9-2016

Three Essays in Intuitionistic Epistemology

Tudor Protopopescu
The Graduate Center, City University of New York

Three Essays in Intuitionistic Epistemology

by

Tudor Protopopescu

A dissertation submitted to the Graduate Faculty in Philosophy in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2016

This manuscript has been read and accepted for the Graduate Faculty in Philosophy in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

(required signature)

_____
Date

_____
Chair of Examining Committee

(required signature)

_____
Date

_____
Executive Officer

Sergei Artemov

_____

Melvin Fitting

_____

Arnold Koslow

_____

Richard Mendelsohn

_____

Graham Priest

_____

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Three Essays in Intuitionistic Epistemology

by

Tudor Protopopescu

Advisor: Sergei Artemov

We present three papers studying knowledge and its logic from an intuitionistic viewpoint.

An Arithmetic Interpretation of Intuitionistic Verification

Intuitionistic epistemic logic introduces an epistemic operator to intuitionistic logic which reflects the intended BHK semantics of intuitionism. The fundamental assumption concerning intuitionistic knowledge and belief is that it is the product of verification. The BHK interpretation of intuitionistic logic has a precise formulation in the Logic of Proofs and its arithmetical semantics. We show here that this interpretation can be extended to the notion of verification upon which intuitionistic knowledge is based. This provides the systems of intuitionistic epistemic logic extended by an epistemic operator based on verification with an arithmetical semantics too. This confirms the conception of verification incorporated in these systems reflects the BHK interpretation.

Intuitionistic Verification and Modal Logics of Verification

The systems of intuitionistic epistemic logic, IEL, can be regarded as logics of intuitionistic verification. The intuitionistic language, however, has expressive limitations. The classical

modal language is more expressive, enabling us to formulate various classical principles which make explicit the relationship between intuitionistic verification and intuitionistic truth, implicit in the intuitionistic epistemic language. Within the framework of the arithmetic semantics for IEL we argue that attempting to base a general verificationism on the properties of intuitionistic verification, as characterised by IEL, yields a view of verification stronger than is warranted by its BHK reading.

Intuitionistic Knowledge and Fallibilism

Fallibilism is the view that knowledge need not guarantee the truth of the proposition known. In the context of a classical conception of truth fallibilism is incompatible with the truth condition on knowledge, i.e. that false propositions cannot be known. We argue that an intuitionistic approach to knowledge yields a view of knowledge which is both fallibilistic and preserves the truth condition. We consider some problems for the classical approach to fallibilism and argue that an intuitionistic approach also resolves them in a manner consonant with the motivation for fallibilism.

Acknowledgements

It hase been my immense good fortune to have worked with my superviser, Sergei Artemov, from whom I have learnt so much more than just what is in this dissertation, or in our papers. More than any specific fact I learnt from him how to think, and how to be a professional. Without his guidance and patience I would not be the scholar and, more importantly, the person I am today.

Doctorates and dissertations are not just intellectual marathons, they are also personal ones. Without the love and support of my wife, Lillian Albert-Gardner, the light of my life, the dark and miserable times would have been unendurable, and none of this possible. There are no adequate words to express how lucky I feel we found each other.

I also would like to record an intellectual debt to Melvin Fitting and Graham Priest.

It was one of Melvin Fitting's courses that prompted me to shift my focus to logic, and also lead me to attend Artemov's Justification Logic course, after which the rest is history. Had it not been for these two events this dissertation would never have come close to happening.

Graham Priest was the first cause of this dissesrtation. His questioning in my prospectus exam and the ensuing discussion was the impetus for the project of formulating intuitionistic epistemic logic, of which the present papers are immediate descendents. But for this I would have written the dissertation I first proposed, not this one.

Contents

Introduction

Intuitionistic epistemology is the study of knowledge based on the intuitionistic conception of truth. Knowledge is valued as highly as it is because it is directed at the truth, we seek it because it tells us what is really the case. The properties of knowledge, consequently, are bound up with what it means for something to be true. The conception of truth of most epistemology, and epistemic logic, is classical; truth is bi-valent, there are only two truth values, true and false, and obeys the law of excluded middle, every proposition is either true or false. On this conception certain epistemic principles are valid, even evidently so, whilst others are obviously invalid. In particular the principle stating the factivity of knowledge, the reflection principle,

$$\mathbf{K}A \rightarrow A \tag{Reflection}$$

is the prime example of the former, while the co-reflection principle,

$$A \rightarrow \mathbf{K}A \tag{Co-Reflection}$$

is a prime example of the latter. The justification for reflection is that it merely expresses the truth condition on knowledge, that knowledge must be true, or that knowledge cannot be false, which is fundamental to any definition of knowledge. Co-Reflection claims that any true proposition is known, which is obviously not the case.

If we shift to a different conception of truth then the relationship between knowledge and truth changes also. Intuitionistic epistemology explores the consequences of thinking of truth along intuitionistic lines, specifically along the lines of the Brouwer-Heyting-Kolmogorov (BHK) semantics for intuitionism. The BHK semantics, formulated by Heyting and Kolmogorov, is acknowledged to be the intended interpretation of intuitionism, capturing the specific version of constructivism put forward by Brouwer.[1] According to BHK a proposition is true only if there is a proof of it. This is, accordingly an 'epistemic' view of truth, because the truth of a proposition depends on the existence of a certain kind of evidence. The classical conception of truth is 'evidence-transcendent' in the sense that the truth of a proposition does not depend on the existence of any kind of evidence.

On an intuitionistic view of truth the situation is strikingly different. The basic assumption of an intuitionistic approach to knowledge is that knowledge is the product of verification, which is understood as a procedure providing sufficient information to justify a claim to knowledge of a proposition without necessarily being a proof of it. Given this assumption, the co-reflection principle is a valid intuitionistic epistemic principle, while the reflection principle is too strong as an expression of the truth condition on knowledge, and is invalid. According to the BHK reading of co-reflection it says, very roughly, that a proof of a proposition can be turned into a proof that the proposition is verified, or known, while reflection says that a proof that there is a, knowledge-producing, verification of a proposition can be turned into a proof of it. There is such a procedure in the first case, proof checking, while there is not necessarily one in the second.

---

[1]We will use the terms 'intuitionistic' and 'constructive' more or less synonymously throughout, though this is not strictly correct. Intuitionism is a species of constructivism; undoubtedly the best known but not exhaustive.

This conception of knowledge and systems of intuitionistic doxastic and epistemic logic were introduced in [3]. The three papers which follow build upon what is outlined there.

The first paper *An Arithmetic Interpretation of Intuitionistic Verification* explores further the claim that the conception of knowledge given in [3] is indeed true to the BHK interpretation of intuitionistic logic. The BHK interpretation of intuitionistic logic speaks of proofs, but does not specify what counts as a proof. Artemov [1, 2] showed that explicit proof in Peano Arithmetic, PA, is the model of provability BHK specifies: intuitionistic logic is an implicit logic of proofs in PA. The paper shows that this arithmetic interpretation can be extended to accommodate the knowledge operator introduced in intuitionistic epistemic logic, and hence that intuitionistic knowledge, as outlined in [3] is indeed BHK-compliant.

The second paper, *Intuitionistic Verification and Modal Logics of Verification* builds on the first paper. Since intuitionistic knowledge is regarded as the product of verification intuitionistic epistemic logic can be regarded as a logic of intuitionistic verification. It is a natural question to ask if this logic might be appropriate for the kind of verificationism put forward in the works of Dummett, Prawitz, Martin-Löf and others. The intuitionistic language, however, has expressive limitations. The classical modal language is more expressive, allowing us to formulate various classical principles which make explicit the relationship between intuitionistic verification and intuitionistic truth (i.e. proof), implicit in the intuitionistic epistemic language. We consider if the arithmetic interpretation given in the previous paper extends to these principles also, and argue that it does not. This suggests that attempting to base a general verificationism on the properties of intuitionistic verification, as characterised by IEL, yields a view of verification stronger than is warranted by its BHK reading.

The third paper, *Intuitionistic Knowledge and Fallibilism*, argues that intuitionistic knowledge in some respects is a better conception of fallible knowledge than knowledge based on classical truth. Fallibilism is the view that knowledge need not guarantee the truth of the proposition known. In the context of a classical conception of truth fallibilism is incompatible with the truth condition on knowledge, i.e. that false propositions cannot be known. The paper argues that an intuitionistic approach to knowledge yields a view of knowledge which is both fallibilistic and preserves the truth condition. We consider some problems for the classical approach to fallibilism and argue that an intuitionistic approach also resolves them in a manner consonant with the motivation for fallibilism.

Note on the Papers

The following dissertation consists of three stand-alone papers, rather than a monograph.

The first paper is a fuller version of a paper presented at *Logical Foundations of Computer Science 2016*, [5]. The second paper is, in some ways, a sequel to the first, clearly building on it, but with a different goal. It is a development of a paper presented at *Logics of Rationality and Interaction 2015*, [4]. Both are intended for a largely technical audience. The third paper is intended for a more generalist, and non-formal, philosopical audience, and contains the fullest exposition of the basics of an intuitionistic conception of knowledge, and an outline of intuitionistic epistemic logic, as given in [3].

Each of the papers are intended to be self-contained and can, in theory, be read in any order. For the reader unfamiliar with intuitionistic epistemic logic probably the best reading order is to start with the third paper, since it contains the fullest exposition of the intuitionistic conception of knowledge and the motivations for intuitionistic epistemic logic,

and then the first and the second.

References

[1]   Sergei Artemov. *Operational Modal Logic*. Tech. rep. MSI 95-29. Cornell University, 1995 (cit. on p. 3).

[2]   Sergei Artemov. "Explicit Provability and Constructive Semantics." In: *Bulletin of Symbolic Logic* 7.1 (2001), pp. 1–36 (cit. on p. 3).

[3]   Sergei Artemov and Tudor Protopopescu. "Intuitionistic Epistemic Logic." In: *Review of Symbolic Logic* (2016). DOI: 10.1017/S1755020315000374 (cit. on pp. 3, 4).

[4]   Tudor Protopopescu. "Intuitionistic Epistemology and Modal Logics of Verification." In: Proceedings. Logics, Rationality and Interaction (LORI 2015). (Oct. 26–29, 2015). Ed. by Wiebe van der Hoek and Wesley Holliday. Lecture Notes in Computer Science 9394. Taipei: Springer, 2015, pp. 295–307. DOI: 10.1007/978-3-662-48561-3_24 (cit. on p. 4).

[5]   Tudor Protopopescu. "An Arithmetical Interpretation of Verification and Intuitionistic Knowledge." In: *ArXiv*. math.LO 1601.03059v1 (Jan. 2016). Post-print correcting various publisher's errors. Published in Artemov, S. and Nerode, A., editors, Logical Foundations of Computer Science, Number 9537 in Lecture Notes in Computer Science, pp. 317-330, Springer. DOI: 10.1007/978-3-319-27683-0_22. ArXiv: 1601.03059v1 (math.LO) (cit. on p. 4).

1. An Arithmetic Interpretation of Intuitionistic Verification

1 Introduction

The intended semantics for intuitionistic logic is the Brouwer-Heyting-Kolmogorov (BHK) interpretation, which holds that a proposition is true if proved [14, 15, 17]. The systems of intuitionistic epistemic logic, the IEL family introduced in [5], extend intuitionistic logic with an epistemic operator and interpret it in a manner reflecting the BHK semantics. The fundamental assumption concerning knowledge interpreted intuitionistically is that knowledge is the product of verification, where a verification is understood to be a justification sufficient to warrant a claim to knowledge which is not necessarily a strict proof.

In [5] the notion of verification was treated intuitively. Here we show that verification can also be given an arithmetical interpretation, thereby showing that the notion of verification assumed in an intuitionistic interpretation of knowledge has an exact model.

Following Gödel [13] it is well known that intuitionistic logic can be embedded into the classical modal logic S4 regarded as a provability logic. Artemov [1, 2] formulated the Logic of Proofs, LP, and showed that S4 in turn can be interpreted in LP, and that LP has an arithmetical interpretation as a calculus of explicit proofs in Peano Arithmetic PA.[1] Accordingly

---

[1] As opposed to provability in PA, the calculus of which is the modal logic GL, see [4, 6]. On the arithmetical

this makes precise the BHK semantics for intuitionistic logic. Intuitionistic logic, then, can be regarded as an implicit logic of proofs, and its extension with an epistemic/verification operator in the systems IEL$^-$ and IEL (given in Section 2) can be regarded as logics of implicit proofs, verification and their interaction.

This is of interest for a number of reasons. It shows that the notion of verification on which intuitionistic epistemic logic is based is coherent and can be made concrete. Moreover this is done in a manner consonant within the context of the provability model of intuitionistic logic, which suggests that this conception of verification is BHK-compliant. Further, given intuitionistic logic's importance in computer science as well as the need for a constructive theory of knowledge, finding a precise provability model for verification and intuitionistic epistemic logic (see Section 5) is well-motivated.

2 Intuitionistic Epistemic Logic

According to the BHK semantics a proposition, A, is true if there is a proof of it and false if the assumption that there is a proof of A yields a contradiction. This is extended to complex propositions by the following clauses:

- a proof of $A \wedge B$ consists in a proof of A and a proof of B;

- a proof of $A \vee B$ consists in giving either a proof of A or a proof of B;

- a proof of $A \rightarrow B$ consists in a construction which given a proof of A returns a proof of B;

- $\neg A$ is an abbreviation for $A \rightarrow \bot$, and $\bot$ is a proposition that has no proof.

---

interpretation of the Logic of Proofs see also [18].

The salient property of verification-based justification, in the context of the BHK semantics, is that it follows from intuitionistic truth, hence

$$A \to \mathbf{K}A \qquad \text{(Co-Reflection)}$$

is valid on a BHK reading. Since any proof is a verification, the intuitionistic truth of a proposition yields that the proposition is verified.

By similar reasoning the converse principle

$$\mathbf{K}A \to A \qquad \text{(Reflection)}$$

is not valid on a BHK reading. A verification need not be, or yield a method for obtaining, a proof, hence does not guarantee the intuitionistic truth of a proposition. Reflection expresses the factivity of knowledge in a classical language, intuitionistically factivity is expressed by

$$\mathbf{K}A \to \neg\neg A. \qquad \text{(Intuitionistic Reflection)}$$

The basic system of intuitionistic epistemic logic, incorporating minimal assumptions about the nature of verification, is the system IEL^-. IEL^- can be seen as the system formalising intuitionistic belief.

Definition 1.1 (IEL^-)**.** The list of axioms and rules of IEL^- consists of:

Axioms.

IE0. Axioms of propositional intuitionistic logic.

IE1. $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$

IE2. $A \to \mathbf{K}A$

Definition 1.2.

Rules. Modus Ponens

It is consistent with IEL^- that false propositions can be verified. It is desirable, however, that false propositions not be verifiable; to be a logic of knowledge the logic should reflect the truth condition on knowledge, i.e. factivity – that it is not possible to know falsehoods. The system IEL incorporates the truth condition and hence can be viewed as an intuitionistic logic of knowledge.

Definition 1.3 (IEL)**.** The list of axioms and rules for IEL are those for IEL^- with the additional axiom:

IE3. $\mathbf{K}A \to \neg\neg A$.

Given co-reflection the idea that it is not possible to know a falsehood can be equivalently expressed by

$$\neg\mathbf{K}\bot.^2$$

For the following we will use this form of the truth condition in place of intuitionistic reflection.

Definition 1.4 (Semantics for $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$)**.** Models for $\mathcal{L}$ are intuitionistic Kripke models, $\langle W, R, \Vdash \rangle$, with an additional accessibility relation E.

IEL^-: An IEL^- model satisfies the following conditions on E, for states u, v, w

IM1. uEv yields uRv;

---

[2]Or indeed, $\neg(\mathbf{K}A \wedge \neg A)$, $\neg A \to \neg\mathbf{K}A$ or $\neg\neg(\mathbf{K} \to A)$, all are equivalent to intuitionistic reflection given co-reflection, see [5]. For other systems of intuitionistic epistemic logic, though not based on the BHK semantics, see [16, 20, 23]

IM2. uRv and vEw yield uEw;

IM3. $u \Vdash \mathbf{K}A$ iff $v \Vdash A$ for all v such that uEv.

IEL: An IEL model is an IEL^- model with the additional condition on E that:

IM4. E is serial, for all u, there is a v such that uEv.

States R-accessible from a given state, u, are logically possible developments of the information available at u, while the set of states E-accessible from u, represent possible verifications given the information at u.

3   Embedding Intuitionistic Epistemic Logic into Classical Modal Logic of Verification

The well known Gödel translation yields a faithful embedding of the intuitionistic propositional calculus, IPC, into the classical modal logic S4.[3] By extending S4 with a verification modality $\mathbf{V}$, the embedding can be extended to IEL^- and IEL, and shown to remain faithful, see [21]. On this reading appending a $\square$ to a proposition is a way of expressing in a classical language that it is constructively true. The translation takes a formula, A, of IPC and returns a formula of S4, $tr(A)$, according to the rule

$$box \; every \; subformula \; of \; A.$$

By extending S4 with a verification modality $\mathbf{V}$, the translation can be extended to each of the logics IEL^- and IEL. We will define the systems $S4V^-$, $S4V$ and show that the Gödel translation yields a faithful embedding of each intuitionistic system into its classical modal

---

[3]The soundness of the translation was proved by Gödel [13] while the faithfulness was proved by McKinsey and Tarski [19]. See [8] for a semantic, and [22] for a syntactic proof.

companion. In this way we interpret intuitionistic truth in a setting where we can make explicit when (and if) a proposition is intuitionistically true, or verified, or some combination of them.

Intuitionistic **K** represents verifications which are not necessarily proofs, which is why intuitionistic reflection can fail. Similarly, **V** represents a verification procedure which is not necessarily factive (unlike $\Box$, which represents proof). This is a realistic assumption given many, if not most, of our justifications are fallible, and hence so is the knowledge based on them.The systems S4V$^-$and S4V may be regarded as systems of proof and verification-based belief or fallible knowledge. $\mathbf{V}A \to A$ could be added to the systems in question to yield systems of verification-based infallible, i.e. factive, knowledge and proof. The embedding results below do not require reflection for **V**, nor would adding reflection alter them.

3.1 Modal Logics S4V$^-$, S4V

Definition 1.5 (S4V$^-$). The list of axioms and rules of S4V$^-$ consists of

A0. The axioms of S4 for $\Box$.

A1. $\mathbf{V}(A \to B) \to (\mathbf{V}A \to \mathbf{V}B)$

A2. $\Box A \to \mathbf{V}A$

R1. Modus Ponens

R2. $\Box$-Necessitation $\dfrac{\vdash A}{\vdash \Box A.}$

S4V$^-$ represents basic, not necessarily consistent, verification, the only requirement of which is that anything which is proved be regarded as verified.

As with IEL we add the further condition that verifications should be consistent.

Definition 1.6 (S4V)**.** S4V is S4V$^-$ with the additional axiom:[4]

    A3. $\neg\Box\mathbf{V}\bot$.

Proposition 1.7. *The rule of* **V***-Necessitation is derivable in* $\mathcal{L}_\Box$*.*

Proof. Assume $\vdash A$, by $\Box$-necessitation $\vdash \Box A$ follows, hence by Axiom A2 $\vdash \mathbf{V}A$.   □

Definition 1.8 (Semantics for S4V$^-$ and S4V)**.** Models for $\mathcal{L}_\Box$ are S4 Kripke models, $\langle W, R_\Box, \Vdash \rangle$, with an additional accessibility relation $R_\mathbf{V}$.

S4V$^-$: An S4V$^-$-model satisfies the following conditions on $R_\mathbf{V}$, for states x, y, z

    M1. $xR_\mathbf{V}y$ yields $xR_\Box y$;

    M2. $x \Vdash \mathbf{V}A$ iff $y \Vdash A$ for all y such that $xR_\mathbf{V}y$.

S4V: An S4V-model is an S4V$^-$-model with the additional condition on $R_\mathbf{V}$ that:

    M3.   for all x there are y and z such that $xR_\Box y$ and $yR_\mathbf{V}z$ (weak seriality).

Proposition 1.9. *The inclusion* S4V$^-$ $\subset$ S4V *is strict.*

Proof. See [5, Theorem 4.5], the model there can be regarded as an S4V$^-$-model in which Axiom A3 is not valid.   □

---

[4][21] presented a stronger version of S4V with $\neg\mathbf{V}\bot$ instead of $\neg\Box\mathbf{V}\bot$. The weaker axiom presented here is sufficient for the embedding; one can readily check that the Gödel translation of $\neg\mathbf{K}\bot$, $\Box\neg\Box\mathbf{V}\Box\bot$, is derivable in S4V as formulated here. The weaker axiom allows for a uniform arithmetical interpretation of verification.

3.2 Sequent Systems for S4V$^-$ and S4V

Since the realisation theorem, Theorems 1.52 and 1.53, which connects $\mathcal{L}_\square$ with their explicit counter-parts, LPV$^-$ and LPV, defined below Definitions 1.34 and 1.35, and hence IEL with its arithmetic interpretation, depends on cut-free sequent proofs in S4V$^-$ and S4V we will give a sequent formulation of these logics. We will denote these by S4V$^-$g, S4Vg respectively. In proving completeness we will also show that the systems are cut-free.

A sequent is a figure, $\Gamma \Rightarrow \Delta$, in which Γ, Δ are multi-sets of formulas.

To keep things simpler we define $\lozenge$ as $\neg\square\neg$. Each is an extension of the system **G1s** from [22] for the S4 part.

Definition 1.10 (S4V$^-$g)**.** The axioms and rules of S4V$^-$g are the following:

Axioms

$$P \Rightarrow P, \ P \text{ atomic} \qquad \bot \Rightarrow$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, X \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow X, \Delta}$$

$$\frac{X, X, \Gamma \Rightarrow \Delta}{X, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, X, X}{\Gamma \Rightarrow \Delta, X}$$

Negation Rules

$$\frac{\Gamma, X \Rightarrow \Delta}{\Gamma \Rightarrow \neg X, \Delta} \qquad \frac{\Gamma \Rightarrow X, \Delta}{\Gamma, \neg X \Rightarrow \Delta}$$

Conjunction Rules

$$\frac{\Gamma, X, Y \Rightarrow \Delta}{\Gamma, X \wedge Y \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow X, \Delta \qquad \Gamma \Rightarrow Y, \Delta}{\Gamma \Rightarrow X \wedge Y, \Delta}$$

Disjunction Rules

$$\frac{\Gamma, \Rightarrow X, Y, \Delta}{\Gamma \Rightarrow X \vee Y, \Delta} \qquad \frac{\Gamma, X \Rightarrow \Delta \qquad \Gamma, Y \Rightarrow \Delta}{\Gamma, X \vee Y \Rightarrow \Delta}$$

Implication Rules

$$\frac{\Gamma, X \Rightarrow Y, \Delta}{\Gamma \Rightarrow X \rightarrow Y, \Delta} \qquad \frac{\Gamma, Y \Rightarrow \Delta \qquad \Gamma \Rightarrow X, \Delta}{\Gamma, X \rightarrow Y \Rightarrow \Delta}$$

□-Rules

$$\frac{\Gamma, X \Rightarrow \Delta}{\Gamma, \Box X \Rightarrow \Delta} \, (\Box \Rightarrow) \qquad\qquad \frac{\Box\Gamma \Rightarrow X}{\Box\Gamma \Rightarrow \Box X} \, (\Rightarrow \Box)$$

V-Rule

$$\frac{\Box\Theta, \Gamma \Rightarrow X}{\Box\Theta, \mathbf{V}\Gamma \Rightarrow \mathbf{V}X} \, (\Rightarrow \mathbf{V})$$

Interaction-Rule

$$\frac{\Gamma, \mathbf{V}X \Rightarrow \Delta}{\Gamma, \Box X \Rightarrow \Delta} \, (\mathbf{V}/\Box \Rightarrow)$$

S4Vg extends S4V⁻g with the extra following rule.

Definition 1.11 (S4Vg)**.** S4Vg consists of the rules and axioms of S4V⁻g as well as the following rule:

Weak Inconsistency Elimination, *WIE*

$$\frac{\Gamma \Rightarrow \Box \mathbf{V} \bot}{\Gamma \Rightarrow} \, (\Rightarrow \Box \mathbf{V})$$

We will show the following are equivalent, for each of $\mathsf{S4V^-g}$ and $\mathsf{S4Vg}$ respectively, by showing that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$:

1. $\mathsf{S4V^-g}/\mathsf{S4Vg} \vdash \Gamma \Rightarrow \Delta$;

2. $\mathsf{S4V^-g}/\mathsf{S4Vg}$ *with* $cut \vdash \Gamma \Rightarrow \Delta$;

3. $\mathsf{S4V^-}/\mathsf{S4V} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$;

4. $\mathsf{S4V^-}/\mathsf{S4V} \Vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$

5. In any finite $\mathsf{S4V^-}/\mathsf{S4V}$ model, $\mathcal{M}$, $\mathcal{M} \Vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.

That 1 yields 2 is obvious. Similarly, that 4 yields 5. That 2 yields 3 is a matter of showing that the rules of $\mathsf{S4Vg}$ are all provable, Hilbert-style, in $\mathsf{S4V}$. Hence it remains to show 3 yields 4 and 5 yields 1.

We start with 3 yields 4 which is the soundness theorem below, Theorem 1.13. 5 yields 1 is the completeness theorem, Theorem 1.26.

Soundness

To prove the soundness of $\mathsf{S4V^-g}$ and $\mathsf{S4Vg}$ we define a mapping from sequents to formulas of each system thus:

Definition 1.12. f is a mapping from sequents to formulas of $\mathsf{S4V}$ defined thus:

$$[X_1, \ldots, X_n \Rightarrow Y_1, \ldots, Y_k]^f = [(X_1 \wedge \cdots \wedge X_n) \to (Y_1 \vee \cdots \vee Y_k)]$$

$$[X_1, \ldots, X_n \Rightarrow]^f = [(X_1 \wedge \cdots \wedge X_n) \to \bot]$$

$$[\Rightarrow Y_1, \ldots, Y_k]^f = [\top \to (Y_1 \vee \cdots \vee Y_k)]$$

We will abbreviate conjunctions $X_1 \wedge \ldots \wedge X_n$ by $\widehat{X}_n$, and disjunctions $Y_1 \vee \ldots \vee Y_k$ by $\widecheck{Y}_k$, so $\widehat{\Box Z}_m$ stands for $\Box Z_1 \wedge \ldots \wedge \Box Z_m$, and similarly for disjunctions and combinations of modalities.

Theorem 1.13 ($\mathsf{S4V^-g}$ Soundness)**.** *If a sequent P is derivable in $\mathsf{S4V^-g}$ then P is $\mathsf{S4V^-}$ valid.*

Proof. By induction on proofs in $\mathsf{S4V^-g}$. The propositional rules, and the $\Box$ rules are well-known already. Let us go through the cases involving $\mathbf{V}$.

Let $\mathcal{S}$ be a sequent which is the conclusion of a sequent rule with either $\mathcal{S}_1$ or both $\mathcal{S}_1$ and $\mathcal{S}_2$ as premises.

Case 1 (($\Rightarrow \mathbf{V}$) Rule)**.**

Let $\mathcal{S} = \Box\Theta, \mathbf{V}\Gamma \Rightarrow \mathbf{V}X$ and $[\mathcal{S}]^f = (\widehat{\Box C}_l \wedge \widehat{\mathbf{V}A}_n) \to \mathbf{V}X$. Assume $\mathcal{S}$ is derived by the ($\Rightarrow \mathbf{V}$) rule.

Hence $\mathcal{S}_1 = \Box\Theta, \Gamma \Rightarrow X$, and $[\mathcal{S}_1]^f = (\widehat{\Box C}_l \wedge \widehat{A}_n) \to X$.

Assume that $[\mathcal{S}]^f$ is not valid. Hence there is a state x such that $x \Vdash \widehat{\mathbf{V}A}_n$ and $x \nVdash \mathbf{V}X$. Hence there is a state y such that $xR_{\mathbf{V}}y$ and $y \nVdash X$ and $y \Vdash \widehat{A}_n$. In which case $y \nVdash [\mathcal{S}_1]^f =$

$(\widehat{\Box C_l} \wedge \widehat{A_n}) \rightarrow X$, and hence $[\mathcal{S}_1]^f$ is not valid.

Case 2 (Interaction Rule).

Let $\mathcal{S} = [\Gamma, \Box X \Rightarrow \Delta]$ and $[\mathcal{S}]^f = ((\widehat{A_n}) \wedge \Box X) \rightarrow (\breve{B}_m)$. Assume $\mathcal{S}$ is derived by the Interaction rule.

Hence $\mathcal{S}_1 = [\Gamma, \mathbf{V} X \Rightarrow \Delta]$, and $[\mathcal{S}]^f = ((\widehat{A_n}) \wedge \mathbf{V} X) \rightarrow (\breve{B}_m)$.

Assume $[\mathcal{S}]^f$ is not valid, hence there is a state, x, such that $x \nVdash ((\widehat{A_n}) \wedge \Box X) \rightarrow (\breve{B}_m)$, hence $x \Vdash \widehat{A_n}$ and $x \Vdash \Box X$. In which case for all y such that $x R_\Box y$ $y \Vdash X$. Now, let z be any state $R_\mathbf{V}$-accessible from x. Since $R_\mathbf{V} \subseteq R_\Box$ then $x R_\Box z$ also, hence $z \Vdash X$, and hence $x \Vdash \mathbf{V} X$ too. In which case $x \nVdash ((\widehat{A_n}) \wedge \mathbf{V} X) \rightarrow (\breve{B}_m)$. Hence $[\mathcal{S}_1]^f$ is not valid either.

$\Box$

Theorem 1.14 (S4Vg Soundness). $\mathsf{S4Vg} \vdash F \implies \mathsf{S4V} \Vdash F$

Proof. We add to the proof of Theorem 1.13 the following case.

Case 1 ($\Box\mathbf{V} \Rightarrow$ Rule). Let $\mathcal{S} = [\Gamma \Rightarrow \quad]$ and $\mathcal{S}^f = [\neg \widehat{A_n}]$.

Hence $\mathcal{S}_1 = [\Gamma \Rightarrow \Box\mathbf{V}\bot]$ and $\mathcal{S}_1^f = [\widehat{A_n} \rightarrow \Box\mathbf{V}\bot]$.

Assume $\mathcal{S}^f$ is not valid, hence there is a state x in a model such that $x \Vdash \widehat{A_n}$. By weak seriality there are y and z s.t. $x R_\Box y$ and $y R_\mathbf{V} z$, and $z \nVdash \bot$. In which case, $x \nVdash A \rightarrow \Box\mathbf{V}\bot$ and $\mathcal{S}_1^f$ is not valid either.

$\Box$

Completeness

We show completeness by an analog of the maximal consistent set construction. Instead of maximal consistent sets, which are infinite, we will construct finite consistent sets from the sub-formulas of an underivable sequent. This will suffice to define a counter-model to the sequent in question. The difference with the canonical model construction is that here we obtain a finite counter-model for each underivable sequent, rather than one infinite counter-model for all underivable sequents. As a corollary we also obtain the finite model property, and cut-elimination.

Definition 1.15. A sequent $\Gamma \Rightarrow \Delta$ is *underivable* (or *consistent*) if $\mathsf{S4Vg} \nvdash \Gamma \Rightarrow \Delta$.

Definition 1.16. A sequent $\Gamma \Rightarrow \Delta$ is *saturated* if

1. $\perp \in \Delta$

2. If $A \wedge B \in \Gamma$ then $A \in \Gamma$ and $B \in \Gamma$

3. If $A \wedge B \in \Delta$ then $A \in \Delta$ or $B \in \Delta$

4. If $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$

5. If $A \vee B \in \Delta$ then $A \in \Delta$ and $B \in \Delta$

6. If $A \rightarrow B \in \Gamma$ then $A \in \Delta$ or $B \in \Gamma$

7. If $A \rightarrow B \in \Delta$ then $A \in \Gamma$ and $B \in \Delta$

8. If $\neg A \in \Gamma$ then $A \in \Delta$

9. If $\neg A \in \Delta$ then $A \in \Gamma$

10. If $\square A \in \Gamma$ then $A \in \Gamma$ and $\mathbf{V} A \in \Gamma$

Definition 1.17 ($Sat(S)$)**.** For a set of formulas S let $Sat(S)$ be the set consisting of the sub-formulas of S, $Sub(S)$ and $\mathbf{V}A \in Sat(S)$ if $\Box A \in Sub(S)$, i.e. $Sat(S) = Sub(S) \cup \{\mathbf{V}X | \Box X \in Sub(S)\}$.

Definition 1.18. A *saturation* $\Gamma' \Rightarrow \Delta'$ of a sequent $\Gamma \Rightarrow \Delta$ is obtained from $\Gamma \Rightarrow \Delta$ by the following procedure: Initial steps:

- Add $\bot$ to Δ

- If $\Gamma \Rightarrow \Delta, \mathbf{V}\bot$ is consistent (not derivable), add $\mathbf{V}\bot$ to Δ.

Remark 1.19. The following observation will be useful later when we prove completeness. This guarantees that at the end of the saturation procedure $\mathbf{V}\bot \in \Delta'$ whenever $\Gamma \Rightarrow \Delta, \mathbf{V}\bot$ is not derivable. Hence if $\mathbf{V}\bot \notin \Delta'$, then $\Gamma \Rightarrow \Delta, \mathbf{V}\bot$ must be derivable.

- If $A \wedge B \in \Gamma$ put A in Γ and B in Γ

- If $A \wedge B \in \Delta$ then put A in Δ or B in Δ, whichever is consistent

- If $A \vee B \in \Gamma$ then put either A in Γ or B in Γ, whichever is consistent

- If $A \vee B \in \Delta$ then put A in Δ and B in Δ

- If $A \to B \in \Gamma$ then either B in Γ or A in Δ, whichever is consistent

- If $A \to B \in \Delta$ then put A in Γ and B in Δ

- If $\neg A \in \Gamma$ then put $A \in \Delta$

- If $\neg A \in \Delta$ then put $A \in \Gamma$

- If $\Box A \in \Gamma$ then put A and $\mathbf{V}A$ in Γ

Lemma 1.20. *If a sequent $\Gamma' \Rightarrow \Delta'$ is obtained from $\Gamma \Rightarrow \Delta$ by the application of one or more of the rules of the saturation procedure, then $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.*

Proof. By induction on the clauses of the definition of the saturation procedure. The only nonstandard step is adding $\mathbf{V}\bot$ to Δ if $\Gamma \Rightarrow \Delta, \mathbf{V}\bot$ is not derivable. It obviously preserves consistency since the resulting sequent $\Gamma \Rightarrow \Delta, \mathbf{V}\bot$ is presumed consistent. $\square$

Lemma 1.21. *If sequent $\mathcal{S} = [\Gamma \Rightarrow \Delta]$ is underivable then the sequent resulting from applying a saturation rule $\mathcal{S}' = [\Gamma' \Rightarrow \Delta']$ is also underivable, i.e. saturation preserves underivability.*

Proof. By induction on the saturation rules. Let us check just the modal case, the rest are standard.

Case 1 ($\Box A \in \Gamma$). $\Gamma = \Theta \cup \Box A$. After application of the saturation rule $\mathcal{S}' = [\Theta \cup \Box A, A, \mathbf{V}A \Rightarrow \Delta]$.

Assume $\Theta \cup \Box A, A, \mathbf{V}A \Rightarrow \Delta'$ is derivable, hence $\Theta \cup \Box A, \Box A, \Box A \Rightarrow \Delta'$ is derivable by $(\Box, \Rightarrow)$ and $(\mathbf{V}, \Rightarrow)$, applied in either order, and then by contraction $\Gamma \Rightarrow \Delta$ is derivable. Hence by contraposition, if $\mathcal{S}$ is underivable so is $\mathcal{S}'$.

$\square$

Definition 1.22. $\Gamma^{\Box} = \{\Box X | \Box X \in \Gamma\}$; $\Gamma^{\mathbf{V}} = \{X | \mathbf{V}X \in \Gamma\} \cup \Gamma^{\Box}$

Definition 1.23 (Canonical Model). The canonical model is a quadruple $\langle W, R_{\Box}, R_{\mathbf{V}}, \Vdash \rangle$ such that:

- $W^C =$ the set of all consistent saturated sequents.

- $R_{\Box} = (\Gamma \Rightarrow \Delta) R_{\Box}^C (\Theta \Rightarrow \Pi)$ iff $\Gamma^{\Box} \subseteq \Theta$.

- $R_{\mathbf{V}} = (\Gamma \Rightarrow \Delta) R_{\mathbf{V}}^C (\Theta \Rightarrow \Pi)$ iff $\Gamma^{\mathbf{V}} \subseteq \Theta$.

- $\Vdash\ =(\Gamma\Rightarrow\Delta)\Vdash p$ iff $p\in\Gamma$.

Lemma 1.24. *The canonical model is an* $\mathsf{S4V}$ *model.*

Proof. We have to show that $R_\square^C$ is reflexive and transitive, $R_{\mathbf{V}}^C\subseteq R_\square^C$, and that weak seriality holds.

Case 1 ($R_\square^C$ is reflexive and transitive)**.** Both hold by the reflexivity and transitivity of $\subseteq$.

Case 2 ($R_{\mathbf{V}}^C\subseteq R_\square^C$)**.** Assume $(\Gamma\Rightarrow\Delta)R_{\mathbf{V}}^C(\Theta\Rightarrow\Pi)$, i.e. $\Gamma^{\mathbf{V}}\subseteq\Theta$, hence $\{\square X|\square X\in\Gamma\}\subseteq\Theta$ so $(\Gamma\Rightarrow\Delta)R_\square^C(\Theta\Rightarrow\Pi)$.

Case 3 (Weak Seriality)**.** Take a sequent $\Gamma\Rightarrow\Delta\in W^C$. Consider $\Gamma^\square\Rightarrow\emptyset$, we claim it is underivable. Assume otherwise, then $\Gamma\Rightarrow\Delta$ is derivable also, which contradicts our assumption. Hence $\Gamma^\square\Rightarrow\emptyset$ is underivable. Let $\Theta\Rightarrow\Pi$ be an underivable saturation of $\Gamma^\square\Rightarrow\emptyset$, hence it is in W^C and $(\Gamma\Rightarrow\Delta)R_\square(\Theta\Rightarrow\Pi)$.

Let $\{X|\mathbf{V}X\in\Theta\}=\Theta_{\mathbf{V}}$, hence $\Theta^{\mathbf{V}}=\Theta^\square\cup\Theta_{\mathbf{V}}$.

Now consider $\Theta^{\mathbf{V}}\Rightarrow\emptyset$, we claim that it too is underivable. Assume it is derivable, i.e. $\Theta^\square\cup\Theta_{\mathbf{V}}\Rightarrow\perp$ is derivable. Hence $\Theta^\square\cup\mathbf{V}(\Theta_{\mathbf{V}})\Rightarrow\mathbf{V}\perp$ is derivable by $(\Rightarrow\mathbf{V})$ and hence so is $\Theta\Rightarrow\Pi,\mathbf{V}\perp$ by weakenings. Since $\Theta\Rightarrow\Pi$ is not derivable it follows that $\mathbf{V}\perp\notin\Pi$. Recall that $\Theta\Rightarrow\Pi$ is the saturation of $\Gamma^\square\Rightarrow\emptyset$, and so by Remark 1.19 since $\mathbf{V}\perp\notin\Pi$ then $\Gamma^\square\Rightarrow\mathbf{V}\perp$ is derivable. Hence so is $\Gamma^\square\Rightarrow\square\mathbf{V}\perp$, hence $\Gamma^\square\Rightarrow\ $ and so $\Gamma\Rightarrow\Delta$ is derivable, which is a contradiction. Hence $\Theta^{\mathbf{V}}\Rightarrow\emptyset$ is not derivable; let $\Psi\Rightarrow\Phi$ be an underivable saturation, and hence in W^C. Now $(\Theta\Rightarrow\Pi)R_{\mathbf{V}}(\Psi\Rightarrow\Phi)$ as desired.

$\square$

Lemma 1.25 (Truth Lemma)**.** *If* $(\Gamma\Rightarrow\Delta)\in W^C$ *then*

    1. $X \in \Gamma$ *implies* $(\Gamma \Rightarrow \Delta) \Vdash^C X$

    2. $X \in \Delta$ *implies* $(\Gamma \Rightarrow \Delta) \nVdash^C X$

Proof. By induction on the construction of X. The propositional cases are standard.

Case 1 ($X = \Box A$). Assume $\Box A \in \Gamma$ and $(\Gamma \Rightarrow \Delta) R_\Box^C (\Theta \Rightarrow \Pi)$ for arbitrary $\Theta \Rightarrow \Pi$. Since $\Gamma^\Box \subseteq \Theta$ it follows that $A \in \Theta$. By the induction hypothesis $(\Theta \Rightarrow \Pi) \Vdash A$, hence $(\Gamma \Rightarrow \Delta) \Vdash \Box A$.

    Assume $\Box A \in \Delta$. Consider the sequent $\Gamma^\Box \Rightarrow A$, it is not derivable. Assume otherwise, then $\Gamma^\Box \Rightarrow \Box A$ is derivable, and hence $\Gamma \Rightarrow \Delta$ is too, which is a contradiction. Let $\Gamma' \Rightarrow \Delta'$ be the consistent saturation of $\Gamma^\Box \Rightarrow A$, which is hence in W^C and $\Gamma^\Box \subseteq \Gamma'$. Hence $(\Gamma \Rightarrow \Delta) R_\Box^C (\Gamma' \Rightarrow \Delta')$, and $A \in \Delta'$. By induction hypothesis $\Gamma' \Rightarrow \Delta' \nVdash A$, hence $\Gamma \Rightarrow \Delta \nVdash \Box A$.

Case 2 ($X = \mathbf{V}A$). Assume $\mathbf{V}A \in \Gamma$ and $(\Gamma \Rightarrow \Delta) R_\mathbf{V}^C (\Theta \Rightarrow \Pi)$ for arbitrary $\Theta \Rightarrow \Pi$. Since $\Gamma^\mathbf{V} \subseteq \Theta$ it follows that $A \in \Theta$. By the induction hypothesis $(\Theta \Rightarrow \Pi) \Vdash A$, hence $(\Gamma \Rightarrow \Delta) \Vdash \mathbf{V}A$.

    Assume $\mathbf{V}A \in \Delta$ and consider the sequent $\Gamma^\mathbf{V} \Rightarrow A$, we claim it is not derivable. Suppose otherwise, then $\Gamma \Rightarrow \Delta$ is derivable; $\Gamma = \Box\Omega \cup \mathbf{V}\Theta \cup \Pi$, hence $\Gamma^\mathbf{V} = \Box\Omega \cup \Theta$. If $\Box\Omega \cup \Theta \Rightarrow A$ is derivable so is $\Box\Omega \cup \mathbf{V}\Theta \cup \Pi \Rightarrow \mathbf{V}A \ = \Gamma \Rightarrow \Delta$ by $(\Rightarrow \mathbf{V})$ and weakening. Let $\Gamma' \Rightarrow \Delta'$ be the saturation of $\Gamma^\mathbf{V} \Rightarrow A$, which is hence in W^C; $\Gamma^\mathbf{V} \subseteq \Gamma'$, hence $(\Gamma \Rightarrow \Delta) R_\mathbf{V} (\Gamma' \Rightarrow \Delta')$ and $A \in \Delta'$. By the induction hypothesis $\Gamma' \Rightarrow \Delta' \nVdash A$ hence $\Gamma \Rightarrow \Delta \nVdash \mathbf{V}A$.

$\square$

    Finally we show that $5 \Rightarrow 1$, specifically that $not - 1 \Rightarrow not - 5$.

Theorem 1.26 (Completeness). *If* $\mathsf{S4V^-}, \mathsf{S4V} \Vdash \bigwedge \Gamma \to \bigvee \Delta$ *then* $\mathsf{S4Vg^-}, \mathsf{S4Vg} \vdash (\Gamma \Rightarrow \Delta)$.

Proof. We prove this for $\mathsf{S4Vg}$, the other case is similar. Assume

$$\mathsf{S4Vg} \nvdash (\Gamma \Rightarrow \Delta).$$

By Lemma 1.21, $\Gamma \Rightarrow \Delta$ can be extended to a saturated consistent sequent $\widetilde{\Gamma} \Rightarrow \widetilde{\Delta} \in W$. By Lemma 1.25, $(\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}) \Vdash X$ for all $X \in \widetilde{\Gamma}$ and $(\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}) \nVdash X$ for all $X \in \widetilde{\Delta}$, hence

$$(\widetilde{\Gamma} \Rightarrow \widetilde{\Delta}) \nVdash \bigwedge \Gamma \to \bigvee \Delta.$$

$\square$

Corollary 1.27 (Cut Free).

Proof. Note that $\mathsf{S4V^-g}$ and $\mathsf{S4Vg}$ are formulated without the cut rule.

$\square$

Corollary 1.28 (Finite Model Property). *If* $\mathsf{S4Vg} \nvdash \Gamma \Rightarrow \Delta$ *then there is a finite* $\mathsf{S4V}$ *model* $\mathcal{M}$ *s.t.* $\mathcal{M} \nVdash \bigwedge \Gamma \to \bigvee \Delta$.

Proof. Let $Sat(\Gamma \cup \Delta)$ be as in Definition 1.17, the set of sub-formulas of $\Gamma \Rightarrow \Delta$ plus $\mathbf{V}X$ if $\Box X \in Sub(\Gamma \cup \Delta)$. If we restrict W^C of Definition 1.23 to be the set of all saturated sequents consisting of formulas from $Sat(\Gamma \cup \Delta)$, then since $Sat(\Gamma \cup \Delta)$ is finite the resulting canonical model, in each case, is finite also. Hence if $\Gamma \Rightarrow \Delta$ is underivable we can construct a finite counter-model for $\bigwedge \Gamma \to \bigvee \Delta$. $\square$

3.3 Embedding Intuitionistic Epistemic Logics into Modal Logics of Provability and Verification

For $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL},\}$ and $\mathcal{L}_\Box \in \{\mathsf{S4V}^-, \mathsf{S4V},\}$, respectively, we will show that

$$\mathcal{L} \vdash F \Leftrightarrow \mathcal{L}_\Box \vdash tr(F)$$

where for each F of the appropriate $\mathcal{L}$ $tr(F)$ is the result of prefixing each sub-formula of F with $\Box$.

Lemma 1.29.

$$\mathcal{L} \vdash F \Rightarrow \mathcal{L}_\Box \vdash tr(F).$$

Proof. By induction on derivations in $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL},\}$.

The case of the propositional intuitionistic axioms IE0 and modus ponens is the embedding of IPC into $\mathsf{S4}$. The cases for each of Axioms IE1 to IE3 are all quite similar involving repeated use of necessitation and distribution; as an example let us check $\mathsf{S4V}^- \vdash \Box(\Box A \to \Box\mathbf{V}\Box A) = tr(A \to \mathbf{K}A)$. To keep notation simple we assume that A is an atomic formula.

1. $\Box\Box A \to \Box\Box\Box A$, $\mathsf{S4}$ Axiom A0;

2. $\Box\Box A \to \mathbf{V}\Box A$, Axiom A2;

3. $\Box\Box\Box A \to \Box\mathbf{V}\Box A$, from 2 $\Box$-necessitation and distribution;

4. $\Box\Box A \to \Box\mathbf{V}\Box A$, from 1,3 by propositional reasoning;

5. $\Box A \to \Box\Box A$, $\mathsf{S4}$ Axiom A0;

6. $\Box A \to \Box\mathbf{V}\Box A$, from 4,5 by propositional reasoning;

7. $\Box(\Box A \to \Box\mathbf{V}\Box A)$, from 6 by necessitation.

$\Box$

To show the converse, consider an $\mathcal{L}$-model $\mathcal{M} = \langle W, R, E, \Vdash \rangle$. We can consider $\mathcal{M}$ to be an $\mathcal{L}_\square$-model $\mathcal{M}' = \langle W, R_\square, R_{\mathbf{V}}, \Vdash' \rangle$ by taking $R = R_\square$, $E = R_{\mathbf{V}}$ and treating $\Vdash$ as a classical forcing $\Vdash'$.

Clearly for all $\mathcal{L}_\square$ $R_\square$ is transitive and reflexive and $R_{\mathbf{V}}$ yields $R_\square$, hence all axioms of $\mathsf{S4V}^-$ hold in $\mathcal{M}'$. Where $\mathcal{M}$ is an IEL-model it is additionally the case that $R_{\mathbf{V}}$ is weakly serial, hence all axioms of $\mathsf{S4V}$ hold in $\mathcal{M}'$.

Lemma 1.30. *For each formula F of $\mathcal{L}$ and each $u \in W$,*

$$\mathcal{M}', u \Vdash' tr(F) \Leftrightarrow \mathcal{M}, u \Vdash F$$

Proof. By induction on F.

Case 1 (F is atomic p). Assume $\mathcal{M}, u \Vdash p$, then for all v such that uRv $\mathcal{M}, v \Vdash p$, hence for all v such that $uR_\square v$ $v \Vdash p$, so $u \Vdash \square p$, i.e. $tr(p)$.

Conversely, assume $\mathcal{M}, u \nVdash p$, then $\mathcal{M}', u \nVdash' \square p$ since $R_\square$ is reflexive, hence $\mathcal{M}', u \nVdash' tr(p)$.

Case 2 (Boolean cases $F = A \wedge B$ and $F = A \vee B$ are standard).

Case 3 ($F = A \rightarrow B$). Assume $\mathcal{M}, u \Vdash A \rightarrow B$, hence for all v such that uRv either $\mathcal{M}, v \nVdash A$ or $v \Vdash B$. By the induction hypothesis $\mathcal{M}', v \nVdash' tr(A)$ or $\mathcal{M}', v, \Vdash' tr(B)$, hence $\mathcal{M}', u \Vdash' \square(tr(A) \rightarrow tr(B))$, and $\mathcal{M}', u \Vdash' tr(A \rightarrow B)$.

Conversely, assume $\mathcal{M}, u \nVdash A \rightarrow B$, hence there is a v such that uRv in which $\mathcal{M}, v \Vdash A$ and $v \nVdash B$. By the induction hypothesis $\mathcal{M}', v \Vdash' tr(A)$ and $\mathcal{M}', v, \nVdash' tr(B)$, hence $\mathcal{M}', v \nVdash' tr(A) \rightarrow tr(B)$. Since $R = R_\square$ $\mathcal{M}', u \nVdash' \square(tr(A) \rightarrow tr(B))$, hence $\mathcal{M}', u \nVdash' tr(A \rightarrow B)$.

Case 4 ($F = \mathbf{K}A$). Assume $\mathcal{M}, u \Vdash \mathbf{K}A$; for any u such that uRv and any w such that vEw

uEw holds by Condition IM2, hence $\mathcal{M}, w \Vdash A$. By the induction hypothesis $\mathcal{M}', w \Vdash' tr(A)$, hence $v \Vdash' \mathbf{V}tr(A)$ and $\mathcal{M}', u \Vdash' \Box\mathbf{V}tr(A)$, hence $\mathcal{M}', u \Vdash' tr(\mathbf{K}A)$

Conversely, assume $\mathcal{M}, u \nVdash \mathbf{K}A$ so there is a v such that uEv in which $v \nVdash A$. By induction hypothesis $\mathcal{M}', v \nVdash' tr(A)$. Since $E = R_{\mathbf{V}}$ $\mathcal{M}', u \nVdash' \mathbf{V}tr(A)$. Since $R_{\Box}$ is reflexive $\mathcal{M}', u \nVdash' \Box\mathbf{V}tr(A)$, hence $\mathcal{M}', u \nVdash' tr(\mathbf{K}A)$.

$\square$

Lemma 1.31.

$$\mathcal{L}_{\Box} \vdash tr(F) \;\Rightarrow\; \mathcal{L} \vdash F.$$

Proof. Assume $\mathcal{L} \nvdash F$. By $\mathcal{L}$-completeness, there is an $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}, \}$-model $\mathcal{M} = \langle W, R, E, \Vdash \rangle$ and a state $u \in W$ such that $u \nVdash F$. By Lemma 1.30, $u \nVdash' tr(F)$ in an $\mathcal{L}_{\Box} \in \{\mathsf{S4V}^-, \mathsf{S4V}, \}$-model $\mathcal{M}'$. By $\mathcal{L}_{\Box}$-soundness, $\mathcal{L}_{\Box} \nvdash tr(F)$. $\square$

Hence for each of IEL^- and IEL, their embedding into $\mathsf{S4V}^-$ and $\mathsf{S4V}$ respectively, is faithful. Lemma 1.29 and Lemma 1.31 yield:

Theorem 1.32 (Embedding). *The Gödel translation faithfully embeds each $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$ into each $\mathcal{L}_{\Box} \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$ respectively:*

$$\mathcal{L} \vdash F \Leftrightarrow \mathcal{L}_{\Box} \vdash tr(F).$$

4   Logics of explicit proofs and verification

Gödel [13] suggested that the modal logic $\mathsf{S4}$ be considered as a provability calculus. This was given a precise interpretation by Artemov, see [2, 3], who showed that explicit proofs in Peano Arithmetic, PA, was the model of provability which $\mathsf{S4}$ described. The explicit counter-part of

S4 is the Logic of Proofs LP in which each □ in S4 is replaced by a term denoting an explicit proof. Since intuitionistic logic embeds into S4 the intended BHK semantics for IPC as an implicit calculus of proofs is given an explicit formulation in LP, and hence an arithmetical semantics. Here we show that this arithmetical interpretation can be further extended to the Logic of Proofs augmented with a verification modality, LPV⁻ and LPV, providing S4V⁻ and S4V, and therefore IEL⁻ and IEL with an arithmetical semantics. Similarly to the foundational picture regarding the relation between IPC, S4 and LP (see [2]) we have that

$$\mathsf{IEL} \ \hookrightarrow \ \mathsf{S4V} \ \hookrightarrow \ \mathsf{LPV}^5$$

The basic system of explicit proofs and verifications LPV⁻ is defined thus:

Definition 1.33 (Explicit Language). The language of LPV⁻ consists of:

1. The language of classical propositional logic;

2. A verification operator **V**;

3. *Proof variables*, denoted by $x, y, x_1, x_2 \ldots$;

4. *Proof constants*, denoted by $a, b, c, c_1, c_2 \ldots$;

5. *Operations on proof terms*, building complex proof terms from simpler ones of three types:

   (a) Binary operation · called *application*;

   (b) Binary operation + called *plus*;

   (c) Unary operation ! called *proof checker*;

---

[5]Similar embeddings hold for IEL⁻, S4V⁻, and LPV⁻.

6. *Proof terms*: any proof variable or constant is a proof term; if t and s are proof terms so are $t \cdot s$, $t + s$ and $!t$.

7. *Formulas*: A propositional letter p is a formula; if A and B are formulas then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\mathbf{V}A$, $t{:}A$.

Formulas of the type $t{:}A$ are read as "t *is a proof* A".

Definition 1.34 (LPV^-). The list of axioms and rules of LPV^- consists of:

E0. Axioms of propositional classical logic.

E1. $t{:}(A \rightarrow B) \rightarrow (s{:}A \rightarrow (t \cdot s){:}B)$

E2. $t{:}A \rightarrow A$

E3. $t{:}A \rightarrow !t{:}t{:}A$

E4. $t{:}A \rightarrow (s + t){:}A$, $t{:}A \rightarrow (t + s){:}A$

E5. $\mathbf{V}(A \rightarrow B) \rightarrow (\mathbf{V}A \rightarrow \mathbf{V}B)$

E6. $t{:}A \rightarrow \mathbf{V}A$

R1. Modus Ponens

R2. Axiom Necessitation: $\dfrac{\vdash A}{\vdash c{:}A}$ where A is any of Axioms E0 to E6 and c is some proof constant.

Definition 1.35 (LPV). The system LPV is LPV^- with the additional axiom:

E7. $\neg t{:}\mathbf{V}\bot$

A *constant specification*, $\mathcal{CS}$, is a set $\{c_1{:}A_1, c_2{:}A_2 \ldots\}$ of formulas such that each A_i is an axiom from the list E0 to E6 above, and each c_i is a proof constant. This set is generated

by each use of the constant necessitation rule in an LPV^- proof. The axiom necessitation rule can be replaced with a 'ready made' constant specification which is added to LPV^- as a set of extra axioms. For such a $\mathcal{CS}$ let LPV^--$\mathcal{CS}$ mean LPV^- minus the axiom necessitation rule plus the members of $\mathcal{CS}$ as additional axioms.

A proof term, t, is called a *ground term* if it contains no proof variables, but is built only from proof constants and operations on those constants.

LPV^- and LPV are able to internalise their own proofs, that is if

$$A_1 \ldots A_n, y_1{:}B_1 \ldots y_n{:}B_n \vdash F$$

then for some term $p(x_1 \ldots x_n, y_1 \ldots y_n)$

$$x_1{:}A_1 \ldots x_n{:}A_n, y_1{:}B_1 \ldots y_n{:}B_n \vdash p(x_1 \ldots x_n, y_1 \ldots y_n){:}F,$$

see [2]. As a consequence LPV^- and LPV have the constructive necessitation rule: for some ground proof term t,

$$\frac{\vdash F}{\vdash t{:}F.}$$

This yields in turn:

Lemma 1.36 (**V** Necessitation)**.** **V***-Necessitation* $\dfrac{\vdash A}{\vdash \mathbf{V}A}$ *is derivable in* LPV^- *and* LPV.

Proof. Assume $\vdash A$, then by constructive necessitation $\vdash t{:}A$ for some ground proof term t, hence by Axiom E6 $\vdash \mathbf{V}A$. $\qquad\square$

Note that the Deduction Theorem holds for both LPV^- and LPV.

5 Arithmetical Interpretation of LPV⁻ and LPV

We give an arithmetical interpretation of LPV⁻ and LPV by specifying a translation of the formulas of LPV⁻ and LPV into the language of Peano Arithmetic, PA. We assume that a coding of the syntax of PA is given. n denotes a natural number and $\overline{n}$ the corresponding numeral. $\overline{\ulcorner F \urcorner}$ denotes the numeral of the Gödel number of a formula F. For readability we suppress the overline for numerals and corner quotes for the Gödel number of formulas, and trust that the appropriate number or numeral, as context requires, can be recovered.[6]

Definition 1.37 (Normal Proof Predicate). A *normal proof predicate* is a provably Δ formula $\mathsf{Prf}(x, y)$ such that for every arithmetical sentence F the following holds:

1. $\mathsf{PA} \vdash F \Leftrightarrow$ for some $n \in \omega, \mathsf{Prf}(n, F)$

2. A proof proves only a finite number of things; i.e. for every k the set $T(k) = \{l | \mathsf{Prf}(k, l)\}$ is finite.[7]

3. Proofs can be joined into longer proofs; i.e. for any k and l there is an n s.t. $T(k) \cup T(l) \subseteq T(n)$.

Example 1.38. An example of a numerical relation that satisfies the definition of $\mathsf{Prf}(x, y)$ is the standard proof predicate $\mathsf{Proof}(x, y)$ the meaning of which is

"x is the Gödel number of a derivation of a formula with the Gödel number y".

Theorem 1.39. *For every normal proof predicate* $\mathsf{Prf}(x, y)$ *there exist recursive functions* $\mathsf{m}(x, y)$, $\mathsf{a}(x, y)$ *and* $\mathsf{c}(x)$ *such that for any arithmetical formulas F and G and all natural numbers k and n the following formulas hold:*

---

[6]E.g. by techniques found in [6] and [9].
[7]I.e. $T(k)$ is the set of theorems proved by the proof k.

1. $(\mathsf{Prf}(k, F \to G) \land \mathsf{Prf}(n, F)) \to \mathsf{Prf}(\mathsf{m}(k, n), G)$

2. $\mathsf{Prf}(k, F) \to \mathsf{Prf}(\mathsf{a}(k, n), F), \quad \mathsf{Prf}(n, F) \to \mathsf{Prf}(\mathsf{a}(k, n), F)$

3. $\mathsf{Prf}(k, F) \to \mathsf{Prf}(\mathsf{c}(k), \mathsf{Prf}(k, F))$.

Proof. See [2]. $\square$

Definition 1.40 (Verification Predicate for LPV^-)**.** A *verification predicate* is a provably Σ formula $\mathsf{Ver}(x)$ satisfying the following properties, for arithmetical formulas F and G:

1. $\mathsf{PA} \vdash \mathsf{Ver}(F \to G) \to (\mathsf{Ver}(F) \to \mathsf{Ver}(G))$

2. For each n, $\mathsf{PA} \vdash \mathsf{Prf}(n, F) \to \mathsf{Ver}(F)$.

These are properties which a natural notion of verification satisfies.

Let $\mathsf{Bew}(x)$ be the standard provability predicate, and $\mathsf{Con}(\mathsf{PA})$ be the statement which expresses that PA is consistent, i.e. $\neg\mathsf{Bew}(\bot)$. $\neg\mathsf{Con}(\mathsf{PA})$ correspondingly is $\mathsf{Bew}(\bot)$.

Example 1.41. The following are examples of a verification predicate $\mathsf{Ver}(x)$:

1. "Provability in PA", i.e. $\mathsf{Ver}(x) = \mathsf{Bew}(x)$; for a formula F $\mathsf{Ver}(F)$ is $\exists x \mathsf{Prf}(x, F)$.

2. "Provability in $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$" i.e. $\mathsf{Ver}(x) = \mathsf{Bew}(\mathsf{Con}(\mathsf{PA}) \to x)$; one example of a formula for which $\mathsf{Ver}(x)$ holds in this sense is just the formula $\mathsf{Con}(\mathsf{PA})$. Such verification is capable of verifying propositions not provable in PA.

3. "Provability in $\mathsf{PA} + \neg\mathsf{Con}(\mathsf{PA})$" i.e. $\mathsf{Ver}(x) = \mathsf{Bew}(\neg\mathsf{Con}(\mathsf{PA}) \to x)$; an example of a verifiable formula which is not provable in PA, is the formula $\neg\mathsf{Con}(\mathsf{PA})$. Such verification is capable of verifying false propositions.

4. $\top$, i.e. $\mathsf{Ver}(x) = \top$; that is for any formula F $\mathsf{Ver}(F) = \top$, hence any F is verified.

Lemma 1.42. $\mathsf{PA} \vdash F \;\Rightarrow\; \mathsf{PA} \vdash \mathsf{Ver}(F)$.

Proof. Assume $\mathsf{PA} \vdash F$, then by Definition 1.37 there is an n such that $\mathsf{Prf}(n, F)$ is true, hence $\mathsf{PA} \vdash \mathsf{Prf}(n, F)$, and by Definition 1.40 part 2 $\mathsf{PA} \vdash \mathsf{Ver}(F)$. $\qquad\square$

We now define an interpretation of the language of LPV^- into the language of Peano Arithmetic. An arithmetical interpretation takes a formula of LPV^- and returns a formula of Peano Arithmetic; we show the soundness of such an interpretation, if F is valid in LPV^- then for any arithmetical interpretation * F^* is valid in PA.[8]

Definition 1.43 (Arithmetical Interpretation for LPV^-)**.** An *arithmetical interpretation* for LPV^- has the following items:

- A normal proof predicate, Prf, with the functions $\mathsf{m}(x, y)$, $\mathsf{a}(x, y)$ and $\mathsf{c}(x)$ as in Definition 1.37 and Theorem 1.39;

- A verification predicate, Ver, satisfying the conditions in Definition 1.40;

- An evaluation of propositional letters by sentences of PA;

- An evaluation of proof variables and constants by natural numbers.

An arithmetical interpretation is given inductively by the following clauses:

$$
\begin{aligned}
(p)^* &= p \text{ an atomic sentence of } \mathsf{PA} & (t \cdot s)^* &= \mathsf{m}(t^*, s^*) \\
\perp^* &= \perp & (t + s)^* &= \mathsf{a}(t^*, s^*) \\
(A \wedge B)^* &= A^* \wedge B^* & (!t)^* &= \mathsf{c}(t^*) \\
(A \vee B)^* &= A^* \vee B^* & (t{:}F)^* &= \mathsf{Prf}(t^*, F^*) \\
(A \to B)^* &= A^* \to B^* & (\mathbf{V} F)^* &= \mathsf{Ver}(F^*)
\end{aligned}
$$

---

[8]A corresponding completeness theorem is left for future work, as is the development of a system with explicit verification terms, in addition to proof terms, realising the verification modality of $\mathsf{S4V}^-$ or $\mathsf{S4V}$.

Let X be a set of LPV^- formulas, then X^* is the set of all F^*'s such that $F \in X$. For a constant specification, $\mathcal{CS}$, a *$\mathcal{CS}$-interpretation* is an interpretation * such that all formulas from $\mathcal{CS}^*$ are true. An LPV^- formula is *valid* if F^* is true under all interpretations *. F is *provably valid* if $\mathsf{PA} \vdash F^*$ under all interpretations *. Similarly, F is *valid under constant specification* $\mathcal{CS}$ if F^* is true under all $\mathcal{CS}$-interpretations, and F is *provably valid under constant specification* $\mathcal{CS}$ if $\mathsf{PA} \vdash F^*$ under any $\mathcal{CS}$-interpretation *.

Theorem 1.44 (Arithmetical Soundness of LPV^-). *For any $\mathcal{CS}$-interpretation * with a verification predicate as in Definition 1.40 any LPV^--$\mathcal{CS}$ theorem, F, is provably valid under constant specification $\mathcal{CS}$:*

$$\mathsf{LPV}^-\text{-}\mathcal{CS} \vdash F \;\Rightarrow\; \mathsf{PA} \vdash F^*.$$

Proof. By induction on derivations in LPV^-. The cases of the LP axioms are proved in [2].

Case 1 $(\mathbf{V}(A \to B) \to (\mathbf{V}A \to \mathbf{V}B))$**.**

$$[\mathbf{V}(A \to B) \to (\mathbf{V}A \to \mathbf{V}B)]^* \equiv \mathsf{Ver}(F \to G) \to (\mathsf{Ver}(F) \to \mathsf{Ver}(G)).$$

But $\mathsf{PA} \vdash \mathsf{Ver}(F \to G) \to (\mathsf{Ver}(F) \to \mathsf{Ver}(G))$ by Definition 1.40.

Case 2 $(t{:}F \to \mathbf{V}F)$**.**

$$[t{:}F \to \mathbf{V}F]^* \equiv \mathsf{Prf}(t^*, F^*) \to \mathsf{Ver}(F^*).$$

Likewise $\mathsf{PA} \vdash \mathsf{Prf}(t^*, F^*) \to \mathsf{Ver}(F^*)$ holds by Definition 2.33.

$\square$

This arithmetical interpretation can be extended to LPV. Everything is as above except to Definition 1.40 we add the following item:

Definition 1.45 (Verification Predicate for LPV)**.**

    3. for any n, $\mathsf{PA} \vdash \neg\mathsf{Prf}(n, \mathsf{Ver}(\bot))$.

    1–3 of Example 1.41 remain examples of a verification predicate which also satisfies the above consistency property. In each case respectively $\mathsf{Ver}(\bot)$ is

    1. $\mathsf{Bew}(\bot)$

    2. $\mathsf{Bew}(\neg\mathsf{Bew}(\bot) \to \bot)$, i.e. $\mathsf{Bew}(\neg\mathsf{Con}(\mathsf{PA}))$

    3. $\mathsf{Bew}(\neg\neg\mathsf{Bew}(\bot) \to \bot)$, i.e. $\mathsf{Bew}(\mathsf{Con}(\mathsf{PA}))$.

All of these are false in the standard model of PA, and hence not provable in PA, hence for each n $\mathsf{PA} \vdash \neg\mathsf{Prf}(n, \mathsf{Ver}(\bot))$.

    4. $\mathsf{Ver}(\bot) = \top$, is not an example of a verification predicate for LPV in the sense of Definition 1.45: $\mathsf{Ver}(\bot)$ would be provable in PA, and hence there would be an n for which $\mathsf{PA} \vdash \mathsf{Prf}(n, \mathsf{Ver}(\bot))$ holds, which contradicts Definition 1.45.

Theorem 1.46 (Arithmetical Soundness of LPV)**.** *For any $\mathcal{CS}$-interpretation * with a verification predicate as in Definition 1.45, if F is an LPV-$\mathcal{CS}$ theorem then it is provably valid under constant specification $\mathcal{CS}$:*

$$\mathsf{LPV}\text{-}\mathcal{CS} \vdash F \;\Rightarrow\; \mathsf{PA} \vdash F^*.$$

Proof. Add to the proof of Theorem 1.44 the following case:

Case 3 ($\neg t{:}\mathbf{V}\bot$)**.**

$$[\neg t{:}\mathbf{V}\bot]^* \equiv \neg\mathsf{Prf}(n, \mathsf{Ver}(\bot)).$$

$\mathsf{PA} \vdash \neg\mathsf{Prf}(n, \mathsf{Ver}(\bot))$ holds by Definition 1.45.

$\square$

6   Realisation of S4V⁻ and S4V

Here we show that each $\square$ in an S4V⁻ or S4V theorem can be replaced with a proof term so that the result is a theorem of LPV⁻ or LPV, and hence that IEL⁻ and IEL each have a proof interpretation. The converse, that for each LPV⁻ or LPV theorem if all the proof terms are replaced with $\square$'s the result is a theorem of S4V⁻ or S4V also holds.

Definition 1.47 (Forgetful Projection). The *forgetful projection*, F^0 of an LPV⁻ or LPV formula is the result of replacing each proof term in F with a $\square$.

Theorem 1.48. LPV⁻, LPV $\vdash F \;\Rightarrow\;$ S4V⁻, S4V $\vdash F^0$ *respectively.*

Proof. By induction on S4V⁻ derivations. The forgetful projections of Axioms E1 to E4 and E6 are $\square(A \to B) \to (\square A \to \square B)$, $\square A \to A$, $\square A \to \square\square A$, $\square A \to \square A$ and $\square A \to \mathbf{V}A$ respectively, which are all provable in S4V⁻. The forgetful projection of $\neg t{:}\mathbf{V}\bot$ is $\neg\square\mathbf{V}\bot$ which is provable in S4V. The rules are obvious. $\qquad\qquad\square$

Definition 1.49 (Realisation). A *realisation*, F^r, of an S4V⁻ or S4V formula F is the result of substituting a proof term for each $\square$ in F, such that if S4V⁻, S4V $\vdash F$ then LPV⁻, LPV $\vdash F^r$ respectively.

Definition 1.50 (Polarity of Formulas). Occurrences of $\square$ in F in $G \to F$, $F \wedge G$, $G \wedge F$, $F \vee G$, $G \vee F$, $\square G$ and $\Gamma \Rightarrow \Delta, F$ have the same polarity as the occurrence of $\square$ in F.

Occurrences of $\square$ in F from $F \to G$, $\neg F$ and $F, \Gamma \Rightarrow \Delta$ have the polarity opposite to that of the occurrence of $\square$ in F.

Definition 1.51 (Normal Realisation). A realisation r is called *normal* if all negative occurrences of $\square$ are realised by proof variables.

The informal reading of the S4 provability modality $\Box$ is existential, $\Box F$ means 'there is a proof of F' (as opposed to the Kripke semantic reading which is universal, i.e. 'F holds in all accessible states'), normal realisations are the ones which capture this existential meaning, see [2].

The realisation theorem, Theorem 1.52, shows that if a formula F is a theorem of S4V$^-$ then there is a substitution of proof terms for every $\Box$ occurring in F such that the result is a theorem of LPV$^-$. This means that every $\Box$ in S4V$^-$ can be thought of as standing for a (possibly complex) proof term in LPV$^-$, and hence, by Theorem 1.44, implicitly represents a specific proof in PA. The proof of the realisation theorem consists in a procedure by which such a proof term can be built, see [1, 2, 7, 10, 11, 12]. Given a (cut-free) proof in S4V$^-$g we show how to assign proof terms to each of the $\Box$'s occurring in the S4V$^-$g proof so that each sequent in the proof corresponds to a formula provable in LPV$^-$; this is done by constructing a Hilbert-style LPV$^-$ proof for the formula corresponding to each sequent, so as to yield the desired realisation.

Occurrences of $\Box$ in an S4V$^-$g derivation can be divided up into *families* of related occurrences. Occurrences of $\Box$ are related if they occur in related formulas of premises and conclusions of rules. A family of related occurrences is given by the transitive closure of such a relation. A family is called *essential* if it contains at least one occurrence of $\Box$ which is introduced by the ($\Rightarrow \Box$) rule. A family is called *positive (respectively negative)* if it consists of positive (respectively negative) occurrences of $\Box$. It is important to note that the rules of S4V$^-$g preserve the polarities of $\Box$. Any $\Box$ introduced by ($\Rightarrow \Box$) is positive, while $\Box$'s introduced by ($\Box \Rightarrow$) and the interaction rule are negative.

Theorem 1.52 ($\mathsf{S4V}^-$ Realisation). *If $\mathsf{S4V}^- \vdash F$ then $\mathsf{LPV}^- \vdash F^r$ for some normal realisation r.*

Proof. If $\mathsf{S4V}^-\mathsf{g} \vdash F$ then there exists a cut-free sequent proof, $\mathcal{S}$, of the sequent $\Rightarrow F$. The realisation procedure described below (following [1, 2]) describes how to construct a normal realisation r for any sequent in $\mathcal{S}$.

Step 1. In every negative family and non-essential positive family replace each occurrence of $\Box B$ by $x{:}B$ for a fresh proof variable x.

Step 2. Pick an essential family, f, and enumerate all of the occurrences of the rule $(\Rightarrow \Box)$ which introduce $\Box$'s in this family. Let n_f be the number of such introductions. Replace all $\Box$'s of family f by the proof term $v_1 + \ldots + v_{n_f}$ where v_i does not already appear as the result of a realisation. Each v_i is called a *provisional* variable which will later be replaced with a proof term.

After this step has been completed for all families of $\Box$ there are no $\Box$'s left in $\mathcal{S}$.

Step 3. This proceeds by induction on the depth of a node in $\mathcal{S}$. For each sequent in $\mathcal{S}$ we show how to construct an LPV^- formula, F^r, corresponding to that sequent, such that $\mathsf{LPV}^- \vdash F^r$.

The realisation of a sequent $\mathcal{G} = \Gamma \Rightarrow \Delta$ is an LPV^- formula, $\mathcal{G}^r$, of the following form:

$$A_1^r \wedge \ldots \wedge A_n^r \to B_1^r \vee \ldots \vee B_m^r$$

The A^r's and B^r's denote realisations already performed. Let Γ^r, Θ^r stand for conjunctions of formulas and Δ^r for disjunctions of formulas; Γ^r prefixed with a $\mathbf{V}$ stands for conjunctions of $\mathbf{V}$'ed formulas, i.e. $\mathbf{V}\Gamma_n^r = \mathbf{V}A_1^r \wedge \ldots \wedge \mathbf{V}A_n^r$. Similarly $\vec{x}{:}\Theta_p^r$ stands for $x_1{:}C_1^r \wedge \ldots \wedge x_p{:}C_p^r$.

The cases realising the rules involving the propositional connectives and $\Box$ are shown in $[2]^9$ (including how to replace provisional variables with terms). Let us check the rules involving $\mathbf{V}$.

Case 1 (Sequent $\mathcal{G}$ is the conclusion of a $(\Rightarrow \mathbf{V})$ rule: $\Box\Theta, \mathbf{V}\Gamma \Rightarrow \mathbf{V}X$).

$$\mathcal{G}^r = (\vec{x}{:}\Theta_p^r \wedge \mathbf{V}\Gamma_n^r) \to \mathbf{V}X^r.$$

Now $\mathsf{LPV}^- \vdash ((\vec{x}{:}\Theta_p^r \wedge \Gamma_n^r) \to X^r) \Rightarrow \mathsf{LPV}^- \vdash ((\vec{x}{:}\Theta_p^r \wedge \mathbf{V}\Gamma_n^r) \to \mathbf{V}X^r)$.

1. $(\vec{x}{:}\vec{\Theta}_p^r \wedge \Gamma_n^r) \to X^r$, assumption;

2. $\mathbf{V}(\vec{x}{:}\vec{\Theta}_p^r \wedge \Gamma_n^r) \to \mathbf{V}X^r$, $\mathbf{V}$-Necessitation and distribution;

3. $(\mathbf{V}\vec{x}{:}\vec{\Theta}_p^r \wedge \mathbf{V}\Gamma_n^r) \to \mathbf{V}X^r$, from 2 by $\mathbf{V}(X \wedge Y) \leftrightarrow (\mathbf{V}X \wedge \mathbf{V}Y)$;

4. $(\mathbf{V}\vec{x}{:}\vec{\Theta}_p^r) \to (\mathbf{V}\Gamma_n^r \to \mathbf{V}X^r)$, from 3 by propositional reasoning;

5. $!\vec{x}{:}\vec{x}{:}\vec{\Theta}_p^r \to \mathbf{V}\vec{x}{:}\vec{\Theta}_p^r$, Axiom E6;

6. $!\vec{x}{:}\vec{x}{:}\vec{\Theta}_p^r \to (\mathbf{V}\Gamma_n^r \to \mathbf{V}X^r)$, from 4 and 5;

7. $\vec{x}{:}\vec{\Theta}_p^r \to !\vec{x}{:}\vec{x}{:}\vec{\Theta}_p^r$, axiom Axiom E3;

8. $\vec{x}{:}\vec{\Theta}_p^r \to (\mathbf{V}\Gamma_n^r \to \mathbf{V}X^r)$ from 6 and 7 by propositional reasoning;

9. $(\vec{x}{:}\vec{\Theta}_p^r \wedge \mathbf{V}\Gamma_n^r) \to \mathbf{V}X^r$, from 8 by propositional reasoning;

By the induction hypothesis the realisation of the premise of the rule, $(\vec{x}{:}\Theta_p^r \wedge \Gamma_n^r) \to X^r$, is provable in LPV^-, and hence:

$$\mathsf{LPV}^- \vdash (\vec{x}{:}\Theta_p^r \wedge \mathbf{V}\Gamma_n^r) \to \mathbf{V}X^r.$$

---

[9]The procedure described in [2] gives an exponential increase in the size of the derivation of the desired F^r. [7] describes a modification of the procedure which gives only a polynomial increase.

Case 2 (Sequent $\mathcal{G}$ is the conclusion of a $(\mathbf{V}/\square \Rightarrow)$ rule: $\Gamma, \square X \Rightarrow \Delta$).

$$\mathcal{G}^r = (\Gamma_n^r \wedge x{:}X^r) \to \Delta_m^r.$$

Since $x{:}A \to \mathbf{V}A$ is provable in LPV^- we have that

$$\mathsf{LPV}^- \vdash ((\Gamma_n^r \wedge \mathbf{V}X^r) \to \Delta_m^r) \Rightarrow \mathsf{LPV}^- \vdash ((\Gamma_n^r \wedge x{:}X^r) \to \Delta_m^r).$$

1. $((\Gamma_n^r \wedge \mathbf{V}X^r) \to \Delta_m^r)$, Assumption;

2. $\mathbf{V}X^r \to (\Gamma_n^r \to \Delta_m^r)$, from 1 propositional reasoning;

3. $x{:}X^r \to \mathbf{V}X^r$, Axiom E6;

4. $x{:}X^r \to (\Gamma_n^r \to \Delta_m^r)$, from 2 and 3;

5. $((\Gamma_n^r \wedge x{:}X^r) \to \Delta_m^r)$, from 4;

By the induction hypothesis the realisation of the formula corresponding to the premise of the rule, $(\Gamma_n^r \wedge \mathbf{V}X^r) \to \Delta_m^r$, is provable, and hence:

$$\mathsf{LPV}^- \vdash (\Gamma_n^r \wedge x{:}X^r) \to \Delta_m^r.$$

Step 4. After applying the above three steps each $\mathcal{G} \in \mathcal{S}$ has been translated into the language of LPV^-, and been shown to be derivable in LPV^-. Hence for the formula corresponding to the root sequent, $\Rightarrow F$, we have that

$$\mathsf{LPV}^- \vdash \top \to F^r.$$

Since $\mathsf{LPV}^- \vdash \top$

$$\mathsf{LPV}^- \vdash F^r.$$

Hence if $\mathsf{S4V}^- \vdash F$ there is a normal realisation r such that $\mathsf{LPV}^- \vdash F^r$.

$\square$

Theorem 1.53 (S4V Realisation). *If* S4V $\vdash F$ *then* LPV $\vdash F^r$ *for some normal realisation* r.

Proof. We simply add the following case to Step 3 of Theorem 1.52. The rest is the same.

Case 3 (Sequent $\mathcal{G}$ is the conclusion of the Weak Inconsistency Elimination: $\Gamma \Rightarrow$)**.**

$$\mathcal{G}^r = \Gamma^r_n \to \bot.$$

LPV $\vdash \Gamma^r_n \to x{:}\mathbf{V}\bot \Rightarrow$ LPV $\vdash \Gamma^r_n \to \bot$, since LPV $\vdash x{:}\mathbf{V}\bot \to \bot$, hence by the induction hypothesis the realisation of the premise of the rule, $\Gamma^r_n \to x{:}\mathbf{V}\bot$, is provable in LPV, and hence:

$$\text{LPV} \vdash \Gamma^r_n \to \bot.$$

$\square$

We are finally in a position to show that the systems of intuitionistic epistemic logic, IEL^- and IEL, do indeed have an arithmetical interpretation.

Definition 1.54. A formula of IEL^- or IEL is called *proof realisable* if $(tr(F))^r$ is LPV^-, respectively LPV, valid under some normal realisation r.

It follows that IEL^- and IEL are sound with respect to proof realisability.

Theorem 1.55. *If* $\mathsf{IEL}^-, \mathsf{IEL} \vdash F$ *then* F *is proof realisable.*

Proof. By Theorem 1.32 if $\mathsf{IEL}^-, \mathsf{IEL} \vdash F$ then $\mathsf{S4V}^-, \mathsf{S4V} \vdash tr(F)$, respectively, and by Theorem 1.52 and Theorem 1.53 if $\mathsf{S4V}^-, \mathsf{S4V} \vdash tr(F)$ then $\mathsf{LPV}^-, \mathsf{LPV} \vdash (tr(F))^r$ respectively.

$\square$

By Theorems 1.44 and 1.46 LPV^- and LPV are sound with respect to their arithmetical interpretation, and hence by Theorem 1.55 so are IEL^- and IEL.

7 Conclusion

Intuitionistic epistemic logic has an arithmetical interpretation, hence an interpretation in keeping with its intended BHK reading. Explicit proofs in Peano Arithmetic is the provability model for the BHK interpretation of intuitionistic logic, and the notion of verification which intuitionistic epistemic logic adds can also be interpreted within this model. Naturally verification in Peano Arithmetic, as outlined above, is not the only interpretation of verification for which the principles of intuitionistic epistemic logic are valid. IEL$^-$ and IEL may be interpreted as logics of the interaction between complete (conclusive) and incomplete but adequate evidence, e.g. mathematical proof vs. probabilistic confirmation, or justification by observation vs. justification by testimony, see for instance the examples in [5, section 2.3.2]. The question about exact interpretations for other intuitive readings of these logics is left for further investigation.

References

[1] Sergei Artemov. *Operational Modal Logic*. Tech. rep. MSI 95-29. Cornell University, 1995 (cit. on pp. 7, 37, 38).

[2] Sergei Artemov. "Explicit Provability and Constructive Semantics." In: *Bulletin of Symbolic Logic* 7.1 (2001), pp. 1–36 (cit. on pp. 7, 27, 28, 30, 32, 34, 37–39).

[3] Sergei Artemov. "The Logic of Justification." In: *Review of Symbolic Logic* (2008) (cit. on p. 27).

[4] Sergei Artemov and Lev D. Beklemishev. "Provability Logic." In: *Handbook of Philosophical Logic*. Vol. 13: *Handbook of Philosophical Logic*. Ed. by D. Gabbay and F. Guenthner. 2nd ed. Springer, 2005, pp. 189–360 (cit. on p. 7).

[5] Sergei Artemov and Tudor Protopopescu. "Intuitionistic Epistemic Logic." In: *Review of Symbolic Logic* (2016). DOI: `10.1017/S1755020315000374` (cit. on pp. 7, 10, 13, 42).

[6] George Boolos. *The Logic of Provability*. Cambridge University Press, 1993 (cit. on pp. 7, 31).

[7] Vladimir N. Brezhnev and R. Kuznets. "Making Knowledge Explicit: How Hard It Is." In: *Theoretical Computer Science* 357.1 (July 2006), pp. 23–34. DOI: `10.1016/j.tcs.2006.03.010` (cit. on pp. 37, 39).

[8] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Clarendon Press, Oxford, 1997 (cit. on p. 11).

[9] Solomon Feferman. "Arithmetization of Metamathematics in a General Setting." In: *Fundamenta Mathematicae* 49.1 (1960) (cit. on p. 31).

[10] Melvin Fitting. "The Logic of Proofs, Semantically." In: *Annals of Pure and Applied Logic* 132 (2005), pp. 1–25 (cit. on p. 37).

[11] Melvin Fitting. "Modal Logics, Justification Logics and Realization." 2015 (cit. on p. 37).

[12] Melvin Fitting. "Quasi-Realization." In: (2015) (cit. on p. 37).

[13] Kurt Gödel. "An Interpretation of the Intuitionistic Propositional Calculus." In: *Collected Works*. Ed. by S Feferman et al. Vol. 1. Oxford University Press, 1933, pp. 301–303 (cit. on pp. 7, 11, 27).

[14]   Arend Heyting. "The Intuitionistic Foundations of Mathematics." In: *Philosophy of Mathematics, Selected Readings.* Ed. by H. Putnam and P. Benacerraf. 1st. Englewood Cliffs, NJ: Prentice-Hall, 1964, pp. 42–49 (cit. on p. 7).

[15]   Arend Heyting. *Intuitionism: an introduction.* North-Holland Amsterdam, 1971 (cit. on p. 7).

[16]   Yoichi Hirai. "An intuitionistic epistemic logic for sequential consistency on shared memory." In: *Logic for Programming, Artificial Intelligence, and Reasoning.* Springer. 2010, pp. 272–289 (cit. on p. 10).

[17]   A. Kolmogorov. "On the Interpretation of Intuitionistic Logic." In: *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* Ed. by P. Mancosu. New York: Oxford University Press, 1932. Chap. 25, pp. 328–334 (cit. on p. 7).

[18]   Roman Kuznets and Thomas Studer. "Weak Arithmetical Interpretations for the Logic of Proofs." In: *Logic Journal of the IGPL* (2016). DOI: `10.1093/jigpal/jzw002` (cit. on p. 8).

[19]   John Charles Chenoweth McKinsey and Alfred Tarski. "Some Theorems About the Sentential Calculi of Lewis and Heyting." English. In: *The Journal of Symbolic Logic* 13.1 (Mar. 1948), pp. 1–15 (cit. on p. 11).

[20]   Carlo Proietti. "Intuitionistic Epistemic Logic, Kripke Models and Fitch's Paradox." In: *Journal of Philosophical Logic* 41.5 (2012). 10.1007/s10992-011-9207-1, pp. 877–900 (cit. on p. 10).

[21]   Tudor Protopopescu. "Intuitionistic Epistemology and Modal Logics of Verification." In: Proceedings. Logics, Rationality and Interaction (LORI 2015). (Oct. 26–29, 2015). Ed. by Wiebe van der Hoek and Wesley Holliday. Lecture Notes in Computer Science 9394. Taipei: Springer, 2015, pp. 295–307. DOI: `10.1007/978-3-662-48561-3_24` (cit. on pp. 11, 13).

[22]   A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 2000 (cit. on pp. 11, 14).

[23]   Timothy Williamson. "On Intuitionistic Modal Epistemic Logic." In: *Journal of Philosophical Logic* 21.1 (1992), pp. 63–89 (cit. on p. 10).

2. Intuitionistic Verification and Modal Logics of Verification

1 Introduction

Intuitionistic epistemic logic is epistemic logic based on the constructive understanding of truth as formulated in the Brouwer-Heyting-Kolmogorov (BHK) semantics. Intuitionistically a propostion is true only if proved; the basic assumption made about intuitionistic knowledge, and belief, is that it is the product of verification. A verification is understood as a process which is sufficient to warrant knowledge but is not necessarily a proof. The basic logics of intuitionistic knowledge and belief are formulated in [7]. These systems are implicit logics of proof, verification, and their interaction; each system incorporating different assumptions about the nature of the intuitionistic epistemic operator. Our purpose here is to study intuitionistic, verification-based, knowledge and belief from a classical perspective. The classical modal language is more expressive, enabling us to make explicit assumptions which the intuitionistic epistemic language cannot express, and thereby gaining us a more nuanced understanding of verification-based knowledge and belief. We also gain a more nuanced view of the nature of the kind of verification which is supposed to be foundational for philosophical verificationism. Our treatment makes explict some basic assumptions of a verificationism

seeking to emulate the properties of intuitionistic verification as characterised by the logics of intuitionistic knowledge and belief.

The BHK semantics is acknowledged as the indended semantics for intuitionistic logic, cf. [12]. Through Gödel's embedding of intuitionistic logic into S4 [10, 23, 37], the realisation of S4 in the Logic of Proofs, LP, and the arithmetical interpretation, in Peano Arithemetic PA of LP [1, 2], the BHK semantics is given a precise meaning. Intuitionistic Epistemic Logic, IEL, extends intuitionistic propositional logic, IPC, by adding an epistemic modality $\mathbf{K}$ asserting a proposition is known (or believed) on the basis of verification. By extending S4 and LP with a verification modality to the systems S4V and LPV, and PA with a verification predicate, this understanding of the BHK semantics can be extended to intuitionistic verification:

$$
\begin{array}{ccccccc}
\mathsf{IPC} & \hookrightarrow & \mathsf{S4} & \hookrightarrow & \mathsf{LP} & \hookrightarrow & \mathsf{PA} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{IEL} & \hookrightarrow & \mathsf{S4V} & \hookrightarrow & \mathsf{LPV} & \hookrightarrow & \mathsf{PA + Ver}
\end{array}
$$

Intuitionistic truth is based essentially on proof in the sense that proof is truth-making, a proposition is intuitionistically true in virtue of its proof. The language of IEL introduces formulas of the form $\mathbf{K}A$ into the intuitionistic setting, the intended meaning of which is 'A is known on the basis of verification' or 'A is verified (and thereby known)'. This format suggests the possibility of also treating verification as truth-making in some more general constructive sense. So $\mathbf{K}A$, can be read in two ways:

it is verified that A holds intuitionistically, i.e. that A has a proof, not necessarily specified

in the process of verification

or

> *It is verified that A holds in some constructive, but not necessarily proof, sense.*

The first reading treats verification as verification of provability, i.e. verification that a truth-making justification exists. The second reading treats verification as itself a truth-making justification. These readings are clearly different, but the language supports both. A classical approach enables us to distinguish these readings and begin to study their differences and consequences.

A classical modal language, by making the proof element explicit, provides a setting in which the idea that a proposition is made true by a proof, or by a verification, can be distinguished and stated explicitly. It provides a means, then, for studying how the second understanding of verification fits within an intuitionistic context. The assumption of philosophical verificationists has been that verification, as a generalisation of the intuitionistic notion of proof, behaves like proof in its capacity as a truth-making justification, see e.g. [16, 17, 19, 29, 32, 33]. Accordingly it should be characterised by intuitionistic logic and the BHK semantics. IEL is an intuitionistically acceptable way of introducing the notion of verification into an intuitionistic setting, and as characterised by its principles verification has certain strong properties. Verification satisfies both positive and negative introspection, and it is monotonic with respect to intuitionistic truth, hence the knowledge and beliefs based on it are indefeasible. It might be natural to suppose then that verification in the second 'direct' sense has these properties. At the same time, being in an intuitionistic context, these properties depend on and interact with intuitionistic truth; these properties come about as a product of these relations. The classical framework shows that the principles formulating these relations,

so that 'direct' verification emulates the properties of intutionistic, IEL, verification, go beyond what is intuitionistically, BHK, acceptable.

2 Intuitionistic Epistemic Logic

According to the BHK semantics a proposition, A, is true only if there is a proof of it and false if the assumption that there is a proof of A yields a contradiction. This is extended to complex propositions by the following clauses:

- a proof of $A \wedge B$ consists in a proof of A and a proof of B;

- a proof of $A \vee B$ consists in giving either a proof of A or a proof of B;

- a proof of $A \rightarrow B$ consists in a construction which given a proof of A returns a proof of B;

- $\neg A$ is an abbreviation for $A \rightarrow \bot$, and $\bot$ is a proposition that has no proof.

The fundamental principle of verification-based intuitionistic knowledge, and belief, is that

$$A \rightarrow \mathbf{K}A \qquad \qquad \text{(Co-Reflection)}$$

is valid on a BHK reading. Intuitionistic truth is based on proof; since any proof is a verification, the intuitionistic truth of a proposition yields a verification and hence knowledge/belief.

By similar reasoning the converse principle,

$$\mathbf{K}A \rightarrow A, \qquad \qquad \text{(Reflection)}$$

is not valid on a BHK reading. A verification may warrant knowledge, but need not be, or yield a method for obtaining, a proof.[1] Co-reflection, along with the distributivity of **K** over implication $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$, forms the basic logic of intuitionistic belief, IEL$^-$.

Definition 2.1 (IEL$^-$)**.** The list of axioms and rules of IEL$^-$ consists of:

IA0.   Axioms of propositional intuitionistic logic;

IA1. $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$;

IA2. $A \to \mathbf{K}A$;

IR0.   Modus Ponens.

Intuitionistic beliefs are beliefs formed on the basis of verification, while allowing that such verification may yield a falsehood. In such a system a proposition like $\mathbf{K}A \wedge \neg A$ is consistent, reflecting that intuitionistic belief allows having a verification for a proposition which itself yields a contradiction, i.e. is false. Evidence confirming scientific theories that turn out to be false are examples of such false but justifiable beliefs.

The difference between intuitionistic knowledge and belief, as in the classical case, is that knowledge obeys the truth condition: falsehoods cannot be known, or only truths can be known. Classically these formulations of the truth condition are equivalent, and the reflection principle expresses them both. Intuitionistically these formulations are distinct. The former, formalisable by the reflection principle is too strong, but a weaker principle expressing the second formulation is intuitionistically valid. The intuitionistically acceptable formulation of

---

[1]For example, interpreting $\mathbf{K}A$ as a 'truncated' or 'squash' type of Intuitionistic Type Theory, [11, 38] yields the invalidity of reflection.

the truth condition is:

$$\mathbf{K}A \to \neg\neg A. \qquad \text{(Intuitionistic Reflection)}$$

adding this yields the basic intuitionistic logic of knowledge, IEL.[2]

Definition 2.2 (IEL)**.** IEL is the system IEL$^-$ with the additional axiom:

IA3. $\mathbf{K}A \to \neg\neg A.$

Intuitionistic reflection says that verification rules out the possibility of refutation. Classically, of course, this is equivalent to reflection; indeed, intuitionistic reflection expresses in an intuitionistically acceptable manner just what classical reflection does. Classical reflection says that knowledge yields truth, but classically truth does not depend on the existence of evidence, hence knowledge does not depend on the existence of such evidence. Classical reflection is consistent with having knowledge without any specific evidence, and this is what intuitionistic reflection asserts: *knowledge yields truth, but without any specific evidence for that truth.*

Definition 2.3 (Semantics for $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$)**.** Models for $\mathcal{L}$ are intuitionistic Kripke models, $\langle W, R, \Vdash \rangle$, with an additional accessibility relation E.

IEL$^-$: An IEL$^-$ model satisfies the following conditions on E, for states u, v, w

IM1. uEv yields uRv;

IM2. uRv and vEw yield uEw;

---

[2]Other acceptable formulations of the truth condition are $\neg A \to \neg \mathbf{K}A$, $\neg(\mathbf{K}A \wedge \neg A)$ $\neg\neg(\mathbf{K}A \to A)$ and $\neg\mathbf{K}\bot$. Adding these as axioms to IEL$^-$ yields equivalent systems, see [7].

IM3. $u \Vdash \mathbf{K}A$ iff $v \Vdash A$ for all v such that uEv.

IEL: An IEL model is an IEL$^-$ model with the additional condition on E that:

IM4. E is serial, for all u, there is a v such that uEv.

Following the standard picture of intuitionistic Kripke models as representing the development of the stock of propositions proved by an ideal mathematician, IEL$^-$ and IEL models can be thought of as modeling the states of information of an ideal researcher, who both proves and verifies. R can be read as the possibilities of proof, from a given state, and E the possibilities of verification.

IEL$^-$ and IEL are each sound and complete, satisfy monotonicity, have the disjunction property, and the rule of $\mathbf{K}$-necessitation is derivable, see [6].

For other formulations of an intuitionistic epistemic logic, though not necessarily from a BHK perspective, see [27, 34, 39]. All these endorse reflection and are arguably too classical in their view of knowledge as a result.

As intuitionistic modal logics IEL$^-$ and IEL (e.g. [9, 15, 41, 42]) are similar to Došen's [14] in that reflection fails and co-reflection holds, but his $\square$ is rather a simulation of classical logic inside intuitionistic logic rather than an epistemic modality.

Proposition 2.4. *For each $\mathcal{L} \in \{\text{IEL}^-, \text{IEL}\}$ $\mathcal{L}$ has the following properties:*

1. *The rule of $\mathbf{K}$-necessitation $\dfrac{\vdash A}{\vdash \mathbf{K}A}$ is derivable.*

2. *$\mathcal{L} \vdash \mathbf{K}A \to \mathbf{K}\mathbf{K}A$*

3. *$\mathcal{L} \vdash \neg\mathbf{K}A \to \mathbf{K}\neg\mathbf{K}A$*

4. For each model of $\mathcal{L}$ and a formula A, if $u \Vdash A$ and uRv then $v \Vdash A$.

5. $\mathcal{L}$ has the disjunction property.

6. Each $\mathcal{L}$ is sound and complete with respect to its class of models.

Proof. See [6]. □

3 Properties of Intuitionistic Verification

Intuitionistic verification has a number of properties which make sense within the first, provability, reading of verification noted above, Section 1. But, as mentioned, on a formal level this reading is not distinguishable from the second, truth-making, reading. The provabilty reading says that verification verifies the existence of a proof of a proposition, that it is intuitionistically true. The intuitionistic notion of truth has it that a proposition is true in virtue of its proof, and a verification shows that such a proof exists. The second reading has it, then, that the non-proof verification itself is what makes a proposition true, rather than a proof. Verification in this sense does not have an acceptable intuitionistic reading. To argue this we show that intuitionistic **K** has certain properties which a truth-making notion of verification does not. We do this via the Gödel translation, which allows us to model the idea that a proposition is intuitionistically true, $\Box A$, or truth-making verified, $\mathbf{V}A$, or intuitionistically verified, $\mathbf{V}\Box A$. We will see that truth-making verification does not have any of the properties characteristic of intuitionistic verification, and that adopting any of them goes beyond what an intuitionistic interpretation of verification allows.

3.1   Stability of Truth and Knowledge

Intuitionistic knowledge and belief are monotonic with respect to truth, which means that both are indefeasible, once $\mathbf{K}A$ is true it can never become false. This is due to the stability of intuitionistic truth, i.e. proof; once a proposition is proved it can never become 'unproved'. The stability of truth is encoded by the definition of $\Vdash$. This stability is extended to $\mathbf{K}$ by Condition IM2, and accounts for the indefeasibility of knowledge and belief, as well as positive and negative introspection. These are essential properties of intuitionistic knowledge and belief precisely because they are aspects of the intuitionistic notion of truth.

In a classical modal framework truth and knowledge are not stable by default, offering the flexibility to assume explicitly the stability of knowledge.[3]

3.2   Positive and Negative Introspection

Proposition 2.4 shows that for intuitionistic knowledge and belief positive and negative introspection, $\mathbf{K}A \rightarrow \mathbf{K}\mathbf{K}A$ and $\neg\mathbf{K}A \rightarrow \mathbf{K}\neg\mathbf{K}A$ are built-in. They are in fact just forms of the co-reflection principle. This means that verifications are checkable, $\mathbf{K}A \rightarrow \mathbf{K}\mathbf{K}A$, if A is verified then that is itself verified. Morover verifications are negatively verifiable, the impossibility of a verification is verifiable, $\neg\mathbf{K}A \rightarrow \mathbf{K}\neg\mathbf{K}A$.

These are very robust; within an intuitionistic context focused on mathematical truth and mathematical knowledge that both positive and negative introspection hold is plausible.[4] In other contexts they can seem implausibly strong; positive introspection in particular has come in for heavy criticism, e.g. [31, 40] amongst others, and critiques of positive introspection

---

[3]For the role of stability in a constructive resolution of the 'knowability paradox' see [5].

[4]See [26] for a defense of positive introspection. In formal epistemology both positive and negative introspection are standard, see e.g. [20]. See also [25].

would seem to apply with even more force to negative introspection. We may wish to model a view of verification-based knowledge which does not validate either, or only one.

As we will see the classical modal language allows us to separate these properties of knowledge explicitly from other properties; allowing us to retain the fundamental intuitionistic idea that proof yields knowledge, the intention of co-reflection, whilst not also automatically 'dragging in' positive and negative introspection. In the classical framework both kinds of introspection have to be assumed explicitly.

4   Modal Logics of Verification and Proof

The well-known Gödel translation yields a faithful embedding of the intuitionistic propositional calculus, IPC, into the classical modal logic S4 (see [10, 23, 30, 37]).

Following Gödel [23] we interpret the $\Box$ of S4 as provability; this reading has been made precise within the framework of the Logic of Proofs [2]. On this reading appending a $\Box$ to a proposition is a way of expressing in a classical language that it is constructively true. The translation takes a formula, A, of IPC and returns a formula of S4, $tr(A)$, according to the rule

$$box\ every\ subformula\ of\ A.$$

By extending S4 with a verification modality $\mathbf{V}$, the translation can be extended to each of the logics IEL⁻, IEL. We will define the systems S4V⁻, S4V and show that the Gödel translation yields a faithful embedding of each intuitionistic system into its classical modal companion. In this way we interpret intuitionistic truth in a setting where we can make explicit when (and if) a proposition is intuitionistically true, or verified, or some combination

of them.

Intuitionistic **K** represents verifications which are not necessarily proofs, which is why intuitionistic reflection can fail. Similarly, **V** represents a verification procedure which is not necessarily factive (unlike $\Box$, which represents proof). This is a realistic assumption given many, if not most, of our justifications are fallible, and hence so is the knowledge based on them.[5] The systems $\mathsf{S4V}^-$, $\mathsf{S4V}$ may be regarded as systems of proof and verification-based belief or fallible knowledge. $\mathbf{V}A \to A$ could be added to the systems in question to yield systems of verification-based infallible, i.e. factive, knowledge and proof. The embedding results below do not require reflection for **V**, nor would adding reflection alter them.

4.1   Modal Logics $\mathsf{S4V}^-$, $\mathsf{S4V}$

$\mathsf{S4V}^-$ is the basic logic of provability and verification.

Definition 2.5 ($\mathsf{S4V}^-$ Axioms)**.** The list of axioms and rules of $\mathsf{S4V}^-$ consists of

Axioms.

    A0.   Axioms of $\mathsf{S4}$;

    A1. $\mathbf{V}(A \to B) \to (\mathbf{V}A \to \mathbf{V}B)$;

    A2. $\Box A \to \mathbf{V}A$;

    R0.   Modus Ponens;

    R1. $\Box$-Necessitation.

---

[5]Fallibilism is a position which "...contemporary [mainstream] epistemologists almost universally agree in endorsing" [28]. See e.g. [25] for an opposing view.

$\mathsf{S4V}^-$ represents basic, not necessarily consistent, verification, the only requirement of which is that anything which is proved be regarded as verified.

Definition 2.6 ($\mathsf{S4V}$). $\mathsf{S4V}$ is $\mathsf{S4V}^-$ with the additional axiom:

  A3. $\neg\Box\mathbf{V}\bot$.

$\mathsf{S4V}$ represents consistent verification, which does not guarantee the truth of the proposition verified.

Proposition 2.7. *The rule of* $\mathbf{V}$*-Necessitation is derivable in* $\mathcal{L}_\Box$.

Proof. Assume $\vdash A$, by $\Box$-necessitation $\vdash \Box A$ follows, hence by Axiom A2 $\vdash \mathbf{V}A$. □

Definition 2.8 (Semantics for $\mathcal{L}_\Box \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$). Models for $\mathcal{L}_\Box$ are $\mathsf{S4}$ Kripke models, $\langle W, R_\Box, \Vdash \rangle$, with an additional accessibility relation $R_\mathbf{V}$.

$\mathsf{S4V}^-$: An $\mathsf{S4V}^-$-model satisfies the following conditions on $R_\mathbf{V}$, for states x, y, z

  M1. $xR_\mathbf{V}y$ yields $xR_\Box y$;

  M2. $x \Vdash \mathbf{V}A$ iff $y \Vdash A$ for all y such that $xR_\mathbf{V}y$.

$\mathsf{S4V}$: An $\mathsf{S4V}$-model is an $\mathsf{S4V}^-$-model with the additional condition on $R_\mathbf{V}$ that:

  M3. $R_\mathbf{V}$ is weakly serial, for all x there are y and z such that $xR_\Box y$ and $yR_\mathbf{V}z$.

Proposition 2.9. *The inclusions* $\mathsf{S4V}^- \subset \mathsf{S4V}$ *is strict.*

Proof. See [6, Theorem 3], the model there can be regarded, respectively, as an $\mathsf{S4V}^-$-model in which Axiom A3 is not valid. □

Theorem 2.10 ($\mathcal{L}_\square$ Soundness and Completeness). *For* $\mathcal{L}_\square \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$,

$$\mathcal{L}_\square \vdash A \Leftrightarrow \mathcal{L}_\square \Vdash A.$$

Proof. See [35, 36]. □

4.2 Embedding Intuitionistic Epistemic Logics into Modal Logics of Provability and Verification

For $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$ and $\mathcal{L}_\square \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$, respectively,

$$\mathcal{L} \vdash F \Leftrightarrow \mathcal{L}_\square \vdash tr(F)$$

where for each F of the appropriate $\mathcal{L}$ $tr(F)$ is the result of prefixing each sub-formula of F with $\square$.

Proof. see [36] □

5 Making Explicit Properties of Intuitionistic Knowledge

There are several assumptions about verification-based knowledge implicit in the intuitionistic epistemic framework; verification is only of provability; truth, hence knowledge and belief, is stable, and consequently both positive and negative introspection hold. These properties do not necessarily hold for the respective classical modal counterparts in $\mathcal{L}_\square$. If we wish to model a view of verification which has any one of these properties then we must assume each explicitly.

5.1   Truth-making Verification

Section 3 outlined two ways in which intuitionistic verification, **K**, can be understood. According to the first verification amounts to a kind of proof-checking – a verification of A is a verification that there is a proof of A. This reading is reflected by the Gödel translation of **K**A which is $\Box\mathbf{V}\Box A$. According to the second reading a non-proof justificatory procedure can suffice for the truth of the proposition. In accepting a proposition as known such evidence may be perfectly adequate, or the only kind practically available.

The modal framework can approximate this latter understanding by extending $\mathcal{L}_\Box$ with the additional principle

$$\mathbf{V}A \to \mathbf{V}\Box A \tag{P}$$

which states that a non-proof verification is sufficiently robust to guarantee the existence of a proof. This makes explicit the relationship between verification and proof that underlies the notion of verification as itself a truth-making process. We often accept informal arguments based on general theoretical reasons or clear examples in place of specific proofs when it is clear that such proofs can be obtained. For instance, we might justify that $\mathsf{IPC} \vdash \neg\neg(A \vee \neg A)$ by reasoning informally on the basis of the BHK interpretation, rather than exhibiting a derivation in IPC (see e.g. [18, Section 1.3] for examples).

Definition 2.11 ($\mathcal{L}_\Box$+ P)**.** $\mathcal{L}_\Box + \mathsf{P}$ is $\mathcal{L}_\Box \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$ with the additional axiom:

   A4. $\mathbf{V}A \to \mathbf{V}\Box A$.

Definition 2.12. A model for $\mathcal{L}_\Box + \mathsf{P}$ is an $\mathcal{L}_\Box$-model with the additional condition

   M4.   For states x, y, z in a model $xR_\mathbf{V}y$ and $yR_\Box z$ yield $xR_\mathbf{V}z$.

Proposition 2.13. *Let $\mathfrak{F} = \langle W, R_\square, R_{\mathbf{V}} \rangle$ be a frame. P holds at all states of a model based on $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies Condition M4.*

Proof. $\Leftarrow$: Assume $\mathfrak{F}$ satisfies Condition M4 and there is some state $a \in W$ s.t. $a \Vdash \mathbf{V}A$, in which case for all b s.t. $aR_{\mathbf{V}}b$ $b \Vdash A$. Assume further that $bR_\square c$ for an arbitrary c; by M4 $aR_{\mathbf{V}}c$, so $c \Vdash A$ also. Hence $a \Vdash \mathbf{V}\square A$.

$\Rightarrow$: By contrapositive. Assume $\mathfrak{F}$ does not satisfy Condition M4. Hence there are states $a, b, c \in W$ such that $aR_{\mathbf{V}}b$ and $bR_\square c$ but $\neg aR_{\mathbf{V}}c$. Define a valuation $V(p) = \{x \in W \mid aR_{\mathbf{V}}x\}$. In the resulting model $a \Vdash \mathbf{V}p$ but $c \nVdash p$, hence $b \nVdash \square p$ and so $a \nVdash \mathbf{V}\square p$. $\square$

Theorem 2.14 ($\mathcal{L}_\square + \mathsf{P}$ Soundness and Completeness)**.**

$$\mathcal{L}_\square + \mathsf{P} \Vdash A \Leftrightarrow \mathcal{L}_\square + \mathsf{P} \vdash A.$$

Proof. Soundness follows from Proposition 2.13. For completeness we verify that the $\mathcal{L}_\square + \mathsf{P}$ canonical model satisfies Condition M4.

Assume $\Gamma R_{\mathbf{V}}^c \Delta$ and $\Delta R_\square^c \Omega$, and that $\mathbf{V}X \in \Gamma$. By maximal consistency $\mathbf{V}X \to \mathbf{V}\square X \in \Gamma$, hence $\mathbf{V}\square X \in \Gamma$. Since $\Gamma R_{\mathbf{V}}^c \Delta$ it follows that $\square X \in \Delta$ and hence $X \in \Omega$. So $\mathbf{V}X \in \Gamma$ yields that $X \in \Omega$, i.e. $\Gamma R_{\mathbf{V}}^c \Omega$. $\square$

Given the equivalence of $\mathbf{V}A$ and $\mathbf{V}\square A$ in $\mathcal{L}_\square + \mathsf{P}$ we can simplify the translations of IEL formulas by substituting $\mathbf{V}A$ for $\mathbf{V}\square A$, hence for example $tr(A \to \mathbf{K}A) = \square(\square A \to \square \mathbf{V}A)$. With this observation and the embedding theorem it is clear that this modified translation holds in the respective systems $\mathcal{L}_\square + \mathsf{P}$.

5.2 Stability of Knowledge

Intuitionistic truth, hence intuitionistic $\mathbf{K}$, is stable, but $\mathbf{V}$ in $\mathcal{L}_\square$ is not.

Theorem 2.15. *Neither truth nor* **V** *are monotonic with respect to* $R_\Box$ *for any* $\mathcal{L}_\Box \in \{$S4V$^-$, S4V$\}$, *i.e. if* $xR_\Box y$ *then 1)* $x \Vdash A$ *does not necessarily yield* $y \Vdash A$ *and 2)* $x \Vdash \mathbf{V}A$ *does not necessarily yield* $y \Vdash \mathbf{V}A$.

Proof. Consider the S4V-model (hence S4V$^-$- and S4V-) $\mathcal{M}_2$:
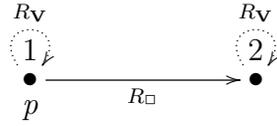
Figure 2.1: S4V-model $\mathcal{M}_1$

1) holds by definition of $\mathcal{M}_2$. For 2) since $1 \Vdash p$ then $1 \Vdash \mathbf{V}p$, and since $2 \nVdash p$ $\ 2 \nVdash \mathbf{V}p$. Hence $\mathbf{V}p$ does not hold at all the $R_\Box$-successors of 1 where $\mathbf{V}p$ holds. $\qquad\square$

To ensure **V** is monotonic we can adopt the principle

$$\mathbf{V}A \to \Box\mathbf{V}A, \tag{M}$$

which says that whenever we have a verification we can prove it to be correct, but such a proof guarantees the verification can never be defeated, so can never be lost. Adding M to a system in $\mathcal{L}_\Box$ yields a logic in which **V** is monotonic with respect to $R_\Box$.

Definition 2.16 ($\mathcal{L}_\Box + $ M). $\mathcal{L}_\Box + $ M is any system $\mathcal{L}_\Box$ with the additional axiom:

A5. $\quad \mathbf{V}A \to \Box\mathbf{V}A$.

Definition 2.17 (**V**-Monotonic Models). A **V**-Monotonic model is an $\mathcal{L}_\Box$-model with the additional condition:

M5. $\quad$ For states x, y, z in a model $xR_\Box y$ and $yR_\mathbf{V}z$ yield $xR_\mathbf{V}z$.

Proposition 2.18. *Let $\mathfrak{F} = \langle W, R_\square, R_\mathbf{V} \rangle$ be a frame. M holds at all states of a model based on $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies Condition M5.*

Proof. Virtually identical to the proof of Proposition 2.13.

$\square$

Theorem 2.19 (Monotonicity)**.** *If a model satisfies Condition M5 then $x \Vdash \mathbf{V}A$ yields that for any y such that $xR_\square y$ $y \Vdash \mathbf{V}A$ holds.*

Proof. Assume there is a state $a \in W$ such that $a \Vdash \mathbf{V}A$. Take an arbitrary b such that $aR_\square b$, and an arbitrary c such that $bR_\mathbf{V}c$; by M5 $aR_\mathbf{V}c$, hence $c \Vdash A$. Hence $b \Vdash \mathbf{V}A$, since c is arbitrary. $\square$

Theorem 2.20 ($\mathcal{L}_\square + \mathsf{M}$ Soundness and Completeness)**.**

$$\mathcal{L}_\square + \mathsf{M} \vdash A \Leftrightarrow \mathcal{L}_\square + \mathsf{M} \Vdash A.$$

Proof. Soundness follows from Proposition 2.18 and the soundness of $\mathcal{L}_\square$. The canonicity of Condition M5 is shown in an identical manner to that of Theorem 2.14.

$\square$

5.3 Positive Introspection and Negative Introspection

In $\mathcal{L}$ positive and negative introspection are instances of the 'proof yields verification' co-reflection principle IA2.

For positive introspection in $\mathcal{L}_\square$ the principle M suffices for the stability of positive verification statements $\mathbf{V}A$, hence yields positive introspection.

Theorem 2.21. $\mathcal{L}_\square + M \vdash \mathbf{V}A \to \mathbf{V}\mathbf{V}A$.

Proof. Argue in $\mathsf{S4V^-} + \mathsf{M}$:

1. $\Box \mathbf{V} A \to \mathbf{V}\mathbf{V} A$, Axiom A2;

2. $\mathbf{V} A \to \Box \mathbf{V} A$, Axiom A5;

3. $\mathbf{V} A \to \mathbf{V}\mathbf{V} A$, propositional reasoning.

$\Box$

We note in passing that positive introspection also holds in $\mathcal{L}_\Box + \mathsf{P}$.

M asserts only that positive verification statements, $\mathbf{V} A$, are stable. To ensure that negative verification statements, $\neg \mathbf{V} A$, are also stable we can adopt the principle

$$\neg \mathbf{V} A \to \Box \neg \mathbf{V} A. \tag{N}$$

which says that the failure of verification is provable, hence where a verification has not succeeded it can never succeed.

Definition 2.22 $(\mathcal{L}_\Box + \mathsf{N})$. $\mathcal{L}_\Box + \mathsf{N}$ is any system $\mathcal{L}_\Box$ with the additional axiom:

A6. $\neg \mathbf{V} A \to \Box \neg \mathbf{V} A$

$\mathcal{L}_\Box + \mathsf{N}$ yields negative introspection by an obvious modification of Theorem 2.21.

Definition 2.23. A model for $\mathcal{L}_\Box + \mathsf{N}$ is an $\mathcal{L}_\Box$ model with the additional condition

M6.   For states x, y, z in a model $x R_\Box y$ and $x R_{\mathbf{V}} z$ yield $y R_{\mathbf{V}} z$.

Proposition 2.24. *Let $\mathfrak{F} = \langle W, R_\Box, R_{\mathbf{V}} \rangle$ be a frame. N holds at all states of a model based on $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies Condition M6.*

Proof. $\Leftarrow$: Assume $\mathfrak{F}$ satisfies Condition M6 and there is some $a \in W$ such that $a \Vdash \neg \mathbf{V} A$

holds, hence there is a $c \in W$ such that $a R_{\mathbf{V}} c$ and $c \nVdash A$. Let b be an arbitrary state such

that $a R_{\square} b$, by M6 $b R_{\mathbf{V}} c$ holds, hence $b \nVdash \mathbf{V} A$, i.e. $b \Vdash \neg \mathbf{V} A$. Since b is arbitrary, $a \Vdash \square \neg \mathbf{V} A$.

$\Rightarrow$: Assume $\mathfrak{F}$ does not satisfy Condition M6, hence there is a model based on $\mathfrak{F}$ with

states a, b and c such that $a R_{\square} b$ and $a R_{\mathbf{V}} c$, but $\neg b R_{\mathbf{V}} c$. Define a valuation such that

$V(p) = \{x \in W | x \neq c\}$; hence $b \Vdash \mathbf{V} p$, hence $b \nVdash \neg \mathbf{V} p$, in which case $a \nVdash \square \neg \mathbf{V} p$. Since

$c \nVdash p$ then $a \nVdash \mathbf{V} p$, hence $a \Vdash \neg \mathbf{V} p$, so $a \nVdash \neg \mathbf{V} p \to \square \neg \mathbf{V} p$. $\qquad \square$

Theorem 2.25 ($\mathcal{L}_{\square} + \mathsf{N}$ Soundness and Completeness)**.**

$$\mathcal{L}_{\square} + \mathsf{N} \vdash A \Leftrightarrow \mathcal{L}_{\square} + \mathsf{N} \Vdash A$$

Proof. Soundness follows from Proposition 2.24 and the soundness of $\mathcal{L}_{\square}$. For completeness

we check that the $\mathcal{L}_{\square} + \mathsf{N}$ canonical model satisfies Condition M6.

Assume $\Gamma R^c_{\square} \Delta$ and $\Gamma R^c_{\mathbf{V}} \Omega$. Suppose $\mathbf{V} A \in \Delta$ but $A \notin \Omega$. Hence $\mathbf{V} A \notin \Gamma$, so $\neg \mathbf{V} A \in \Gamma$;

by maximal consistency $\neg \mathbf{V} A \to \square \neg \mathbf{V} A \in \Gamma$ so $\square \neg \mathbf{V} A \in \Gamma$, and so $\neg \mathbf{V} A \in \Delta$, which is a

contradiction. Hence if $\mathbf{V} A \in \Delta$ then $A \in \Omega$, i.e. $\Delta R^c_{\mathbf{V}} \Omega$.

$\qquad \square$

6 Logics of explicit proofs and verification

The BHK semantics is the intended interpretation of intuitionistic logic, and intuitionistic

reasoning more generally. BHK speaks of proofs but it does not specify any further what

counts as a proof. Gödel [23] gave part of an answer when he specified $\mathsf{S4}$ as the, classical,

provability calculus into which IPC can be embedded. In turn, however, what counted as

a proof which the S4 □ represents was left unspecified.[6] Artemov [2] showed that explicit proofs in Peano Arithmetic was the model of provability S4, and hence IPC, described, and thereby provided a precising of the notion of proof in the BHK interpretation. The link between S4 and PA is the Logic of Proofs, LP.

S4 can be regarded as an implicit logic of proofs LP is its explicit counter-part. In LP each □ in S4 is replaced by a term denoting an explicit proof; formulas of the form $\Box F$, which are read as 'F has a proof' are replaced with ones of the form $t{:}F$ and are read as 't is a proof F'. Implicit and explicit provability are connected by the realisation theorem, [1, 2]:

$$\text{if } \mathsf{S4} \vdash F \text{ then there is a proof term } t \text{ s.t. } \mathsf{LP} \vdash t{:}F$$

[36] and Paper 1 above showed that the arithmetical interpretation of LP can be extended to IEL⁻ and IEL. Since $\mathcal{L} \in \{\mathsf{IEL}^-, \mathsf{IEL}\}$ is an extension of IPC and $\mathcal{L}_\Box \in \{\mathsf{S4V}^-, \mathsf{S4V}\}$ is an extension of S4 into which $\mathcal{L}$ can be embedded, the same relation obtains between $\mathcal{L}$ and $\mathcal{L}_\Box$ as does between IPC and S4. $\mathcal{L}_\Box$ is a classical provability calculus extended with verifications, where verifications are not necessarily PA proofs. $\mathcal{L}_\Box$ hence allows for the verification of statements on grounds more general than strict PA proofs. Given the realisation of S4 into LP we know that each $\mathcal{L}_\Box$ □ stands for a specific explicit proof. Systems of explicit proofs augmented with (implicit) verifications, called $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$ (defined below), corresponding to $\mathcal{L}_\Box$ were constructed and shown to be realisations of $\mathcal{L}_\Box$. This suggests that the notion of verification underlying the epistemic modality of IEL⁻ and IEL is indeed BHK-compliant.

Our aim here is to consider the explicit versions of the verification principles M, P, N, and

---

[6]Gödel suggested a solution in [24], but this was not published until 1995.

determine whether the BHK interpretation of IEL can be extended to them. The principles are realisable, but their arithmetic reading shows the properties of verification they assert are stronger than those of intuitionistic verification. First we define the systems LPV$^-$ and LPV.

Definition 2.26 (Explicit Language)**.** The language of LPV$^-$ and LPV consists of:

1. The language of classical propositional logic;

2. A verification operator $\mathbf{V}$;

3. *Proof variables*, denoted by $x, y, x_1, x_2 \ldots$;

4. *Proof constants*, denoted by $a, b, c, c_1, c_2 \ldots$;

5. *Operations on proof terms*, building complex proof terms from simpler ones of three types:

    (a) Binary operation $\cdot$ called *application*;

    (b) Binary operation $+$ called *plus*;

    (c) Unary operation ! called *proof checker*;

6. *Proof terms*: any proof variable or constant is a proof term; if t and s are proof terms so are $t \cdot s$, $t + s$ and $!t$.

7. *Formulas*: A propositional letter p is a formula; if A and B are formulas then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\mathbf{V}A$, $t{:}A$.

Definition 2.27 (LPV$^-$)**.**

Axioms.

E0.   Axioms of propositional classical logic.

E1. $t{:}(A \to B) \to (s{:}A \to (t \cdot s){:}B)$

E2. $t{:}A \to A$

E3. $t{:}A \to !t{:}t{:}A$

E4. $t{:}A \to (s + t){:}A$, $t{:}A \to (t + s){:}A$

E5. $\mathbf{V}(A \to B) \to (\mathbf{V}A \to \mathbf{V}B)$

E6. $t{:}A \to \mathbf{V}A$

Rules.

R1. Modus Ponens $\dfrac{A \quad A \to B}{B}$.

R2. Axiom Necessitation: $\vdash c{:}A$ where A is any of Axioms E0 to E6 and c a proof constant.

Definition 2.28 (LPV Axioms)**.** The system LPV contains all the rules and axioms of LPV$^-$ with the additional axiom:

A7. $\neg t{:}\mathbf{V}\bot$.

Formulas of the type $t{:}A$ are read as "*t is a proof A*". Constants should be thought of as primitive proofs or justifications which the agent accepts as given. "$(t \cdot s)$" is the proof resulting from applying the proof t to the proof s, e.g. through a step of modus ponens. Axiom E1, then, says that if t is a proof of $A \to B$ and s is a proof of A then t applied to s is a proof of B. "$!t$" represents proof-checking, so Axiom E3 says that if t is a proof of A then the result $!t$ of checking that proof is a proof that t is a proof of A. Suppose t is part of one's stock of proofs, then adding another proof s to this stock, represented by "$(s + t)$" does

not change what t proves; hence Axiom E4 says that if t is a proof of A then an enlarged stock of proofs $(s + t)$ remains a proof of A.

Finally we outline what a constant specification is, though it will not play a role in what follows. A *constant specification*, $\mathcal{CS}$, is a set $\{c_1{:}A_1, c_2{:}A_2 \dots\}$ of formulas such that each A_i is an axiom from the list E0 to E6 above, and each c_i is a proof constant. This set is generated by each use of the constant necessitation rule in an LPV$^-$ proof. The axiom necessitation rule can be replaced with a 'ready made' constant specification which is added to LPV$^-$ as a set of extra axioms. For such a $\mathcal{CS}$ let LPV$^-$-$\mathcal{CS}$ mean LPV$^-$ minus the axiom necessitation rule plus the members of $\mathcal{CS}$ as additional axioms.[7]

We now add to $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$ the explicit counter-parts of M, P, N, the principles EM, EP, EN respectively.

The explicit counter-part of M is:

$$\mathbf{V}A \to t{:}\mathbf{V}A \tag{EM}$$

which says that if A is verified then any t is a proof of this.

Likewise the counter-part of P is:

$$\mathbf{V}A \to \mathbf{V}t{:}A, \tag{EP}$$

which says that if A is verified then it is verified that t is a proof of A, for any t.

Finally the explicit counter-part of N is:

$$\neg\mathbf{V}A \to t{:}\neg\mathbf{V}A. \tag{EN}$$

This asserts that if there is no verification of A then any t is a proof of this.

---

[7]For more on the detail of the Logic of Proofs, and more generally Justification Logics, see [2, 3, 4, 21].

Hence we have the following extensions of $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$.

Definition 2.29 ($\mathcal{L}_{t:} + \mathsf{EM}$)**.** The systems $\mathcal{L}_{t:} + \mathsf{EM}$ are one of the systems $\mathcal{L}_{t:}$ with the additional axiom:

    EM. $\mathbf{V}A \to t{:}\mathbf{V}A$

Definition 2.30 ($\mathcal{L}_{t:} + \mathsf{EP}$)**.** The systems $\mathcal{L}_{t:} + \mathsf{EP}$ are the systems $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$ with the additional axiom:

    EP. $\mathbf{V}A \to \mathbf{V}t{:}A$

Definition 2.31 ($\mathcal{L}_{t:} + \mathsf{EN}$)**.** The systems $\mathcal{L}_{t:} + \mathsf{EN}$ are the systems $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$ with the additional axiom:

    EN. $\neg\mathbf{V}A \to t{:}\neg\mathbf{V}A$

6.1   Arithmetical Interpretation

First let us outline the arithemetical interpretation of LPV^-, and LPV.

The arithmetical interpretation of LPV^- and LPV specifies a translation of the formulas of LPV^- and LPV into the language of PA. The interpretation consists of the following (for readability we assume a coding of the syntax is given and suppress details of Gödel numbering etc.):

Definition 2.32 (Normal Proof Predicate)**.** A *normal proof predicate* is a provably Δ formula $\mathsf{Prf}(x, y)$ such that for every arithmetical sentence F the following holds:

    1. $\mathsf{PA} \vdash F \Leftrightarrow$ for some $n \in \omega, \mathsf{Prf}(n, F)$

2. A proof proves only a finite number of things; i.e. for every k the set $T(k) = \{l \,|\, \mathsf{Prf}(k, l)\}$ is finite.[8]

3. Proofs can be joined into longer proofs; i.e. for any k and l there is an n s.t. $T(k) \cup T(l) \subseteq T(n)$.

4. $(\mathsf{Prf}(k, F \to G) \wedge \mathsf{Prf}(n, F)) \to \mathsf{Prf}(\mathsf{m}(k, n), G)$

5. $\mathsf{Prf}(k, F) \to \mathsf{Prf}(\mathsf{a}(k, n), F), \quad \mathsf{Prf}(n, F) \to \mathsf{Prf}(\mathsf{a}(k, n), F)$

6. $\mathsf{Prf}(k, F) \to \mathsf{Prf}(\mathsf{c}(k), \mathsf{Prf}(k, F))$.

Definition 2.33 (Verification Predicate for LPV^-)**.** A *verification predicate* is a provably Σ formula $\mathsf{Ver}(x)$ satisfying the following properties, for arithmetical formulas F and G:

1. $\mathsf{PA} \vdash \mathsf{Ver}(F \to G) \to (\mathsf{Ver}(F) \to \mathsf{Ver}(G))$

2. For each n, $\mathsf{PA} \vdash \mathsf{Prf}(n, F) \to \mathsf{Ver}(F)$.

Definition 2.34 (Verification Predicate for LPV)**.** As above plus:

3. for any n, $\mathsf{PA} \vdash \neg \mathsf{Prf}(n, \mathsf{Ver}(\bot))$.

Definition 2.35 (Arithmetical Interpretation for LPV^-)**.** An *arithmetical interpretation* for LPV^- has the following items:

- A normal proof predicate, Prf, with the functions $\mathsf{m}(x, y)$, $\mathsf{a}(x, y)$ and $\mathsf{c}(x)$ as in Definition 2.32;

- A verification predicate, Ver, satisfying the conditions in Definition 2.33;

- An evaluation of propositional letters by sentences of PA;

- An evaluation of proof variables and constants by natural numbers.

---

[8]I.e. $T(k)$ is the set of theorems proved by the proof k.

An arithmetical interpretation is given inductively by the following clauses:

$$(p)^* = p \text{ an atomic sentence of } \mathsf{PA} \qquad\qquad (t \cdot s)^* = \mathsf{m}(t^*, s^*)$$
$$\perp^* = \perp \qquad\qquad (t + s)^* = \mathsf{a}(t^*, s^*)$$
$$(A \wedge B)^* = A^* \wedge B^* \qquad\qquad (!t)^* = \mathsf{c}(t^*)$$
$$(A \vee B)^* = A^* \vee B^* \qquad\qquad (t{:}F)^* = \mathsf{Prf}(t^*, F^*)$$
$$(A \to B)^* = A^* \to B^* \qquad\qquad (\mathbf{V}F)^* = \mathsf{Ver}(F^*)$$

Example 2.36. An example of a numerical relation that satisfies the definition of $\mathsf{Prf}(x, y)$ is the standard proof predicate $\mathsf{Proof}(x, y)$ the meaning of which is

"x is the Gödel number of a derivation of a formula with the Gödel number y".

Example 2.37. The following are examples of a verification predicate $\mathsf{Ver}(x)$ for either LPV^- or LPV:

1. "Provability in PA", i.e. $\mathsf{Ver}(x) = \mathsf{Bew}(x)$; for a formula F $\mathsf{Ver}(F)$ is $\exists x \mathsf{Prf}(x, F)$.

2. "Provability in $\mathsf{PA} + \mathsf{Con}(\mathsf{PA})$" i.e. $\mathsf{Ver}(x) = \mathsf{Bew}(\mathsf{Con}(\mathsf{PA}) \to x)$; one example of a formula for which $\mathsf{Ver}(x)$ holds in this sense is just the formula $\mathsf{Con}(\mathsf{PA})$. Such verification is capable of verifying propositions not provable in PA.

LPV^- and LPV are both sound on this interpretation, and hence via the Gödel embedding and the realisation theorem, IEL^- and IEL are sound for this interpretation also, and hence have a precise BHK interpretation, see [36].

6.2 Arithmetical Interpretation of Explicit Principles

We can now consider whether the principles P, M, N have a plausible intuitionistic meaning. Does the interpretation of LPV^- and LPV extend to the principles outlined above? The answer is apparently not, what each asserts about verifications, about the kind of information,

they are supposed to embody is very strong. When read arithmetically these principles are too strong for a general-purpose notion of truth-making verification. This suggests that the attempt to pattern truth-making verification on intuitionistic verification, as characterised by the principles of IEL, requires more thought.

Let us then read each principle in accordance with the arithmetical semantics of Section 6.1.

Principle EP

The principle EP

$$\mathbf{V}A \rightarrow \mathbf{V}t{:}A$$

says that if A is verified then it is verified that any proof, t, is a proof of A.

Interpreted arithmetically it is interpreted as:

$$\mathsf{Ver}(A) \rightarrow \mathsf{Ver}(\mathsf{Prf}(t, A)).$$

An example of a verification predicate, Ver, is the provability predicate Bew, see Example 2.37, hence a possible reading of this is:

$$\mathsf{Bew}(A) \rightarrow \mathsf{Bew}(\mathsf{Prf}(t, A)).$$

But from the fact that A is provable in PA, it does not follow that it is provable that any PA proof will be a proof of A. A related principle asserting that verification yields verification that *some* t is a proof of A is more plausible, but this would require extending the machinery of $\mathcal{L}_{t:} \in \{\mathsf{LPV}^-, \mathsf{LPV}\}$ with quantifiers over proof terms, see [22] and [13].

How should we understand what EP asserts given we wished to adopt it as a principle of verification? Verifications provide such strong evidence for a proposition that any proof will

be a proof of the proposition; or rather if the proposition is verified then it can be asserted at any stage of a proof. Verification, then, shows that the proposition verified has the status of an axiom or a tautology. This amounts to holding that verification is in some senses stronger than proof, since a verification suffices to ensure that A can be the conclusion of any proof, i.e. is a logical truth.

Principle **EM**

The principle **EM**

$$\mathbf{V}A \to t{:}\mathbf{V}A$$

says that if A is verified then any proof, t, is a proof of the fact that A is verified.

Interpreted arithmetically **EM** comes out as:

$$\mathsf{Ver}(A) \to \mathsf{Prf}(t, \mathsf{Ver}(A)).$$

And again taking provability in **PA** as an example of **Ver** suffices to show the unsoundness of **EM**:

$$\mathsf{Bew}(A) \to (\mathsf{Prf}(t, \mathsf{Bew}(A))).$$

If A is provable it does not follow that any proof, t, is a proof of $\mathsf{Bew}(A)$. There might *some* t which is such a proof, but not every one will be. **EM** says that a verification is the kind of justification that once established can be asserted in any proof. Once $\mathbf{V}A$ becomes true it takes on the status of an axiom. In a sense such verifications are self-certifying; once a proposition is verified the fact that this verification holds becomes universally assertable.[9]

---

[9] Descartes' 'I think therefore I am', might be just such a thing. Once you verify that it is true, you also see that such a verification is universally correct.

Principle EN

The principle EN

$$\neg \mathbf{V} A \to t{:}\neg \mathbf{V} A$$

says that if A is not verified, then any proof t is a proof of this fact.

Arithmetically EN it is interpreted as:

$$\neg \mathsf{Ver}(A) \to \mathsf{Prf}(t, \neg \mathsf{Ver}(A)).$$

Bew, again, serves as an example:

$$\neg \mathsf{Bew}(A) \to \mathsf{Prf}(t, \neg \mathsf{Bew}(A)).$$

But similar to EM from the fact that A is not provable it does not follow that any proof is a proof of this fact. There might be *some* t, for some A, which provides such a proof, but not any one. As a principle about verification it asserts a kind of completeness with respect to the the verification of statements, it asserts that the lack of verification is self-certifying. If a verification is missing then this fact is universally assertable.

In terms of the BHK intepretation the principles P, M, N go beyond what is intuitionistically acceptable. Intuitionistic verification is verification of the provability of a proposition, and the properties of verification, stability, positive and negative introspection depend on this fact. This is not evident in the intuitionistic language, but can be made explicit (doubly so, as it were, see [8]) in the context of S4V and LPV. A verificationism that attempted to "lift" the properties of verification naively from IEL would result in, arguably, implausibly strong conceptions of verifications.

7 Conclusion

The logics in $\mathcal{L}_\Box$ and their extensions offer a more nuanced way of understanding verification-based epistemic-doxastic states than do the logics in $\mathcal{L}$. Intuitionistically proof is a truth-making justification, a proposition is true in virtue of its proof. The incorporation of verification into the intuitionistic language suggests that **K** might similarly be read as such a truth-making justification. We argue that this is not the case, at least not straightforwardly. The translation of the intuitionistic epistemic language into a classical one shows that verification is verification of provability, verification that a truth-making justification exists, but is not such a justification itself. If we attempt to formulate a notion of verification that behaves in this manner, emulating the properties intuitionistic **K** has, the result goes beyond what is intuitionistically acceptable in terms of its intended, BHK, semantics. A formulation of a conception of truth-making verification that fits within an intuitionistic framework, a desideratum of philosophical verificationists, cannot be straightforwardly extracted from the results of IEL.

References

[1] Sergei Artemov. *Operational Modal Logic*. Tech. rep. MSI 95-29. Cornell University, 1995 (cit. on pp. 46, 64).

[2] Sergei Artemov. "Explicit Provability and Constructive Semantics." In: *Bulletin of Symbolic Logic* 7.1 (2001), pp. 1–36 (cit. on pp. 46, 54, 64, 67).

[3] Sergei Artemov. "The Logic of Justification." In: *Review of Symbolic Logic* (2008) (cit. on p. 67).

[4] Sergei Artemov and Melvin Fitting. "Justification Logic." In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2012. 2012 (cit. on p. 67).

[5] Sergei Artemov and Tudor Protopopescu. "Discovering Knowability: A Semantical Analysis." In: *Synthese* 190.16 (2013), pp. 3349–3376. DOI: `10.1007/s11229-012-0168-x` (cit. on p. 53).

[6] Sergei Artemov and Tudor Protopopescu. *Intuitionistic Epistemic Logic*. Tech. rep. Dec. 2014. arXiv: `1406.1582v2 [math.LO]` (cit. on pp. 51, 52, 56).

[7] Sergei Artemov and Tudor Protopopescu. "Intuitionistic Epistemic Logic." In: *Review of Symbolic Logic* (2016). DOI: `10.1017/S1755020315000374` (cit. on pp. 45, 50).

[8] Johann van Benthem. *Implicit and Explicit Stances in Logic*. Tech. rep. 2013 (cit. on p. 73).

[9] M. Božić and Kosta Došen. "Models for Normal Intuitionistic Modal Logics." In: *Studia Logica* 43.3 (1984), pp. 217–245 (cit. on p. 51).

[10] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Clarendon Press, Oxford, 1997 (cit. on pp. 46, 54).

[11] R. Constable. "Types in Logic, Mathematics and Programming." In: *Handbook of Proof Theory*. Ed. by S. Buss. Elsevier, 1998, pp. 683–786 (cit. on p. 49).

[12] Dirk van Dalen. "Intuitionistic Logic." In: *Handbook of Philosophical Logic*. 2nd ed. Vol. 5. Springer, 2002, pp. 1–114 (cit. on p. 46).

[13] Walter Dean and Hidenori Kurokawa. "From the Knowability Paradox to the Existence of Proofs." In: *Synthese* (2009) (cit. on p. 71).

[14] Kosta Došen. "Intuitionistic Double Negation as a Necessity Operator." In: *Publications de L'Institute Mathématique (Beograd)(NS)* 35.49 (1984), pp. 15–20 (cit. on p. 51).

[15] Kosta Došen. "Models for Stronger Normal Intuitionistic Modal Logics." English. In: *Studia Logica, An International Journal for Symbolic Logic* 44.1 (1985), pp. 39–70 (cit. on p. 51).

[16] Michael Anthony Eardley Dummett. "The Justification of Deduction." In: Cambridge: Harvard University Press, 1973 (cit. on p. 47).

[17] Michael Anthony Eardley Dummett. "The Philosophical Basis of Intuitionistic Logic." In: *Truth and Other Enigmas*. Cambridge, Mass.: Harvard University Press, 1973, pp. 215–247 (cit. on p. 47).

[18] Michael Anthony Eardley Dummett. *Elements of Intuitionism*. Clarendon Press Oxford, 1977 (cit. on p. 58).

[19] Michael Anthony Eardley Dummett. *The Logical Basis of Metaphysics*. The William James Lectures, 1976. Cambridge: Harvard University Press, 1991 (cit. on p. 47).

[20] R. Fagin et al. *Reasoning about Knowledge*. MIT Press, 1996 (cit. on p. 53).

[21] Melvin Fitting. "The Logic of Proofs, Semantically." In: *Annals of Pure and Applied Logic* 132 (2005), pp. 1–25 (cit. on p. 67).

[22] Melvin Fitting. "A Quantified Logic of Evidence." In: *Annals of Pure and Applied Logic* 152 (2008), pp. 67–83 (cit. on p. 71).

[23] Kurt Gödel. "An Interpretation of the Intuitionistic Propositional Calculus." In: *Collected Works*. Ed. by S Feferman et al. Vol. 1. Oxford University Press, 1933, pp. 301–303 (cit. on pp. 46, 54, 63).

[24] Kurt Gödel. "Lecture at Zilsel's." In: *Collected Works Volume III. Unpublished essays and lectures*. Ed. by Solomon Feferman et al. New York: Oxford University Press, 1938, pp. 86–113 (cit. on p. 64).

[25] V. F. Hendricks. *Formal and Mainstream Epistemology*. Cambridge University Press, 2006 (cit. on pp. 53, 55).

[26] Jaakko Hintikka. *Knowledge and belief; an introduction to the logic of the two notions*. 23 cm. Contemporary philosophy. Ithaca, N.Y., Cornell University Press, 1962 (cit. on p. 53).

[27] Yoichi Hirai. "An intuitionistic epistemic logic for sequential consistency on shared memory." In: *Logic for Programming, Artificial Intelligence, and Reasoning*. Springer. 2010, pp. 272–289 (cit. on p. 51).

[28] Adam Leite. "Fallibilism." In: *A Companion to Epistemology*. Ed. by Jonathan Dancy, Ernest Sosa, and Matthias Steup. Second Edition. Blackwell, 2010, pp. 370–375 (cit. on p. 55).

[29] Per Martin-Löf. "Verificationism Then and Now." In: *Judgement and the Epistemic Foundation of Logic*. Ed. by M. van der Schaar. Logic, Epistemology, and the Unity of Science. Springer, 2013. Chap. 1, pp. 1–14 (cit. on p. 47).

[30] John Charles Chenoweth McKinsey and Alfred Tarski. "Some Theorems About the Sentential Calculi of Lewis and Heyting." English. In: *The Journal of Symbolic Logic* 13.1 (Mar. 1948), pp. 1–15 (cit. on p. 54).

[31] R. Nozick. *Philosophical Explanations.* Cambridge, Massachusetts: Harvard University Press, 1981 (cit. on p. 53).

[32] Dag Prawitz. "Meaning and Proofs: on the Conflict Between Classical and Intuitionistic Logic." In: *Theoria* 43.1 (1977), pp. 2–40 (cit. on p. 47).

[33] Dag Prawitz. "Truth and Objectivity from a Verificationist Point of View." In: *Truth in Mathematics.* Ed. by H.G. Dales and G. Olivieri. Oxford University Press, 1998. Chap. 2, pp. 41–51 (cit. on p. 47).

[34] Carlo Proietti. "Intuitionistic Epistemic Logic, Kripke Models and Fitch's Paradox." In: *Journal of Philosophical Logic* 41.5 (2012). 10.1007/s10992-011-9207-1, pp. 877–900 (cit. on p. 51).

[35] Tudor Protopopescu. "Intuitionistic Epistemology and Modal Logics of Verification." In: Proceedings. Logics, Rationality and Interaction (LORI 2015). (Oct. 26–29, 2015). Ed. by Wiebe van der Hoek and Wesley Holliday. Lecture Notes in Computer Science 9394. Taipei: Springer, 2015, pp. 295–307. DOI: 10.1007/978-3-662-48561-3_24 (cit. on p. 57).

[36] Tudor Protopopescu. "An Arithmetical Interpretation of Verification and Intuitionistic Knowledge." In: *ArXiv.* math.LO 1601.03059v1 (Jan. 2016). Post-print correcting various publisher's errors. Published in Artemov, S. and Nerode, A., editors, Logical Foundations of Computer Science, Number 9537 in Lecture Notes in Computer Science, pp. 317-330, Springer. DOI: 10.1007/978-3-319-27683-0_22. ArXiv: 1601.03059v1 (math.LO) (cit. on pp. 57, 64, 70).

[37] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 2000 (cit. on pp. 46, 54).

[38] Univalent Foundations Program. *Homotopy Type Theory. Univalent Foundations of Mathematics.* Univalent Foundations Program, 2013 (cit. on p. 49).

[39] Timothy Williamson. "On Intuitionistic Modal Epistemic Logic." In: *Journal of Philosophical Logic* 21.1 (1992), pp. 63–89 (cit. on p. 51).

[40] Timothy Williamson. *Knowledge and Its Limits.* Oxford: Oxford University Press, 2000 (cit. on p. 53).

[41] F. Wolter and Michael Zakharyaschev. "Intuitionistic Modal Logics as Fragments of Classical Bimodal Logics." In: *Logic at Work.* Ed. by E. Orlowska. Springer-Verlag, 1999, pp. 168–186 (cit. on p. 51).

[42] Frank Wolter and Michael Zakharyaschev. "Intuitionistic Modal Logic." In: *Logic and Foundations of Mathematics.* Ed. by E. Casari, A. Cantini, and P. Minari. Kluwer Academic Publishers, 1999, pp. 227–238. DOI: 10.1007/978-94-017-2109-7_17 (cit. on p. 51).

3. Intuitionistic Knowledge and Fallibilism

1 Introduction

The argument of this paper is that an intuitionistic conception of knowledge, introduced in [4], yields a natural reconceptualisation of fallibilism. An intuitionistic approach to knowledge makes clear a conceptual tension in fallibilism as it is typically defined. Knowledge, conceptualised on the basis of a constructive, 'epistemic', notion of truth resolves this tension in a manner that preserves the motivation for fallibilism.

Fallibilism is the view that knowledge need not logically guarantee the truth of the proposition known. It is, however, a necessary condition of knowledge that the proposition known is true; it is not possible to know falsehoods, yet fallibilism seems open to this possibility. Fallibilism apparently violates a necessary (*the* necessary) condition for knowledge – the truth condition – and hence would appear to be inconsistent. An intuitionistic approach to knowledge does not suffer from this inconsistency.

Fallibilism is motivated by a sense of epistemic modesty – our justifications and evidence gathering are rarely, if ever, so good as to *guarantee* truth. The problem is that if a proposition is known then by definition the possibility of being mistaken is foreclosed – the truth condition

does not seem to allow for epistemic modesty.[1] Fallibilism, then, would appear to present a dilemma; give up on modesty, which amounts to a commitment to infallibilism, or give up on the truth condition on knowledge, which allows that it is possible to know falsehoods.

This dilemma is the product of a classical conception of truth and knowledge; a classical framework is not very discriminating and is unable to make certain distinctions, e.g. like that between a proposition A and its double negation $\neg\neg A$. In a classical framework, consequently, it is not possible to distinguish the truth condition from infallibilism. An intuitionistic conception of truth and knowledge is more discriminating, and hence allows for a way to maintain the sense of modesty that motivates fallibilism, whilst at the same time maintaining the truth condition on knowledge.

The intuitionistic, or constructive, view of truth is that a proposition is true only if there is a proof of it, and false if the assumption that there is a proof yields a contradiction. This, provability, interpretation, and its application to the interpretation of the logical connectives, is known as the Brouwer-Heyting-Kolmogorov (BHK) interpretation for intuitionistic logic. An intuitionistic conception of knowledge is one which interprets the meaning of a knowledge operator in accordance with the BHK semantics. As we will see the properties of a knowledge operator differ in significant ways in this context from those of an operator with a classical interpretation. BHK is acknowledged as the intended interpretation of intuitionistic logic. On the BHK reading a proposition's not being true is not equivalent to it being false; a proposition might not be true because it lacks a proof, but it is false only if there is a proof of its refutability. This distinction enables us to separate the truth condition, a proposition

---

[1]Hence fallibilism might also be seen as a response to Kripke's Dogmatism Paradox [29, 39]; if one really knows then one is not mistaken, hence one is justified in discounting all counter-evidence, but that does not appear to be a rational attitude. If one might be mistaken then there is reason not to be dogmatic.

cannot be false, from the infallibility of knowledge, known propositions must be true.

First we will give an outline of fallibilism and consider some of the problems the view faces, Section 2. Next, Section 3, we will consider the inconsistency between fallibilism and the the classical view of truth, specifically how that leads to a formulation of the truth condition which is inconsistent with fallibilism. Then we will describe what an intuitionistic conception of knowledge entails, Section 4, and specify the basic intuitionistic logic of knowledge, see [4], Section 5. Next we will discuss how this view of knowledge is fallibilistic, how it meets the definition of fallibilism and how it reflects the basic motivation for fallibilism, Section 6. Finally we will explain how the intuitionistic view of knowledge deals with the problems for fallibilism described earlier, Section 7.

2 Fallibilism

2.1 Definitions

What, more precisely, is fallibilism regarding knowledge? The following are some standard definitions of fallibilistic knowledge:

> It is possible for S to know that p even if S does not have logically conclusive evidence to justify believing that p [24].

or

> S fallibly knows that $p =_{df}$ (1) S knows that p on the basis of justification j even though (2) j does not entail that S's belief that p is true [62].

and

(1) A given belief of yours is fallibly justified $=_{df}$. (i) You belief is justified... (ii) A belief's being justified in the way referred to in (i) is compatible with its being false. (2) At least some of justification is fallibilist, as defined in (1) [32].

What all these have in common is the idea that knowledge, specifically the justification on which it is grounded, does not logically entail the truth of the proposition believed.

Another common way of defining fallibilism is to say that the known proposition could have been false. For example,

p is known fallibly just in case p is known on the basis of justification j and this belief on the basis of j could have been false [63].

If a proposition is known on the basis of a justification and it is possible to have that same justification while the proposition is false, then one's justification, and hence knowledge, does not entail the truth of the proposition.

To make this more precise let us express what is common to these definitions in the language of classical epistemic logic, i.e. classical propositional logic with an epistemic modality $\mathbf{K}$, see [23, 50]. For this purpose any epistemic logic containing the system T will do, that is both $\mathbf{K}(A \to B) \to (\mathbf{K}A \to \mathbf{K}B)$ and $\mathbf{K}A \to A$ are valid.

The thesis that knowledge, or a justification sufficient for knowledge, guarantees truth may be expressed formally by the *reflection principle*.

$$\mathbf{K}A \to A \tag{Reflection}$$

The reflection principle, hence, serves as a formalisation of infallibilism. The reflection principle is also what separates (classical) logics of knowledge from logics of belief, because it

is also serves as a formalisation of the truth condition.

The basic thesis of fallibilism, hence, may be expressed by denying the universal validity of the reflection principle. There are instances of reflection which are false. Consequently, fallibilism can be formulated by saying that there are propositions which satisfy the formula[2]

$$\mathbf{K}A \wedge \neg A.$$

Fallibilism in this sense has a weaker and stronger version. The weaker version holds that some, maybe even most, knowledge is fallible; some, or most, of our justifications on which the knowledge is based does not guarantee the truth of the known proposition. It allows however that some justifications do guarantee truth, and hence are infallible; knowledge of mathematical truths justified by proof is an obvious candidate for such infallible knowledge.

The stronger version has it that all knowledge is fallible; no justification ever guarantees the truth of the proposition because every process of justification has the potential for mistake. Hence even the best candidates for infallible knowledge, like mathematical knowledge based on proof, fall short of guaranteeing truth.

2.2 Motivations

The primary motivation for fallibilism is a sense of epistemic modesty. Humans are imperfect epistemic agents, our cognitive and reasoning processes and our means for acquiring evidence are not perfect, and accordingly we make mistakes or are simply unable to take into account all the relevant evidence. Nevertheless, according to the fallibilist, we are to some degree successful at knowing. Fallibilism reflects the recognition of this situation by holding that

---

[2]Either in this or some other possible world, it does not make a difference.

our justifications, in general (or indeed always), are not perfect in the sense of guaranteeing the truth of what they justify, and yet it is still possible that they can be sufficient to ground knowledge.

Related to this is the recognition that an infallibilist view of knowledge, i.e. one which holds that only conclusive justification can yield knowledge, is not a very accurate model of much of the knowledge we take ourselves to have. Many things we take ourselves to know were justified by means which we acknowledge could have led us to a mistaken belief – we have knowledge because as a matter of fact they did not, but we see there are circumstances in which the same evidence would not yield a true belief. This is not a acknowledgement an infallibilist can make, and hence infallibilism does not accord with the knowledge we take ourselves to have.

2.3 Problems with Fallibilism

Fallibilism appears to have several counter-intuitive, or unwelcome, consequences. We will argue that approaching knowledge from an intuitionistic perspective provides responses to each of these problems.

Skepticism

First it seems that fallibilism implies a certain kind of skepticism; in particular the kind that insists one's justification must guarantee the truth of what is justified in order to count as knowledge, and where that guarantee is lacking a claim to knowledge is not warranted. The argument for this is roughly the following. By the very concept, knowledge implies the impossibility of being mistaken, since it guarantees truth. Yet fallibilism is precisely

the view that such mistake is always possible, indeed that possibility is essential to making one's knowledge fallible. Accordingly it is not possible, on a fallibilistic view, to meet the requirements of knowledge in any domain where there is this possibility of mistake. Hence wherever knowledge is fallible it leads to skepticism. Since most epistemologists, indeed most people, think skepticism is false, even in those domains where knowledge is fallible, it would seem to follow that fallibilism cannot be the case.

This is not necessarily as question-begging as it might initially appear. The charge of skepticism can be seen as an instance, or a consequence, of a more general reaction to fallibilism which holds that it is just obviously wrong. The idea is that knowledge, by definition, cannot be mistaken; that knowledge is factive, and reliably so, is precisely what we mean by knowledge and the reason it is so highly valued. It is a further question whether any mental state satisfies these criteria, of course, but a mental state which does not cannot legitimately be called knowledge. Given the history of this line of thought, then, it does not seem like unwarranted burden shifting to criticise fallibilism in this way (see Page 88).

Necessary Truth

Another counter-intuitive consequence of fallibilism is that it seems to preclude the possibility of knowing necessary truths, for instance mathematical truths. This holds particularly for the strong, universal, version of fallibilism. Necessary truths cannot be false, and hence no justification leading to knowledge of a necessary proposition can be mistaken, because there is no possible situation in which one is justified but the proposition is false. Hence, if every justification, and the knowledge based on it, is fallible then necessary propositions cannot be known; because for the belief to count as knowledge it must be possible for it be false,

which it will never be in the case when the proposition is necessarily true. Put another way, a proposition is known fallibly if the reflection principle fails for it in some possible situation; if the proposition is necessary there is no possible situation in which reflection fails for it, because the consequent is true in all situations. Since some necessary truths are known then not all knowledge is fallible, contrary to the strong fallibilist thesis.

Mutatis mutandis this applies to the weak fallibilist who allows that some necesary truths can be known only fallibly, even if some are known infallibly.

Gettier Problem

One of Gettier's assumptions is that the kind of justification which is at issue in the justified true belief analysis of knowledge is fallible; it is possible to be justified in believing a false proposition, [26]. Hence the kind of justification required for knowledge need not guarantee the truth of the proposition, so having knowledge, i.e. a justified true belief, is not sufficient to guarantee the truth of the proposition. This is precisely what Gettier's counter-examples are designed to make evident. Hence Gettier's counter-examples are counter-examples to a fallibilist conception of knowledge. Given that, it is particularly important that a fallibilist account of knowledge have a response to the Gettier cases.[3]

The Lottery Paradox

The lottery paradox, due to Kyburg [41, 42], reveals a tension between plausible principles of fallible knowledge.

---

[3]Indeed, it seems that Gettier counter-examples can only be about fallibilistic knowledge. Would an infallibilist notion of knowledge be subject to the Gettier examples? If justification guaranteed truth, then the agents in Gettier's cases would not have the problematic justified true beliefs, because they could not have the intial justified false beliefs from which they are inferred.

Assume there are one thousand tickets in a fair lottery, and that one and only one will win. According to the fallibilist conception a proposition justified with a probability of .99 or more counts as known. It is certain that one ticket will win, since the lottery is fair. Now, for any given ticket, since the probability of it losing is more than .99 it is fallibly known that it will lose. If all the conjuncts of a conjunction are known then the conjunction is known, hence since it is known that 'ticket 1 will lose' and known that 'ticket 2 will lose' and so on for every ticket, it is known that every ticket will lose. But this contradicts the knowledge that one and only one ticket will win.

The belief that each ticket will lose is a paradigmatic instance of fallible knowledge. The probability of each one being true is, .999, but it does not guarantee the truth of the belief. The problem is that this allows for inconsistent knowledge; seemingly, one can fallibly know both a proposition and its negation. The paradox seems to entail that if we are to be fallibilist we have to give up on some basic assumptions about knowledge.[4]

We will argue that an intuitionistic approach to knowledge yields a fallibilistic view of knowledge, and offers a reasonable response to each of the problems outlined above.

3   The Classical Truth Condition vs. Fallibilism

Every definition of knowledge extant holds that knowledge is some species of true belief, be it justified (or justified plus ...), reliable, safe, certain, 'relevantly alternative' etc.[5] All these definitions have it that truth is a necessary condition for knowledge,[6] no one has ever argued

---

[4]"...it seems that fallibilism will require some modification of our basic assumptions governing knowledge" [63, p. 590].

[5]See [73] for a view not fitting this mould.

[6]Even for Williamson [73] this is true, "knowledge is the most basic factive state".

that it is possible to know a false proposition. It is universally acknowledged that it is not possible to know false propositions, hence we can formulate the *truth condition on knowledge*:

Only true propositions can be known

or put negatively,

False propositions cannot be known.

Within a classical epistemic logical context, where truth is bi-valent and the law of excluded middle holds, these formulations are equivalent. The formalisation of each of these is, respectively:

$$\mathbf{K}A \to A$$

and

$$\neg(\mathbf{K}A \land \neg A)$$

The first, reflection, expresses the idea that truth is a necessary condition for knowledge, while the second says that it is not the case that a false proposition is known. As principles in classical epistemic logic these are equivalent.

But as we have seen already fallibilism may be fairly formalised as the thesis that the reflection principle $\mathbf{K}A \to A$ is not universally valid. There is some instance, p, such that $\neg(\mathbf{K}p \to p)$, and this is equivalent to $\mathbf{K}p \land \neg p$. On the fallibilist view, then, false propositions can be known. *The truth condition and fallibilism are not classically compatible.*

We have already mentioned the idea that there is an essential connection between knowledge and getting it right in connection with the skepticism problem for fallibilism, Section 2.3. The idea that knowledge is fallible, does not logically imply truth, has been taken

by some as simply self-contradictory. For instance Lewis says "to speak of fallible knowledge, of knowledge despite uneliminated possibilities of error, just sounds like a contradiction" [43, p. 367]. Hendricks claims that "knowledge must be *infallible* by definition. . . [a] fallible notion of knowledge is not much different from a concept of belief potentially allowing the agent to 'know' a falsehood, severing the connection between knowledge and truth" [31, p.8].

But these criticisms only make sense when the notion of truth in question is classical. It is a classical presupposition to think that less-than-conclusive, or imperfect, evidence implies that it can be mistaken. Likewise it is a classical presupposition to hold that knowledge by definition cannot be wrong, and that therefore one's evidence must guarantee the truth of the proposition in question. The problem is that thesis that knowledge logically guarantees truth and the truth condition *are the same* thing in a classical context. If one gives up the entailment thesis, denies reflection, in a classical context then one must accept that knowledge of falsehoods can be known. This threatens the very foundation of fallibilism as a theory of knowledge since the truth condition is an absolute *sine qua non* of knowledge.

A constructive, intuitionistic, notion of truth and an understanding of knowledge based on it does not suffer from this problem because it can consistently maintain the failure of reflection with the validity of the truth condition.

4 Intuitionistic Knowledge

Intuitionistic knowledge is knowledge viewed on the basis of the intended semantics for intuitionism, the Brouwer-Heyting-Kolmogorov (BHK) semantics [9, 10, 33, 37]. The BHK semantics for intuitionistic logic holds that a proposition, A, is true only if there is a proof it,

and A is false if the supposition that A has a proof yields a contradiction. This understanding of truth is extended to the logical connectives in the following manner:

Definition 3.1 (BHK Semantics)**.**

- a proof of $A \wedge B$ consists in a proof of A and a proof of B;

- a proof of $A \vee B$ consists in giving either a proof of A or a proof of B;

- a proof of $A \to B$ consists in a construction which given a proof of A returns a proof of B;

- $\neg A$ is an abbreviation for $A \to \bot$, and $\bot$ is a proposition that has no proof.

How should the epistemic state of an agent, like belief and knowledge, be understood within this context? How should a proposition of the type $\mathbf{K}A$ be interpreted, where $\mathbf{K}$ is a knowledge or belief operator?

We propose that intuitionistic belief and knowledge be understood as the product of a verification process which is adequate for practical purposes to warrant a claim to knowledge, but which need not be a strict proof. The idea that knowledge is the product of non-proof verification is one with various antecedents, e.g. [72]. Note that proofs are especially strict and pure types of verification. This is a fundamental property of proofs and verifications with profound consequences for the conception of knowledge.

The interpretation of $\mathbf{K}A$ in the BHK context is:

- a proof of $\mathbf{K}A$ is conclusive evidence of a verification that A has a proof, see [4].

Knowledge of a proposition means having conclusive evidence that there exists a proof of A, i.e. A is true, without necessarily being in possession of this proof. A verification

certifies such a proof exists, but need not contain enough information to yield even a method for finding such a proof.[7] For example, zero-knowledge protocols are methods of verifying something without thereby obtaining any information as to why or how it is true. The verifier verifies that an agent possesses, for instance, a pin number, or encryption key, without themselves having that pin or key. More prosaic, verification via expert, or insider, testimony is another example. An expert testifies that A holds, they have conclusive evidence for A. Their testimony is sufficient to warrant a non-expert to claim knowledge of A[8] but the non-expert need have no access at all to this conclusive evidence.

From the fact that intuitionistic truth is based on proof, and the assumption that intuitionistic knowledge is the product of verification we can draw the following conclusions about the properties of knowledge, interpreted from an intuitionistic standpoint:

1. proof yields verification-based knowledge (co-reflection);

2. verification-based knowledge does not yield proof, hence truth (reflection).

A BHK-compliant epistemology accepts 1 and rejects 2.

1 can be formalised as

$$A \to \mathbf{K}A \qquad \text{(co-reflection)}$$

Interpreted intuitionistically this is valid, since proofs are verifications, indeed a particularly strict kind. The co-reflection principle should be read as expressing the constructive nature of intuitionistic truth. It says that *given a proof of A one can always construct a proof of*

---

[7]Accordingly verifications are not the neo-verificationist's 'canonical proofs' or generalisations thereof, see [7, 11, 15, 17, 18, 19, 21, 46, 54, 55, 58, 59, 64, 65, 66, 68]

[8]Beyond a reasonable doubt even.

KA. This follows from the BHK reading of implication, and indeed such a construction is always possible since proof-checking is a universally valid operation on proofs. A proof of A can always be proof-checked to yield a proof that A has a proof, and since all proofs are verifications it follows that A is verified, and hence known. A proof of A yields knowledge of A and proof-checking yields a proof of **K**A. Co-reflection is a foundational property of intuitionistic knowledge, expressing a truism about intuitionistic truth and knowledge.[9]

The informal principle 2 above can be expressed formally by holding that reflection is not a valid epistemic principle, that is **K**$A \nrightarrow A$. On the BHK interpretation reflection is not universally valid. If not all verifications are proofs then it follows that verification-based knowledge need not always yield a proof. It is possible to have knowledge of a proposition without thereby having a proof of that proposition. According to BHK the reflection principle says that *given a proof of* **K**A *it is always possible to construct a proof of* A. But this is not true in general, the evidence that A has a proof does not necessarily contain enough information to construct that proof. The zero-knowledge protocols mentioned above are an example of such a scenario; the testimony of an expert is another such example.

The reflection principle is practically definitive of knowledge. Every definition and characterisation of knowledge would seem to validate it. How is the intuitionistic approach to knowledge not committed to the possibility of knowledge of false propositions?

The reflection principle is a natural formalisation of the truth condition on knowledge i.e. that *known propositions must be true*, or equivalently that *false propositions cannot be known*. When the truth in question is classical these are equivalent, but they are not so

---

[9]See [12, 30, 36, 48, 51, 52, 70, 71, 74] for arguments (not always endorsed) that co-reflection, when read intuitionistically, is valid.

intuitionistically, and this non-equivalence makes it possible to deny that known propositions must be true, as we have just seen, whilst not committing oneself to holding that false propositions can be known.

Intuitionistically the truth condition on knowledge holds that

false propositions cannot be known

which can be formalised by the principle of intuitionistic reflection:

$$\mathbf{K}A \to \neg\neg A. \qquad \text{(intuitionistic reflection)}$$

According to the BHK semantics this says that *given a proof of* $\mathbf{K}A$ *it is always possible to construct a proof that the assumption that a proof of A yields a contradiction itself yields a contradiction.* More succinctly this can be read as

if A is known then it is impossible that A is false,

which is just a way of stating the truth condition.

One might object that knowledge should guarantee truth, and so while the intuitionistic formulation of the truth condition captures something important it is too weak. It is arguable however that intuitionistic reflection expresses just as much as the reflection principle does when it is read classically. The double negation translation of classical logic into intuitionistic logic can be regarded as a means by which an intuitionist can approximate the classical truth of a proposition, see [5, 8, 10, 13, 27, 34, 38]. This is because a formula of the form $\neg\neg A$ can be intuitionistically true without an explicit proof of A; establishing the impossibility of a refutation of A is not equivalent to proving A. The case of disjunction is,

of course, representative; establishing the impossibility of $\neg(A \vee B)$ need not provide any specific information as to which one of A or B is true. Hence intuitionistic reflection can be understood as claiming that knowledge yields classical truth, i.e. truth which does not have a specific witness. Accordingly intuitionistic reflection appears to be just as strong as classical reflection, and expresses the same thing. Hence it should not be supposed that the intuitionistic truth condition is weaker than the classical, it is rather the case that reflection read intuitionistically expresses a property of knowledge which is in fact stronger than the truth condition.

Intuitionistic reflection says that knowing involves having sufficient information to rule out the possibility of a refutation of the proposition in question, i.e. having enough information to conclude that it is impossible for the proposition to be false. Hence the justification upon which knowledge is based establishes the logical possibility of a proof of, or more generally of conclusive specific evidence for, a proposition.

The following are other intuitionistically equivalent ways to express the truth condition on knowledge, see [4].

ITC1. $\neg(\mathbf{K}A \wedge \neg A)$;

ITC2. $\neg A \rightarrow \neg \mathbf{K}A$;

ITC3. $\neg\neg(\mathbf{K}A \rightarrow A)$;

ITC4. $\neg\mathbf{K}\bot$.

Each can be understood as claiming that false propositions cannot be known. Given the double negation translation of classical logic to intuitionistic logic Item ITC3 can also be

read as stating that reflection is a classically valid principle, or that it is not ruled out that verification yields proof.

5 The Logic of Intuitionistic Knowledge

Given the above considerations the basic intuitionistic logic of knowledge (intuitionistic epistemic logic IEL) is the following.

Definition 3.2 (IEL)**.** The list of axioms and rules of IEL consists of [4]:

IA0. Axioms of propositional intuitionistic logic;

IA1. $\mathbf{K}(A \rightarrow B) \rightarrow (\mathbf{K}A \rightarrow \mathbf{K}B)$;

IA2. $A \rightarrow \mathbf{K}A$;

IA3. $\mathbf{K}A \rightarrow \neg\neg A$.

IR0. Modus Ponens.

Given the co-reflection principle the rule of $\mathbf{K}$ necessitation is derivable.[10]

The difference between knowledge and belief is the truth condition; it is possible to believe falsely, but not to know falsely. Accordingly if Axiom IA3 is dropped from IEL the basic intuitionistic logic of belief, IEL$^-$, is obtained.[11]

The argument of the previous section was that these principles of intuitionistic $\mathbf{K}$ characterise knowledge based on the BHK semantics. Confirmation that these informal considerations are correct is given by the fact that IEL (and IEL$^-$) both have an arithmetical interpretation

---

[10]See [72], [60] and [35] for other formulations of an epistemic logic based on intuitionistic logic, see [4, section 6.2] for discussion.

[11]Roughly, IEL$^-$ is to IEL as classical D is to T, understood as epistemic logics.

which extends that of the intuitionistic propositional calculus IPC, see [61] and Paper 1 above. The BHK semantics, Definition 3.1, is informal, it speaks of 'proof' without making precise what constitutes a proof. Kolmogorov [37] suggested that this be understood as proofs in classical mathematics. Gödel [28] showed that IPC can be embedded into the classical modal logic S4 interpreted as a provability calculus.[12] Gödel's result also left unspecified what kind of provability the S4 $\Box$ represented.[13] Artemov [1, 2] showed that explicit proofs in Peano Arithmetic, i.e. $Proof_{\mathsf{PA}}(p, A)$, is the provability model of S4 via the realisation of S4 into the Logic of Proofs and its arithmetic interpretation.[14] Hence a precising of 'proof' in the BHK clauses is explicit proof in PA; IPC can be interpreted as an implicit logic of proofs in PA. This arithmetic interpretation can be extended to IEL (and IEL$^-$) where $\mathbf{K}$ is interpreted by a verification predicate Ver in PA. Standard provability in PA is one example of such a verification predicate, as is provability in PA+Consis(PA), i.e. $\neg\mathsf{Prov}(\bot)$, which is a proper extension of PA, cf. [61] and Paper 1 above.

IEL and IEL$^-$ also have a Kripke model-theoretic interpretation, which while not in the BHK spirit has illuminating features.

Definition 3.3 (Semantics for IEL)**.** Models for IEL are intuitionistic Kripke models, $\langle W, R, \Vdash \rangle$, with an additional accessibility relation E, satisfying the following conditions, for states u v and w:

---

[12][49] showed the embedding is faithful. For an appropriate translation, $tr(A)$ of an intuitionistic formula A:

$$\mathsf{IPC} \vdash A \Leftrightarrow \mathsf{S4} \vdash tr(A).$$

See [5, 67] for proofs.

[13]Though he did show that it cannot be 'provability in a given formal system T', i.e. $\exists x Proof_T(x, y)$. $\mathsf{S4} \vdash \Box(\Box A \to A)$ but $Provable_T(Provable_T(\bot) \to \bot)$ is false by Gödel's second incompleteness theorem.

[14]See [3] for a survey of these issues.

IM1. R is reflexive, transitive and anti-symmetric (i.e. a partial order);

IM2. if uEv then uRv;

IM3. if uRv and vEw then uEw;

IM4. for all u, there is a v such that uEv.

Truth for the propositional connectives is defined inductively, for any u and v:

- if $u \Vdash A$ then for all v such that uRv $v \Vdash A$

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$

- $u \Vdash A \vee B$ iff either $u \Vdash A$ or $u \Vdash B$

- $u \Vdash A \rightarrow B$ iff for all v such that uRv either $v \nVdash A$ or $v \Vdash B$

- $u \nVdash \bot$

- $u \Vdash \neg A$ iff $v \nVdash A$ for all v such that uRv [15]

- $u \Vdash \mathbf{K}A$ iff $v \Vdash A$ for all v such that uEv.

An IEL^- model is obtained by dropping Condition IM4.

The heuristic reading of intuitionistic Kripke models is that each state represents the state of information of an ideal researcher, and the accessibility relation the development of those states, so each accessible state is a logically possible development of the state it is accessed from. More specifically, in an intuitionistic context a model can be regarded as representing the lines of development of the stock of proved propositions of an ideal mathematician, see [9]. For IEL models we can think of an agent who both proves and verifies propositions for a given proposition. For a given state u the set of states intuitionistically, R, accessible from it

---

[15]I.e. $u \Vdash A \rightarrow \bot$ iff for all v such that uRv either $v \nVdash A$ or $v \Vdash \bot$, and by the previous clause the latter is impossible.

represent the logically possible developments of u, what is provable given u, while the set of states epistemically, E, accessible from it (the 'audit' set of u) represent possible verifications given u.

The following are some notable theorems of IEL:[16]

1. Each of ITC1 to ITC4 is derivable in IEL (but not IEL$^-$).

2. $\vdash \mathbf{K}\neg A \rightarrow \neg A$

3. $\vdash \neg\mathbf{K}A \leftrightarrow \mathbf{K}\neg A$

4. $\vdash \neg\mathbf{K}A \leftrightarrow \neg A$

5. $\neg(\neg\mathbf{K}A \wedge \neg\mathbf{K}\neg A)$

6 Intuitionistic Fallibilism

With the details of the logic of intuitionistic knowledge established we can now discuss how an intuitionistic conception of knowledge can be both fallibilistic while also maintaining the truth condition on knowledge. Intuitionistic knowledge does not logically guarantee the truth of the proposition known, since reflection is not valid, and hence satisfies the definition of fallible knowledge. At the same time the truth condition is maintained because it is not possible to know falsehoods intuitionistically, since intuitionistic reflection is valid. Reflection is too strong a statement, intuitionistically, for the truth condition, but intuitionistic reflection, a strictly weaker statement, is not. Indeed intuitionistic reflection appears to capture nicely the content of the reflection principle read classically as a formalisation of the truth condition.

The proximate reason for this is the difference between intuitionistic and classical negation.

---

[16]See [4] for proofs and further details. See [40] for a proof-theoretic study of IEL.

Classically there is no distinction between a proposition not being true, model-theoretically $\nVdash A$, and a proposition being false $\Vdash \neg A$. By contrast in an intuitionistic model $\nVdash A$ does not imply $\Vdash \neg A$. This makes it possible to be in a situation where $\mathbf{K}A$ holds but A does not, i.e. $\nVdash A$, but not possible to be in a situation where $\mathbf{K}A$ and also $\neg A$. The following IEL model illustrates this point nicely. 1) $W : \{x, y\}$; 2) $R : xRy$, R is transitive and reflexive, i.e. loops around x and y are suppressed; 3) $E : xEy,\ yEy$; 4) $\Vdash: y \Vdash p$.
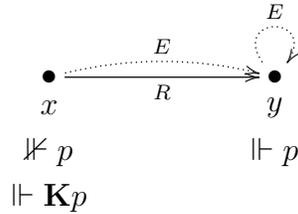
Figure 3.1: Model $\mathcal{M}_2$

By our definitions for $x \Vdash \neg p$ to be the case $\nVdash p$ must be the case at every state accessible from x, so in particular at y. In which case it would be not be possible for $x \Vdash \mathbf{K}p$ to hold. But $x \nVdash A$ and $x \Vdash \mathbf{K}p$ is possible. Moreover $x \Vdash \mathbf{K}p$ implies that there is some state R accessible from x, in this case y, where p holds. At x p is known because there is sufficient evidence to warrant that p has a proof, or that p will be proved at some logically possible state, but it does not require that x be that state.

The deeper reason for the possibility of distinguishing between infallibilism and the truth condition is, of course, the different conceptions of truth, from which follow the differences about negation. The question is what do we mean by fallibilism in this context? The different conception of truth has profound consequences on how to conceive of knowledge and its relation to truth; how does this effect what we should think of as fallibilism? If intuitionistic knowledge does not allow one to know false propositions, then in what sense is it *fallible* (or

fallibilistic)?

As mentioned the guiding idea of fallibilism is the idea that it is possible to have knowledge on the basis of evidence which is less than conclusive. But if one's evidence is less than conclusive then it is natural to conclude that one might have such justification and nevertheless still believe falsely. Less-than-conclusive evidence is fallible because it does not guarantee the truth of the proposition it justifies, and hence does not preclude the possibility of mistake.

Intuitionistic truth avoids this. An important feature of intuitionistic knowledge which it does not share with classical fallibilistic knowledge is: *it cannot be mistaken.* Intuitionistic knowledge shares with the infallibilist conception of knowledge the idea that knowledge precludes the possibility that one can know a proposition which is false. This is what intuitionistic reflection expresses. At the same time it shares with the fallibilist conception the idea that one's evidence can be less than conclusive; intuitionistically 'less than conclusive' is not equivalent to 'possibly mistaken'. The question is how can a proposition be less than conclusively justified, without also leaving open the possibility that the proposition justified is false despite the evidence that it is not? Or put another way, why is this not an infallibilist view of knowledge after all?

6.1   Constructive Truth

The difference stems from the constructivity of truth. The constructive view has it that the truth of a proposition consists in there being a proof of it. This is the basic idea of the BHK semantics, Definition 3.1, above. The intended understanding of this is that proof means some kind of mathematical proof, this was the context within which intuitionism and the BHK semantics first came about. But, as noted, the BHK semantics does not specify what is

meant by a proof, it simply relates the truth of a proposition to this unspecified notion of proof, and then lays down ways in which the propositional connectives behave in relation to proof. The notion of proof however is not a uniquely mathematical one, the word 'proof', at least as used in English, is a synonym for 'conclusive evidence'.[17] So we can say that, in general, the constructive, epistemic, view of truth is that a proposition is true if there is conclusive evidence for it.

The traditional understanding of what this means[18] is that for a proposition to count as true this conclusive evidence must be in possession of the agent in question. A looser understanding, somewhat more concessive to classical (realist, Platonist) intuitions, has it that such evidence can exist independently of the agent but must always, in principle, be accessible.[19]

Abstracting from these views what they have in common we get that the existence of this conclusive evidence is necessary for the truth of the proposition. And properly speaking it is this which is the core of the constructive, aka the epistemic, view of truth.

In what sense necessary? In the sense that the existence, and hence the accessibility, of this evidence is what makes the proposition true. And it is this necessity which makes this evidence conclusive; being in possession of this kind of truth-making evidence leaves no room for error because one would be in possession of the thing which constitutes the truth of the

---

[17]Indeed, as a matter of English usage the mathematical use of the word 'proof' is a specification of the more general notion. The analogy of construction is mathematical in origin; the idea that the truth does not exist until it is "built" fits particularly well with the practice of mathematical proof; a theorem is not true until the chain of reasoning has no gaps, the same way that the floor of a building does not exist until all the floors beneath it have been built.

[18]E.g. Heyting: [34, p.2] "In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed.'" Dummett [14, 22] also takes this approach.

[19]E.g. the views of Prawitz and Martin-Löf which characterise constructive truth as knowability: [45, 46, 47, 53, 56, 57]. See [12] for a discussion of these issues.

proposition. On the constructive view such evidence is not a representation of, or a pointer to, an independent truth, in a sense it is the truth.[20]

An illustrative example is the proof-theoretic explanation, in terms of 'canonical' proofs or verifications, of the meaning and truth of the logical connectives which serves as the basis of verificationist views. This is based on Gentzen's explanation of the meanings of the logical connectives in terms of the rules by means of which the connective is introduced and eliminated, cf. [21, 25]. These rules are what make the connectives what they are, they constitute the meanings of the connectives. And they also specify the evidence which must obtain in order for statements involving the relevant connective to be true. Take, for example, the rules for conjunction. A conjunction can be introduced when the conjuncts are already proved, and likewise it can be eliminated to yield any of the conjuncts. That a connective is introduced and eliminated in this way is what characterises it as conjunction, and also specifies what it means for a conjunction to be true.

The proofs which serve as the basis of the introduction rules, because of their meaning constitutive role, are what are called 'canonical', or 'direct', proofs. The distinction between canonical and non-canonical proofs, or more generally verifications, is precisely the role the former play in determining the truth of the proposition involving the relevant connective. The possibility of canonical proof is essential for counting the proposition as true, whereas non-canonical proofs are not; indeed, the justification of non-canonical proofs or verification

---

[20]NB: Yes, that 'in a sense' is a place-holder for the value of a big IOU. It is not our purpose here, however, to elaborate on the metaphysics of constructive truth, or to examine its fine-structure. This is a big topic, with a long history and a voluminous literature. Our purpose here is to outline those characteristics of constructive truth which help explain in what sense intuitionistic knowledge can be considered fallibilistic.

What bearing the intuitionistic conception of knowledge outlined here, and in [4], has on these larger issues is left for future studies.

is that they serve as assurance that canonical proofs or verifications are possible, see [16]. There are many ways a conjunction might be established, e.g. as the conclusion of a reductio ad absurdum, or as the result of the elimination of an implication by modus ponens. But these do not provide information as to why the conjunction is true. That would be because there are, canonical, proofs of the conjuncts.

In general, then, a proposition is constructively true when there is canonical evidence for it. Intuitionistic, verification-based, knowledge then consists in having sufficient evidence that such canonical, truth-making, evidence exists, or is in principle attainable. But the knowledge-producing verification need not be this truth-making evidence itself, it is sufficient that it points to such evidence. This is why intuitionistic reflection is valid. If one has sufficient information to conclude that a proposition has a conclusive justification, even if one knows nothing of its nature, then one can conclude that it is not possible for the proposition in question to be refuted, i.e. one has ruled out the possibility of conclusive refuting evidence. This is the sense in which intuitionistic knowledge eliminates the possibility of error. Consequently it also follows that intuitionistic knowledge is indefeasible. If one really has knowledge then any counter-evidence is, in fact, misleading.[21]

6.2 Fallibilism

Intuitionistic knowledge is fallibilistic because it is based on less-than-conclusive evidence. In the constructive case non-conclusive does not mean that one might be mistaken, it means

---

[21] So one is justified in being dogmatic, see [29, 39]? Not necessarily, being sure in the requisite way requires knowing that you know, that is having sufficient information to conclude that there exists conclusive evidence that one's information does eliminate the possibility that one believes falsely. If one really knows then such evidence must exist, but showing such evidence exists might be incredibly hard to achieve, where that is the case one cannot pronounce one's dogmatism, even if it is in fact justified.

rather than one's evidence is not truth-making or canonical. Intuitionistic fallibilism allows that non-truth-making evidence can suffice for a claim to knowledge. It can do so because for such evidence to rise to the level of justifying a belief as knowledge it must be good enough to ensure the existence of conclusive evidence, hence truth.

The aspect of fallibilism which intuitionistic knowledge reflects, then, is that less-than-conclusive evidence may suffice for knowledge, rather than that it leaves open the possibility of mistake. These are equivalent only on a classical conception of truth and knowledge but not on an intuitionistic one.

Take for instance the difference between knowing by direct observation that A is true, and learning that A is true by, accurate, truthful, testimony. The existence of the evidence an observer gains is necessary for the truth of A; the observation is the "canonical" form of evidence for A in that were it not to exist, i.e. nothing to observe, it would not be possible to claim the proposition is true. In contrast testimony indicates that the possibility of observation exists (or did exist). But it is not itself in any way essential to the truth of the proposition, the possibility of such testimony has no bearing on the possibility of the observation. Testimony might not be possible because the opportunity for direct observation has passed, but this does not imply that the phenomenon is (and always was and will be), in principle, unobservable.

The distinction between conclusive and less-than-conclusive evidence also suggests how to explain the sense that the possibility of mistake has not been eliminated. Less-than-conclusive evidence is a pointer to the existence of truth-making evidence, which by hypothesis is sufficiently reliable to warrant knowledge, but is not itself truth-making. This difference

leaves room for doubt, accounting for the sense that one might be mistaken, see Footnote 21.

7 Solutions

We turn now to showing how an intuitionistic conception of knowledge, based on constructive truth, handles the problems for fallibilism outlined in Section 2.3.

7.1 Skepticism

The solution to the skeptical problem is simply an application of the intuitionistic truth condition. The skeptical problem for fallibilism is premised on the idea that knowledge, to be genuine, must yield the impossibility of being mistaken, i.e. to know means that what one knows cannot be false: skepticism insists on the truth-condition for knowledge. Since fallibilism, as normally, classically, defined allows for violations of the truth condition, it must have skeptical consequences.

Intuitionistic knowledge does not allow violations of the truth condition. Knowing does yield it is not possible to be mistaken regarding what is known. This is precisely what the intuitionistic truth condition, intuitionistic reflection,

$$\mathbf{K}A \rightarrow \neg\neg A$$

says. If A is known then it is impossible for A to be false.

At the same time the intuitionistic approach allows that knowledge producing justifications can be less-than-conclusive, just as the fallibilist argues.

7.2   Necessary Truths

Fallibilism, particularly of the strong variety, holds that knowledge does not guarantee the truth of the proposition in question, hence it is possible for any fallibly known proposition to be false. Since necessary propositions cannot be false, they cannot be known fallibly, because there is no possibility of being mistaken, which is essential for the knowledge to be fallible, i.e. reflection can never fail.

Intuitionistically a necessary truth can be known fallibly, because reflection can fail, even if the proposition in question is necessary. Assume A is some necessary truth, which is known via a knowledge-producing justification which is less-than-conclusive, for this to be the case one need not be in possession of the conclusive, truth-making, justification for A, hence reflection need not hold. On the intuitionistic view of fallible knowledge it is not required that it be possible for the proposition in question to be false, in order that there be some possible situation in which the proposition is justified (and hence known) but false. Since the intuitionistic view decouples the possibility of mistake from knowledge by less-than-conclusive means, the fact that A cannot be false no longer has bearing on the possibility of it being known by less-than-conclusive justifications.

7.3   The Gettier Problem

In his paper, [26], outlining counter-examples to the justified true belief analysis of knowledge Gettier makes two assumptions: that justification is closed under implication, and that knowledge-producing justification is not necessarily factive, it is possible to be justified in believing a false proposition. Accordingly the knowledge under discussion is fallible, so

Gettier's counter-examples are problems for fallibilism.

The key feature of Gettier-type cases is the accidental nature of how the justified true belief condition for knowledge is satisfied. An agent is justified in believing a false proposition, from this they validly infer, and hence believe, a true proposition, which is true for a reason different from that of their justification. Since justification is closed under entailment, the agent ends up with a justified but accidentally true belief, and hence without knowledge.

Consider Gettier's second case. Smith has evidence, and believes on the basis of this evidence, that 'Jones owns a Ford' J, but it is not true that Jones owns a Ford. He infers from J that 'Jones owns a Ford or Brown is in Barcelona', $J \vee B$. As it happens Brown is in Barcelona, hence $J \vee B$ is true, but Smith has no evidence concerning this, hence though he has a belief which is true and justified he does not know $J \vee B$.

On a constructive view of truth and knowledge this kind of accidental 'knowing' is not possible. A proposition is constructively true if there is a conclusive justification for it. Knowledge that a proposition is true requires that one have evidence that such a conclusive justification exists, either by being in possession of it, or by having evidence sufficient to rule out the falsity of the proposition. But this conclusive evidence is canonical or truth-making, it is constitutive of the truth of the proposition, hence the kind of accidental truth of a proposition upon which Gettier-cases depend cannot come up in a constructive setting.

The hallmark of classical, realist, truth, in contra-distinction to the constructive view, is that it is 'verification transcendent'. This means that it is possible for a proposition to be true independently of the existence of any justification for it. Hence, classically, a justification for a proposition, has no necessary connection with what makes the proposition true. By

contrast, constructively there is such a necessary connection between evidence and truth, there can be no truth without the existence of such evidence. Accordingly, having a justified constructively true belief, means one has evidence that conclusive, truth-making, evidence exists; having a justification implies one is aware of the existence of what actually makes the proposition true.

Hence in Gettier's second case Smith does not have a justified true belief, because he possesses no evidence as to the existence of what makes the proposition he believes true. $J \vee B$ is true because B is, and constructively this means that there is truth-making conclusive evidence for B, for the existence of which Smith has no evidence, and hence no justification. But then Smith has no evidence for the truth of $J \vee B$ either.

This point is even better exemplified in Gettier's first case. Here Smith and Jones have applied for a job. Smith has evidence that 'Jones will get the job and Jones has ten coins in his pocket', from which he infers that 'the man who will get the job has ten coins in his pocket'. Unbeknown to Smith he in fact will get the job and he also has ten coins in his pocket. Treating what Smith believes as a definite description, i.e. an (intuitionistic) existential generalisation, requires Smith to have a witness for the proposition he believes, and his evidence regarding Jones and the state of his pockets is not such a witness, having no essential connection with the truth of the proposition believed. Accordingly, Smith does not have a justified constructively true belief.

On the intuitionistic view of knowledge Gettier cases are not the problem for a fallibilistic view of knowledge as they are for the classical view.

7.4   The Lottery Paradox

The lottery paradox suggests that fallibilistic knowledge is inconsistent. Plausible principles of epistemic reasoning are mutually incompatible, precisely for a conception of knowledge which allows that less-than-conclusive evidence can be sufficient for knowledge.

Assuming a fair thousand ticket lottery, with only one winning ticket, the reasoning of the paradox may be formalised as follows. Let 1, 2 . . . etc. represent 'ticket 1 wins' etc.

Basic epistemic principles:

A. $(\mathbf{K}X_1 \wedge \ldots \wedge \mathbf{K}X_n) \to \mathbf{K}(X_1 \wedge \ldots \wedge X_n)$, *Conjunction Principle.*[22]

B. $(\mathbf{K}(X \to Y) \wedge \mathbf{K}X) \to \mathbf{K}Y$, *Weak Deduction Principle.*

C. $\neg\mathbf{K}\bot$ *Consistency.*

The first just asserts that if a set of propositions are each known then their conjunction is known. The second asserts that the consequences of known propositions are known. The third asserts that knowledge is consistent, contradictions are not knowable.

The basic assumptions of the lottery:

1. $\mathbf{K}(1 \vee \ldots \vee 1000)$ – it is known one of the tickets will win.

2. $\mathbf{K}\neg 1 \wedge \ldots \wedge \mathbf{K}\neg 1000$ – it is known 1 will lose, and known 2 will lose . . . etc.

Premise 2 depends on knowledge being fallible. If the evidence for a proposition says that the probability of it being true is .99 or greater then that justification is knowledge-producing.

From these premises and the principles of reasoning about knowledge, we reason as follows:

3. $\mathbf{K}(\neg 1 \wedge \ldots \wedge \neg 1000)$ – from 2 and the conjunction principle;

---

[22]The names are Kyburg's, [42].

4. $\mathbf{K}(1 \lor \ldots \lor 1000) \land \mathbf{K}(\neg 1 \land \ldots \land \neg 1000)$ – from 1 and 3;

5. $\mathbf{K}((1 \lor \ldots \lor 1000) \land (\neg 1 \land \ldots \land \neg 1000))$ – from 4 by the conjunction principle;

6. $(\neg 1 \land \ldots \land \neg 1000) \rightarrow \neg(1 \lor \ldots \lor 1000)$ – theorem of IPC;

7. $\mathbf{K}((\neg 1 \land \ldots \land \neg 1000) \rightarrow \neg(1 \lor \ldots \lor 1000))$ – from 6 since it is a theorem;

8. $\mathbf{K}((1 \lor \ldots \lor 1000) \land \neg(1 \lor \ldots \lor 1000))$ – from 7 by the weak deduction principle;

9. $\bot$, from 8 and Consistency.

The obvious response to the lottery paradox is that premise 2 is false; it is not known of any of the tickets that it will lose.[23] This is the very point of a fair lottery, that it is not known each ticket will lose precisely because every ticket has some chance of winning.[24] The problem is that this requires giving up on fallibilism. An intuitionistic approach to knowledge can preserve this argument without conceding to infallibilism.

Since for each ticket it is possible that it will win, because it is a lottery and it is fair, it is not known of any given ticket that it will lose. In the case of each ticket there is a defeater to the claim that it is known it will lose, namely the possibility in which it wins.

Consider the following IEL model, which models an agent's state of knowledge before and after the drawing of the lottery. Let $n \in \{1 \ldots 1000\}$. 1) $W = \{u, t_1 \ldots t_{1000}\}$; 2) $R = uRt_n$; 3) $E = uEt_n, \ t_nEt_n$; 4) $t_n \Vdash n$.[25]

---

[23]See [6, 69] for responses along this line.

[24]The oft-repeated trope which exemplifies this point: the New York State lottery's motto is 'you never know', because if you did buying a ticket would be nothing more than a voluntary donation to the state.

[25]We trust this abuse of notation will cause no confusion.
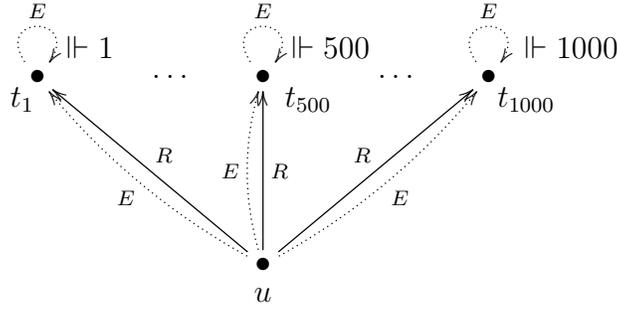
Figure 3.2: Model $\mathcal{M}_3$

State u is the state which the lottery paradox describes, states $t_1 \ldots t_{1000}$ each represent one possible outcome of the lottery.

Since for each n $t_n \Vdash n$ it follows that $t_n \Vdash 1 \vee \ldots \vee 1000$ and so $u \Vdash \mathbf{K}(1 \vee \ldots \vee 1000)$. It is known a ticket will win.

For intuitionistic knowledge to hold it must be that there is evidence that conclusive evidence for the proposition exists; if not available currently, it is potentially available. This is not the case for any of the propositions $\neg 1$ to $\neg 1000$; since for each ticket there exists the possibility of conclusive evidence that it wins.

Since $t_n \Vdash n$, for each n, $t_n \Vdash \mathbf{K}n$, hence $t_n \nVdash \mathbf{K}\neg 1 \wedge \ldots \wedge \mathbf{K}\neg 1000$. Moreover, since uEt_n $u \nVdash \mathbf{K}\neg 1 \wedge \ldots \wedge \mathbf{K}\neg 1000$. Hence $u \Vdash \neg(\mathbf{K}\neg 1 \wedge \ldots \wedge \mathbf{K}\neg 1000)$. The supposition that it is known that each ticket will lose yields a contradiction. Why? Because each ticket might win.

The lottery paradox, hence, is not a problem for a fallibilist view of knowledge, from an intuitionistic point of view. Intuitionistic knowledge allows us to maintain that knowledge is fallible, along with the obvious response to the lottery paradox that one does not know each ticket will lose, because there is a chance that it wins, and therefore $\neg(\mathbf{K}\neg 1 \wedge \ldots \wedge \mathbf{K}\neg 1000)$. Furthermore, there is also no need to give up on the epistemic principles A, B, or C.

8   Conclusion

The central idea of fallibilism is that less-than-conclusive justification can suffice for knowledge. Specifically this means that it is possible to know a proposition even if the justification does not logically guarantee the truth of the proposition. In the context of a classical view of truth and knowledge this implies that one might know falsely. The truth condition on knowledge can be expressed, in classical epistemic logic, by the reflection principle $\mathbf{K}A \to A$. Fallibilism, on the other hand, can be defined by the view that reflection is not a universally valid epistemic principle. Classically, then, some instance of $\mathbf{K}A \wedge \neg A$ is true. The problem is this violates the truth-condition. Fallibilism and the truth condition are classically inconsistent. Intuitionistic knowledge does not suffer from this problem. Reflection is not valid intuitionistically, but intuitionistic reflection $\mathbf{K}A \to \neg\neg A$ is, and this suffices to express the truth-condition: falsehoods cannot be known. Hence, intuitionistic knowledge, though fallibilistic, does preclude the possibility of error.

On this view the term 'fallibilism' appears to be a classically informed misnomer. Intuitionistic knowledge is not fallible, rather it can be justified by less-than-conclusive evidence which is nevertheless still adequate to justify a claim to knowledge. Such justification can be less than perfect, partial, incomplete, and it can leave room for doubt, which might be reasonable, but is unwarranted. But this accords with the motivation for fallibilism. The intuitive and attractive core of fallibilism is the acknowledgement of our epistemic limitations and imperfections, not that falsehoods can be known; an intuitionistic view of knowledge satisfies this.

References

[1]   Sergei Artemov. *Operational Modal Logic*. Tech. rep. MSI 95-29. Cornell University, 1995 (cit. on p. 95).

[2]   Sergei Artemov. "Explicit Provability and Constructive Semantics." In: *Bulletin of Symbolic Logic* 7.1 (2001), pp. 1–36 (cit. on p. 95).

[3]   Sergei Artemov. "Kolmogorov and Gödel's approach to Intuitionistic Logic: Current Developments." In: *Russian Mathematical Surveys* 59.2 (2004) (cit. on p. 95).

[4]   Sergei Artemov and Tudor Protopopescu. "Intuitionistic Epistemic Logic." In: *Review of Symbolic Logic* (2016). DOI: 10.1017/S1755020315000374 (cit. on pp. 78, 80, 89, 93, 94, 97, 101).

[5]   Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Clarendon Press, Oxford, 1997 (cit. on pp. 92, 95).

[6]   Stewart Cohen. "How to Be a Fallibilist." In: *Philosophical Perspectives* 2 (1988), pp. 91–123 (cit. on p. 109).

[7]   P. Contu. "The Justification of the Logical Laws Revisited." English. In: *Synthese* 148.3 (2006), pp. 573–588 (cit. on p. 90).

[8]   Dirk van Dalen. "Intuitionistic Logic." In: *Handbook of Philosophical Logic*. 2nd ed. Vol. 5. Springer, 2002, pp. 1–114 (cit. on p. 92).

[9]   Dirk van Dalen. *Logic and Structure*. Springer, 2004 (cit. on pp. 88, 96).

[10] Dirk van Dalen and A.S. Troelstra. *Constructivism in Mathematics An Introduction, vol. I*. Studies in Logic and the Foundations of Mathematics vol. 121. Elsevier, 1988 (cit. on pp. 88, 92).

[11] Dirk van Dalen and A.S. Troelstra. *Constructivism in Mathematics An Introduction, vol. II*. Studies in Logic and the Foundations of Mathematics vol. 121. Elsevier, 1988 (cit. on p. 90).

[12] Walter Dean and Hidenori Kurokawa. "From the Knowability Paradox to the Existence of Proofs." In: *Synthese* (2009) (cit. on pp. 91, 100).

[13] Kosta Došen. "Intuitionistic Double Negation as a Necessity Operator." In: *Publications de L'Institute Mathématique (Beograd)(NS)* 35.49 (1984), pp. 15–20 (cit. on p. 92).

[14] Michael Anthony Eardley Dummett. "Truth." In: *Truth and Other Enigmas*. Cambridge, MA: Harvard University Press, 1959, pp. 1–24 (cit. on p. 100).

[15] Michael Anthony Eardley Dummett. "Realism." In: *Truth and Other Enigmas*. Cambridge: Harvard University Press, 1963, pp. 145–165 (cit. on p. 90).

[16] Michael Anthony Eardley Dummett. "The Justification of Deduction." In: Cambridge: Harvard University Press, 1973 (cit. on p. 102).

[17] Michael Anthony Eardley Dummett. "The Philosophical Basis of Intuitionistic Logic." In: *Truth and Other Enigmas*. Cambridge, Mass.: Harvard University Press, 1973, pp. 215–247 (cit. on p. 90).

[18] Michael Anthony Eardley Dummett. "What is a Theory of Meaning II." In: *The Seas of Language*. New York: Oxford University Press, 1976, pp. 34–93 (cit. on p. 90).

[19] Michael Anthony Eardley Dummett. *Elements of Intuitionism*. Clarendon Press Oxford, 1977 (cit. on p. 90).

[20] Michael Anthony Eardley Dummett. *Truth and Other Enigmas*. Cambridge: Harvard University Press, 1978.

[21] Michael Anthony Eardley Dummett. *The Logical Basis of Metaphysics*. The William James Lectures, 1976. Cambridge: Harvard University Press, 1991 (cit. on pp. 90, 101).

[22] Michael Anthony Eardley Dummett. "Truth from the Constructive Standpoint." In: *Theoria* 64.2-3 (1998), pp. 122–138. DOI: 10.1111/j.1755-2567.1998.tb00184.x (cit. on p. 100).

[23] R. Fagin et al. *Reasoning about Knowledge*. MIT Press, 1996 (cit. on p. 81).

[24] Richard Feldman. "Fallibilism and Knowing that One Knows." In: *The Philosophical Review* 90.2 (Apr. 1981), pp. 266–282 (cit. on p. 80).

[25] Gerhard Gentzen. "Investigations into Logical Deduction." In: *The Collected Papers of Gerhard Gentzen*. Ed. by M. E. Zsabo. Amsterdam: North-Holland, 1935 (cit. on p. 101).

[26] Edmund Gettier. "Is Knowledge Justified True Belief?" In: *The Theory of Knowledge*. Ed. by L.P. Pojman. Belmont, California: Wadsworth Thompson, 1963, pp. 125–127 (cit. on pp. 85, 105).

[27] Valentin Glivenko. "On Some Points of the Logic of Mr. Brouwer." In: *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's*. Ed. by P. Mancosu. Oxford University Press, 1929. Chap. 22, pp. 301–305 (cit. on p. 92).

[28] Kurt Gödel. "An Interpretation of the Intuitionistic Propositional Calculus." In: *Collected Works*. Ed. by S Feferman et al. Vol. 1. Oxford University Press, 1933, pp. 301–303 (cit. on p. 95).

[29] Gilbert Harman. *Thought*. Princeton University Press, 1973, pp. 256–258 (cit. on pp. 79, 102).

[30]   W.D. Hart. "Access and Inference." In: *Proceedings of the Aristotelian Society* LIII (1979), pp. 153–166 (cit. on p. 91).

[31]   V. F. Hendricks. *Formal and Mainstream Epistemology.* Cambridge University Press, 2006 (cit. on p. 88).

[32]   Stephen Cade Hetherington. *Knowledge Puzzles. An Introduction to Epistemology.* Westview Press, 1996 (cit. on p. 81).

[33]   Arend Heyting. "On Intuitionistic Logic." In: *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* Ed. by P. Mancosu. Oxford University Press, 1930. Chap. 23, pp. 306–310 (cit. on p. 88).

[34]   Arend Heyting. *Intuitionism: An Introduction.* 2nd Revised. Vol. 41. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland, 1966 (cit. on pp. 92, 100).

[35]   Yoichi Hirai. "An intuitionistic epistemic logic for sequential consistency on shared memory." In: *Logic for Programming, Artificial Intelligence, and Reasoning.* Springer. 2010, pp. 272–289 (cit. on p. 94).

[36]   D. Khlentzos. *Naturalistic Realism and the Anti-Realist Challenge.* MIT Press, 2004 (cit. on p. 91).

[37]   A. Kolmogorov. "On the Interpretation of Intuitionistic Logic." In: *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* Ed. by P. Mancosu. New York: Oxford University Press, 1932. Chap. 25, pp. 328–334 (cit. on pp. 88, 95).

[38]   Andrei N. Kolmogorov. "On the Principle of Excluded Middle." In: *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931.* Ed. and trans. by Jean van Heijenoort. Harvard University Press, 1925, pp. 415–437 (cit. on p. 92).

[39]   Saul Aaron Kripke. "Two Paradoxes of Knowledge." In: *Philosophical Troubles. Collected Papers Vol I.* Ed. by Saul A. Kripke. Oxford University Press, 2011, pp. 27–51 (cit. on pp. 79, 102).

[40]   Vladimir N. Krupski and Alexey Yatmanov. "Sequent Calculus for Intuitionistic Epistemic Logic IEL." In: *Logical Foundations of Computer Science. International Symposium.* Proceedings. Logical Foundations of Computer Science. LFCS 2016 (Dearfield Beach, Florida, USA, Jan. 4–6, 2016). Ed. by Sergei Artemov and Anil Nerode. Lecture Notes in Computer Science 9537. Springer, 2016, pp. 187–201. DOI: 10.1007/978-3-319-27683-0 (cit. on p. 97).

[41]   Henry E. Kyburg Jr. *Probability and the Logic of Rational Belief.* Vol. 34. Wesleyan University Press, 1961, pp. 283–285 (cit. on p. 85).

[42]   Henry E. Kyburg Jr. "Conjunctivitis." In: *Epistemology and Inference.* University of Minnesota Press, 1970, pp. 232–254 (cit. on pp. 85, 108).

[43]   David Lewis. "Elusive Knowledge." In: *Australasian Journal of Philosophy* 74.4 (1996), pp. 549–567 (cit. on p. 88).

[44] P. Mancosu, ed. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* Oxford University Press, 1998.

[45] Per Martin-Löf. "A Path from Logic to Metaphyiscs." In: *Atti del Congresso Nuovi problemi della logica e della filosofia della scienza.* Viareggio: Società Italiana di Logica e Filosofia delle Scienze, 1990, pp. 141–149 (cit. on p. 100).

[46] Per Martin-Löf. "Truth and Knowability on the Principles C and K of Michael Dummett." In: *Truth in Mathematics.* Ed. by H.G. Dales and G. Olivieri. Oxford University Press, 1998. Chap. 5, pp. 105–114 (cit. on pp. 90, 100).

[47] Per Martin-Löf. "Verificationism Then and Now." In: *Judgement and the Epistemic Foundation of Logic.* Ed. by M. van der Schaar. Logic, Epistemology, and the Unity of Science. Springer, 2013. Chap. 1, pp. 1–14 (cit. on p. 100).

[48] E. Martino and G. Usberti. "Temporal and Atemporal Truth in Intuitionistic Mathematics." In: *Topoi* 13.2 (1994), pp. 83–92. DOI: `10.1007/BF00763507` (cit. on p. 91).

[49] John Charles Chenoweth McKinsey and Alfred Tarski. "Some Theorems About the Sentential Calculi of Lewis and Heyting." English. In: *The Journal of Symbolic Logic* 13.1 (Mar. 1948), pp. 1–15 (cit. on p. 95).

[50] J.-J. Ch. Meyer and W. van der Hoek. *Epistemic Loigc for AI and Computer Science.* Cambridge Tracts in Theoretical Computer Science 41. Cambridge University Press, 1995 (cit. on p. 81).

[51] J. Murzi. "Knowability and Bivalence: Intuitionistic Solutions ot the Paradox of Knowability." In: *Synthese* (2009) (cit. on p. 91).

[52] P. Percival. "Fitch and Intuitionistic Knowability." In: *Analysis* 50.3 (1990), pp. 182–187 (cit. on p. 91).

[53] Dag Prawitz. "Meaning and Proofs: on the Conflict Between Classical and Intuitionistic Logic." In: *Theoria* 43.1 (1977), pp. 2–40 (cit. on p. 100).

[54] Dag Prawitz. "Intuitionistic Logic: A Philosophical Challenge." In: *Logic and Philosophy.* Ed. by G. H. von Wright. Hague: Martinus Nijhoff, 1980, pp. 1–10 (cit. on p. 90).

[55] Dag Prawitz. "Truth and Objectivity from a Verificationist Point of View." In: *Truth in Mathematics.* Ed. by H.G. Dales and G. Olivieri. Oxford University Press, 1998. Chap. 2, pp. 41–51 (cit. on p. 90).

[56] Dag Prawitz. "Truth From a Constructive Perspective." In: *Truth in Perspective, Recent Issues in Logic, Representation and Ontology.* Ed. by C. Martinez, U. Rivas, and L. Villegas-Forero. Brookfield, Vermont: Ashgate, 1998. Chap. 2, pp. 23–35 (cit. on p. 100).

[57] Dag Prawitz. "Problems for a Generalization of a Verificationist Theory of Meaning." In: *Topoi* 21 (2002), pp. 87–92 (cit. on p. 100).

[58] Dag Prawitz. "Logical Consequence From a Constructive Point of View." In: *The Oxford Handbook of Philosophy of Mathematics and Logic.* Ed. by S. Shapiro. Oxford University Press, 2005. Chap. 22, pp. 671–695 (cit. on p. 90).

[59] Dag Prawitz. "Meaning Approached via Proofs." English. In: *Synthese* 148.3 (2006), pp. 507–524 (cit. on p. 90).

[60] Carlo Proietti. "Intuitionistic Epistemic Logic, Kripke Models and Fitch's Paradox." In: *Journal of Philosophical Logic* 41.5 (2012). 10.1007/s10992-011-9207-1, pp. 877–900 (cit. on p. 94).

[61] Tudor Protopopescu. "An Arithmetical Interpretation of Verification and Intuitionistic Knowledge." In: *ArXiv.* math.LO 1601.03059v1 (Jan. 2016). Post-print correcting various publisher's errors. Published in Artemov, S. and Nerode, A., editors, Logical Foundations of Computer Science, Number 9537 in Lecture Notes in Computer Science, pp. 317-330, Springer. DOI: 10.1007/978-3-319-27683-0_22. ArXiv: 1601.03059v1 (math.LO) (cit. on p. 95).

[62] Baron Reed. "How to Think About Fallibilism." In: *Philosophical Studies. An International Journal for Philosophy in the Analytic Tradition* 107.2 (Jan. 2002), pp. 143–157 (cit. on p. 80).

[63] Baron Reed. "Fallibilism." In: *Philosophy Compass* 9.7 (2012), pp. 585–596. DOI: 10.1111/j.1747-9991.2012.00502.x (cit. on pp. 81, 86).

[64] Peter Schroeder-Heister. "Validity Concepts in Proof-Theoretic Semantics." English. In: *Synthese* 148.3 (2006), pp. 525–571 (cit. on p. 90).

[65] Göran Sundholm. "Proof theory and Meaning." In: *Handbook of Philosophical Logic.* Ed. by D. Gabbay and F. Guenther. 2nd. Vol. 9. Kluwer, 2002, pp. 165–198 (cit. on p. 90).

[66] Neil Tennant. *The Taming of the True.* Oxford University Press, 1997 (cit. on p. 90).

[67] A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 2000 (cit. on p. 95).

[68] G. Usberti. "Towards a Semantics Based on the Notion of Justification." English. In: *Synthese* 148.3 (2006), pp. 675–699 (cit. on p. 90).

[69] Jonathan Vogel. "Are There Counterexamples to the Closure Principle? Contemporary Perspectives on Skepticism." In: *Doubting. Contemporary Perspectives on Skepticism.* Ed. by Michael D. Roth and Glenn Ross. Vol. 48. Philosophical Studies 48. Kluwer Academic Publishers, 1990, pp. 13–28 (cit. on p. 109).

[70] Timothy Williamson. "Intuitionism Disproved?" In: *Analysis* 42.4 (1982), pp. 203–207 (cit. on p. 91).

[71] Timothy Williamson. "Knowability and Constructivism." In: *Philosophical Quarterly* 38.153 (1988), pp. 422–432 (cit. on p. 91).

[72] Timothy Williamson. "On Intuitionistic Modal Epistemic Logic." In: *Journal of Philosophical Logic* 21.1 (1992), pp. 63–89 (cit. on pp. 89, 94).

[73] Timothy Williamson. *Knowledge and Its Limits.* Oxford: Oxford University Press, 2000 (cit. on p. 86).

[74]   Crispin Wright. "Can A Davidsonian Meaning-theory be Construed in Terms of Assertibility?" In: *Realism, Meaning and Truth*. 2nd. Cambridge, MA: Blackwell, 1993. Chap. 14, pp. 403–432 (cit. on p. 91).