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Fractals in complexity and geometry

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Fractals in complexity and geometry

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9AM, November 19, 2009, I was walking from the Memorial Union to Atanasoff Hall across the lawn at the central campus. Breathing the lightly chilly morning breeze filled with the familiar scent of fresh grass, I could not hold back the joy inside me. There are only a a few places in the world I would call home, and this is one of them. Iowa State, I am back. An hour ahead was my final exam. By noon, the good news came out as the sun broke the cloud.

Nine years at Iowa State and six years on my dissertation research, I could not have gone through without the people around me. First and foremost, I would like to thank my advisor Jack Lutz for his invaluable help during my graduate study. He never told me what I should do, but he always steered me in the right direction when I was lost. Without his encouragement and guidance, I could not have persevered through all those years.

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Last but most importantly, I owe the most to my parents. It is their guidance in many pivotal times during my earlier years that made it all possible.

By the time this is written, I had already left Iowa State to start my new adventures in California. The California sun might provide some warmth when Iowa is chilled by the winter freeze. But what warms my heart is feeling at home. I hope I have found a new one.

1 INTRODUCTION

Fractal phenomena exist everywhere in the physical world. The British coast line, the shape snow flake crystals, the shape of tree leaves, etc [9]. In computer science, people have investigated phenomena of similar nature in terms of dimensionality [19]. For these works, classical dimensions were used to study the fractal structure of complexity classes. Due to the restrictions of classical dimensions (or more precisely the lack of computational restrictions on classical dimensions) the usefulness of dimension-theoretic techniques was very limited, since most of the complexity classes are themselves countable sets, which has dimension 0 for all the classical dimensions we are concerned with.

This situation changed dramatically when Lutz [62, 63] first characterized Hausdorff dimension using gales and extended this most famous fractal dimension notion to resource-bounded dimensions for computational complexity classes and investigated the fractal phenomena therewithin. For example, he proved that the class of languages decidable by using boolean circuit of size at most $\alpha 2^n/n$ for each input size n has dimension α in ESPACE, i.e.,

$$\dim(\mathrm{SIZE}(\alpha\frac{2^n}{n})|\mathrm{ESPACE}) = \alpha.$$

Note that $\dim_{\mathrm{H}}(\mathrm{ESPACE}) = 0$, as ESPACE is the set of languages that can be decided by deterministic Turing machines using tape space that is polynomial in the size of input encoded in binary and it is thus a countable set. Therefore resource-bounded dimensions are stronger than their classical counterparts in the sense that resource-bounded dimensions are meaningful in spaces that have classical dimension 0 themselves. Due to the strength of the resource-bounded dimensions, it is even meaningful for individual languages (singleton sets) now. With resource-bounded dimension, more quantitative analysis have been done for the structure of complexity classes and close connections have been identified among dimensionality, Kolmogorov complexity, and compressibility for both complexity classes [63, 7] and individual languages [64, 26]. We further investigate the relationship between dimensionality and computational complexity and compressibility using resource-bounded dimension in chapters 3 and 4.

Besides dimensionality, another well-known aspect of fractal phenomena is geometry. With the tools of resource-bounded dimensions, we are able to investigate the dimensionality of individual sequences in Canter spaces, which, through simple encoding, makes it possible to study the individual points in Euclidean spaces. Questions about the roles of dimensionality of individual points in geometry becomes valid. In chapter 5, we investigate questions of this kind and the relationship between individual points and computable curves of finite length. Although many curves of interests are not fractals in terms of dimensionality, they do have many geometric features that a true fractal geometric construct exhibits.

In the following, we give a summary of technical contributions in chapters 3, 4, and 5.

1.1 Fractals in Complexity Classes

In chapter 3, we investigate fractals in complexity classes by extending some results in resource-bounded measure to resource-bounded dimension. We focus on two aspects of fractals related to computational complexity. One is the measurement of the relative size of complexity classes. The other is power of fractals in the sense of derandomization.

1.1.1 Dimensions of Polynomial-Size Circuits

Circuit-size complexity is one of the most investigated topics in computer science. In particular, much effort has been centered on the relationship between polynomial size circuits and uniform complexity classes. Since the 1970s, it has been known that $\mathrm{ESPACE} \not\subseteq \mathrm{P/poly}^{\mathrm{i.o.}}$ [92, 89, 52, 93], i.e., that there exists a language in ESPACE that does not have polynomial size circuits, even on only infinitely many lengths.

When Lutz invented resource-bounded measure [60], one of his first resource-bounded measure result was the *quantitative* separation that

$$\mu(\mathrm{P/poly}^{\mathrm{i.o.}} | \mathrm{ESPACE}) = 0,$$

which means that it is *typical* for a language in ESPACE not to have polynomial size circuits even on only infinitely many lengths. Lutz also showed that for all $c > 0$,

$$\mu(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)|\mathrm{EXP}) = \mu_{\mathrm{p}_2}(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)) = 0 \tag{1.1.1}$$

and

$$\mu(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \mu_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = 0, \tag{1.1.2}$$

where $\mathrm{E}_3 = \mathrm{DTIME}(2^{2^{\mathrm{poly}\log n}})$.

Measure theory does not distinguish among measure 0 sets. In classical analysis, Hausdorff dimension [42] and packing dimension [95, 94] serve as refined measurements that complement this limitation of measure. In computational complexity, Lutz et al. effectivized these two dimension notions as the resource-bounded dimension and strong dimension to examine the structure inside resource-bounded measure 0 sets [63, 7]. Very soon after the effectivization, Hitchcock, Lutz and Mayordomo [47] further generalized these dimensions to scaled dimensions to reveal subtle relationships that cannot be addressed without scaling [47]. At the same time, resource-bounded dimension and strong dimension for individual sequences were defined to measure the "level of randomness" for individual sequences [64].

Hitchcock and Vinodchandran [48] recently extended Lutz's measure results (1.1.1) and (1.1.2) with dimension measurements of P/poly. They proved that, for all $c > 0$,

$$\dim(\mathrm{SIZE}(n^c)|\mathrm{EXP}) = \dim_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0 \tag{1.1.3}$$

and

$$\dim(\mathrm{P/poly}|\mathrm{E}_3) = \dim_{\mathrm{p}_3}(\mathrm{P/poly}) = 0. \tag{1.1.4}$$

Recent results by Allender et al. [1, 2] regarding time-bounded Kolmogorov complexity KT and circuit size complexity of strings enable us to measure the class of polynomial size circuits even more precisely. In section 3.1, we take advantage of their results to prove that

$$\dim(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)|\mathrm{EXP}) = \dim_{\mathrm{p}_2}(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)) = \frac{1}{2} \tag{1.1.5}$$

and

$$\dim(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \dim_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = \frac{1}{2}. \tag{1.1.6}$$

Note that (1.1.5) and (1.1.6) strengthen (1.1.1) and (1.1.2), respectively. They also show that (1.1.3) and (1.1.4) cannot be extended to the corresponding i.o.-classes.

Additionally, we prove the strong dimension result

$$\mathrm{Dim}(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \mathrm{Dim}_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = 1. \tag{1.1.7}$$

In order to prove the lower bound on the dimension and strong dimension of $\mathrm{P/poly}^{\mathrm{i.o.}}$, we establish a Supergale Dilation Theorem, which extends to dimension theory the measure theoretic martingale dilation technique introduced by Ambos-Spies, Terwijn, and Zheng implicitly in [4] and made explicit by Juedes and Lutz in [51].

We also improve Hitchcock and Vinodchandran's recent results (1.1.3) and (1.1.4) from dimension to scaled strong dimension by showing that, for all $c > 0$ and all $i \in \mathbb{N}$,

$$\mathrm{Dim}^{(i)}(\mathrm{SIZE}(n^c)|\mathrm{E}_2) = \mathrm{Dim}^{(i)}_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0 \tag{1.1.8}$$

and

$$\mathrm{Dim}^{(i)}(\mathrm{P/poly}|\mathrm{E}_3) = \mathrm{Dim}^{(i)}_{\mathrm{p}_3}(\mathrm{P/poly}) = 0. \tag{1.1.9}$$

1.1.2 Fractals and Derandomization

One of the most used formulation of randomized algorithms is to have a time-bounded Turing machine with access to some random input bits in addition to the given input of the computation. In such formulation, the distribution of the random inputs bit sequence induces a distribution on the outcome of the computation on the given input. When the space of the outcomes is binary, the outcome that carries higher probability is typically identified as the outcome of the randomized algorithm. (Typically, we require the probability of the outcome to be higher than $\frac{2}{3}$, which is difficult to guarantee syntactically if possible at all. This is why such problems are called promise problems. General promise problems were introduced by Grollman and Selman [37].)

Many important randomized complexity classes are defined in this way, e.g., BPP, AM, etc. Given a randomized complexity class $\mathcal{C}$ defined in this manner, one can defined a non-randomized version $\mathcal{C}^S_0$ of $\mathcal{C}$ by forcing the distribution of the random bit sequence to have a singleton support $\{S\}$. This, in effect, replaces the random input with a fixed sequence of bits. Namely, the randomness of the computation is replaced with access to a fixed oracle. One can ask the question whether $\mathcal{C} \subseteq \mathcal{C}_0^S$ or more quantitatively, how weak an assumption we can place on an oracle S and still be assured that $\mathcal{C} \subseteq \mathcal{C}_0^S$. For example, how weak an assumption can we place on an oracle S and still be assured that $\mathrm{BPP} \subseteq \mathrm{P}^S$? For this particular question, it was a result of folklore that $\mathrm{BPP} \subseteq \mathrm{P}^S$ holds for every oracle S that is algorithmically random in the sense of Martin-Löf [67]; it was shown by Lutz [61] that $\mathrm{BPP} \subseteq \mathrm{P}^S$ holds for every oracle S that is pspace-random; and it was shown by Allender and Strauss [3] that $\mathrm{BPP} \subseteq \mathrm{P}^S$ holds for every oracle S that is p-random, or even random relative to a particular sublinear-time complexity class.

In the results mentioned above, the oracle S is required to be random with respect to some resource bound. Such oracles have full resource-bounded dimensions and therefore are not fractals. We extend this line of inquiry by considering oracles S that are proper fractals, i.e, oracles that have *positive dimension* with respect to various resource bounds. Specifically, we prove that every oracle S that has positive Δ_3^{p}-dimension (hence every oracle S that has positive pspace-dimension) satisfies $\mathrm{BPP} \subseteq \mathrm{P}^S$.

This result is a generalization of this fact that applies to randomized promise classes at various levels of the polynomial-time hierarchy. The randomized promise class Promise-BPP was introduced by Buhrman and Fortnow [17] and shown by Fortnow [33] to characterize a "strength level" of derandomization hypotheses. The randomized promise class Promise-AM was introduced by Moser [74].) For every integer $k \geq 0$, we show that, for every oracle S with positive $\Delta_{k+3}^{\mathrm{p}}$-dimension, every $\mathrm{BP} \cdot \Sigma_k^{\mathrm{P}}$ promise problem is $\Sigma_k^{\mathrm{P},S}$-separable. In particular, if S has positive Δ_3^{p}-dimension, then every BPP promise problem is P^S-separable, and, if S has positive Δ_4^{p}-dimension, then every AM promise problem is NP^S-separable.

We use our results to investigate classes of the form

$$\text{dimalmost-}\mathcal{C} = \left\{ A \;\middle|\; \dim_{\mathrm{H}}(\left\{ B \;\middle|\; A \notin \mathcal{C}^B \right\}) = 0 \right\}$$

for various complexity classes $\mathcal{C}$. It is clear that dimalmost-$\mathcal{C}$ is contained in the extensively investigated class

$$\text{almost-}\mathcal{C} = \left\{ A \;\middle|\; \text{Prob}[A \notin \mathcal{C}^B] = 0 \right\},$$

where the probability is computed according to the uniform distribution (Lebesgue measure) on the set of all oracles B. We show that

$$\text{dimalmost-}\Sigma_k^{\text{P}}\text{-Sep} = \text{almost-}\Sigma_k^{\text{P}}\text{-Sep} = \text{Promise-BP} \cdot \Sigma_k^{\text{P}}$$

holds for every integer $k \geq 0$, where Σ_k^{P}-Sep is the set of all Σ_k^{P}-separable pairs of languages. This implies that

$$\text{dimalmost-P} = \text{BPP},$$

refining the proof by Bennett and Gill [11] that almost-P = BPP. Also, for all $k \geq 1$,

$$\text{dimalmost-}\Sigma_k^{\text{P}} = \text{BP} \cdot \Sigma_k^{\text{P}},$$

refining the proof by Nisan and Wigderson [76] that almost-$\Sigma_k^{\text{P}} = \text{BP} \cdot \Sigma_k^{\text{P}}$.

It is worth noting that Bennet and Gill's technique cannot be extended to obtain these characterizations and that derandomization plays a more significant role in the proof of our results than in that of their almost-classes counterparts. The 1997 derandomization method of Impagliazzo and Wigderson [49] is central to our arguments. Moreover, Nisan and Wigderson's proof that AM $\subseteq$ almost-NP is elementary, while we prove the inclusion AM $\subseteq$ dimalmost-NP relies on Impagliazzo and Wigderson's derandomization.

1.2 Fractals in Individual Sequences and Saturated Sets

In chapter 4, we study the the fractal structures of sets of sequences with respect to certain structures in terms of digits. We will focus on two kinds of sets. One is a kind of sets that are *saturated*, i.e., they contain all the sequences satisfying certain asymptotic properties of distribution of digits. The other is singleton sets of sequences formed by concatenating digits of numbers from a subset of natural numbers. The former is inspired by the study of the collective properties of the set of all Borel normal numbers and the latter is inspired by the study of Borel normal numbers as individual objects.

1.2.1 Saturated Sets with Prescribed Limit Frequencies of Digits

Borel, in search for a good definition of intrinsic randomness, defined normal numbers as real numbers that have base-k extensions in which all finite strings have a *fair* asymptotic distribution [13]. He proved that such numbers are very abundant, namely, the set of all such real numbers have Lebesgue measure 1, or

$$\mu(\{\text{normal numbers}\}) = 1.$$

The invention of fractal dimension allowed the similar investigations into sets that contain all real numbers whose base-k ($k \in \mathbb{N}$) extensions have prescribed frequencies of digits [12, 36, 30]. For example, Besicovitch [12] proved that for each $\beta \in [0, \frac{1}{2}]$,

$$\dim_{\mathrm{H}}(\mathrm{FREQ}^{\leq\beta}) = \mathcal{H}(\beta),$$

where $\mathrm{FREQ}^{\leq\beta}$ is the set of all infinite binary sequences that has fewer than β fraction of 0's in its finite prefixes asymptotically. Good and Eggleston [36, 30] also proved similar results. Their results share a common feature, namely, the Hausdorff dimensions of the sets are all the maximum of the entropies of the limit distributions of digits. (The limit of the distributions of digits or their entropies need not exist.) Volkmann [97] made such observations in a kind of fractal dimension that is defined in probability spaces, which is now called the Billingsley dimension. Volkmann's student Cajar studied such phenomena systematically in his Ph.D. thesis [20]. The key observation they made was that this kind of sets share the property that they are *saturated* in the sense that they contain *all* real numbers with some prescribed restrictions on the asymptotic behavior of the distributions of digits in their base-k extensions. Cajar realized that such sets have a very natural decomposition into an uncountable collection of subsets, each of which has a dimension that is relatively easy to calculate and the whole set takes as the dimension the supremum of the dimensions of all the subsets in the collection. He also noted that the Hausdorff dimension only has such property over a union of countable collection in general.

Some recent works on Hausdorff and packing dimensions of saturated sets used sophisticated multifractal and ergodic theoretic techniques [10, 78, 79, 80, 81, 82, 83]. We extend this line of research to finite-state dimensions [26]. We calculate the finite-state dimensions of some saturated sets with partial constraints on the asymptotic distributions of digits. We prove in Theorem 4.4.13 that for any $X \subseteq \mathbf{C}_m$ that is saturated,

$$\dim_{\mathrm{FS}}(X) = H \text{ and } \mathrm{Dim}_{\mathrm{FS}}(X) = P \tag{1.2.1}$$

and

$$\dim_{\mathrm{FS}}(X) = \dim_{\mathrm{H}}(X) \text{ and } \mathrm{Dim}_{\mathrm{FS}}(X) = \dim_{\mathrm{P}}(X), \tag{1.2.2}$$

where $H = \sup_{S \in X} \liminf_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S,n))$ and $P = \sup_{S \in X} \limsup_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S,n))$, $\mathcal{H}_m$ is the m-ary entropy, and $\pi(S,n)$ is the empirical distribution of digits in the length n prefix of S.

With (1.2.1), we affirm the maximum entropy principle for the finite-state dimensions of saturated sets. With (1.2.2), we obtain a correspondence principle for the finite-state dimensions of saturated sets, namely, the finite-state dimension and strong dimension of saturated sets corresponds to their Hausdorff dimension and packing dimension.

It is also worth noting that (1.2.1) also gives us a point-wise characterization of the finite-state dimensions of saturated sets in terms of the maximum entropy of the empirical distribution of digits of the individual sequences rather than the finite-state dimensions of the individual sequences. Last but not the least, with finite-state dimensions, the state of the union is less fortunate, as, in general, finite-state dimensions are only stable under finite unions [26, 7]. Nevertheless, our point-wise results tell us that the uncountable stability observed by Cajar for the Hausdorff dimension of saturated sets remains true for finite-state dimensions.

1.2.2 The Copeland-Erdős Sequences

Knowing that normal numbers are very abundant did not make it easy to give us examples of such numbers. It was not until 1933, Champernowne [21] gave first example of a normal number. His number is simply

$$S_1 = 0.123456789101112\ldots \tag{1.2.3}$$

formed by concatenating the decimal expansions of the positive integers in order. Champernowne's argument is not specific to decimal numbers. What he proved was that for any $k \geq 2$, the sequence formed by concatenating the base-k expansions of the positive integers in order is normal over the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$. Champernowne conjectured that instead of concatenating all the positive integers, concatenating the base-k expansions of the essence of all the positive integers, i.e., of all the prime numbers, would give rise to a normal number too. In 1946, Copeland and Erdős [24] proved that this number

$$S_2 = 0.235711131719232931\ldots \tag{1.2.4}$$

is indeed normal. What is curious about this new example of a specific normal number is not the fact that it is a normal number but the proof Copeland and Erdős used to show the normality.

Let A be an infinite set of positive integers and an integer $k \geq 2$, the *base-k Copeland-Erdős sequence* $\mathrm{CE}_k(A)$ of A over the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$ is the sequence formed by concatenating the base-k expansions of the numbers in A in order. With this notation, $S_1 = \mathrm{CE}_k(\mathbb{Z}^+)$ and $S_2 = \mathrm{CE}_k(\mathrm{PRIMES})$. What Copeland and Erdős proved was that for any A that is sufficiently dense, $\mathrm{CE}_k(A)$ is normal. More specifically, if $A \subseteq \mathbb{Z}^+$ satisfies the condition that for every $\alpha < 1$ and all sufficiently large $n \in \mathbb{Z}^+$ $|A \cap \{1, 2, \ldots, n\}| > n^\alpha$, then $\mathrm{CE}_k(A)$ is normal for all $k \geq 2$. The normality of S_2 follows by the Prime Number Theorem saying that

$$\lim_{n\to\infty} \frac{|\mathrm{PRIMES} \cap \{1, 2 \ldots, n\}| \ln n}{n} = 1.$$

The condition used by Copeland and Erdős stated in terms of zeta-dimension is that $\dim_\zeta(A) > \alpha$ for all $\alpha < 1$, which is equivalent to saying that $\dim_\zeta(A) = 1$. Therefore, Copeland-Erdős's result. As it is already known now that normality is equivalent to finite-state dimension 1 [86, 14], what they proved is really the fact that

$$\dim_\zeta(A) = 1 \implies \dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = 1.$$

What we are able to achieve is a general relationship between zeta-dimensions and finite-state dimensions, namely, we prove that for all infinite $A \subseteq \mathbb{Z}^+$ and $k \geq 2$,

$$\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \dim_\zeta(A), \tag{1.2.5}$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \mathrm{Dim}_\zeta(A). \tag{1.2.6}$$

Moreover, we also prove that these bounds are tight in the following strong sense. Let $A \subseteq \mathbb{Z}^+$ be infinite, let $k \geq 2$, and let $\alpha = \dim_\zeta(A)$, $\beta = \mathrm{Dim}_\zeta(A)$, $\gamma = \dim_{\mathrm{FS}}(\mathrm{CE}_k(A))$, $\delta = \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A))$. Then, by (1.2.5), (1.2.6), and elementary properties of these dimensions, we must have the inequalities

$$\begin{array}{ccccc} & \gamma & \leq & \delta & \leq 1 \\ & \vee\!\mid & & \vee\!\mid & \\ 0 \leq & \alpha & \leq & \beta. & \end{array} \tag{1.2.7}$$

Our main theorem also shows that, for *any* α, β, γ, δ satisfying (1.2.7) and any $k \geq 2$, there is an infinite set $A \subseteq \mathbb{Z}^+$ such that $\dim_\zeta(A) = \alpha$, $\mathrm{Dim}_\zeta(A) = \beta$, $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \gamma$, and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \delta$. Thus the inequalities

$$\begin{array}{ccccc} & \dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) & \leq & \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) & \leq 1 \\ & \vee\!\mid & & \vee\!\mid & \\ 0 \leq & \dim_\zeta(A) & \leq & \mathrm{Dim}_\zeta(A). & \end{array} \tag{1.2.8}$$

are the *only* constraints that these four quantities obey in general.

1.3 Effective Fractals in Geometry

In chapter 5, we shift out attention to curves in Euclidean spaces. We investigate what kind of points can be on a curve. In a Euclidean space $\mathbb{R}^n$, a curve is the range Γ of a continuous function $f : [a, b] \to \mathbb{R}^n$ for some $a < b$. As any bounded Euclidean space can be filled by some infinite curve, for the subject to be interesting, we only consider curves of finite length (rectifiable) with finite parametrizations that are computable, namely, computable curves of finite length. In Section 5.3, we characterize exactly those points that can be on computable curves of finite length by extending Jones and Okikiolu's results regarding the "analyst's traveling salesman problem" [50, 77]. (See also the monographs [68, 35].) In the proof of our result and Jones and Okikiolu's, the constructed parametrization of a curve may not avoid crossing itself (retracing), even when the curve is simple. We also explore the dimensionality of points in connection to this question. We investigate this phenomenon in Section 5.4 in the settings of computable curves. We also explore the relation between parametrization of curves and their length. Since any non-degenerate curve has one-dimensional Hausdorff measure 1, they are not fractal in terms of dimension. Nevertheless, in terms of their shape, they do share many characteristics with fractals.

1.3.1 Points on Computable Curves

The "analyst's traveling salesman problem" of geometric measure theory is the problem of characterizing those sets $K \subseteq \mathbb{R}^n$ that can be traversed by curves of finite length. In 1990, Jones solved this problem for $\mathbb{R}^2$ [50]. In 1992, Okikiolu solved this problem for higher-dimensional Euclidean spaces [77]. Their result – the "analyst's traveling salesman theorem", says that a bounded set K is contained in some curve of finite length if and only if a certain "square beta sum" $\beta^2(K)$, involving the "width of K" in each element of an infinite system of overlapping "tiles" of descending size, is finite. Formally, let $K \subseteq \mathbb{R}^n$ be bounded. Then K is contained in some rectifiable curve if and only if $\beta^2(K) < \infty$ [50, 77].

The question we want to answer here is the following: What are the points that lie on computable curves of finite length? The classical analogy of this question that has a completely trivial answer, since every point is a degenerate curve of length 0. This is indeed an interesting question when we restrict ourselves to computable curves of finite length, as we know that an algorithmic random point in the plane is not on any computable curve of finite length, though the proof is not trivial.

We characterize those *points* of Euclidean space that lie on *computable* curves of finite length by formulating and proving a computable extension of the analyst's traveling salesman theorem. Our extension, the *computable analyst's traveling salesman theorem*, says that a point in Euclidean space lies on some computable curve of finite length if and only if it is "permitted" by some computable "Jones constriction". A Jones constriction here is an explicit assignment of a rational cylinder to each of the above-mentioned tiles in such a way that, when the radius of the cylinder corresponding to a tile is used in place of the "width of K" in each tile, the square beta sum is finite. A point is permitted by a Jones constriction if it is contained in the cylinder assigned to each tile containing the point. The main part of our proof is the construction of a computable curve of finite length traversing all the points permitted by a given Jones constriction. Our construction uses the main ideas of Jones's "farthest insertion" construction, but takes a very different form, because, having no direct access to the points permitted by the Jones constriction, our algorithm must work exclusively with the constriction itself.

We also study some other properties of the points on computable rectifiable curves relating to constructive dimension. We show that any point on a computable rectifiable curve has dimension at most 1, while points that are not on any computable rectifiable curve can have dimension 0.

1.3.2 Computable Curves and Their Lengths

The proof of the computable analyst's traveling salesman theorem constructs a computable curve that a set of points permitted by a computable Jones constriction. As we mentioned earlier, the constructed parametrization is not guaranteed to be one-one even if the given set is the subset a simple curve. In the classical case, we know that this is an artifact of the construction, since every simple curve has a one-one parametrization. When we are restricted to the computable parametrizations, things are more interesting. We prove that there are simple curves can have a geometry that is complex enough that none of the computable parametrization is one-one. We do so by exhibiting a polynomial-time computable, rectifiable, and simple (i.e., it has a one-one parametrization) plane curve $\mathbf{\Gamma}$ that *must be retraced* in the sense that every computable parametrization $f : [a, b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ is *not* one-one. In fact, for every $m > 0$, there are points on $\mathbf{\Gamma}$ that has to be retraced at least m times by f. More precisely, for every positive integer m, there exist disjoint, closed subintervals $I_0, \ldots, I_m$ of $[a, b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$.

A fundamental and useful theorem of classical analysis states that every simple, rectifiable curve Γ has a *normalized constant-speed parametrization*, which is a one-to-one parametrization

$f : [0, 1] \to \mathbb{R}^n$ of Γ with the property that $f([0, t])$ has arclength tL for all $0 \leq t \leq 1$, where L is the length of Γ. (A simple, rectifiable curve Γ has exactly two such parametrizations, one in each direction, and standard terminology calls either of these *the* normalized constant-speed parametrization $f : [0, 1] \to \mathbb{R}^n$ of Γ. The constant-speed parametrization is also called the *parametrization by arclength* when it is reformulated as a function $f : [0, L] \to \mathbb{R}^n$ that moves with constant speed 1 along Γ.) Since the constant-speed parametrization does not retrace any part of the curve, our main theorem implies that this classical theorem is not entirely constructive. Even when a simple, rectifiable curve has an efficiently computable parametrization, the constant-speed parametrization need not be computable. Yet, we do prove that every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has the following two properties.

I. The length of Γ is lower semicomputable.

II. The constant-speed parametrization of Γ is computable relative to the length of Γ.

These two things are not hard to prove if the computable parametrization is one-to-one, (in fact, they follow from results of Müller and Zhao [75] in this case) but our results hold even when the computable parametrization retraces portions of the curve many times.

Taken together, I and II have the following two consequences.

1. The curve $\mathbf{\Gamma}$ of our main theorem has a finite length that is lower semi-computable but not computable. (The existence of polynomial-time computable curves with this property was first proven by Ko [55].)

2. Every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has a constant-speed parametrization that is Δ^0_2-computable, i.e., computable relative to the halting problem. Hence, the existence of a constant-speed parametrization for computable rectifiable curves, while not entirely constructive, is constructive relative to the halting problem.

2 PRELIMINARIES

2.1 Languages, Complexity Classes, Resource Bounds

An *alphabet* is a finite set of symbols. A *string* is a finite sequence of symbols. a *sequence* is an infinite sequence of symbols. Given an alphabet Σ, Σ^* denotes the set of all strings using symbols from Σ and Σ^∞ is the set of all infinite sequences of symbols from Σ. For $m \in \mathbb{Z}^+$, we use Σ_m for the m-symbol alphabet – $\{0, \ldots, m-1\}$. The empty string is denoted by λ. A *language* is a set of finite binary strings, i.e., subsets of $\{0,1\}^*$. The *length* $|w|$ of a string w is the number of occurrences of symbols in w and in particular $|\lambda| = 0$. We fix a standard enumeration of all binary strings as $s_0 = \lambda$, $s_1 = 0$, $s_2 = 1$, $s_3 = 00$, etc. $\mathbf{C}_m = \Sigma_m^\infty$ and $\mathbf{C} = \mathbf{C}_2$ is the *Cantor space*, i.e., $\{0,1\}^\infty$.

For a language A, we also identify it with its characteristic sequence $\chi_A \in \mathbf{C}$ such that $\chi_A = [\![s_0 \in A]\!][\![s_1 \in A]\!][\![s_2 \in A]\!]\cdots$, where $[\![\cdot]\!]$ is the boolean evaluation function. We use A for χ_A whenever the meaning is clear from the context. We also call such A an *oracle* when it is given to a Turing machine or a boolean circuit so that the membership of strings in A can be queried for free. With this interpretation, $\mathbf{C}$ is the set of all languages. For integers $0 \leq i, j < |w|$, $w[i..j] = w[i]w[i+1]\cdots w[j]$ and λ if $j < i$. We use the same convention to identify a finite consecutive part of a sequence also. If string x is prefix of string y, we write $x \sqsubseteq y$. If a string w is a prefix of a sequence S, we write $w \sqsubseteq S$. For any language A, $A^n = A \cap \{0,1\}^n$. For any class $\mathcal{C} \subseteq \mathbf{C}$, $\mathcal{C}^{\text{i.o.}} = \{A \mid (\exists L \in \mathcal{C})(\exists^\infty n) A^n = L^n\}$. $\Delta(\Sigma_m)$ is the set of all probability measures on Σ_m.

Regarding circuit-size complexity, $\text{SIZE}(f(n)) = \{A \in \mathbf{C} | (\forall^\infty n) CS_A(n) \leq f(n)\}$, where $CS_A(n)$ is the number of wires in the smallest n-input Boolean circuit that decides A^n. For $x \in \{0,1\}^*$, if $|x| = 2^k$ for some $k \in \mathbb{N}$, then define $\text{SIZE}(x)$ as the size of the smallest k-input circuit whose truth table is x. $\mathrm{P/poly} = \bigcup_{c\in\mathbb{N}} \mathrm{SIZE}(n^c)$.

Let s be a time-constructible function. DTIME(s) is the class of languages decidable in time $O(s)$ by deterministic Turing machines and DTIMEF(s) is the class of functions computable in time $O(s)$ by deterministic Turing transducers. DSPACE(s) and DSPACEF(s) are defined similarly.

We use Δ to represent a function class that serves as a resource bound. (To be precise, a resource bound is a class of type-2 functional in order to have a complete theory of resource-bounded measure and measurability [40]. In here, we take the measurability for granted and only discuss measure, in particular, measure 0 and avoid type-2 computation by doing so.)

In our discussion, Δ may be one of the following:

$\mathrm{all} = \{f \mid f : \{0,1\}^* \to \{0,1\}^*\}$.

$\mathrm{p} = \{f \in \mathrm{all} \mid f \text{ is computable in } n^{O(1)} \text{ time}\}$.

$\mathrm{p}_2 = \mathrm{DTIMEF}(2^{(\log n)^{O(1)}}) = \mathrm{DTIMEF}(n^{(\log n)^{O(1)}})$.

$\mathrm{p}_3 = \mathrm{DTIMEF}(2^{2^{(\log\log n)^{O(1)}}})$.

$\Delta^{\mathrm{p}}_k = \mathrm{p}^{\Sigma^{\mathrm{P}}_{k-1}}$ for $k \geq 2$.

$\mathrm{pspace} = \{f \in \mathrm{all} \mid f \text{ is computable in } n^{O(1)} \text{ space}\} = \mathrm{DSPACEF}(n^{O(1)})$.

$\mathrm{comp} = \{f \in \mathrm{all} \mid f \text{ is computable}\}$.

Lutz defined resource-bounded constructors [59, 60, 63] that generate complexity classes. For a resource bound Δ, the corresponding *result class* is denoted as $R(\Delta)$. The correspondences between resource bounds and complexity classes that we use are:

$R(\mathrm{all}) = \mathbf{C}$.

$R(\mathrm{p}) = \mathrm{E} = \mathrm{DTIME}(2^{\mathrm{linear}})$.

$R(\mathrm{p}_2) = \mathrm{E}_2 = \mathrm{EXP} = \mathrm{DTIME}(2^{n^{O(1)}})$.

$R(\mathrm{p}_3) = \mathrm{E}_3 = \mathrm{DTIME}(2^{2^{(\log n)^{O(1)}}})$.

$R(\Delta^{\mathrm{p}}_k) = \Delta^{\mathrm{E}}_k = \mathrm{E}^{\Sigma^{\mathrm{P}}_{k-1}}$.

$R(\mathrm{pspace}) = \mathrm{ESPACE} = \mathrm{DSPACE}(2^{O(n)}) = \mathrm{DSPACE}(2^{\mathrm{linear}})$.

$R(\mathrm{comp}) = \mathrm{DEC}$.

When using these resource bounds on the computation of real-valued functions, there are specific semantics.

A real-valued function $f : \{0,1\}^* \to [0,\infty)$ is Δ-*computable* if there is a function $\hat{f} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that $\hat{f} \in \Delta$ (where the input $(w,r) \in \{0,1\}^* \times \mathbb{N}$ is suitably encoded with r in unary) and, for all $w \in \{0,1\}^*$ and $r \in \mathbb{N}$, $|\hat{f}(w,r) - f(w)| \leq 2^{-r}$.

A slightly differently defined class of real-valued functions is the lower semicomputable functions. A real-valued function $f : \{0,1\}^* \to [0,\infty)$ is *lower semicomputable* (a.k.a. *constructive*) if there is a function $\hat{f} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ such that $\hat{f} \in \mathrm{comp}$ for all $w \in \{0,1\}^*$ and $r \in \mathbb{N}$,

$$\hat{f}(w,r) \leq \hat{f}(w,r+1) \leq f(w)$$

and

$$\lim_{r\to\infty} \hat{f}(w,r) = f(w).$$

2.2 Measure, Dimension and Category

In this section, we summarize some concepts and theorems about measures and dimensions that we will use in the development of our results.

Definition. Let $\Sigma = \Sigma_m$ be an alphabet. Let $s \in [0,\infty)$. An s-supergale is a function $d : \Sigma^* \to [0,\infty)$ such that $d(\lambda) \in (0,\infty)$ and for all $w \in \Sigma^*$

$$d(w) \geq \frac{1}{|\Sigma|^s} \sum_{a\in\Sigma} d(wa). \tag{2.2.1}$$

An s-*gale* is an s-supergale with equality in (2.2.1). A *supermartingale* is a 1-supergale and a *martingale* is a 1-gale. The *success set* of an s-supergale d is

$$S^\infty[d] = \left\{ S \in \mathbf{C}_m \;\middle|\; \limsup_{n\to\infty} d(S[0..n-1]) = \infty \right\}.$$

We say that d *succeeds* on $S \in \mathbf{C}_m$ if $S \in S^\infty[d]$. The *strong success set* of d is

$$S^\infty_{\mathrm{str}}[d] = \left\{ S \in \mathbf{C}_m \;\middle|\; \liminf_{n\to\infty} d(S[0..n-1]) = \infty \right\}.$$

We say that d *succeeds strongly* on $S \in \mathbf{C}$ if $S \in S^\infty_{\mathrm{str}}[d]$.

An s-supergale can be regarded as a betting strategy over sequences in $\mathbf{C}_m$. It starts with $d(\lambda)$, a finite amount of initial capital, and bets on the successive bits of a string w. The payoff of the betting is defined by the $d(w)$. The parameter s gauges the fairness of the betting environment. When $s = 1$, the betting environment is fair. We can then see from (2.2.1) that for any martingale d, if the house uniformly at random pick a string $w \in \Sigma^n$, the expected amount of payoff we can get from betting according to d is $d(\lambda)$. By the Markov inequality, the probability that we can make $k \cdot d(\lambda)$ amount of money is at most $\frac{1}{k}$. Then, intuitively, the probability that we can make unbounded amount of money is thus 0. This intuition gives rise to the definition of measure 0. And if we impose resource bound on the computation of martingales, we have resource-bounded measure.

Definition (Lutz [60])**.** Let $X \subseteq \mathbf{C}$. X has Δ-*measure* 0, and we write $\mu_\Delta(X) = 0$ if there exists a Δ-computable supermartingale d such that $X \subseteq S^\infty[d]$. X has Δ-*measure* 1 if X^c has Δ-measure 0. X has *measure* 0 *in* $R(\Delta)$ if $\mu_\Delta(X \cap R(\Delta)) = 0$. X has *measure* 1 *in* $R(\Delta)$ if $\mu_\Delta(X^c \cap R(\Delta)) = 0$.

For all these definition to make sense, it is essential that $R(\Delta)$ does not have measure 0. It is indeed so as affirmed by the following *measure conservation theorem*.

Theorem 2.2.1 (Lutz [60])**.** *$R(\Delta)$ does not have measure 0 in $R(\Delta)$.*

When $\Delta = \text{all}$, the measure defined by all Δ computable martingales coincides with the classical Lebesgue measure on $\mathbf{C}$ [60].

It turns out that the fairness parameter s of gales can be used to characterize the classical Hausdorff and packing dimensions [63, 7]. In here, we use the gale characterizations as definitions, since this provides us with unified definitions of resource-bounded dimensions and classical dimensions.

Definition (Lutz [63], Athreya, Hitchcock, Lutz, and Mayordomo [7])**.** Let $X \subseteq \mathbf{C}$. The Δ-*dimension* of X is

$$\dim_\Delta(X) = \inf\{s \in [0, \infty) | X \subseteq S^\infty[d] \text{ for some } \Delta\text{-computable } s\text{-supergale } d\}.$$

The Δ-*strong dimension* of X, denoted $\mathrm{Dim}_\Delta(X)$, is defined similarly with respect to strong success. The *dimension* of X in $R(\Delta)$ is $\dim(X|R(\Delta)) = \dim_\Delta(X \cap R(\Delta))$. The *strong dimension* of X in $R(\Delta)$ is $\mathrm{Dim}(X|R(\Delta)) = \mathrm{Dim}_\Delta(X \cap R(\Delta))$.

When Δ is the set of all functions (with no computational restriction), the above definitions of dimension and strong dimension give us the classical Hausdorff dimension $\dim_{\mathrm{H}}$ and packing dimension $\dim_{\mathrm{P}}$, respectively. When Δ is the set of all lower semi-computable functions, we get the notions of constructive dimension $\mathrm{cdim}(X)$ and strong dimension $\mathrm{cDim}(X)$.

Observation 2.2.2 (Lutz [63], Athreya, Hitchcock, Lutz, and Mayordomo [7])**.** *1. For all* $X \subseteq \mathbf{C}$ *and all resource bounds* Δ*, if* $\dim_\Delta(X) < 1$ *then* $\mu_\Delta(X) = 0$.

2. For all $X \subseteq \mathbf{C}$ *and all resource bounds* Δ*,* $\dim_\Delta(X) \leq \mathrm{Dim}_\Delta(X)$*.*

3. For all $X \subseteq Y$ *and all resource bounds* Δ*,* $\dim_\Delta(X) \leq \dim_\Delta(Y)$*.*

4. Let Δ*,* Δ' *be resource bounds such that* $\Delta \subseteq \Delta'$*. Then for all* $X \subseteq \mathbf{C}$*,* $\dim_{\Delta'}(X) \leq \dim_\Delta(X)$*, and* $\mathrm{Dim}_{\Delta'}(X) \leq \mathrm{Dim}_\Delta(X)$*.*

In contrast to classical measure and dimension theory, when resource bounds are enforced on the computation of gales, dimensions of individual sequences become meaningful.

Definition. Let $S \in \mathbf{C}$ be an infinite binary sequence. The Δ-*dimension* of S is $\dim_\Delta(S) = \dim_\Delta(\{S\})$. The Δ-*strong dimension* of S is $\mathrm{Dim}_\Delta(S) = \mathrm{Dim}_\Delta(\{S\})$.

Hitchcock, Lutz, and Mayordomo also introduced a theory of resource-bounded scaled dimension that has more distinguishing power for some problems in complexity theory.

Definition (Hitchcock, Lutz, and Mayordomo [47])**.** A *scale* is a continuous function $g : H \times [0, \infty) \to \mathbb{R}$ such that $H = (a, \infty)$ for some $a \in \mathbb{R} \cup \{-\infty\}$; $g(m, 1) = m$ for all $m \in H$; $g(m, 0) = g(m', 0) \geq 0$ for all $m, m' \in H$; for every sufficiently large $m \in H$, the function $s \mapsto g(m, s)$ is nonnegative and strictly increasing; and for all $s' > s \geq 0$, $\lim_{m\to\infty}[g(m, s') - g(m, s)] = \infty$.

Definition (Hitchcock, Lutz, and Mayordomo [47])**.** Let $g : H \times [0, \infty) \to \mathbb{R}$ be a scale, and let $s \in [0, \infty)$. A *g-scaled s-supergale* (*$s^{(g)}$-supergale*) is a function $d : \{0, 1\}^* \to [0, \infty)$ such that for all $w \in \{0,1\}^*$ with $|w| \in H$,

$$d(w) \geq \frac{d(w0) + d(w1)}{2^{\Delta g(|w|,s)}}, \tag{2.2.2}$$

where $\Delta g(m,s) = g(m+1,s) - g(m,s)$.

The definitions for scaled dimensions are identical to those of regular dimensions except that they use scaled supergales. In corresponding notations, we use superscript $^{(g)}$ to indicate the scale as in $\dim_{\Delta}^{(g)}(\cdot)$, $\mathrm{Dim}_{\Delta}^{(g)}(\cdot)$.

Some commonly used scales are defined as follows.

Definition (Hitchcock, Lutz, and Mayordomo[47])**.** Let $g : H \times [0,\infty) \to \mathbb{R}$ be a scale.

1. The *first rescaling* of g is the scale $g^{\#} : H^{\#} \times [0,\infty) \to \mathbb{R}$ defined by

$$H^{\#} = \{2^m | m \in H\},$$
$$g^{\#}(m,s) = 2^{g(\log m, s)}.$$

2. For each $i \in \mathbb{N}$, $a_0 = -\infty$, $a_{i+1} = 2^{a_i}$.

3. For each $i \in \mathbb{N}$, the ith scale $g_i : (a_i,\infty) \times [0,\infty) \to \mathbb{R}$ is defined such that

   (a) $g_0(m,s) = sm$.

   (b) For $i \geq 0$, $g_{i+1} = g_i^{\#}$.

When these scales are used, we use superscript $^{(i)}$ instead of $^{(g_i)}$. We call $\dim^{(i)}$ and $\mathrm{Dim}^{(i)}$ the ith-order scaled dimension and the ith-order scaled strong dimension, respectively. Resource-bounded 0th scaled dimensions and strong dimensions coincide with the regular dimensions and strong dimensions. With the scales defined above, it was shown that the scaled dimensions exhibit the following monotonicity with respect to the order of the scale.

Theorem 2.2.3 (Hitchcock, Lutz, and Mayordomo [47])**.** *Let* $i \in \mathbb{N}$ *and let* $X \subseteq \mathbf{C}$. *If* $\dim_{\Delta}^{(i+1)}(X) < 1$, *then* $\dim_{\Delta}^{(i)}(X) = 0$.

When we study the fractals in Euclidean spaces $\mathbb{R}^n$, we need to be able to properly encode points in Euclidean spaces using infinite sequences. One of several equivalent ways to achieve this is to expand the coordinates of each point $x = (x_1, ..., x_n) \in \mathbb{R}^n$ in base 2. If the expansions of the fractional parts of these coordinates are $S_1, ..., S_n \in \mathbf{C}$, respectively, then $S(x)$ is the interleaving of these sequences, i.e.,

$$S(x) = S_1[0]S_2[0]...S_n[0]S_1[1]S_2[1]...S_n[1]S_1[2]S_2[2]....$$

For each $X \subseteq \mathbb{R}^n$, $S(X) = \{S(x) \mid x \in X\}$. Then the (Hausdorff, computable, constructive) dimension of $S(X)$ is n times the (Hausdorff, computable, constructive) dimension of $S(X)$ [65].

For each $\alpha \in [0, n]$ we denote as $\mathrm{DIM}^{=\alpha}$ the set $\{x \mid \{x\}$ has constructive dimension $\alpha\}$. Note that Hausdorff dimension of any countable set is 0, while effective dimension of a singleton set may be as large as the dimension of the space.

The following fact is easily verified: if Δ is any of the *countable* resource bounds above, then

$$\dim_{\mathrm{H}}(\{S \mid \dim_{\Delta}(S) = 0\}) = 0. \tag{2.2.3}$$

Next we define category in Cantor space in terms of the Banach-Mazur game, more information can be found in [84].

Definition.

1. A *constructor* is a function $h : \{0,1\}^* \to \{0,1\}^*$ such that for every $w \in \{0,1\}^*$, $w \underset{\neq}{\sqsubset} h(w)$.
2. If h is a constructor, $R(h)$ is the only element in $\mathbf{C}$ such that $h^i(\lambda) \sqsubseteq R(h)$ for all $i \in \mathbb{N}$.
3. If g and h are constructors then $R(g, h) = R(h \circ g)$
4. $X \subseteq \mathbf{C}$ is meager if for every constructor g there is a constructor h such that $R(g, h) \notin X$.
5. $X \subseteq \mathbf{C}$ is co-meager if X^c is meager.

A useful property is that a countable union of meager sets is meager. Equivalently, if a $\cup X_i$ is co-meager then at least one of the X_i is co-meager.

Category in Euclidean space can be defined through the above identification of $X \subseteq \mathbb{R}^n$, with $S(X) \subseteq \mathbf{C}$. Notice that in this case co-meager sets are dense in some interval.

3 FRACTALS IN COMPLEXITY CLASSES

In this chapter, we examine some fractals in complexity classes from two aspects. In Section 3.1, we examine the dimensions of some circuit complexity classes. In Section 3.2, we look at the power of fractals, in particular, the derandomization power of non-trivial fractals and use such power to characterize some important complexity classes.

3.1 Dimensions of Polynomial-Size Circuits

Our starting point is the following theorem regarding polynomial size circuits.

Theorem 3.1.1 (Lutz[60]). *For all $c > 0$,*

$$\mu(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)|\mathrm{EXP}) = \mu_{\mathrm{p}_2}(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)) = 0$$

and

$$\mu(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \mu_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = 0.$$

This result was improved to dimension as follows by Hitchcock and Vinodchandran.

Theorem 3.1.2 ([48]). *For all $c \geq 1$,*

$$\dim(\mathrm{SIZE}(n^c)|\mathrm{EXP}) = \dim_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0$$

and

$$\dim(\mathrm{P/poly}|\mathrm{E}_3) = \dim_{\mathrm{p}_3}(\mathrm{P/poly}) = 0.$$

We use the relationship between the following time bounded Kolmogorov complexity and circuit complexity to give a more thorough analysis of the dimensions of polynomial size circuits.

Definition (Allender [1])**.** Let U be a universal Turing machine. Define $\mathrm{KT}(x)$ to be

$$\min\{|p| + t| \text{ for all } i \leq |x|, U(p,i) = x_i \text{ in at most } t \text{ steps}\}.$$

Theorem 3.1.3 (Allender [1], Allender, Buhrman, Koucký, van Melkebeek, and Ronneburger [2])**.** $\mathrm{SIZE}(x) = O((\mathrm{KT}(x))^4)$*, and* $\mathrm{KT}(x) = O((\mathrm{SIZE}(x))^2 \cdot (\log(\mathrm{SIZE}(x))^2 + \log|x|))$*.*

Lemma 3.1.4. *Let* $A \subseteq \{0,1\}^*$*.*

1. $A \in \mathrm{P/poly}^{\mathrm{i.o.}}$ *if and only if for some integer* $c \in \mathbb{N}$*,* $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) \leq n^c$ *for infinitely many* $n \in \mathbb{N}$*.*

2. $A \in \mathrm{P/poly}$ *if and only if for some integer* $c \in \mathbb{N}$*,* $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) \leq n^c$ *for all but finitely many* $n \in \mathbb{N}$*.*

Proof. Both follow from Theorem 3.1.3. □

Using this lemma, we first establish the following two theorems for individual languages concerning $\mathrm{P/poly}^{\mathrm{i.o.}}$ and $\mathrm{P/poly}$.

Theorem 3.1.5. *Let* $A \subseteq \{0,1\}^*$ *be a language such that* $\dim_{\mathrm{p}_2}(A) > \frac{1}{2}$*. Then* $A \notin \mathrm{P/poly}^{\mathrm{i.o.}}$*.*

Proof. We prove the contrapositive. Assume that $A \in \mathrm{P/poly}^{\mathrm{i.o.}}$. Then by Lemma 3.1.4, $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) < n^c$ for infinitely many n and some fixed constant c. It suffices to show that $\dim_{\mathrm{p}_2}(A) \leq \frac{1}{2}$.

Let $r > \frac{1}{2}$ be a polynomial-time computable real number. It suffices to show that there exists a p_2-computable r-supergale d that succeeds on A.

For $i \geq 1$ and $w \in \{0,1\}^*$, let

$$C_i = \{x \in \{0,1\}^{2^i} \mid \mathrm{KT}(x) < i^c\}$$

$$C_i^w = \{x \in C_i \mid w[2^i - 1..|w| - 1] \sqsubseteq x\},$$

and let d_i be such that

$$d_i(w) = \begin{cases} 2^{(r-1)|w|} & \text{if } |w| < 2^i \\ d_i(w[0..2^i - 2])2^{r(|w|-(2^i-1))}\frac{|C_i^w|}{|C_i|} & \text{if } 2^i \leq |w| \leq 2^{i+1} - 1 \\ 2^{(r-1)(|w|-(2^{i+1}-1))}d_i(w[0..2^{i+1} - 2]) & \text{if } |w| > 2^{i+1} - 1. \end{cases}$$

We compute d_i by simulating the universal Turing machine U to enumerate C_i by cycling all programs of length up to i^c and all bit indices less than or equal to 2^i within running time less than i^c. For every such program, a valid simulation generates 2^i bits and by concatenating them, we get an output string of length 2^i in C_i. During the enumeration, d_i counts the number of strings in C_i and in C_i^w to get $|C_i|$ and $|C_i^w|$. Note that $|C_i| \leq 2^{i^c}$.

Let $d = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i$. It is easy to verify that d is a p_2-computable r-supergale.

For any $n \geq 1$ such that $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) < n^c$, we have

$$\begin{aligned} d(A[0..2^{n+1}-2]) &\geq \frac{1}{2^n} d_n(A[0..2^{n+1}-2]) \\ &= \frac{1}{2^n} d_n(A[0..2^n-2]) 2^{r2^n} \frac{\left|C_n^{A[0..2^{n+1}-2]}\right|}{|C_n|} \\ &\geq \frac{1}{2^n} 2^{(r-1)(2^n-1)} 2^{r2^n} \frac{1}{2^{n^c}} \\ &= \frac{2^{(2r-1)2^n - r + 1}}{2^n \cdot 2^{n^c}}. \end{aligned}$$

Since $r > \frac{1}{2}$ and $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) < n^c$ for infinitely many n, it follows that the value that the r-supergale d can obtain along A is unbounded, and thus $\dim_{\mathrm{p}_2}(A) \leq r$. Since polynomial-time computable real numbers are dense in $\mathbb{R}$, it follows that $\dim_{\mathrm{p}_2}(A) \leq \frac{1}{2}$. □

Corollary 3.1.6. *For $c > 0$,*

$$\dim(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)|\mathrm{EXP}) \leq \frac{1}{2} \text{ and } \dim_{\mathrm{p}_2}(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)) \leq \frac{1}{2}$$

and

$$\dim(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) \leq \frac{1}{2} \text{ and } \dim_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) \leq \frac{1}{2}.$$

Proof. By Theorem 3.1.5 and standard universal simulation techniques, $\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)$ is a p_2-union of sets of p_2-dimension at most $\frac{1}{2}$, and $\mathrm{P/poly}^{\mathrm{i.o.}}$ is a p_3-union of sets of p_2-dimension (hence p_3-dimension) at most $\frac{1}{2}$. The corollary then follows by the effective stability of resource-bounded dimension (Lemma 4.11 of [63]). □

By changing the simulation in the proof of Theorem 3.1.5 from cycling programs of length exactly i to cycling programs of length at most i, we can establish an analogous result regarding P/poly, but now with strong dimension.

Theorem 3.1.7. *Let* $A \subseteq \{0,1\}^*$ *be a language such that* $\mathrm{Dim}_{\mathrm{p}_2}(A) > 0$. *Then* $A \notin \mathrm{P/poly}$.

Proof. We prove the contrapositive. Assume that $A \in \mathrm{P/poly}$. Then by Lemma 3.1.4, $\mathrm{KT}(A[2^n-1..2^{n+1}-2]) < n^c$ for all but finitely many $n \in \mathbb{N}$ and some fixed constant c. It suffices to show that $\mathrm{Dim}_{\mathrm{p}_2}(A) = 0$.

Let $r > 0$ be a polynomial-time computable real number. It suffices to show that there exists a p_2-computable r-supergale d that succeeds on A.

For $i \geq 1$ and $w \in \{0,1\}^*$, let

$$C_{\leq i} = \{x \in \{0,1\}^{2^{i+1}-1} \mid \mathrm{KT}(x[2^k-1..2^{k+1}-2]) < k^c, 0 < k \leq i\}$$

$$C^w_{\leq i} = \{x \in C_{\leq i} \mid w \sqsubseteq x\},$$

and let d_i be such that

$$d_i(w) = \begin{cases} 2^{r|w|}\frac{|C^w_{\leq i}|}{|C_{\leq i}|} & \text{if } |w| \leq 2^{i+1}-1 \\ 2^{(r-1)(|w|-(2^{i+1}-1))}d_i(w[0..2^{i+1}-2]) & \text{if } |w| > 2^{i+1}-1. \end{cases}$$

We compute d_i by simulating the universal Turing machine U to enumerate $C_{\leq i}$ by cycling all programs of length at most k^c and all bit indices less than or equal to 2^k within running time less than k^c for $k = 0, 1, \ldots, i$ in a depth first fashion. For every such k and a particular program, a valid simulation generates 2^k bits and by concatenating them, we get an output string of length 2^k. By concatenating the outputs for k from 0 to i, we get a string of length $2^{i+1}-1$ in $C_{\leq i}$. $|C_{\leq i}|$ and $|C^w_{\leq i}|$ are obtained respectively by counting the number of strings in $C_{\leq i}$ and the number of those strings with w as a prefix. Note that $|C_{\leq i}| \leq 2^{i^{c+1}}$.

Let $d = \sum_{i=1}^{\infty}\frac{1}{2^i}d_i$. It is easy to verify that d is a p_2-computable r-supergale.

For any $n > 1$ and $0 < k \leq 2^n$, we have

$$\begin{aligned} d(A[0..2^n-2+k]) &\geq \frac{1}{2^n}d_n(A[0..2^n-2+k]) \\ &= \frac{1}{2^n}2^{r(2^n-1+k)}\frac{\left|C^{A[0..2^n-2+k]}_{\leq n}\right|}{|C_{\leq n}|} \\ &\geq \frac{1}{2^n}2^{r(2^n-1+k)}\frac{1}{2^{n^{c+1}}}. \end{aligned}$$

Since $r > 0$ and $k > 0$, it follows that the value that the r-supergale d can obtain along A goes to infinity, i.e.,

$$\liminf_{n\to\infty} d(A[0..n-1]) = \infty.$$

So the r-supergale d succeeds strongly on A, and hence the $\mathrm{Dim}_{\mathrm{p}_2}(A) \leq r$. By the density of polynomial-time computable real numbers, $\mathrm{Dim}_{\mathrm{p}_2}(A) = 0$. □

Our next theorem shows that scaled dimension can be used to significantly relax the hypothesis of Theorem 3.1.7. We first give an observation about the transformation between different scaled supergales that simplifies the calculation of scaled dimensions.

Observation 3.1.8. *Let g_1, g_2 be two scales and $s_1, s_2 \in [0, \infty)$. Let $d : \{0,1\}^* \to [0,\infty)$ be a g_1-scaled s_1-supergale ($s_1^{(g_1)}$-supergale), i.e.,*

$$d(w) \geq \frac{d(w0) + d(w1)}{2^{\Delta g_1(|w|, s_1)}}.$$

Then the function $d' : \{0,1\}^* \to [0,\infty)$ defined by $d'(w) = d(w)2^{g_2(|w|,s_2) - g_1(|w|,s_1)}$ is an $s_2^{(g_2)}$-supergale.

Proof. This follows from easy verification of the $s_2^{(g_2)}$-supergale condition (2.2.2). □

Now we use Observation 3.1.8 to extend Theorem 3.1.7 to scales of arbitrary nonnegative order.

Theorem 3.1.9. *Let $j \in \mathbb{N}$ and $A \subseteq \{0,1\}^*$ be a language such that $\mathrm{Dim}^{(j)}_{\mathrm{p}_2}(A) > 0$. Then $A \notin \mathrm{P/poly}$.*

Proof. We prove the contrapositive. Assume that $A \in \mathrm{P/poly}$. Then by Lemma 3.1.4, $\mathrm{KT}(A[2^n - 1..2^{n+1} - 2]) < n^c$ for all but finitely many $n \in \mathbb{N}$ and some fixed constant c. It suffices to show that $\mathrm{Dim}^{(j)}_{\mathrm{p}_2}(A) = 0$.

Let $s > 0$ be a polynomial-time computable real number. It suffices to show that there exists a p_2-computable $s^{(j)}$-supergale that succeeds on A.

Let $r > 0$ be a polynomial-time computable real number. For all $i \in \mathbb{N}$, let d_i be defined as in the proof of Theorem 3.1.7 and similarly let $d = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i$. It is clear that d is a p_2-computable r-supergale.

Define d' such that

$$d'(w) = d(w)2^{g_j(|w|,s) - g_0(|w|,r)}.$$

By Observation 3.1.8, d' is a p_2-computable $s^{(j)}$-supergale and

$$\begin{aligned} d'(A[0..2^n - 2 + k]) &\geq \frac{1}{2^n} d_n(A[0..2^n - 2 + k]) \frac{2^{g_j(2^n - 1 + k, s)}}{2^{r(2^n - 1 + k)}} \\ &\geq \frac{1}{2^n} \frac{2^{r(2^n-1+k)}}{2^{n^{c+1}}} \frac{2^{g_j(2^n - 1 + k, s)}}{2^{r(2^n - 1 + k)}} \\ &= \frac{2^{g_j(2^n - 1 + k, s)}}{2^n \cdot 2^{n^{c+1}}}. \end{aligned}$$

Since $s > 0$, $c \in \mathbb{N}$, $k > 0$, the growth rate of the function $g_j(2^n - 1 + k, s)$ is higher than that of the function n^{c+1}. It follows that $\liminf_{n \to \infty} d'(A[0..2^n - 2 + k]) = \infty$, i.e., $\mathrm{Dim}^{(j)}_{\mathrm{p}_2}(A) = 0$. □

By using Theorem 3.1.7 and Theorem 3.1.9 together with the same techniques used in the proof of Corollary 3.1.6, we obtain the following theorem.

Theorem 3.1.10. *For all $c > 0$ and all $i \in \mathbb{N}$,*

$$\mathrm{Dim}^{(i)}(\mathrm{SIZE}(n^c)|\mathrm{EXP}) = \mathrm{Dim}^{(i)}_{\mathrm{p}_2}(\mathrm{SIZE}(n^c)) = 0$$

and

$$\mathrm{Dim}^{(i)}(\mathrm{P/poly}|\mathrm{E}_3) = \mathrm{Dim}^{(i)}_{\mathrm{p}_3}(\mathrm{P/poly}) = 0.$$

Jack Lutz suggested that the upper bounds for dimensions in Corollary 3.1.6 are tight. We prove a general theorem on dimension lower bound of infinitely-often classes, which is then used to show that the inequalities in Corollary 3.1.6 may be replaced by equalities. In the proof, we will use the supergale dilation technique, which is an extension of the martingale dilation technique introduced by Ambos-Spies, Terwijn, and Zheng implicitly [4] and made explicit by Juedes and Lutz [51]. In the following, we only state and prove the case for nonnegative scales of dimensions and strong dimensions. Both the theorem and the corollary generalize to negative scales.

Definition. Let $f : \{0,1\}^* \to \{0,1\}^*$. We call f a *dilation* if for all $x, y \in \{0,1\}^*$ with $x \sqsubseteq y$, $f(x) \sqsubseteq f(y)$, and for all x, there exists $x \sqsubseteq x'$ such that $f(x) \sqsubsetneq f(x')$, and $|f(x0)| = |f(x1)| \leq |f(x)| + 1$ for all $x \in \{0,1\}^*$.

Let f be a dilation. For $A \in \mathbf{C}$, let $f(A) = S \in \mathbf{C}$ such that $f(A[0..n-1]) \sqsubseteq S$ for all $n \in \mathbb{N}$. We call $f(A)$ the *f-dilation* of A. For $x \in \{0,1\}^*$, define the *collision set of f on x* as

$$\mathrm{Col}(f,x) = \big\{0 \leq n < |x| \;\big|\; f(x[0..n-1]0) = f(x[0..n-1]1) \neq f(x[0..n-1])\big\}.$$

Theorem 3.1.11 (Supergale Dilation Theorem)**.** *Let $\mathcal{C} \subseteq \mathbf{C}$, Δ be a resource bound, $i, j \in \mathbb{N}$ and $s, s' \in [0,1]$. Let f be a Δ-computable dilation.*

1. *If $\dim_\Delta^{(i)}(f(\mathcal{C})) < s$ and for every $A \in \mathcal{C}$,*

$$g_i(|f(A[0..n-1])|, s) + |\mathrm{Col}(f, A[0..n-1])| \leq g_j(n, s') - n + |f(A[0..n-1])| \qquad (3.1.1)$$

for all but finitely many n, then

$$\dim_\Delta^{(j)}(\mathcal{C}) \leq s'.$$

2. *If $\mathrm{Dim}_\Delta^{(i)}(f(\mathcal{C})) < s$ and for every $A \in \mathcal{C}$, (3.1.1) holds for infinitely many n, then*

$$\dim_\Delta^{(j)}(\mathcal{C}) \leq s'.$$

3. *If $\mathrm{Dim}_\Delta^{(i)}(f(\mathcal{C})) < s$ and for every $A \in \mathcal{C}$, (3.1.1) holds for all but finitely many n, then*

$$\mathrm{Dim}_\Delta^{(j)}(\mathcal{C}) \leq s'.$$

Note that in contrast to [4] and [51], we are looking at the dilation from a different perspective. In [4] and [51], the dilation is defined in terms of strings (in languages). Here, the dilation is defined in terms of the prefixes of characteristic sequences of languages. It is easy to verify that every dilation that is consistent with [51] can be written in a way that is consistent with the definition we have here. But the converse is not true.

Proof. We prove 1; the proofs of 2 and 3 are similar. Since $\dim_\Delta^{(j)}(\mathcal{C}) \leq 1$, the theorem is true when $s' \geq 1$. Assume $s' < 1$, $\dim_\Delta^{(i)}(f(\mathcal{C})) < s$ and (3.1.1). Then, by Observation 3.1.8, there exists a Δ-computable supermartingale d such that for every $A \in \mathcal{C}$ and some $\epsilon > 0$,

$$d(f(A[0..n-1])) \geq 2^{|f(A[0..n-1])| - g_i(|f(A[0..n-1])|, s-\epsilon)} \qquad (3.1.2)$$

for infinitely many n. Define d' with the following recursion.

$$\begin{cases} d'(\lambda) = d(f(\lambda)) \\ d'(wb) = 2d'(w)\frac{d(f(wb))}{d(f(w0))+d(f(w1))}. \end{cases}$$

Since f is Δ-computable, it is clear that d' is a Δ-computable martingale. Note that

$$d'(A[0..n-1]) = d(f(A[0..n-1])) \prod_{i=0}^{n-2} \frac{d(f(A[0..i])) \cdot 2}{d(f(A[0..i]0)) + d(f(A[0..i]1))}.$$

Since d is a martingale, for each $i \notin \mathrm{Col}(f, A[0..n-1])$

$$\frac{d(f(A[0..i])) \cdot 2}{d(f(A[0..i]0)) + d(f(A[0..i]1))} = 1$$

and $i \in \mathrm{Col}(f, A[0..n-1])$

$$\frac{d(f(A[0..i])) \cdot 2}{d(f(A[0..i]0)) + d(f(A[0..i]1))} \geq \frac{1}{2}.$$

Therefore

$$d'(A[0..n-1]) \geq \frac{d(f(A[0..n-1]))}{2^{|\mathrm{Col}(f,A[0..n-1])|}}.$$

Since (3.1.2) holds for infinitely many n, and (3.1.1) holds for all but finitely many n, we have that, for infinitely many n,

$$\begin{aligned} d'(A[0..n-1]) &\geq \frac{2^{|f(A[0..n-1])|-g_i(|f(A[0..n-1])|,s-\epsilon)}}{2^{g_j(n,s')-n+|f(A[0..n-1])|-g_i(|f(A[0..n-1])|,s)}} \\ &> 2^{n-g_j(n,s')}. \end{aligned}$$

Since $\lim_{n\to\infty} 2^{n-g_j(n,s')} = \infty$ for $s' < 1$, $\dim_\Delta^{(j)}(\mathcal{C}) \leq s'$. □

Corollary 3.1.12. *Let $\mathcal{C} \subseteq \mathbf{C}$, Δ be a resource bound, $i, j \in \mathbb{N}$ and $s, s' \in [0,1]$. Let f be a Δ-computable dilation.*

1. *If $\dim^{(i)}(f(\mathcal{C})|R(\Delta)) < s$ and for every $A \in \mathcal{C}$, (3.1.1) holds for all but finitely many n, then*

$$\dim^{(j)}(\mathcal{C}|R(\Delta)) \leq s'.$$

2. *If $\mathrm{Dim}^{(i)}(f(\mathcal{C})|R(\Delta)) < s$ and for every $A \in \mathcal{C}$, (3.1.1) holds for infinitely many n, then*

$$\dim^{(j)}(\mathcal{C}|R(\Delta)) \leq s'.$$

3. *If* $\mathrm{Dim}^{(i)}(f(\mathcal{C})|R(\Delta)) < s$ *and for every* $A \in \mathcal{C}$, (3.1.1) *holds for all but finitely many* n, *then*

$$\mathrm{Dim}^{(j)}(\mathcal{C}|R(\Delta)) \leq s'.$$

Proof. We prove 1; the proofs of 2 and 3 are similar.

Note that $f(\mathcal{C} \cap R(\Delta)) \subseteq R(\Delta)$ and $f(\mathcal{C} \cap R(\Delta)) \subseteq f(\mathcal{C})$. Therefore $f(\mathcal{C} \cap R(\Delta)) \subseteq f(\mathcal{C}) \cap R(\Delta)$.

Since $\dim_\Delta^{(i)}(f(\mathcal{C}) \cap R(\Delta)) = \dim^{(i)}(f(\mathcal{C})|R(\Delta)) < s$, $\dim_\Delta^{(i)}(f(\mathcal{C} \cap R(\Delta))) < s$. Now apply Theorem 3.1.11 and we have

$$\dim_\Delta^{(j)}(\mathcal{C} \cap R(\Delta)) < s',$$

i.e., $\dim^{(j)}(\mathcal{C}|R(\Delta)) \leq s'$. □

Theorem 3.1.13. *Let* $\mathcal{C}$ *be a language class that contains the trivial language* $\varnothing$. *Then for all* $\Delta \supseteq \mathrm{p}$, $\dim(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) \geq 1/2$ *and* $\mathrm{Dim}(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) = 1$.

Proof. Let $f : \{0,1\}^* \to \{0,1\}^*$ be defined such that for all $x \in \{0,1\}^*$, $|f(x)| = |x|$ and for all $i < |x|$,

$$f(x)[i] = \begin{cases} 0 & |s_i| = 2^k \text{ for some } k \\ x[i] & \text{otherwise.} \end{cases}$$

It is clear that f is a p-computable dilation.

By the construction of f, it is easy to see that $f(R(\Delta)) \subseteq \mathcal{C}^{\mathrm{i.o.}}$. Also note that $f(R(\Delta)) \subseteq R(\Delta)$.

Let

$$\#n = \left|\left\{i < n \;\middle|\; |s_i| = 2^k \text{ for some } k\right\}\right|.$$

Note that for all $n \in \mathbb{N}$ and all $A \in \mathbf{C}$,

$$|\mathrm{Col}(f, A[0..n-1])| = \#n$$

and

$$|f(A[0..n-1])| = n.$$

It is easy to verify that for every $A \in R(\Delta)$,

$$|\mathrm{Col}(f, A[0..n-1])| \leq n/2 + 2\sqrt{n/2}$$

for all but finitely many n and

$$|\mathrm{Col}(f, A[0..n-1])| \leq \sqrt{n}$$

for infinitely many n. Let $\epsilon > 0$. Now we have that, for all but finitely many n,

$$\begin{aligned}(1/2 - 2\epsilon)\, n + |\mathrm{Col}(f, A[0..n-1])| &\leq n/2 - 2\epsilon n + n/2 + 2\sqrt{n/2} \\ &= (1-2\epsilon)n + 2\sqrt{n/2} \\ &\leq (1-\epsilon)n,\end{aligned}$$

i.e.,

$$g_0(n, 1/2 - 2\epsilon) + |\mathrm{Col}(f, A[0..n-1])| \leq g_0(n, 1-\epsilon) \text{ for all but finitely many } n. \tag{3.1.3}$$

And similarly

$$g_0(n, 1 - 2\epsilon) + |\mathrm{Col}(f, A[0..n-1])| \leq g_0(n, 1-\epsilon) \text{ for infinitely many } n. \tag{3.1.4}$$

Note that $\dim(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) < 1/2$ implies that $\dim_\Delta(f(R(\Delta))) = \dim(f(R(\Delta))|R(\Delta)) < 1/2$. By Theorem 3.1.11 and (3.1.3), $\dim(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) < 1/2$ then implies that $\dim_\Delta(R(\Delta)) < 1$, which by Observation 2.2.2, implies $\mu_\Delta(R(\Delta)) = 0$. By the Measure Conservation Theorem, we know that $\mu_\Delta(R(\Delta)) = 1$. Thus $\dim(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) \geq 1/2$.

Similarly, $\mathrm{Dim}(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) < 1$ implies that $\mathrm{Dim}_\Delta(f(R(\Delta))) = \mathrm{Dim}(f(R(\Delta))|R(\Delta)) < 1$. By Theorem 3.1.11 and (3.1.4), $\mathrm{Dim}(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) < 1$ then implies that $\dim_\Delta(R(\Delta)) < 1$ and thus $\mu_\Delta(R(\Delta)) = 0$. Again by the Measure Conservation Theorem, $\mathrm{Dim}(\mathcal{C}^{\mathrm{i.o.}}|R(\Delta)) = 1$. □

Corollary 3.1.14. *Let $\mathcal{C}$ be a language class that contains the trivial language $\varnothing$. Then for all $\Delta \supseteq \mathrm{p}$, $\dim_\Delta(\mathcal{C}^{\mathrm{i.o.}}) \geq \frac{1}{2}$ and $\mathrm{Dim}_\Delta(\mathcal{C}^{\mathrm{i.o.}}) = 1$.*

Corollary 3.1.15. *Let $\mathcal{C}$ be a language class that contains the trivial language $\varnothing$. Then Hausdorff dimension $\dim_{\mathrm{H}}(\mathcal{C}^{\mathrm{i.o.}}) \geq \frac{1}{2}$ and packing dimension $\dim_{\mathrm{P}}(\mathcal{C}^{\mathrm{i.o.}}) = 1$.*

Proof. Let Δ be all functions from $\{0,1\}^* \to \{0,1\}^*$. This follows immediately. $\square$

Now by Observation 2.2.2 and Corollary 3.1.6, we obtain the following theorem.

Theorem 3.1.16. *For all* $c > 0$

$$\dim(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)|\mathrm{EXP}) = \dim_{\mathrm{p}_2}(\mathrm{SIZE}^{\mathrm{i.o.}}(n^c)) = \frac{1}{2},$$

$$\dim(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \dim_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = \tfrac{1}{2}$$

and

$$\mathrm{Dim}(\mathrm{P/poly}^{\mathrm{i.o.}}|\mathrm{E}_3) = \mathrm{Dim}_{\mathrm{p}_3}(\mathrm{P/poly}^{\mathrm{i.o.}}) = 1.$$

By Theorem 2.2.3, the 0th scale is the best scale for evaluating scaled p_3-dimension of $\mathrm{P/poly}^{\mathrm{i.o.}}$. We cannot obtain more informative strong dimension results about $\mathrm{P/poly}^{\mathrm{i.o.}}$ and it is not hard to show that for any infinitely-often class, the scaled strong dimension is 1 for every scale g_i (even for $i < 0$, see [47]). The statement involving strong dimension of infinitely often classes in Theorem 3.1.13 also generalizes to all scales.

3.2 Fractals and Derandomization

In last section, we calculated the some dimensions of the classes of polynomial-size circuits. In this section, we will look at some complexity-theoretic consequences of sequences having non-zero dimensions.

3.2.1 Resource-Bounded Dimension and Relativized Circuit Complexity

We first review and develop those aspects of resource-bounded dimension and its relationship to relativized circuit-size complexity that are needed here. It is convenient to use entropy rates as an intermediate step in this development.

We use a recent result of Hitchcock and Vinodchandran [48] relating entropy rates to dimension. Entropy rates were studied by Chomsky and Miller [22], Kuich [56], Staiger [90, 91], Hitchcock [44], and others.

Definition. The *entropy rate* of a language $A \subseteq \{0,1\}^*$ is

$$H_A = \limsup_{n\to\infty} \frac{\log |A_{=n}|}{n},$$

where $A_{=n} = A \cap \{0,1\}^n$.

Definition. Let $\mathcal{C}$ be a class of languages, and let $X \subseteq \mathbf{C}$. The $\mathcal{C}$-*entropy rate* of X is

$$\mathcal{H}_{\mathcal{C}}(X) = \inf \left\{ H_A \;\middle|\; A \in \mathcal{C} \text{ and } X \subseteq A^{\text{i.o.}} \right\},$$

where

$$A^{\text{i.o.}} = \{ S \in \mathbf{C} \mid (\exists^\infty n) S[0..n-1] \in A \}.$$

The following result is a routine relativization of Theorem 5.5 of [48].

Theorem 3.2.1. *(Hitchcock and Vinodchandran [48]). For all $X \subseteq \mathbf{C}$ and $k \in \mathbb{Z}^+$,*

$$\dim_{\Delta^{\mathrm{P}}_{k+2}}(X) \le \mathcal{H}_{\Sigma^{\mathrm{P}}_k}(X).$$

Definition.

1. ([99]) For $f : \{0,1\}^n \to \{0,1\}$ and $A \subseteq \{0,1\}^*$, $\mathrm{size}^A(f)$ is the minimum size of (i.e., number of wires in) an n-input oracle circuit γ such that γ^A computes f.

2. For $x \in \{0,1\}^*$ and $A \subseteq \{0,1\}^*$, $\mathrm{size}^A(x) = \mathrm{size}^A(f_x)$, where $f_x : \{0,1\}^{\lceil \log |x| \rceil} \to \{0,1\}$ is defined by

$$f_x(w_i) = \begin{cases} x[i] & \text{if } 0 \le i < |x| \\ 0 & \text{if } i \ge |x|, \end{cases}$$

$w_0, \ldots, w_{2^{\lceil \log |x| \rceil}-1}$ lexicographically enumerate $\{0,1\}^{\lceil \log |x| \rceil}$, and $x[i]$ is the ith bit of x.

Lemma 3.2.2. *For all $A, S \in \mathbf{C}$,*

$$\mathcal{H}_{\mathrm{NP}^A}(\{S\}) \le \liminf_{n\to\infty} \frac{\mathrm{size}^A(S[0..n-1]) \log n}{n}.$$

Proof. Assume that

$$\alpha > \beta > \liminf_{n\to\infty} \frac{\mathrm{size}^A(S[0..n-1]) \log n}{n}.$$

Since α and β are arbitrary, it suffices to show that $\mathcal{H}_{\mathrm{NP}^A}(\{S\}) \le \alpha$.

Let B be the set of all strings x such that $\text{size}^A(x) < \beta\frac{|x|}{\log|x|}$. By standard circuit-counting arguments (e.g., see [66]), there is a constant $c \in \mathbb{N}$ such that, for all sufficiently large n, if we choose $m \in \mathbb{N}$ with $2^{m-1} \leq n < 2^m$ and write $\gamma = 2^{-m}n$, so that

$$\beta\frac{n}{\log n} = \beta\frac{\gamma 2^m}{\log(\gamma 2^m)} \leq \beta\gamma\frac{2^m}{m-1},$$

then

$$|B_{=n}| \leq c\left(4e\beta\gamma\frac{2^m}{m-1}\right)^{\beta\gamma\frac{2^m}{m-1}},$$

so

$$\begin{aligned}\log|B_{=n}| &\leq \log c + \beta\gamma\frac{2^m}{m-1}\log\left(4e\beta\gamma\frac{2^m}{m-1}\right)\\ &= \log c + \beta\gamma 2^m\left[\frac{m}{m-1} + \frac{\log 4e\beta\gamma - \log(m-1)}{m-1}\right]\\ &\leq \alpha n,\end{aligned}$$

whence

$$H_B = \limsup_{n\to\infty}\frac{\log|B_{=n}|}{n} \leq \alpha.$$

By our choice of β, $S \in B^{\text{i.o.}}$. Since $B \in \text{NP}^A$, it follows that $\mathcal{H}_{\text{NP}^A}(\{S\}) \leq \alpha$. □

Notation. For $k \in \mathbb{N}$ and $x \in \{0,1\}^*$, we write

$$\text{size}^{\Sigma_k^{\text{P}}}(x) = \text{size}^{K^k}(x),$$

where K^k is the canonical Σ_k^{P}-complete language [8].

By Theorem 3.2.1 and Lemma 3.2.2, we have the following connection between a language's dimension and its circuit complexity.

Theorem 3.2.3. *For all $S \in \mathbf{C}$ and $k \in \mathbb{N}$,*

$$\dim_{\Delta_{k+3}^{\text{P}}}(S) \leq \liminf_{n\to\infty}\frac{\text{size}^{\Sigma_k^{\text{P}}}(S[0..n-1])\log n}{n}.$$

3.2.2 Probabilistic Promise Problems

Definition. Given a class $\mathcal{C}$ of languages, an ordered pair $A = (A^+, A^-)$ of (disjoint) languages is $\mathcal{C}$-*separable* if there exists a language $C \in \mathcal{C}$ such that $A^+ \subseteq C$ and $A^- \cap C = \varnothing$. We write

$$\mathcal{C}\text{-Sep} = \left\{(A^+, A^-) \;\middle|\; (A^+, A^-) \text{ is } \mathcal{C}\text{-separable}\right\}.$$

Definition. Fix a standard paring function $\langle , \rangle : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$.

1. A *witness configuration* is an ordered pair $\mathcal{B} = (B, g)$ where $B \subseteq \{0,1\}^*$ and $g : \mathbb{N} \to \mathbb{N}$.

2. Given a witness configuration $\mathcal{B} = (B, g)$, the $\mathcal{B}$-*critical event* for a string $x \in \{0,1\}^*$ is the set

$$\mathcal{B}_x = \left\{ w \in \{0,1\}^{g(|x|)} \mid \langle x, w \rangle \in B \right\},$$

   interpreted as an event in the sample space $\{0,1\}^{g(|x|)}$ with the uniform probability measure. (That is, the probability of $\mathcal{B}_x$ is $\Pr(\mathcal{B}_x) = 2^{-g(|x|)}|\mathcal{B}_x|$.)

3. Given a class $\mathcal{C}$ of languages, we define the class Promise-BP $\cdot\, \mathcal{C}$ to be the set of all ordered pairs $A = (A^+, A^-)$ of languages for which there is a witness configuration $\mathcal{B} = (B, q)$ with the following four properties.

   (i) $B \in \mathcal{C}$.

   (ii) q is a polynomial.

   (iii) For all $x \in A^+$, $\Pr(\mathcal{B}_x) \geq \frac{2}{3}$.

   (iv) For all $x \in A^-$, $\Pr(\mathcal{B}_x) \leq \frac{1}{3}$.

Note that Promise-BP is an operator that maps a class $\mathcal{C}$ of languages to a class Promise-BP$\cdot$ $\mathcal{C}$ of disjoint pairs of languages. In particular,

$$\text{Promise-BP} \cdot \text{P} = \text{Promise-BPP}$$

is the class of *BPP promise problems* investigated by Buhrman and Fortnow [17] and Moser [73], and

$$\text{Promise-BP} \cdot \text{NP} = \text{Promise-AM}$$

is the class of *Arthur-Merlin promise problems* investigated by Moser [74].

3.2.3 Positive-Dimension Derandomization

We first state the main theorem regarding the relationship between dimensionality and derandomization.

Theorem 3.2.4. *For every $S \in \mathbf{C}$ and $k \in \mathbb{N}$,*

$$\dim_{\Delta^{\mathrm{P}}_{k+3}}(S) > 0 \implies \text{Promise-BP} \cdot \Sigma^{\mathrm{P}}_k \subseteq \Sigma^{\mathrm{P},S}_k\text{-Sep}.$$

The proof of Theorem 3.2.4 uses the lower bound on the circuit complexity of S provided by Theorem 3.2.3, to derandomize the probabilistic computation via Impagliazzo and Wigderson's pseudorandom generator [49]. Before proving Theorem 3.2.4, we derive some of its consequences. First, the cases $k = 0$ and $k = 1$ are of particular interest:

Corollary 3.2.5. *For every $S \in \mathbf{C}$,*

$$\dim_{\Delta^{\mathrm{P}}_3}(S) > 0 \implies \text{Promise-BPP} \subseteq \mathrm{P}^S\text{-Sep}$$

and

$$\dim_{\Delta^{\mathrm{P}}_4}(S) > 0 \implies \text{Promise-AM} \subseteq \mathrm{NP}^S\text{-Sep}.$$

We next note that our results for promise problems imply the corresponding results for decision problems. (Note, however, that the results of Fortnow [33] suggest that the results on promise problems are in some sense stronger.)

Corollary 3.2.6. *For every $S \in \mathbf{C}$ and $k \in \mathbb{N}$,*

$$\dim_{\Delta^{\mathrm{P}}_{k+3}}(S) > 0 \implies \mathrm{BP} \cdot \Sigma^{\mathrm{P}}_k \subseteq \Sigma^{\mathrm{P},S}_k.$$

In particular,

$$\dim_{\Delta^{\mathrm{P}}_3}(S) > 0 \implies \mathrm{BPP} \subseteq \mathrm{P}^S \tag{3.2.1}$$

and

$$\dim_{\Delta^{\mathrm{P}}_4}(S) > 0 \implies \mathrm{AM} \subseteq \mathrm{NP}^S. \tag{3.2.2}$$

Intuitively, (3.2.1) says that even an oracle S with Δ_3^{p}-dimension 0.001 – which need not be random relative to any reasonable distribution – "contains enough randomness" to carry out a deterministic simulation of BPP. To put the matter differently, to prove that P = BPP, we need "only" show how to dispense with such an oracle S.

For each relativizable complexity class $\mathcal{C}$ (of languages or pairs of languages), recall the *dimension-almost-class* dimalmost-$\mathcal{C}$ and the *almost-class* almost-$\mathcal{C}$ defined in the introduction.

Theorem 3.2.7. *For every $k \in \mathbb{N}$,*

$$\text{dimalmost-}\Sigma_k^{\mathrm{P}}\text{-Sep} = \text{almost-}\Sigma_k^{\mathrm{P}}\text{-Sep} = \text{Promise-BP} \cdot \Sigma_k^{\mathrm{P}}.$$

Nisan and Wigderson's unconditional pseudorandom generator for constant depth circuits is used in the proof for Theorem 3.2.7. We state it here.

Theorem 3.2.8 (Nisan and Wigderson [76])**.** *Let $d \in \mathbb{Z}^+$. There exists a function G^{NW} : $\{0,1\}^* \to \{0,1\}^*$ defined by a collection $\{G_n : \{0,1\}^{l_n} \to \{0,1\}^n\}$ such that $l_n = O((\log n)^{2d+6})$, G_n is computable by a logspace uniform family of circuits of polynomial size and depth $d+4$, and for any circuit family $\{C_n : \{0,1\}^n \to \{0,1\}\}$ of polynomial size and depth d,*

$$|\Pr[C_n(x) = 1] - \Pr[C_n(G_n(y))]| \leq 1/n.$$

Proof of Theorem 3.2.7. We only prove this for $k > 0$; the proof is easier when $k = 0$. Since every set of Hausdorff dimension less than 1 has Lebesgue measure 0, it is clear that dimalmost-Σ_k^{P}-Sep $\subseteq$ almost-Σ_k^{P}-Sep.

To see that almost-Σ_k^{P}-Sep $\subseteq$ Promise-BP $\cdot\, \Sigma_k^{\mathrm{P}}$, we use Nisan and Wigderson's proof that almost-Σ_k^{P} is contained in BP$\cdot\Sigma_k^{\mathrm{P}}$. Let $A = (A^+, A^-) \in$ almost-Σ_k^{P}-Sep. Then by the Lebesgue density theorem, there exists a Σ_k^{P} oracle machine N' with time bound n^m such that

$$\Pr_R[N'^R \text{ separates A}] \geq 3/4.$$

Note that when an input x is fixed and $|x| = n$, the computation of $N'(x)$ may be represented as a depth $k+2$ circuit of size at most $2^{(k+1)n^m}$ with at most $2^{(k+1)n^m}$ oracle queries as input. This is a linear size (with respect to oracle input length) depth $k+2$ circuit. We will use

Theorem 3.2.8 to reduce the exponential number of queries on random oracle to $n^{(2k+10)m}$ random oracle queries.

Let G^{NW} be the Nisan-Wigderson pseudorandom generator. Let $l_n = n^{(2k+10)m}$. Let N be the following Turing machine.

> **input** x
>
> $n = |x|$
>
> **input** $s \in \{0,1\}^{l_n}$
>
> let $\tilde{R} = G^{NW}(s)$
>
> simulate $N'^{\tilde{R}}(x)$
>
> output the output of the simulation

Note that in the above Turing machine, we do not compute $\tilde{R}$ as a whole. When a bit of $\tilde{R}$ is queried, we compute that bit individually, which can be done in polynomial time. And for each computation path of $N'^{\tilde{R}}(x)$, there are only polynomially many queries to $\tilde{R}$.

For all $x \in A^+$, $\Pr_R[N'^R(x) = 1] \geq 3/4$. By the pseudorandomness of G^{NW}, then,

$$\Pr_{s\in\{0,1\}^{l_n}}[N'^{G^{NW}(s)}(x) = 1] \geq 2/3. \tag{3.2.3}$$

Similarly, for all $x \in A^-$,

$$\Pr_{s\in\{0,1\}^{l_n}}[N'^{G^{NW}(s)}(x) = 1] \leq 1/3. \tag{3.2.4}$$

Let

$$B = \{\langle x, s\rangle \mid N(\langle x, s\rangle) = 1\}.$$

It is clear that $B \in \Sigma_k^{\mathrm{P}}$. Also by (3.2.3), for all $x \in A^+$,

$$\Pr(B_x) \geq 2/3,$$

and, by (3.2.4), for all $x \in A^-$,

$$\Pr(B_x) \leq 1/3.$$

Then $(B, n^{(2k+10)m})$ is a witness configuration for A, so $A \in \text{Promise-BP} \cdot \Sigma_k^{\text{P}}$.

To see that $\text{Promise-BP} \cdot \Sigma_k^{\text{P}} \subseteq \text{dimalmost-}\Sigma_k^{\text{P}}\text{-Sep}$, let $A \in \text{Promise-BP} \cdot \Sigma_k^{\text{P}}$. Let

$$X = \left\{ S \;\middle|\; A \notin {\Sigma_k^{\text{P}}}^{S}\text{-Sep} \right\}.$$

By Theorem 3.2.4, every element of X has Δ_{k+3}^{P}-dimension 0. By (2.2.3), this implies that $\dim_{\text{H}}(X) = 0$, whence $A \in \text{dimalmost-}\Sigma_k^{\text{P}}\text{-Sep}$. □

Corollary 3.2.9. *For every $k \in \mathbb{N}$,*

$$\text{dimalmost-}\Sigma_k^{\text{P}} = \text{BP} \cdot \Sigma_k^{\text{P}}.$$

In particular,

$$\text{dimalmost-P} = \text{BPP} \tag{3.2.5}$$

and

$$\text{dimalmost-NP} = \text{AM}. \tag{3.2.6}$$

We now turn to the proof of Theorem 3.2.4. We use the following well-known derandomization theorem.

Theorem 3.2.10 (Impagliazzo and Wigderson [49])**.** *For each $\epsilon > 0$, there exists constants $c' > c > 0$ such that, for every $A \subseteq \{0,1\}^*$ and integer $n > 1$, the following holds. If $f : \{0,1\}^{\lfloor c \log n \rfloor} \to \{0,1\}$ is a Boolean function that cannot be computed by an oracle circuit of size at most $n^{c\epsilon}$ relative to A, then the generator $G_f^{IW97} : \{0,1\}^{\lfloor c' \log n \rfloor} \to \{0,1\}^n$ has the property that, for every oracle circuit γ with size at most n,*

$$\left| \Pr_{r \in U_n}[\gamma^A(r) = 1] - \Pr_{x \in U_{\lfloor c' \log n \rfloor}}[\gamma^A(G_f^{IW97}(x)) = 1] \right| < \tfrac{1}{n},$$

where U_m denotes $\{0,1\}^m$ with the uniform probability measure.

Proof of Theorem 3.2.4. We prove the theorem for $k > 0$, since the proof is easier when $k = 0$.

Assume that $\dim_{\Delta_{k+3}^p}(S) = \alpha > 0$. It suffices to show that for every $A \in \text{Promise-BP} \cdot \Sigma_k^{\text{P}}$, $A \in \Sigma_k^{\text{P},S}\text{-Sep}$. Note that $\text{Promise-BP} \cdot \Sigma_k^{\text{P}}$ does not have oracle access to S. So we actually prove $A \in \text{NP}^{\Sigma_{k-1}^{\text{P}}, S}\text{-Sep}$.

By Theorem 3.2.3, we have that $\text{size}^{\Sigma_k^{\mathrm{P}}}(S[0..n-1]) > \frac{\alpha n}{2\log n}$ for all but finitely many n.

Let $A = (A^+, A^-) \in \text{Promise-BP} \cdot \Sigma_k^{\mathrm{p}}$. There exist $B \in \Sigma_k^{\mathrm{P}}$ and a polynomial q such that (B, q) is a witness configuration for A. Therefore, there exist polynomial-time oracle Turing machine M and a polynomial p such that, for all $x \in A^+$,

$$\Pr_r[(\exists w \in \{0,1\}^{p(|x|)}) M^{K^{k-1}}(x, r, w) = 1] \geq 2/3 \tag{3.2.7}$$

and, for all $x \in A^-$,

$$\Pr_r[(\exists w \in \{0,1\}^{p(|x|)}) M^{K^{k-1}}(x, r, w) = 1] \leq 1/3. \tag{3.2.8}$$

Let n^d be the upper bound on the running time of M on x of length n with r and w of corresponding lengths.

Let $\epsilon = \alpha/2$, let c', c be fixed in Theorem 3.2.10, and let $f : \{0,1\}^{\lfloor cd \log n \rfloor} \to \{0,1\}$ be (the Boolean function whose truth table is) given by the first $2^{\lfloor cd \log n \rfloor}$ bits of S.

By Theorem 3.2.10, G_f^{IW97} derandomizes linear size circuits with Σ_k^{P} oracles and linear size nondeterministic circuits with $\Sigma_{k-1}^{\mathrm{P}}$ oracles. Let $N^{K^{k-1},S}$ be the following nondeterministic Turing machine with oracles K^{k-1} and S.

input x

$n = |x|$

guess $w_1, w_2, \ldots, w_{2^{\lfloor c'd \log n \rfloor}} \in \{0,1\}^{p(n)}$

query the first $2^{\lfloor cd \log n \rfloor}$ bits of S

Let $f : \{0,1\}^{\lfloor cd \log n \rfloor} \to \{0,1\}$ be given by the first $2^{\lfloor cd \log n \rfloor}$ bits of S

for each string $s_i \in \{0,1\}^{\lfloor c'd \log n \rfloor}$ **do**

  Let $r_i = G_f^{IW97}(s_i)$

end for

Let $r = 0$

for each r_i

  if $M^{K^{k-1}}(x, r_i, w_i) = 1$ **then** $r = r + 1$

end for

if $\frac{r}{2^{\lfloor c'd \log n \rfloor}} \geq 1/2$ **then** output 1

else output 0.

By Theorem 3.2.10, (3.2.7), and (3.2.8), for all $x \in A^+$, there exists a tuple of witnesses $\langle w_1, w_2, \ldots, w_{2^{\lfloor c'd \log n \rfloor}} \rangle$ such that $N^{K^{k-1},S}(x) = 1$, and, for all $x \in A^-$, such witnesses do not exist. Therefore, the $\mathrm{NP}^{K^{k-1}}$ machine we constructed above separates A with oracle S, and hence $A \in \Sigma_k^{\mathrm{P},S}$-Sep. □

It should be noted that derandomization plays a significantly larger role in the proof of Corollary 3.2.9 than in the proofs of the analogous results for almost-classes. For example, the proof by Bennett and Gill [11] that almost-P = BPP uses the easily proven fact that the set $X = \left\{ S \;\middle|\; \mathrm{P}^S \neq \mathrm{BPP}^S \right\}$ has Lebesgue measure 0. Hitchcock [46] has recently proven that this set has Hausdorff dimension 1, so the Bennett-Gill argument does not extend to a proof of (3.2.5). Instead, our proof of (3.2.5) relies, via (3.2.1), on Theorem 3.2.10 to prove that the set $Y = \left\{ S \;\middle|\; \mathrm{BPP} \nsubseteq \mathrm{P}^S \right\}$ has Hausdorff dimension 0. Similarly, the proof by Nisan and Wigderson [76] that almost-NP ⊆ AM uses derandomization, but their proof that AM ⊆ almost-NP is elementary. In contrast, *both* directions of the proof of (3.2.6) use derandomization: The inclusion dimalmost-NP ⊆ AM relies on the fact that almost-NP ⊆ AM (hence on derandomization), and our proof that AM ⊆ dimalmost-NP relies, via (3.2.2), on Theorem 3.2.10.

4 Fractals in Individual Sequences and Saturated Sets

In this chapter, we investigate the fractal phenomenon at the finite-state level. In particular, we study two very different kinds of sets. One is singleton sets that contains exactly one individual sequence that we call the Copeland-Erdős sequences and the other is sets with certain saturation properties. We start with a review of the finite-state dimensions.

4.1 Finite-State Dimensions

Finite-state dimension and strong dimension are finite-state counterparts of classical Hausdorff dimension [42] and packing dimension [69, 94] introduced in early 2000s in the Cantor space $\mathbf{C}$ [26, 7]. Finite-state dimensions are defined by using the gale characterizations of the Hausdorff dimension [63] and the packing dimension [7] by restricting the gales to the ones whose underlying betting strategies can be carried out by finite-state gamblers. In the following, we give the definitions of the finite-state dimensions in space $\mathbf{C}_m$ and review their basic properties. First, we define finite-state gamblers on alphabet Σ_m, which is the fundamental construct in defining finite-state dimensions. Finite-state gamblers were investigated by Schnorr and Stimm [87], Feder [32], and others in connection to finite-state data compression and normality. The definition here was given by Dai, Lathrop, Lutz, and Mayordomo [26].

Definition. A *finite-state gambler* (*FSG*) is a 5-tuple $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ such that Q is a non-empty finite set of *states*; Σ_m is the input alphabet; $\delta : Q \times \Sigma_m \to Q$ is the *state transition function*; $\vec{\beta} : Q \to \Delta(\Sigma_m)$ is the *betting function*; $q_0 \in Q$ is the *initial state*.

The extended transition function $\delta^* : Q \times \Sigma_m^* \to Q$ is defined such that

$$\delta^*(q, wa) = \begin{cases} q & \text{if } w = a = \lambda, \\ \delta(\delta^*(q, w), a) & \text{if } w \neq \lambda. \end{cases}$$

We use δ for δ^* and $\delta(w)$ for $\delta(q_0, w)$ for convenience.

The betting function $\vec{\beta} : Q \to \Delta(\Sigma_m)$ specifies the bets $\beta_i(q)$ the FSG places on each input symbol i in Σ_m with respect to a state $q \in Q$.

Definition. ([26]). Let $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ be an FSG. The *s-gale* of G is the function $d_G^{(s)} : \Sigma_m^* \to [0, \infty)$ defined by the recursion

$$d_G^{(s)}(wb) = \begin{cases} 1 & \text{if } w = b = \lambda, \\ m^s d_G^{(s)}(w) \beta_i(\delta(w))(b) & \text{if } b \neq \lambda, \end{cases}$$

for all $w \in \Sigma_m^*$ and $b \in \Sigma_m \cup \{\lambda\}$. For $s \in [0, \infty)$, a function $d : \Sigma_m^* \to [0, \infty)$ is a *finite-state s-gale* if it is the s-gale of some finite-state gambler.

Note that in the original definition of a finite-state gambler the range of the betting function $\vec{\beta}$ is $\Delta(\{0,1\}) \cap \mathbb{Q}^2$ [26, 7]. It was shown in [39] that allowing the range of $\vec{\beta}$ to have irrational probability measures does not change the notions of finite-state dimension and strong dimension.

The definitions of finite-state dimensions are straightforward.

Definition. ([26, 7]). Let $X \subseteq \mathbf{C}_m$. The *finite-state dimension* of X is

$$\dim_{\mathrm{FS}}(X) = \inf \{ s \in [0, \infty) \mid X \subseteq S^\infty[d] \text{ for some finite-state } s\text{-gale } d \}$$

and the *finite-state strong dimension* of X is

$$\mathrm{Dim}_{\mathrm{FS}}(X) = \inf \{ s \in [0, \infty) \mid X \subseteq S^\infty_{\mathrm{str}}[d] \text{ for some finite-state } s\text{-gale } d \}.$$

We will use the following basic properties of the Hausdorff, packing, finite-state, and strong finite-state dimensions.

Theorem 4.1.1 ([26, 7])**.** *Let $X, Y, X_i \subseteq \Sigma_m^\infty$ for $i \in \mathbb{N}$.*

1. $0 \le \dim_{\mathrm{H}}(X) \le \dim_{\mathrm{FS}}(X) \le 1$, $0 \le \dim_{\mathrm{P}}(X) \le \mathrm{Dim}_{\mathrm{FS}}(X) \le 1$.

2. $\dim_{\mathrm{H}}(X) \le \dim_{\mathrm{P}}(X)$, $\dim_{\mathrm{FS}}(X) \le \mathrm{Dim}_{\mathrm{FS}}(X)$.

3. *If $X \subseteq Y$, then the dimension of X is at most the dimension of Y.*

4. $\dim_{\mathrm{FS}}(X \cup Y) = \max\{\dim_{\mathrm{FS}}(X), \dim_{\mathrm{FS}}(Y)\}$.

5. $\mathrm{Dim}_{\mathrm{FS}}(X \cup Y) = \max\,\{\mathrm{Dim}_{\mathrm{FS}}(X), \mathrm{Dim}_{\mathrm{FS}}(Y)\}$.

6. $\dim_{\mathrm{H}}\,(\bigcup_{i=0}^{\infty} X_i) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{H}}(X_i)$, $\dim_{\mathrm{P}}(\bigcup_{i=0}^{\infty} X_i) = \sup_{i\in\mathbb{N}} \dim_{\mathrm{P}}(X_i)$.

We repeatedly use the obvious fact that $d_G^{(s)}(w) \le k^{s|w|}$ holds for all s and w.

We now develop a measure of the size of a finite-state gambler so that we can study the limitation of finite-state gambler in the context of some lower bound arguments we use later. This size notion depends on the alphabet size, the number of states, and the least common denominator of the values of the betting function in the following way.

Definition. The *size* of an FSG $G = (Q, \Sigma_k, \delta, \beta, q_0)$ is

$$\mathrm{size}(G) = (k + l)|Q|,$$

where $l = \min\{l \in \mathbb{Z}^+ \mid (\forall q \in Q)(\forall i \in \Sigma_k) l\beta(q)(i) \in \mathbb{Z}\}$.

Observation 4.1.2. *For each $k \ge 2$ and $t \in \mathbb{Z}^+$, there are, up to renaming of states, fewer than $t^2(2t)^t$ finite-state gamblers G with* $\mathrm{size}(G) \le t$*.*

Proof. Given $k, l, m \in \mathbb{Z}^+$ with $k \ge 2$, let $\mathcal{G}_{k,l,m}$ be the set of all FSGs $G = (\Sigma_m, \Sigma_k, \delta, \beta, q_0)$ satisfying $l\beta(q)(i) \in \mathbb{Z}$ for all $q \in \Sigma_m$ and $i \in \Sigma_k$. Equivalently, $\mathcal{G}_{k,l,m}$ is the set of all FSGs $G = (Q, \Sigma_k, \delta, \beta, q_0)$ such that $Q = \{0, \ldots, m-1\}$ and $\beta : Q \to \Delta_{\mathbb{Q}_l}(\Sigma_k)$, where

$$\Delta_{\mathbb{Q}_l}(\Sigma_k) = \{\pi \in \Delta_{\mathbb{Q}}(\Sigma_k) \mid (\forall i \in \Sigma_k) l\pi(i) \in \mathbb{Z}\}\,.$$

Since $|\Delta_{\mathbb{Q}_l}(\Sigma_k)| = \binom{k+l-1}{k-1}$, it is easy to see that

$$|\mathcal{G}_{k,l,m}| = m^{km+1}\binom{k+l-1}{k-1}^m. \tag{4.1.1}$$

Now fix $k \geq 2$ and $t \in \mathbb{Z}^+$, and let $\mathcal{G}_t$ be the set of all FSGs $G = (\Sigma_m, \Sigma_k, \delta, \beta, q_0)$ with $\text{size}(G) \leq t$. Our objective is to show that $|\mathcal{G}_t| < t^2(2t)^t$. For each $1 \leq j \leq t$, there are at most j pairs (l, m) such that $(k + l)m = j$, and, for each of these pairs (l, m), (4.1.1) tells us that $|\mathcal{G}_{k,l,m}| < (2j)^j$, so

$$|\mathcal{G}_t| < \sum_{j=1}^{t} j(2j)^j < t^2(2t)^t.$$

□

In general, an *s-gale* is a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = k^{-s} \sum_{a=0}^{k-1} d(wa)$$

for all $w \in \Sigma_k^*$ [63]. It is clear that $d_G^{(s)}$ is an s-gale for every FSG G and every $s \in [0, \infty)$. The case $k = 2$ of the following lemma was proven in [63]. The extension to arbitrary $k \geq 2$ is routine.

Theorem 4.1.3 (Lutz [63], Dai, Lathrop, Lutz, and Mayordomo [26])**.** *Let d be an s-supergale, where $s \in [0, \infty)$. Then for all $w \in \Sigma_m^*$, $l \in \mathbb{N}$, and $0 < \alpha \in \mathbb{R}$, there are fewer than $\frac{m^{sl}}{\alpha}$ strings $u \in \Sigma_m^l$ for which $d(wu) > \alpha d(w)$.*

The following lemma is an extension of the above theorem that bound the number of profitable strings when multiple gales are used together.

Lemma 4.1.4. *For each $s, \alpha \in (0, \infty)$ and $k, n, t \in \mathbb{Z}^+$ with $k \geq 2$, there are fewer than*

$$\frac{k^{2s} n^s t^2 (2t)^t}{\alpha(k^s - 1)}$$

integers $m \in \{1, \ldots, n\}$ for which

$$\max_{\text{size}(G) \leq t} d_G^{(s)}(\sigma_k(m)) \geq \alpha,$$

where the maximum is taken over all FSGs $G = (Q, \Sigma_k, \delta, \beta, q_0)$ with $\text{size}(G) \leq t$.

Proof. Let s, α, k, n, and t be as given, and let $\mathcal{G}_t$ be the set of all FSGs $G = (\Sigma_m, \Sigma_k, \delta, \beta, q_0)$ with $\text{size}(G) \leq t$. For each $j \in \mathbb{Z}^+$ and $G \in \mathcal{G}_t$, Theorem 4.1.3 tells us that there are fewer than $\frac{k^{sj}}{\alpha}$ strings $u \in \Sigma_k^*$ of length j for which $d_G^{(s)}(u) \geq \alpha$. It follows by Observation 4.1.2 that, for each $j \in \mathbb{Z}^+$, there are fewer than $t^2(2t)^t \frac{k^{sj}}{\alpha}$ strings $u \in \Sigma_k^*$ of length j for which

$$\max_{G \in \mathcal{G}_t} d_G^{(s)}(u) \geq \alpha$$

holds. Since

$$\sum_{j=1}^{|\sigma_k(n)|} t^2(2t)^t \frac{k^{sj}}{\alpha} = \frac{t^2(2t)^t}{\alpha} \sum_{j=1}^{1+\lfloor \log_k n \rfloor} k^{sj} \leq \frac{k^{2s} n^s t^2 (2t)^t}{\alpha(k^s - 1)},$$

the lemma follows. □

4.2 Zeta-dimension

The Zeta-dimension is a quantitative measure of the logarithmic asymptotic density of a set A of positive integers. It has been discovered several times by researchers in various areas over the past few decades.

Definition. The *zeta-dimension* of a set $A \subseteq \mathbb{Z}^+$ is

$$\mathrm{Dim}_\zeta(A) = \inf \{ s \mid \zeta_A(s) < \infty \},$$

where the *A-zeta function* $\zeta_A : [0, \infty) \to [0, \infty]$ is defined by

$$\zeta_A(s) = \sum_{n \in A} n^{-s}.$$

It is easy to see (and was proven by Cahen [18] in 1894; see also [6, 41]) that zeta-dimension admits the "entropy characterization"

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \to \infty} \frac{\log |A \cap \{1, \ldots, n\}|}{\log n}. \tag{4.2.1}$$

It is then natural to define the *lower zeta-dimension* of A to be

$$\dim_\zeta(A) = \liminf_{n \to \infty} \frac{\log |A \cap \{1, \ldots, n\}|}{\log n}. \tag{4.2.2}$$

Various properties of zeta-dimension and lower zeta-dimension, along with extensive historical references, appear in the recent paper [27], but none of this material is needed to follow our technical arguments here. In the following, we will develop some properties of the zeta-dimensions that we will use here.

The following lemma gives useful characterizations of the zeta-dimensions in terms of the increasing enumeration of A.

Lemma 4.2.1. *Let $A = \{a_1 < a_2 < \cdots\}$ be an infinite set of positive integers.*

1. $\dim_\zeta(A) = \inf\left\{t \geq 0 \mid (\exists^\infty n) a_n^t > n\right\} = \inf\left\{t \geq 0 \mid (\exists^\infty n) a_n^t \geq n\right\}$
$= \sup\left\{t \geq 0 \mid (\forall^\infty n) a_n^t < n\right\} = \sup\left\{t \geq 0 \mid (\forall^\infty n) a_n^t \leq n\right\}.$

2. $\mathrm{Dim}_\zeta(A) = \inf\left\{t \geq 0 \mid (\forall^\infty n) a_n^t > n\right\} = \inf\left\{t \geq 0 \mid (\forall^\infty n) a_n^t \geq n\right\}$
$= \sup\left\{t \geq 0 \mid (\exists^\infty n) a_n^t < n\right\} = \sup\left\{t \geq 0 \mid (\exists^\infty n) a_n^t \leq n\right\}.$

Proof. Let A be as given. For each $R \in \{<, \leq, >, \geq\}$, define the sets

$$I_R = \left\{t \geq 0 \mid (\exists_n^\infty) a_n^t \; R \; n\right\},$$

$$J_R = \left\{t \geq 0 \mid (\forall^\infty n) a_n^t \; R \; n\right\}.$$

Our task is then to prove that

$$\dim_\zeta(A) = \inf I_> = \inf I_\geq = \sup J_< = \sup J_\leq \tag{4.2.3}$$

and

$$\mathrm{Dim}_\zeta(A) = \inf J_> = \inf J_\geq = \sup I_< = \sup I_\leq. \tag{4.2.4}$$

Note that each of the pairs $(J_<, I_\geq)$, $(J_\leq, I_>)$, $(I_<, J_\geq)$, $(I_\leq, J_>)$ partitions $[0, \infty)$ into two nonempty subsets with every element of the left component less than every element of the right component, the left components satisfying

$$0 \in J_< \subseteq J_\leq \cap I_< \subseteq J_\leq \cup I_< \subseteq I_\leq,$$

and the right components satisfying

$$(1, \infty) \subseteq J_> \subseteq J_\geq \cap I_> \subseteq J_\geq \cup I_> \subseteq I_\geq.$$

It follows immediately from this that

$$\sup J_< = \inf I_\geq \leq \sup J_\leq = \inf I_>$$

and

$$\sup I_< = \inf J_\geq \leq \sup I_\leq = \inf J_>.$$

Hence, to prove (4.2.3) and (4.2.4), it suffices to show that

$$\inf I_> \leq \dim_\zeta(A) \leq \inf I_\geq \tag{4.2.5}$$

$$\inf J_> \leq \mathrm{Dim}_\zeta(A) \leq \inf J_\geq. \tag{4.2.6}$$

To see that $\inf I_> \leq \dim_\zeta(A)$, let $t > \dim_\zeta(A)$. Fix t' with $t > t' > \dim_\zeta(A)$. Then, by the definition of $\dim_\zeta(A)$, there exist infinitely many $n \in \mathbb{Z}^+$ such that

$$|A \cap \{1, \ldots, n\}| < n^{t'}. \tag{4.2.7}$$

If n satisfies (4.2.7) and is large enough that $n^t \geq n^{t'} + 1$, fix k such that $a_k \leq n < a_{k+1}$. Then we have

$$a_{k+1}^t > n^t \geq n^{t'} + 1 > |A \cap \{1, \ldots, n\}| + 1 = k + 1.$$

It follows that there exist infinitely many k such that $a_k^t > k$, i.e., that $t \in I_>$, whence $\inf I_> \leq t$. Since this holds for all $t > \dim_\zeta(A)$, it follows that $\inf I_> \leq \dim_\zeta(A)$.

To see that $\dim_\zeta(A) \leq \inf I_\geq$, let $t > \inf I_\geq$. Then there exist infinitely many $n \in \mathbb{Z}^+$ such that $a_n^t \geq n$. For each of these n, we have

$$|A \cap \{1, \ldots, a_n\}| = n \leq a_n^t,$$

so there exist infinitely many $m \in \mathbb{Z}^+$ such that

$$|A \cap \{1, \ldots, , m\}| \leq m^t.$$

This implies that

$$\dim_\zeta(A) = \liminf_{m \to \infty} \frac{\log|A \cap \{1, \ldots, m\}|}{\log m} \leq t.$$

Since this holds for all $t > \inf I_\geq$, it follows that $\dim_\zeta(A) \leq \inf I_\geq$. This completes the proof that (4.2.5) holds.

The proof that (4.2.6) holds is similar. □

4.3 Dimensions of Copeland-Erdős Sequences

Now we are ready to establish the connection between the zeta-dimensions of A and the finite-state dimensions of $\mathrm{CE}_k(A)$.

Theorem 4.3.1. *Let $k \geq 2$. For every infinite set $A \subseteq \mathbb{Z}^+$,*

$$\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \dim_\zeta(A) \tag{4.3.1}$$

and

$$\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \mathrm{Dim}_\zeta(A). \tag{4.3.2}$$

Proof. Let $A = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{Z}^+$ be infinite. Fix $0 < s < t < 1$, let

$$J_t = \left\{ n \in \mathbb{Z}^+ \;\middle|\; a_n^t < n \right\},$$

and let $G = (Q, \Sigma_k, \delta, \beta, q_0)$ be an FSG. Let $n \in \mathbb{Z}^+$, and consider the quantity $d_G^{(s)}(w_n)$, where

$$w_n = \sigma_k(a_1) \cdots \sigma_k(a_n).$$

There exist states $q_1, \ldots, q_n \in Q$ such that

$$d_G^{(s)}(w_n) = \prod_{i=1}^{n} d_{G_{q_i}}^{(s)}(\sigma_k(a_i)),$$

where $G_{q_i} = (Q, \Sigma_k, \delta, \beta, q_i)$. Let $B = \left\{ 1 \leq i \leq n \;\middle|\; d_{G_{q_i}}^{(s)}(\sigma_k(a_i)) \geq \frac{1}{k} \right\}$, and let $B^c = \{1, \ldots, n\} - B$. Then

$$d_G^{(s)}(w_n) = \left(\prod_{i \in B} d_{G_{q_i}}^{(s)}(\sigma_k(a_i)) \right) \left(\prod_{i \in B^c} d_{G_{q_i}}^{(s)}(\sigma_k(a_i)) \right). \tag{4.3.3}$$

By our choice of B,

$$\prod_{i \in B^c} d_{G_{q_i}}^{(s)}(\sigma_k(a_i)) \leq k^{|B| - n}. \tag{4.3.4}$$

By Lemma 4.1.4,

$$|B| \leq \frac{c k^{2s+1} a_n^s}{k^s - 1}, \tag{4.3.5}$$

where $c = \mathrm{size}(G)^2 (2\mathrm{size}(G))^{\mathrm{size}(G)}$. Since $d_{G_{q_i}}^{(s)}(u) \leq k^{s|u|}$ must hold in all cases, it follows that

$$\prod_{i \in B} d_{G_{q_i}}^{(s)}(\sigma_k(a_i)) \leq k^{s|B||\sigma_k(a_n)|} \leq k^{s|B|(1 + \log_k a_n)}. \tag{4.3.6}$$

By (4.3.3), (4.3.4), (4.3.5), and (4.3.6), we have

$$\log_k d_G^{(s)}(w_n) \leq \tau(1 + s + s\log_k a_n)a_n^s - n, \tag{4.3.7}$$

where $\tau = \frac{ck^{2s+1}}{k^s - 1}$. If n is sufficiently large, and if $n+1 \in J_t$, then (4.3.7) implies that

$$\begin{aligned}
\log_k d_G^{(s)}(w_n) &\leq \tau(1 + s + s\log_k a_n)a_n^s - 2(n+1)^{\frac{s+t}{2t}} \\
&\leq \tau(1 + s + s\log_k a_n)a_n^s - 2a_{n+1}^{\frac{s+t}{2}} \\
&\leq \tau(1 + s + s\log_k a_n)a_n^s - a_n^{\frac{s+t}{2}} - s(1 + \log_k a_{n+1}) \\
&\leq -s(1 + \log_k a_{n+1}) \\
&\leq -s|\sigma_k(a_{n+1})|.
\end{aligned}$$

We have now shown that

$$d_G^{(s)}(w_n) \leq k^{-s|\sigma_k(a_{n+1})|} \tag{4.3.8}$$

holds for all sufficiently large n with $n+1 \in J_t$.

To prove (4.3.1), let $s < t < \dim_\zeta(A)$. It suffices to show that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq s$. Since $t < \dim_\zeta(A)$, Lemma 4.2.1 tells us that the set J_t is cofinite. Hence, for every sufficiently long prefix $w \sqsubseteq \mathrm{CE}_k(A)$, there exist n and $u \sqsubseteq \sigma_k(a_{n+1})$ such that $w = w_n u$ and (4.3.8) holds, whence

$$d_G^{(s)}(w) \leq k^{-s|\sigma_k(a_{n+1})|}k^{s|u|} \leq 1.$$

This shows that the s-gale of G does not succeed on $\mathrm{CE}_k(A)$, whence $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq s$.

To prove (4.3.2), let $s < t < \mathrm{Dim}_\zeta(A)$. It suffices to show that $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq s$. Since $t < \mathrm{Dim}_\zeta(A)$, Lemma 4.2.1 tells us that the set J_t is infinite. For the infinitely many n for which $n+1 \in J_t$ and (4.3.8) holds, we then have $d_G^{(s)}(w_n) < 1$. This shows that the s-gale of G does not strongly succeed on $\mathrm{CE}_k(A)$, whence $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq s$. □

The above theorem may also be proved using Ziv and Lempel's result [100] and the equivalence between finite-state compression ratios and finite-state dimension [26, 7].

In the following, we establish the tightness of the bounds in the above theorem. In order to achieve this, we first establish the following relationship between entropy of a probability distribution and the abundance of the strings whose symbols satisfy the distribution.

Lemma 4.3.2. *For every $n \geq k \geq 2$ and every partition $\vec{a} = (a_0, \ldots, a_{k-1})$ of n, there are more than*

$$k^{n\mathcal{H}_k(\frac{\vec{a}}{n})-(k+1)\log_k n}$$

integers m with $|\sigma_k(m)| = n$ and $\#(i, \sigma_k(m)) = a_i$ for each $i \in \Sigma_k$.

Proof. Let $n \geq k \geq 2$, and let $\vec{a} = (a_0, \ldots, a_{k-1})$ be a partition of n. Define the sets

$$B = \{u \in \Sigma_k^n \mid (\forall i \in \Sigma_k)\#(i, u) = a_i\},$$

$$C = \left\{m \in \mathbb{Z}^+ \mid \sigma_k(m) \in B\right\}.$$

Define an equivalence relation $\sim$ on B by

$$u \sim v \iff (\exists x, y \in \Sigma_k^*)[u = xy \text{ and } v = yx].$$

Then each $\sim$-equivalence class has at most n elements and contains $\sigma_k(m)$ for at least one $m \in C$, so

$$|C| \geq \frac{1}{n}|B|.$$

Using multinomial coefficients and the well-known estimate $e(\frac{t}{e})^t < t! < et(\frac{t}{e})^t$, valid for all $t \in \mathbb{Z}^+$, we have

$$|B| = \binom{n}{a_0, \ldots, a_{k-1}} = \frac{n!}{\prod_{i=0}^{k-1} a_i!} > \frac{1}{e^{k-1}\prod_{i=0}^{k-1} a_i} \prod_{i=0}^{k-1} \left(\frac{n}{a_i}\right)^{a_i}.$$

Since the geometric mean is bounded by the arithmetic mean,

$$\prod_{i=0}^{k-1} a_i \leq \left(\frac{1}{k}\sum_{i=0}^{k-1} a_i\right)^k = \left(\frac{n}{k}\right)^k.$$

Putting this all together, we have

$$|C| > \frac{k^k}{e^{k-1}n^{k+1}} \prod_{i=0}^{k-1} \left(\frac{n}{a_i}\right)^{a_i} \geq \frac{1}{n^{k+1}} \prod_{i=0}^{k-1} \left(\frac{n}{a_i}\right)^{a_i},$$

whence

$$\begin{aligned}\log_k|C| &> \left(\log_k \prod_{i=0}^{k-1} \left(\frac{n}{a_i}\right)^{a_i}\right) - (k+1)\log_k n \\ &= n\mathcal{H}_k\left(\frac{\vec{a}}{n}\right) - (k+1)\log_k n.\end{aligned}$$

□

Theorem 4.3.3. *Let $k \geq 2$. For any four real numbers $\alpha, \beta, \gamma, \delta$ satisfying the inequalities*

$$\begin{array}{ccccc} & & \gamma & \leq & \delta & \leq 1 \\ & & \vee\!| & & \vee\!| & \\ 0 & \leq & \alpha & \leq & \beta, & \end{array} \tag{4.3.9}$$

there exists an infinite set $A \subseteq \mathbb{Z}^+$ such that $\dim_\zeta(A) = \alpha$, $\mathrm{Dim}_\zeta(A) = \beta$, $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \gamma$, and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \delta$.

Proof. Let α, β, γ, and δ be real numbers satisfying (4.3.9). We will explicitly construct an infinite set $A \subseteq \mathbb{Z}^+$ with the indicated dimensions. Intuitively, the values of $\dim_\zeta(A)$ and $\mathrm{Dim}_\zeta(A)$ will be achieved by controlling the density of A; the upper bounds on $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A))$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A))$ will be achieved by constructing A from integers whose base-k expansions have controlled frequencies of digits (such integers being abundant by Lemma 4.3.2); and the lower bounds on $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A))$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A))$ will be achieved by avoiding use of the very few (by Lemma 4.1.4) integers on whose base-k expansions a finite-state gambler can win.

We first define some useful probability measures on Σ_k, all expressed as vectors. Let $\vec{\mu} = (\frac{1}{k}, \ldots, \frac{1}{k}) \in \Delta(\Sigma_k)$ be the uniform probability measure, and let $\vec{\nu} = (1, 0, \ldots, 0) \in \Delta(\Sigma_k)$ be the degenerate probability measure that concentrates all probability on 0. Define the function $g : [0, 1] \to \Delta(\Sigma_k)$ by

$$g(r) = r\vec{\mu} + (1 - r)\vec{\nu}.$$

Then g defines a line segment from a corner $g(0) = \vec{\nu}$ to the centroid $g(1) = \vec{\mu}$ of the simplex $\Delta(\Sigma_k)$. Also, $\mathcal{H}_k \circ g : [0, 1] \to [0, 1]$ is strictly increasing and continuous, with $\mathcal{H}_k(g(0)) = 0$ and $\mathcal{H}_k(g(1)) = 1$. Let $r_\gamma = (\mathcal{H}_k \circ g)^{-1}(\gamma)$, $r_\delta = (\mathcal{H}_k \circ g)^{-1}(\delta)$, $\vec{\pi} = g(r_\gamma)$, and $\vec{\tau} = g(r_\delta)$, so that

$$\mathcal{H}_k(\vec{\pi}) = \gamma, \mathcal{H}_k(\vec{\tau}) = \delta.$$

Then let $\vec{\pi}^{(k)}$, $\vec{\pi}^{(k+1)}$, $\vec{\pi}^{(k+2)}$, $\ldots$ and $\vec{\tau}^{(k)}$, $\vec{\tau}^{(k+1)}$,$\vec{\tau}^{(k+2)}$, $\ldots$ be sequences in $\Delta_{\mathbb{Q}}(\Sigma_k)$ with the following properties.

(i) For each $n \geq k$, $n\vec{\pi}^{(n)}$ and $n\vec{\tau}^{(n)}$ are partitions of n, with each $n\vec{\pi}^{(n)}(i) \geq \sqrt{n}$ and $n\vec{\tau}^{(n)}(i) \geq \sqrt{n}$ for $n \geq k^2$.

(ii) $\lim_{n\to\infty} \vec{\pi}^{(n)} = \vec{\pi}$ and $\lim_{n\to\infty} \vec{\tau}^{(n)} = \vec{\tau}$.

Note that (i) ensures that

$$\mathcal{H}_k(\vec{\pi}^{(n)}) \geq \frac{k-1}{2\sqrt{n}} \log_k n, \quad \mathcal{H}_k(\vec{\tau}^{(n)}) \geq \frac{k-1}{2\sqrt{n}} \log_k n \tag{4.3.10}$$

hold for all $n \geq k^2$.

For each $u \in \Sigma_k^*$ and $s \in [0, \infty)$, let $\mathcal{G}_u$ be the set of all FSGs G with size$(G) \leq \log_k \log_k |u|$, and let

$$d_{\max}^{(s)}(u) = \max_{G \in \mathcal{G}_u} d_G^{(s)}(u).$$

Define the sets

$$\begin{aligned}
U &= \left\{ a \geq k^{k-1} \;\middle|\; d_{\max}^{(\mathcal{H}_k(\vec{\pi}^{(|\sigma_k(a)|)}))}(\sigma_k(a)) > |\sigma_k(a)|^{k+2} \right\}, \\
V &= \left\{ a \geq k^{k-1} \;\middle|\; d_{\max}^{(\mathcal{H}_k(\vec{\tau}^{(|\sigma_k(a)|)}))}(\sigma_k(a)) > |\sigma_k(a)|^{k+2} \right\}, \\
C &= \left\{ a \geq k^{k-1} \;\middle|\; (\forall i \in \Sigma_k) \#(i, \sigma_k(a)) = |\sigma_k(a)| \vec{\pi}^{(|\sigma_k(a)|)}(i) \right\}, \\
D &= \left\{ a \geq k^{k-1} \;\middle|\; (\forall i \in \Sigma_k) \#(i, \sigma_k(a)) = |\sigma_k(a)| \vec{\tau}^{(|\sigma_k(a)|)}(i) \right\}, \\
C' &= C - U, \\
D' &= D - V.
\end{aligned}$$

Then, for all $n \geq k$, we have

$$|U_{=n}| = \left\{ a \in \mathbb{Z}^+_{=n} \;\middle|\; d_{\max}^{(\mathcal{H}_k(\vec{\pi}^{(n)}))}(\sigma_k(a)) > n^{k+2} \right\},$$

so Lemma 4.1.4 tells us that

$$|U_{=n}| < \frac{k^{2\mathcal{H}_k(\vec{\pi}^{(n)}) + n\mathcal{H}_k(\vec{\pi}^{(n)})} t^2 (2t)^t}{n^{k+2}(k^{\mathcal{H}_k(\vec{\pi}^{(n)})} - 1)}$$

for all $n \geq k$, where $t = \log_k \log_k n$. It follows easily from this that

$$|U_{=n}| = o(k^{n\mathcal{H}_k(\vec{\pi}^{(n)}) - (k+1)\log_k n}) \tag{4.3.11}$$

as $n \to \infty$. By Lemma 4.3.2, we have

$$|C_{=n}| \geq k^{n\mathcal{H}_k(\vec{\pi}^{(n)}) - (k+1)\log_k n}. \tag{4.3.12}$$

(By (4.3.10), this is positive for all sufficiently large n.) Putting (4.3.11) and (4.3.12) together with our choice of the $\vec{\pi}^{(n)}$ gives us

$$|C'_{=n}| \geq \max\{1, k^{(\alpha - o(1))n}\} \tag{4.3.13}$$

as $n \to \infty$. A similar argument shows that

$$|D'_{=n}| \geq \max\{1, k^{(\beta - o(1))n}\} \tag{4.3.14}$$

as $n \to \infty$. It follows that we can fix sets $C'' \subseteq C'$ and $D'' \subseteq D'$ such that

$$\max\{1, k^{(\alpha - o(1))n}\} \leq |C''_{=n}| \leq k^{(\alpha + o(1))n} \tag{4.3.15}$$

and

$$\max\{1, k^{(\beta - o(1))n}\} \leq |D''_{=n}| \leq k^{(\beta + o(1))n} \tag{4.3.16}$$

as $n \to \infty$.

Now define $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ by the recursion

$$T(1) = k, T(l+1) = k^{T(l)},$$

so that $T(l)$ is an "exponential tower" $k^{k^{\cdot^{\cdot^{\cdot^{k}}}}}$ of height l. For each $n \geq k$, let $T^{-1}(n)$ be the unique l such that $T(l) \leq n < T(l+1)$. Let

$$C^* = \bigcup_{T^{-1}(n) \text{ even}} C''_{=n}, \quad D^* = \bigcup_{T^{-1}(n) \text{ odd}} D''_{=n},$$

and let

$$A = C^* \cup D^*.$$

This is our set A.

We now note the following.

1. By (4.3.15),

$$\begin{aligned}
&|A \cap \{1, \ldots, k^{T(2l+1)-1} - 1\}| \\
&= \sum_{n=1}^{T(2l)-1} |A_{=n}| + \sum_{n=T(2l)}^{T(2l+1)-1} |A_{=n}| \\
&\leq \sum_{n=0}^{T(2l)-1} k^n + \sum_{n=T(2l)}^{T(2l+1)-1} k^{(\alpha+o(1))n} \\
&\leq k^{T(2l)} + k^{(\alpha+o(1))T(2l+1)} \\
&= k^{(\alpha+o(1))T(2l+1)}
\end{aligned}$$

as $l \to \infty$, so (4.2.2) tells us that

$$\begin{aligned}
\dim_\zeta(A) &\leq \liminf_{l\to\infty} \frac{\log_k |A \cap \{1, \ldots, k^{T(2l+1)-1} - 1\}|}{\log_k k^{T(2l+1)-2}} \\
&\leq \liminf_{l\to\infty} \frac{(\alpha + o(1))T(2l+1)}{T(2l+1) - 2} = \alpha.
\end{aligned}$$

2. By (4.3.15), (4.3.16), and the fact that $\alpha \leq \beta$,

$$\begin{aligned}
|A \cap \{1, \ldots, m\}| &\geq \sum_{n=1}^{|\sigma_k(m)|-1} |A_{=n}| \\
&\geq \sum_{n=1}^{|\sigma_k(m)|-1} k^{(\alpha-o(1))n} \\
&= k^{(\alpha-o(1))|\sigma_k(m)|} \\
&= m^{\alpha-o(1)}
\end{aligned}$$

as $m \to \infty$, so (4.2.2) tells us that $\dim_\zeta(A) \geq \alpha$.

3. By (4.3.15), (4.3.16), and the fact that $\alpha \leq \beta$,

$$\begin{aligned}
|A \cap \{1, \ldots, m\}| &\leq \sum_{n=1}^{|\sigma_k(m)|} |A_{=n}| \\
&\leq \sum_{n=1}^{|\sigma_k(m)|} k^{(\beta+o(1))n} \\
&= k^{(\beta+o(1))|\sigma_k(m)|} \\
&= m^{\beta+o(1)}
\end{aligned}$$

as $m \to \infty$, so (4.2.1) tells us that $\mathrm{Dim}_\zeta(A) \le \beta$.

4. By (4.2.1) and (4.3.16),

$$\begin{aligned}\mathrm{Dim}_\zeta(A) &\ge \limsup_{n\to\infty} \frac{\log_k |A_{=n}|}{\log_k(k^n-1)} \\ &\ge \limsup_{n\to\infty} \frac{\log_k k^{(\beta-o(1))n}}{\log_k(k^n-1)} = \beta.\end{aligned}$$

These four things together show that $\dim_\zeta(A) = \alpha$ and $\mathrm{Dim}_\zeta(A) = \beta$.

Our next objective is to prove that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \ge \gamma$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \ge \delta$. For this, let $G = (Q, \Sigma_k, \delta, \beta, q_0)$ be an FSG, and let $s \in [0, \infty)$. It suffices to prove that

$$s < \gamma \Rightarrow \text{the } s\text{-gale of } G \text{ does not succeed on } \mathrm{CE}_k(A) \tag{4.3.17}$$

and

$$s < \delta \Rightarrow \text{the } s\text{-gale of } G \text{ does not strongly succeed on } \mathrm{CE}_k(A). \tag{4.3.18}$$

Write $A = \{a_1 < a_2 < \cdots\}$, so that

$$\mathrm{CE}_k(A) = \sigma_k(a_1)\sigma_k(a_2)\sigma_k(a_3)\cdots.$$

There is a sequence $q_1, q_2, q_3, \ldots$ of states $q_i \in Q$ such that, for any $m \ge 0$ and any proper prefix $u \sqsubsetneq \sigma_k(a_{m+1})$,

$$d_G^{(s)}(\sigma_k(a_1)\cdots\sigma_k(a_m)u) = \left(\prod_{i=0}^{m-1} d_{G_{q_i}}^{(s)}(\sigma_k(a_{i+1}))\right) d_{G_{q_m}}^{(s)}(u), \tag{4.3.19}$$

where $G_q = (Q, \Sigma_k, \delta, \beta, q)$. Let $c = \mathrm{size}(G)$. Note that, for all $q \in Q$, $\mathrm{size}(G_q) = c$, so

$$\begin{aligned}a \ge k^{k^{k^c}} &\Rightarrow c \le \log_k \log_k \log_k a \le \log_k \log_k |\sigma_k(a)| \\ &\Rightarrow G_q \in \mathcal{G}_{\sigma_k(a)}.\end{aligned}$$

Since $C^* \cap U = \varnothing$, it follows that, for all $q \in Q$,

$$k^{k^{k^c}} \le a \in C^*_{=n} \Rightarrow d_{G_q}^{(\mathcal{H}_k(\vec{\pi}^{(n)}))}(\sigma_k(a)) \le n^{k+2}.$$

Using the identity $d_{G_q}^{(s)}(x) = k^{(s-s')|x|} d_{G_q}^{(s')}(x)$ and the facts that $\mathcal{H}_k(\vec{\pi}^{(n)}) = \gamma + o(1)$ and $n^{k+2} = k^{o(n)}$ as $n \to \infty$, we then have, for all $q \in Q$,

$$a \in C^*_{=n} \Rightarrow d_{G_q}^{(s)}(\sigma_k(a)) \le k^{(s-\gamma+o(1))n} \tag{4.3.20}$$

as $n \to \infty$. A similar argument shows that, for all $q \in Q$,

$$a \in D^*_{=n} \Rightarrow d^{(s)}_{G_q}(\sigma_k(a)) \leq k^{(s-\delta+o(1))n} \tag{4.3.21}$$

as $n \to \infty$.

To verify (4.3.17), assume that $s < \gamma$. Then, since $\gamma \leq \delta$, (4.3.20) and (4.3.21) tell us that

$$d^{(s)}_{G_{q_i}}(\sigma_k(a_{i+1})) \leq k^{(s-\gamma+o(1))|\sigma_k(a_{i+1})|}$$

as $i \to \infty$. It follows by (4.3.19) that, for any prefix $w \sqsubseteq \mathrm{CE}_k(A)$, if we write $w = \sigma_k(a_1)\cdots\sigma_k(a_m)u$, where $u \underset{\neq}{\sqsubseteq} \sigma_k(a_{m+1})$, then $|u| = o(|w|)$ as $|w| \to \infty$, so

$$\begin{aligned} d^{(s)}_G(w) &\leq \left(\prod_{i=0}^{m-1} k^{(s-\gamma+o(1))|\sigma_k(a_{i+1})|}\right) k^{s|u|} \\ &= k^{(s-\gamma+o(1))(|w|-|u|)+s|u|} \\ &= k^{(s-\gamma+o(1))|w|} \end{aligned}$$

as $|w| \to \infty$. Since $s < \gamma$, it follows that

$$\limsup_{n\to\infty} d^{(s)}_G(\mathrm{CE}_k(A)[0..n-1]) = 0,$$

affirming (4.3.17).

To verify (4.3.18), assume that $s < \delta$. For each $l \in \mathbb{Z}^+$, let

$$v_l = \sigma_k(a_{i_l})\sigma_k(a_{i_l+1})\cdots\sigma_k(a_{i_{l+1}-1}),$$

where i_l is the least i such that $|\sigma_k(a_i)| = T(l)$, and let

$$w_l = v_1 v_2 \cdots v_{l-1},$$

noting that each $w_l \sqsubseteq \mathrm{CE}_k(A)$. Then $|w_l| = o(|v_l|)$ as $l \to \infty$, so

$$\begin{aligned} d^{(s)}_G(w_{2l}) &= d^{(s)}_G(w_{2l-1}) \prod_{i=i_{2l-1}}^{i_{2l}-1} d^{(s)}_{G_{q_{i-1}}}(\sigma_k(a_i)) \\ &\leq k^{s|w_{2l-1}|} \prod_{i=i_{2l-1}}^{i_{2l}-1} k^{(s-\delta+o(1))|\sigma_k(a_i)|} \\ &= k^{s|w_{2l-1}|+(s-\delta+o(1))|v_{2l-1}|} \\ &= k^{(s-\delta+o(1))|v_{2l-1}|} \end{aligned}$$

as $l \to \infty$. Since $s < \delta$, this affirms (4.3.18) and concludes the proof that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \gamma$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \delta$.

All that remains is to prove that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq \gamma$ and $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq \delta$. For each rational $r \in \mathbb{Q} \cap [0,1]$, let G_r be the 1-state FSG whose bets are given by $g(r)$, where $g : [0,1] \to \Delta(\Sigma_k)$ is the function defined earlier in this proof. That is, for all $s \in [0,\infty)$, $w \in \Sigma_k^*$, and $a \in \Sigma_k$, we have

$$d_{G_r}^{(s)}(wa) = k^s g(r)(a) d_{G_r}^{(s)}(w).$$

If we write $\theta_w(a) = \frac{\#(a,w)}{|w|}$ for all $w \in \Sigma_k^+$ and $a \in \Sigma_k$, then this implies that, for all $w \in \Sigma_k^+$,

$$d_{G_r}^{(s)}(w) = k^{s|w|} \prod_{a \in \Sigma_k} g(r)(a)^{\#(a,w)},$$

whence

$$\begin{aligned}
\log_k d_{G_r}^{(s)}(w) &= s|w| + \sum_{a \in \Sigma_k} \#(a,w) \log_k g(r)(a) \\
&= |w| \left(s - \sum_{a \in \Sigma_k} \theta_w(a) \log_k \frac{1}{g(r)(a)} \right) \\
&= |w| \left(s - \mathrm{E}_{\theta_w} \log_k \frac{1}{g(r)(a)} \right) \\
&= |w| \left(s - \mathrm{E}_{\theta_w} \log_k \frac{1}{\theta_w(a)} - \mathrm{E}_{\theta_w} \log_k \frac{\theta_w(a)}{g(r)(a)} \right) \\
&= |w| \left(s - \mathcal{H}_k(\theta_w) - \mathcal{D}_k(\theta_w \parallel g(r)) \right).
\end{aligned}$$

We have thus shown that

$$d_{G_r}^{(s)}(w) = k^{(s - \mathcal{H}_k(\theta_w) - \mathcal{D}_k(\theta_w \| g(r)))|w|} \tag{4.3.22}$$

holds for all $r \in \mathbb{Q} \cap [0,1]$, $s \in [0,\infty)$, and $w \in \Sigma_k^+$.

We now note a useful property of the function g. If we fix $r \in (0,1]$, then

$$\frac{d}{dx}[\mathcal{H}_k(g(x)) + \mathcal{D}_k(g(x) \parallel g(r))] = \frac{k-1}{k} \log_k \frac{k+r-kr}{r} \geq 0,$$

so

$$q \leq r \Rightarrow \mathcal{H}_k(g(q)) + \mathcal{D}_k(g(q) \parallel g(r)) \leq \mathcal{H}_k(g(r)). \tag{4.3.23}$$

For each $n \in \mathbb{Z}^+$, let $\theta_n^A = \theta_{w_n}$, where $w_n = \mathrm{CE}_k(A)[0..n-1]$ is the string consisting of the first n symbols in $\mathrm{CE}_k(A)$. Then $\theta_1^A, \theta_2^A, \ldots$ is an infinite sequence of probability vectors in the simplex $\Delta(\Sigma_k)$. For every n such that $T^{-1}(n)$ is even, $A_{=n} = C^*_{=n}$ consists entirely of integers a for which $\theta_{\sigma_k(a)} = \vec{\pi}^{(n)}$, and for every n such that $T^{-1}(n)$ is odd, $A_{=n} = D^*_{=n}$ consists entirely of integers a for which $\theta_{\sigma_k(a)} = \vec{\tau}^{(n)}$. Since $\vec{\pi}^{(n)}$ converges to $g(r_\gamma)$, $\vec{\tau}^{(n)}$ converges to $g(r_\delta)$, and T grows very rapidly, it follows easily that the set of limit points of the sequence $\theta_1^A, \theta_2^A, \ldots$ is precisely the closed line segment $g([r_\gamma, r_\delta])$ (which is a point if $\gamma = \delta$).

To see that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq \gamma$, assume that $\gamma < s \leq 1$. It suffices to show that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq s$. For this, fix $r \in \mathbb{Q} \cap (r_\gamma, (\mathcal{H}_k \circ g)^{-1}(s))$. Since $g(r_\gamma)$ is a limit point of $\theta_1^A, \theta_2^A, \ldots$, there is a sequence $n_1 < n_2 < \cdots$ of positive integers such that $\lim_{i\to\infty} \theta_{n_i}^A = g(r_\gamma)$. By (4.3.22), (4.3.23), and the continuity of $\mathcal{H}_k(\vec{x}) + \mathcal{D}_k(\vec{x} \parallel g(r))$ as a function of $\vec{x}$, we then have

$$\begin{aligned} d_{G_r}^{(s)}(w_{n_i}) &= k^{(s-\mathcal{H}_k(\theta_{n_i}^A)-\mathcal{D}_k(\theta_{n_i}^A \| g(r)))n_i} \\ &= k^{(s-\mathcal{H}_k(g(r_\gamma))-\mathcal{D}_k(g(r_\gamma)\|g(r))-o(1))n_i} \\ &\geq k^{(s-\mathcal{H}_k(g(r))-o(1))n_i} \end{aligned}$$

as $i \to \infty$. Since $\mathcal{H}_k(g(r)) < s$, it follows that the s-gale of G_r succeeds on $\mathrm{CE}_k(A)$, whence $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq s$.

To see that $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq \delta$, assume that $\delta < s \leq 1$. It suffices to show that $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq s$. For this, fix $r \in \mathbb{Q} \cap (r_\delta, (\mathcal{H}_k \circ g)^{-1}(s))$. For each $n \in \mathbb{Z}^+$, let $g(q_n)$ be the point on the line segment $g([r_\gamma, r_\delta])$ that is closest to θ_n^A. Since $g([r_\gamma, r_\delta])$ contains every limit point of θ_1^A, $\theta_2^A, \ldots$, $\Delta(\Sigma_k)$ is compact, and $\mathcal{H}_k(\vec{x}) + \mathcal{D}_k(\vec{x} \parallel g(r))$ is a continuous function of $\vec{x}$, we have

$$\mathcal{H}_k(\theta_n^A) + \mathcal{D}_k(\theta_n^A \parallel g(r)) = \mathcal{H}_k(g(q_n)) + \mathcal{D}_k(g(q_n) \parallel g(r)) + o(1) \tag{4.3.24}$$

as $n \to \infty$. By (4.3.22), (4.3.23), and (4.3.24),

$$\begin{aligned} d_{G_r}^{(s)}(w_n) &= k^{(s-\mathcal{H}_k(\theta_n^A)-\mathcal{D}_k(\theta_n^A \| g(r)))n} \\ &= k^{(s-\mathcal{H}_k(g(q_n))-\mathcal{D}_k(g(q_n)\|g(r))-o(1))n} \\ &\geq k^{(s-\mathcal{H}_k(g(r))-o(1))n} \end{aligned}$$

as $n \to \infty$. Since $\mathcal{H}_k(g(r)) < s$, it follows that the s-gale of G_r strongly succeeds on $\mathrm{CE}_k(A)$, whence $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \leq s$. □

The original Copeland-Erdős theorem is a special case of our Theorem 4.3.1.

Corollary 4.3.4. (Copeland and Erdős [24]). *Let $k \geq 2$ and $A \subseteq \mathbb{Z}^+$. If, for all $\alpha < 1$, for all sufficiently large $n \in \mathbb{Z}^+$, $|A \cap \{1, \ldots, n\}| > n^\alpha$, then the sequence $\mathrm{CE}_k(A)$ is normal over the alphabet Σ_k. In particular, the sequence $\mathrm{CE}_k(\mathrm{PRIMES})$ is normal over the alphabet Σ_k.*

Proof. The hypothesis implies that $\dim_\zeta(A) \geq \alpha$ for all $\alpha < 1$, i.e., that $\dim_\zeta(A) = 1$. By Theorem 4.3.1, this implies that $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = 1$, which is equivalent [87, 14] to the normality of $\mathrm{CE}_k(A)$. □

4.4 Saturated Sets with Prescribed Limit Frequencies of Digits

In last section, we studied the finite-state dimensions of a particular kind of singleton sets. In this section, we turn our attention to a very different kind of sets – sets that are saturated with sequences with certain asymptotic properties in terms of relative frequencies of digits.

4.4.1 Relative Frequencies of Digits

Given a probability measure π on Σ_m, define the frequency class

$$\mathrm{FREQ}^\pi = \left\{ S \in \mathbf{C}_m \;\middle|\; (\forall i \in \Sigma_m) \lim_{n\to\infty} \pi_i(S, n) = \pi(i) \right\}.$$

In the particular case $m = 2$, we also write FREQ^π as FREQ^β, where $\beta = \pi(0)$. For $\beta \in [0, \frac{1}{2}]$, we also define the class

$$\mathrm{FREQ}^{\leq\beta} = \left\{ S \in \mathbf{C} \;\middle|\; \limsup_{n\to\infty} \pi_0(S, n) \leq \beta \right\}.$$

The Hausdorff dimension has been used to study these sets.

Theorem 4.4.1 (Besicovitch[12]). *For each* $\beta \in [0, \frac{1}{2}]$,

$$\dim_{\mathrm{H}}(\mathrm{FREQ}^{\leq \beta}) = \mathcal{H}(\beta).$$

Theorem 4.4.2 (Eggleston [30]). *For each* $\pi \Delta(\Sigma_m)$,

$$\dim_{\mathrm{H}}(\mathrm{FREQ}^{\pi}) = \mathcal{H}_m(\pi).$$

In particular, if $m = 2$*, then, for each* $\beta \in [0, 1]$,

$$\dim_{\mathrm{H}}(\mathrm{FREQ}^{\beta}) = \mathcal{H}(\beta).$$

We now first calculate the finite-state dimension of some more exotic sets that contain m-adic sequences that satisfy certain conditions placed on the frequencies of digits. These calculations use straightforward constructions of finite-state gamblers. Both the constructions and analysis use completely elementary techniques.

Let $\mathcal{H}_{\beta,m}(\alpha) = -(\alpha \log_m \alpha + \beta\alpha \log_m \beta\alpha + (1 - \alpha - \beta\alpha) \log_m \frac{1-\alpha-\beta\alpha}{m-2})$. Let

$$\alpha^*(x) = \begin{cases} \frac{1}{m} & \text{if } x < 1 \\ \frac{1}{1+x+(m-2)x^{\frac{x}{x+1}}} & \text{otherwise.} \end{cases}$$

Note that

$$\mathcal{H}_{\beta,m}(\alpha^*(\beta)) = \sup_{\alpha \in [0, \frac{1}{1+\beta}]} \mathcal{H}_{\beta,m}(\alpha) = \begin{cases} 1 & \text{if } \beta < 1, \\ \log_m(m - 2 + \frac{1+\beta}{\beta^{\frac{\beta}{\beta+1}}}) & \text{otherwise.} \end{cases}$$

Theorem 4.4.3. *Let* $\beta' \geq \beta \geq 0$*. Let*

$$X = \left\{ S \;\middle|\; \liminf_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta \text{ and } \limsup_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta' \right\}.$$

Then $\dim_{\mathrm{H}}(X) = \dim_{\mathrm{FS}}(X) = \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ *and* $\dim_{\mathrm{P}}(X) = \mathrm{Dim}_{\mathrm{FS}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

Proof. We assume that $\beta' \geq \beta \geq 1$, since when either of these values is less than 1, the proof is essentially looking at the subset of X where their values are replaced by 1. When S is clear from the context, let $\alpha_n = \pi_0(S,n)$ and $\beta_n = \pi_1(S,n)$. Let $\alpha' = \alpha^*(\beta')$ and let $\alpha = \alpha^*(\beta)$.

First, we prove the lower bounds for the dimensions. For Hausdorff dimension and finite-state dimension, let

$$Y = \left\{ S \;\middle|\; \lim_{n\to\infty} \alpha_n = \alpha', \lim_{n\to\infty} \beta_n = \beta'\alpha', \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha'-\beta'\alpha'}{m-2} \right\}.$$

By Eggleston's theorem, we have $\dim_{\mathrm{H}}(Y) = \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$. Since $\beta' \geq \beta \geq 1$ and $Y \subseteq X$,

$$\dim_{\mathrm{FS}}(X) \geq \dim_{\mathrm{H}}(X) \geq \dim_{\mathrm{H}}(Y) = \mathcal{H}_{\beta',m}(\alpha^*(\beta')).$$

For packing dimension and finite-state strong dimension, let

$$Z = \left\{ S \;\middle|\; \lim_{n\to\infty} \alpha_n = \alpha, \lim_{n\to\infty} \beta_n = \beta\alpha, \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha-\beta\alpha}{m-2} \right\}.$$

Now we construct from Z a set $Z' \subseteq X$ by interpolating the sequences in Z. First let $l_0 = 2$ and, for every $i \in \mathbb{N}$, $l_{i+1} = 2^{l_i}$. Define $f_0 : \Sigma_m^* \to \Sigma_m^*$ be such that $f_0(w) = w$ for all $w \in \Sigma_m^*$. Let $\rho = \frac{1}{\alpha\beta' - \alpha\beta + 1}$. For each $n > 0$, define $f_n : \Sigma_m^* \to \Sigma_m^*$ such that, for every $w \in \Sigma_m^*$, $|f_n(w)| = |w|$ and for every $i < |w|$,

$$f_n(w)[i] = \begin{cases} f_{n-1}(w)[i] & \text{if } i \leq l_{n-1} \\ w[i] & \text{if } i \leq \lceil \rho l_n \rceil \text{ and } i > l_{n-1} \\ 1 & \text{if } i > \lceil \rho l_n \rceil \text{ and } i \leq l_n \\ w[i] & \text{if } i > l_n. \end{cases}$$

Define $f : \Sigma_m^* \to \Sigma_m^*$ such that, for all $w \in \Sigma_m^*$,

$$f(w) = f_{n(w)}(w),$$

where $n(w) = \min\{n \in \mathbb{N} \mid l_n \geq |w|\}$. Also, extend f to $f : \Sigma_m^\infty \to \Sigma_m^\infty$ such that, for all $S \in \Sigma_m^\infty$,

$$f(S) = \lim_{n\to\infty} f(S[0..n-1]).$$

Let

$$Z' = f(Z).$$

By the construction of f and choice of ρ, it is clear that f is a dilation (see Theorem 3.1.11) and, for all $n \in \mathbb{N}$, $|\mathrm{Col}(f, S[0..\lceil \rho l_n \rceil - 1])| \leq \log l_n$. Thus, for all $\epsilon > 0$, there are infinitely many n such that

$$|\mathrm{Col}(f, S[0..n-1])| < \epsilon n. \tag{4.4.1}$$

Note that, by Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. Then by Theorem 3.1.11 and (4.4.1), $\dim_{\mathrm{P}}(Z') \geq \mathcal{H}_{\beta,m}(\alpha^*(\beta))$.

It is easy to verify that, for every $S \in Z'$,

$$\liminf_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta \text{ and } \limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta'.$$

So $Z' \subseteq X$. Therefore,

$$\mathrm{Dim}_{\mathrm{FS}}(X) \geq \dim_{\mathrm{P}}(X) \geq \mathcal{H}_{\beta,m}(\alpha^*(\beta)).$$

Now, we prove that $\mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ is an upper bound for $\dim_{\mathrm{H}}(X)$ and $\dim_{\mathrm{FS}}(X)$.

When $\beta' < 1$, $\mathcal{H}_{\beta',m}(\alpha^*(\beta')) = 1$ and the upper bound holds trivially. So assume that $\beta' \geq 1$.

Let $\alpha = \alpha^*(\beta')$. Let $s > \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$. Define

$$d(\lambda) = 1$$

$$d(wb) = \begin{cases} m^s \alpha d(w) & \text{if } b = 0 \\ m^s \beta' \alpha d(w) & \text{if } b = 1 \\ m^s \frac{1-\alpha-\beta'\alpha}{m-2} d(w) & \text{if } b \geq 2 \end{cases}.$$

It is clear that d is a finite-state s-gale.

Let

$$B = \beta'^{\frac{\beta'}{\beta'+1}}.$$

Let

$$\epsilon = \frac{s - \mathcal{H}_{\beta',m}(\alpha^*(\beta'))}{2\log_m B}.$$

Let $S \in X$ and let $\delta > 0$ be such that $\delta \leq \min(\epsilon\beta'^2/2, 1/2)$. Since

$$\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \geq \beta',$$

there exists an infinite set $J \subseteq \mathbb{N}$ such that for all $n \in J$

$$\frac{\beta_n}{\alpha_n} \geq \beta' - \delta.$$

By the choice of δ, for all $n \in J$

$$\frac{\alpha_n}{\beta_n} \leq \frac{1}{\beta' - \delta} = \frac{1}{\beta'} + \frac{\delta}{(\beta' - \delta)\beta'} \leq \frac{1}{\beta'} + \epsilon;$$

i.e.,

$$\alpha_n + \beta_n \leq \frac{\beta' + 1}{\beta'}\beta_n + \epsilon. \tag{4.4.2}$$

Now note that

$$m^s B^{1-\epsilon} = (1 + \beta' + (m-2)B)B^{\epsilon}, \tag{4.4.3}$$

since

$$\begin{aligned} m^s B^{1-\epsilon} &= m^s B^{1 - \frac{s - \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\ &= B^{1 + \log_B m^s - \frac{\log_m m^s - \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\ &= B^{1 + \frac{2\log_m m^s - \log_m m^s + \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\ &= B^{1 + \frac{\log_m m^s + \log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\ &= B^{1 + \frac{s - \log_m(m-2+\frac{1+\beta'}{B}) + 2\log_m(m-2+\frac{1+\beta'}{B})}{2\log_m B}} \\ &= B^{1 + \epsilon + \log_B(m-2+\frac{1+\beta'}{B})}. \end{aligned}$$

For all $n \in J$,

$$\begin{aligned} d(S[0..n-1]) &= m^{sn} \alpha^{n\alpha_n} (\beta'\alpha)^{n\beta_n} \left(\frac{1 - \alpha - \beta'\alpha}{m-2}\right)^{n(1-\alpha_n-\beta_n)} \\ &= \left[\frac{m^s \beta'^{\beta_n} B^{1-\alpha_n-\beta_n}}{1 + \beta' + (m-2)B}\right]^n \\ &\geq^{(4.4.2)} \left[\frac{m^s \beta'^{\beta_n} B^{1 - \frac{\beta'+1}{\beta'}\beta_n - \epsilon}}{1 + \beta' + (m-2)B}\right]^n \\ &= \left[\frac{m^s B^{1-\epsilon}}{1 + \beta' + (m-2)B}\right]^n \\ &=^{(4.4.3)} B^{\epsilon n}. \end{aligned}$$

Since J is an infinite set,

$$\limsup_{n\to\infty} d(S[0..n-1]) = \infty;$$

i.e., $S \in S^{\infty}[d]$. Since $s > \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$ is arbitrary and d is finite-state s-gale, $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{FS}}(X) \leq \mathcal{H}_{\beta',m}(\alpha^*(\beta'))$.

An essentially identical argument gives us $\dim_{\mathrm{P}}(X) \leq \mathrm{Dim}_{\mathrm{FS}}(X) \leq \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. □

Corollary 4.4.4 (Barreira, Saussol, and Schmeling [10])**.** *Let* $\beta \geq 0$. *Let*

$$X = \left\{ S \;\middle|\; \lim_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} = \beta \right\}.$$

Let $\beta' = \max\{\beta, 1/\beta\}$. *Then*

$$\dim_{\mathrm{H}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta')) = \log_m \left(m - 2 + \frac{1+\beta'}{\beta^{\frac{\beta'}{\beta'+1}}} \right).$$

Proof. We prove the case where $\beta' = \beta$. The other case is similar by switching 0's and 1's in the sequences. Let $Y = \left\{ S \;\middle|\; \liminf_{n\to\infty} \frac{\pi_1(S,n)}{\pi_0(S,n)} \geq \beta \right\}$. Let

$$Z = \left\{ S \;\middle|\; \begin{array}{l} \lim_{n\to\infty} \pi_0(S,n) = \alpha^*(\beta), \lim_{n\to\infty} \pi_1(S,n) = \beta\alpha^*(\beta), \\ \text{and } (\forall i > 1) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha^*(\beta)-\beta\alpha^*(\beta)}{m-2} \end{array} \right\}.$$

By Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. Since $Z \subseteq X \subseteq Y$, it follows immediately from Theorem 4.4.3 that $\dim_{\mathrm{H}}(X) = \mathcal{H}_{\beta,m}(\alpha^*(\beta))$. □

Note that Theorem 4.4.3 gives more than Corollary 4.4.4, since it also implies that $\dim_{\mathrm{P}}(X)$, $\dim_{\mathrm{FS}}(X)$, and $\mathrm{Dim}_{\mathrm{FS}}(X)$ have the value $\dim_{\mathrm{H}}(X)$.

Theorem 4.4.5. *Let* $\alpha \geq 1/m$. *Let*

$$X = \left\{ S \;\middle|\; \lim_{n\to\infty} \pi_0(S,n) = \alpha \right\}$$

and

$$Y = \left\{ S \;\middle|\; \liminf_{n\to\infty} \pi_0(S,n) \geq \alpha \right\}.$$

Then

$$\dim_{\mathrm{P}}(X) = \dim_{\mathrm{H}}(X) = \dim_{\mathrm{P}}(Y) = \dim_{\mathrm{H}}(Y) = \log_m \left[\alpha^{-\alpha} \left(\frac{1-\alpha}{m-1} \right)^{\alpha-1} \right].$$

Proof. The results are clear for $\alpha = 1/m$, so we assume that $\alpha > 1/m$. Let

$$H_{\alpha,m} = \log_m \left[\alpha^{-\alpha} \left(\frac{1-\alpha}{m-1} \right)^{\alpha-1} \right].$$

We first show that $\dim_{\mathrm{P}}(Y) \le H_{\alpha,m}$. For $s > H_{\alpha,m}$, define

$$d(\lambda) = 1$$

$$d(wb) = \begin{cases} m^s \alpha d(w) & \text{if } b = 0 \\ m^s \frac{1-\alpha}{m-1} d(w) & \text{if } b \neq 0. \end{cases}$$

It is clear that d is an s-gale. Let

$$\epsilon = \frac{s - H_{\alpha,m}}{2 \log_m \frac{\alpha(m-1)}{1-\alpha}}. \tag{4.4.4}$$

Note that $\frac{\alpha(m-1)}{1-\alpha} > 1$. Let $S \in Y$; i.e., $\liminf\limits_{n\to\infty} \pi_0(S,n) \ge \alpha$. So there exists $J \subseteq \mathbb{N}$ such that J is co-finite and, for every $n \in J$, $\pi_0(S,n) \ge \alpha - \epsilon$. Now

$$\begin{aligned} d(S[0..n-1]) &= \left[m^s \alpha^{\pi_0(S,n)} \left(\frac{1-\alpha}{m-1} \right)^{1-\pi_0(S,n)} \right]^n \\ &=^{(4.4.4)} \left[\left(\frac{\alpha(m-1)}{1-\alpha} \right)^{2\epsilon} \alpha^{-\alpha} \left(\frac{1-\alpha}{m-1} \right)^{\alpha-1} \alpha^{\pi_0(S,n)} \left(\frac{1-\alpha}{m-1} \right)^{1-\pi_0(S,n)} \right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha} \right)^{2\epsilon} \alpha^{\pi_0(S,n)-\alpha} \left(\frac{1-\alpha}{m-1} \right)^{\alpha-\pi_0(S,n)} \right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha} \right)^{2\epsilon} \left(\frac{\alpha(m-1)}{1-\alpha} \right)^{\pi_0(S,n)-\alpha} \right]^n \\ &= \left[\left(\frac{\alpha(m-1)}{1-\alpha} \right)^{2\epsilon+\pi_0(S,n)-\alpha} \right]^n. \end{aligned}$$

Then, for every $n \in J$,

$$d(S[0..n-1]) \ge \left[\frac{\alpha(m-1)}{1-\alpha} \right]^{\epsilon n}.$$

Since $\frac{\alpha(m-1)}{1-\alpha} > 1$, $S \in S^{\infty}_{\mathrm{str}}[d]$ and $\dim_{\mathrm{H}}(Y) \le \dim_{\mathrm{P}}(Y) \le H_{\alpha,m}$. Note that $X \subseteq Y$, so $\dim_{\mathrm{H}}(X) \le \dim_{\mathrm{P}}(X) \le H_{\alpha,m}$.

Now it suffices to show that $\dim_{\mathrm{H}}(X) \ge H_{\alpha,m}$. Let

$$Z = \left\{ S \;\middle|\; \lim_{n\to\infty} \pi_0(S[0..n-1]) = \alpha \text{ and } (\forall i > 0) \lim_{n\to\infty} \pi_i(S[0..n-1]) = \frac{1-\alpha}{m-1} \right\}.$$

By Eggleston's theorem, $\dim_{\mathrm{H}}(Z) = H_{\alpha,m}$. Since $Z \subseteq X \subseteq Y$, $\dim_{\mathrm{H}}(Y) \geq \dim_{\mathrm{H}}(X) \geq H_{\alpha,m}$. □

Theorem 4.4.6 (Barreira, Saussol, and Schmeling [10]). *Let Σ_m be the m-ary alphabet. Let $k < m$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in [0,1]$ be such that $\alpha = \sum_{i=0}^{k-1} \alpha_i \leq 1$. Let*

$$X = \left\{ S \;\middle|\; \lim_{n\to\infty} \pi_i(S,n) = \alpha_i, 0 \leq i \leq k \right\}.$$

Then $\dim_{\mathrm{H}}(X)$ *is*

$$\mathcal{H}_m\left(\alpha_0, \ldots, \alpha_{k-1}, \tfrac{1-\alpha}{m-k}, \ldots, \tfrac{1-\alpha}{m-k}\right) = \log_m \left[\alpha_0^{-\alpha_0} \cdots \alpha_{k-1}^{-\alpha_{k-1}} \left(\tfrac{1-\alpha}{m-k}\right)^{-(1-\alpha)} \right]$$

and

$$\dim_{\mathrm{FS}}(X) = \mathrm{Dim}_{\mathrm{FS}}(X) = \dim_{\mathrm{P}}(X) = \dim_{\mathrm{H}}(X).$$

Proof. We insist that $0^0 = 1$ and $0/0 = 1$ in this proof.

Let

$$H = \mathcal{H}_m\left(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \frac{1-\alpha}{m-k}, \ldots, \frac{1-\alpha}{m-k}\right).$$

For $s > H$, define

$$d(\lambda) = 1$$

$$d(wb) = \begin{cases} m^s d(w) \alpha_b & \text{if } b < k \\ m^s d(w) \frac{1-\alpha}{m-k} & \text{otherwise.} \end{cases}$$

It is clear that d is a finite-state s-gale. Let

$$\delta = \frac{s - H}{-2 \log_m (\alpha_0 \cdots \alpha_{k-1} \frac{1-\alpha}{m-k})}.$$

For $S \in X$,

$$\lim_{n\to\infty} \pi_i(S,n) = \alpha_i, 0 \leq i \leq k.$$

So there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $|\pi_i(S,n) - \alpha_i| < \delta$ for all $i < k$ and that

$$\left| \alpha - \sum_{i=0}^{k-1} \pi_i(S,n) \right| < \delta.$$

Then, for all $n \geq n_0$,

$$\begin{aligned}
d(S[0..n-1]) &= \left[m^s \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)}\right]^n \\
&= \left[m^{s-H} m^H \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)}\right]^n \\
&= \left[m^{s-H} \alpha_0^{-\alpha_0} \cdots \alpha_{k-1}^{-\alpha_{k-1}} \left(\frac{1-\alpha}{m-k}\right)^{-(1-\alpha)} \left(\frac{1-\alpha}{m-k}\right)^{1-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)}\right]^n \\
&= \left[m^{s-H} \left(\frac{1-\alpha}{m-k}\right)^{\alpha-\sum_{i=0}^{k-1}\pi_i(S,n)} \prod_{i=0}^{k-1} \alpha_i^{\pi_i(S,n)-\alpha_i}\right]^n \\
&\geq \left[m^{s-H} \left(\alpha_0 \cdots \alpha_{k-1} \frac{1-\alpha}{m-k}\right)^{\delta}\right]^n = \left[m^{s-H} m^{\frac{H-s}{2}}\right]^n \\
&= m^{\frac{s-H}{2}n}.
\end{aligned}$$

So $S \in S^{\infty}_{\text{str}}[d]$, and thus $\dim_{\text{FS}}(X) \leq \text{Dim}_{\text{FS}}(X) \leq H$.

Let

$$Z = \left\{ S \;\middle|\; (\forall i < k) \lim_{n\to\infty} \pi_i(S,n) = \alpha_i \text{ and } (\forall i \geq k) \lim_{n\to\infty} \pi_i(S,n) = \frac{1-\alpha}{m-k} \right\}.$$

By Eggleston's theorem, $\dim_{\text{H}}(Z) = H$. The theorem then follows from the monotonicity of dimensions. □

4.4.2 Saturated Sets and the Maximum Entropy Principle

In Section 4.4.1, we calculated the finite-state dimensions of many sets defined using properties on asymptotic frequencies of digits. They are all saturated sets. Now we formally define saturated sets and investigate their collective properties.

Let $\Pi_n(S) = \{\vec{\pi}(S,k) \mid k \geq n\}$ for all $n \in \mathbb{N}$. Let $\bar{\Pi}_n(S) = \overline{\Pi_n(S)}$; i.e., $\bar{\Pi}_n(S)$ is the closure of $\Pi_n(S)$. Define $\Pi : \mathbf{C}_m \to \mathcal{P}(\Delta(\Sigma_m))$ such that for all $S \in \mathbf{C}_m$, $\Pi(S) = \bigcap_{n\in\mathbb{N}} \bar{\Pi}_n(S)$.

Definition. Let $X \subseteq \mathbf{C}_m$. We say that X is *saturated* if for all $S, S' \in \mathbf{C}_m$,

$$\Pi(S) = \Pi(S') \Rightarrow [S \in X \iff S' \in X].$$

When we determine an upper bound on the finite-state dimensions of a set $X \subseteq \mathbf{C}_m$, it is in general not possible to use a single probability measure as the betting strategy, even when

X is saturated. However, when certain conditions are true, a simple 1-state gambler may win on a huge set of sequences with different empirical digit distribution probability measures.

In the following, we formalize such a condition and reveal some relationships between betting and the Kullback-Leibler distance (relative entropy) [25]. Note that the m-dimensional Kullback-Leibler distance $\mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha})$ is defined as

$$\mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) = \mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}}.$$

Definition. Let $\vec{\alpha}, \vec{\beta} \in \Delta(\Sigma_m)$. We say that $\vec{\alpha}$ *ϵ-dominates* $\vec{\beta}$, denoted as $\vec{\alpha} \gg^{\epsilon} \vec{\beta}$, if $\mathcal{H}_m(\vec{\alpha}) \geq \mathcal{H}_m(\vec{\beta}) + \mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) - \epsilon$. We say that $\vec{\alpha}$ *dominates* $\vec{\beta}$, denoted as $\vec{\alpha} \gg \vec{\beta}$, if $\vec{\alpha} \gg^{0} \vec{\beta}$.

Note that $\mathcal{H}_m(\vec{\beta}) + \mathcal{D}_m(\vec{\beta} \parallel \vec{\alpha}) = \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\beta}} + \mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}} = \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}}$, where $\mathrm{E}_{\vec{\beta}} \log_m \frac{\vec{\beta}}{\vec{\alpha}} = \sum_{i=0}^{m-1} \beta_i \log_m \frac{\beta_i}{\alpha_i}$. It is very easy to see that the uniform probability measure dominates all probability measures.

Observation 4.4.7. *If $\vec{\alpha} = (\frac{1}{m}, \ldots, \frac{1}{m})$ and $\vec{\beta} \in \Delta(\Sigma_m)$, then $\vec{\alpha} \gg \vec{\beta}$.*

Here, we give a few interesting properties of the domination relation.

Theorem 4.4.8. *Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}), \vec{\beta} = (\beta_0, \ldots, \beta_{m-1}) \in \Delta(\Sigma_m)$. If $\beta_j = 1$ for some $j \in \Sigma_m$, then $\vec{\alpha} \gg \vec{\beta}$ and $\mathcal{H}_m(\vec{\beta}) = 0$.*

Proof. It is easy to see that $\mathcal{H}_m(\vec{\beta}) = 0$. It suffices to show that

$$\mathcal{H}_m(\vec{\alpha}) \geq \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}}.$$

Fix $j \in \Sigma_m$ such that $\beta_j = 1$. Then

$$\begin{aligned}
\mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} &= \sum_{i=0}^{m-1} \beta_i \log_m \frac{1}{\alpha_i} = \beta_j \log_m \frac{1}{\alpha_j} \\
&= \log_m \frac{1}{\alpha_j} \leq \sum_{i=0}^{m-1} \alpha_i \log_m \frac{1}{\alpha_i} \\
&= \mathcal{H}_m(\vec{\alpha}).
\end{aligned}$$

□

Theorem 4.4.9. *Let $\vec{\alpha}, \vec{\beta} \in \Delta(\Sigma_m)$, $\epsilon \geq 0$, and $r \in [0,1]$. If $\vec{\alpha} \gg^{\epsilon} \vec{\beta}$, then $\vec{\alpha} \gg^{\epsilon} r\vec{\alpha} + (1-r)\vec{\beta}$.*

Proof. Assume $\vec{\alpha} \gg^{\epsilon} \vec{\beta}$. It suffices to show that

$$\mathcal{H}_m(\vec{\alpha}) \geq \mathrm{E}_{r\vec{\alpha}+(1-r)\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} - \epsilon.$$

This holds because

$$\begin{aligned}
\mathrm{E}_{r\vec{\alpha}+(1-r)\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} - \epsilon &= \sum_{i=0}^{m-1} (r\alpha_i + (1-r)\beta_i) \log_m \frac{1}{\alpha_i} - \epsilon \\
&= \sum_{i=0}^{m-1} r\alpha_i \log_m \frac{1}{\alpha_i} + \sum_{i=0}^{m-1} (1-r)\beta_i \log_m \frac{1}{\alpha_i} - \epsilon \\
&= r\mathcal{H}_m(\vec{\alpha}) + (1-r)\mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} - (1-r)\epsilon - r\epsilon \\
&\leq \mathcal{H}_m(\vec{\alpha}).
\end{aligned}$$

□

Theorem 4.4.10. *Let $\vec{\mu} = (\frac{1}{m}, \ldots, \frac{1}{m}) \in \Delta(\Sigma_m)$ be the uniform probability measure. Let $\vec{\beta} \in \Delta(\Sigma_m)$. Let $s \in [0,1]$. Let $\vec{\alpha} = s\vec{\mu} + (1-s)\vec{\beta}$. Then $\vec{\alpha} \gg \vec{\beta}$.*

Proof. Let $A = \{i \mid \mu_i \geq \beta_i\}$, and let $B = \{i \mid \mu_i < \beta_i\}$. Then $A \cap B = \varnothing$ and $A \cup B = [0..m-1]$. Note that, for any $i \in A$, $\mu_i = \frac{1}{m} \geq \beta_i$ and $\log_m \frac{1}{s\mu_i+(1-s)\beta_i} \geq 1$, and, for any $i \in B$, $\mu_i = \frac{1}{m} < \beta_i$ and $\log_m \frac{1}{s\mu_i+(1-s)\beta_i} < 1$. Since $\sum_{i=0}^{m-1} s(\mu_i - \beta_i) = 0$, we have $\sum_{i\in A} s(\mu_i - \beta_i) = -\sum_{i\in B} s(\mu_i - \beta_i)$. It follows that

$$\begin{aligned}
&\mathrm{E}_{\vec{\alpha}} \log_m \frac{1}{\vec{\alpha}} - \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}} \\
&\quad = \mathrm{E}_{s(\vec{\mu}-\vec{\beta})} \log_m \frac{1}{s\vec{\mu} + (1-s)\vec{\beta}} \\
&\quad = \sum_{i=0}^{m-1} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} \\
&\quad = \sum_{i\in A} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} + \sum_{i\in B} s(\mu_i - \beta_i) \log_m \frac{1}{s\mu_i + (1-s)\beta_i} \\
&\quad \geq \sum_{i\in A} s(\mu_i - \beta_i) \cdot 1 + \sum_{i\in B} s(\mu_i - \beta_i) \cdot 1 \\
&\quad \geq 0.
\end{aligned}$$

Therefore,

$$\mathrm{E}_{\vec{\alpha}} \log_m \frac{1}{\vec{\alpha}} \geq \mathrm{E}_{\vec{\beta}} \log_m \frac{1}{\vec{\alpha}};$$

i.e., $\vec{\alpha} \gg \vec{\beta}$. ☐

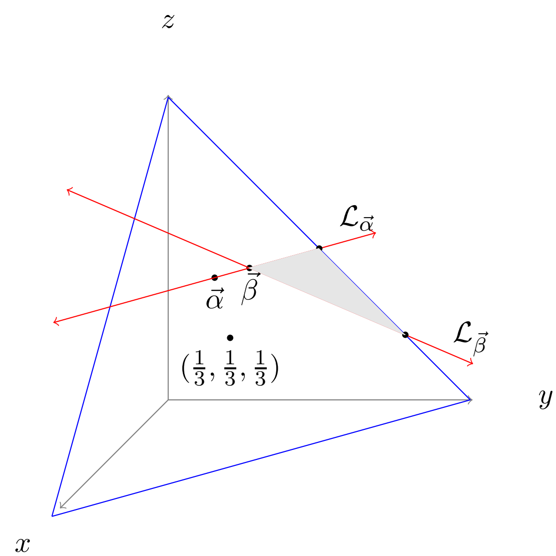

Figure 4.4.1 Domination relationships

Theorem 4.4.11. *The domination relation $\gg$ is not transitive.*

Proof. We give a counterexample with $m = 3$, explaining the idea geometrically so that it easily extends to higher dimensions.

Recall that $\Delta(\Sigma_3)$ is a 2-dimensional simplex in $\mathbb{R}^3$. (See Figure 4.4.1.) The centroid of this simplex is the uniform probability measure $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We first choose *any* probability measure $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$ that is not the centroid and does not lie on the boundary of $\Delta(\Sigma_3)$. For definiteness, say that $\vec{\alpha} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Now, for any $\vec{\delta} = (\delta_0, \delta_1, \delta_2) \in \Delta(\Sigma_m)$,

$$\begin{aligned}\vec{\alpha} \gg \vec{\delta} &\iff \mathcal{H}_3(\vec{\alpha})) \geq \mathcal{H}_3(\vec{\delta}) + \mathcal{D}_3(\vec{\delta}||\vec{\alpha}) \\ &\iff \delta_0 \log_3 \frac{1}{\alpha_0} + \delta_1 \log_3 \frac{1}{\alpha_1} + \delta_2 \log_3 \frac{1}{\alpha_2}.\end{aligned}$$

That is, if we define the line

$$\mathcal{L}_{\vec{\alpha}} = \left\{ \vec{\delta} \;\middle|\; \delta_0 \log_3 \tfrac{1}{\alpha_0} + \delta_1 \log_3 \tfrac{1}{\alpha_1} + \delta_2 \log_3 \tfrac{1}{\alpha_2} = \mathcal{H}_3(\vec{\alpha}) \right\}$$

(which goes through $\vec{\alpha}$), then $\vec{\alpha} \gg \vec{\delta}$ holds if and only if $\vec{\delta}$ lies on $\mathcal{L}_{\vec{\alpha}}$ or on the far side of $\mathcal{L}_{\vec{\alpha}}$ from the centroid.

If we now let $\vec{\beta} = (\beta_0, \beta_1, \beta_2)$ be any point on $\mathcal{L}_{\vec{\alpha}}$ that is not $\vec{\alpha}$ and does not lie on the boundary of $\Delta(\Sigma_3)$, say, $\vec{\beta} = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$, then $\vec{\beta}$ similarly determines a line

$$\mathcal{L}_{\vec{\beta}} = \left\{ \vec{\delta} \;\middle|\; \delta_0 \log_3 \tfrac{1}{\beta_0} + \delta_1 \log_3 \tfrac{1}{\beta_1} + \delta_2 \log_3 \tfrac{1}{\beta_2} = \mathcal{H}_3(\vec{\beta}) \right\}$$

through $\vec{\beta}$ such that $\vec{\beta} \gg \vec{\delta}$ holds if and only if $\vec{\delta}$ lies on $\mathcal{L}_{\vec{\beta}}$ or on the far side of $\mathcal{L}_{\vec{\beta}}$ from the centroid.

Now $\mathcal{L}_{\vec{\alpha}}$ and $\mathcal{L}_{\vec{\beta}}$ both go through $\vec{\beta}$; $\mathcal{L}_{\vec{\alpha}}$ and $\mathcal{L}_{\vec{\beta}}$ have different slopes; and $\vec{\beta}$ is strictly interior to the simplex $\Delta(\Sigma_3)$. It follows from these three things that there is a nonempty region of $\Delta(\Sigma_3)$ (the shaded region in Figure 4.4.1) consisting of probability measures on the far side of $\mathcal{L}_{\vec{\beta}}$ from the centroid and *strictly* on the near side of $\mathcal{L}_{\vec{\alpha}}$ from the centroid. If we choose any $\vec{\gamma}$ in this region, say, $\vec{\gamma} = (0, 0.6, 0.4)$, then $\vec{\alpha} \gg \vec{\beta}$ and $\vec{\beta} \gg \vec{\gamma}$, but $\vec{\alpha} \not\gg \vec{\gamma}$. □

The following theorem relates the domination relation to finite-state dimensions.

Theorem 4.4.12. *Let $\vec{\alpha} \in \Delta(\Sigma_m)$ and $X \subseteq \Sigma_m^\infty$.*

1. *If $\vec{\alpha} \gg^\epsilon \vec{\pi}(S, n)$ for infinitely many n for every $\epsilon > 0$ and every $S \in X$, then $\dim_{\mathrm{FS}}(X) \leq \mathcal{H}_m(\vec{\alpha})$.*

2. *If $\vec{\alpha} \gg^\epsilon \vec{\pi}(S, n)$ for all but finitely many n for every $\epsilon > 0$ and every $S \in X$, then $\mathrm{Dim}_{\mathrm{FS}}(X) \leq \mathcal{H}_m(\vec{\alpha})$.*

Proof. Let $G = (Q, \Sigma_m, \delta, \vec{\beta}, q_0)$ be an FSG such that $Q = \{q_0\}$, $\delta(q_0, b) = q_0$ for all $b \in \Sigma_m$, and $\vec{\beta}(q_0) = \vec{\alpha}$.

Let $s > \mathcal{H}_m(\vec{\alpha}) + \epsilon$. The s-gale $d_G^{(s)}$ of G is defined by the following recursion,

$$d_G^{(s)}(\lambda) = 1,$$

$$d_G^{(s)}(wb) = m^s d_G^{(s)}(w)\alpha_b$$

for all $w \in \Sigma_m^*$ and $b \in \Sigma_m$. Let $S \in X$. Then

$$\begin{aligned} d_G^{(s)}(S[0..n-1]) &= m^{sn} \prod_{i=0}^{m-1} \alpha_i^{n\pi_i(S,n)} \\ &= m^{sn} m^{n \sum_{i=0}^{m-1} \pi_i(S,n) \log_m \alpha_i} \\ &= \left(m^{s - \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}}} \right)^n . \end{aligned}$$

Thus $S \in S^{\infty}[d_G^{(s)}]$ and $\dim_{\mathrm{FS}}(S) \leq s$, when the domination condition holds for infinitely many n. Similarly, $S \in S^{\infty}_{\mathrm{str}}[d_G^{(s)}]$ and $\mathrm{Dim}_{\mathrm{FS}}(S) \leq s$, when the domination condition holds for all but finitely many n. The theorem then follows, since ϵ can be arbitrarily small. □

Theorem 4.4.12 tells us that a probability measure $\vec{\alpha}$ that dominates the empirical frequencies of elements of a set $X \subseteq \mathbf{C}_m$ can be used to infer $\mathcal{H}_m(\vec{\alpha})$ as an upper bound on the finite-state dimension of X. If we insist on doing this with only a single $\vec{\alpha}$, this upper bound may not be a good approximation of the finite-state dimension. (For example, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the only probability measure dominating all of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, so this could give the upper bound 1 on a set of dimension 0.) Nevertheless, the following theorem uses the compactness of $\Delta(\Sigma_m)$ to give a general method for finding the dimensions of saturated sets. It says that the dimension of a saturated set is the supremum of the asymptotic entropies of the empirical frequencies of digits.

Theorem 4.4.13. *Let $X \subseteq \mathbf{C}_m$ be saturated. Let*

$$H = \sup_{S \in X} \liminf_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S,n))$$

and

$$P = \sup_{S \in X} \limsup_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S,n)).$$

Then

$$\dim_{\mathrm{FS}}(X) = H \text{ and } \mathrm{Dim}_{\mathrm{FS}}(X) = P$$

and

$$\dim_{\mathrm{FS}}(X) = \dim_{\mathrm{H}}(X) \text{ and } \mathrm{Dim}_{\mathrm{FS}}(X) = \dim_{\mathrm{P}}(X).$$

In order to prove this theorem, we need the following result, which is a restatement of Lemma 4.3.2 in terms of strings instead of integers.

Lemma 4.4.14. *For every $n \geq m \geq 2$ and every partition $\vec{a} = (a_0, \ldots, a_{m-1})$ of n, there are more than*

$$m^{n\mathcal{H}_m(\frac{\vec{a}}{n}) - (m+1)\log_m n}$$

strings u of length n with $\#(i,u) = a_i$ for each $i \in \Sigma_m$.

Proof of Theorem 4.4.13. First we prove $\dim_{\mathrm{H}}(X) \geq H$. It suffices to show that, for all $s < H$, $\dim_{\mathrm{H}}(X) \geq s$.

Let $s < H$. Let d be an arbitrary s-supergale. Let $s' = (H+s)/2$. Let $n_0 \in \mathbb{N}$ be such that $\sqrt{m} < n_0(H - s')$ and $m^{s'n_0 - (m+1)\log_m n_0} > 2^{sn_0+1}$. Fix an $S \in X$ such that $\liminf\limits_{n\to\infty} \mathcal{H}_m(\vec{\pi}(S,n)) > s'$.

For each $i \geq n_0$, let $\{\vec{\beta}_{i,1}, \ldots, \vec{\beta}_{i,c_i}\} \subseteq \Delta(\Sigma_m)$ be such that, for each $j \in [1..c_i]$, $\vec{\beta}_{i,j} = \frac{\vec{a}}{n}$ for some partition $\vec{a} \in \mathbb{Z}^m$ of n and $\mathcal{H}_m(\vec{\beta}_{i,j}) > s'$; for all $\vec{\beta} \in \Pi(S)$ there exists $j \in [1..c_i]$ such that $|\vec{\beta}_{i,j} - \vec{\beta}| < 1/i$; for all $j \in [1..c_i]$, there exists $\vec{\beta} \in \Pi(S)$ such that $|\vec{\beta}_{i,j} - \vec{\beta}| < 1/i$; for all $j \in [1..c_i - 1]$, $|\vec{\beta}_{i,j} - \vec{\beta}_{i,j+1}| < \frac{1}{i}$; for all $i \geq n_0$, $|\vec{\beta}_{i+1,0} - \vec{\beta}_{i,c_i}| < \frac{1}{i+1}$. This is possible because $\Pi(S)$ is a compact set.

Now, we first construct a sequence $S' \in \Sigma_m^\infty$ by building its prefixes inductively. Let w_0 be such that $|w_0| = 2^{n_0}$. Note that the choice of w_0 does not affect the argument, since w_0 does not change the asymptotic behavior of the sequence. Without loss of generality, assume $\vec{\pi}(w_0, |w_0|) = \beta_{n_0,1}$.

For all $n > 0$, assume that w_{n-1} is already constructed. Let $w_{n,0} = w_{n-1}$. We construct inductively $w_{n,1}, \ldots, w_{n,c_n}$ and then let $w_n = w_{n,c_n}$. For $j > 0$, assume that $w_{n,j-1}$ is already constructed. Let $l = n_0 + n - 1$. For each l, j, let

$$B_{l,j} = \left\{ u \in \Sigma_m^l \;\middle|\; \vec{\pi}(u,l) = \vec{\beta}_{l,j} \right\}.$$

For each $l \geq n_0$ and $w \in \Sigma_m^*$, let

$$W_{l,w} = \left\{ u \in \Sigma_m^l \;\middle|\; d(wu) \leq \frac{1}{m} d(w) \right\}.$$

Since d is an s-supergale, by Theorem 4.1.3, for all $w \in \Sigma_m^*$, there are fewer than m^{sl+1} strings $u \in \Sigma_m^l$ for which $d(wu) > \frac{1}{m} d(w)$. By the choice of n_0, $\vec{\beta}_{l,j}$, and Lemma 4.4.14,

$$|B_{l,j}| > m^{sl+1};$$

i.e., $W_{l,w} \cap B_{l,j} \neq \varnothing$.

Let $u_1 \in W_{l,w} \cap B_{l,j}$. For all $i \in [2..2^{|w_{n,j-1}|}]$, let $u_i \in W_{l,wu_1\ldots u_{i-1}} \cap B_{l,j}$. Let $w_{n,j} = w_{n,j-1} u_1 \ldots u_{2^{|w_{n,j-1}|}}$. Let

$$S' = \lim_{n\to\infty} w_n.$$

Note that, when w_n is being constructed, $l \leq \lfloor \log_m |w_{n,j-1}| \rfloor$. It is then easy to verify that $S' \notin S^{\infty}[d]$.

Now we verify that $\Pi(S) = \Pi(S')$, from which we can conclude that $S' \in X$, since X is defined by asymptotic frequency.

Let $\vec{\beta} \in \Pi(S)$ be arbitrary. For each $l = n_0 + n - 1$, there exists some j_l such that $|\vec{\beta} - \vec{\beta}_{l,j_l}| < \frac{1}{l}$. Then, by the construction,

$$|\vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) - \vec{\beta}_{l,j_l}| < \sqrt{m}\frac{2}{|w_{l,j_l}|} < \frac{1}{l}.$$

So it is clear that

$$|\vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) - \vec{\beta}| < \frac{2\sqrt{m}}{l}.$$

Thus

$$\lim_{l \to \infty} \vec{\pi}(w_{l,j_l}, |w_{l,j_l}|) = \vec{\beta}.$$

Since $w_{l,j_l} \sqsubseteq S'$ for all $l = n_0 + n - 1$. So we have for all $n \in \mathbb{N}$, $\vec{\beta} \in \bar{\Pi}_n(S')$, hence $\vec{\beta} \in \Pi(S')$. Therefore $\Pi(S) \subseteq \Pi(S')$.

We prove $\Pi(S') \subseteq \Pi(S)$ by proving its contrapositive. Now, let $\vec{\beta} \notin \Pi(S)$. Since $\Pi(S)$ is closed, there exists $\delta > 0$ such that, for all $\vec{\beta}' \in \Pi(S)$, $|\vec{\beta} - \vec{\beta}'| > \delta$. Let n_1 be such that $l_1 = n_0 + n_1 - 1 > \frac{8m}{\delta}$. By construction, for all $l \geq l_1$, all $j \in [1..c_l]$, and all $|w_{l,j-1}| \leq k \leq |w_{l,j}|$,

$$|\vec{\pi}(w_{l,j}, |w_{l,j}|) - \vec{\pi}(w_{l,j}, k)| < \frac{2\sqrt{m}}{l}.$$

Also, for all $l \geq l_1$ and all $j \in [1..c_l]$, there exists $\vec{\beta}' \in \Pi(S)$ such that

$$|\vec{\pi}(w_{l,j}, |w_{l,j}|) - \vec{\beta}'| < \frac{2\sqrt{m}}{l}.$$

Thus, for all $k > |w_{l_1,1}|$, there exists $\vec{\beta}' \in \Pi(S)$ such that

$$|\vec{\pi}(S, k) - \vec{\beta}'| < \frac{4m}{l}.$$

Therefore, for all $k > |w_{l_1,1}|$,

$$|\vec{\pi}(S, k) - \vec{\beta}'| < \frac{4m}{l_1} < \frac{\delta}{2}.$$

Thus, for all sufficiently large k,

$$|\vec{\pi}(S, k) - \vec{\beta}| > \frac{\delta}{2}.$$

So there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, $\vec{\beta} \notin \bar{\Pi}_n$, i.e., $\vec{\beta} \notin \Pi(S')$.

Now we have that $S' \in X$. Since $S' \notin S^\infty[d]$, $s < H$ is arbitrary, and d is an arbitrary s-supergale, it follows that

$$\dim_{\mathrm{H}}(X) \geq H.$$

By a similar construction, we may prove that

$$\dim_{\mathrm{P}}(X) \geq P.$$

In the following, we prove the finite-state dimension upper bounds. Given $\vec{\alpha} \in \Delta(\Sigma_m)$, define $B(\vec{\alpha}, r)$ as

$$B(\vec{\alpha}, r) = \Delta(\Sigma_m) \cap \left\{ \vec{\beta} \in \mathbb{R}^m \;\middle|\; (\forall i)[\beta_i < \alpha_i m^r \text{ and } \beta_i > \alpha_i m^{-r}] \right\}.$$

Let

$$F(X) = \left\{ \vec{\alpha} \in \Delta(\Sigma_m) \;\middle|\; \mathcal{H}_m(\vec{\alpha}) = H \right\}.$$

Let $\epsilon > 0$. Let

$$\mathcal{C} = \left\{ B(\vec{\alpha}, \tfrac{\epsilon}{2}) \;\middle|\; \vec{\alpha} \in F(X) \right\}.$$

It is clear that $\mathcal{C}$ is an open cover of $F(X)$. Since $F(X)$ is compact, there exists $C \subseteq \Delta(\Sigma_m)$ such that $|C| < \infty$ and

$$F(X) \subseteq \bigcup_{\vec{\alpha} \in C} B(\vec{\alpha}, \tfrac{\epsilon}{2}).$$

Let $S \in X$. Then $\liminf\limits_{n \to \infty} \mathcal{H}_m(\vec{\pi}(S, n)) \leq H$. Since the entropy function is continuous in its domain, there exists $\vec{\alpha}^* \in F(X)$ that is a convex combination of the uniform probability measure and $\vec{\pi}(S, n)$. By Theorem 4.4.10, $\vec{\alpha}^* \gg \vec{\pi}(S, n)$ for infinitely many $n \in \mathbb{N}$. Then, by the construction of C, there exists $\vec{\alpha} \in C$ such that $\vec{\alpha}^* \in B(\vec{\alpha}, \frac{\epsilon}{2})$. Now, we have that, for infinitely many $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{H}_m(\vec{\alpha}) = \mathcal{H}_m(\vec{\alpha}^*) &\geq \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}^*} - \frac{\epsilon}{2} \\
&= \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}} + \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{\vec{\alpha}}{\vec{\alpha}^*} - \frac{\epsilon}{2}.
\end{aligned}$$

By the definition of $B(\vec{\alpha}, \frac{\epsilon}{2})$,

$$\mathcal{H}_m(\vec{\alpha}) \geq \mathrm{E}_{\vec{\pi}(S,n)} \log_m \frac{1}{\vec{\alpha}} - \epsilon;$$

i.e., $\vec{\alpha} \gg^{\epsilon} \vec{\pi}(S,n)$ for infinitely many $n \in \mathbb{N}$. Since $S \in X$ is arbitrary, we may partition X as $X = \bigcup_{\vec{\alpha} \in C} X_{\vec{\alpha}}$ such that, for every $\vec{\alpha} \in C$,

$$X_{\vec{\alpha}} = \{S \in X \mid \vec{\alpha} \gg^{\epsilon} \vec{\pi}(S,n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Since $\epsilon > 0$ is arbitrary, Theorem 4.4.12 tells us that $\dim_{\mathrm{FS}}(X_{\vec{\alpha}}) \leq \mathcal{H}_m(\vec{\alpha}) = H$ for every $\vec{\alpha} \in C$. Since $|C| < \infty$, Theorem 4.1.1 tells us that $\dim_{\mathrm{FS}}(X) \leq H$. Similarly, $\mathrm{Dim}_{\mathrm{FS}}(X) \leq P$. □

Theorem 4.4.13 automatically gives a *pointwise* solution for finding an upper bounds for dimensions of arbitrary X.

Corollary 4.4.15. *Let $X \subseteq \mathbf{C}_m$, and let H and P be defined as in Theorem 4.4.13. Then $\dim_{\mathrm{FS}}(X) \leq H$ and $\mathrm{Dim}_{\mathrm{FS}}(X) \leq P$.*

In the following, we derive the dimensions of a few interesting saturated sets using Theorem 4.4.13.

Let $H_{\alpha,m} = \log_m[\alpha^{-\alpha}(\frac{1-\alpha}{m-1})^{\alpha-1}]$.

Theorem 4.4.16. *Let $\underline{\alpha}, \bar{\alpha} \in [0,1]$ such that $1/m < \underline{\alpha} \leq \bar{\alpha}$ and let*

$$M_k^{\underline{\alpha},\bar{\alpha}} = \{S \in \Sigma_m^{\infty} \mid \liminf_{n\to\infty} \pi_k(S,n) = \underline{\alpha} \text{ and } \limsup_{n\to\infty} \pi_k(S,n) = \bar{\alpha}\}.$$

Then $\dim_{\mathrm{H}}(M_k^{\underline{\alpha},\bar{\alpha}}) = H_{\bar{\alpha},m}$ and $\dim_{\mathrm{P}}(M_k^{\underline{\alpha},\bar{\alpha}}) = H_{\underline{\alpha},m}$.

Proof. It is easy to check that $M_k^{\underline{\alpha},\bar{\alpha}}$ is saturated. We prove this theorem for $k = 0$. For other values of k, the proof is essentially identical.

Let $\vec{\rho}_0 = (\underline{\alpha}, \frac{1-\underline{\alpha}}{m-1})$. Let $\vec{\rho}_1 = (\bar{\alpha}, \frac{1-\bar{\alpha}}{m-1})$. Note that $H_{\underline{\alpha},m} = \mathcal{H}_m(\vec{\rho}_0)$ and $H_{\bar{\alpha},m} = \mathcal{H}_m(\vec{\rho}_1)$. It is easy to verify that

$$H_{\bar{\alpha},m} = \inf_{\vec{\rho} \in \Delta(\Sigma_m) \cap [\underline{\alpha},\bar{\alpha}] \times \mathbb{R}^{m-1}} \mathcal{H}_m(\vec{\rho}),$$

and that

$$H_{\underline{\alpha},m} = \sup_{\vec{\rho} \in \Delta(\Sigma_m) \cap [\underline{\alpha},\underline{\alpha}] \times \mathbb{R}^{m-1}} \mathcal{H}_m(\vec{\rho}).$$

The theorem follows from Theorem 4.4.13 by easily confirming that there exists a sequence $S \in M_k^{\underline{\alpha},\bar{\alpha}}$ such that $\vec{\rho}_0 \in \Pi(S)$ and $\vec{\rho}_1 \in \Pi(S)$. □

Corollary 4.4.17. *Let* $\underline{\alpha}, \bar{\alpha} \in [0,1]$ *such that* $\underline{\alpha} \le \bar{\alpha}$ *and let*

$$M_k^{\underline{\alpha},\bar{\alpha}} = \{S \in \mathbf{C}_m \mid \liminf_{n\to\infty} \pi_k(S,n) = \underline{\alpha} \text{ and } \limsup_{n\to\infty} \pi_k(S,n) = \bar{\alpha}\}.$$

Then

$$\dim_{\mathrm{H}}(M_k^{\underline{\alpha},\bar{\alpha}}) = \inf_{\alpha\in[\underline{\alpha},\bar{\alpha}]} H_{\alpha,m} = \min(H_{\underline{\alpha},m}, H_{\bar{\alpha},m})$$

and

$$\dim_{\mathrm{P}}(M_k^{\underline{\alpha},\bar{\alpha}}) = \sup_{\alpha\in[\underline{\alpha},\bar{\alpha}]} H_{\alpha,m} = \begin{cases} 1 & \text{if } \underline{\alpha} \le 1/m \le \bar{\alpha}, \\ \max(H_{\underline{\alpha},m}, H_{\bar{\alpha},m}) & \text{otherwise.} \end{cases}$$

Proof. If $\underline{\alpha} \le 1/m \le \bar{\alpha}$, then for some $S \in M_k^{\underline{\alpha},\bar{\alpha}}$, $\limsup_{n\to\infty} \mathcal{H}_m(\vec{\pi}(S,n)) = 1$. □

Corollary 4.4.18 (Barreira, Saussol, and Schmeling [10])**.** *Let* $k \in \Sigma_m$ *and let*

$$M_k = \{S \in \mathbf{C}_m \mid \liminf_{n\to\infty} \pi_k(S,n) < \limsup_{n\to\infty} \pi_k(S,n)\}.$$

Then

$$\dim_{\mathrm{H}}(\cap_{k=0}^{m-1} M_k) = 1.$$

Proof. Let $M = \cap_{k=0}^{m-1} M_k$. For all $\epsilon \in (0, \frac{1}{m})$ and all $k \in \Sigma_m$, $M_k^{\frac{1}{m}-\epsilon,\frac{1}{m}+\epsilon} \subseteq M_k$. Let $M_\epsilon = \cap_{k\in M} M_k^{\frac{1}{m}-\epsilon,\frac{1}{m}+\epsilon}$. It is clear that $M_\epsilon \neq \varnothing$, $M_\epsilon \subseteq M$, and M_ϵ is saturated. By Corollary 4.4.17, $\dim_{\mathrm{H}}(M_\epsilon) = H_{\frac{1}{m}-\epsilon,m}$. Then by the monotonicity of dimension (Theorem 4.1.1),

$$\dim_{\mathrm{H}}(M) \ge H_{\frac{1}{m}-\epsilon,m}. \tag{4.4.5}$$

Note that (4.4.5) holds for all ϵ. Therefore,

$$\dim_{\mathrm{H}}(M) \ge \sup_{\epsilon\in(0,\frac{1}{m})} H_{\frac{1}{m}-\epsilon,m} = \lim_{\epsilon\to 0} H_{\frac{1}{m}-\epsilon,m} = 1.$$

□

Theorem 4.4.19. *Let* A *be a* $d \times m$ *matrix and* $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$. *Let*

$$K^{\mathrm{i.o.}}(A,b) = \{S \in \mathbf{C}_m \mid (\exists \{k_n\} \subseteq \mathbb{N}) \lim_{n\to\infty} k_n = \infty \text{ and } \lim_{n\to\infty} A(\vec{\pi}(S,k_n))^T = b\}$$

and let

$$K(A,b) = \{S \in \mathbf{C}_m \mid \lim_{n\to\infty} A(\vec{\pi}(S,n))^T = b\}.$$

Then

$$\dim_{\mathrm{FS}}(K^{\mathrm{i.o.}}(A,b)) = \dim_{\mathrm{H}}(K^{\mathrm{i.o.}}(A,b)) = \sup_{\substack{\vec{\alpha}\in\Delta(\Sigma_m)\\ A\vec{\alpha}^T=b}} \mathcal{H}_m(\vec{\alpha}),$$

$\dim_{\mathrm{P}}(K^{\mathrm{i.o.}}(A,b)) = 1,$ *and* $\dim_{\mathrm{H}}(K(A,b)) = \mathrm{Dim}_{\mathrm{FS}}(K(A,b)) = \sup_{\substack{\vec{\alpha}\in\Delta(\Sigma_m)\\ A\vec{\alpha}^T=b}} \mathcal{H}_m(\vec{\alpha}).$

Proof. It is easy to check that $K^{\mathrm{i.o.}}(A,b)$ and $K(A,b)$ are both saturated. □

Many more examples of the application of Theorem 4.4.13 can be easily enumerated and such examples can be very exotic and the determination of the actual value of the fractal dimensions can still be very difficult. A tool like Theorem 4.4.13 significantly reduces the difficulty of determining fractal dimensions by connecting the dimension of a set to the dimensions of individual elements in the set. However, in practice, the mere difficulty in determining what elements belong to the set under consideration can be prohibitive.

5 FRACTALS IN GEOMETRY

In this chapter, we study computable curves of finite length. The set of all the points that are on some computable curve of finite length form a set $\mathcal{R}$, or the computable transit network.

5.1 Curves and Computability

We fix an integer $n \geq 2$ and work in the Euclidean space $\mathbb{R}^n$. A *tour* is a continuous function $f : [a, b] \to \mathbb{R}^n$ for some real numbers $a < b$. A *curve* is the range of a tour and we say that the tour is a *parametrization* of the curve. We very often choose $a = 0$ and $b = 1$ for convenience. The *length* of a tour f is

$$\text{length}(f) = \sup_{\vec{a}} \sum_{i=0}^{k-1} |f(a_{i+1}) - f(a_i)|,$$

where $|x|$ is the Euclidean norm of a point $x \in \mathbb{R}^n$ and the supremum is taken over all *dissections* $\vec{a}$ of $[a, b]$, i.e., all $\vec{a} = (a_0, \ldots, a_k)$ with $0 = a_0 < a_1 < \cdots < a_k = 1$. Note that length$(f)$ is the length of the actual path traced by f. If f is one-to-one (i.e., the tour is *simple*), then length(f) coincides with $\mathcal{H}^1(f([0, 1]))$, which is the length (i.e., the one-dimensional Hausdorff measure [31]) of the range of f, but, in general, f may "retrace" parts of its range, so length(f) may exceed $\mathcal{H}^1(f([0, 1]))$. A tour f is *rectifiable* if length$(f) < \infty$. A curve is rectifiable if it is the range of some tour f that is rectifiable. A curve is simple if it is the range of some simple tour.

A function f is the *tour of a set* $K \subseteq \mathbb{R}^n$ if f is a tour such that $K \subseteq f([0, 1])$.

Since tours are continuous, the extended computability notion introduced by Braverman [15] coincides with the computability notion formulated in the 1950s by Grzegorczyk [38] and Lacombe [57] and exposited in the recent paper by Braverman and Cook [16] and in the monographs [85, 54, 98]. Specifically, a tour $f : [0,1] \to \mathbb{R}^n$ is *computable* if there is an oracle Turing machine M with the following property. For all $t \in [0,1]$ and $r \in \mathbb{N}$, if M is given a function oracle $\varphi_t : \mathbb{N} \to \mathbb{Q}$ such that, for all $k \in \mathbb{N}$, $|\varphi_t(k) - t| \leq 2^{-k}$, then M, with oracle φ_t and input r, outputs a rational point $M^{\varphi_t}(r) \in \mathbb{Q}^n$ such that $|M^{\varphi_t}(r) - f(t)| \leq 2^{-r}$. A curve Γ is computable if there exists a computable tour f such that $\Gamma = \text{range}(f)$.

A point $x \in \mathbb{R}^n$ is *computable* if there is a computable function $\psi_x : \mathbb{N} \to \mathbb{Q}^n$ such that, for all $r \in \mathbb{N}$, $|\psi_x(r) - x| \leq 2^{-r}$. It is well known and easy to see that, if $f : [0,1] \to \mathbb{R}^n$ and $t \in [0,1]$ are computable, then $f(t)$ is computable.

5.2 The Computable Transit Network

We use $\mathcal{R}$ to denote the computable transit network, i.e., points that lie on rectifiable computable curves. Here we briefly discuss the structure of $\mathcal{R}$, referring freely to existing literature on fractal geometry [31] and effective dimension [63, 64, 28].

For each rectifiable tour f, we have $\mathcal{H}^1(f([0,1])) \leq \text{length}(f) < \infty$, so the Hausdorff dimension of $f([0,1])$ is 1, unless $f([0,1])$ is a single point (in which case the Hausdorff dimension is 0). Since $\mathcal{R}$ is the union of countably many such sets $f([0,1])$, it follows by countable stability [31] that $\mathcal{R}$ has Hausdorff dimension 1. This implies that $\mathcal{R}$ is a Lebesgue measure 0 subset of $\mathbb{R}^n$, i.e., that almost every point in $\mathbb{R}^n$ lies in the complement of $\mathcal{R}$.

Since $\mathcal{R}$ contains every computable point in $\mathbb{R}^n$, $\mathcal{R}$ is dense in $\mathbb{R}^n$. Also, if $x \in f([0,1])$ and $y \in g([0,1])$, where f and g are rectifiable computable tours, then we can use f, g, and the segment from $f(1)$ to $g(0)$ to assemble a rectifiable computable tour h such that $x, y \in h([0,1])$. Hence, $\mathcal{R}$ is path-connected in the strong sense that any two points in $\mathcal{R}$ lie in a *single* rectifiable computable tour.

For each rectifiable computable tour f, the set $f([0,1])$ is a computably closed (i.e., Π^0_1) subset of $\mathbb{R}^n$ [72]. Since $\mathcal{R}$ is the union of all such $f([0,1])$, it follows by Hitchcock's correspondence principle [45] that the constructive dimension of $\mathcal{R}$ coincides with its Hausdorff dimension, which we have observed to be 1. (It is worth mention here that $\mathcal{R}$ can easily be shown *not* to have computable measure 0, whence $\mathcal{R}$ has computable dimension n [63]. By

Staiger's correspondence principle [91, 45], this implies that $\mathcal{R}$ is not a Σ^0_2 set.) It follows that each point $x \in \mathcal{R}$ has dimension at most 1 (in the sense that $\{x\}$ has constructive dimension 1 [64]). It might be reasonable to conjecture that this actually characterizes points in $\mathcal{R}$, but the following example shows that this is not the case.

Construction 5.2.1. Given an infinite binary sequence R, define a sequence $A_0, A_1, A_2, \ldots$ of closed squares in $\mathbb{R}^2$ by the following recursion. First, $A_0 = [0,1]^2$. Next, assuming that A_n has been defined, let a and b be the $2n$th and $(2n+1)$st bits, respectively of R. Then A_{n+1} is the ab-most closed subsquare of A_n with $\text{area}(A_{n+1}) = \frac{1}{16}\text{area}(A_n)$, where $00 =$"lower left", $01 =$"lower right", $10 =$ "upper left", and $11 =$"upper right". Let x_R be the unique point in $\mathbb{R}^2$ such that $x_R \in A_n$ for all $n \in \mathbb{N}$.

It is well known [71, 35] that the set K consisting of all such points x_R is a bounded set with positive, finite one-dimensional Hausdorff measure (and hence with Hausdorff dimension 1), but that K is not contained in any rectifiable curve. The next lemma is a constructive extension of this fact.

Lemma 5.2.2. *For any sequence R that is random (in the sense of Martin-Löf [67]; see also [58, 28]), the point x_R of Construction 5.2.1 has dimension 1 and does not lie on any computable curve of finite length.*

We will need the following claim about geometry to prove Lemma 5.2.2.

Claim. *Let $n \in \mathbb{Z}^+$. Let X be a set of points such that for each $x \in X$, there exists $w_x \in \{0,1\}^{2n}$ with $x \in A_n(w_x)$ and for $x \neq y \in X$, $w_x \neq w_y$. (Note that $|X| \leq 4^n$.) Then the length of any curve that traverse X is at least*

$$\frac{6}{4} 4^{-n} |X| \log_4 |X|.$$

Proof. We prove this by induction on n. For $n = 1$, $|X| \leq 4$ and the claim can be easily verified by using the triangle inequality of the Euclidean plane.

Let $1 < n \in \mathbb{Z}^+$. Assume the claim for the case of $n-1$.

Let $w \in \{0,1\}^{2n}$. The sidelength of $A_n(w)$ is 4^{-n}. For each $a, b \in \{0,1\}$, let

$$X_{ab} = \{A_{n-1}(w) \mid A_n(abw) \in X\}.$$

Then we have that $X = \bigcup_{a,b\in\{0,1\}} X_{ab}$ and $X_{ab} \subseteq A_1(ab)$. Note that for each $a, b \in \{0,1\}$, $A_1(ab)$ is a $\frac{1}{4}$ scaling of the unit square $A_0(\lambda)$. Regard, $A_1(ab)$ as the unit square, then it is clear that the assumption of this claim holds for X_{ab} such that for each $x \in X_{ab}$, there exists a distinct $w_x \in \{0,1\}^{2(n-1)}$ with $x \in A_{n-1}(w_x)$. By the induction hypothesis, the length required to traverse X_{ab} is $\frac{1}{4} \cdot \frac{6}{4} 4^{-(n-1)} |X_{ab}| \log_4 |X_{ab}|$ (note the scaling factor $\frac{1}{4}$ in front).

By the triangle inequality of the Euclidean plane, we know that it uses less length if we connect each non-empty X_{ab} internally and then make $c-1$ connections (each of length at least $\frac{1}{2}$) between different X_{ab}'s, where $c \le 4$ is the number of non-empty X_{ab}'s. So the length required to connect all points in X is

$$\begin{aligned}
&\frac{c-1}{2} + \sum_{\substack{a,b\in\{0,1\}\\ X_{ab}\neq\varnothing}} \left(\frac{1}{4}\cdot\frac{6}{4}4^{-(n-1)}|X_{ab}|\log_4|X_{ab}|\right)\\
\ge&\frac{c-1}{2} + \frac{1}{4}\cdot\frac{6}{4}4^{-(n-1)} \sum_{\substack{a,b\in\{0,1\}\\ X_{ab}\neq\varnothing}} |X_{ab}|\log_4|X_{ab}|\\
\ge^{Jensen's}&\frac{c-1}{2} + \frac{6}{4}4^{-n}c\frac{|X|}{c}\log_4\frac{|X|}{c}\\
\ge&\frac{6}{4}4^{-n}|X|\log_4|X| + \frac{c-1}{2} - \frac{6}{4}4^{-n}|X|\log_4 c\\
\ge&\frac{6}{4}4^{-n}|X|\log_4|X|.
\end{aligned}$$

Note that for $1 \le c \le 4$, $|X| \le c4^{n-1}$ and it can be verified that the above inequality holds for each $1 \le c \le 4$. □

Proof of Lemma 5.2.2. Let $\gamma : [0,1] \to \mathbb{R}^2$ be a computable curve. Let $X_n \subseteq \{0,1\}^{2n}$ be the set of all strings $w \in \{0,1\}^{2n}$ such that the distance between $A_n(w)$ and $\gamma([0,1])$ is less than $2^{-2^{2^n}}$. We defined martingales $d_m : \{0,1\}^* \to [0,\infty)$ such that for all strings of length

2^{m+1}

$$d_m(w) = \begin{cases} 2^{2\cdot 2^m} \frac{|\{x \in X_{2^m} \mid w \sqsubseteq x\}|}{|X_{2^m}|} & |w| \leq 2^{m+1} \\ d(w[0..2^{m+1}-1]) & \text{otherwise.} \end{cases}$$

Note that since γ is computable and bounded, it can be sampled to any precision computably. Therefore, d_m is computable for each m. Let $d = \sum_{m=1}^{\infty} \frac{1}{m^2} d_m$. It is clear that d is constructive.

Let R be (Martin-Löf) random such that $x_R \in \gamma([0,1])$. Then $R[0..2n-1] \in X_n$ for all $n \in \mathbb{N}$ and there exists $c \in \mathbb{N}$ such that $d(R[0..2n-1]) < c$ for all $n \in \mathbb{N}$. Therefore, for each $m > 0$,

$$\frac{1}{m^2} d_m(R[0..2^{m+1}-1]) < d(R[0..2^{m+1}-1]) < c,$$

i.e.,

$$\frac{1}{m^2} 2^{2\cdot 2^m} \frac{\left| \left\{ x \in X_{2^m} \;\middle|\; R[0..2^{m+1}-1] \sqsubseteq x \right\} \right|}{|X_{2^m}|} < c.$$

Note that $\left| \left\{ x \in X_{2^m} \;\middle|\; R[0..2^{m+1}-1] \sqsubseteq x \right\} \right| = 1$. We have

$$|X_{2^m}| > \frac{2^{2^{m+1}}}{cm^2},$$

there are more than points in (or extremely close to) $\frac{2^{2^{m+1}}}{cm^2}$ blocks of the form $A_{2^m}(w)$. Since all these blocks are traversed by γ, by the Claim, the length of γ is at least

$$\frac{6}{4} 4^{-2^m} \frac{2^{2^{m+1}}}{cm^2} \log_4 \left(\frac{2^{2^{m+1}}}{cm^2} \right) = \frac{3}{2cm^2} (2^m - \log_4(cm^2)) \to \infty \text{ as } m \to \infty.$$

Therefore, γ cannot have finite length. □

The following theorem shows that more is true, although the proof, a Baire category argument, does not yield such a concrete example.

Theorem 5.2.3. *The complement of $\mathcal{R}$ contains points of arbitrarily small dimension, including* 0.

Lemma 5.2.4. $\mathrm{DIM}^{=0} \cap [0,1]$ *is co-meager in* $[0,1]$ *and* $\mathrm{DIM}^{=0} \cap [0,1]^2$ *is co-meager in* $[0,1]^2$.

Proof. We prove in the coding space instead of $[0,1]$ and $[0,1]^2$. Since the proof for $[0,1]$ and $[0,1]^2$ are almost identical, we only prove it for the $[0,1]$ case. Note that the constructive dimension of a point in $\mathbb{R}^n$ is n times the dimension of its coding sequence [65].

Let $h : \{0,1\}^* \to \{0,1\}^*$ be a constructor such that

$$h(w) = w0^{2^{|w|}}$$

for all $w \in \{0,1\}^*$. Let $g : \{0,1\}^* \to \{0,1\}^*$ be an arbitrary constructor. Let $S = R(g, h)$.

We claim that $S \in \mathrm{DIM}^{=0}$.

Let $\epsilon > 0$ be rational. Let $d_\epsilon : \{0,1\}^* \to [0,\infty)$ be such that $d_\epsilon(\lambda) = 1$ and for all $\lambda \neq w \in \{0,1\}^*$,

$$d_\epsilon(w0) = 2^s d_\epsilon(w)(1-\epsilon) \text{ and } d_\epsilon(w1) = 2^s d_\epsilon(w)\epsilon.$$

It is easy to verify that d_ϵ is a computable (hence constructive) s-gale. Let $f(n) = (f_0(n), f_1(n)) \in [0,1] \times [0,1]$ be such that $f_0(n)$ is the frequency of 0's in $S[0..n-1]$ and $f_1(n)$ is the frequency of 1's in $S[0..n-1]$. Let $n = |(h \circ g)^i(\lambda)|$.

$$\begin{aligned} d_\epsilon(S[0..n-1]) &= 2^{sn}(1-\epsilon)^{nf_0(n)}\epsilon^{nf_1(n)} \\ &= 2^{(s+f_0(n)\log(1-\epsilon)+f_1(n)\log\epsilon)n} \\ &= 2^{(s-f_0(n)\log(1-\epsilon)^{-1}+f_1(n)\log\epsilon^{-1})n}. \end{aligned}$$

Note that by the definition of h,

$$\lim_{i\to\infty} f_0(|(h \circ g)^i(\lambda)|) = 1.$$

By the continuity of the function $(x, y) \mapsto x \log y + (1-x)\log(1-y)$, for all $\epsilon > 0$, there is a $\delta > 0$ such that for all sufficiently large $n = |(h \circ g)^i(\lambda)|$, $f_0(n)\log(1-\epsilon)^{-1} + f_1(n)\log\epsilon^{-1} < \delta$ and $\delta \to 0$ as $\epsilon \to 0$.

By the above analysis, d_ϵ witnesses that the constructive dimension of S is no more than 2δ. By taking limit of ϵ approaching 0, we have that the dimension of S is 0, i.e., $S \in \mathrm{DIM}^{=0}$. Therefore, $\mathrm{DIM}^{=0}$ is co-meager. □

The following lemma is used in the proof of Theorem 5.2.3. As we mentioned, it may be proven using Hitchcock's correspondence principle [45].

Lemma 5.2.5. *Every point in $\mathcal{R}$ has dimension at most* 1.

In the following, we prove Theorem 5.2.3 in $\mathbb{R}^2$. The proof for the general case in $\mathbb{R}^n$ is very similar.

Proof of Theorem 5.2.3. Let $\alpha \geq 0$. Without loss of generality, we prove that $\mathrm{DIM}^{=\alpha} \cap [0,1]^2$ is not contained in $\mathcal{R}$.

We first prove the case where $\alpha > 0$.

We use the Cantor space $\mathbf{C} = \{0,1\}^\infty$ in place of $[0,1]$ for this proof. Let $r \in \mathrm{RAND} \cap \mathbf{C}$. Let $b = f(r)$, where $f : \mathbf{C} \to \mathbf{C}$ is defined such that for all $S \in \mathbf{C}$ and all $n \in \mathbb{N}$,

$$f(S)[2^n - 1..2^n - 1 + \lfloor \alpha 2^n \rfloor - 1] = S[2^n - 1..2^n - 1 + \lfloor \alpha 2^n \rfloor - 1]$$

and

$$f(S)[2^n - 1 + \lfloor \alpha 2^n \rfloor..2^{n+1} - 2] = 0^{2^n - \lfloor \alpha 2^n \rfloor}.$$

It is clear by the definition of f that $\dim(b) = \alpha$. Let $L_b = \{(x,b) \mid x \in [0,1]\}$. Let $L'_b = \{(x,b) \mid x \in \mathrm{DIM}^{=0} \cap [0,1]\}$. Note that every point in L'_b has dimension α in $\mathbb{R}^2$.

Suppose every point in $\mathrm{DIM}^{=\alpha} \cap \mathbb{R}^2 \subseteq \mathcal{R}$, then every point in L'_b is on some computable rectifiable curve. Since there are only countably many computable curves — $\Gamma_0, \Gamma_1, \ldots,$

$$L'_b \subseteq \bigcup_{i=0}^{\infty} (\Gamma_i \cap L'_b).$$

For $A \subseteq \mathbb{R}^2$, let $P(A) = \{x \mid (x,y) \in A\}$. Then we have

$$P(L'_b) \subseteq \bigcup_{i=0}^{\infty} P(\Gamma_i \cap L'_b).$$

Note that $P(L'_b) = \mathrm{DIM}^{=0} \cap [0,1]$. By Lemma 5.2.4, we have that for some $n_0 \in \mathbb{N}$, $P(\Gamma_{n_0} \cap L'_b)$ is dense in some interval $I \subseteq [0,1]$. Since Γ_{n_0} is compact, $I \times \{b\} \subseteq \Gamma_{n_0} \cap (I \times \{b\})$. Let RAND^r be the subset of $[0,1]$ that contains all real numbers that are random relative to r. Since RAND^r is dense in $[0,1]$, there is a real number $r' \in \mathrm{RAND}^r \cap I$. Since r' is random relative to r, r is random relative to r'. Hence r' is random relative to b and b has dimension α relative to r'. Therefore $\dim((r', b)) = 1 + \alpha$, which contradicts Lemma 5.2.5. Therefore, some point of dimension α is not on any computable rectifiable curve.

For the case $\alpha = 0$, the proof is simpler.

Let $f : [0,1] \to \mathbb{R}^2$ be an arbitrary computable rectifiable curve in the plane. Since $f([0,1])$ is compact, $f([0,1])$ is nowhere dense, since otherwise $f([0,1])$ would have covered part of the plane with a positive area. Since there are only countably many computable curves, the union of all their images is a countable union of nowhere dense set, hence meager. (For basic properties of Baire Category, refer to [84].) Therefore, $\mathcal{R}$ is meager. As in Lemma 5.2.4, $\mathrm{DIM}^{=0} \cap [0,1]^2$ is co-meager, hence $\mathrm{DIM}^{=0} \cap [0,1]^2 \cap \mathcal{R}^c \neq \varnothing$. □

5.3 Points on Rectifiable Computable Curves

In last section, we have shown that it is not possible to use constructive dimension to characterize points in $\mathcal{R}$. In this section, we characterize points in $\mathcal{R}$ by extending the famous "analyst's traveling salesman theorem" of geometric measure theory to a theorem in computable analysis. We begin by describing the classical "analyst's traveling salesman theorem" in detail.

For each $m \in \mathbb{Z}$, let $\mathcal{Q}_m$ be the set of all *dyadic cubes of order* m, which are half-closed, half-open cubes

$$Q = [a_1, a_1 + 2^{-m}) \times \cdots \times [a_n, a_n + 2^{-m})$$

in $\mathbb{R}^n$ with $a_1, \ldots, a_n \in 2^{-m}\mathbb{Z}$. Note that such a cube Q has sidelength $\ell(Q) = 2^{-m}$ and all its vertices in $2^{-m}\mathbb{Z}^n$. Let $\mathcal{Q} = \bigcup_{m\in\mathbb{Z}} \mathcal{Q}_m$ be the set of all dyadic cubes of all orders. We regard each dyadic cube Q as an "address" of the larger cube $3Q$, which has the same center as Q and sidelength $\ell(3Q) = 3\ell(Q)$. The analyst's traveling salesman theorem is stated in terms of the resulting system $\{3Q \mid Q \in \mathcal{Q}\}$ of overlapping cubes.

Let K be a bounded subset of $\mathbb{R}^n$. For each $Q \in \mathcal{Q}$, let $r(Q)$ be the least radius of any infinite closed cylinder in any direction in $\mathbb{R}^n$ that contains all of $K \cap 3Q$. Then the *Jones beta-number* of K at Q is

$$\beta_Q(K) = \frac{r(Q)}{\ell(Q)},$$

and the *Jones square beta-number* of K is

$$\beta^2(K) = \sum_{Q\in\mathcal{Q}} \beta_Q(K)^2 \ell(Q)$$

(which may be infinite). With these notations, the analyst's traveling salesman theorem can be stated precisely as follows.

Theorem 5.3.1. *(Jones [50], Okikiolu [77]). Let $K \subseteq \mathbb{R}^n$ be bounded. Then K is contained in some rectifiable curve if and only if $\beta^2(K) < \infty$.*

Jones's proof of the "if" direction of Theorem 5.3.1 is an intricate "farthest insertion" construction of a curve containing K, together with an amortized analysis showing that the length of this curve is finite. This proof works in any Euclidean space $\mathbb{R}^n$. However, Jones's proof of the "only if" direction of Theorem 5.3.1 uses nontrivial methods from complex analysis and only works in the Euclidean plane $\mathbb{R}^2$ (regarded as the complex plane $\mathbb{C}$). Okikiolu's subsequent proof of the "only if" direction is a clever geometric argument that works in any Euclidean space $\mathbb{R}^n$. (It should also be noted that these papers establish a *quantitative* relationship between $\beta^2(K)$ and the infimum length of a curve containing K, and that the constants in this relationship have been improved in the recent thesis by Schul [88]. In contrast, we are only concerned with the *qualitative* question of the *existence* of a rectifiable curve containing K here.)

Theorem 5.3.1 is generally regarded as a solution of the "analyst's traveling salesman problem" (analyst's TSP), which is to characterize those sets $K \subseteq \mathbb{R}^n$ that can be traversed by curves of finite length. It is then natural to pose the *computable analyst's TSP*, which is to characterize those sets $K \subseteq \mathbb{R}^n$ that can be traversed by *computable* curves of finite length. While the analyst's TSP is only interesting for infinite sets K (because *every* finite set K is contained in a rectifiable curve), the computable analyst's TSP is interesting for arbitrary sets K including singleton sets.

5.3.1 The Computable Analyst's Traveling Salesman Theorem

To solve the computable analyst's TSP, we first replace the Jones square beta-number of the arbitrary set K with a data structure that can be required to be computable. To this end, we define a *cylinder assignment* to be a function γ assigning to each dyadic cube Q an (infinite) closed *rational* cylinder $\gamma(Q)$, by which we mean that $\gamma(Q)$ is a cylinder whose axis passes through two (hence infinitely many) points of $\mathbb{Q}^n$ and whose radius $\rho(Q)$ is rational. (If $\rho(Q) = 0$, the cylinder is a line; if $\rho(Q) < 0$, the cylinder is empty.) The *set permitted by* a cylinder assignment γ is the (closed) set $\kappa(\gamma)$ consisting of all points $x \in \mathbb{R}^n$ such that, for all $Q \in \mathcal{Q}$,

$$x \in (3Q)^o \Rightarrow x \in \gamma(Q),$$

where $(3Q)^o$ is the interior of $3Q$.

There is one technical point that needs to be addressed here. If γ is a cylinder assignment that, at some $Q \in \mathcal{Q}$, prohibits a subcube $3Q'$ of $3Q$ (i.e., $\gamma(Q) \cap (3Q')^o = \varnothing$), then $\kappa(\gamma)$ contains no interior point of $3Q'$, so it is pointless and misleading for γ to assign Q' a cylinder $\gamma(Q')$ that meets $(3Q')^o$. We define a cylinder assignment γ to be *persistent* if it does not make such pointless assignments, i.e., if, for all $Q, Q' \in \mathcal{Q}$ with $Q' \subseteq Q$ and $\gamma(Q) \cap (3Q')^o = \varnothing$, we have $\gamma(Q') \cap (3Q')^o = \varnothing$. It is easy to transform a cylinder assignment γ into a persistent cylinder assignment γ' that is equivalent to γ in the sense that $\kappa(\gamma) = \kappa(\gamma')$, with γ' computable if γ is.

Definition. Let γ be a cylinder assignment.

1. The *Jones beta-number* of γ *at a cube* $Q \in \mathcal{Q}$ is

$$\beta_Q(\gamma) = \frac{\rho(Q)}{\ell(Q)}.$$

2. The *Jones square beta-number* of γ is

$$\beta^2(\gamma) = \sum_{Q \in \mathcal{Q}} \beta_Q(\gamma)^2 \ell(Q).$$

Note that $\beta^2(\gamma)$ may be infinite.

Definition. A *Jones constriction* is a persistent cylinder assignment γ for which $\beta^2(\gamma) < \infty$.

We can now state the computable analyst's traveling salesman theorem.

Theorem 5.3.2. *Let $K \subseteq \mathbb{R}^n$ be bounded. Then K is contained in some rectifiable computable curve if and only if there is a computable Jones constriction γ such that $K \subseteq \kappa(\gamma)$.*

Theorem 5.3.2 solves the computable analyst's TSP, and thus immediately gives us a characterization of $\mathcal{R}$.

Corollary 5.3.3. *A point $x \in \mathbb{R}^n$ is rectifiable if and only if x is permitted by some computable Jones constriction. That is,*

$$\mathcal{R} = \bigcup_{\text{computable } \gamma} \kappa(\gamma),$$

where the union is taken over all computable Jones constrictions.

It should be noted that (the proof of) Theorem 5.3.2 relativizes to arbitrary oracles, so it implies Theorem 5.3.1. This is the sense in which our computable analyst's traveling salesman theorem is an extension of the analyst's traveling salesman theorem.

Our proof of the "only if" direction of Theorem 5.3.2 is easy, because we are able to use the corresponding part of Theorem 5.3.1 as a "black box". However, our proof of the "if" direction is somewhat involved. Given an arbitrary computable Jones constriction γ, we construct a rectifiable computable tour containing $\kappa(\gamma)$. In this construction, we are able to follow the broad outlines of Jones's "farthest insertion" construction and to use its key ideas, but we have an additional obstacle to overcome. The analyst's TSP does not require an algorithm, so Jones's proof can simply "choose" elements of the given set K according to various criteria at each stage of the construction (often moving these points later as needed). However, even if γ is computable, neither the set $\kappa(\gamma)$ nor its elements need be computable. Hence the algorithm for our computable tour cannot directly choose points in (or even reliably near) $\kappa(\gamma)$. Our proof succeeds by carefully separating the algorithm/construction from the amortized analysis of the length of the tour that it computes. The construction is discussed in section 5.3.2 and the analysis is in section 5.3.3.

Before we go into the details, we first summarize our proof. Since a version of the Pythagorean Theorem is center to the proof, we state it first:

Theorem 5.3.4. *Let $m \in \mathbb{Z}$ and $A > 9$. Let a, b, c be the lengths of three line segments that form a triangle inside a cylinder of length $l = A2^{1-m}$ and width $w < \frac{l}{A^3\sqrt{n}}$ such that*

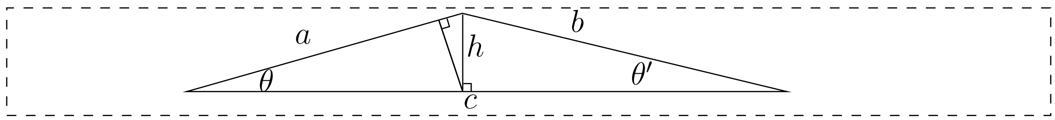

Figure 5.3.1 Pythagorean Theorem

$2^{1-m} \geq a, b \geq 2^{-m}$ *and* $c \geq 2^{1-m}$*, where* n *is the dimension of the space. Let* $\beta = \frac{w}{l}$*. Then*

$$a + b \leq c + 2A\beta^2 l.$$

Proof. Let θ be the small angle determined by line segments a and c. Let θ' be the small angle determined by line segments b and c. Let h be the distance from the intersection of line segments a and b to line segment c.

$$\begin{aligned} a + b - c &\leq h \sin\theta + h \sin\theta' = h \cdot \frac{h}{a} + h \cdot \frac{h}{b} \\ &= a \cdot \left(\frac{h}{a}\right)^2 + b \cdot \left(\frac{h}{b}\right)^2 \leq 2A \left(\frac{w}{l}\right)^2 \cdot l \\ &= 2A\beta^2 l. \end{aligned}$$

□

This version of Pythagorean Theorem easily generalizes to the case where more line segments are involved in the setting.

We first dispose of the "only if" direction. If we are given a rectifiable computable tour f and a rational $\epsilon > 0$, it is routine to construct a computable a cylinder assignment γ such that $f([0,1]) \subseteq \kappa(\gamma)$ and $\beta^2(\gamma) \leq \beta^2(f([0,1])) + \epsilon$. The "only if" direction of Theorem 5.3.2 hence follows easily from the "only if" direction of Theorem 5.3.1. We thus focus our attention on proving the "if" direction of Theorem 5.3.2.

As pointed out by Jones [50], the analyst's TSP is significantly different from the classical TSP in that it typically involves uncountably many points at locations that are not explicitly specified. In his construction, he has the privilege to "know" whether a point is in the set K or not, since he is concerned only with the existence of a tour and not with the computability of the tour. This is no longer true in our situation, since we work with only a computable constriction, from which we may not computably determine whether a point is in the set. Although the situations differ by so much, ideas with a flavor of the "farthest insertion" and "nearest insertion" heuristics that are used in Jones's argument and the classical TSP are essential to our solution.

Given a computable Jones constriction γ, we construct computably a tour $f : [0, 1] \to \mathbb{R}^n$ of the set $K = \kappa(\gamma)$ permitted by γ such that $\kappa(\gamma) \subseteq f([0, 1])$ and the length of the tour is finite.

Our construction proceeds in stages. In each stage $m \in \mathbb{N}$, a set of points with regulated density is chosen according to the constriction and a tour f_m of these points is constructed so that every point in K is at most roughly 2^{-m} from the tour. Every tour is constructed by patching the previous tour locally so that the sequence of tours $\{f_m\}$ converges computably.

During the tour patching at each stage, the insertion ideas mentioned earlier are applied at different parts of the set K according to the local topology given by the constriction. Note that it is not completely clear that the use of "farthest insertion" is absolutely necessary. However, it greatly facilitates the associated amortized analysis of length, which is as crucial in our proof as it is in Jones's. In the following, we describe in more detail how and when these ideas are applied in the algorithmic construction of the tour.

In each stage $m \in \mathbb{N}$, we look at cubes Q of sidelength $A2^{-m}$, where $A = 2^{k_0}$ is a sufficiently large universal constant. We pick points so that they are at least 2^{-m} from each other and every point in K is at most 2^{-m} from some of those chosen points. Based on the value of $\beta_Q(\gamma)$, which measures the relative width of $3Q \cap K$, we divide cubes into "narrow" ones ($\beta_Q(\gamma) < \epsilon_0$) and "fat" ones ($\beta_Q(\gamma) \geq \epsilon_0$), where ϵ_0 is a small universal constant.

The fat cubes are easy to process, since the associated square beta-number is large. We connect the points in those cubes to nearby surrounding points, some of which are guaranteed to be in the previous tour due to the density of the points in the tour. Since the points are chosen with regulated density, the number of connections we make here is bounded by a universal constant. The length of each connection is proportional to the sidelength of the cube, which is proportional to 2^{-m}. Thus the total length we add to the tour is bounded by $c_0 \cdot \epsilon_0^2 \ell(Q)$, which is then bounded by $c_0 \cdot \beta_Q^2(\gamma)\ell(Q)$, where c_0 is a sufficiently large universal constant.

For the narrow cubes, we carry out either "farthest insertion" or "nearest insertion" depending on the local topology around each insertion point.

Suppose that we are about to patch the existing tour to include a point x. Since from stage to stage, the points are picked with increasing density, there is always a point z_1 already in the tour inside the cube that contains x. However, there are two possibilities for the neighborhood of x. One is that there is another point z_2 already in the tour and z_2 is inside the cube that contains x. The other possibility is that z_1 is the only such point.

In the first case, point x lies in a narrow cube and there are points z_1 and z_2 in the narrow cube such that x is between z_1 and z_2. Points z_1 and z_2 are in the existing tour and are connected directly with a line segment in the tour. In this case, we apply "nearest insertion" by letting z_1 and z_2 be the closest two neighbors of x in the existing tour, breaking the line segment between z_1, z_2, and connecting z_1 to x and x to z_2. The increment of the length of the tour is $\ell([z_1, x]) + \ell([x, z_2]) - \ell([z_1, z_2])$, which is bounded by $c_1 \cdot \beta_Q^2(\gamma)\ell(Q)$ by an application of the Pythagorean Theorem, since the cube is very narrow.

In the second case, point z_1 is the only point in the existing tour that is in the same cube as x. It is not guaranteed that x can be inserted between two points in the existing tour. Even when it is possible, the other point in the existing tour would be outside the cube that we are looking at and thus it might require backtracking an unbounded number of stages to bound the increment of length, which would make the proof extremely complicated (if even possible). Therefore, we keep the patching for every point local and, in this case, we make sure x is locally the "farthest" point from z_1 and connect x directly to z_1. (Note that the actual situation is slightly more involved and is addressed in the full proof.) In this case, the Pythagorean Theorem cannot be used and thus we cannot use the Jones square beta-number to directly bound the increment of length. To remedy this, we employ amortized analysis and save spare square beta-numbers in a savings account over the stages and use the saved values to bound the length increment. In order for this to work, we choose ϵ_0 so small that at a particular neighborhood, "farthest insertion" does not happen very frequently and we always have the time to save up enough of the square beta-number before we need to use it.

5.3.2 The Construction Of The Tour

In this section, we fully describe the construction of f, a computable tour that contains $K = \kappa(\gamma)$.

Note that by the definition of constriction, the set $K = \kappa(\gamma)$ permitted by constriction γ is compact. We assume $K \subseteq [0, 1/\sqrt{n}]^n$, $(0, \ldots, 0) \in K$, and $(1/\sqrt{n}, \ldots, 1/\sqrt{n}) \in K$. We do not lose generality by imposing this assumption, since scaling of a function can be easily computed. Let $A = 2^{k_0} > 9$. Let $\epsilon_0 < \frac{1}{A^3\sqrt{n}}$ be a fixed small constant, where n is the dimension of the Euclidean space we are working with.

In the construction, we inductively build point sets $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m \cdots$ in stages with the following properties.

C1: $|z_j - z_k| \geq 2^{-m} - \sqrt{n}2^{-2^m}$, for $z_j, z_k \in L_m$, $j \neq k$.

C2: For $m \in \mathbb{N}$ and every $x \in K$, there exists $z \in L_m$ such that $|x - z| \leq 2^{-m} + \sqrt{n}2^{-2^m}$.

Note that for each $m \in \mathbb{N}$, $L_m \subseteq K_m$, where K_m is the union of dyadic cubes of sidelength 2^{-2^m} permitted by γ. However, the points in L_m are not specified by explicit coordinates. Instead, every point in L_m is specified by an algorithm, which when given a precision parameter r, outputs the coordinates of the dyadic cube of sidelength at most 2^{-r} that the point lies in. At stage m, we use $r = 2^m$. Although the points we pick may not have rational coordinates, at each stage m, we only look at them with precision r and treat them as if they all have rational coordinates. The dyadic cube determined by the coordinates is a sub-cube of the dyadic cube given by smaller precision parameter m. Thus the point is specified by a nested chain of dyadic cubes of progressively smaller sizes. When, for some m, such a dyadic cube is not permitted by γ, the output of the algorithm remains to be the coordinates given by the algorithm with the largest precision parameter that leads to an output of a dyadic cube that is permitted by γ. Thus it is possible that a point in L_m is not in K.

In stage $m \in \mathbb{N}$, we look at cubes Q of sidelength $A2^{-m}$. For each Q, we use $3Q$ to denote the cube of side length $3A2^{-m}$ centered at the center of Q. For the sake of precision, we look at the resolution level of K_m. Let $\beta(Q) = \beta_Q(\gamma) = \frac{\rho(Q)}{\ell(Q)}$. Note that Jones square beta-number $\beta^2(\gamma)$ of set K is $\sum_{Q \in \mathcal{Q}} \beta^2(Q)\ell(Q)$. For each term in the sum, we call $\beta^2(Q)\ell(Q)$ the local square beta-number at Q. We build a tour $f_m : [0,1] \to \mathbb{R}^n$ of L_m by patching the tour f_{m-1} locally according to the local topology of K_m given by the constriction so that the sequence of tours $\{f_m\}$ converges computably.

Since the tour we build is computable, which requires parameterized approximation, the approximation scheme in computing the points in L_m is not harmful.

As we mentioned earlier, points in L_m may not lie in K, thus it is possible that, at some stage, a point chosen earlier is discovered to be outside K. However, when this happens, we don't remove the point. Instead, we keep such points in order to maintain the convergence of the parameterizations of the sequence of tours. Therefore, due to the inability to computably choose points strictly from K, we may introduce extra length to the tours. However the extra length turns out to be bounded by the local square beta-numbers and thus the access to the set K in Jones's original construction is a nonessential feature of the analyst's traveling salesman problem and our characterization using Jones constriction is a proper relaxation of Jones's characterization. However, we also note that in Jones's world, using K is equivalent to using the constriction.

Before getting into the construction, we describe some sub-routines that we will use in the construction to patch the tours.

First note again that, at each stage m, we use a precision parameter of $r = 2^m$ for points and treat them as if they have dyadic rational coordinates. It is also easy to make sure that for each f_m, for all $p \in [0,1]$ such that $f_m(p) \in L_m \implies p \in [0,1] \cap \mathbb{Q}$. Thus, we may keep a table of all $p \in [0,1]$ with $f_m(p) \in L_m$.

The first procedure is $\text{attach}(f, z, x, m)$ with $z \in L_{m-1}$ or $z \in L_m$ being already explicitly traversed by f. This procedure modifies f so that the output $f' = \text{attach}(f, z, x, m)$ traverses line segment $[z, x]$ in addition to the set f originally traverses and for all $p \in [0,1]$, $|f(p) -$

$f'(p)| \leq 2^{1-m}$.

The procedure first looks up the table and finds $q \in [0,1]$ such that $f(q) = z$. Then it finds $a \in \mathbb{Q} \cap (0,1)$ such that $|f(q-2a) - f(q)| < 2^{1-m}$, $|f(q+2a) - f(q)| < 2^{1-m}$, and z is the only point in $L_{m-1} \cap f([q-2a, q+2a])$ and it appears only once. The output f' is such that for all $p \in [0,1] \setminus [q-2a, q+2a]$, $f'(p) = f(p)$; f' maps $[q-2a, q-a]$ to $f([q-2a, q])$ linearly; f' maps $[q-a, q]$ to $[z, x]$ linearly; f' maps $[q, q+a]$ to $[x, z]$ linearly; f' maps $[q+a, q+2a]$ to $f([q, q+2a])$ linearly.

The second procedure is reconnect$(f, z_1, z_2, x_0, \ldots, x_N, m)$ with the assumption that f traverses line segment $[z_1, z_2]$ from one end to the other. This procedure first looks up the table and, without loss of generality, we assume that it finds the smallest interval $[p, q] \subseteq [0,1]$ such that $f(p) = z_1$ and $f(q) = z_2$ and $f([p, q]) = [z_1, z_2]$. We obtain f' by reparameterizing f to include $x_0, \ldots, x_N$ in order. First we pick rational points $q_0, \ldots, q_N$ such that for each $i \in [0..N]$, $|f(q_i) - x_i| \leq 2\epsilon_0 3A2^{-m}$. Then we let f' map $[p, q_0]$ to $[z_1, x_0]$ and let f' map $[q_N, q]$ to $[x_N, z_2]$. For $i \in [0..N-1]$, let f' map $[q_i, q_{i+1}]$ to $[x_i, x_{i+1}]$. Note that if all these points involved lie in a very narrow strip, it is guaranteed that the newly added line segments are very close to the longer line segment they replace. The distance between the new parameterization and the old one is bounded by $2\epsilon_0 3A2^{-m}$.

Note that in each of the above procedures, when f is reparameterized to obtain f', the table that saves the information on the preimages of points in L_{m-1} and L_m is updated to reflect the changes.

The construction proceeds as follows:

Stage 0: $m = 0$ and the size of Q we consider is $\ell(Q) = A$. L_0 contains the two diagonal points of $[0, 1/\sqrt{n}]^n$, i.e., $L_0 = \{(0, \ldots, 0), (1/\sqrt{n}, \ldots, 1/\sqrt{n})\}$. Let f_0 map $[0,1]$ linearly to the line segment $[(0, \ldots, 0), (1/\sqrt{n}, \ldots, 1/\sqrt{n})]$.

Stage m: For any point z and x with $z \neq x$, let

$$E_{z,x} = \{y \mid y - z \text{ is at most } \tfrac{2}{3}\pi \text{ from } x - z \ \}.$$

For all $x \in K$, let Q_x be such that $x \in Q_x$ and $Q_x \in \mathcal{Q}_{m-k_0}$. Let $z_x \in L_{m-1}$ be the closest neighbor of x ($2^{-m} - \sqrt{n}2^{-2^m} \leq |x - z_x| \leq 2^{1-m} + \sqrt{n}2^{-2^{m-1}}$).

First we build a set of points that we eventually add into L_{m-1} to form L_m. The following piece of code first finds new points in K_m that correspond to the cases where "farthest insertion" is required. Note that in this case, as long as the point we pick is sufficiently close to the farthest point, the construction will work. (By "sufficiently close", we mean that the point we pick is close enough to a farthest point so that another instance of "farthest insertion" does not happen within k_0 stages in that neighborhood.) This allows us to computably pick points for "farthest insertion" without worrying about not being able to pick the actual farthest points.

$L' \subseteq K_m$ be a set of points such that $L_{m-1} \cup L'$ satisfies conditions C1 and C2;

$L' = L' \cap \{x \in K_m \mid \beta(Q_x) < \epsilon_0$ and $L_{m-1} \cap B_{A2^{1-m}+\sqrt{n}2^{-2^m}}(z_x) \cap E_{z_x,x} \setminus \{z_x\} = \varnothing\}$;

$\hat{L} = \varnothing$;

for all $x_0 \in L'$ **do**

  if $\ell([x_0, z_{x_0}]) \geq \max\{\ell([x, z_x]) \mid x \in E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0}) \cap K_m\} - \sqrt{n}2^{-2^m}$;

  then

    $\hat{L} = \hat{L} \cup \{x_0\}$;

  else

    let $x_0' \in K_m$ be such that

      $\ell([x_0', z_{x_0}]) = \max\{\ell([x, z_{x_0}]) \mid x \in E_{z_{x_0},x_0} \cap K_m \cap B_{2^{1-m}}(z_{x_0})\} - \sqrt{n}2^{-2^m}$;

    /* $z_{x_0'} \equiv z_{x_0}$ */

    $\hat{L} = \hat{L} \cup \{x_0'\}$;

  end if

end for

Let $\hat{L}_1 = \hat{L}$ /* $\hat{L}_1$ contains all the "farthest insertion" points */

Greedily add more points into $\hat{L}$ so that $\hat{L}$ satisfies conditions C1 and C2;

We connect every point in $\hat{L}$ to some points in L_{m-1} by reparameterizing f_{m-1} to get f_m. Initially, let $L_m = L_{m-1}$ and $f_m = f_{m-1}$. We divide the process into 3 steps.

Step 1: Farthest Insertion

for all $x_0 \in \hat{L}_1$ **do** /* $\beta(Q_{x_0}) < \epsilon_0$ */

if $|\hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-n}}(z_{x_0}) \setminus \{x_0\}| = 0$

then

$L_m = L_m \cup \{x_0\}$;

$f = \text{attach}(f, z_{x_0}, x_0, m)$;

else /* $|\hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0}) \setminus \{x_0\}| = 1$ */

Let $x_1 \in \hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0})$ with $x_1 \neq x_0$;

$L_m = L_m \cup \{x_0, x_1\}$;

$f = \text{attach}(f, z_{x_1}, x_1, m)$; $f = \text{attach}(f, x_1, x_0, m)$;

end if

end for

Step 2: Nearest Insertion

for $x_0 \in \hat{L}$ with $\beta(Q_{x_0}) < \epsilon_0$ that are not processed yet **do**

Let z_1 be the closest neighbor of x_0 in $L_{m-1} \cap B_{A2^{1-m}}(z_{x_0}) \cap E_{z_{x_0},x_0} \setminus \{z_{x_0}\}$;

/* Note that f already explicitly traverses $[z_{x_0}, z_1]$ */

Let $\{x_*, x_1, \ldots, x_N\} = \hat{L} \cap E_{z_{x_0},x_0} \cap B_{\ell([z_{x_0},z_1])}(z_{x_0})$ be ordered by x component;

if $x_* \neq x_0$ **then continue**; **end if**

$f = \text{reconnect}(f, z_{x_0}, z_1, x_0, \ldots, x_N, m)$;

$L_m = L_m \cup \{x_0, x_1, \ldots, x_N\}$;

mark $x_0, x_1, \ldots, x_N$ as processed and never process again;

end for

Step 3:

for all $x_0 \in \hat{L}$ with $\beta(Q_{x_0}) \geq \epsilon_0$ **do**

if $[z_{x_0}, x_0]$ is not explicitly traversed by f **then** $f = \text{attach}(f, z_{x_0}, x_0, m)$;

for all $x_1 \in 3Q_{x_0} \cap (\hat{L} \cup L_{m-1})$ **do**

  if $[x_0, x_1]$ is not explicitly traversed by f **then** $f = \text{attach}(f, x_0, x_1, m)$;

end for

$L_m = L_m \cup \{x_0\}$;

end for

By construction, for every $m \in \mathbb{N}$, the distance between f_m and f_{m+1} is bounded by $\sqrt{n}3A2^{-m}$. So by the convergence of the geometric series, $\{f_m\}$ is a convergent sequence of bounded continuous functions. Thus $f = \lim_{m\to\infty} f_m$ exists and is actually computable, since each f_m is computable from the computable constriction and the modulus of computation may be obtained by using the geometric series for the distance between f_m and f_{m+1}.

5.3.3 The Amortized Analysis Of The Construction

In this section, we analyze the construction and prove that if Jones square beta-number of γ is finite, then $K = \kappa(\gamma) \subseteq f([0,1])$ and $\text{length}(f) < \infty$.

Proof. In order to make the analysis possible, we associate with each $z \in \bigcup_{m\in\mathbb{N}} L_m$ a variable $M(z)$ and a variable $V(z)$. Variables M may be taken as a savings account where local square beta-numbers are saved at times when they are not used up. The saved values are then used to cover the cost at times when new local square beta-numbers may not cover the cost. Variables V are used to keep track of the information about the local environment of each point $z \in \bigcup_{m\in\mathbb{N}} L_m$ during the construction. The initial value of $M(z)$ before the first assignment is 0 and that of $V(z)$ is $\varnothing$. $M(z)$ only changes when a new assignment occurs. The values of the variables may change over stages and during the various steps of the construction in a single stage, so $M(z)$ and $V(z)$ always refer to their respective current values.

In the following, we describe how the values of variables M and variables V are updated during each stage and each step of the construction. We also analyze the construction and argue that, any at time during the construction, the increment to M values is bounded by corresponding local square beta-numbers and M values are always sufficient to cover the construction cost when local square beta-numbers may not be used. Since M values come from local square beta-numbers, the increase of the length is again bounded by local square beta-numbers, though indirectly. During the construction, whenever we use M values, we decrement M values accordingly to ensure that M values are not used repeatedly.

Since the construction is inductive, the analysis is also inductive. We will show that the following two properties hold during the construction for all $z \in L_m$, $m \in \mathbb{N}$.

P1: For all $z' \in V(z)$, let $\{y_1, \ldots, y_N\} = V(z)$ be arranged in the order of their projections on the line determined by $[z, z']$. Then for all $j \leq N-1$, $[y_j, y_{j+1}]$ is a direct line segment in f_m.

P2: $V(z) \neq \varnothing$ and one of the following is true.

(1) If there are at least two points $z_1, z_2 \in V(z)$ such that the angle between $[z, z_1]$ and $[z, z_2]$ is at least $2\pi/3$, then $M(z) \geq \sum_{z' \in V(z)} \ell([z, z'])$.

(2) If for some $u \neq z$, $E_{z,u} \cap V(z) = \varnothing$ and $V(z) \neq \varnothing$, then we have both of the following.

(a) $M(z) \geq 2^{1-m} + \sum_{z' \in V(z)} \ell([z, z'])$.

(b) For all $k \geq 0$, if $B_{2^{-m-k}}(z) \cap E_{z,u} \neq B_{2^{1-m}}(z) \cap E_{z,u}$ (at the resolution of K_m), then $M(z) \geq A 2^{1-m-k} + \sum_{z' \in V(z)} \ell([z, z'])$.

We verify that the properties are true initially and that if the properties are true at any time, after any legal step of construction the properties are still true.

Stage 0: Initially, M values are all 0 and V values are all $\varnothing$, so the properties trivially hold.

Let the two diagonal points be z_1, z_2. Note that $\ell([z_1, z_2]) = 1$. Let $M(z_1) = A + 1$ and $M(z_2) = A + 1$. Let $V(z_1) = \{z_2\}$ and $V(z_2) = \{z_1\}$. Note that this assignment may be regarded as a special case for step 3 in the construction. Without loss of generality, assume z_1 is added before z_2. It is easy to check that property P1 and property P2 (part (2)) are true after z_1 is added and remain true when z_2 is added.

Stage m: We give different assignment rules for M values for each of the 3 steps in the construction. For clarity, we keep the code for the construction and give the assignment rules in annotations.

Step 1: Farthest Insertion

for all $x_0 \in \hat{L}_1$ **do** /* $\beta(Q_{x_0}) < \epsilon_0$ */

  if $|\hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0})| = 1$

  then

    $L_m = L_m \cup \{x_0\}$;

    $f = \text{attach}(f, z_{x_0}, x_0, m)$;

    @ $V(x_0) = V(x_0) \cup \{z_{x_0}\}$;

    @ **if** $V(z_{x_0}) \cap E_{z_{x_0},x_0} \neq \varnothing$

    @ **then**

    @ $V(z_{x_0}) = V(z_{x_0}) \setminus (V(z_{x_0}) \cap E_{z_{x_0},x_0})$;

    @ **end if**

    @ $V(z_{x_0}) = V(z_{x_0}) \cup \{x_0\}$;

    @ $M(z_{x_0}) = M(z_{x_0}) - A2^{1-m} + 2^{1-m}$;

    @ $M(x_0) = 2 \cdot 2^{1-m}$;

  else /* $|\hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0}) \setminus \{x_0\}| = 1$ */

    Let $x_1 \in \hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0})$ with $x_1 \neq x_0$;

    $L_m = L_m \cup \{x_0, x_1\}$;

    $f = \text{attach}(f, z_{x_1}, x_1, m)$; $f = \text{attach}(f, x_1, x_0, m)$;

    @ $V(x_0) = V(x_0) \cup \{x_1\}$;

    @ $V(x_1) = V(x_1) \cup \{z_{x_0}, x_0\}$;

    @ **if** $V(z_{x_0}) \cap E_{z_{x_0},x_0} \neq \varnothing$

    @ **then**

    @ $V(z_{x_0}) = V(z_{x_0}) \setminus (V(z_{x_0}) \cap E_{z_{x_0},x_0})$;

    @ **end if**

@ $V(z_{x_0}) = V(z_{x_0}) \cup \{x_1\}$;

@ $M(z_{x_0}) = M(z_{x_0}) - A2^{1-m} + 2^{1-m} + 2\sqrt{n}2^{-2^{m-1}}$;

@ $M(x_0) = 2(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}})$;

@ $M(x_1) = 2(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}})$;

end if

end for

Whenever "farthest insertion" is involved, the point x_0 under consideration always lies in a narrow cube that contains x_0, z_{x_0}, and possibly x_1. Therefore, P1 is satisfied at x_0 due to the narrowness of the cube. For z_{x_0}, P1 is maintained due to the removal of points in $V(z_{x_0}) \cap E_{z_{x_0},x_0}$ from $V(z_{x_0})$.

In every stage $m \in \mathbb{N}$, the tour f_m traverses a set of line segments. By the construction, every line segment is traversed at most twice. Therefore, for each $m \in \mathbb{N}$, length$(f_m) \leq 2\ell(f_m([0,1]))$, where $\ell(f_m([0,1]))$ is the one dimensional Hausdorff measure of the set $f_m([0,1])$. In the following analysis, we bound $\ell(f_m([0,1]))$ instead of length(f_m).

The length of each line segment that we add in this case is at most $2^{1-m} + 2\sqrt{n}2^{-2^{m-1}}$ (taking into consideration the approximation of the locations of end points), and we add at most 2 line segments. The total of M values for z, x_0, and x_1 (if it exists) is bounded by $5(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}})$. So the sum of added length and M values is bounded by $7 \cdot 2^{1-m}$.

Since $A > 9$, it suffices to show that we may use $A2^{1-m}$ from old M value to cover the cost here.

Before this step of construction involving x_0 and z_{x_0}, z_{x_0} satisfied property P2.

If part (1) of property P2 was satisfied before this step, there is a point $z' \in V(z_{x_0}) \cap E_{z_{x_0},x_0}$ such that $\ell([z_{x_0}, z']) > A2^{1-m}$. Since z' is removed from $V(z_{x_0})$, the reduction of $A2^{1-m}$ from $M(z_{x_0})$ is used to cover the cost and is balanced by the removal of z'.

If after the addition of either x_0 or x_1 to $V(z_{x_0})$, the condition of part (1) in property P2 is true, then since the addition to $M(z_{x_0})$, which is $2^{1-m}+2\sqrt{n}2^{-2^{m-1}} \geq \ell([z_{x_0}, x_0])$ (or in case $|\hat{L}_1 \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0}) \setminus \{x_0\}| = 1$, $2^{1-m} + 2\sqrt{n}2^{-2^{m-1}} \geq \ell([z_{x_0}, x_1])$), part (1) in property P2 remains true.

If after the addition of either x_0 or x_1 to $V(z_{x_0})$, the condition of part (2) in property P2 is true, then since the addition to $M(z_{x_0})$ is $2^{1-m} + 2\sqrt{n}2^{-2^{m-1}}$, part (2)-(a) in property P2 is satisfied at z_{x_0}. Since $\beta(Q_{x_0}) < \epsilon_0$, on the side of z_{x_0} (given by z' in the P2) where $V(z_{x_0}) \cap E_{x_{x_0},z'}$ is empty, there will not be further construction within less than k_0 stages, i.e., the condition of part (2)-(b) of property P2 will not be true within k_0 stages. Together with the fact that $2^{1-m} \geq A2^{1-m-k_0}$, part (2)-(b) of property P2 is satisfied at z_{x_0}.

$V(x_0)$ contains only one point whose distance from x_0 is between $2^{-m} - 2^{-2^{m-1}}$ and $2^{1-m} + 2^{-2^{m-1}}$. So part (2)-(a) of property P2 is satisfied at x_0. Since $\beta(Q_{x_0}) < \epsilon_0$, there will be no further construction within less than k_0 stages on the empty side of $V(x_0)$, i.e., the condition of part (2)-(b) of property P2 will not be true within k_0 stages. Therefore, part (2)-(b) of property P2 is satisfied at x_0.

If x_1 is added to L_m in this step, since $\beta(Q_{x_0}) < \epsilon_0$, x_1 is between z_{x_0} and x_0, part (1) of property P2 is satisfied at x_1.

If part (2) was satisfied before this step, we have two possibilities.

One possibility is that $E_{z_{x_0},x_0} \cap V(z_{x_0}) = \varnothing$. Then since we have a "farthest insertion" construction at x_0, $B_{2^{-m}}(z_{x_0}) \cap E_{z_{x_0},x_0} \neq B_{2^{1-m}}(z_{x_0}) \cap E_{z_{x_0},x_0}$, i.e., the condition for part (2)-(b) of property P2 is true and thus $M(z_{x_0}) \geq A2^{1-m} + \sum_{z' \in V(z_{x_0})} \ell([z_{x_0}, z'])$. Now the extra $A2^{1-m}$ may be used to cover the cost and is the amount that is deducted from $M(z_{x_0})$. After we add x_0 to $V(z_{x_0})$, since $\beta(Q_{x_0}) < \epsilon_0$, the condition of part (1) of property P2 is true. Since $2^{1-m}+2\sqrt{n}2^{-2^{m-1}} \geq \ell([z_{x_0}, x_0])$ (or in case $|\hat{L} \cap E_{z_{x_0},x_0} \cap B_{2^{1-m}}(z_{x_0}) \setminus \{x_0\}| = 1$, $2^{1-m} + 2\sqrt{n}2^{-2^{m-1}} \geq \ell([z_{x_0}, x_1])$), part (1) of property P2 is satisfied at z_{x_0}.

The other possibility is that $E_{z_{x_0},x_0} \cap V(z_{x_0}) \neq \varnothing$. Then there is a point $u \in V(z_{x_0}) \cap E_{z_{x_0},x_0}$ such that $\ell([z_{x_0}, u]) > A2^{1-m}$. Now the analysis will be the same as in the case when part (1) of property P2 was satisfied before this step except that we need to note that although $V(z_{x_0})$ changes, the amount $M(z_{x_0}) - \sum_{u \in V(z_{x_0})} \ell([z_{x_0}, u])$ does not decrease during the process. Therefore part (2) of property P2 remains true and thus P2 remains true.

The analysis of the properties at x_0 and x_1 are the same as in the case when part (1) of property P2 was satisfied before this step.

Also note that we never make variable V empty.

Step 2: Nearest Insertion

for all $x_0 \in \hat{L}$ with $\beta(Q_{x_0}) < \epsilon_0$ that are not processed yet **do**

Let z_1 be the closest neighbor of x_0 in $L_{m-1} \cap B_{A2^{1-m}}(z_{x_0}) \cap E_{z_{x_0},x_0} \setminus \{z_{x_0}\}$;

/* Note that $[z_{x_0}, z_1]$ is traversed explicitly by f_{m-1} */

Let $\{x_*, x_1, \ldots, x_N\} = \hat{L} \cap E_{z_{x_0},x_0} \cap B_{\ell([z_{x_0},z_1])}(z_{x_0})$ be ordered by x component;

if $x_* \neq x_0$ **then continue**; **end if**

$f = \text{reconnect}(f, z_{x_0}, z_1, x_0, \ldots, x_N, m)$;

@ $V(z_{x_0}) = V(z_{x_0}) \cup \{x_0\} \setminus \{z_1\}$;

@ $M(z_{x_0}) = M(z_{x_0}) - \ell([z_{x_0}, z_1]) + \ell([z_{x_0}, x_0])$;

@ $V(x_0) = V(x_0) \cup \{z_{x_0}\}$;

@ $M(x_0) = M(x_0) + \ell([z_{x_0}, x_0])$;

@ $V(z_1) = V(z_1) \cup \{x_N\} \setminus \{z_{x_0}\}$;

@ $M(z_1) = M(z_1) - \ell([z_{x_0}, z_1]) + \ell([x_N, z_1])$;

@ $V(x_N) = V(x_N) \cup \{z_1\}$;

@ $M(x_N) = M(x_N) + \ell([x_N, z_1])$;

for $i = 0$ **to** $N - 1$ **do**

@ $V(x_i) = V(x_i) \cup \{x_{i+1}\}$;

@ $M(x_i) = M(x_i) + \ell([x_i, x_{i+1}])$;

@ $V(x_{i+1}) = V(x_{i+1}) \cup \{x_i\}$;

@ $M(x_{i+1}) = M(x_{i+1}) + \ell([x_i, x_{i+1}])$;

end for

$L_m = L_m \cup \{x_0, x_1, \ldots, x_N\}$;

mark $x_0, x_1, \ldots, x_N$ as processed and never process again;

end for

Since in this case the points we work with are all located along a very narrow and long cylinder, by the Pythagorean Theorem, we have that the length added is bounded by

$$C_3 \sum_{\beta(Q)<\epsilon_0} \beta(Q)^2 \ell(Q).$$

Note that if we make ϵ_0 smaller, constant C_3 can also be chosen smaller. Since we don't need to increase C_3, we may fix C_3 large enough for all sufficiently small ϵ_0 so that C_3 does not depend on the choice of ϵ_0 or the choice of A. Also since the changes happen in a narrow cylinder, P1 is maintained.

For $j \in [0..N]$, $M(x_j)$ satisfies P2, in particular part (1) of P2, since each of them is connected to 2 other points that are more than $2\pi/3$ angle apart.

For z_{x_0}, in this case, $z_1 \in V(z_{x_0})$ before we make the changes. So $E_{z_{x_0},x_0} \cap V(z_{x_0}) \neq \varnothing$, and after we make the changes to $M(z_{x_0})$, since $V(z_{x_0})$ is changed accordingly, the value $M(z_{x_0}) - \sum_{z' \in V(z_{x_0})} \ell([z_{x_0}, z'])$ does not decrease. Therefore P2 remains true after this step regardless of whether part (1) or part (2) was true. The same argument tells us that P2 remains true at z_1.

Due to the way we assign M values, the total increment of M values in this case is bounded by at most 2 times the total increase of length, i.e.,

$$2 \cdot C_3 \sum_{\beta(Q)<\epsilon_0} \beta(Q)^2 \ell(Q).$$

Step 3:

for all $x_0 \in \hat{L}$ with $\beta(Q_{x_0}) \geq \epsilon_0$ **do**

if $[z_{x_0}, x_0]$ is not explicitly traversed by f **then**

$f = \text{attach}(f, z_{x_0}, x_0, m)$;

@ $V(x_0) = V(x_0) \cup \{z_{x_0}\}$;

@ $M(x_0) = M(x_0) + \ell([x_0, z_{x_0}])$;

@ $V(z_{x_0}) = V(z_{x_0}) \cup \{x_0\}$;

@ $M(z_{x_0}) = M(z_{x_0}) + \ell([x_0, z_{x_0}])$;

end if

for all $x_1 \in 3Q_{x_0} \cap (\hat{L} \cup L_{m-1})$ **do**

if $[x_0, x_1]$ is not explicitly traversed by f

then

$f = \text{attach}(f, x_0, x_1, m)$;

@ $V(x_0) = V(x_0) \cup \{x_1\}$;

@ $M(x_0) = M(x_0) + \ell([x_0, x_1])$;

@ $V(x_1) = V(x_1) \cup \{x_0\}$;

@ $M(x_1) = M(x_1) + \ell([x_0, x_1])$;

end if

end for

$L_m = L_m \cup \{x_0\}$;

@ $M(x_0) = M(x_0) + A2^{-m}$;

end for

It is easy to verify that property P1 is maintained for each involved point.

Since we assign $A2^{-m}$ to $M(x_0)$ in addition to the sum of length of connected line segments, P2 is true for every x_0. For those $x_1 \in L_{m-1}$ that are involved in this case, $M(x_1)$ value is incremented by the length of the line segment for each of the added line segment. The value $M(x_1) - \sum_{z' \in V(x_1)} \ell([x_1, z'])$ does not decrease. Therefore, P2 remains true after the changes.

Let C_1 be an upper bound of the maximum number of points that can be fit into $3Q$ and satisfy property C1 for L_m. Let C_2 be an upper bound of the maximum number of points in $L_m \setminus L_{m-1}$ that can fit into $3Q$. Note that C_1 and C_2 can be made to be independent of L_m and to be functions of only n – the dimension of the Euclidean space we are working with. So both the total length we add to f_m and for each point in L_m, the total increment of M value are bounded by

$$\begin{aligned} C_1 \cdot A2^{-m} + C_1 \cdot 2 \sum_{\beta(Q)\geq\epsilon_0} C_2 \cdot 3\sqrt{n}\ell(Q) &= \frac{9 \cdot C_1 \cdot C_2\sqrt{n}}{\epsilon_0^2} \sum_{\beta(Q)\geq\epsilon_0} \epsilon_0^2 \ell(Q) \\ &\leq \frac{9 \cdot C_1 \cdot C_2\sqrt{n}}{\epsilon_0^2} \sum_{\beta(Q)\geq\epsilon_0} \beta(Q)^2 \ell(Q). \end{aligned}$$

We have, by now, established case by case a bound on length increment in every stage. Now we put all these things together and bound the length of the tour we obtain.

Let

$$M_m = \sum_{z\in L_m} M(z),$$

where $M(z)$ takes the value at the end of stage m. So $M_0 = 2A + 2$.

Let l_m be the total increment of length from f_{m-1} to f_m introduced by "farthest insertion" and $l_0 = 0$.

Let $C = \max\left(\frac{9\cdot C_1\cdot C_2\sqrt{n}}{\epsilon_0^2}, 2 \cdot C_3\right)$.

Let $M_{m,1}$ be the total reduction of M values in stage m in "farthest insertion". Let $M_{m,23}$ be the total increment of M values in stage m in Steps 2 and 3. By the construction, $M_{m,23} \leq C \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2\ell(Q)$.

Note that in an instance of "farthest insertion", the increment of length Δl is bounded by $2(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}})$, i.e., $\Delta l \leq 2(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}}) \leq 3 \cdot 2^{1-m}$. For the involved point $z \in L_{m-1} \subset L_m$ and $x_0, x_1 \in L_m \setminus L_{m-1}$, the increment of M values at z, x_0, and x_1 is at most by $5(2^{1-m} + 2\sqrt{n}2^{-2^{m-1}}) \leq 7 \cdot 2^{1-m}$ and the loss of M value at z is $A2^{1-m}$. Note that x_1 may not be present in the construction. Since we give an upper bound here, we use the worst case and assume x_1 is present. So the total reduction in M value involved in such an instance of "farthest insertion", $\Delta M(z)$ is at least $(A-5)2^{-m+1}$. So for each individual instance of "farthest insertion" in stage m, the ratio between the reduction in M values and the increment of length is

$$\frac{\Delta M(z)}{\Delta l} \geq \frac{A-7}{3}.$$

So $M_{m,1} \geq \frac{A-7}{3} l_m$.

Note that in the following, we are combining the $\beta(Q) \geq \epsilon_0$ part and the $\beta(Q) < \epsilon_0$ part of the sum of local square beta-numbers, i.e., the sums for Step 2 and Step 3 are combined.

$$M_m - M_{m-1} = M_{m,23} - M_{m,1} < C \sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) - \frac{A-7}{3} l_m.$$

Note that due to property P2, for all $m_0 \in \mathbb{N}$, $M_{m_0} \geq 0$. So

$$0 \leq M_{m_0} = M_0 + \sum_{m=1}^{m_0} (M_m - M_{m-1}) < M_0 + \sum_{m=1}^{m_0} \left(C \sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) - \frac{A-7}{3} l_m \right).$$

Therefore

$$\sum_{m=1}^{m_0} \frac{A-7}{3} l_m < M_0 + \sum_{m=1}^{m_0} \left(C \sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) \right).$$

And thus

$$\sum_{m=1}^{\infty} \frac{A-7}{3} l_m \leq M_0 + C \sum_{m=1}^{\infty} \left(\sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) \right).$$

So

$$\sum_{m=1}^{\infty} l_m \leq \frac{3M_0}{A-7} + \frac{3C}{A-7} \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q).$$

By our construction, $\ell(f_m) - \ell(f_{m-1})$ consists of the increments in Step 1, Step 2, and Step 3. So

$$\ell(f_m) - \ell(f_{m-1}) \leq l_m + C \sum_{Q \in \mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q).$$

Now we have that the one dimensional Hausdorff measure of $f([0,1])$ is

$$\begin{aligned}
\lim_{m\to\infty} \ell(f_m) &= \ell(f_0) + \sum_{m=1}^{\infty} \left(\ell(f_m) - \ell(f_{m-1})\right) \\
&\le \ell(f_0) + \sum_{m=1}^{\infty} \left(l_m + C \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) \right) \\
&= \ell(f_0) + C \sum_{m=1}^{\infty} \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) + \sum_{m=1}^{\infty} l_m \\
&\le \ell(f_0) + C \sum_{m=1}^{\infty} \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) + \frac{3M_0}{A-7} + \frac{3C}{A-7} \sum_{m=1}^{\infty} \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q) \\
&= \ell(f_0) + \frac{3M_0}{A-7} + C\left(1 + \frac{3}{A-7}\right) \sum_{m=1}^{\infty} \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q).
\end{aligned}$$

Therefore

$$\text{length}(f) \le 2 \cdot \mathcal{H}^1(f([0,1])) \le 2\ell(f_0) + \frac{6M_0}{A-7} + 2C\left(1 + \frac{3}{A-7}\right) \sum_{m=1}^{\infty} \sum_{Q\in\mathcal{Q}_{m-k_0}} \beta(Q)^2 \ell(Q).$$

Since the square beta-number $\beta^2(\gamma) < \infty$, $\text{length}(f) < \infty$. □

5.4 Computable Curves and Their Lengths

As mentioned in the introduction, the tour constructed in Section 5.3 may retrace part of the curve. In Section 5.4.1, we show that retracing is unavoidable in general. In contrast, we show in Section 5.4.2 that every computable simple curve of finite length has a constant-speed (hence non-retracing) parametrization that is computable relative to the halting problem.

5.4.1 An Efficiently Computable Curve That Must Be Retraced

In the following, we construct a smooth, rectifiable, simple plane curve $\mathbf{\Gamma}$ that is parametrizable in polynomial time but not computably parametrizable in any amount of time without unbounded retracing. We begin with a precise construction of the curve $\mathbf{\Gamma}$ by describing it as if we are modeling the movement of a particle in the plane. We then give a brief intuitive discussion of this construction.

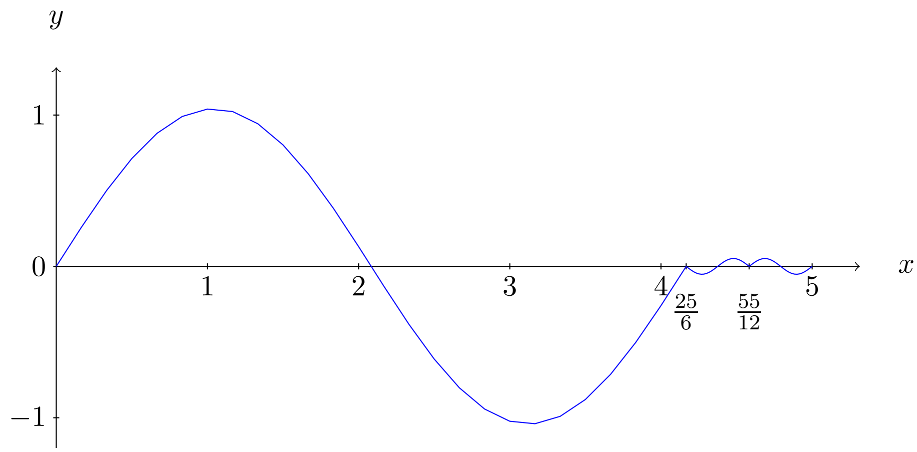

Figure 5.4.1 $\psi_{0,5,1}$

Construction 5.4.1. (1) For each $a, b \in \mathbb{R}$ with $a < b$, define the functions $\varphi_{a,b}, \xi_{a,b} : [a, b] \to \mathbb{R}$ by

$$\varphi_{a,b}(t) = \frac{b-a}{4} \sin \frac{2\pi(t-a)}{b-a}$$

and

$$\xi_{a,b}(t) = \begin{cases} -\varphi_{a,\frac{a+b}{2}}(t) & \text{if } a \leq t \leq \frac{a+b}{2} \\ \varphi_{\frac{a+b}{2},b}(t) & \text{if } \frac{a+b}{2} \leq t \leq b. \end{cases}$$

(2) For each $a, b \in \mathbb{R}$ with $a < b$ and each positive integer n, define the function $\psi_{a,b,n} : [a, b] \to \mathbb{R}$ by

$$\psi_{a,b,n}(t) = \begin{cases} \varphi_{a,d_0}(t) & \text{if } a \leq t \leq d_0 \\ \xi_{d_{i-1},d_i}(t) & \text{if } d_{i-1} \leq t \leq d_i, \end{cases}$$

where

$$d_i = \frac{a+5b}{6} + i\frac{b-a}{6n}$$

for $0 \leq i \leq n$. (See Figure 5.4.1.)

(3) Fix a standard enumeration $M_1, M_2, \ldots$ of (deterministic) Turing machines that take positive integer inputs. For each positive integer n, let $\tau(n)$ denote the number of steps

executed by M_n on input n. It is well known that the *diagonal halting problem*

$$K = \{n \in \mathbb{Z}^+ \mid \tau(n) < \infty\}$$

is undecidable.

(4) Define the horizontal and vertical acceleration functions $a_x, a_y : [0,1] \to \mathbb{R}$ as follows. For each $n \in \mathbb{N}$, let

$$t_n = \int_0^n e^{-x} dx = 1 - e^{-n},$$

noting that $t_0 = 0$ and that t_n converges monotonically to 1 as $n \to \infty$. Also, for each $n \in \mathbb{Z}^+$, let

$$t_n^- = \frac{t_{n-1} + 4t_n}{5}, \quad t_n^+ = \frac{6t_n - t_{n-1}}{5},$$

noting that these are symmetric about t_n and that $t_n^+ \leq t_{n+1}^-$.

(i) For $0 \leq t \leq 1$, let

$$a_x(t) = \begin{cases} -2^{-(n+\tau(n))}\xi_{t_n^-,t_n^+}(t) & \text{if } t_n^- \leq t < t_n^+ \\ 0 & \text{if no such } n \text{ exists,} \end{cases}$$

where $2^{-\infty} = 0$.

(ii) For $0 \leq t < 1$, let

$$a_y(t) = \psi_{t_{n-1},t_n,n}(t),$$

where n is the unique positive integer such that $t_{n-1} \leq t < t_n$.

(iii) Let $a_y(1) = 0$.

(5) Define the horizontal and vertical velocity and position functions $v_x, v_y, s_x, s_y : [0,1] \to \mathbb{R}$ by

$$v_x(t) = \int_0^t a_x(\theta)d\theta, \quad v_y(t) = \int_0^t a_y(\theta)d\theta,$$
$$s_x(t) = \int_0^t v_x(\theta)d\theta, \quad s_y(t) = \int_0^t v_y(\theta)d\theta.$$

(6) Define the vector acceleration, velocity, and position functions $\vec{a}, \vec{v}, \vec{s} : [0,1] \to \mathbb{R}^2$ by

$$\vec{a}(t) = (a_x(t), a_y(t)),$$

$$\vec{v}(t) = (v_x(t), v_y(t)),$$

$$\vec{s}(t) = (s_x(t), s_y(t)).$$

(7) Let $\mathbf{\Gamma} = \text{range}(\vec{s})$.

Intuitively, a particle at rest at time $t = a$ and moving with acceleration given by the function $\varphi_{a,b}$ moves forward, with velocity increasing to a maximum at time $t = \frac{a+b}{2}$ and then decreasing back to 0 at time $t = b$. The vertical acceleration function a_y, together with the initial conditions $v_y(0) = s_y(0) = 0$ implied by (5), thus causes a particle to move generally upward (i.e., $s_y(t_0) < s_y(t_1) < \cdots$), coming to momentary rests at times $t_1, t_2, t_3, \ldots$. Between two consecutive such stopping times t_{n-1} and t_n, the particle's vertical acceleration is controlled by the function $\psi_{t_{n-1},t_n,n}$. This function causes the particle's vertical motion to do the following between times t_{n-1} and t_n.

(i) From time t_{n-1} to time $\frac{t_{n-1}+5t_n}{6}$, move upward from elevation $s_y(t_{n-1})$ to elevation $s_y(t_n)$.

(ii) From time $\frac{t_{n-1}+5t_n}{6}$ to time t_n, make n round trips to a lower elevation $s \in (s_y(t_{n-1}), s_y(t_n))$.

In the meantime, the horizontal acceleration function a_x, together with the initial conditions $v_x(0) = s_x(0) = 0$ implied by (5), ensure that the particle remains on or near the y-axis. The deviations from the y-axis are simply described: The particle moves to the right from time $\frac{t_{n-1}+4t_n}{5}$ through the completion of the n round trips described in (ii) above and then moves to the y-axis between times t_n and $\frac{6t_n-t_{n-1}}{5}$. The amount of lateral motion here is regulated by the coefficient $2^{-(n+\tau(n))}$. If $\tau(n) = \infty$, then there is no lateral motion, and the n round trips in (ii) are retracings of the particle's path. If $\tau(n) < \infty$, then these n round trips are "forward" motion along a curvy part of $\mathbf{\Gamma}$. In fact, $\mathbf{\Gamma}$ contains points of arbitrarily high curvature, but the particle's motion is kinematically realistic in the sense that the acceleration vector $\vec{a}(t)$ is polynomial time computable, hence continuous and bounded on the interval $[0,1]$. Figure

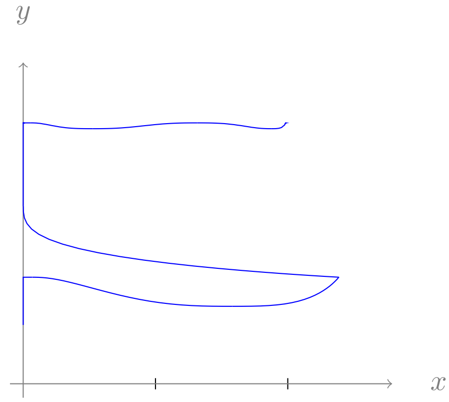

Figure 5.4.2   Example of $\vec{s}(t)$ from t_0 to t_2

5.4.1 illustrates the path of the particle from time t_{n-1} to t_{n+1} with $n = 1$ and hypothetical (model dependent!) values $\tau(1) = 1$ and $\tau(2) = 2$.

The rest of this section is devoted to proving the following theorem concerning the curve $\mathbf{\Gamma}$.

Theorem 5.4.2. *Let $\vec{a}, \vec{v}, \vec{s}$, and $\mathbf{\Gamma}$ be as in Construction 5.4.1.*

1. *The functions $\vec{a}, \vec{v}$, and $\vec{s}$ are Lipschitz and computable in polynomial time, hence continuous and bounded.*

2. *The total length, including retracings, of the parametrization $\vec{s}$ of $\mathbf{\Gamma}$ is finite and computable in polynomial time.*

3. *The curve $\mathbf{\Gamma}$ is simple, rectifiable, and smooth except at one endpoint.*

4. *Every computable parametrization $f : [a, b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ has* unbounded retracing.

For the remainder of this section, we use the notation of Construction 5.4.1.

The following two observations facilitate our analysis of the curve $\mathbf{\Gamma}$. The proofs are routine calculations.

Observation 5.4.3. *For all $n \in \mathbb{Z}^+$, if we write*

$$d_i^{(n)} = \frac{t_{n-1} + 5t_n}{6} + i\frac{t_n - t_{n-1}}{6n}$$

and

$$e_i^{(n)} = d_i^{(n)} + \frac{t_n - t_{n-1}}{12n}$$

for all $0 \le i < n$*, then*

$$t_{n-1} < t_n^- < d_0^{(n)} < e_0^{(n)} < d_1^{(n)} < e_1^{(n)} < \cdots < d_{n-1}^{(n)} < e_{n-1}^{(n)} < t_n < t_n^+ < t_{n+1}^-.$$

Observation 5.4.4. *For all* $a, b \in \mathbb{R}$ *with* $a < b$*,*

$$\int_a^b \int_a^t \varphi_{a,b}(\theta) d\theta dt = \frac{(b-a)^3}{8\pi}.$$

We now proceed with a quantitative analysis of the geometry of $\mathbf{\Gamma}$. We begin with the horizontal component of $\vec{s}$.

Lemma 5.4.5. *1. For all* $t \in [0,1] - \bigcup_{n \in K}(t_n^-, t_n^+)$*,* $v_x(t) = s_x(t) = 0$*.*

2. *For all* $n \in K$ *and* $t \in (t_n^-, t_n)$ *,* $v_x(t) > 0$*.*
3. *For all* $n \in K$ *and* $t \in (t_n, t_n^+)$*,* $v_x(t) < 0$*.*
4. *For all* $n \in \mathbb{Z}^+$*,* $s_x(t_n) = \frac{(e-1)^3}{1000\pi e^{3n}} 2^{-(n+\tau(n))}$*.*
5. $s_x(1) = 0$*.*

Proof. Parts 1-3 are routine by inspection and induction. For $n \in \mathbb{Z}^+$, Observation 5.4.4 tells us that

$$\begin{aligned} s_x(t_n) &= \frac{(t_n - t_n^-)^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(\frac{1}{5}(t_n - t_{n-1}))^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(\frac{1}{5}((e-1)e^{-n}))^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(e-1)^3}{1000\pi e^{3n}} 2^{-(n+\tau(n))} \end{aligned}$$

so 4 holds. This implies that $s_x(t_n) \to 0$ as $n \to \infty$, whence 5 follows from 1,2, and 3. □

The following lemma analyzes the vertical component of $\vec{s}$. We use the notation of Observation 5.4.3, with the additional proviso that $d_n^{(n)} = t_n$.

Lemma 5.4.6. *1. For all $n \in \mathbb{Z}^+$ and $t \in (t_{n-1}, d_0^{(n)})$, $v_y(t) > 0$.*

2. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in (d_i^{(n)}, e_i^{(n)})$, $v_y(t) < 0$.

3. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in (e_i^{(n)}, d_{i+1}^{(n)})$, $v_y(t) > 0$.

4. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in \{e_i^{(n)}, d_i^{(n)}, t_n\}$, $v_y(t) = 0$.

5. For all $n \in \mathbb{Z}^+$ and $0 \le i \le n$, $s_y(d_i^{(n)}) = s_y(d_0^{(n)})$.

6. For all $n \in \mathbb{Z}^+$ and $0 \le i < n$, $s_y(e_i^{(n)}) = s_y(e_0^{(n)})$.

7. For all $n \in \mathbb{N}$, $s_y(t_n) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi} \sum_{i=1}^n \frac{1}{e^{3i}}$.

8. For all $n \in \mathbb{Z}^+$, $s_y(e_0^{(n)}) = s_y(t_n) - \frac{(e-1)^3}{12^3 n^3 8\pi e^{3n}}$.

9. $s_y(1) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi(e^3-1)}$.

Proof. Parts 1-6 are clear by inspection and induction. By 4. and Observation 5.4.4,

$$\begin{aligned} s_y(t_n) - s_y(t_{n-1}) &= s_y(d_0^{(n)}) - s_y(t_{n-1}) \\ &= \frac{[\frac{5}{6}(t_n - t_{n-1})]^3}{8\pi} = \frac{[\frac{5}{6}((e-1)e^{-n})]^3}{8\pi} \\ &= \frac{5^3(e-1)^3}{6^3 \cdot 8\pi e^{3n}} \end{aligned}$$

for all $n \in \mathbb{Z}^+$, so 6 holds by induction. Also by 4 and Observation 5.4.4,

$$\begin{aligned} s_y(t_n) - s_y(e_0^{(n)}) &= s_y(d_0^{(n)}) - s_y(e_0^{(n)}) \\ &= \frac{[\frac{1}{12n}(t_n - t_{n-1})]^3}{8\pi} = \frac{[\frac{1}{12n}((e-1)e^{-n})]^3}{8\pi} \\ &= \frac{(e-1)^3}{12^3 n^3 8\pi e^{3n}}, \end{aligned}$$

so 7 holds. Finally, by 6,

$$s_y(1) = \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)},$$

i.e., 8 holds. ∎

By Lemmas 5.4.5 and 5.4.6, we see that $\vec{s}$ parametrizes a curve from $\vec{s}(0) = (0,0)$ to $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.

The proofs of Lemmas 5.4.5 and 5.4.6 are included in the appendix.

It is clear from Observation 5.4.3 and Lemmas 5.4.5 and 5.4.6 that the curve Γ does not intersect itself. We thus have the following.

Corollary 5.4.7. $\mathbf{\Gamma}$ *is a simple curve from* $\vec{s}(0) = (0,0)$ *to* $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.

Proof. Let $\vec{s}' : [0,1] \to \mathbb{R}^2$ be such that

$$\vec{s}'(t) = \begin{cases} \vec{s}(t_n^+)\frac{t-t_n^-}{t_n^+-t_n^-} + \vec{s}(t_n^-)\frac{t_n^+-t}{t_n^+-t_n^-} & t \in (t_n^-, t_n^+), n \notin K, \\ \vec{s}(t) & \text{otherwise.} \end{cases}$$

Note that by construction of $\vec{s}$, retracing happens along y-axis between $(0, \vec{s}(t_n^-))$ and $(0, \vec{s}(t_n^+))$ only when $t \in (t_n^-, t_n^+)$ for $n \notin K$. In $\vec{s}'$, for all $n \notin K$, $\vec{s}'$ maps (t_n^-, t_n^+) to the vertical line segment between $(0, \vec{s}(t_n^-))$ and $(0, \vec{s}(t_n^+))$ linearly. Otherwise, $\vec{s}'(t) = \vec{s}(t)$. Hence, $\vec{s}'(0) = (0,0)$, $\vec{s}'(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$, and $\vec{s}'$ is a one-to-one parametrization of $\Gamma = \text{range}(\vec{s})$, although $\vec{s}'$ is not computable. Therefore Γ is a simple curve. □

Lemma 5.4.8. *The functions* $\vec{a}, \vec{v}$, *and* $\vec{s}$ *are Lipschitz, hence continuous, on* $[0,1]$.

Proof. It is clear by differentiation that $Lip(\varphi_{a,b}) = \frac{\pi}{2}$ for all $a, b \in \mathbb{R}$ with $a < b$. It follows by inspection that $Lip(a_x) \le \frac{\pi}{4}$ and $Lip(a_y) = \frac{\pi}{2}$, whence

$$Lip(\vec{a}) \le \sqrt{Lip(a_x)^2 + Lip(a_y)^2} \le \frac{\pi\sqrt{5}}{4}.$$

Thus $\vec{a}$ is Lipschitz, hence continuous (and bounded), on $[0,1]$. It follows immediately that $\vec{v}$ and $\vec{s}$ are Lipschitz, hence continuous, on $[0,1]$. □

Since every Lipschitz parametrization has finite total length [5], and since the length of a curve cannot exceed the total length of any of its parametrizations, we immediately have the following.

Corollary 5.4.9. *The total length, including retracings, of the parametrization* $\vec{s}$ *is finite. Hence the curve* $\mathbf{\Gamma}$ *is rectifiable.*

Lemma 5.4.10. *The curve* $\mathbf{\Gamma}$ *is smooth except at the endpoint* $\vec{s}(1)$*.*

Proof. We have seen that $\mathbf{\Gamma}([0, t_1^-])$ is simply a segment of the y-axis, and that the vector velocity function $\vec{v}$ is continuous on $[0, 1]$. Since the set

$$Z = \{t \in (0,1) \mid \vec{v}(t) = 0\}$$

has no accumulation points in $(0, 1)$, it therefore suffices to verify that, for each $t^* \in Z$,

$$\lim_{t \to t^{*-}} \frac{\vec{v}(t)}{|\vec{v}(t)|} = \lim_{t \to t^{*+}} \frac{\vec{v}(t)}{|\vec{v}(t)|}, \tag{5.4.1}$$

i.e., that the left and right tangents of $\mathbf{\Gamma}$ coincide at $\vec{s}(t^*)$. But this is clear, because Lemmas 5.4.5 and 5.4.6 tell us that

$$Z = \left\{t_n \;\middle|\; n \in \mathbb{Z}^+ \text{ and } \tau(n) = \infty\right\},$$

and both sides of (5.4.1) are $(0, 1)$ at all t^* in this set. □

Lemma 5.4.11. *The functions* $\vec{a}, \vec{v}$, *and* $\vec{s}$ *are computable in polynomial time. The total length including retracings, of* $\vec{s}$ *is computable in polynomial time.*

Proof. This follows from Observation 5.4.4, Lemmas 5.4.5 and 5.4.6, and the polynomial time computability of $f(n) = \sum_{i=1}^{n} e^{-3i}$. □

Definition. A *modulus of uniform continuity* for a function $f : [a, b] \to \mathbb{R}^n$ is a function $h : \mathbb{N} \times \mathbb{N}$ such that, for all $s, t \in [a, b]$ and $r \in \mathbb{N}$,

$$|s - t| \leq 2^{-h(r)} \implies |f(s) - f(t)| \leq 2^{-r}.$$

It is well known (e.g., see [54]) that every computable function $f : [a, b] \to \mathbb{R}^n$ has a modulus of uniform continuity that is continuous.

Lemma 5.4.12. *Let* $f : [a, b] \to \mathbb{R}^2$ *be a parametrization of* $\mathbf{\Gamma}$*. If* f *has bounded retracing and a computable modulus of uniform continuity, then* $\mathrm{K} \leq_{\mathrm{T}} f_y$*, where* f_y *is the vertical component of* f*.*

Proof. Assume the hypothesis. Then there exist $m \in \mathbb{Z}^+$ and $h : \mathbb{N} \to \mathbb{N}$ such that f does not have m-fold retracing and h is a computable modulus of uniform continuity for f. Note that h is also a modulus of uniform continuity for f_y.

Let M be an oracle Turing machine that, given an oracle $\mathcal{O}_g$ for a function $g : [a, b] \to \mathbb{R}$, implements the algorithm in Figure 5.4.3. The key properties of this algorithm's choice of r and Δ are that the following hold when $g = f_y$.

(i) For each time t with $f_y(t) = s_y(t_n)$, there is a nearby time τ_j with j high. Similarly for $f_y(t) = s_y(e_0^{(n)})$ and j low.

(ii) For each high j, $|f_y(\tau_j) - s_y(t_n)| \leq 3 \cdot 2^{-r}$. Similarly for each low j and $s_y(e_0^{(n)})$.

(iii) No j can be both high and low.

Now let $n \in \mathbb{Z}^+$. We show that $M^{\mathcal{O}_{f_y}}(n)$ accepts if $n \in \mathrm{K}$ and rejects if $n \notin \mathrm{K}$. This is clear if $n \leq m$, so assume that $n > m$.

If $n \in \mathrm{K}$, then Observation 5.4.3, Lemma 5.4.5, and Lemma 5.4.6 tell us that $M^{\mathcal{O}_{f_y}}(n)$ accepts. If $n \notin \mathrm{K}$, then the fact that f does not have m-fold retracing tells us that $M^{\mathcal{O}_{f_y}(n)}$ rejects. □

Proof of Theorem 5.4.2. Part 1 follows from Lemmas 5.4.8 and 5.4.11. Part 2 follows from Lemma 5.4.11. Part 3 follows from Corollaries 5.4.7 and 5.4.9 and Lemma 5.4.10. Part 4 follows from Lemma 5.4.12, the fact that every computable function $g : [a, b] \to \mathbb{R}^2$ has a computable modulus of uniform continuity, and the fact that A is decidable wherever $A \leq_{\mathrm{T}} g$ and g is computable. □

5.4.2 Lower Semicomputability of Length

In this section we prove that every computable curve Γ has a lower semicomputable length. Our proof is somewhat involved, because our result holds even if every computable parametrization of Γ is retracing.

Construction 5.4.13. Let $f : [0, 1] \to \mathbb{R}^n$ be a computable function. Given an oracle Turing machine M that computes f and a computable modulus $m : \mathbb{N} \to \mathbb{N}$ of the uniform continuity

input $n \in \mathbb{Z}^+$;
if $n \leq m$ **then**
use a finite lookup table to accept if $n \in \mathrm{K}$ and reject if $n \notin \mathrm{K}$
else
begin
  r:= the least positive integer such that $2^{3-r} < s_y(t_n) - s_y(e_0^{(n)})$;
  Δ:=$2^{-h(r)}$;
  for $0 \leq j \leq (b-a)/\Delta$ **do**
  begin
    τ_j:=$a + \Delta_j$;
    call j **high** if $|\mathcal{O}_g(\tau_j, r) - s_y(t_n)| < 2^{1-r}$
    call j **low** if $|\mathcal{O}_g(\tau_j, r) - s_y(e_0^{(n)}| < 2^{1-r}$
  end;
  if there are $0 < j_0 < j_1 < \cdots < j_m$ in which j_i is high for all even i
        and low for all odd i
  then accept
  else reject
end.

Figure 5.4.3 Algorithm for $M^{\mathcal{O}_g}(n)$ in the proof of Lemma 5.4.12.

of f, the (M,m)-cautious polygonal approximator of range(f) is the function $\pi_{M,m} : \mathbb{N} \to \{\textit{polygonal paths}\}$ computed by the following algorithm.

input $r \in \mathbb{N}$;

$S := \{\}$; // S may be a multi-set

for i:=0 **to** $2^{m(r)}$ **do**

$a_i := i2^{-m(r)}$;

use M to compute x_i with

$|x_i - f(a_i)| \leq 2^{-(r+m(r)+1)}$;

add x_i to S;

output a longest path inside a minimum spanning tree of S.

Definition. Let (X,d) be a metric space. Let $\Gamma \subseteq X$ and $\epsilon > 0$. Let

$$\Gamma(\epsilon) = \left\{ p \in X \;\middle|\; \inf_{p' \in \Gamma} d(p,p') \leq \epsilon \right\}$$

be the *Minkowski sausage* of Γ with radius ϵ.

Let $d_{\mathrm{H}} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ be such that for all $\Gamma_1, \Gamma_2 \in \mathcal{P}(X)$

$$d_{\mathrm{H}}(\Gamma_1,\Gamma_2) = \inf \left\{ \epsilon \;|\; \Gamma_1 \subseteq \Gamma_2(\epsilon) \text{ and } \Gamma_2 \subseteq \Gamma_1(\epsilon) \right\}.$$

Note that d_{H} is the *Hausdorff distance* function.

Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a metric space [29].

Theorem 5.4.14. *(Frink [34], Michael [70]). Let (X,d) be a compact metric space. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a compact metric space.*

Definition. Let $\mathcal{RC}$ be the set of all simple rectifiable curves in $\mathbb{R}^n$.

Theorem 5.4.15. *([96] page 55). Let $\Gamma \in \mathcal{RC}$. Let $\{\Gamma_n\}_{n\in\mathbb{N}} \subseteq \mathcal{RC}$ be a sequence of rectifiable curves such that $\lim_{n\to\infty} d_{\mathrm{H}}(\Gamma_n,\Gamma) = 0$. Then $\mathcal{H}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n)$.*

This theorem has the following consequence.

Theorem 5.4.16. *Let $\Gamma \in \mathcal{RC}$. For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\Gamma' \in \mathcal{RC}$, if $d_{\mathrm{H}}(\Gamma, \Gamma') < \delta$, then $\mathcal{H}^1(\Gamma') > \mathcal{H}^1(\Gamma) - \epsilon$.*

In the following, we prove a few technical lemmas that lead to Lemma 5.4.21, which plays an important role in proving Theorem 5.4.22.

Lemma 5.4.17. *Let $\Gamma \in \mathcal{RC}$. Let $p_0, p_1, \in \Gamma$ be its two endpoints. Let $\Gamma' \subsetneq \Gamma$ such that $p_0, p_1 \in \Gamma'$. Then $\Gamma' \notin \mathcal{RC}$.*

Proof. If Γ' is not closed, then we are done. Assume that Γ' is closed. Let γ be a parametrization of Γ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.

Since $\Gamma' \neq \Gamma$ and $p_0, p_1 \in \Gamma'$, $\gamma^{-1}(\Gamma') \subseteq I_0 \cup I_1$, where $I_0 \subseteq [0,1]$ and $I_1 \subseteq [0,1]$ are closed and disjoint.

It is easy to see that $\gamma(I_0)$ and $\gamma(I_1)$ are closed and disjoint. And thus, for any continuous function $\gamma' : [0,1] \to \mathbb{R}^n$, $\gamma'^{-1}(\gamma(I_0))$ and $\gamma'^{-1}(\gamma(I_1))$ are closed and disjoint. Therefore, for any continuous function $\gamma' : [0,1] \to \mathbb{R}^n$, $\gamma^{-1}(\Gamma') \neq [0,1]$, i.e., $\Gamma' \notin \mathcal{RC}$. □

Lemma 5.4.18. *Let $\Gamma \in \mathcal{RC}$. Let $\Gamma' \subseteq \Gamma$ be a connected compact set. Then $\Gamma' \in \mathcal{RC}$.*

Proof. Let γ be the parametrization of Γ.

Let $a = \inf\{\gamma^{-1}(\Gamma')\}$ and let $b = \sup\{\gamma^{-1}(\Gamma')\}$.

Let $\gamma' : [0,1] \to \mathbb{R}^n$ be such that for all $t \in [0,1]$

$$\gamma'(t) = \gamma(a + t(b-a)).$$

Then γ' defines a curve and we show that $\gamma'([0,1]) = \Gamma'$.

It is clear that $\Gamma' \subseteq \gamma'([0,1])$. Since Γ' is compact, we know that $\gamma'(0), \gamma'(1) \in \Gamma'$.

Suppose for some $t' \in (0,1)$, $\gamma'(t') \notin \Gamma'$. Since Γ' is compact, there exists $\epsilon > 0$ such that $\gamma'([t'-\epsilon, t'+\epsilon]) \cap \Gamma' = \varnothing$. Then $\Gamma' \subseteq \gamma'([0, t'-\epsilon)) \cup \gamma'((t'+\epsilon, 1])$. Since γ' is one-one,

$$d_{\mathrm{H}}(\gamma'([0, t'-\epsilon)), \gamma'((t'+\epsilon, 1])) > 0.$$

Hence,

$$d_{\mathrm{H}}(\Gamma' \cap \gamma'([0, t' - \epsilon)), \Gamma' \cap \gamma'((t' + \epsilon, 1])) > 0.$$

Thus, Γ' cannot be connected.

Therefore, if Γ' is connected, then $\Gamma' = \gamma'([0,1])$ and hence $\Gamma' \in \mathcal{RC}$. ☐

Lemma 5.4.19. *Let $\Gamma_0, \Gamma_1, \ldots$ be a convergent sequence of compact sets in compact metric space (X, d) that is eventually connected. Let $\Gamma = \lim_{n\to\infty} \Gamma_n$. Then Γ is connected.*

Proof. We prove the contrapositive.

Assume that Γ is not connected. Then there exists open sets $A, B \subseteq X$ such that $A \cap B = \varnothing$, $\Gamma \cap A \neq \varnothing$, $\Gamma \cap B \neq \varnothing$, and $\Gamma \subseteq A \cup B$.

Then $(\Gamma \cap A) \cap (\Gamma \cap B) = \varnothing$, thus $d_{\mathrm{H}}(\Gamma \cap A, \Gamma \cap B) > 0$. Let

$$\delta = d_{\mathrm{H}}(\Gamma \cap A, \Gamma \cap B).$$

Since $\lim_{n\to\infty} \Gamma_n = \Gamma$, let n_0 be such that for all $n \geq n_0$,

$$d_{\mathrm{H}}(\Gamma_n, \Gamma) \leq \tfrac{\delta}{3}.$$

It is clear that

$$(\Gamma \cap A)(\tfrac{\delta}{3}) \cap \Gamma_n \neq \varnothing,$$

$$(\Gamma \cap B)(\tfrac{\delta}{3}) \cap \Gamma_n \neq \varnothing,$$

and

$$\Gamma_n \subseteq (\Gamma \cap A)(\tfrac{\delta}{3}) \cup (\Gamma \cap B)(\tfrac{\delta}{3}).$$

By the definition of δ,

$$d_{\mathrm{H}}((\Gamma \cap A)(\tfrac{\delta}{3}), (\Gamma \cap B)(\tfrac{\delta}{3})) \geq \tfrac{\delta}{3}.$$

Thus Γ_n is not connected for all $n \geq n_0$. ☐

Lemma 5.4.20. *Let $\Gamma \in \mathcal{RC}$ and let $f : [0,1] \to \Gamma$ be a parametrization of Γ. Let*

$$L(\Gamma, \epsilon) = \inf \left\{ \mathcal{H}^1(\Gamma') \;\middle|\; \Gamma' \in \mathcal{RC} \text{ and } \Gamma' \subseteq \Gamma(\epsilon) \text{ and } f(0), f(1) \in \Gamma' \right\}.$$

Then

$$\lim_{\epsilon \to 0^+} L(\Gamma, \epsilon) = \mathcal{H}^1(\Gamma).$$

Proof. It is clear that $\lim_{\epsilon \to 0^+} L(\Gamma, \epsilon) \leq \mathcal{H}^1(\Gamma)$. It suffices to show that $\lim_{\epsilon \to 0^+} L(\Gamma, \epsilon) \geq \mathcal{H}^1(\Gamma)$.

Let $\delta > 0$. For each $i \in \mathbb{N}$, let

$$S_i = \left\{ \Gamma' \in \mathcal{RC} \;\middle|\; \Gamma' \subseteq \Gamma(\tfrac{1}{i}) \text{ and } \gamma(0), \gamma(1) \in \Gamma' \right\},$$

where γ is a parametrization of Γ. Note that if $i_2 < i_1$, then $S_{i_1} \subseteq S_{i_2}$.

Let $\Gamma_0, \Gamma_1, \ldots$ be an arbitrary sequence such that for all $i \in \mathbb{N}$, $\Gamma_i \in S_{k_i}$, and $k_0, k_1, \cdots \in \mathbb{N}$ is a strictly increasing sequence.

Since for all $i \in \mathbb{N}$, Γ_i is compact and connected, by Theorem 5.4.14 and Lemma 5.4.19, there is at least one cluster point and every cluster point is a connected compact set. Let Γ' be a cluster point. It is clear that $\Gamma' \subseteq \Gamma$. Then by Lemma 5.4.18, $\Gamma' \in \mathcal{RC}$.

It is also clear that $\gamma(0), \gamma(1) \in \Gamma'$ by definition of S_i. Thus by Lemma 5.4.17, $\Gamma' = \Gamma$.

By Theorem 5.4.15, $\liminf_{n \to \infty} \mathcal{H}^1(\Gamma_n) \geq \mathcal{H}^1(\Gamma') = \mathcal{H}^1(\Gamma)$. Then by Theorem 5.4.16, this implies that for all sufficiently large $i \in \mathbb{N}$,

$$(\forall \Gamma'' \in S_i) \mathcal{H}^1(\Gamma'') \geq \mathcal{H}^1(\Gamma) - \delta.$$

Therefore, for all sufficiently large $i \in \mathbb{N}$, $L(\Gamma, \frac{1}{i}) \geq \mathcal{H}^1(\Gamma) - \delta$. Since $\delta > 0$ is arbitrary,

$$\lim_{\epsilon \to 0^+} L(\Gamma, \epsilon) \geq \mathcal{H}^1(\Gamma).$$

□

Lemma 5.4.21. *Let $\Gamma \in \mathcal{RC}$ and let $f : [0, 1] \to \Gamma$ be a parametrization of Γ. Let*

$$L(\Gamma, \epsilon, p_1, p_2) = \inf \left\{ \mathcal{H}^1(\Gamma') \;\middle|\; \Gamma' \in \mathcal{RC} \text{ and } \Gamma' \subseteq \Gamma(\epsilon) \text{ and } p_1, p_2 \in \Gamma' \right\}.$$

Then

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) = \mathcal{H}^1(\Gamma).$$

Proof. For every $p \in \Gamma(\epsilon)$, there exists a point $p' \in \Gamma$ such that $\|p, p'\| \leq \epsilon$ and line segment $[p, p'] \subseteq \Gamma(\epsilon)$. Thus it is clear that for all $p_1, p_2 \in \Gamma(\epsilon)$, $L(\Gamma, \epsilon, p_1, p_2) \leq 2\epsilon + \mathcal{H}^1(\Gamma)$. Therefore,

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) \leq \mathcal{H}^1(\Gamma).$$

For the other direction, observe that

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) \geq \lim_{\epsilon \to 0^+} L(\Gamma, \epsilon).$$

Applying Lemma 5.4.20 completes the proof. □

Theorem 5.4.22. *Let $\Gamma \in \mathcal{RC}$ such that $\Gamma = \gamma([0,1])$, where γ is a continuous function. (Note that γ may not be one-one.) Let $S(a) = \{\gamma(a_i) \mid a_i \in a\}$ for all dissection a. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of dissections of Γ such that*

$$\lim_{n \to \infty} \operatorname{mesh}(a_n) = 0.$$

Then

$$\lim_{n \to \infty} \mathcal{H}^1(LMST(a_n)) = \mathcal{H}^1(\Gamma),$$

where $LMST(a)$ is the longest path inside the Minimum Euclidean Spanning Tree of $S(a)$.

Proof. For all $n \in \mathbb{N}$, let

$$\epsilon_n = 2d_{\mathrm{H}}(\Gamma, S(a_n)).$$

Note that since γ is uniformly continuous and $\lim_{n \to \infty} \operatorname{mesh}(a_n) = 0$, $\lim_{n \to \infty} \epsilon_n = 0$.

Let $w = 2\epsilon_n$.

Claim. *Let T be a Euclidean Spanning Tree of $S(a)$. If T has an edge that is not inside $\Gamma(w)$, then T is not a minimum spanning tree.*

Proof of Claim. Let E be an edge of T such that $E \not\subseteq \Gamma(w)$. Then $\mathcal{H}^1(E) > 2w$. Removing E from T will break T into two subtrees T_1, T_2. By the definition of ϵ_n and the continuity of γ, there exists $s_1, s_2 \in S(a)$ with $\|s_1 - s_2\| \leq \epsilon_n$ such that $s_1 \in T_1$ and $s_2 \in T_2$.

It is clear that $T_1 \cup T_2 \cup \{(s_1, s_2)\}$ is also a Euclidean Spanning Tree of $S(a)$ and $\mathcal{H}^1(T_1 \cup T_2 \cup \{(s_1, s_2)\}) < \mathcal{H}^1(T)$, i.e., T is not minimum. □

Let T be a Minimum Euclidean Spanning Tree of $S(a)$. Let L be the longest path inside T. Then $L \subseteq T \subseteq \Gamma(w)$.

Note that $\mathcal{H}^1(L) \leq \mathcal{H}^1(\Gamma)$.

Let p_0, p_1 be the two endpoints of Γ.

Since L is the longest path inside T and p_0, p_1 are each within ϵ_n distance to some point in $S(a_n)$,

$$L(\Gamma, w, p_0, p_1) \leq 2\epsilon_n + \mathcal{H}^1(L).$$

By Lemma 5.4.21,

$$\lim_{w \to 0^+} L(\Gamma, w, p_0, p_1) = \mathcal{H}^1(\Gamma).$$

Then

$$\lim_{n \to \infty} \mathcal{H}^1(LMST(a_n)) = \mathcal{H}^1(\Gamma).$$

□

This result implies that when the sampling density is high, the number of leaves in the minimum spanning tree is asymptotically smaller than the total number of nodes.

We now have the machinery to prove the main result of this section.

Theorem 5.4.23. *Let $\gamma : [0,1] \to \mathbb{R}^n$ be computable such that $\Gamma = \gamma([0,1]) \in \mathcal{RC}$. Then $\mathcal{H}^1(\Gamma)$ is lower semicomputable.*

Proof. Let the function f, M, and m in Construction 5.4.13 be γ, a computation of γ, and its computable modulus respectively.

For each input $r \in \mathbb{N}$, $\pi_{M,m}(r)$ is the longest path L_r in $MST(S_r)$, where S_r is the set of points sampled by $\pi_{M,m}(r)$.

Let $l_r = \mathcal{H}^1(L_r) - 2^{-r}$. Note that l_r is computable from $r \in \mathbb{N}$.

We show that for all $r \in \mathbb{N}$, $l_r \leq \mathcal{H}^1(\Gamma)$ and $\lim_{r\to\infty} l_r = \mathcal{H}^1(\Gamma)$.

Let $\tilde{f}$ be a one-one parametrization of Γ. Let $\pi : \{0, \ldots, 2^{m(r)}\} \to \{0, \ldots, 2^{m(r)}\}$ be a permutation of $\{0, \ldots, 2^{m(r)}\}$ such that for all $i, j \in \{0, \ldots, 2^{m(r)}\}$,

$$i < j \implies \tilde{f}^{-1}(f(a_{\pi(i)})) < \tilde{f}^{-1}(f(a_{\pi(j)})).$$

Let $\hat{\Gamma}_r$ be the polygonal curve connecting the points $f(a_{\pi(0)}), f(a_{\pi(1)}), \ldots, f(a_{\pi(2^{m(r)})})$ in order. Then $\hat{\Gamma}_r$ is a polygonal approximation of Γ and $\mathcal{H}^1(\hat{\Gamma}_r) \leq \mathcal{H}^1(\Gamma)$.

Let $\bar{\Gamma}_r$ be the polygonal curve connecting the points in S_r in the order of $x_{\pi(0)}$, $x_{\pi(1)}$, $\ldots, x_{\pi(2^{m(r)})}$.

Due to the approximation induced by the computation in Construction 5.4.13,

$$\mathcal{H}^1(\bar{\Gamma}_r) \leq \mathcal{H}^1(\hat{\Gamma}_r) + 2^{-r}.$$

Then it is clear that

$$\mathcal{H}^1(L_r) = \mathcal{H}^1(LMST(S_r)) \leq \mathcal{H}^1(\bar{\Gamma}_r) \leq \mathcal{H}^1(\hat{\Gamma}_r) + 2^{-r}.$$

Thus

$$l_r \leq \mathcal{H}^1(\hat{\Gamma}_r).$$

Let $\hat{S}_r = \{f(a_0), f(a_1), \ldots, f(a_{2^{m(r)}})\}$. Note that $\hat{S}_r$ may be a multi-set. By Theorem 5.4.22,

$$\lim_{r\to\infty} LMST(\hat{S}_r) = \mathcal{H}^1(\Gamma).$$

Let

$$\epsilon_r = 2d_{\mathrm{H}}(\Gamma, S_r).$$

By Contruction 5.4.13,

$$\lim_{r\to\infty} \epsilon_r = 0.$$

Let $w_r = 2\epsilon_r$.

Let T_r be a Minimum Euclidean Spanning Tree of S_r. Let L_r be the longest path inside T_r. By the Claim in Theorem 5.4.22, $L \subseteq T \subseteq \Gamma(w_r)$.

By an essentially identical argument as the one in the proof of Theorem 5.4.22,

$$\lim_{r\to\infty} l_r = \lim_{r\to\infty} \mathcal{H}^1(LMST(S_r)) = \mathcal{H}^1(\Gamma),$$

which completes the proof. □

In the following, we use Ko's curve fitting construction [23, 55] to prove that every positive constructive real number may be the length of a computable curve in Theorem 5.4.25. We use the following lemma to fit the curves.

Lemma 5.4.24. *Let $k, n \in \mathbb{Z}^+$ be such that $k < n$. Let $a \in \mathbb{Z}^+$ be such that $a < 2^k$. Then there exists a polynomial-time computable curve $f \equiv F_{a2^{-k},2^{-n}} : [0,1] \to \mathbb{R}^2$ such that $f([0,1]) \subseteq [0, 2^{-n}]^2$, $f(0) = 0$, $f(1) = (0, 2^{-n})$, and length of the curve is $a2^{-k}$.*

Proof. We construct the curve using length unit 2^{-n-2}. Since we need the curve to have total length $a2^{-k}$, the number of unit length is

$$T = \frac{a2^{-k}}{2^{-n-2}} = a2^{n-k+2}.$$

Let $x_1 = 0$, $x_2 = 2^{-n-2}$, $x_T = 2^{-n}$, $x_{T-1} = 2^{-n} - 2^{-n-2}$, $x_{T-2} = 2^{-n} - 2^{-n-1}$. Let m be the smallest integer such that $2^m > T - 4$. For each positive integer $i \leq T - 5$, let $x_{2+i} = x_2 + i2^{-n-2-m}$.

Let $L_0 = [(x_1, 0), (x_2, 0)]$. For each positive integer $i \leq (T-4)/2$, let

$$L_i = [(x_{2i}, 0), (x_{2i}, 2^{-n-2})] \cup [(x_{2i}, 2^{-n-2}), (x_{2i+1}, 2^{-n-2})] \cup [(x_{2i+1}, 2^{-n-2}), (x_{2i+1}, 0)].$$

For each positive integer $i < (T-4)/2$, let

$$L'_i = [(x_{2i+1}, 0), (x_{2i+2}, 0)].$$

Let $L'_{(T-4)/2} = \varnothing$. Let $L_e = [(x_{T-3}, 0), (2^{-n}, 0)]$. Let

$$L = L_0 \cup L_e \cup \bigcup_{i \in [1..(T-4)/2]} (L_i \cup L'_i).$$

Define $f : [0,1] \xrightarrow[onto]{1-1} L$ to be a continuous mapping with $f(0) = (0,0)$ and $f(1) = (2^{-n}, 0)$ that is parameterized by curve length. It is clear that f is a polynomial-time computable curve with length $a2^{-k}$. □

Theorem 5.4.25. *Every positive constructive real number is the length of a polynomial-time computable rectifiable curve.*

Proof. Let α be a positive c.e. real number. Let $\alpha(0) = 0$, $\alpha(1)$, $\alpha(2)$, ..., $\alpha(n)$, ... be a computable sequence (using Turing machine M) of dyadic rationals with the properties that $\alpha(i+1) > \alpha(i)$ for all $i \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha(n) = \alpha$. Let $t(n)$ be the number of steps M takes to print $\alpha(n)$ on input n. Let $T(n) = \sum_{i=1}^{n} t(i)$ for all $n > 0$ and $T(0) = 0$.

For $n \in \mathbb{Z}^+$ and $x \in [1 - 2^{-T(n-1)}, 1 - 2^{-T(n)}]$, let

$$\begin{aligned} f_n(x) = {} & (0, \sum_{i=1}^{n-1} 2^{-T(i)-2}) \\ & + F_{\alpha(n)-\alpha(n-1), 2^{-T(n)-2}} \left(\frac{x - (1 - 2^{-T(n-1)})}{2^{-T(n-1)} - 2^{-T(n)}} \right), \end{aligned}$$

where $F_{\alpha(n)-\alpha(n-1), 2^{-T(n)-2}}$ is the function defined in Lemma 5.4.24.

For all $n \in \mathbb{Z}^+$ and $x \in [0, 1]$, if $f_n(x)$ is not specified above, then $f_n(x) = (0, 0)$.

Note that for all $n \in \mathbb{Z}^+$, $f_n(x)$ on $[1 - 2^{-T(n-1)}, 1 - 2^{-T(n)}]$ defines a curve of length $\alpha(n) - \alpha(n-1)$.

Let $f : [0, 1] \to \mathbb{R}^2$ be such that for all $x \in [0, 1)$

$$f(x) = \sum_{i=1}^{\infty} f_i(x)$$

and

$$f(1) = \sum_{i=1}^{\infty} 2^{-T(i)-2}.$$

It is easy to verify that f is continuous on $[0, 1]$. It is clear that for all $x \in [1 - 2^{-T(n-1)}, 1 - 2^{-T(n)}]$

$$f(x) = f_n(x).$$

Note that for all $n \in \mathbb{Z}^+$,

$$|f(1) - f_n(1 - 2^{-T(n)})| = \sum_{i=n+1}^{\infty} 2^{-T(i)-2} \leq 2^{-T(n+1)-1}$$

and

$$\begin{aligned} f([1 - 2^{-T(n)}, 1]) & \subseteq (0, \sum_{i=1}^{n} 2^{-T(i)-2}) + [0, |f(1) - f_n(1 - 2^{-T(n)})|] \times [0, 2^{-T(n+1)-2}] \\ & \subseteq (0, \sum_{i=1}^{n} 2^{-T(i)-2}) + [0, 2^{-T(n+1)-1}] \times [0, 2^{-T(n+1)-2}]. \end{aligned}$$

Thus for every point $x \in f([1-2^{-T(n)},1])$,

$$|x - f(1-2^{-T(n)})| \leq \sqrt{2} \cdot 2^{-T(n+1)-1} < 2^{-T(n+1)}.$$

Define $T^{-1} : \mathbb{N} \to \mathbb{N}$ be such that $T^{-1}(k) = n_0$ with $T(n_0) \leq k < T(n_0+1)$. Let $\hat{f} : \mathbb{Q} \cap [0,1] \times \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ be such that

$$\hat{f}(q,r) = \begin{cases} \sum_{i=1}^{T^{-1}(k)} f_i(q) & 0 \leq q \leq 1 - 2^{-T(T^{-1}(k))} \\ (0, \sum_{i=1}^{T^{-1}(k)} 2^{-T(i)-2}) & 1 - 2^{-T(T^{-1}(k))} < q \leq 1. \end{cases}$$

It is clear that $\hat{f}$ is computable in time polynomial to $|q| + r$ and $\hat{f}$ is a computation of f.

The length of the curve defined by f is

$$\sum_{n=1}^{\infty} \text{length}(f_n) = \sum_{n=1}^{\infty} (\alpha(n) - \alpha(n-1)) = \lim_{n\to\infty} \alpha(n) = \alpha.$$

□

5.4.3 Δ_2^0-Computability of the Constant-Speed Parametrization

In this section we prove that every computable curve Γ has a constant speed parametrization that is Δ_2^0-computable.

Theorem 5.4.26. *Let $\Gamma = \gamma^*([0,1]) \in \mathcal{RC}$. ($\gamma^*$ may not be one-one.) Let $l = \mathcal{H}^1(\Gamma)$ and O_l be an oracle such that for all $n \in \mathbb{N}$, $|O_l(n) - l| \leq 2^{-n}$. Let f be a computation of γ^* with modulus m. Let γ be the constant speed parametrization of Γ. Then γ is computable with oracle O_l.*

Proof. On input k as the precision parameter for computation of the curve and a rational number $x \in [0,1] \cap \mathbb{Q}$, we output a point $f_k(x) \in \mathbb{R}^n$ such that $|f_k(x) - \gamma(x)| \leq 2^{-k}$.

Without loss of generality, assume that $\mathcal{H}^1(\Gamma) > 1000 \cdot 2^{-k}$.

Let $\delta = 2^{-(4+k)}$.

Run f as in Construction 5.4.13 with increasingly larger precision parameter $r > -\log \delta$ until

$$\mathcal{H}^1(LMST(a)) > \mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}$$

and the shortest distance between the two endpoints of $LMST(a)$ inside the polygonal sausage around $LMST(a)$ with width $2d = 2 \cdot 2^{-r}$ is at least $\mathcal{H}^1(\Gamma) - \frac{\delta}{2}$. This can be achieved by using Euclidean shortest path algorithms [53, 43].

Let $d_k \leq 2^{-(4+k)}$ be the largest d such that the above conditions are satisfied, which is assured by Theorem 5.4.23 and Lemma 5.4.21. Let $\mathcal{S}$ be the polygonal sausage around $LMST(a)$ with width $2d_k$.

For $p_1, p_2 \in \mathcal{S}$, let $d_{\mathcal{S}}(p_1, p_2)$ = the shortest distance between p_1 and p_2 inside $\mathcal{S}$. Note that $\mathcal{S}$ is connected.

Let f_k be the constant speed parametrization of $LMST(a)$ and γ be the constant speed parametrization of Γ. Without loss of generality, assume that $\|\gamma(0) - f_k(0)\| < \|\gamma(1) - f_k(0)\|$ and $\|\gamma(1) - f_k(1)\| < \|\gamma(0) - f_k(1)\|$, since we can hardcode approximate locations of $\gamma(0)$ and $\gamma(1)$ such that when d_k is sufficiently small, we can decide wehther a sampled point is closer to $\gamma(0)$ or $\gamma(1)$. As we now prove

$$\lim_{k \to \infty} \{f_k(0), f_k(1)\} = \{\gamma(0), \gamma(1)\}.$$

Note that for each $s \in S$ such that $s \notin LMST(a)$, there exists $p \in LMST(a) \cap S$ such that the shortest path from s to p in $MST(a)$ has length less than $\frac{\delta}{2}$, i.e., $d_{MST(a)}(s, p) < \frac{\delta}{2}$, since $\mathcal{H}^1(LMST(a)) > \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$ and $\mathcal{H}^1(MST(a)) \leq \mathcal{H}^1(\Gamma)$.

Let $\delta_0 = d_{\mathcal{S}}(\gamma(0), f_k(0))$. Let s_0 be the closest point to $\gamma(0)$ in $S \cap LMST(a)$. Then $d_{\mathcal{S}}(\gamma(0), s_0) \leq \frac{\delta}{2} + d_k$. Then $d_{LMST(a)}(s_0, f_k(0)) \geq \delta_0 - \frac{\delta}{2} - d_k$. Since $s_0 \in S \cap LMST(a)$ and we assume $\mathcal{H}^1(\Gamma) > 1000 \cdot 2^{-k}$,

$$d_{\mathcal{S}}(s_0, \gamma(1)) \leq \mathcal{H}^1(LMST(a)) - \delta_0 + \tfrac{\delta}{2} + d_k + \tfrac{\delta}{2} + d_k = \mathcal{H}^1(LMST(a)) - \delta_0 + \delta + 2d_k.$$

Then

$$\begin{aligned} d_{\mathcal{S}}(\gamma(0), \gamma(1)) &\leq \mathcal{H}^1(LMST(a)) - \delta_0 + \delta + 2d_k + \tfrac{\delta}{2} + d_k \\ &< \mathcal{H}^1(LMST(a)) - \delta_0 + \tfrac{3\delta}{2} + 3d_k. \end{aligned}$$

And hence

$$d_{\mathcal{S}}(\gamma(0), \gamma(1)) \le \mathcal{H}^1(\Gamma) - \delta_0 + 2\delta + 3d_k. \tag{5.4.2}$$

By the choice of d_k, we have that $d_{\mathcal{S}}(f_k(0), f_k(1)) \ge \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$. Now, note that for any two points $p_1, p_2 \in \Gamma$,

$$d_{\mathcal{S}}(p_1, p_2) \le \frac{\mathcal{H}^1(\Gamma) + d_{\mathcal{S}}(\gamma(0), \gamma(1))}{2},$$

since we can put them in half of a loop. Therefore

$$d_{\mathcal{S}}(f_k(0), f_k(1)) \le \frac{\mathcal{H}^1(\Gamma) + d_{\mathcal{S}}(\gamma(0), \gamma(1))}{2}.$$

Thus

$$d_{\mathcal{S}}(\gamma(0), \gamma(1)) \ge \mathcal{H}^1(\Gamma) - \delta. \tag{5.4.3}$$

By (5.4.2) and (5.4.3), we have

$$\delta_0 \le 3\delta + 3d_k \le 6\delta < 2^{-k}, \tag{5.4.4}$$

i.e.,

$$\|f_k(0) - \gamma(0)\| \le d_{\mathcal{S}}(f_k(0), \gamma(0)) \le 6\delta < 2^{-k}. \tag{5.4.5}$$

Similarly,

$$\|f_k(1) - \gamma(1)\| \le d_{\mathcal{S}}(f_k(1), \gamma(1)) \le 6\delta < 2^{-k}. \tag{5.4.6}$$

Now we proceed to show that for all $t \in (0, 1)$, $\|f_k(t) - \gamma(t)\| < 10\delta$ with $f(0)$ being at most 6δ from $\gamma(0)$ inside $\mathcal{S}$ and $f(1)$ being at most 6δ from $\gamma(1)$ inside $\mathcal{S}$.

Let $\Delta_k = \|f_k(t) - \gamma(t)\|$.

Let $s_f \in S \cap LMST(a)$ be such that $|f_k^{-1}(s_f) - t|$ is minimized. Then $d_{LMST(a)}(f_k(t), s_f) \le d_k$, since every edge in $MST(a)$ is at most d_k long.

Let $s'_\gamma \in S \cap \Gamma$ be such that $|\gamma^{-1}(s'_\gamma) - t|$ is minimized. Then $d_\Gamma(\gamma(t), s'_\gamma) \le d_k$, since we sample S using d_k as the density parameter.

Let $s_\gamma \in S \cap LMST(a)$ such that $d_{MST(a)}(s_\gamma, s'_\gamma)$ is minimized. Then $d_{MST(a)}(s_\gamma, s'_\gamma) \le \frac{\delta}{2}$, since $\mathcal{H}^1(MST(a)) \ge \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$.

Then $\|f_k(t) - s_\gamma\| \ge \Delta_k - (\frac{\delta}{2} + d_k) = \Delta_k - \frac{\delta}{2} - d_k$.

Note that $d_{LMST(a)}(s_f, s_\gamma) \geq \|s_f - s_\gamma\| \geq \Delta_k - \frac{\delta}{2} - 2d_k$.

Without loss of generality, assume that distance from s_γ to $f_k(0)$ along $LMST(a)$ is $\Delta_k - \frac{\delta}{2} - d_k$ more than the distance from $f_k(t)$ to $f_k(0)$. Otherwise, we simply look from the $\gamma(1)$ and $f_k(1)$ side instead.

The path traced by γ from $\gamma(0)$ to $\gamma(t)$ has length $t \cdot \mathcal{H}^1(\Gamma)$.

The shortest distance between $\gamma(t)$ to s_γ inside $\Gamma \cup MST(a)$ is at most $d_k + \frac{\delta}{2}$.

The path traced by f_k from s_γ to $f_k(1)$ has length

$$d_{LMST(a)}(s_\gamma, f_k(1)) \leq \mathcal{H}^1(LMST(a)) - [t(\mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}) - d_k + \Delta_k - \tfrac{\delta}{2} - d_k].$$

The shortest distance from $\gamma(1)$ to $f_k(1)$ inside $\mathcal{S}$ is at most 6δ.

Then the distance from $\gamma(0)$ to $\gamma(1)$ inside $\mathcal{S}$ is at most

$$\begin{aligned}
&t \cdot \mathcal{H}^1(\Gamma) + d_k + \tfrac{\delta}{2} + \mathcal{H}^1(LMST(a)) - [t(\mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}) - d_k + \Delta_k - \tfrac{\delta}{2} - d_k] + 6\delta \\
&\leq \mathcal{H}^1(LMST(a)) + 3d_k + 8\delta - \Delta_k \\
&\leq \mathcal{H}^1(\Gamma) + 11\delta - \Delta_k.
\end{aligned}$$

By (5.4.3), we have

$$\Delta_k \leq 12\delta < 2^{-k}.$$

□

Corollary 5.4.27. *Let Γ be a curve with the property described in property 4 of Theorem 5.4.2. Then the length $\mathcal{H}^1(\Gamma)$ of Γ is not computable.*

Proof. We prove the contrapositive. Let Γ be a curve with a computable parametrization with a computable length $\mathcal{H}^1(\Gamma)$. Then by Theorem 5.4.26, we can use the Turing machine that computes $\mathcal{H}^1(\Gamma)$ as the oracle in the statement of Theorem 5.4.26 and obtain a Turing machine that computes the constant speed parametrization of Γ. Therefore, Γ does not have the property described in item 4 of Theorem 5.4.2. □

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