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**Start-up manufacturing firms:
Operations for survival**

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Abstract

Start-up firms play an important role in the economy. Statistics show that a large percent of start-up firms fail after few years of establishment. Raising capital, which is crucial to success, is one of the difficulties start-up firms face. This Ph.D thesis aims to draw suggestions for start-up firm survival from mathematical models and numerical investigations. Instead of the commonly held profit maximizing objective, this thesis assumes that a start-up firm aims to maximize its survival probability during the planning horizon. A firm fails if it runs out of capital at a solvency check. Inventory management in manufacturing start-up firms is discussed further with mathematical theories and numerical illustrations, to gain insight of the policies for start-up firms. These models consider specific inventory problems with total lost sales, partial backorders and joint inventory-advertising decisions. The models consider general cost functions and stochastic demand, with both lead time zero and one cases.

The research in this thesis provides quantitative analysis on start-up firm survival, which is new to the literature. From the results, a threshold exists on the initial capital requirement to start-up firms, above which the increase of capital has little effect on survival probability. Start-up firms are often risk-averse and cautious about spending. Entering the right niche market increases their chance of survival, where the demand is more predictable, and start-ups can obtain higher backorder rates and product price. Sensitivity tests show that selling price, purchasing price and overhead cost have the most impact on survival probability. Lead time has a negative effect on start-up firms, which can be offset by increasing the order frequent. Advertising, as an investment in goodwill, can increase start-up firms' survival. The advertising strategies vary according to both goodwill and inventory levels, and the policy is more flexible in start-up firms. Externally, a slightly less frequency solvency check gives start-up firms more room for fund raising and/or operation adjustment, and can increase the survival probability. The problems are modelled using Markov decision processes, and numerical illustrations are implemented in Java.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Kuangyi Liu

To my parents

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Chapter 1

Introduction

Start-up firms are important to a nation's economy. They "boost productivity, increase competition and innovation, create employment and prosperity, and revitalise the communities" [Bank of England, 2004]. However, start ups often find themselves in the risk of insolvency due to reasons from economic and industrial conditions, to firm finance and management. The gathering of initial capital is often difficult for start-up firms due to their low bank credit and low survival rate. From statistics, the start-up capital is mostly from the founders' own savings and their families and friends. It can be partly from external bodies such as angel capital, trace credit and bank loans, the amount of which is often limited. Meanwhile, due to the high risk of failure, start-up firms often obtain external funding with high interest rates or strict liquidation policies. The capital condition is different between start ups and well-established firms. It may result in variation of business planning strategies and policies, which may have been observed in some of the start-up firms. However there has not been much a work that discusses these differences and better business solutions for start-ups. Start-up firms also differ from well-established firms due to the fact that, most of the owners of start-up firms are young entrepreneurs or experienced professionals, whose goals may be "non-financial in order to achieve more personal objectives" [Keasey and Watson, 1993, pp. 101]. Therefore, the most important matter is whether or not the firm is in operation, in which case the general profit maximizing objective and the results under such an assumption may not be appropriate to all start-up firms.

In the literature, discussions on start-ups consider general management, the funding and investment in venture capital, and bank lending and liquidation policies. Research is also presented on the factors that affect firm survival across industries and regions.

The research is mostly accounting based empirical measurements, which according to Keasey and Watson [1993, pp. 95],

... are far from perfect, particularly in relation to the small firm sector. Due to the fact that accounting based measures (e.g. profits growth, returns on assets, etc) are generated from past transactions and the methods chosen by accountants to report them, they are frequently poor proxies for the economically relevant variables...

This thesis investigates start-up firm survival from a quantitative point of view. Instead of the commonly held profit maximizing objective, this thesis assumes that start-up firms are concerned about survival and have the goal of maximizing the survival probability over a planning horizon [Archibald et al., 2002]. It models the inventory management in manufacturing start-up firms and provides numerical analysis to support their decision making, which is new to the current literature.

Inventory management has been a topic in operational research and management science for decades. The benefits from inventory management have become obvious from mid-1980s. In recent years, in some of the developed countries such as Britain and the United States, manufacturing is no longer the attention of the press and it seems to be “on a path of distinction”. However, some fundamental industries, such as textile, machine tool, food industry, and some others like microprocessors, computers, are considered to be vital to a nation. Inventory management is still an important element to the modern economy. To developed countries, inventory management is as the lifeblood of many financial services and consulting firms who are the domain of their economy. For those manufacturing firms, inventory management can be “a source of competitive advantage” to a business, it can also be “the cause of the firm’s decline” if ignored. Although there has been intensive studies on inventory management, in real-time businesses, firms seem not be “fully understand the complexities of inventory management or production planning and scheduling” [Silver et al., 1998, pp 1-12]. This can also be one of the issues for start-up firms, where the management has little experience in running the business and due to the different financial situations, the developed theories for established firms may not apply as well.

Meanwhile, other departments such as finance and marketing work together with operations in a firm as whole. Kotler [1971, pp. 192-196] defines the different objectives and decisions of each department within a firm. Take the manufacturing and marketing departments for instance. A manufacturing department aims at minimizing the costs

and benefits from low inventory levels, reasonable product quality, stable demand rate, etc. The marketing department on the other hand, aims to achieve as much sales as possible and benefits from high inventory levels, high quality products, and promotion that could generate as much demand as possible. The conflict of interests between departments often leads to a “suboptimization” of firm profits [Kotler, 1971]. This joint decision problem will also be considered in the thesis.

This thesis proposes a general framework for modelling operations decisions in start-up manufacturing firms and comparing the optimal policies with those in well-established firms. The thesis then considers three distinct inventory models which differ in assumptions about shortages and goodwill. The first inventory model assumes lost sales with general cost functions and stochastic demand. It discusses the effect of each operation parameter, order frequency and solvency checks on a start-up firm’s survival. The partial backorder models generalize the total lost sales models, considering the probability of each unsatisfied customer accepting the backorder offer as fixed. This way of modelling backorders has not been considered in the inventory literature. The shortages cause loss of goodwill which reduces customer demand in future. The last inventory model considers a firm that advertises to improve the goodwill. This model extends the inventory problems to a wider view of business planning, which is different from most of the works on start-up firms in literature. For comparison with the start-up firms, this thesis also includes models on well-established firms who maximize the average profit. This thesis makes quantitative suggestions for start-up firms by studying the inventory models, discussing the policies and the effect of a large range of factors that affect start-up firm survival. Numerical examples are presented for each model, the parameters in which are initialized taking consideration of both the industry standard and the special features of start-up firms.

The whole thesis develops as follows. Chapter 2 introduces the mathematical models in this thesis, i.e. Markov chain and Markov decision processes, and reviews the literature in inventory control models, firm goodwill, quantitative advertising models and research on start-up firms. Chapter 3 presents general models of start-up firms who maximize survival probability, and of established firms who maximize average profits. This chapter links the survival maximizing model with the average reward model such that, if a survival maximizing firm uses the same optimal policy as a profit maximizing firm with non-negative average profits, the firm has a non-negative chance of survival. However, the firm could have a higher survival probability since the policy it used may

not be optimal for survival. The specific inventory problems from Chapter 4 further discuss start-up firms' decisions and survival. Each chapter has a similar structure: introduction, model description, theoretical results, numerical investigation, and conclusion and discussions. Chapter 4 models total lost sales problems, where the demand from any unsatisfied customers is lost. It compares the optimal order decisions in both start-up and well-established firms. Focusing on start-ups, the chapter performs sensitivity tests on revenue and cost parameters in both lead time zero and one cases, and analyses the effect of the frequency of solvency checks and order opportunity on a firm's survival. Chapter 5 considers a mixture of lost sales and backorders. Each unsatisfied customer has a fixed probability of accepting the backorder offer. Results show the positive effect of backorders on start-ups' chance of survival, and stresses the importance of choosing the right niche market for start-up firms. Chapter 6 models joint inventory and advertising decisions. Advertising, as an investment in goodwill, is decided each period after orders, and affects the goodwill in the following period. The goodwill decays over time and is negatively affected by poor customer service. The average demand is estimated from the goodwill level at the start of the period. Firms plan advertising policies in consideration of both the inventory and goodwill level. Start-up firms have much more flexible advertising strategies than profit maximizing firms, due to the nature of finance. Results across the models prove that the optimal decisions are different in the two firms. While in most of the cases, the survival maximizing firms tend to be cautious about spending, in order to meet the solvency checks, they can be risk-seeking when facing difficult situation, and make one last try to achieve sales. There exists a threshold on the initial capital requirement, which quantifies the effect of capital on start-up firm survival and provides a tool of calculating the suitable level of capital initialization. Chapter 8 concludes the findings in this thesis, and provides suggestions for future research, and to practitioners and public policy makers.

In terms of research method, this thesis uses experimental quantitative models, which "lends clarity, visibility, flexibility, and generality to assumptions and relationships deemed to be important in the phenomena" [Kotler, 1971]. The problems are modelled using discrete Markov decision processes which is an efficient decision making tool for stochastic problems. The models are implemented using Java programming. Analysis is conducted by theoretical proof and numerical examples, in such a way to draw suggestions for start-up firm management and policy makers.

Chapter 2

Literature review

2.1 Stochastic models: definitions and terminologies

This section introduces the mathematical techniques used in this thesis; Markov chain, random walk and Markov decision processes. The models in this thesis are built using discrete Markov decision processes (MDPs) which are powerful tools for decision making in stochastic problems. MDPs can be seen as a series of Markov chains with decisions and rewards. A Markov chain is a stochastic process where the current states depend on the directly preceding state only. Random walk is a special case of Markov chain and is used in the analysis.

2.1.1 Stochastic processes: Markov chain and random walk

Stochastic processes evolve over time in probabilistic manners. A general stochastic process can be thought of as a collection of indexed variables $\{X_t\}$ where X_t is a random variable that represents the state of the system at time t . In discrete problems, t is a positive integer. The probability of $X_{t+1} = j$ may depend on the entire history of the process up to and including time t , i.e. X_0, \dots, X_t . A Markov chain and a random walk are two special cases of such a process that are of interest. More details about Stochastic processes could be found in Feller [1968] or Hillier and Lieberman [2001]. The following introduction to Markov chain and random walk is based on Feller [1968], except where noted.

Markov chain

A Markov chain is a special case of a general stochastic process in which the future states depend only on the present state. This is called the *Markovian property* and can be stated as $P\{X_{t+1}|X_t, X_{t-1}, X_{t-2}, \dots, X_0\} = P\{X_{t+1}|X_t\}$, where X_t is the set of possible states at time t [Hillier and Lieberman, 2001]. The transition probabilities for a Markov chain, p_{ij} , define the probability of the process moving from state i at time t to state j at time $t+1$, for all i, j and t . The transition probabilities can be arranged in a transition probability matrix, $P = \{p_{ij}\}$. A transition probability is called *stationary* if it does not change over time, i.e. $P\{X_{t+1} = j|X_t = i\} = P\{X_1 = j|X_0 = i\} = p_{ij}$ which is independent on t , for $t = 1, 2, \dots$. In a Markov chain, the n -step transition probability p_{ij}^n is the probability of transition from state i to state j in n steps, i.e. $p_{ij}^n = P\{X_{t+n} = j|X_t = i\}$. The transition probabilities satisfy that $\sum_j p_{ij}^n = 1$, for all i and $n = 0, 1, 2, \dots$.

The study of a Markov chain can start by considering the set of states that the process can visit. In a Markov chain, a *closed set* is a set of states with the property that no state outside the closed set can be reached from the states in this set. A state is called *absorbing* if this state forms a closed set. If a Markov chain has a closed set, this subchain can be studied separately from other states. A Markov chain is *irreducible* if and only if it contains no smaller closed set. In a process, the return to the initial state forms a *recurrent event*. The study of a Markov chain is equivalent to the study of the recurrent events.

States in a Markov chain can be classified into persistent states and transient states. Define f_{ij}^n to be the probability that a process starting from state i first enters state j at the n th step. Let $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$. A state i is *persistent* if $f_{ii} = 1$ and is *transient* if $f_{ii} < 1$. Each persistent state belongs to an irreducible set in which the states have the same characteristics. These irreducible sets have their own behavior and can be studied independently. In an irreducible Markov chain, $\lim_{n \rightarrow \infty} p_{ij}^n$ exists and represents the probability that the system is finally in state j , given initial state i . Let $u_j = \lim_{n \rightarrow \infty} p_{ij}^n$. It can be shown that u_j is independent on the initial state i . Furthermore, $\sum_j u_j = 1$ and $u_j = \sum_i u_i p_{ij}$. The probability distribution of state, $\{u_j\}$, is called the *stationary distribution* for the given Markov chain.

Random walk

A random walk is a stochastic process often studied in the content of gambler's ruin problems. The classical ruin problem considers a game between two players. After each round, a gambler wins or loses a pound with probability p or q , where $p + q = 1$. Assume one gambler starts with capital z , and the other gambler starts with capital $a - z$. The total capital for the game is a . The game continues until one of the players loses all the capital. The classical ruin problem is interested in the probability of the gambler's ruin and the duration of the game.

A classical ruin problem can be seen as a walk along the x -axis, where the step is one unit in every movement. Assume the gambler starts at position z . The gambler moves to the right (moves towards winning) with probability p , and moves to the left (moves towards ruin) with probability q , where $p + q = 1$. The classical ruin problem is a special case of Markov chains where the transition probability follows that, $p_{ij} = p$ for $j = i + 1$, $p_{ij} = q$ for $j = i - 1$ and $p_{ij} = 0$ otherwise. Denote the gambler's current position as x . The process has two absorbing barriers which are at $x = 0$ where the gambler is ruined, and $x = a$ where the gambler wins. Let q_z be the probability of ultimate ruin and p_z be the probability of winning, and allow $p_z + q_z = 1$. The boundary conditions on two absorbing states are $q_0 = 1$ and $q_a = 0$. After the first step, there is

$$q_z = pq_{z+1} + qq_{z-1}. \quad (2.1)$$

Solving the difference equations above, the ultimate ruin probability at initial position z is found that,

$$q_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1}, \text{ for } q \neq p, \text{ and} \quad (2.2)$$

$$q_z = 1 - \frac{z}{a}, \text{ for } q = p = \frac{1}{2}. \quad (2.3)$$

In a generalized one-dimensional random walk, the step can be of any integer value k . Define the probability of changing the position from z to x is p_{x-z} . The probability of ruin in the first step is $r_z = p_{-z} + p_{-z-1} + p_{-z-2} + \dots$. Let u_z be the ruin probability at z , it follows

$$u_z = \sum_{x=1}^{a-1} u_x p_{x-z} + r_z. \quad (2.4)$$

The calculation of u_z is further discussed in Lemma 3.4.4 in Chapter 3.

2.1.2 Markov decision processes

Markov decision process (MDP) is widely used in sequential decision making models, such as inventory models, communication models. It has an early root which can be traced back to the calculation of variations problems in the 17th century. The modern study of stochastic sequential problems began in the Second World War, with Wald [1947]. Bellman [1954] became the first major player through his work on “functional equations, dynamic programming, and the principle of optimality” [Puterman, 1994]. More details about Markov decision processes, including the other MDP models, the mathematics and examples, could be found in Puterman [1994] or Tijms [1994]. The following introduction to MDPs and the model explanations in later chapters are based on the work by Puterman [1994], except where noted.

A Markov decision process is a series of Markov chains, with decisions and rewards in each period. MDPs have the Markovian property that, the state of the process in the future depends only on the current states. In an MDP, a *decision rule* prescribes an action to select in each state at a particular decision epoch. There are four classes of decision rules: deterministic Markovian rules (MD), randomized Markovian rules (MR), deterministic history dependent rules (HD) and randomized history dependent rules (HR). In Markovian rules the action selected depends only on the current state while in history dependent rules the action selected depends on the past history. In deterministic rules the decision selected is determined by the state information while in randomized rules the decision is selected according to a probability distribution. Deterministic Markovian rules (MD) are the most focused and commonly used decision rules, as they are the easiest to implement and evaluate. Randomized history dependent rules (HR) are the most general decision rules. *Policies* prescribes the decision rule to be used at each decision epoch. Let Π^K denote the set of all policies of class K , where $K = \{HR, MD, MR, HD\}$. A *stationary policy* is one that uses the same decision rule at every decision epoch in the horizon, denoted by $\pi = \{d, d, \dots, d\}$. The models in this thesis all use deterministic Markovian stationary policies.

Markov decision processes have five elements. Based on MD stationary policies, the five elements are defined as follows.

1. A set of decision epochs, $T = \{0, 1, 2, \dots, N\}$, where $N < \infty$ in finite horizon problems and $N \rightarrow \infty$ in infinite horizon problems. Decisions are made at each epoch, except that no decisions are made at the N th epoch where the process

ends. See Figure 2.1, let n be the number of periods to the end of planning horizon, decision epoch n is the epoch when there are n periods to the end.

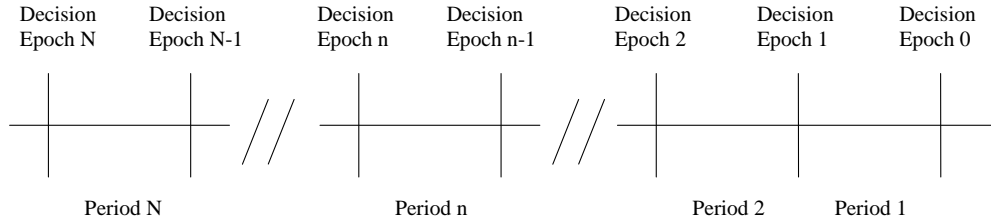


Figure 2.1: Markov decision processes: Decision epochs and periods

2. A set of system states, S . Decisions are taken after observing the system state s at each epoch, where $s \in S$.
3. A set of possible actions, $A = \cup_{\{s \in S\}} A_s$, where A_s is the set of possible actions when the system is in state s .
4. A set of transition probabilities, depending on the state and the action, $P = \{p(j|s, a), s, j \in S, a \in A_s\}$.
5. A set of rewards which depends on the state and the action taken in the epoch, $R = \{r(s, a), s \in S, a \in A_s\}$.

They will be further discussed in later chapters when models are presented.

The models in this thesis are built on two kinds firms by using different MDP models. The start-up firm model maximizes a firm's survival probability, using a total

expected reward MDP model. The established firm model maximizes a firm's expected average profit, using average expected reward MDP model. This thesis considers discrete problems and all the variables are integers. The state space is either finite or infinite countable. These two MDP models will be explained in detail next.

Expected total reward model

The expected total reward model can be of a finite horizon or an infinite horizon. In a finite horizon problem, $v_N^\pi(s)$ denotes the expected total reward of a policy π over N periods of planning horizon, where $N < \infty$, $s \in S$ and $\pi \in \Pi^{HR}$.

$$v_N^\pi(s) \equiv E_s^\pi \left\{ \sum_{t=1}^{N-1} r(X_t, Y_t) + r(X_N) \right\}, \quad (2.5)$$

where $r(X_t, Y_t)$ is the instant reward at epoch t , given the state X_t and action taken Y_t . $r(X_N)$ is the reward at epoch N where there is no decision to be made. E_s^π denotes expectation given π and s . The value of the Markov decision process is

$$v_N^*(s) = \sup_{\pi \in \Pi^{HR}} v_N^\pi(s). \quad (2.6)$$

In an infinite horizon problem, the expected total reward of a policy $\pi \in \Pi^{HR}$, $v^\pi(s)$, is calculated by

$$v^\pi(s) \equiv E_s^\pi \left\{ \sum_{t=1}^{\infty} r(X_t, Y_t) \right\} = \lim_{N \rightarrow \infty} v_N^\pi(s). \quad (2.7)$$

The value of the Markov decision process, v^* , by the optimal policy π^* , is calculated by

$$v^*(s) \equiv \sup_{\pi \in \Pi^{HR}} v^\pi(s). \quad (2.8)$$

Define $v_+^\pi(s) \equiv E_s^\pi \left\{ \sum_{t=1}^{\infty} r^+(X_t, Y_t) \right\}$, where $r^+(s, a) \equiv \max\{r(s, a), 0\}$. A *positive bounded model* is an expected total reward MDP model such that, for each $s \in S$, there exists an action $a \in A_s$ for which $r(s, a) \geq 0$, and $v_+^\pi(s) < \infty$ for all $\pi \in \Pi^{HR}$. By Puterman [1994, pp. 124–125, 279], there exists a Markovian deterministic policy Π^{MD} for such models, which gives the same expected total reward as by a history dependent randomized policy Π^{HR} , i.e. $v^*(s) = \sup_{\pi \in \Pi^{HR}} v^\pi(s) = \sup_{\pi \in \Pi^{MD}} v^\pi(s)$.

Average reward model

An average reward model assumes stationary rewards and transition probabilities, bounded rewards and finite state spaces. The average expected reward $g^\pi(s)$ of a policy π is

$$g^\pi(s) = \lim_{N \rightarrow \infty} \frac{1}{N} E_s^\pi \left\{ \sum_{t=1}^N r(X_t, Y_t) \right\}, \quad (2.9)$$

where $r(X_t, Y_t)$ is the reward by action Y_t at epoch t .

Let $r(s)$ denote a reward function, given the state s , by a fixed policy. In Markov decision processes, each stationary policy d^∞ generates a Markov reward process $\{(X_t, r(X_t)); t = 1, 2, \dots\}$. The transition probability matrix P has the property that, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N P^{t-1} = P^*$. The gain of the reward process,

$$g(s) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} E_s \left\{ \sum_{t=1}^N r(X_t) \right\} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N P_\pi^{t-1} r_{d_n}(s) = P^* r(s), \quad (2.10)$$

exists when the reward is bounded and P^* is stochastic. The gain is the average reward for a system in a steady state and, the value of which is referred as the *stationary reward*. The bias term $h(s)$ represents the expected difference between the total reward and the stationary reward, such that

$$h(s) = E_s \left\{ \sum_{t=1}^{\infty} [r(X_t) - g(X_t)] \right\}. \quad (2.11)$$

In vector form, there is,

$$v_{N+1} = Ng + h + o(1).$$

$o(1)$ denotes a vector with components approaching 0 as $N \rightarrow \infty$.

A Markov decision process is a *unichain* if the transition matrix corresponding to every deterministic stationary policy is unichain. In such models, there exists a constant gain g which is independent on the initial state, i.e. $\max_{d \in D} \{(P_d - I)g\} = 0$, where D denotes a set of deterministic Markovian policies.

2.2 Literature review on inventory models

Inventory control has been a popular research topic since the 1950s and is widely believed to be important to business and the economy. There has been great development in inventory models, from the classic constant demand economic order quantity (EOQ)

problems, to stochastic demand lot-sizing problems and perishable goods newsvendor problems, and then to more complex multi-product, multi-echelon problems. For details of these inventory models, including typical assumptions, order policies, and use in practice, see for example Hadley and Whitin [1963] or Silver et al. [1998]. This literature review will focus on partial backorder models and joint decision models as these are closest to the research interests of this thesis.

2.2.1 Inventory systems: a short introduction

Inventory is essential to manufacturing firms. Firms manage the inventory in the aim of satisfying most of the customers at the least costs, in consideration of the stock, lead time, customer preferences when stockout appears, etc. Before the literature review, an introduction is presented to explain some of the essential terminologies in inventory models, following Silver et al. [1998, pp. 233-241]

Lost sales versus backorders

When there are stockouts, a *lost sale* happens if the unsatisfied customer goes elsewhere, a *backorder* happens if the customer waits or agrees to return to be served.

Inventory level versus inventory position

Inventory level (also known as on-hand stock) is stock physically available on shelf. *Inventory position* (also available stock) is inventory level plus the amount on-order but not arrived, minus the number of backorders. Firms make orders by checking stock status, i.e. the number of items in stock, the number of items on order, and the number of backorders.

Continuous review versus periodic review

In *continuous review* systems, the stock status is assumed to be always known. In *periodic review* systems, the stock status is checked every R units of time. For example, R could be 4 hours, one day, or whatever the review time is in the system.

Replenishment lead time

Lead time, is the time between a request for a replenishment order being sent and the products ordered being available for customers. It is often considered as either constant

or stochastic in classic inventory problems. Stock is therefore important in such cases to meet variable demands.

Order systems

In the inventory literature, there are four commonly used order systems: (s, Q) , (s, S) , (R, S) and (R, s, S) .

Order-point, order-quantity (s, Q) system It is used in continuous review systems. s is called order point, Q is order quantity. An order of Q items is made if the inventory position drops to s or lower.

Order-point, order-up-to-level (s, S) system It is also used in continuous review systems. s is the order point, S is called order-up-to-level. If the inventory position drops to s or below, an order is made to bring the inventory position to level S . If the demand is unit-sized, this policy is equivalent to the (s, Q) policy letting $S = s + Q$. Otherwise, the order quantity in (s, S) system can vary.

Periodic-review, order-up-to-level (R, S) system It is commonly used in periodic review systems. R is the time between review periods, S is the order-up-to-level. An inventory review is made every R time units, orders are made at each review time to bring the inventory position to level S .

(R, s, S) system It is used in periodic review systems. The policy is a combination of (s, S) policy and (R, S) policy. Reviews are made every R time units. Orders are made to bring the inventory position to S if it drops to s or below.

Just-in-time production

The Japanese experience of just-in-time (JIT) production has become popular in modern production management. In such a system, any form of waste, such as inventory or lead time, is minimized. JIT is a dynamic system where there are continuous improvements. It shares the same idea with other modern systems such as lean production, zero inventory. In research, lead time is treated as a control variable in a JIT system. A “crashing cost” is a composition of administrative costs, transport costs and supplier’s speed-up cost and is charged on each unit of waiting time.

2.2.2 Optimal inventory control models

Total backorder/lost sale models

Scarf [1960] gives one of the first models taking backorders into consideration. The model considers a stochastic demand continuous review inventory problem with a fixed ordering cost, K . Scarf defines a K -convex cost function which is thereafter widely used in inventory models with a fixed ordering cost, see for example Chen et al. [2006]. By Scarf, a function $f(x)$ is K -convex if

$$K + f(a + x) - f(x) - af'(s) \geq 0, \text{ for all } a > 0 \text{ and all } x,$$

given that $f(x)$ is a differentiable function. If differentiability is not assumed, $f(x)$ is K -convex if

$$K + f(a + x) - f(x) - a\left[\frac{f(x) - f(x - b)}{b}\right] \geq 0 \text{ for all } a, b > 0 \text{ and all } x.$$

Scarf proves that, if the holding cost and the shortage cost are linear, or more generally, if the expected total cost is K -convex on inventory level, an (s, S) policy is always optimal to the problem. The conclusion applies in all constant lead time systems with total backorders.

Veinott and Wagner [1965] use renewal theory and stationary analysis to analyse the optimal policy in a periodic review inventory system, with constant lead time, stochastic demand and a single-period discount factor. Unsatisfied demands are backordered. The total cost is a function of fixed set-up cost, purchasing cost, expected holding cost and backorder cost. The objective is to minimize the average cost. A (R, s, S) policy is found to be optimal.

Ehrhardt [1984] considers stochastic lead time in a periodic review system with backorders, for both finite and infinite planning horizon problems. Lead time is an identically distributed random variable, ranging from zero up to a fixed level. The model has the property that replenishment orders do not cross in time and the lead time is independent of outstanding orders. Demands in successive periods are independent but not necessarily identically distributed. Optimal policies could be found to minimize the total expected cost, if the functions of expected holding cost, purchasing cost and shortage cost are convex for all time. When there is no set-up cost, a myopic (R, S) policy is optimal; when there is a constant set-up cost, a (R, s, S) policy is optimal.

A policy is called myopic if it is "the optimal policy for a single period model that is defined explicitly in terms of the original model parameters" [see Porteus, 1990, pp. 628].

Chiang [2005] considers periodic review models with either backorders or lost sales. In the backorder model, shortage cost is charged on the duration of shortages. A (R, S) policy is optimal for the total backorder problem. In the lost sale model, the optimal order quantity is a function of the inventory level. The results are found to hold in systems with both constant and stochastic lead time. However, no fixed ordering cost is considered in either of the models.

In the literature, lost sale problems are considered to be more difficult to solve than those with backorders. In inventory models with total backorders and fixed lead time, L say, the inventory level at time t is equal to the inventory position at time $t-1$ minus the demand during the interval $[t-1, t]$. Whereas with lost sales, the relationship does not apply since the sales from the outstanding orders during the interval may be lost, which in turn affects the inventory level at time t . Inventory position is no longer all that matters, the remaining lead times of ourstanding orders are now relevant. Therefore, it is no longer as simple to derive optimal order policies for lost sales models. Only when those variables external to cost minimization function, for example product price, are assigned to be optimal, could profit maximization problems be equivalent to cost minimization problems [Whitin, 1955].

Hill and Johansen [2006] present lost sale problems in both periodic and continuous review models. The models have fixed ordering costs, Poisson-distributed demand, and consider both fixed and variable lead times. The objective is to minimize the long-run average cost per unit time. Profit maximization objective is said to differ from cost minimization by a fixed term, if the purchasing cost and sales price can be appropriately incorporated to the unit lost sale cost. Neither (s, S) nor (s, Q) policies are optimal for the problems. A near-optimal policy is found by solving a semi-Markov decision process.

Partial backorder models

In practice, rather than having total backorders or total lost sales, firms are more likely to face partial backorders. This is generally for the reason that each customer has his/her own preference. When there are stockouts, some customers would like to wait

for backorders and some others would prefer other providers or substitutes.

Montgomery et al. [1973] are the first to consider partial backorder problems. In their model, a fixed portion of customers accept backorders, the others are lost. Both deterministic and stochastic demand cases are studied in the paper. In the deterministic demand model, the optimal order quantity is found to minimize the overall costs. The stochastic demand model considers continuous review and periodic review problems and discusses the optimal (s, Q) and (R, S) policies in the two systems, respectively. Montgomery et al argue that the assumption of total lost sales or total backorders would affect total cost significantly if the real problem is a mixture of the two. The functions of expected total cost derived in this paper are often referred in later research, see for example, Hariga and Ben-Daya [1999] and Zequeira et al. [2005].

Ouyang et al. [1996] consider partial backorders in a just-in-time (JIT) system, where backorders are a constant percent of total unfilled demands. Lead time demand is normally distributed. The reorder level is fixed as the average lead time demand plus the safety stock which is a fixed multiple of the standard deviation of lead time demand. Solutions are to find the optimal decisions on crashing lead time and order quantity. For other examples of JIT models, see for example Liao and Shyu [1991] and Ben-Daya and Raouf [1994].

In Montgomery et al. [1973], backorders are assumed to be a fixed proportion of all shortages. In later research, the backorder rate is taken as a function of other parameters considering customers' impatience. Waiting time is one of those factors widely used in the literature. Abad [2001] considers easily perishable goods which decay over time and cannot be salvaged at the end of an inventory cycle. The inventory cycle is separated into two intervals: one length of time during which the net stock is positive and demand is totally satisfied, plus one length of time during which the net stock is zero or negative and stockouts may occur. Decisions are made on finding the optimal time of order placement. The backorder rate is a decreasing function of waiting time. A revenue based approach is used, where product price is assumed to vary in order to maximize the instant revenue and no shortage costs are considered. The price within an inventory cycle is constant and is independent of the lot size. The model is decomposed as a pricing model plus a lot-sizing sub-problem, where demand is a function of price. Abad argues that the solution is applicable as backorder cost and lost sale cost are difficult to estimate in practice.

Zequeira et al. [2005] consider a case where customers are impatient for backorders.

Demand follows a Poisson process, backorder is a function of waiting time. Decisions are the optimal lead time, reorder point and order quantity. Two waiting time functions are discussed. One case is an exponential distributed function where the backorder probability depends exponentially on the time until the next replenishment. The other case is a piecewise function, in which the total waiting time is divided into several intervals, the backordering probability in each interval decreases with the waiting time, i.e. the longer the waiting time, the less probable a backorder is completed. These two functions are shown to be equivalent. The piecewise function can be applied in practice as the backorder rate and demand can be treated as constant during the interval, and can be estimated by the equivalent exponential function.

San-Jose et al. [2005] extend the partial backorder models, specifying opportunity cost and goodwill cost from lost sale cost. The lost sale cost is the goodwill cost plus the opportunity cost per item lost. The backorder cost is a fixed cost plus a part proportional to units of backorders and waiting time. The backorder rate is defined as a two-piece function, which is a linear function increasing with time until the waiting time reaches a maximum level beyond which the function vanishes. The inventory system is of continuous review and with no lead time, demand is constant per unit of time. Solutions are found to find the optimal inventory cycle and lot-size quantity, to minimize average total cost. The model is said to be a generalized form of previous inventory models. San-Jose et al. [2006] present another partial backorder model where the backorder rate is a negative exponential function of waiting time.

Lodree [2007] has a continuous review, partial backorder, stochastic demand model with sale contracts. Backorders are offered when there are stockouts. The backorder rate is a piecewise function of waiting time and backorders are canceled if the waiting time exceeds an upper level. If backorders are canceled, the buyer will not make a contract with the firm, i.e. the buyer will not consider buying products from the same supplier for a fixed number of periods. The contract is treated as an additional strict penalty on lost of backorders, with the same unit cost on lost sales. Optimal order quantity is found to minimize total expected cost.

2.2.3 Joint decision inventory models

In most of the literature, inventory management is treated separately without coordination with other departments. Since a firm is one single organization with several

departments, decisions should be coordinated to achieve the final goal. The functions of other departments affect an inventory system and so the accuracy of inventory decisions. For example, marketing affects expected demand which is often used to inform order decisions. Poor communication between these two departments could result in unexpected high demand and inappropriate inventory control. In the literature, only a few papers consider joint decisions of inventory and other areas such as pricing and marketing. The current research emphasizes the importance of coordination between departments.

Inventory-pricing models have been of interest in the literature. Whitin [1955] gives the first model considering pricing effects. Whitin presents a lot-sizing lost sale model in which the demand is formulated as a linear function of product price. Costs considered in the model are a fixed ordering cost, purchasing cost and holding cost. The objective is to maximize the average profit, by choosing the optimal price. The profit maximization problem is equivalently replaced by a cost minimization problem. The deterministic demand model is extended for style goods, where there is a probability of product sales and a goodwill loss. Other possible extensions such as multi-stage production, are also discussed in the paper, by combining economic theories and businesses.

Thomas [1974] gives the first multi-period price control model. The model considers a discounted periodic review inventory problem with a fixed setup cost and total backorders. Demand is a decreasing function of price. A two-level ordering and optimal pricing policy named as (s, S, p) policy, is derived to be optimal when the price is continuous in a feasible range. By the (s, S, p) policy, reorder level s and order-up-to-level S are determined first in each period. If the inventory level x is above s , the firm makes no order and sets the price at $p(x)$ for x . If the inventory level is below s , the firm orders up to level S , and charges the price for $p(S)$. Thomas adds that the policy may fail if price is restricted to a discrete set.

More recently, Chen and Simchi-Levi [2004a] present a finite-horizon periodic review inventory-pricing model with total backorders. The demand is a decreasing function of price plus a random component that is independent of price. This is called an additive demand function (for more details about additive demand functions, see Petruzzi and Dada [1999]). This model differs from that of Thomas [1974] by considering lower and upper bounds on price. Decisions on ordering and pricing are made at the beginning of each review period to maximize the expected profit over the planning horizon. An (s, S, p) policy is found to be optimal for an additive demand function. However, for

a general demand function, i.e. additive plus multiplicative function (see Petruzzi and Dada [1999]), the cost function is no longer necessarily K -convex and so (s, S, p) policy may not be optimal. Chen and Simchi-Levi define a term “symmetric- K -convex” which is a weaker version of K -convex. An optimal policy named (s, S, A, p) is derived on condition of a symmetric- K -convex cost function. Under such a policy, there is an $A \subset [s, (s+S)/2]$. If the inventory level x is less than s or in set A , an order of size $S-x$ is made. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of a review period. In their infinite-horizon model [Chen and Simchi-Levi, 2004b], a stationary (s, S, p) policy is found to be optimal for both discounted profit and average profit problems, with a general demand function.

Chen et al. [2006] consider a periodic review inventory-pricing model. Unsatisfied demands are lost and leftover inventory has a salvage value. Demand is an additive function of price. (s, S, p) policy is proved to be optimal for profit maximization in single-period newsvendor problems and discounted finite-horizon problems. Both fixed ordering cost and lost sale cost are found to have significant effects on the policies, as well as the rewards.

A few other inventory models take consideration of joint decisions in areas such as production, marketing and finance. Doshi et al. [1978] present an inventory-production decision model. The producer switches between two production levels, to minimize the expected costs. A named (y_1, y_2) production rule is used in operation, where y_1 and y_2 are low and high inventory levels, respectively. Following this rule, the producer switches from a high production level (level-2) to a low production level (level-1) if the inventory reaches the high level y_2 , and switches from the low level to the high level if the inventory falls below y_1 , or chooses no production if the inventory level is higher than an upper inventory bound. Each switch has a fixed cost and happens instantaneously. Demand follows a Poisson process. Demand is totally backordered up to a limit after which any further unfilled demand is lost. Renewal theory and Markov chains are used to analyse the expected average cost and stationary distribution of the inventory level.

Luo [1998] considers pricing and advertising effects in a single epoch inventory system for perishable goods without salvage value. Backorders are offered when there are stockouts. In each cycle, production is first made to meet the backorders carried from the previous period and then for demands in the present period. The production is stopped at a point leaving some storage. During the production cycle, the inventory

value decays over time following a Weibull distribution. Decisions are on economic production quantity (EPQ) and backorder level, in the objective of maximizing total profit. In the inventory system, orders are delivered instantly, costs include a fixed setup cost, holding cost, shortage cost and decay cost. Demand increases with advertising frequency and decreases with product price. Both advertising frequency and price are pre-determined.

Cheng and Sethi [1999] develop a joint inventory-promotion decision model using Markov decision processes. It is one of the very few papers considering marketing decisions in an inventory model. Cheng and Sethi assume that a firm makes decisions on the order quantity and whether or not to make a promotion on the product, to maximize the total profit over a finite planning horizon. Decisions are made at the beginning of each review period, when inventory level and demand are known. Demand moves from one state to another each time a promotion decision is made. Unsatisfied demand is totally backordered. Replenishment arrives instantly after orders. Promotion has a fixed cost, inventory costs include purchasing cost, holding cost and backorder cost. By the end of a planning horizon, leftover products are salvaged at a unit price. A two-base-stock order and multi-threshold promotion policy (S^0, S^1, P) is found optimal for such a problem. Under such a policy, each demand state has an inventory threshold P . The promotion decision is a 0 – 1 decision such that, the product is promoted if the initial inventory position is not lower than P . An order is made to bring inventory position to S^1 if the initial position is below S^1 and the product is promoted, an order is made to bring inventory position to S^0 if the initial inventory position is below S^0 and the product is not promoted, where $S^0 \leq S^1$.

Berling and Rosling [2005] discuss the financial risks, i.e. the determination of the discounted rate, in an inventory system. The firm uses a (R, Q) policy, to maximize its present market value. A firm's present market value is a function of its monetary value and the expected annual return, following the Consumption Capital Asset Pricing Model (C-CAPM) (for more about C-CAPM model, see Breeden [1979]). In the model, demand and purchasing price are considered as two systematic risks, both of them follow a logarithmic Wiener process. Shortages are met from external sources with cost on each item and also a discount as compensation. The discount rate is formulated as a function of risk-free interest rate plus a risk premium which is the price of risk times the covariance of the aggregate per capita consumption and the stochastic variables in monetary value. The problem is modelled in both a newsvendor problem with positive

lead time, shortage cost and a setup cost, and an infinite-horizon problem. The results show that systematic purchasing price risk has the most effect on inventory decisions. The effect of systematic demand risk seems to be negligible.

Table 2.1 lists the models in the inventory literature, in terms of system lead time, lost sales/backorders, and review system.

2.3 Literature review on goodwill in general

Goodwill has been an important topic in business for a long time. According to Falk and Gordon [1977], there are four frequently discussed definitions:

1. Super (Excess) profit: goodwill is the present value of profits in excess of normal returns on the identifiable assets owned by a firm.
2. Master valuation: goodwill equals the “difference” between the purchase consideration given for a firm and the net value of the identifiable assets (tangible and intangible) taken over.
3. Momentum: goodwill represents some initial momentum (or push) to the acquiring firm.
4. Results of imperfect competition: intangibles are “conditions of imperfect competition impinging on the operation of the business”.

For whatever the definition, goodwill represents a firm’s “above-average ability” to make profits. Since goodwill is a commercial valuable, it is regarded as an asset. As goodwill is not physically visible, it is then an intangible asset. However, goodwill differs from other intangible assets in that, it is “not protected by special legislation or by legal instrument” [Hughes, 1982, pp. 7-8]. Hughes makes analysis on goodwill’s characteristics and gets a conclusion of its properties that [Hughes, 1982, pp. 175-195]

- Goodwill has no physical existence.
- It is inseparable from a firm.
- Its creation and maintenance are a function of a firm’s absolute and relative size.
- As an intangible asset, goodwill arises in successful businesses. There exists limitation upon its recognition.

Table 2.1: A summary of the inventory literature

Author(s)	Lead Time	Backorders	Review
Abad(2001)	zero	mixture, function of waiting time	continuous
Ben-Daya et al (1994)	decision	no shortages	continuous
Berling et al (2005)	zero	backorders	continuous
Chen et al (2004 a,b)	zero	backorders	periodic
Chen et al (2006)	zero	lost sales	periodic
Cheng et al (1999)	zero	backorders	periodic
Chiang (2005)	constant	backorders/lost sales	periodic
Doshi et al (1978)	zero	mixture, backorders up-to a limit	periodic
Ehrhardt (1984)	stochastic	backorders	periodic
Hariga et al (1999)	decision	mixture, fixed rate	continuous/periodic
Hill et al (2006)	constant/variable	lost sales	continuous/periodic
Liao et al (1991)	decision	no shortages	continuous
Lodree (2007)	zero	mixture, function of waiting time	single epoch
Luo (1998)	zero	decision	single epoch
Montgomery et al (1973)	zero/constant	mixture, fixed rate	continuous/periodic
Ouyang et al (1996)	decision	mixture, fixed rate	continuous
San-Jose et al (2005)	zero	mixture, function of waiting time	continuous
San-Jose et al (2006)	zero	mixture, function of waiting time	continuous
Scarf (1960)	zero/constant	backorders	continuous
Thomas (1974)	zero	backorders	periodic
Veinott et al (1965)	constant	backorders	periodic
Whitin (1955)	zero	lost sales	single epoch
Zequeira et al (2005)	decision	mixture, function of waiting time	continuous

Based on the imperfect competition concept, Falk and Gordon [1977] examined 17 characteristics that “related to business combination decisions and resulting goodwill”, which are primarily related to five imperfection markets. They are,

1. Three characteristics related to the imperfections of the financial markets. They are (1) large cash reserves of the acquired firm, (2)* increased ability of the newly combined entity to raise more funds, (3) increased ability of the newly combined entity to raise funds at a lower cost.
2. Six characteristics related to the imperfections of the capital goods markets. They are (4)* production economies accruing to the newly combined entity, (5) avoidance of transaction costs, (6) reduction of inventory holding costs, (7)* assurance of supply, (8)* reduction of seasonal or cyclical fluctuations, (9)* access to otherwise unavailable technology.
3. Four characteristics related to the imperfections of the labour markets. They are (10)* managerial talent, (11) good labour relations, (12) training programmes, (13) organisational structure of the acquired firm.
4. Three characteristics related to the imperfections of the product markets. They are (14) good relationships with the general public, (15) good relationships with government officials and authorities, (16)* the existence of an established brand name.
5. One characteristic related to market imperfections resulting from government regulatory activities.

The characteristics with asteria are found to have the highest degree of importance. Falk and Gordon further classify these 17 characteristics into four groups according to empirical research.

- “increasing short-run cash flow” (1, 2*, 3, 4*, 5, 6, 17),
- “stability” (7*, 8*, 15),
- “human factor” (10*, 11, 12, 13),
- “exclusiveness” (9*, 16*).

From both theory and practice, goodwill is regarded as a winning power but at the same time, has invisible effect to businesses. There have been discussions on the valuation of goodwill. For instance, Mueller and Supina [2002] build a model on calculating goodwill capital. In the paper, goodwill capital is defined as the firm's market value less the firm's physical capital stock and other tangible assets, its intangible Research and Development (R&D) capital stock and its intangible capital stock. Statistical models are built to measure firms' goodwill capital.

Some other works focus on the relationship between goodwill and profitability. For instance, Chauvin and Hirschey [1993] study the empirical relevance of a number of "firm specific characteristics as sources of goodwill", using samples from both manufacturing and non-manufacturing sectors. The study shows that both advertising and R&D expenditure have primary influences on goodwill. Intangible assets have positive effects on goodwill in the R&D intensive manufacturing sector. Firm size, measured by tangible assets, also has an important influence on goodwill. Accounting goodwill numbers are found to have a mixed influence on net income in manufacturing and non-manufacturing sectors. The influence is not significant in either sectors, which is said to depend on the balance of positive economic goodwill effects and negative reporting effects. In their another paper, Chauvin and Hirschey [1994] study the size advantages in advertising and R&D. They argue that advertising and R&D are more profitable to large firms than to small firms, due to economies of scale. However, small firms can also get high profits from well-targeted advertising and R&D.

Furthermore, mathematical models are built on finding the change of goodwill by advertising. Examples are the classic advertising goodwill model by Nerlove and Arrow [1962], and some other optimal advertising control models such as Monahan [1983], Weber [2005], Nair and Narasimhan [2006] and Marinelli [2007]. These will be introduced in the next literature review on advertising models.

2.4 Literature review on advertising control models

Optimal advertising control is a popular topic in marketing science. Models are built to discuss advertising effects, advertising response curves and optimal control decisions. Sethi [1977] and Feichtinger et al. [1994] give two comprehensive reviews on the development of mathematical advertising models. This literature review focuses on monopolistic firm problems. It covers three parts, i.e. classic advertising models,

advertising control models and joint decision advertising models.

2.4.1 Classic advertising models

The following five advertising models focus on different advertising effects and are widely implemented in advertising models in the literature. The Nerlove-Arrow model considers advertising effect on goodwill, Vidale-Wolfe model addresses advertising effect on sales, BRANDAID model is an aggregated model on sales, Bass model focuses on word-of-mouth diffusion, and Lancaster model is a two-player competition model.

Advertising capital model (N-A model)

Nerlove-Arrow model [Nerlove and Arrow, 1962] is one of the most important advertising models [Sethi, 1977]. According to Nerlove and Arrow, advertising expenditure shifts the demand function by generating new customers, and may also change the shape of the demand function as well by affecting the tastes and preferences of the customers. This effect appears at present and lasts in subsequent periods. Goodwill $A(t)$ is introduced as a stock of advertising effect on demand, at time t . One unit of goodwill has a monetary value, say \$1. The same as other capital, goodwill depreciates over time at a constant proportional rate δ . The change of goodwill over time is formulated as

$$\dot{A} + \delta A = \alpha, \quad (2.12)$$

where α is the advertising outlay on goodwill. Formula 2.12 explains that, the unit change of goodwill is the difference between advertising investment and stock depreciation. The demand is a function of product price, firm's goodwill and other functions that are not under control of the firm. Total production cost is a function of output. The objective is to maximize the total present value of revenue net of production cost and advertising expenditure. Decisions are made on pricing and advertising.

The model decides an optimal goodwill level A^* . If the initial goodwill level A_0 is greater than A^* , the firm should do no advertising and let the goodwill decrease to A^* and then, spend δA^* to keep it at the optimal level. If the initial goodwill is no more than A^* , the firm should spend as much as possible to bring the goodwill up to A^* and then maintain this level by spending δA^* during the whole planning horizon. It is a "bang-bang control followed by a singular control" [Sethi, 1977].

The N-A model focuses on marketing decisions and no production and inventory

conditions are considered in detail. Firms are assumed to be monopolistic and, the production amount equals exactly the number of items demanded.

Advertising response model (V-W model)

Vidale-Wolfe model [Vidale and Wolfe, 1957] considers the sales response to advertising, based on experimental results. Three parameters are found to describe advertising effect.

- The sales decay constant λ . It is the percentage of sales rate that is decreased due to product obsolescence, competing advertising, etc, given no advertising expenditure on the product. Vidale and Wolfe introduce the exponential decay constant λ , to describe the change of sales rate $s(t)$ so that, $s(t) = s(0) \exp(-\lambda t)$, where $s(0)$ is the initial sales rate of an unpromoted product.
- Saturation level M . It is the limit of the advertising effect on sales. This limit depends on the product and advertising medium used. It defines the part of the market that an advertising campaign can capture. The saturation level is the most significant difference between the V-W model and N-A model [Sethi, 1977].
- Response constant r . It describes the sales generated by a unit of expense on advertising, when there are no sales initially. Generally, the sales rate generated by a unit of advertising decreases as the value of current sales increases. In the model, the sales rate generated per unit of expenditure on advertising when the current sales rate is s , is expressed as $\frac{r(M-s)}{M}$.

The sales response to advertising is modelled as,

$$\dot{s} = rA(t) \frac{(M-s)}{M} - \lambda s, \quad (2.13)$$

where s is the sales rate and $A(t)$ is the advertising expenditure at time t . Equation 2.13 quantifies how the sales rate is affected by advertising, $rA(t) \frac{(M-s)}{M}$, and depreciation, λs .

Vidale and Wolfe argue that, sales rate has the most dramatic increase at $t = 0$ and then the effect decreases with time. As a result, the first dollar spent on advertising has the most effect, the second dollar has the next most effect, so on and so forth. For equal advertising expenditures, a protracted campaign is more profitable than a short, intense campaign. However, there is no example on the conclusion.

BRANDAID model

Little [1975] presents an aggregated marketing model named BRANDAID. This model considers a few marketing actions, i.e. advertising, promotion, pricing, salesmen and retail distribution, each of them forms a submodel on sales. In the advertising submodel, Little considers the sales response of advertising $e(t)$. Little assumes that having other performances not changed, the sales would remain the same if advertising is kept at a reference rate. The sales rate decreases if advertising rate is below this reference rate and it increases if advertising rate is above the reference rate. The sales response function is formulated as

$$e(t) = \alpha e(t-1) + (1-\alpha)r[a(t)], \quad (2.14)$$

where $a(t)$ is the advertising rate at time t , $r[a(t)]$ is the long-run sales response to advertising, and α determines how advertising changes sales in the long-run. Little [1979] points out that BRANDAID model accommodates V-W model and N-A model as special cases.

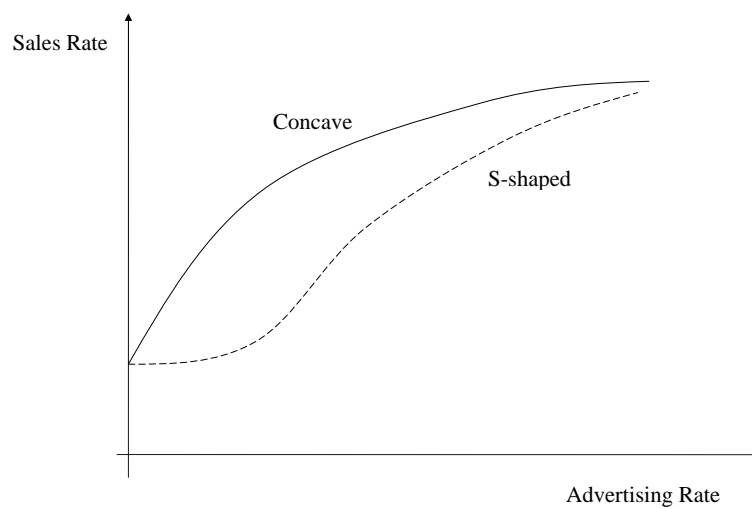


Figure 2.2: Advertising response curve to sales

Little explains that, whether advertising expenditure should be spent evenly or in

pulses, depends on the shape of advertising curve (see Figure 2.2). If advertising has a concave response curve to sales, an even policy is optimal. If advertising response curve is S-shaped, a pulsing policy is optimal. The shape of advertising curve, as well as the optimal policy, has been in great discussion. This will be shown in later literature.

Bass diffusion model

Bass [1969] models the timing of initial purchase of new products based on adoption and diffusion theory. Customers are divided into two categories, innovators and imitators. The difference between these two groups of customers is that, innovators make independent decisions on purchases, while imitators are influenced by others. Denote the initial purchasing time is T , the probability of the initial purchase, denoted as $P(T)$, is modelled such that,

$$P(T) = p + \frac{q}{m}Y(T). \quad (2.15)$$

$Y(T)$ is the total number purchasing in period $(0, T)$ and, $Y(0) = 0$. $p, q/m$ are constants. p is the probability of an initial purchase from innovators at time $T = 0$. q is the coefficient of imitators and m is the total initial purchase of the product over duration T . $\frac{q}{m}Y(T)$ reflects the purchase pressure on imitators from other buyers. The expected sales at time T , $S(T)$, is derived as,

$$S(T) = mf(T) = pm + (q - p)Y(T) - (q - p)[Y(T)]^2, \quad (2.16)$$

where $f(T)$ is the likelihood of purchase at time T , $F(T)$ is its cumulative function, and $Y(T) = mF(T)$.

Bass model is a model on word-of-mouth effect and is considered as “the earliest model emphasizing the importance of cumulative sales” [Feichtinger et al., 1994]. For its good description of empirical data [Bass and Krishnan, 1994], Bass diffusion model is widely used in areas modelling new product and new technology, see for example [Krishnan and Jain, 2006].

Lanchester competition model

Lanchester competition model was first established for military compacts in World War II. It has four submodels each considering a specific problem between two military forces, named Force 1 and Force 2. Lanchester model is described to be universally valid

in modelling competitions. Model 4 has been successfully implemented in advertising competitions [Kimball, 1957]. Assume there are a pair of competitors, Firm 1 and Firm 2, in one market. Let k_1 and k_2 denote their advertising expenses, and n_1 and n_2 denote the number of customers each firm owns, respectively. The governing equations of this competition are

$$\frac{dn_1}{dt} = k_1n_2 - k_2n_1, \quad \frac{dn_2}{dt} = k_2n_1 - k_1n_2, \quad (2.17)$$

where $n_1 + n_2$ is the total number of customers in the market and it is a constant. This model interprets that, each firm wins customers from its rival at a rate proportional to the number of customers the rival has, the rate depends on the firm's own advertising effort.

2.4.2 Optimal advertising control models

The research on optimal advertising decisions is of great interest in recent years. Some models are built on finding optimal advertising expenditures, and some others focus on the shape of advertising response curve and the corresponding optimal advertising policies. This section will give a separate review on these two areas.

Optimal advertising expenditure

Monahan [1983] represents a model for new or growing products by using Markov decision processes (MDPs). Monahan assumes that demand in a single period is a function of goodwill and the cumulative sale-to-date. Goodwill, w_t , is a stock for advertising expenditure, a_t , such that, $w_t = \sum_{k=0}^{t-1} a_{t-k}\theta^k = a_t + \theta w_{t-1}$, where $1 - \theta$ is the advertising depreciation rate in each period. Goodwill has an upper bound which presents its saturation level. It also has a salvage value by the end of planning horizon. The objective is to maximize a firm's discounted expected profit plus salvage value, by choosing an optimal goodwill level in each period. Monahan concludes that advertising expenditure is affected by sales and goodwill. It is optimal to advertise less if either goodwill or cumulative sales increases, in both finite horizon and infinite horizon problems. No specific optimal solution is derived.

Weber [2005] analyses an infinite-horizon model on durable goods, considering both N-A advertising-goodwill model and Vidale-Wolfe advertising-sales model. The model assumes decreasing return to scale from advertising. Demand for new products at

any instant generated by advertising D is modelled as $D(z, y) = (1 - z)y$, where z is the evolution to the installed base and changes according to V-W model and y is the cumulative advertising effect following N-A model. The demand is assumed equal to supply. The objective is to maximize total discounted profit in an infinite horizon, by deciding the optimal advertising effort. Weber shows that there exists an optimal policy for the infinite horizon model and focuses on techniques for solving the resulting non-convex optimization problem.

Nair and Narasimhan [2006] model a differential game in which goodwill depends on product quality and advertising. Demand is a multiplicatively separable function of price and goodwill, which decreases with price and increases with goodwill. Price function is assumed to be linear, where the demand is decreasing with price. Goodwill function is assumed to be quadratic and concave. Each rival's goodwill is a function of their own product quality plus advertising effect minus the depreciation of their own goodwill and, their rival's product quality and advertising effort. The competitor's factor has less effect on goodwill than the firm's own efforts. Each of them maximizes the discounted profit over an infinite horizon. Nair and Narasimhan show that, advertising rate decreases with increasing goodwill level and, product quality increases with goodwill level. Nair and Narasimhan argue that a decrease in the rate of demand does not necessarily show that goodwill level is decreasing. Investments on both advertising and product quality are inversely related.

Marinelli [2007] presents a stochastic goodwill model with multiple objectives. A firm has the goals of maximizing the discounted awareness of product and minimizing its expected advertising effort at a given launch time. The reward function is assumed to be concave and continuous and the loss function is convex and continuous. The evolution of goodwill is a stochastic perturbation of the N-A model. Marinelli discusses optimal advertising strategy in two specific reward and loss functions. When there is a linear reward and loss function, a bang-bang policy is optimal: the firm does not advertise till a certain time t^* , after which it advertises at a maximum rate. Optimal advertising effort is derived, and, optimal launch time and minimum cost are also discussed.

Advertising effect and policy

In the literature, advertising is assumed either to have a concave response curve or an S-shaped response curve (see Figure 2.15). A concave curve indicates that the effect of advertising diminishes right after advertising campaign. It is theoretically found in microeconomics as the diminishing return to input. An S-shaped curve represents advertising that initially has an increasing effect on sales, but reaches a point after which the effect decreases with advertising rate. This S-shaped curve is often referred to as “the curve of learning” [Simon and Arndt, 1980] and is supported by the majority in advertising area. There have been debates on finding which effect is happening in practice, but no conclusion is made. Simon and Arndt [1980] argue that there is “no S-shaped response function over the normal operating range”. Mahajan and Muller [1986] argue that this is because the pulsing policy, which is optimal for an S-shaped response function, linearizes the convex part of the curve and makes it not clearly observed from empirical studies.

In the literature, some works focus on finding the optimal advertising policy, based on the advertising effect. A few of them, such as ADPULS model [Simon, 1982] and M-M model [Mahajan and Muller, 1986], give further development on advertising effects based on the classic N-A and V-W models. The advertising policies which are often referred in the discussions, are summarized by Mahajan and Muller [1986] as follows.

- Even policy: the firm advertises at a constant level throughout the planning horizon;
- Pulsing policy: the firm alternates between high and zero levels of advertising expenditure;
- Chattering policy: a theoretical policy by Sasieni [1971], the firm alternates between high and zero advertising levels infinite times during a finite planning horizon. It can be seen as a special case of pulsing policy with infinite frequency;
- Blitz policy (one pulse policy): the firm concentrates all its efforts in one interval of the planning horizon, advertising at a constant level throughout this interval. It is seen as a special case of pulsing policy with only one pulsation;
- Pulsing maintenance policy: the firm combines any of the above policies with a low level of advertising, usually a maintenance level.

Tapiero [1978] proposes a discrete stochastic model on advertising which extends the N-A goodwill model. Advertising is an investment on goodwill and it reflects a firm's attitude towards risk. The model considers recalling, which increases goodwill due to the positive effect of advertising on purchasing, and forgetting, which decreases goodwill due to customers forgetting past advertising efforts. In each period after the advertising decisions, goodwill level changes as a result of recalling and forgetting. The goodwill effect changes according to a stationary transition probability. The probabilistic effect on goodwill is said to bring risks to management. Tapiero discusses the optimal advertising expenditure by both an open-loop advertising strategy and a feedback strategy. Under an open-loop strategy, advertising expenditure is a function of time only, say $a(x, t) = a(t)$, where a is the advertising expenditure at time t and x is the goodwill level. Under a feedback advertising strategy, advertising is decided on observing both time and goodwill level, i.e. $a(x, t) = a_1(t)x + a_2(t)$. The equivalent solutions are found for these two advertising control problems. Tapiero shows that a firm's attitude towards risk is an important role when there is a low forgetting rate. In such cases, a risk-seeking firm would invest more on advertising, but a risk-averse firm would invest less on advertising.

Simon [1982] proposes the ADPULS model which considers the wearout effect of advertising. A wearout effect means that sales increase quickly with an increasing advertising rate but fall off over time even if the advertising rate is unchanged. This idea is generated from adaptation theory. In adaptation theory, effect happens at the time an action is taken but diminishes right away afterwards. According to Simon, this wearout effect can be seen on both new and old products. Pulsing is found optimal in this model, either with or without advertising budget constraints. However, Simon's model does not consider costs. The optimality of pulsing policy may be changed when operation costs are considered.

Mahajan and Muller [1986] compare the five advertising policies on new products. Advertising expenditure is chosen to maximize the product awareness, with a total spending budget. Awareness is formulated according to learning and forgetting phenomena in terms of time, basing on psychology research results. For a new product, the change of its market equals the advertising effects on the new customers (who are not aware of the product) less the decrease of awareness in the "old" customers. This is equivalent to learning effect minus forgetting effect. Advertising is assumed to have an S-shaped response function under which pulsing policy is found to be optimal. This

model is used in models introducing new projects, relating advertising to awareness. Mahajan and Muller argue that the results may be different if with an objective of profit maximization. This model is referred as “M-M model” in later papers.

Following the M-M model, Sasieni [1989] gives an awareness generation model. The change of product awareness is a function of awareness and advertising rate which increases with advertising and decreases with awareness. It is said to be a general function of N-A model, V-W model and M-M model. Under the assumption of an S-shaped advertising curve, chattering policy is used to replace the original increasing part of S-shaped curve with a straight line lying above it. This chattering policy is found to be optimal theoretically. Analysis shows that awareness (or sales) maximization with a budget constraint is equivalent to profit maximization after advertising, with or without a budget constraint.

Feinberg [1992] represents a deterministic, continuous time model, with an S-shaped advertising curve. Advertising is said to have two phases: a start-up period during which sales are brought to some level, and a long-run optimum period in which the firm aims to maintain the sales by a certain advertising policy. A filter, which is widely used in economics, is applied in the model to smooth the chattering policy to an even policy exponentially. A simulation is run on a V-W model, where with an exponential filtering mechanism, the optimal policies may include pulsing, besides chattering.

Mesak [1992] considers the wearout effects of advertising, based on the model by Simon [1982]. The model is based on V-W model, with some modifications to accommodate both concave and S-shaped response functions. Decisions on advertising rate are made to maximize total discounted profit. Pulsing policy and even policy are compared in the model in condition of the discount rate which decides the superiority of these two policies. Pulsing policy is superior to even policy when the discount rate is zero, and when the discount rate is positive and the initial sales is the same as with an even policy. Even policy is better when there is a large discount rate. This conclusion applies for both concave and S-shaped response functions. Mesak [2002] utilizes a modified V-W model to discuss the initial sales effects on pulsation policies by dynamic programming. Blitz and pulsing maintenance policies are analysed. Both low-high and high-low cycle sequences are considered for a pulsing maintenance policy. Results show that initial non-zero sales affect advertising policies, regardless of the shape of the advertising response function. For a blitz policy, more profits can be generated if advertising is placed at the end of the planning horizon. For a pulsing

maintenance policy, a low-high sequence is better than a high-low sequence.

Bronnenberg [1998] discusses a constrained advertising model using a two-dimension discrete Markov process. The model considers two kinds of brand, i.e. advertised brands and all remaining brands. The probability of transition within and between these two groups of brand depends on advertising expenditure. Market share is used to express the brand status and the total market share is fixed. There is a budget on advertising. Buildup and decay are explained as the asymmetric effects of advertising purchasing and are considered when making advertising decisions. Buildup has a greater effect when there are more customers repurchasing than switching, or else decay effect is greater than buildup effect. The optimal advertising policies are discussed depending on the dominate advertising effects.

2.4.3 Joint decision advertising models

There are only a few papers considering joint decisions on advertising and other operation decisions such as pricing and production. Albright and Winston [1979] present a model with joint advertising and pricing decisions, in both non-competitive market and competitive duopoly market. In the non-competitive model, Markov decision process is used for analysis. The objective is to maximize the total discounted profit. Decisions are made at the beginning of each period, observing the firm's market position. Market position is defined as the number of loyal customers. The transition probability of market position depends on the firm's decisions in each period. Three sub-problems are discussed in detail. The first problem is a general case in which both advertising and pricing decisions are considered. Analysis shows that both optimal advertising expenditure and pricing are either increasing or decreasing with market position, depending on the properties of the transition probabilities. In the second problem, advertising expenditure is the only decision. Customers are divided into two groups, one group of customers are always loyal to the product and the other group switch between loyal and nonloyal preferences according to a probability. In this case, more budget should be spent on advertising if there are more loyal customers. The third problem also considers the advertising decision only. The difference is that, the wealth not used on advertising must be spent on investment or consumption to increase customer satisfaction. There are conditions under which optimal advertising expenditure increases with market position and current wealth. In the competitive model, two firms are competing

for customers in a fixed size market. A non-zero sum stochastic game is used to find the equilibrium.

Ulusoy and Yazgac [1995] represent an aggregated multi-period multi-product model with simultaneous advertising and production decisions, to maximize expected profit. Decisions are made on advertising efficiency and product price. Demand is defined as a function in consideration of the price and advertising efficiency effects of each product. Price has a negative effect on demand, while advertising efficiency has a positive effect on demand. The model considers a lagging effect from advertising, which represents the fact that advertising affects future demand and future profit. Total cost is assumed to be a concave function of number of products and includes holding cost, production cost and backorder cost. Ulusoy and Yazgac emphasize the importance of coordination between production and marketing departments where decisions on pricing and advertising are made to smooth production as well as demand. Different conditions on fluctuating/smooth demand and inelastic/elastic price are discussed in the examples.

Khouja and Robbins [2003] consider advertising effect on demand in a single-period newsvendor problem. Decision is made on order quantity. Two different objectives are discussed in the model: maximizing profit and maximizing the probability of achieving a target profit. In the model, demand is a concave function of advertising expenditure. The mean of demand is a constant part plus a variable part. The constant part of the function represents the demand with no advertising and the variable part is proportional to a value which depends on advertising expenditure. Three models for the variance of demand are considered:

1. Advertising has not effect on variance;
2. Advertising changes variance with a constant coefficient;
3. Advertising affects variance with an increasing coefficient.

Results show that with the objective of profit maximization, advertising increases both order quantity and profit. The scale of increase depends on the effect of advertising on demand variance. There is less increase in order quantity and profit if variance of demand is more affected by advertising expenditure. Advertising has a similar positive effect on maximizing the probability of achieving a target profit. Optimal order quantity and advertising expenditure are derived.

Tan and Mookerjee [2005] consider the online advertising and IT capacity for electronic retailers. Tan and Mookerjee point out that for online retailing, advertising increases the demand which may conflict with the internet capacity. They consider joint decisions on advertising and IT capacity allocation, to maximize the expected profit. Demand follows a stationary Poisson process, processing time is generic distributed. Customers leave randomly, each customer has a time budget which is exponentially distributed. The service time for each demand has a constant mean and has a logistic distribution following Johansson [1979]. Total cost includes advertising expenditure and IT capacity which includes a constant setup cost and an operating cost by unit of capacity. There is a centralized model which gives a general joint decision solution. Tan and Mookerjee argue that there is an advertising threshold above which the retailer should not increase the demand. This threshold strictly increases with market size. Coordination is important due to the fact that, retailers always “overadvertise” because of the ignorance of IT capacity and cost of losing customers. They suggest two solutions: “Reduced session value” is to reduce the net unit revenue to a lower level which lowers optimal advertising level; “processing contract” builds a contract between the marketing and IT departments by requiring the marketing department pay a processing fee to the IT department based on average demand. Uncertainty of average demand, as well as advertising and capacity allocation lagging time, are discussed as well. It is proved that uncertainty reduces maximum profit.

Table 2.2 lists a summary of the models in advertising literature by publishing time, in terms of advertising effects, decisions and modelling methods.

2.5 Literature review on start-up firms

Start-up firms play an important role in the economy. They “boost productivity, increase competition and innovation, create employment and prosperity, and revitalise the communities” [Bank of England, 2004]. However, start-up firms face a high failure rate in their first few years of establishment. There is evidence that around half of the start-up firms fail after three years of establishment [Bank of England, 2004]. The empirical research on start-up firm survival can be classified as firm-specific and industry-specific [Lin and Huang, 2008]. Firm-specific research discusses the effect of firm characteristics such as entry size, firm age and employee conditions on firms’ survival. Some of the results find that start-up survival rate is positively affected by firms’ entry size,

Table 2.2: A summary of the advertising literature

Author(s)	Advertising Effect	Decision(s)	Mathematical Method
Albright, Winston (1979)	effect on market position	joint, advertising and pricing	MDP, non-zero stochastic game
Bronnenberg (1998)	effect on demand	advertising policy	discrete Markov processes
Feinberg (1992)	sales, V-W	advertising policy	continuous model
Khouja, Robbins (2003)	effect on demand	joint, advertising and inventory	numerical analysis
Mahajan, Muller (1986)	advertising and awareness	advertising policy	continuous model
Martinelli (2007)	N-A	optimal expenditure	multi-objective optimization
Mesek (1992)	V-W	advertising policy	continuous model
Mesek (2002)	V-W	advertising policy	dynamic programming
Monahan (1983)	N-A	optimal expenditure	Markov decision processes
Nair, Narasimhan (2006)	N-A	optimal expenditure	differential game
Sasieni (1989)	advertising and awareness	advertising policy	continuous model
Simon, Arndt (1980)	shape of response function	empirical research	statistics
Simon (1982)	wearout effect	optimal advertising policy	numerical analysis
Tan, Mookerjee (2005)	logistic effect on demand	joint, advertising and IT capacity	queueing theory
Tapiero (1978)	N-A	advertising policy	stochastic process
Ulusoy, Yazgac (1995)	effect on demand	joint, advertising and pricing	Non-linear integer programming
Weber (2005)	N-A, V-W	optimal expenditure	Hamilton-Pontryagin function

age, number of plants, access to resources, employees' higher education level, see for example Persson [2004] (Sweden) and Mata and Portugal [1994] (Portugal). However, Arauzo-Carod and Segarra-Blasco [2005] (Italy) argue that there is no evidence linking survival to firm size.

On the other hand, industry-specific research focuses on market structure, entry barriers, industry characteristics, etc. A few approaches are commonly used for post-entry performance analysis [Audretsch et al., 1999]. Sunk cost in the industry indicates how easy the market is for firm entry. An industry with high sunk cost is difficult to enter and so relates to a high failure rate, but firms who stay could have a high growth rate. Low sunk cost on the other hand relates to a low failure rate and low growth rate, resulting from the easy entry and market competitiveness. Degree of scale economics, also referred to as minimum entry scale (MES), has a similar effect. An industry with high MES has high failure rate and high growth rate (if the firm stays in the industry), and low MES relates to low failure rate and low growth rate. For innovative environments, less risk-averse firms prefer to enter high innovation industries which have high failure rate but high growth. Other factors such as high market demand and industry growth rate have positive impact, while industry dynamics have negative impact on survival (Bartik [1989] (US), Watson and Everett [1996] (Australia), Fritsch et al. [2006] and Strotmann [2007] (Germany)).

In respect to strategic planning, Romanelli [1989] has a study on environmental effects and organization strategies on start-up firms' survival, with a longitudinal study from the minicomputer industry in the US. Romanelli shows that generally, specialists and aggressive strategies increase the likelihood of survival. A specialist is the kind of organization that concentrates on the specification of its products. A firm has an aggressive strategy if it seeks to control many resources as fast as possible, and takes risks to get more resources.

Smith [1998] discusses the strategies of start-up firms. The research is based on longitudinal evidence of 150 start-up firms in Scotland. The firms are classified into three groups, high, medium and low performance. The measurements are growth, profitability and productivity. Results show that, high performance firms tend to have clear long-term objectives. They target a specific niche market, produce high quality products and provide good customer service. Financially, these firms monitor cash flow on a daily basis and stick to their initial budgets.

Start-up failure is widely discussed in empirical research. According to Watson and

Everett [1996], the five commonly held definitions of start-up failure are, discontinuance of businesses, change of ownership, bankruptcy/loss to creditors, disposed of to prevent further losses, and failing to “make a go of it”. Watson and Everett point out that calculation of failure rate is different under these definitions, among which change of ownership gives the upper extreme of failure rate and bankruptcy/loss to creditors gives the lower extreme. For individual research, the choice of definition of failure is more dependent on data availability.

Hazard rate, which is used as a prediction of firm failure, is often found to be an inverted U-shape as a result of “liability of adolescence” [Mahmood, 2000]. Start-up firms have a low hazard rate at the very early stage of foundation during which they rely on the stock of initial resources. The hazard rate increases soon after when firms are about to use up the initial capital and performance depends on firms’ ability of earning profit. Following this, hazard rate falls again as firms start to become established. This can be observed from other research, for example Audretsch et al. [1999], where in manufacturing sector, the hazard rate first goes up reaching the highest in the first year and decreases later from the second year.

As to firm finance, most of the funds to start-ups are from internal sources, i.e. the founders’ own savings and their family/friends’ equity. External funds are available in the form of venture capital, angel capital (direct funding from high net worth individuals, i.e. “angel”s), and debt from financial institutions (commercial banks and finance companies), nonfinancial business and government, and individuals [Berger and Udell, 1998]. In their paper, Berger and Udell present extensive research on private equity and debt markets for small firms. Generally speaking, during the start-up stage, firms rely on initial insider finance, trace credit and/or angel finance. The access to finance becomes easier as the firm grows. Only 54.23% of the start-up firms have any loans or leases from financial institutions, of which 86.95% of small businesses take commercial banks as their “primary” financial institution. When the economy is in hard times, the regulations and supervision of banks are justified in part to keep them safe and sound, by reducing the bank credit to small bank-dependent businesses. The reduction of bank loans slows the macroeconomy and affects the growth and investment of small manufacturing firms more than established firms.

Huyghebaert et al. [2007] compares start-up firms’ choice between bank debt and trace credit. Bank debt has lower interest rates but a more strict liquidation policy (which sometimes could be premature liquidation), than trace credit. Many en-

trepreneurs who have high risk ventures prefer trace credit and its high interest rates so as to avoid the risk of defaulting on a bank loan. Those with less risky ventures, choose banks to reduce the cost of debt.

In the area of quantitative research on start-up firms, Archibald et al. [2002] first propose an inventory model in which start-up manufacturing firms have a different objective from well-established firms. Due to the fact of limited capital, start-up firms are mainly concerned about survival rather than profits. They maximize the chance of survival over a certain period of time. The model is on manufacturing start-up firms and their inventory decisions. The authors consider a newsvendor problem in a periodic review inventory system. Demand is stochastic and any excess demand is lost. System lead time is one period. Parameters considered in the model are selling price, fixed overhead cost and purchasing cost. Holding cost is assumed to be on the loss of capital and considered in the form of capital constraint. A firm makes order decisions at the beginning of each review period, observing its inventory level and capital available. A start-up firm is defined to be bankrupt if it runs out of capital and, it is considered to be well-established, where the survival probability is one, when the capital is over a deterministic level. This survival model is compared with an average profit maximizing model for well-established firms. Results show that, under the pressure of limited capital, start-up firms tend to operate more conservatively than established firms. Properties of the survival probability and order policy are studied in the paper.

Possani et al. [2003] extend this model by allowing loans and having a more generalized inventory model with a fixed ordering cost and lost sale cost. System lead time is fixed as one period. Loans are evaluated as a constant percentage of purchasing value of the components in stock. Loans are found to improve the survival probability of start-up firms, the larger the collateral rate is, the higher chance of survival. High shortage cost and ordering cost make it hard for a start-up firm to survive. Archibald et al. [2007] extend the model further by considering return policies, in a zero lead time newsvendor problem. Unsold products are returned to the supplier with a salvage price. Costs in the model are purchasing cost, overhead cost, holding cost and salvage cost. Under this condition, start-up firms are not necessarily more cautious than established firms. Start-up firms take riskier ordering decisions when they have low capital level and/or when the salvage price is low. On the other hand, start-up firms are more cautious than well-established firms when the capital level is high and/or the salvage

price is high.

For the other few decision models on start-up firms, most of them are on start-up firm financing. Traditionally, large firms are assumed to be in perfect markets, where operational and financial decisions are seen as two independent decision variables [Modigliani and Miller, 1958]. However, in practice, the market is imperfect, especially for start-up firms, where the roles of a chief operations officer and a chief finance officer are usually delegated to one single person. Thus operation decisions and finance decisions should better be made simultaneously [Babich and Sobel, 2004]. Based on this idea, Babich and Sobel build a joint decision model on start-up firms who take initial public offering (IPO) as a cash-out opportunity and maximize the expected present value of the proceeds from an IPO. The value of proceeds is a function of the firm's current assets, previous period sales and previous period profit. Decisions are made in each period on inventory capacity expansion, production quantity, bank loan and whether or not to offer an IPO. Demand follows a Markov process. As start-up firms always produce unique products, excess demand is totally backordered. The loan rate is a function of the firm's capacity level, current assets level, demand and the short-term risk free interest rate. There is a fixed IPO fee and no IPO is offered if the current assets level is below the IPO fee. This problem is modelled as an infinite horizon discounted Markov decision process. The IPO event is the stopping time of the process. Results show that there exists an optimal threshold stopping rule and the value of the threshold is monotone with the state variables. No explicit solution is given.

Buzacott and Zhang [2004] present an asset-based financing model. In asset-based financing, lenders offer to loan an amount based on the firm's assets in the form of cash, inventory and account receivable. The cash available in each period is a function of assets and liabilities. The firm makes joint decisions on ordering, production and shipment, and financing, say making loans. Both deterministic demand and stochastic demand problems are considered. Buzacott and Zhang emphasize the importance of simultaneous decision making. They show that it is not always optimal for a firm to borrow up to the loan limit. This asset-based loan allows a firm to continue growing without renegotiating the loan. As for a bank, asset-based loan reduces the risk of not receiving the repayment and improves the return.

Li et al. [2005] build a model on control of dividends. They assume that instead of maximizing profit, the start-up firm maximizes the expected present value of dividends. Three decisions, say short-term loan, production level and dividend amount, are made

at the beginning of each period. The firm pays the loan with fixed interest rate. The procurement cost is proportional to the amount of product produced. There is a default penalty which has to be paid if the retained earnings at the beginning of a period is negative. Demand is stochastic, and is totally backordered. The optimal policy is found to be myopic. The optimal base-stock inventory level is found to be lower in this dividend maximizing model than in a profit maximizing model.

Swinney et al. [2005] present a competition model in a duopoly market. They adopt the idea in the paper by Archibald et al. [2002], where start-up firms maximize the survival probability and established firms maximize the average profit. Three kinds of two-player games are analysed: games between two start-up firms, between two established firms and between one start-up firm and one established firm. Start ups might even get more profits than established firms when they are aggressive in competitions. The objective of survival probability maximization is found to have a significant influence on the nature of competition.

From the current literature, the major research on start-up firm survival is empirical. Although empirical research provides a review of the management issues and investigates the factors for small business success, it does not give any further support on decision making, especially in complex quantitative problems, such as operations, marketing, finance. There is a lack of research focusing on specific decision making models [Keasey and Watson, 1993]. The main work in this area so far follows the series of survival models [Archibald et al., 2002], [Possani et al., 2003] and [Archibald et al., 2007]. However, the present models consider only inventory problems with total lost sales. No further consideration is given to operations conditions such as liquidation policy and order frequency. The models assume that demand distribution is stationary during the planning horizon. The research for this thesis follows the idea of survival maximizing for start-up firms, and considers a complex inventory environment with general cost functions and operations conditions. Furthermore, this thesis introduces advertising decisions into the models, which influences the firms' future demand and relaxes the assumption of a fixed demand distribution.

Chapter 3

General models for start-up firms

3.1 Problem assumptions and description

From statistics [Bank of England, 2004], around 50% of start-up firms fail by their third year of foundation. Capital is one of the crucial difficulties for start-up firms. The largest proportion of start-up finance is internal, from savings of owner-managers and other shareholders, and family and friends of owner-managers. The internal investment is a contribution to the founders' motivation and business ambitions. The investor expects some level (but not necessarily high level) of return. External finance, in the form of overdraft, bank debt, asset finance, equity, is accessible. However, the amount of both internal and external finance is often limited. Besides finance, the strategy adopted by the management plays an important role in a firm's success. External factors, such as economy, market policy, customer behavior, which are independent of firms' decisions, also affect firms' success. This model considers start-up manufacturing firms in general, focusing on their survival in relation to available capital, operation, policy, and uncertain external factors. Some of the operation factors, decisions and related external uncertainties are outlined in Table 3.1.

This thesis assumes that in consideration of the financial situation, start-up firms aim to maximize the chance of survival rather than profit, the latter of which is commonly assumed to be the objective of well-established firms. Generally speaking, firms hold periodic reviews in operation. Start-up firms have an N -period planning horizon, where N can be finite or infinite. To survive a period, start ups are required to pass a solvency check at the end of each review period. The models in this thesis assume that a start-up firm fails if it has negative capital at solvency check. This definition of

firm failure is regarded as the lower extreme case from the definitions of firm failure commonly used in research [Watson and Everett, 1996].

Table 3.1: Operations: Examples of states, decisions and uncertainty factors

Operation factors	Decisions	Uncertainty Factors
Inventory position, Production level, Sales	Order quantity, Plant expansion, Pricing	Customer behavior (demand, backorder, loyalty, etc)
Equipment status	Maintenance and replacement	Equipment deterioration
Goodwill level	Marketing	Customer satisfaction
Employee skills	Employee training	Market uncertainty
Capital	Investment and loans	Financial risk

The research in this thesis considers manufacturing firms in general, operating in stable economic periods of time. Therefore, no specific economic factors, such as features of the industry, market competition, financial environment, are discussed in particular. Start-up firms begin with a certain number of customers who may be drawn from the founders' previous business, friends' and family's contacts, or by doing low budget, local advertising and promotions, etc. The models assume that the start-up firms have arranged physical resources such as plant, equipment, and intellectual properties such as techniques, research on product specifications, as well as human resources such as employment. The models assume that the firms have arranged an amount of investment and debt which is fixed during the planning horizon. This is the initial capital of the firm. The firm is able to make orders from suppliers using trade credit and, all financial transactions are processed at the end of each review period. The research in this thesis will investigate the operation process in start-up firms, discussing how a firm's survival probability and decisions are affected by operation factors. The thesis expects to draw general suggestions to start-up operation from the study of inventory problems in manufacturing firms.

The state of the firm at the beginning of a period is defined to be (i, x) where x denotes the capital available to the firm and i is a vector which includes one or more of the other operation factors to the firm's performance, see Table 3.1 for examples of factors. Capital availability is treated separately from the other factors to facilitate the comparison of the survival maximizing and profit maximizing objectives. The

vector is bounded such that $i \in \mathcal{I}$ where \mathcal{I} is finite. The bounds could be defined according to operation limitations, such as storage capacity on inventory, saturation limit on goodwill, maximum number of production lines, etc. At the beginning of each period, after observing the current state of the firm, the management has to make a number of decisions. The decisions taken are defined to be a vector k which includes decisions on how much of the available capital to allocate to each of the operational factors. It is assumed that the capital currently available to the firm does not limit the range of decisions that can be taken. Hence during a period, a firm can plan to use the capital available to it at the beginning of the period plus an additional amount based on projected revenues. This assumption models the budgeting decisions and cash flow of the firm. The decision vector is bounded such that $k \in K_i$, where K_i is the set of decisions that can be taken in state i and is finite. The decision bounds could be inventory capacity, investment budget, etc. After the decision is taken, the uncertain factors for the period are realized. The realization of the uncertain factors is defined to be the vector d , bounded such that $d \in \mathcal{D}$ where \mathcal{D} is finite. If decisions k are taken when the state is i , the uncertainties d occur with probability $p(i, k, d)$, where $\sum_{d \in \mathcal{D}} p(i, k, d) = 1$. The function $\mathcal{F}(i, k, d)$ defines the change in state i and the function $\mathcal{G}(i, k, d)$ defines the operating profit during a period in which decisions k are taken when the state is i and the uncertainties d occur. In the solvency check at the end of each period, the firm is said to fail if the capital available to it is negative. Hence, when decisions k are taken in state (i, x) and the uncertainties d occur, the firm will fail the solvency check if $x + \mathcal{G}(i, k, d) < 0$. Otherwise, the state of the firm at the beginning of the next period will be $(\mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d))$. This chapter compares two models of the firm, one with limited capital and the objective of maximizing the probability of survival and the other with no capital constraint and the objective of maximizing the average profit. The two models are formulated using Markov decision processes (MDPs) and are explained in detail in the following two sections. The properties of these two models are discussed in Section 3.4.

3.2 Maximizing survival probability

Define $q(n, i, x)$ to be the maximum probability of the firm surviving the planning horizon given that the state of the firm is (i, x) and there are n periods to the end of the planning horizon. It follows that $q(n, i, x) \in [0, 1]$ for $i \in \mathcal{I}$, $x \geq 0$ and $n \in [0, N]$.

The start-up firm fails if it has negative capital at a solvency check, so $q(n, i, x) = 0$ for $x < 0$. This problem is formulated as an expected total reward problem in Markov decision processes.

1. **Decision epochs and periods.** This model has discrete decision epochs. Let $T \equiv \{0, 1, 2, \dots, N\}$ denote the set of decision epochs, where N is the length of the planning horizon. Decision epoch $t \in T$ occurs when there are t epochs remaining in the planning horizon. N can be a finite positive integer in a finite horizon problem, or an infinite positive integer, say $N \rightarrow \infty$, in an infinite horizon problem.
2. **State and action sets.** At each decision epoch, the firm is in some state $s = (i, x)$ where $s \in S = \{(i, x); i \in \mathcal{I}, x \in [0, 1, \dots]\}$. It is assumed that all monetary terms are expressed as a multiple of a common unit so that the variable x , representing the capital available to the firm, is a discrete variable. On observing the state $s = (i, x)$, an action $k \in K_i$ is chosen. No action is taken at the end of the planning horizon (i.e. decision epoch 0). The state space S is infinite countable and the action space K_i is finite for each $i \in \mathcal{I}$.
3. **Reward and transition probabilities.** Since the objective is to maximize the probability of the firm surviving the planning horizon, the only reward is generated at the end of the planning horizon. Therefore the reward during each period is zero. If action k is chosen when the firm is in state $s = (i, x)$ and the uncertainty vector d occurs, the system makes a transition from the current state, (i, x) , to state at the next decision epoch $(\mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d))$. This event occurs with probability $p(i, k, d)$.
4. **Decision rules.** Since the state space \mathcal{I} , and the action space, K_i , for each $i \in \mathcal{I}$, are both finite, $K \equiv \cup K_i$ is finite. According to Puterman [1994, pp. 277–284], this is a positive bounded total reward model. By Theorem 7.1.9 in Puterman [1994, pp. 284], there exists a Markovian deterministic decision rule (MD) which brings an optimal reward to this model.
5. **Iteration and boundaries.** Backward iteration is used for calculation. Iteration starts from the last period of the planning horizon. In a finite horizon problem, iteration starts at epoch 0 and stops at epoch N . In an infinite horizon problem,

iteration starts at epoch 0 and stops when the value of maximum survival probability converges, i.e. when the difference between the survival probability for every state in two successive periods is below a pre-defined limit ϵ , where $\epsilon > 0$ [Puterman, 1994, pp.280–282]. For computational reasons, X is introduced as the upper bound on capital. A start-up firm is said to be established, i.e. be sure to survive, if its capital is above the bound. X can be either finite ($X < \infty$) or infinite ($X \rightarrow \infty$). An infinite capital limit describes the situation where there are always risks on survival. In addition to the previous boundary conditions, the start-up firm model has that,

$$q(n, i, x) = 1, \text{ if } x > X, \text{ for } i \in \mathcal{I}, n > 0.$$

The maximum survival probability is determined as follows,

$$q(n, i, x) = \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(n-1, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\},$$

for $i \in \mathcal{I}, x \in [0, X], n > 0$ (3.1)

with terminal values

$$q(0, i, x) = \begin{cases} 1, & \text{for } \forall i \in \mathcal{I}, x \geq 0, \\ 0, & \text{for } \forall i \in \mathcal{I}, x < 0. \end{cases}$$

and boundary condition

$$q(n, i, x) = \begin{cases} 0, & \text{for } \forall i \in \mathcal{I}, x < 0, n > 0, \\ 1, & \text{for } \forall i \in \mathcal{I}, x > X, n > 0. \end{cases}$$

3.3 Maximizing average profit

The profit maximizing problem is formulated as an average reward MDP model. It is assumed that there is no practical constraint on the capital available to the firm. At the beginning of each period, the firm takes decision, k , after observing the state, i . The model is assumed to be unichain (See Section 2.1.2) so all stationary policies have a constant gain [Puterman, 1994, pp.348–353]. Define the maximum average profit of the firm to be g and the bias term to be $v(i)$ which depends on the firm's state, i .

- 1. Decision epochs and period.** Decisions are made at the beginning of each period. This average reward model is an infinite horizon model and the set of decision epochs is $T \equiv \{0, 1, 2, \dots\}$.
- 2. State and action sets.** The state vector is i , where $i \in \mathcal{I}$. On observing the state, i , at a decision epoch, an action k is chosen from a finite action set such that $k \in K_i$.
- 3. Reward and transition probabilities.** The reward in each period is the firm's instant earnings in the period which is given by $\mathcal{G}(i, k, d)$ for state i , action k , and uncertainties, d . The firm makes a transition from the current state, i , to state $\mathcal{F}(i, k, d)$ with probability $p(i, k, d)$.
- 4. Decision rules.** The model is unichain and there exists a stationary Markovian deterministic policy (MD) which brings a constant average reward.

The average reward model is formulated as follows:

$$g + v(i) = \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) (\mathcal{G}(i, k, d) + v(\mathcal{F}(i, k, d))) \right\}, \text{ for } i \in \mathcal{I}. \quad (3.2)$$

The state, i , action, k , and uncertainties, d , are bounded. Assume the instant reward $\mathcal{G}(i, k, d)$ is also bounded, i.e. $|\mathcal{G}(i, k, d)| < \infty$. Moreover, the action set K_i is finite. By Theorem 8.4.5 Puterman [1994, pp. 361], in such a unichain model, there exists a stationary optimal policy which brings the Markov chain to a unique stationary distribution, $\pi(i)$, such that,

$$g = \sum_{i \in \mathcal{I}} \pi(i) \sum_{d \in \mathcal{D}} p(i, k^*(i), d) \mathcal{G}(i, k^*(i), d), \quad (3.3)$$

where $\sum_{i \in \mathcal{I}} \pi(i) = 1$ and $k^*(i)$ is the action in state i under a stationary optimal policy.

3.4 General properties of the models

Lemma 3.4.1. $q(n, i, x)$ is non-increasing in n .

Proof. Prove that $q(n, i, x) - q(n + 1, i, x) \geq 0$ for $n \geq 0$ by induction on n .

When $n = 0$, for $\forall i \in \mathcal{I}$,

$$q(0, i, x) = \begin{cases} 1, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

By the definitions of firm survival and failure, there is

$$q(1, i, x) \begin{cases} = 1, & \text{for } x > X, \\ \in [0, 1], & \text{for } x \in [0, X], \\ = 0, & \text{for } x < 0. \end{cases}$$

Thus $q(0, i, x) - q(1, i, x) \geq 0$ and the result holds for $n = 0$.

Assume the results holds for some $n \geq 0$, i.e. $q(n, i, x) - q(n + 1, i, x) \geq 0$. Note that

$$q(n + 1, i, x) = q(n + 2, i, x) = \begin{cases} 1, & \text{for } x > X, \\ 0, & \text{for } x < 0. \end{cases}$$

Thus $q(n + 2, i, x) - q(n + 1, i, x) = 0$, for $x < 0$ or $x > X$. For $x \in [0, X]$, by $\max_i \{a_i\} - \max_i \{b_i\} \leq \max_i \{a_i - b_i\}$, there is,

$$\begin{aligned} & q(n + 2, i, x) - q(n + 1, i, x) \\ &= \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(n + 1, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\} \\ & \quad - \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\} \\ &\leq \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) \left[q(n + 1, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right. \right. \\ & \quad \left. \left. - q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right] \right\} \\ &\leq 0, \text{ by the inductive hypothesis since } p(i, k, d) \geq 0 \text{ for all } d \in \mathcal{D}. \end{aligned}$$

The hypothesis holds for $n + 1$.

Therefore, $q(n, i, x) - q(n + 1, i, x) \geq 0$ for $n \geq 0$, for all $i \in \mathcal{I}$ and $x \in [0, X]$. \square

Lemma 3.4.2. $q(n, i, x)$ is non-decreasing in x .

Proof. Prove that $q(n, i, x) - q(n, i, x + y) \leq 0$ for $n \geq 0$ by induction on n , where y is a positive integer, i.e. $y > 0$. When $n = 0$, $\forall i \in \mathcal{I}$,

$$q(0, i, x) = \begin{cases} 1, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Hence $q(0, i, x) - q(0, i, x + y) \leq 0$, for $y > 0$.

Assume the result holds for some $n \geq 0$. Thus $q(n, i, x) - q(n, i, x + y) \leq 0$, for all $i \in \mathcal{I}$ and all x . For $n + 1$,

$$q(n + 1, i, x) = \begin{cases} = 1, & \text{for } x > X, \\ \in [0, 1], & \text{for } x \in [0, X], \\ = 0, & \text{for } x < 0; \end{cases}$$

$$\text{and } q(n + 1, i, x + y) \begin{cases} = 1, & \text{for } x > X - y, \\ \in [0, 1], & \text{for } x \in [-y, X - y], \\ = 0, & \text{for } x < -y. \end{cases}$$

Thus $q(n + 1, i, x) - q(n + 1, i, x + y) \leq 0$ for $x < 0$ or $x > X - y$. For $x \in [-y, X - y]$, by $\max_i \{a_i\} - \max_i \{b_i\} \leq \max_i \{a_i - b_i\}$,

$$\begin{aligned} & q(n + 1, i, x) - q(n + 1, i, x + y) \\ &= \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\} \\ & \quad - \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(n, \mathcal{F}(i, k, d), x + y + \mathcal{G}(i, k, d)) \right\} \\ &\leq \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) \left[q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right. \right. \\ & \quad \left. \left. - q(n, \mathcal{F}(i, k, d), x + y + \mathcal{G}(i, k, d)) \right] \right\} \\ &\leq 0, \text{ by the inductive hypothesis since } p(i, k, d) \geq 0 \text{ for all } d \in \mathcal{D}. \end{aligned}$$

The hypothesis holds for $n + 1$.

Therefore, $q(n, i, x) - q(n, i, x + y) \leq 0$, for $n \geq 0$, $i \in \mathcal{I}$ and $x \in [0, X]$, with $y > 0$. \square

Theorem 3.4.3. $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$ exists and is non-increasing in x .

$q(i, x)$ satisfies the optimality equation:

$$q(i, x) = \max_{k \in K_i} \left\{ \sum_{d \in \mathcal{D}} p(i, k, d) q(\mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\}, \text{ for } i \in \mathcal{I}. \quad (3.4)$$

with boundary conditions

$$q(i, x) = \begin{cases} 1, & \text{for } x > X, \\ 0, & \text{for } x < 0. \end{cases}$$

Proof. From Lemma 3.4.1, $q(n, i, x)$ is non-increasing in n and $q(n, i, x)$ is bounded within the range $[0, 1]$. It is known that bounded, non-increasing sequences converge. Hence there is a $q(i, x) \in [0, 1]$ such that

$$q(i, x) \equiv \lim_{n \rightarrow \infty} q(n, i, x).$$

The other results follow by taking $n \rightarrow \infty$ in Lemma 3.4.2 and equation 3.1. \square

An interesting question to consider is whether there exists a solution to equation 3.4 for which $q(i, x) > 0$ for some i and x , particularly when $X \rightarrow \infty$. In other words, is it possible for the firm to survive in the long-run, particularly when there are always risks to survival. The following results provide an answer to this question and link the survival of the firm in the existence of a positive maximum average reward in the profit maximizing model.

Lemma 3.4.4. *Consider the generalized one-dimensional random walk in which the size of step taken in each period is an integer in the range $[-\nu, \mu]$ where ν and μ are the maximum steps to the left and right respectively. Let p_k be the probability that the step size is k , for integer k . Assume there is a left absorbing barrier at -1 , so that the random walk terminates (or is absorbed by the barrier) if it ever reaches a point $x \leq -1$, where x is the current position. Let u_x be the probability that the random walk eventually ruins, i.e. the random walk is absorbed by the barrier at -1 . If the average step size, i.e. $\sum_{k=-\nu}^{\mu} p_k k > 0$, then $u_x < 1$ for all $x \geq 0$ and hence there is a non-zero probability that the random walk escapes the left barrier at -1 . If $\sum_{k=-\nu}^{\mu} p_k k \leq 0$, then $u_x = 1$ for all $x \geq 0$.*

Proof. (The proof follows Feller [1968, pp. 342–366]).

Assume $p_k > 0$ for some $k > 0$ and for some $k < 0$, as otherwise $u_x = 1$ or $u_x = 0$ for all $x \geq 0$ trivially. Introduce a right absorbing barrier at $X + 1$, so that the random walk is absorbed by this barrier if it ever reaches a point $x \geq X + 1$. The two barriers are written as:

$$u_x = \begin{cases} 1, & \text{for } x \in [-\nu, -1], \\ 0, & \text{for } x \in [X + 1, X + \mu]. \end{cases} \quad (3.5)$$

Given the random walk starts at position z , with $0 \leq z \leq X$, the probability of ultimate ruin is calculated as

$$u_z = \sum_{k=-\nu}^{\mu} p_k u_{z+k}. \quad (3.6)$$

Using the boundary conditions 3.5,

$$u_z = \sum_{k=-z}^{X-z} p_k u_{z+k} + \sum_{k=-\nu}^{-z-1} p_k, \text{ or}$$

$$u_z = \sum_{x=0}^X p_{x-z} u_x + b_z, \quad (3.7)$$

where b_z is the probability of ruin at the first step. Equation (3.7) is a system of $X + 1$ linear equations in $X + 1$ unknowns. In vector form, equation (3.7) can be written as:

$$\mathbf{u} = Q\mathbf{u} + \mathbf{b} \text{ or } (I - Q)\mathbf{u} = \mathbf{b}, \quad (3.8)$$

where $Q = \{q_{ij}\}$ such that $q_{ij} = p_{j-i}$ for $0 \leq i, j \leq X$. Hence, $Q \geq 0$. Note that the sum of the elements of row i of $Q = \sum_{j=0}^X p_{j-i} = \sum_{j=-i}^{X-i} p_j \leq 1$, since $\{p_k\}$ is the probability distribution of step size.

To show that the linear system (3.8) has a unique solution, the following proves that the associated homogeneous system $\mathbf{u} = Q\mathbf{u}$ has no solution except zero. Consider a solution u^* to $\mathbf{u} = Q\mathbf{u}$. Let k be any positive integer such that $p_{-k} > 0$, let $\bar{u} = \max_j \{u_j^*\}$ and let $u_r^* = \bar{u}$. It follows that $\bar{u} = \sum_{j=0}^X p_{j-r} u_r^*$. From the definition of \bar{u} , this is only possible if the coefficients $\sum_{j=0}^X p_{j-r} = 1$ and $u_j^* = \bar{u}$ whenever $p_{j-r} > 0$. Hence $u_{r-k}^* = \bar{u}$. Repeating this argument, $u_{r-nk}^* = \bar{u}$ for integer $n > 0$. Since $u_z = 0$ for $z < 0$, \bar{u} must equal 0. If \mathbf{u}^* is a solution to $\mathbf{u} = Q\mathbf{u}$, it follows that $-\mathbf{u}^*$ is also a solution. Hence $\mathbf{u}^* = 0$ is the only solution. This shows that 1 is not an eigenvalue of Q .

By Theorem 8.3.1 in Horn and Johnson [1987, pp.503], the spectral radius of Q , $\rho(Q)$ is an eigenvalue of matrix Q . By Corollary in Horn and Johnson [1987, pp.346], $\rho(Q) \leq \max_i \sum_j q_{ij} \leq 1$. Hence $\rho(I - Q) < 1$. By definition [Horn and Johnson, 1991, pp.113], $I - Q$ is a Z -matrix. [Note: a Z -matrix is defined as a non-real matrix $A = \{a_{i,j}\}$ satisfying $a_{i,j} \leq 0$ if $i \neq j$, $1 \leq i, j \leq n$.] By Theorem 2.5.8 in Horn and Johnson [1991, pp.114], for any Z -matrix A , the following are equivalent:

1. A is an M -matrix;
2. $A = I - Q$, $Q \geq 0$, $\rho(Q) < 1$;
3. A is non-singular and $A^{-1} \geq 0$.

It follows that $I - Q$ is non-singular, $(I - Q)^{-1} \geq 0$ and equation (3.8) has a unique solution $\mathbf{u} = (I - Q)^{-1}\mathbf{b}$.

Let u_z^* be the solution of (3.7) and \bar{u}_z be a solution of (3.6) with boundary conditions

$$u_z \geq \begin{cases} 1, & \text{for } z \in [-\nu, -1] \\ 0, & \text{for } x \in [X + 1, X + \mu]. \end{cases} \quad (3.9)$$

Note that $\sigma_z = \bar{u}_z - u_z^*$ also satisfies (3.6), so

$$\sigma_z = \sum_{x=-\nu}^{-1} p_{x-z}\sigma_x + \sum_{x=0}^X p_{x-z}\sigma_x + \sum_{x=X+1}^{X+\mu} p_{x-z}\sigma_x = \sum_{x=0}^X p_{x-z}\sigma_x + c_z$$

where $c_z \geq 0$ since $\bar{u}_z - u_z^* \geq 0$ for $z \in [-\nu, -1]$ and $z \in [X + 1, X + \mu]$. Hence, $\boldsymbol{\sigma} = (I - Q)^{-1}\mathbf{c} \geq 0$ and \bar{u}_z is an upper bound on u_z^* for all z . Let \underline{u}_z be a solution of (3.6) with boundary conditions

$$u_z \leq \begin{cases} 1, & \text{for } z \in [-\nu, -1] \\ 0, & \text{for } x \in [X + 1, X + \mu]. \end{cases} \quad (3.10)$$

In a similar manner to above, it can be shown that $u_z^* - \underline{u}_z \geq 0$ and hence \underline{u}_z is a lower bound on u_z^* .

Following the method of particular solutions, consider the characteristic equation for (3.6):

$$\sum_{k=-\nu}^{\mu} p_k \sigma^k = 1. \quad (3.11)$$

If σ is a root of equation (3.11), then $u_z = A\sigma^z$ is a formal solution of (3.6) for all z

where A is any constant. Let $f(\sigma) = \sum_{k=-\nu}^{\mu} p_k \sigma^k - 1$. Clearly, $f(1) = 0$, so equation (3.11) has a root at $\sigma = 1$. Note that $f'(1) = \sum_{k=-\nu}^{\mu} k p_k$ which is the expected step size.

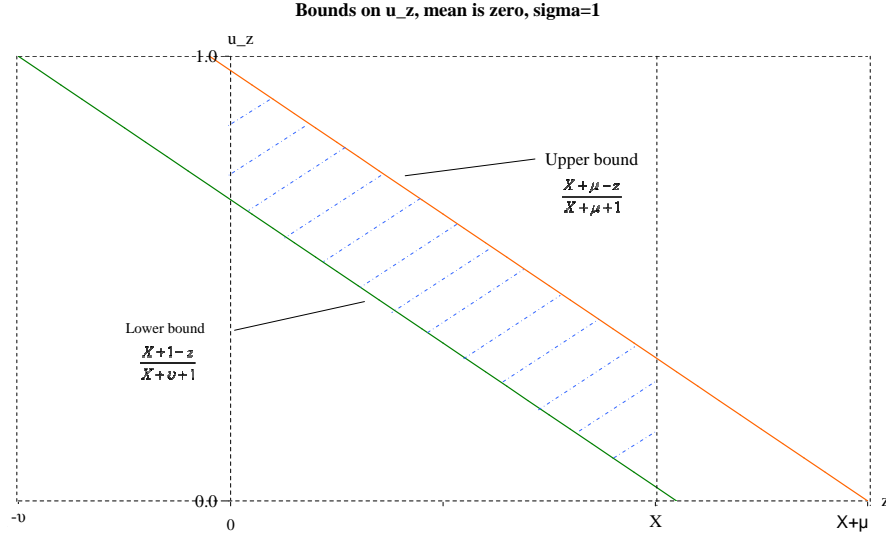


Figure 3.1: Numerical illustration of bounds on u_x in Lemma 3.4.4 when expected step size is 0

If the expected step size is 0, then equation (3.11) has a double root at $\sigma = 1$. Then $u_z = A + Bz$ is a solution to (3.6) which can be made to satisfy boundary conditions (3.9) by requiring

$$A + Bz = \begin{cases} 1, & \text{if } z = -1 \\ 0, & \text{if } z = X + \mu \end{cases}$$

and boundary conditions (3.10) by requiring

$$A + Bz = \begin{cases} 1, & \text{if } z = -\nu \\ 0, & \text{if } z = X + 1. \end{cases}$$

Hence, $\bar{u}_z = \frac{X+\mu-z}{X+\mu+1}$ is an upper bound on the ruin probability u_z for $0 \leq z \leq X$ and $\underline{u}_z = \frac{X+1-z}{X+\nu+1}$ is a lower bound. See Figure 3.1, u_z lies in the shaded area between the two bounding functions. Note that as the right barrier, X tends to infinity, the upper and lower bounds on the survival probability both tend to be 1. Hence, as the right

barrier tends to infinity, the ruin probability tends to 1.

If the expected step size is not equal to 0, there is a simple root of equation (3.11) at $\sigma = 1$. Since $p_k > 0$ for some $k > 0$ and for some $k < 0$, $\lim_{\sigma \rightarrow \infty} f(\sigma) = \infty$ and $\lim_{\sigma \rightarrow 0} f(\sigma) = \infty$. Note that $f''(\sigma) = \sum_{k=-\nu}^{\mu} k(k-1)p_k\sigma^{k-2} \geq 0$ for $\sigma > 0$. For positive σ , $f(\sigma)$ is continuous and convex with one root at $\sigma = 1$. There exists exactly one more root in $(0, \infty)$, denote this root as σ_1 . By applying similar arguments to the case above, $u_z = A + B\sigma_1^z$ is a solution to equation (3.6) which can be made to satisfy boundary conditions (3.9) by requiring

$$A + B\sigma_1^z = \begin{cases} 1, & \text{if } z = -1 \\ 0, & \text{if } z = X + \mu \end{cases}$$

and boundary conditions (3.10) by requiring

$$A + B\sigma_1^z = \begin{cases} 1, & \text{if } z = -\nu \\ 0, & \text{if } z = X + 1. \end{cases}$$

Hence, the ruin probability u_z is bounded for $0 \leq z \leq X$ such that:

$$\underline{u}_z = \frac{\sigma_1^{X+1} - \sigma_1^z}{\sigma_1^{X+1} - \sigma_1^{-\nu}} \leq u_z \leq \frac{\sigma_1^{X+\mu} - \sigma_1^z}{\sigma_1^{X+\mu} - \sigma_1^{-1}} = \bar{u}_z. \quad (3.12)$$

If $f'(1) < 0$, $\sigma_1 > 1$ and as the right barrier, X , tends to infinity, \underline{u}_z and \bar{u}_z both tend to 1. See Figure 3.2(a) for an example of the bounding functions in this case. If $f'(1) > 0$, $\sigma_1 < 1$ and $\lim_{X \rightarrow \infty} \bar{u}_z = \sigma_1^{z+1} < 1$. Hence as the right barrier tends to infinity, u_z is bounded above by a value less than 1 and there is a non-zero probability that the random walk escapes the left barrier at -1 . See Figure 3.2(b) for an example of the bounding functions in this case. \square

Theorem 3.4.5. *Let g be the maximum average reward and π be the unique stationary distribution corresponding to the optimal policy for the average reward model formulated in equation (3.2). Assume the firm is initially in state $i \in \mathcal{I}$. If*

(i) *there exists a policy which, with non-zero probability p_0 , takes the state of the firm to the distribution π in finite time t_0 and at a total cost (ignoring revenue) bounded above by X_0 ; and*

(ii) *$g > 0$,*

then with initial capital $x \geq X_0$, the probability that the firm survives in the long

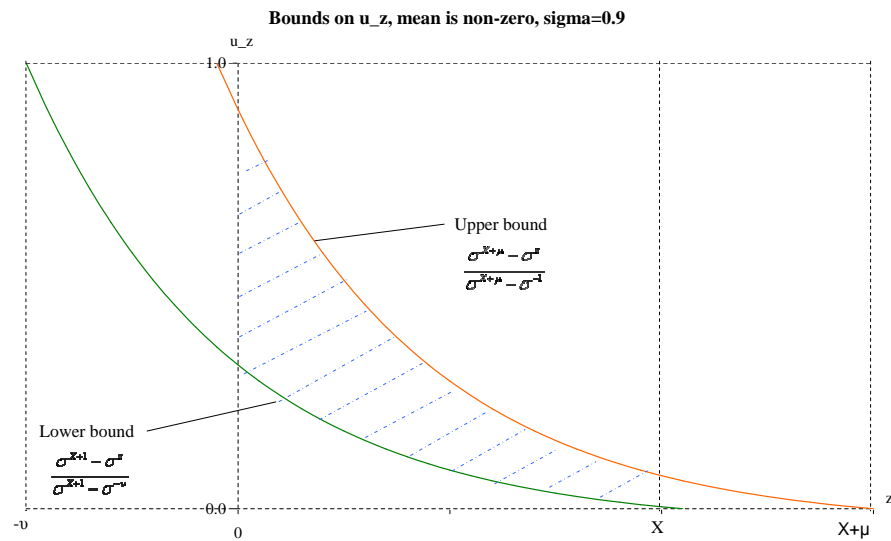
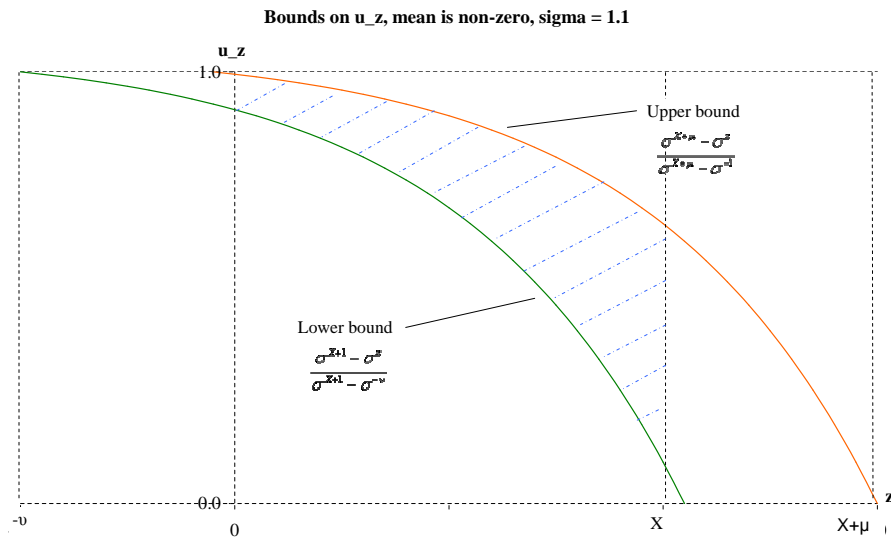


Figure 3.2: Numerical illustration of bounds on u_x in Lemma 3.4.4 when the expected step size is not 0

run, $q(i, x)$ from equation (3.6), is greater than zero.

Proof. Suppose the firm starts with capital $x \geq X_0$ and follows the policy described in (i). With probability $p_0 > 0$, the firm will survive to periods (the firm's capital at all times during this interval is at least $x - X_0 \geq 0$) by which time the distribution of the state of the firm is π .

From this point on, suppose the firm follows the optimal policy for the average reward model (i.e. choose decision $k^*(i)$ in state i whatever the value of x). Since π is the unique stationary distribution corresponding to this policy, the distribution of the state of the firm is unchanged. The process can then be thought of as a random walk on the capital available to the firm. At each period the process takes a step of size $\mathcal{G}(i, k^*(i), d)$ with probability $\pi(i)p(i, k^*(i), d)$. Since $\mathcal{G}(i, k^*(i), d)$ is bounded, it follows from Lemma 3.4.4 that the process escapes the absorbing barrier at $x = -1$ with non-zero probability provided

$$\begin{aligned} 0 &< \sum_{i \in \mathcal{I}, d \in \mathcal{D}} \pi(i)p(i, k^*(i), d)\mathcal{G}(i, k^*(i), d) \\ &= \sum_{i \in \mathcal{I}} \pi(i) \sum_{d \in \mathcal{D}} p(i, k^*(i), d)\mathcal{G}(i, k^*(i), d) = g. \end{aligned}$$

□

Chapter 4

Modelling inventory decisions with total lost sales

4.1 Problem assumptions and description

This chapter considers a start-up manufacturing firm and its inventory management. For simplicity, the models assume that the firm produces a single product at a single location and the product is manufactured from a single component. The firm may carry an inventory of components but, as the product is manufactured to order, the firm never carries an inventory of manufactured products. This may be for the reason that the customers choose the exact specification of the products, or the products cannot be stored, such as easily perishable products. Inventory level, i , is limited by the storage capacity I , where $I < \infty$. In real life, the assumption is applicable to start-up firms who in most cases have limited capital to keep large inventories. In the cases where short-term borrowing is possible, the firm may be able to increase the level of borrowing during the planning horizon. However, this is not considered in the model studied in this thesis. Often requests for additional finance will be rejected unless the firm has achieved its initial goals. The models in this thesis could be considered as modelling the success of the initial business plan. Interest rates are not considered explicitly in the models. The financial costs would form part of the overhead cost.

The firm uses a periodic review policy to manage the inventory of components. The review period is defined to be one unit of time. The firm operates in the following way. At the beginning of each review period, it has the opportunity to order k components from the supplier, observing its inventory level, i , and capital available, x . The firm

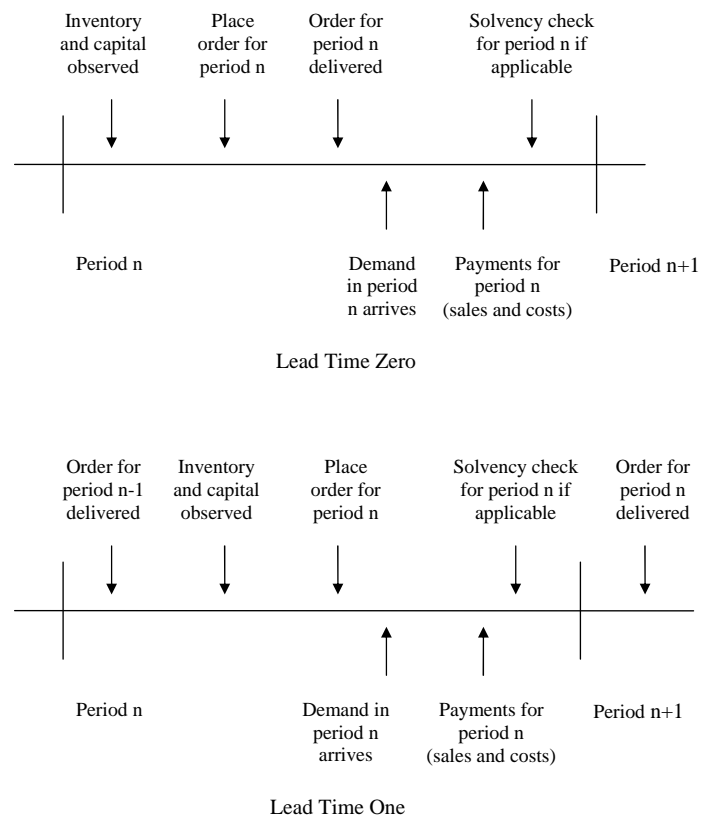
ensures that the inventory level is within the capacity I , i.e. $0 \leq k \leq I - i$. The orders are totally filled, with no damages or shortages. The lead time is denoted as L . Both lead time zero ($L = 0$) and lead time one ($L = 1$) cases are considered in this thesis. In $L = 0$, the products can be ready for sale in a very short time after components are ordered, compared to the review period, and so production time can be negligible. In $L = 1$, the products are ready for sale at the beginning of the following period, before a new order is made. The firm does not offer backorders, customers are lost if the demand is unsatisfied on arrival.

Customers arrive after the firm places the order. Customer demand, d , is the only uncertain factor. This thesis assumes that each arriving customer orders one unit of product. Therefore, the number of customers is equivalent to the number of products in demand. The demand in each period is bounded such that $d \in [0, D]$, where D is finite is an independent and identically distributed discrete random variable, with probability distribution $p(d)$ where $\sum_{d=0}^D p(d) = 1$. The demand distribution is stationary in the planning horizon and is independent of factors such as goodwill and customer service. Customers are served on a first-come, first-served basis. Stockouts, which occur when demand cannot be met by production during the period, result in lost sales. .

The capital available to the firm, x , is updated to reflect all costs and revenues that occur in each period. All costs and revenues are expressed as a multiple of a common unit, so that the capital can be counted in integer values. All the financial transactions arising in a review period are made at the end of the period. This gives the firm instant access to the earnings and in the meantime, allows the firm to make orders with no concern about capital at the start of each period. A start-up firm faces a solvency check at the end of each period, after financial transactions are made. The firm continues to operate only if it has non-negative capital, i.e. $x \geq 0$. Otherwise the firm is said to have failed. See Figure 4.1 for the inventory operations in $L = 0$ and $L = 1$.

Define the earnings in each period to be $\mathcal{G}(i, k, d)$ which is a function of inventory level, i , order quantity, k , and demand, d . A firm's profit is affected by the following factors in this lost sales model [Silver et al., 1998, pp.44–48]. These parameters will be included in the models developed in the later chapters with adjustments where necessary.

- Product selling price S . It is the price at which products are sold to customers. The price is fixed in the planning horizon;

Figure 4.1: Inventory operations, total lost sales, $L = 0$ and $L = 1$

- Fixed overhead cost H . An overhead cost covers operation costs associated with activities, including lease of equipment, salaries, finance charges, etc;
- Component purchasing price C . It is the price at which the firm orders from supplier plus any cost incurred for the sale of each product;
- Fixed ordering cost c . An ordering cost includes all the costs of making orders, such as order forms, postage, phone calls, authorization, invoices, etc;
- Unit stock holding rate h . It includes the opportunity cost of the capital invested in the inventory (which equals the largest profit that the firm could get from investing the capital elsewhere), the bills of running a warehouse, managing stocks, the cost of product deterioration, obsolescence, theft and damage, insurance and tax;
- Unit shortage cost due to lost sales r_L . It is due to the loss of future customers, or say, the loss of goodwill, in a comparatively short term.

In this lost sales model, the profit earned in one period, $\mathcal{G}(i, k, d)$, is calculated as,

$$\mathcal{G}(i, k, d) = \begin{cases} S \min\{i + k, d\} - H - hi - c\delta(k) - Ck \\ -r_L \max\{d - i - k, 0\}, & L = 0, \\ S \min\{i, d\} - H - hi - c\delta(k) - Ck \\ -r_L \max\{d - i, 0\}, & L = 1, \end{cases}$$

where

$$\delta(f) \equiv \begin{cases} 0, & \text{for } f \leq 0, \\ 1, & \text{for } f > 0. \end{cases}$$

In $\mathcal{G}(i, k, d)$, in the case of $L = 0$, $S \min\{i+k, d\}$ is the revenue from selling $\min\{i+k, d\}$ units of products, and $H + hi + c\delta(k) + Ck + r_L \max\{d - i - k, 0\}$ is the total cost on condition of inventory level i , order k and demand d . The pattern is similar to $L = 1$ given the number of items available for sale satisfying $\min\{i, d\}$. The inventory level changes in one period according to $\mathcal{F}(i, k, d)$ which is a function of inventory level, i , order quantity, k , and demand, d ,

$$\mathcal{F}(i, k, d) = \begin{cases} (i + k - d)^+, & \text{for } L = 0, \\ (i - d)^+ + k, & \text{for } L = 1 \end{cases}$$

where $f^+ \equiv \max\{f, 0\}$.

4.2 Maximizing survival probability

Start-up firms have limited capital and need to pass the solvency check at the end of each review period. Define $q(n, i, x)$ to be the firm's maximum survival probability when there are n periods to the end of the planning horizon, given inventory level, i , and capital available, x . It follows that $q(n, i, x) \in [0, 1]$, for all $n \geq 0$, $i \in [0, I]$ and $x \geq 0$. The start-up firm is said to fail if it has negative capital at the time of a solvency check and so, $q(n, i, x) = 0$ for $x < 0$. This survival maximizing model is formulated as an expected total reward problem in Markov decision processes as follows:

1. **Decision epochs and periods.** This model has discrete decision epochs. The decision epoch set is $T \equiv \{0, 1, 2, \dots, N\}$. N is the planning horizon and can be either a finite positive integer ($N < \infty$) in a finite planning problem, or an infinite positive integer ($N \rightarrow \infty$) in an infinite horizon problem.
2. **State and action sets.** At each decision epoch, the firm is in some state $s = (i, x)$, where $s \in S = \{(i, x) : i \in \{0, 1, 2, \dots, I\}, x \in \{0, 1, 2, \dots\}\}$. Until the later numerical investigation (Section 4.5), it is assumed that the capital available is unbounded ($X \rightarrow \infty$) so that there are always risks to survival. The firm makes decision, k , on order quantity on observing the state (i, x) , in condition that the inventory level does not exceed the capacity, i.e. $k \in K_i = \{0, 1, \dots, I - i\}$. No action is taken at the end of the planning horizon (i.e. decision epoch 0). The state space S is infinite countable, and the action space K_i is finite for each $i \in [0, I]$.
3. **Reward and transition probabilities.** Since the objective is to maximize the probability of the firm surviving the planning horizon, the only reward is generated at the end of the planning horizon. Therefore the reward during each period is zero. The probability that the demand in a period is d is given by $p(d)$, which is independent of the state and decision. Given the demand d and action k , the system makes a transition from state (i, x) to state $(\mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d))$ at the next decision epoch, with probability $p(d)$.
4. **Decision rules.** The state space S is infinite countable. The action space K_i is finite for each $i \in [0, I]$, and so $K \equiv \cup K_i$ is finite. According to Puterman [1994, pp. 277–284], this is a positive bounded total reward model. By Theorem 7.1.9 in

Puterman [1994, pp.284], there exists a Markovian deterministic (MD) decision rule which brings an optimal reward.

The survival probability maximizing problem is formulated as follows,

$$q(n, i, x) = \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) q(n-1, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\},$$

for $i \in [0, I]$ and $x \geq 0$, (4.1)

with terminal values

$$q(0, i, x) = \begin{cases} 1, & \text{for all } i \in [0, I], x \geq 0, \\ 0, & \text{for all } i \in [0, I], x < 0. \end{cases}$$

and boundary condition

$$q(n, i, x) = 0, \text{ for all } i \in [0, I], x < 0, n > 0.$$

Furthermore, define $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$ to be a start-up firm's long-run survival probability which is of particular interest in this thesis.

4.3 Maximizing average profit

The firms who maximize the average profit operate under the same circumstances as the start-up firms who maximize their survival probability. The only difference is that, these firms do not have to pass a solvency check at the end of each review period. This effectively means that there is no limit on the capital available to the firm. This profit maximizing model is formulated as an expected average reward MDP model [Puterman, 1994, pp. 348–353]. Let g be the average profit and $v(i)$ be the bias term depending on the state i . The profit maximizing problem is defined as follows:

- 1. Decision epochs and periods.** Decisions are made at the beginning of each review period. Average reward model is an infinite horizon model and the decision epoch set is $T \equiv \{0, 1, 2, \dots\}$.
- 2. State and action sets.** Inventory level, i , is the only state variable in this model, i.e. $s = (i)$. The state space $S = \{0, 1, \dots, I\}$ is finite. At the beginning of each

period, an action is chosen on order quantity, k , from an action set K_i such that $k \in K_i = \{0, 1, \dots, I - i\}$.

3. Reward and transition probabilities. The reward in each period is the firm's instant earnings in the period, $\mathcal{G}(i, k, d)$, which is a function of inventory level, i , order quantity, k , and demand, d . The firm makes a transition from state (i) to the state $(\mathcal{F}(i, k, d))$, with probability $p(d)$.

4. Decision rules. The state, s , action, k , and demand, d , are bounded, thus the reward is bounded, i.e. $|\mathcal{G}(i, k, d)| < \infty$. In most practical cases, the weak unichain assumption holds [Tijms, 1994, pp. 199], where there exists a stationary Markovian deterministic policy (MD) which brings a constant average reward.

The average reward model for the lost sales problem is formulated as follows,

$$g + v(i) = \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) \left(\mathcal{G}(i, k, d) + v(\mathcal{F}(i, k, d)) \right) \right\}, \text{ for } i \in [0, I]. \quad (4.2)$$

Moreover, by Theorem 8.4.5 Puterman [1994, pp. 361], there exists a stationary optimal policy under which the Markov chain has a unique stationary distribution of the state of the process, $\pi(i)$, such that,

$$g = \sum_{i=0}^I \pi(i) \sum_{d=0}^D p(d) \mathcal{G}(i, k^*(i), d), \quad (4.3)$$

where $k^*(i)$ is the decision chosen in state i under stationary policy k^* and

$$\sum_{i=0}^I \pi(i) = 1.$$

4.4 Properties of the lost sales models

Theorem 4.4.1. *The maximum finite horizon survival probability of a start-up firm under the assumption of the partial backorder model, $q(n, i, x)$ from equation 4.1, satisfies the following properties:*

- (i) $q(n, i, x)$ is non-increasing in n ;
- (ii) $q(n, i, x)$ is non-decreasing in x ;
- (iii) $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$ exists and is non-decreasing in x .

Proof. Note that the total lost sales model has the same structure as the general model of Chapter 3. The results are then an immediate consequence of Lemma 3.4.1, Lemma

3.4.2 and Theorem 3.4.5. □

Theorem 4.4.2. *Under the assumption of the total lost sales model, if the maximum average reward, g from equation (4.3), is greater than 0, then there exists a finite capital level X_0 such that the maximum infinite horizon survival probability satisfies $q(0, x) > 0$ for all $x \geq X_0$.*

Proof.

Let d_0 be any level of demand such that $p(d_0) > 0$. Let $\pi(i)$ be the stationary distribution of the Markov chain corresponding to the profit maximizing policy, $k^*(i)$. Assume the firm starts with zero inventory (i.e. $i = 0$) and consider the policy that in the first period orders $k_0(i)$ with probability $\pi(i)$, where

$$k_0(i) = \begin{cases} i + d_0, & \text{for } L = 0, \\ i, & \text{for } L = 1. \end{cases}$$

If the demand in the first period is d_0 , the inventory level at the end of the first period is i with probability $\pi(i)$. Further, the cost incurred in the first period is at most

$$H + c + C(I + d_0), \text{ for } L = 0, \text{ and}$$

$$H + c + CI + r_L d_0, \text{ for } L = 1.$$

It follows from Theorem 3.4.5 that if $g > 0$, $q(0, x) > 0$ for $x \geq X_0 = H + c + C(I + d_0)$ for $L = 0$ and for $x \geq X_0 = H + c + CI + r_L d_0$ for $L = 1$. □

4.5 Design of test problems

The purpose of this numerical investigation in this thesis is to illustrate how the models might be applied to investigate the survival chances of a start-up firm and to suggest some apparent general trends. If the model were to be applied to a particular start-up firm, the problem parameters would have to be carefully estimated for the specific characteristics of the firm and the environment in which it operates. This is beyond the scope of this research.

In the present thesis presentation, the following criteria are considered in choosing the parameters for the numerical investigation of the models proposed in this chapter.

- Review period length: The examples in the experiment take monthly review, i.e. one period is equivalent to one month.
- Average profit: To compare the survival probability across different operating circumstances, market conditions and levels of growth, four maximum average profit levels are considered, i.e. $g = 1, 2, 5$ and 11 . This is the maximum average monthly profit, expressed in a suitable currency unit.
- Demand per period: Both Poisson and Uniform demand distributions are considered in the experiments. Poisson distribution, which has the property that the mean equals the variance, is widely used for demand estimation. Uniform distribution on the other hand, has the property that the probability of each discrete event is equally distributed. It has a relatively large variance, which can be used as interpretation of the dispersion of demand in the entry market.
- Holding cost per item per period, h : The holding cost is an evaluation of the opportunity cost of the capital spent on stock. According to Brown [1967, pp. 29-31], it is more of a top management policy variable that can be changed from time to time to meet the change of environment. The “correct” evaluation of holding cost is the one that considers whole inventory system, where aggregate investment, total number of orders per year, and the overall customer service level are in agreement with corporation, operation, marketing and strategy. In these experiments the amount of holding cost is considered to be 10%, 20% and 30% of the unit purchasing cost, C . Therefore, $h = \zeta C/12$ where $\zeta = 0.1, 0.2, 0.3$.
- Lost sales cost per item, r_L : Lost sales cost, treated as the penalty cost of losing customers, is set such that the service level in a profit maximizing firm is guaranteed to be no lower than a target level which is set as 98% for the experiments.
- Purchasing cost per component, C : The purchasing cost is fixed and equal to 10 (i.e. $C = 10$).
- Selling price per item, S : The selling price is determined from the purchasing price, C , by the mark-up factor γ . Two mark-up factors are considered in the experiments, $\gamma = 1.4$ and $\gamma = 1.6$. $S = \gamma C$.
- Fixed order cost, c : Two fixed order costs are considered, $c = 0.2$ and $c = 1.0$, representing relatively low (between 1.8% and 20% of maximum average monthly

profit) and relatively high (between 9% and 100% of maximum average monthly profit) values respectively. Fixed order costs are generally low in modern systems, but maybe higher in start-up firms due to less well-established procedures.

- Fixed overhead cost, H : The fixed overhead cost is calculated to give the required maximum average monthly profit for the problem given the other price/cost parameters.

Flowchart 4.2 indicates the procedure for calculating H and r_L .

Table 4.1: Combination of parameters, $L = 0$ and $L = 1$

Index/Parameters*	c	γ	ζ
1	0.2	1.4	0.1
2	1	1.4	0.1
3	0.2	1.6	0.1
4	1	1.6	0.1
5	0.2	1.4	0.2
6	1	1.4	0.2
7	0.2	1.6	0.2
8	1	1.6	0.2
9	0.2	1.4	0.3
10	1	1.4	0.3
11	0.2	1.6	0.3
12	1	1.6	0.3

$$* S = \gamma C, h = \zeta C/12$$

A list of the combinations of parameters in each lead time case is provided in Table 4.1. These combinations of parameters are used for examples with each average profit $g = 1, 2, 5$ and 11. Two demand distributions, $p(d) \sim \text{Uniform}[0, D]$, and, $p(d) \sim \text{Poisson}(\lambda)$, are discussed in the examples, where $D = 19$ and $\lambda = 9.5$ units of demand.

The examples of two extreme maximum average profits, i.e. $g = 1$ and $g = 11$, are indicated in Figures 4.3, 4.4, 4.5 and 4.6. The survival probability curves are in two groups according to the mark-up parameter, γ , in most of the examples except the case of $L = 1$ with Uniform distributed demand. Table 4.2 and 4.3 provide the overhead cost in each example, the value of which is relevant to the expected maximum average profit, g , and the mark-up parameter, γ . The examples with $\gamma = 1.4$ require lower overhead cost to meet the expected average profit than those with $\gamma = 1.6$. Both the selling price and overhead cost result in the two groups of examples. With Uniform distributed demand, the curves are sparse in $L = 1$, in which case survival is more difficult due to the relatively large variance.

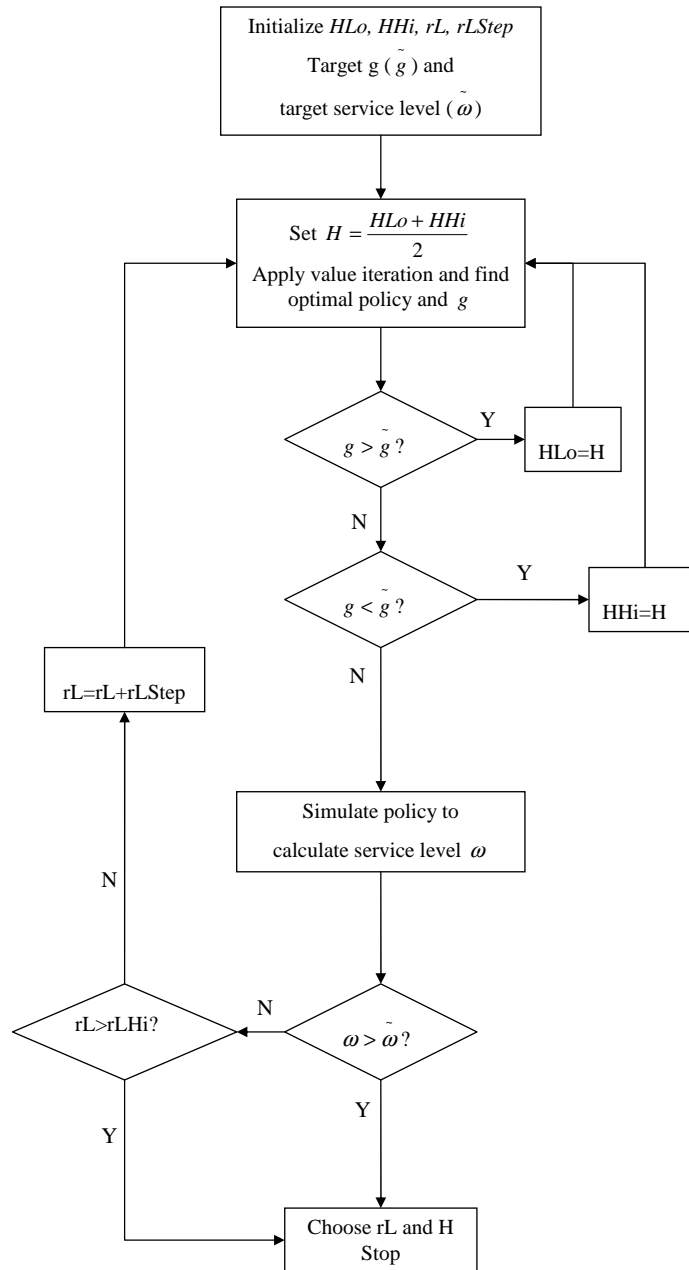
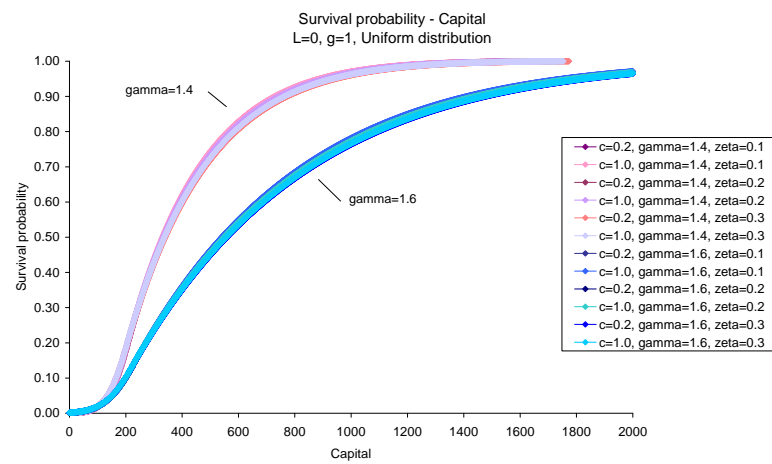
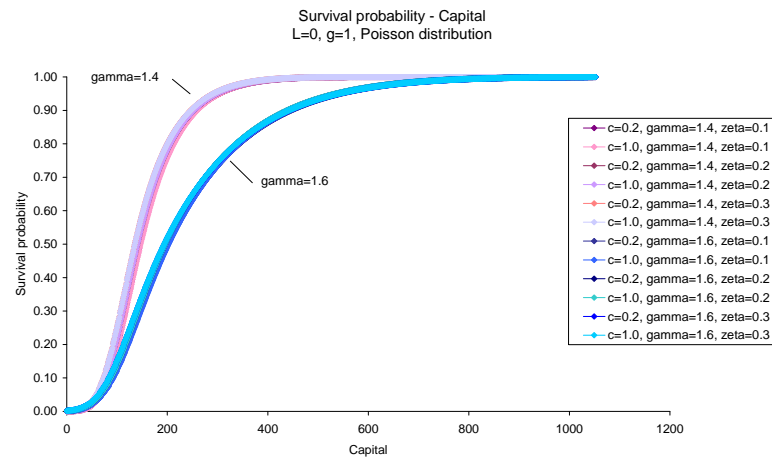
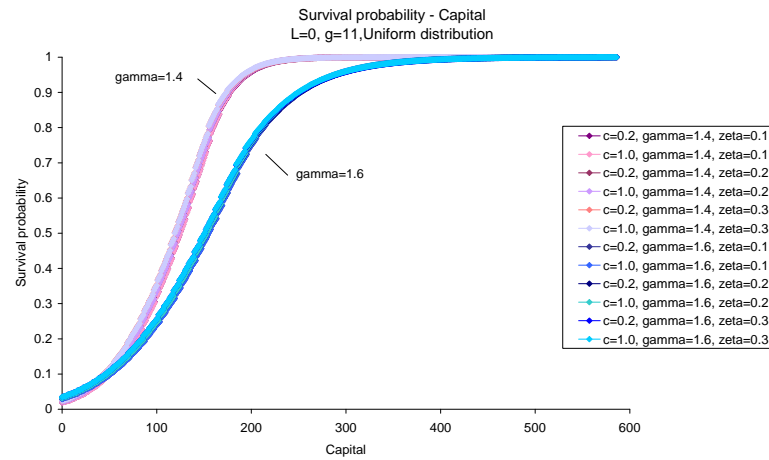
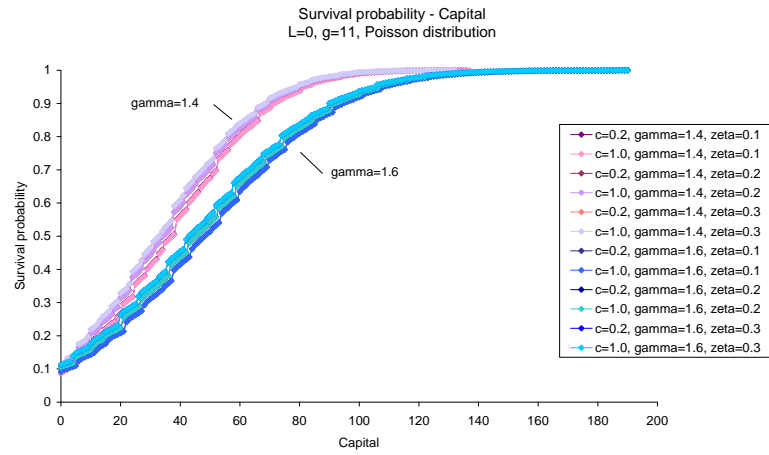
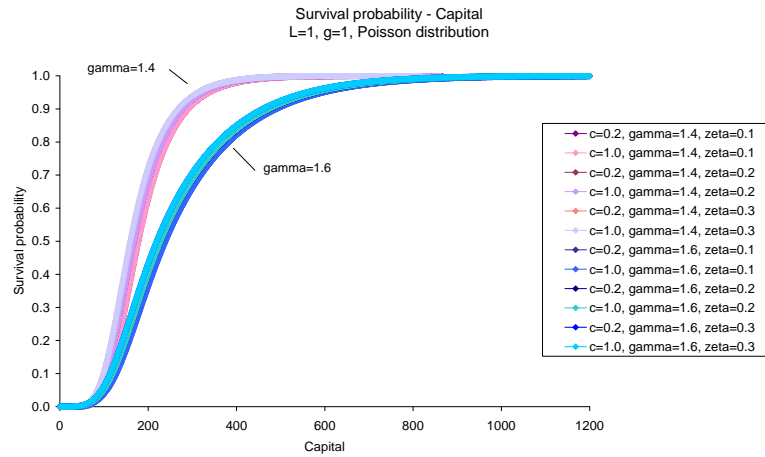


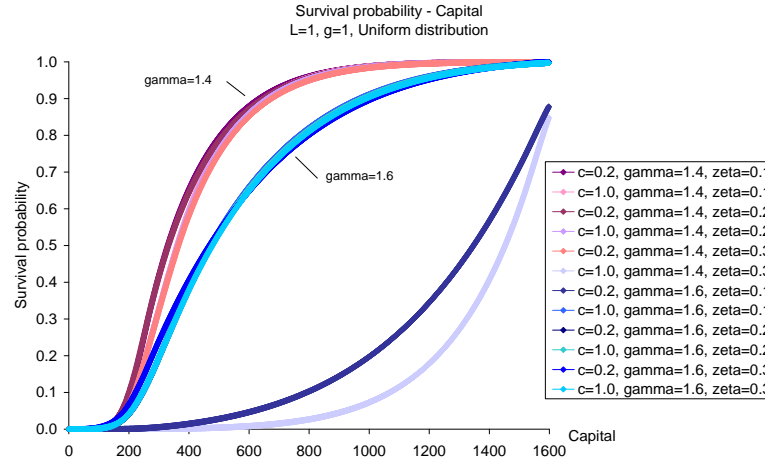
Figure 4.2: Algorithm: Parameter initialization

Figure 4.3: Survival probability - capital, $L = 0$, $g = 1$

Figure 4.4: Survival probability - capital, $L = 0$, $g = 11$

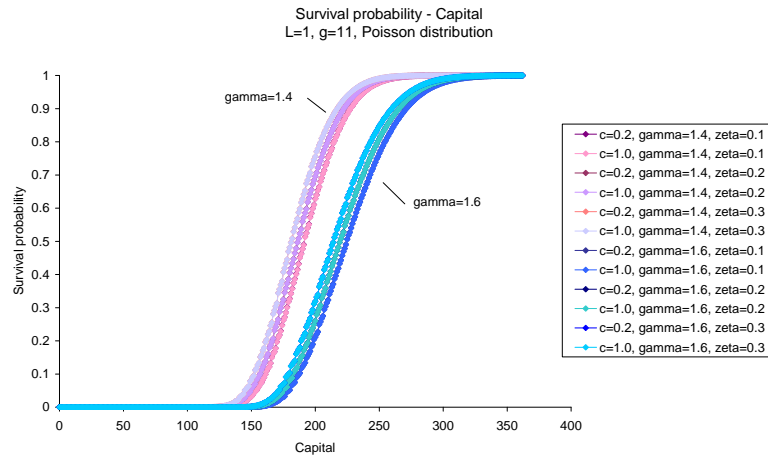


(a) Poisson distribution

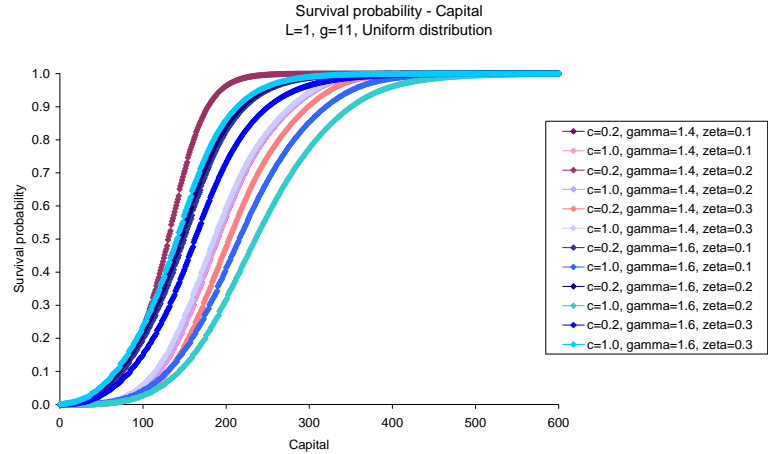


(b) Uniform distribution

Figure 4.5: Survival probability - capital, $L = 1$, $g = 1$



(a) Poisson distribution



(b) Uniform distribution

Figure 4.6: Survival probability - capital, $L = 1$, $g = 11$

Table 4.2: Estimated overhead cost, H , $L = 0$

$g, p(d)$	$S = 14$						$S = 16$					
	$h = 0.08$		$h = 0.17$		$h = 0.25$		$h = 0.08$		$h = 0.17$		$h = 0.25$	
	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$
$g=11, P$	26.01	25.25	25.42	24.62	24.87	24.08	44.95	44.18	44.33	43.53	43.78	42.98
$g=11, U$	26.02	25.29	25.23	24.50	24.44	23.72	45.02	44.28	44.23	43.50	43.44	42.71
$g=5, P$	32.01	31.25	31.42	30.62	30.87	30.08	50.95	50.18	50.33	49.53	49.78	48.98
$g=5, U$	32.02	31.29	31.23	30.50	30.44	29.72	51.02	50.28	50.23	49.50	49.44	48.71
$g=2, P$	35.01	34.25	34.42	33.62	33.87	33.08	53.95	53.18	53.33	52.53	52.78	51.98
$g=2, U$	35.02	34.29	34.23	33.50	33.44	32.72	54.02	53.28	53.23	52.50	52.44	51.71
$g=1, P$	36.01	35.25	35.42	34.62	34.87	34.08	54.95	54.18	54.33	53.53	53.78	52.98
$g=1, U$	36.02	35.29	35.23	34.50	34.44	33.72	55.02	54.28	54.23	53.50	53.44	52.71

Table 4.3: Estimated overhead cost, H , $L = 1$

$g, p(d)$	$S = 14$						$S = 16$					
	$h = 0.08$		$h = 0.17$		$h = 0.25$		$h = 0.08$		$h = 0.17$		$h = 0.25$	
	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$
$g=11, P$	24.94	24.16	23.37	22.57	21.89	21.09	43.86	43.08	42.25	41.46	40.73	39.93
$g=11, U$	20.47	19.71	21.41	17.99	19.69	16.26	38.99	38.23	37.26	39.17	38.20	34.77
$g=5, P$	30.94	30.16	29.37	28.57	27.89	27.09	49.86	49.08	48.25	47.46	46.73	45.93
$g=5, U$	29.14	25.71	26.75	26.67	23.02	22.26	44.99	44.23	45.93	42.50	44.20	40.77
$g=2, P$	33.94	33.16	32.37	31.57	30.89	30.09	52.86	52.08	51.25	50.46	49.73	48.93
$g=2, U$	29.47	31.39	30.41	26.99	28.69	27.95	50.66	47.23	48.93	48.17	47.20	43.77
$g=1, P$	34.94	34.16	33.37	32.57	31.89	31.09	53.86	53.08	52.25	51.46	50.73	49.93
$g=1, U$	33.14	29.71	31.41	27.99	27.02	28.95	51.66	48.23	47.26	46.15	48.20	44.77

In the later numerical investigation section, sensitivity tests are first presented to give an insight of the impact that each price and cost parameter has on start-up firms' survival. Following this, analysis will focus on Firm F1 and its survival in each lead time and demand case. Comparison of order policies between survival and profit maximizing firms will be presented last.

4.6 Numerical investigation

Consider two firms, named F1 and F2. F1 maximizes the chance of survival and F2 maximizes the average expected profit. Both firms operate in the same business environment, i.e. with the same prices, costs, demand and lead time.

Examples with average profit $g = 5$, $\gamma = 1.4$ are used for analysis in both lead time $L = 0$ and $L = 1$. The initial inventory level is $i = 0$ in $L = 0$ and $i = 9$ in $L = 1$, and initial capital for survival maximizing firm is $x = 50$ where applicable. The reason for choosing different initial inventory levels is that in $L = 1$, F1 can hardly survive with no stock given $x = 50$, see Figure 4.7. Where noted, this setting of parameters will be used throughout this chapter.

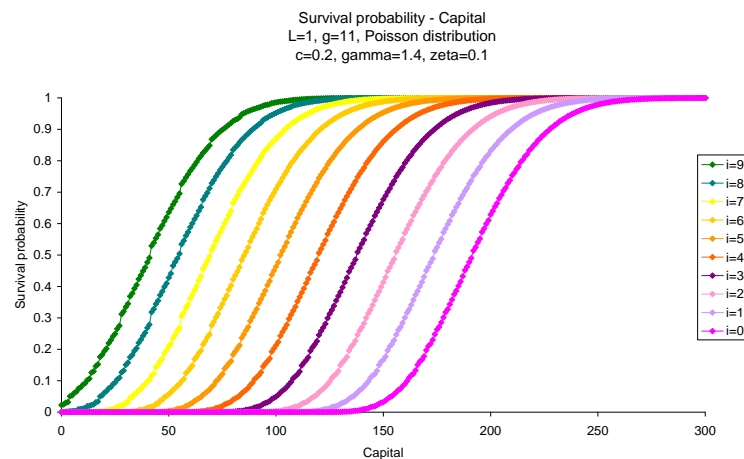


Figure 4.7: Survival probability - capital, $L = 1$, $i = [0, 9]$

4.6.1 Sensitivity test

A sensitivity test (Figures 4.8 and 4.9) is performed on the operation parameters, i.e. selling price, S , purchasing cost, C , overhead cost, H , fixed order cost, c , holding cost, h , and lost sales cost, r_L . Define the elasticity of survival probability $q(i, x)$ to variable z as $e_z(i, x)$, where

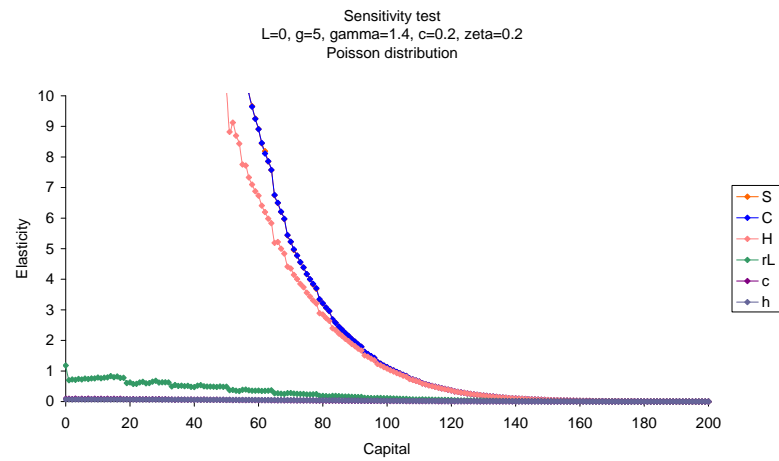
$$e_z(i, x) = \frac{\% \text{ change of } q(i, x)}{\% \text{ change of } z}, \text{ i.e. } e_z(i, x) = \frac{\Delta q(i, x)/q(i, x)}{\Delta z/z},$$

given inventory level, i , and capital, x . A elasticity level of 1 is used in economics where a variable is said to be inelastic if the elasticity level is below 1.

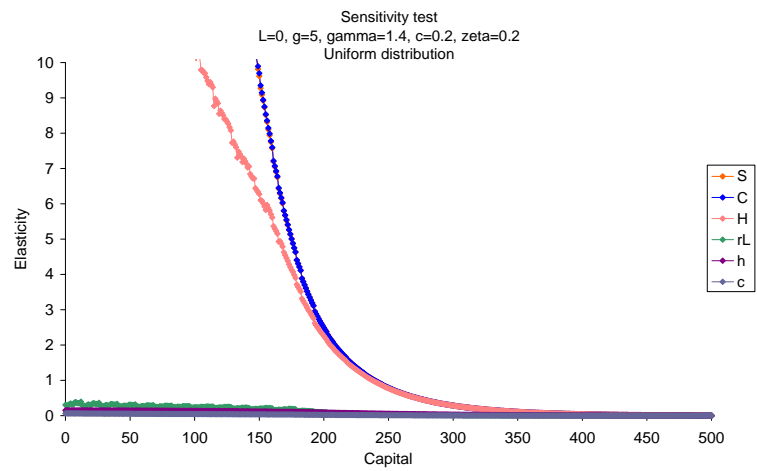
Across the examples of different demand distributions and lead times, the three parameters with the most significant effect on start-up survival in order of impact are, selling price, purchasing cost and overhead cost. In $L = 0$, the rest of the parameters have an elasticity below 1 which means that the influence they have on survival is not significant. In $L = 1$ however, following the top three parameters, both lost sales cost and holding cost have some level of influence, which should be taken into account when considering survival. The elasticity rate decreases with the capital, and at some capital level, the rates all drop below 1, at which point the start-up firm's survival probability is not very much changed from either increasing the selling price or decreasing any of the costs. Looking at the later experiments (Figures 4.13 and 4.14), the survival probability approaches 1 at high capital levels, which means that the firm has little risk in terms of survival.

4.6.2 Start-up survival in time, capital and inventory

Figures 4.10 and 4.11 illustrates how the survival probability decreases with time (of Theorem 4.4.1). The survival probability has a sharp drop in the first few periods of establishment and becomes flat afterwards. The first one year of establishment is crucial to a start-up firm's survival. The survival curve with Uniform demand distribution has a steeper slope and a lower survival probability in the long-term, compared to the cases with Poisson demand distribution in $L = 0$ and $L = 1$. This indicates the difficulty of a firm surviving with a large-variance demand distribution. In $L = 1$ when $i = 9$, both ordering and holding costs have an impact on the order policy (see the groups of orders in Figure 4.12), which results in the separation of the probability curves.

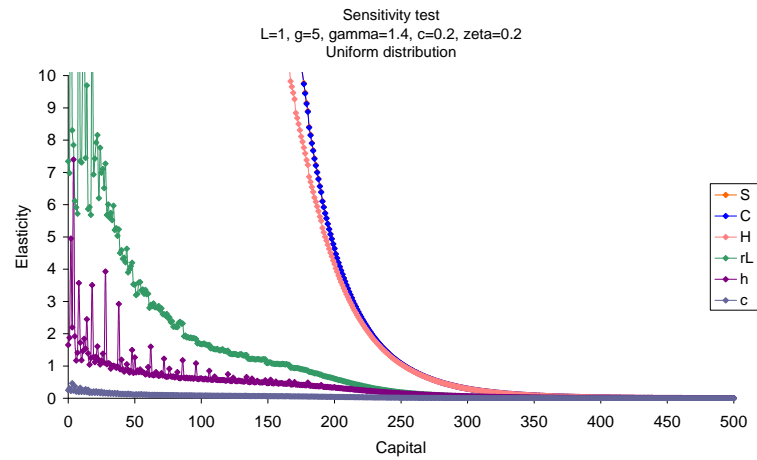
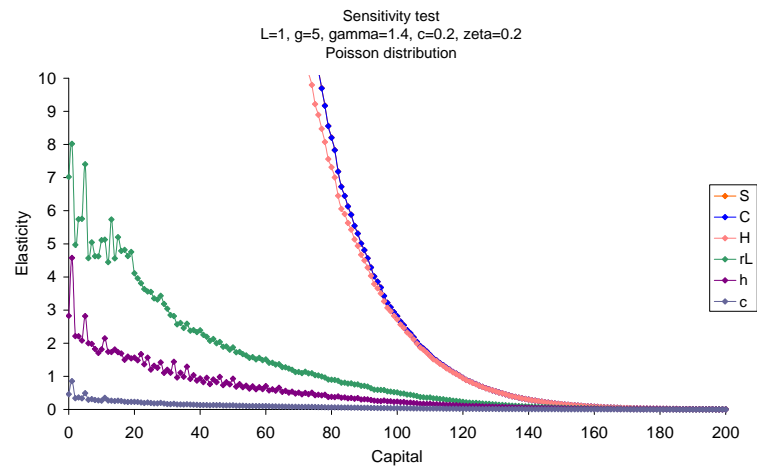


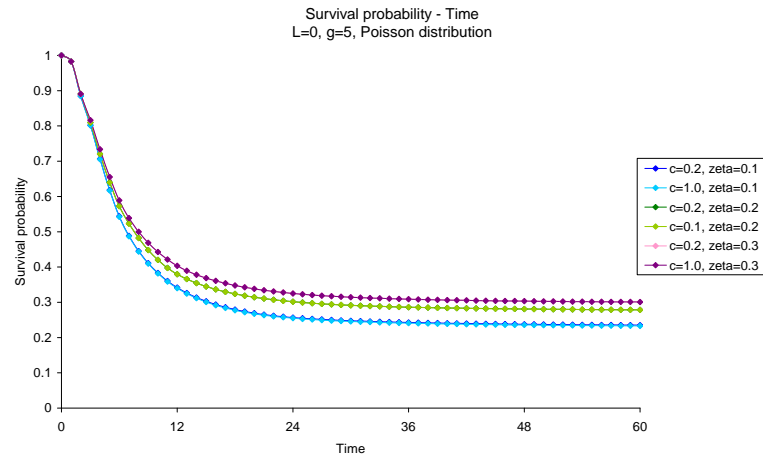
(a) Poisson distribution



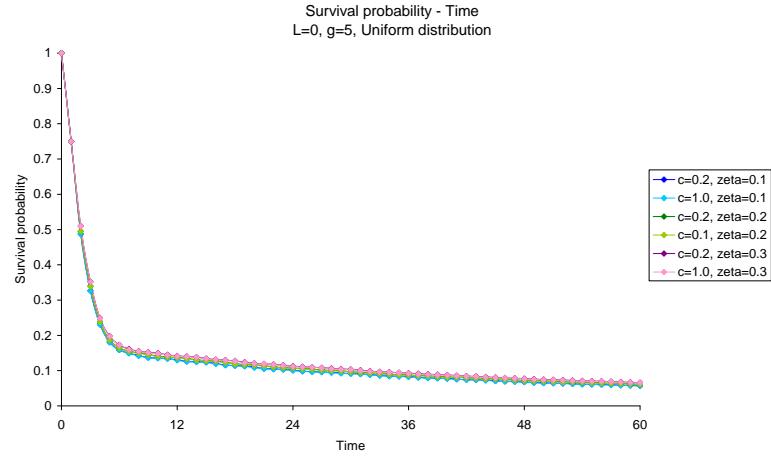
(b) Uniform distribution

Figure 4.8: Sensitivity test, $L = 0$

Figure 4.9: Sensitivity test, $L = 1$

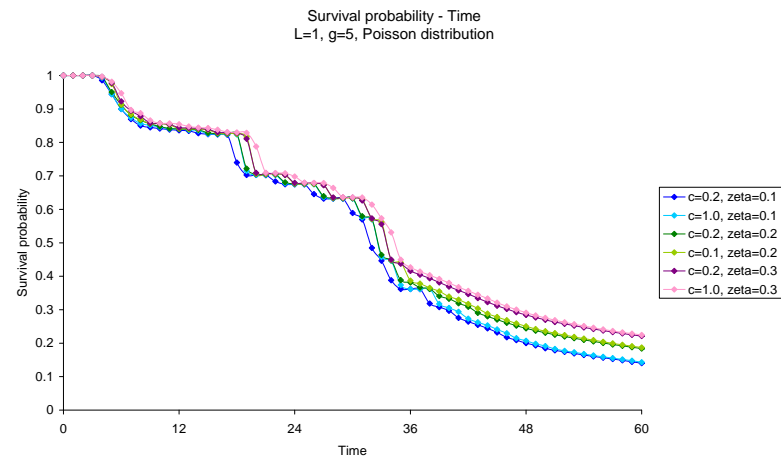


(a) Poisson distribution

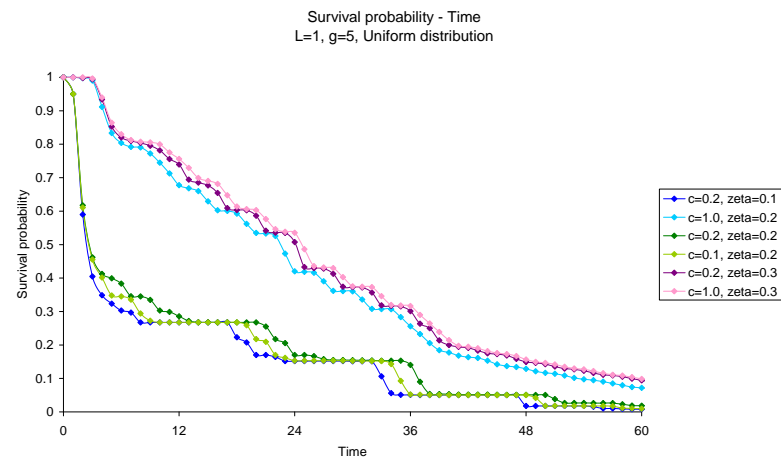


(b) Uniform distribution

Figure 4.10: Survival probability - planning horizon, $L = 0$, $i = 0$

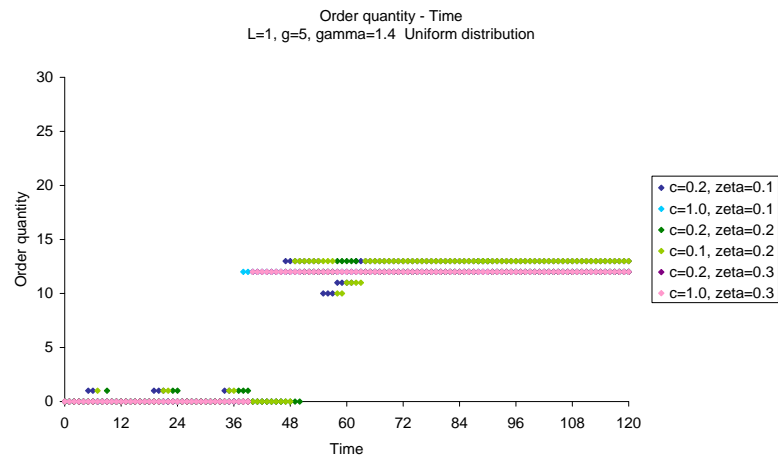


(a) Poisson distribution

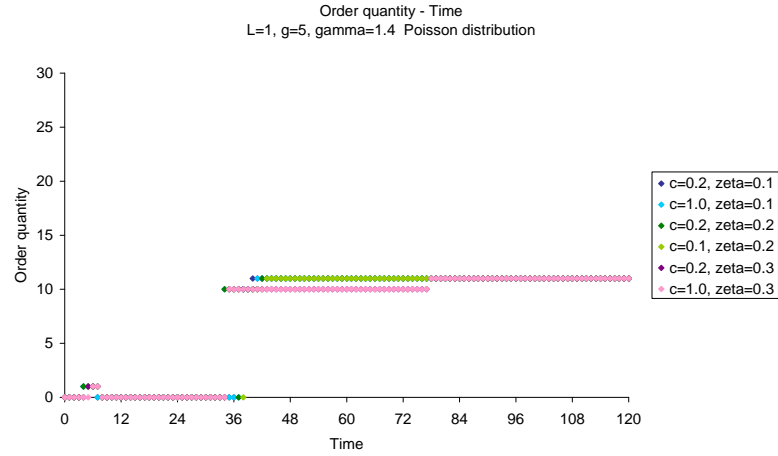


(b) Uniform distribution

Figure 4.11: Survival probability - planning horizon, $L = 1$, $i = 9$



(a) Poisson distribution



(b) Uniform distribution

Figure 4.12: Order quantity - planning horizon, $L = 1$, $i = 9$

Figures 4.13 and 4.14 illustrate how survival probability increases with capital (of Theorem 4.4.1). and maximum average profit. It is interesting to note that in all cases there exists a level of capital above which the firm is almost certain to survive and below which the survival probability rapidly falls towards 0. This threshold level of capital decreases as the average profit increases, but the rate of decrease is not proportional to the increase in average profit (Table 4.4). As average profit increases, the change in the threshold level decreases. This is consistent with observation in the empirical literature on growth rates. When average profits are high, the start-up firm does not rely so much on initial capital stock. However, when average profits are low, the firm relies more on capital stock to pay costs and keep the business going. Hence, a greater level of initial capital is required to give the firm the same chance of success. Hence, the proposed model provides a quantitative tool to investigate the impact of growth rate on the capital required by the start-up firm.

Table 4.4: Capital requirement for survival probability at 99.99%

	Avg. profit	Pois	Slope	Unif	Slope
$L = 0$	$g = 1$	571	571	1732	1732
	$g = 2$	377	194	1128	604
	$g = 5$	205	57	600	176
	$g = 11$	134	12	331	45
$L = 1$	$g = 1$	687	687	2411	2411
	$g = 2$	424	263	811	811
	$g = 5$	221	68	620	327
	$g = 11$	138	14	317	51

^{NB:} X_i = capital requirement, g_i = average reward,
 $\text{Slope}_{i+1} = -(X_{i+1} - X_i)/(g_{i+1} - g_i)$, for $i = 2, 3$
and 4, and $\text{Slope}_1 = X_1/g_1$.

In all cases considered in this chapter, the survival probability is non-decreasing with inventory level (see Figures 4.15 and 4.16 for the case of $\zeta = 0.2$). In other examples (excluded in the results) where the annual holding cost is about half of the purchasing cost, the survival probability drops if the inventory level reaches 30. Therefore, survival probability does not increase with inventory level in all conditions. In the examples presented, the values of fixed order and holding cost have little effect on the survival probability except for the case of $L = 1$ with Uniform distributed demand. For the latter case, overhead and lost sales cost affect the survival (see Table 4.5).

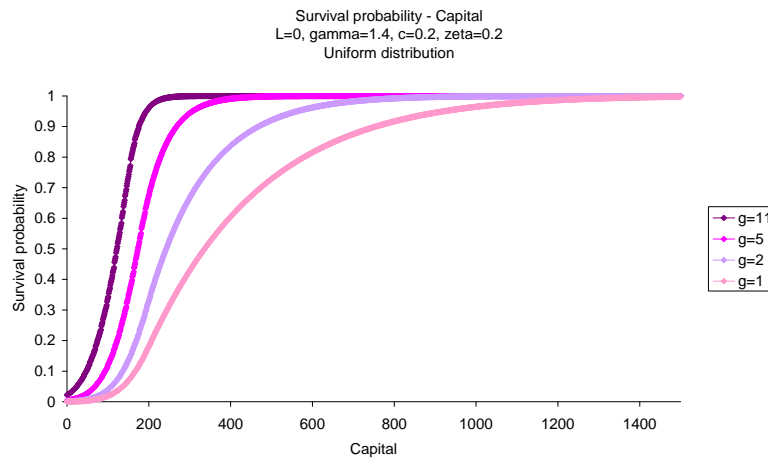
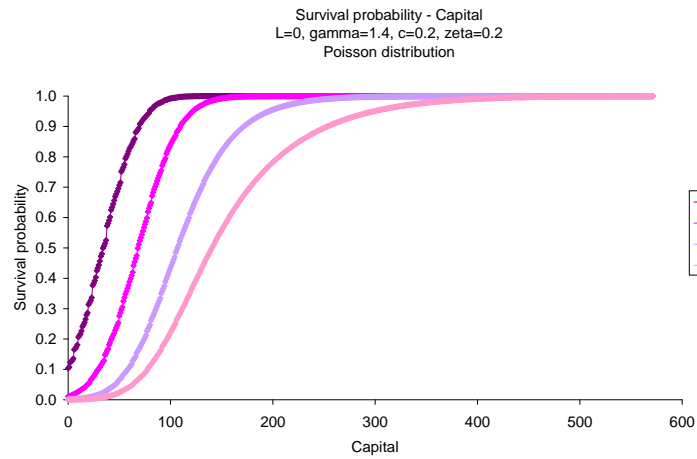
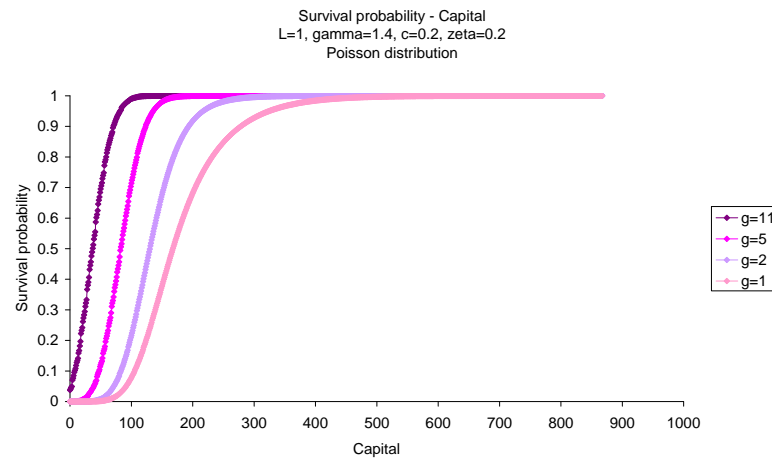


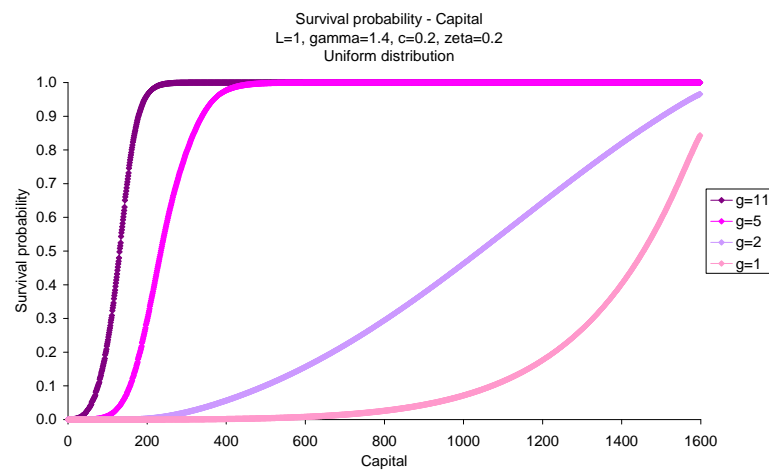
Figure 4.13: Survival probability - capital, examples of profit, $L = 0$, $i = 0$

Table 4.5: $L = 1$ Uniform distribution, values of overhead and lost sales cost

	$\zeta = 0.1$		$\zeta = 0.2$		$\zeta = 0.3$	
	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$	$c = 0.2$	$c = 1.0$
H	29.14	25.71	24.75	26.67	23.02	22.26
r_L	15	4	15	15	4	4

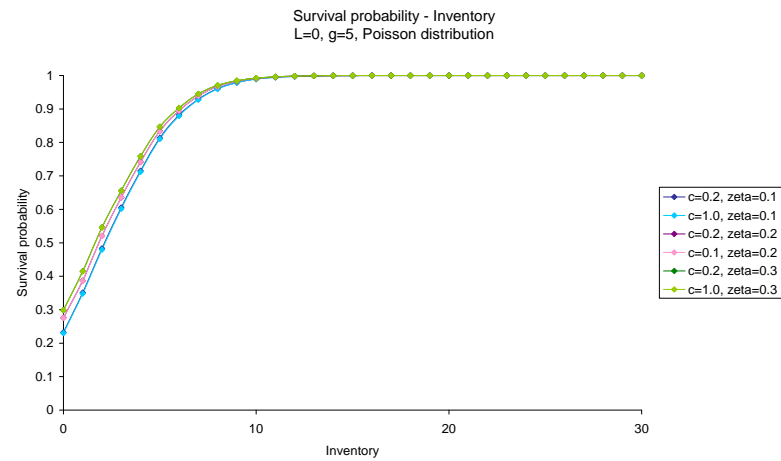


(a) Poisson distribution

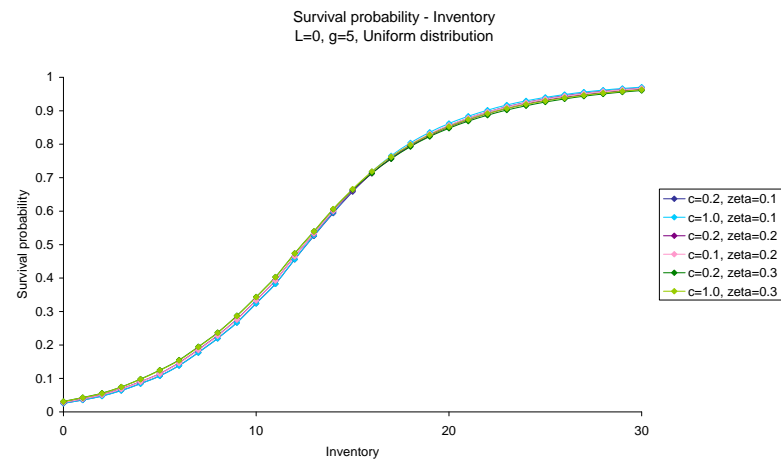


(b) Uniform distribution

Figure 4.14: Survival probability - capital, examples of profit, $L = 1$, $i = 9$

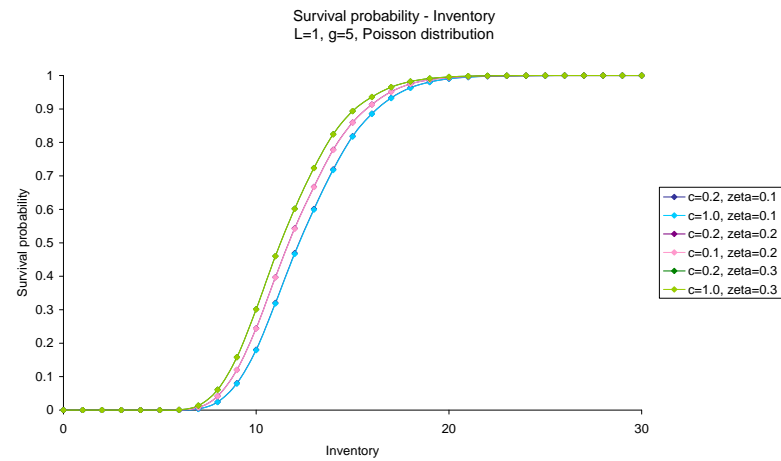


(a) Poisson distribution

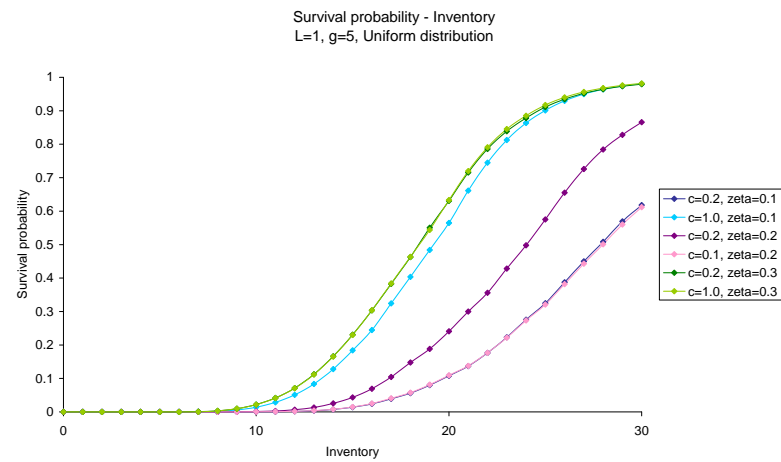


(b) Uniform distribution

Figure 4.15: Survival probability - inventory level, $L = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 4.16: Survival probability - inventory level, $L = 1$

4.6.3 The market and demand distribution

In the previous examples, the demand distribution influences the firm's policy and survival. Figure 4.17 presents F1's chance of survival with Poisson and Uniform distributed demand. In both lead time cases, the firm has a much higher survival probability with Poisson distributed demand than with Uniform distributed demand. The amount of capital that assures firm survival is distinguishably different under the two distributions, where F1 needs much less capital stock if with a Poisson demand (Table 4.4). The demand distribution can be regarded as an indication of the entry market: the firm is in a strong position in a market with Poisson distribution where demand is more concentrated around the mean and can be predicted with lower variance. On the other hand, the firm is in a weak position if the demand follows Uniform distribution of which the variance is very large and all levels of demand occur with equal probability. This emphasizes the importance of the entry market, where a start-up firm benefits from high survival probability and low capital stock requirement from finding the right niche market.

4.6.4 Comparison between objectives: Survival and profit

In general from the experiments (Figures 4.18 and 4.19), given capital $x = 50$, F1 who maximizes survival, orders less than F2 who maximizes the profit. This is due to the requirement of regular solvency check, which can also be reflected from the lower service level that F1 has (see later Figure 4.22). Across the capital (Figures 4.20 and 4.21), at inventory level $i = 0$ in $L = 0$ and $i = 9$ in $L = 1$, F1 varies the order policy depending on capital level. In the first three cases, the order level goes up and then down again when the capital is at a level close to the amount that assures survival. When the firm has a good amount of capital, it increases the orders for more sales. It reduces the order quantity, when it accumulates enough capital which can almost guarantee survival. In such cases, the firm should change its objective, to profit maximizing for example, and adjust its policy accordingly. In $L = 1$ with Uniform demand, F1 finally orders more than F2 does to meet the difficult supply and demand situation.

In terms of service level (Figure 4.22), F1 sacrifices the service level for capital reserve to meet the regular solvency check. F1 places more orders if the demand is of Uniform distribution, in which case it has a slightly higher service level. Even so, the survival probability is not higher than the case with Poisson distribution, since

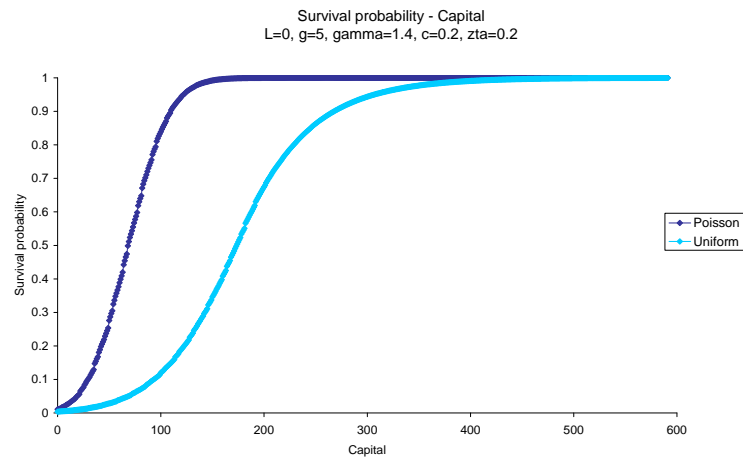
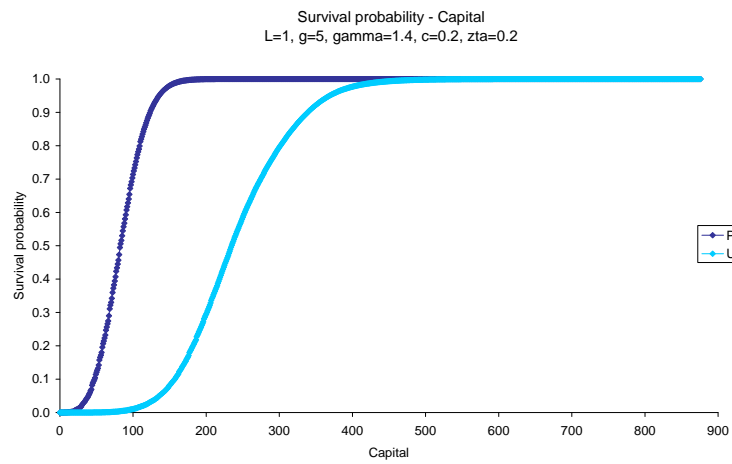
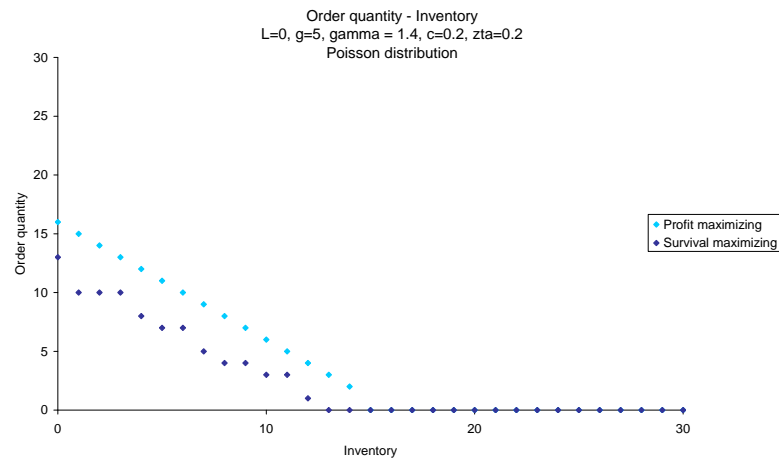
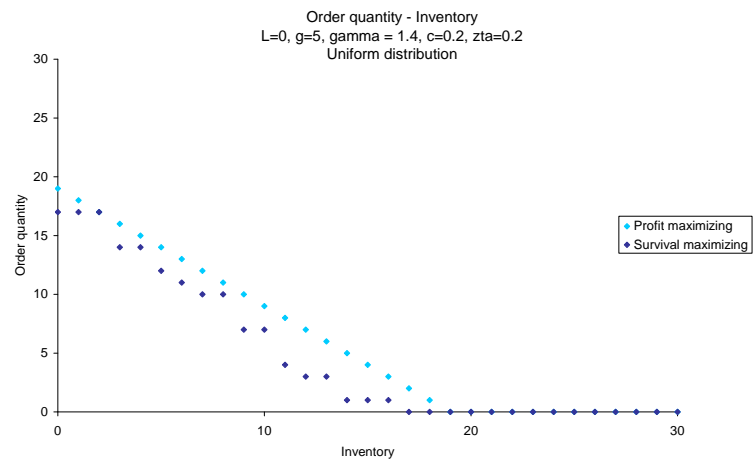
(a) $L = 0, X = 600$ (b) $L = 1, X = 900$

Figure 4.17: Demand distribution: Survival probability - capital

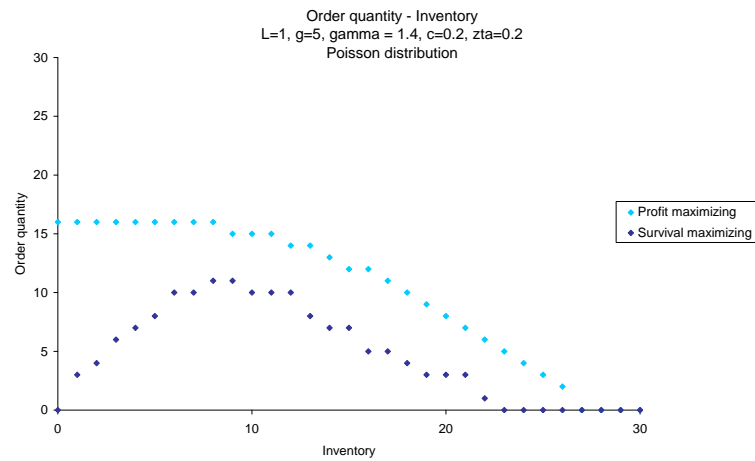


(a) Poisson distribution

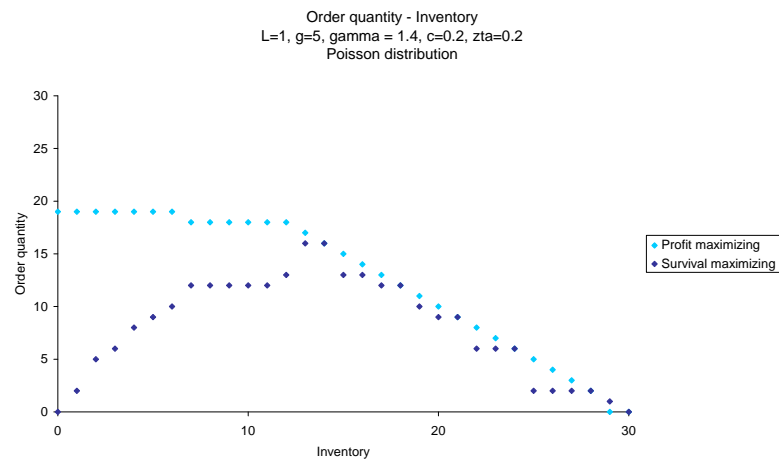


(b) Uniform distribution

Figure 4.18: Comparison between objectives: Order quantity - inventory level, $L = 0$

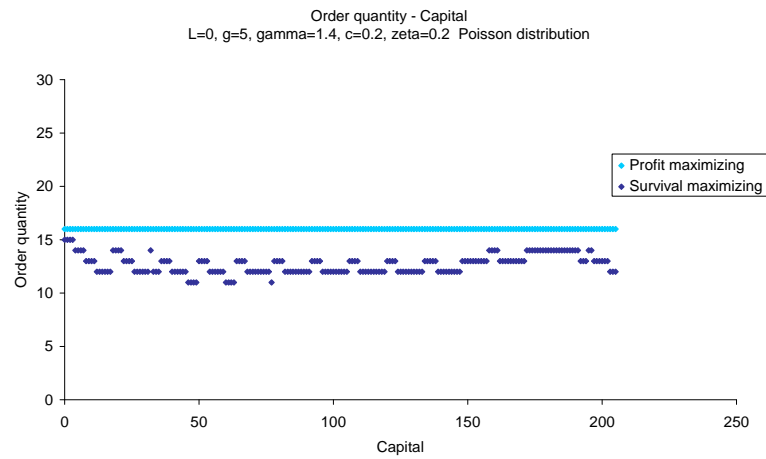


(a) Poisson distribution

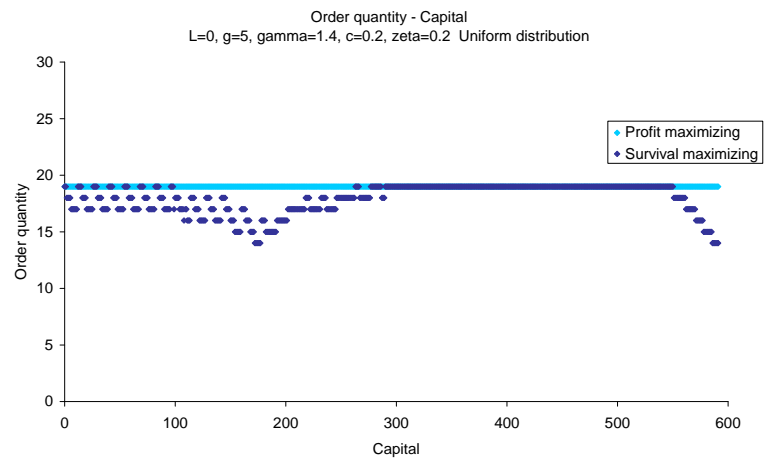


(b) Uniform distribution

Figure 4.19: Comparison between objectives: Order quantity - inventory level, $L = 1$

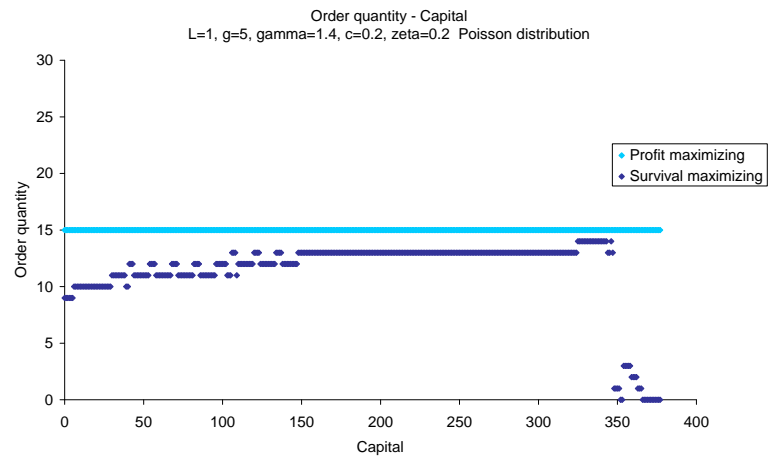


(a) Poisson distribution

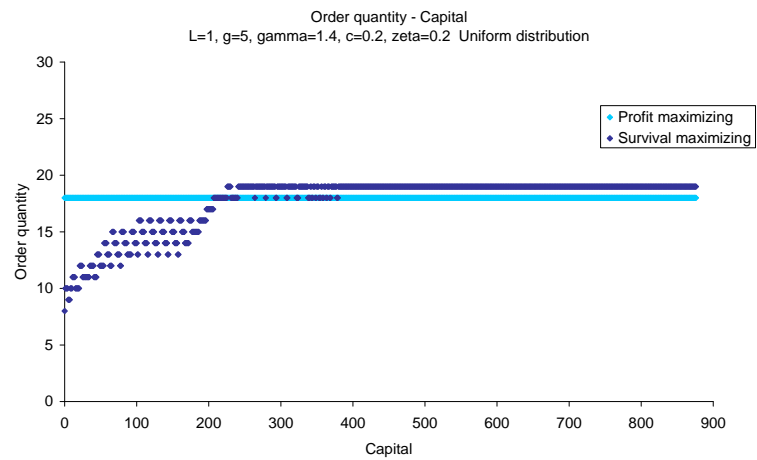


(b) Uniform distribution

Figure 4.20: Comparison between objectives: Order quantity - capital, $L = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 4.21: Comparison between objectives: Order quantity - capital, $L = 1$

the firm has some of the capital tied up to the inventory stock and holds less cash flow on average. Note that the service level is calculated by simulation given the firm starts with the assumed inventory level. Hence, the service levels in the two lead time examples are not comparable.

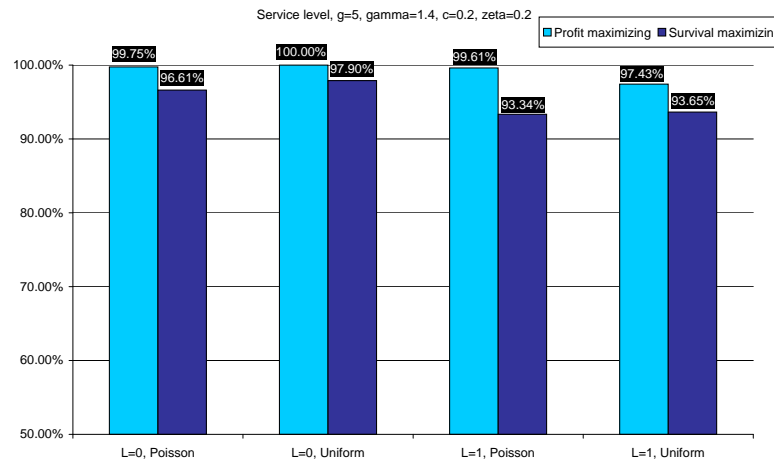


Figure 4.22: Comparison between objectives: Service level, $L = 0$, $i = 0$ and $L = 1$, $i = 9$

4.7 Two extensions to the basic survival model

This section extends the basic survival model, considering changes to the solvency check frequency and order frequency, for insight on the impact of those operations on a start-up firm's survival.

4.7.1 Solvency check

In the original survival maximizing model, F1 must satisfy a solvency check at the end of each review period and can only keep on going if the capital is non-negative. This section relaxes the solvency check criteria, letting the check take place every M periods, where M is a positive integer. In principle, there is no limit on the capital available to the firm between solvency checks. However, in practice, there is likely to be a limit on the debt that the firm can accumulate. For example, there is a limit to the level of credit a supplier is likely to grant a firm. Moreover, for computational reasons, there

needs to be an upper bound on the firm's debt so that the state space is finite. In the relaxed model, there is a debt limit, Y , which F1 must remain within between solvency checks. F1's capital level is therefore bounded such that $x \in [-Y, X]$. The optimality equation for the relaxed model is as follows:

$$q(n, i, x) = \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) q(n-1, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\},$$

for $0 \leq i \leq I$ and $x \in [-Y, X]$, (4.4)

with terminal values

$$q(0, i, x) = \begin{cases} 1, & \text{for all } i \in [0, I], x \geq 0, \\ 0, & \text{for all } i \in [0, I], x < 0. \end{cases}$$

and boundary conditions

$$q(n, i, x) = 1, \text{ for } \forall i \in [0, I], x > X, n > 0, \text{ and}$$

$$q(n, i, x) = \begin{cases} 0 & \text{for all } i \in [0, I], x < 0, \text{ if } n \% M = 0, \\ 0 & \text{for all } i \in [0, I], x < -Y, \text{ if } n \% M \neq 0 \end{cases}$$

where $n \% M$ stands for the remainder of $\frac{n}{M}$. It is expected that F1 is more likely to survive as a result of the looser capital requirement.

Property of solvency check model

Lemma 4.7.1. $q(mM + k, i, x)$ is non-increasing in m , for integer $M \geq 1$ and $k \in [0, M - 1]$.

Proof. Prove $q(n + M, i, x) \leq q(n, i, x)$ by induction on n . When $n = 0$, $q(M, i, x) \leq q(0, i, x)$ because

$$q(0, i, x) = \begin{cases} 1, & \text{for } \forall i \in [0, I], x \geq 0, \\ 0, & \text{for } \forall i \in [0, I], x < 0. \end{cases}$$

and

$$q(n, i, x) \begin{cases} \in [0, 1] & \text{for } i \in [0, I], x \geq 0, n > 0, \\ 0 & \text{for } i \in [0, I], x < 0, n > 0 \end{cases}$$

Assume that the result holds for some $n \geq 0$, i.e. $q(n + M, i, x) \leq q(n, i, x)$. For

$n + 1$, either $q(n + 1 + M, i, x) - q(n + 1, i, x) = 0$ by boundary conditions or, by $\max_i \{a_i\} - \max_i \{b_i\} \leq \max_i \{a_i - b_i\}$,

$$\begin{aligned}
& q(n + 1 + M, i, x) - q(n + 1, i, x) \\
&= \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) q(n + M, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\} \\
&\quad - \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right\} \\
&\leq \max_{0 \leq k \leq I-i} \left\{ \sum_{d=0}^D p(d) \left[q(n + M, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) - \right. \right. \\
&\quad \left. \left. q(n, \mathcal{F}(i, k, d), x + \mathcal{G}(i, k, d)) \right] \right\} \\
&\leq 0, \text{ since } p(d) \geq 0 \text{ for all } d.
\end{aligned}$$

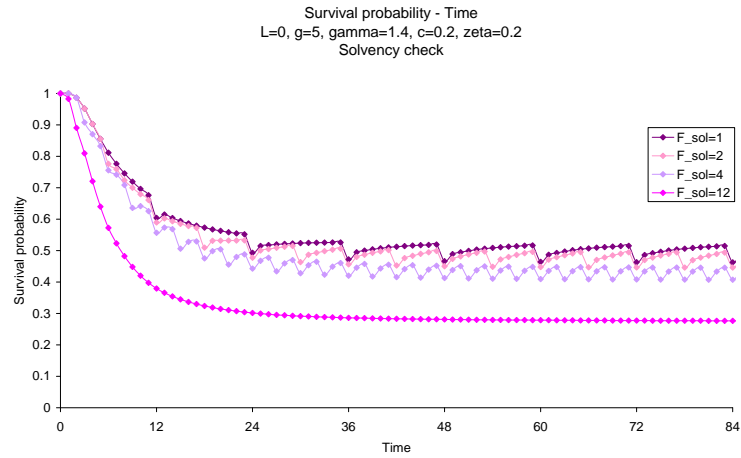
The hypothesis holds for $n + 1$. Therefore, the result holds. \square

Numerical investigation

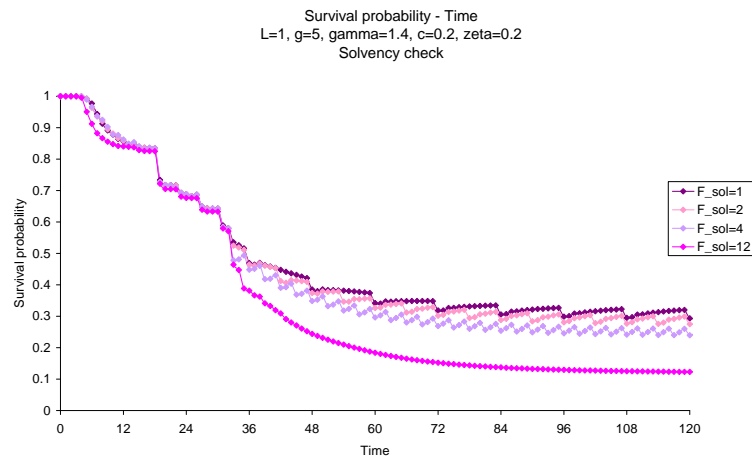
In the initial models, solvency checks are applied at the end of each review month, i.e. the frequency of solvency check $F_{sol} = 12$ per year. In this section, three other frequencies are considered, namely $F_{sol} = 4$, $F_{sol} = 2$ and $F_{sol} = 1$, in which case, solvency checks are carried out quarterly, half-yearly and yearly respectively. Demand is Poisson distributed, with mean 9.5. The other parameters used are $g = 5$, $c = 0.2$, $\gamma = 1.4$ and $\zeta = 0.2$.

Reducing the solvency check frequency increases F1's survival probability (Figure 4.23), the effect of which diminishes with the check frequency. The survival probability is largely improved when the solvency check is reduced to quarterly. This type of analysis would be useful when designing the terms of a bank loan for a start-up firm by providing insight on the impact of different liquidation policies on the probability of default. In terms of planning horizon, the impact of less solvency checks increases with time in the first few periods. The frequency does not change the survival probability much in the first year when the firm has a decreasing survival probability. This is especially obvious in $L = 1$. Overall, reducing the solvency check from monthly to quarterly allows the start-up firm to find extra funding in the short term, or gives the

firm time to recover from temporary debt or shortages.



(a) $L = 0$



(b) $L = 1$

Figure 4.23: Solvency check: Survival probability - time

4.7.2 Order frequency

In the initial models, orders are made monthly, i.e. order frequency $F_{ord} = 12$. In the this extension model, two other order frequencies, $F_{ord} = 26$ and $F_{ord} = 52$ are introduced, where orders are made half-monthly and weekly respectively, while solvency

checks are taken monthly for all the examples. Demand is Poisson distributed. Accordingly, parameters such as average demand, λ , overhead cost, H , and holding cost, h , are a half/quarter of the values in the initial model. The parameters are listed in Table 4.6.

Table 4.6: Parameters in the extension models: Order frequency

F_{ord} /Parameters	Demand distribution $p(d)$	g	c	S	h
26	Poisson(4.75)	5	0.2	14	0.0083
52	Poisson(2.375)	5	0.2	14	0.0042

The survival probability is greatly improved by having more frequent orders each month, and the effect increases with the order frequency (Figure 4.24). This is to be expected as the firm can invest less capital in inventory and hence keep large reserves to meet the regular solvency checks. The model provides a tool to estimate the impact of more frequent orders on the capital requirements of the firm. If the reduction in capital required is greater than the investment required to arrange more reorder opportunities (e.g. by negotiation with suppliers, improving supply chain efficiency, etc) then this would benefit the firm overall. Since orders are delivered right after it is sent in $L = 0$, the increase of order frequency reduces the cost of holding stocks, but does not change the service level much. In $L = 1$ on the other hand, more order opportunities in fact shortens the lead time and reduces the chance of losing customers. As a result the firm has a higher service level. Taking $i = 0$ for example, the service level is increased from 0 (in which case the firm was not able to survive) to 94.96% with half-monthly orders and to 97.95% with weekly orders.

4.8 Conclusion and discussions

This chapter presents models of manufacturing start-up firms and their decisions, based on total lost sales inventory models. Two lead time cases $L = 0$ and $L = 1$, and two demand distributions, Poisson and Uniform, are considered in the numerical investigations. A solvency check is placed at the end of each operation period, i.e. by the end of each month in the examples included, and a start-up firm is said to have failed if it has negative capital at a solvency check. A parallel study is on firms maximizing the average profit who operate in the same environment.

Generally speaking, survival maximizing firms are cautious on decisions. They vary

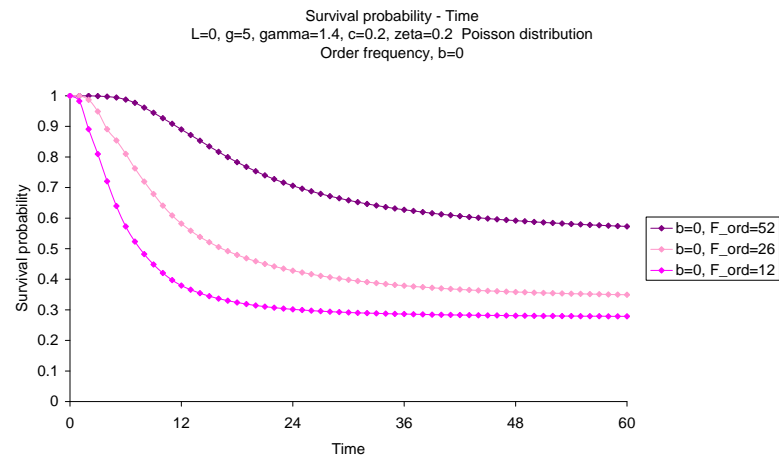
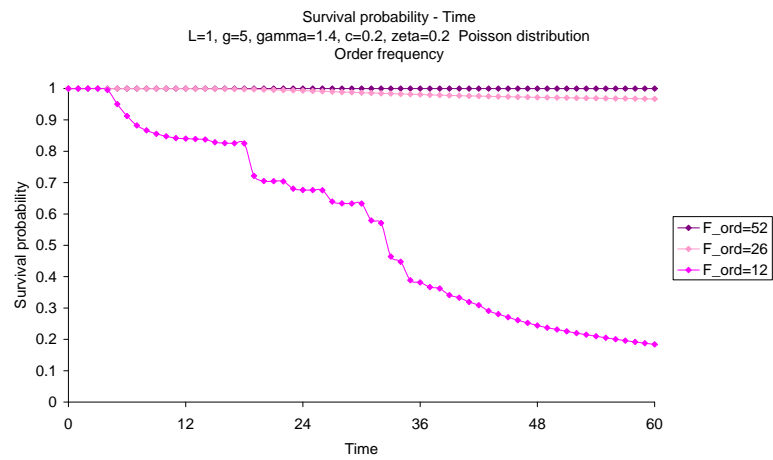
(a) $L = 0, i = 0$ (b) $L = 1, i = 9$

Figure 4.24: Order frequency: Survival probability - time

order quantity according to the financial condition, and sacrifice some service level to meet the regular solvency checks. The first year is crucial to a start-up firm, during which the survival probability drops the fastest in the whole planning horizon. The survival probability gradually becomes constant along the time axis, after a few years of establishment. In terms of sales and profit, the firm operating in a high growth market needs less capital stock to ensure a good chance of survival. On the other hand, firms in markets with low average profit need a large amount of capital to keep the business going while it becomes established. The profit effect on survival has a decreasing return to scale, the first sum of increase on profit has the most significant impact on firm survival.

Within a manufacturing firm, lead time results in delay of delivery of products and so lost sales. A firm in such circumstances needs to keep high stock in order to meet the demand, otherwise the firm has little chance to survive. This negative effect of lead time on survival can be reduced by having more frequent orders, which can be achieved by negotiating with suppliers, improving procurement process efficiency, etc. As a result, the firm is able to meet more demand and the survival probability is greatly increased.

While the models developed verify intuitive trends in the behaviour of survival probability with changes in capital, length of planning horizon, etc, the most important feature of the models is their ability to quantify the affect of such changes. Assuming the parameters of the models can be evaluated, this would provide important insights for founders of and investors in a start-up firm. Analysis of the sensitivity of survival probability to changes in price and cost parameters shows that, for a range of problems, the top three parameters are selling price, purchasing cost and overhead cost, in both lead time cases. Lost sales and holding cost also have some level of influence on survival in $L = 1$ case. This analysis helps start-up firms to prioritize when seeking ways to improve survival probability.

From the financial point of view, a less strict solvency check policy, i.e. conducting quarterly checks rather than monthly checks, brings flexibility and will give start-up firms more room to adjust operations to meet short-term losses and reduce the failure rate to some extent. Note that reducing solvency checks had the most significant effect when changing from monthly to quarterly. This suggests a slightly looser liquidation policy will save the start-ups who are able to proceed. In terms of entry market, from the two demand distributions, firms have easier survival conditions with Poisson

distributed demand, in which case customer demand is concentrated closer to the mean with a relatively small deviation. In contrast, firms having Uniform distributed demand face more uncertainties and have a lower chance of survival. Choosing the niche market is crucial to start-up survival.

Overall, the models in this chapter presents the policies and survival conditions of start-up firms, and introduces sensitivity test, service level, solvency check, order frequency, to bring further analysis of firm performance. In the next chapter, the models will generalize the stockout situation to a mixture of lost sales and backorders, and discuss the impact of having backorders on firms' decisions and survival.

Chapter 5

Modelling inventory decisions with partial backorders

5.1 Problem assumptions and description

In practice, when there are stockouts, firms will often offer customers the possibility of backorders. Backorders are treated as a way to achieve sales and maintain the customer service at some level. Customers will be more likely to wait for backorders if, for example, the product has a special feature(s) and/or an outstanding price to quality ratio, the product comes with good customer service, or the firm has positive goodwill in the market. On the other hand, customers will not accept backorders if, for example, they can easily find similar products or alternatives elsewhere, they do not have time to wait for backorders, or they find themselves not wanting the product any more.

This chapter considers inventory problems with a mixture of lost sales and backorders. Each unsatisfied customer has a probability p_b of accepting a backorder; and a probability $1 - p_b$ of declining the offer. The backorder rate, p_b , is a pre-determined constant, where $p_b \in [0, 1]$. Negative inventory level, i.e. $i < 0$, is used to denote the number of outstanding backorders.

This chapter assumes that at each ordering opportunity the firm automatically orders enough components to satisfy outstanding backorders. Therefore, the decision is how many components to order in addition to this, to meet demand from coming customers. Backorders have the priority of production and should all be satisfied at the earliest time possible. For instance, at the beginning of period n , when the inventory

level is $i < 0$, there are $(-i)^+$ backorders carried forward from period $n - 1$, where $i^+ \equiv \max\{i, 0\}$. The firm orders $(-i)^+$ items for these backorders and another k items to meet new arrivals. In $L = 0$, these $(-i)^+ + k$ units arrive instantaneously. Hence, it takes the customers who accept backorders one period to be served. In $L = 1$, these $(-i)^+ + k$ units do not arrive until the beginning of period $n + 1$. Hence, customers need to wait for up to two periods to get the products. During these two periods, there may be more backorders from the customers who arrive in period n . This group of backorders may not be totally satisfied until period $n + 2$. In such a situation, during one period, there could be two groups of waiting customers. The total number of outstanding backorders in $L = 1$ can be up to $2D$, while the outstanding backorders are up to D units in $L = 0$.

In each period, the total number of unsatisfied customers is a function of inventory level, i , demand, d , and order quantity, k , denoted by

$$B(i, d, k) = \begin{cases} \max\{d - i^+ - k, 0\}, & \text{for } L = 0, \\ \max\{d - i^+, 0\}, & \text{for } L = 1. \end{cases}$$

$B(i, d, k)$ can be up to the maximum demand, i.e. $B(i, d, k) \in [0, D]$. In the rest of the chapter, notation B is used as a short form of $B(i, d, k)$ for convenience. For a total of B unsatisfied customers, the probability that b ($b \in [0, B]$) customers accept backorders follows a binomial distribution, i.e.

$$Pr\{b \text{ out of } B \text{ customers accept backorders}\} = \binom{B}{b} p_b^b (1 - p_b)^{B-b}.$$

In this thesis, any cost relating to the time spent waiting for a backorder is assumed to be included in the unit backorder cost and is not considered in particular. All financial transactions in a period are completed by the end of the same period. The revenue from sales of backorders is received instantly when the customers accept the backorder offers. The payment for components used to satisfy backorders is made in the period in which the order is sent. Hence, for the purposes of the model, a backorder is effectively satisfied as soon as a component is ordered to meet it. Therefore, the inventory level is in the range $i \in [-D, I]$, in both $L = 0$ and $L = 1$ cases.

The shortage cost rates for lost sales and backorders are denoted as r_L and r_B , respectively, and are defined below [Silver et al., 1998, pp.47–48].

- Unit shortage cost on lost sales r_L . It is the loss of a firm's goodwill in a comparatively short term;
- Unit shortage cost on backorders r_B . It is the loss of some goodwill plus the cost spent on making emergency orders, changing machineries, attendant costs, etc.

Lost sales have a higher unit cost rate than backorders, i.e. $r_L > r_B$.

The firm's earnings in one period, $\mathcal{G}(i, k, d, b)$, is a function of the inventory level, i , order quantity, k , demand, d , and backorder quantity, b , such that,

$$G(i, k, d, b) = \begin{cases} S(\min\{i^+ + k, d\} + b) - H - c\delta(k + (-i)^+) \\ -C(k + (-i)^+) - hi^+ - r_B b - r_L(B - b), & L = 0, \\ S(\min\{i^+, d\} + b) - H - c\delta(k + (-i)^+) \\ -C(k + (-i)^+) - hi^+ - r_B b - r_L(B - b), & L = 1. \end{cases} \quad (5.1)$$

In $G(i, k, d, b)$, in the case of $L = 0$, $S(\min\{i^+ + k, d\} + b)$ is the revenue from selling $\min\{i^+ + k, d\}$ items and agreeing b backorders in the period. The total costs contain overhead cost H , ordering cost $c\delta(k + (-i)^+)$, purchasing cost $C(k + (-i)^+)$, holding cost hi^+ and shortage costs $r_B b + r_L(B - b)$. The pattern is similar to $L = 1$ given that the number of items available to sell is $\min\{i^+, d\}$. The inventory level in the next period, $\mathcal{F}(i, k, d, b)$, also depends on backorder quantity, such that,

$$\mathcal{F}(i, k, d, b) = \begin{cases} (i^+ + k - d)^+ - b, & \text{for } L = 0, \\ (i^+ - d)^+ + k - b, & \text{for } L = 1, \end{cases}$$

5.2 Maximizing survival probability

Start-up firms maximize the chance of survival and offer backorders which are guaranteed to be served as soon as products are available. At the beginning of each period, a firm orders, k items, observing its inventory level, i , and capital available, x . This order quantity is made in addition to the $(-i)^+$ items required to satisfy backorders. Demand, d , arrives after orders are sent to suppliers, with probability $p(d)$. Each unsatisfied customer accepts the backorder with a probability p_b .

Define the maximum survival probability given inventory level, i , and capital available, x , when there are n periods to the end of planning horizon, to be $q(n, i, x)$. The

firm is said to fail if it is unable to satisfy the solvency requirement, i.e. $q(n, i, x) = 0$ for $x < 0$. The survival probability maximizing model is formulated as an expected total reward MDP model as follows:

1. Decision epochs and periods. This model has discrete decision epochs. Let $T = \{0, 1, 2, \dots, N\}$ denote the set of decision epochs, where N is the planning horizon. N can be either a finite positive integer ($N < \infty$) or an infinite positive integer ($N \rightarrow \infty$).

2. State and action set. At each decision epoch, the firm is in some state $s = (i, x)$, where i is the inventory level and x is the capital available. The state space is $S = \{(i, x) : i \in \{-D, -D + 1, \dots, -1, 0, 1, \dots, I\}, x \in \{0, 1, 2, \dots\}\}$. On observing the state (i, x) , an action $k \in K_i = \{0, 1, \dots, I - i^+\}$ is chosen representing the order quantity in addition to the items required for backorders. In total, the quantity ordered is $(-i)^+ + k$. No action is taken at the end of the planning horizon (i.e. decision epoch 0). The state space S is infinite countable. The action space K_i is finite for each $i \in [-D, I]$.

3. Reward and transition probabilities. Since the objective is to maximize the probability of the firm surviving the planning horizon, the only reward is generated at the end of the planning horizon. Therefore the reward during each period is zero. The uncertain events during a period are the demand, d , and backorders, b . Given the demand d , backorders b and action k , the system makes a transition from state (i, x) to state $(\mathcal{F}(i, k, d, b), x + \mathcal{G}(i, k, d, b))$ at the next decision epoch. The probability that this event happens is the product of the probabilities of d units of demand and b backorders, i.e. $p(d) \binom{B}{b} p_b^b (1 - p_b)^{B-b}$.

4. Decision rules. The state space S is infinite countable. The action space K_i is finite for each $i \in [-D, I]$, and so $K \equiv \cup K_i$ is finite. According to Puterman [1994, pp. 277–284], this is a positive bounded total reward model. By Theorem 7.1.9 in Puterman [1994, pp. 284], there exists a Markovian deterministic decision policy (MD) which brings an optimal reward.

The survival probability maximizing problem is formulated as below,

$$\begin{aligned}
 & q(n, i, x) \\
 &= \max_{0 \leq k \leq I - i^+} \left\{ \sum_{d=0}^D p(d) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} q(n-1, \mathcal{F}(i, k, d, b), x + \mathcal{G}(i, k, d, b)) \right\}, \\
 & \text{for } -D \leq i \leq I \text{ and } x \geq 0, \quad (5.2)
 \end{aligned}$$

with terminal values

$$q(0, i, x) = \begin{cases} 1, & \text{for all } i \in [-D, I], x \geq 0, \\ 0, & \text{for all } i \in [-D, I], x < 0, \end{cases}$$

and boundary condition

$$q(n, i, x) = 0, \text{ for all } i \in [-D, I], x < 0, n > 0.$$

Furthermore, define $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$ to be a start-up firm's long-run survival probability which is of particular interest in this thesis.

5.3 Maximizing average profit

Profit maximizing firms do not have to meet regular solvency checks and aim to maximize the average profit. Similar to start-up firms, these firms offer backorders to unsatisfied customers. At the beginning of each period, they order k items in addition to those $(-i)^+$ items required for backorders. Demand d arrives, with probability $p(d)$. When there are stockouts, each unsatisfied customer has a probability p_b of accepting the backorder offer. Let g be the maximum average profit and $v(i)$ be the bias term depending on the state i . The profit maximizing model is an expected average reward MDP model as follows:

- 1. Decision epochs and period.** Decisions are made at the beginning of each review period. The expected average reward model is an infinite horizon model. The decision epoch set is denoted as $T = \{0, 1, 2, \dots\}$.
- 2. State and action set.** Inventory level i is the only state variable in this problem. Let negative inventory level stand for the number of outstanding backorders which is bounded by the maximum demand D . The state space is $S = \{(i) : i \in$

$\{-D, -D + 1, \dots, -1, 0, 1, \dots, I\}$. Decisions are inventory orders such that $k \in K_i = \{0, 1, \dots, I - i^+\}$ which is in addition to the $(-i)^+$ items required for backorders. This is a finite state and finite action model.

3. Reward and transition probabilities. The reward in each period is the firm's instant earnings $\mathcal{G}(i, k, d, b)$, which is a function of inventory level i , order quantity k , demand d and backorders b . The firm makes a transition from state (i) to $(\mathcal{F}(i, k, d, b))$ depending on the probabilities of demand $p(d)$ and backorders $\binom{B}{b} p_b^b (1 - p_b)^{B-b}$.

4. Decision rules. The state, s , order quantity, k , demand, d , and backorders, b , are bounded, thus the reward is bounded, i.e. $|\mathcal{G}(i, k, d, b)| < \infty$. In most practical cases, the weak unichain assumption holds [Tijms, 1994, pp.199], where there exists a stationary Markovian deterministic policy (MD) which brings a constant average reward.

The model is formulated as

$$g + v(i) = \max_{0 \leq k \leq I - i^+} \left\{ \sum_{d=0}^D p(d) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} (\mathcal{G}(i, k, d, b) + v(\mathcal{F}(i, k, d, b))) \right\},$$

for $i \in [-D, I]$. (5.3)

By Theorem 8.4.5 Puterman [1994, pp.361], there exists a stationary optimal order k^* with a unique stationary distribution $\pi(i)$, such that

$$g = \sum_{i=-D}^I \pi(i) \sum_{d=0}^D p(d) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \mathcal{G}(i, k^*(i), d, b), \text{ for } i \in [-D, I], \quad (5.4)$$

where $k^*(i)$ is the action chosen in state i under policy k^* .

5.4 Properties of the partial backorders models

Theorem 5.4.1. *The maximum finite horizon survival probability of a start-up firm under the assumption of the partial backorder model, $q(n, i, x)$ from equation 5.2, satisfies the following properties:*

- (i) $q(n, i, x)$ is non-increasing in n ;
- (ii) $q(n, i, x)$ is non-decreasing in x ;

(iii) $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$ exists and is non-decreasing in x .

Proof. Note that the partial backorder model has the same structure as the general model of Chapter 3. The results are then an immediate consequence of Lemma 3.4.1, Lemma 3.4.2 and Theorem 3.4.3. \square

Theorem 5.4.2. *Under the assumption of the partial backorder model, if the maximum average reward, g from equation (5.3), is greater than 0, then there exists a finite capital level X_0 such that the maximum infinite horizon survival probability satisfies $q(0, x) > 0$ for all $x \geq X_0$.*

Proof. If $p_b = 0$, stockouts always result in lost sales and the result follows from Theorem 4.4.2.

Assume $p_b > 0$. Let d_0 be any level of demand such that $p(d_0) > 0$. Let $\pi(i)$ be the stationary distribution of the Markov chain corresponding to the profit maximizing policy, $k^*(i)$. Assume the firm starts with zero inventory (i.e. $i = 0$) and consider the policy that in the first period orders

$$k_0(0) = \begin{cases} i + d_0, & \text{with probability } \pi(i) \text{ where } i \in (0, I], \\ d_0, & \text{with probability } \sum_{j=-D}^0 \pi(j) \end{cases}$$

and in the second period orders

$$k_1(i) = \begin{cases} k^*(i), & \text{if } i \in (0, I], \\ k^*(j), & \text{with probability } \pi(j) \text{ where } j \in [-D, 0], \text{ if } i = 0. \end{cases}$$

With probability $p_0 \geq p(d_0)p_b^{d_0} > 0$, the demand in the first period is d_0 and all stockouts (which only arise when $L = 1$) result in backorders.

In this situation, the inventory level at the end of the first period is i with probability $\pi(i)$ for $0 < i \leq I$ and 0 otherwise. In the second period, the policy orders $k^*(i)$ in state $i > 0$ with probability $\pi(i)$ and orders enough to satisfy any outstanding backorders plus an additional $k^*(j)$ with probability $\pi(j)$ where $-D \leq j \leq 0$. The effect is the same as applying the profit maximizing policy when the distribution of the inventory level is π . Hence, at the end of the second period, the distribution of inventory level is π .

Further, the cost incurred in the first period is at most

$$H + c + C(I + d_0) + r_B d_0$$

and the cost incurred in the second period is at most

$$H + c + CI + hI + r_L D.$$

It follows from Theorem 3.4.5 that if $g > 0$, $q(0, x) > 0$ for $x \geq X_0 = 2H + 2c + 2CI + d_0C + hI + r_B d_0 + r_L D$. \square

5.5 Numerical investigation

Compared with the lost sales model of Chapter 4, the partial backorder model has two additional parameters, namely the backorder cost r_B and the backorder rate $p(b)$. The examples from Chapter 4 are extended for the partial backorder model by setting $r_B = 10h$ (a convention that is sometimes used in models in the literature) and considering three backorder rates, $p(b) = 0.4, 0.7$ and 1 . A further backorder rate, $p(b) = 0$, has effectively already been considered in the lost sales model. The four specific examples shown in Table 5.1 are used in this section to illustrate the results of the model. Many more examples have been considered and similar results recorded. The capital level and inventory level used on observation are $x = 50$ for both lead time cases, and $i = 0$ in $L = 0$ and $i = 9$ in $L = 1$.

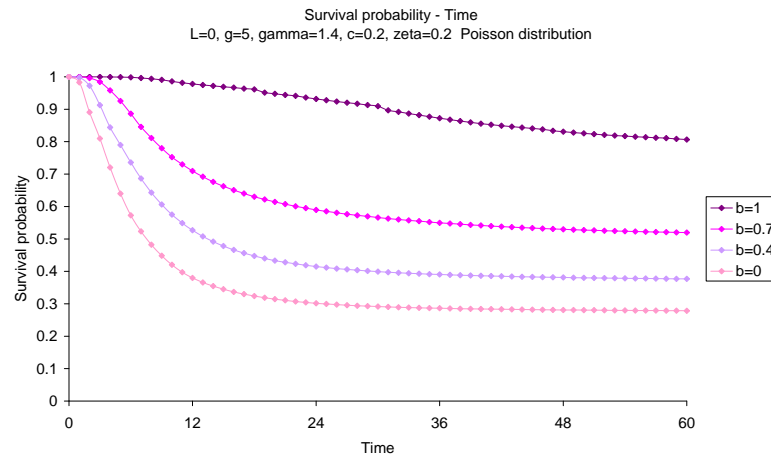
Table 5.1: Parameters in the partial backorder models

	Demand distribution $p(d)$	g	C	c	S	h	H	r_L	r_B
$L = 0$	Poisson(9.5)	5	10	0.2	14	0.17	30.87	4	1.67
	Uniform[0, 19]	5	10	0.2	14	0.17	30.44	4	1.67
$L = 1$	Poisson(9.5)	5	10	0.2	14	0.17	27.89	4	1.67
	Uniform[0, 19]	5	10	0.2	14	0.17	23.02	4	1.67

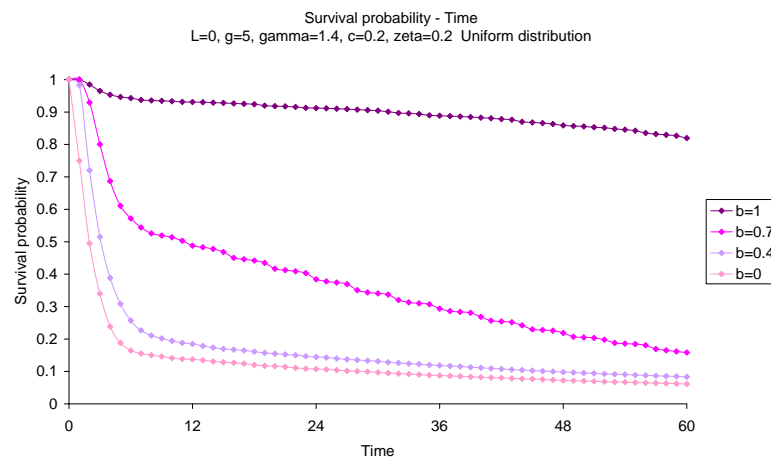
The later section will focus on the effect partial backorders have on survival maximizing firms. It then introduces hazard rate to estimate the likelihood of start-up firms' failure after launch. Changes of solvency check and order frequency will also be discussed in the backorder models.

5.5.1 Start-up survival in time, capital and inventory

Figures 5.1 and 5.2 present the survival probability over time, for different partial back-order rates. As expected, higher backorder rate, $p(b)$, increases the survival probability in all cases.

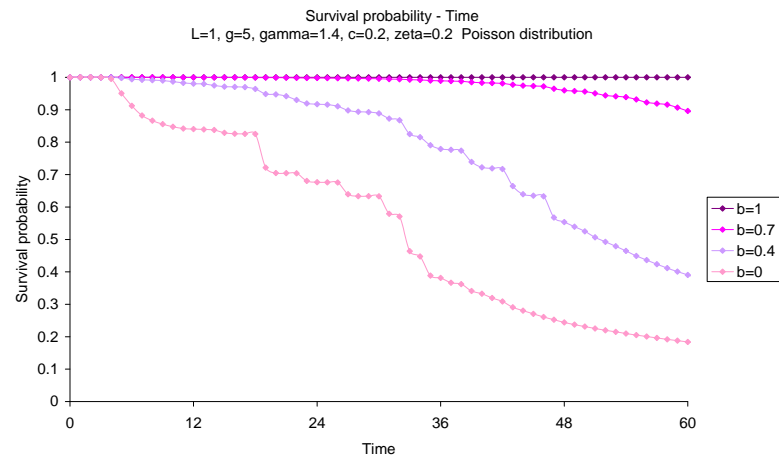


(a) Poisson distribution

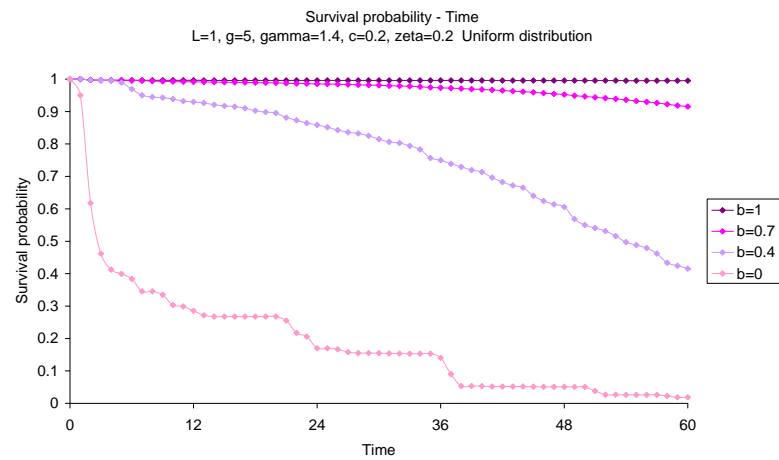


(b) Uniform distribution

Figure 5.1: Partial backorders: Survival probability - time, $L = 0$, $i = 0$



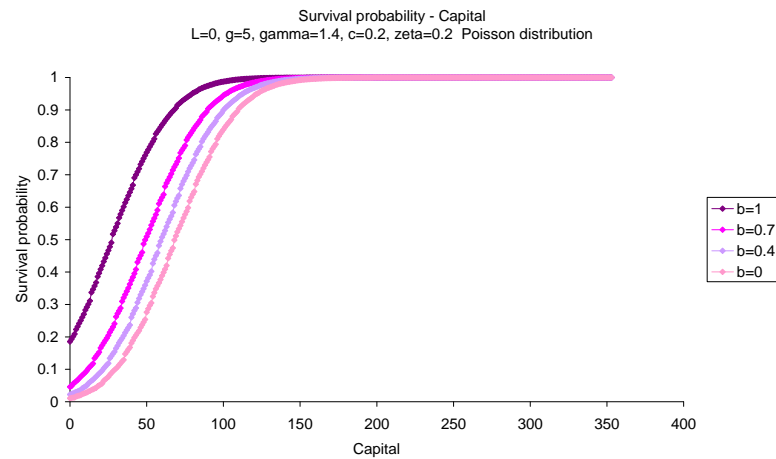
(a) Poisson distribution



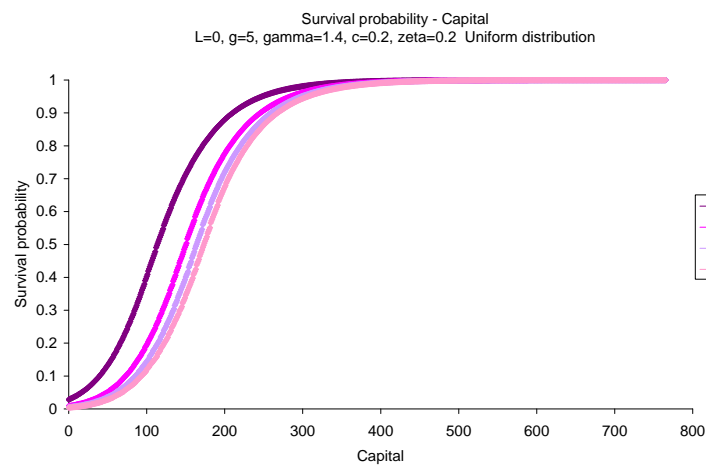
(b) Uniform distribution

Figure 5.2: Partial backorders: Survival probability - time, $L = 1$, $i = 9$

Partial backorders shifts the survival curve to the left along the capital axis (Figures 5.3 and 5.4). The total backorder curves stand out from the examples as the firm eventually meets all the customer demand. Comparing across lead times and demand distributions, the effect of partial backorder rate is more significant where the operational conditions are difficult for survival, i.e. $L = 1$, and with Uniform distributed demand.

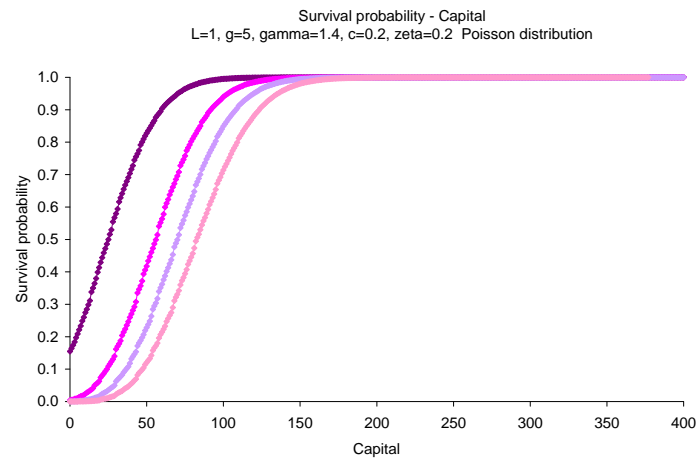


(a) Poisson distribution

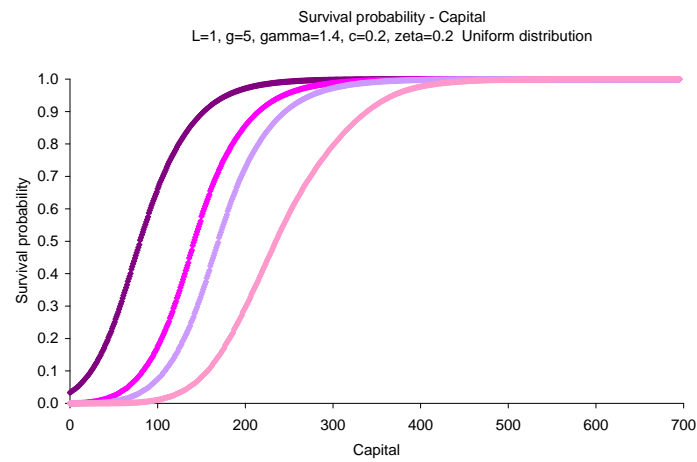


(b) Uniform distribution

Figure 5.3: Partial backorders: Survival probability - capital, $L = 0$, $i = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 5.4: Partial backorders: Survival probability - capital, $L = 1$, $i = 9$

Backorders reduce the inventory level required to satisfy a given level of survival probability, see Figures 5.5 and 5.6. Figure 5.7 concludes the inventory level at which the survival probability first achieves 80%. This stock level varies across examples and in each case, it decreases with the backorder rate. Again the difference is greater when $L = 1$ and demand is Uniform.

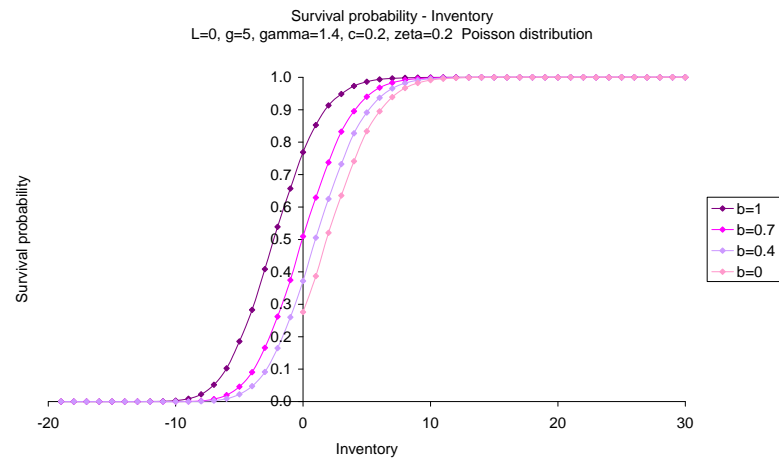
Overall, increased backorder rates reduce both the inventory and capital stock a firm requires to have a given chance of survival. These effects are more noticeable in cases where survival is harder, for example due to lead time or variance of demand. The model provides a tool to quantify the effects of increased backorders. There are ways that firms can stimulate backordering products, for example by providing products with special features, particular designs, outstanding customer service, etc.

5.5.2 Comparison between objectives; Survival and profit

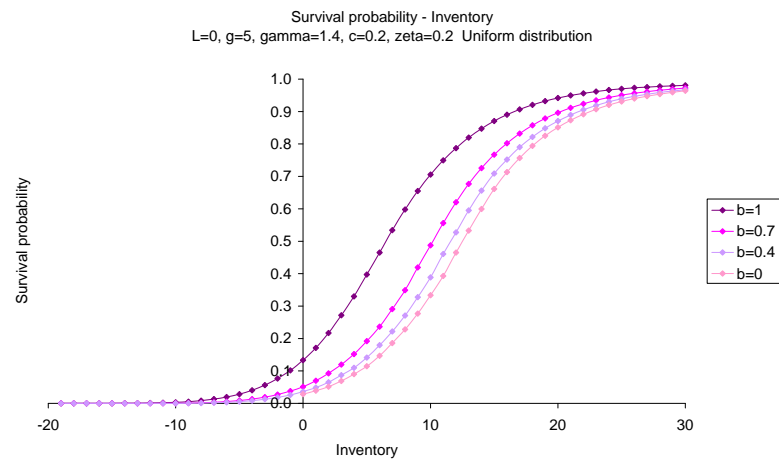
The order policies of the two firms in the four partial backorder examples are shown in Figures 5.8 and 5.9. Table 5.2 further outlines the order each firm makes at $i = 0$ in $L = 0$, and $i = 9$ in $L = 1$, where for F1, the quantity is at the level where the order quantity becomes stable along the capital axis. F1 orders less than F2 does, for each backorder rate in most of the cases. With Uniform distributed demand, F1 reduces the order whereas F2 increases the order as backorder rate increases in each case, In addition, Table 5.3 lists the average profit that F2 earns in each backorder example. The increasing slope indicates the increasing impact that backorder rate has on F2's average profit.

Table 5.2: Comparison: Order quantity, $i = 0$ in $L = 0$ and $i = 9$ in $L = 1$

	Demand	Objective	$p(b) = 0$	$p(b) = 0.4$	$p(b) = 0.7$	$p(b) = 1$
$L = 0$	Poisson	Profit	16	16	15	14
		Survival	13	13	12	11
	Uniform	Profit	19	19	19	18
		Survival	19	18	18	16
$L = 1$	Poisson	Profit	15	16	16	16
		Survival	14	14	13	10
	Uniform	Profit	18	20	21	21
		Survival	19	19	19	15

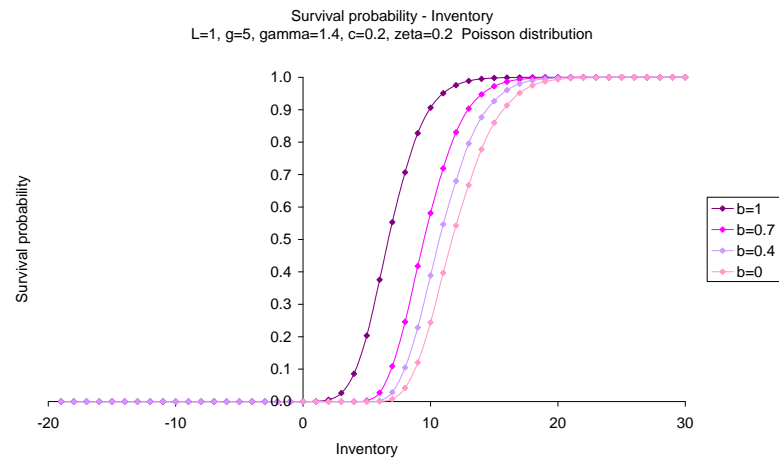


(a) Poisson distribution

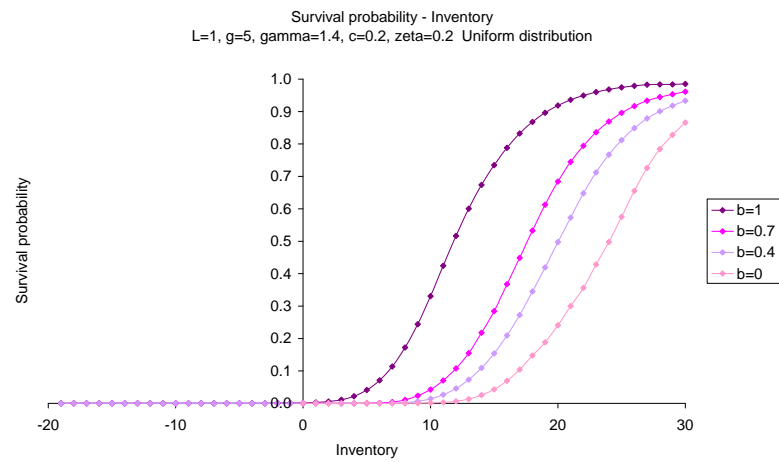


(b) Uniform distribution

Figure 5.5: Partial backorders: Survival probability - inventory, $L = 0, x = 50$



(a) Poisson distribution



(b) Uniform distribution

Figure 5.6: Partial backorders: Survival probability - inventory, $L = 1, x = 50$

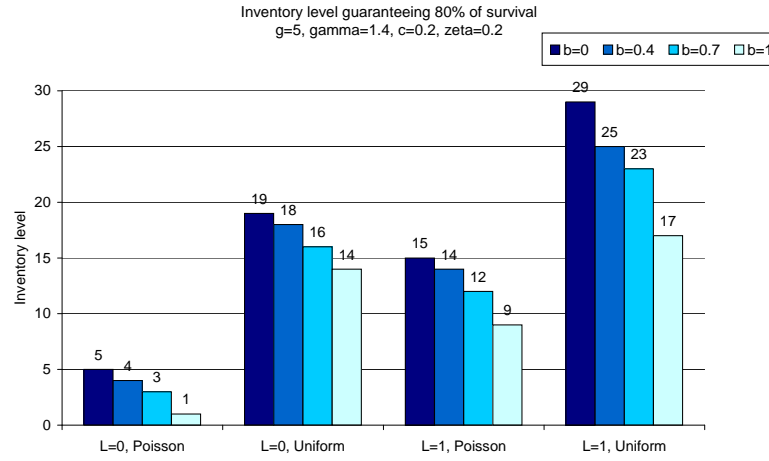
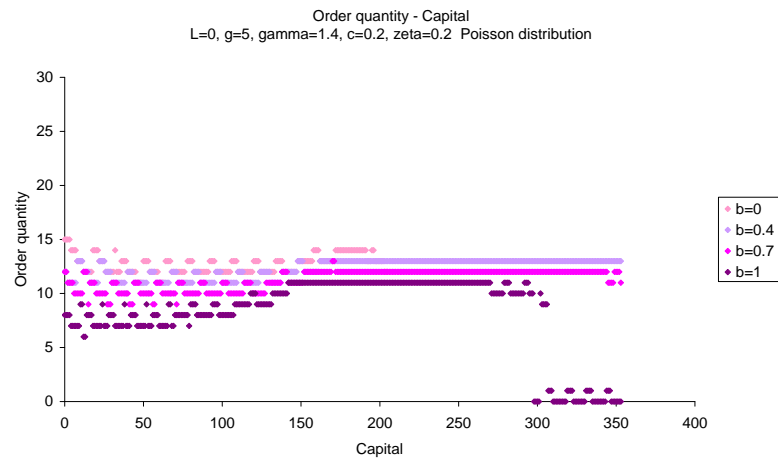


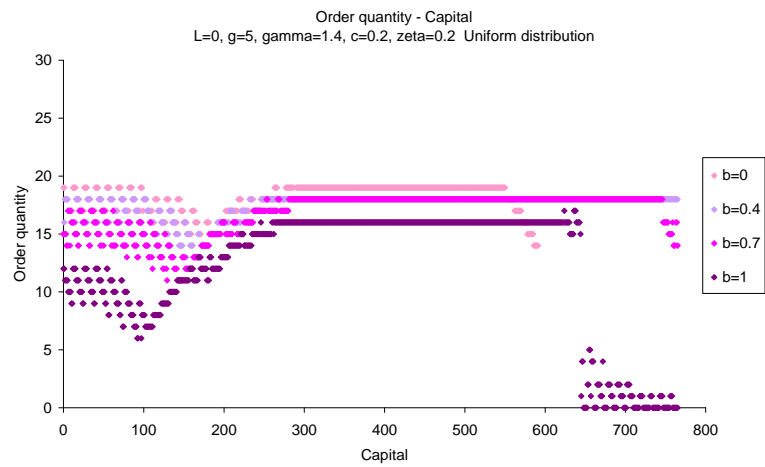
Figure 5.7: Inventory level, at which survival probability achieves 80%

Table 5.3: Comparison: Impact of backorder rate on average profit, $i = 0$ in $L = 0$ and $i = 9$ in $L = 1$

	Demand	g/slope	$p(b) = 0$	$p(b) = 0.4$	$p(b) = 0.7$	$p(b) = 1$
$L = 0$	Poisson	g	5	5.0038	5.0085	5.0866
		slope	-	0.0095	0.0157	0.2603
	Uniform	g	5	5.0638	5.1601	5.3281
		slope	-	0.1595	0.3210	0.5600
$L = 1$	Poisson	g	5	5.1024	5.2215	5.4584
		slope	-	0.2560	0.3970	0.7897
	Uniform	g	5	5.5412	5.9810	6.4922
		slope	-	1.3530	1.4660	1.7040

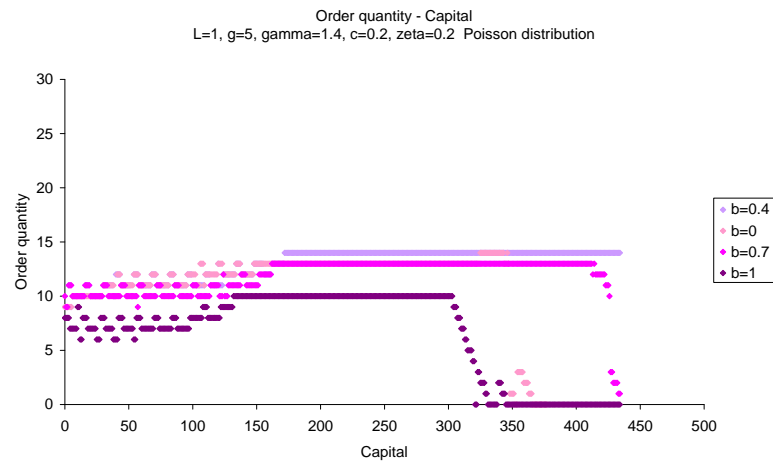


(a) Poisson distribution

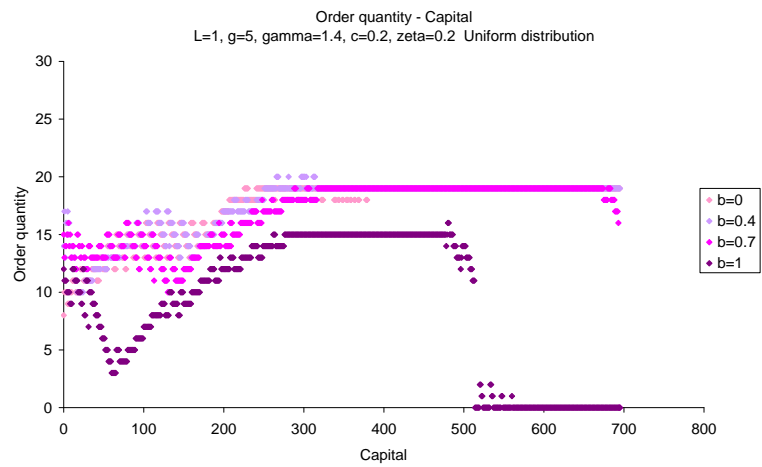


(b) Uniform distribution

Figure 5.8: Partial backorders: Order quantity - capital, $L = 0$, $i = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 5.9: Partial backorders: Order quantity - capital, $L = 1$, $i = 9$

A sensitivity test is performed for each example, Figures 5.10 and 5.11. The sequence of the top four parameters that have the most impact on start-up survival is the same as in the lost sales model. The backorder cost has little influence, compared to other parameters.

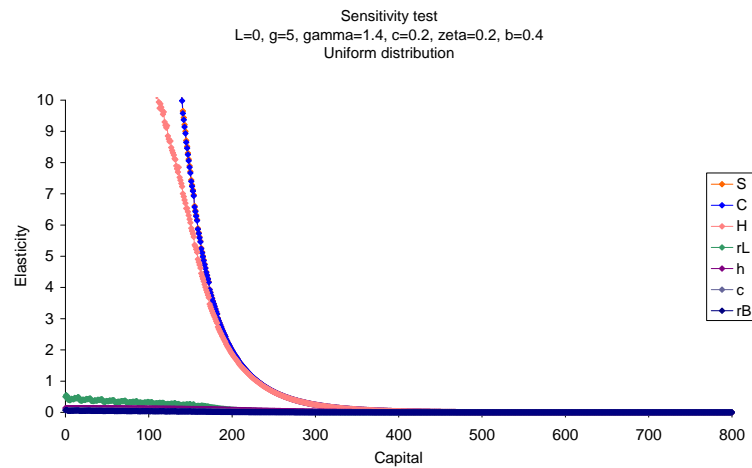
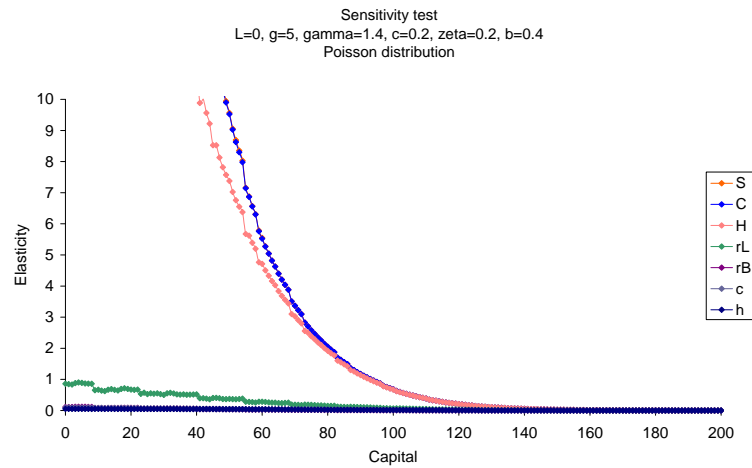
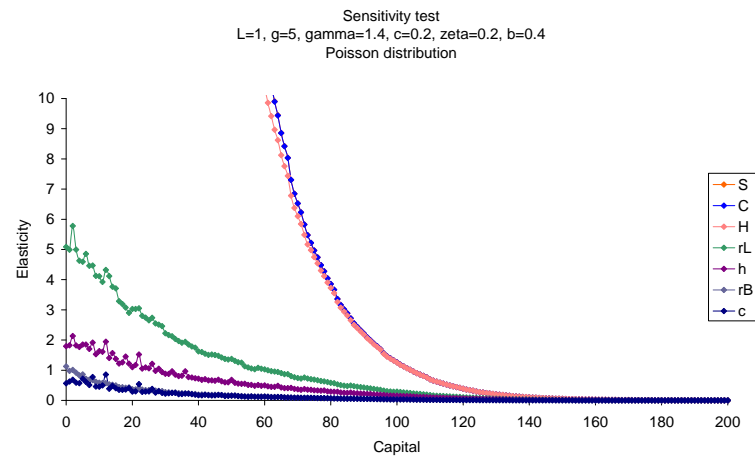
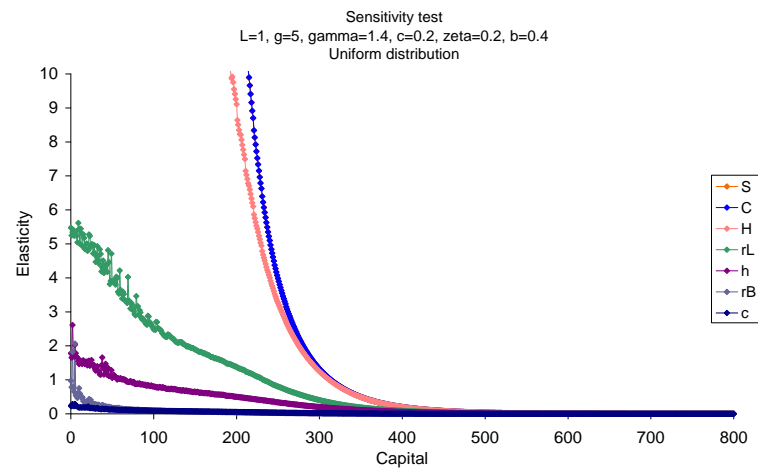


Figure 5.10: Sensitivity test, $b = 0.4, L = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 5.11: Sensitivity test, $b = 0.4, L = 1$

5.5.3 Hazard rate and survival

Hazard rate, which was first used by Barlow et al. [1963], is also known as “failure rate” in reliability theory. It measures the risk of occurrence of an event, such as equipment failure, death, disease, etc. In recent years, hazard rate is widely used in business and economics, in fields such as insurance and credit risk analysis. Hazard rate is used in this thesis to measure the risk of a start-up firm’s failure.

In a continuous distribution, given a random variable x , let f_t be the density function that failure happens in period t , and F_t be the distribution function, where $t \in [0, T]$ with T being the time limit of the process. The failure rate, r_t , is defined as the conditional probability density that failure happens at time t , given failure did not happen until time t . It is calculated as [Ross, 1997, pp. 239–240],

$$r_t = \frac{f_t}{1 - F_t}. \quad (5.5)$$

In a discrete distribution, given p_n is the probability that failure happens in period n , $n \in \{0, 1, \dots\}$, hazard rate is calculated as [Rolski et al., 1998, pp. 43-45],

$$r_n = \frac{p_n}{\sum_{j \geq n} p_j}. \quad (5.6)$$

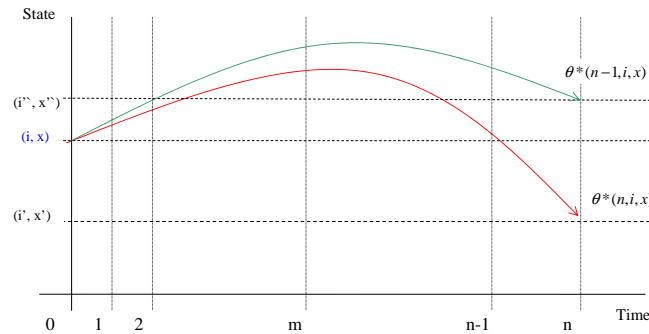
Hazard rates in discrete distributions are often monotonically increasing or decreasing with time [Rolski et al., 1998, pp. 43-45].

Let $\theta(n, i, x)$ be F1’s survival probability under the optimal infinite horizon stationary policy, given state (i, x) , when there are n periods to the end of the planning horizon. Alternatively, $\theta(n, i, x)$ can be considered as the survival probability over an n -period planning horizon, given F1 starts in state (i, x) and follows the optimal infinite horizon stationary survival policy. $\theta(n - 1, i, x) - \theta(n, i, x)$ is the probability that F1 fails in the n th period, given the firm starts in state (i, x) . This is illustrated in Figure 5.12. According to the definition of hazard rate in (5.6), the hazard rate of Firm F1 in period n , given it starts in state (i, x) , is such that,

$$r_n(i, x) = \frac{\theta(n - 1, i, x) - \theta(n, i, x)}{\sum_{j \geq n} \theta(j - 1, i, x) - \theta(j, i, x)} = \frac{\theta(n - 1, i, x) - \theta(n, i, x)}{\theta(n - 1, i, x)}, \quad (5.7)$$

where $i \in [-D, I]$ and $x \in [0, X]$. In this section, the hazard rate is calculated using equation (5.7). Hazard rate at time n , r_n , indicates how likely a start-up firm is to fail

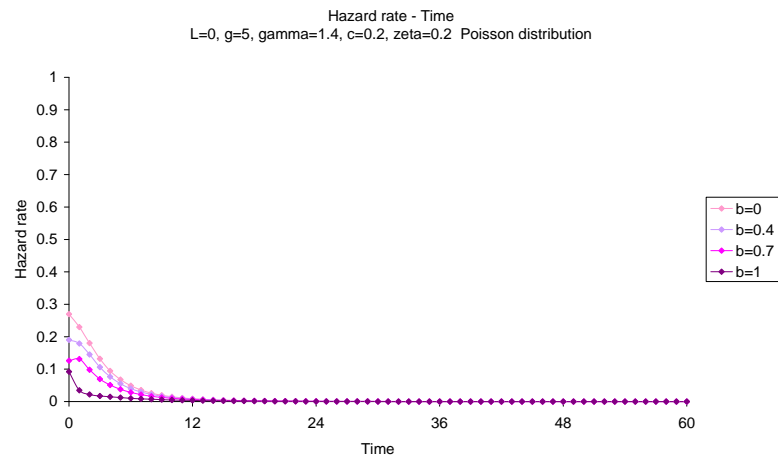
from time n onwards, given it starts with inventory i and capital x and survives until the end of period $n - 1$.



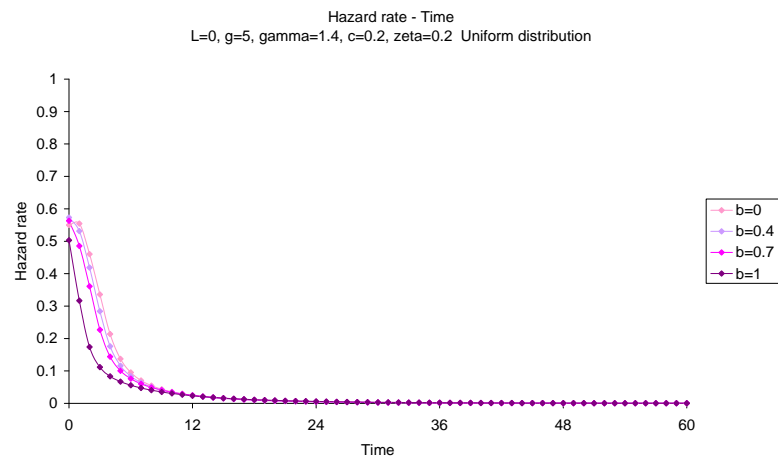
- Note: 1. (i, x) represents the initial state of firm F1 which is assumed to be known. (i', x') and (i'', x'') represent the state of firm F1 after $n - 1$ and n periods respectively and are uncertain. $i, i', i'' \in [-D, I]$ and $x, x', x'' \in [0, X]$.
 2. $\theta^*(n, i, x)$ represents the maximum probability firm F1 survives n periods of operations given it starts in state (i, x) .

Figure 5.12: Illustration: Hazard rate calculation in the survival models

Figures 5.13 and 5.14 present the hazard rate in each example. Overall, the hazard rate is the highest in the first year. Looking across the backorder rate examples, the firm has the highest risk of failure if it has only lost sales, and the hazard rate decreases with the backorder rate. Comparing between demand distributions, firms with Uniform distributed demand have a higher hazard rate. Going along the time line, some of the hazard rates display an inverted U-shape, which may be because the firm depends on the initial capital stock for survival in the first few months of establishment. The firm has the highest risk of failure as soon as the initial capital runs out. The failure rate decreases thereafter. The result in this section corresponds to the empirical research in literature, see for example [Mahmood, 2000] and [Audretsch et al., 1999]. There are a few cases where the hazard rate decreases from the start, which happens where the firm has either total lost sales or total backorders. In some cases this may be because it is very difficult to survive, i.e. with only lost sales, or Uniform distributed demand and the initial capital has little impact on firm survival. In other cases, the hazard rate maybe generally low, such as with total backorders and Poisson distributed demand.

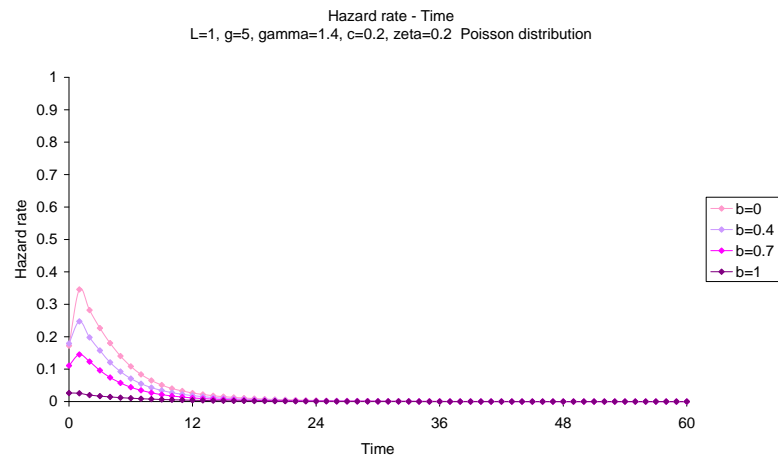


(a) Poisson distribution

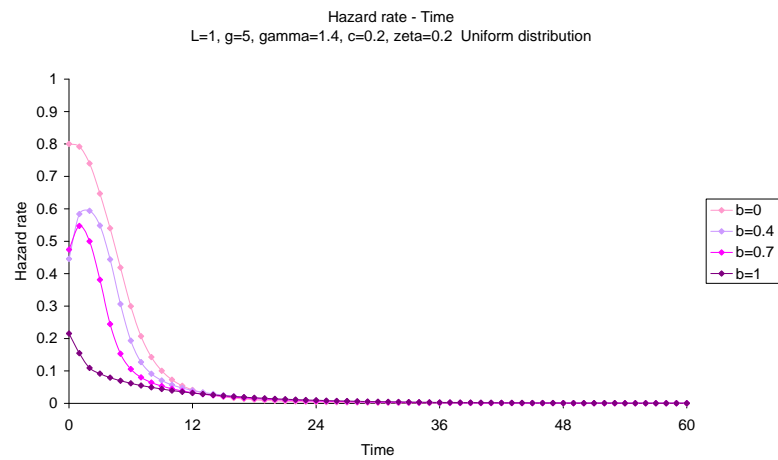


(b) Uniform distribution

Figure 5.13: Partial backorders: Hazard rate - time, $L = 0$



(a) Poisson distribution



(b) Uniform distribution

Figure 5.14: Partial backorders: Hazard rate - time, $L = 1$

5.5.4 Two extensions: Solvency check and order frequency

The models follow the idea of extensions in the lost sales model. Examples in this section consider Poisson distributed demand only, other parameters are as in the details given in Table 5.1.

Solvency check

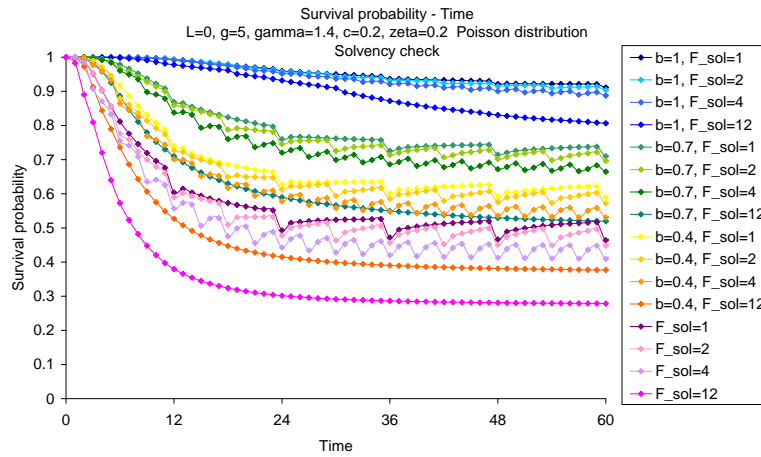
Three solvency frequencies, i.e. $F_{sol} = 4$, $F_{sol} = 2$ and $F_{sol} = 1$, are considered to compare with the initial monthly ($F_{sol} = 12$) solvency check (Figure 5.15). As expected, frequent solvency checks improve a firm's chance of survival. Looking through the four backorder rate cases, this effect diminishes with the backorder rate. Unlike in $L = 0$, the effect of less frequent solvency check is not always apparent over short planning horizons. Again the most significant observation is that the biggest change in survival probability occurs when switching from monthly to quarterly solvency checks. This suggests that a small relaxation in a lender's liquidation policy can have a significant effect on a firm's survival.

Order frequency

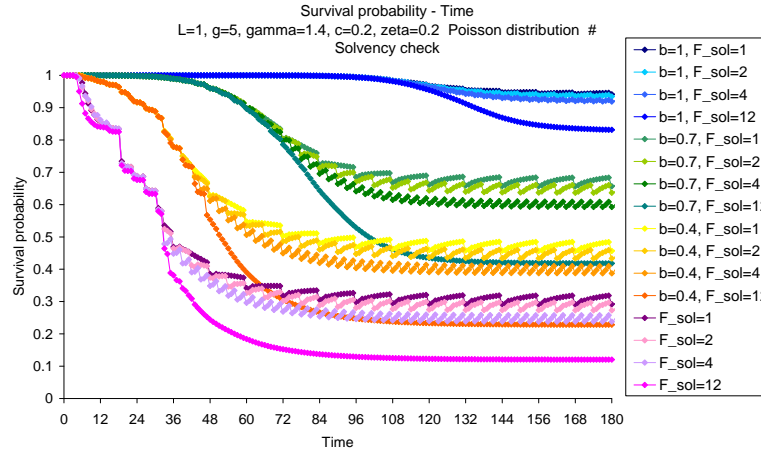
Two other order frequencies, i.e. half-monthly orders, $F_{ord} = 26$, and weekly orders, $F_{ord} = 52$, are compared with the original monthly order example where $F_{ord} = 12$, see Figure 5.16. Higher order frequency shifts the survival curve to the left towards lower capital levels. Figure 5.17 presents the capital stock, X , at which F1 has a survival probability greater or equal to 99.99%. In $L = 1$, the capital level is dramatically reduced if the firm is able to make more frequent orders. It is also worth noticing that the amount of capital stock required with a lower backorder rate can be below that level in a higher backorder rate case, if the firm makes more frequent orders, which implies more order opportunities has a positive impact on start-up survival.

5.6 Conclusion and discussions

This chapter examines a mixture of lost sales and backorders in manufacturing start-up firms. Each customer unsatisfied on arrival has a probability of accepting the backorder offer, otherwise the demand is lost. Experimental results confirm that backorders have a positive effect on start-up firms' survival. This effect has more influence in the cases



(a) $L = 0$



(b) $L = 1$

Figure 5.15: Partial backorders, solvency check: Survival probability - time

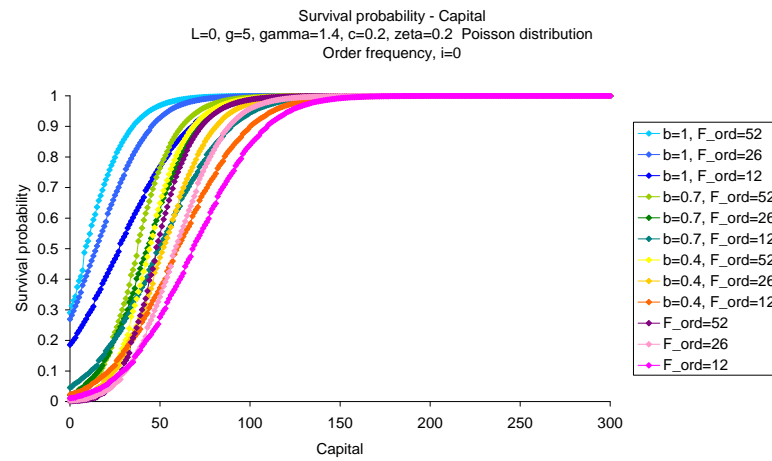
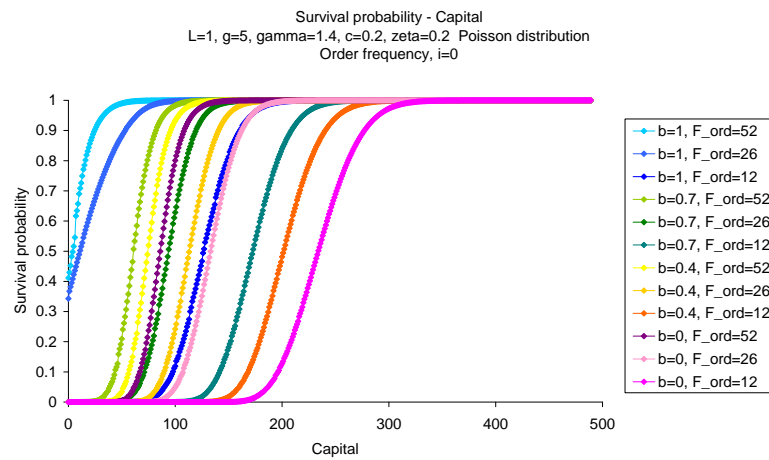
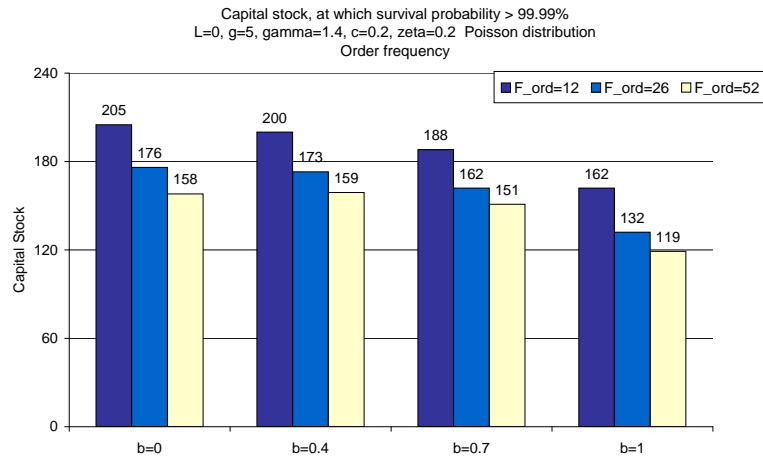
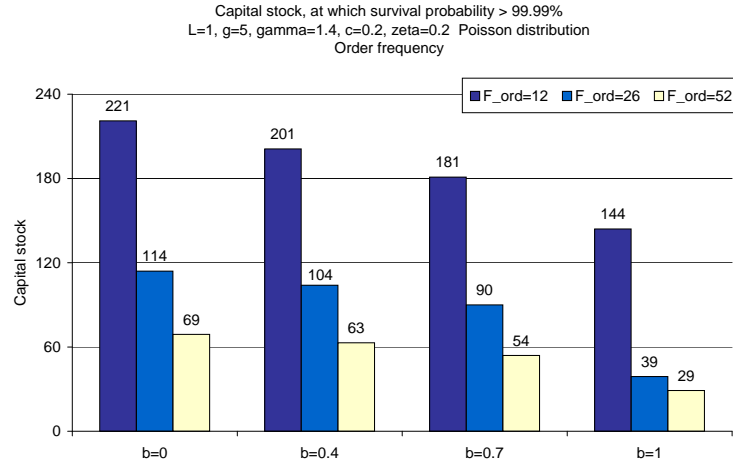
(a) $L = 0$ (b) $L = 1$

Figure 5.16: Partial backorders, order frequency: Survival probability - capital

(a) $L = 0$ (b) $L = 1$ Figure 5.17: Partial backorders, order frequency: Capital at $\geq 99.99\%$ of survival

where the firm is in a difficult operational environment, such as high variance of demand or positive lead time. The amount of capital and inventory needed to achieve a targeted survival probability decreases with the backorder rate. The model allows all of these effects to be quantified which is vital for investment decisions.

Hazard rate is introduced into the partial backorder model as to estimate a start-up's failure over time. In many cases, the hazard rate has an 'inverted U-shape', which is also found in empirical research. Start-up firms can rely on initial capital for survival in the first few periods of establishment. Hazard rate therefore increases as initial capital runs out. Revenue from sales and other activities are reserved for longer survival. In difficult operation condition such as total lost sales and sparse demand, the initial capital stock does not have much impact and the firm's hazard rate may decrease from the start. As an extension to the initial models, change of solvency check and order frequency are implemented as well. Either reducing solvency check frequency to a quarterly level or increasing order opportunity to twice a month can have a large positive impact on firms' survival probability. The model suggests that the benefits of further changes to the frequencies of solvency checks and order opportunities have little affect across a large range of problem parameters.

Both the survival curves and hazard rate, as well as other results, identify the positive effect that backorders have on start-up firms' survival. Therefore it is worth start-ups investing some capital on improving the backorder rate of its product. The firm will benefit from the investment as long as the amount is within a range which can be estimated using the difference of the capital stock requirement between the two cases with present and target backorder rates. The improvement can be achieved by investing in areas such as research and development (R&D), marketing, customer service, etc. According to experience, products that meet consumers' needs, bring new concepts, have original design and functions, are often found to be popular in the market. Meanwhile, the products/brands that are supported by successful advertising campaigns and those that have been in the market for a long time with high goodwill are more likely to have loyal customers. Last but not least, products with outstanding pre-sale and after-sale service which meets customers' needs, tend to be popular in the market. An investment in any of these areas could make the firm stand out from the crowd and be appealing to the mass market. As a result, when there are stockouts, customers will be more likely to wait for backorders.

When there are stockouts, either lost sales or backorders, customers' demand is not

met on arrival, which results in goodwill loss to the firm. Goodwill is a key intangible asset and is regarded as an earning power to a business. It has invisible and long-term effect on businesses. Apart from the loss in operations, goodwill itself decays over time due to people forgetting. In the next chapter, the models introduce goodwill into the inventory model, and consider advertising as an investment on goodwill.

Chapter 6

Inventory models with joint advertising decisions

6.1 Problem assumptions and description

Advertising is considered as an investment on goodwill which is an intangible asset and a key factor to a business. Advertising changes customers' perceptions on products and shifts a firm's demand outward [Mueller and Supina, 2002]. Almost every firm considers advertising activities such as promotion, branding, maintaining firm image, in the hope of bringing in more customers and so profits. Meanwhile, advertising plan is not to be made alone, since it is connected to other areas such as production and finance. Too much advertising may generate demands exceeding the firm's production capacity, which would lead to shortages and customer dissatisfaction, and damage a firm's goodwill. High advertising budget may also cause financial problems, especially to start-up firms who have limited capital.

The models in this chapter consider advertising expenditure as a joint decision with orders. Goodwill is treated as a signal of firm performance and product quality, and determines the average demand in the same period. Due to people forgetting, goodwill itself decays over time. Advertising improves a firm's goodwill, following the well-known Nerlove-Arrow (N-A) Model [Nerlove and Arrow, 1962]. Shortages on the other hand, cause customer dissatisfaction and therefore, cost the firm goodwill per unit of lost sales or backorders, which is consistent with the goodwill loss models in Chapter 6. Overall, a firm's goodwill changes as a consequence of the following three factors, depreciation, advertising investment and shortages.

The scenario of the inventory-advertising models are as follows. At the beginning of each review period, a start-up firm observes the inventory level, i , goodwill level, w , and capital available, x . It expects to have λ customers, where λ is estimated according to the goodwill level observed such that $\lambda = \lfloor \frac{w}{\eta} \rfloor$. $\eta > 0$ is a constant which converts units of goodwill to units of average demand. The firm first orders k items, in addition to the $(-i)^+$ items for backorders. To ensure the model is computationally tractable, it is assumed that, when making the order decisions, the average demand remains constant at the level estimated at the beginning of each period for the remainder of the planning horizon. The implications of this assumption will be investigated in the subsequent analysis. After the order decisions, the firm decides the advertising expenditure a which in turn increases the goodwill in the following period by αa . α is the advertising layout rate on goodwill. The advertising expenditure can be up to the maximum budget, A , i.e. $a \in [0, A]$. After the decisions, d customers arrive with probability $p(d, \lambda)$. The demand probability follows a Poisson distribution such that $p(\lambda) = Poisson(\lambda)$, given $d \in [0, D]$. Backorders are offered when stockouts appear and are up to $B(i, k, d)$ items, where

$$B(i, k, d) = \begin{cases} \max\{d - i^+ - k, 0\}, & \text{for } L = 0, \\ \max\{d - i^+, 0\}, & \text{for } L = 1. \end{cases}$$

Notation B is used as a short for $B(i, k, d)$ for convenience. Each unmet customer accepts the backorder offer with probability p_b . Overall, b out of B unsatisfied customers will take backorders with probability $\binom{B}{b} p_b^b (1 - p_b)^{B-b}$. Lost sales and backorders cost the firm goodwill in unit of shortages at rate β_L and β_B per item, respectively. The goodwill costs are deducted from the total goodwill to forecast the demand in the following period. Each backorder also costs r_E for emergent production which is deducted instantly from the capital. By the idea of saturation level (Vidale and Wolfe [1957]), this model considers an upper bound on goodwill W , which limits the firm's goodwill level in the short-term. Financial transactions are made by the end of each period before the solvency check. A start-up firm is said to fail if it has negative capital at the solvency check. The process is shown in detail in Figure 6.1. In comparison to the partial backorder model in Chapter 5, this model has an advertising decision after orders. The average demand in a period depends on the goodwill at the start of the period and is independent of changes in goodwill during the period. Shortages and advertising affect goodwill over time.

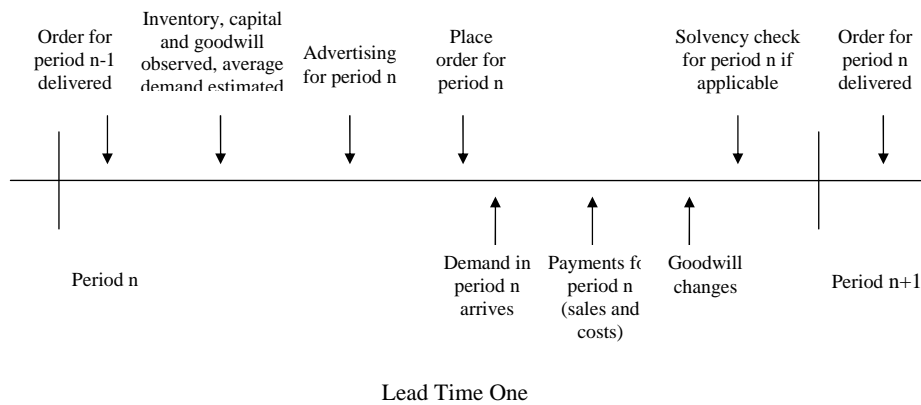
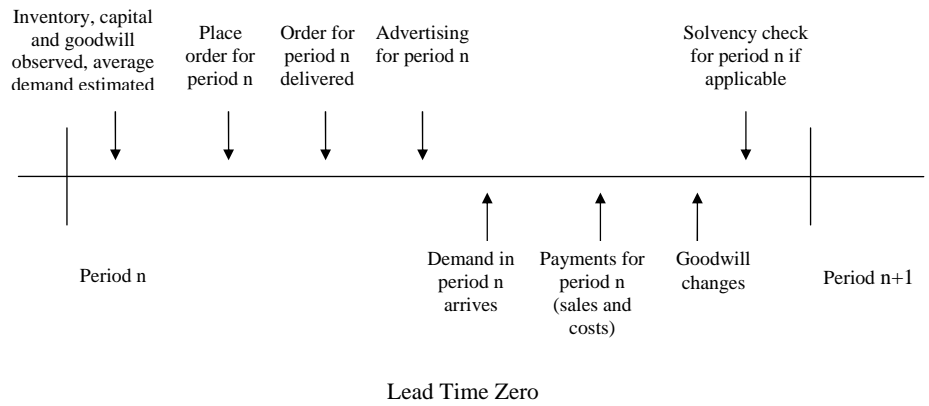


Figure 6.1: Operations with joint inventory-advertising decisions

The goodwill level, w , is a new state variable, the value of which can be evaluated by accounting methods. Goodwill decays at a fixed rate β where $0 \leq \beta \leq 1$, a goodwill of value w at present is worth βw in the next period. Advertising improves the goodwill at rate α . So an amount a spent on advertising increases the goodwill by αa in the next period. Meanwhile, shortages decrease the firm's goodwill level. Lost sales reduce goodwill at unit rate β_L and backorders reduce goodwill at unit rate β_B . The firm's goodwill changes following $\mathcal{W}(w, a, b, B)$ such that,

$$\mathcal{W}(w, a, b, B) = \min\{\lfloor \beta w + \alpha a - \beta_B b - \beta_L(B - b) \rfloor, W\}. \quad (6.1)$$

Since the model is discrete, calculation is rounded to the nearest integer not exceeding its value, by $\lfloor \cdot \rfloor$ operator where $\lfloor y \rfloor$ denotes the nearest integer not exceeding y . $\beta_B b + \beta_L(B - b)$ are the shortage costs on goodwill, where b is the number of backorders and $B - b$ is the number of lost sales. Equation 6.1 introduces the three factors that affect a firm's goodwill, i.e. depreciation, advertising investment and shortages. The goodwill is bounded by saturation level, such that,

$$\mathcal{W}(w, a, b, B) = W, \text{ if } \lfloor \beta w + \alpha a - \beta_B b - \beta_L(B - b) \rfloor > W.$$

The firm's instant earning is calculated following $\mathcal{G}(i, k, a, d, b)$ such that,

$$\mathcal{G}(i, k, a, d, b) = \begin{cases} S(\min\{i^+ + k, d\} + b) - a - H - c\delta(k + (-i)^+) \\ -C(k + (-i)^+) - hi^+ - r_E b, & L = 0, \\ S(\min\{i^+, d\} + b) - a - H - c\delta(k + (-i)^+) \\ -C(k + (-i)^+) - hi^+ - r_E b, & L = 1. \end{cases}$$

In $\mathcal{G}(i, k, a, d, b)$, for $L = 0$, $S(\min\{i^+ + k, d\} + b)$ is the revenue from selling $\min\{i^+ + k, d\} + b$ products, a is the advertising expenditure, H is the overhead cost, $c\delta(k + (-i)^+)$ is the one-off ordering cost, $C(k + (-i)^+)$ is the purchasing cost, hi^+ is the holding cost, and $r_E b$ is the instant emergency cost for backorders. In $L = 1$, the calculation follows the similar pattern, only that the number of items available to sell is $\min\{i^+, d\} + b$. Inventory level changes following $\mathcal{F}(i, k, d, b)$ as in Chapter 5 such that,

$$\mathcal{F}(i, k, d, b) = \begin{cases} (i^+ + k - d)^+ - b, & \text{for } L = 0, \\ (i^+ - d)^+ + k - b, & \text{for } L = 1, \end{cases}$$

6.2 Maximizing survival probability

In this inventory-advertising model, a start-up firm makes decisions on order quantity, k , and advertising expenditure, a , at the beginning of each review period, to maximize its chance of survival. The uncertain variables, such as demand, d , and backorders, b , are modelled in the same way as in the partial backorder model in Chapter 5.

Define the maximum survival probability to be $q(n, i, w, x)$, given the inventory level, i , goodwill level, w , and capital available, x , when there are n periods to the end of planning horizon. It follows that $q(n, i, w, x) \in [0, 1]$, for $i \in [-D, I]$, $w \in [0, W]$, and $x \geq 0$. The start-up firm is said to fail if it has negative capital at the solvency check, i.e. $q(n, i, w, x) = 0$ for $x < 0$. This survival probability maximizing model is formulated as an expected total reward MDP model as follows:

1. **Decision epochs and periods.** This model has discrete decision epochs. Let $T = \{0, 1, 2, \dots, N\}$ denote the set of decision epochs. N is the planning horizon and can be either a finite positive integer ($N < \infty$) in a finite planning problem, or an infinite positive integer ($N \rightarrow \infty$) in an infinite horizon problem.
2. **State and action set.** At each decision epoch, the firm is in some state $s = (i, w, x)$, where $s \in S = \{(i, w, x) : i \in \{-D, -D + 1, \dots, -1, 0, 1, \dots, I\}, w \in \{0, 1, \dots, W\}, x \in \{0, 1, 2, \dots\}\}$. Inventory level is within the range $[-D, I]$, where D is the maximum demand and is thus the maximum number of backorders, and I is the inventory capacity. Goodwill is non-negative and upper bounded and, capital is non-negative. An action $(k, a) \in K_i = \{(k, a); k \in \{0, 1, \dots, I - i^+\}, a \in \{0, 1, \dots, A\}\}$ is chosen observing state (i, w, x) . Orders are made in condition that the inventory is within the storage capacity, I , and all outstanding backorders can be satisfied. Advertising expenditure is within the maximum budget, A . No action is taken at the end of the planning horizon (i.e. decision epoch 0). The state space is infinite countable and the action space is finite.
3. **Reward and transition probabilities.** Since the objective is to maximize the probability of the firm surviving the planning horizon, the only reward is generated at the end of the planning horizon. Therefore the reward during each period is zero. Given the demand d , backorders b and action (k, a) , the system makes a transition from state (i, w, x) to state $(\mathcal{F}(i, k, d, b), \mathcal{W}(w, a, b, B), x + \mathcal{G}(i, k, a, d, b))$

at the next decision epoch, depending on the probabilities of demand, $p(d, \lambda)$, and backorders, $\binom{B}{b} p_b^b (1 - p_b)^{B-b}$.

4. Decision rules. The state space is infinite countable for each $s \in S$, the action space is finite for each K_i and so $K \equiv \cup K_i$ is finite. According to Puterman [1994, pp. 277–284], this is a positive bounded total reward model. By Theorem 7.1.9 in Puterman [1994, pp. 284], there exists a Markovian deterministic (MD) decision rule which brings an optimal reward.

To ensure that the model is computationally tractable, the decision making process is divided into two steps. At the first step, an order quantity is determined for each combination of inventory level, i , goodwill level, w , and capital available, x . This is similar to the partial backorder model in Chapter 5, only that the average demand is determined by the goodwill level observed. Let $\theta(n, i, w, x)$ be F1's survival probability under the stationary order policy, given (i, w, x) . This is modelled as,

$$\theta(n, i, w, x) = \max_{0 \leq k \leq I-i+} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. \theta(n-1, \mathcal{F}(i, k, d, b), w, x + \mathcal{G}_B(i, k, d, b)) \right\}, \\ \text{for all } i \in [-D, I], w \in [0, W], \text{ and } x \geq 0, \quad (6.2)$$

with terminal values

$$\theta(0, i, w, x) = \begin{cases} 1, & \text{for all } i \in [-D, I], w \in [0, W], x \geq 0, \\ 0, & \text{for all } i \in [-D, I], w \in [0, W], x < 0. \end{cases}$$

and boundary condition

$$\theta(n, i, w, x) = 0, \text{ for all } i \in [-D, I], w \in [0, W], x < 0, n > 0.$$

Note that the method of choosing an order quantity assumes that the goodwill level does not change (note that w appears on both sides of equation (6.2)) and therefore is necessarily optimal. However, this possibly reflects the situation in many firms where inventory decisions are made based on demand forecast. The instant earnings $\mathcal{G}_B(i, k, d, b)$ are the same as in the backorder models, say (5.1), with $r_L = \beta_L$, $r_B = \beta_B + r_E$. The average demand $\lambda = \lfloor \frac{w}{\eta} \rfloor$. Let $k^*(i, w, x)$ denote the stationary optimal order policy derived from (6.2), such that $k^*(i, w, x) = \arg\{\theta(N, i, w, x)\}$, where N is

the time when iteration stops. At this step, the firm makes orders following the same pattern as in the partial backorder model. The earnings are not calculated until all the financial transactions are processed before the solvency check.

Next, decisions are made on advertising expenditure, given the inventory order was made following $k^*(i, w, x)$. This is modelled as,

$$q(n, i, w, x) = \max_{0 \leq a \leq A} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. q(n-1, \mathcal{F}(i, k^*(i, w, x), d, b), \mathcal{W}(w, a, b, B), x + \mathcal{G}(i, k^*(i, w, x), a, d, b)) \right\},$$

for $i \in [-D, I]$, $w \in [0, W]$, and $x \geq 0$, (6.3)

with terminal values

$$q(0, i, w, x) = \begin{cases} 1, & \text{for all } i \in [-D, I], w \in [0, W], x \geq 0, \\ 0, & \text{for all } i \in [-D, I], w \in [0, W], x < 0. \end{cases}$$

and boundary conditions

$$q(n, i, w, x) = \begin{cases} q(n, i, W, x), & \text{for all } i \in [-D, I], w > W, x \geq 0, n > 0, \\ 0, & \text{for all } i \in [-D, I], w \in [0, W], x < 0, n > 0. \end{cases}$$

The capital by the end of each period is calculated following $x + G(i, k, a, d, b)$, based on which the solvency check is made.

Theorem 6.2.1. *The maximum finite horizon survival probability of a start-up firm under the assumption of the inventory-advertising model, $q(n, i, w, x)$ from equation (6.3), satisfies the following properties:*

- (i) $q(n, i, w, x)$ is non-increasing in n ;
- (ii) $q(n, i, w, x)$ is non-decreasing in x ;
- (iii) $q(i, w, x) = \lim_{n \rightarrow \infty} q(n, i, w, x)$ exists and is non-decreasing in x .

Proof. Note that the inventory-advertising model has the same structure as the general model of Chapter 3. The results are then an immediate consequence of Lemma 3.4.1, Lemma 3.4.2 and Theorem 3.4.3. □

6.3 Maximizing average profit

This model examines the firms who maximize the average profit over an infinite horizon. At the beginning of each period, a firm makes order decisions, k , and then advertising decisions, a , observing the inventory level, i , and goodwill level, w . All the other parameters, such as inventory capacity, advertising budget, demand distribution, and backorder probability, are the same as in the survival probability maximizing model. The profit maximizing model is an expected average reward MDP model. Let g be the average profit and $v(i, w)$ be the bias term given the state (i, w) . The profit maximizing problem is as follows:

1. **Decision epochs and period.** Decisions are made at the beginning of each review period. The expected average reward model is an infinite horizon model and decision epoch set is $T = \{0, 1, 2, \dots\}$.
2. **State and action set.** The state is $s = (i, w)$, where $s \in S = \{(i, w); i \in \{-D, -D+1, \dots, -1, 0, 1, \dots, I\}, w \in \{0, 1, \dots, W\}\}$. An action $(k, a) \in K_i = \{(k, a); k \in \{0, 1, \dots, I - i^+\}, a \in \{0, 1, \dots, A\}\}$ is chosen given the state (i, w) . This is a finite state and finite action model.
3. **Reward and transition probabilities.** The reward in each period is the firm's instant earning $\mathcal{G}(i, k, a, d, b)$ as a function of inventory level, i , order decision, k , advertising expenditure, a , demand, d , and backorder quantity, b . In each period, the firm makes a transition from state (i, w) to state $(\mathcal{F}(i, k, d, b), \mathcal{W}(w, a, b, B))$ depending on the probabilities of demand $p(d, \lambda)$, and backorders, $\binom{B}{b} p_b^b (1 - p_b)^{B-b}$.
4. **Decision rules.** The state, (i, w) , order quantity, k , demand, d , and backorders, b , are bounded, thus the reward is bounded, i.e. $|\mathcal{G}(i, w, k, d, b)| < \infty$. In most practical cases, the weak unichain assumption holds [Tijms, 1994, pp. 199], where there exists a stationary Markovian deterministic policy (MD) which brings a constant average reward.

Similar to the survival probability maximizing model, the firm makes sequential decisions on order quantity and advertising expenditure. Order decisions are made in the similar way as in the partial backorder model in Chapter 5 such that, for a fixed

goodwill level, w , the order quantity $k^*(i, w)$ is chosen to satisfy:

$$g(w) + v(i, w) = \max_{0 \leq k \leq I-i^+} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. (\mathcal{G}_B(i, k, d, b) + v(\mathcal{F}(i, k, d, b), w)) \right\}, \text{ for } i \in [-D, I], w \in [0, W]. \quad (6.4)$$

As before, the method choosing the order quantity assumes that the goodwill level does not change. $p(d, \lambda)$ is the demand distribution, with mean $\lambda = \lfloor \frac{w}{\eta} \rfloor$. $g(w)$ is the average profit given the goodwill level is w .

Next, decisions are made on advertising expenditure, which is modelled as follows,

$$g + v(i, w) = \max_{0 \leq a \leq A} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. (\mathcal{G}(i, k^*(i, w), a, d, b) + v(\mathcal{F}(i, k^*(i, w), d, b), \mathcal{W}(w, a, b, B))) \right\}, \\ \text{for } i \in [-D, I], w \in [0, W]. \quad (6.5)$$

The order $k^*(i, w)$ is stationary, and the inventory level, backorders, demand, and the advertising expenditure are bounded. Hence, $\mathcal{G}(i, k^*(i, w), d, b, a)$ is also bounded, i.e. $|\mathcal{G}(i, k^*(i, w), a, d, b)| < \infty$. By Theorem 8.4.5 Puterman [1994, pp. 361], there exists a stationary optimal advertising policy $a^*(i, w)$ with a unique stationary distribution of the state of the process $\pi(i, w)$.

$$g = \sum_{i=-D}^I \sum_{w=0}^W \pi(i, w) \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \mathcal{G}(i, w, k^*(i, w), a^*(i, w), d, b), \\ \text{for } i \in [-D, I], w \in [0, W]. \quad (6.6)$$

6.4 Numerical investigation

For computational reasons, an upper bound on capital, X , is introduced into the model so that $x \in [0, X]$. This bound is a positive integer such that the survival probability is defined to be 1 if the capital level is beyond X . The value X is pre-determined by calculation test in finite horizon problems where iteration may stop before the survival probability converges to a constant. In infinite horizon problems, X is initialized at a large positive integer and is increased by one unit after each iteration till the survival probability converges to a constant. The additional capital boundary condition on the

survival probability model is that, for the inventory order decisions in equation (6.2),

$$\theta(n, i, w, x) = 1, \text{ if } x > X, \text{ for } i \in [-D, I], w \in [0, W], n > 0. \quad (6.7)$$

and for advertising decisions in equation (6.3),

$$q(n, i, w, x) = 1, \text{ if } x > X, \text{ for } i \in [-D, I], w \in [0, W], n > 0. \quad (6.8)$$

The iteration of start-up firms in finite and infinite horizon is illustrated in Figures 6.2 and 6.3.

6.4.1 Simultaneous decision model

This chapter models firms making sequential decisions on order quantity and advertising expenditure. Another possibility is that they make simultaneous decisions on orders and advertising. The probability maximizing model with simultaneous decisions is as follows,

$$q_{SM}(n, i, w, x) = \max_{0 \leq k \leq I-i^+, 0 \leq a \leq A} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. q_{SM}(n-1, \mathcal{F}(i, k, d, b), \mathcal{W}(w, a, b, B), x + \mathcal{G}(i, k, a, d, b)) \right\}, \\ \text{for } i \in [-D, I], w \in [0, W], \text{ and } x \geq 0, \quad (6.9)$$

with terminal values

$$q_{SM}(0, i, w, x) = \begin{cases} 1, & \text{for all } i \in [-D, I], w \in [0, W], x \geq 0, \\ 0, & \text{for all } i \in [-D, I], w \in [0, W], x < 0, \end{cases}$$

and boundary conditions

$$q_{SM}(n, i, w, x) = q(n, i, W, x), \text{ for all } i \in [-D, I], x \in [0, X], w > W, n > 0, \text{ and}$$

$$q_{SM}(n, i, w, x) = \begin{cases} 1, & \text{for all } i \in [-D, I], w \in [0, W], x > X, n > 0, \\ 0, & \text{for all } i \in [-D, I], w \in [0, W], x < 0, n > 0. \end{cases}$$

Theorem 6.4.1. *The maximum finite horizon survival probability of a start-up firm under the assumption of the simultaneous advertising decision model, $q_{SM}(n, i, w, x)$ from equation 6.4.1, satisfies the following properties:*

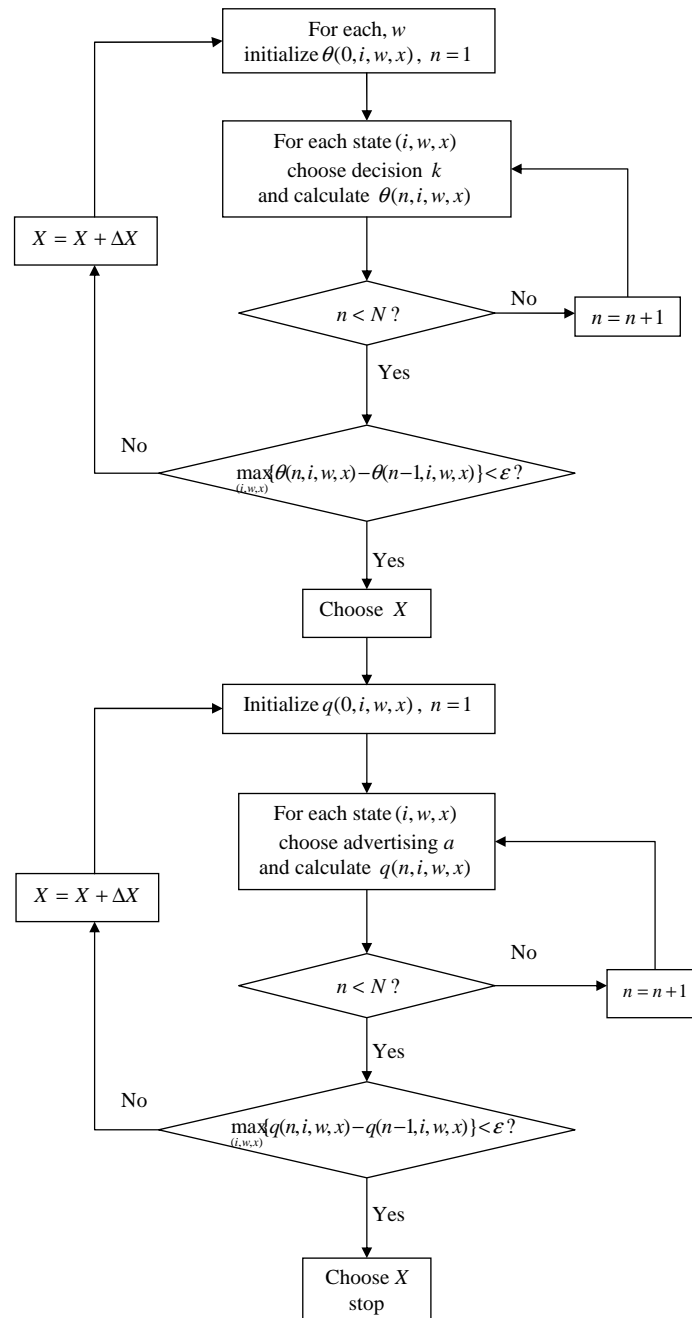


Figure 6.2: Algorithm: Value iteration in finite horizon, joint decision survival probability maximizing models

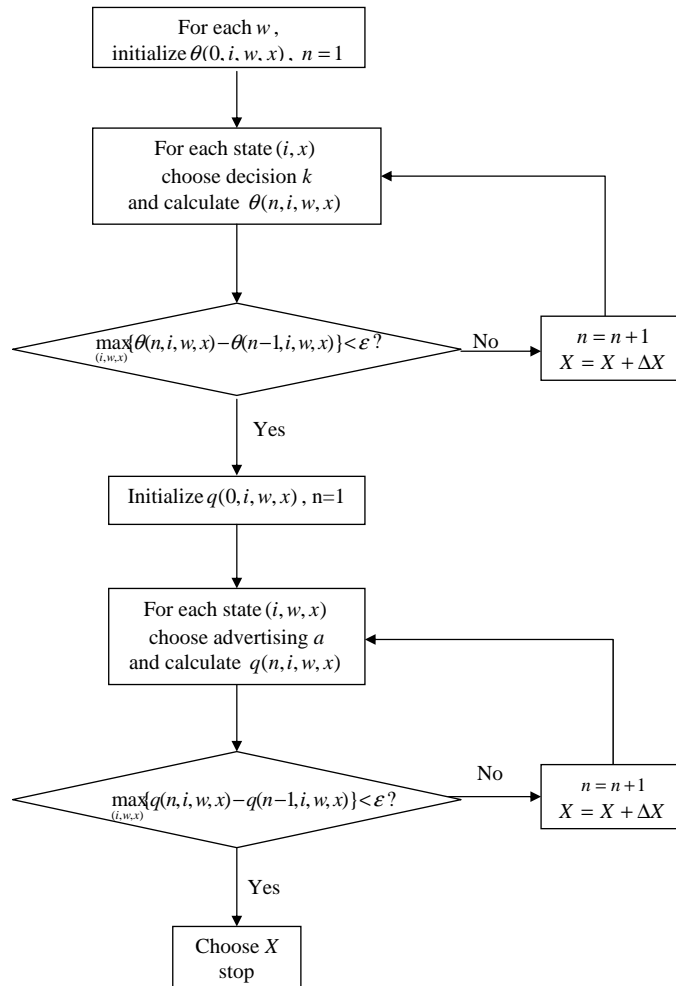


Figure 6.3: Algorithm: Value iteration in infinite horizon, joint decision survival probability maximizing models

- (i) $q_{SM}(n, i, w, x)$ is non-increasing in n ;
- (ii) $q_{SM}(n, i, w, x)$ is non-decreasing in x ;
- (iii) $q_{SM}(i, w, x) = \lim_{n \rightarrow \infty} q_{SM}(n, i, w, x)$ exists and is non-decreasing in x .

Proof. Note that the simultaneous inventory-advertising model has the same structure as the general model of Chapter 3. The results are then an immediate consequence of Lemma 3.4.1, Lemma 3.4.2 and Theorem 3.4.3. \square

The profit maximizing model with simultaneous decisions is as follows,

$$g + v(i, w) = \max_{0 \leq k \leq I-i^+, 0 \leq a \leq A} \left\{ \sum_{d=0}^D p(d, \lambda) \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \right. \\ \left. (\mathcal{G}(i, k, a, d, b) + v(\mathcal{F}(i, k, d, b), \mathcal{W}(w, a, b, B))) \right\}, \\ \text{for } i \in [-D, I], w \in [0, W]. \quad (6.10)$$

Since $\mathcal{G}(i, k, a, d, b)$ is bounded, i.e. $|\mathcal{G}(i, k, a, d, b)| < \infty$, there exists a stationary model which is optimal with a unique stationary distribution [Puterman, 1994, pp. 353-354].

$$g = \sum_{i=-D}^I \sum_{w=0}^W \pi_g(i, w) \sum_{d=0}^D p(d, \lambda) \\ \sum_{b=0}^B \binom{B}{b} p_b^b (1 - p_b)^{B-b} \mathcal{G}(i, w, k_g^*(i, w), a_g(i, w), d, b), \\ \text{for } i \in [-D, I], w \in [0, W], \quad (6.11)$$

where $k_g^*(i, w)$ and $a_g(i, w)$ are the decisions with probability $\pi_g(i, w)$.

The simultaneous decision models are not the main interest of this thesis due to the long computation time. Nevertheless, they are implemented to allow comparison with the sequential decision models. The later results show the efficiency of the sequential decision models adopted in this thesis.

6.4.2 Presentation of parameters

This research was initialized in part by the founder of a start-up manufacturing firm who observed that traditional inventory models would result in excessive inventory levels and ultimately in the failure of the firm due to the resulting investment in inventory. In the results of Chapters 4 and 5, holding cost does not have a major affect on the survival of the firm. It is possible that due to increased cost of capital, the appropriate

holding cost for a start-up firm is much higher than in traditional inventory models. Indeed, if the firm is limited by a tight capital constraint, it could be argued that the holding cost is even infinite. For this reason, a much higher holding cost is used in the examples considered in this chapter.

The models in this chapter introduce the concept of goodwill and the effect that advertising has on increasing firms' goodwill. The numerical illustration in this section aims to look into the question how the goodwill can be related to start-up firm survival and furthermore, what advertising strategy can be applied to start-up firms. Since there is no standard on advertising spending in start-up firms, the experiments in this chapter consider all possible spending, in the hope of presenting a big picture and testing the workability of the proposed joint decision model.

This joint advertising inventory model introduces goodwill, w , as a new state variable, together with the capital, x , and inventory level, i . It considers goodwill cost of lost sales, β_L , and backorders, β_B , in account of the effect that shortages have on firms' goodwill, and the production cost of backorders, r_E , on each item of backorders. Table 6.1 lists the parameters used in this section. Backorder rate $p(b) = 0.7$ is used in all the examples. The annual holding cost is larger than 30% of the purchasing cost. Due to the financial limitation, a larger holding cost/purchasing cost ratio is reasonable. Due to the capital constraint in start-up firms, the opportunity cost of holding stock may be very high for start-up firms. From the sensitivity test in earlier examples, holding cost does not have a large influence on start-up survival. This is contrary to the experience of founders of start-up firms. So larger holding costs are considered in this chapter.

Table 6.1: Examples in the inventory-advertising models, $L = 0$ and $L = 1$

	S	H	h	C	c	β_L	β_B	r_E	i_0	w_0	x_0
$L = 0$	9	8	4	5	5	3	1	1	0	20	30
$L = 1$	12	8	2	4	5	4	2	1	0	20	50

Other parameters include demand which is Poisson distributed, i.e. $\text{Poisson}(\lambda)$, where λ is estimated from the goodwill level at the beginning of each review period using $\lambda = w/\eta$. The maximum advertising budget, $A = 30$, is roughly equal to the gross profit each period in $L = 0$, and is about half of the gross profit in $L = 1$. The upper goodwill level is set at 30. Three values are considered for the goodwill decay rate, $\beta = 0.4, 0.7$ and 1, and four values for the advertising outlay rate on goodwill is $\alpha = 0.4, 0.5, 0.7$ and 1.

6.4.3 Average profit maximizing and advertising policy

This section presents experiments on profit maximizing models, comparing the effect of sequential and simultaneous decisions on firms' policies and profit.

Sequential decisions - average profit and advertising policy

F2's average profit is non-decreasing with goodwill decay rate, β , (Table 6.2), and is non-decreasing with advertising layout on goodwill, α , (Table 6.3). These trends indicate that it is more profitable if the goodwill has a long-term effect and if the advertising campaign is more effective on improving the goodwill. The goodwill decay rate and advertising layout rate could depend on various factors. For instance, whether the firm is in a consumable or durable product market, whether or not the market is competitive, whether the advertising is for short-term promotion or long-term branding, etc. From experience, goodwill has a long-term effect if the product is valuable to customers, the market is competitive where substitutes can be found with little extra effort, and advertising is conducted to build up goodwill.

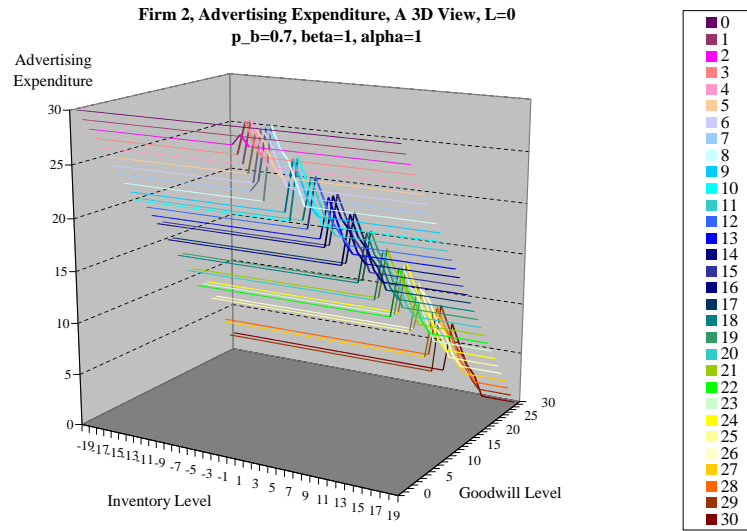
Table 6.2: F2: Sequential decisions, average profit with different β , $\alpha = 1$

Parameters	$\beta = 0.4$	$\beta = 0.7$	$\beta = 1$
$L = 0$	13.5937	22.0499	30.7891
$L = 1$	21.6993	29.3906	37.5183

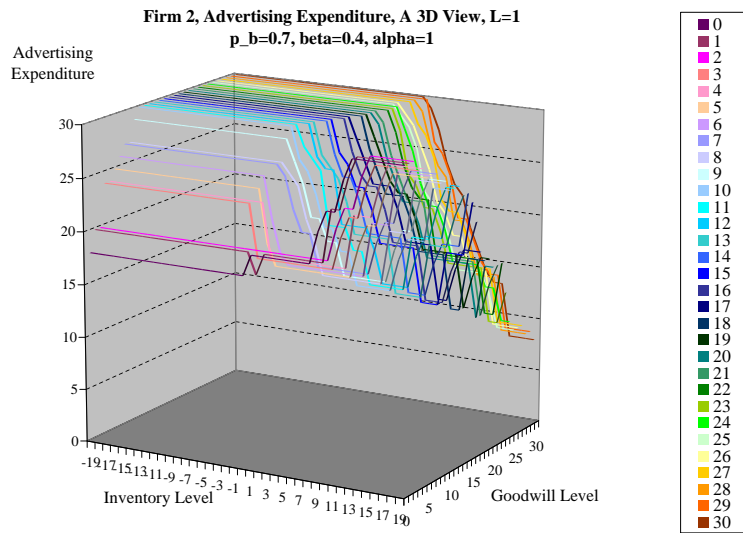
Table 6.3: F2: Sequential decisions, average profit with different α , $\beta = 0.7$

Parameters	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.7$
$L = 0$	7.9780	13.8355	19.9079
$L = 1$	14.2265	26.0319	37.5697

Before discussing F2's order policy, define $\beta w + \alpha a$ to be the advertising-controlled goodwill. Note that in equation (6.1), where $\mathcal{W}(w, a, b, B) = \lfloor \beta w + \alpha a - \beta_B b - \beta_L(B - b) \rfloor$, only $\beta w + \alpha a$ is directly changed by the firm's advertising decisions. $\beta_B b + \beta_L(B - b)$ is the cost of shortages on goodwill and is directly connected to inventory. $\beta w + \alpha a$ will be referred to as "future goodwill level" for convenience and will be used in later discussions.



(a) L=0



(b) L=1

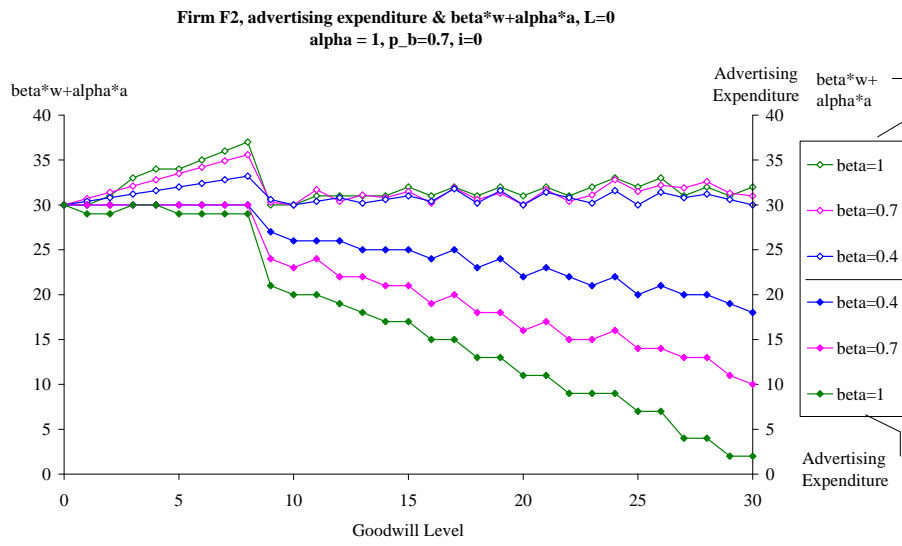
Figure 6.4: F2: Advertising with sequential inventory-advertising decisions

F2 plans the advertising expenditure based on both the goodwill and inventory level. In $L = 0$, Figure 6.4 F2 uses an order-up-to-level inventory policy, in which case the inventory level after ordering remains a constant until the firm orders nothing when the inventory level is high. In consideration of the stock, F2 spends a constant amount on advertising to keep the goodwill at the same level until it has a relatively high inventory level and stops making orders. At this point the firm increases the advertising expenditure to offset the goodwill loss from shortages until inventory reaches a threshold. The advertising expenditure then drops again as inventory increases. The firm keeps the advertising spending at the same level for higher inventory levels. Therefore, F2 plans advertising expenditure in account of both the stock and demand. It is used to keep the goodwill and thus the demand, at a stable level whereas the inventory level is either at the order-up-to level or higher. The spending is increased to overcome the shortages when the inventory is not enough to meet the demand.

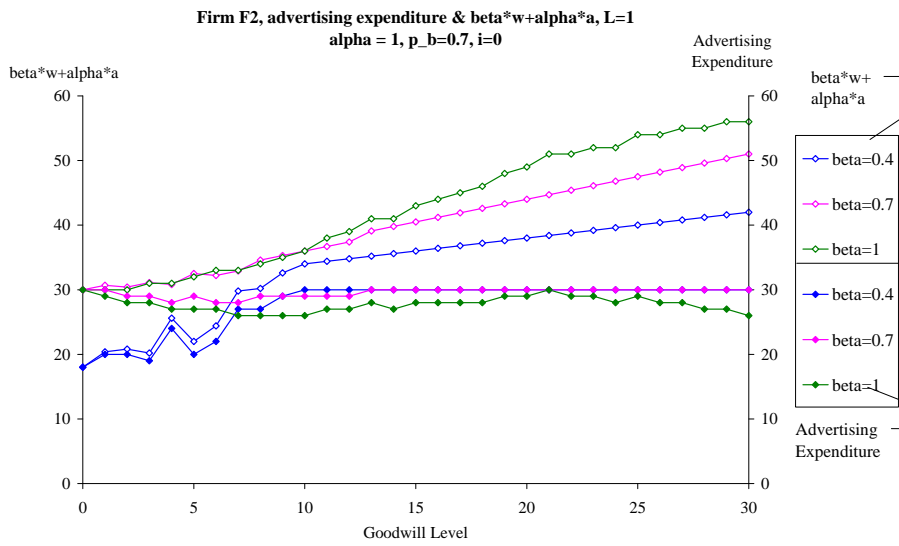
This advertising policy also applies to $L = 1$, in which case it is more complex. The advertising policy is flexible with the inventory and goodwill level, and it has a ‘twist’ along the inventory axis. When there are low inventory and high goodwill levels, F2 spends on advertising to offset the inevitable high loss of goodwill due to low customer service in the period. With high inventory in stock, F2 uses advertising to balance the inventory and goodwill level in the next period: it spends a comparatively high amount on advertising if the goodwill is low to generate more demand, and a low amount on advertising otherwise so that it does not generate demand beyond the supply capability.

Figure 6.5 takes a special inventory case $i = 0$ for further discussion. In $L = 0$, F2 spends at the maximum budget ($a = A = 30$) when the goodwill level is low, say $w \leq 8$, expecting to increase the future goodwill. When F2 has a comparatively moderate goodwill level, say $w \geq 9$, it reduces advertising expenditure in the way to maintain the future goodwill level at the maximum level $\beta w + \alpha a = 30$. This could be defined as a one-pulse maintenance policy. In $L = 1$ however, the advertising policy depends on the goodwill decay rate β . F2 tends to have a more stable advertising policy on goodwill level if there is a high β than it does with a low β . In general, F2 maintains the future goodwill level when the goodwill level is low ($w \leq 7$). It has a constant budget on advertising as the goodwill level increases ($w \geq 8$), which in return increases the future goodwill level. This could be named as a pulsing maintenance policy.

So far, results are on sequential inventory-advertising decisions, where F2 makes



(a) $L=0$



(b) $L=1$

Figure 6.5: F2: Advertising with sequential inventory-advertising decisions, $i = 0$

advertising decisions after the inventory orders. The next results are on examples with simultaneous decisions. For convenience, SQ is used for models with sequential decisions and SM is used for models with simultaneous decisions.

Simultaneous decisions - a comparison

Table 6.4: F2: Simultaneous decisions, average profit with different β , $\alpha = 1$

Parameters	$\beta = 0.4$	$\beta = 0.7$	$\beta = 1$
$L = 0, g_{SM}$	13.5952	22.1757	30.9587
$L = 0, 100(g_{SM} - g_{SQ})/g_{SM} \%$	0.0113%	0.5673%	0.5476%
$L = 1, g_{SM}$	22.4216	29.9185	37.9406
$L = 1, 100(g_{SM} - g_{SQ})/g_{SM} \%$	3.2215%	1.7645%	1.1133%

By a simultaneous decision policy (SM), F2 makes orders and advertising decisions at the same time, observing the inventory level and goodwill level. Results in Table 6.4 show that, there is not a significant difference on F2's average profit between an SM policy and an SQ policy. However, as to computing time, the SQ model needs to consider

$$\begin{aligned} & (1 + I)(1 + W)(1 + D)(1 + B)(1 + I + D) + (1 + I)(1 + W)(1 + D)(1 + B)(1 + A) \\ & = (1 + I)(1 + W)(1 + D)(1 + B)(2 + I + D + A) \end{aligned}$$

state-action combinations, while the SM model needs to consider

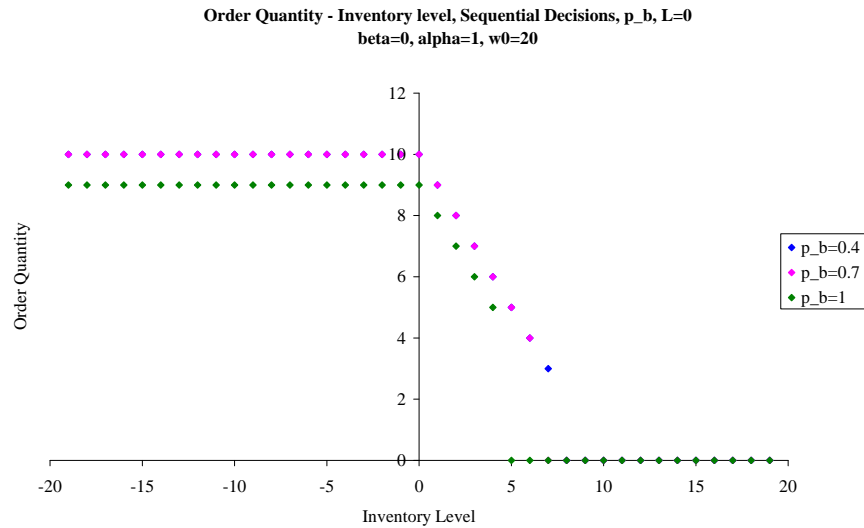
$$(1 + I)(1 + W)(1 + D)(1 + B)(I + D + 1)(1 + A)$$

state-action combinations. Hence the time saved by the approach of the SQ model can be estimated as

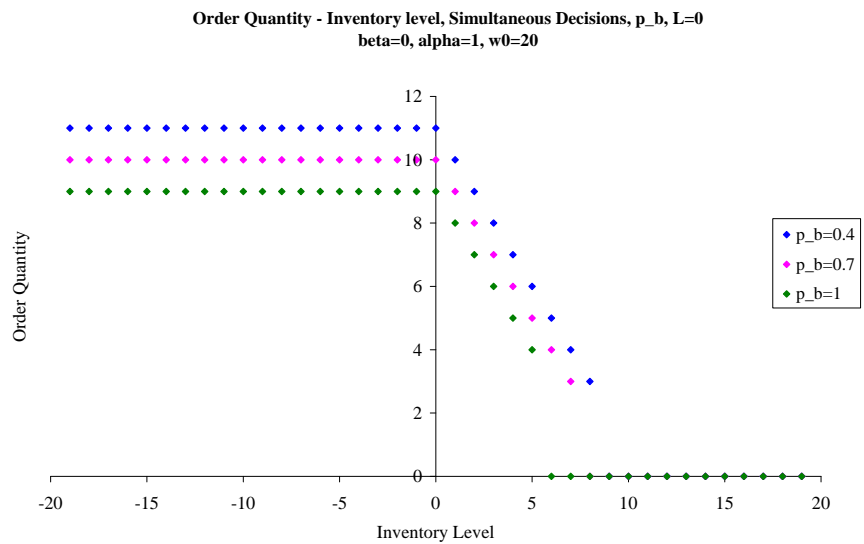
$$\frac{(I + D + 1)(1 + A) - (2 + I + D + A)}{(I + D + 1)(1 + A)} = 1 - \left(\frac{1}{I + D + 1} + \frac{1}{1 + A} \right)$$

of the time needed for the SM model. Therefore, SQ models are more computationally efficient. In survival probability maximizing models, due to the need to model the solvency check on F1, the complexity of both the SM and the SQ models is increased

by a factor of $X + 1$. Therefore, the percentage saving is the same, but the time saved is increased by a factor of $X + 1$. Due to the computational complexity of the SQ model, it was found to be unsuitable for the F1 model in the thesis.

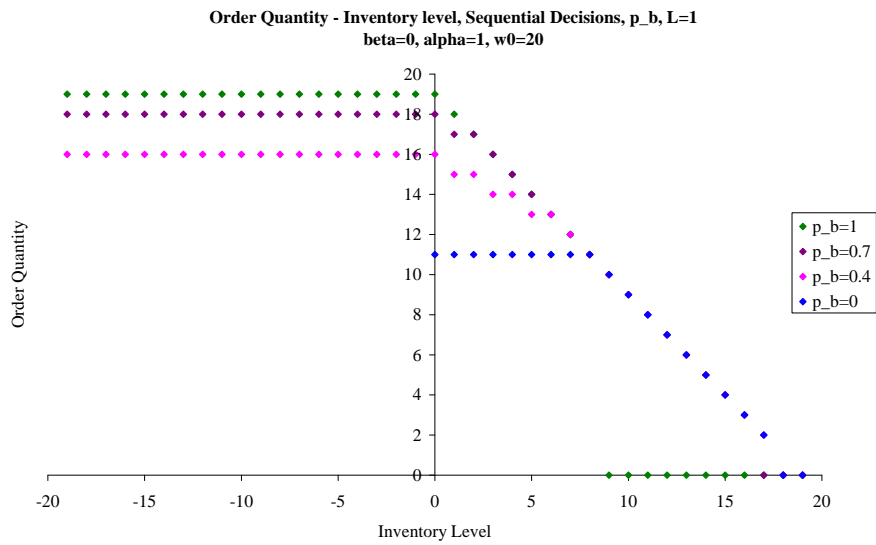


(a) Sequential inventory - advertising decisions

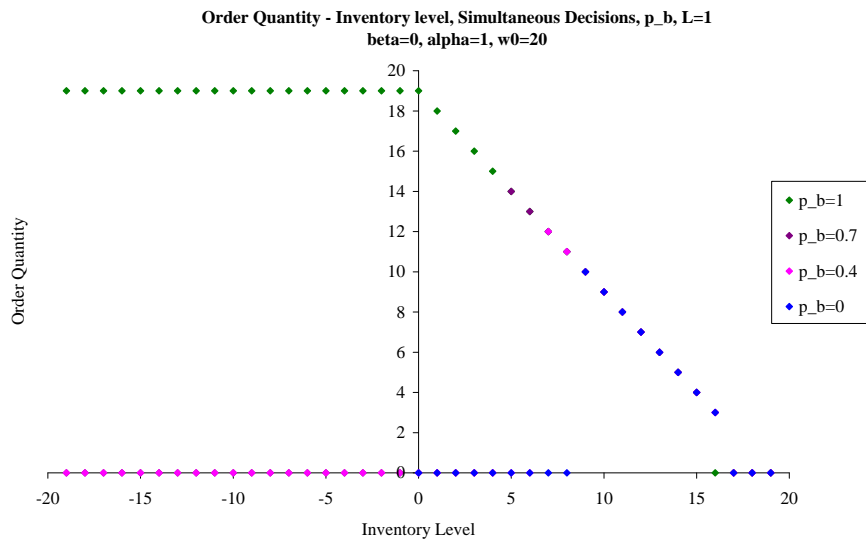


(b) Simultaneous inventory - advertising decisions

Figure 6.6: F2: Inventory policy in $L = 0$



(a) Sequential inventory - advertising decisions



(b) Simultaneous inventory - advertising decisions

Figure 6.7: F2: Inventory policy in $L = 1$

Nevertheless, it is still of interest to see F2's inventory and advertising policies in the simultaneous decision models. Figure 6.6 and Figure 6.7 compare F2's order

decisions in the SM and SQ models. In $L = 0$, F2 uses (s, S) policies in both SQ and SM models. The reorder level, s , and order-up-to-level, S , in these two models are different, see Table 6.5. With simultaneous decisions, F2 orders no less than with sequential decisions. In $L = 1$, F2 uses a combination of (s, S) and fixed order order policy in SQ model, and uses a combination of no-order and (s, S) policy in SM model.

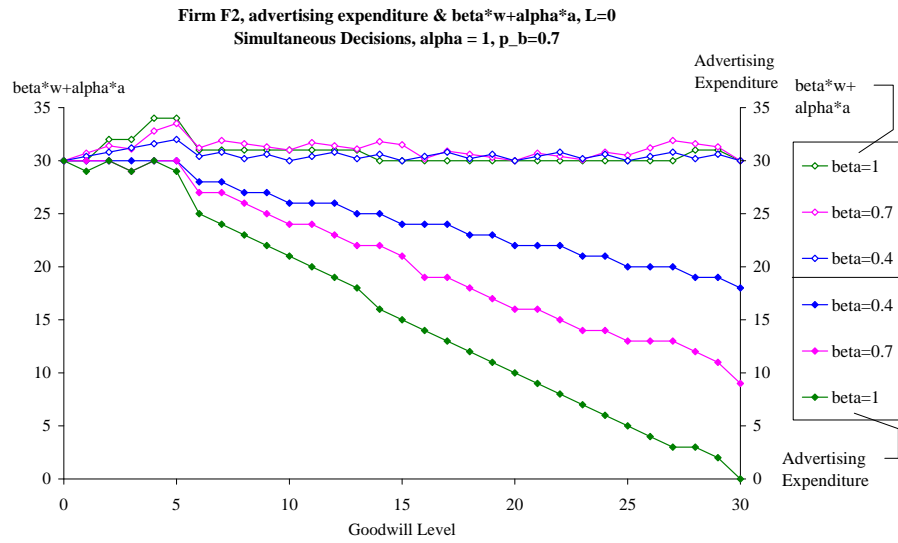
Table 6.5: F2: The (s, S) policies in sequential and simultaneous decision models, $L = 0$

Parameters	$p_b = 0.4$	$p_b = 0.7$	$p_b = 1$
SQ Model	(7,10)	(6,10)	(4,9)
SM Model	(8,11)	(7,10)	(5,9)

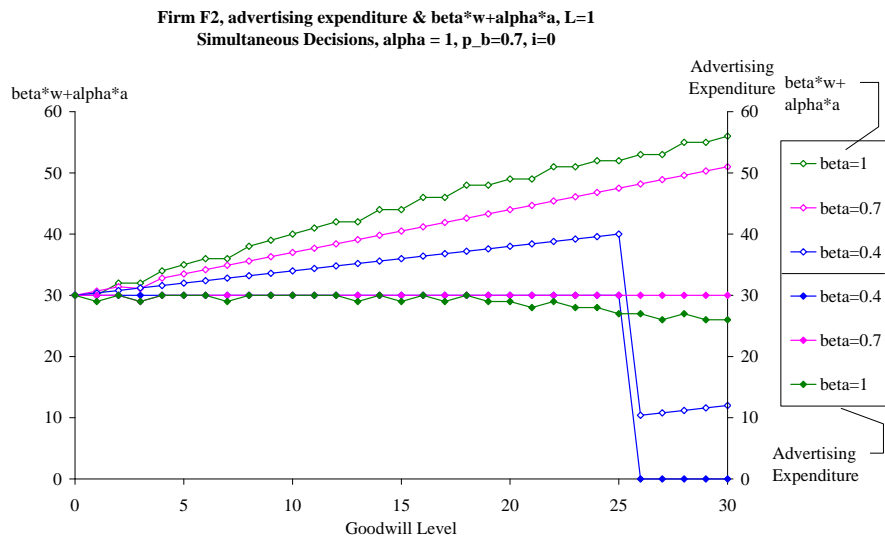
Figure 6.8 shows the advertising policies in the SM model. In $L = 0$, F2 spends as much as possible when $w \leq 5$, and then keeps the goodwill level at its maximum. Compared to the advertising policy in the SQ model (Figure 6.5), the SM model also uses a one-pulse maintenance policy, only that it is comparatively less aggressive on advertising expenditure when goodwill level is low (i.e. $w \leq 8$). In $L = 1$, F2 uses a maintenance policy to keep the future goodwill at the maximum level. The difference in decisions is a result of the decision process and indicates the coordination between production and advertising departments.

6.4.4 Survival probability and advertising policy

As a start-up firm, F1 has to take a solvency check at the end of each review period. In this inventory-advertising model, F1's advertising expenditure increases the goodwill level, which in consequence determines the firm's average demand in the following period. The demand directly affects F1's sales and so its survival. It is expected that F1 would consider using advertising as a lever to sales for a higher chance of survival. Meanwhile, F1 should be careful about spending, otherwise it may put itself into the risk of insolvency. Many factors, such as lead time, goodwill decay rate, as well as the state variables, i.e. inventory level, goodwill level and capital available, would affect F1's decisions and survival. Comparing between objectives, F1 may have different advertising policies from F2. The following results will present evidence to the expectations.



(a) $L=0$



(b) $L=1$

Figure 6.8: F2: Advertising with simultaneous inventory-advertising decisions, $i = 0$

Survival probability

As a result of the positive effect on goodwill and demand, advertising increases F1's chance of survival, in the case where the firm has a low goodwill level, see Figure 6.9. When w is small, the firm basically has few customers and could hardly survive without advertising. In contrast, the firm has a non-zero survival probability if it advertises the product. In the special case with $w = 0$, the firm has no demand at the beginning and therefore no risk of stockouts. It does not order anything and thus has little cost. Instead, the firm spends as much as possible on advertising to generate demand in order to survive (see the future goodwill level Figure 6.10 for $w = 0$). In other cases with non-zero goodwill, advertising has a positive effect on F1's survival when goodwill level is low. The effect is more significant if the decay rate is high, i.e. if the goodwill has a long-term effect. With high goodwill levels, advertising does not have much effect on goodwill which is a result of saturation. The survival probability with advertising is lower than that without advertising, due to the "extra" cost of advertising.

Advertising policy

In Figure 6.10, for both of the lead time cases, F1 advertises when $w = 0$, in which case the firm is certain about having no customers. It has no more cost than paying for the overheads and has no profits either. The firm would spend as much as possible to generate demand. When the goodwill level is low, F1 invests less on advertising than it does with $w = 0$. In $L = 0$, having little sales, the firm has a tight capital. It spends some budget on advertising for a slightly higher future goodwill level $\beta w + \alpha a$. When the goodwill level is higher, say when $w \geq 6$, F1 spends a constant amount on advertising which keeps its future goodwill growing. When the goodwill is even greater, the firm increases the budget on advertising again and keeps it at a fixed amount to increase the future goodwill level. The exact spending policy depends on the decay rate, β .

In $L = 1$, when the goodwill level is low, F1 advertises to maintain the future goodwill at a constant level. When the goodwill is above some level, i.e. $w = 5$ in this example, the firm increases the advertising expenditure to keep the future goodwill growing roughly at a constant. When the goodwill level is higher, say $w \geq 18$, the firm decreases the advertising budget which can be down to zero in the $\beta = 0.4$ and $\beta = 0.7$ examples. Advertising at a high goodwill level is equivalent to generating demand that

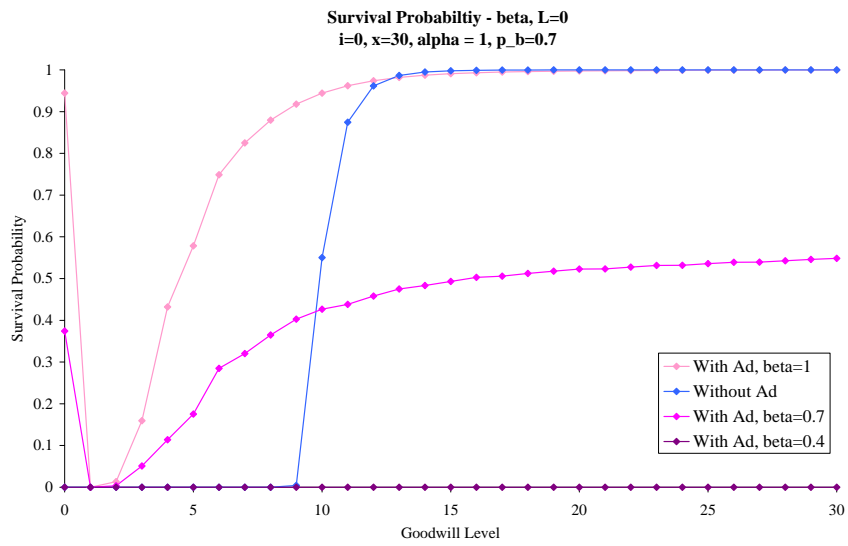
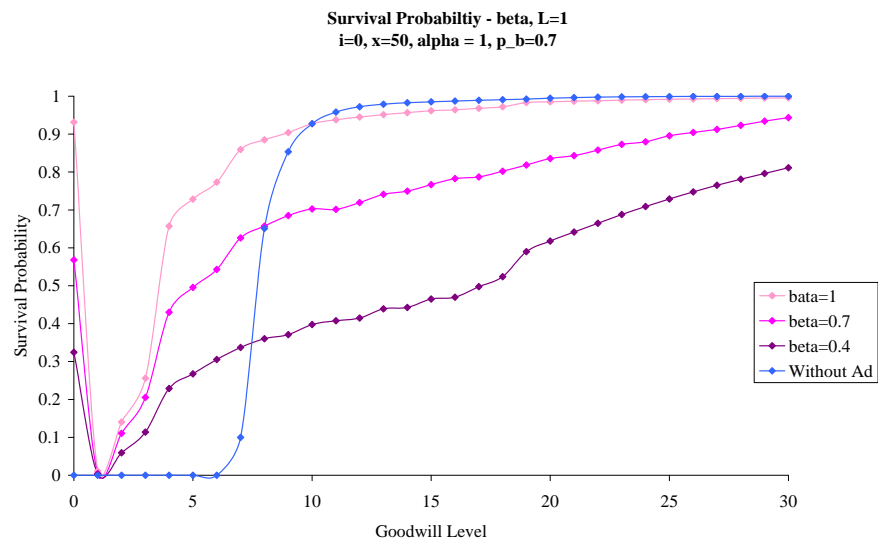
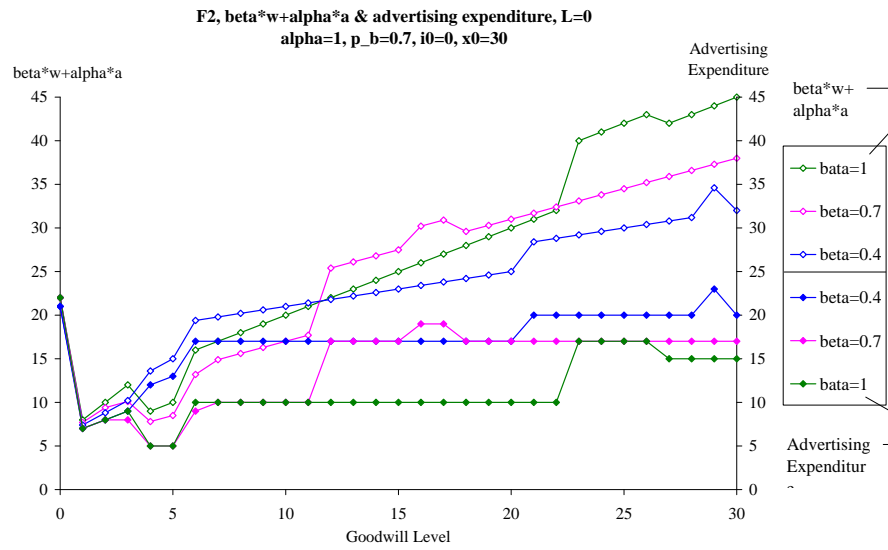
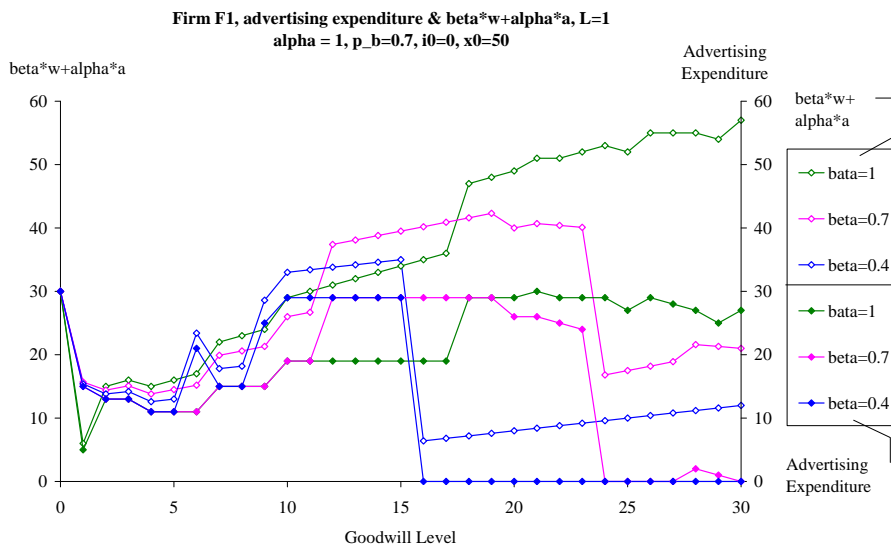
(a) $L=0$ (b) $L=1$

Figure 6.9: F1: Survival probability changing with goodwill



(a) $L=0$



(b) $L=1$

Figure 6.10: F1: Advertising policy changing with goodwill

may be beyond the firm's production capability. Exceeding demand causes shortages which result in customer dissatisfaction and cost the firm future goodwill. In the circumstance where F1 has a survival probability close to 1 and tends to decrease the advertising budget to suit its production capacity, the firm may consider plant expansion for long-term development, which will in turn allow it to meet more demand and increase profits.

6.4.5 Advertising in different inventory lead times

For both firms, advertising decisions are made on condition of all the state variables, i.e. inventory level and goodwill level, and (for F1) capital available. Therefore, there may not be one single golden rule for all the circumstances. Figure 6.4 and Figure 6.11 show how each firm's advertising policy varies with the inventory and goodwill levels, in both $L = 0$ and $L = 1$ cases. In general, lead time has an impact on both firms' advertising policies. In $L = 0$, firms advertise to maintain the future goodwill level, $\beta w + \alpha a$, roughly at a constant. F2 keeps the future goodwill level at 30 for most of the states. F1 keeps the future goodwill within a range which depends on the inventory level. In $L = 1$, firms advertise to improve the goodwill when there is a large number of backorders (the inventory level is negative), and to generate demand when the expected demand is low and the inventory level is high.

Comparing the two firms, F1 is more flexible with advertising than F2. F1's advertising expenditure pulses with inventory level, given the goodwill is fixed, which may be a result of the limited capital. F1 takes chances to advertise at times but does not spend the same amount on advertising all the time, in order to keep enough capital for the solvency check. Generally speaking, F1 advertises with careful consideration of the production capability and the expected demand. F2 looks to be more positive and confident about the spending.

6.5 Conclusion and discussions

This chapter introduced goodwill into the partial backorder inventory model. Shortages lead to customer dissatisfaction and reduce firm goodwill. Moreover, being an intangible asset, goodwill level decreases over time. Advertising decisions are considered as an investment on goodwill. Advertising expenditure increases the goodwill level which determines the average demand in the following period. The numerical tests demonstrate

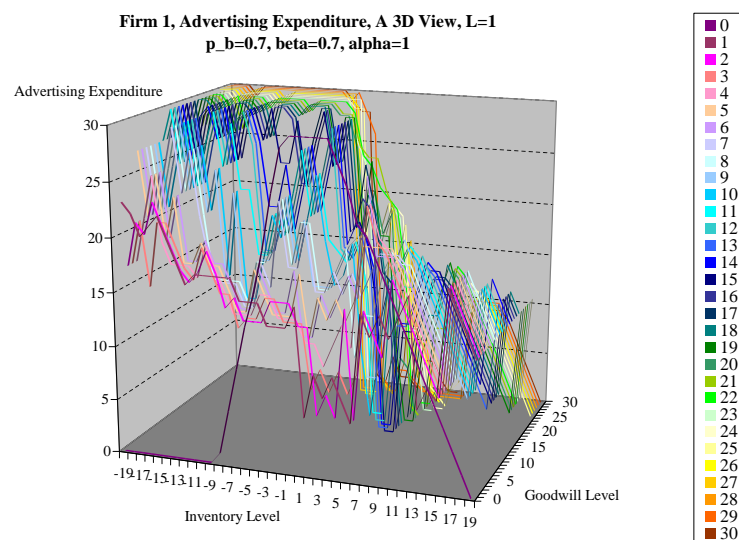
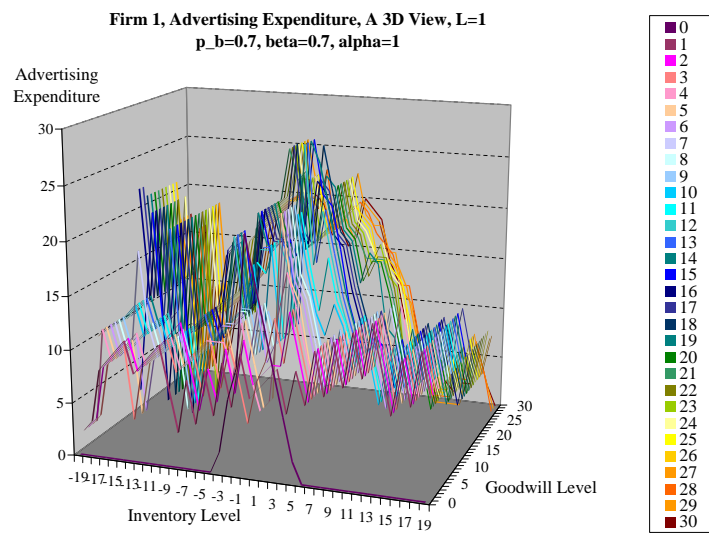


Figure 6.11: F1: Advertising policy, a 3-dimension view

the applicability of the models proposed.

From the examples considered, the survival probability can be increased by placing advertising when the goodwill level is low. As expected, the probability increases with the goodwill decay rate. The examples considered show that advertising policy can be a complex function depending on goodwill level, inventory level and lead time. Overall, the survival maximizing firm has a more flexible policy than the profit maximizing firm. However, there are few clear trends. The value of this model would be as a tool to investigate the impact of different advertising policies on survival probability.

For future work, the proposed models can be considered for real-time problems, discussing the implementation and possible improvement of the inventory and advertising policies in start-up firms. It is also worth discussing heuristic algorithms in such complex problems to improve computational efficiency.

Chapter 7

Conclusions and further research

Start-up firms play an important role to a nation's economy. Many firms are established each year holding the founders' ambition and great expectations. However, a large percent of them do not survive after a few years of foundation. This is due to many factors, such as the limited financial resources to start-up firms, lack of management experience. Motivated by from such difficulties to start-up survival, the research in this thesis proposed mathematical models on start-up firms, stressing the importance of objective to start-up firm survival. Instead of the commonly held profit focused objective, this thesis assumes that the crucial issue to start-up firms is survival. All the planning and operation decisions are thus implemented following the objective of survival maximization. A start-up firm is said to fail if it has negative capital at a solvency check.

In this thesis, a general model was first presented, discussing the features of start-up firms. It builds up the link between the survival maximizing and profit maximizing models. The optimal decision policies in a profit maximizing firm can be used in a start-up firm and give a non-negative survival probability, given that the firm earns non-negative profit under such a policy. However, this policy may not be optimal for a survival maximizing firm. Alternatively speaking, the widely applied decision policies which are generated based on mature firms, may not be optimal for start-up firms. A start-up firm may obtain a higher chance of survival if it utilizes a more suitable policy. Following the general model, this thesis then took the inventory management in manufacturing firms to further discuss the start-up firms' decisions and factors affecting their survival. The three inventory problems considered are lost sales, partial backorders and joint inventory-advertising models.

Each of the three inventory models looked into a specific operating circumstance in start-up firms. They provided quantitative analysis and insightful decision making criteria for start-up firm survival, which has little been done in the current literature. To a start-up firm, the initial capital is crucial to its survival. The capital's impact on start-up firm survival is not monotonic. The existence of threshold on capital requirement is found in all examples of models. The survival probability has a steep curve if the capital is below the threshold, and is flat if the capital is above the threshold in which case the firm is close to being established. This indicates that every little amount of capital helps if the firm has little resources at firm foundation. The models proposed can be used as a tool helping start-up firms find the suitable level of capital requirement. Meanwhile, this capital requirement is related to firms' average profit. Firms who are capable of earning high profit can rely on relatively less initial capital compared to those who struggle with profits. The experiments indicated that aiming at survival, firms tend to be risk-averse and make careful use of the resources in order to have enough cash flow for operations and to meet solvency checks. In hard survival situation, the firms may be risk-seeking in order to grasp the last chance of survival.

In terms of the operation parameters, from the sensitivity tests, the top three significant factors to start-up firm survival in order of impact are, selling price, purchasing cost and overhead cost. This applies to both lead time zero and one cases. It is worth noticing how much these top three parameters can change a firm's survival. If a firm provides unique products, in respect of design, technology, or even service, the firm makes itself stand out from the crowd and can therefore charge a reasonably high price for its product. The managers can seek cheaper material suppliers, which may depend on their bargaining power and on the production requirements. The managers can also improve the system efficiency to reduce the overhead costs. The unit lost sales and holding cost in lead time one has some impact on survival as well. These cost rates are more intangible compared to other costs. It might be possible to consider shortening the inventory cycle and making more efficient use of capital, due to reduced number of lost sales and opportunity cost of holding stock.

Positive lead time has a negative effect on firm survival. The delay of material delivery can result in shortages, loss of profit and customer dissatisfaction. To overcome this, firms keep large stocks where possible, which as a result, tightens start-up firms' cash flow and reduces the chance of survival. The lead time is due to many factors such as the administration process, supply channel, and may not be easily altered. On

the other hand, firms can find ways to make more frequent orders which reduces the interval between each delivery. Firms are then able to hold less inventory and meet more demand, which consequently increases the firms' operation efficiency and survival probability.

The hazard rate is widely used as a measurement of start-up firm survival. As also found in the empirical research, the hazard rate is often an inverted U-shaped curve of time. The hazard rate increases with time at the first stages of foundation where the firms relies on the initial resources. The hazard rate comes to a peak as soon as the initial resources are used up and the firms will then depend on profits to pay for all the costs and meet the solvency checks. The hazard rate drops down gradually during which time the firms grow to maturity.

From the external point of view, the entry market is crucial to start-ups' survival. It is more difficult to survive in a competitive market where the demand has a high deviation. On the other hand, a firm benefits from finding the right niche market where the demand is concentrated closer to the mean. The market demand with a low deviation is helpful for production planning and inventory control. Meanwhile, customers in such a market are more likely to backorder products since there are few similar suppliers or alternative options. Finding the right niche market would require extensive research in the market, knowing the current products, suppliers, demand and trend. It can be the pre-foundation starting point and sets the strategic plan for the first several years of operation. Investing on research and design of the products is equally important to start-up firms, which keeps the products in demand and helps the firm to compete in the market.

Advertising is also necessary in start-up firms. It is an investment in improving market awareness, firm goodwill, and therefore the market demand in the future. The experiments show the positive effect of advertising on firm survival. Spending on advertising should be carefully considered in start-up firms. Start-up firms have different advertising strategies from those profit maximizing firms. The advertising expenditure is considered in close relation to firms' goodwill and inventory levels. Advertising is used to increase the goodwill if there is low average demand, and is used to maintain the goodwill if the demand is comparatively high. In either case the exact spending depends on the inventory level.

In terms of start-up firm finance, a solvency check is made at the end of each review period. A firm is said to fail if it has negative capital. The models suggested that,

slightly looser liquidation policies, i.e. less frequent solvency checks, give the start-up firms more time looking for external funding or making business process improvement and thus a higher survival probability. The change of review from monthly to quarterly makes a distinct difference to firm survival, whereas half-yearly checks make a little further improvement. It is also worth noticing that the change of policy is more crucial in the early age where the hazard rate varies the most. It may not affect the firm much after a few years of foundation when the management is more experienced and the firm is going to maturity. As to policy makers, the liquidation policy can possibly be tailored for individual start-up firms. Balanced scorecards can be used for performance measurement. The fields in the scorecards can include firm finance, and other factors such as economy, industry, region, entry size, firm age, employee and top management. From the results in this thesis, the firm management can be assessed in evaluation of the manager's previous experience, aim of foundation, knowledge of the industry and market, risk attitude, leadership, etc.

Overall, this thesis proposed mathematical models on start-up firm survival, and provided quantitative analysis for insights into firm operations. The models were established under a few assumptions. The economy environment was not taken into particular discussion while it can play a crucial role in firm survival. From empirical research, a good economy gives the start-up firms a comparatively easy environment to survive, while a difficult economy affects start-up firms more than those established firms. The operations planning, such as the employment, number of equipment, capacity of the plant, were assumed to be settled and excluded in the models. Firm finance, such as ways of external funding, taxes, insurances, were assumed to be a part of either initial capital or overhead cost, and were not discussed in the models. All the assumptions were made to provide a modelling environment for the inventory problems in start-up firm, but should be taken into careful consideration in real-time cases.

As for later research, the models can be expanded with some of the assumptions relaxed. It will be worth working with models on real data, whereas the estimation of parameter values and simplification of the complex situation should be carefully discussed in detail. This thesis has covered joint inventory-advertising decisions, the experiments of which invoked the need of heuristic algorithms in solving the problems, which can contribute to computation efficiency and enlarge the problem size for work on real-time data. Other joint decisions such as with finance, human resources, can also be implemented into the current models, presuming that the computation difficulties

are solved. With accomplishing of the start-up models on individual problems, it is also worth considering building up an organic system in which models could interact with each other, to provide overall support to start-up businesses. The research can be used for practitioners as a reference for founding business, strategic planning and decision making. It will be helpful to take the conclusions as a guidance in assistance to the firm management, though detailed decisions can also be obtained from implementing the models with real-time data. The quantitative research presents all possible outcomes, which provides policy makers with a big picture about start-up firms and can be used to support policy establishment or improvement.

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