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# Steady rolling contact and resistance of a cylinder on a viscoelastic foundation by an integral equation algorithm

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**Steady rolling contact and resistance of a cylinder on a viscoelastic foundation by an  
integral equation algorithm**

by

**Pallavi Padyala**

A thesis submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
**MASTER OF SCIENCE**

Major: Aerospace Engineering

Program of Study Committee:  
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J. Adin Mann

Iowa State University

Ames, Iowa

2009

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## **DEDICATION**

This work is dedicated to my mom and dad.

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**ABSTRACT**

A first-kind integral equation relating the surface contact pressure to surface displacement is used to form an iterative algorithm which efficiently solves the problem of steady rolling of a cylinder on a viscoelastic half-space. The immediate solution is the contact pressure, but the quantity of interest is the friction coefficient (ratio of drag force to normal force), or effective resistance to rolling motion, which is readily obtained from the contact pressure. The so-called generalized N-parameter Maxwell model is used to characterize the rheology and is quite general and capable of representing realistic polymeric, elastomeric and rubber compounds. An iterative algorithm, like the Newton Raphson method, is used to obtain consistent equilibrium values of the contact stress profile and surface displacement for a given total resultant load. The friction coefficient is subsequently determined by integration of the interface pressure about the center of the rolling cylinder. As the employed integral equation formulation is in terms of the creep functions, a conversion of the creep parameters to relaxation parameter is also required. Results are provided for a typical rubber compound, showing the effects of velocity and applied load on friction coefficient.



## CHAPTER 1. INTRODUCTION

### 1.1 Motivation

Energy losses in rolling motion can account for most of the power required to drive many common processes, such as automobile tires on highways, train wheels on steel tracks, drawing or rolling of metals in manufacturing processes, advancement of paper through printing rolls or the motion of conveyors over supporting idlers. This type of energy loss in rolling motions is especially pronounced if the material of either the roller or rolled material undergoes plastic or viscoelastic deformation during the process, such as in metal drawing or rolling of polymer materials. Since most materials do exhibit properties of viscoelasticity under specific circumstances and conditions, the need for rigorous study of these processes is warranted.

Analytical models and specific studies of rolling friction as specialized to various problems are not new, and many of these models can be quite sophisticated and involved, such as those used in the design of automobile tires to minimize energy losses at the tire/road interface. The rubber tire industry has been keenly interested in this issue since the invention of pneumatic tires, where a rubber compounds is the energy losing medium. The same issue is present in the design of large conveyor belts, where a rubber “backing” layer is vulcanized over the surface of the belt that rolls over the carrying idlers. This interface layer is usually a rubber compound which is specifically tailored to minimize energy absorption as it deforms, and rubber compounds generally exhibit viscoelastic behavior as they are deformed. As the steel rolls (idlers) of a conveyor are much stiffer than the rubber belt backing, a simple model of this problem would be that of rigid cylinder rolling on a viscoelastic layer.

The focus of this work is to build on one of the most efficient analytical models already available for this problem and extend it so as to make it a practical design tool for conveyor systems. Specifically, an integral equation formulation of the problem of uniform rolling motion of a cylinder over a viscoelastic half-space is used as a basis of an iterative algorithm to solve this nonlinear contact problem. The result is an efficient

computational methodology that can be effectively used in the design and analysis of conveyor systems.

## 1.2 Literature Review

Several numerical and analytical methods have been developed to solve problems involving rolling and sliding contact with elastic and viscoelastic materials. This work presents an extension of the work of several authors including Gonzalez and Abascal<sup>1</sup>, Kong and Wang<sup>2</sup>, Chertok, Golden and Graham<sup>3</sup>, Lynch<sup>4</sup>, Goryacheva and Sadeghi<sup>5</sup>, Nuttall, Lodewijks and Breteler<sup>6</sup>, Rudolphi and Reicks<sup>7</sup> and Goodier and Loutzenheiser<sup>8</sup>.

Gonzalez and Abascal formulated the problem of steady-state moving loads along the boundary of a 2D viscoelastic material by integral equations of the second kind (boundary elements). They used a three-parameter rheological model (the standard linear solid) along with the boundary element formulation [1].

Similarly, a boundary element method approach for steady rolling contact over a viscoelastic solid was provided by Kong and Wang. The main focus in their work was on calculating the normal and tangential contact pressures [2].

Chertok, Golden and Graham emphasized the determination of the hysteretic friction for the rolling contact problem, where also the standard linear solid model was employed. They developed a numerical algorithm to study the variation of contact interval length, pressure and coefficient of hysteretic friction with varying loads and varying speeds of the indenter [3].

Lynch presented a complete numerical method for solving steady state linear viscoelastic stress analysis problem. The emphasis of this work was a mixed boundary value formulation with special case of viscoelastic sheet rolling to obtain a numerical solution algorithm [4].

An analytical model was developed by Goryacheva and Sadeghi to investigate the effects of viscoelastic layer coatings. The effects of geometry and mechanical properties of a thin viscoelastic layer bonded to an elastic semi-infinite plane in contact with a

rolling/sliding elastic cylinder were investigated. Variation of normal and tangential stress distribution and indentation of the contact zone were presented [5].

Nuttall, Lodewijks and Breteler studied the problem of rolling friction due to hysteresis and the relationship between traction and slip in wheel driven belt conveyors. In this report they have concentrated on comparing the rolling friction at varying speeds with the variation in the impressed normal force. Three-parameter and seven-parameter models are used and variation of traction and rolling friction with respect to the complexity of the model was studied. Important conclusions were made in this report, such as when considering applied traction, there is no significant difference between three-parameter and seven-parameter models, but when predicting the rolling friction due to hysteresis, the complexity of the model has a significant effect.

Rudolphi and Reicks used the mechanical model of cylinder on a one-dimensional viscoelastic bed (Winkler foundation), along with a generalized N-parameter Maxwell model for the viscoelastic behavior, for predicting the indentation and rolling resistance of a linear viscoelastic material backing on a conveyor belt. In that report they have developed a method that can be directly used to design and minimize the power required for conveyor systems. Emphasis was also made on temperature dependent material modeling and variation of indentation resistance with speed of the belt.

As most of the above mentioned researchers have considered only a simplified model of the viscoelastic foundation, or a three-parameter material model practical behavior is not simulated. Real polymeric materials are expected to have a high load rate dependence, for which a three parameter model is deficient. Hence, considering a more complex model is important in order to predict realistic viscoelastic behavior. In the current work, the viscoelastic rolling contact problem is addressed by incorporating a generalized N-parameter Maxwell model.

### **1.3 Thesis Organization**

The first chapter of this thesis provides an introduction to the current work and references some important works that form a foundation in this field. Chapter 2 explains various rheological models that are capable of representing the behavior of realistic viscoelastic materials. Conversion of the relaxation modulus into creep compliance is also described in Chapter 2. Chapter 3 deals with developing the viscoelastic formulation from the classic Boussinesq problem of linear elasticity. The most part of Chapter 3 is a summary of a technical report by Goodier and Loutzenheiser [8], which forms the analytical and integral equation formulation, and foundation for the present work. Chapter 4 presents results for the developed numerical process, providing the contact pressure and also the variation of friction coefficient according to the velocity and also the normal load applied on the backing material. Finally, a discussion and conclusions drawn from this work are presented in Chapter 5.

## CHAPTER 2. VISCOELASTIC PROPERTIES

### 2.1 Viscoelasticity

Viscoelasticity is used to describe the property of a material that exhibits both viscous and elastic properties. Due to the viscous element of this behavior, the relation between stress and strain in such materials depends on time. Mathematical models of viscous materials are often referred to as rheological models and are central to all analytical descriptions of viscoelasticity. Mechanical analogies of these rheological models consist of series or parallel combination of springs and dashpots, where the springs account for the elastic behavior and dashpots represent the time dependent or viscous behavior. Each rheological model differs in the arrangement of these elements.

The elastic components, as mentioned previously, can be modeled as linear springs which obey Hooke's law.

$$\sigma = E\varepsilon \tag{2.1}$$

where  $\sigma$  is the stress,  $E$  is the elastic modulus of the material and  $\varepsilon$  is the strain that occurs under the given stress,  $\sigma$ .

The viscous components are modeled as linear dashpots which obeys Newton's law

$$\sigma = \eta \frac{d\varepsilon}{dt} \tag{2.2}$$

where  $\sigma$  is the stress,  $\eta$  is the viscosity of the material and  $d\varepsilon / dt$  is the strain rate.

#### Rheological Models

Various rheological models of viscoelastic material behavior can be defined by simple arrangements of the mechanical elements of springs and dashpots. The two

fundamental building blocks are the Maxwell element (spring and dashpot in series) and

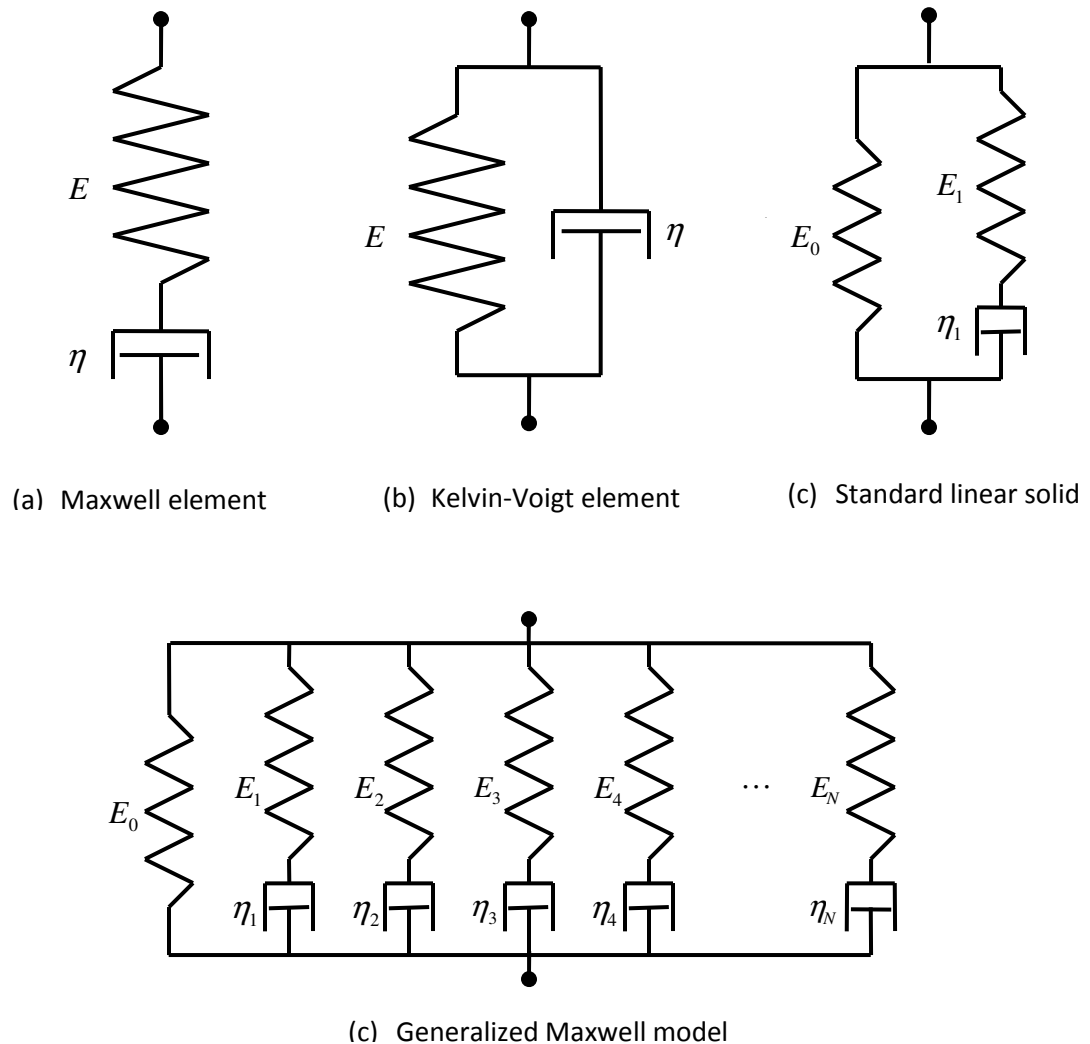


Figure 1: Rheological models for viscoelastic solids

the Kelvin-Voigt element (spring and dashpot in parallel) as shown in Figure 1 (a) and (b), respectively. The Maxwell element serves as simple model for a viscoelastic fluid, since the element can deform indefinitely under stress. The Kelvin-Voigt serves as a simple model for a viscoelastic solid, since indefinite deformation under stress is prevented by the spring. Another arrangement, and what is known as the standard linear solid, is shown in Figure 1(c) and a generalization of that, known as the N-parameter Maxwell model is shown in Figure 1(d). The standard linear solid contains more

flexibility than the Kelvin-Voigt element and is often used to model viscoelastic solids with a narrow range of response times, as permitted by the single viscous element. For realistic polymeric and rubber materials, that simple model is severely limited, but for the generalized Maxwell mode, and by suitable selection of the  $2N + 1$  parameters (spring and dashpot strengths), a real viscoelastic material can be represented. Of course, the choice of the parameters must be done to match the experimentally determined stress/strain/strain rate behavior of the actual material.

The analytical relationships appropriate to the three models of Figure 1, or constitutive equations, may be determined through the following rules:

- (a) For elements connected in parallel, their strains are equal and the total stress is equal to the sum of the individual stresses.
- (b) For elements connected in series, their stress is equal and the total strain is equal to the sum of the individual strains.

Using these principles, the constitutive equation for the Maxwell element is readily established to be

$$\frac{d\varepsilon}{dt} = \frac{1}{E} \frac{d\sigma}{dt} + \frac{\sigma}{\eta} \quad (2.3)$$

while for the Kelvin-Voigt element, one has

$$\sigma = E\varepsilon + \eta \frac{d\varepsilon}{dt} \quad (2.4)$$

Similarly, for the standard linear solid, the constitutive equation is

$$\sigma + \frac{\eta_1}{E_1} \frac{d\sigma}{dt} = E_0\varepsilon + \left( \frac{E_0 + E_1}{E_1/\eta_1} \right) \frac{d\varepsilon}{dt} \quad (2.5)$$

For the generalized Maxwell model, equation (2.5) is generalized to

$$\sigma + \frac{1}{\sum_{i=1}^N \frac{E_i}{\eta_i}} \frac{d\sigma}{dt} = NE_0\varepsilon + \frac{\sum_{i=0}^N E_i}{\sum_{i=1}^N \frac{E_i}{\eta_i}} \frac{d\varepsilon}{dt} \quad (2.6)$$

For all these constitutive equations (2.3)-(2.6) it is noted that there is the explicit appearance of the strain rate term, which of course makes the response of any of the models a function of time.

Creep and stress relaxation are the two important characteristic properties of the viscoelastic materials. Creep is the slow, progressive deformation of the material under constant stress. The strain increases with time in these materials. The ratio of  $\varepsilon(t)$  and  $\sigma_0$  is called the creep compliance.

$$\gamma(t) = \frac{\varepsilon(t)}{\sigma_0} \quad (2.7)$$

Stress relaxation is defined as the property of the material where the stress decreases when held at constant strain. The ratio of  $\sigma(t)$  to  $\varepsilon_0$  is called the relaxation modulus.

$$E(t) = \frac{\sigma(t)}{\varepsilon_0} \quad (2.8)$$

For the generalized N-Maxwell model that is considered in the present work the relaxation modulus is represented in the form of a Prony series as given by the equation below.

$$E(t) = E_e + \sum_{i=1}^N E_i e^{-\frac{t}{\rho_i}} \quad (2.9)$$

In the above equation constants  $E_e$  and  $E_i$  are calculated experimentally and depend entirely on the material properties and behavior,  $t$  is the time and the parameter  $\rho_i$  is called the relaxation time and is defined as

$$\rho_i = \frac{\eta_i}{E_i} \quad (2.10)$$



## 2.2 Conversion of Relaxation Modulus to Creep Compliance

In formulating problems of viscoelasticity, there are several common methods to incorporate the viscoelastic effects. One is to use the relaxation spectrum, or relaxation form of the storage and loss moduli, through equations like (2.9). Another would be through a similar form called the creep spectrum. As the formulation to be subsequently used in the contact problem herein is in the form of the creep spectrum, and the common experimentally-determined values are the relaxation values, it is necessary to convert between the two. The numerical method developed by S. W. Park and R. A. Schapery [9] is used in the present work for this conversion and is summarized in the following section. It is noted that there are various ways to perform this conversion, but that of Park and Schapery is especially accurate and straight forward.

### NUMERICAL METHOD:

Consider the relaxation modulus in the form of Prony series and given by the eq. the creep compliance is given by

$$\gamma(t) = \gamma_g + \frac{t}{\eta_0} + \sum_{j=1}^N \gamma_j (1 - e^{-t/\tau_j}) \quad (2.11)$$

where  $\gamma_g$  is the glassy compliance,  $\eta_0$  the shear zero or the long time viscosity,  $\gamma_j$  is the retardation strength and  $\tau_j$  is the retardation time. In the above equation  $\eta_0 \rightarrow \infty$  for the case of a viscoelastic solid. Hence, as the solid case is of current interest, the second term of equation (2.11) will be absent, and the creep compliance will be of the form

$$\gamma(t) = \gamma_g + \sum_{j=1}^N \gamma_j (1 - e^{-t/\tau_j}) \quad (2.12)$$

Now, the important relation between the relaxation and creep spectrum is in the form of an integral relationship, relating  $E(t)$  and  $\gamma(t)$ . The relationship is

$$\int_0^t E(t-\tau) \frac{d\gamma(\tau)}{d\tau} d\tau = 1 \quad (t > 0) \quad (2.13)$$

This convolution relationship can then be used to determine the unknown creep compliance  $\gamma(t)$  from the known relaxation modulus  $E(t)$ . Substituting equations (2.3) and

(2.5) into (2.13) and carrying out integration will give the required creep constants (creep spectrum). One has

$$\int_0^t \left( E_e + \sum_{i=1}^m E_i e^{-(t-\tau/\rho_i)} \right) \left( \gamma_g \delta(\tau) + \sum_{j=1}^N \frac{\gamma_j}{\tau_j} e^{-(\tau/\tau_j)} \right) d\tau = 1 \quad (2.14)$$

where  $\delta(\cdot)$  denotes the Dirac delta function.

$$\gamma_g \left( E_e + \sum_{i=1}^m E_i e^{-(t/\rho_i)} \right) + E_e \sum_{j=1}^N \frac{\gamma_j}{\tau_j} \int_0^t e^{-(\tau/\tau_j)} d\tau + \sum_{i=1}^m \sum_{j=1}^N \frac{E_i \gamma_j}{\tau_j} e^{-(t/\rho_i)} \int_0^t e^{-[(\tau/\tau_j) - (\tau/\rho_i)]} d\tau = 1 \quad (2.15)$$

The integrations in the above equation can be evaluated as

$$\int_0^t e^{-(\tau/\tau_j)} d\tau = \tau_j (1 - e^{-(t/\tau_j)}) \quad (2.16)$$

$$\int_0^t e^{-[(\tau/\tau_j) - (\tau/\rho_i)]} d\tau = \begin{cases} \frac{\rho_i \tau_j}{\rho_i - \tau_j} (1 - e^{-[(t/\tau_j) - (t/\rho_i)]}) , & \rho_i \neq \tau_j \\ t , & \rho_i = \tau_j \end{cases} \quad (2.17)$$

so substituting (2.10) and (2.11) into (2.9) and rearranging gives

$$\sum_{j=1}^N \left[ \sum_{i=1}^m \frac{\rho_i E_i}{\rho_i - \tau_j} \left( e^{-(t/\rho_i)} - e^{-(t/\tau_j)} \right) + E_e \left( 1 - e^{-(t/\rho_i)} \right) \right] \gamma_j = 1 - \gamma_g \left( E_e + \sum_{i=1}^m E_i e^{-(t/\rho_i)} \right), \rho_i \neq \tau_j \quad (2.18)$$

and

$$\sum_{j=1}^N \left[ \sum_{i=1}^m \frac{t E_i}{\tau_j} e^{-(t/\rho_i)} + E_e \left( 1 - e^{-(t/\tau_j)} \right) \right] \gamma_j = 1 - \gamma_g \left( E_e + \sum_{i=1}^m E_i e^{-(t/\rho_i)} \right), \rho_i = \tau_j \quad (2.19)$$

In equation (2.12) and (2.13),  $\gamma_g$  can be expressed in terms of  $E_e$  and  $E_i$ 's as

$$\gamma_g = \frac{1}{E_e + \sum_{i=1}^m E_i} \quad (2.20)$$

In matrix form equation (2.1) is reduced to

$$[A]\{\gamma\} = \{B\}$$

or

$$A_{kj}\gamma_j = B_k, j=1,2,\dots,n; k=1,2,\dots,p$$

(2.21)

$$A_{kj} = \begin{cases} E_e(1 - e^{-(t_k/\tau_j)}) + \sum_{i=1}^m \frac{\rho_i E_i}{\rho_i - \tau_j} (e^{-(t_k/\rho_i)} - e^{-(t_k/\tau_j)}), & \rho_i \neq \tau_j \\ E_e(1 - e^{-(t_k/\tau_j)}) + \sum_{i=1}^m \frac{t_k E_i}{\tau_j} (e^{-(t_k/\rho_i)}) & , \rho_i = \tau_j \end{cases} \quad (2.22)$$

$$B_k = 1 - (E_e + \sum_{i=1}^m E_i e^{-(t_k/\rho_i)}) / (E_e + \sum_{i=1}^m E_i) \quad (2.23)$$

If in the above equation, one knows  $E_e$ ,  $E_i$ ,  $\rho_i$ ,  $t_k$  and  $\tau_j$  the only unknown is  $\gamma_j$ . To solve this equation, the collocation method is used when  $p = N$  and the least squares method is used when  $p > N$ . The values of  $\tau_j$  are specified and the  $t_k$  values depend on the method used. In the collocation method  $t_k$  can be calculated from  $t_k = a\tau_k$  where  $a$  can be 1 or  $1/2$ . In the least squares method, values of  $t_k$  can be considered at equidistant intervals on the log  $t$  axis.

### CHAPTER 3. THE VISCOELASTIC FORMULATION

This chapter deals with the formulation of a viscoelastic contact problem in which a rigid circular roller is rolling on the initially flat viscoelastic semi-infinite base. In order to achieve this, a brief description of elastic solution has been addressed first.

#### 3.1 Elastic Solution

The starting point to develop the integral equation formulation of the viscoelastic contact problem is the well known Boussinesq problem, (displacement due to a distributed force on an elastic half-space) is readily available (Love, p. 192 [10]). The vertical (normal to the surface) displacement in this problem is given by

$$u_z^e(x, y, 0) = \frac{1-\nu}{\mu} P(x, y, 0) \quad (3.1)$$

where  $u_z^e$  is the displacement in the z-direction, the superscript e denotes the elastic solution,  $\nu$  is Poisson's ratio,  $\mu$  the shear stress modulus and P is the potential of the contact pressure distribution. The potential is given by

$$P(x, y, z) = \frac{1}{2\pi} \iint_A \frac{q(x', y')}{r'} dx' dy' \quad (3.2)$$

where  $r' = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$  and q is the normal pressure distribution

In a two dimensional contact problem, the load does not vary with y, so the problem can then be considered in only the (x, z) direction with load q(x) and displacement in z-direction. The two dimensional elastic solution is then

$$P(x, 0, 0) = \frac{1}{2\pi} \int_0^l \int_{-\infty}^{\infty} \frac{q(x')}{\sqrt{(x-x')^2 + (y')^2}} = \frac{1}{\pi} \int_0^l q(x') [\log|x-x'|] dx' + \text{constant} \quad (3.3)$$

and the vertical displacement is given by

$$u_z^e = \frac{1-\nu}{\mu} \frac{1}{\pi} \int_0^l q(x') \log|x-x'| dx' + \text{constant} \quad (3.4)$$

The arbitrary constant can be eliminated by referring the displacement to a specific point on the surface, so that the above equation can be written

$$u_z^e(x) - u_z^e(x_0) = \frac{1-\nu}{\mu} \int_0^l q(x') [\log|x_0 - x'| - \log|x - x'|] dx' \quad (3.5)$$

This equation relates the surface displacement to the surface pressure, or stress, in the form of what is called an integral equation of the first kind. If the pressure  $q(x)$  were known, then one could directly determine the surface displacement from the equation by quadrature, or integration. However, if the displacement were known, then one would be faced with the determination of the pressure  $q(x)$ , which is then truly an integral equation problem. Also, since in this case, the unknown is entirely under the integral sign, the equation is referred to one of the first kind, as opposed to one of the second kind, where the unknown is present both under the integral sign and outside of it. It is also noted that the logarithmic term under the integral is called the kernel of the equation and plays a very important role in how easily and accurately this type of equation can be solved.

### 3.2 Viscoelastic Solution

As discussed in the previous section, the stress-strain relations of a viscoelastic material depend on time. Hence, viscoelastic solutions can be found by removing the time dependency by a Laplace's transform of the applicable equations, which reduces the problem to an associated elastic problem. The solution of this problem can be found similar to the elastic solution with an additional  $s$  parameter. Once the solution for an associated elastic problem is found, a viscoelastic solution is just the inverse Laplace transform of the elastic with the various elastic parameters replace by their viscoelastic counterparts. The process is known as the correspondence principle of viscoelasticity. Following this method, the two dimensional viscoelastic surface displacement is given by

$$u_z(x, y, \tau) = u_z^e(x, y) + \int_0^\tau \gamma(\zeta) u_z^e(x + VT\zeta, y) d\zeta \quad (3.6)$$

In the above equation  $u_z$  is the viscoelastic displacement and  $u_z^e$  is the elastic displacement. Viscoelastic material properties and behavior are included in the above

equation in the form of  $\gamma(\tau)$  function.  $\gamma(\tau)$  for materials represented in the form of mathematical models is given by

$$\gamma(\tau) = \sum_{i=1}^N f_i e^{-b_i \tau} \quad (3.7)$$

where,

$$f_i = \left( \frac{\gamma_j}{\gamma_g} \right) \left( \frac{T}{\tau_j} \right), \quad b_i = \left( \frac{T}{\tau_j} \right) \quad \& \quad T = \tau_1 \quad (3.8)$$

Boundary conditions used in solving the above three dimensional viscoelastic problem are

$$\tau_{xz}(x, y, 0) = \tau_{yz}(x, y, 0) = 0 \text{ for all } x \text{ and } y \quad (3.9)$$

$$\sigma_z(x, y, 0) = 0 \text{ outside the contact area} \quad (3.10)$$

$$u_z(x, y, 0) = \delta + \alpha x + \beta y + w^*(x - b, y - c) \text{ within the contact area} \quad (3.11)$$

where  $\tau_{xz}$  and  $\tau_{yz}$  are the shear stresses,  $\sigma_z$  is the normal stress,  $\delta$ ,  $\alpha$  and  $\beta$  are the constants that depend on the contact area and  $w^*$  is the shape of the contact surface.

Once the pressure distribution is known inside the contact area, the friction coefficient as defined by

$$\mathcal{X} = \frac{F^*}{N^*} \quad (3.12)$$

can be determined, where  $F^*$  is the resultant horizontal force of the contact stress distribution, as given by

$$F^* = - \iint_A q(x, y) \frac{\partial u_z(x, y)}{\partial x} dx dy \quad (3.13)$$

and the resultant vertical force of the contact stress distribution as defined by

$$N^* = \iint_A q(x, y) dx dy \quad (3.14)$$

By the above definition of the friction coefficient, it is analogous to the coefficient of sliding friction for a block on a plane or the Coloumb friction coefficient. To simplify the solution, it is desirable to non-dimensionalize all the above equations. This can be achieved using a characteristic length parameter of VT, where, V is the velocity of the

moving load and  $T$  is a characteristic time constant. Similarly, the following non-dimensional parameters are introduced

$$\begin{aligned}\xi &= \frac{x}{VT} \\ \eta &= \frac{y}{VT} \\ v^e(\xi, \eta) &= \frac{u_z(x, y, 0)}{VT} \\ Q(\xi, \eta) &= \frac{1-\nu_0}{\mu_0} \frac{1}{\pi} q(x, y)\end{aligned}$$

Using these non-dimensional parameters, the non-dimensional elastic displacement would then be

$$v^e(\xi, \eta) = \frac{1}{2} \iint_A \frac{Q(\xi', \eta')}{\sqrt{(\xi - \xi')^2 + (\eta - \eta')^2}} d\xi' d\eta' \quad (3.15)$$

By the viscoelastic correspondence principle, the viscoelastic displacement would then be

$$\begin{aligned}v(\xi, \eta, \tau) &= v^e(\xi, \eta) + \int_0^\tau \gamma(\zeta) v^e(\xi + \zeta, \eta) d\zeta \\ v(\xi, \eta, \tau) &= \frac{1}{2} \iint_A Q(\xi', \eta') \left[ \frac{1}{\sqrt{(\xi - \xi')^2 + (\eta - \eta')^2}} + \int_0^\tau \frac{\gamma(\zeta) d\zeta}{\sqrt{(\xi + \zeta - \xi')^2 + (\eta - \eta')^2}} \right] d\xi' d\eta'\end{aligned} \quad (3.16)$$

Non-dimensional forms of forces  $N^*$  and  $F^*$  are represented by  $N$  and  $F$  respectively, i.e.

$$\begin{aligned}N &= \frac{1-\nu_0}{\mu_0} \frac{1}{\pi} \frac{N^*}{(VT)^2} = \iint_A Q(\xi, \eta) d\xi d\eta \\ F &= \frac{1-\nu_0}{\mu_0} \frac{1}{\pi} \frac{F^*}{(VT)^2} = -\iint_A Q(\xi, \eta) \frac{\partial v(\xi, \eta)}{\partial \xi} d\xi d\eta\end{aligned} \quad (3.17)$$

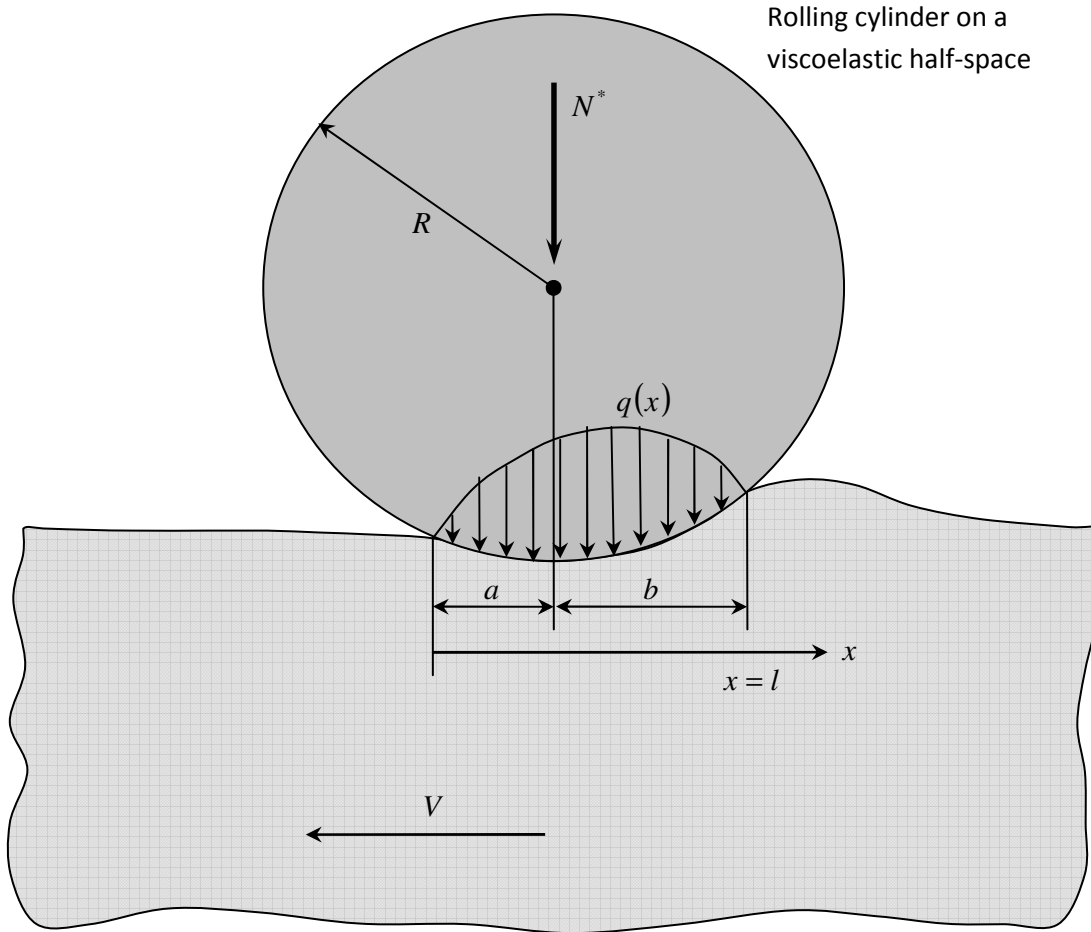


Figure 2. Two dimensional viscoelastic contact-cylinder on a viscoelastic strip.

In the current work a steady state two dimensional contact problem is solved. Assuming that the equations above are independent of  $\eta$  would then convert the problem to be two dimensional.

$$v(\xi, \tau) - v(\xi_0, \tau) = \int_0^\lambda Q(\xi') \left[ \log \left| \frac{\xi_0 - \xi'}{\xi - \xi'} \right| + \int_0^\tau \gamma(\zeta) \log \left| \frac{\xi_0 + \zeta - \xi'}{\xi + \zeta - \xi'} \right| d\zeta \right] d\xi' \quad (3.18)$$

is the equation that gives the displacement of a 2D viscoelastic problem where

$$\lambda = \frac{l}{VT}$$

is the non dimensional form of the contact length  $l=a+b$ .

Every viscoelastic material reaches a steady state after an infinite amount of time. Hence, a steady state solution would then be



$$v(\xi) - v(\xi_0) = \int_0^\lambda Q(\xi') [K(\xi_0 - \xi') - K(\xi - \xi')] d\xi' \quad (3.19)$$

$$K(\xi) = \log|\xi| + \int_0^\infty \gamma(\zeta) \log|\xi + \zeta| d\zeta$$

The horizontal and vertical forces will then be

$$N = \frac{1 - \nu_0}{\mu_0} \frac{1}{\pi VT} N^* = \int_0^\lambda Q(\xi) d\xi \quad (3.20)$$

$$F = \frac{1 - \nu_0}{\mu_0} \frac{1}{\pi VT} F^* = - \int_0^\lambda Q(\xi) \frac{\partial v(\xi)}{\partial \xi} d\xi$$

To satisfy the boundary condition mentioned in equation (3.11), the viscoelastic displacement is written as

$$v(\xi) = v(\xi) - v(0) = \alpha\xi + w(\xi - \beta) - w(-\beta) \quad (3.21)$$

where  $\alpha$  is the angle of tilt of the contacting rigid profile and  $\beta$  is the profile itself. For a rolling cylinder, the surface of the cylinder at the contact region can be approximated to be a parabola where,

$$w(\xi - \beta) = -\frac{1}{2\rho} (\xi - \beta)^2$$

and where  $\rho$  is the dimensionless form of the radius R given by

$$\rho = \frac{R}{VT}$$

Equation (3.21) with a zero-angle of tilt is

$$v(\xi) = \frac{1}{\rho} \left( \xi\beta - \frac{1}{2} \xi^2 \right) \quad (3.22)$$

Equations (3.19) and (3.22) both give the displacement inside the contact region hence

$$\int_0^\lambda Q(\xi') \left[ \log \left| \frac{\xi'}{\xi - \xi'} \right| + \int_0^\infty \gamma(\zeta) \log \left| \frac{\zeta - \xi'}{\xi + \zeta - \xi'} \right| d\zeta \right] d\xi' = \frac{1}{\rho} \left( \xi\beta - \frac{1}{2} \xi^2 \right) \quad (3.23)$$

Introducing now, new dimensionless quantities

$$Q'(\xi) = \rho Q(\xi) = \frac{R}{VT} \frac{1-\nu_0}{\mu_0} \frac{1}{\pi} q(x)$$

$$N' = 2\rho N = \frac{R}{VT} \frac{1-\nu_0}{\mu_0} \frac{2}{\pi} \frac{N^*}{VT} = 2 \int_0^\lambda Q'(\xi) d\xi \quad (3.24)$$

equation (3.23) then becomes

$$\int_0^\lambda Q'(\xi') [K(-\xi') - K(\xi - \xi')] d\xi' = \xi\beta - \frac{1}{2}\xi^2 \quad (3.25)$$

This equation (3.25) represents the integral equation (of the first kind) which can subsequently be used to solve the rolling contact problem. The integral equation is then converted into N number of linear algebraic equations and can be written as

$$\sum_{i=1}^N \{Q'(\xi_i) [K(-\xi_i) - K(\xi - \xi_i)]\} - \xi\beta = -\frac{1}{2}\xi^2 \quad (3.26)$$

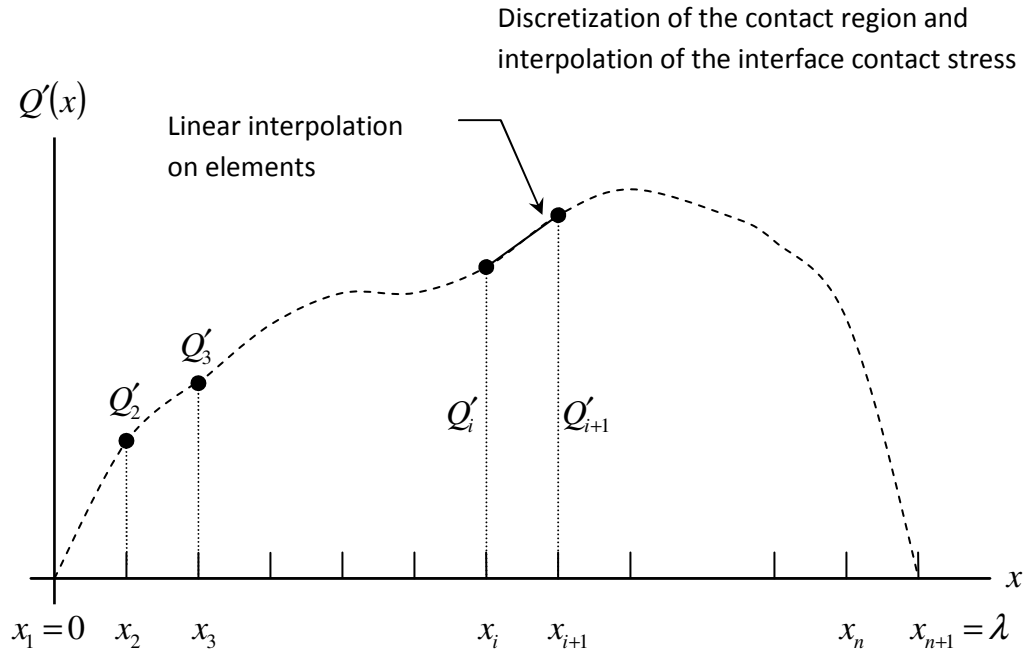


Figure 3. Discretization of the interface stress

In figure 3, the contact pressure at the ends of the contact length are known and therefore there are  $(n-1)$  number of  $Q'$  unknowns. The unknown of the equation (3.26) is the normalized interface pressure  $Q'$  and contact profile  $\beta$  when  $\lambda$  is given. However, in the actual situation, what is actually prescribed is the normal load resultant  $N^*$ , so, like most contact problems, an iteration process is required to determine a consistent load/stress/contact length values that satisfy the equation. This iterative, numerical algorithm is developed in the following chapter.

## CHAPTER 4. A NUMERICAL/ITERATIVE ALGORITHM FOR CONTACT

Having derived all the pertinent equations required to solve the viscoelastic contact problem, this chapter describes a numerical and iterative solution. All the viscoelastic equations derived earlier uses the material properties in the form of creep function of the material. As discussed in the earlier chapters, expressing material properties in terms of relaxation parameters is the most common way to characterize the material, since these properties are a direct result of strain controlled experimental measurements. Hence, the first step in obtaining a solution is to convert the relaxation parameters into the corresponding creep parameters. If the relaxation and creep functions represented in the form of equations (2.3) and (2.6) then creep parameters can be expressed in terms of relaxation parameters using equations (2.14) and (2.15).

The equations derived in chapter 3, specifically equation (3.19)-(3.22), determine the pressure distribution, resultant horizontal and vertical forces, and friction coefficient for a given contact length, radius of the cylinder, velocity and time constant. But, the contact length is always dependent on the vertical force acting. Hence, an iterative process is required which makes an initial guess of contact length according to the given vertical force.

Let, the normalized load acting on the backing material be  $N'$ , the contact length  $\lambda$  and let  $N_0'$  denote the desired load on the cylinder or roller. The following algorithm, similar to the Newton-Raphson method for solving nonlinear equations, forms a process to iterate to a consistent load and contact length.

1. Given a target load value  $N'_0$
2. Estimate initial contact length from long-term (elastic) value  $\lambda_1 = 2\sqrt{N'_0}$
3. Set initial values  $\lambda_0 = \lambda_1 / 2$  and  $N_0 = 0$
4. Set error tolerance  $Tol$  and iteration index  $i = 0$

while

increment index  $i = i + 1$

generate grid for this  $\lambda = \lambda_i$

solve the integral equation for  $Q'$

integrate to get  $N'_i$

converged:  $N'_0 - N_0 \leq Tol$

yes: exit loop

no: estimate new contact length  $\lambda_i = \lambda_{i-1} + \left( \frac{N_{i-1} - N'_0}{N_{i-1} - N_{i-2}} \right) (\lambda_{i-2} - \lambda_{i-1})$

continue

end

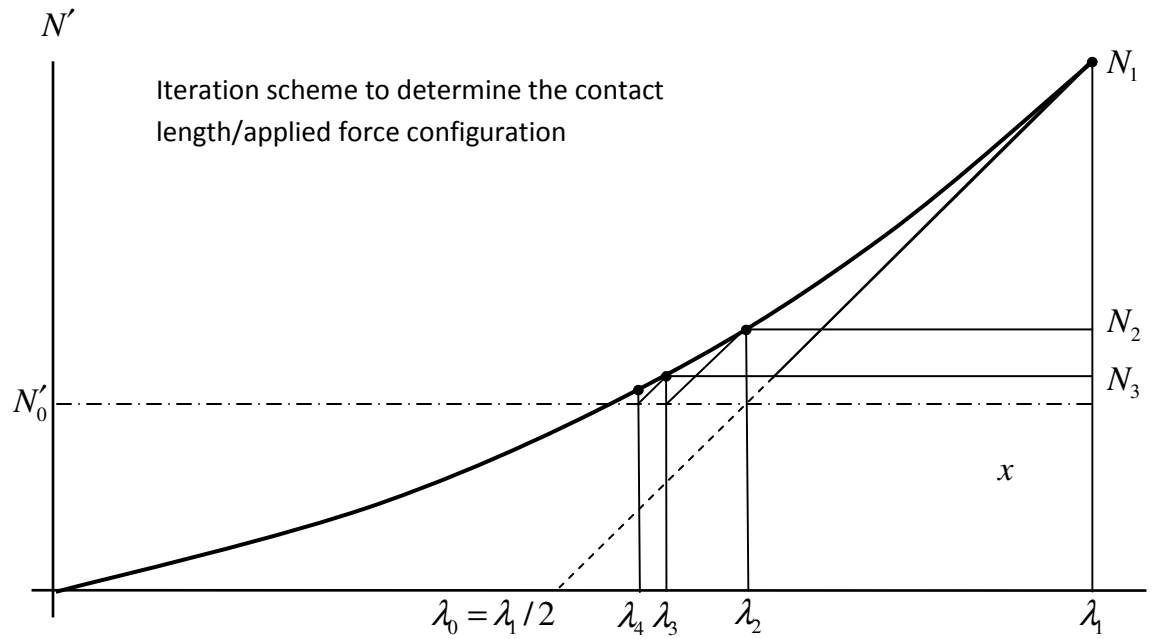


Figure 4. The iteration process.

Once, the appropriate contact length is found the pressure distribution and the friction coefficient are found using equations (3.24) and (3.25). In this way, the developed program calculates the pressure distribution and coefficient of friction from the given data of vertical force, radius of cylinder, velocity and time constant.

## CHAPTER 5. RESULTS

In this chapter results like contact pressure and friction coefficient have been presented for a contact problem with a rigid cylinder rolling on a realistic viscoelastic backing material.

### 5.1 Material Properties

As mentioned earlier, the behavior of the material is represented in the form of relaxation modulus. Figure 5 shows the master curve for a typical rubber compound used in conveyor belt coverings. It shows the storage and loss moduli of the material as a function of frequency, as constructed from data measured from frequency sweeps, at various fixed temperatures, over a limited range of frequencies as is practical in a dynamic mechanical analyzer (DMA). Typical temperature ranges are  $-80^{\circ}\text{C}$  to  $+80^{\circ}\text{C}$  in increment of  $10^{\circ}\text{C}$  and typical frequency ranges are about 0.1 Hz. to 100 Hz. The data at a fixed temperature are then shifted horizontally on the frequency axis to overlay on a single plot to form a graph like that shown. Justification for extending the data to a broad range of frequencies, hence time scales, is called the correspondence principle of viscoelasticity, and allows measured properties at moderate frequencies to be extended to a much broader time and frequency scale by taking data at a range of temperatures.

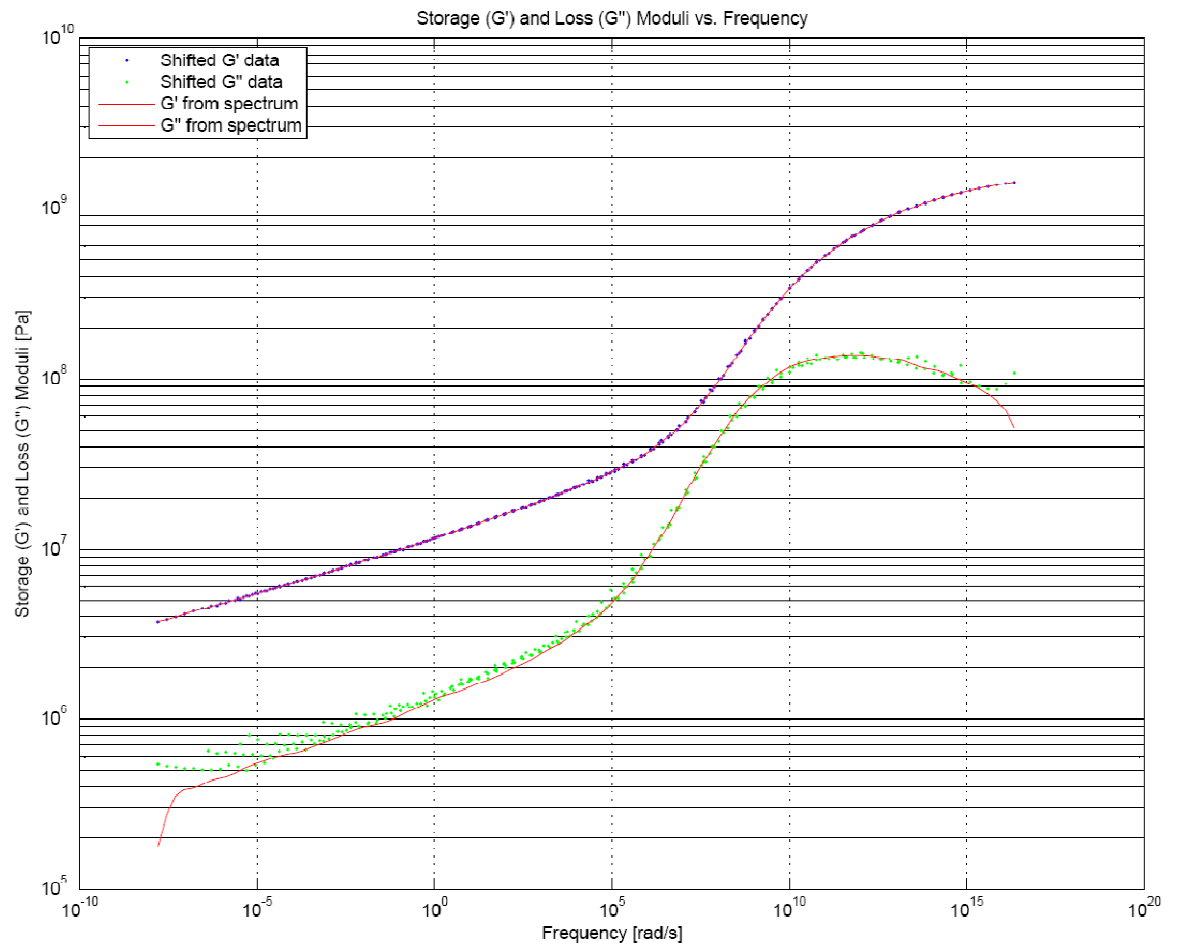


Figure 5. A master curve of the viscoelastic storage and loss moduli

From this plot the relaxation strengths and relaxation times may be determined and Figure 6 shows the plot of relaxation strengths and the corresponding relaxation times.



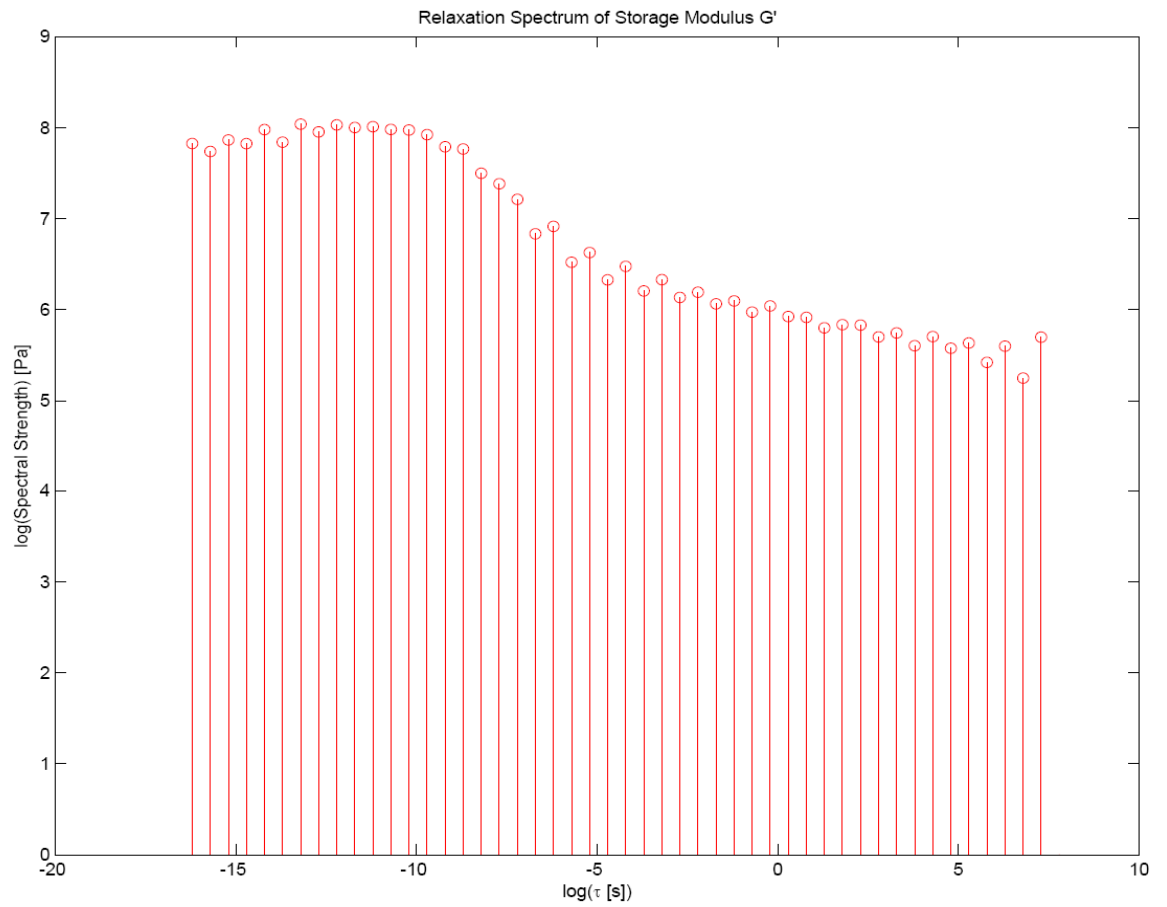


Figure 6. A discrete relaxation spectrum

This relaxation spectrum is converted to creep compliance by the process outlined in chapter 2.2 and Figure 7 shows the source relaxation modulus and computed creep compliance for the material of figure 6.

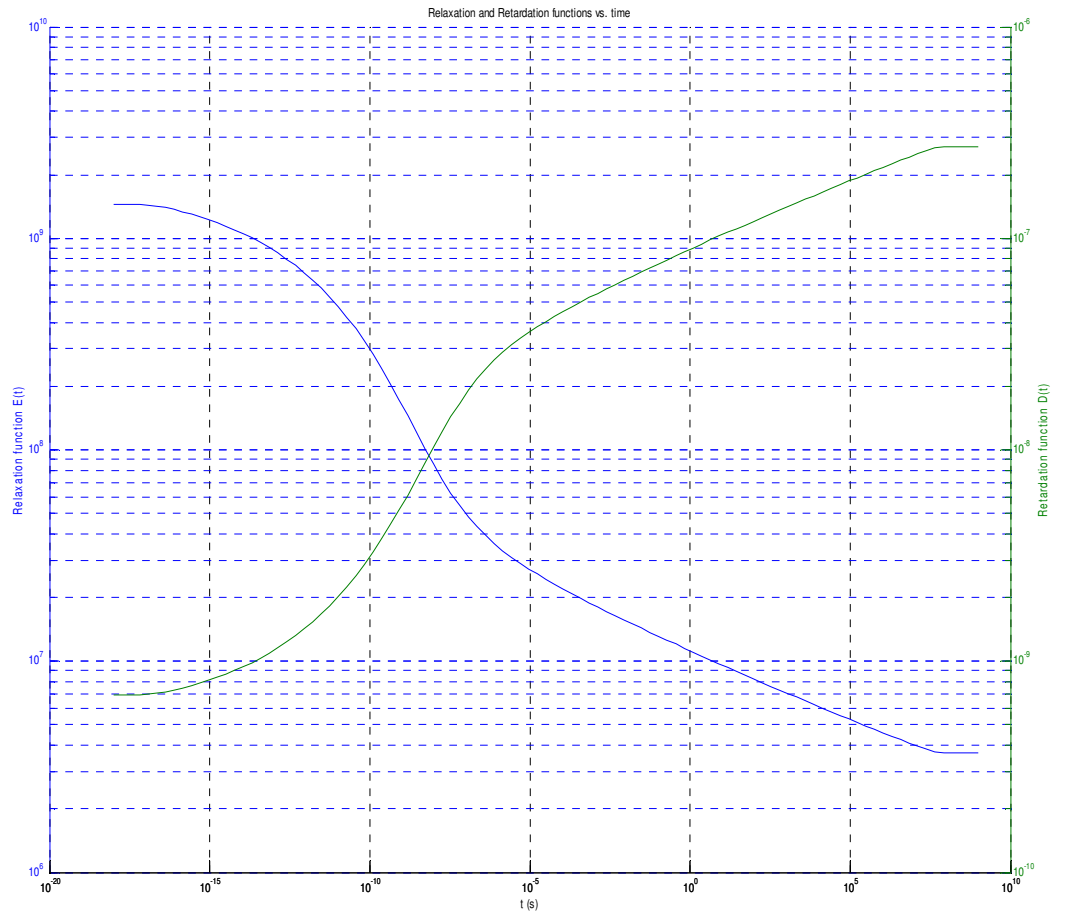


Figure 7. Relaxation and creep function vs time.

## 5.2 Two Dimensional Rolling Cylinder Problem

In this section the results for the contact problem are presented. As check cases, 2 sets of results, corresponding to the results presented in Goodier and Loutzenheiser [8], are obtained first. In these check cases results are obtained for one-parameter and five-parameter models. Plots shown below depict the contact pressure profile for a one parameter (Figure 8) with different material properties (varying  $f$ ) and a five parameter (Figure 9) models.

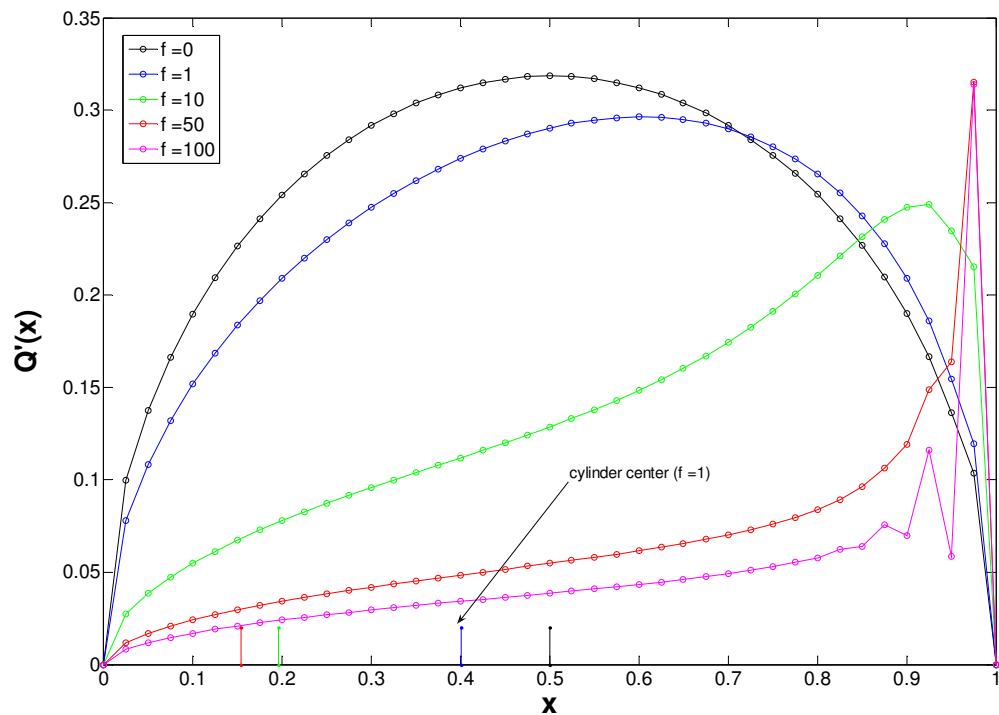


Figure 8. Contact pressure profiles for the one-parameter model

Figure 8 shows the contact pressure profiles for a one parameter model with different material properties represented by  $f$ .

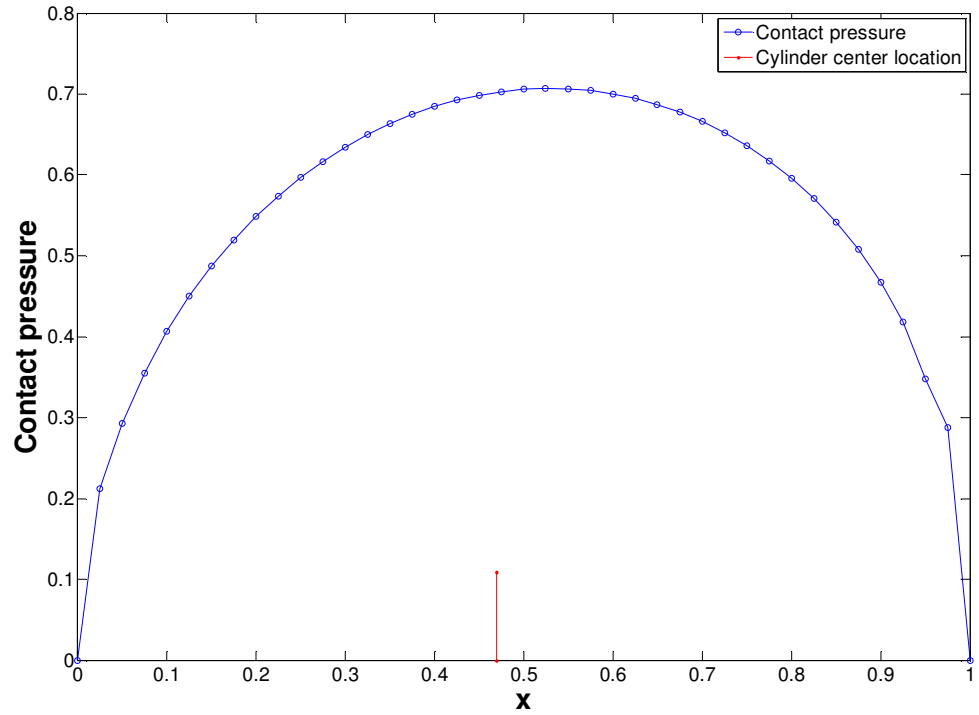


Figure 9. Contact pressure profile for the five-parameter model.

Table 1.

Number of parameters	$\beta/\lambda$	$\beta/\lambda$ (Goodier & Loutzenheiser)	Friction coefficient	Friction coefficient (Goodier & Loutzenheiser)
1 (with $f=10$ and $n=40$ )	0.1959	0.1952	0.43093	0.4308
5 ( $n=40$ )	0.46992	0.4694	0.040523	0.0402

Table 1 shows the contact length for one parameter and five parameter models compared with the results from Goodier & Loutzenheiser.

Goodier and Loutzenheiser's formulation calculates the contact stress and normal force acting with the given contact length. This is further modified in the current work to calculate the contact pressure and contact length with the given normal force. Contact pressure is calculated for normal force of 2000 N/m and a velocity of 5 m/s. Plot shown below (Figure 10) represents the variation of contact pressure along the contact length.

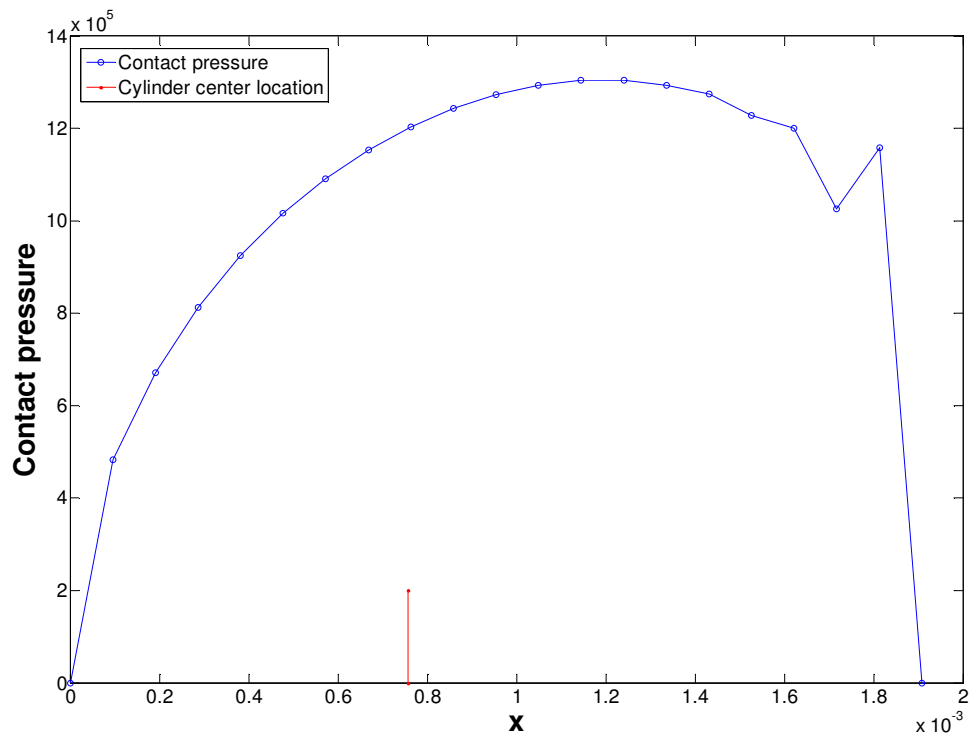


Figure 10. Contact pressure along the contact surface.

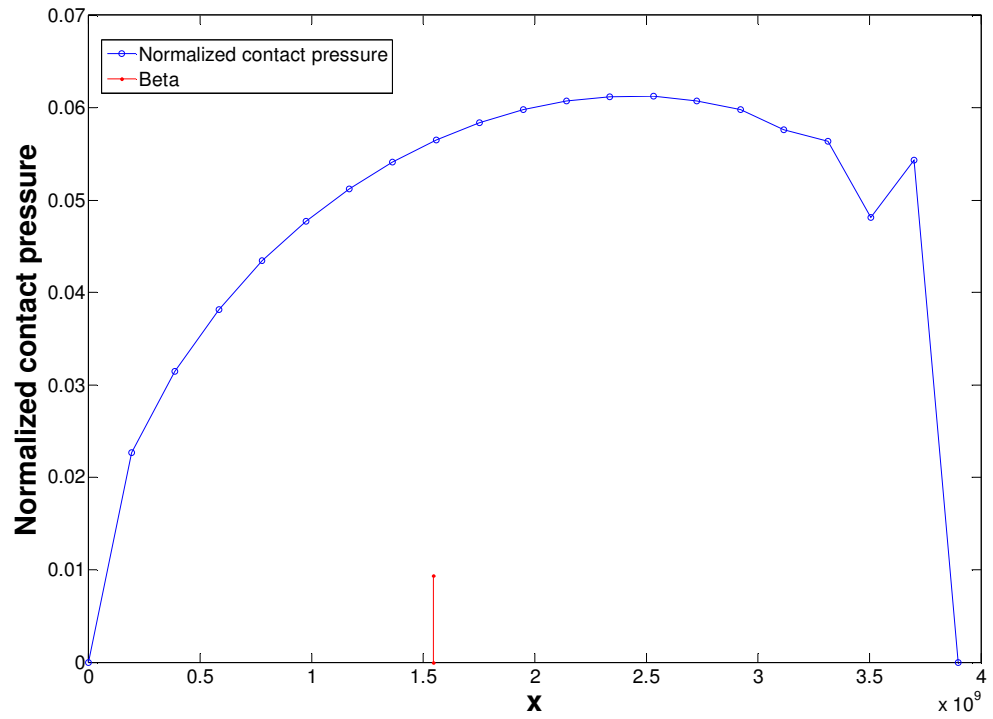


Figure 11. Normalized contact pressure along the contact length.

Friction coefficient is plotted by varying the normal load acting on the backing material and also the velocity. Figure 12 shows the plot of friction coefficient varying according to the normal load and velocity.

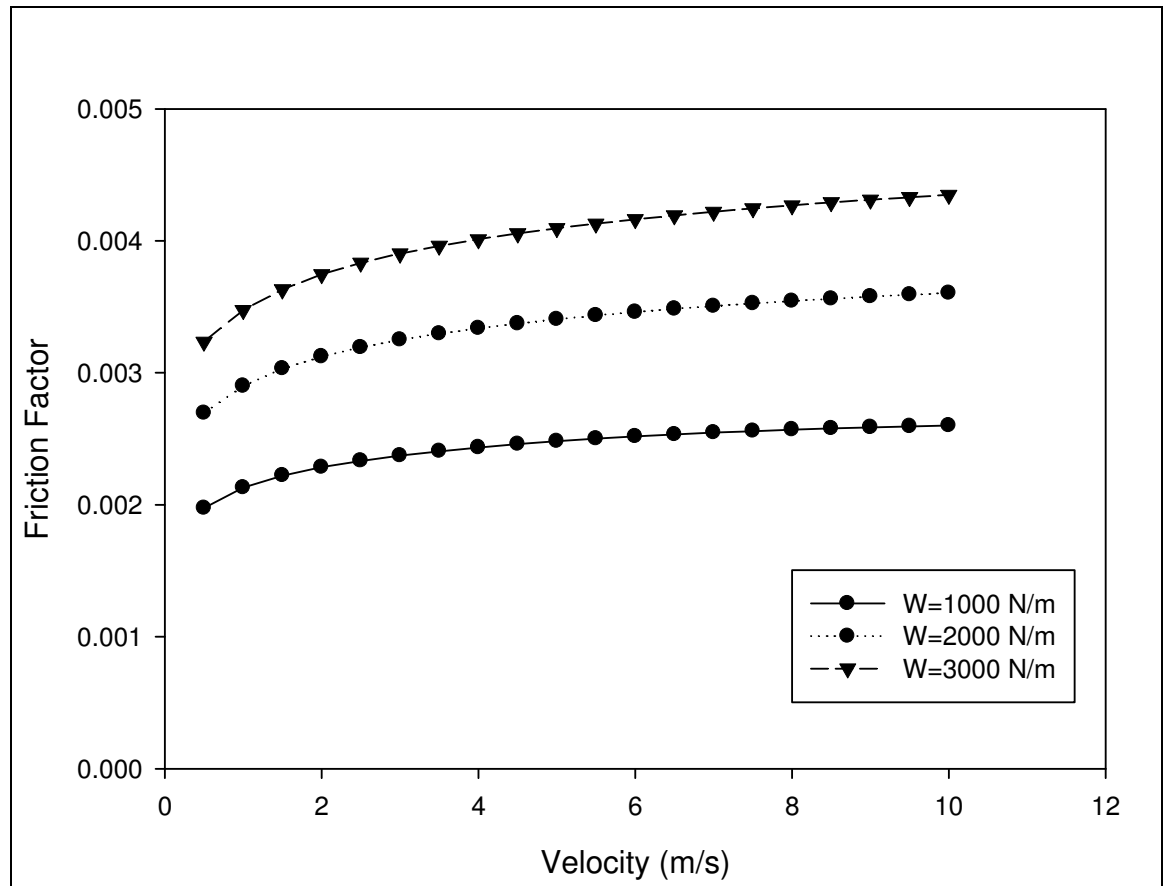


Figure 12. Variation of friction coefficient with velocity and normal load.

## CHAPTER 6. DISCUSSION & CONCLUSIONS

From the results and example problems there are several conclusions evident from this study and implementation of the viscoelastic rolling contact formulation and solution.

Although the issue of converting the relaxation modulus to creep compliance is a peripheral one to the main goal of determination of the rolling resistance, the conversion methodology provided by Park and Schapery [9] is a very efficient and accurate way to perform the conversion. Furthermore, for any given viscoelastic material of the substrate and the relaxation parameters, this conversion need only be done once prior to the contact iteration solution.

The integral equation formulation of the contact problem as given by Goodier and Loutzenheiser [8] forms a very elegant and ultimately efficient numerical formulation. When combined with a simple iterative algorithm to determine a consistent contact length/applied force pair, for a given material, speed and the roll radius, the solution is relatively quick and accurate. In all the cases examined here, the solution converged with less than 10 iterations for an error tolerance of  $10^{-4}$ . The integral equation method, requiring integration only over the length of the contact, requires only a discretization of the contact length, the remainder of the domain of the problem – the semi-infinite viscoelastic base – and the far-field boundary conditions are all taken into account by the kernel of the integral equation.

As a consequence of the integral equation formulation being one of the first kind, the numerical solution suffers from the non-uniqueness associated with numerical



solution of these type of resulting equations. This effect reveals itself in the jagged behavior of the solution near the front (where the contact first occurs, right side) edge of the contact region for certain viscoelastic material parameters and speeds. Goodier and Loutzenheiser [8] allude to and discuss this problem in their report, even doing some numerical experiments to remove this effect from the numerical solution, but to no particular avail. Although this effect is also present in this study with realistic viscoelastic materials, its effect in the rolling resistance factor is not expected to be serious, since some oscillatory behavior of the interface pressure in the local vicinity of the leading point of contact does not effect the integrated moment of that pressure profile, which determines both the contact resultant force and moment about the roller center. Tolerance of this oscillatory behavior is evident in the simple test problems with single parameter materials.

The rolling friction coefficient as predicted by the present approach also agrees qualitatively with simpler models, i.e., from the last example above, the friction coefficient shows a weak dependency on the speed, and approaches zero as the speed goes to zero. The dependence on the impressed contact force is more dramatic and follows more weak power dependence on the force, all other parameters held constant. This is qualitatively evident from the last example. Simpler solutions show this dependency to be proportional to  $(\text{Impressed force})^{1/3}$ .

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