 \int_0^1 d\xi_2 \xi_1^{D/2 + \nu_1 - \nu_{123} - 1 - \omega} \xi_2^{D/2 - \nu_{123} + \nu_2 - 1} (1 - \xi_1)^{D/2 - \nu_1 - 1 + \omega} (1 - \xi_2)^{\nu_3 - 1 + \omega}. \end{aligned} \quad (2.57)$$

After integrating out  $\xi_1$  and  $\xi_2$  we get;

$$\begin{aligned} I_3^D &= \frac{(1-)^{D/2}}{2\pi i} \int_{-\infty}^{+\infty} d\omega \frac{\Gamma(\nu_{123} - D/2 + \omega)\Gamma(-\omega)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} (p_1^2)^\omega (p_2^2)^{D/2 - \nu_{123} - \omega} \\ &\times \frac{\Gamma(D/2 - \nu_{123} + \nu_2)\Gamma(\nu_3 + \omega)}{\Gamma(D/2 - \nu_1 + \omega)} \frac{\Gamma(D/2 - \nu_1 + \omega)\Gamma(D/2 - \nu_{123} + \nu_1 - \omega)}{\Gamma(D - \nu_{123})}. \end{aligned} \quad (2.58)$$

This is a typical Mellin-Barnes integral.

$$\begin{aligned} I_3^D &= \frac{(-1)^{D/2} (p_2^2)^{D/2 - \nu_{123}} \Gamma(D/2 - \nu_{123} + \nu_2)}{2\pi i \Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(D - \nu_{123})} \\ &\times \int_{-\infty}^{+\infty} d\omega \Gamma(\nu_{123} - D/2 + \omega)\Gamma(\nu_3 + \omega)\Gamma(D/2 - \nu_{123} + \nu_1 - \omega)\Gamma(-\omega) \left(\frac{p_1^2}{p_2^2}\right)^\omega \end{aligned} \quad (2.59)$$

which we close to the right (corresponding to  $p_1^2 < p_2^2$ ). There are two solutions due to series of poles at  $\omega = n$  and  $\omega = n + D/2 - \nu_{23}$  respectively;

$$I_3^D = I_{3,1}^D + I_{3,2}^D \quad (2.60)$$

Each of them is evaluated as follows:

$$I_{3,1}^D = (-1)^{D/2} (p_2^2)^{D/2 - \nu_{123}} \frac{\Gamma(D/2 - N + \nu_2) \Gamma(D/2 - \nu_{123} + \nu_1) \Gamma(\nu_{123} - D/2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D - \nu_{123})} \\ \times {}_2F_1 \left( \nu_{123} - D/2, \nu_3; 1 + \nu_{123} - D/2 - \nu_1; \frac{p_1^2}{p_2^2} \right)$$

and

$$I_{3,2}^D = (-1)^{D/2} \frac{(p_1^2)^{D/2 - \nu_{123} + \nu_1} (p_2^2)^{-\nu_1} \Gamma(D/2 - \nu_{123} + \nu_2) \Gamma(\nu_{123} - \nu_1 - D/2) \Gamma(D/2 - \nu_2)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(D - \nu_{123})} \\ \times {}_2F_1 \left( D/2 - \nu_2, \nu_1; 1 - \nu_{123} + \nu_1 + D/2; \frac{p_1^2}{p_2^2} \right) \quad (2.61)$$

## 2.4 Negative Dimension Integration Method

The Negative Dimension Integration Method (NDIM) was first developed by Halliday and Ricotta in 1987 with the amazing assertion that the dimension  $D$  can be considered as negative [22]. Their proposition was to perform the loop integration in a negative number of dimensions and analytically continue to positive dimensions *after* evaluating the integrals. Since loop integrals are analytic in the number of dimension  $D$ , this turns out to be a valid method. This method starts from the Schwinger parameterization of the integral, but thereafter another path is followed to evaluate the integrals. This method naturally introduces with infinite sums leading to hypergeometric and other transcendental functions.

Some new concepts such as constraint equations, analytic continuation and possible kinematic region solutions, come into play during the calculations [23–25]. The unique feature of this method is that solutions valid in different kinematic regions can be found simultaneously.

The momentum integral expression that represents any diagram in  $D = 4 - 2\epsilon$  dimensional Minkowski space is:

$$I^D = \int \frac{d^D k_1}{i\pi^{D/2}} \cdots \frac{d^D k_L}{i\pi^{D/2}} \frac{1}{(B_1^2 - M_1^2 + i0)^{\nu_1}} \cdots \frac{1}{(B_N^2 - M_N^2 + i0)^{\nu_N}} \quad (2.62)$$

where  $N$  is the number of propagators and  $L$  is the number of loops. After introducing Schwinger's representation, it is possible to solve the momenta integrals in

terms of Gaussian integrals for which the general case is given by:

$$I^D = \frac{(-1)^{LD/2}}{\prod_{j=1}^N \Gamma(\nu_j)} \int_0^\infty d\vec{x} \frac{\exp(-\mathcal{M}) \exp\left(-\frac{\mathcal{P}}{\mathcal{Q}}\right)}{\mathcal{Q}^{D/2}}. \quad (2.63)$$

To simplify we have introduced  $d\vec{x} = dx_1 \dots dx_N \prod_{j=1}^N x_j^{\nu_j-1}$  where  $\mathcal{P}$  and  $\mathcal{Q}$  are the  $L$ -linear and  $(L+1)$ -multilinears introduced in the previous section. The determination of  $\mathcal{P}$  and  $\mathcal{Q}$  depends on the topology of the related Feynman diagram. The mass function  $\mathcal{M}$  is defined by

$$\mathcal{M} = - \sum_{j=1}^N x_j M_j^2.$$

The key element of the method comes from the integral  $\Leftrightarrow$  Kronecker delta function in the  $D$ -dimensional the Gaussian integral:

$$\int \frac{d^D k}{i\pi^{D/2}} \exp(\alpha k^2) = \frac{1}{\alpha^{D/2}}. \quad (2.64)$$

To derive the relevant identity, we make a Taylor series expansion,

$$\int \frac{d^D k}{i\pi^{D/2}} \exp(\alpha k^2) = \sum_{n=0}^{\infty} \int \frac{d^D k}{i\pi^{D/2}} \frac{(\alpha k^2)^n}{n!}$$

and use the identity,

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{(k^2)^n}{n!} = \delta_{n+D/2,0}. \quad (2.65)$$

The Taylor series requires that  $n \geq 0$  and therefore  $\frac{D}{2} \leq 0$ , which is then the origin of the name of this integration technique. Inserting the delta function into the Taylor series immediately gives the correct result.

### 2.4.1 The Algorithm

We begin our description of this method with (2.62). We introduce a Schwinger parameter  $x_i$  for each propagator:

$$\frac{1}{A_i^{\nu_i}} = \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1} \exp(x_i A_i) \quad (2.66)$$

so that eq. (2.66) becomes:

$$I^D(\{\nu_i\}; \{Q_i^2\}; \{M_i^2\}) = \int \mathcal{D}x \int \frac{d^D q_1}{i\pi^{D/2}} \dots \int \frac{d^D q_N}{i\pi^{D/2}} \exp\left(\sum_{i=1}^n x_i A_i\right) \quad (2.67)$$

with the shorthand;

$$\int \mathcal{D}x = (-1)^\sigma \left( \prod_{i=1}^n \frac{1}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1} \right). \quad (2.68)$$

If we expand the exponentials and simplify we have:

$$\begin{aligned} I^D(\{\nu_i\}; \{Q_i^2\}; \{M_i^2\}) &= \int \mathcal{D}x \sum_{n_1, \dots, n_n=0}^\infty \int \frac{d^D q_1}{i\pi^{D/2}} \cdots \int \frac{d^D q_n}{i\pi^{D/2}} \prod_{i=1}^n \frac{(x_i A_i)^{n_i}}{n_i!} \\ &= \int \mathcal{D}x \sum_{n_1, \dots, n_n=0}^\infty I^D(-n_1, \dots, -n_n; \{Q_i^2\}; \{M_i^2\}) \prod_{i=1}^n \frac{x_i^{n_i}}{n_i!} \end{aligned} \quad (2.69)$$

$$(2.70)$$

Likewise we expand the exponentials in eq. (2.66) using the Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1, \dots, n_m=0}^\infty \frac{n!}{n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m} \delta_{n, n_1 + \cdots + n_m} \quad (2.71)$$

for the expressions  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{M}$  in the intermediate steps of the calculations.

$$I^D(\{\nu_i\}; \{Q_i^2\}; \{M_i^2\}) = \int \mathcal{D}x \sum_{n=0}^\infty \frac{\mathcal{Q}^n \mathcal{P}^{-n-D/2}}{n!} \sum_{m=0}^\infty \frac{(-\mathcal{M})^m}{m!} \quad (2.72)$$

With the Multinomial theorem in hand we can expand the terms in equation (2.72):

$$\begin{aligned} \mathcal{Q}^n &= \sum_{q_1, \dots, q_q=0}^\infty \frac{\mathcal{Q}_1^{q_1}}{q_1!} \cdots \frac{\mathcal{Q}_q^{q_q}}{q_q!} (q_1 + \cdots + q_q)! \\ \mathcal{P}^{-n-D/2} &= \sum_{p_1, \dots, p_n=0}^\infty \frac{x_1^{p_1}}{p_1!} \cdots \frac{x_n^{p_n}}{p_n!} (p_1 + \cdots + p_n)! \\ (-\mathcal{M})^m &= \sum_{m_1, \dots, m_n=0}^\infty \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \cdots \frac{(-x_n M_n^2)^{m_n}}{m_n!} (m_1 + \cdots + m_n)! \end{aligned} \quad (2.73)$$

Inserting the expansions of the quantities into eq. (2.72) we find:

$$\begin{aligned} I^D(\{\nu_i\}; \{Q_i^2\}; \{M_i^2\}) &= \\ \int \mathcal{D}x \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^\infty &\frac{\mathcal{Q}_1^{q_1} \cdots \mathcal{Q}_q^{q_q}}{q_1! \cdots q_q!} \frac{x_1^{p_1} \cdots x_n^{p_n}}{p_1! \cdots p_n!} \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \cdots \frac{(-x_n M_n^2)^{m_n}}{m_n!} (p_1 + \cdots + p_n)! \end{aligned} \quad (2.74)$$

with the constraints.

$$\sum_{i=1}^q q_i = n, \quad \sum_{i=1}^n p_i = -n - \frac{D}{2}, \quad \sum_{i=1}^n m_i = m. \quad (2.75)$$

The constraints come into play because of the delta function in the multinomial expansion and are very important for generating the full solution. There are many infinite sums and the constraints which will be used together fix the summation parameters in different ways.

In fact we are only expanding two representations of the same integral  $I^D$ . Therefore eqs. (2.70) and (2.72) must be equivalent.

$$\sum_{n_1, \dots, n_n=0}^{\infty} I_n^D(-n_1, \dots, -n_n; \{Q_i^2\}, \{M_i^2\}) \prod_{i=1}^n \frac{x_i^{n_i}}{n_i!} = \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} \frac{Q_1^{q_1} \dots Q_q^{q_q} x_1^{p_1} \dots x_n^{p_n} (-x_1 M_1^2)^{m_1} \dots (-x_n M_n^2)^{m_n}}{q_1! \dots q_q! p_1! \dots p_n! m_1! \dots m_n!} (p_1 + \dots + p_n)! \quad (2.76)$$

At this stage comparing the terms in the left-hand-side with the right-hand-side we deduce that the power of  $x_i$  variables should be negative. There is a further constraint obtained by adding the first and the second term in the constraint equations which will be useful in later steps of the calculation.

$$p_1 + \dots + p_n + q_1 + \dots + q_q = -\frac{D}{2} \quad (2.77)$$

Writing up a general solution is not possible in NDIM method since  $\mathcal{P}$ 's and  $\mathcal{Q}$ 's are process dependent. Nevertheless, we can extract the momentum scale  $Q_i^2$  from each of the  $\mathcal{Q}_i$  and find the coefficient of this term;

$$\sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} (Q_1^2)^{q_1} \dots (Q_q^2)^{q_q} (-M_1^2)^{m_1} \dots (-M_n^2)^{m_n} \times \left( \prod_{i=1}^n \frac{1}{\Gamma(1+m_i)\Gamma(1+p_i)} \right) \left( \prod_{i=1}^q \frac{1}{\Gamma(1+q_i)} \right) \Gamma\left(1 + \sum_{k=1}^n p_k\right) \quad (2.78)$$

In order to find the solutions, one has to evaluate the sums using the existing constraints among the indexes of the sums. The number of different ways to evaluate the loop integral  $I^D$  is given by combinatorics:

$$C_{\delta}^{\sigma} = \frac{\sigma!}{\delta!(\sigma - \delta)!} \quad (2.79)$$

where  $\sigma$  is the number of the summations and  $\delta$  is the number of the constraints. Although the total number of solutions is fixed by the combinatorics of the particular

system, some of the solution sets will be empty due to the nature of the system. In the example below we will see how the method works and give an idea of the form of the solutions. We will also use some properties of the  $\Gamma$ -functions to present the solutions in a nice way.

### 2.4.2 Example: One-Loop Massless Triangle Diagram

To clarify the concepts of the last section we aim to give an example of this method with the same loop integral we calculated in previous sections, the one-loop triangle integral with one off-shell leg and no internal masses.

First we write down the  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{M}$  functions of the triangle diagram in its full generality - three internal masses,  $M_i$  for  $i = 1, \dots, 3$  and three off-shell legs  $p_i^2 = Q_i^2$  for  $i = 1, \dots, 3$ ;

$$\begin{aligned}\mathcal{P} &= x_1 + x_2 + x_3 \\ \mathcal{Q} &= x_2x_3Q_1^2 + x_3x_1Q_2^2 + x_1x_2Q_3^2 \\ \mathcal{M} &= x_1M_1^2 + x_2M_2^2 + x_3M_3^2\end{aligned}\tag{2.80}$$

It is very handy to construct a template solution of the diagram;

$$I^D(\nu_1, \nu_2, \nu_3, Q_1^2, Q_2^2, Q_3^2, M_1^2, M_2^2, M_3^2)$$

The constraint equations of the diagram is evaluated by equating the  $x_i$ 's in the left-hand-side and right-hand-side of the expressions with using the Multinomial theorem:

$$\begin{aligned}q_2 + q_3 + p_1 + m_1 &= -\nu_1 \\ q_1 + q_3 + p_2 + m_2 &= -\nu_2 \\ q_1 + q_2 + p_3 + m_3 &= -\nu_3 \\ p_1 + p_2 + p_3 + q_1 + q_2 + q_3 &= -\frac{D}{2}\end{aligned}\tag{2.81}$$

The generic solution has nine summation variables and four constraints which gives us  $\binom{9!}{4!5!} = 126$  solutions of which forty-five are empty sets due to the nature of the system. The difference between the number of summation variables (nine) and the

number of constraints (four) is five which means five-fold sums. In the literature there are some special hypergeometric functions describing up to three-fold sums but none for four-fold or more-fold sums.

In our special case we have three massless propagators and two external particles are on-shell. So this reduces the number of solutions to only one  $\binom{4!}{4!0!} = 1$ .

$$M_1 = M_2 = M_3 = 0 \quad Q_2^2 = Q_3^2 = 0.$$

Putting this in our template solution, we immediately recover the same result as in the previous chapters.

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, 0) = (-1)^{D/2} (Q_1^2)^{D/2-N} \times \frac{\Gamma(\frac{D}{2} - \nu_1 - \nu_2) \Gamma(\frac{D}{2} - \nu_1 - \nu_3) \Gamma(N - \frac{D}{2})}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - N)} \quad (2.82)$$

where  $N = \nu_1 + \nu_2 + \nu_3$ .

## 2.5 Differential Equations Method

This method was first introduced by A.V. Kotikov [26, 27] and successfully applied by Gehrmann and Remiddi for the evaluation of double box two-loop master integrals [28–32]. In this approach, we avoid the explicit integration of the loop momenta by deriving differential equations in the internal propagator masses or in external momenta for the master integral, and then solve these differential equations with appropriate boundary conditions. The Integration-By-Parts (IBP) and Lorentz Invariance (LI) identities allow us to express integrals as a combination of few integrals which are less complicated than the original integral. These few integrals which are not reducible further are called master integrals for the related Feynman diagram. Differential equations method is one of the other methods to evaluate the master integrals of a given topology.

Since there is no way of having general identities for this method we begin with a pedagogical example to show how it works. In the case of four point functions with

one external off-shell leg and no internal masses we have three differential equations.

$$\begin{aligned}
s_{12} \frac{\partial}{\partial s_{12}} &= \frac{1}{2} \left( +p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \\
s_{13} \frac{\partial}{\partial s_{13}} &= \frac{1}{2} \left( +p_1^\mu \frac{\partial}{\partial p_1^\mu} - p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \\
s_{23} \frac{\partial}{\partial s_{23}} &= \frac{1}{2} \left( -p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} \right)
\end{aligned} \tag{2.83}$$

The derivatives in the invariants are expressed by the external momenta  $s_{ij} = (p_i + p_j)^2$ . The derivatives on the right hand side of eq. (2.83) are not linearly independent but related to each other under rescaling of the external momenta. Let's suppose we have an integral  $\mathcal{I}_{t,r,s}(s_{12}, s_{13}, s_{23}, D)$  where  $t$  being the number of different propagators,  $r$  sum of powers of all propagators and  $s$  sum of powers of all scalar products, one finds a scaling relation;

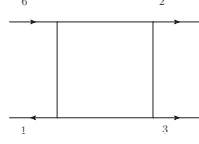
$$I_{t,r,s}(s_{12}, s_{13}, s_{23}, D) = \lambda^{-\alpha(d,r,s)} I_{t,r,s}(\lambda^2 s_{12}, \lambda^2 s_{13}, \lambda^2 s_{23}, d) \tag{2.84}$$

$\alpha(d, r, s)$  is the mass dimension of the integral. The above equation yields the rescaling relation;

$$\left[ -\frac{\alpha}{2} + s_{12} \frac{\partial}{\partial s_{12}} + s_{13} \frac{\partial}{\partial s_{13}} + s_{23} \frac{\partial}{\partial s_{23}} \right] I_{t,r,s}(s_{12}, s_{13}, s_{23}, d) = 0 \tag{2.85}$$



Let us now consider the diagram we are interested in and apply the method to the four point function with one off-shell external leg and massless propagators;



$$\mathcal{I} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2}$$

The derivatives in the external momenta for this diagram are:

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} \mathcal{I} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \left( \frac{2p_1^\mu(k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} \right), \quad (2.86)$$

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} \mathcal{I} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \left( \frac{2p_2^\mu(k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_2^\mu(k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} + \frac{2p_2^\mu(k-p_2)_\mu}{(k-p_2)^2} \right) \quad (2.87)$$

$$p_3^\mu \frac{\partial}{\partial p_3^\mu} \mathcal{I} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \left( \frac{2p_3^\mu(k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_3^\mu(k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} \right). \quad (2.88)$$

When we look at the right hand side terms of the equations we see some of the propagators are squared and some are pinched (eliminated or reduced in power). With the help of the IBP relations we can reduce these integrals to simpler ones and thereby make the differential equations simpler. To get a compact representation we introduce a shorthand notation for our integral;

$$\mathcal{I}(\nu_1, \nu_2, \nu_3, \nu_4, D) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2}$$

Our scalar integral reads  $\mathcal{I}(1, 1, 1, 1, D)$ . The differential equations are simplified by using the IBP equations and one obtains;

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} \mathcal{I}(1, 1, 1, 1, D) = -\mathcal{I}(1, 1, 1, 1, D) + \mathcal{I}(1, 1, 0, 2, D) \quad (2.89)$$

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} \mathcal{I}(1, 1, 1, 1, D) = -\mathcal{I}(1, 1, 1, 1, D) + \mathcal{I}(2, 0, 1, 1, D) \quad (2.90)$$

$$p_3^\mu \frac{\partial}{\partial p_3^\mu} \mathcal{I}(1, 1, 1, 1, D) = (D-6)\mathcal{I}(1, 1, 1, 1, D) - \mathcal{I}(1, 1, 0, 2, D) - \mathcal{I}(2, 0, 1, 1, D) \quad (2.91)$$

The pinched and squared integrals on the right hand side of the equations can be further reduced less complicated and simpler integrals are obtained.

$$\mathcal{I}(1, 1, 0, 2, D) = \frac{D-3}{p_2 \cdot (p_1 + p_3)} \left[ \frac{1}{(p_1 + p_2 + p_3)^2} \mathcal{B}(1, 1, p_{123}) - \frac{1}{(p_1 + p_3)^2} \mathcal{B}(1, 1, p_{13}) \right] \quad (2.92)$$

where  $\mathcal{B}$  is a bubble type integral.

$$\begin{aligned} \mathcal{B}(1, 1, p^2) &= \left[ \frac{(4\pi)^{\frac{4-D}{2}} \Gamma(3-D/2) \Gamma^2(D/2-1)}{16\pi^2 \Gamma(D-3)} \right] \frac{-2i}{(D-4)(D-3)} (-p^2)^{\frac{D-4}{2}} \\ &\equiv A_{2,LO}(-p^2)^{\frac{D-4}{2}} \end{aligned}$$

Inserting the necessary expressions to our set of differential equations in (2.89)-(2.91)

we have;

$$\begin{aligned} s_{12} \frac{\partial}{\partial s_{12}} \mathcal{I}(1, 1, 1, 1, D) &= \frac{D-4}{2} \mathcal{I}(1, 1, 1, 1, D) \\ &+ \frac{2(D-3)}{s_{12} + s_{13}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{23}} \mathcal{B}(1, 1, p_{23}) \right] \\ &+ \frac{2(D-3)}{s_{12} + s_{23}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{13}} \mathcal{B}(1, 1, p_{13}) \right] \quad (2.93) \end{aligned}$$

$$\begin{aligned} s_{13} \frac{\partial}{\partial s_{13}} \mathcal{I}(1, 1, 1, 1, D) &= \frac{D-6}{2} \mathcal{I}(1, 1, 1, 1, D) \\ &- \frac{2(D-3)}{s_{12} + s_{13}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{23}} \mathcal{B}(1, 1, p_{23}) \right] \quad (2.94) \end{aligned}$$

$$\begin{aligned} s_{23} \frac{\partial}{\partial s_{23}} \mathcal{I}(1, 1, 1, 1, D) &= \frac{D-6}{2} \mathcal{I}(1, 1, 1, 1, D) \\ &- \frac{2(D-3)}{s_{12} + s_{23}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{13}} \mathcal{B}(1, 1, p_{13}) \right] \quad (2.95) \end{aligned}$$

where  $s_{123} = s_{12} + s_{13} + s_{23}$ . Equations (2.93)-(2.95) are linear first order inhomogeneous equations which can be solved by introducing an integrating factor. We obtain boundary conditions by setting the invariants to zero. Setting  $s_{ij} = 0$  in the above equations:

$$\mathcal{I}(1, 1, 1, 1, D)_{s_{12}=0} = \frac{4(D-3)}{D-4} \frac{1}{s_{13}s_{23}} [\mathcal{B}(1, 1, p_{123}) - \mathcal{B}(1, 1, p_{13}) - \mathcal{B}(1, 1, p_{23})] \quad (2.96)$$

$$\mathcal{I}(1, 1, 1, 1, D)_{s_{13}=0} = \frac{4(D-3)}{D-6} \frac{1}{s_{12}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{23}} \mathcal{B}(1, 1, p_{23}) \right] \quad (2.97)$$

$$\mathcal{I}(1, 1, 1, 1, D)_{s_{23}=0} = \frac{4(D-3)}{D-6} \frac{1}{s_{12}} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{13}} \mathcal{B}(1, 1, p_{13}) \right] \quad (2.98)$$

In principle the solutions for the differential equations can be obtained from any of the equations (2.96)-(2.98). We seek a general result with respect to the scale  $q^2 = s_{123} = s_{12} + s_{13} + s_{23}$  and, by making a change of variables, we find the differential equation in  $s_{123}$  reads;

$$\begin{aligned} \frac{\partial}{\partial s_{123}} \mathcal{I}(1, 1, 1, 1, D) + \frac{D-4}{2(s_{123} - s_{13} - s_{23})} \mathcal{I}(1, 1, 1, 1, D) = & \quad (2.99) \\ & + \frac{2(D-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{23}} \mathcal{B}(1, 1, p_{23}) \right] \\ & + \frac{2(D-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[ \frac{1}{s_{123}} \mathcal{B}(1, 1, p_{123}) - \frac{1}{s_{13}} \mathcal{B}(1, 1, p_{13}) \right] \end{aligned}$$

Eq. (2.99) is first order linear inhomogeneous differential equation of the form;

$$\frac{\partial y(x)}{\partial x} + f(x)y(x) = g(x)$$

which can be solved with standard techniques by introducing an integrating factor

$$M(x) = e^{\int f(x)dx}$$

The general solution of the differential equations of these forms is;

$$y(x) = \frac{1}{M(x)} \left( \int g(x)M(x)dx + C \right)$$

where C is an integrating constant. In eq. (2.104) we have the integrating factor;

$$M(s_{123}) = (s_{13} + s_{23} - s_{123})^{\frac{D-4}{2}}$$

Finally the non-trivial solution of the differential equation in the invariants reads;

$$\begin{aligned} \mathcal{I}(1, 1, 1, 1, D)_{\{s_{123}, s_{13}, s_{23}\}} = & 2(D-3)A_{2,LO}(s_{13} + s_{23} - s_{123})^{2-D/2} \\ & \times \int^{s_{123}} d\acute{s}_{123} (s_{13} + s_{23} - \acute{s}_{123})^{D/2-3} \\ & \times \left[ \frac{(-s_{13})^{D/2-3}}{s_{13} - \acute{s}_{123}} + \frac{(-s_{23})^{D/2-3}}{s_{23} - \acute{s}_{123}} - \frac{2\acute{s}_{123} - s_{13} - s_{23}}{(s_{13} - \acute{s}_{123})(s_{23} - \acute{s}_{123})} (-\acute{s}_{123})^{\frac{D}{2}-3} \right] \end{aligned} \quad (2.100)$$

In order to evaluate the integrals we change the boundaries in the relevant integrals. For the first two integrals the integration variable is shifted to  $\acute{s}_{123} - s_{13} - s_{23}$  and for the last one it is shifted to  $\acute{s}_{123}(\acute{s}_{123} - s_{13} - s_{23})$ . After performing the integrals

with the shifted boundaries we obtain hypergeometric functions of the type  ${}_2F_1$ . The result for one-loop box integral with massless propagators and one off-shell leg is:

$$\begin{aligned} \mathcal{I}(1, 1, 1, 1, D) = & -\frac{4(D-3)}{D-4} A_{2,LO} \frac{1}{s_{13}s_{23}} \\ & \left[ \left( \frac{s_{13}s_{23}}{s_{13}-s_{123}} \right)^{\frac{D}{2}-2} {}_2F_1 \left( D/2-2, D/2-2; D/2-1; \frac{s_{123}-s_{13}-s_{23}}{s_{123}-s_{13}} \right) \right. \\ & + \left( \frac{s_{13}-s_{23}}{s_{23}-s_{123}} \right)^{\frac{D}{2}-2} {}_2F_1 \left( D/2-2, D/2-2; D/2-1; \frac{s_{123}-s_{13}-s_{23}}{s_{123}-s_{23}} \right) \\ & \left. - \left( \frac{-s_{123}s_{13}s_{23}}{(s_{13}-s_{123})(s_{23}-s_{123})} \right)^{\frac{D}{2}-2} {}_2F_1 \left( D/2-2, D/2-2; D/2-1; \frac{s_{123}(s_{123}-s_{13}-s_{23})}{(s_{123}-s_{13})(s_{123}-s_{23})} \right) \right] \end{aligned} \quad (2.101)$$

where  $A_{2,LO}$  is a bubble type integral.

Since we have encountered some Hypergeometric functions in the results, it's best to represent some of them here. More information can be found in the relevant books [33, 34]. The Hypergeometric functions of one variable are sums of Pochhammer symbols over a single summation parameter  $m$ .

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma, x) &= \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)} \frac{x^m}{m!} \\ {}_3F_2(\alpha, \beta, \beta', \gamma, \gamma', x) &= \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)(\beta', m)}{(\gamma, m)(\gamma', m)} \frac{x^m}{m!} \end{aligned}$$

where Pochhammer symbol  $(z, n)$  is defined:

$$(z, n) \equiv \frac{\Gamma(z+n)}{\Gamma(z)}$$

# Chapter 3

## Integration By Parts

So far we introduced how scalar integrals are evaluated with different methods in dimensional regularization. The techniques introduced in the previous chapter handle the problem in different ways and some of them may not be suitable for the process to be calculated. Besides analytical evaluations there are also numerical methods to calculate the integrals. In this thesis we are most concerned with the analytical approach.

In any typical QCD calculation after applying the Feynman rules, one usually ends up with a set of very many tensor integrals. The evaluation of these integrals with the methods outlined before is very hard. We need some other relational identities between scalar integrals and tensor integrals. Our aim is to find a recursive method which relates tensor integrals to scalars. The idea is to write down various equations for integrals of derivatives with respect to loop momenta and use this set of relations between Feynman integrals to express a general Feynman integral of the given class in terms of a small number of master integrals. The reduction can be stopped whenever one arrives at sufficiently simple integrals. One could also try to solve to reduce a given integral to true *irreducible* integrals (master integrals) which cannot be reduced further. This technique was first introduced and applied by Chetyrkin, K. G. and Tkachov, F. V. in 1981 to calculate Beta functions at four loops. [35]

Although there are various algorithms which were devised to do the reduction in the literature [36,37], the Laporta Algorithm [38] has become the standard approach.

We will give the basics of the reduction mechanism and apply it to some Feynman integrals in the next section. Before that it is good to mention another type of identities called Lorentz invariance identities (*LI*) [28]. Lorentz Invariance identities (*LI*) complement the IBP identities and sometimes become very useful. In fact it was proved in a recent paper by R.Lee [39] that the LI identities can be derived from IBP directly.

### 3.1 Integration By Parts and Lorentz Invariance Identities

We begin by exploring a way of expressing the IBP and LI identities for the loop integrals in a symbolic and general manner. Consider the general scalar  $m$ -loop diagram in  $D$  dimensions with  $n$  propagators  $\frac{1}{A_i}$  raised to arbitrary powers  $\nu_i$  and  $p_i, \dots, p_r$  external momenta.

$$\mathcal{I}^D = \int \frac{d^D k_1}{i\pi^{D/2}} \cdots \int \frac{d^D k_m}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1} \dots A_n^{\nu_n}} \quad (3.1)$$

If we differentiate with respect to the loop momenta  $a^\mu = k_1^\mu, \dots, k_m^\mu$ , the surface terms disappear and we get:

$$\int \frac{d^D k_1}{i\pi^{D/2}} \cdots \int \frac{d^D k_m}{i\pi^{D/2}} \frac{\partial}{\partial a_\mu} \frac{b_\mu}{A_1^{\nu_1} \dots A_n^{\nu_n}} \equiv 0, \quad (3.2)$$

$b^\mu$  can be internal loop momenta or external momenta of the diagram.

$$b^\mu = k_1^\mu, \dots, k_m^\mu, p_1^\mu, \dots, p_{r-1}^\mu$$

The total derivative of the integrand yields two types of terms:

$$\frac{\partial}{\partial a^\mu} \frac{b^\mu}{A_1, \dots, A_n} = \begin{cases} \frac{\partial b^\mu}{\partial a^\mu} \frac{1}{A_1, \dots, A_n} & \mathbf{A} \\ \frac{b^\mu}{A_1 \dots A_{i-1} A_{i+1} \dots A_n} \frac{\partial}{\partial a^\mu} \left( \frac{1}{A_i} \right) & \mathbf{B} \end{cases} \quad (3.3)$$

The terms in **A** are always zero unless  $a = b$  in which case the result is  $D$ . The terms in **B** are a bit complicated and may result in two types of contribution. The derivative of the loop momenta in the denominator may either create *reducible* or *irreducible* terms in the numerator. Suppose we have a denominator of the form:

$$\frac{1}{A} = \frac{1}{(\sum \xi_j k_j + q)^2 + i0}$$

Differentiation gives ;

$$b^\mu \frac{\partial}{\partial a^\mu} \left( \frac{1}{A^\nu} \right) = -\nu \frac{\sum \xi_j k_j \cdot b + q \cdot b}{A^{\nu+1}}. \quad (3.4)$$

If the scalar products in the numerator can be cast in terms of the propagators  $A_i$  it is called *reducible* otherwise it is *irreducible*. If one can obtain reducible numerators after differentiation, there are cancellations in the denominator and the integrals turn out to be simpler than the parent one. It is a trade off between obtaining simpler integrals and getting many integrals at the same time. The irreducible terms can be often be simplified and eliminated by some of the IBP identities and reduced to reducible forms.

We shall devise a symbolic representation for raising and lowering the powers of the terms in the denominator by this operation;

$$i^+ \mathcal{I} = \mathcal{I}(\dots, \nu_i + 1, \dots)$$

$$i^- \mathcal{I} = \mathcal{I}(\dots, \nu_i - 1, \dots)$$

The operator  $i^\pm$  acts on the  $i$ th term in the denominator and raises or lowers the power by one.

The Lorentz invariance identities [28] are sometimes used to complement the IBP identities and might be useful in some calculations. As we know the Feynman integral is a function of the scalar products of the external momenta and it is invariant under Lorentz transformations. However in a recent paper by R.Lee [39], it was proved that these identities can be derived by the IBP identities. Nevertheless it might be good to mention them here. The Lorentz transformation (rotation) is of the form:

$$\acute{p}_i^\mu = \Lambda_\nu^\mu p_i^\nu, \quad \Lambda_{\mu\nu} = g_{\mu\nu} + \delta \epsilon_{\mu\nu}, \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}.$$

Here  $\delta$  is a small parameter. The Feynman integral can be transformed as;

$$\int \frac{d^D k_1}{i\pi^{D/2}} \cdots \int \frac{d^D k_m}{i\pi^{D/2}} f(k_j, p_i) = \int \frac{d^D k_1}{i\pi^{D/2}} \cdots \int \frac{d^D k_m}{i\pi^{D/2}} f(k_j, \acute{p}_i). \quad (3.5)$$

Expanding the right hand side of eq. (3.5) around  $\delta = 0$ , we get

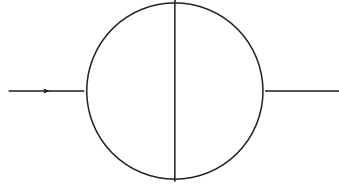
$$\int \frac{d^D k_i}{i\pi^{D/2}} \cdots \int \frac{d^D k_m}{i\pi^{D/2}} \sum_{a=1}^{r-1} \frac{\partial f(k_j, p_i)}{\partial p_a^\mu} \epsilon_\nu^\mu p_a^\nu = 0 \quad (3.6)$$

$r - 1$  being the independent external momenta. The number of the LI identities one can build is;

$$\mathcal{N}_{LI} = \frac{1}{2}(r - 1) \times (r - 2)$$

### 3.1.1 Example: One

We begin with a simple example to make the concepts of the last section more explicit.



$$\mathcal{I} = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{A_1 A_2 A_3 A_4 A_5}$$

where

$$A_1 = k^2, \quad A_2 = (k + p)^2, \quad A_3 = (l + p)^2, \quad A_4 = l^2, \quad A_5 = (l - k)^2.$$

Taking the total derivative of the tensor integral with  $(k - l)^\mu$  in the numerator with respect to the loop momenta  $k^\mu$  we find the identity

$$\mathcal{I} = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} \frac{(k - l)^\mu}{k^2 (k + p)^2 l^2 (l + p)^2 (k - l)^2} = 0 \quad (3.7)$$

Expanding the integrand, we find

$$0 = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \left\{ D - (k - l)^\mu \left[ \frac{2(l + p)_\mu}{(l + p)^2} + \frac{2l_\mu}{l^2} + \frac{2(k - l)_\mu}{(k - l)^2} \right] \right\} \times \frac{1}{k^2 (k + p)^2 (l + p)^2 l^2 (k - l)^2} \quad (3.8)$$

where we have used the following relations:

$$\begin{aligned} \frac{d}{dx^\mu} \frac{x^\mu}{f(x^\mu)} &= \frac{1}{f(x^\mu)} \frac{d}{dx^\mu} x^\mu - \frac{x^\mu}{f^2(x^\mu)} \frac{df(x)}{dx^\mu} \\ \frac{d}{dx^\mu} x^\mu &= g_\mu^\mu = \delta_\mu^\mu = D \\ \frac{d}{dx^\mu} x^2 &= \frac{d}{dx^\mu} g^{\rho\sigma} x_\rho x_\sigma \\ &= g^{\rho\sigma} (g^{\rho\mu} x_\sigma + g_{\mu\sigma} x_\rho) \\ &= 2x_\mu \end{aligned}$$



The expressions in the square brackets were simplified according to another relation;

$$2(a + b) \cdot (a + c) = (a + c)^2 + (a + b)^2 - (b - c)^2$$

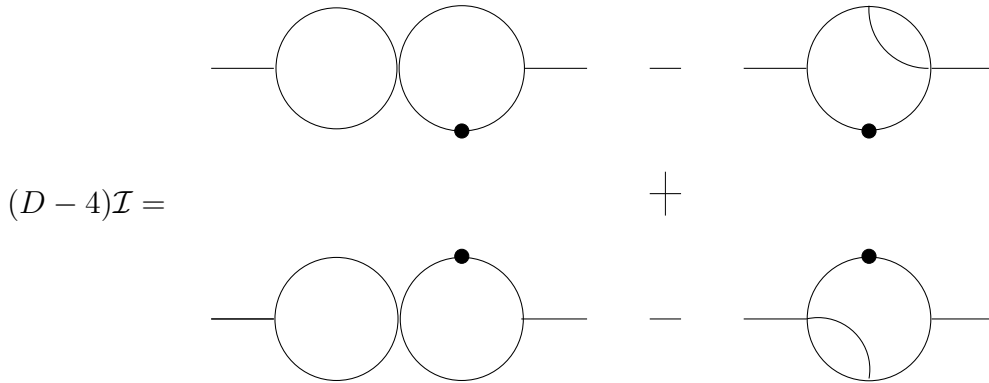
After some rearrangements we obtain:

$$0 = \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \left\{ D - 4 - \frac{(k - l)^2}{(l + p)^2} - \frac{(k - l)^2}{l^2} + \frac{k^2}{l^2} + \frac{(k + p)^2}{(l + p)^2} \right\} \times \frac{1}{k^2(k + p)^2(l + p)^2 l^2 (k - l)^2} \quad (3.9)$$

Our original integral takes the reduced form;

$$\mathcal{I} = \frac{1}{D - 4} \int \frac{d^D k}{i\pi^{D/2}} \int \frac{d^D l}{i\pi^{D/2}} \left\{ \frac{1}{k^2(k + p)^2[(l + p)^2]l^2} + \frac{1}{k^2(k + p)^2(l + p)^2[l^2]^2} - \frac{1}{(k + p)^2(l + p)^2[l^2]^2(k - l)^2} - \frac{1}{k^2[(l + p)^2]l^2(k - l)^2} \right\} \quad (3.10)$$

In a diagrammatic form the integral looks like:

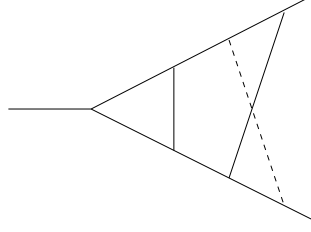


The dots on the diagrams represent the squared propagators. Now the integral is decomposed into a sum of four less complicated integrals which can be easily solved compared to the original one.

### 3.1.2 Example: Two

Our next example is a rather more complicated three-loop non-planar nine-propagator integral and it is one of the scalar integrals in our database in this project. We will only give the output for the integral obtained using the IBP package FIRE [40]. FIRE has been used in our reductions extensively and it took a while for the calculation of this integral (seed). The details of the program and algorithm will be

explained in the following sections. Since it is a  $1 \rightarrow 2$  three-loop topology with two external legs on-shell, there are 15 IBP equations.



$$\mathcal{I} = \int \frac{d^D j}{2\pi^{D/2}} \int \frac{d^D k}{2\pi^{D/2}} \int \frac{d^D l}{2\pi^{D/2}} \frac{1}{A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9}$$

where  $p_1^2 = p_2^2 = 0$ ,  $(p_1 + p_2)^2 = s_{12}$ ,

$$\begin{aligned} A_1 &= j & A_2 &= j + p_1 + p_2 & A_3 &= k \\ A_4 &= k + p_1 & A_5 &= l & A_6 &= l + p_1 + p_2 \\ A_7 &= j - k & A_8 &= l - j & A_9 &= j - k + p_2 \end{aligned} \quad (3.11)$$

After the reduction the integral reads;

$$\begin{aligned} \mathcal{I} = & - \frac{32(D-3)(2D-7)(2D-5)(3D-10)(3D-8)(13D-57)}{3(D-4)^5(2D-9)s_{12}^5} A_{41} \\ & - \frac{128(D-3)^3(2D-7)}{3(D-4)^4 s_{12}^4} B_{52} \\ & - \frac{64(D-3)^3(2D-7)(3D-14)}{3(D-4)^4(2D-9)s_{12}^4} A_{52} \\ & - \frac{8(D-3)^2(3D-10)(3D-8)}{(D-4)^3(2D-9)s_{12}^4} B_{51} \\ & - \frac{32(D-3)^2(2D-7)(3D-8)(19D-84)}{3(D-4)^4(2D-9)s_{12}^4} A_{51} \\ & - \frac{4(D-4)}{(2D-9)s_{12}^2} A_{73} \\ & + \frac{8(3D-13)(3D-11)}{(D-4)(2D-9)s_{12}^2} A_{74} \\ & + \frac{10(D-3)(5D-22)}{(D-4)(2D-9)s_{12}^2} A_{71} \\ & - \frac{2(D-3)}{(D-4)s_{12}} C_{81} \end{aligned}$$

The quantities ( $A_{41}, A_{51}, A_{52}, B_{51}, B_{52}, A_{71}, A_{73}, A_{74}, C_{81}$ ) represent the master (irreducible) integrals in our database. They correspond to 4-5-6-7-8 propagator type integrals and we have many of them in this study. Their topologies and definitions will be discussed in the next section.

We have seen the reduction of one integral produces many different master integrals. In a typical pQCD process one has to deal with thousands of integrals and bookkeeping turns out to be very important in these calculations.

## 3.2 Laporta Algorithm

The seminal paper by S.Laporta [38] introduced a very efficient technique to systematically organise the IBP equations and use them to write an integral in terms of a (small) set of masters. Since then it has become the standard algorithm in the reduction process of scalar and tensor integrals. It solves the set of equations in a systematic way. The strategy is similar to pyramid hierarchy which begins from bottom and goes up to top. Although a detailed explanation of the algorithm is given in the paper [38], we would like to mention the basic steps here.

### The Algorithm

We shall have compact symbolic representations of the numbers related to the generic Feynman Integral. The input of the algorithm consists of four terms.

$[DenSet, MaxDen, MaxNum, SolutionSet]$ .

In the beginning, our solution set is empty  $\{\}$ . We store our previous solutions in this set as the program goes on calculating different integrals (seeds). DenSet is the set of denominators which correspond to our propagators in the diagram. It may have positive and negative values in the entries. MaxDen and MaxNum are defined as;

$$M_d = \sum_i (\nu_i - 1) \quad M_p = \sum_j (\nu_j) \quad (3.12)$$

where  $i$  can only be positive and  $j$  can be zero or negative. Let us define  $N_m$  as the number of loops in the diagram and  $N_d$  as the number of denominators of the integral in the denominator set.

- Set  $N_m = 0$  in the beginning
- Let  $n = N_m + 1$  generate all combinations from  $n$  to  $N_d$  and store them set  $\mathcal{A}$
- Select the  $n$  denominator topologies from  $\mathcal{A}$  and store them in set  $\mathcal{B}$

- Select the first seed from the set  $\mathcal{B}$  and let  $M_d = 0$   $M_p = 0$
- Generate all combinations in which the sum of the powers of the  $n$  denominators is  $M_d + n$  and the sum of the powers of the numerators is  $M_p$  and store them in set  $\mathcal{C}$
- take the 1<sup>st</sup> member of the set  $\mathcal{C}$  and generate the IBP equations for that seed
- Solve the system of IBP equations for that seed and store them in the solution set
- Take the next member and repeat the procedure, union the set
- End the Loop on set  $\mathcal{C}$  and take the next member of the set  $\mathcal{B}$
- End the Loop on  $\mathcal{B}$  and take the next member of the set  $\mathcal{A}$
- Terminate  $\rightarrow$  Solution Set

In perturbative calculations one has to find a way handling lots of integrals to the required level which we call master (irreducible) integrals. In order to achieve that, many attempts have been done with using Laporta algorithm. Currently there are two public codes available in order to do the reduction using the Laporta algorithm - FIRE and AIR. In this study we chose to use FIRE rather than AIR. This is because FIRE makes better use of the computer memory. AIR uses less memory, but can slow down when writing large numbers of small files. The differences will become clear in the comparison benchmark below.

- AIR (Automated Integral Reduction) [41] was written by Charalampos Anastasiou and Achilleas Lazopoulos in Maple. The details and the usage of the code can be found in their paper [41].
- FIRE (*F*eynman *I*ntegral *R*eduction) [40] was written in Mathematica [42] by A.V.Smirnov and became public in July 2008. It has lots of versions and the most stable one is FIRE3.4.0. Most of our calculations were done with FIRE3.4.0 and some parts were crosschecked with AIR.

### 3.3 The FIRE (*F*eynman *I*ntegral *R*Eduction) Package

The FIRE code interfaces with two external programs - Flink and Qlink - to speed up the calculations that are slow in the built-in routines in the Mathematica working environment.

#### **FLink**

FLink tool links the Fermat code for polynomials to FIRE.

#### **QLink**

Qlink tool was written in C and links the Database programs to FIRE. Unlike AIR, FIRE stores all the information in the memory. For heavy calculations the program may exceed the physical memory of the system. In most situations it becomes inevitable to use a database. The previous versions of FIRE applied QDMB database then changed to TokyoCabinet because of the efficiency problems. In the latest version TokyoCabinet is recommended.

One can work with FIRE in two modes - either pure Laporta or employing Gröbner bases. We preferred to use the pure Laporta mode instead of using Gröbner bases mainly because the Gröbner bases are less well defined. More information can be found in [43] and the algorithm for Gröbner bases is described in [37].

In the program FIRE some definitions are generated and applied for solving the system of equations in an efficient way. The terms; directions, sectors, regions are used to do the ordering and priority of the integrals are devised. Basically they aim to shorten the evaluation of the required integrals. The termination of the over-constrained system is guaranteed by masking some of the IBPs and doing the necessary substitutions effectively. This is because the set of linear orderings in the Laporta algorithm is big enough so that only a subset of these is used in practice.

The program first creates the sectors according to the proper expressions of the system and finds the number of sub-integrals to be evaluated. It scans all the sectors one by one and in each sector evaluates the sub-sectors from one corner to another. Everytime a sector finishes it stores the relevant information in the memory and constantly crosscheck with the ongoing sectors. This crosscheck is vital for the sake

of termination time and memory or harddrive of the system. To construct a proper expression for a given integral FIRE:

- checks the integral whether it is zero or not by the boundary conditions
- checks the integral whether it is zero or not by the parity conditions
- looks at the symmetry conditions of the given integral
- looks for manual rules- if there are any - for mappings onto other integrals

### Benchmarks

To illustrate the terminating times for the two codes, we consider a simple massless one-loop box diagram with four legs on-shell. Only the tensor type seeds were given as inputs to both codes.  $F(a_1, a_2, a_3, a_4)$  where  $a_i \leq 1$  and  $-\sum_i \max(a_i, 0) \leq N$  for a given  $N$ . The termination time also depends on some other factors such as CPU

N	AIR	FIRE (pure Laporta)
10	56	8
15	126	15
20	331	26
30	1375	70
50	14137	336

Table 3.1: Comparison of AIR and FIRE (times given in seconds)

speed and harddrive speed. But as seen from the table FIRE is faster than AIR in all respects. [40]

**Usage Of the Code FIRE** It is simpler to explain the instructions of the code with the help of one of our inputs. It will make the performance of the code easier to understand. We preferred to use the code in two steps and the output of the first step becomes the input of the second. We show the generic input for calculating planar vertex type  $1 \rightarrow 2$  diagrams with two legs on-shell at three-loops.

INPUT 1:

```
(1) Get["FIRE_3.4.0.m"];
```

```

(2)  Get["IBP.m"];
(3)  UsingFermat=True;
(4)  DirectIBP=False;
(5)  LeeIdeas=True;
(6)  Internal = {k1, k2, k3};
(7)  External = {p1, p2};
(8)  Propagators = {k12, (k1 + p1)2, (k1+p1+p2)2,
k22, (k2+p1)2, (k2+p1+p2)2, k32, (k3+p1)2, (k3+p1+p2)2,
(k1-k2)2, (k2-k3)2, (k3-k1)2};
(9)  PrepareIBP[];
(10) reps = {p12 → 0, p22 → 0, p1*p2 → s12/2};
(11) startinglist = {IBP[k1,k1], IBP[k1,k1+p1], IBP[k1,k1+p1+p2],
IBP[k2,k2], IBP[k2, k2+p1], IBP[k2,k2+p1+p2], IBP[k3,k3],
IBP[k3,k3+p1], IBP[k3,k3+p1+p2], IBP[k1,k1-k2],
IBP[k1,k3-k1], IBP[k2,k1-k2], IBP[k2,k2-k3],
IBP[k3,k2-k3], IBP[k3,k3-k1]} /. reps
(12) r=Get["zero"];
(13) RESTRICTIONS = r;
(14) SYMMETRIES = {{4,5,6,1,2,3,7,8,9,10,12,11},
{7,8,9,4,5,6,1,2,3,11,10,12},{1,2,3,7,8,9,4,5,6,12,11,10},
{4,5,6,7,8,9,1,2,3,11,12,10},{7,8,9,1,2,3,4,5,6,12,10,11}
{3,2,1,6,5,4,9,8,7,10,11,12}}};
(15) Prepare[];
(16) SaveStart["loop3.4"];
(17) Burn[];
(18) SaveData["loop3.4"];
(19) Quit[]

```

In our first input we prepare the two files-loop3.4.start and loop3.4.data ready for the second stage. All the mathematica packages should be in the same folder otherwise you need to locate the files in the code. Let's explain briefly what the code does in each line;

1. The code reads the main package.
2. Gets the auxiliary package for composing the Integration By Parts Equations.
3. The package option whether Fermat code is used or not. We strictly recommend this option. Set to TRUE.
4. This turns of the Gröbner bases off. Set to FALSE for pure Laporta Mode.
5. It eliminates some of the IBP equations according to R.Lee ideas [37]
6. Number of loops in the diagram. In our case we have three.
7. Number of external legs.
8. The propagators of the diagram.
9. It initiates of the auxiliary package.
10. On-shell conditions for the diagram.
11. All possibilities of differentiation according to the loop momenta. The number of IBP's is  $\mathcal{N}_{IBP} = m \times (m + n - 1)$
12. The file contains the integrals which are zero-boundary conditions.
13. The code reads the zeros and skips the unnecessary sectors for the calculation. This saves a big amount of time during the calculations.
14. If a diagram contains some certain symmetries it is better to specify them in order to save time. For the planar case our auxiliary diagram is symmetric when the loop momenta is shifted.
15. The code prepares the .start file. It contains the information about all sectors.
16. Saves the .start file
17. Main routine of the program.
18. Saves the .data file



## 19. Quit

There are two outputs of this program `loop3.4.start` and `loop3.4.data`. The termination time is less than 3 minutes. These two files contain the necessary information for the reduction process. Our second input evaluates the integrals.

```
(1)  INPUT 2:
(2)  Get["FIRE_3.4.0.m"];
(3)  UsingIBP=True;
(4)  UsingFermat=True;
(5)  DirectIBP=False;
(6)  LeeIdeas=True;
(7)  LoadStart["loop3.4",1];
(8)  Burn[];
(9)  r=Get["int1"];
(10) EvaluateAndSave[r,"int1.Tables"];
(11) Quit[]
```

The steps from 1 to 6 are the same as in the first input file.

7. Loads the `.start` file.
8. Performs some internal optimizations for speeding up the algorithm.
9. Read the integrals to be evaluated from the file.
10. Main routine to perform the reductions. It calculates and saves the results as tables.
11. Quit

The evaluation time depends on the number and complexity of the integrals. Step 2 provides a `.Tables` file as an output. All the reductions and master integrals are stored in that file.

At this stage one can determine and label the master integrals. The master integrals appear as  $G[1, \{\dots\}]$  form. If there is no symmetry in the system some  $G$  integrals may correspond to the same master integral. It is best to try to draw the

master integrals and make a global definition of them. If some symmetry conditions are given, the code successfully maps the integrals onto the same master integral. The optimum way of running FIRE is to enter as many as boundary and symmetry conditions if possible for achieving less termination times. In our case we had the difficulty of dealing with the tensor integrals with up to 5 powers in the numerator and there was no symmetry in the diagram.

Another problem is related with the storage capability. If one uses QLink to avoid the memory problems in the system, we expose a second (hardware) problem. The efficiency of reading and writing from the hard drive decreases as long as the calculations continue. To avoid this problem and keep the efficiency at a certain level, we had to employ a **SSD**(Solid State Disc) which is essentially almost fast as RAM. Since the code can perform millions of integrals, one of the possible improvements in the future is to enable the parallelization of the code.

# Chapter 4

## Quark and Gluon Form Factors up to Two Loops

### 4.1 History and definitions

The form factors are of considerable interest for phenomenology: In fact, they naturally appear as building blocks in the computation of some of the ‘gold-plated’ observables such as Drell-Yan production, Deep Inelastic Scattering and Higgs boson hadroproduction.

The one-loop corrections to the quark form factor  $F_q$  was evaluated by Schwinger in 1949 [44], and the two-loop calculations became available 20 years ago [45–49]. Progress in determining the gluon form factor  $F_g$  (in the large top mass limit) has been more recent. The one-loop corrections were established in 1990/1991 [50, 51] and the two-loop corrections were computed in 2000 [49].

In recent years, attention has been focused on evaluating the three-loop form factor [52–55]. The first calculation of the three-loop form factors was carried out in refs. [52, 53] where only pole parts of  $F_q$  and  $F_g$  were given.

The purpose of this section is to establish the definitions of the form factors and test whether our code is working properly or not by re-calculating the one-loop and two-loop form factors again and comparing our results with those in the literature [46, 56–59].

### 4.1.1 The quark form factor

The sum of all diagrams contributing to the on-shell quark form factor is given by :

$$\bar{u}(p_1)\Gamma_q^\mu v(p_2) = \bar{u}(p_1)\gamma^\mu v(p_2)\mathcal{F}_q(q^2) \quad (4.1)$$



Figure 4.1: Vertex diagram for the process

$\gamma^* \rightarrow q\bar{q}$

Here  $u$  and  $v$  are quark and anti-quark spinors,  $\Gamma_q^\mu$  is the vertex function and  $\mathcal{F}_q(q^2)$  is the scalar quark form factor. The most general possible structure of the vertex function looks like:

$$\Gamma_q^\mu = A\gamma^\mu + B(p_1^\mu + p_2^\mu) + C(p_1^\mu - p_2^\mu) \quad (4.2)$$

With the help of the Ward identity

$$p_{1,\mu}\Gamma_q^\mu = p_{2,\mu}\Gamma_q^\mu = 0$$

and the Dirac equation of motion we conclude that

$$B = C = 0$$

$\mathcal{F}_q(q^2)$  can be extracted using a  $D$ -dimensional projection operator such that,

$$\mathcal{F}_q(q^2) = \sum_{Q=1}^{N_F N} \frac{1}{2(2-D)(-ie e_Q)q^2} \text{Tr}(\gamma_\mu \not{p}_1 \Gamma_q^\mu \not{p}_2) \Big|_{p_1^2=p_2^2=0} \quad (4.3)$$

where  $D = 4 - 2\epsilon$ ,  $q = p_1 + p_2$ ,  $N_F$  is the number of quark flavors,  $N$  is the number of colours and  $e_Q$  is the colour charge of the quarks.



Figure 4.2: Vertex diagram for the process

 $H \rightarrow gg$ 

### 4.1.2 The gluon form factor

In the limit where the top quark is very heavy, the  $H$ -gluon interactions are obtained from the effective interaction,

$$\mathcal{L}_{eff} = -\frac{H}{v} C_1 (G_{\mu\nu}^a)^2. \quad (4.4)$$

where  $G_{\mu\nu}^a$  denotes the field strength tensor and  $v = 246 \text{ GeV}$  is the vacuum expectation value. The coefficient function  $C_1$  is computed up to  $\alpha_s^4$  in [60], with  $n_l = 5$ ,  $N_c = 3$  and  $m_t$  being the  $\overline{MS}$  top-quark mass renormalized at the scale  $\mu$ .

$$\begin{aligned} C_1 = & -\frac{1}{12} \frac{\alpha_s^{(n_l+1)}(\mu)}{\pi} \left\{ 1 + \frac{\alpha_s^{(n_l+1)}(\mu)}{\pi} \left( \frac{11}{4} - \frac{1}{6} \ln \frac{\mu^2}{m_t^2} \right) \right. \\ & + \left( \frac{\alpha_s^{(n_l+1)}(\mu)}{\pi} \right)^2 \left[ \frac{2821}{288} - \frac{3}{16} \ln \frac{\mu^2}{m_t^2} + \frac{1}{36} \ln^2 \frac{\mu^2}{m_t^2} + n_l \left( -\frac{67}{96} + \frac{1}{3} \ln \frac{\mu^2}{m_t^2} \right) \right] \\ & + \left( \frac{\alpha_s^{(n_l+1)}(\mu)}{\pi} \right)^3 \left[ -\frac{4004351}{62208} + \frac{1305893}{13824} \zeta(3) - \frac{859}{288} \ln \frac{\mu^2}{m_t^2} + \frac{431}{144} \ln^2 \frac{\mu^2}{m_t^2} - \frac{1}{216} \ln^3 \frac{\mu^2}{m_t^2} \right. \\ & + n_l \left( \frac{115607}{62208} - \frac{110779}{13824} \zeta(3) + \frac{641}{432} \ln \frac{\mu^2}{m_t^2} + \frac{151}{288} \ln^2 \frac{\mu^2}{m_t^2} \right) \\ & \left. \left. + n_l^2 \left( -\frac{6865}{31104} + \frac{77}{1728} \ln \frac{\mu^2}{m_t^2} - \frac{1}{18} \ln^2 \frac{\mu^2}{m_t^2} \right) \right] \right\} \end{aligned} \quad (4.5)$$

The most general possible structure of the  $Hgg$  vertex is:

$$\Gamma_g^{\mu\nu} = \mathcal{F}_g(q^2) (A g^{\mu\nu} p_1 \cdot p_2 + B p_1^\mu p_2^\nu + C p_1^\nu p_2^\mu) \quad (4.6)$$

where  $\Gamma_g^{\mu\nu}$  represents the vertex function obtained from Feynman diagrams.

Using the Ward identity

$$p_{1,\mu}\Gamma_g^{\mu\nu} = p_{2,\mu}\Gamma_g^{\mu\nu} = 0,$$

we find that  $B = C = -A$ . By choosing  $A = 1$  and using an appropriate projector  $(g_{\mu\nu}p_1 \cdot p_2 - p_{1,\mu}p_{2,\nu} - p_{1,\nu}p_{2,\mu})$  acting on both sides of eq. (4.6),

$$\begin{aligned} & (g_{\mu\nu}p_1 \cdot p_2 - p_{1,\mu}p_{2,\nu} - p_{1,\nu}p_{2,\mu})\Gamma_g^{\mu\nu} = \\ & (g_{\mu\nu}p_1 \cdot p_2 - p_{1,\mu}p_{2,\nu} - p_{1,\nu}p_{2,\mu})(g^{\mu\nu}p_1 \cdot p_2 - p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)\mathcal{F}_g^{\mu\nu}(q^2) \end{aligned}$$

we can extract the gluon form factor,

$$\mathcal{F}_g(q^2) = \frac{4\Gamma_g^{\mu\nu}(g_{\mu\nu}p_1 \cdot p_2 - p_{1,\mu}p_{2,\nu} - p_{1,\nu}p_{2,\mu})}{(D-2)q^4}. \quad (4.7)$$

## 4.2 General Algorithm for Form Factor Calculation

In order to evaluate the quark and gluon form factors, all relevant diagrams for the relevant processes are summed up. For this purpose QGRAF program is employed [61]. Table 4.3 shows the number of diagrams generated by QGRAF. We have set the external legs to be free of self energy insertions and the graphs free of tadpoles. The QGRAF program needs three input files to process the diagrams(data, style

Process	$\gamma^* \rightarrow qq$	$H \rightarrow gg$
1-loop	1	4
2-loop	13	69
3-loop	244	1586

Table 4.1: Number of Feynman diagrams generated by QGRAF

and model files). Examples of the source code for these input files can be found in the appendix.

The general layout of the form factor calculations is depicted in fig. (4.1). After the generation of the diagrams, a specifically designed FORM program is employed to calculate the amplitude of the relevant process [62]. Before passing the QGRAF output to our FORM code, it has to be prepared in such a way that FORM can handle and process this file. For this purpose a small code was devised in MAPLE. The main FORM code performs the sums over colour and spin as well as the Dirac traces. During the calculation of the Dirac traces the program creates thousands of integrals which have to be reduced and evaluated. The gluon polarisations are summed over by ensuring polarisation states are physical (transverse). In this thesis we applied Feynman gauge for the internal gluons and axial gauge for the external gluons.

$$\sum_{pols} \epsilon_i^\mu \epsilon_i^{\nu*} = -g^{\mu\nu} + \frac{\eta^\mu p^\nu + \eta^\nu p^\mu}{\eta \cdot p} - \frac{\eta^2 p^\mu p^\nu}{(\eta \cdot p)^2}, \text{ Axial gauge}$$

$$\sum_{pols} \epsilon_i^\mu \epsilon_i^{\nu*} = -g^{\mu\nu}, \text{ Feynman gauge}$$

The final result does not depend on the choice of the light-like momentum  $n^\mu$ .

All the integrals appearing in the output file are collected and separated according to their topological structure and then stored. Integrals which are trivially zero are identified and eliminated at this stage.

Integrals belonging to a particular topology are then fed into the FIRE package. In our study we preferred to split the planar and non-planar integrals and run them separately. Naturally the time for the difficult tensor reductions takes a long cpu time and memory.

As mentioned earlier several crosschecks were made between the outputs of FIRE and AIR just to be sure of getting the correct expressions of the integrals.

After running FIRE, the expressions for the integrals are included in our original FORM program. In between these, the integrals which point the same topology should be identified and named in order prevent the master integral inflation. As can be seen in the appendix A.3 for the three-loop form factor, lots of integrals point the same topology. By doing the necessary simplifications and groupings in FORM we get the raw result in the form of polynomials in  $D$  and  $q^2 = s_{12}$  multiplying the master integrals.

All L-loop integrals have an overall factor of,

$$s_{12}^n \frac{(-s_{12} - i0)^{-L\epsilon}}{\Gamma(1 - \epsilon)^L}$$

where  $n$  is fixed by dimensional arguments. Unlike Refs. [63–65], there is no  $(-1)^n$  factor.

At the final stage the expansion of the integrals in  $\epsilon$  are inserted and the result is represented as compactly as possible.

The general layout of the entire process can be depicted in the flowchart diagram shown in fig. (4.1).



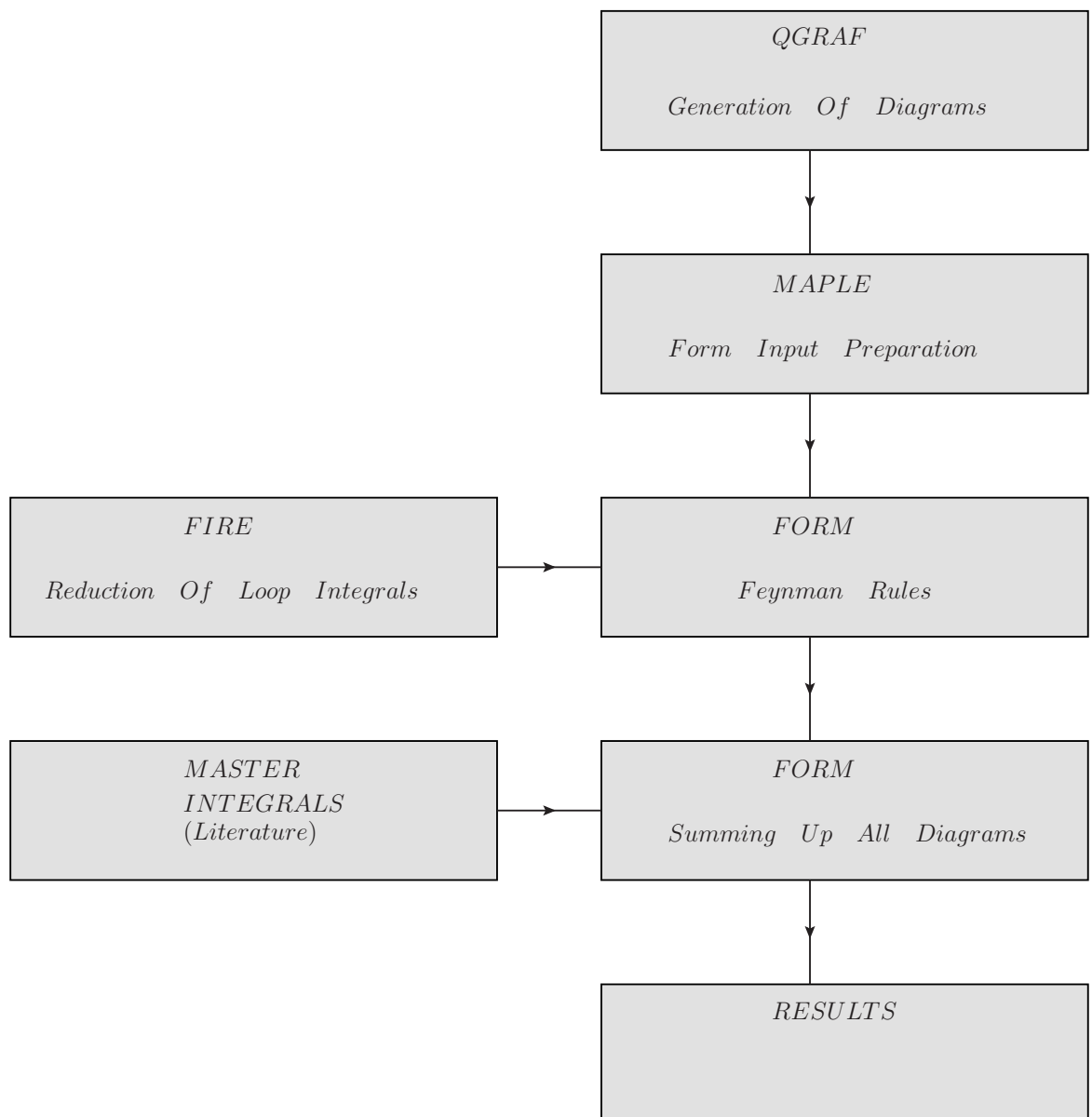


Figure 4.3: General layout of the form factor calculations

## 4.3 Master Integrals for One and Two-Loop Form Factors

The master integrals relevant for the one- and two-loop form factors are given in this section. All the integrals are expanded up to the desired order in  $\epsilon$ . In the figures, the dark lines represent the incoming and outgoing momenta of the loop integral. The incoming momentum is  $p_{12} = p_1 + p_2$  while the outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ . The actual QGRAF output is different than these diagrams e.g two external lines come out instead of one at the outer vertex.

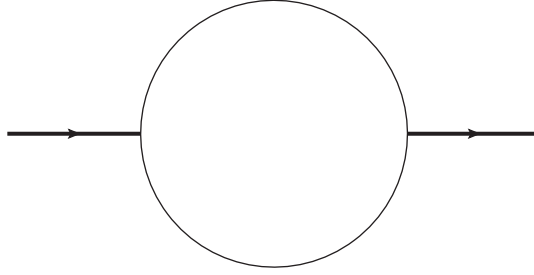


Figure 4.4: B21: One-loop master integral with two massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B21 &= \int d\mathcal{K}_1 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2} & (4.8) \\
 &= + \frac{1}{\epsilon} + 2 + 4\epsilon - \epsilon^2 (+2\zeta_3 - 8) \\
 &\quad - \epsilon^3 \left( +\frac{6\zeta_2^2}{5} + 4\zeta_3 - 16 \right) \\
 &\quad - \epsilon^4 \left( +\frac{12\zeta_2^2}{5} + 8\zeta_3 + 6\zeta_5 - 32 \right) \\
 &\quad - \epsilon^5 \left( +\frac{16\zeta_2^3}{7} + \frac{24\zeta_2^2}{5} - 2\zeta_3^2 + 16\zeta_3 + 12\zeta_5 64 \right) \\
 &\quad + \epsilon^6 b21
 \end{aligned}$$

where the integral measure is represented as :

$$d\mathcal{K}_i = \int \frac{d^D k_i}{(2\pi)^D}, \quad i = 1, 2, 3$$

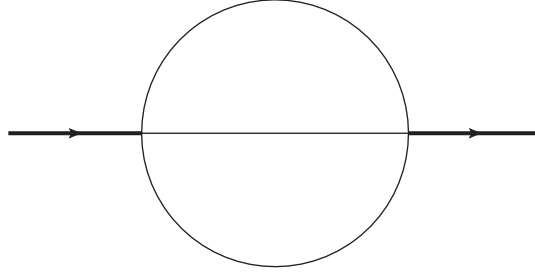


Figure 4.5: B31: Two-loop master integral with three massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B31 &= \int d\mathcal{K}_1 d\mathcal{K}_2 \frac{1}{(k_1 + p_1 + p_2)^2 (k_1 - k_2)^2 k_2^2} & (4.9) \\
 &= -\frac{1}{4\epsilon} \\
 &\quad -\frac{13}{8} \\
 &\quad -\frac{115\epsilon}{16} \\
 &\quad + \epsilon^2 \left( +\frac{5\zeta_3}{2} - \frac{865}{32} \right) \\
 &\quad + \epsilon^3 \left( +\frac{3\zeta_2^2}{2} + \frac{65\zeta_3}{4} - \frac{5971}{64} \right) \\
 &\quad + \epsilon^4 \left( +\frac{39\zeta_2^2}{4} + \frac{575\zeta_3}{8} + \frac{27\zeta_5}{2} - \frac{39193}{128} \right) \\
 &\quad + \epsilon^5 \left( +\frac{44\zeta_2^3}{7} + \frac{345\zeta_2^2}{8} - \frac{25\zeta_3^2}{2} + \frac{4325\zeta_3}{16} + \frac{351\zeta_5}{4} - \frac{249355}{256} \right) \\
 &\quad - \frac{\epsilon^6 b31}{4}
 \end{aligned}$$

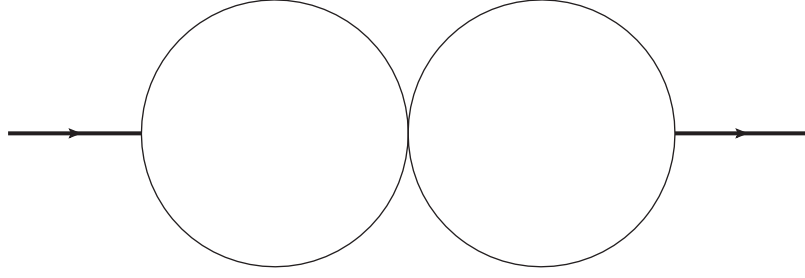


Figure 4.6: B42: Two-loop master integral with four massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B_{42} &= \int d\mathcal{K}_1 d\mathcal{K}_2 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_2 + p_1 + p_2)^2} & (4.10) \\
 &= + \frac{1}{\epsilon^2} \\
 &\quad + \frac{4}{\epsilon} \\
 &\quad + 12 \\
 &\quad - \epsilon (+4\zeta_3 - 32) \\
 &\quad - \epsilon^2 \left( +\frac{12\zeta_2^2}{5} + 16\zeta_3 - 80 \right) \\
 &\quad - \epsilon^3 \left( +\frac{48\zeta_2^2}{5} + 48\zeta_3 + 12\zeta_5 - 192 \right) \\
 &\quad - \epsilon^4 \left( +\frac{32\zeta_2^3}{7} + \frac{144\zeta_2^2}{5} - 8\zeta_3^2 + 128\zeta_3 + 48\zeta_5 - 448 \right) \\
 &\quad + \epsilon^5 b_{42}
 \end{aligned}$$

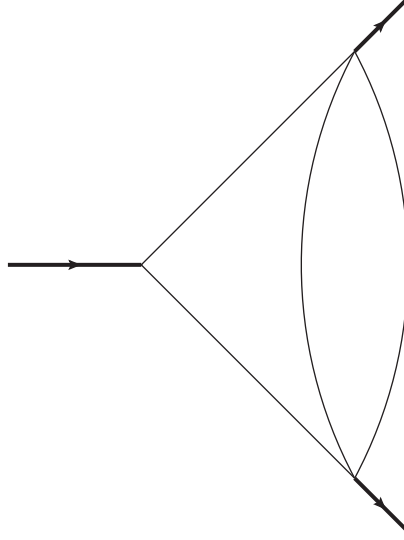


Figure 4.7: C41: Two-loop master integral with four massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
C41 &= \int d\mathcal{K}_1 d\mathcal{K}_2 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_2 + p_1)^2} & (4.11) \\
&= + \frac{1}{2\epsilon^2} \\
&\quad + \frac{5}{2\epsilon} \\
&\quad + \left( +\zeta_2 + \frac{19}{2} \right) \\
&\quad + \epsilon \left( +5\zeta_2 - 4\zeta_3 + \frac{65}{2} \right) \\
&\quad - \epsilon^2 \left( +\frac{6\zeta_2^2}{5} - 19\zeta_2 + 20\zeta_3 - \frac{211}{2} \right) \\
&\quad - \epsilon^3 \left( +6\zeta_2^2 + 8\zeta_2\zeta_3 - 65\zeta_2 + 76\zeta_3 + 24\zeta_5 - \frac{665}{2} \right) \\
&\quad - \epsilon^4 \left( +\frac{528\zeta_2^3}{35} + \frac{114\zeta_2^2}{5} + 40\zeta_2\zeta_3 - 16\zeta_3^2 - 211\zeta_2 + 260\zeta_3 + 120\zeta_5 - \frac{2059}{2} \right) \\
&\quad + \frac{\epsilon^5 c41}{2}
\end{aligned}$$

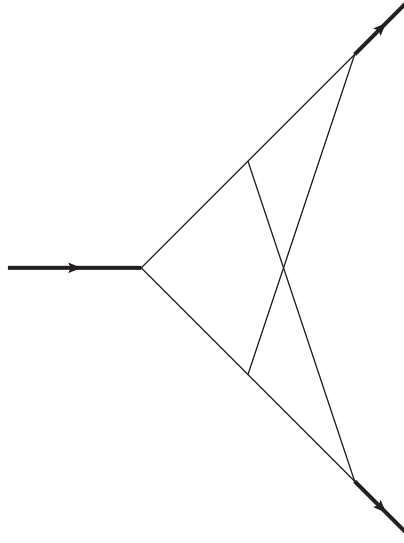


Figure 4.8: C62: Two-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
C62 = & \int d\mathcal{K}_1 d\mathcal{K}_2 \frac{1}{(k_1 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_1 - k_2 + p_2)^2 (k_2 + p_1)^2 k_1^2 k_2^2} \quad (4.12) \\
& + \frac{1}{\epsilon^4} \\
& - \frac{5\zeta_2}{\epsilon^2} \\
& - \frac{27\zeta_3}{\epsilon} \\
& - 23\zeta_2^2 \\
& + \epsilon (+48\zeta_2\zeta_3 - 117\zeta_5) \\
& - \epsilon^2 \left( +\frac{456\zeta_2^3}{35} - 267\zeta_3^2 \right) \\
& + \epsilon^3 c62
\end{aligned}$$

## 4.4 Notation for the Form Factor

We present our results in terms of the expansion coefficients of the bare (unrenormalised) form factor,

$$\mathcal{F}_b^a(\alpha_s^b, s_{12}) = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_s^b}{4\pi} \right)^n \left( \frac{-s_{12}}{\mu^2} \right)^{-n\epsilon} \mathcal{F}_n^a, \quad (4.13)$$

where  $a = q, g$ .

## 4.5 Results at one-loop

Written in terms of the one-loop bubble integral, the unrenormalised one-loop form factors are given by

$$\begin{aligned} \mathcal{F}_1^q &= C_F B_{21} \frac{D^2 - 7D + 16}{(D - 4)} \\ \mathcal{F}_1^g &= -C_A B_{21} \frac{D^3 - 16D^2 + 68D - 88}{(D - 4)(D - 2)} \end{aligned}$$

which agree with eqs. (8) and (9) of ref. [56] respectively.

Inserting the expansion of the one-loop master integrals and keeping terms through to  $\mathcal{O}(\epsilon^4)$ , we find that

$$\begin{aligned} \mathcal{F}_1^q &= C_F \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + (+\zeta_2 - 8) + \epsilon \left( +\frac{3\zeta_2}{2} + \frac{14\zeta_3}{3} - 16 \right) + \epsilon^2 \left( +\frac{47\zeta_2^2}{20} + 4\zeta_2 + 7\zeta_3 - 32 \right) \right. \\ &\quad \left. + \epsilon^3 \left( +\frac{141\zeta_2^2}{40} - \frac{7\zeta_2\zeta_3}{3} + 8\zeta_2 + \frac{56\zeta_3}{3} + \frac{62\zeta_5}{5} - 64 \right) \right. \\ &\quad \left. + \epsilon^4 \left( +\frac{949\zeta_2^3}{280} + \frac{47\zeta_2^2}{5} - \frac{7\zeta_2\zeta_3}{2} - \frac{49\zeta_3^2}{9} + 16\zeta_2 + \frac{112\zeta_3}{3} + \frac{93\zeta_5}{5} - 128 \right) \right] \\ \mathcal{F}_1^g &= C_A \left[ -\frac{2}{\epsilon^2} + \zeta_2 + \epsilon \left( +\frac{14\zeta_3}{3} - 2 \right) + \epsilon^2 \left( +\frac{47\zeta_2^2}{20} - 6 \right) \right. \\ &\quad \left. + \epsilon^3 \left( +\frac{7\zeta_2\zeta_3}{3} - \zeta_2 - \frac{62\zeta_5}{5} + 14 \right) + \epsilon^4 \left( +\frac{949\zeta_2^3}{280} - \frac{49\zeta_3^2}{9} + 3\zeta_2 + \frac{14\zeta_3}{3} - 30 \right) \right] \end{aligned}$$

which agrees with eq. (7) of ref. [52].

## 4.6 Results at two-loops

Written in terms of the two-loop master integrals, the unrenormalised two-loop gluon form factor is given by

$$\begin{aligned}
\mathcal{F}_2^g = & C_F^2 \left[ \begin{aligned} & +B42 \frac{(D^2 - 7D + 16)^2}{(D - 4)^2} \\ & -C41 \left( +\frac{7D^2}{8} - \frac{983D}{48} - \frac{565}{32(2D - 7)} - \frac{20}{9(3D - 8)} - \frac{28}{(D - 4)} \right. \\ & \quad \left. - \frac{40}{(D - 4)^2} + \frac{10693}{288} \right) \\ & +\frac{B31}{s12} \left( +\frac{27D^2}{8} - \frac{1293D}{16} + \frac{3955}{32(2D - 7)} - \frac{17}{2(D - 3)} - \frac{476}{(D - 4)} \right. \\ & \quad \left. - \frac{456}{(D - 4)^2} - \frac{288}{(D - 4)^3} + \frac{581}{32} \right) \\ & \left. -s12^2 C62 \frac{D^3 - 20D^2 + 104D - 176}{8(2D - 7)} \right] \\ & +C_F C_A \left[ \begin{aligned} & -C41 \left( +\frac{D^2}{16} + \frac{77D}{32} + \frac{565}{64(2D - 7)} + \frac{12}{5(3D - 8)} + \frac{23}{15(D - 1)} \right. \\ & \quad \left. + \frac{8}{3(D - 4)} + \frac{16}{(D - 4)^2} + \frac{163}{64} \right) \\ & -\frac{B31}{s12} \left( +\frac{75D^2}{16} - \frac{1837D}{32} + \frac{3955}{64(2D - 7)} + \frac{3}{4(D - 3)} - \frac{186}{(D - 4)} \right. \\ & \quad \left. - \frac{144}{(D - 4)^2} - \frac{96}{(D - 4)^3} + \frac{3845}{64} \right) \\ & \left. +s12^2 C62 \frac{D^3 - 20D^2 + 104D - 176}{16(2D - 7)} \right] \\ & +C_F N_F \left[ \begin{aligned} & -C41 \frac{(D - 2)(3D^3 - 31D^2 + 110D - 128)}{(3D - 8)(D - 4)(D - 1)} \end{aligned} \right] \\ \\
\mathcal{F}_2^g = & C_A^2 \left[ \begin{aligned} & +B42 \left( +D^2 - 20D - \frac{48}{(D - 2)} + \frac{32}{(D - 4)} + \frac{16}{(D - 2)^2} \right. \\ & \quad \left. + \frac{16}{(D - 4)^2} + 100 \right) \\ & +C41 \left( +\frac{27D}{2} + \frac{119}{48(2D - 5)} + \frac{75}{16(2D - 7)} + \frac{10}{3(D - 1)} + \frac{80}{(D - 2)} \right. \\ & \quad \left. + \frac{103}{3(D - 4)} - \frac{32}{(D - 2)^2} + \frac{24}{(D - 4)^2} - \frac{609}{8} \right) \end{aligned} \right]
\end{aligned}$$



$$\begin{aligned}
& + \frac{B_{31}}{s_{12}} \left( +24D + \frac{107}{144(2D-5)} + \frac{525}{16(2D-7)} + \frac{116}{9(D-1)} + \frac{96}{(D-2)} \right. \\
& \quad - \frac{2}{(D-3)} - \frac{1175}{3(D-4)} - \frac{32}{(D-2)^2} - \frac{1388}{3(D-4)^2} - \frac{192}{(D-4)^3} \\
& \quad \left. - \frac{1955}{8} \right) \\
& + s_{12}^2 C_{62} \frac{3(3D-8)(D-3)}{4(2D-5)(2D-7)} \Big] \\
+ C_A N_F \Big[ & + C_{41} \left( +\frac{7D}{8} + \frac{119}{12(2D-5)} + \frac{35}{48(2D-7)} + \frac{20}{3(D-1)} - \frac{40}{3(D-2)} \right. \\
& \quad \left. - \frac{2}{(D-4)} - \frac{45}{16} \right) \\
& - \frac{B_{31}}{s_{12}} \left( +\frac{19D}{8} - \frac{107}{36(2D-5)} - \frac{245}{48(2D-7)} - \frac{232}{9(D-1)} + \frac{40}{3(D-2)} \right. \\
& \quad \left. - \frac{3}{2(D-3)} + \frac{8}{9(D-4)} - \frac{8}{(D-4)^2} - \frac{61}{16} \right) \\
& + s_{12}^2 C_{62} \frac{(2D^3 - 25D^2 + 94D - 112)(D-4)}{8(D-2)(2D-5)(2D-7)} \Big] \\
+ C_F N_F \Big[ & - C_{41} \frac{(46D^4 - 545D^3 + 2395D^2 - 4606D + 3248)(D-6)}{2(2D-7)(2D-5)(D-4)(D-2)} \\
& + \frac{B_{31}}{s_{12}} \left( +\frac{35D}{4} - \frac{107}{18(2D-5)} - \frac{245}{24(2D-7)} + \frac{8}{3(D-2)} - \frac{1}{(D-3)} \right. \\
& \quad \left. - \frac{448}{9(D-4)} - \frac{112}{3(D-4)^2} - \frac{333}{8} \right) \\
& - s_{12}^2 C_{62} \frac{(2D^3 - 25D^2 + 94D - 112)(D-4)}{4(D-2)(2D-5)(2D-7)} \Big]
\end{aligned}$$

which, after re-expressing in terms of  $N$  and  $N_F$  agrees with eqs. (10) and (11) of ref. [56].

Inserting the expansion of the two-loop master integrals and keeping terms through to  $\mathcal{O}(\epsilon^2)$ , we find that

$$\begin{aligned}
\mathcal{F}_2^q = C_F^2 \Big[ & + \frac{2}{\epsilon^4} + \frac{6}{\epsilon^3} - \frac{1}{\epsilon^2} \left( +2\zeta_2 - \frac{41}{2} \right) - \frac{1}{\epsilon} \left( +\frac{64\zeta_3}{3} - \frac{221}{4} \right) \\
& - \left( +13\zeta_2^2 - \frac{17\zeta_2}{2} + 58\zeta_3 - \frac{1151}{8} \right) \\
& - \epsilon \left( +\frac{171\zeta_2^2}{5} - \frac{112\zeta_2\zeta_3}{3} - \frac{213\zeta_2}{4} + \frac{839\zeta_3}{3} + \frac{184\zeta_5}{5} - \frac{5741}{16} \right) \\
& + \epsilon^2 \left( +\frac{223\zeta_2^3}{5} - \frac{3401\zeta_2^2}{20} + 54\zeta_2\zeta_3 + \frac{2608\zeta_3^2}{9} + \frac{1839\zeta_2}{8} \right)
\end{aligned}$$

$$\begin{aligned}
& \left. -\frac{6989\zeta_3}{6} - \frac{462\zeta_5}{5} + \frac{27911}{32} \right) \Bigg] \\
& + C_F C_A \left[ -\frac{11}{6\epsilon^3} + \frac{1}{\epsilon^2} \left( +\zeta_2 - \frac{83}{9} \right) - \frac{1}{\epsilon} \left( +\frac{11\zeta_2}{6} - 13\zeta_3 + \frac{4129}{108} \right) \right. \\
& \quad + \left( +\frac{44\zeta_2^2}{5} - \frac{119\zeta_2}{9} + \frac{467\zeta_3}{9} - \frac{89173}{648} \right) \\
& \quad + \epsilon \left( +\frac{1891\zeta_2^2}{60} - \frac{89\zeta_2\zeta_3}{3} - \frac{6505\zeta_2}{108} + \frac{6586\zeta_3}{27} + 51\zeta_5 - \frac{1775893}{3888} \right) \\
& \quad - \epsilon^2 \left( +\frac{809\zeta_2^3}{70} - \frac{2639\zeta_2^2}{18} + \frac{397\zeta_2\zeta_3}{9} + \frac{569\zeta_3^2}{3} + \frac{146197\zeta_2}{648} \right. \\
& \quad \quad \left. - \frac{159949\zeta_3}{162} - \frac{3491\zeta_5}{15} + \frac{33912061}{23328} \right) \Bigg] \\
& + C_F N_F \left[ +\frac{1}{3\epsilon^3} + \frac{14}{9\epsilon^2} + \frac{1}{\epsilon} \left( +\frac{\zeta_2}{3} + \frac{353}{54} \right) + \left( +\frac{14\zeta_2}{9} - \frac{26\zeta_3}{9} + \frac{7541}{324} \right) \right. \\
& \quad - \epsilon \left( +\frac{41\zeta_2^2}{30} - \frac{353\zeta_2}{54} + \frac{364\zeta_3}{27} - \frac{150125}{1944} \right) \\
& \quad \left. - \epsilon^2 \left( +\frac{287\zeta_2^2}{45} + \frac{26\zeta_2\zeta_3}{9} - \frac{7541\zeta_2}{324} + \frac{4589\zeta_3}{81} + \frac{242\zeta_5}{15} - \frac{2877653}{11664} \right) \right]
\end{aligned}$$

which agrees with eq. (3.6) of ref. [53].

Similarly we find that the two-loop expansion of the gluon form factor is given by

$$\begin{aligned}
\mathcal{F}_2^g = C_A^2 \left[ +\frac{2}{\epsilon^4} - \frac{11}{6\epsilon^3} - \frac{1}{\epsilon^2} \left( +\zeta_2 + \frac{67}{18} \right) + \frac{1}{\epsilon} \left( +\frac{11\zeta_2}{2} - \frac{25\zeta_3}{3} + \frac{68}{27} \right) \right. \\
\quad - \left( +\frac{21\zeta_2^2}{5} - \frac{67\zeta_2}{6} - \frac{11\zeta_3}{9} - \frac{5861}{162} \right) \\
\quad - \epsilon \left( +\frac{77\zeta_2^2}{60} - \frac{23\zeta_2\zeta_3}{3} - \frac{106\zeta_2}{9} + \frac{1139\zeta_3}{27} - \frac{71\zeta_5}{5} - \frac{158201}{972} \right) \\
\quad + \epsilon^2 \left( +\frac{2313\zeta_2^3}{70} - \frac{1943\zeta_2^2}{60} - \frac{55\zeta_2\zeta_3}{3} + \frac{901\zeta_3^2}{9} + \frac{481\zeta_2}{54} \right. \\
\quad \quad \left. - \frac{26218\zeta_3}{81} + \frac{341\zeta_5}{15} + \frac{3484193}{5832} \right) \Bigg] \\
+ C_A N_F \left[ +\frac{1}{3\epsilon^3} + \frac{5}{9\epsilon^2} - \frac{1}{\epsilon} \left( +\zeta_2 + \frac{26}{27} \right) - \left( +\frac{5\zeta_2}{3} + \frac{74\zeta_3}{9} + \frac{808}{81} \right) \right. \\
\quad - \epsilon \left( +\frac{51\zeta_2^2}{10} + \frac{16\zeta_2}{9} + \frac{604\zeta_3}{27} + \frac{23131}{486} \right) \\
\quad \left. - \epsilon^2 \left( +\frac{257\zeta_2^2}{18} - \frac{50\zeta_2\zeta_3}{3} - \frac{28\zeta_2}{27} + \frac{3962\zeta_3}{81} + \frac{542\zeta_5}{15} + \frac{540805}{2916} \right) \right]
\end{aligned}$$

$$+C_{FN_F} \left[ -\frac{1}{\epsilon} + \left( +8\zeta_3 - \frac{67}{6} \right) + \epsilon \left( +\frac{16\zeta_2^2}{3} + \frac{7\zeta_2}{3} + \frac{92\zeta_3}{3} - \frac{2027}{36} \right) + \epsilon^2 \left( +\frac{184\zeta_2^2}{9} - \frac{40\zeta_2\zeta_3}{3} + \frac{209\zeta_2}{18} + \frac{1124\zeta_3}{9} + 32\zeta_5 - \frac{47491}{216} \right) \right]$$

which agrees with eq. (8) of ref. [52].

# Chapter 5

## Quark and Gluon Form Factors At Three Loops

In this chapter we extend our study of quark and gluon form factors to three loops. As in the one and two loop cases, the amplitudes are reduced to a small set of master integrals by means of algebraic reduction techniques. Since the number of the diagrams for three loops are significantly more than two loops, one expects and finds more master integrals.

We therefore need to find a compact and general representation for all three-loop vertex integrals. To this end we have created (fictitious) general planar and non-planar auxiliary diagrams. In these diagrams, each of the propagators is labelled by an integer and carries a specific momentum conserving the momenta throughout. The auxiliary diagrams help us to represent our Feynman diagrams in the real process. All possible distributions of momenta in Feynman diagrams with different sets of propagators, can be obtained by pinching some of the propagators from these diagrams.

## 5.1 Auxiliary Planar Diagram

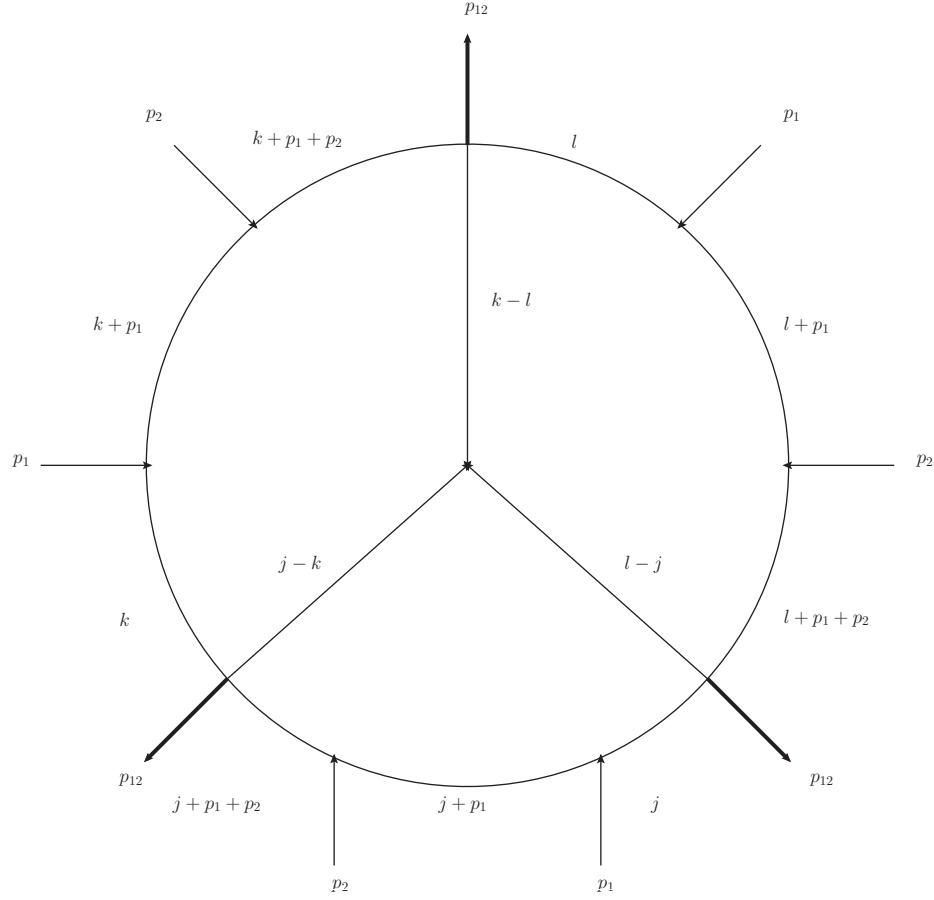


Figure 5.1: The twelve-propagator general planar diagram A with  $p_1$  and  $p_2$  incoming and  $p_{12}$  outgoing

The most general planar integral that we can consider is:

$$I^D(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8, \nu_9, \nu_{10}, \nu_{11}, \nu_{12}, s_{12}) = \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7} A_8^{\nu_8} A_9^{\nu_9} A_{10}^{\nu_{10}} A_{11}^{\nu_{11}} A_{12}^{\nu_{12}}} \quad (5.1)$$

$$\begin{aligned} A_1 &= k_1^2 & A_4 &= k_2^2 & A_7 &= k_3^2 & A_{10} &= (k_1 - k_2)^2 \\ A_2 &= (k_1 + p_1)^2 & A_5 &= (k_2 + p_1)^2 & A_8 &= (k_3 + p_1)^2 & A_{11} &= (k_2 - k_3)^2 \\ A_3 &= (k_1 + p_1 + p_2)^2 & A_6 &= (k_2 + p_1 + p_2)^2 & A_9 &= (k_3 + p_1 + p_2)^2 & A_{12} &= (k_3 - k_1)^2 \end{aligned}$$

$(k_1, k_2, k_3)$  corresponds to  $(j, k, l)$  in the auxiliary diagram and  $d\mathcal{K}_\dots$  is represented

as before:

$$d\mathcal{K}_i = \frac{d^D k_i}{(2\pi)^D} \quad i = 1, 2, 3$$

The integral is symmetric under any of the exchanges,

$$\begin{aligned} k_1 &\leftrightarrow k_2, \\ k_2 &\leftrightarrow k_3, \\ k_3 &\leftrightarrow k_1, \end{aligned} \tag{5.2}$$

or equivalently,

$$\begin{aligned} \{\nu_1, \nu_2, \nu_3, \nu_{10}, \nu_{12}\} &\leftrightarrow \{\nu_4, \nu_5, \nu_6, \nu_{10}, \nu_{11}\}, \\ \{\nu_4, \nu_5, \nu_6, \nu_{10}, \nu_{11}\} &\leftrightarrow \{\nu_7, \nu_8, \nu_9, \nu_{12}, \nu_{11}\}, \\ \{\nu_7, \nu_8, \nu_9, \nu_{11}, \nu_{12}\} &\leftrightarrow \{\nu_1, \nu_2, \nu_3, \nu_{10}, \nu_{12}\}. \end{aligned} \tag{5.3}$$

There is a further symmetry when  $\nu_i \geq 0 \forall i$ ,

$$k_1 \leftrightarrow -k_1 - p_1 - p_2, \quad k_2 \leftrightarrow -k_2 - p_1 - p_2, \quad k_3 \leftrightarrow -k_3 - p_1 - p_2 \tag{5.4}$$

or equivalently,

$$\{\nu_1, \nu_3, \nu_4, \nu_6, \nu_7, \nu_9\} \leftrightarrow \{\nu_3, \nu_1, \nu_6, \nu_4, \nu_9, \nu_7\}. \tag{5.5}$$

Any planar diagram of the QGRAF output can be mapped into our general planar auxiliary diagram by suitable momentum shiftings. As an example the propagators of the 98th diagram of the QGRAF output are:

$(-k_1, k_1 + p_1, -k_1 + p_2, -k_1 + k_3 + p_2, -k_3, -k_2)$ . This is a seven propagator planar diagram and in our notation it is represented as:

$$(-k_1, k_1 + p_1, -k_1 + p_2, -k_1 + k_3 + p_2, -k_3, -k_2) \rightarrow \text{int1}(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, s_{12})$$

By pinching five propagators from our auxiliary diagram we get the desired diagram. As mentioned, by doing lots of momentum shiftings for each diagram of QGRAF output we are able to obtain all planar diagrams.

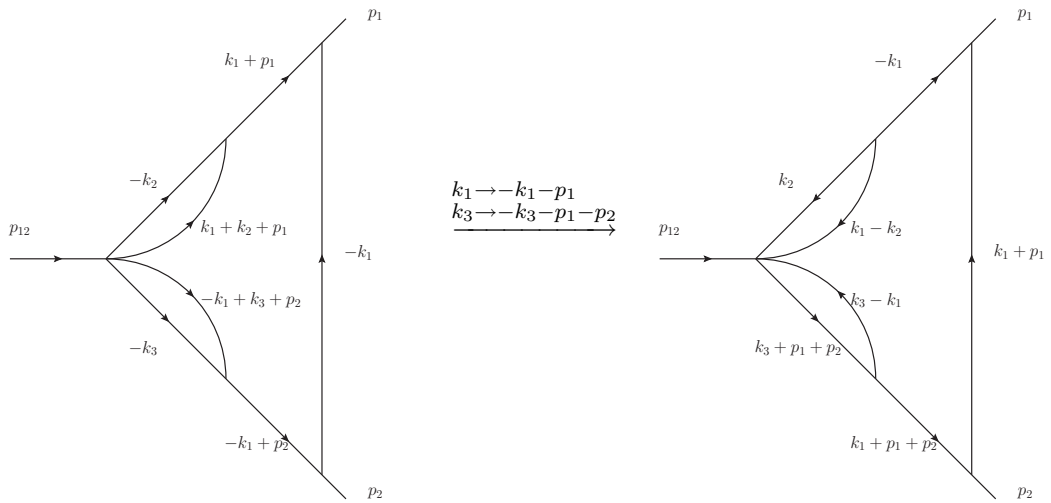


Figure 5.2: Mapping of the seven propagator planar diagram

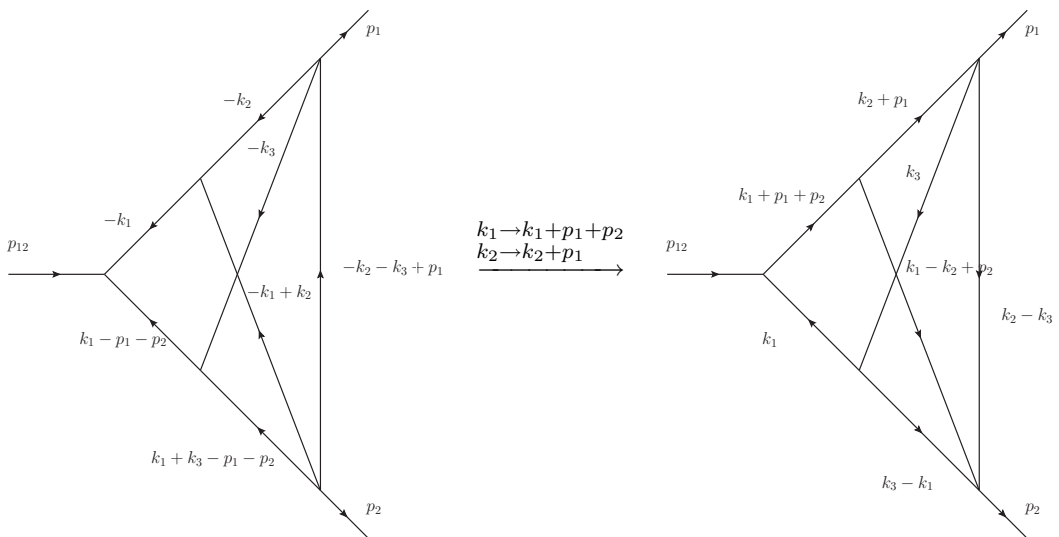


Figure 5.3: Mapping of the seven propagator non-planar diagram

## 5.2 Auxiliary Non-Planar Diagrams

However not all the diagrams are planar. Some non-planar diagrams do not fit into our propagator set. We need two more auxiliary topologies in order to cast all non-planar diagrams in a compact notation similar to eq. (5.1).

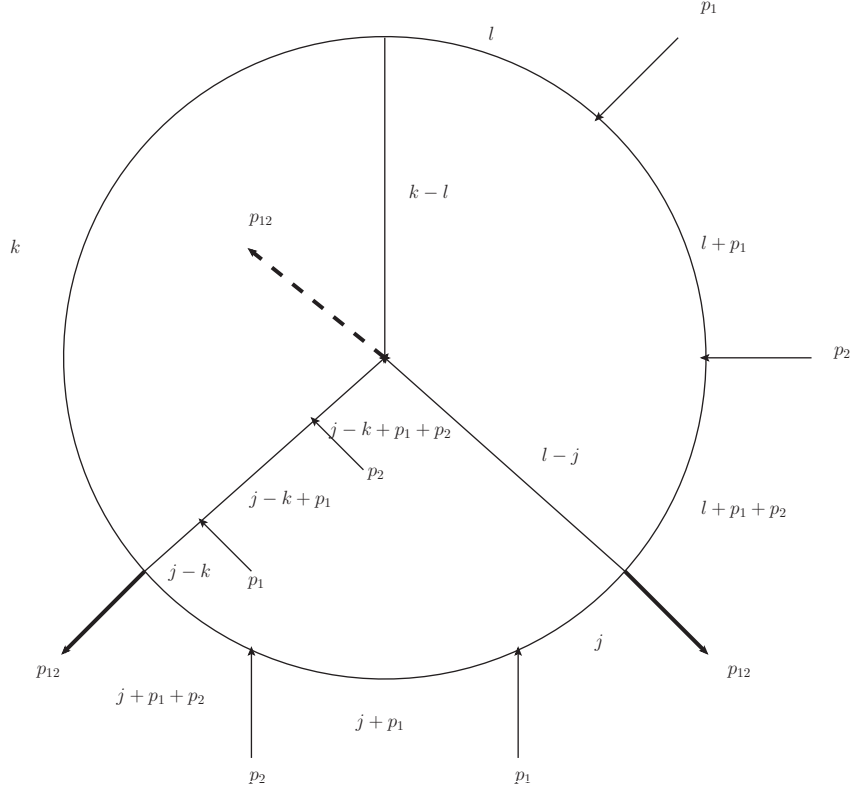


Figure 5.4: The twelve propagator general non-planar diagram B with  $p_1$  and  $p_2$  incoming and  $p_{12}$  outgoing

The most general non-planar integral B that we can consider is:

$$I^D(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8, \nu_9, \nu_{10}, \nu_{11}, \nu_{12}, s_{12}) = \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7} A_8^{\nu_8} A_9^{\nu_9} A_{10}^{\nu_{10}} A_{11}^{\nu_{11}} A_{12}^{\nu_{12}}} \quad (5.6)$$

$$\begin{aligned} A_1 &= k_1^2 & A_4 &= k_2^2 & A_7 &= (k_3 + p_1 + p_2)^2 & A_{10} &= (k_3 - k_1)^2 \\ A_2 &= (k_1 + p_1)^2 & A_5 &= k_3^2 & A_8 &= (k_1 - k_2)^2 & A_{11} &= (k_1 - k_2 + p_1)^2 \\ A_3 &= (k_1 + p_1 + p_2)^2 & A_6 &= (k_3 + p_1)^2 & A_9 &= (k_2 - k_3)^2 & A_{12} &= (k_1 - k_2 + p_1 + p_2)^2 \end{aligned}$$



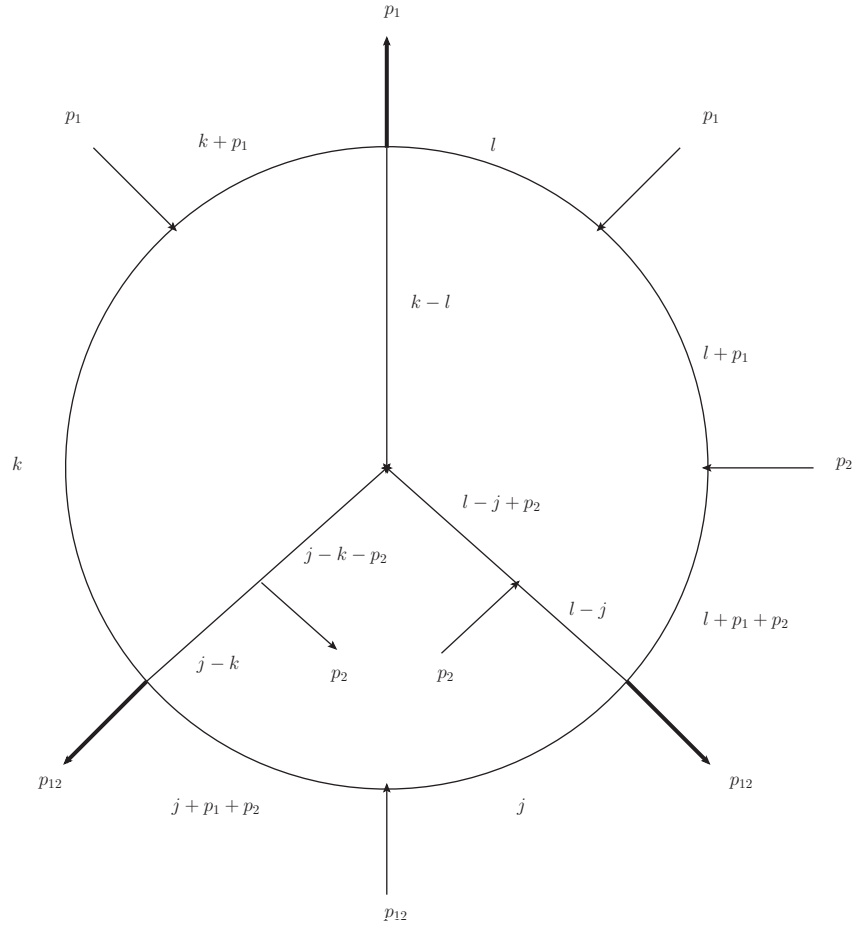


Figure 5.5: The twelve propagator general non-planar diagram C with  $p_1$  and  $p_2$  incoming and  $p_{12}$  outgoing

The most general non-planar integral C that we can consider is:

$$I^D(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8, \nu_9, \nu_{10}, \nu_{11}, \nu_{12}, s_{12}) = \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{A_1^{\nu_1} A_2^{\nu_2} A_3^{\nu_3} A_4^{\nu_4} A_5^{\nu_5} A_6^{\nu_6} A_7^{\nu_7} A_8^{\nu_8} A_9^{\nu_9} A_{10}^{\nu_{10}} A_{11}^{\nu_{11}} A_{12}^{\nu_{12}}} \quad (5.7)$$

$$\begin{aligned} A_1 &= k_1^2 & A_4 &= (k_2 + p_1)^2 & A_7 &= (k_3 + p_1 + p_2)^2 & A_{10} &= (k_3 - k_1)^2 \\ A_2 &= (k_1 + p_1 + p_2)^2 & A_5 &= k_3^2 & A_8 &= (k_1 - k_2)^2 & A_{11} &= (k_1 - k_2 + p_2)^2 \\ A_3 &= k_2^2 & A_6 &= (k_3 + p_1)^2 & A_9 &= (k_2 - k_3)^2 & A_{12} &= (k_3 - k_1 - p_2)^2 \end{aligned}$$

## 5.3 IBP Identities for Auxiliary Integrals

We know from chapter three, that the number of IBP equations depends on the number of loops and external independent momenta. For our auxiliary integrals we have  $m \times (m + n - 1) = 15$  IBP identities. Applying the IBP method to the planar and non-planar auxiliary diagrams we get the following fifteen identities for each of them. In order to keep the expressions compact we preferred arranging them as operators acting on the function. In each IBP identities  $\mathcal{I}$  belongs to its propagator set. As mentioned before not all of these identities are used in the actual calculation if one sets the option of Lee's ideas to true. That option reduces the number of IBP identities and saves time in the calculations. Unlike AIR, FIRE evaluates the identities in an other module (IBP.m) and prepares the sectors for the main algorithm. The IBP identities for the non-planar auxiliary diagrams have more terms than the planar one because of the non-planar structure of the propagator configuration. In addition of having no symmetry of the non-planar diagrams, the calculation time needed for the evaluation is quite longer than the planar one.

## 5.3.1 IBP Identities for the Planar Auxiliary Diagram A

$$\begin{aligned}
& \left( D - 2\nu_1 - \nu_2 - \nu_3 - \nu_{10} - \nu_{12} + s_{12}\nu_3\mathbf{3}^+ - \nu_2\mathbf{2}^+\mathbf{1}^- \right. \\
& \quad \left. - \nu_3\mathbf{3}^+\mathbf{1}^- - \nu_{10}\mathbf{10}^+\mathbf{1}^- - \nu_{12}\mathbf{12}^+\mathbf{1}^- + \nu_{10}\mathbf{10}^+\mathbf{4}^- + \nu_{12}\mathbf{12}^+\mathbf{7}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_1 - 2\nu_2 - \nu_3 - \nu_{10} - \nu_{12} - \nu_1\mathbf{1}^+\mathbf{2}^- - \nu_3\mathbf{3}^+\mathbf{2}^- \right. \\
& \quad \left. - \nu_{10}\mathbf{10}^+\mathbf{2}^- - \nu_{12}\mathbf{12}^+\mathbf{2}^- + \nu_{10}\mathbf{10}^+\mathbf{5}^- + \nu_{12}\mathbf{12}^+\mathbf{8}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_1 - \nu_2 - 2\nu_3 - \nu_{10} - \nu_{12} + s_{12}\nu_1\mathbf{1}^+ - \nu_1\mathbf{1}^+\mathbf{3}^- \right. \\
& \quad \left. - \nu_2\mathbf{2}^+\mathbf{3}^- - \nu_{10}\mathbf{10}^+\mathbf{3}^- - \nu_{12}\mathbf{12}^+\mathbf{3}^- + \nu_{10}\mathbf{10}^+\mathbf{6}^- + \nu_{12}\mathbf{12}^+\mathbf{9}^- \right) \mathcal{I} = 0 \\
& \left( D - 2\nu_4 - \nu_5 - \nu_6 - \nu_{10} - \nu_{11} + s_{12}\nu_6\mathbf{6}^+ + \nu_{10}\mathbf{10}^+\mathbf{1}^- \right. \\
& \quad \left. - \nu_5\mathbf{5}^+\mathbf{4}^- - \nu_6\mathbf{6}^+\mathbf{4}^- - \nu_{10}\mathbf{10}^+\mathbf{4}^- - \nu_{11}\mathbf{11}^+\mathbf{4}^- + \nu_{11}\mathbf{11}^+\mathbf{7}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_4 - 2\nu_5 - \nu_6 - \nu_{10} - \nu_{11} + \nu_{10}\mathbf{10}^+\mathbf{2}^- - \nu_4\mathbf{4}^+\mathbf{5}^- \right. \\
& \quad \left. - \nu_6\mathbf{6}^+\mathbf{5}^- - \nu_{10}\mathbf{10}^+\mathbf{5}^- - \nu_{11}\mathbf{11}^+\mathbf{5}^- + \nu_{11}\mathbf{11}^+\mathbf{8}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_4 - \nu_5 - 2\nu_6 - \nu_{10} - \nu_{11} + s_{12}\nu_4\mathbf{4}^+ + \nu_{10}\mathbf{10}^+\mathbf{3}^- \right. \\
& \quad \left. - \nu_4\mathbf{4}^+\mathbf{5}^- - \nu_5\mathbf{5}^+\mathbf{6}^- - \nu_{10}\mathbf{10}^+\mathbf{6}^- - \nu_{11}\mathbf{11}^+\mathbf{6}^- + \nu_{11}\mathbf{11}^+\mathbf{9}^- \right) \mathcal{I} = 0 \\
& \left( D - 2\nu_7 - \nu_8 - \nu_9 - \nu_{11} - \nu_{12} + s_{12}\nu_9\mathbf{9}^+ + \nu_{12}\mathbf{12}^+\mathbf{1}^- \right. \\
& \quad \left. + \nu_{11}\mathbf{11}^+\mathbf{4}^- - \nu_8\mathbf{8}^+\mathbf{7}^- - \nu_9\mathbf{9}^+\mathbf{7}^- - \nu_{11}\mathbf{11}^+\mathbf{7}^- - \nu_{12}\mathbf{12}^+\mathbf{7}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_7 - 2\nu_8 - \nu_9 - \nu_{11} - \nu_{12} + \nu_{12}\mathbf{12}^+\mathbf{2}^- + \nu_{11}\mathbf{11}^+\mathbf{5}^- \right. \\
& \quad \left. - \nu_7\mathbf{7}^+\mathbf{8}^- - \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_{11}\mathbf{11}^+\mathbf{8}^- - \nu_{12}\mathbf{12}^+\mathbf{8}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_7 - \nu_8 - 2\nu_9 - \nu_{11} - \nu_{12} + s_{12}\nu_7\mathbf{7}^+ + \nu_{11}\mathbf{11}^+\mathbf{6}^- \right. \\
& \quad \left. - \nu_7\mathbf{7}^+\mathbf{9}^- - \nu_8\mathbf{8}^+\mathbf{9}^- - \nu_{11}\mathbf{11}^+\mathbf{9}^- - \nu_{12}\mathbf{12}^+\mathbf{9}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_1 - \nu_2 - \nu_3 - 2\nu_{10} - \nu_{12} + \nu_1\mathbf{1}^+\mathbf{4}^- + \nu_2\mathbf{2}^+\mathbf{5}^- + \nu_3\mathbf{3}^+\mathbf{6}^- \right. \\
& \quad \left. - \nu_1\mathbf{1}^+\mathbf{10}^- - \nu_2\mathbf{2}^+\mathbf{10}^- - \nu_3\mathbf{3}^+\mathbf{10}^- - \nu_{12}\mathbf{12}^+\mathbf{10}^- - \nu_{12}\mathbf{12}^+\mathbf{11}^- \right) \mathcal{I} = 0
\end{aligned}$$

$$\begin{aligned}
& \left( -D + \nu_1 + \nu_2 + \nu_3 + \nu_{10} + 2\nu_{12} - \nu_1 \mathbf{1}^+ \mathbf{7}^- - \nu_2 \mathbf{2}^+ \mathbf{8}^- \right. \\
& \quad \left. - \nu_3 \mathbf{3}^+ \mathbf{9}^- - \nu_{10} \mathbf{10}^+ \mathbf{11}^- + \nu_1 \mathbf{1}^+ \mathbf{12}^- + \nu_2 \mathbf{2}^+ \mathbf{12}^- + \nu_3 \mathbf{3}^+ \mathbf{12}^- + \nu_{10} \mathbf{10}^+ \mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_4 + \nu_5 + \nu_6 + 2\nu_{10} + \nu_{11} - \nu_4 \mathbf{4}^+ \mathbf{1}^- - \nu_5 \mathbf{5}^+ \mathbf{2}^- \right. \\
& \quad \left. - \nu_6 \mathbf{6}^+ \mathbf{3}^- + \nu_4 \mathbf{4}^+ \mathbf{10}^- + \nu_5 \mathbf{5}^+ \mathbf{10}^- + \nu_6 \mathbf{6}^+ \mathbf{10}^- + \nu_{11} \mathbf{11}^+ \mathbf{10}^- - \nu_{11} \mathbf{11}^+ \mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_4 - \nu_5 - \nu_6 - \nu_{10} - 2\nu_{11} + \nu_4 \mathbf{4}^+ \mathbf{7}^- + \nu_5 \mathbf{5}^+ \mathbf{8}^- \right. \\
& \quad \left. + \nu_6 \mathbf{6}^+ \mathbf{9}^- - \nu_4 \mathbf{4}^+ \mathbf{11}^- - \nu_5 \mathbf{5}^+ \mathbf{11}^- - \nu_6 \mathbf{6}^+ \mathbf{11}^- - \nu_{10} \mathbf{10}^+ \mathbf{11}^- + \nu_{10} \mathbf{10}^+ \mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_7 + \nu_8 + \nu_9 + 2\nu_{11} + \nu_{12} - \nu_7 \mathbf{7}^+ \mathbf{4}^- - \nu_8 \mathbf{8}^+ \mathbf{5}^- \right. \\
& \quad \left. - \nu_9 \mathbf{9}^+ \mathbf{6}^- - \nu_{12} \mathbf{12}^+ \mathbf{10}^- + \nu_7 \mathbf{7}^+ \mathbf{11}^- + \nu_8 \mathbf{8}^+ \mathbf{11}^- + \nu_9 \mathbf{9}^+ \mathbf{11}^- + \nu_{12} \mathbf{12}^+ \mathbf{11} \right) \mathcal{I} = 0 \\
& \left( D - \nu_7 - \nu_8 - \nu_9 - \nu_{11} - 2\nu_{12} + \nu_7 \mathbf{7}^+ \mathbf{1}^- + \nu_8 \mathbf{8}^+ \mathbf{2}^- \right. \\
& \quad \left. + \nu_9 \mathbf{9}^+ \mathbf{3}^- + \nu_{11} \mathbf{11}^+ \mathbf{10}^- - \nu_7 \mathbf{7}^+ \mathbf{12}^- - \nu_8 \mathbf{8}^+ \mathbf{12}^- - \nu_9 \mathbf{9}^+ \mathbf{12}^- - \nu_{11} \mathbf{11}^+ \mathbf{12}^- \right) \mathcal{I} = 0
\end{aligned} \tag{5.8}$$

From the symmetry of the planar auxiliary topology (5.3), we see that these IBP identities are related. As an example the sixth, fifth and fourth ones can be obtained from the third, second and first ones by using eq. (5.3).

## 5.3.2 IBP Identities for the Non-Planar Auxiliary Diagram

B

$$\begin{aligned} & \left( D - 2\nu_1 - \nu_2 - \nu_8 - \nu_{10} - \nu_{12} + s_{12}\nu_2\mathbf{2}^+ + s_{12}\nu_{11}\mathbf{11}^+ + s_{12}\nu_{12}\mathbf{12}^+ - \nu_2\mathbf{2}^+1^- - \nu_8\mathbf{8}^+1^- \right. \\ & \quad - \nu_{10}\mathbf{10}^+1^- - \nu_{11}\mathbf{11}^+1^- - \nu_{12}\mathbf{12}^+1^- + \nu_8\mathbf{8}^+3^- + \nu_{11}\mathbf{11}^+3^- + \nu_{10}\mathbf{10}^+5^- \\ & \quad + \nu_{12}\mathbf{12}^+5^- + \nu_{12}\mathbf{11}^+6^- + \nu_{12}\mathbf{12}^+6^- - \nu_{11}\mathbf{11}^+7^- - \nu_{12}\mathbf{12}^+7^- \\ & \quad \left. - \nu_{11}\mathbf{11}^+8^- + \nu_{11}\mathbf{11}^+10^- - \nu_{11}\mathbf{11}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( D - \nu_1 - 2\nu_2 - \nu_{10} - \nu_{11} - \nu_{12} + s_{12}\nu_1\mathbf{1}^+ - \nu_1\mathbf{1}^+2^- - \nu_8\mathbf{8}^+2^- - \nu_{10}\mathbf{10}^+2^- \right. \\ & \quad - \nu_{11}\mathbf{11}^+2^- - \nu_{12}\mathbf{12}^+2^- + \nu_8\mathbf{8}^+4^- + \nu_{11}\mathbf{11}^+4^- - \nu_8\mathbf{8}^+6^- + \nu_{12}\mathbf{12}^+6^- \\ & \quad \left. + \nu_8\mathbf{8}^+7^- + \nu_{10}\mathbf{10}^+7^- - \nu_8\mathbf{8}^+10^- - \nu_8\mathbf{8}^+11^- + \nu_8\mathbf{8}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( -D + \nu_1 + \nu_2 + \nu_{10} + 2\nu_{12} - s_{12}\nu_1\mathbf{1}^+ - \nu_1\mathbf{1}^+5^- - \nu_1\mathbf{1}^+6^- \right. \\ & \quad - \nu_2\mathbf{2}^+6^- + \nu_1\mathbf{1}^+7^- - \nu_8\mathbf{8}^+9^- - \nu_{11}\mathbf{11}^+9^- + \nu_8\mathbf{8}^+10^- + \nu_8\mathbf{8}^+11^- \\ & \quad \left. + \nu_1\mathbf{1}^+12^- + \nu_2\mathbf{2}^+12^- + \nu_{10}\mathbf{10}^+12^- + \nu_{11}\mathbf{11}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( D - 2\nu_3 - \nu_4 - \nu_8 - \nu_9 - \nu_{11} - s_{12}\nu_{11}\mathbf{11}^+ + \nu_8\mathbf{8}^+1^- + \nu_{11}\mathbf{11}^+1^- - \nu_4\mathbf{4}^+3^- - \nu_8\mathbf{8}^+3^- \right. \\ & \quad \left. - \nu_9\mathbf{9}^+3^- - \nu_{11}\mathbf{11}^+3^- + \nu_9\mathbf{9}^+5^- - \nu_{11}\mathbf{11}^+6^- - \nu_{11}\mathbf{11}^+10^- + \nu_{11}\mathbf{11}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( D - \nu_3 - 2\nu_4 - \nu_8 - \nu_9 - \nu_{11} + \nu_8\mathbf{8}^+2^- + \nu_{11}\mathbf{11}^+2^- - \nu_3\mathbf{3}^+4^- - \nu_8\mathbf{8}^+4^- - \nu_9\mathbf{9}^+4^- \right. \\ & \quad \left. - \nu_{11}\mathbf{11}^+4^- + \nu_8\mathbf{8}^+6^- + \nu_9\mathbf{9}^+6^- - \nu_8\mathbf{8}^+7^- + \nu_8\mathbf{8}^+10^- - \nu_8\mathbf{8}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( D + \nu_3 + \nu_4 + \nu_8 + \nu_9 + 2\nu_{11} + s_{12}\nu_3\mathbf{3}^+ - \nu_3\mathbf{3}^+1^- - \nu_4\mathbf{4}^+2^- + \nu_3\mathbf{3}^+6^- - \nu_3\mathbf{3}^+7 \right. \\ & \quad \left. + \nu_3\mathbf{3}^+10^- + \nu_3\mathbf{3}^+11^- + \nu_4\mathbf{4}^+11^- + \nu_8\mathbf{8}^+11^- + \nu_9\mathbf{9}^+11^- - \nu_3\mathbf{3}^+12^- - \nu_9\mathbf{9}^+12^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned} & \left( D - 2\nu_5 - \nu_6 - \nu_7 - \nu_9 - \nu_{10} + s_{12}\nu_7\mathbf{7}^+ - s_{12}\nu_{12}\mathbf{12}^+ + \nu_{10}\mathbf{10}^+1^- \right. \\ & \quad + \nu_{12}\mathbf{12}^+1^- + \nu_9\mathbf{9}^+3^- - \nu_6\mathbf{6}^+5^- - \nu_7\mathbf{7}^+5^- - \nu_9\mathbf{9}^+5^- \\ & \quad \left. - \nu_{10}\mathbf{10}^+5^- - \nu_{12}\mathbf{12}^+5^- - \nu_{12}\mathbf{12}^+6^- + \nu_{12}\mathbf{12}^+7^- - \nu_{12}\mathbf{12}^+10^- \right) \mathcal{I} = 0 \end{aligned}$$

$$\begin{aligned}
& \left( D - \nu_5 - 2\nu_6 - \nu_7 - \nu_9 - \nu_{12} + \nu_{10}\mathbf{10}^+\mathbf{2}^- + \nu_{12}\mathbf{12}^+\mathbf{2}^- + \nu_9\mathbf{9}^+\mathbf{4}^- - \nu_5\mathbf{5}^+\mathbf{6}^- \right. \\
& \quad \left. - \nu_7\mathbf{7}^+\mathbf{6}^- - \nu_9\mathbf{9}^+\mathbf{6}^- - \nu_{12}^+\mathbf{6}^- - \nu_{10}\mathbf{10}^+\mathbf{7}^- - \nu_{10}\mathbf{10}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_5 - \nu_6 - 2\nu_7 - \nu_9 - \nu_{10} + \nu_5\mathbf{5}^+ + \nu_{10}\mathbf{10}^+\mathbf{2}^- + \nu_{12}\mathbf{12}^+\mathbf{2}^- + \nu_9\mathbf{9}^+\mathbf{4}^- \right. \\
& \quad \left. - \nu_9\mathbf{9}^+\mathbf{6}^- - \nu_{12}\mathbf{12}^+\mathbf{6}^- - \nu_5\mathbf{5}^+\mathbf{7}^- - \nu_6\mathbf{6}^+\mathbf{7}^- - \nu_{10}\mathbf{10}^+\mathbf{7}^- \right. \\
& \quad \left. + \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_9\mathbf{9}^+\mathbf{10}^- - \nu_{12}\mathbf{12}^+\mathbf{10}^- - \nu_9\mathbf{9}^+\mathbf{11}^- - \nu_9\mathbf{9}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_1 - \nu_2 - 2\nu_8 - \nu_{10} - \nu_{11} + \nu_1\mathbf{1}^+\mathbf{3}^- + \nu_2\mathbf{2}^+\mathbf{4}^- - \nu_2\mathbf{2}^+\mathbf{6}^- + \nu_2\mathbf{2}^+\mathbf{7}^- \right. \\
& \quad \left. - \nu_1\mathbf{1}^+\mathbf{8}^- - \nu_{10}\mathbf{10}^+\mathbf{8}^- - \nu_{11}\mathbf{11}^+\mathbf{8}^- + \nu_{10}\mathbf{10}^-\mathbf{9}^- + \nu_{12}\mathbf{12}^+\mathbf{9}^- - \nu_2\mathbf{2}^+\mathbf{10}^- \right. \\
& \quad \left. - \nu_{12}\mathbf{12}^+\mathbf{10}^- - \nu_2\mathbf{2}^+\mathbf{11}^- - \nu_{12}\mathbf{12}^+\mathbf{11}^- + \nu_2\mathbf{2}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_1 + \nu_2 + \nu_8 + 2\nu_{10} + \nu_{12} - \nu_1\mathbf{1}^+\mathbf{5}^- - \nu_2\mathbf{2}^+\mathbf{7}^- + \nu_{11}\mathbf{11}^+\mathbf{8}^- - \nu_8\mathbf{8}^+\mathbf{9}^- \right. \\
& \quad \left. - \nu_{11}\mathbf{11}^+\mathbf{9}^- + \nu_1\mathbf{1}^+\mathbf{10}^- + \nu_2\mathbf{2}^+\mathbf{10}^- + \nu_8\mathbf{8}^+\mathbf{10}^- + \nu_{12}\mathbf{12}^+\mathbf{10}^- + \nu_{11}\mathbf{11}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_3 + \nu_4 + 2\nu_8 + \nu_9 + \nu_{11} - \nu_3\mathbf{3}^+\mathbf{1}^- - \nu_4\mathbf{4}^+\mathbf{2}^- - \nu_4\mathbf{4}^+\mathbf{6}^- + \nu_4\mathbf{4}^+\mathbf{7}^- \right. \\
& \quad \left. + \nu_3\mathbf{3}^+\mathbf{8}^- + \nu_4\mathbf{4}^+\mathbf{8}^- + \nu_9\mathbf{9}^+\mathbf{8}^- + \nu_{11}\mathbf{11}^+\mathbf{8}^- - \nu_4\mathbf{4}^+\mathbf{10}^- - \nu_9\mathbf{9}^+\mathbf{10}^- + \nu_4\mathbf{4}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_3 - \nu_4 - \nu_8 - 2\nu_9 - \nu_{11} + \nu_3\mathbf{3}^+\mathbf{5}^- + \nu_4\mathbf{4}^+\mathbf{6}^- - \nu_3\mathbf{3}^+\mathbf{9}^- \right. \\
& \quad \left. - \nu_8\mathbf{8}^+\mathbf{9}^- - \nu_4\mathbf{4}^+\mathbf{9}^- - \nu_{11}\mathbf{11}^+\mathbf{9}^- + \nu_8\mathbf{8}^+\mathbf{10}^- + \nu_{11}\mathbf{11}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_5 + \nu_6 + 2\nu_9 + \nu_{10} + \nu_{12} - \nu_5\mathbf{5}^+\mathbf{3}^- - \nu_6\mathbf{6}^+\mathbf{4}^- - \nu_7\mathbf{7}^+\mathbf{4}^- + \nu_7\mathbf{7}^+\mathbf{6}^- \right. \\
& \quad \left. - \nu_7\mathbf{7}^+\mathbf{8}^- - \nu_{10}\mathbf{10}^+\mathbf{8}^- + \nu_5\mathbf{5}^+\mathbf{9}^- + \nu_6\mathbf{6}^+\mathbf{9}^- + \nu_7\mathbf{7}^+\mathbf{9}^- + \nu_{10}\mathbf{10}^+\mathbf{9}^- \right. \\
& \quad \left. + \nu_{10}\mathbf{10}^+\mathbf{9}^- + \nu_{12}\mathbf{12}^+\mathbf{9}^- + \nu_7\mathbf{7}^+\mathbf{10}^- + \nu_7\mathbf{7}^+\mathbf{11}^- - \nu_{12}\mathbf{12}^+\mathbf{11}^- - \nu_7\mathbf{7}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_5 - \nu_7 - \nu_9 - 2\nu_{10} - \nu_{12} + \nu_5\mathbf{5}^+\mathbf{1}^- + \nu_6\mathbf{6}^+\mathbf{2}^- + \nu_7\mathbf{7}^+\mathbf{2}^- \right. \\
& \quad \left. - \nu_6\mathbf{6}^+\mathbf{7}^- + \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_5\mathbf{5}^+\mathbf{10}^- - \nu_7\mathbf{7}^+\mathbf{10}^- - \nu_9\mathbf{9}^+\mathbf{10}^- - \nu_{12}\mathbf{12}^+\mathbf{10}^- - \nu_6\mathbf{6}^+\mathbf{12}^- \right) \mathcal{I} = 0
\end{aligned} \tag{5.9}$$

## 5.3.3 IBP Identities for the Non-Planar Auxiliary Diagram

C

$$\left( D - 2\nu_1 - \nu_2 - \nu_3 - \nu_8 - \nu_{10} + s_{12}\nu_3\mathbf{3}^+ + s_{12}\nu_{12}\mathbf{12}^+ - \nu_2\mathbf{2}^+\mathbf{1}^- - \nu_3\mathbf{3}^+\mathbf{1}^- \right. \\ \left. - \nu_8\mathbf{8}^+\mathbf{1}^- - \nu_{10}\mathbf{10}^+\mathbf{1}^- - \nu_{11}\mathbf{11}^+\mathbf{2}^- - \nu_{12}\mathbf{12}^+\mathbf{3}^- + \nu_8\mathbf{8}^+\mathbf{4}^- \right. \\ \left. + \nu_{11}\mathbf{11}^+\mathbf{4}^- + \nu_{12}\mathbf{12}^+\mathbf{4}^- + \nu_{10}\mathbf{10}^+\mathbf{5}^- - \nu_{11}\mathbf{11}^+\mathbf{8}^- - \nu_{12}\mathbf{12}^+\mathbf{8}^- \right) \mathcal{I} = 0$$

$$\left( D - \nu_1 - 2\nu_2 - \nu_3 - \nu_{10} - \nu_{11} - \nu_8\mathbf{8}^+\mathbf{1}^- - \nu_1\mathbf{1}^+\mathbf{2}^- - \nu_3\mathbf{3}^+\mathbf{2}^- \right. \\ \left. - \nu_{10}\mathbf{10}^+\mathbf{2}^- - \nu_{11}\mathbf{11}^+\mathbf{2}^- - \nu_{12}\mathbf{12}^+\mathbf{3}^- + \nu_8\mathbf{8}^+\mathbf{4}^- \right. \\ \left. + \nu_{11}\mathbf{11}^+\mathbf{4}^- + \nu_{12}\mathbf{12}^+\mathbf{4}^- + \nu_{10}\mathbf{10}^+\mathbf{6}^- - \nu_8\mathbf{8}^+\mathbf{11}^- - \nu_{12}\mathbf{12}^+\mathbf{11}^- \right) \mathcal{I} = 0$$

$$\left( D - \nu_1 - \nu_2 - 2\nu_3 - \nu_{10} - \nu_{12} - s_{12}\nu_1\mathbf{1}^+ + s_{12}\nu_8\mathbf{8}^+ - \nu_{11}\mathbf{11}^+\mathbf{2}^- \right. \\ \left. - \nu_1\mathbf{1}^+\mathbf{3}^- - \nu_2\mathbf{2}^+\mathbf{3}^- - \nu_{10}\mathbf{10}^+\mathbf{3}^- - \nu_{12}\mathbf{12}^+\mathbf{3}^- + \nu_8\mathbf{8}^+\mathbf{4}^- \right. \\ \left. + \nu_{11}\mathbf{11}^+\mathbf{4}^- + \nu_{12}\mathbf{12}^+\mathbf{4}^- + \nu_{10}\mathbf{10}^+\mathbf{7}^- - \nu_8\mathbf{8}^+\mathbf{12}^- - \nu_{11}\mathbf{11}^+\mathbf{12}^- \right) \mathcal{I} = 0$$

$$\left( D - 2\nu_4 - \nu_8 - \nu_9 - \nu_{11} - \nu_{12} + \nu_8\mathbf{8}^+\mathbf{1}^- + \nu_{11}\mathbf{11}^+\mathbf{2}^- + \nu_{12}\mathbf{12}^+\mathbf{3}^- \right. \\ \left. - \nu_8\mathbf{8}^+\mathbf{4}^- - \nu_9\mathbf{9}^+\mathbf{4}^- - \nu_{11}\mathbf{11}^+\mathbf{4}^- - \nu_{12}\mathbf{12}^+\mathbf{4}^- + \nu_9\mathbf{9}^+\mathbf{5}^- \right) \mathcal{I} = 0$$

$$\left( -D + \nu_4 + \nu_8 + \nu_9 + 2\nu_{11} + \nu_{12} + \nu_9\mathbf{9}^+\mathbf{1}^- - \nu_4\mathbf{4}^+\mathbf{2}^- - \nu_9\mathbf{9}^+\mathbf{2}^- - \nu_9\mathbf{9}^+\mathbf{5}^- \right. \\ \left. + \nu_9\mathbf{9}^+\mathbf{6}^- - \nu_9\mathbf{9}^+\mathbf{10}^- + \nu_4\mathbf{4}^+\mathbf{11}^- + \nu_8\mathbf{8}^+\mathbf{11}^- + \nu_9\mathbf{9}^+\mathbf{11}^- + \nu_{12}\mathbf{12}^+\mathbf{11}^- \right) \mathcal{I} = 0$$

$$\left( -D + \nu_4 + \nu_8 + \nu_9 + \nu_{11} + 2\nu_{12} - s_{12}\nu_8\mathbf{8}^+ - s_{12}\nu_9\mathbf{9}^+ + \nu_9\mathbf{9}^+\mathbf{1}^- \right. \\ \left. - \nu_4\mathbf{4}^+\mathbf{3}^- - \nu_9\mathbf{9}^+\mathbf{3}^- - \nu_9\mathbf{9}^+\mathbf{5}^- + \nu_9\mathbf{9}^+\mathbf{7}^- - \nu_9\mathbf{9}^+\mathbf{10}^- \right. \\ \left. - \nu_4\mathbf{4}^+\mathbf{12}^- + \nu_8\mathbf{8}^+\mathbf{12}^- + \nu_9\mathbf{9}^+\mathbf{12}^- + \nu_{11}\mathbf{11}^+\mathbf{12}^- \right) \mathcal{I} = 0$$

$$\left( D - 2\nu_5 - \nu_6 - \nu_7 - \nu_9 - \nu_{10} + s_{12}\nu_7\mathbf{7}^+ + \nu_{10}\mathbf{10}^+\mathbf{1}^- \right. \\ \left. + \nu_9\mathbf{9}^+\mathbf{4}^- - \nu_6\mathbf{6}^+\mathbf{5}^- - \nu_7\mathbf{7}^+\mathbf{5}^- - \nu_9\mathbf{9}^+\mathbf{5}^- - \nu_{10}\mathbf{10}^+\mathbf{5}^- \right) \mathcal{I} = 0$$

$$\left( D - \nu_5 - 2\nu_6 - \nu_7 - \nu_{10} - \nu_9\mathbf{9}^+\mathbf{1}^- + \nu_9\mathbf{9}^+\mathbf{2}^- + \nu_{10}\mathbf{10}^+\mathbf{2}^- + \nu_9\mathbf{9}^+\mathbf{4}^- \right. \\ \left. - \nu_5\mathbf{5}^+\mathbf{6}^- - \nu_7\mathbf{7}^+\mathbf{6}^- - \nu_9\mathbf{9}^+\mathbf{6}^- - \nu_{10}\mathbf{10}^+\mathbf{6}^- + \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_9\mathbf{9}^+\mathbf{11}^- \right) \mathcal{I} = 0$$

$$\begin{aligned}
& \left( D - \nu_5 - \nu_6 - 2\nu_7 - \nu_9 - \nu_{10} + s_{12}\nu_5\mathbf{5}^+ + s_{12}\nu_9\mathbf{9}^+ \right. \\
& \quad - \nu_9\mathbf{9}^+\mathbf{1}^- \nu_9\mathbf{9}^+\mathbf{3}^- + \nu_{10}\mathbf{10}^+\mathbf{3}^- + \nu_9\mathbf{9}^+\mathbf{4}^- - \nu_5\mathbf{5}^+\mathbf{7}^- \\
& \quad \left. \nu_6\mathbf{6}^+\mathbf{7}^- - \nu_9\mathbf{9}^+\mathbf{7}^- - \nu_{10}\mathbf{10}^+\mathbf{7}^- + \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_9\mathbf{9}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_1 - 2\nu_8 - \nu_{10} - \nu_{11} - \nu_{12} + s_{12}\nu_3\mathbf{3}^+ + s_{12}\nu_{12}\mathbf{12}^+ - \nu_2\mathbf{2}^+\mathbf{1}^- \right. \\
& \quad \left. \nu_3\mathbf{3}^+\mathbf{1}^- + \nu_1\mathbf{1}^+\mathbf{4}^- + \nu_2\mathbf{2}^+\mathbf{4}^- + \nu_3\mathbf{3}^+\mathbf{4}^- - \nu_1\mathbf{1}^+\mathbf{8}^- - \nu_{10}\mathbf{10}^+\mathbf{8}^- \right. \\
& \quad \left. \nu_{11}\mathbf{11}^+\mathbf{8}^- - \nu_{12}\mathbf{12}^+\mathbf{8}^- + \nu_{10}\mathbf{10}^+\mathbf{9}^- - \nu_2\mathbf{2}^+\mathbf{11}^- - \nu_3\mathbf{3}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_1 + \nu_2 + \nu_3 + \nu_8 + 2\nu_{10} - \nu_{11}\mathbf{11}^+\mathbf{1}^- - \nu_{12}\mathbf{12}^+\mathbf{1}^- + \nu_{11}\mathbf{11}^+\mathbf{2}^- + \nu_{12}\mathbf{12}^+\mathbf{3}^- \right. \\
& \quad - \nu_1\mathbf{1}^+\mathbf{5}^- + \nu_{11}\mathbf{11}^+\mathbf{5}^- + \nu_{12}\mathbf{12}^+\mathbf{5}^- - \nu_2\mathbf{2}^+\mathbf{6}^- - \nu_{11}\mathbf{11}^+\mathbf{6}^- - \nu_3\mathbf{3}\mathbf{6}^+\mathbf{7}^- \\
& \quad - \nu_{12}\mathbf{12}^+\mathbf{7}^- + \nu_{11}\mathbf{11}^+\mathbf{8}^- + \nu_{12}\mathbf{12}^+\mathbf{8}^- - \nu_8\mathbf{8}^+\mathbf{9}^- - \nu_{11}\mathbf{11}^+\mathbf{9}^- - \nu_{12}\mathbf{12}^+\mathbf{9}^- \\
& \quad \left. + \nu_1\mathbf{1}^+\mathbf{10}^- + \nu_2\mathbf{2}^+\mathbf{10}^- + \nu_3\mathbf{3}^+\mathbf{10}^- + \nu_8\mathbf{8}^+\mathbf{10}^- + \nu_{11}\mathbf{11}^+\mathbf{10}^- + \nu_{12}\mathbf{12}^+\mathbf{10}^- \right) \mathcal{I} = 0 \\
& \left( -D + \nu_4 + 2\nu_8 + \nu_9 + \nu_{11} + \nu_{12} - s_{12}\nu_{12}\mathbf{12}^+ - \nu_4\mathbf{4}^+\mathbf{1}^- \right. \\
& \quad \left. + \nu_4\mathbf{4}^+\mathbf{8}^- + \nu_9\mathbf{9}^+\mathbf{8}^- + \nu_{11}\mathbf{11}^+\mathbf{8}^- + \nu_{12}\mathbf{12}^+\mathbf{8}^- - \nu_9\mathbf{9}^+\mathbf{10}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_4 - \nu_8 - 2\nu_9 - \nu_{11} - \nu_{12} + s_{12}\nu_{12}\mathbf{12}^+ - \nu_{11}\mathbf{11}^+\mathbf{1}^- - \nu_{12}\mathbf{12}^+\mathbf{1}^- + \nu_{11}\mathbf{11}^+\mathbf{2}^- \right. \\
& \quad + \nu_{12}\mathbf{12}^+\mathbf{3}^- + \nu_4\mathbf{4}^+\mathbf{5}^- + \nu_{11}\mathbf{11}^+\mathbf{5}^- + \nu_{12}\mathbf{12}^+\mathbf{5}^- - \nu_{11}\mathbf{11}^+\mathbf{6}^- - \nu_{12}\mathbf{12}^+\mathbf{7}^- - \nu_4\mathbf{4}^+\mathbf{9}^- \\
& \quad \left. - \nu_8\mathbf{8}^+\mathbf{9}^- - \nu_{11}\mathbf{11}^+\mathbf{9}^- - \nu_{12}\mathbf{12}^+\mathbf{9}^- + \nu_8\mathbf{8}^+\mathbf{10}^- + \nu_{11}\mathbf{11}^+\mathbf{10}^- + \nu_{12}\mathbf{12}^+\mathbf{10}^- \right) \mathcal{I} = 0 \\
& \left( D + \nu_5 + \nu_6 + \nu_7 + 2\nu_9 + \nu_{10} - s_{12}\nu_7\mathbf{7}^+ + \nu_6\mathbf{6}^+\mathbf{1}^- + \nu_7\mathbf{7}^+\mathbf{1}^- - \nu_6\mathbf{6}^+\mathbf{2}^- \right. \\
& \quad - \nu_7\mathbf{7}^+\mathbf{3}^- - \nu_5\mathbf{5}^+\mathbf{4}^+ - \nu_6\mathbf{6}^+\mathbf{4}^- - \nu_7\mathbf{7}^+\mathbf{4}^- - \nu_6\mathbf{6}^+\mathbf{8}^- - \nu_7\mathbf{7}^+\mathbf{8}^- - \nu_{10}\mathbf{10}^+\mathbf{8}^- \\
& \quad \left. + \nu_5\mathbf{5}^+\mathbf{9}^- + \nu_6\mathbf{6}^+\mathbf{9}^- + \nu_7\mathbf{7}^+\mathbf{9} + \nu_{10}\mathbf{10}^+\mathbf{9}^- + \nu_6\mathbf{6}^+\mathbf{11}^- + \nu_7\mathbf{7}^+\mathbf{12}^- \right) \mathcal{I} = 0 \\
& \left( D - \nu_5\nu_6 - \nu_7 - \nu_9 - 2\nu_{10} + \nu_5\mathbf{5}^+\mathbf{1}^- + \nu_6\mathbf{6}^+\mathbf{2}^- + \nu_7\mathbf{7}^+\mathbf{3}^- \right. \\
& \quad \left. + \nu_9\mathbf{9}^+\mathbf{8}^- - \nu_5\mathbf{5}^+\mathbf{10}^- - \nu_6\mathbf{6}^+\mathbf{10}^- - \nu_7\mathbf{7}^+\mathbf{10}^- - \nu_9\mathbf{9}^+\mathbf{10}^- \right) \mathcal{I} = 0
\end{aligned} \tag{5.10}$$



Since there is no symmetry in the non-planar diagrams we don't have inter-relations between IBP identities. In the next section we list all the master integrals necessary for the evaluation of the quark and gluon form factors. Some of the master integrals have been known for sometime and the harder ones were calculated recently. [63–65]. We present the master integrals expanded in Laurent series in  $\epsilon$ .

All L-loop integrals have an overall factor of,

$$s_{12}^n \frac{(-s_{12} - i0)^{-L\epsilon}}{\Gamma(1 - \epsilon)^L}$$

where  $n$  is fixed by dimensional arguments. Unlike Refs. [63–65], there is no  $(-1)^n$  factor.

## 5.4 Master Integrals for Three-Loop Form Factors

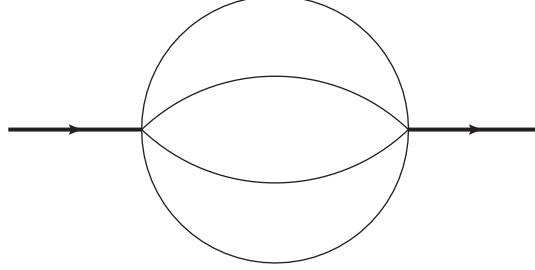


Figure 5.6: B41: Three-loop master integral with four massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B41 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2} & (5.11) \\
 &= -\frac{1}{36\epsilon} \\
 &\quad -\frac{71}{216} \\
 &\quad -\epsilon \left( \frac{3115}{1296} \right) \\
 &\quad + \epsilon^2 \left( -\frac{109403}{7776} + \frac{7\zeta_3}{9} \right) \\
 &\quad + \epsilon^3 \left( -\frac{3386467}{46656} + \frac{7\pi^4}{540} + \frac{497\zeta_3}{54} \right) \\
 &\quad + \epsilon^4 \left( -\frac{96885467}{279936} + \frac{497\pi^4}{3240} + \frac{21805\zeta_3}{324} + 7\zeta_5 \right) \\
 &\quad + \epsilon^5 \left( -\frac{2631913075}{1679616} + \frac{4361\pi^4}{3888} + \frac{4\pi^6}{243} + \frac{765821\zeta_3}{1944} - \frac{98\zeta_3^2}{9} + \frac{497\zeta_5}{6} \right) \\
 &\quad + \epsilon^6 a41
 \end{aligned}$$

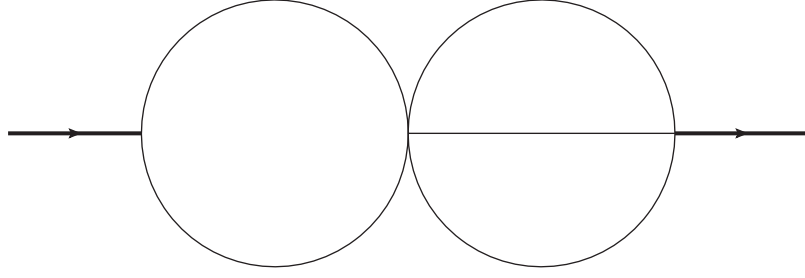


Figure 5.7: B51: Three-loop master integral with five massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
B51 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 (k_2 + p_1 + p_2)^2 (k_3)^2 (k_1 - k_2)^2} \quad (5.12) \\
&= + \frac{1}{4\epsilon^2} \\
&\quad + \frac{17}{8\epsilon} \\
&\quad + \frac{183}{16} \\
&\quad + \epsilon \left( -3\zeta_3 + \frac{1597}{32} \right) \\
&\quad + \epsilon^2 \left( -\frac{51\zeta_3}{2} - \frac{\pi^4}{20} + \frac{12359}{64} \right) \\
&\quad + \epsilon^3 \left( -\frac{549\zeta_3}{4} - 15\zeta_5 - \frac{17\pi^4}{40} + \frac{88629}{128} \right) \\
&\quad + \epsilon^4 \left( -\frac{4791\zeta_3}{8} - \frac{255\zeta_5}{2} - \frac{183\pi^4}{80} - \frac{2\pi^6}{63} + 18\zeta_3^2 + \frac{603871}{256} \right) \\
&\quad + \frac{\epsilon^5 b51}{4}
\end{aligned}$$

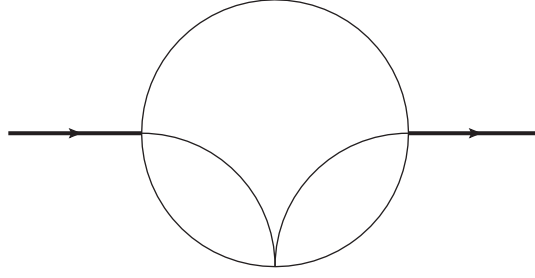


Figure 5.8: B52: Three-loop master integral with five massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B52 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2} & (5.13) \\
 &= + \frac{1}{3\epsilon^2} \\
 &\quad + \frac{10}{3\epsilon} \\
 &\quad + \frac{64}{3} \\
 &\quad + \epsilon \left( +112 - \frac{22\zeta_3}{3} \right) \\
 &\quad + \epsilon^2 \left( +528 - \frac{220\zeta_3}{3} - \frac{11\pi^4}{90} \right) \\
 &\quad + \epsilon^3 \left( +2336 - \frac{1408\zeta_3}{3} - 70\zeta_5 - \frac{11\pi^4}{9} \right) \\
 &\quad + \epsilon^4 \left( -\frac{352\pi^4}{45} - 2464\zeta_3 - 700\zeta_5 + \frac{29824}{3} - \frac{94\pi^6}{567} + \frac{242\zeta_3^2}{3} \right) \\
 &\quad + \frac{\epsilon^5 b52}{3}
 \end{aligned}$$

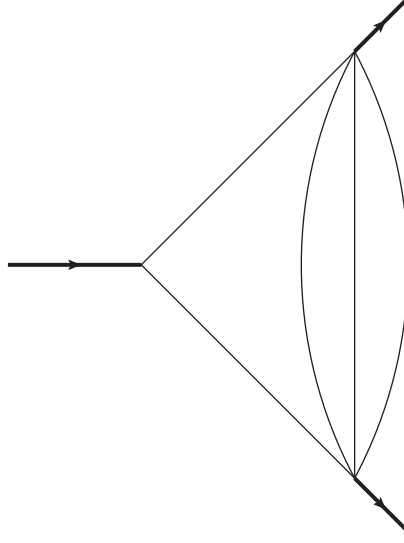


Figure 5.9: A51: Three-loop master integral with five massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A51 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2} & (5.14) \\
&= -\frac{1}{24\epsilon^2} \\
&\quad -\frac{19}{48\epsilon} \\
&\quad + \left( -\frac{233}{96} - \frac{\pi^2}{24} \right) \\
&\quad + \epsilon \left( -\frac{2363}{192} - \frac{19\pi^2}{48} + \frac{11\zeta_3}{12} \right) \\
&\quad + \epsilon^2 \left( -\frac{7227}{128} - \frac{233\pi^2}{96} - \frac{\pi^4}{80} + \frac{209\zeta_3}{24} \right) \\
&\quad + \epsilon^3 \left( -\frac{62641}{256} - \frac{2363\pi^2}{192} - \frac{19\pi^4}{160} + \frac{2563\zeta_3}{48} + \frac{11\pi^2\zeta_3}{12} + \frac{35\zeta_5}{4} \right) \\
&\quad + \epsilon^4 \left( -\frac{1575481}{1536} - \frac{7227\pi^2}{128} - \frac{233\pi^4}{320} + \frac{919\pi^6}{45360} + \frac{25993\zeta_3}{96} + \frac{209\pi^2\zeta_3}{24} - \frac{121\zeta_3^2}{12} + \frac{665\zeta_5}{8} \right) \\
&\quad - \epsilon^5 a51
\end{aligned}$$

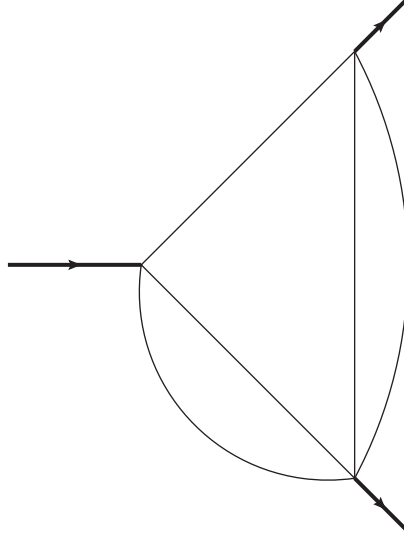


Figure 5.10: A52: Three-loop master integral with five massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A52 &= \int d\mathcal{K}d\mathcal{K}_2d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2(k_2 + p_1)^2k_3^2(k_1 - k_2)^2(k_3 - k_1)^2} & (5.15) \\
&= + \frac{1}{6\epsilon^2} \\
&\quad + \frac{5}{3\epsilon} \\
&\quad + \left( + \frac{32}{3} + \frac{\pi^2}{12} \right) \\
&\quad + \epsilon \left( + 56 + \frac{5\pi^2}{6} - \frac{11\zeta_3}{3} \right) \\
&\quad + \epsilon^2 \left( + 264 + \frac{16\pi^2}{3} - \frac{19\pi^4}{720} - \frac{110\zeta_3}{3} \right) \\
&\quad + \epsilon^3 \left( + 1168 + 28\pi^2 - \frac{19\pi^4}{72} - \frac{704\zeta_3}{3} - \frac{11\pi^2\zeta_3}{6} - 35\zeta_5 \right) \\
&\quad + \epsilon^4 \left( + \frac{14912}{3} + 132\pi^2 - \frac{76\pi^4}{45} - \frac{9011\pi^6}{90720} - 1232\zeta_3 - \frac{55\pi^2\zeta_3}{3} + \frac{121\zeta_3^2}{3} - 350\zeta_5 \right) \\
&\quad - \epsilon^5 a52
\end{aligned}$$

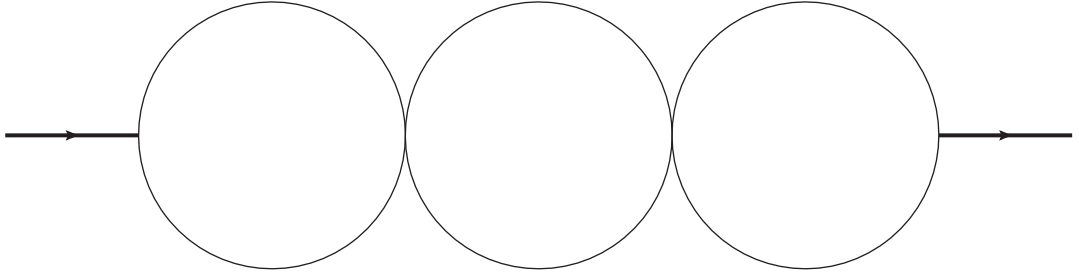


Figure 5.11: B61: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B61 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_2 + p_1 + p_2)^2 k_3^2 (k_3 + p_1 + p_2)^2} \quad (5.16) \\
 &= -\frac{1}{\epsilon^3} \\
 &\quad -\frac{6}{\epsilon^2} \\
 &\quad -\frac{24}{\epsilon} \\
 &\quad + \left( -80 + 6\zeta_3 \right) \\
 &\quad + \epsilon \left( -240 + 36\zeta_3 + \frac{\pi^4}{10} \right) \\
 &\quad + \epsilon^2 \left( -672 + 144\zeta_3 + 18\zeta_5 + \frac{3\pi^4}{5} \right) \\
 &\quad + \epsilon^3 \left( -1792 + 480\zeta_3 + 108\zeta_5 + \frac{12\pi^4}{5} + \frac{2\pi^6}{63} - 18\zeta_3^2 \right) \\
 &\quad - \epsilon^4 b61
 \end{aligned}$$

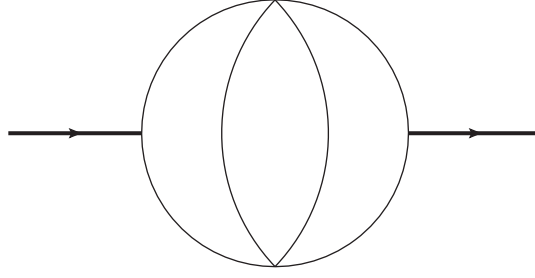


Figure 5.12: B62: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
B62 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_3^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_2 - k_3)^2} \quad (5.17) \\
&= -\frac{1}{3\epsilon^3} \\
&\quad -\frac{7}{3\epsilon^2} \\
&\quad -\frac{31}{3\epsilon} \\
&\quad + \left( -\frac{8\zeta_3}{3} - \frac{103}{3} \right) \\
&\quad + \epsilon \left( -\frac{235}{3} - \frac{56\zeta_3}{3} - \frac{2\pi^4}{45} \right) \\
&\quad + \epsilon^2 \left( -\frac{19}{3} - 120\zeta_5 - \frac{320\zeta_3}{3} - \frac{14\pi^4}{45} \right) \\
&\quad + \epsilon^3 \left( +\frac{3953}{3} - 840\zeta_5 - \frac{1832\zeta_3}{3} - \frac{16\pi^4}{9} - \frac{176\pi^6}{567} + \frac{292\zeta_3^2}{3} \right) \\
&\quad + \epsilon^4 b62
\end{aligned}$$



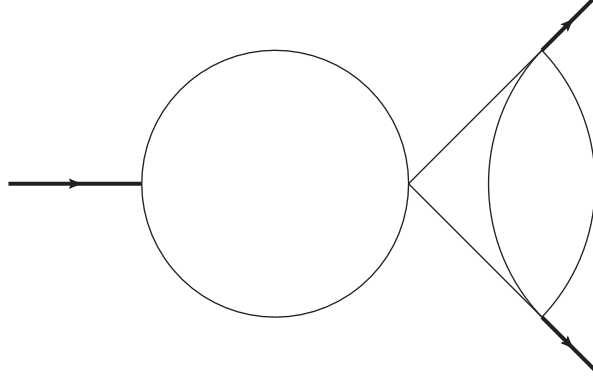


Figure 5.13: C61: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
C61 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_2 + p_1 + p_2)^2 (k_3 + p_1)^2 (k_2 - k_3)^2} \quad (5.18) \\
&= -\frac{1}{2\epsilon^3} \\
&\quad -\frac{7}{2\epsilon^2} \\
&\quad + \frac{1}{\epsilon} \left( -\frac{\pi^2}{6} - \frac{33}{2} \right) \\
&\quad + \left( -\frac{7\pi^2}{6} + 5\zeta_3 - \frac{131}{2} \right) \\
&\quad + \epsilon \left( -\frac{11\pi^2}{2} + 35\zeta_3 + \frac{\pi^4}{20} - \frac{473}{2} \right) \\
&\quad + \epsilon^2 \left( -\frac{131\pi^2}{6} + \frac{5\pi^2\zeta_3}{3} + 165\zeta_3 + 27\zeta_5 + \frac{7\pi^4}{20} - \frac{1611}{2} \right) \\
&\quad + \epsilon^3 \left( -\frac{473\pi^2}{6} + \frac{35\pi^2\zeta_3}{3} + 655\zeta_3 + 189\zeta_5 + \frac{33\pi^4}{20} + \frac{61\pi^6}{756} - 25\zeta_3^2 - \frac{5281}{2} \right) \\
&\quad - \frac{\epsilon^4 c61}{2}
\end{aligned}$$

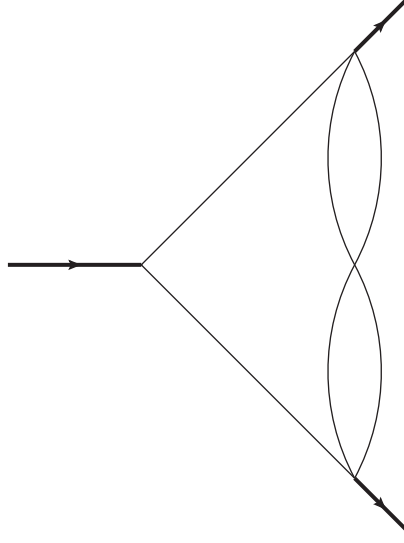


Figure 5.14: A61: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A61 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 (k_2 + p_1)^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_3 - k_1)^2} r \\
&= -\frac{1}{3\epsilon^3} \\
&\quad -\frac{1}{3\epsilon^2} \\
&\quad + \frac{1}{\epsilon} \left( -\frac{44}{3} - \frac{\pi^2}{3} \right) \\
&\quad + \left( -\frac{8\pi^2}{3} - \frac{208}{3} + \frac{16\zeta_3}{3} \right) \\
&\quad + \epsilon \left( -304 - \frac{44\pi^2}{3} - \frac{2\pi^4}{15} + \frac{128\zeta_3}{3} \right) \\
&\quad + \epsilon^2 \left( -1280 - \frac{208\pi^2}{3} - \frac{16\pi^4}{15} + \frac{704\zeta_3}{3} + \frac{16\pi^2\zeta_3}{3} + 56\zeta_5 \right) \\
&\quad + \epsilon^3 \left( -\frac{15808}{3} - 304\pi^2 - \frac{88\pi^4}{15} + \frac{55\pi^6}{567} + \frac{3328\zeta_3}{3} + \frac{128\pi^2\zeta_3}{3} - \frac{128\zeta_3^2}{3} + 448\zeta_5 \right) \\
&\quad + \epsilon^4 a61
\end{aligned} \tag{5.19}$$

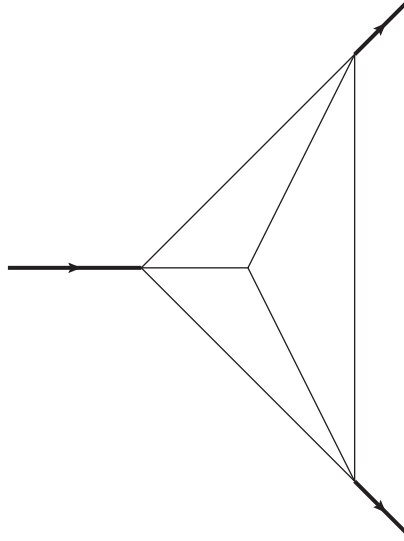


Figure 5.15: A62: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A62 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_2 + p_1 + p_2)^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_3 - k_1)^2} \\
 &= -\frac{2\zeta_3}{\epsilon} + \left( -\frac{7\pi^4}{180} - 18\zeta_3 \right) \\
 &\quad + \epsilon \left( -\frac{7\pi^4}{20} - 122\zeta_3 + \frac{2\pi^2\zeta_3}{3} - 10\zeta_5 \right) \\
 &\quad + \epsilon^2 \left( -\frac{427\pi^4}{180} + \frac{163\pi^6}{7560} - 738\zeta_3 + 6\pi^2\zeta_3 + 76\zeta_3^2 - 90\zeta_5 \right) \\
 &\quad + \epsilon^3 a62
 \end{aligned} \tag{5.20}$$

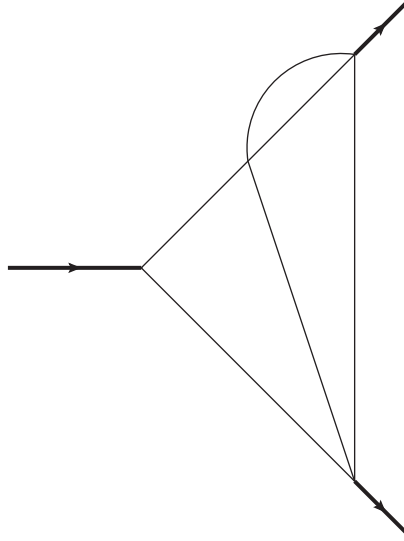


Figure 5.16: A63: Three-loop master integral with six massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A63 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 (k_2 + p_1)^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_3 - k_1)^2} \\
&= -\frac{1}{6\epsilon^3} \\
&\quad -\frac{3}{2\epsilon^2} \\
&\quad + \frac{1}{\epsilon} \left( -\frac{55}{6} - \frac{\pi^2}{6} \right) + \left( -\frac{3\pi^2}{2} - \frac{95}{2} + \frac{17\zeta_3}{3} \right) \\
&\quad + \epsilon \left( -\frac{1351}{6} - \frac{55\pi^2}{6} - \frac{\pi^4}{90} + 51\zeta_3 \right) \\
&\quad + \epsilon^2 \left( -\frac{2023}{2} - \frac{95\pi^2}{2} - \frac{\pi^4}{10} + \frac{935\zeta_3}{3} + \frac{10\pi^2\zeta_3}{3} + 65\zeta_5 \right) \\
&\quad + \epsilon^3 \left( -\frac{26335}{6} - \frac{1351\pi^2}{6} - \frac{11\pi^4}{18} + \frac{7\pi^6}{54} + 1615\zeta_3 + 30\pi^2\zeta_3 - \frac{268\zeta_3^2}{3} + 585\zeta_5 \right) \\
&\quad + \epsilon^4 a63
\end{aligned} \tag{5.21}$$

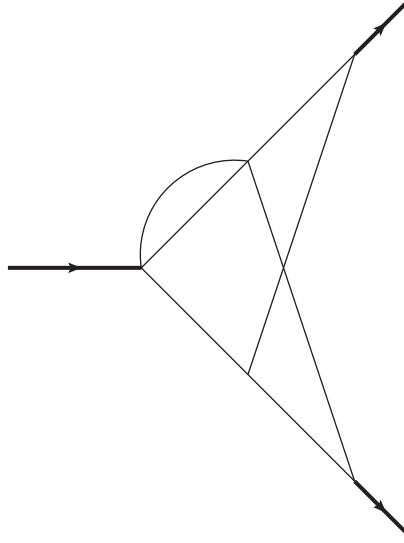


Figure 5.17: A71: Three-loop master integral with seven massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A71 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_2 + p_1)^2 k_3^2 (k_3 + p_1)^2 (k_3 - k_1)^2 (k_1 - k_2 + p_2)^2 (k_3 - k_1 - p_2)^2} \\
 &= -\frac{1}{4\epsilon^5} \\
 &\quad -\frac{1}{2\epsilon^4} \\
 &\quad + \frac{1}{\epsilon^3} \left( -1 + \frac{\pi^2}{6} \right) \\
 &\quad + \frac{1}{\epsilon^2} \left( -2 + \frac{\pi^2}{3} + 10\zeta_3 \right) \\
 &\quad + \frac{1}{\epsilon} \left( -4 + \frac{2\pi^2}{3} + \frac{11\pi^4}{45} + 20\zeta_3 \right) \\
 &\quad + \left( +\frac{22\pi^4}{45} + \frac{4\pi^2}{3} - \frac{14\pi^2\zeta_3}{3} - 8 + 40\zeta_3 + 88\zeta_5 \right) \\
 &\quad + \epsilon \left( -16 + \frac{8\pi^2}{3} + \frac{44\pi^4}{45} + \frac{943\pi^6}{7560} + 80\zeta_3 - \frac{28\pi^2\zeta_3}{3} - 196\zeta_3^2 + 176\zeta_5 \right) \\
 &\quad - \epsilon^2 a71
 \end{aligned} \tag{5.22}$$

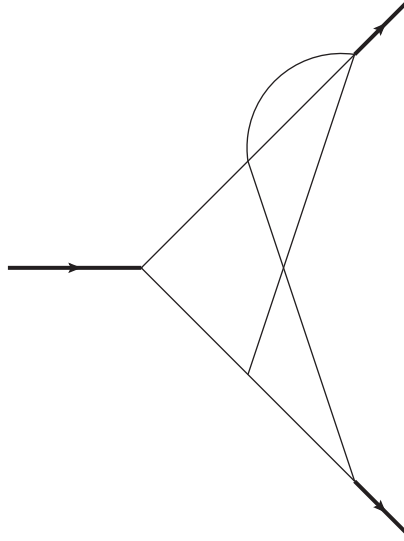


Figure 5.18: A72: Three-loop master integral with seven massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A72 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_3^2 (k_3 + p_1)^2 (k_2 - k_3)^2 (k_3 - k_1)^2 (k_1 - k_2 + p_2)^2} \\
 &= -\frac{\pi^2}{12\epsilon^3} \\
 &\quad + \frac{1}{\epsilon^2} \left( -\frac{\pi^2}{6} - 2\zeta_3 \right) \\
 &\quad + \frac{1}{\epsilon} \left( -\frac{\pi^2}{3} - \frac{83\pi^4}{720} - 4\zeta_3 \right) \\
 &\quad + \left( -\frac{2\pi^2}{3} - \frac{83\pi^4}{360} - 8\zeta_3 + \frac{5\pi^2\zeta_3}{3} - 15\zeta_5 \right) \\
 &\quad + \epsilon \left( -\frac{4\pi^2}{3} - \frac{83\pi^4}{180} - \frac{2741\pi^6}{90720} - 16\zeta_3 + \frac{10\pi^2\zeta_3}{3} + 73\zeta_3^2 - 30\zeta_5 \right) \\
 &\quad - \epsilon^2 a72
 \end{aligned} \tag{5.23}$$

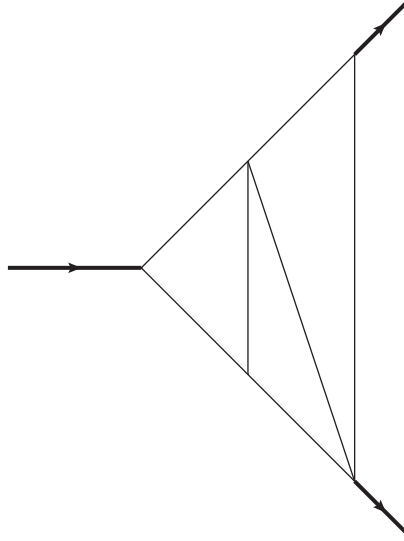


Figure 5.19: A73: Three-loop master integral with seven massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A73 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_3 + p_1)^2 (k_3 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_2 - k_3)^2} \\
 &= + \frac{1}{\epsilon} \left( + \frac{\pi^2 \zeta_3}{6} + 10\zeta_5 \right) \\
 &\quad + \left( + \frac{119\pi^6}{2160} + \frac{31\zeta_3^2}{2} \right) \\
 &\quad - \epsilon a73
 \end{aligned} \tag{5.24}$$

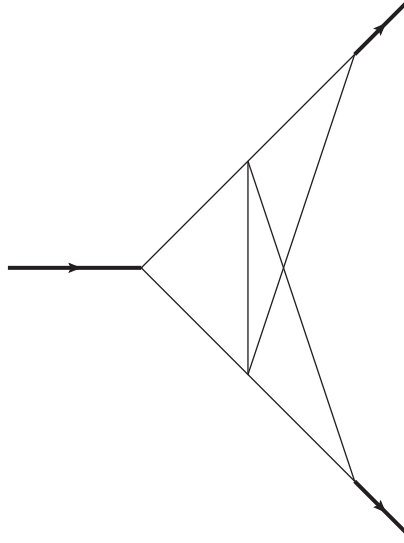


Figure 5.20: A74: Three-loop master integral with seven massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A74 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_3^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_1 - k_2 + p_2)^2} \\
 &= -\frac{6\zeta_3}{\epsilon^2} \\
 &\quad + \frac{1}{\epsilon} \left( -\frac{11\pi^4}{90} - 36\zeta_3 \right) \\
 &\quad + \left( -216\zeta_3 + 2\pi^2\zeta_3 - \frac{11\pi^4}{15} - 46\zeta_5 \right) \\
 &\quad + \epsilon \left( -\frac{22\pi^4}{5} + \frac{19\pi^6}{270} - 1296\zeta_3 + 12\pi^2\zeta_3 + 282\zeta_3^2 - 276\zeta_5 \right) \\
 &\quad - \epsilon^2 a74
 \end{aligned} \tag{5.25}$$



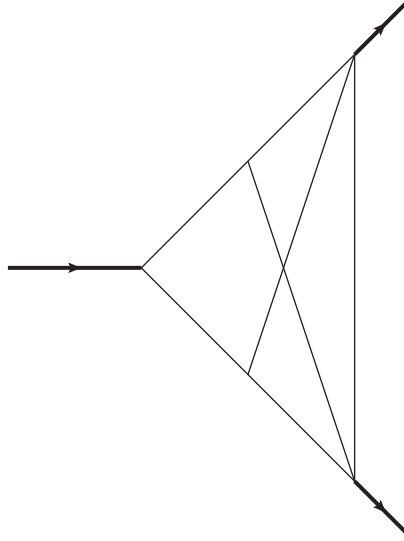


Figure 5.21: A75: Three-loop master integral with seven massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 A75 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2 (k_1 + p_1 + p_2)^2 k_2^2 (k_3 + p_1)^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_3 - k_1 - p_2)^2} \\
 &= + \left( -2\pi^2 \zeta_3 - 10\zeta_5 \right) \\
 &\quad + \epsilon \left( -\frac{11\pi^6}{162} - 12\pi^2 \zeta_3 - 18\zeta_3^2 - 60\zeta_5 \right) \\
 &\quad - \epsilon^2 a75
 \end{aligned} \tag{5.26}$$

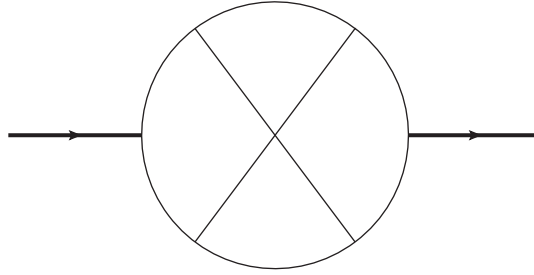


Figure 5.22: B81: Three-loop master integral with eight massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
 B81 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2 k_2^2 k_3^2 (k_3 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_3 - k_1)^2} \\
 &\quad \times \frac{1}{(k_1 - k_2 + p_1 + p_2)^2} \tag{5.27} \\
 &= -20\zeta_5 \\
 &\quad - \epsilon \left( +68\zeta_3^2 + 40\zeta_5 + \frac{10\pi^6}{189} \right) \\
 &\quad - \epsilon^2 \left( +136\zeta_3^2 + \frac{34\pi^4\zeta_3}{15} + 80\zeta_5 + \frac{20\pi^6}{189} + 450\zeta_7 \right) \\
 &\quad + \epsilon^3 b81
 \end{aligned}$$

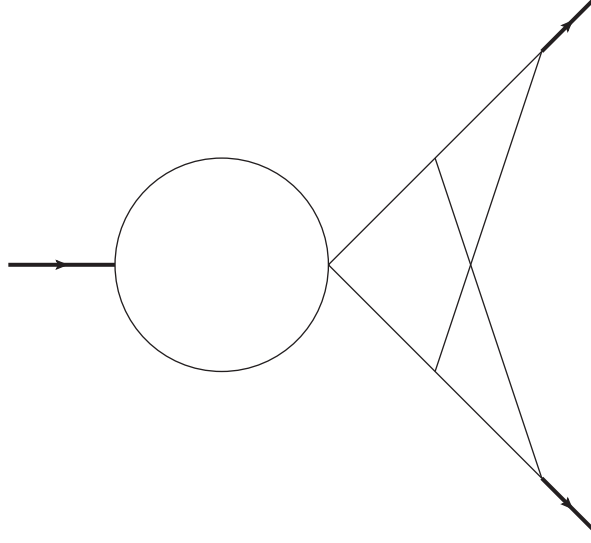


Figure 5.23: C81: Three-loop master integral with eight massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
C81 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2(k_1 + p_1 + p_2)^2 k_2^2(k_2 + p_1)^2 k_3^2(k_3 + p_1 + p_2)^2} \\
&\quad \times \frac{1}{(k_1 - k_2)^2(k_1 - k_2 + p_2)^2} \\
&= -\frac{1}{\epsilon^5} \\
&\quad -\frac{2}{\epsilon^4} \\
&\quad + \frac{1}{\epsilon^3} \left( +\frac{5\pi^2}{6} - 4 \right) \\
&\quad + \frac{1}{\epsilon^2} \left( -8 + \frac{5\pi^2}{3} + 29\zeta_3 \right) \\
&\quad + \frac{1}{\epsilon} \left( -16 + \frac{10\pi^2}{3} + 58\zeta_3 + \frac{121\pi^4}{180} \right) \\
&\quad + \left( -32 + \frac{20\pi^2}{3} - \frac{29\pi^2\zeta_3}{3} + 116\zeta_3 + 123\zeta_5 + \frac{121\pi^4}{90} \right) \\
&\quad + \epsilon \left( -\frac{58\pi^2\zeta_3}{3} + 232\zeta_3 + 246\zeta_5 + \frac{40\pi^2}{3} - 323\zeta_3^2 + \frac{121\pi^4}{45} - 64 + \frac{163\pi^6}{3780} \right) \\
&\quad - \epsilon^2 c81
\end{aligned} \tag{5.28}$$

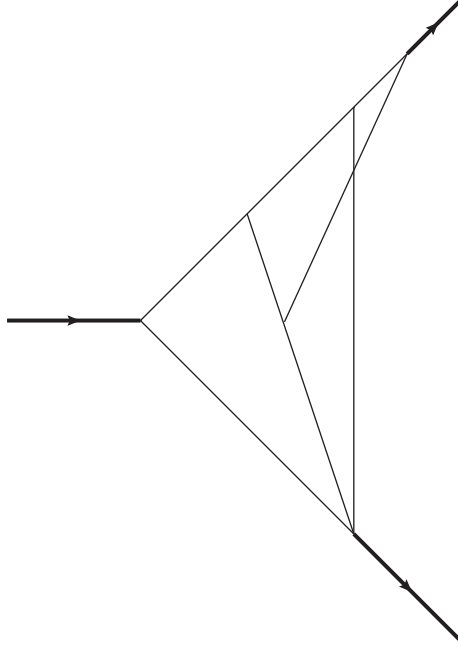


Figure 5.24: A81: Three-loop master integral with eight massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A81 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1 + p_2)^2 (k_2 + p_1)^2 k_3^2 (k_3 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_2 - k_3)^2} \\
&\quad \times \frac{1}{(k_3 - k_1)^2 (k_1 - k_2 + p_2)^2} \tag{5.29} \\
&= + \frac{8\zeta_3}{3\epsilon^2} \\
&\quad + \frac{1}{\epsilon} \left( + \frac{5\pi^4}{27} - 8\zeta_3 \right) \\
&\quad + \left( - \frac{5\pi^4}{9} + 24\zeta_3 - \frac{52\pi^2\zeta_3}{9} + \frac{352\zeta_5}{3} \right) \\
&\quad + \epsilon \left( + \frac{5\pi^4}{3} + \frac{1709\pi^6}{8505} - 72\zeta_3 + \frac{52\pi^2\zeta_3}{3} - \frac{332\zeta_3^2}{3} - 352\zeta_5 \right) \\
&\quad + \epsilon^2 a81
\end{aligned}$$

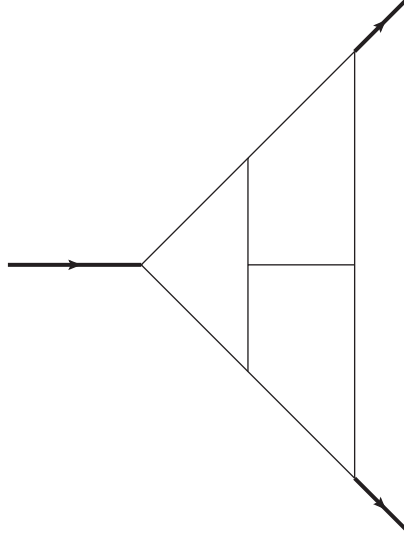


Figure 5.25: A91: Three-loop master integral with nine massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A91 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2(k_1 + p_1 + p_2)^2(k_2 + p_1)^2(k_2 + p_1 + p_2)^2 k_3^2(k_3 + p_1)^2} \\
&\quad \times \frac{1}{(k_1 - k_2)^2(k_2 - k_3)^2(k_3 - k_1)^2} \\
&= + \frac{1}{18\epsilon^5} \\
&\quad - \frac{1}{2\epsilon^4} \\
&\quad + \frac{1}{\epsilon^3} \left( + \frac{53}{18} + \frac{4\pi^2}{27} \right) \\
&\quad + \frac{1}{\epsilon^2} \left( - \frac{29}{2} - \frac{22\pi^2}{27} + 2\zeta_3 \right) \\
&\quad + \frac{1}{\epsilon} \left( + \frac{8\pi^2}{3} - \frac{158\zeta_3}{9} + \frac{20\pi^4}{81} + \frac{129}{2} \right) \\
&\quad + \left( - \frac{322\pi^4}{405} - 6\pi^2 + \frac{14\pi^2\zeta_3}{3} - \frac{537}{2} + \frac{578\zeta_3}{9} + \frac{238\zeta_5}{3} \right) \\
&\quad + \epsilon \left( + \frac{2133}{2} + 4\pi^2 + \frac{302\pi^4}{135} + \frac{2398\pi^6}{5103} - 158\zeta_3 + \frac{26\pi^2\zeta_3}{3} + \frac{466\zeta_3^2}{3} - \frac{826\zeta_5}{3} \right) \\
&\quad - \epsilon^2 a91b
\end{aligned} \tag{5.30}$$

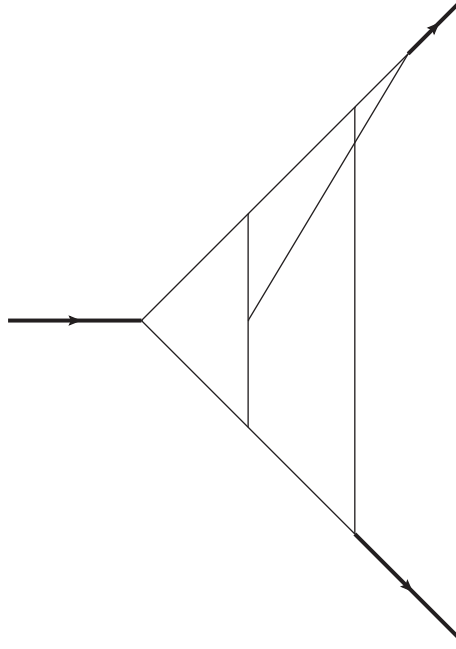


Figure 5.26: A92: Three-loop master integral with nine massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A92 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{(k_1 + p_1)^2 (k_1 + p_1 + p_2)^2 k_2^2 k_3^2 (k_3 + p_1 + p_2)^2 (k_1 - k_2)^2 (k_2 - k_3)^2} \\
&\quad \times \frac{1}{(k_3 - k_1)^2 (k_1 - k_2 + p_1)^2} \tag{5.31} \\
&= -\frac{2}{9\epsilon^6} \\
&\quad -\frac{5}{6\epsilon^5} \\
&\quad + \frac{1}{\epsilon^4} \left( +\frac{20}{9} + \frac{7\pi^2}{27} \right) \\
&\quad + \frac{1}{\epsilon^3} \left( -\frac{50}{9} + \frac{17\pi^2}{27} + \frac{91\zeta_3}{9} \right) \\
&\quad + \frac{1}{\epsilon^2} \left( -\frac{4\pi^2}{3} + \frac{166\zeta_3}{9} + \frac{373\pi^4}{1080} + \frac{110}{9} \right) \\
&\quad + \frac{1}{\epsilon} \left( -\frac{494\zeta_3}{9} - \frac{179\pi^2\zeta_3}{27} + 167\zeta_5 + \frac{16\pi^2}{9} + \frac{187\pi^4}{540} - \frac{170}{9} \right) \\
&\quad - a_{92a} \\
&\quad - \epsilon a_{92b}
\end{aligned}$$

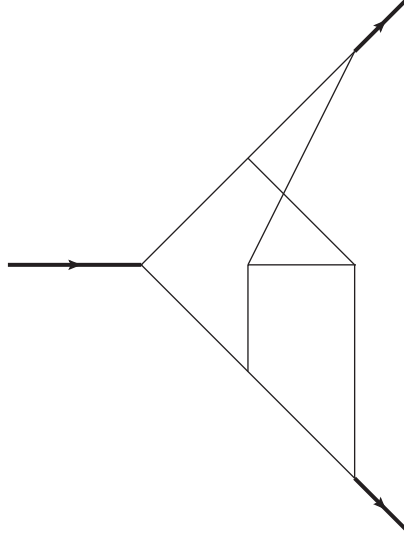


Figure 5.27: A94: Three-loop master integral with nine massless propagators. The incoming momentum is  $p_{12} = p_1 + p_2$ . Outgoing momenta are taken to be on-shell and massless,  $p_1^2 = p_2^2 = 0$ .

$$\begin{aligned}
A94 &= \int d\mathcal{K}_1 d\mathcal{K}_2 d\mathcal{K}_3 \frac{1}{k_1^2(k_1 + p_1 + p_2)^2 k_2^2(k_2 + p_1)^2 k_3^2(k_2 - k_3)^2 (k_3 - k_1)^2} \\
&\quad \times \frac{1}{(k_1 - k_2 + p_2)^2 (k_3 - k_1 - p_2)^2} \tag{5.32} \\
&= -\frac{1}{9\epsilon^6} \\
&\quad -\frac{8}{9\epsilon^5} \\
&\quad + \frac{1}{\epsilon^4} \left( +1 + \frac{10\pi^2}{27} \right) \\
&\quad + \frac{1}{\epsilon^3} \left( +\frac{14}{9} + \frac{47\pi^2}{27} + 12\zeta_3 \right) \\
&\quad + \frac{1}{\epsilon^2} \left( -17 - \frac{71\pi^2}{27} + \frac{200\zeta_3}{3} + \frac{47\pi^4}{810} \right) \\
&\quad - \frac{1}{\epsilon} \left( -84 - \frac{7}{18}\pi^2 + \frac{940}{9}\zeta_3 - \frac{671}{540}\pi^4 - \frac{11}{3}\zeta_2 - \frac{692}{9}\zeta_5 + \frac{652}{27}\pi^2\zeta_3 \right) \\
&\quad - a_{94a} \\
&\quad + \epsilon a_{94b}
\end{aligned}$$

# Chapter 6

## Results

### 6.1 Results at three-loops

In this section we give the results for the three loop form factor. At first we present our results in closed form without expansion of the master integrals in  $D$  dimensions. Then we will provide the final result in  $4 - 2\epsilon$  dimensions as usual. We present our results in terms of the expansion coefficients of the bare (unrenormalised) form factor,

$$\mathcal{F}_b^a(\alpha_s^b, s_{12}) = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_s^b}{4\pi} \right)^n \left( \frac{-s_{12}}{\mu^2} \right)^{-n\epsilon} \mathcal{F}_n^a, \quad (6.1)$$

where  $a = q, g$ .

$$\begin{aligned} \mathcal{F}_3^q &= C_F^3 X_{C_F^3}^q + C_F^2 C_A X_{C_F^2 C_A}^q + C_F C_A^2 X_{C_F C_A^2}^q + C_F^2 N_F X_{C_F^2 N_F}^q \\ &\quad + C_F C_A N_F X_{C_F C_A N_F}^q + C_F N_F^2 X_{C_F N_F^2}^q + C_F N_{F,V} \left( \frac{N^2 - 4}{N} \right) X_{C_F N_{F,V}}^q \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_3^g &= C_A^3 X_{C_A^3}^g + C_A^2 N_F X_{C_A^2 N_F}^g + C_A C_F N_F X_{C_A C_F N_F}^g + C_F^2 N_F X_{C_F^2 N_F}^g \\ &\quad + C_A N_F^2 X_{C_A N_F^2}^g + C_F N_F^2 X_{C_F N_F^2}^g \end{aligned} \quad (6.2)$$

with  $s_{12} = (p_1 + p_2)^2$ .

$$\begin{aligned} X_{C_F^3}^q &= \\ &+ \frac{B41}{s_{12}^2} \left( + \frac{489406D^3}{625} - \frac{43304589D^2}{3125} + \frac{615952127D}{7500} + \frac{34015}{4(2D-7)} - \frac{109222498}{75(2D-9)} \right) \end{aligned}$$



$$\begin{aligned}
& + \frac{50720}{9(3D-10)} + \frac{6816654}{11(3D-14)} + \frac{89728}{25(D-2)} - \frac{12581}{12(D-3)} + \frac{6489724}{15(D-4)} \\
& + \frac{19326056092}{7734375(5D-16)} - \frac{7186019918}{78125(5D-18)} + \frac{643118017984}{703125(5D-22)} - \frac{1024}{3(D-2)^2} \\
& - \frac{779}{12(D-3)^2} + \frac{884312}{5(D-4)^2} + \frac{1187096}{15(D-4)^3} + \frac{745376}{15(D-4)^4} + \frac{91648}{5(D-4)^5} \\
& - \frac{53258146831}{562500} \Big) \\
+ \frac{A51}{s_{12}} & \left( + \frac{54568D^3}{625} - \frac{16060301D^2}{9375} + \frac{135964099D}{11250} - \frac{7315}{2(2D-7)} - \frac{59657807}{1300(2D-9)} \right. \\
& - \frac{36208}{27(3D-10)} + \frac{142784}{75(D-2)} - \frac{106}{3(D-3)} + \frac{770008}{15(D-4)} + \frac{3481535536}{3046875(5D-16)} \\
& + \frac{2887120096}{78125(5D-18)} - \frac{32265012416}{234375(5D-22)} + \frac{83104}{3(D-4)^2} + \frac{12800}{(D-4)^3} + \frac{26112}{5(D-4)^4} \\
& \left. - \frac{35332079719}{1687500} \right) \\
- \frac{A52}{s_{12}} & \left( + \frac{87316D^3}{1875} - \frac{8532244D^2}{9375} + \frac{15436454D}{3125} + \frac{418}{(2D-7)} + \frac{3273151}{325(2D-9)} \right. \\
& + \frac{9920}{27(3D-10)} + \frac{2913100}{99(3D-14)} + \frac{400}{27(3D-8)} + \frac{152056}{225(D-2)} - \frac{70952}{15(D-4)} \\
& - \frac{9365062376}{33515625(5D-16)} - \frac{368980436}{78125(5D-18)} - \frac{37325556352}{703125(5D-22)} - \frac{512}{3(D-2)^2} \\
& \left. - \frac{97088}{15(D-4)^2} - \frac{19392}{5(D-4)^3} - \frac{22016}{15(D-4)^4} - \frac{3578943149}{421875} \right) \\
+ \frac{B51}{s_{12}} & \left( + \frac{46827D^3}{5000} - \frac{7169631D^2}{50000} + \frac{221676243D}{100000} + \frac{177975}{64(2D-7)} - \frac{2274503}{130(2D-9)} \right. \\
& + \frac{9728}{15(D-2)} - \frac{151}{2(D-3)} + \frac{44476}{5(D-4)} - \frac{50689072}{3046875(5D-16)} + \frac{648026848}{78125(5D-18)} \\
& \left. + \frac{1055401984}{78125(5D-22)} + \frac{53792}{5(D-4)^2} + \frac{33824}{5(D-4)^3} + \frac{19072}{5(D-4)^4} - \frac{3774996391}{1000000} \right) \\
- \frac{B52}{s_{12}} & \left( + \frac{105167D^3}{1875} - \frac{33225224D^2}{28125} + \frac{792891607D}{84375} - \frac{1654184}{65(2D-9)} - \frac{4912}{81(3D-10)} \right. \\
& + \frac{29696}{15(D-2)} + \frac{150868}{15(D-4)} + \frac{500415488}{3046875(5D-16)} + \frac{470084528}{78125(5D-18)} \\
& \left. + \frac{3580901184}{78125(5D-22)} - \frac{29936}{3(D-4)^2} + \frac{22112}{3(D-4)^3} + \frac{46592}{15(D-4)^4} - \frac{21148347004}{1265625} \right) \\
+ A61 & \left( + \frac{2207D^3}{375} - \frac{7837D^2}{50} + \frac{62616143D}{56250} + \frac{291310}{297(3D-14)} + \frac{2252}{9(D-2)} \right. \\
& + \frac{23}{(D-3)} + \frac{3136}{15(D-4)} + \frac{1553908}{61875(5D-16)} - \frac{5323758}{15625(5D-18)} \\
& \left. - \frac{74762464}{46875(5D-22)} + \frac{496}{5(D-4)^2} + \frac{192}{5(D-4)^3} - \frac{183504334}{84375} \right)
\end{aligned}$$

$$\begin{aligned}
& -A62 \left( +\frac{39857D^3}{3000} - \frac{4628009D^2}{15000} + \frac{22107268D}{9375} - \frac{627}{16(2D-7)} + \frac{235409}{600(2D-9)} \right. \\
& \quad + \frac{98768}{225(D-2)} - \frac{4993}{280(D-3)} + \frac{1024}{15(D-4)} + \frac{36333448}{140625(5D-16)} \\
& \quad \left. + \frac{446887648}{984375(5D-22)} + \frac{1024}{3(D-2)^2} + \frac{58}{3(D-3)^2} + \frac{256}{15(D-4)^2} - \frac{817543919}{150000} \right) \\
& +A63 \left( +\frac{3422D^3}{125} - \frac{2017249D^2}{3750} + \frac{835107683D}{225000} + \frac{1045}{8(2D-7)} + \frac{4880379}{5200(2D-9)} \right. \\
& \quad + \frac{80}{27(3D-8)} + \frac{28736}{25(D-2)} - \frac{161}{4(D-3)} + \frac{4336}{5(D-4)} + \frac{30753004}{203125(5D-16)} \\
& \quad - \frac{1526704}{3125(5D-18)} - \frac{85442816}{15625(5D-22)} + \frac{576}{(D-4)^2} + \frac{3392}{15(D-4)^3} - \frac{10891722217}{1350000} \left. \right) \\
& -B61 \frac{(D^2 - 7D + 16)^3}{(D-4)^3} \\
& +B62 \left( +\frac{7D^3}{8} - \frac{1109D^2}{48} + \frac{29395D}{288} + \frac{8475}{64(2D-7)} + \frac{200}{27(3D-8)} \right. \\
& \quad \left. - \frac{264}{(D-4)} - \frac{152}{(D-4)^2} - \frac{160}{(D-4)^3} - \frac{374753}{1728} \right) \\
& +C61 \left( +\frac{7D^3}{8} - \frac{1109D^2}{48} + \frac{29395D}{288} + \frac{8475}{64(2D-7)} + \frac{200}{27(3D-8)} \right. \\
& \quad \left. - \frac{264}{(D-4)} - \frac{152}{(D-4)^2} - \frac{160}{(D-4)^3} - \frac{374753}{1728} \right) \\
& +s_{12}A71 \left( +\frac{21D^3}{50} - \frac{5907D^2}{500} + \frac{523857D}{5000} - \frac{1213}{12(2D-7)} + \frac{29539}{624(2D-9)} \right. \\
& \quad \left. + \frac{64}{3(D-2)} + \frac{80}{(D-4)} - \frac{655856}{121875(5D-16)} - \frac{388064}{9375(5D-18)} - \frac{2151447}{10000} \right) \\
& -s_{12}A72 \left( +\frac{15D^3}{16} - \frac{733D^2}{32} + \frac{228267D}{1600} + \frac{42745}{288(2D-7)} + \frac{232399}{8320(2D-9)} \right. \\
& \quad \left. + \frac{488}{45(D-2)} - \frac{128}{(D-4)} - \frac{2821088}{14625(5D-16)} + \frac{47928}{125(5D-18)} - \frac{4633049}{16000} \right) \\
& -s_{12}A73 \left( +\frac{601D^3}{1250} - \frac{60199D^2}{6250} + \frac{760189D}{6250} - \frac{1213}{12(2D-7)} + \frac{29539}{2340(2D-9)} \right. \\
& \quad + \frac{496}{(D-2)} + \frac{8329}{210(D-3)} - \frac{909167104}{3046875(5D-16)} + \frac{101225984}{703125(5D-18)} \\
& \quad \left. - \frac{15661504}{546875(5D-22)} + \frac{21}{(D-3)^2} - \frac{8803773}{15625} \right) \\
& +s_{12}A74 \left( +\frac{2489D^3}{5000} - \frac{686707D^2}{50000} + \frac{7042751D}{60000} + \frac{865}{72(2D-7)} + \frac{235409}{2880(2D-9)} \right. \\
& \quad + \frac{556}{45(D-2)} - \frac{4397}{420(D-3)} - \frac{16}{5(D-4)} + \frac{93131696}{703125(5D-16)} \\
& \quad \left. - \frac{9430916}{703125(5D-18)} - \frac{365471216}{4921875(5D-22)} + \frac{1}{3(D-3)^2} - \frac{277480707}{1000000} \right)
\end{aligned}$$

$$\begin{aligned}
& +s_{12}A75 \left( +\frac{2489D^3}{5000} - \frac{686707D^2}{50000} + \frac{7042751D}{60000} + \frac{865}{72(2D-7)} + \frac{235409}{2880(2D-9)} \right. \\
& \quad + \frac{556}{45(D-2)} - \frac{4397}{420(D-3)} - \frac{16}{5(D-4)} + \frac{93131696}{703125(5D-16)} \\
& \quad - \frac{9430916}{703125(5D-18)} - \frac{365471216}{4921875(5D-22)} + \frac{1}{3(D-3)^2} - \left. \frac{277480707}{1000000} \right) \\
& -s_{12}^2A81 \left( +\frac{3411D^3}{80000} - \frac{758793D^2}{800000} + \frac{3243781D}{320000} + \frac{33573}{1024(2D-7)} + \frac{32}{(D-2)} \right. \\
& \quad - \frac{3015}{448(D-3)} + \frac{4}{5(D-4)} + \frac{3411716}{78125(5D-16)} - \frac{8269536}{78125(5D-18)} \\
& \quad - \left. \frac{1425936}{546875(5D-22)} + \frac{4389}{256(2D-7)^2} - \frac{663954073}{16000000} \right) \\
& +s_{12}^2B81 \frac{(D^3 - 20D^2 + 104D - 176)(D^2 - 7D + 16)}{8(2D-7)(D-4)} \\
& +s_{12}^2C81 \frac{(D^3 - 20D^2 + 104D - 176)(D^2 - 7D + 16)}{8(2D-7)(D-4)} \\
& -s_{12}^3A91 \left( +\frac{243D^3}{1250} - \frac{14661D^2}{3125} + \frac{257769D}{6250} + \frac{256}{5(D-2)} - \frac{225}{14(D-3)} - \frac{48}{5(D-4)} \right. \\
& \quad + \frac{4083992}{78125(5D-16)} + \frac{6463488}{78125(5D-18)} + \frac{2127008}{546875(5D-22)} - \left. \frac{4086513}{31250} \right) \\
& +s_{12}^3A92 \frac{3(3D-14)(D^6 - 41D^5 + 661D^4 - 4992D^3 + 19276D^2 - 37104D + 28288)}{10(5D-16)(5D-18)(5D-22)(D-3)} \\
& +s_{12}^3A94 \left( +\frac{567D^3}{80000} - \frac{125091D^2}{800000} + \frac{1808937D}{1600000} + \frac{4067}{2304(2D-7)} + \frac{232399}{998400(2D-9)} \right. \\
& \quad - \frac{16}{75(D-2)} - \frac{225}{448(D-3)} + \frac{8388688}{3046875(5D-16)} - \frac{574016}{234375(5D-18)} \\
& \quad - \left. \frac{7557808}{4921875(5D-22)} - \frac{38866491}{16000000} \right)
\end{aligned}$$

$$X_{C_F^2 C_A}^q =$$

$$\begin{aligned}
& -\frac{B41}{s_{12}^2} \left( +\frac{225717D^3}{250} - \frac{9995657D^2}{625} + \frac{893831341D}{9375} + \frac{45685}{24(2D-7)} - \frac{990312631}{975(2D-9)} \right. \\
& \quad + \frac{116080}{21(3D-10)} + \frac{707967}{(3D-14)} - \frac{371482}{6237(D-1)} + \frac{125032}{25(D-2)} - \frac{875}{4(D-3)} \\
& \quad + \frac{172908907}{405(D-4)} - \frac{5622111978}{2234375(5D-16)} - \frac{345702357}{3125(5D-18)} - \frac{5032168544}{46875(5D-22)} \\
& \quad - \frac{1280}{3(D-2)^2} + \frac{9}{4(D-3)^2} + \frac{22959056}{135(D-4)^2} + \frac{3875224}{45(D-4)^3} + \frac{2622656}{45(D-4)^4} \\
& \quad + \left. \frac{340096}{15(D-4)^5} - \frac{2728978211}{25000} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{A51}{s_{12}} \left( +\frac{71304D^3}{625} - \frac{12785517D^2}{6250} + \frac{156484099D}{12500} - \frac{17815}{3(2D-7)} - \frac{97578701}{2600(2D-9)} \right. \\
& \quad - \frac{73232}{63(3D-10)} + \frac{2467480}{27027(D-1)} + \frac{108736}{75(D-2)} - \frac{1}{(D-3)} + \frac{6422264}{135(D-4)} \\
& \quad + \frac{9290021216}{11171875(5D-16)} + \frac{26729196704}{1015625(5D-18)} - \frac{10782693376}{78125(5D-22)} + \frac{179840}{9(D-4)^2} \\
& \quad \left. + \frac{130816}{15(D-4)^3} + \frac{50176}{15(D-4)^4} - \frac{23118311953}{1125000} \right) \\
& + \frac{A52}{s_{12}} \left( +\frac{25782D^3}{625} - \frac{24183202D^2}{28125} + \frac{437583827D}{84375} + \frac{2036}{3(2D-7)} + \frac{7969573}{650(2D-9)} \right. \\
& \quad + \frac{190048}{567(3D-10)} + \frac{100850}{3(3D-14)} + \frac{16}{(3D-8)} - \frac{2178524}{81081(D-1)} + \frac{71828}{75(D-2)} \\
& \quad - \frac{1092208}{405(D-4)} - \frac{9583273136}{33515625(5D-16)} - \frac{4168502038}{1015625(5D-18)} - \frac{14910736064}{234375(5D-22)} \\
& \quad \left. - \frac{640}{3(D-2)^2} - \frac{523712}{135(D-4)^2} - \frac{90176}{45(D-4)^3} - \frac{35072}{45(D-4)^4} - \frac{21872512759}{2531250} \right) \\
& - \frac{B51}{s_{12}} \left( +\frac{114279D^3}{10000} - \frac{22958287D^2}{100000} + \frac{448177891D}{200000} + \frac{177975}{128(2D-7)} - \frac{326249}{26(2D-9)} \right. \\
& \quad + \frac{6128}{15(D-2)} - \frac{2}{(D-3)} + \frac{35294}{5(D-4)} - \frac{25589872}{3046875(5D-16)} + \frac{831628448}{78125(5D-18)} \\
& \quad \left. - \frac{130364416}{78125(5D-22)} + \frac{7272}{(D-4)^2} + \frac{17696}{5(D-4)^3} + \frac{10624}{5(D-4)^4} - \frac{7313613927}{2000000} \right) \\
& + \frac{B52}{s_{12}} \left( +\frac{44339D^3}{625} - \frac{4405736D^2}{3125} + \frac{287753909D}{28125} - \frac{237272}{13(2D-9)} + \frac{16}{27(3D-10)} \right. \\
& \quad + \frac{25456}{15(D-2)} + \frac{154624}{15(D-4)} - \frac{76140512}{3046875(5D-16)} + \frac{650239528}{78125(5D-18)} \\
& \quad \left. + \frac{1665432384}{78125(5D-22)} + \frac{47624}{5(D-4)^2} + \frac{32416}{5(D-4)^3} + \frac{41984}{15(D-4)^4} - \frac{7361965928}{421875} \right) \\
& - A61 \left( +\frac{4678D^3}{625} - \frac{609667D^2}{3125} + \frac{25888633D}{18750} + \frac{10085}{9(3D-14)} + \frac{545}{54(D-1)} \right. \\
& \quad + \frac{5366}{15(D-2)} + \frac{18}{(D-3)} + \frac{31492}{135(D-4)} + \frac{5042024}{78125(5D-16)} - \frac{23129849}{78125(5D-18)} \\
& \quad \left. - \frac{142238992}{78125(5D-22)} + \frac{1264}{9(D-4)^2} + \frac{512}{15(D-4)^3} - \frac{777739429}{281250} \right) \\
& + A62 \left( +\frac{9671D^3}{1200} - \frac{1150061D^2}{6000} + \frac{27382277D}{18000} - \frac{509}{8(2D-7)} + \frac{5481191}{15600(2D-9)} \right. \\
& \quad + \frac{728}{25(D-2)} - \frac{257261}{1680(D-3)} + \frac{3164}{15(D-4)} + \frac{92725544}{121875(5D-16)} \\
& \quad \left. + \frac{20493136}{65625(5D-22)} + \frac{1280}{3(D-2)^2} + \frac{1}{2(D-3)^2} + \frac{3952}{45(D-4)^2} - \frac{379243601}{112500} \right) \\
& - A63 \left( +\frac{60679D^3}{2500} - \frac{33322501D^2}{75000} + \frac{436696447D}{150000} + \frac{2545}{12(2D-7)} + \frac{14641137}{10400(2D-9)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{16}{5(3D-8)} - \frac{250462}{19305(D-1)} + \frac{37352}{75(D-2)} - \frac{1}{4(D-3)} + \frac{106784}{135(D-4)} \\
& + \frac{3533209912}{33515625(5D-16)} - \frac{433927224}{1015625(5D-18)} - \frac{487676544}{78125(5D-22)} + \frac{27568}{45(D-4)^2} \\
& + \frac{640}{3(D-4)^3} - \frac{2875843347}{500000} \Big) \\
+ B62 & \left( + \frac{D^3}{16} + \frac{71D^2}{32} - \frac{283D}{64} - \frac{8475}{128(2D-7)} - \frac{8}{(3D-8)} \right. \\
& \left. - \frac{46}{9(D-1)} + \frac{1000}{9(D-4)} + \frac{80}{3(D-4)^2} + \frac{64}{(D-4)^3} + \frac{1587}{128} \right) \\
+ C61 & \left( + \frac{D^3}{16} + \frac{71D^2}{32} - \frac{283D}{64} - \frac{8475}{128(2D-7)} - \frac{8}{(3D-8)} \right. \\
& \left. - \frac{46}{9(D-1)} + \frac{1000}{9(D-4)} + \frac{80}{3(D-4)^2} + \frac{64}{(D-4)^3} + \frac{1587}{128} \right) \\
- s_{12}A71 & \left( + \frac{27D^3}{50} - \frac{7009D^2}{500} + \frac{584559D}{5000} - \frac{301}{6(2D-7)} + \frac{21185}{624(2D-9)} \right. \\
& \left. + \frac{32}{(D-2)} + \frac{80}{(D-4)} - \frac{6069872}{121875(5D-16)} - \frac{827368}{9375(5D-18)} - \frac{2492249}{10000} \right) \\
+ s_{12}A72 & \left( + \frac{517D^3}{800} - \frac{142459D^2}{8000} + \frac{9023129D}{80000} + \frac{42745}{192(2D-7)} + \frac{697197}{16640(2D-9)} \right. \\
& \left. - \frac{1124}{143(D-1)} - \frac{76}{15(D-2)} - \frac{112}{(D-4)} - \frac{106444064}{446875(5D-16)} + \frac{9644612}{40625(5D-18)} \right. \\
& \left. - \frac{33226167}{160000} \right) \\
+ s_{12}A73 & \left( + \frac{601D^3}{1250} - \frac{29899D^2}{6250} + \frac{152417D}{3125} - \frac{301}{6(2D-7)} + \frac{4237}{468(2D-9)} \right. \\
& \left. + \frac{544}{3(D-2)} + \frac{3683}{210(D-3)} + \frac{10424832}{1015625(5D-16)} + \frac{54045184}{703125(5D-18)} \right. \\
& \left. - \frac{12403904}{546875(5D-22)} - \frac{3}{(D-3)^2} - \frac{15077947}{62500} \right) \\
- s_{12}A74 & \left( + \frac{19D^3}{80} - \frac{458683D^2}{60000} + \frac{40603349D}{600000} - \frac{235}{32(2D-7)} + \frac{5481191}{74880(2D-9)} \right. \\
& \left. + \frac{118}{15(D-2)} - \frac{26393}{840(D-3)} - \frac{24}{5(D-4)} + \frac{46026288}{203125(5D-16)} \right. \\
& \left. - \frac{501158}{140625(5D-18)} - \frac{21760904}{328125(5D-22)} - \frac{62067409}{400000} \right) \\
- s_{12}A75 & \left( + \frac{19D^3}{80} - \frac{458683D^2}{60000} + \frac{40603349D}{600000} - \frac{235}{32(2D-7)} + \frac{5481191}{74880(2D-9)} \right. \\
& \left. + \frac{118}{15(D-2)} - \frac{26393}{840(D-3)} - \frac{24}{5(D-4)} + \frac{46026288}{203125(5D-16)} \right. \\
& \left. - \frac{501158}{140625(5D-18)} - \frac{21760904}{328125(5D-22)} - \frac{62067409}{400000} \right)
\end{aligned}$$

$$\begin{aligned}
& +s_{12}^2 A81 \left( +\frac{1197D^3}{160000} - \frac{979611D^2}{1600000} + \frac{8338443D}{640000} + \frac{29027}{2048(2D-7)} + \frac{160}{3(D-2)} \right. \\
& \quad - \frac{13665}{896(D-3)} - \frac{34}{5(D-4)} + \frac{23109548}{234375(5D-16)} - \frac{3147936}{78125(5D-18)} \\
& \quad \left. - \frac{1018736}{546875(5D-22)} + \frac{3563}{128(2D-7)^2} - \frac{2065843091}{32000000} \right) \\
& -s_{12}^2 B81 \frac{(D^3 - 20D^2 + 104D - 176)(D^2 - 7D + 16)}{16(2D-7)(D-4)} \\
& -s_{12}^2 C81 \frac{(D^3 - 20D^2 + 104D - 176)(D^2 - 7D + 16)}{16(2D-7)(D-4)} \\
& +s_{12}^3 A91 \left( +\frac{243D^3}{1250} - \frac{14661D^2}{3125} + \frac{257769D}{6250} + \frac{256}{5(D-2)} - \frac{225}{14(D-3)} \right. \\
& \quad - \frac{48}{5(D-4)} + \frac{4083992}{78125(5D-16)} + \frac{6463488}{78125(5D-18)} \\
& \quad \left. + \frac{2127008}{546875(5D-22)} - \frac{4086513}{31250} \right) \\
& -s_{12}^3 A92 \frac{(3D-14)(3D^6 - 108D^5 + 1586D^4 - 11304D^3 + 41928D^2 - 78208D + 57984)}{10(5D-16)(5D-18)(5D-22)(D-3)} \\
& -s_{12}^3 A94 \left( +\frac{1701D^3}{160000} - \frac{375273D^2}{1600000} + \frac{5426811D}{3200000} + \frac{4067}{1536(2D-7)} + \frac{232399}{665600(2D-9)} \right. \\
& \quad - \frac{8}{25(D-2)} - \frac{675}{896(D-3)} + \frac{4194344}{1015625(5D-16)} - \frac{287008}{78125(5D-18)} \\
& \quad \left. - \frac{3778904}{1640625(5D-22)} - \frac{116599473}{32000000} \right)
\end{aligned}$$

$$X_{C_F C_A^2}^q =$$

$$\begin{aligned}
& +\frac{B41}{s_{12}^2} \left( +\frac{153701D^3}{625} - \frac{45111262D^2}{9375} + \frac{3307905503D}{112500} - \frac{7045}{6(2D-7)} - \frac{140183197}{975(2D-9)} \right. \\
& \quad - \frac{2060}{27(3D-10)} + \frac{9383166}{55(3D-14)} - \frac{165455}{2673(D-1)} + \frac{44164}{25(D-2)} + \frac{659}{12(D-3)} \\
& \quad + \frac{416301857}{4860(D-4)} - \frac{138263099401}{402187500(5D-16)} - \frac{5093619454}{234375(5D-18)} - \frac{144904142656}{703125(5D-22)} \\
& \quad - \frac{128}{(D-2)^2} + \frac{143}{12(D-3)^2} + \frac{8793673}{405(D-4)^2} + \frac{915068}{135(D-4)^3} + \frac{345128}{45(D-4)^4} \\
& \quad \left. + \frac{20864}{5(D-4)^5} - \frac{31855488829}{843750} \right) \\
& +\frac{A51}{s_{12}} \left( +\frac{4277D^3}{125} - \frac{243647D^2}{375} + \frac{106547887D}{28125} - \frac{49315}{24(2D-7)} - \frac{18960447}{2600(2D-9)} \right. \\
& \quad + \frac{357584}{1323(3D-10)} + \frac{7618840}{567567(D-1)} + \frac{10064}{25(D-2)} + \frac{10}{3(D-3)} + \frac{3703898}{405(D-4)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{332977894}{1340625(5D-16)} + \frac{563608248}{203125(5D-18)} - \frac{1415791832}{46875(5D-22)} + \frac{3488}{189(D-1)^2} \\
& + \frac{81212}{45(D-4)^2} + \frac{11584}{45(D-4)^3} + \frac{256}{3(D-4)^4} - \frac{452116231}{67500} \Big) \\
- \frac{A52}{s_{12}} & \left( + \frac{20594D^3}{1875} - \frac{6630308D^2}{28125} + \frac{25700042D}{16875} + \frac{1409}{6(2D-7)} + \frac{2348211}{650(2D-9)} \right. \\
& - \frac{11944}{567(3D-10)} + \frac{801980}{99(3D-14)} - \frac{10360820}{243243(D-1)} + \frac{75128}{225(D-2)} - \frac{559813}{1215(D-4)} \\
& - \frac{1022764469}{33515625(5D-16)} - \frac{1052157372}{1015625(5D-18)} - \frac{11705152928}{703125(5D-22)} - \frac{64}{(D-2)^2} \\
& \left. - \frac{306428}{405(D-4)^2} - \frac{43696}{135(D-4)^3} - \frac{1408}{15(D-4)^4} - \frac{3344023858}{1265625} \right) \\
+ \frac{B51}{s_{12}} & \left( + \frac{7497D^3}{1250} - \frac{394839D^2}{3125} + \frac{10983001D}{12500} - \frac{75999}{40(2D-9)} + \frac{1424}{15(D-2)} \right. \\
& - \frac{1}{(D-3)} + \frac{1055}{(D-4)} - \frac{712201}{234375(5D-16)} + \frac{200430972}{78125(5D-18)} \\
& - \frac{146129984}{78125(5D-22)} + \frac{4176}{5(D-4)^2} + \frac{296}{(D-4)^3} + \frac{1152}{5(D-4)^4} - \frac{189456643}{125000} \Big) \\
- \frac{B52}{s_{12}} & \left( + \frac{13666D^3}{625} - \frac{4406452D^2}{9375} + \frac{92206756D}{28125} - \frac{13818}{5(2D-9)} + \frac{320}{27(3D-10)} \right. \\
& + \frac{2536}{5(D-2)} + \frac{10325}{6(D-4)} - \frac{3748279}{156250(5D-16)} + \frac{169244232}{78125(5D-18)} + \frac{122152296}{78125(5D-22)} \\
& + \frac{22774}{15(D-4)^2} + \frac{15736}{15(D-4)^3} + \frac{7552}{15(D-4)^4} - \frac{2504444962}{421875} \Big) \\
+ A61 & \left( + \frac{17033D^3}{7500} - \frac{1493603D^2}{25000} + \frac{95481487D}{225000} + \frac{80198}{297(3D-14)} + \frac{1711}{288(D-1)} \right. \\
& + \frac{1090}{9(D-2)} + \frac{113}{32(D-3)} + \frac{479}{18(D-4)} + \frac{294544271}{15468750(5D-16)} - \frac{3011532}{78125(5D-18)} \\
& - \frac{104077288}{234375(5D-22)} + \frac{1853}{144(D-1)^2} + \frac{136}{9(D-4)^2} - \frac{64}{15(D-4)^3} - \frac{1505097247}{1687500} \Big) \\
- A62 & \left( + \frac{4249D^3}{6000} - \frac{322949D^2}{15000} + \frac{61126331D}{300000} - \frac{1409}{64(2D-7)} + \frac{403479}{5200(2D-9)} \right. \\
& - \frac{872}{297(D-1)} + \frac{5584}{225(D-2)} - \frac{117431}{1680(D-3)} + \frac{9704}{135(D-4)} + \frac{5192329489}{20109375(5D-16)} \\
& + \frac{41976608}{984375(5D-22)} + \frac{128}{(D-2)^2} - \frac{25}{12(D-3)^2} + \frac{176}{5(D-4)^2} - \frac{846754451}{1800000} \Big) \\
+ A63 & \left( + \frac{3073D^3}{625} - \frac{831416D^2}{9375} + \frac{41933917D}{75000} + \frac{7045}{96(2D-7)} + \frac{4880379}{10400(2D-9)} \right. \\
& - \frac{13867}{429(D-1)} + \frac{9004}{75(D-2)} + \frac{13}{4(D-3)} + \frac{293}{30(D-4)} + \frac{1849361647}{67031250(5D-16)} \\
& - \frac{49983292}{1015625(5D-18)} - \frac{118945472}{78125(5D-22)} + \frac{236}{5(D-4)^2} + \frac{96}{5(D-4)^3} - \frac{51691069}{46875} \Big)
\end{aligned}$$

$$\begin{aligned}
& +s_{12}A71 \left( +\frac{33D^3}{200} - \frac{8111D^2}{2000} + \frac{645261D}{20000} + \frac{3}{16(2D-7)} + \frac{329}{64(2D-9)} \right. \\
& \quad \left. + \frac{32}{3(D-2)} + \frac{20}{(D-4)} - \frac{220844}{9375(5D-16)} - \frac{105556}{3125(5D-18)} - \frac{2833051}{40000} \right) \\
& -s_{12}A72 \left( +\frac{71D^3}{800} - \frac{25417D^2}{8000} + \frac{1658227D}{80000} + \frac{42745}{576(2D-7)} + \frac{232399}{16640(2D-9)} \right. \\
& \quad - \frac{562}{143(D-1)} - \frac{236}{45(D-2)} - \frac{24}{(D-4)} - \frac{285048488}{4021875(5D-16)} + \frac{928156}{40625(5D-18)} \\
& \quad \left. - \frac{5030461}{160000} \right) \\
& +s_{12}A73 \left( +\frac{3D^3}{625} - \frac{2023D^2}{12500} - \frac{41819D}{12500} - \frac{3}{16(2D-7)} - \frac{329}{240(2D-9)} \right. \\
& \quad - \frac{44}{3(D-2)} - \frac{263}{105(D-3)} - \frac{2204608}{234375(5D-16)} - \frac{33404}{78125(5D-18)} \\
& \quad \left. + \frac{2035336}{546875(5D-22)} + \frac{7}{4(D-3)^2} + \frac{421802}{15625} \right) \\
& -s_{12}A74 \left( +\frac{57D^3}{10000} + \frac{116647D^2}{300000} - \frac{2694797D}{600000} + \frac{3845}{576(2D-7)} - \frac{134493}{8320(2D-9)} \right. \\
& \quad - \frac{38}{45(D-2)} + \frac{1833}{140(D-3)} + \frac{8}{5(D-4)} - \frac{732913468}{9140625(5D-16)} \\
& \quad - \frac{368278}{234375(5D-18)} + \frac{71838976}{4921875(5D-22)} + \frac{1}{12(D-3)^2} + \frac{16428169}{2000000} \left. \right) \\
& -s_{12}A75 \left( +\frac{57D^3}{10000} + \frac{116647D^2}{300000} - \frac{2694797D}{600000} + \frac{3845}{576(2D-7)} - \frac{134493}{8320(2D-9)} \right. \\
& \quad - \frac{38}{45(D-2)} + \frac{1833}{140(D-3)} + \frac{8}{5(D-4)} - \frac{732913468}{9140625(5D-16)} \\
& \quad - \frac{368278}{234375(5D-18)} + \frac{71838976}{4921875(5D-22)} + \frac{1}{12(D-3)^2} + \frac{16428169}{2000000} \left. \right) \\
& +s_{12}^2A81 \left( +\frac{1107D^3}{160000} + \frac{110409D^2}{1600000} - \frac{2547331D}{640000} + \frac{2273}{2048(2D-7)} - \frac{56}{3(D-2)} \right. \\
& \quad + \frac{5325}{896(D-3)} + \frac{18}{5(D-4)} - \frac{8995987}{234375(5D-16)} - \frac{493416}{78125(5D-18)} \\
& \quad \left. + \frac{152884}{546875(5D-22)} - \frac{9863}{1024(2D-7)^2} + \frac{700944509}{32000000} \right) \\
& -s_{12}^3A91 \left( +\frac{243D^3}{5000} - \frac{62289D^2}{50000} + \frac{283527D}{25000} + \frac{88}{5(D-2)} - \frac{285}{56(D-3)} \right. \\
& \quad - \frac{2}{(D-4)} + \frac{1549163}{78125(5D-16)} + \frac{1250592}{78125(5D-18)} + \frac{406132}{546875(5D-22)} - \frac{4692843}{125000} \left. \right) \\
& +s_{12}^3A92 \frac{(3D-14)(3D^6-93D^5+1189D^4-7632D^3+26028D^2-45104D+31104)}{40(5D-16)(5D-18)(5D-22)(D-3)} \\
& +s_{12}^3A94 \left( +\frac{567D^3}{160000} - \frac{125091D^2}{1600000} + \frac{1808937D}{3200000} + \frac{4067}{4608(2D-7)} + \frac{232399}{1996800(2D-9)} \right)
\end{aligned}$$



$$\begin{aligned} & -\frac{8}{75(D-2)} - \frac{225}{896(D-3)} + \frac{4194344}{3046875(5D-16)} - \frac{287008}{234375(5D-18)} \\ & - \left( \frac{3778904}{4921875(5D-22)} - \frac{38866491}{32000000} \right) \end{aligned}$$

$$X_{C_F^2 N_F}^q =$$

$$\begin{aligned} & -\frac{B41}{s_{12}^2} \left( +\frac{72D^2}{5} - \frac{11524D}{75} - \frac{37120}{63(3D-10)} - \frac{742964}{6237(D-1)} + \frac{4}{(D-3)} \right. \\ & \quad + \frac{10576}{81(D-4)} + \frac{319872}{1375(5D-16)} - \frac{3088}{27(D-4)^2} + \frac{1024}{9(D-4)^3} + \frac{256}{3(D-4)^4} \\ & \quad \left. + \frac{412948}{1125} \right) \\ & + \frac{A51}{s_{12}} \left( +\frac{1216D^2}{75} - \frac{83936D}{375} - \frac{512}{3(3D-10)} - \frac{2293120}{11583(D-1)} - \frac{19648}{81(D-4)} \right. \\ & \quad + \frac{297024}{6875(5D-16)} + \frac{3698688}{8125(5D-18)} - \frac{6272}{27(D-4)^2} - \frac{1024}{9(D-4)^3} + \frac{983968}{1875} \left. \right) \\ & + \frac{A52}{s_{12}} \left( +\frac{4D^2}{75} - \frac{78712D}{1125} - \frac{6656}{189(3D-10)} - \frac{32}{9(3D-8)} - \frac{4357048}{81081(D-1)} \right. \\ & \quad - \frac{20464}{81(D-4)} + \frac{131376}{6875(5D-16)} + \frac{1585152}{8125(5D-18)} - \frac{5312}{27(D-4)^2} - \frac{1024}{9(D-4)^3} \\ & \quad \left. + \frac{2009608}{16875} \right) \\ & + A61 \frac{(D^2 - 7D + 16)(6D^3 - 65D^2 + 238D - 288)(D-2)}{2(D-3)(D-1)(D-4)^2} \\ & - A63 \left( +\frac{28D^2}{25} - \frac{2868D}{125} - \frac{32}{45(3D-8)} - \frac{500924}{19305(D-1)} - \frac{400}{27(D-4)} \right. \\ & \quad \left. - \frac{39984}{6875(5D-16)} - \frac{176128}{8125(5D-18)} - \frac{128}{9(D-4)^2} + \frac{334516}{5625} \right) \\ & + B62 \frac{(D^2 - 7D + 16)(3D^3 - 31D^2 + 110D - 128)(D-2)}{(D-1)(3D-8)(D-4)^2} \\ & + C61 \frac{(D^2 - 7D + 16)(3D^3 - 31D^2 + 110D - 128)(D-2)}{(D-1)(3D-8)(D-4)^2} \\ & + s_{12} A72 \frac{8(D-2)(D^4 - 28D^3 + 220D^2 - 696D + 784)}{(5D-18)(D-1)(5D-16)} \end{aligned}$$

$$X_{C_F C_A N_F}^q =$$

$$-\frac{B41}{s_{12}^2} \left( +\frac{24D^2}{5} - \frac{396D}{25} - \frac{250}{3(3D-10)} + \frac{330910}{2673(D-1)} - \frac{170}{3(D-2)} \right)$$

$$\begin{aligned}
& + \frac{34408}{243(D-4)} + \frac{92064}{1375(5D-16)} + \frac{32}{3(D-2)^2} + \frac{14624}{81(D-4)^2} + \frac{3904}{27(D-4)^3} \\
& + \frac{256}{9(D-4)^4} + \frac{4364}{125} \Big) \\
-A51 & \left( \frac{208D^2}{75} - \frac{3656D}{125} + \frac{25352}{441(3D-10)} - \frac{4990592}{63063(D-1)} + \frac{184}{3(D-2)} \right. \\
& - \frac{992}{9(D-4)} + \frac{148512}{6875(5D-16)} + \frac{1849344}{8125(5D-18)} - \frac{6976}{189(D-1)^2} - \frac{16}{(D-2)^2} \\
& \left. + \frac{256}{27(D-4)^2} - \frac{512}{9(D-4)^3} + \frac{205952}{5625} \right) \\
-A52 & \left( \frac{84D^2}{25} - \frac{6984D}{125} + \frac{460}{63(3D-10)} - \frac{20721640}{243243(D-1)} + \frac{20}{(D-2)} \right. \\
& - \frac{24568}{243(D-4)} - \frac{24312}{6875(5D-16)} + \frac{792576}{8125(5D-18)} - \frac{16}{3(D-2)^2} - \frac{5024}{81(D-4)^2} \\
& \left. - \frac{1024}{27(D-4)^3} + \frac{700768}{5625} \right) \\
-A61 & \left( + \frac{3D^2}{2} - \frac{47D}{4} - \frac{3073}{72(D-1)} + \frac{86}{3(D-2)} + \frac{11}{8(D-3)} \right. \\
& \left. + \frac{8}{9(D-4)} - \frac{109}{4(D-1)^2} - \frac{8}{(D-2)^2} + \frac{16}{3(D-4)^2} + \frac{133}{4} \right) \\
-A62 & \left( \frac{90D^7 - 1803D^6 + 15301D^5 - 70848D^4}{9(5D-16)(D-3)(D-2)^2(D-1)(D-4)} \right. \\
& \left. + \frac{191676D^3 - 299024D^2 + 242976D - 74880}{9(5D-16)(D-3)(D-2)^2(D-1)(D-4)} \right) \\
+A63 & \frac{42D^7 - 656D^6 + 3854D^5 - 11430D^4 + 24896D^3 - 65144D^2 + 134560D - 113856}{3(D-1)(D-2)^2(5D-18)(5D-16)} \\
-s_{12}A72 & \frac{4(D-2)(D^4 - 28D^3 + 220D^2 - 696D + 784)}{(5D-18)(D-1)(5D-16)}
\end{aligned}$$

$$\begin{aligned}
X_{C_F N_F^2}^q & = \\
& -A61 \frac{(6D^3 - 65D^2 + 238D - 288)(D-2)^2}{6(D-4)(D-3)(D-1)^2}
\end{aligned}$$

$$\begin{aligned}
X_{C_F N_{F,V}}^q & = \\
& -\frac{B41}{s_{12}^2} \left( + \frac{119132D}{625} + \frac{380}{9(2D-7)} - \frac{50245888}{975(2D-9)} + \frac{280}{3(3D-10)} + \frac{9088443}{242(3D-14)} \right. \\
& + \frac{9269061200}{722007(D-1)} - \frac{4169543}{675(D-2)} + \frac{182}{3(D-3)} + \frac{1194157}{54(D-4)} - \frac{57818921783}{265443750(5D-16)} \\
& \left. + \frac{808885693}{243750(5D-18)} + \frac{221659776}{53125(5D-22)} + \frac{15748}{9(D-2)^2} + \frac{35}{3(D-3)^2} + \frac{750554}{45(D-4)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{608}{3(D-2)^3} + \frac{25148}{3(D-4)^3} + \frac{2048}{(D-4)^4} + \frac{12730856}{9375} \Big) \\
& -\frac{A51}{s_{12}} \left( +\frac{12243D}{625} + \frac{665}{9(2D-7)} - \frac{448624}{325(2D-9)} - \frac{56}{3(3D-10)} + \frac{92219200}{65637(D-1)} \right. \\
& \quad -\frac{83479}{225(D-2)} + \frac{10}{3(D-3)} + \frac{148888}{135(D-4)} - \frac{14475728}{446875(5D-16)} - \frac{29074423}{40625(5D-18)} \\
& \quad -\frac{292095986}{53125(5D-22)} - \frac{1496}{3(D-2)^2} - \frac{14608}{45(D-4)^2} + \frac{256}{(D-2)^3} - \frac{2048}{15(D-4)^3} \\
& \quad \left. -\frac{2436926}{9375} \right) \\
& +\frac{A52}{s_{12}} \left( +\frac{14644D}{1875} - \frac{76}{9(2D-7)} + \frac{40784}{325(2D-9)} + \frac{56}{9(3D-10)} + \frac{215775}{121(3D-14)} \right. \\
& \quad +\frac{4075274240}{2166021(D-1)} - \frac{1082473}{675(D-2)} + \frac{34666}{405(D-4)} - \frac{176454986}{132721875(5D-16)} \\
& \quad -\frac{831538}{40625(5D-18)} - \frac{99059576}{53125(5D-22)} + \frac{8024}{9(D-2)^2} + \frac{17912}{135(D-4)^2} - \frac{640}{3(D-2)^3} \\
& \quad \left. +\frac{2176}{45(D-4)^3} + \frac{2370412}{28125} \right) \\
& -\frac{B51}{s_{12}} \left( +\frac{3474D}{625} - \frac{196273}{325(2D-9)} - \frac{25123840}{65637(D-1)} + \frac{144542}{225(D-2)} - \frac{2}{(D-3)} \right. \\
& \quad +\frac{93338}{135(D-4)} + \frac{1437614}{4021875(5D-16)} - \frac{9370564}{40625(5D-18)} - \frac{39796848}{53125(5D-22)} \\
& \quad \left. -\frac{1984}{3(D-2)^2} + \frac{2900}{9(D-4)^2} + \frac{224}{(D-2)^3} + \frac{1664}{15(D-4)^3} + \frac{190269}{3125} \right) \\
& +\frac{B52}{s_{12}} \left( +\frac{37136D}{1875} - \frac{285488}{325(2D-9)} + \frac{80}{9(3D-10)} + \frac{28564480}{65637(D-1)} + \frac{163559}{450(D-2)} \right. \\
& \quad +\frac{150403}{135(D-4)} - \frac{25678283}{4021875(5D-16)} - \frac{13477879}{81250(5D-18)} + \frac{10007466}{53125(5D-22)} \\
& \quad \left. -\frac{2972}{3(D-2)^2} + \frac{32612}{45(D-4)^2} + \frac{480}{(D-2)^3} + \frac{3328}{15(D-4)^3} + \frac{4577488}{28125} \right) \\
& +A61 \left( +\frac{49D}{625} - \frac{14385}{242(3D-14)} - \frac{58221088}{722007(D-1)} + \frac{931}{18(D-2)} + \frac{1}{2(D-3)} \right. \\
& \quad +\frac{418}{135(D-4)} - \frac{3759722}{3403125(5D-16)} - \frac{432134}{40625(5D-18)} + \frac{3537842}{53125(5D-22)} \\
& \quad \left. -\frac{16}{(D-2)^2} - \frac{64}{45(D-4)^2} + \frac{100399}{18750} \right) \\
& +A62 \left( +\frac{17533D}{3000} + \frac{19}{24(2D-7)} + \frac{10196}{975(2D-9)} + \frac{1382570}{1683(D-1)} - \frac{411274}{675(D-2)} \right. \\
& \quad -\frac{1623}{56(D-3)} + \frac{1088}{45(D-4)} + \frac{18324568}{2413125(5D-16)} + \frac{1024224}{74375(5D-22)} + \frac{3680}{9(D-2)^2} \\
& \quad \left. -\frac{11}{6(D-3)^2} - \frac{256}{3(D-2)^3} + \frac{42883}{1875} \right)
\end{aligned}$$

$$\begin{aligned}
& -A_{63} \left( +\frac{26253D}{2500} - \frac{95}{36(2D-7)} + \frac{41793584}{21879(D-1)} - \frac{13967}{9(D-2)} + \frac{3}{(D-3)} \right. \\
& \quad + \frac{3548}{45(D-4)} - \frac{359662}{103125(5D-16)} + \frac{883413}{40625(5D-18)} - \frac{12129744}{53125(5D-22)} \\
& \quad \left. + \frac{2048}{3(D-2)^2} + \frac{352}{15(D-4)^2} - \frac{64}{(D-2)^3} - \frac{243387}{12500} \right) \\
& -s_{12}A_{71} \left( +\frac{9D}{100} + \frac{37}{18(2D-7)} + \frac{2549}{1560(2D-9)} + \frac{280}{143(D-1)} + \frac{37}{90(D-2)} \right. \\
& \quad \left. - \frac{144076}{160875(5D-16)} + \frac{6313}{9750(5D-18)} + \frac{4}{(D-2)^2} + \frac{3927}{1000} \right) \\
& +s_{12}A_{72} \frac{5D^6 - 93D^5 + 892D^4 - 4656D^3 + 12528D^2 - 15472D + 6080}{(D-1)(D-2)^2(5D-18)(5D-16)} \\
& +s_{12}A_{73} \left( +\frac{211D}{625} + \frac{37}{18(2D-7)} + \frac{2549}{5850(2D-9)} - \frac{5424}{187(D-1)} + \frac{2566}{75(D-2)} \right. \\
& \quad + \frac{2}{3(D-3)} + \frac{26995456}{4021875(5D-16)} - \frac{79688}{28125(5D-18)} - \frac{56582}{53125(5D-22)} \\
& \quad \left. - \frac{328}{3(D-2)^2} - \frac{2}{(D-3)^2} + \frac{128}{(D-2)^3} + \frac{4576}{3125} \right) \\
& -s_{12}A_{74} \left( +\frac{162D}{625} - \frac{37}{72(2D-7)} + \frac{2549}{1170(2D-9)} - \frac{22736}{2431(D-1)} + \frac{754}{45(D-2)} \right. \\
& \quad - \frac{323}{84(D-3)} - \frac{2696828}{4021875(5D-16)} + \frac{97442}{365625(5D-18)} - \frac{1877744}{1115625(5D-22)} \\
& \quad \left. - \frac{8}{(D-2)^2} - \frac{1}{12(D-3)^2} + \frac{273763}{75000} \right) \\
& -s_{12}A_{75} \left( +\frac{162D}{625} - \frac{37}{72(2D-7)} + \frac{2549}{1170(2D-9)} - \frac{22736}{2431(D-1)} + \frac{754}{45(D-2)} \right. \\
& \quad - \frac{323}{84(D-3)} - \frac{2696828}{4021875(5D-16)} + \frac{97442}{365625(5D-18)} - \frac{1877744}{1115625(5D-22)} \\
& \quad \left. - \frac{8}{(D-2)^2} - \frac{1}{12(D-3)^2} + \frac{273763}{75000} \right) \\
& -s_{12}^2A_{81} \left( +\frac{1107D}{80000} + \frac{1643}{768(2D-7)} - \frac{8085}{884(D-1)} + \frac{349}{45(D-2)} + \frac{1305}{448(D-3)} \right. \\
& \quad + \frac{22363}{28125(5D-16)} - \frac{77484}{40625(5D-18)} + \frac{85352}{1115625(5D-22)} + \frac{133}{384(2D-7)^2} \\
& \quad \left. - \frac{8}{(D-2)^2} - \frac{1511991}{800000} \right) \\
& +s_{12}^3A_{91} \left( +\frac{243D}{5000} + \frac{24255}{884(D-1)} - \frac{24}{(D-2)} - \frac{75}{56(D-3)} - \frac{10076}{9375(5D-16)} \right. \\
& \quad \left. - \frac{83136}{40625(5D-18)} + \frac{252703}{1115625(5D-22)} + \frac{16}{(D-2)^2} + \frac{5103}{25000} \right) \\
& -s_{12}^3A_{92} \frac{3(3D-14)(D-4)(D^5 + 25D^4 - 290D^3 + 1036D^2 - 1560D + 928)}{20(D-1)(D-2)(D-3)(5D-22)(5D-18)(5D-16)}
\end{aligned}$$

$$\begin{aligned}
X_{C_A^3}^g = & \\
& -\frac{B41}{s_{12}^2} \left( +282D^2 - \frac{2592693D}{125} - \frac{4480}{D} - \frac{447830}{63(2D-7)} + \frac{46485469184}{16575(2D-9)} \right. \\
& + \frac{106048}{21(3D-10)} - \frac{35562618}{55(3D-14)} + \frac{128}{5(3D-8)} - \frac{37565622628}{1216215(D-1)} + \frac{8017184}{225(D-2)} \\
& + \frac{1120}{3(D-3)} - \frac{635092036}{1215(D-4)} + \frac{1440}{(D-6)} - \frac{29989344832}{30121875(5D-14)} \\
& + \frac{5361643396}{4021875(5D-16)} - \frac{6939682834}{121875(5D-18)} - \frac{10119933824}{3125(5D-22)} - \frac{55039}{108(D-1)^2} \\
& - \frac{102944}{5(D-2)^2} - \frac{1787}{4(D-3)^2} - \frac{130962128}{405(D-4)^2} + \frac{3328}{(D-2)^3} - \frac{21706624}{135(D-4)^3} \\
& \left. - \frac{359296}{9(D-4)^4} + \frac{2048}{3(D-4)^5} + \frac{600552286}{9375} \right) \\
& + \frac{A51}{s_{12}} \left( +\frac{2114656D}{375} + \frac{608}{5(2D-5)} + \frac{13520}{9(2D-7)} - \frac{415048832}{5525(2D-9)} - \frac{256192}{2205(3D-10)} \right. \\
& - \frac{6487807808}{567567(D-1)} + \frac{8756344}{225(D-2)} + \frac{32}{(D-3)} + \frac{5951984}{81(D-4)} + \frac{5760}{7(D-6)} \\
& + \frac{262210816}{4303125(5D-14)} + \frac{7942807088}{28153125(5D-16)} + \frac{776706728}{40625(5D-18)} \\
& - \frac{561369536}{3125(5D-22)} - \frac{22016}{189(D-1)^2} - \frac{95808}{5(D-2)^2} + \frac{160064}{3(D-4)^2} \\
& \left. + \frac{3072}{(D-2)^3} + \frac{1263488}{45(D-4)^3} + \frac{23552}{3(D-4)^4} - \frac{993704848}{28125} \right) \\
& + \frac{A52}{s_{12}} \left( +\frac{1594502D}{1125} - \frac{576}{D} - \frac{37696}{1485(2D-5)} + \frac{10816}{63(2D-7)} - \frac{37731712}{5525(2D-9)} \right. \\
& - \frac{136352}{945(3D-10)} - \frac{1013180}{33(3D-14)} + \frac{448}{15(3D-8)} - \frac{1468089542}{405405(D-1)} + \frac{4190624}{225(D-2)} \\
& + \frac{8039792}{1215(D-4)} + \frac{2304}{7(D-6)} + \frac{3429250816}{90365625(5D-14)} + \frac{7639927408}{28153125(5D-16)} \\
& + \frac{1610174348}{446875(5D-18)} + \frac{141383424}{3125(5D-22)} - \frac{994}{27(D-1)^2} - \frac{12736}{(D-2)^2} + \frac{5071552}{405(D-4)^2} \\
& \left. + \frac{3072}{(D-2)^3} + \frac{1332992}{135(D-4)^3} + \frac{28160}{9(D-4)^4} - \frac{902670584}{84375} \right) \\
& + \frac{B51}{s_{12}} \left( +204D^2 - \frac{495373D}{250} + \frac{5681}{72(2D-5)} + \frac{13125}{8(2D-7)} - \frac{181583864}{5525(2D-9)} \right. \\
& - \frac{634790}{81(D-1)} + \frac{943816}{75(D-2)} + \frac{10}{(D-3)} + \frac{3438808}{135(D-4)} + \frac{206859136}{4303125(5D-14)} \\
& - \frac{415744}{40625(5D-16)} - \frac{12552904}{3125(5D-18)} + \frac{407493632}{9375(5D-22)} - \frac{36224}{5(D-2)^2} \\
& \left. + \frac{188464}{5(D-4)^2} + \frac{1280}{(D-2)^3} + \frac{81856}{3(D-4)^3} + \frac{7680}{(D-4)^4} + \frac{14613393}{3125} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{B52}{s_{12}} \left( +\frac{437838D}{125} + \frac{3078304}{10395(2D-5)} - \frac{264121984}{5525(2D-9)} + \frac{19136}{105(3D-10)} + \frac{128}{5(3D-8)} \right. \\
& -\frac{3842871566}{405405(D-1)} + \frac{2331716}{75(D-2)} + \frac{2263616}{135(D-4)} + \frac{288}{7(D-6)} - \frac{97794176}{4303125(5D-14)} \\
& +\frac{100409312}{1340625(5D-16)} - \frac{1661408308}{446875(5D-18)} + \frac{325457216}{3125(5D-22)} + \frac{8534}{27(D-1)^2} \\
& \left. -\frac{93088}{5(D-2)^2} + \frac{3165056}{135(D-4)^2} + \frac{3840}{(D-2)^3} + \frac{132736}{9(D-4)^3} + \frac{9728}{3(D-4)^4} - \frac{182624344}{9375} \right) \\
& +A61 \left( +D^2 + \frac{6324D}{125} + \frac{101318}{99(3D-14)} + \frac{1186}{81(D-1)} + \frac{716}{3(D-2)} \right. \\
& +\frac{796}{3(D-4)} + \frac{6706784}{253125(5D-14)} - \frac{623884}{34375(5D-16)} - \frac{38922}{3125(5D-18)} \\
& \left. -\frac{5049408}{3125(5D-22)} - \frac{787}{9(D-1)^2} - \frac{384}{5(D-2)^2} + \frac{528}{5(D-4)^2} - \frac{5861159}{28125} \right) \\
& -A62 \left( +\frac{112372D}{225} - \frac{640}{D} - \frac{1102}{21(2D-5)} + \frac{338}{21(2D-7)} + \frac{9432928}{16575(2D-9)} \right. \\
& -\frac{103424}{297(D-1)} + \frac{365056}{75(D-2)} + \frac{16544}{105(D-3)} - \frac{23576}{135(D-4)} - \frac{64}{7(D-6)} \\
& -\frac{1060567456}{2008125(5D-14)} - \frac{5265096}{625625(5D-16)} + \frac{3843264}{4375(5D-22)} - \frac{3328}{(D-2)^2} \\
& \left. -\frac{40}{(D-3)^2} - \frac{1280}{9(D-4)^2} - \frac{14730632}{5625} \right) \\
& +A63 \left( +4D^2 + \frac{133488D}{125} - \frac{3380}{63(2D-7)} + \frac{1216}{15(3D-8)} + \frac{9376286}{19305(D-1)} \right. \\
& +\frac{525424}{45(D-2)} + \frac{50}{(D-3)} + \frac{97276}{45(D-4)} - \frac{118752128}{590625(5D-14)} + \frac{11452108}{309375(5D-16)} \\
& +\frac{77648}{40625(5D-18)} - \frac{17312256}{3125(5D-22)} - \frac{42752}{5(D-2)^2} + \frac{12496}{15(D-4)^2} + \frac{1536}{(D-2)^3} \\
& \left. +\frac{128}{(D-4)^3} - \frac{74684696}{9375} \right) \\
& +B61 \left( +4D^3 - 120D^2 + 1200D + \frac{2112}{(D-2)} - \frac{768}{(D-4)} \right. \\
& \left. -\frac{1152}{(D-2)^2} - \frac{768}{(D-4)^2} + \frac{256}{(D-2)^3} - \frac{256}{(D-4)^3} - 4096 \right) \\
& +B62 \left( +54D^2 - \frac{1689D}{2} + \frac{2261}{72(2D-5)} + \frac{625}{8(2D-7)} - \frac{1400}{9(D-1)} \right. \\
& -\frac{3152}{(D-2)} - \frac{2440}{3(D-4)} + \frac{2048}{(D-2)^2} - \frac{2800}{3(D-4)^2} - \frac{512}{(D-2)^3} \\
& \left. -\frac{384}{(D-4)^3} + 3530 \right) \\
& +C61 \left( +54D^2 - \frac{1689D}{2} + \frac{2261}{72(2D-5)} + \frac{625}{8(2D-7)} - \frac{1400}{9(D-1)} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{3152}{(D-2)} - \frac{2440}{3(D-4)} + \frac{2048}{(D-2)^2} - \frac{2800}{3(D-4)^2} - \frac{512}{(D-2)^3} \\
& - \frac{384}{(D-4)^3} + 3530 \Big) \\
+ s_{12} A71 & \left( + \frac{54D}{5} + \frac{304}{77(2D-5)} - \frac{4792}{63(2D-7)} + \frac{294779}{3315(2D-9)} - \frac{9072}{715(D-1)} \right. \\
& - \frac{1396}{45(D-2)} + \frac{80}{(D-4)} + \frac{48}{35(D-6)} + \frac{288}{119(5D-14)} - \frac{226736}{32175(5D-16)} \\
& \left. + \frac{557732}{10725(5D-18)} + \frac{32}{(D-2)^2} + \frac{873}{25} \right) \\
- s_{12} A72 & \left( + \frac{784D}{25} - \frac{12392}{715(D-1)} + \frac{104}{(D-2)} - \frac{160}{(D-4)} - \frac{192}{35(D-6)} \right. \\
& - \frac{704}{125(5D-14)} + \frac{21216}{1925(5D-16)} + \frac{488216}{1625(5D-18)} - \frac{64}{(D-2)^2} - \frac{18472}{125} \Big) \\
- s_{12} A73 & \left( + \frac{26064D}{125} + \frac{228}{(2D-5)} - \frac{4792}{63(2D-7)} + \frac{1179116}{49725(2D-9)} - \frac{2008}{(D-1)} \right. \\
& + \frac{1444072}{225(D-2)} + \frac{17032}{35(D-3)} - \frac{56929472}{74375(5D-14)} - \frac{20013568}{73125(5D-16)} \\
& + \frac{707192}{5625(5D-18)} - \frac{656704}{13125(5D-22)} - \frac{7360}{(D-2)^2} - \frac{40}{(D-3)^2} \\
& \left. + \frac{2048}{(D-2)^3} - \frac{1315312}{625} \right) \\
+ s_{12} A74 & \left( + \frac{8401D}{250} - \frac{19}{4(2D-5)} + \frac{1198}{63(2D-7)} + \frac{1179116}{9945(2D-9)} + \frac{43}{15(D-1)} \right. \\
& + \frac{4936}{45(D-2)} - \frac{717}{35(D-3)} - \frac{16}{5(D-6)} + \frac{44793584}{1115625(5D-14)} - \frac{4939984}{365625(5D-16)} \\
& - \frac{478744}{28125(5D-18)} - \frac{7045984}{65625(5D-22)} - \frac{64}{(D-2)^2} + \frac{1}{(D-3)^2} - \frac{1219407}{12500} \Big) \\
+ s_{12} A75 & \left( + \frac{8401D}{250} - \frac{19}{4(2D-5)} + \frac{1198}{63(2D-7)} + \frac{1179116}{9945(2D-9)} + \frac{43}{15(D-1)} \right. \\
& + \frac{4936}{45(D-2)} - \frac{717}{35(D-3)} - \frac{16}{5(D-6)} + \frac{44793584}{1115625(5D-14)} - \frac{4939984}{365625(5D-16)} \\
& - \frac{478744}{28125(5D-18)} - \frac{7045984}{65625(5D-22)} - \frac{64}{(D-2)^2} + \frac{1}{(D-3)^2} - \frac{1219407}{12500} \Big) \\
+ s_{12}^2 A81 & \left( + \frac{837D}{250} - \frac{463}{21(2D-7)} + \frac{3376}{45(D-2)} + \frac{30}{7(D-3)} - \frac{16}{(D-4)} \right. \\
& - \frac{318032}{21875(5D-14)} + \frac{450604}{28125(5D-16)} + \frac{380688}{3125(5D-18)} + \frac{320272}{65625(5D-22)} \\
& \left. + \frac{169}{24(2D-7)^2} - \frac{64}{(D-2)^2} - \frac{807543}{25000} \right) \\
+ s_{12}^2 B81 & \frac{3(D-3)(3D-8)(D^3-16D^2+68D-88)}{(2D-5)(D-4)(2D-7)(D-2)}
\end{aligned}$$

$$\begin{aligned}
& +s_{12}^2 C81 \frac{3(D-3)(3D-8)(D^3-16D^2+68D-88)}{(2D-5)(D-4)(2D-7)(D-2)} \\
& -s_{12}^3 A91 \left( +\frac{729D}{125} + \frac{528}{5(D-2)} + \frac{15}{7(D-3)} - \frac{8}{(D-4)} - \frac{65856}{3125(5D-14)} \right. \\
& \quad \left. + \frac{55176}{3125(5D-16)} + \frac{7488}{3125(5D-18)} + \frac{120224}{21875(5D-22)} - \frac{384}{5(D-2)^2} - \frac{156321}{3125} \right) \\
& +s_{12}^3 A92 \left( \frac{6(3D-14)}{5(D-3)(5D-22)(5D-16)(5D-14)(5D-18)(D-2)} \right. \\
& \quad \left. \times \frac{(75D^6-1048D^5+5956D^4-17776D^3+30208D^2-29440D+13952)}{5(D-3)(5D-22)(5D-16)(5D-14)(5D-18)(D-2)} \right)
\end{aligned}$$

$$\begin{aligned}
X_{C_A^2 N_F}^g = & \\
& +\frac{B41}{s_{12}^2} \left( +\frac{148672D^2}{125} - \frac{11618238D}{625} + \frac{8960}{3D} + \frac{69850}{63(2D-7)} + \frac{416494232}{3315(2D-9)} \right. \\
& \quad +\frac{44648}{21(3D-10)} - \frac{9272016}{385(3D-14)} + \frac{256}{45(3D-8)} + \frac{77206856351}{1216215(D-1)} - \frac{1315066}{45(D-2)} \\
& \quad -\frac{41}{3(D-3)} - \frac{34916536}{1215(D-4)} + \frac{720}{(D-6)} - \frac{2255988208}{6024375(5D-14)} + \frac{2153109412}{4021875(5D-16)} \\
& \quad +\frac{33333874}{40625(5D-18)} - \frac{4831233056}{28125(5D-22)} + \frac{7637}{9(D-1)^2} + \frac{26960}{3(D-2)^2} - \frac{151}{(D-3)^2} \\
& \quad \left. +\frac{3133072}{405(D-4)^2} - \frac{3584}{3(D-2)^3} + \frac{1253792}{135(D-4)^3} + \frac{117248}{45(D-4)^4} + \frac{111929368}{5625} \right) \\
& +\frac{A51}{s_{12}} \left( +\frac{27524D^2}{125} - \frac{15192506D}{5625} + \frac{2432}{5(2D-5)} + \frac{2150}{9(2D-7)} + \frac{27315013}{5525(2D-9)} \right. \\
& \quad -\frac{850144}{6615(3D-10)} - \frac{13162938752}{567567(D-1)} + \frac{959648}{75(D-2)} + \frac{32}{(D-3)} - \frac{12064}{3(D-4)} \\
& \quad -\frac{2880}{7(D-6)} - \frac{76196704}{4303125(5D-14)} - \frac{640942528}{5630625(5D-16)} + \frac{270310544}{40625(5D-18)} \\
& \quad +\frac{40938352}{9375(5D-22)} - \frac{44032}{189(D-1)^2} - \frac{19712}{3(D-2)^2} - \frac{250816}{135(D-4)^2} + \frac{1024}{(D-2)^3} \\
& \quad \left. -\frac{77056}{45(D-4)^3} + \frac{183292489}{16875} \right) \\
& -\frac{A52}{s_{12}} \left( +\frac{30752D^2}{375} - \frac{3289648D}{1875} + \frac{384}{D} + \frac{150784}{1485(2D-5)} - \frac{1720}{63(2D-7)} \right. \\
& \quad -\frac{9958772}{5525(2D-9)} - \frac{82592}{945(3D-10)} - \frac{264160}{231(3D-14)} + \frac{896}{135(3D-8)} \\
& \quad +\frac{988080008}{135135(D-1)} - \frac{444172}{75(D-2)} + \frac{1509632}{1215(D-4)} + \frac{1152}{7(D-6)} + \frac{269380672}{3614625(5D-14)} \\
& \quad \left. +\frac{295117696}{9384375(5D-16)} - \frac{549255572}{446875(5D-18)} + \frac{78770048}{28125(5D-22)} + \frac{3976}{27(D-1)^2} \right)
\end{aligned}$$



$$\begin{aligned}
& + \frac{2016}{(D-2)^2} + \frac{500992}{405(D-4)^2} - \frac{1024}{3(D-2)^3} + \frac{17920}{27(D-4)^3} + \frac{15434708}{3375} \Big) \\
- \frac{B51}{s_{12}} & \left( + \frac{1397D^2}{250} + \frac{646933D}{2500} - \frac{5681}{18(2D-5)} - \frac{6125}{24(2D-7)} - \frac{8679748}{5525(2D-9)} \right. \\
& + \frac{1269580}{81(D-1)} - \frac{2167064}{225(D-2)} - \frac{2}{(D-3)} - \frac{7448}{45(D-4)} + \frac{198571648}{4303125(5D-14)} \\
& - \frac{1954816}{365625(5D-16)} + \frac{4984}{125(5D-18)} + \frac{57180032}{9375(5D-22)} + \frac{12544}{3(D-2)^2} \\
& \left. + \frac{12544}{45(D-4)^2} - \frac{512}{(D-2)^3} + \frac{5248}{15(D-4)^3} - \frac{11790061}{5000} \right) \\
- \frac{B52}{s_{12}} & \left( + \frac{3484D^2}{75} - \frac{6867848D}{5625} + \frac{12313216}{10395(2D-5)} + \frac{12625088}{5525(2D-9)} - \frac{1024}{945(3D-10)} \right. \\
& - \frac{256}{45(3D-8)} - \frac{7346825032}{405405(D-1)} + \frac{2768314}{225(D-2)} - \frac{61552}{405(D-4)} - \frac{144}{7(D-6)} \\
& + \frac{6733664}{53125(5D-14)} + \frac{551540128}{28153125(5D-16)} + \frac{177548474}{446875(5D-18)} - \frac{27136592}{3125(5D-22)} \\
& + \frac{2008}{3(D-1)^2} - \frac{21392}{3(D-2)^2} - \frac{84416}{135(D-4)^2} + \frac{896}{(D-2)^3} - \frac{18688}{45(D-4)^3} + \frac{75081472}{16875} \Big) \\
+ A61 & \left( + \frac{394D^2}{25} - \frac{5986D}{25} - \frac{26416}{693(3D-14)} - \frac{17332}{81(D-1)} - \frac{636}{5(D-2)} \right. \\
& + \frac{176}{45(D-4)} - \frac{15400288}{354375(5D-14)} + \frac{18256}{4125(5D-16)} - \frac{46252}{625(5D-18)} \\
& \left. + \frac{70784}{1875(5D-22)} - \frac{1772}{9(D-1)^2} - \frac{32}{(D-4)^2} + \frac{4813796}{5625} \right) \\
- A62 & \left( + \frac{238D^2}{25} - \frac{84531D}{250} - \frac{1280}{3D} - \frac{4408}{21(2D-5)} + \frac{215}{84(2D-7)} \right. \\
& - \frac{93023}{1950(2D-9)} - \frac{206848}{297(D-1)} - \frac{77056}{75(D-2)} - \frac{8366}{105(D-3)} - \frac{4336}{135(D-4)} \\
& + \frac{32}{7(D-6)} + \frac{152148512}{118125(5D-14)} - \frac{645725648}{5630625(5D-16)} - \frac{1324688}{39375(5D-22)} \\
& \left. + \frac{1088}{3(D-2)^2} + \frac{16}{(D-3)^2} - \frac{256}{3(D-2)^3} + \frac{40336667}{22500} \right) \\
+ A63 & \left( + \frac{921D^2}{25} - \frac{1571239D}{2250} - \frac{1075}{126(2D-7)} - \frac{4757193}{22100(2D-9)} - \frac{2432}{135(3D-8)} \right. \\
& + \frac{18752572}{19305(D-1)} - \frac{465736}{225(D-2)} - \frac{12}{(D-3)} + \frac{448}{45(D-4)} + \frac{556167296}{2008125(5D-14)} \\
& - \frac{5952208}{804375(5D-16)} - \frac{8552}{1625(5D-18)} + \frac{80896}{625(5D-22)} + \frac{128}{(D-2)^2} - \frac{96}{5(D-4)^2} \\
& \left. + \frac{512}{(D-2)^3} + \frac{182032831}{67500} \right) \\
+ B62 & \left( + \frac{7D^2}{2} - \frac{185D}{4} + \frac{2261}{18(2D-5)} + \frac{875}{72(2D-7)} - \frac{2800}{9(D-1)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2368}{9(D-2)} + \frac{248}{9(D-4)} - \frac{640}{3(D-2)^2} + \frac{32}{(D-4)^2} + \frac{793}{8} \Big) \\
+ C_{61} & \left( + \frac{7D^2}{2} - \frac{185D}{4} + \frac{2261}{18(2D-5)} + \frac{875}{72(2D-7)} - \frac{2800}{9(D-1)} \right. \\
& + \frac{2368}{9(D-2)} + \frac{248}{9(D-4)} - \frac{640}{3(D-2)^2} + \frac{32}{(D-4)^2} + \frac{793}{8} \Big) \\
+ s_{12} A_{71} & \left( + \frac{16D^2}{25} - \frac{1437D}{125} + \frac{1216}{77(2D-5)} - \frac{902}{63(2D-7)} - \frac{28181}{6630(2D-9)} \right. \\
& - \frac{36992}{2145(D-1)} - \frac{526}{15(D-2)} - \frac{24}{35(D-6)} + \frac{9012656}{223125(5D-14)} \\
& + \frac{152185504}{5630625(5D-16)} - \frac{14146606}{268125(5D-18)} - \frac{80}{3(D-2)^2} + \frac{10761}{250} \Big) \\
- s_{12} A_{72} & \left( + \frac{17D^2}{20} - \frac{2237D}{100} - \frac{71585}{2016(2D-7)} - \frac{226533}{35360(2D-9)} - \frac{24784}{715(D-1)} \right. \\
& + \frac{568}{45(D-2)} + \frac{96}{35(D-6)} + \frac{252352}{14875(5D-14)} + \frac{420416}{375375(5D-16)} \\
& + \frac{222744}{1625(5D-18)} - \frac{128}{3(D-2)^2} + \frac{35181}{400} \Big) \\
- s_{12} A_{73} & \left( + \frac{44D^2}{125} - \frac{19864D}{625} + \frac{912}{(2D-5)} - \frac{902}{63(2D-7)} - \frac{56362}{49725(2D-9)} \right. \\
& - \frac{4016}{(D-1)} + \frac{566216}{225(D-2)} - \frac{808}{5(D-3)} + \frac{252301792}{371875(5D-14)} + \frac{6857728}{73125(5D-16)} \\
& + \frac{517192}{28125(5D-18)} + \frac{11792}{9375(5D-22)} - \frac{1920}{(D-2)^2} + \frac{8}{(D-3)^2} + \frac{512}{(D-2)^3} \\
& + \frac{209752}{625} \Big) \\
+ s_{12} A_{74} & \left( + \frac{21D^2}{250} - \frac{15043D}{1250} - \frac{19}{(2D-5)} + \frac{6805}{1008(2D-7)} - \frac{93023}{9360(2D-9)} \right. \\
& + \frac{86}{15(D-1)} - \frac{3116}{45(D-2)} + \frac{934}{105(D-3)} + \frac{8}{5(D-6)} + \frac{4403312}{65625(5D-14)} \\
& - \frac{10487648}{365625(5D-16)} - \frac{17612}{5625(5D-18)} + \frac{1574056}{196875(5D-22)} + \frac{1}{(D-3)^2} + \frac{916069}{15000} \Big) \\
+ s_{12} A_{75} & \left( + \frac{21D^2}{250} - \frac{15043D}{1250} - \frac{19}{(2D-5)} + \frac{6805}{1008(2D-7)} - \frac{93023}{9360(2D-9)} \right. \\
& + \frac{86}{15(D-1)} - \frac{3116}{45(D-2)} + \frac{934}{105(D-3)} + \frac{8}{5(D-6)} + \frac{4403312}{65625(5D-14)} \\
& - \frac{10487648}{365625(5D-16)} - \frac{17612}{5625(5D-18)} + \frac{1574056}{196875(5D-22)} + \frac{1}{(D-3)^2} + \frac{916069}{15000} \Big) \\
+ s_{12}^2 A_{81} & \left( + \frac{113787D}{20000} - \frac{2377}{448(2D-7)} + \frac{2848}{45(D-2)} - \frac{45}{7(D-3)} \right. \\
& - \frac{679432}{21875(5D-14)} + \frac{423016}{28125(5D-16)} + \frac{51216}{3125(5D-18)} - \frac{13424}{65625(5D-22)} \Big)
\end{aligned}$$

$$\begin{aligned}
& + \frac{215}{192(2D-7)^2} - \frac{920019}{20000} \Big) \\
& + s_{12}^2 B81 \frac{(2D^3 - 25D^2 + 94D - 112)(D^3 - 16D^2 + 68D - 88)}{2(2D-5)(2D-7)(D-2)^2} \\
& + s_{12}^2 C81 \frac{(2D^3 - 25D^2 + 94D - 112)(D^3 - 16D^2 + 68D - 88)}{2(2D-5)(2D-7)(D-2)^2} \\
& - s_{12}^3 A91 \left( + \frac{81D^2}{250} - \frac{9369D}{1250} - \frac{176}{3(D-2)} + \frac{45}{14(D-3)} + \frac{32928}{625(5D-14)} \right. \\
& \quad \left. - \frac{160688}{9375(5D-16)} + \frac{11712}{3125(5D-18)} + \frac{12424}{65625(5D-22)} + \frac{54801}{1250} \right) \\
& - s_{12}^3 A92 \left( \frac{(3D-14)(D-4)}{5(D-3)(5D-22)(5D-16)(5D-14)(5D-18)(D-2)} \right. \\
& \quad \left. \times \frac{(135D^5 - 2200D^4 + 15156D^3 - 54336D^2 + 99776D - 74112)}{5(D-3)(5D-22)(5D-16)(5D-14)(5D-18)(D-2)} \right) \\
& + s_{12}^3 A94 \left( + \frac{81D^2}{4000} - \frac{513D}{2500} - \frac{973}{2304(2D-7)} - \frac{75511}{1414400(2D-9)} - \frac{16}{75(D-2)} \right. \\
& \quad + \frac{3024}{10625(5D-14)} - \frac{56144}{121875(5D-16)} + \frac{4672}{3125(5D-18)} + \frac{8936}{28125(5D-22)} \\
& \quad \left. + \frac{36117}{80000} \right)
\end{aligned}$$

$$X_{C_A C_F N_F}^g =$$

$$\begin{aligned}
& - \frac{B41}{s_{12}^2} \left( + \frac{676832D^2}{125} - \frac{114134744D}{1875} - \frac{4480}{3D} + \frac{17860}{7(2D-7)} + \frac{4283546344}{16575(2D-9)} \right. \\
& \quad - \frac{226592}{63(3D-10)} + \frac{36336222}{385(3D-14)} - \frac{6656}{45(3D-8)} + \frac{79424288}{405405(D-1)} - \frac{13370848}{675(D-2)} \\
& \quad - \frac{4552}{3(D-3)} + \frac{1922612}{135(D-4)} + \frac{8647206224}{30121875(5D-14)} + \frac{3472406764}{2413125(5D-16)} \\
& \quad + \frac{1338137458}{24375(5D-18)} - \frac{6431698208}{9375(5D-22)} + \frac{53504}{9(D-2)^2} - \frac{76}{3(D-3)^2} + \frac{2219552}{15(D-4)^2} \\
& \quad \left. + \frac{1886176}{15(D-4)^3} + \frac{187136}{5(D-4)^4} + \frac{3507215464}{28125} \right) \\
& - \frac{A51}{s_{12}} \left( + \frac{103212D^2}{125} - \frac{43508066D}{5625} + \frac{4864}{5(2D-5)} + \frac{9560}{9(2D-7)} + \frac{64410511}{5525(2D-9)} \right. \\
& \quad - \frac{255904}{945(3D-10)} - \frac{55040}{567(D-1)} + \frac{1180528}{225(D-2)} + \frac{160}{3(D-3)} - \frac{2847392}{405(D-4)} \\
& \quad - \frac{45502976}{478125(5D-14)} + \frac{26533088}{73125(5D-16)} + \frac{38430896}{3125(5D-18)} - \frac{28747264}{3125(5D-22)} \\
& \quad \left. - \frac{14848}{3(D-2)^2} - \frac{23104}{135(D-4)^2} - \frac{77312}{45(D-4)^3} + \frac{1255139429}{84375} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{A52}{s_{12}} \left( + \frac{109888D^2}{375} - \frac{27257176D}{5625} - \frac{192}{D} + \frac{301568}{1485(2D-5)} - \frac{7648}{63(2D-7)} \right. \\
& \quad - \frac{28282268}{5525(2D-9)} - \frac{109744}{945(3D-10)} + \frac{1035220}{231(3D-14)} - \frac{3968}{27(3D-8)} \\
& \quad - \frac{8757872}{81081(D-1)} - \frac{5629112}{675(D-2)} + \frac{6848}{(D-4)} + \frac{4586010176}{30121875(5D-14)} \\
& \quad - \frac{1929846656}{12065625(5D-16)} - \frac{1857022964}{446875(5D-18)} - \frac{40768}{375(5D-22)} + \frac{26176}{9(D-2)^2} \\
& \quad \left. + \frac{287104}{45(D-4)^2} + \frac{41984}{15(D-4)^3} + \frac{497787076}{28125} \right) \\
& - \frac{B51}{s_{12}} \left( + \frac{7331D^2}{125} + \frac{205803D}{1250} + \frac{5681}{9(2D-5)} + \frac{6125}{12(2D-7)} + \frac{1080464}{325(2D-9)} \right. \\
& \quad + \frac{162816}{25(D-2)} + \frac{76}{(D-3)} + \frac{42832}{15(D-4)} - \frac{5625(5D-14)}{40625(5D-16)} - \frac{41344}{40625(5D-16)} \\
& \quad - \frac{12023872}{3125(5D-18)} - \frac{47710208}{1875(5D-22)} - \frac{3712}{(D-2)^2} - \frac{896}{3(D-4)^2} - \frac{768}{5(D-4)^3} \\
& \quad \left. - \frac{52885663}{12500} \right) \\
& + \frac{B52}{s_{12}} \left( + \frac{29636D^2}{75} - \frac{21234136D}{5625} + \frac{24626432}{10395(2D-5)} + \frac{1571584}{325(2D-9)} + \frac{114112}{945(3D-10)} \right. \\
& \quad + \frac{64}{9(3D-8)} + \frac{7687616}{27027(D-1)} + \frac{1695824}{225(D-2)} + \frac{518816}{81(D-4)} + \frac{53516512}{253125(5D-14)} \\
& \quad + \frac{4170656384}{28153125(5D-16)} - \frac{736492144}{446875(5D-18)} - \frac{19509056}{625(5D-22)} - \frac{21376}{3(D-2)^2} \\
& \quad \left. + \frac{475904}{135(D-4)^2} + \frac{42496}{45(D-4)^3} + \frac{449511616}{84375} \right) \\
& - A61 \left( + \frac{21284D^2}{375} - \frac{1360516D}{1875} + \frac{103522}{693(3D-14)} - \frac{1720}{27(D-1)} - \frac{9092}{15(D-2)} \right. \\
& \quad + \frac{712}{3(D-4)} - \frac{6580208}{590625(5D-14)} - \frac{3293512}{103125(5D-16)} - \frac{243682}{3125(5D-18)} \\
& \quad \left. - \frac{595952}{3125(5D-22)} + \frac{1088}{15(D-4)^2} + \frac{68305472}{28125} \right) \\
& + A62 \left( + \frac{1489D^2}{25} - \frac{334171D}{375} + \frac{640}{3D} - \frac{8816}{21(2D-5)} + \frac{239}{21(2D-7)} \right. \\
& \quad - \frac{3947149}{33150(2D-9)} - \frac{1846496}{675(D-2)} + \frac{16262}{105(D-3)} + \frac{1216}{9(D-4)} + \frac{676472528}{669375(5D-14)} \\
& \quad - \frac{495934112}{1535625(5D-16)} - \frac{931856}{13125(5D-22)} + \frac{2176}{9(D-2)^2} + \frac{5}{3(D-3)^2} + \frac{14824063}{3750} \left. \right) \\
& - A63 \left( + \frac{24791D^2}{125} - \frac{26566007D}{11250} - \frac{2390}{63(2D-7)} - \frac{14271579}{22100(2D-9)} + \frac{6080}{27(3D-8)} \right. \\
& \quad - \frac{364768}{225(D-2)} - \frac{8}{(D-3)} + \frac{53864}{45(D-4)} - \frac{585166784}{3346875(5D-14)} + \frac{16210664}{365625(5D-16)} \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{800736}{3125(5D-18)} - \frac{2043264}{3125(5D-22)} + \frac{6208}{15(D-4)^2} + \frac{2491573703}{337500} \Big) \\
-B62 & \left( +23D^2 - \frac{729D}{2} + \frac{2261}{9(2D-5)} + \frac{875}{36(2D-7)} - \frac{5056}{9(D-2)} \right. \\
& \left. + \frac{1520}{9(D-4)} - \frac{128}{3(D-2)^2} + \frac{320}{3(D-4)^2} + \frac{5401}{4} \right) \\
-C61 & \left( +23D^2 - \frac{729D}{2} + \frac{2261}{9(2D-5)} + \frac{875}{36(2D-7)} - \frac{5056}{9(D-2)} \right. \\
& \left. + \frac{1520}{9(D-4)} - \frac{128}{3(D-2)^2} + \frac{320}{3(D-4)^2} + \frac{5401}{4} \right) \\
-s_{12}A71 & \left( + \frac{64D^2}{25} - \frac{3084D}{125} + \frac{2432}{77(2D-5)} - \frac{556}{21(2D-7)} - \frac{1754}{195(2D-9)} \right. \\
& \left. + \frac{6976}{429(D-1)} - \frac{4768}{45(D-2)} + \frac{40448}{525(5D-14)} + \frac{290701952}{5630625(5D-16)} \right. \\
& \left. - \frac{36841312}{268125(5D-18)} - \frac{128}{3(D-2)^2} + \frac{43662}{625} \right) \\
+s_{12}A72 & \left( + \frac{83D^2}{20} - \frac{1307D}{20} - \frac{71585}{672(2D-7)} - \frac{679599}{35360(2D-9)} - \frac{984}{5(D-2)} \right. \\
& \left. + \frac{905568}{14875(5D-14)} + \frac{30848}{4875(5D-16)} + \frac{51544}{125(5D-18)} + \frac{645603}{2000} \right) \\
+s_{12}A73 & \left( + \frac{676D^2}{125} - \frac{96D}{5} + \frac{1824}{(2D-5)} - \frac{556}{21(2D-7)} - \frac{7016}{2925(2D-9)} \right. \\
& \left. + \frac{62816}{225(D-2)} + \frac{1888}{15(D-3)} + \frac{5730624}{21875(5D-14)} - \frac{30838784}{365625(5D-16)} \right. \\
& \left. + \frac{191648}{5625(5D-18)} + \frac{704}{1875(5D-22)} - \frac{7168}{3(D-2)^2} + \frac{4}{(D-3)^2} - \frac{953744}{3125} \right) \\
-s_{12}A74 & \left( + \frac{313D^2}{250} - \frac{10751D}{750} - \frac{38}{(2D-5)} + \frac{1807}{112(2D-7)} - \frac{3947149}{159120(2D-9)} \right. \\
& \left. - \frac{6724}{45(D-2)} + \frac{1817}{105(D-3)} + \frac{48387368}{371875(5D-14)} - \frac{26422144}{365625(5D-16)} \right. \\
& \left. - \frac{228988}{28125(5D-18)} + \frac{444664}{21875(5D-22)} + \frac{128}{3(D-2)^2} + \frac{5}{3(D-3)^2} + \frac{5654087}{75000} \right) \\
-s_{12}A75 & \left( + \frac{313D^2}{250} - \frac{10751D}{750} - \frac{38}{(2D-5)} + \frac{1807}{112(2D-7)} - \frac{3947149}{159120(2D-9)} \right. \\
& \left. - \frac{6724}{45(D-2)} + \frac{1817}{105(D-3)} + \frac{48387368}{371875(5D-14)} - \frac{26422144}{365625(5D-16)} \right. \\
& \left. - \frac{228988}{28125(5D-18)} + \frac{444664}{21875(5D-22)} + \frac{128}{3(D-2)^2} + \frac{5}{3(D-3)^2} + \frac{5654087}{75000} \right) \\
-s_{12}^2A81 & \left( + \frac{37071D}{5000} - \frac{1853}{112(2D-7)} + \frac{4528}{45(D-2)} - \frac{60}{7(D-3)} \right. \\
& \left. - \frac{1474512}{21875(5D-14)} + \frac{809248}{28125(5D-16)} + \frac{125664}{3125(5D-18)} - \frac{5872}{13125(5D-22)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{239}{48(2D-7)^2} - \frac{1622997}{25000} \Big) \\
& -s_{12}^2 B81 \frac{(2D^3 - 25D^2 + 94D - 112)(D^3 - 16D^2 + 68D - 88)}{(2D-5)(2D-7)(D-2)^2} \\
& -s_{12}^2 C81 \frac{(2D^3 - 25D^2 + 94D - 112)(D^3 - 16D^2 + 68D - 88)}{(2D-5)(2D-7)(D-2)^2} \\
& +s_{12}^3 A91 \left( + \frac{162D^2}{125} - \frac{13878D}{625} - \frac{576}{5(D-2)} + \frac{30}{7(D-3)} + \frac{197568}{3125(5D-14)} \right. \\
& \quad \left. - \frac{23232}{3125(5D-16)} - \frac{51456}{3125(5D-18)} + \frac{7008}{4375(5D-22)} + \frac{350478}{3125} \right) \\
& -s_{12}^3 A92 \frac{12(D-4)(3D-14)(15D^5 - 266D^4 + 1708D^3 - 5016D^2 + 6624D - 2944)}{5(5D-18)(5D-14)(5D-16)(5D-22)(D-3)(D-2)} \\
& -s_{12}^3 A94 \left( + \frac{243D^2}{4000} - \frac{1539D}{2500} - \frac{973}{768(2D-7)} - \frac{226533}{1414400(2D-9)} - \frac{16}{25(D-2)} \right. \\
& \quad + \frac{9072}{10625(5D-14)} - \frac{56144}{40625(5D-16)} + \frac{14016}{3125(5D-18)} + \frac{8936}{9375(5D-22)} \\
& \quad \left. + \frac{108351}{80000} \right)
\end{aligned}$$

$$X_{C_F^2 N_F}^g =$$

$$\begin{aligned}
& + \frac{B41}{s_{12}^2} \left( + \frac{752476D^2}{125} - \frac{123631802D}{1875} - \frac{93029552}{5525(2D-9)} - \frac{1120}{9(3D-10)} + \frac{53622972}{385(3D-14)} \right. \\
& \quad - \frac{640}{9(3D-8)} - \frac{93104}{25(D-2)} - \frac{998}{3(D-3)} + \frac{3093496}{45(D-4)} - \frac{6135374944}{3346875(5D-14)} \\
& \quad + \frac{19420431464}{4021875(5D-16)} + \frac{318829588}{9375(5D-18)} - \frac{9146303296}{28125(5D-22)} + \frac{256}{(D-2)^2} \\
& \quad \left. - \frac{106}{3(D-3)^2} + \frac{2295872}{15(D-4)^2} + \frac{541632}{5(D-4)^3} + \frac{144384}{5(D-4)^4} + \frac{1185475208}{9375} \right) \\
& + \frac{A51}{s_{12}} \left( + \frac{108328D^2}{125} - \frac{6789932D}{625} + \frac{14951178}{5525(2D-9)} + \frac{224}{9(3D-10)} - \frac{359072}{75(D-2)} \right. \\
& \quad - \frac{128}{3(D-3)} + \frac{1131328}{135(D-4)} - \frac{65923808}{286875(5D-14)} + \frac{6002112}{40625(5D-16)} \\
& \quad + \frac{8723232}{3125(5D-18)} - \frac{195901888}{9375(5D-22)} + \frac{378752}{45(D-4)^2} + \frac{34816}{15(D-4)^3} + \frac{174596354}{5625} \Big) \\
& - \frac{A52}{s_{12}} \left( + \frac{85268D^2}{375} - \frac{5132552D}{1875} - \frac{16310376}{5525(2D-9)} - \frac{1376}{27(3D-10)} + \frac{509240}{77(3D-14)} \right. \\
& \quad - \frac{1600}{27(3D-8)} - \frac{187888}{225(D-2)} + \frac{119488}{45(D-4)} + \frac{566390144}{3346875(5D-14)} \\
& \quad \left. + \frac{292125376}{1340625(5D-16)} - \frac{6431928}{3125(5D-18)} - \frac{25131904}{5625(5D-22)} + \frac{128}{(D-2)^2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{34304}{15(D-4)^2} + \frac{1024}{(D-4)^3} + \frac{611242648}{84375} \Big) \\
& + \frac{B51}{s_{12}} \left( + \frac{28956D^2}{125} - \frac{1786684D}{625} - \frac{7136}{5(D-2)} - \frac{44}{(D-3)} + \frac{739904}{135(D-4)} \right. \\
& \quad - \frac{8126272}{84375(5D-14)} + \frac{10304}{3125(5D-16)} - \frac{7519008}{3125(5D-18)} - \frac{10203648}{625(5D-22)} \\
& \quad \left. + \frac{150784}{45(D-4)^2} + \frac{18944}{15(D-4)^3} + \frac{25627244}{3125} \right) \\
& - \frac{B52}{s_{12}} \left( + \frac{46436D^2}{75} - \frac{2905928D}{375} - \frac{2912}{27(3D-10)} - \frac{1600}{27(3D-8)} - \frac{19792}{5(D-2)} \right. \\
& \quad + \frac{221152}{27(D-4)} + \frac{468416}{3375(5D-14)} + \frac{50336}{625(5D-16)} - \frac{532368}{625(5D-18)} \\
& \quad \left. - \frac{2060352}{125(5D-22)} + \frac{53120}{9(D-4)^2} + \frac{25088}{15(D-4)^3} + \frac{15350288}{675} \right) \\
& + A61 \left( + \frac{18928D^2}{375} - \frac{1201712D}{1875} + \frac{50924}{231(3D-14)} - \frac{21832}{45(D-2)} + \frac{10096}{45(D-4)} \right. \\
& \quad - \frac{6678656}{196875(5D-14)} - \frac{1159792}{61875(5D-16)} - \frac{2932}{125(5D-18)} - \frac{2663584}{9375(5D-22)} \\
& \quad \left. + \frac{256}{3(D-4)^2} + \frac{19570384}{9375} \right) \\
& - A62 \left( + \frac{2026D^2}{25} - \frac{430946D}{375} - \frac{226533}{5525(2D-9)} - \frac{616064}{225(D-2)} + \frac{392}{3(D-3)} \right. \\
& \quad + \frac{832}{15(D-4)} + \frac{6648608}{6375(5D-14)} - \frac{20564992}{73125(5D-16)} + \frac{35744}{5625(5D-22)} \\
& \quad \left. - \frac{256}{(D-2)^2} - \frac{10}{3(D-3)^2} + \frac{8774231}{1875} \right) \\
& + A63 \left( + \frac{28662D^2}{125} - \frac{1856633D}{625} - \frac{4757193}{11050(2D-9)} - \frac{242144}{75(D-2)} + \frac{36}{(D-3)} \right. \\
& \quad + \frac{17584}{15(D-4)} + \frac{8105216}{159375(5D-14)} + \frac{224944}{24375(5D-16)} + \frac{135008}{625(5D-18)} \\
& \quad \left. - \frac{3044096}{3125(5D-22)} + \frac{2048}{5(D-4)^2} + \frac{64132247}{6250} \right) \\
& + s_{12} A71 \frac{64(D-3)(D^2-7D+16)(5D^3-62D^2+236D-288)}{(D-2)(5D-16)(5D-14)(5D-18)} \\
& - s_{12} A72 \left( + \frac{49D^2}{10} - \frac{3181D}{50} - \frac{71585}{1008(2D-7)} - \frac{226533}{17680(2D-9)} - \frac{6224}{45(D-2)} \right. \\
& \quad \left. + \frac{862656}{14875(5D-14)} + \frac{11008}{2925(5D-16)} + \frac{26192}{125(5D-18)} + \frac{50413}{200} \right) \\
& - s_{12} A73 \left( + \frac{1676D^2}{125} - \frac{132544D}{625} - \frac{1312}{(D-2)} + \frac{4504}{105(D-3)} + \frac{4678464}{3125(5D-14)} \right. \\
& \quad \left. + \frac{241664}{3125(5D-16)} - \frac{66464}{9375(5D-18)} - \frac{14016}{4375(5D-22)} - \frac{28}{(D-3)^2} + \frac{3149844}{3125} \right)
\end{aligned}$$

$$\begin{aligned}
& +s_{12}A74 \left( +\frac{271D^2}{125} - \frac{48887D}{1875} + \frac{3197}{504(2D-7)} - \frac{75511}{8840(2D-9)} - \frac{2488}{45(D-2)} \right. \\
& \quad + \frac{12}{(D-3)} - \frac{2778544}{74375(5D-14)} - \frac{11700224}{365625(5D-16)} - \frac{5256}{3125(5D-18)} \\
& \quad \left. + \frac{196592}{28125(5D-22)} - \frac{4}{3(D-3)^2} + \frac{47263}{500} \right) \\
& +s_{12}A75 \left( +\frac{271D^2}{125} - \frac{48887D}{1875} + \frac{3197}{504(2D-7)} - \frac{75511}{8840(2D-9)} - \frac{2488}{45(D-2)} \right. \\
& \quad + \frac{12}{(D-3)} - \frac{2778544}{74375(5D-14)} - \frac{11700224}{365625(5D-16)} - \frac{5256}{3125(5D-18)} \\
& \quad \left. + \frac{196592}{28125(5D-22)} - \frac{4}{3(D-3)^2} + \frac{47263}{500} \right) \\
& -s_{12}^3A91 \left( +\frac{162D^2}{125} - \frac{13878D}{625} - \frac{576}{5(D-2)} + \frac{30}{7(D-3)} + \frac{197568}{3125(5D-14)} \right. \\
& \quad \left. - \frac{23232}{3125(5D-16)} - \frac{51456}{3125(5D-18)} + \frac{7008}{4375(5D-22)} + \frac{350478}{3125} \right) \\
& +s_{12}^3A94 \left( +\frac{81D^2}{2000} - \frac{513D}{1250} - \frac{973}{1152(2D-7)} - \frac{75511}{707200(2D-9)} - \frac{32}{75(D-2)} \right. \\
& \quad + \frac{6048}{10625(5D-14)} - \frac{112288}{121875(5D-16)} + \frac{9344}{3125(5D-18)} + \frac{17872}{28125(5D-22)} \\
& \quad \left. + \frac{36117}{40000} \right)
\end{aligned}$$

$$X_{C_A N_F^2}^g =$$

$$\begin{aligned}
& -\frac{B41}{s_{12}^2} \left( +\frac{3872D}{25} + \frac{2860}{21(3D-10)} - \frac{3387058}{81081(D-1)} - \frac{220}{3(D-2)} - \frac{74}{3(D-3)} \right. \\
& \quad - \frac{49984}{81(D-4)} + \frac{48224}{3375(5D-14)} + \frac{83328}{1375(5D-16)} + \frac{1197504}{1625(5D-18)} + \frac{9217}{27(D-1)^2} \\
& \quad \left. + \frac{16}{(D-2)^2} + \frac{1}{(D-3)^2} - \frac{6016}{9(D-4)^2} - \frac{3584}{9(D-4)^3} - \frac{187024}{375} \right) \\
& +\frac{A52}{s_{12}} \left( +\frac{952D}{75} - \frac{272}{63(3D-10)} - \frac{11224376}{81081(D-1)} + \frac{320}{3(D-2)} - \frac{1024}{81(D-4)} \right. \\
& \quad + \frac{85184}{10125(5D-14)} - \frac{9984}{1375(5D-16)} - \frac{93312}{1625(5D-18)} - \frac{3976}{27(D-1)^2} - \frac{32}{(D-2)^2} \\
& \quad \left. - \frac{1024}{27(D-4)^2} - \frac{16384}{1125} \right) \\
& -\frac{B52}{s_{12}} \left( +\frac{616D}{75} - \frac{128}{63(3D-10)} - \frac{1113304}{81081(D-1)} - \frac{512}{81(D-4)} - \frac{68288}{10125(5D-14)} \right. \\
& \quad \left. - \frac{8832}{1375(5D-16)} - \frac{46656}{1625(5D-18)} + \frac{2008}{27(D-1)^2} - \frac{512}{27(D-4)^2} - \frac{41632}{1125} \right)
\end{aligned}$$



$$\begin{aligned}
& -A61 \frac{4(5D-16)(D-4)}{3(D-1)^2(D-2)^2} \\
& +s_{12}A71 \frac{32(D-3)(D-4)(5D^3-62D^2+236D-288)}{(D-1)(5D-16)(5D-14)(5D-18)}
\end{aligned}$$

$$\begin{aligned}
X_{CFN_F^2}^g = & \\
& + \frac{B41}{s_{12}^2} \left( + \frac{9344D}{25} - \frac{5440}{21(3D-10)} - \frac{256}{15(3D-8)} + \frac{81051008}{135135(D-1)} - \frac{17024}{9(D-4)} \right. \\
& + \frac{96448}{3375(5D-14)} + \frac{166656}{1375(5D-16)} + \frac{2395008}{1625(5D-18)} - \frac{2304}{(D-4)^2} - \frac{1024}{(D-4)^3} \\
& \left. - \frac{634048}{375} \right) \\
& - \frac{A52}{s_{12}} \left( + \frac{7904D}{75} + \frac{640}{63(3D-10)} - \frac{128}{9(3D-8)} + \frac{17515744}{81081(D-1)} - \frac{4096}{27(D-4)} \right. \\
& + \frac{170368}{10125(5D-14)} - \frac{19968}{1375(5D-16)} - \frac{186624}{1625(5D-18)} - \frac{1024}{9(D-4)^2} - \frac{196256}{375} \left. \right) \\
& + \frac{B52}{s_{12}} \left( + \frac{544D}{25} + \frac{640}{63(3D-10)} + \frac{128}{9(3D-8)} + \frac{5925536}{81081(D-1)} - \frac{2048}{27(D-4)} \right. \\
& - \frac{136576}{10125(5D-14)} - \frac{17664}{1375(5D-16)} - \frac{93312}{1625(5D-18)} - \frac{512}{9(D-4)^2} - \frac{143264}{1125} \left. \right) \\
& -s_{12}A71 \frac{64(D-3)(D-4)(5D^3-62D^2+236D-288)}{(D-1)(5D-16)(5D-14)(5D-18)}
\end{aligned}$$

Inserting the expansion of the three-loop master integrals and keeping terms through to  $\mathcal{O}(\epsilon^0)$ , we find that the three-loop coefficients are given by

$$\begin{aligned}
\mathcal{F}_3^q = & C_F^3 \left[ -\frac{4}{3\epsilon^6} - \frac{6}{\epsilon^5} + \frac{1}{\epsilon^4} (+2\zeta_2 - 25) - \frac{1}{\epsilon^3} \left( +3\zeta_2 - \frac{100\zeta_3}{3} + 83 \right) \right. \\
& + \frac{1}{\epsilon^2} \left( +\frac{213\zeta_2^2}{10} - \frac{77\zeta_2}{2} + 138\zeta_3 - \frac{515}{2} \right) \\
& + \frac{1}{\epsilon} \left( +\frac{1461\zeta_2^2}{20} - \frac{214\zeta_2\zeta_3}{3} - \frac{467\zeta_2}{2} + \frac{2119\zeta_3}{3} + \frac{644\zeta_5}{5} - \frac{9073}{12} \right) \\
& - \left( -\frac{1961387\zeta_2^3}{12600} - \frac{19075\zeta_2^2}{24} - \frac{30883\zeta_2\zeta_3}{15} - \frac{2669\zeta_3^2}{3} + \frac{24x_{91}}{5} \right. \\
& \left. - \frac{24x_{92}}{5} - \frac{6x_{94}}{5} + \frac{95137\zeta_2}{60} - \frac{5569\zeta_3}{5} + \frac{16642\zeta_5}{5} - \frac{26871}{8} \right) \Big] \\
& + C_F^2 C_A \left[ +\frac{11}{3\epsilon^5} - \frac{1}{\epsilon^4} \left( +2\zeta_2 - \frac{431}{18} \right) - \frac{1}{\epsilon^3} \left( +\frac{7\zeta_2}{6} + 26\zeta_3 - \frac{6415}{54} \right) \right. \\
& - \frac{1}{\epsilon^2} \left( +\frac{83\zeta_2^2}{5} - \frac{1487\zeta_2}{36} + 210\zeta_3 - \frac{79277}{162} \right) \\
& - \frac{1}{\epsilon} \left( +\frac{9839\zeta_2^2}{72} - \frac{215\zeta_2\zeta_3}{3} - \frac{38623\zeta_2}{108} + \frac{6703\zeta_3}{6} + 142\zeta_5 - \frac{1773839}{972} \right) \\
& + \left( -\frac{1118529\zeta_2^3}{2800} - \frac{11155817\zeta_2^2}{10800} - \frac{92554\zeta_2\zeta_3}{45} - \frac{36743\zeta_3^2}{30} + \frac{24x_{91}}{5} \right. \\
& \left. - \frac{16x_{92}}{5} - \frac{9x_{94}}{5} + \frac{4239679\zeta_2}{1620} - \frac{121753\zeta_3}{30} + \frac{610462\zeta_5}{225} + \frac{20003431}{29160} \right) \Big] \\
& + C_F C_A^2 \left[ -\frac{242}{81\epsilon^4} + \frac{1}{\epsilon^3} \left( +\frac{88\zeta_2}{27} - \frac{6521}{243} \right) - \frac{1}{\epsilon^2} \left( +\frac{88\zeta_2^2}{45} + \frac{553\zeta_2}{81} - \frac{1672\zeta_3}{27} + \frac{40289}{243} \right) \right. \\
& + \frac{1}{\epsilon} \left( +\frac{802\zeta_2^2}{15} - \frac{88\zeta_2\zeta_3}{9} - \frac{68497\zeta_2}{486} + \frac{12106\zeta_3}{27} - \frac{136\zeta_5}{3} - \frac{1870564}{2187} \right) \\
& - \left( -\frac{4741699\zeta_2^3}{50400} - \frac{4042277\zeta_2^2}{10800} - \frac{5233\zeta_2\zeta_3}{12} - \frac{63043\zeta_3^2}{180} + x_{91} \right. \\
& \left. - \frac{2x_{92}}{5} - \frac{3x_{94}}{5} + \frac{3486997\zeta_2}{2916} - \frac{3062512\zeta_3}{1215} + \frac{202279\zeta_5}{450} + \frac{88822328}{32805} \right) \Big]
\end{aligned}$$

$$\begin{aligned}
& +C_F^2 N_F \left[ -\frac{2}{3\epsilon^5} - \frac{37}{9\epsilon^4} - \frac{1}{\epsilon^3} \left( +\frac{\zeta_2}{3} + \frac{545}{27} \right) - \frac{1}{\epsilon^2} \left( +\frac{133\zeta_2}{18} - \frac{146\zeta_3}{9} + \frac{6499}{81} \right) \right. \\
& \quad + \frac{1}{\epsilon} \left( +\frac{337\zeta_2^2}{36} - \frac{2849\zeta_2}{54} + \frac{2557\zeta_3}{27} - \frac{138865}{486} \right) \\
& \quad \left. + \left( +\frac{8149\zeta_2^2}{216} - \frac{343\zeta_2\zeta_3}{9} - \frac{45235\zeta_2}{162} + \frac{51005\zeta_3}{81} + \frac{278\zeta_5}{45} - \frac{2732173}{2916} \right) \right] \\
& +C_F C_A N_F \left[ +\frac{88}{81\epsilon^4} - \frac{1}{\epsilon^3} \left( +\frac{16\zeta_2}{27} - \frac{2254}{243} \right) + \frac{1}{\epsilon^2} \left( +\frac{316\zeta_2}{81} - \frac{256\zeta_3}{27} + \frac{13679}{243} \right) \right. \\
& \quad - \frac{1}{\epsilon} \left( +\frac{44\zeta_2^2}{5} - \frac{11027\zeta_2}{243} + \frac{6436\zeta_3}{81} - \frac{623987}{2187} \right) \\
& \quad \left. - \left( +\frac{1093\zeta_2^2}{27} - \frac{368\zeta_2\zeta_3}{9} - \frac{442961\zeta_2}{1458} + \frac{45074\zeta_3}{81} + \frac{208\zeta_5}{3} - \frac{8560052}{6561} \right) \right] \\
& +C_F N_F^2 \left[ -\frac{8}{81\epsilon^4} - \frac{188}{243\epsilon^3} - \frac{1}{\epsilon^2} \left( +\frac{4\zeta_2}{9} + \frac{124}{27} \right) - \frac{1}{\epsilon} \left( +\frac{94\zeta_2}{27} - \frac{136\zeta_3}{81} + \frac{49900}{2187} \right) \right. \\
& \quad \left. - \left( +\frac{83\zeta_2^2}{135} + \frac{62\zeta_2}{3} - \frac{3196\zeta_3}{243} + \frac{677716}{6561} \right) \right] \\
& +C_F N_{F,V} \left[ -\left( +\frac{2\zeta_2^2}{5} - 10\zeta_2 - \frac{14\zeta_3}{3} + \frac{80\zeta_5}{3} - 4 \right) \right] \left( \frac{N^2 - 4}{N} \right) \tag{6.3}
\end{aligned}$$

where the last term is generated by graphs where the virtual gauge boson does not couple directly to the final-state quarks. This contribution is denoted by  $N_{F,V}$  and is proportional to the charge weighted sum of the quark flavours. In the case of purely electromagnetic interactions, we find,

$$N_{F,\gamma} = \frac{\sum_q e_q}{e_q}. \tag{6.4}$$

The pole contributions of  $\mathcal{F}_3^q$  are given in eq. (3.7) of ref. [53] while the finite parts of the  $N_F^2$ ,  $C_A N_F$  and  $C_F N_F$  contributions are given in eq. (6) of ref. [52]. The remaining finite contributions are given in eqs. (8) and (9) of ref. [66].

Similarly, the expansion of the gluon form factor at three-loops is given by

$$\begin{aligned}
\mathcal{F}_3^g = & C_A^3 \left[ -\frac{4}{3\epsilon^6} + \frac{11}{3\epsilon^5} + \frac{361}{81\epsilon^4} - \frac{1}{\epsilon^3} \left( +\frac{517\zeta_2}{54} - \frac{22\zeta_3}{3} + \frac{3506}{243} \right) \right. \\
& + \frac{1}{\epsilon^2} \left( +\frac{247\zeta_2^2}{90} + \frac{481\zeta_2}{162} - \frac{209\zeta_3}{27} - \frac{17741}{243} \right) \\
& - \frac{1}{\epsilon} \left( +\frac{3751\zeta_2^2}{360} + \frac{85\zeta_2\zeta_3}{9} - \frac{20329\zeta_2}{243} - \frac{241\zeta_3}{9} + \frac{878\zeta_5}{15} + \frac{145219}{2187} \right) \\
& - \left( +\frac{33539\zeta_2^3}{224} - \frac{280069\zeta_2^2}{2160} - \frac{1821\zeta_2\zeta_3}{4} - \frac{545\zeta_3^2}{36} + x_{91} \right. \\
& \left. \left. - 2x_{92} - \frac{384479\zeta_2}{2916} + \frac{370649\zeta_3}{486} + \frac{66421\zeta_5}{90} - \frac{14423912}{6561} \right) \right] \\
& + C_A^2 N_F \left[ -\frac{2}{3\epsilon^5} - \frac{2}{81\epsilon^4} + \frac{1}{\epsilon^3} \left( +\frac{47\zeta_2}{27} + \frac{1534}{243} \right) - \frac{1}{\epsilon^2} \left( +\frac{425\zeta_2}{81} - \frac{518\zeta_3}{27} - \frac{4280}{243} \right) \right. \\
& + \frac{1}{\epsilon} \left( +\frac{2453\zeta_2^2}{180} - \frac{7561\zeta_2}{243} + \frac{1022\zeta_3}{81} - \frac{92449}{2187} \right) \\
& \left. + \left( +\frac{437\zeta_2^2}{60} - \frac{439\zeta_2\zeta_3}{9} - \frac{37868\zeta_2}{729} - \frac{754\zeta_3}{27} + \frac{3238\zeta_5}{45} - \frac{10021313}{13122} \right) \right] \\
& + C_A C_F N_F \left[ +\frac{20}{9\epsilon^3} - \frac{1}{\epsilon^2} \left( +\frac{160\zeta_3}{9} - \frac{526}{27} \right) - \frac{1}{\epsilon} \left( +\frac{176\zeta_2^2}{15} + \frac{22\zeta_2}{3} + \frac{224\zeta_3}{27} - \frac{2783}{81} \right) \right. \\
& \left. - \left( +\frac{16\zeta_2^2}{5} - 48\zeta_2\zeta_3 + \frac{41\zeta_2}{3} - \frac{11792\zeta_3}{81} - \frac{32\zeta_5}{9} + \frac{155629}{486} \right) \right] \\
& + C_F^2 N_F \left[ +\frac{2}{3\epsilon} + \left( +\frac{296\zeta_3}{3} - 160\zeta_5 + \frac{304}{9} \right) \right] \\
& + C_A N_F^2 \left[ -\frac{8}{81\epsilon^4} - \frac{80}{243\epsilon^3} + \frac{1}{\epsilon^2} \left( +\frac{20\zeta_2}{27} + \frac{8}{9} \right) + \frac{1}{\epsilon} \left( +\frac{200\zeta_2}{81} + \frac{664\zeta_3}{81} + \frac{34097}{2187} \right) \right. \\
& \left. + \left( +\frac{797\zeta_2^2}{135} + \frac{76\zeta_2}{27} + \frac{11824\zeta_3}{243} + \frac{1479109}{13122} \right) \right] \\
& + C_F N_F^2 \left[ +\frac{8}{9\epsilon^2} - \frac{1}{\epsilon} \left( +\frac{32\zeta_3}{3} - \frac{424}{27} \right) - \left( +\frac{112\zeta_2^2}{15} + \frac{16\zeta_2}{3} + \frac{704\zeta_3}{9} - \frac{10562}{81} \right) \right]
\end{aligned} \tag{6.5}$$

The divergent parts agree with eq. (8) of ref. [52] while the finite contributions agree with eq. (10) of ref. [66]. The relations between  $(x_{92}, x_{94})$  and  $(a_{92a}, a_{94b})$  are as follows:

$$x_{92} = -a_{92a} - \frac{55}{18}\pi^2 + \frac{50}{9}\zeta_3 + \frac{83}{216}\pi^4 - \frac{43637}{544320}\pi^6 - \frac{92}{9}\zeta_3^2 + \frac{1}{2}\zeta_5 - \frac{1177}{216}\pi^2\zeta_3 \tag{6.6}$$

$$x_{94} = -a_{94b} + \frac{17}{4}\pi^2 - \frac{14}{9}\zeta_3 + \frac{2939}{4320}\pi^4 - \frac{1285}{217728}\pi^6 - \frac{217}{18}\zeta_3^2 + \frac{8}{15}\zeta_5 - \frac{503}{27}\pi^2\zeta_3 \quad (6.7)$$

where  $x_{91}$  is:

$$x_{91} = \frac{2133}{2} - \frac{97}{8}\pi^2 + \frac{4717}{2880}\pi^4 + \frac{76801}{186624}\pi^6 - \frac{287}{2}\zeta_3 + \frac{2969}{216}\pi^2\zeta_3 + \frac{5521}{36}\zeta_3^2 - \frac{8251}{30}\zeta_5 \quad (6.8)$$

# Chapter 7

## Conclusions and Outlook

The main purpose of this thesis is the calculation of the quark and gluon form factors at three loops. It is one of the main ingredients necessary for the calculation of the standard model Drell-Yan and Higgs production via gluon fusion cross sections at NNNLO. The Higgs production process is particularly important since gluon fusion is expected to be the largest of the four Higgs production mechanisms at the LHC. Because of the very large NLO and NNLO corrections to Higgs gluo-production, the NNNLO corrections may be necessary for the determination of the Higgs properties in the experiments at the Large Hadron Collider.

The three-loop form factors represent highly non-trivial calculations, but with the help of the new techniques and powerful computer supplies they became manageable and possible.

In the first chapter we gave a brief introduction to the basics of QCD. The Feynman rules derived from the QCD Lagrangian were given and the concept of regularization explained. We used Dimensional Regularization (DR) and showed the steps going from four to  $D$  dimensions. The divergences are exposed as poles in  $\epsilon$ . In order to absorb the singularities and get an UV-finite Green's functions we employed multiplicative renormalization scheme with using Modified Minimal subtraction. At the end it was shown how the strong interaction coupling  $\alpha_s$  runs with the scale  $\mu$  known as asymptotic freedom.

Loop integrals are introduced with some simple examples in the second chapter. We introduce some basic methods for evaluating loop integrals which are efficient

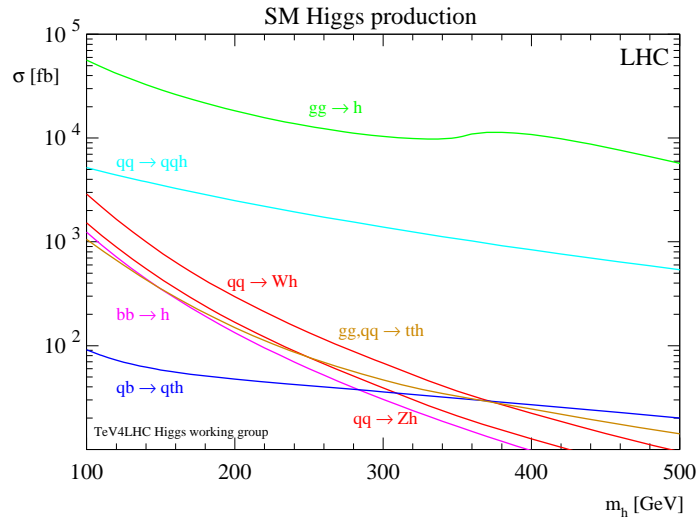


Figure 7.1: Standard Model Higgs production at the LHC [2]

and applicable for most of the calculations. As mentioned in chapter two, the Mellin-Barnes technique turned out to be widely accepted in multi-loop calculations and there some packages publicly available for this purpose [19–21, 67–70]. At one-loop and beyond one naturally encounters complicated functions such as hypergeometric functions and Zeta functions and we expect to see these functions frequently in the future calculations.

The three-loop quark and gluon form factor calculations involve a huge number of scalar and tensor integrals. In chapter three, the derivation of the integration by parts (IBP) procedure explained and showed how it works in practice with some examples. In order to deal with the relatively difficult tensor integrals we employed the Laporta algorithm in the reduction process. In our calculations we used the mathematica package FIRE. All the inputs and outputs can be found in the appendix section for both form factors. The main disadvantage of the code is that it can only use a single cpu. For difficult tensor integrals the termination time is very long and one may encounter hardware problems. We hope that for the future applications the code would be parallelized.

In the fourth chapter, we gave the definitions of the form factors and explained how to extract them from Feynman diagrams using projection operators. We calculated the one-loop and two-loop form factors again to make sure of that our

codes are working properly and cross checked our results with those in the literature [45–49, 56, 71].

Our approach for the calculation of the form factors at three-loops was based on the auxiliary diagrams introduced in chapter five. We found three auxiliary diagrams, figs. (5.1), (5.4) and (5.5), from which all three-loop integrals can be extracted out by pinching the relevant propagators. This is the core of our calculation. By mapping all the integrals onto these auxiliary integrals, we managed to reduce them to scalar integrals. Explicit expressions for all the master integrals are given up to the desired order in  $\epsilon$ . The coefficients of the  $\epsilon^0$  terms in the last two master integrals  $A92$  and  $A94$  are numerically known [66], but not analytically. We expect that these two unknown coefficients will be evaluated soon.

As we mentioned in the results chapter, wherever there is an overlap, our findings agree with the previous works. The pole and finite parts of the three-loop quark and gluon form factors coincide with those in refs. [52–55, 63–66].

The results presented in this thesis are only a small contribution to the great efforts by the particle physics community in studying the fundamental constituents of nature. We think that our result is an important building block for some physical applications. Among them are the hadronic cross section in  $e^+e^-$  collisions, lepton pair production via the Drell-Yan mechanism and Higgs boson gluon-production in hadron colliders. The three loop gluon form factor is an ingredient to the NNNLO Higgs cross section at the LHC and may be important for hunting for the Higgs boson.

We finish this thesis by listing the matrix elements that would be needed for computing the Higgs boson production via gluon fusion at this high accuracy. The theoretical uncertainties at NNNLO level will be smaller than the existing NNLO estimates thereby enabling improved descriptions of the high energy phenomena [72–76]. At this order, one encounters the three-loop  $Hgg$ , the two-loop  $Hggg$ , the one loop  $Hgggg$  and the tree-level  $Hggggg$  amplitudes sketched in Fig. (7.2). All the other contributions at this order can be found in various papers [71, 75, 77–87] and references therein.



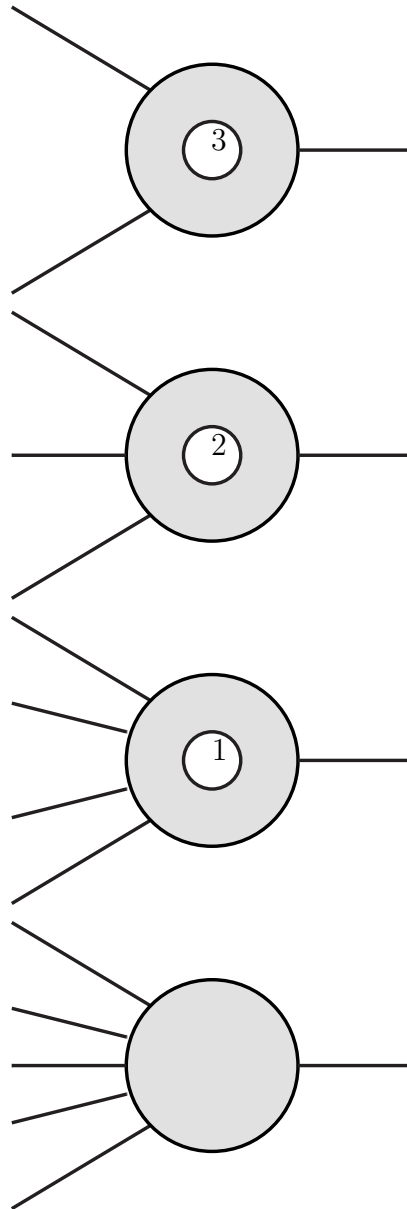


Figure 7.2: Amplitudes that contribute to the NNNLO contributions to Higgs boson production via gluon fusion.

# Appendix A

## Trace and Color Structure of QCD Scattering Amplitudes

### A.1 Dirac Algebra in Dimension $D$

The definition of  $\gamma$  in  $D$  dimension is:

$$\gamma_0, \gamma_{\mu 1}, \dots, \gamma_{\mu D} \quad (\text{A.1.1})$$

$$\gamma^0 \text{ is Hermitian, } \gamma^i, i > 0 \text{ anti-Hermitian :} \quad (\text{A.1.2})$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0. \quad (\text{A.1.3})$$

The algebra is defined through the usual anti-commutation relations;

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \hat{\mathbf{1}}. \quad (\text{A.1.4})$$

$$g_\mu^\mu = D \text{ and therefore } \gamma_\mu^\mu = D \hat{\mathbf{1}}. \quad (\text{A.1.5})$$

Using the relations above, one can derive some useful identities:

$$\gamma^\mu \gamma_\alpha \gamma_\mu = (2 - D) \gamma_\alpha \quad (\text{A.1.6})$$

$$\gamma^\mu \gamma_\alpha \gamma_\beta \gamma_\mu = (D - 4) \gamma_\alpha \gamma_\beta + 4g_{\alpha\beta} \quad (\text{A.1.7})$$

$$\gamma^\mu \gamma_\alpha \gamma_\beta \gamma_\delta \gamma_\mu = -2\gamma_\delta \gamma_\beta \gamma_\alpha - (D - 4) \gamma_\alpha \gamma_\beta \gamma_\delta \quad (\text{A.1.8})$$

$$\gamma^\mu \sigma_{\alpha\beta} \gamma_\mu = (D - 4) \sigma_{\alpha\beta} \quad (\text{A.1.9})$$

$$\gamma^\mu \sigma_{\alpha\mu} = i(D - 1) \gamma_\alpha \quad (\text{A.1.10})$$

$$\sigma_{\mu\alpha} \gamma^\mu = i(1 - D) \gamma_\alpha \quad (\text{A.1.11})$$

where we have in the usual way:

$$\begin{aligned}\sigma^{\mu\nu} &= \frac{i}{2} [\gamma^\mu, \gamma^\nu] \\ \gamma_\mu \gamma_\nu &= g_{\mu\nu} - i\sigma_{\mu\nu}\end{aligned}\tag{A.1.12}$$

The trace over an odd number of  $\gamma_5$  matrices vanishes:

$$\text{tr}\gamma^\mu = 0\tag{A.1.13}$$

$$\text{tr}\gamma^\mu \gamma^\nu = 4g^{\mu\nu}\tag{A.1.14}$$

$$\text{tr}\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})\tag{A.1.15}$$

The definition of  $\gamma_5$  in 4-dimension is:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\tag{A.1.16}$$

whereas the  $\epsilon_{\mu\nu\rho\sigma}$  tensor is:

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1, & \text{even permutation of (0123)} \\ -1, & \text{odd permutation of (0123)} \\ 0, & \text{otherwise} \end{cases}\tag{A.1.17}$$

From this one gets the following commutation relations:

$$[\gamma_5, \gamma_\mu] = 0 : \mu = 0, 1, 2, 3\tag{A.1.18}$$

In  $D \neq 4$  dimensions, the definition of  $\gamma_5$  is more complicated, but in this thesis we didn't need to use it.

## A.2 Colour Interactions

The gauge transformations of *QCD* satisfy the  $SU(N)$  Lie group properties. The  $SU(N)$  group consists of determinant one acting on a complex  $N$  dimensional vector space. The number of generators of the group is determined by the free parameters of a  $N \times N$  dimensional space. Since the determinant of the group is one, the number of the free parameters is given by  $N^2 - 1$ . The generators of a  $SU(N)$  Lie group obey the following identity:

$$[T^a, T^b] = if^{abc}T^c\tag{A.2.19}$$

$f^{abc}$  is the structure constant. The combination of the commutators in such a way that it gives zero is called *Jacobi Identity*:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \quad (\text{A.2.20})$$

and using equation A2.19, A2.20 can be written in the form:

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \quad (\text{A.2.21})$$

There are two representations of the Lie algebra for the  $SU(N)$ . In the fundamental representation the generators are denoted by  $T_{ij}^a$  where  $a$  is a colour index and runs from 1 to  $N^2 - 1$ , and  $i, j$  are the group indices run from 1 to  $N$ . *Pauli* and *Gell-mann* matrices are the examples of this representation. It has the following properties:

$$\text{tr} [T^a] = T_{ij}^a = 0 \quad (\text{A.2.22})$$

$$(T_{ij}^a)^\dagger = T_{ij}^a \quad (\text{A.2.23})$$

In the adjoint representation the generators are denoted by  $F^{abc}$  where  $a, b, c$  are all colour indices and run from 1 to  $N^2 - 1$ . They are defined as:

$$F^{abc} = i f^{abc} \quad (\text{A.2.24})$$

The conversion from the adjoint to the fundamental representation is:

$$f^{abc} = -2i \text{tr} \{ [T^a, T^b] T^c \} \quad (\text{A.2.25})$$

The generators are normalized using the following identities,

$$\text{tr} [T^a T^b] = \frac{1}{2} \delta^{ab} \quad (\text{A.2.26})$$

$$\text{tr} [F^a F^b] = N \delta^{ab} \quad (\text{A.2.27})$$

where we have dropped any indices which are summed over. The products of two generators in the fundamental representation is called the *Fierz Identity*,

$$T_{ij}^a T_{kl}^a = \frac{1}{2} (\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl}) \quad (\text{A.2.28})$$

If each generator has only one free index, is known as *Casimir operators*,

$$T_{ij}^a T_{ik}^a = \frac{N^2 - 1}{2N} \delta_{jk} \quad (\text{A.2.29})$$

and in the adjoint representation,

$$F^{acd} F^{bcd} = N \delta^{ab} \quad (\text{A.2.30})$$

## A.3 FIRE Input Codes for Auxiliary Integrals

In this section we list the input files for the planar fig. (5.1) and non-planar figs. (5.4-5.5) auxiliary diagrams. As mentioned in Chapter three, we run FIRE in three steps. First input prepares the .start and .data files for the actual run of the program. Second input is the main code for the evaluation of the master integrals. In the third input we substitute and name of the master integrals that can be processed by our FORM code.

### A.3.1 Planar Diagram A

INPUT 1:

```
(1)  Get["FIRE_3.4.0.m"];
(2)  Get["IBP.m"];
(3)  UsingFermat=True;
(4)  DirectIBP=False;
(5)  LeeIdeas=True;
(6)  Internal = {k1, k2, k3};
(7)  External = {p1, p2};
(8)  Propagators = {k12, (k1 + p1)2, (k1+p1+p2)2,
k22, (k2+p1)2, (k2+p1+p2)2, k32, (k3+p1)2, (k3+p1+p2)2,
(k1-k2)2, (k2-k3)2, (k3-k1)2};
(9)  PrepareIBP[];
(10) reps = {p12 → 0, p22 → 0, p1*p2 → s12/2};
(11) startinglist = {IBP[k1,k1], IBP[k1,k1+p1], IBP[k1,k1+p1+p2],
IBP[k2,k2], IBP[k2, k2+p1], IBP[k2,k2+p1+p2], IBP[k3,k3],
IBP[k3,k3+p1], IBP[k3,k3+p1+p2], IBP[k1,k1-k2],
IBP[k1,k3-k1], IBP[k2,k1-k2], IBP[k2,k2-k3],
IBP[k3,k2-k3], IBP[k3,k3-k1]} /. reps
(12) r=Get["zero"];
(13) RESTRICTIONS = r;
(14) SYMMETRIES = {
```

```
{7,8,9,4,5,6,1,2,3,11,10,12},{1,2,3,7,8,9,4,5,6,12,11,10},  
{4,5,6,7,8,9,1,2,3,11,12,10},{7,8,9,1,2,3,4,5,6,12,10,11},  
{3,2,1,6,5,4,9,8,7,10,11,12},{4,5,6,1,2,3,7,8,9,10,12,11}};  
(15) Prepare[];  
(16) SaveStart["loop3int1"];  
(17) Burn[];  
(18) SaveData["loop3int1.data"];  
(19) Quit[]
```

INPUT 2:

```
(1) Get["FIRE_3.4.0.m"];  
(2) UsingIBP=True;  
(3) UsingFermat=True;  
(4) DirectIBP=False;  
(5) LeeIdeas=True;  
(6) LoadStart["loop3int1",1];  
(7) Burn[];  
(8) r=Get["int1"];  
(9) EvaluateAndSave[r,"int1.Tables"];  
(10) Quit[]
```

INPUT 3:

```
(1) Get["FIRE_3.4.0.m"];  
(2) UsingIBP=True;  
(3) UsingFermat=True;  
(4) DirectIBP=False;  
(5) LeeIdeas=True;  
(6) LoadStart["loop3int1",1];  
(7) Burn[];
```

```

(8)   r=Get["int1"];
(9)   LoadTables["int1.Tables"];
(10)  st=OpenWrite["int1-r",PageWidth→Infinity];
(11)  Do[Write[st,Id,r[[i,2]], "=",F[1,r[[i,2]]]/.{
G[1, {0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1}]→A41,
G[1, {0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1}]→A52,
G[1, {0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1}]→A62,
G[1, {0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1}]→B52,
G[1, {0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}]→A51,
G[1, {0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1}]→A63,
G[1, {1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1}]→A61,
G[1, {0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1}]→B51,
G[1, {0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1}]→A73,
G[1, {1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1}]→C61,
G[1, {0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1}]→B62,
G[1, {1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0}]→B61,
G[1, {0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1}]→A91,
s→s12,d→D},";"],{i,1,Length[r]};
(12)  Quit[]

```

### A.3.2 Non-Planar Diagram B

INPUT 1:

```

(1)   Get["FIRE_3.4.0.m"];
(2)   Get["IBP.m"];
(3)   UsingFermat=True;
(4)   DirectIBP=False;
(5)   LeeIdeas=True;
(6)   Internal = {k1, k2, k3};
(7)   External = {p1, p2};
(8)   Propagators = {k12, (k1 + p1 + p2)2, k22,

```

```

(k2+p1)2, k32, (k3+p1)2, (k3+p1+p2)2,
(k1-k2)2, (k2-k3)2, (k3-k1)2
(k1-k2+p2)2, (k3-k1-p2)2};
(9) PrepareIBP[];
(10) reps = {p12 → 0, p22 → 0, p1*p2 → s12/2};
(11) startinglist = {IBP[k1,k1], IBP[k1,k1+p1+p2], IBP[k1,k3-k1-p2],
IBP[k2,k2], IBP[k2,k2+p1], IBP[k2,k1-k2+p2], IBP[k3,k3],
IBP[k3,k3+p1], IBP[k3,k3+p1+p2], IBP[k1,k1-k2],
IBP[k1,k3-k1], IBP[k2,k1-k2], IBP[k2,k2-k3],
IBP[k3,k2-k3], IBP[k3,k3-k1]} /. reps
(12) r=Get["zero"];
(13) RESTRICTIONS = r;
(14) SYMMETRIES ={}
(15) Prepare[];
(16) SaveStart["loop3int2"];
(17) Burn[];
(18) SaveData["loop3int2.data"];
(19) Quit[]

```

INPUT 2:

```

(1) Get["FIRE_3.4.0.m"];
(2) UsingIBP=True;
(3) UsingFermat=True;
(4) DirectIBP=False;
(5) LeeIdeas=True;
(6) LoadStart["loop3int2",1];
(7) Burn[];
(8) r=Get["int2"];
(9) EvaluateAndSave[r,"int2.Tables"];
(10) Quit[]

```



INPUT 3:

```
(1)  Get["FIRE_3.4.0.m"];
(2)  UsingIBP=True;
(3)  UsingFermat=True;
(4)  DirectIBP=False;
(5)  LeeIdeas=True;
(6)  LoadStart["loop3int2",1];
(7)  Burn[];
(8)  r=Get["int2"];
(9)  LoadTables["int2.Tables"];
(10) st=OpenWrite["int2-r",PageWidth→Infinity];
(11) Do[Write[st,Id,r[[i,2]], "=",F[1,r[[i,2]]]/.{
G[1,{0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0}]→A41,
G[1,{0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0}]→A41,
G[1,{0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1}]→A41,
G[1,{0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0}]→A41,
G[1,{0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0}]→A41,
G[1,{1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0}]→A41,
G[1,{1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1}]→A41,
G[1,{0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0}]→A51,
G[1,{0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0}]→A51,
G[1,{0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0}]→A51,
G[1,{1, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0}]→A51,
G[1,{1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0}]→A51,
G[1,{1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0}]→A51,
G[1,{1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1}]→A51,
G[1,{0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0}]→A52,
G[1,{0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0, 0}]→A52,
G[1,{0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1}]→A52,
```

$G[1, \{0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0\}] \rightarrow A52,$   
 $G[1, \{1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A52,$   
 $G[1, \{1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1\}] \rightarrow A52,$   
 $G[1, \{1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1\}] \rightarrow A52,$   
 $G[1, \{1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0\}] \rightarrow A52,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0\}] \rightarrow A52,$   
 $G[1, \{1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0\}] \rightarrow B52,$   
 $G[1, \{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0\}] \rightarrow B52,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0\}] \rightarrow B51,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0\}] \rightarrow B51,$   
 $G[1, \{0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0\}] \rightarrow A63,$   
 $G[1, \{0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0\}] \rightarrow A63,$   
 $G[1, \{1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0\}] \rightarrow A63,$   
 $G[1, \{1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1\}] \rightarrow A63,$   
 $G[1, \{1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A63,$   
 $G[1, \{0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A62,$   
 $G[1, \{0, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0\}] \rightarrow A62,$   
 $G[1, \{0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A62,$   
 $G[1, \{0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0\}] \rightarrow A62,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A62,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1\}] \rightarrow A62,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1\}] \rightarrow A62,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A62,$   
 $G[1, \{1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A62,$   
 $G[1, \{1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1\}] \rightarrow A63,$

$G[1, \{1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0\}] \rightarrow A63,$   
 $G[1, \{1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0\}] \rightarrow C61,$   
 $G[1, \{1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0\}] \rightarrow C61,$   
 $G[1, \{1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 1\}] \rightarrow A61,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0\}] \rightarrow A73,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0\}] \rightarrow A73,$   
 $G[1, \{0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0\}] \rightarrow A73,$   
 $G[1, \{1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0\}] \rightarrow A73,$   
 $G[1, \{0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0\}] \rightarrow A74,$   
 $G[1, \{0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1\}] \rightarrow A71,$   
 $G[1, \{0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0\}] \rightarrow A71,$   
 $G[1, \{1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0\}] \rightarrow A71,$   
 $G[1, \{1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0\}] \rightarrow A72,$   
 $G[1, \{1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A72,$   
 $G[1, \{1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0\}] \rightarrow A74,$   
 $G[1, \{1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1\}] \rightarrow A72,$   
 $G[1, \{1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A75,$   
 $G[1, \{1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1\}] \rightarrow A72,$   
 $G[1, \{1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A75,$   
 $G[1, \{1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1\}] \rightarrow A74,$   
 $G[1, \{1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A72,$   
 $G[1, \{0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0\}] \rightarrow A81,$   
 $G[1, \{1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0\}] \rightarrow C81,$   
 $G[1, \{1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1\}] \rightarrow A94,$   
 $G[1, \{0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0\}] \rightarrow A62,$   
 $G[1, \{0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0\}] \rightarrow B52,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0\}] \rightarrow A75,$   
 $G[1, \{0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0\}] \rightarrow A73,$   
 $G[1, \{1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1\}] \rightarrow B52,$

```

G[1,{1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0}]→A52,
G[1,{1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0}]→A62,
G[1,{1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1}]→A71,
s→s12,d→D}, " ;"],{i,1,Length[r]}}]
(12) Quit[]

```

### A.3.3 Non-Planar Diagram C

INPUT 1:

```

(1) Get["FIRE_3.4.0.m"];
(2) Get["IBP.m"];
(3) UsingFermat=True;
(4) DirectIBP=False;
(5) LeeIdeas=True;
(6) Internal = {k1, k2, k3};
(7) External = {p1, p2};
(8) Propagators = {k12, (k1 + p1)2, (k1 + p1 + p2)2,
k22, k32, (k3+p1)2, (k3+p1+p2)2,
(k1-k2)2, (k2-k3)2, (k3-k1)2
(k1-k2+p1)2, (k1-k2+p1+p2)2};
(9) PrepareIBP[];
(10) reps = {p12 → 0, p22 → 0, p1*p2 → s12/2};
(11) startinglist = {IBP[k1,k1], IBP[k1,k1+p1], IBP[k1,k1+p1+p2],
IBP[k2,k2], IBP[k2,k1-k2+p1], IBP[k2,k1-k2+p1+p2], IBP[k3,k3],
IBP[k3,k3+p1], IBP[k3,k3+p1+p2], IBP[k1,k1-k2],
IBP[k1,k3-k1], IBP[k2,k1-k2], IBP[k2,k2-k3],
IBP[k3,k2-k3], IBP[k3,k3-k1]} /. reps
(12) r=Get["zero"];
(13) RESTRICTIONS = r;
(14) SYMMETRIES ={}
(15) Prepare[];

```

```
(16) SaveStart["loop3int8"];
(17) Burn[];
(18) SaveData["loop3int8.data"];
(19) Quit[]
```

INPUT 2:

```
(1) Get["FIRE_3.4.0.m"];
(2) UsingIBP=True;
(3) UsingFermat=True;
(4) DirectIBP=False;
(5) LeeIdeas=True;
(6) LoadStart["loop3int8",1];
(7) Burn[];
(8) r=Get["int8"];
(9) EvaluateAndSave[r,"int8.Tables"];
(10) Quit[]
```

INPUT 3:

```
(1) Get["FIRE_3.4.0.m"];
(2) UsingIBP=True;
(3) UsingFermat=True;
(4) DirectIBP=False;
(5) LeeIdeas=True;
(6) LoadStart["loop3int8",1];
(7) Burn[];
(8) r=Get["int8"];
(9) LoadTables["int8.Tables"];
(10) st=OpenWrite["int8-r",PageWidth→Infinity];
(11) Do[Write[st,Id,r[[i,2]], "=",F[1,r[[i,2]]]}/. {
```

$G[1, \{0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0\}] \rightarrow A41,$   
 $G[1, \{0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1\}] \rightarrow A41,$   
 $G[1, \{0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1\}] \rightarrow A41,$   
 $G[1, \{0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0\}] \rightarrow A41,$   
 $G[1, \{0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0\}] \rightarrow A41,$   
 $G[1, \{0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0\}] \rightarrow A41,$   
 $G[1, \{0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1\}] \rightarrow B51,$   
 $G[1, \{0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1\}] \rightarrow B51,$   
 $G[1, \{0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1\}] \rightarrow B51,$   
 $G[1, \{0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0\}] \rightarrow B51,$   
 $G[1, \{1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1\}] \rightarrow B51,$   
 $G[1, \{1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1\}] \rightarrow B51,$   
 $G[1, \{0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1\}] \rightarrow B52,$   
 $G[1, \{0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1\}] \rightarrow B52,$   
 $G[1, \{0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0\}] \rightarrow B52,$   
 $G[1, \{0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0\}] \rightarrow B52,$   
 $G[1, \{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1\}] \rightarrow B52,$   
 $G[1, \{0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1\}] \rightarrow A52,$   
 $G[1, \{0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0\}] \rightarrow A52,$   
 $G[1, \{0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0\}] \rightarrow A52,$   
 $G[1, \{0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\}] \rightarrow A51,$   
 $G[1, \{0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0\}] \rightarrow A51,$   
 $G[1, \{0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1\}] \rightarrow A51,$   
 $G[1, \{0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0\}] \rightarrow A51,$   
 $G[1, \{0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0\}] \rightarrow A51,$   
 $G[1, \{0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1\}] \rightarrow A52,$   
 $G[1, \{0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0\}] \rightarrow A52,$   
 $G[1, \{0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0\}] \rightarrow A52,$   
 $G[1, \{0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0\}] \rightarrow A52,$

```

G[1,{0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0}]→A52,
G[1,{0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1}]→B62,
G[1,{0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1}]→B62,
G[1,{0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0}]→A62,
G[1,{0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1}]→A62,
G[1,{0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0}]→A62,
G[1,{0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0}]→A62,
G[1,{0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0}]→A62,
G[1,{0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0}]→A62,
G[1,{0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0}]→A63,
G[1,{0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0}]→A63,
G[1,{1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1}]→B61,
G[1,{0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1}]→A73,
G[1,{0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 1}]→A73,
G[1,{0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0}]→A73,
G[1,{0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1}]→A73,
G[1,{0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0}]→A73,
G[1,{0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0}]→A73,
G[1,{0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1}]→A73,
G[1,{0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1}]→A75,
G[1,{0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0}]→A75,
G[1,{0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0}]→A72,
G[1,{0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0}]→A74,
G[1,{0, 1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0}]→A71,
G[1,{0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1}]→B81,
G[1,{0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0}]→A81,
G[1,{0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0}]→A92,
s→s12,d→D},";"],{i,1,Length[r]}};

```

(12) Quit[]

## A.4 QGRAF Input Output Files

In this section we present the input and output files of the processes which were evaluated in the thesis.

```
Higgs-Gluon-Gluon .dat Input:
output= 'test-hgg' ;
style= 'array.sty' ;
model= 'hgg';
in= higgs,gluon,gluon ;
out= ;
loops= 3;
loop_momentum=k ;
options= nosnail,onepi,onshell
```

Since the output of the process Higgs-gluon-gluon is very lengthy, we thought to write down only the first diagram of the output to show how it looks like. (+1/4) is the symmetry factor of the particular diagram and the abbreviations are as follows: `pol` : polarization , `prop` : propagator , `VRTX` : vertex .

```
a(1):= (+1/4)*
pol(higgs(-1,p1))*
pol(gluon(-2,q1))*
pol(gluon(-4,q2))*
prop(gluon(1,k1+k2+q1))*
prop(gluon(3,-k1))*
prop(gluon(5,-k2))*
prop(gluon(7,k2+k3+q1))*
prop(gluon(9,-k3))*
VRTX(gluon(-2,-q1),gluon(5,-k2),gluon(1,k1+k2+q1),gluon(3,-k1))*
VRTX(gluon(7,k2+k3+q1),gluon(9,-k3),gluon(2,-k1-k2-q1),gluon(4,k1))*
VRTX(higgs(-1,p1),gluon(-4,-q2),gluon(6,k2),gluon(8,-k2-k3-q1),gluon(10,k3))
```



Photon-Quark-Anti-Quark .dat Input:

```

output= 'test-pqq' ;
style= 'array.sty' ;
model= 'pqq';
in= Photon,Quark,Qbar ;
out= ;
loops= 3;
loop_momentum=k ;
options= nosnail,onepi,onshell

```

(+1/6) is the symmetry factor and the abbreviations are same as well.

```

a(1):= (+1/6)*
pol(Photon(-1,p1))*
pol(Quark(-2,q1))*
pol(Qbar(-4,q2))*
prop(Quark(1,k1))*
prop(Quark(3,k1-p1))*
prop(gluon(5,-k1+q1))*
prop(gluon(7,k1-q1))*
prop(gluon(9,-k1+k2+k3+q1))*
prop(gluon(11,-k2))*
prop(gluon(13,-k3))*
vrtx(Qbar(2,-k1),Z(-1,p1),Quark(3,k1-p1))*
vrtx(Qbar(-2,-q1),gluon(5,-k1+q1),Quark(1,k1))*
vrtx(Qbar(4,-k1+p1),gluon(7,k1-q1),Quark(-4,-q2))*
vrtx(gluon(6,k1-q1),gluon(9,-k1+k2+k3+q1),gluon(11,-k2),gluon(13,-k3))*
vrtx(gluon(8,-k1+q1),gluon(10,k1-k2-k3-q1),gluon(12,k2),gluon(14,k3));

```

# Bibliography

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