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Master Thesis

A Convex Optimization Approach to Synthesizing Low Order l_1 / H_p Controllers for Autopilot Pitch Aircraft

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ABSTRACT

During the last decade, a large research effort has been devoted to the problem of designing robust controllers capable of guaranteeing stability in the face of plant uncertainty. In spite of large research efforts, this problem is not completely resolved.

Alternatively, mixed objective control problems have attracted much attention lately since they allow for directly capturing different performance specifications without resorting to approximations or the use of weighting functions; thus, eliminating the need for trial and error type iterations. Of course, it is not always possible to capture all the desired performance specifications in a single norm constraint, and so a number of researchers have considered mixed norm minimization problems, including H_2/H_∞, l_1/H_∞, l_1/H_2 and H_2/l_1.

This thesis, considers the problem of minimizing the l_1 norm of a certain closed-loop transfer function, while maintaining the H_2 norm (mixed l_1/H_2), or the H_∞ norm (mixed l_1/H_∞), of a different transfer function below a pre-specified level for an autopilot pitch aircraft.

The main results of this thesis show that a two-stage process can synthesize suboptimal controllers, involving a convex optimization problem, which optimizes the l_1 norm, internal characteristic of controller and H_∞ or H_2 optimization that optimize the external characteristic of the system. Furthermore, this approach also provides a CVX-based parameterization of all suboptimal output feedback controllers, including reduced order, for mixed l_1/H_∞ and l_1/H_2 problems. The developed approach is tested by designing a control with multi-objective autopilot controlling the pitch of an aircraft.

ملخص :

منذ العقد السابق ، أجريت أبحاث ومجهودات كبيرة لتصميم وإنجاز متحكم متماسك وله القدرة على ضمان إستقرارية عالية أمام التغيرات المفاجئة التي تحدث نتيجة الإرتيابات التي يمكن أن تستجد على نظام التحكم بأكمله . وكنتيجة لهذا الهدف تم تطوير طرق فعالة ودقيقة مثل H_2 و H_∞ حيث تم توجيه أبحاث الإستقرارية وصلابة النظام بوجود إرتيابات في نظام ذو طويلة-محدودة. و كنتيجة الحصول على فعالية عالية باستخلاص الحلول باستعمال الحاسوب.

بالمقابل ، إشكالية التحكم ذات الهدف المزدوج شدت الكثير من الإنتباه مأخراً حيث انها مكنت من الحصول مباشرة على مميزات مختلفة بدون اللجوء إلى التقريب أو إستعمال دوال الوزن، وبالتالي عدم الحاجة إلى عملية التجريب والخطأ . بالطبع لايمكن دائما الحصول على كل الخصائص بإستعمال طويلة واحدة ملزمة, ومنه العديد من الابحاث أخذت بعين الإعتبار إشكالية تصغير الطويلة المزدوجة ، وتتظمن الابحاث H_2/H_∞ ، l_1/H_∞ ، l_1/H_2 و H_2/l_1.

في هذا البحث أخذنا بعين الإعتبار إشكالية تصغير طويلة l_1 لدالة تحكم ذات حلقة تحكم مغلقة، مع المحافظة على الطويلة H_2 (المزيج l_1/H_2)، أو الطويلة H_∞ (المزيج l_1/H_∞)، لمختلف دوال الإنتقال تحت مستويات معرفة مسبقا.

في نظرية التحكم الكلاسيكية، فإن قياس الكفائة لنظام تحكم ذو دائرة مغلقة تتمركز حول قدرة النظام على إلغاء (في حالة الإستقرار) الإظطرابات المعروفة، الضجيج، أو قياسات الخطأ التي يمكن أن تظهر في مختلف قطاعات الحلقة المغلقة، حيث أن المجسات، المشغل الميكانيكي، أو مخارج التحكم في بعض المشاكل لاتملك دالة حل رياضي، ومنه الحاجة إلى طرق حل أمثل.

الأمثلية هي الجواب؛ LMI هي تقنية الحل الأمثل. ومن هنا يدخل مفهوم الحل الأمثل، وبالتالي تضييق الإشكالية إلى حل أمثل محدب . إحدى الطرق لحل هذا النوع من الإشكاليات هي LMI (المصفوفة الخطية الغير متساوية) أوCVX الحل الامثل لدالة محدبة بهدف الوصول إلى حل امثل إن كان موجودا أصلا، وتصميم متحكم أمثل يقلل من تأثيرات إشارات الضجيج، وخطأ القياسات أو يلغي تماما إشارات الضجيج.

النتائج الاساسية لهذا البحث تظهر أن مستويين من المعالجة يمكن أن تركب المتحكم دُوَيْنَ الأَمْثَل، مما يستدعي حل أمثل محدب للإشكالية الذي يجد الحل الامثل لطويلة l_1 باعتبارها المميزة الداخلية للمتحكم و الطويلة H_∞, H_2 التي تجد الحل الامثل باعتباره المميزة الخارجية للنظام. بالإظافة إلى أن هذه الطريقة توفر CVX مرتكزة على الحلول الرياضية لكل مخارج دُوَيْنَ الأَمْثَل للتغذية الخلفية للمتحكم بالإظافة إلى تصغير رتبة كل منها , لكل من l_1/H_∞ وl_1/H_2 . وكمثال على التطبيق سوف نصمم جهاز طيران تلقائي الذي سوف يتحكم في زاوية (pitch) لطائرة .

DEDICATION

To My Parents
To my wife
To My daughters Batoul and Sara

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Firstly, I would like to thank my supervisor, DR. Hatem Elaydi, for his guidance and expertise, for his insistence on the highest standards, and especially for his invaluable input into this thesis.
Throughout the course of my master, I benefited from many interesting discussions with DR. Hatem Elaydi on the subject of applications of mixed objective Control.

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CHAPTER 1 INTRODUCTION

The principle of control and feedback control are more and more penetrated in our daily life. Technologies based on complexity, precision, performance and security of systems give big place to build-in regulators and control loops. Control is, generally speaking, the activity that affects a system to behave in a desired way. In feedback control, the system behavior is continuously measured or monitored, and compared to the desired behavior. Yet, the world as we know is unimaginable without control.

1.1. Background

To design a controller, we should approximate our system to model-based approach. This model approach becomes complex whilst the control engineering addresses complex tasks. The model-based approach is mathematical formulations that describe the characteristic and behavior of our system. These mathematical models are used to predict, take decision, and define a control method to our system.

It is necessary we use the simulation to study the system behavior and prediction before implementing our controller. The difference result between the practical model and the mathematical model may be called uncertainty. The uncertainty of the system should be taken into account when building-in our controller to ensure the stability of the system over plant uncertainty and disturbances. These studies lead to the field of robust control.

If multi-objective design is desired, we are oriented to use more powerful techniques to attract distinctive performances possible. These studies lead to the field of linear matrices inequality (LMI) by utilizing convex optimization (CVX).

1.2. Robust Disturbance Attenuation

Skogestad, Postlethwaite (2005) [1] and Sanchez-Pena, Sznaier (1998) [2] gave good introductions and overviews on robust control. The majority of the results in this area considered quadratic-type performance and stability criteria. Well-known examples are least squares, L_2 signal norms, H_2 and H_∞ system norms, and integral quadratic constraints. Related performance frameworks like H_∞ control were applied to many real problems in academic and industry. The Interpretations of the corresponding system behavior in terms of energy, dissipativity, or frequency-domain properties contributed to the attraction of quadratic criteria.

In practice, it is more desirable to influence directly the maximum control error, the response overshoot, the maximum values of control inputs, or other time domain properties of system response. Such goals can be achieved generally by quadratic-type approaches in principle, but often by indirectly and with numerous design iterations.

Norms are considered as measures that can be utilized to constraint the behavior of systems. Examples of such measures are L_1, L_2, L_∞, H_2 and H_∞ norms.

To address the mentioned time-domain properties of a system response more directly, it is natural to consider performance in terms of the L_∞ signal norm

$$\left\| v \right\|_\infty = \sup_t \max_t \left| v_i(t) \right| \tag{1.1}$$

The L_∞-norm measures the maximum amplitude of the components v_i of a signal vector v over time t. To obtain a corresponding measure for a stable system G, the so-called amplitude gain or L_∞-gain is used.

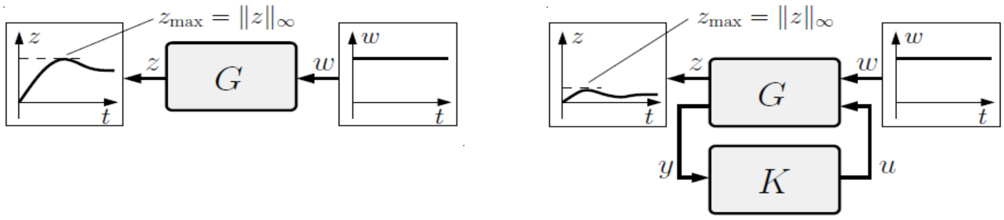
$$\left\| G \right\|_{\infty - ind} := \sup_{0<\left\| w \right\|_\infty<\infty} \frac{\left\| Gw \right\|_\infty}{\left\| w \right\|_\infty} \tag{1.2}$$

This gain characterizes the worst-case amplitude of the system output $z = Gw$ normalized by the maximum amplitude of the input w under the assumption of zero initial conditions. In other words, the L_∞-gain describes how well a system attenuates persistent disturbances. The gain notion is shown in Figure 1.1(a). One speaks of L_∞-gain based disturbance attenuation if a stabilizing controller is look for such L_∞-gain of the closed-loop system is minimized or bounded.

Where:

$$G(s) = \left[\begin{array}{c|c} G_{11}(s) & G_{12}(S) \\ \hline G_{21}(s) & G_{22}(s) \end{array} \right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \tag{1.3}$$

As a common situation, think of z being the control error, the amplitude of which is supposed to be maintained as small as possible as showed in Figure 1.1(b). It can be shown that; for LTI systems, the L_∞-gain is equal to the L_1-norm of the system's impulse response. Therefore, the name L_1- optimal control is used for the field of L_∞-gain based disturbance attenuation. Except the rejection of disturbances, many other control goals can be formulated in such a framework.



(a) L_∞ -gain Open-loop without feedback (b) L_∞ -gain with output feedback.

Figure (1.1): L_∞ -gain

In Figure 1.1 (a) the input w of the system G is assumed to be the worst-case input in terms of the L_∞ -gain. Then the L_∞ -gain $\|G\|_\infty$ is equal to the maximum amplitude of the corresponding output signal Z divided by the maximum amplitude of w. In Figure 1.1 (b) If G is compensated by a controller K such that the L_∞ -gain decreases, then the maximum amplitude of Z normalized by the worst-case input w is smaller than in Figure 1.1 (a) or equal.

Set point control, following of reference commands, minimization of resource consumption, or filtering problems are examples of control problems. The related literature treats almost only discrete-time design methods. Since in this case, it is possible to formulate tractable synthesis conditions that can be solved on a computer. The corresponding measures for discrete-time signals and systems are denoted L_∞ - and L_1 -norms, respectively. The same idea carries over to performance considerations for uncertain systems. With help of the above norm descriptions, it is furthermore possible to quantify uncertainties in terms of their input/output behavior and their maximum gain. If control goals are formulated in terms of several norm constraints, or when different norms are used within one control design problem, we call that multi-objective control.

While the vast amount of contributions in the field of robust linear control is concerned with H_∞ and H_2 control, the L_1 paradigm has also seen a number of basic and promising results. The L_1 control literature treats analysis and synthesis of systems both with and without uncertainties. The available analysis results are straightforward norm computations in the nominal case, whereas for models with uncertainties, small-gain theory in combination with scaling is applied. The synthesis methods proposed so far treat nominal control design in terms of linear programs (LPs). The literature moreover discusses robust design of LTI controllers with respect to structured dynamic uncertainties using iterations over LPs.

L_1 Performance objectives allow to specify desired control goals in the time-domain and to address robustness issues. Although there have been a number of basic results, the literature has paid less attention to L_1 control than to quadratic-type performance frameworks.

1.3. Statement of the Problem:

We have two problems: The first problems is concerned with l_1 / H_2 problem where the system is controllable (A, B_1) and observable$(A,\ C_2)$ We wish to find an internally stabilizing controller such that the mixed objective l_1 / H_2 is minimized.

In the l_1 problem, the design of an internally stabilizing controller minimizes the l_∞ norm of the regulated output due to the worst case magnitude bounded disturbances.

The H_2 problem is minimizing the energy contained in the pulse response of the closed loop transfer function from disturbance to the measured regulated output.

Here we address the problem of minimizing the two norm of an input-output transfer function while keeping the two induced norm of another transfer function below a certain level.

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1.4. Literature Review

- In the mid to late 70’s, optimal control enjoyed tremendous success solving variety of control application problems. The modern optimal control paradigm for feedback design, the LQG problem, however, had relatively little impact on practical control design.

- At about the same time, singular values or the H_∞, norm was proposed for robustness analysis of multivariable systems. This point of view added necessity to the small gain methods of the 1960s [3, 4, 5]. That is, whereas small gain gave sufficient conditions for stability for a set of uncertainty, the robust control interpretation was that the same condition was necessary and sufficient for a particular set. One of the motivations for the original introduction of H_∞, methods by Zames [6] was to emphasize plant uncertainty. The H_∞, norm was found to be appropriate for specifying both the level of plant uncertainty and the signal gain from disturbance inputs to error outputs in the controlled system. The H_∞, norm gives the maximum energy gain (the induced l_2 system gain), or sinusoidal gain of the system.

- Simple state space H_∞ controller formulae were first announced in Glover and Doyle [7]. However, the very simplicity of the new formulae and their similarity with the H_2 ones suggested a more direct approach.

- Independent encouragement for a simpler approach to the H_∞ problem came from papers by Khargonekar, Petersen, Rotea, and Zhou [8, 9], Zhou et al. [10]. They showed that for the state feedback H_∞ problem one can choose a constant gain as an (sub) optimal controller. In addition, a formula for the state-feedback gain matrix was given in terms of an algebraic Riccati equation. Also, these papers established connections between H_∞-optimal control, quadratic stabilization, and linear-quadratic differential games. They showed that the state-feedback H_∞ problem can be solved by solving an algebraic Riccati equation and completing the square. Relations between H_∞ was established with many other topics in modern control: e.g. risk sensitive control of Whittle (1981, 1990); differential games (see Bagar and Bernhard (1991) [11], Limebeer et al (1992) [12], Green and Limebeer (1995) [13]. The state-space theory of H_∞ was also carried much further, by generalizing time-invariant to time varying, infinite horizon to finite horizon, and finite dimensional to infinite dimensional and even to some nonlinear results. Most of these results used fairly conventional modern optimal control techniques.

- In Sadati study (2003) [14], the author used a new approach for controller order reduction based on minimization of the rank of a matrix variable, subject to linear matrix inequality constraints. The proposed approach was applied to an H_∞ high order controller, which was designed for an active suspension system. The performance and stability achieved by the reduced order controller was compared with those achieved by the high-order controller. The comparison was based on both simulation and experimental results obtained by digital controller implementation.

- In Alireza Khosravi (2007) [15], the main purpose of this paper, was performing a new solution on the basis of Linear Matrix Inequality (LMI) for designing induced L_∞ optimal controllers. Induced L_∞ optimal control allowed directly time-domain specifications into the controller synthesis procedure and furnished a complete solution to the robust performance problem. The new technique, which was proposed as an algorithm, combined the original concept of peak-to-peak gain of designed system with optimal control theory and employed a free design parameter allowing for a flexible management of the tradeoff between robustness to disturbance signals and magnitude of the worst peak-to-peak gain of the designed system. For the convergence of this algorithm, a scope was found on the basis of the H_∞ norm. If the length of this interval was small, we had a good estimate of the actual optimal peak-to-peak gain that was achievable by control.

- The l_1 control problem was formulated in Barabanov and Granichin (1984) [16], Vidyasagar (1986) [17]. The l_1 control literature generally treats discrete-time problems, since only these lead to numerically tractable synthesis conditions. l_1 framework is attractive for controller synthesis. More elaborate discussions can be found in Vidyasagar (1986) [17]; Dahleh and Khammash (1993) [18]; Dahleh and Diaz- Bobillo (1995) [19].

- Current drawbacks of the l_1 framework are the often high order of the resulting controllers, and the possibly large size of the LPs. Accounts on how the l_1 framework can be used for practical applications are found in e.g. Spillman and Ridgely (1997) [20]; Tadeo et al. (1998) [21]; Malaterre and Khammash (2000) [22]; Rieber et al. (2005) [23]; Stemmer et al. (2005) [29]; Rieber and Allgower (2006) [24].

- Some interesting properties of l_1-optimal synthesis are that the optimal controller may be nonunique, that the optimal controller may be dynamic even in the state-feedback case, that nonlinear static state-feedback performs as well as linear dynamic feedback, and that nonlinear controllers may result in better performance than the optimal linear controller (Diaz-Bobillo and Dahleh (1992) [25]; Dahleh and Shamma (1992) [26]; Shamma (1993) [27]; Blanchini and Sznaier (1995) [28]; Stoorvogel (1995) [29]; Shamma (1996) [30]. Relations to continuous-time controllers and sampled-data implementations are established in

Dullerud and Francis (1992) [31]; Ohta et al. (1992) [32]; Blanchini and Sznaier (1994) [33]; Chen and Francis (1995) [34].

- The l_∞ -gain of linear systems subject to time-varying parametric uncertainties was analysed by Rieber et al. (2006) [35]; Rieber et al. (2007) [36]. The time-domain response of an uncertain system is characterized in terms of linear fractional transformations. Such a characterization enables robustness analysis of uncertain system responses.

- Introductions and overviews on the topics of H_2 , H_∞, l_1, and multi-objective control are given in Dahleh and Diaz-Bobillo (1995) [37]; Zhou et al. (1996)[38]; Sanchez-Pena and Sznaier (1998) [39]; Skogestad and Postlethwaite (2005) [40]; Rieber and Allgower (2006) [41].

- In Murti V. Salapaka et.al. (1995) [42], the problem of minimizing the l_1 norm for internally stabilizing controllers while keeping its H_2 norm ($\mathrm{L}_1/\mathrm{H}_2$) under a specified level was considered in this paper. The problem was analysed for the discrete-time, SISO, linear time invariant case. Duality theory was employed to show that any optimal solution is a finite impulse response sequence and an a priori bound is given on its length. The problem was reduced to a finite dimensional convex optimization problem with an a priori determined dimension. Finally it was shown that, in the region of interest of the H_2 constraint level the optimal was unique and continuous with respect to changes in the constraint level. However, the paper did not tackle the continuous MIMO system.

- In Mario Szneir et.al. (1998) [43], a methodology for designing mixed $\mathrm{L}_1/\mathrm{H}_\infty$ controllers for MIMO systems was considered in this paper. These controllers allow for minimizing the worst case peak output due to persistent disturbances, while at the same time satisfying an H_∞-norm constraint upon a given closed loop transfer function. The main results of the paper showed that 1) contrary to the case H_2/H_∞, the $\mathrm{L}_1/\mathrm{H}_\infty$ problem admits a solution in l_1, and 2) rational suboptimal controllers can be obtained by solving a sequence of problems, each one consisting of a finite-dimensional convex optimization and a four-block H_∞ problem. Moreover, this sequence of controllers converges in the l_1 topology to an optimum. However, the paper did not tackle the continues time.

- In Takeshi Amishima et.al. (1998) [44], the problem of minimizing the l_1 norm of a closed-loop transfer function while keeping its H_2 norm ($\mathrm{L}_1/\mathrm{H}_2$) under a specified level was considered in this paper. It was showed that the optimal closed-loop impulse response has finite support, and thus a non-rational Laplace transform. To solve this difficulty a method for synthesizing rational controllers with performance arbitrarily close to optimal was proposed. However, the paper did not tackle the MIMO system.

- In M. Sznaier et.al. (2000) [45], a solution to general continuous-time mixed H_2/H_∞ problems, based upon constructing a family of approximating problems was considered, obtained by solving an equivalent discrete-time problem. Each of these approximations can be solved efficiently, and the resulting controllers converge strongly in the H_2 topology to the optimal solution. However, the paper did not tackle the continuous time.

- In Jun Wu et.al.(2002) [46], the general discrete-time single-input single-output (SISO) mixed H_2/L_1 control problem was considered in this paper. It was found that the existing results of duality theory could not directly be applied to this infinite dimensional optimization problem; however, the approach based on duality theory was useful in research on the mixed H_2/L_1 optimization problem, as it was often that the dual problem could be solved for more easily than the primal problem. However, the paper did not tackle the MIMO system.

- In Xiaofu Ji et.al. (2009) [47], the mixed L_1/H_∞ control problem for a class of uncertain linear singular systems was considered using a matrix inequality approach. The purpose was to design a state feedback control law such that the resultant closed-loop system is regular, impulse-free, stable and satisfies some given mixed L_1/H_∞ performance. A sufficient condition for the existence of such control law was given in terms of a set of matrix inequalities by the introduction of inescapable set and norms. When these matrix inequalities are feasible, an explicit expression of the desired state feedback control law was given. However, the paper did not tackle the MIMO system.

1.5. Motivation

Continuous-time and discrete time multiobjective l_1/H_2 and l_1/H_∞ problem can be solved using the youla parameterization; however, the order of the optimal controller is not bounded by the order of the plant [Salapaka et al. (1995), sznaier & Bu (1998)] [42, 44].

Recent development in convex optimization motivated this work to cast this problem using bounds on the l_1, H_2 or H_∞ norms to produce low order suboptimal controllers.

1.6. Thesis contribution

The main contribution of this thesis is the application of mixed objective problems L_1 / H_2 and L_1 / H_∞ for SISO system, linear time invariant to designing controller for Autopilot pitch Aircraft. In this thesis, the Lagrange duality principle methodology proposed by Slapaka and Dahleh (1995) [42] will be used, but with introducing some change to this method as follow:

First, this thesis will consider the minimizing sensitivity of the output system to the input disturbance by the method of 2-norm and infinity norm, this minimization will considered as primal Lagrange multiplier. The reason of using $\min_{K \in s} \|T_{ZW}\|_2$ and $\min_{K \in s} \|T_{ZW}\|_\infty$ is to minimize or reject the effect of disturbance to our system.

In the Second objective of the dual Lagrange multiplier problem, this thesis will consider the minimizing of bounded norm output of the controller designed by using the L_1 norm theory, this lead to use convex optimization approach. The solution of this problem can be obtainable by using CVX-toolbox built by Stephen Boyd. The reason for using the bounded-norm output as dual problem is to minimize the cost function of the output controller and limit the brusque variation introduced to the system by the input control, which lead to BIBO system.

The formation of the mixed objective problems are resolved by using the primal problem as constraint to the dual problem for the two methods L_1 / H_2 and L_1 / H_∞.

1.7. Preliminaries and Notation

This section briefly introduces some preliminary definitions and basic notation to enable a compact and precise statement of problem formulations and results. Related definitions are found in Dahleh and Diaz-Bobillo (1995) [37]; Zhou et al. (1996) [38]; Skogestad and Postlethwaite (2005) [40].

1.7.1. General

$|\cdot|$ is the absolute value of a number, and $\|v\| = \sqrt{v^T v}$ denotes the Euclidean vector norm. Co(S) represents the convex hull of a set S.

1.7.2. Matrices

The symbols **I** and **0** denote the identity and zero matrices of appropriate dimension, respectively. To address components of matrices, we use M_i to denote the ith row of a matrix M, M_j for the jth column, and M_{ij} for the element with index (i, j). M^T and M^* denote transposition and complex conjugate transposition of M, respectively, and M^{-T} represents the transpose of the inverse of M. A square matrix M is called symmetric if M $= M^T$, and Hermitian if M = M^*. $M\dagger(M\ddagger)$ is a left-inverse (right-inverse) of M with the property $M\dagger M = I(MM\ddagger = I)$. A Hermitian matrix is said to be negative (semi-)definite, denoted by M < 0 (M ≤ 0), if $x^*Mx < 0$ ($x^*Mx \leq 0$) for all nonzero x, which is equivalent to all eigenvalues of M being less than (less than or equal to) zero. Analogue definitions for positive (semi)definiteness hold. The notation M < N stands for $\mathrm{M} - \mathrm{N} < 0$, and $[*]^{\mathrm{T}}\mathrm{MN} < 0$ is used to abbreviate $\mathrm{N}^{\mathrm{T}}\mathrm{MN} < 0$.The notation $\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ * & \mathrm{C} \end{bmatrix}$ represents $\begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B}^{\mathrm{T}} & \mathrm{C} \end{bmatrix}$, $\mathrm{diag}(\mathrm{M}_1, \mathrm{M}_2)$ abbreviates the block-diagonal form $\begin{bmatrix} \mathrm{M}_1 & 0 \\ 0 & \mathrm{M}_2 \end{bmatrix}$, and $\mathrm{col}(\mathrm{M}_1, \mathrm{M}_2)$ represents $\begin{bmatrix} \mathrm{M}_1 \\ \mathrm{M}_2 \end{bmatrix}$, similarly for more than two arguments. $\lambda max(M)$ $(\lambda min(M))$ is the largest (smallest) eigenvalue of a Hermitian matrix M. $\sigma_{max}(M) := \sqrt{\lambda_{\max}(M^*M)}$ denotes the maximum singular value of a matrix $M.\rho(M) := max_i \mid \lambda_i(M) \mid$ is the spectral radius of a matrix M. The kernel and the image of a matrix M are denoted by ker(M) and im(M), respectively.

1.7.3. Systems and Interconnections

An operator (or a map) representing a dynamic system is denoted by a capital letter such as G. If G acts on an object w, the outcome is denoted by z = Gw. The impulse response corresponding to an operator G is also denoted by G (with a slight abuse of notation), whereas the corresponding transfer function (if existing) is called $\hat{G}$. An operator G is said to be causal (or proper) if $P_kG = P_kGP_k \; for \; all \; k \geq 0$. An operator G is said to be time-invariant if $SG = GS$. A state-space realization of a transfer function $\hat{G}$ is occasionally written as:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}(z) := C(zI - A)^{-1}B + D = \hat{G}(z).$$

Regular letters $(\mathrm{G}, \mathrm{A}, \mathrm{B}, \ldots)$ are used for open-loop systems, whereas script letters $(\mathcal{G}, \mathcal{A}, \mathcal{B}, \ldots)$ are used for closed-loop systems with a controller connected.

The upper linear fractional transformation (LFT) of two matrices Δ and M with appropriate partitioning shown in Figure (1.2) (a) is defined as:

$$f_u(M,\Delta) = \Delta * \begin{bmatrix} A & B \\ C & D \end{bmatrix} := D + C(I - \Delta A)^{-1}\Delta B.$$

Provided the inverse $(I - \Delta A)^{-1}$ exists. The lower LFT shown in Figure (1.2) (b) is defined similarly as

$$f_l(M,\Delta) = \Delta * \begin{bmatrix} A & B \\ C & D \end{bmatrix} := A + B(I - \Delta D)^{-1}\Delta C.$$

LFTs are a way of describing feedback interconnections as depicted in Figure (1.2) and are special cases of the star product, see e.g. Zhou et al. (1996, Section 10.4) [10]. Maps are also used to represent interconnections of systems. For example, G1G2 stands for a series connection of two systems represented by the maps G1 and G2 as depicted in Figure (1.3)(a), G1 +G2 for a parallel connection as depicted in Figure (1.3)(b), and G1*G2 for an LFT interconnection, like it is common for transfer functions.

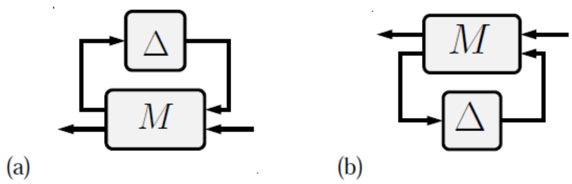


Figure (1.2): LFT (a) upper LFT Δ*M. (b) lower LFT M*Δ.

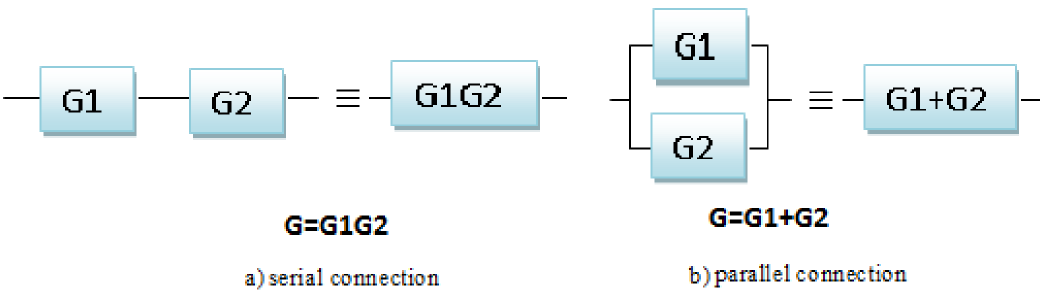


Figure (1.3): connections a) serial connection b) parallel connection

1.8. Thesis Outline (Structure of the Thesis)

Chapter 2 gives details on norms of signals and systems. It also defines the l_1, l_∞, H_2 and H_∞ norms and shows how compute them using CVX optimization toolbox and setup the problem formulation. Chapter 3 covers methodologies and approach. Chapter 4 covers a practical example for an autopilot that controls the pitch angle of an aircraft. Chapter 5 concludes this thesis and suggests recommendations for future work.

CHAPTER 2 NORMS OF SIGNALS AND SYSTEMS

2.1. Norms of signals and systems

Norms of signals and systems are used to quantify the performance and robustness of a control system. They are used in robust optimal control theory.

Vector and Matrix Norms

In finite-dimensional vector spaces, it is very useful to define norms to measure the length of vectors, and matrix norms to measure the maximum "gain" of the matrix. The 2-norm (or Euclidean norm) of an n-dimensional complex vector $x \in C^n$ is defined as:

$$\begin{aligned} \|x\|_2 &= (x^* x)^{1/2} \\ &= (|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2)^{1/2} \end{aligned} \qquad (2.1)$$

where x^* denotes the conjugate transpose of x. The spectral norm of an $n \times m$ complex matrix. $Q \in C^{n\times m}$ is defined as its maximum singular value σ_{max}:

$$\|Q\| = \sigma_{max}(Q) = [\lambda_{\max}(Q^* Q)]^{1/2} \qquad (2.2)$$

Where λ_{max} denotes the maximum eigenvalue. This matrix norm represents the maximum input-output gain in terms of 2-norms of input and output vectors. One can show that, with $x \in R^m$:

$$\|Q\| = \max_{x \neq 0} \frac{\|Qx\|_2}{\|x\|_2} = \max_{\|x\|_2 \leq 1} \|Qx\|_2 = \max_{\|x\|_2 = 1} \|Qx\|_2 \qquad (2.3)$$

2.2. L_2 Norm for Finites Energy Signals

The L_2-norm (or 2 norm) of signal x (t) is the square root of its total energy over $-\infty < t < +\infty$ and is defined as:

$$\|x\|_2 = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt\right)^{1/2} \qquad (2.4)$$

The set of all finite energy signals is called the L_2 space:

$$L_2 := \{x : \|x\|_2 < +\infty\} \qquad (2.5)$$

A "large" signal would have a large L_2-norm, therefore it is a measure of the size of a signal. In a servo system, the objective is to minimize the tracking error signal $e(t) = yd\,(t) - y(t)$. It makes sense to try to minimize its L_2-norm $\|e\|_2$ if the reference signal $yd\,(t)$ belongs to L_2.

The following result allows us to compute the L_2-norm in the frequency domain using the Fourier transform $\hat{X}(jw)$ of the signal.

Parseval's Theorem

$$\|x\|_2^2 = \int_{-\infty}^{\infty} \|x(t)\|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{x}(jw)\|^2\, dw. \tag{2.6}$$

2.3. L_2 Norm of LTI Systems and the Spaces H_2 of Stable Causal Systems

We consider the class of LTI causal systems. The input-output equation for such systems has the form of a convolution showed in Figure (2.1):

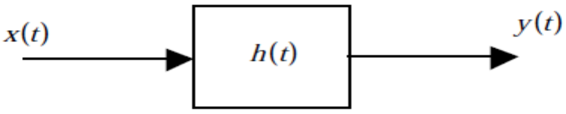


Figure (2.1): Input-output system representation

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\, d\tau \tag{2.7}$$

Where $h(t)$ is the impulse response of the system. For MIMO systems, $h(t)$ is a matrix function. The transfer function of the system is given by:

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \tag{2.8}$$

and its frequency response is simply $H(s)\,|_{s=jw} = H(jw)$. Recall that in the Laplace domain and in the frequency domain, we have much simpler input-output relationships given by:

$$\hat{y}\left(s\right) = H\left(s\right)\hat{x}\left(s\right) \tag{2.9}$$

$$\hat{y}(jw) = H(jw)\hat{x}(jw) \tag{2.10}$$

We will consider finite-dimensional differential LTI systems so that their transfer functions are rational. We say that:

$H(s)$ is proper if $H(j\infty)$ is finite.

$H(s)$ is strictly proper if $H(j\infty) = 0$.

$H(s)$ is biproper if $H(s)$ and $H^{-1}(s)$ are both proper (if $0 < H(j\infty) < \infty$).

Also, recall that $H(s)$ is BIBO stable if and only if all of its poles are in the left half-plane and it is proper.

The L_2-norm (or 2-norm) of a system is defined as:

$$\|H\|_2 = (\frac{1}{2\pi}\int_{-\infty}^{+\infty} trace\{H(jw)^* H(jw)\}dw)^{1/2} \tag{2.11}$$

The set of all systems with finite L_2 -norm is called L_2 : mathematically it is the same space as defined by (2.6). Parseval's theorem gives us a way to compute the L_2 -norm in the time domain from the impulse response matrix:

$$\|H\|_2 = (\frac{1}{2\pi}\int_{-\infty}^{+\infty} trace\{H(jw)^* H(jw)\}dw)^{1/2} = (\int_{-\infty}^{+\infty} trace\{h^*(t)h(t)\}dt)^{1/2} \tag{2.12}$$

If $H(s)$ causal, then

$$\|H\|_2 = (\frac{1}{2\pi}\int_{-\infty}^{+\infty} trace\{H(jw)^* H(jw)\}dw)^{1/2} = (\int_{0}^{+\infty} trace\{h^*(t)h(t)\}dt)^{1/2} \tag{2.13}$$

The space H_2 is the space of all stable, causal systems with finite L_2 -norm:

$$H_2 := \{H \text{ causal, stable} : \|H\|_2 < \infty\}. \tag{2.14}$$

Another way to define H_2 is to say that, it is the subspace of systems in L_2 that are analytic in the closed RHP. The orthogonal complement of H_2 is denoted as $H_2^{\perp}$. It consists of systems in L_2 that are analytic in the closed LHP, so that $L_2 = H_2 \oplus H_2^{\perp}$. The systems in $H_2^{\perp}$ are actually anticausal ($h(t) = 0$, $t < 0$), stable systems with finite L_2 -norm.

2.4. How to Compute the L_2-Norm of Stable Systems

Suppose that $H(s)$ is stable and strictly proper (so that it has a finite L_2 -norm). Further assume that we have a state-space realization (A, B, C, D) of $H(s)$. Define the controllability and observability grammian matrices:

$$L_C := \int_0^{+\infty} e^{At} BB^T e^{A^T t} dt. \tag{2.15}$$

$$L_O := \int_0^{+\infty} e^{A^T t} C^T C e^{At} dt. \tag{2.16}$$

It can be shown that L_O and L_C satisfies the Lyapunov equation:

$$AL_O + L_O A^T + C^T C = 0 \tag{2.17}$$

$$AL_C + L_C A^T + BB^T = 0 \tag{2.18}$$

Then a formula to compute the L_2-norm of the system (also called H_2-norm since the system is stable and hence belongs to H_2) is given by:

$$\|H\|_2 = [trace(CL_o C^T)]^{1/2} = [trace(B^T L_c B)]^{1/2} \tag{2.19}$$

Thus, the procedure consists of computing the controllability and observability grammian matrix L_O *and* L_C by solving the Lyapunov equation (2.15) and (2.16) and then to compute $\|H\|_2$ using (2.19).

2.5. L_∞ Norm of LTI Systems and the Space H_∞ of Stable Systems

The L_∞-norm (or ∞-norm) of a system is defined as:

$$\|H\|_\infty = \sup_{w \in R} |H(jw)| \tag{2.20}$$

It is the maximum gain of the frequency response of the system. The set of all systems with finite L_∞-norms is called L_∞ and is defined by

$$L_{\infty:} = \{H : \|H\|_\infty < +\infty\} \tag{2.21}$$

The space $\|H\|_\infty$ is the space of all causal, stable systems with finite L_∞-norm:

$$H_{\infty:} = \{H \text{ causal, stable}: \|H\|_\infty < +\infty\}. \tag{2.22}$$

2.6. How to Compute the L_∞-Norm of Stable Systems

Suppose that *H*(*s*) is stable and proper, and assume that we have a state-space realization (*A*,*B*,*C*,*D*) of *H*(*s*) . Define the 2*n* ×2*n* Hamiltonian matrix:

$$J = \begin{bmatrix} A & BB^T \\ -C^T C & -A^T \end{bmatrix} \tag{2.23}$$

We have the following result telling us whether the L_∞-norm of the system (also called H_∞-norm since the system is stable and hence belongs to H_∞) is less than 1.

Theorem:

$\|H\|_\infty < 1$ if and only if J has no eigenvalues on the $j\omega$-axis.

Proof can be found in [2] ■

This result suggests a bisection search to find the H_∞-norm of the transfer matrix: Try a large positive value γ_0 first to see if $\|H\|_\infty < \gamma_0$ which is equivalent to $\|\gamma_0^{-1}H\|_\infty < 1$. That is, check if

$$J(\gamma_0) := \begin{bmatrix} A & \gamma_0^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix} \tag{2.24}$$

has no eigenvalues on the $j\omega$-axis. If it doesn't, then select a new $\gamma_1 = \frac{1}{2}\gamma_0$ and check again if $J(\gamma_1)$ has no eigenvalues on the $j\omega$-axis. If it doesn't, then reduce gamma by half again. if it does have eigenvalues on the $j\omega$-axis, then select the middle value $\gamma_2 = \frac{1}{2}(\gamma_0 + \gamma_1)$, and continue the iteration until two consecutive values of gamma representing lower and upper bounds on $\|H\|_\infty$ are found to be close enough.

2.7. Relationships Between System Norms

The maximum gain of a system from the L_2-norm of its input signal $x(t)$ to the L_2-norm of its output signal $y(t)$ is given by the H_∞-norm of its transfer matrix:

$$\|H\|_\infty = \sup_{X\neq 0}\frac{\|Y\|_2}{\|X\|_2} = \max_{\|X\|_2\leq 1}\|H(jw)\hat{X}(jw)\|_2 = \max_{\|X\|_2=1}\|H(jw)\hat{X}(jw)\|_2 \tag{2.25}$$

It turns out that the H_∞-norm is also the maximum power gain of the system:

$$\|H\|_\infty = \sup_{pow(X)\neq 0}\frac{pow(Y)}{pow(X)} = \max_{pow(X)\leq 1} pow[H(jw)\hat{X}(jw)] = \max_{pow(X)=1} pow[H(jw)\hat{X}(jw)]. \tag{2.26}$$

For SISO systems, this means that the H_∞ -norm, seen as the peak value of the magnitude of the Bode plot at some frequency w_0 , is the maximum amplification of a sinusoidal input (a power signal) at frequency w_0 .

The H_2 -norm of a system equals the L_2 -norm of the output $\|Y\|_2$ for an impulse δ(*t*) at its input .

2.8. Parameterization of Stabilizing Controllers

The basic configuration of the feedback systems considered in this chapter is an LFT as shown in Figure (2.2).

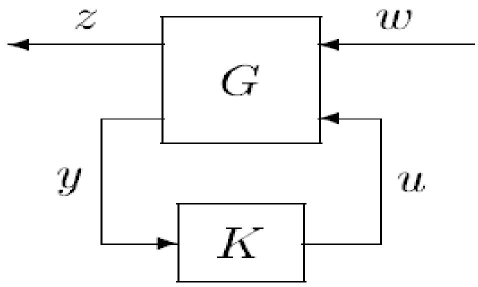


Figure (2.2): General System Interconnection

Where G is the generalized plant with two sets of inputs: the exogenous inputs w, which include disturbances and commands, and control inputs u. The plant G also has two sets of outputs: the measured (or sensor) outputs y and the regulated outputs z. K is the controller to be designed. A control problem in this setup is either to analyze some specific properties, stability or performance, of the closed-loop or to design the feedback control K such that the closed-loop system is stable in some appropriate sense, the error signal z is specified, and some performance condition is satisfied. We are only concerned with the basic internal stabilization problems. Suppose that a given feedback system is feedback stabilizable, then the problem we are mostly interested is parameterizing all controllers that stabilize the system. The construction of the controller parameterization is done via considering a sequence of special problems, which are so-called full information (FI) problems, disturbance feedforward (DF) problems, full control (FC) problems and output estimation (OE) problems.

2.8.1. Existence of Stabilizing Controllers

Consider a system described by the standard block diagram in Figure (2.2). Assume that G(s) has a stabilizable and detectable realization of the form

$$G(s)=\left[\begin{array}{c|c} G_{11}(s) & G_{12}(S) \\ \hline G_{21}(s) & G_{22}(s) \end{array}\right]=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.27}$$

The stabilization problem is to find feedback mapping K such that the closed-loop system is internally stable; the well-posedness is required for this interconnection. This general synthesis question will be called the output feedback (OF) problem.

Definition: A proper system G is said to be stabilizable through output feedback if there exists a proper controller K internally stabilizing G as shown in Figure (2.2). Moreover, a proper controller K(s) is said to be admissible if it internally stabilizes G.

Can be found in [10].■

Lemma: There exists a proper K achieving internal stability iff $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ is detectable. Further, let F and L be such that $A+B_2F$ and $A+LC_2$ are stable, then an observer-based stabilizing controller is given by

$$K(S)=\begin{bmatrix} A+B_2F+LC_2+LD_{22}F & -L \\ F & 0 \end{bmatrix} \tag{2.28}$$

The proof can be found in [10].■

The stabilizability and detectability conditions of $(A,\ B_2,\ C_2)$ are assumed. It follows that the realization for G_{22} is stabilizable and detectable, and these assumptions are enough to yield the following result:

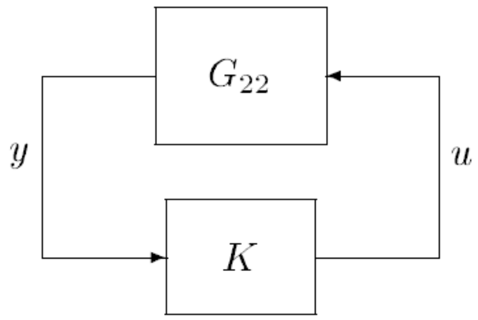


Figure (2.3): Equivalent Stabilization Diagram

Lemma: Suppose $(A,\ B_2,\ C_2)$ is stabilizable and detectable and $G_{22}=\begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}$. Then the system in Figure (2.2) is internally stable iff the one in Figure (2.3) is internally stable. In other words, K(s) internally stabilizes G(s) if and only if it internally stabilizes G_{22}.

The Lemma and proof can be found in [10].■

2.8.2. Duality and Special Problems

We will discuss four problems from which the output feedback solutions are constructed via a separation argument. These special problems are fundamental to the approach taken for synthesis in this thesis, and, as we shall see, they are also of importance in their own right.

Algebraic Duality and Special Problems

The notion of duality can be generalized to a general setting. Consider a standard system block diagram

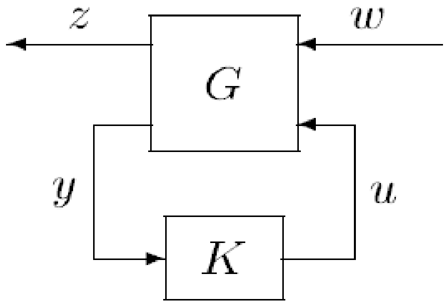


Figure (2.4): General System Interconnection

Where the plant G and controller K are assumed to be linear time invariant. Now consider another system shown below

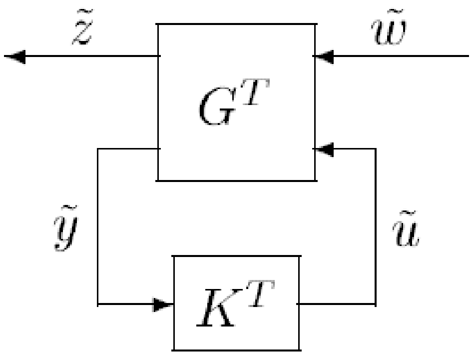


Figure (2.5): Dual General System Interconnection

Whose plant and controller are obtained by transposing G and K. We can check easily that $T_{Zw}^T = [F_l(G,\ \mathrm{K})]^T = F_l(G^T,\ \mathrm{K}^T) = T_{\tilde{Z}\tilde{w}}$. It is not difficult to see that K internally stabilizes G iff K^T internally stabilizes G^T. And we say that these two control structures are algebraically dual, especially, G^T and K^T which are dual objects of G and K, respectively.

The special problems considered here all pertain to the standard block diagram, but to different structures of G. The problems are labeled as:

FI. Full information, with the corresponding plant

$$G_{FI}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \qquad (2.29)$$

FC. Full control, with the corresponding plant

$$G_{FC}(s) = \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \qquad (2.30)$$

DF. Disturbance feedforward, with the corresponding plant

$$G_{DF}(s) = \left[\begin{array}{c|cc} A_1 & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right] \qquad (2.31)$$

OE. Output estimation, with the corresponding plant

$$G_{OE}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right] \qquad (2.32)$$

We say that these special problems are special cases of OF problems in the sense that their structures are specified in comparison to OF problems.The structure of transfer matrices shows clearly that FC, OE (and OI) are duals of FI, DF (and SF), respectively. These relationships are shown in the following diagram:

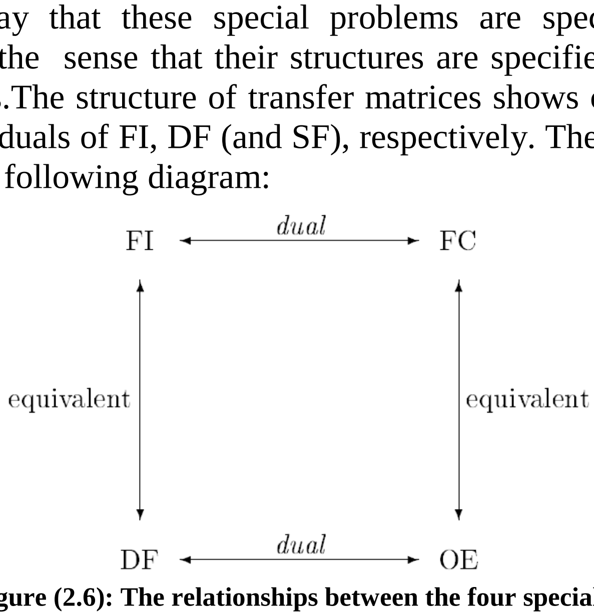


Figure (2.6): The relationships between the four special problems

2.8.3. Full Information (FI) and Disturbance Feedforward (DF)

In the FI problem, the controller is provided with Full Information since $y(t)=\begin{bmatrix} X(t) \\ w(t) \end{bmatrix}$. For the FI problem, we only need to assume that $(A,\ B_2)$ is stabilizable to guarantee the solvability. It is clear that if any output feedback control problem is to be solvable then the corresponding FI problem has to be solvable.

To motivate the name Disturbance Feedforward, consider the special case with $C_2 = 0$. Then there is no feedback and the measurement is exactly w, where w is generally regarded as disturbance to the system. Only the disturbance w, is fed through directly to the output. As we shall see, the feedback caused by $C_2 \neq 0$ does not affect the transfer function from w to the output z, but it does affect internal stability. In fact, the conditions for the solvability of the DF problem are that $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ is detectable.

Now we examine the connection between the DF problem and the FI problem and show the meaning of their equivalence. Suppose that we have controllers K_{FI} and K_{DF} and let T_{FI} and T_{DF} denote the closed-loop T_{zw} in

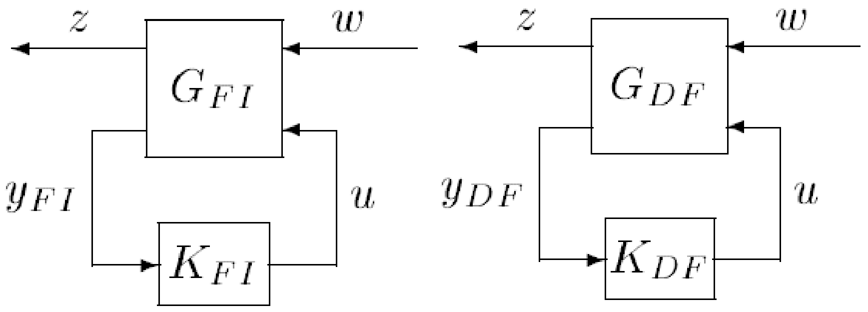


Figure (2.7): The connection between the DF problem and the FI problem

Given either the K_{FI} or the K_{DF} controller, can we construct the other in such a way that $T_{FI} = T_{DF}$. Actually, we have the following:

Lemma: Let G_{FI} and G_{DF} be given as above. Then

i. $G_{DF}(S) = \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & I \end{bmatrix} G_{FI}(S)$.

ii. $G_{FI} = S(G_{DF}, P_{DF})$ (where S(. , .) denotes the star-product)

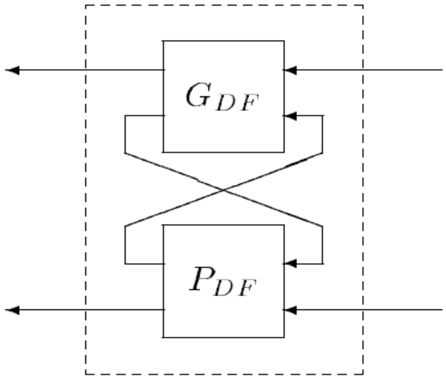


Figure (2.8): Star-product S (G_{DF}, P_{DF})

$$P_{DF}(s) = \left[\begin{array}{c|cc} A - B_1 C_2 & B_1 & B_2 \\ \hline 0 & 0 & I \\ \begin{bmatrix} I \\ -C_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right] \tag{2.33}$$

The proof can be found in [10].■

Theorem: Consider G_{FI} , G_{DF} and P_{DF} be given as above.

i. $K_{FI} = K_{DF}[C_2 \;\; I]$ internally stabilizes G_{FI} if K_{DF} internally stabilizes G_{DF} .Furthermore, $F_l(G_{FI}, K_{DF}[C_2 \;\; I]) = F_l(G_{DF}, K_{DF})$.

ii. Suppose that $A - B_1C_2$ is stable. Then $K_{DF} = F_l(P_{DF}, K_{FI})$ as shown below

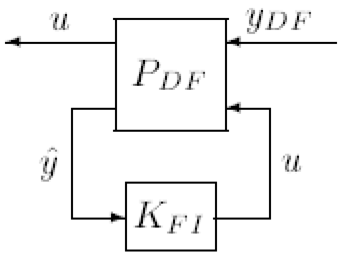


Figure (2.9): $K_{DF} = F_l(P_{DF}, K_{FI})$ **diagram**

Internally stabilizes G_{DF} if $K_{FI}(s)$ internally stabilizes G_{FI} . Furthermore, $F_l(G_{FI}, K_{FI}) = F_l[G_{DF}, F_l(G_{DF}, K_{FI})]$

The proof can be found in [2].■

2.8.4.Full Control and Output Estimation

The term Full Control is used because the controller has full access to both the state through output injection and the output z. The only restriction on the controller is that it must work with the measurement y. This problem is dual to the FI case and has the dual solvability condition to the FI problem, which is also guaranteed by the assumptions on OF problems.

The solutions to this kind of control problem can be obtained by first transposing G_{FC}, and solving the corresponding FI problem, and then transposing back.

On the other hand, problem OE is dual to DF. Thus the discussion of the DF problem is relevant here, when appropriately dualized. And the solvability conditions for the OE problem are that $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ is detectable. To examine the physical meaning of output estimation, first note that

$$Z = C_1 x + D_{11} w + u$$

where z is to be controlled by an appropriately designed control u. In general, our control objective will be to find a u that will estimate $C_1 x + D_{11} w$ in such defined mathematical sense. So this kind of control problem can be regarded as an estimation problem. We are focusing on this particular estimation problem because it is the one that arises in solving the output feedback problem. A more conventional estimation problem would be the special case where no internal stability condition is imposed and $B_2 = 0$. Then the problem would be that of estimating the output z given the measurement y. This special case motivates the term output estimation and can be obtained immediately from the results obtained for the general case.

We will explain the meaning of equivalence between FC and OE problems. Consider the following FC and OE diagrams:

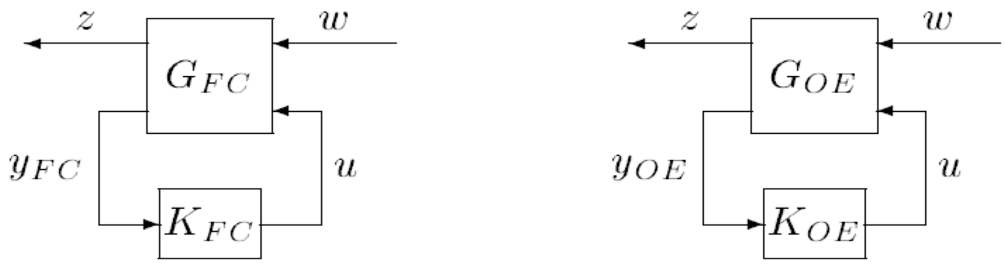


Figure (2.10): Equivalence between FC and OE problems

Lemma: Let G_{FC} and G_{OE} be given as above. Then

i. $G_{OE}(s) = G_{FC}(s)\begin{bmatrix} I & 0 \\ 0 & B_2 \\ 0 & I \end{bmatrix}$

ii. $G_{FC} = S(G_{OE},\ P_{OE})$, where P_{OE} is given by

$$P_{OE}(s) = \left[\begin{array}{c|cc} A - B_2C_1 & 0 & [I - B_2] \\ \hline C_1 & 0 & [0 \quad I] \\ C_2 & I & [0 \quad I] \end{array}\right] \qquad (2.34)$$

The proof can be found in [10].■

Theorem: Let G_{FC}, G_{OE} and P_{OE} be given as above.

i. $K_{FC} = \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}$ internally stabilizes G_{FC} if K_{OE} internally stabilizes G_{OE}.

Furthermore, $F_l(G_{FC}, \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{OE}) = F_l(G_{OE}, K_{OE})$.

ii. Suppose $A\text{-}B_2C_1$ is stable. Then $K_{OE} = F_l(P_{OE}, K_{FC})$, as shown below

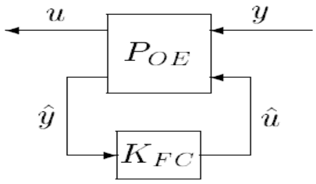


Figure (2.11): $K_{OE} = F_l(P_{OE}, K_{FC})$ **diagram**

internally stabilizes G_{OE} if K_{FC} internally stabilizes G_{FC}. Furthermore, $F_l(G_{OE}, F_l(P_{OE}, K_{FC})) = F_l(G_{FC}, K_{FC})$.

The proof can be found in [10].■

2.9. H_2 Optimal Control

H_2 Optimal control is a theory to design finite-dimensional LTI controllers that minimize the H_2-norm of the closed-loop system. But first we will study the algebraic Riccati equation which is present everywhere in optimal control theory.

2.9.1. Algebric Riccati Equations

Let A, Q, R be real $n \times n$ matrices with Q, R symmetric. Then an algebraic Riccati equation (ARE) is the following matrix equation:

$$A^*X + XA + XRX + Q = 0 \qquad (2.35)$$

Associated with this ARE is the $2n \times 2n$ Hamiltonian matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix} \qquad (2.36)$$

We consider X_1 and X_2 are solutions of algebraic Riccati equation, If X_1 is nonsingular, then we can define $X := X_2X_1^{-1}$ and the Hamiltonian matrix *H* uniquely defines *X*. where X_1 and X_2 are the eigenvectors that corresponds to the stable eigenvalues. These eigenvectors is divided into X_1 upper square matrix (nxn) and the rest (lower part) is termed X_2 .

The following $X := X_2X_1^{-1}$ result states that is a solution to the algebraic Riccati equation:

Theorem: ARE

Suppose that $H \in$dom{Ric}, and X =Ric(H) . Then:

(i) X is real symmetric,
(ii) X satisfies the ARE,
(iii) $A + RX$ *is* stable (all of its eigenvalues are in the open LHP).

The proof can be found in [2].■

2.9.2. Example Using H_2 Norm

Consider the general block diagram of a feedback control system shown in Figure (2.12).

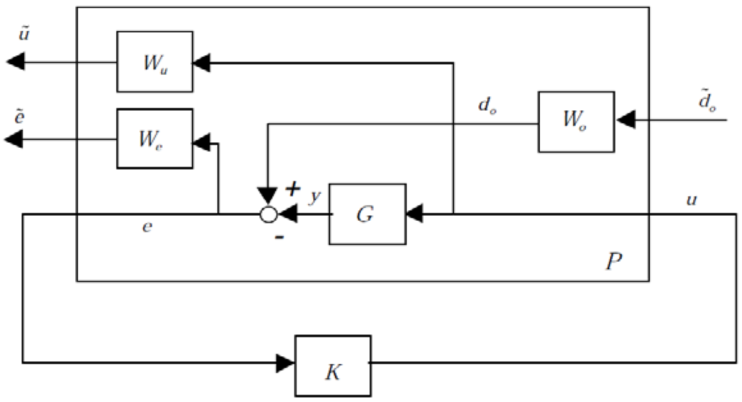


Figure (2.12): Typical setup for H_2 -optimal control

This system can be recast as a linear fractional transformation (LFT) as mentioned in Figure (2.13), with P(s) given such that

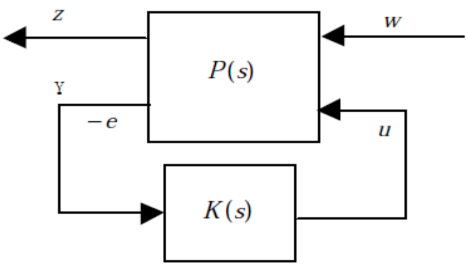


Figure (2.13): Standard LFT diagram for H_2 -optimal control design

$$P(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.37}$$

We consider the four special cases of the general structure (2.37), commonly referred to as full information (FI), disturbance feedforward (DF), full control (FC), and output estimation (OE). The desired parameterization for the general output feedback case will be obtained by combining these cases.

Considering the equation (2.37), where $P(s):=\left[\begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array}\right]$, and the transfer matrix entries of this generalized plant are readily obtained from the paths relating each input signal to each output signal. Here, we have:

$$\begin{aligned} P_{11}(s) &= 0 \\ P_{12}(s) &= \begin{bmatrix} W_u \\ W_e G \end{bmatrix} \\ P_{21}(s) &= W_o \\ P_{22}(s) &= -G \end{aligned} \tag{2.38}$$

The weighting function Wu(s) can be used to constrain the control signal while We(s) can be used to reduce the sensitivity at low frequencies. Weighting function Wo(s) can be used to model the power spectral density or energy-density spectrum of the output disturbance. Once the control system is put in the form of the so-called standard H_2 problem (in LFT form), the minimization problem becomes:

$$\min_{K\in s} \left\| T_{ZW} \right\|_2 \tag{2.39}$$

Where

$$T_{ZW}(s)=F_L[P(s),K(s)]=P_{11}(s)+P_{12}(s)K(s)[I-P_{22}(s)K(s)]^{-1}P_{21}(s) \tag{2.40}$$

Is the closed-loop transfer matrix from the exogenous input w to the output z. One can then use the following result on H_2 optimal control (very similar to LQG).

Suppose that a state-space realization of P(s) is given by:

$$P(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{2.41}$$

Notice the special off-diagonal structure of D: D_{22} is assumed to be 0 so that $P_{22}(s)$ is strictly proper, and D_{11} is assumed to be 0 so that $P_{11}(s)$ is also strictly proper (which is a necessary condition for $P_{11}(s)$ to be in H_2).

First define $R_1 = D_{12}^* D_{12}$ and $R_2 = D_{21}^* D_{21}$, and the two Hamiltonian matrices:

$$H_2 := \begin{bmatrix} A - B_2 R_1^{-1} D_{12}^* C_1 & -B_2 R_1^{-1} B_2^* \\ -C_1^*(I - D_{12} R_1^{-1} D_{12}^*) C_1 & -(A - B_2 R_1^{-1} D_{12}^* C_1)^* \end{bmatrix} \qquad (2.42)$$

$$J_2 := \begin{bmatrix} (A - B_1 D_{21}^* R_2^{-1} C_2)^* & -C_2^* R_2^{-1} C_2 \\ -B_1(I - D_{21}^* R_2^{-1} D_{21}) B_1^* & -(A - B_1 D_{21}^* R_2^{-1} C_2)^* \end{bmatrix} \qquad (2.43)$$

Note that H_2, $J_2 \in$ dom (Ric) and X_2:=Ric(H_2) ≥0, Y_2:=Ric(J_2) ≥ 0. Let us introduce the concepts of stabilizability and detectability. These are weaker versions of controllability and observability: they only require that the unstable modes be controllable and observable.

The proof can be found in [2].■

Definition: The pair (A, B_1) is said to be stabilizable if there exists a state feedback gain matrix K such that $A + B_1 K$ is stable (all eigenvalues have a negative real part).

Definition can be found in [2].■

Definition: The pair (A, C_2) is said to be detectable if there exists an observer gain matrix L such that $A + LC_2$ is stable.

Definition can be found in [2].■

2.9.3.Theorem: H_2-Optimal Controller:

If the following assumptions hold:

1. The pair (A, B_2) is stabilizable and the pair (A, C_2) is detectable
2. $R_1 = D_{12}^* D_{12} > 0$ (meaning that all of its eigenvalues are positive) and $R_2 = D_{21}^* D_{21} > 0$
3. $\begin{bmatrix} A - jwI & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω
4. $\begin{bmatrix} A - jwI & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω

Then, the unique H_2-optimal controller minimizing $\|T_{ZW}\|_2$ is given by:[

$$K_{opt}(s) = \begin{bmatrix} \tilde{A}_2 & -L_2 \\ F_2 & 0 \end{bmatrix} \quad (2.44)$$

Where matrix L_2 is given by $L_2 := -(Y_2C_2^* + B_1D_{21}^*)R_2^{-1}$, and F_2 is given by $F_2 := -R_1^{-1}(B_2^*X_2 + D_{12}^*C_1)$, and $\hat{A}_2 := A + B_2F_2 + L_2C_2$.

Proof can be found in [2].■

Notes:

- Assumptions 3 and 4 ensure that H_2 , $J_2 \in$ dom(Ric)
- The assumptions usually hold when the problem is well posed. For example, there should always be biproper weighting functions on the control signals; otherwise, the optimal controller would produce infinite control signals. This corresponds to matrix D_{12} having full column rank. Likewise, there should be an output disturbance or a measurement noise defined that couples right into the measured signal used by the controller. This corresponds to matrix D_{12} having full column rank.

2.10. H_∞ Optimal Control

H_∞ optimal control is a theory to design finite-dimensional stabilizing LTI controllers that minimize the H_∞-norm of the closed-loop system. H_∞-norm methods are used in control theory to synthesize controllers achieving robust performance or stabilization. To use H_∞-norm methods, a control designer expresses the control problem as a mathematical optimization problem and then finds the controller that solves this.

H_∞ optimization of control systems deals with the minimization of the peak value of certain closed-loop frequency response functions. To clarify this, consider the basic SISO feedback system of Figure (2.14).

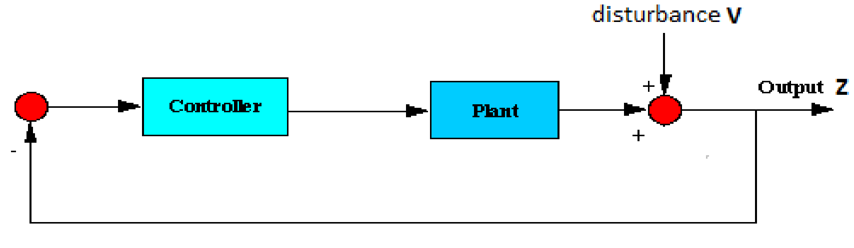


Figure (2.14): Siso feedback loop

The plant has transfer function P and the compensator has transfer function ***C***. The signal ***v*** represents disturbance acting on the system and ***z*** is the control system output. Then from the signal balance equation $\hat{z} = \hat{v} - pc\hat{z}$ with the circumflex denoting the Laplace transform, it follows that : $\hat{Z} = S\hat{V}$ where: $s = \frac{1}{1+pc}$

Is the sensitivity function of the feedback system. As the name implies, it's characterizes the sensitivity of the control system output to disturbances. Ideally, s = 0

The problem originally considered by Zames (1979, 1981) is that of finding a compensator C that makes the closed-loop system stable and minimizes the peak value of the sensitivity function. This peak value (See Figure (2.15) is defined as:

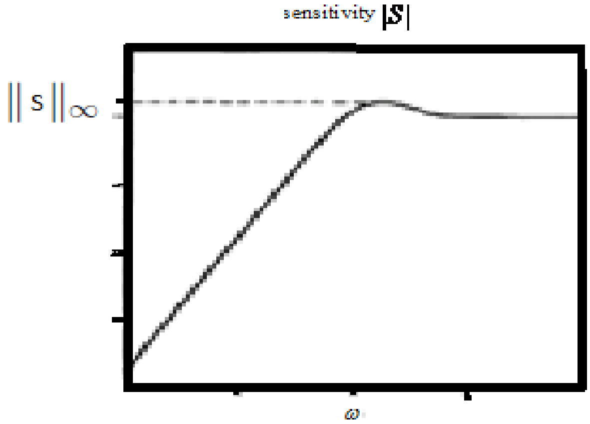


Figure (2.15): $\|S\|_\infty$ **Peak value**

$$\|S\|_\infty = \max_{\omega \in R} |S(j\omega)| \quad (2.45)$$

Where **R** denote the set of real numbers. Because for some functions the peak value may not be assumed for any finite frequency, we replace the maximum here and in the following by the supremum or least upper bound, so that:

$$\|S\|_\infty = \sup_{\omega \in R} |S(j\omega)| \quad (2.46)$$

The justification of this problem is that if the peak value $\|S\|_\infty$ of the sensitivity function S is small, then the magnitude of S necessarily is small for all frequencies, so that disturbances are uniformly attenuated over all frequencies.

Minimization of $\|S\|_\infty$ is worst-case optimization, because it amounts to minimizing the effect on the output of the worst disturbance (namely, a harmonic disturbance at the frequency where $|S|$ has its peak value).

The worst-case model has an important mathematical interpretation. Suppose that the disturbance **v** has unknown frequency content, but finite energy $\|S\|_2^2$. The number

$$\|v\|_2 = \sqrt{\int_{-\infty}^{+\infty} |v(t)^2|\, dt} \tag{2.47}$$

is known as the 2-norm of the disturbance **v**. The energy of **v** is the square of the 2-norm. Then the norm $\|S\|$ of the system S as in Figure (2.16) with input **v** and output **z** induced by the 2-norm is defined as

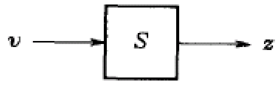


Figure (2.16): System Input-output related with sensitivity

$$\|S\| = \sup_{v:\|v\|_2 \prec \infty} \frac{\|z\|_2}{\|v\|_2} \tag{2.48}$$

Hence, in engineering terms the norm is directly related to the energy gain for the input with the worst possible frequency distribution. Using Parseval's Theorem, it is not difficult to recognize that :

$$\|S\| = \|S\|_\infty \tag{2.49}$$

Hence, the peak value is precisely the norm of the system induced by the 2-norms on the input and output signals. This norm is known as the ∞-norm of the system.

2.10.1. Example Using H_∞ Norm

Consider the block diagram of a feedback control system shown in Figure (2.17).

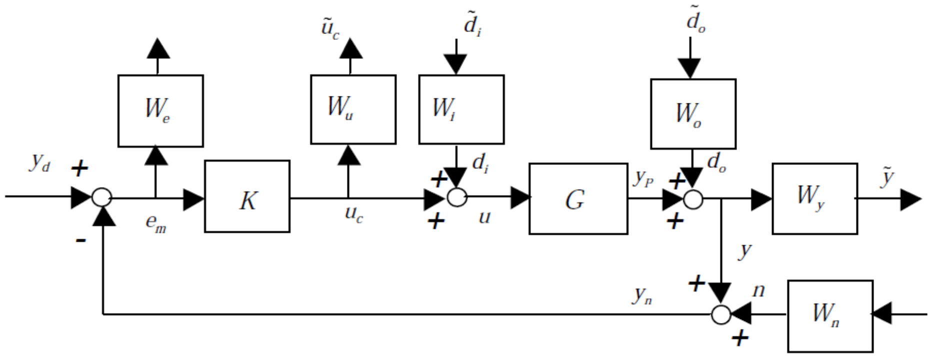


Figure (2.17): Typical feedback control system

Again, an important step in the H_∞ controller design process is to select reasonable weighting functions W_d, W_e, W_y, W_u . These weighting functions have a clearer meaning as design parameters than they do in H_2 control because of the definition of the H_∞ norm of a system. For instance, if the H_∞ norm of a weighted closed-loop transfer matrix is less than some positive real number γ , if $\|W(s)T_{ZW}(s)\|_\infty < \gamma$, where $W(s)$ is a scalar transfer function, then we have the bound at each frequency $\|T_{ZW}(jw)\| < \gamma |W^{-1}(jw)|$. The weighting functions are used to achieve a good tradeoff between concurrent/conflicting closed-loop objectives such as sensitivity minimization and reduction of measurement noise. For simplicity, we again assume that $\tilde{d}_i = 0,\ \tilde{n}=0$, and we consider the regulator problem where the effect of the output disturbance $\tilde{d}_0$ on the weighted output $\tilde{Y}$ must be minimized. After simplification, the system become as mentioned in Figure (2.18).

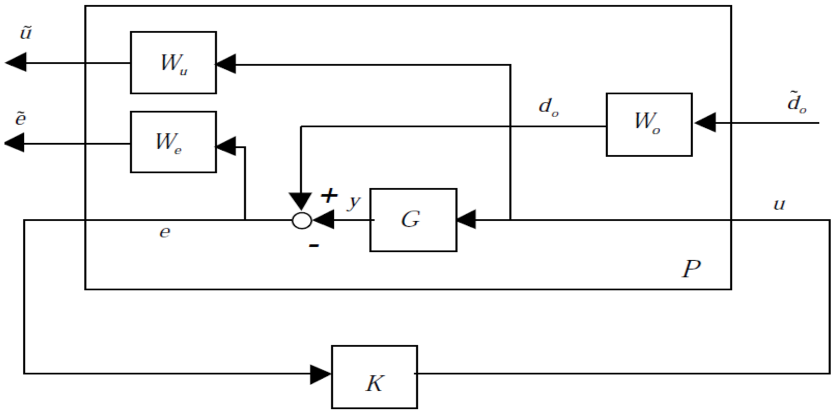


Figure (2.18): Typical setup for H_∞-optimal control design

This system can be recast as a linear fractional transformation (LFT) as shown in Figure (2.19).

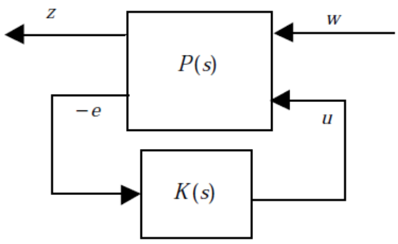


Figure (2.19): Standard LFT diagram for H_∞-optimal control design

Where $P(s) := \left[\begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array}\right]$, and the transfer matrix entries of this generalized plant were given in (2.30) and repeated for convenience

$$
\begin{aligned}
&P_{11}(s) = 0 \\
&P_{12}(s) = \begin{bmatrix} W_u \\ W_e G \end{bmatrix} \\
&P_{21}(s) = W_o \\
&P_{22}(s) = -G
\end{aligned} \tag{2.50}
$$

The weighting function *Wu(s)* can be used to constrain the control signal while *We(s)* can be used to reduce the sensitivity at low frequencies. Weighting function *Wo(s)* can be used to model the power spectral density or energy-density spectrum of the output disturbance. Once the control system is put in the form of the so-called standard H_∞ problem (in LFT form), the minimization problem becomes:

$$
\min_{K \in s} \left\| T_{ZW} \right\|_\infty \tag{2.51}
$$

Where $T_{ZW}(s) = F_L[P(s), K(s)]$ is the closed-loop transfer matrix from the exogenous input *w* to the output *z* . The optimization of (2.51) is very difficult theoretically and numerically. Virtually everybody uses the solution to the suboptimal H_∞ problem stated as Given $\gamma > 0$, find an admissible controller (if there exists any) such that $\left\| T_{ZW} \right\|_\infty < \gamma$. We will present the solution to this problem, and it should be clear that an iterative bisection procedure for reducing γ while checking that a suboptimal controller exists will lead to a controller as close to the optimal controller as desired.

2.10.2. Solution to Simplified H_∞ Suboptimal

The solution to the simplified suboptimal H_∞ problem is obtained from the solutions of a pair of Riccati equations. However, the difference with the H_2 problem is that these Riccati equations cannot be solved independently from one another, making the H_∞ problem more difficult. But first, let's discuss the simplifying assumptions that we will use here. The general problem is more involved mathematically, and does not provide much more insight. Therefore, we will stick with the simplified problem.

Suppose that a state-space realization of the generalized plant *P*(*s*) is given by:

$$
P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \tag{2.52}
$$

Notice the special off-diagonal structure assumed for *D* (just like the H_2 case). Given $\gamma > 0$, define the two Hamiltonian matrices:

$$
H_\infty := \begin{bmatrix} A & \gamma^{-2} B_1 B_1^* - B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{bmatrix} \tag{2.53}
$$

$$J_\infty := \begin{bmatrix} A^* & \gamma^{-2}C_1^*C_1 - C_2^*C_2 \\ -B_1B_1^* & -A \end{bmatrix} \quad (2.54)$$

Assume that:

1. The pair (A, B_1) is stabilizable and the pair (A, C_1) is detectable.
2. The pair (A, B_2) is stabilizable and the pair (A, C_2) is detectable.
3. $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$ (meaning that D_{12} is orthogonal to C_1) and no coupling in D_{12}
4. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (meaning D_{21} is orthogonal to B_1) and no coupling in D_{21}

The proof can be found in [2].■

Notes:

- Assumption 2 is required if we want to stabilize the plant with the controller
- Assumption 1 simplifies the theoretical developments and usually holds in practice
- Assumptions 3 and 4 are also made for technical reasons and practical problems can be set up so that these assumptions hold.

2.10.3. Theorem: H_∞ Controller:

There exists an admissible controller such that $\|T_{ZW}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in dom(Ric) \;\; and \;\; X_\infty := Ric(H_\infty) \geq 0;$
2. $J_\infty \in dom(Ric) \;\; and \;\; Y_\infty := Ric(J_\infty) \geq 0;$
3. $\rho(X_\infty Y_\infty) < \gamma^2$ (the spectral radius of the product $X_\infty Y_\infty$).

When these conditions hold, one such controller is

$$K_\infty(s) := \begin{bmatrix} \tilde{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix} \quad (2.55)$$

Where

$$\tilde{A}_\infty := A + \gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 \quad (2.56)$$

$$F_\infty := -B_2^*X_\infty \quad (2.57)$$

$$L_\infty := -Y_\infty C_2^* \quad (2.58)$$

$$Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1} \quad (2.59)$$

The proof can be found in [2].■

Notes:

- Solutions: $X_\infty := Ric(H_\infty), Y_\infty := Ric(J_\infty)$ of the Riccati equations.
- The theorem suggests an iterative way to find a controller that minimizes the H_∞-norm of the closed-loop system, based on the bisection idea to compute an H_∞ -norm given earlier. Namely, given a large enough starting value for γ, solve the two Riccati equations and check whether the spectral radius $X_\infty Y_\infty$ is less than γ^2 . Then reduce gamma by half in a bisection scheme, backtracking if needed. Continue the iteration until two consecutive values of gamma representing lower and upper bounds on $\|T_{ZW}\|_\infty$ are found to be close enough. Finally, the controller can be computed using the state-space matrices given in (2.55).

2.11.Convex Optimization Problems

2.11.1. Optimization Problems

We use the notation

$$\begin{array}{ll} minimize & f_0(x) \\ subject\ to & f_i(x) \leq 0, \quad i=1,\ldots,m \\ & h_i(x) = 0, \quad i=1,\ldots,p \end{array} \tag{2.60}$$

To describe the problem of finding an x that minimizes $f_0(x)$ among all x that satisfy the conditions:

$$f_i(x) \leq 0, \quad i=1,\ldots,m \qquad ; \quad h_i(x) = 0, \quad i=1,\ldots,p$$

$x \in R^n$ the optimization variable.

$f_0 : R^n \to R$ the objective function or cost function.

$f_i(x) \leq 0$ inequality constraints.

$f_i : R^n \to R$ the inequality constraint functions.

$h_i(x) = 0$ are called the equality constraints and corresponding functions $h_i : R^n \to R$ the equality constraint functions.

If there are no constraints (m = p = 0) we say the problem (2.60) is unconstrained.

The set of points for which the objective and all constrained functions are defined,

$$D = \bigcap_{i=0}^{m} dom\ f_i \ \cap \ \bigcap_{i=1}^{p} dom\ h_i \tag{2.61}$$

And D is called the domain of the optimization problem (2.60). A point $x \in D$ is feasible if it satisfies all constraints. The problem (2.60) is said to be feasible if there exists at least one feasible point, and infeasible otherwise. The set of all feasible points is called the feasible set or the constraint set.

The optimal value is defined as:

$$p^* = inf \{f_0(x) \mid f_i(x) \leq 0, i = 1, . . . ,m, h_i(x) = 0, i = 1, . . . , p\} \quad (2.62)$$

We allow p^* to take on the extended values ±∞. If the problem is infeasible, we have $p^* = \infty$

If there are feasible points x_k with $f_0(x_k) \rightarrow -\infty$ as k → ∞, then $p^* = -\infty$ and we say the problem (2.60) is unbounded below.

2.11.2. Globally and Locally Optimal Points:

A fundamental property of convex optimization problems is that any locally optimal point is also (globally) optimal. Suppose that x is locally optimal for a convex optimization problem, x is feasible and

$$f_0(x) = inf \{f_0(z) \mid z\ feasible,\ \|z - x\|_2 \leq R\} \quad (2.63)$$

for some R > 0. Now suppose that x is not globally optimal, there is a feasible y such that $f_0(y) < f_0(x)$. evidently $\|y - x\|_2 > R$, since otherwise $f_0(x) \leq f_0(y)$. Consider the point z given by:

$$z = (1-\theta)x + \theta y, \qquad \theta = \frac{R}{2\|y - x\|_2} \quad (2.64)$$

Then we have $\|z - x\|_2 = R / 2 < R$, and by convexity of the feasible set, z is feasible. By convexity of f_0 we have:

$$f_0(z) \leq (1\text{-}\theta)f_0(x) + \theta f_0(y) < f_0(x) \quad (2.65)$$

Which contradicts (2.63) . Hence there exists no feasible y with $f_0(y) < f_0(x)$. x is globally optimal.

2.11.3. An Optimality Criterion for Differentiable f_0:

Suppose that the objective f_0 in a convex optimization problem is differentiable, so that for all x , $y \in$ dom f_0,

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y-x) \tag{2.66}$$

Let **X** denote the feasible set, Then x is optimal if and only if $x \in$ X and:

$$\nabla f_0(x)^T \ (y-x) \geq 0 \text{ for all } y \in X \tag{2.67}$$

This optimality criterion can be understood geometrically: If $\nabla f_0(x) \neq 0$, it means that $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x .

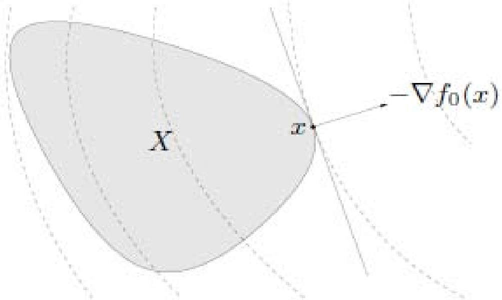


Figure (2.20): Geometric interpretation of the optimality condition [48]

Figure (2.20) shows the feasible set X as shaded area. Some level curves f_0 are shown as dashed lines. The point x is optimal: $-\nabla f_0(x)$ define a supporting hyperplane (shown as a solid line) to X at x .

2.11.4. Quasiconvex Optimization:

A quasiconvex optimization problem has the standard form:

$$\begin{aligned} &maximize \;\; f_0(x) \\ &subject\ to \;\; f_i(x) \leq 0, \;\; i = \;1, \; . \; . \; . \; , m \\ &\qquad\qquad\;\; Ax = b, \end{aligned} \tag{2.68}$$

Where the inequality constraint functions $f_1, \; . \; . \; . \; , \; f_m$ are convex, and the objective f_0 is quasiconvex.

2.11.5. Locally Optimal Solutions and Optimality Conditions

The most important difference between convex and quasiconvex optimization is that a Quasiconvex optimization problem can have locally optimal solutions that are not (globally) optimal.

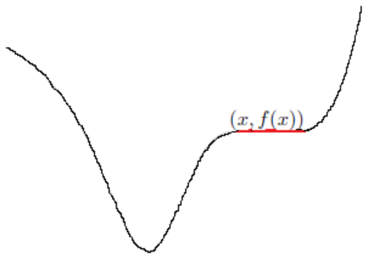


Figure (2.21): A Quasiconvex function f on R, with a locally optimal point x

2.12.Problem Formulation

Now we present a Lagrange duality theorem that applies to the minimization of a convex functional subject to both equality and inequality constraints. Asensitivity result which follows directly from the Lagrange duality is presented. First we need the following definitions.

Definition 1: Let P be a convex cone in a vector space X. We write $x \geq y$ if $x - y \in P$. We write $x > 0$ if $x \in \text{int}(P)$.

Similarly $x \leq y$ if $x - y \in -P := N$. and $x < 0$ if $x \in \text{int}(N)$.

Definition 2: Let X be a vector space and Z be a vector space with positive cone P. A mapping G: X→Z is convex if

$G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with 0< t <1.

The following is lagrange duality theorem where we denote the interior of a set by *int*.

Theorem 1: Let X be a Banach space,**Ω** be a convex subset of X,Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, H: X⟶Y be an affine linear map and $0 \in int[range(H)]$. Define

$$\mu_0 = inf\ \{f(x) : g(x) \ \leq \ 0,\ H(x)=0,\ x \in \Omega\} \tag{2.69}$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) \ \leq \ 0$ and $H(x_1)=0$ and suppose μ_0 is finite. Then,

$$\mu_0 = \max\{\varphi(z^*,\ y) :\ \ z^* \geq 0,\ z^* \in Z^*,\ y \in Y\} \tag{2.70}$$

Where $\varphi(z^*,\ y) := inf\{f(x)+ < g(x), z^* > + < H(x), y >: x \in \Omega\}$ and the maximum is achieved for some $z_0^* \geq 0,\ z_0^* \in Z^*,\ y_0 \in Y$.

Furthermore if infinimum in (3.69) is achieved for some $x_0 \in \Omega$ then

$$< g(x_0), z_0^* > + < H(x_0), y_0 >= 0 \quad (2.71)$$

And x_0 minimizes

$$f(x) + < g(x_0), z_0^* > + < H(x_0), y_0 >= 0 \text{ over all } x \in \Omega \quad (2.72)$$

We refer to (3.69) as the **primal** problem and (3.70) as the **Dual** problem .

Corollary: Let X, Y, Z, f, H, g, Ω be as in Theorem 1. Let x_0 be the solution to the problem

$$\begin{aligned} &\textit{minimize } f(x) \\ &\textit{subject to } x \in \Omega, \ \mathrm{H}(x)=0, \ \mathrm{g}(x) \le z_0 \end{aligned}$$

With (z_0^*, y_0) as the dual solution. Let x_1 be the solution to the problem

$$\begin{aligned} &\textit{minimize } f(x) \\ &\textit{subject to } x \in \Omega, \ \mathrm{H}(x)=0, \ \mathrm{g}(x) \le z_1 \end{aligned}$$

With (z_1^*, y_1) as the dual solution. Then,

$$< z_1 - z_0, z_1^* > \le f(x_0) - f(x_1) \le < z_1 - z_0, z_0^* > \quad (2.73)$$

2.12.1. Problem Formulation for l_1 / H_2

Consider the standard feedback problem represented in Figure (2.13). Where ***P*** and ***K*** are the plant and the controller respectively. Let *w* represent the exogenous input, ***z*** represent the output of interest, y is the measured output and ***u*** is the control input where ***z***, ***w*** are assumed scalar. Let ∅ be the closed loop map which maps $w \rightarrow z$. From Youla parametrization it is known that all achievable closed loop maps under stabilizing controllers are given by

∅ = *h* - u*q (* denotes convolution), where *h, u* , q ∈ l_1; *h, u* depend only on the plant *P* and q is a free parameter in l_1.

Let the zeros of u which are inside the unit disc be given by $\mathrm{z}_1, \ \mathrm{z}_2, \ \mathrm{z}_3, \ldots, \ \mathrm{z}_n$. Let

$$\Theta := \{\phi \in l_1 : \phi \text{ there exists } \mathrm{q} \in l_1 \text{ with } \phi = h - u * q\}.$$

Θ is the set of all achievable closed loop maps under stabilizing controllers. Let $A : l_1 \rightarrow R^n$ be given by:

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 \dots \\ 1 & z_2 & z_2^2 & z_2^3 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 \dots \end{pmatrix}$$

and $b \in R^n$ be given by:

$$b = \begin{pmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{pmatrix}$$

Theorem: the following is true

$\Theta = \{\phi \in l_1 : \hat{\phi}(z) = \hat{h}(z_i)$ for all i=1,...,n}.
$=\{\phi \in l_1 : A\phi = b\}$.

Proof can be found in [42].■

The following problem

$$\begin{aligned} v_\infty &:= inf\{\|h - u * q\|_1 : q \in l_1\} \\ &= inf\{\|\phi\|_1 : \phi \in l_1 \text{ and } A\phi = b\} \end{aligned} \tag{2.74}$$

Is the standard l_1 problem. This problem has a solution which is possibly non-unique. Optimal solutions are shown to be finite impulse response sequences. Let

$$\begin{aligned} \mu_\infty &:= inf\{\|h - u * q\|_2^2 : q \in l_1\}, \\ &= inf\{\|\phi\|_2^2 : \phi \in l_1 \text{ and } A\phi = b\} \end{aligned} \tag{2.75}$$

which is the standard H_2 problem. The solution to this problem is unique but the solution is an infinite impulse response sequence. Define

$$m_1 := \inf_{A\phi = b, \|\phi\|_2^2 \le \mu_\infty} \|\phi\|_1 \tag{2.76}$$

which is the l_1 norm of the unique optimal solution of the standard H_2 problem. Let

$$m_2 := \inf_{A\phi=b, \|\phi\|_1 \le \nu_\infty} \|\phi\|_2^2 \tag{2.77}$$

which is the infimum over the l_2 norms of the optimal solutions of the standard l_1 problem.

The problem of interest is : Given a positive constant $\gamma > \mu_\infty$ obtain a solution to the following mixed objective problem:

$$\begin{aligned} \nu_\infty &:= \inf\{\|h - u * q\|_1 : q \in l_1 \text{ and } <h - u * q, h - u * q> \le \gamma\} \\ &= \inf\{\|\phi\|_1 : \phi \in l_1 \text{ and } A\phi = b \text{ and } <\phi, \phi> \le \gamma\}. \end{aligned} \tag{2.78}$$

Where $< ., . >$ is the inner product associated with l_2.

2.12.2. Problem Formulation for l_1 / H_∞

The following problem

$$\begin{aligned} \nu_\infty &:= \inf\{\|h - u * q\|_1 : q \in l_1\} \\ &= \inf\{\|\phi\|_1 : \phi \in l_1 \text{ and } A\phi = b\} \end{aligned} \tag{2.79}$$

Is the standard l_1 problem. This problem has a solution which is possibly non-unique. Optimal solutions are shown to be finite impulse response sequences. Let

$$\begin{aligned} \mu_\infty &:= \inf\{\|h - u * q\|_\infty : q \in l_1\}, \\ &= \inf\{\|\phi\|_\infty : \phi \in l_1 \text{ and } A\phi = b\} \end{aligned} \tag{2.80}$$

which is the standard H_2 problem. The solution to this problem is unique but the solution is an infinite impulse response sequence. Define

$$m_1 := \inf_{A\phi=b, \|\phi\|_\infty \le \mu_\infty} \|\phi\|_1 \tag{2.81}$$

which is the l_1 norm of the unique optimal solution of the standard H_∞ problem. Let

$$m_2 := \inf_{A\phi=b, \|\phi\|_1 \le \nu_\infty} \|\phi\|_\infty \tag{2.82}$$

which is the infimum over the l_2 norms of the optimal solutions of the standard l_1 problem.

The problem of interest is: Given a positive constant $\gamma > \mu_\infty$ obtain a solution to the following mixed objective problem:

$$\begin{aligned} \nu_\infty &:= inf\{\|h-u*q\|_1 : q \in l_1 \text{ and } <h-u*q, h-u*q> \leq \gamma\} \\ &= inf\{\|\phi\|_1 : \phi \in l_1 \text{ and } \mathrm{A}\phi = b \text{ and } <\phi,\phi> \leq \gamma\}. \end{aligned} \tag{2.83}$$

Where $<.,.>$ is the inner product.

CHAPTER 3 METHODOLOGY AND APPROACH

3.1. Problem Setup:

3.1.1. Problem Setup for H_2 Optimal Control

Figure (3.1) show the standard LFT diagram for H_2 optimal control design

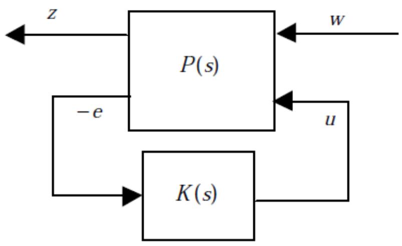


Figure (3.1): Standard LFT diagram for H_2 optimal control design

First, we desire to minimize the H_2 norm of the system:

We have

$$P(s) := \left[\begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array}\right] \tag{3.1}$$

Where

$$\begin{aligned} P_{11}(s) &= 0 \\ P_{12}(s) &= \begin{bmatrix} W_u \\ W_e G \end{bmatrix} \\ P_{21}(s) &= W_o \\ P_{22}(s) &= -G \end{aligned} \tag{3.2}$$

The control system is put in the form of the so-called standard H_2 problem (in LFT form), the minimization problem becomes: $\min_{K \in s} \left\| T_{ZW} \right\|_2$

3.1.2. Problem Setup for H_∞ Optimal Control

Figure (3.2) show the standard LFT diagram for H_∞ optimal control design

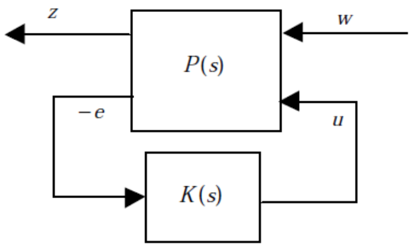


Figure (3.2): Standard LFT diagram for H_∞ optimal control design

First, we desire to minimize the H_∞ norm of the system:

We have

$$P(s) := \left[\begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array}\right] \tag{3.3}$$

Where

$$\begin{aligned} P_{11}(s) &= 0 \\ P_{12}(s) &= \begin{bmatrix} W_u \\ W_e G \end{bmatrix} \\ P_{21}(s) &= W_o \\ P_{22}(s) &= -G \end{aligned} \tag{3.4}$$

We want to design finite-dimensional stabilizing LTI controllers that minimize the H_∞-norm of the closed-loop system:

Consider the block diagram as mentioned before in Figure (3.2). The control system is put in the form of the so-called standard H_∞ problem (in LFT form), the minimization problem becomes: $\min_{K \in s} \left\| T_{ZW} \right\|_\infty$

The solution to the suboptimal H_∞ problem stated as given $\gamma > 0$, find an admissible controller (if there exists any) such that $\left\| T_{ZW} \right\|_\infty < \gamma$. We will present the solution to this problem, and it should be clear that an iterative bisection procedure for reducing γ while checking that a suboptimal controller exists will lead to a controller as close to the optimal controller as desired.

3.1.3. Problem Setup for l_1 Optimal Control

Definition 1: A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty \quad as \quad r \to \infty$.

Definition2: A mapping $H : L_e^m \to L_e^q$ is L stable if there exist a class K function α, defined on $[0, \infty)$ and a nonnegative constant β such that

$$\|(Hu)_\tau\|_L \le \alpha \|u_\tau\|_L + \beta \tag{3.5}$$

and for all $u \in L_e^m$ *and* $\tau \in [0,\infty)$.

It is finite gain L stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_L \le \gamma \|u_\tau\|_L + \beta \tag{3.6}$$

for all $u \in L_e^m$ *and* $\tau \in [0,\infty)$.

The constant β is called the bias term. It is included in the definition to allow for systems where Hu does not vanish at u = 0. When inequality (3.6) is satisfied, we are usually interested in the smallest γ for which there is β such that (3.6) is satisfied. When this value of γ is well defined, we will call it the gain of the system.

Then, the problem setup for l_1 norm is to minimize γ to its optimal value, this is called optimal solution of problem.

Remark: γ is the l_1 norm.

3.2. Approach

3.2.1. Approach for H_2 Optimal Control

We consider the state space realization witch named full information realization (FI)

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3.7}$$

with $D_{11} = D_{22} = 0$, which mean that $P_{11}(s)$ and $P_{22}(s)$ is strictly proper. We define $R_1 = D_{12}^* D_{12}$ and $R_2 = D_{21}^* D_{21}$, and the two Hamiltonian matrices:

$$H_2 := \begin{bmatrix} A - B_2 R_1^{-1} D_{12}^* C_1 & -B_2 R_1^{-1} B_2^* \\ -C_1^*(I - D_{12} R_1^{-1} D_{12}^*) C_1 & -(A - B_2 R_1^{-1} D_{12}^* C_1)^* \end{bmatrix} \tag{3.8}$$

$$J_2 := \begin{bmatrix} (A - B_1 D_{21}^* R_2^{-1} C_2)^* & -C_2^* R_2^{-1} C_2 \\ -B_1(I - D_{21}^* R_2^{-1} D_{21}) B_1^* & -(A - B_1 D_{21}^* R_2^{-1} C_2)^* \end{bmatrix} \tag{3.9}$$

Note that

$$H_2, J_2 \in dom\left(Ric\right) \text{ and } X_2 := \mathrm{Ric}(H_2) \ \ge\ 0, Y_2 := \mathrm{Ric}(J_2) \ \ge\ 0.$$

The proof can be found in [2].■

Theorem: H_2 -Optimal Controller:

If the following assumptions hold:

1. The pair (A, B_2) is stabilizable and the pair (A, C_2) is detectable.
2. $R_1 = D_{12}^* D_{12} > 0$ (meaning that all of its eigenvalues are positive). and $R_2 = D_{21}^* D_{21} > 0$
3. $\begin{bmatrix} A - jwI & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all w
4. $\begin{bmatrix} A - jwI & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all w

Then, the unique H_2 -optimal controller minimizing $\|T_{ZW}\|_2$ is given by

$$K_{opt}(s) = \begin{bmatrix} \tilde{A}_2 & \text{-}L_2 \\ F_2 & 0 \end{bmatrix} \tag{3.10}$$

Where matrix L_2 is given by $L_2 := -(Y_2 C_2^* + B_1 D_{21}^*) R_2^{-1}$, matrix F_2 is given by $F_2 := -R_1^{-1}(B_2^* X_2 + D_{12}^* C_1)$, and $\hat{A}_2 := A + B_2 F_2 + L_2 C_2$.
The proof can be found in [2].■

3.2.2. Approach for H_∞ Optimal Control

We consider the state space realization witch named full information realization (FI)

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3.11}$$

Notice the special off-diagonal structure assumed for D (just like the H_2 case). Given γ > 0, define the two Hamiltonian matrices:

$$H_\infty := \begin{bmatrix} A & \gamma^{-2} B_1 B_1^* - B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{bmatrix} \tag{3.12}$$

$$J_\infty := \begin{bmatrix} A^* & \gamma^{-2} C_1^* C_1 - C_2^* C_2 \\ -B_1 B_1^* & -A \end{bmatrix} \tag{3.13}$$

The proof can be found in [2].■

Theorem: H_∞ **Controller**

There exists an admissible controller such that $\|T_{ZW}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in dom(Ric) \;\; and \;\; X_\infty := Ric(H_\infty) \geq 0$;
2. $J_\infty \in dom(Ric) \;\; and \;\; Y_\infty := Ric(J_\infty) \geq 0$;
3. $\rho(X_\infty Y_\infty) < \gamma^2$ (the spectral radius of the product $X_\infty Y_\infty$).

When these conditions hold, one such controller is

$$K_\infty(s) := \begin{bmatrix} \tilde{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix} \tag{3.14}$$

Where

$$\begin{aligned} \tilde{A}_\infty &:= A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\ F_\infty &:= -B_2^* X_\infty \\ L_\infty &:= -Y_\infty C_2^* \\ Z_\infty &:= (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \end{aligned} \tag{3.15}$$

The proof can be found in [2].■

3.2.3. The Approach for $l_1 / H_2 \;\; and \;\; l_1 / H_\infty$ Optimal Control

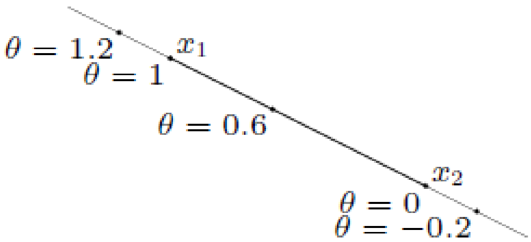
After minimizing $\min_{K\in s} \|T_{ZW}\|_2 \;\; and \;\; \min_{K\in s} \|T_{ZW}\|_\infty$, at this step we have optimized the external characteristic of the system. Now we want to realize the mixed objective by fixing the values obtained for H_2-norn and H_∞-norm which means fixing the external characteristic and minimizing l_1-norm which means internal characteristic of the system.

We use convex optimization approach to find the optimal solution l_1 that minimize the cost function.

3.2.4. Affine set

Line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta) x_2 \qquad (\theta \in \mathrm{R}) \tag{3.16}$$



affine set contains the line through any two distinct points in the set.

Example: solution set of linear equations {x | Ax = b}

Conversely, every affine set can be expressed as solution set of system of linear equations.

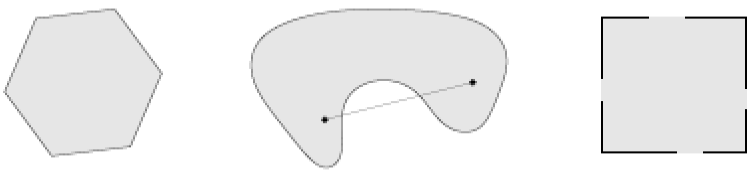
3.2.5. Convex Set

Line segment between x_1 and x_2 : all points

$x = \theta x_1 + (1-\theta)x_2 \qquad 0 \le \theta \le 1$. Convex set contains line segment between any two points in the set

$$x_1,\ x_2 \in \mathrm{C},\ 0 \le \theta \le 1 \ \Rightarrow \theta x_1 \ + \ (1-\theta)x_2 \in \mathrm{C} \qquad (3.17)$$

Examples (one convex, two nonconvex sets)

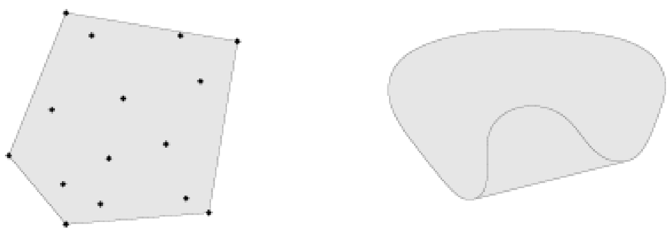


3.2.6. Convex Combination and Convex hull

Convex combination of $x_1, \ldots, x_k$ of any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \text{ With } \theta_1 + \theta_2 + \cdots + \theta_k = 1 \ , \theta_i \ge 0 \qquad (3.18)$$

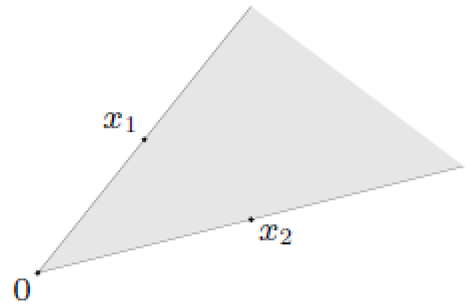
Convex hull of a convex Set Is a set of all convex combinations of points in Set.



3.2.7. Convex Cone

Conic (nonnegative) combination of x_1 and x_2 is any point of the form:

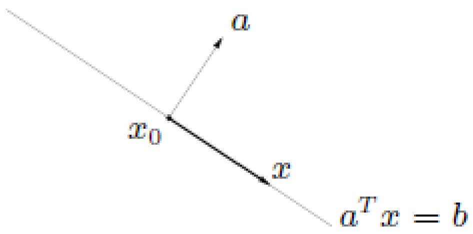
$$x = \theta_1 x_1 + \theta_2 x_2 \qquad \text{with } = \theta_1 \geq 0 \text{ , } \theta_2 \geq 0 \tag{3.19}$$



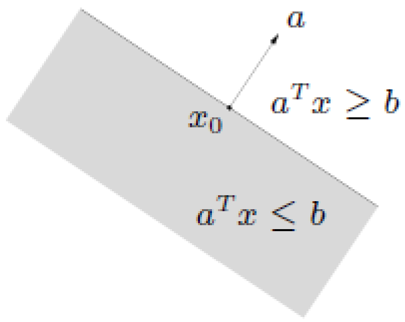
Convex cone is set that contains all conic combinations of points in the set.

3.2.8. Hyperplanes and Halfspaces

Hyperplane is set of the form $\{x \mid a^T x = b\}(a \neq 0)$



Halfspace is set of the form $\{x \mid a^T x \leq b\}(a \neq 0)$



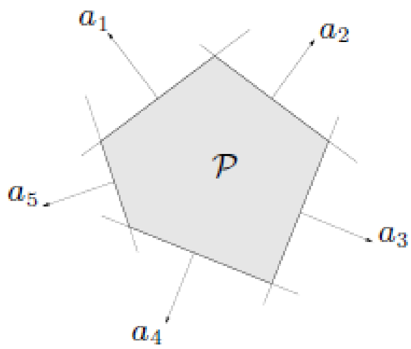
1. a is the normal vector.
2. hyperplanes are affine and convex.
3. halfspaces are convex.

3.2.9. Polyhedra

Polyhedron is a solution of set of finitely many linear inequalities and equalities.

$$Ax \leq b, \qquad Cx = d$$

$(A \in R^{m \times n}, C \in R^{p \times n})$, ≤ is componentwise inequality



polyhedron is intersection of finite number of halfspaces and hyperplanes.

3.2.10. Linear Optimization Problems

When the objective and constraint functions are all affine, the problem is called a linear program (LP). A general linear program has the form:

$$\begin{aligned} &minimize \quad c^T x + d; \\ &subject\ to \quad Gx \leq h; \\ &\qquad\qquad\quad Ax = b; \end{aligned} \tag{3.20}$$

Where $A \in R^{p \times n}$ and $G \in R^{m \times n}$. Linear programs are, of course, convex optimization problems.

It is common to omit the constant d in the objective function, since it does not affect the optimal (or feasible) set. Since we can maximize an affine objective $c^T x + d$, by minimizing $-c^T x - d$ (which is still convex), we also refer to a maximization problem with affine objective and constraint functions as an LP.

The geometric interpretation of an **LP** is illustrated in Figure (3.3). The feasible set of the LP (3.3) is a polyhedron **P**; the problem is to minimize the affine function $c^T x + d$ (or, equivalently, the linear function $c^T x$) over **P**.

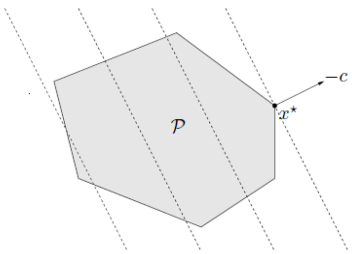


Figure (3.3): Geometric interpretation of an LP [48]

In Figure (3.3) the objective is linear, so its level curves are hyperplanes orthogonal to C. the point x^* is optimal; it is the point in P as far as possible in the direction –C

CHAPTER 4 SIMULATIONS AND RESULTS

4.1. Introduction

In This thesis, the application of mixed objective problems L_1/H_2 and L_1/H_∞ are considered for SISO system, linear time invariant to designing controller for Autopilot pitch Aircraft. The Lagrange duality principle methodology proposed by Slapaka and Dahlah (1995) [42] is used, but with introducing some change to this method as follow:

This thesis considers the problem of minimizing the sensitivity of the output system to the input disturbance by the method of 2-norm and infinity-norm, this minimization has considered as primal Lagrange multiplier. The reason of use $\min_{K\in s}\|T_{ZW}\|_2$ and $\min_{K\in s}\|T_{ZW}\|_\infty$ is to minimize or reject the effect of disturbance to our system.

In the dual Lagrange multiplier problem minimizing the bounded norm output of the controller designed by using the L_1 norm theory is considered, this lead to use convex optimization approach. To solve this problem, the CVX-toolbox built by Stephen Boyd is used. The reason for using the bounded-norm output as dual problem is to minimize the cost function of the output controller and limit the brusque variation introduced to the system by the input control, which lead to BIBO system.

The realization of the mixed objective are achieved by using the primal problem as constraint to the dual problem for the two methods L_1/H_2 and L_1/H_∞.

4.2. Physical Setup and System Equations autopilot pitch Aircraft

The equations governing the motion of an aircraft are a very complicated set of six non-linear coupled differential equations. However, under certain assumptions, they can be decoupled and linearized into the longitudinal and lateral equations. Pitch control is a longitudinal problem, and in this example, we will design an autopilot that controls the pitch of an aircraft.

The basic coordinate axes and forces acting on an aircraft are shown in the Figure (4.1) and Figure (4.2):

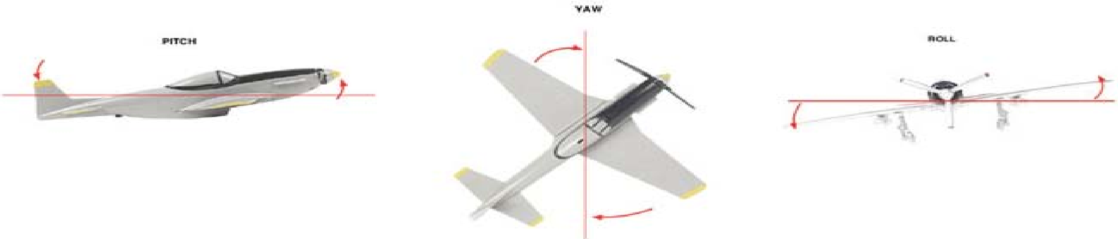


Figure (4.1): Pitch-yaw-roll coordinate axes of aircraft [59]

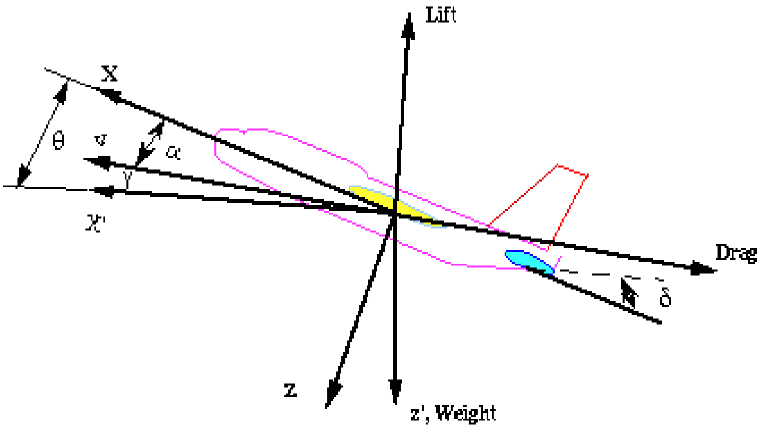


Figure (4.2): The basic coordinate axes and forces acting on an aircraft [59]

Assume that the aircraft is in steady-cruise at constant altitude and velocity; thus, the thrust and drag cancel out and the lift and weight balance out each other. Also, assume that change in pitch angle does not change the speed of an aircraft under any circumstance (unrealistic but simplifies the problem a bit). Under these assumptions, the longitudinal equations of motion of an aircraft can be written as:

$$\dot{\alpha} = \mu\Omega\sigma[-(C_{L\alpha} + C_D)\alpha + (1/\mu - C_{Lq})q - (C_{w\theta}\sin\gamma_e)\theta + C_{LR}]$$
$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\{[C_{M\alpha} - \eta(C_{L\alpha} + C_D)]\alpha + [C_{Mq} + \sigma C_{Ms}(1 - \mu C_L)]q + (\eta C_w \sin\gamma_e)\delta_e\} \quad (4.1)$$
$$\dot{\theta} = \Omega q$$

Equations can be found [59].

And the variables used in the pitch controller modeling equations

α= Angle of attack

q= Pitch rate

θ= Pitch angle

δe= Elevator deflection angle

$$\mu = \frac{\rho_e S\overline{c}}{4m}$$

ρ_e = Density of the surrounding air

S = Planform area of the wing

$\overline{c}$ = Average chord length

m = Mass of the aircraft

$$\Omega = \frac{2U}{\overline{c}}$$

U = Equilibrium flight speed

C_T = Coefficient of thrust

C_D = Coefficient of drag

C_Z = Coefficient of lift

C_W = Coefficient of weight

C_M = Coefficient of pitch moment

γ_e = Flight path angle

$\sigma = \frac{1}{1+\mu C_{Za}}$ = Constant sigma

$i_{,,}$ = Normalized moment of inertia

$\eta = \mu\sigma C_{jk}$ = Constant nu

For this system, the input will be the elevator deflection angle, and the output will be the pitch angle.

4.3. Design Requirements

The next step is to set some design criteria. We want to design a feedback controller so that the output has an overshoot of less than 10%, rise time of less than 2 seconds, settling time of less than 10 seconds, and the steady-state error of less than 2%. For example, if the input is 0.2 rad (11 degress), then the pitch angle will not exceed 0.22 rad, reaches 0.2 rad within 2 seconds, settles 2% of the steady-state within 10 seconds, and stays within 0.196 to 0.204 rad at the steady-state. [59].

Overshoot: Less than 10%

Rise time: Less than 2 seconds

Settling time: Less than 10 seconds

Steady-state error: Less than 2%

4.4. Transfer Function and the State-Space of aircraft

Before finding transfer function and the state-space model, let's plug in some numerical values to simplify the modeling equations (4.1) shown above.

$$\begin{aligned} \dot{\alpha} &= -0.313\alpha + 56.7q + 0.232\delta_e \\ \dot{q} &= -0.0139\alpha - 0.246q + 0.0203\delta_e \\ \dot{\theta} &= 56.7q \end{aligned} \tag{4.2}$$

These values are taken from the data from one of the Boeing's commercial aircraft.

Equations can be found [59].

4.5. Transfer Function of aircraft

To find the transfer function of the above system, we need to take the Laplace transform of the above modeling equations (4.2). When finding a transfer function, zero initial conditions must be assumed. The Laplace transform of the above equations are shown below.

$$\begin{aligned} s\alpha(s) &= -0.313\alpha(s) + 56.7q(s) + 0.232\delta_e(s) \\ sq(s) &= -0.0139\alpha(s) - 0.426q(s) + 0.0203\delta_e(s) \\ s\theta(s) &= 56.7q(s) \end{aligned} \tag{4.3}$$

After few steps of algebra, we should obtain the following transfer function. [59].

$$\frac{\theta(s)}{\delta_e(s)} = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921s} \tag{4.4}$$

4.6. State-Space of aircraft

Knowing the fact that the modeling equations (4.3) are already in the state-variable form, we can rewrite them into the state-space model.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \begin{bmatrix} \delta_e \end{bmatrix} \tag{4.5}$$

Since our output is the pitch angle, the output equation is:

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \delta_e \end{bmatrix} \tag{4.6}$$

4.7. Original Representation of Open-Loop Response of aircraft

The open loop representation of the air craft without controller are shown in Figure (4.3)

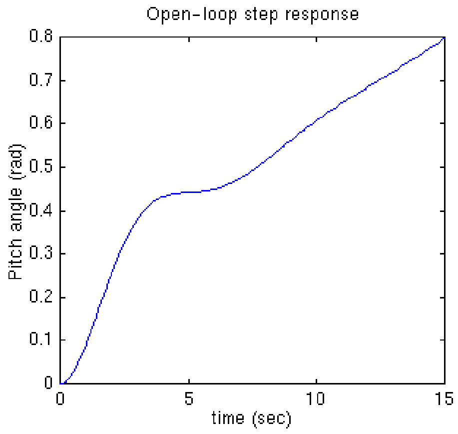


Figure (4.3): Original open-loop response

From the plot, we see that the open-loop response does not satisfy the design criteria at all. In fact, the open-loop response is unstable.

4.8. Original Representation of Close-Loop Response of aircraft

The original close loop response of the system without any controller are shown in Figure (4.4)

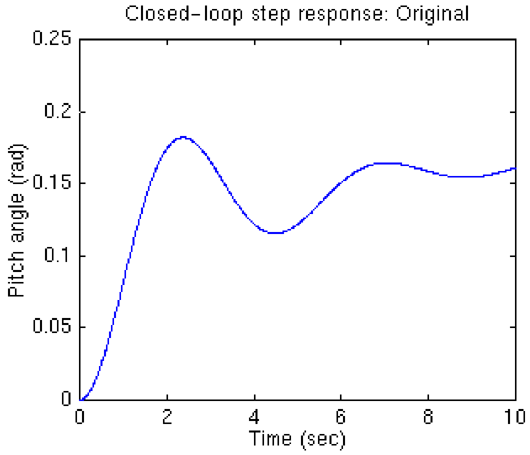


Figure (4.4): Original closed-loop step response

From the plot, we see that the closed-loop response does not satisfy the design criteria at all. In fact, the closed-loop response is stable but with high steady state error.

4.9. Original Root-Locus Plot of aircraft

A root-locus plot shows all possible closed-loop pole locations for a pure proportional controller. Since not all poles are acceptable. The two arguments Natural frequency (Wn) and damping ratio (zeta) can be determined from the rise time, the settling time, and the overshoot requirements and the three equations shown below.

$$\xi\omega_n \geq \frac{4.6}{T_s};\ \omega_n \geq \frac{1.8}{T_r};\ \xi \geq \sqrt{\frac{\left(\frac{\ln M_p}{\pi}\right)^2}{1+\left(\frac{\ln M_p}{\pi}\right)^2}} \tag{4.7}$$

Equations can be found [59].
Where
Wn=Natural frequency
zeta=Damping ratio
Ts=Settling time
Tr=Rise time
Mp=Maximum overshoot

From these three equations, we can determine that the natural frequency (Wn) must be greater than 0.9 and the damping ratio (zeta) must be greater than 0.52.

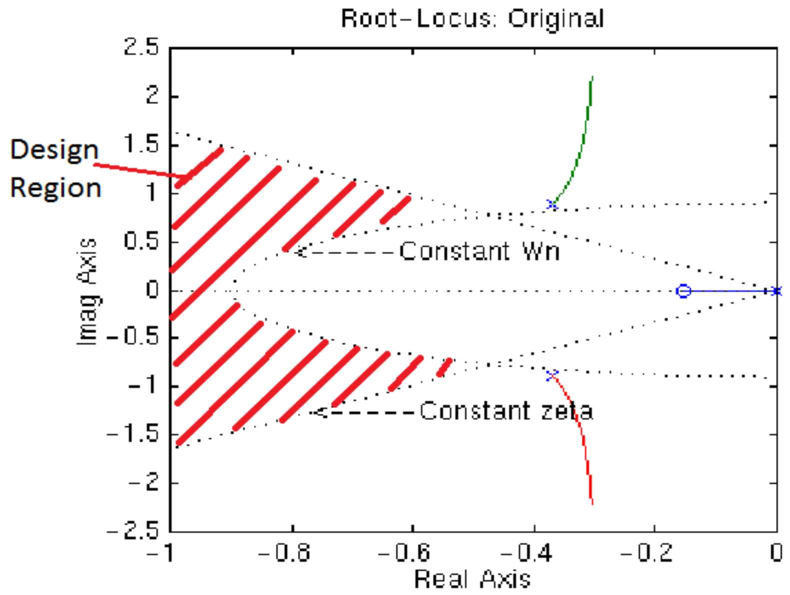


Figure (4.5): Original root-locus

From the Figure (4.5), we see the two dotted lines in an angle indicate the locations of constant damping ratio, and the damping ratio is greater than 0.52 in between these lines. The dotted semi-ellipse indicates the locations of constant natural frequency, and the natural frequency is greater than 0.9 outside the semi-ellipse. As you may notice, there is no root-locus plotted in our desired region. We need to bring the root-locus in between two dotted lines and outside the semi-ellipse by modifying the controller.

4.10.Original Bode Plot of aircraft

Our system is unstable in open loop; however, we can still design the feedback system via frequency response method. You should see a Bode plot similar to the one showed in Figure (4.6):

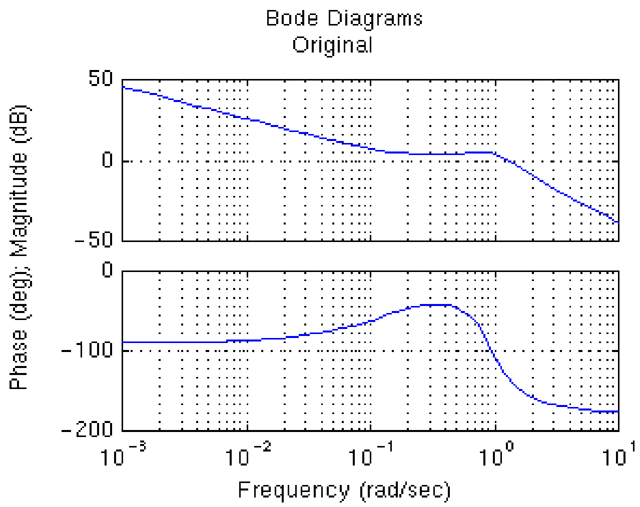


Figure (4.6): Original bode diagrams

4.11. L_1 / H_2 Simulation

4.11.1. Closed-Loop Transfer Function

To solve this problem, a feedback controller will be added to improve the system performance. Figure (4.7) shown below is the block diagram of a typical unity feedback system.

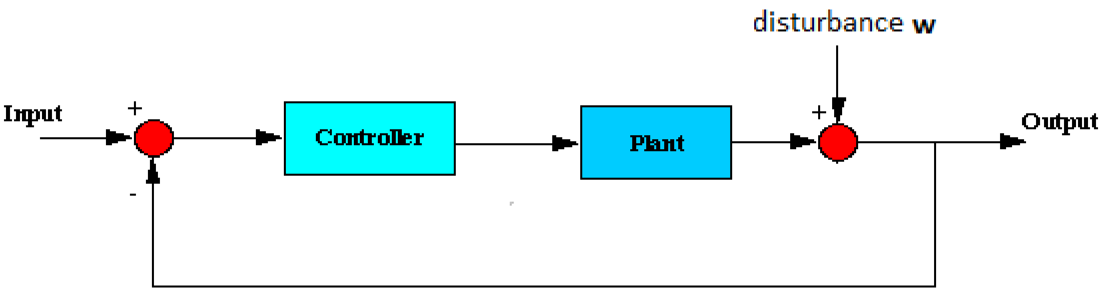


Figure (4.7): Close-loop feedback with controller and plant

A dynamic controller needs to be designed so that the step response satisfies all design requirements.

The state space equations of the controller are:

$$\dot{X}(t) = A_k * X(t) + B_k * u(t)$$
$$Y(t) = C_k * X(t) + D_k * u(t)$$

Where:

$$Ak = \begin{bmatrix} -0.4226 & -1.1547 & 0 & 0 & 0.0000 \\ 1.1547 & -1.5774 & 0 & 0 & -0.0000 \\ 0.0000 & -0.0000 & -0.3097 & 56.6233 & -0.6448 \\ 0.0000 & -0.0000 & -0.0136 & -0.4327 & -0.0042 \\ 0 & 0 & 0 & 56.7000 & -0.6655 \end{bmatrix} ;\ Bk = \begin{bmatrix} 0.5373 \\ -0.5373 \\ -0.6409 \\ -0.0039 \\ -0.6655 \end{bmatrix}$$

$$Ck = \begin{bmatrix} 0.0001 & -0.0001 & 0.0141 & -0.3308 & -0.0166 \end{bmatrix} ;\ Dk = \begin{bmatrix} 0 \end{bmatrix}$$

From the state space model, we get the transfer function of the controller:

$$K(s) = \frac{0.003393s^4 + 0.009139s^3 + 0.01448s^2 + 0.01069s + 0.005914}{s^5 + 3.408s^4 + 6.455s^3 + 6.274s^2 + 3.637s + 0.358} \quad (4.8)$$

The transfer functions of the plant with considering the weighted function of signal disturbance:

$$P(s) = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921s}$$

Where the weighted function of signal disturbance is:

$$W_o = 1$$

Then the transfer function of the output Z to disturbance w is:

$$T_{zw} = \frac{1}{1 + P(s) * K(s)} = [1 + P(s) * K(s)]^{-1}$$

4.11.2 Building the program H_2-optimal controller

First, we build the program H_2-optimal controller, this program can find the optimal external optimization of our system.

1. Use the values of state space system (A, B, C, D) we generate G(s).
2. Define the weighting functions for the system showed in Figure (2.12).
3. Build the system interconnection to obtain the generalized plant P(s).
4. Extact the portioned state-space matrices from the plant P(s) and we got [Ap, Bp, Cp, Dp].
5. Decompose of Ap, Bp, Cp and Dp using linear fractional transformation to Bp1, Bp2, Cp1, Cp2, Dp11, Dp12, Dp21 and Dp22.

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] \Leftrightarrow \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right]$$

6. Test stabilizability of (Ap, Bp2) and (Ap, Bp1).
7. Test detectability of (Cp2, Ap) and (Cp1, Ap).
8. Test the matrix $\begin{bmatrix} A - jwI & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.
9. Test the matrix $\begin{bmatrix} A - jwI & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.
10. Compute H_2 optimal controller using the function H2sys we extract all characteristics of systems as output of this function, this function has to work as follow:

[K, Tzw, Kfi, CL, GAM, INFO]=H2SYN (P, NMEAS, NCON)

This function is from robust toolbox, it takes as input the matrix P and number of measurements and gives us all characteristic of the system. After execution of this function, the results obtained are:

- H_2 optimal controller:

$$K = \begin{bmatrix} -0.4226 & -1.1547 & 0 & 0 & 0.0000 & 0.5373 & 5.0000 \\ 1.1547 & -1.5774 & 0 & 0 & -0.0000 & -0.5373 & 0 \\ 0.0000 & -0.0000 & -0.3097 & 56.6233 & -0.6448 & -0.6409 & 0 \\ 0.0000 & -0.0000 & -0.0136 & -0.4327 & -0.0042 & -0.0039 & 0 \\ 0 & 0 & 0 & 56.7000 & -0.6655 & -0.6655 & 0 \\ 0.0001 & -0.0001 & 0.0141 & -0.3308 & -0.0166 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -Inf \end{bmatrix}$$

- The H_2-norm optimal controller $\|T_{zw}\|_2$:

$$
Tzw = \begin{bmatrix}
-0.4226 & -1.1547 & 0 & 0 & -0.5373 & 0 & 0 & 0 & 0 & 0 & 0.5373 & 0 & 10.0000 \\
1.1547 & -1.5774 & 0 & 0 & 0.5373 & 0 & 0 & 0 & 0 & 0 & -0.5373 & 0 & 0 \\
0 & 0 & -0.3130 & 56.7000 & 0 & 0.0000 & -0.0000 & 0.0033 & -0.0767 & -0.0039 & 0 & 0.2320 & 0 \\
0 & 0 & -0.0139 & -0.4260 & 0 & 0.0000 & -0.0000 & 0.0003 & -0.0067 & -0.0003 & 0 & 0.0203 & 0 \\
0 & 0 & 0 & 56.7000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5373 & -0.4226 & -1.1547 & 0 & 0 & 0.0000 & 0.5373 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5373 & 1.1547 & -1.5774 & 0 & 0 & -0.0000 & -0.5373 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6409 & 0.0000 & -0.0000 & -0.3097 & 56.6233 & -0.6448 & -0.6409 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0039 & 0.0000 & -0.0000 & -0.0136 & -0.4327 & -0.0042 & -0.0039 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6655 & 0 & 0 & 0 & 56.7000 & -0.6655 & -0.6655 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0016 & -0.0024 & 0.4235 & -9.9236 & -0.4993 & 0 & 0 & 0 \\
0.5373 & 0.5373 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Inf
\end{bmatrix}
$$

- Full information-state feedback control law:

$$
Kfi = \begin{bmatrix} 1.7948 & 0.1628 & 1.8021 & 0 \end{bmatrix}
$$

- Full information-state feedback closed-loop system:

$$
Gfi = \begin{bmatrix} 0.0001 & -0.0001 & 0.0141 & -0.3308 & -0.0166 \end{bmatrix}
$$

- Closed-loop system CL= LFT (P,K):

$$
CL = \begin{bmatrix}
-0.4226 & -1.1547 & 0 & 0 & -0.5373 & 0.5373 & 0 & 5.0000 \\
1.1547 & -1.5774 & 0 & 0 & 0.5373 & -0.5373 & 0 & 0 \\
0.0000 & -0.0000 & -0.3097 & 56.6233 & -0.0039 & 0 & 0.2320 & 0 \\
0.0000 & -0.0000 & -0.0136 & -0.4327 & -0.0003 & 0 & 0.0203 & 0 \\
0 & 0 & 0 & 56.7000 & 0 & 0 & 0 & 0 \\
0.0016 & -0.0024 & 0.4235 & -9.9236 & -0.4993 & 0 & 0 & 0 \\
0.5373 & 0.5373 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -Inf
\end{bmatrix}
$$

- X Hamiltonian matrix HAMX:

$$
hamx = \begin{bmatrix}
-0.4226 & -1.1547 & 0 & 0 & -0.5373 & 0 & 0 & 0 & 0 & 0 \\
1.1547 & -1.5774 & 0 & 0 & 0.5373 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.3130 & 56.7000 & 0 & 0 & 0 & -0.0001 & -0.0000 & 0 \\
0 & 0 & -0.0139 & -0.4260 & 0 & 0 & 0 & -0.0000 & -0.0000 & 0 \\
0 & 0 & 0 & 56.7000 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.2887 & -0.2887 & 0 & 0 & 0 & 0.4226 & -1.1547 & 0 & 0 & 0 \\
-0.2887 & -0.2887 & 0 & 0 & 0 & 1.1547 & 1.5774 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3130 & 0.0139 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -56.7000 & 0.4260 & -56.7000 \\
0 & 0 & 0 & 0 & 0 & 0.5373 & -0.5373 & 0 & 0 & 0
\end{bmatrix}
$$

- Y Hamiltonian matrix HAMY:

$$hamy = \begin{bmatrix} -0.4226 & 1.1547 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.1547 & -1.5774 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.3130 & -0.0139 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 56.7000 & -0.4260 & 56.7000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0000 \\ 0 & 0 & 0 & 0 & 0 & 0.4226 & 1.1547 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.1547 & 1.5774 & 0 & 0 & 0 \\ 0 & 0 & -0.0538 & -0.0047 & 0 & 0 & 0 & 0.3130 & -56.7000 & 0 \\ 0 & 0 & -0.0047 & -0.0004 & 0 & 0 & 0 & 0.0139 & 0.4260 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -56.7000 & 0 \end{bmatrix}$$

From the results output program we plot all different diagrams who clarify characteristic of system: the diagrams that clarify the system characteristics are as follows:

- **Output Plant and Controller Response:**

The plot of the plant and controller response when introducing a disturbance signal is shown in Figure (4.8).

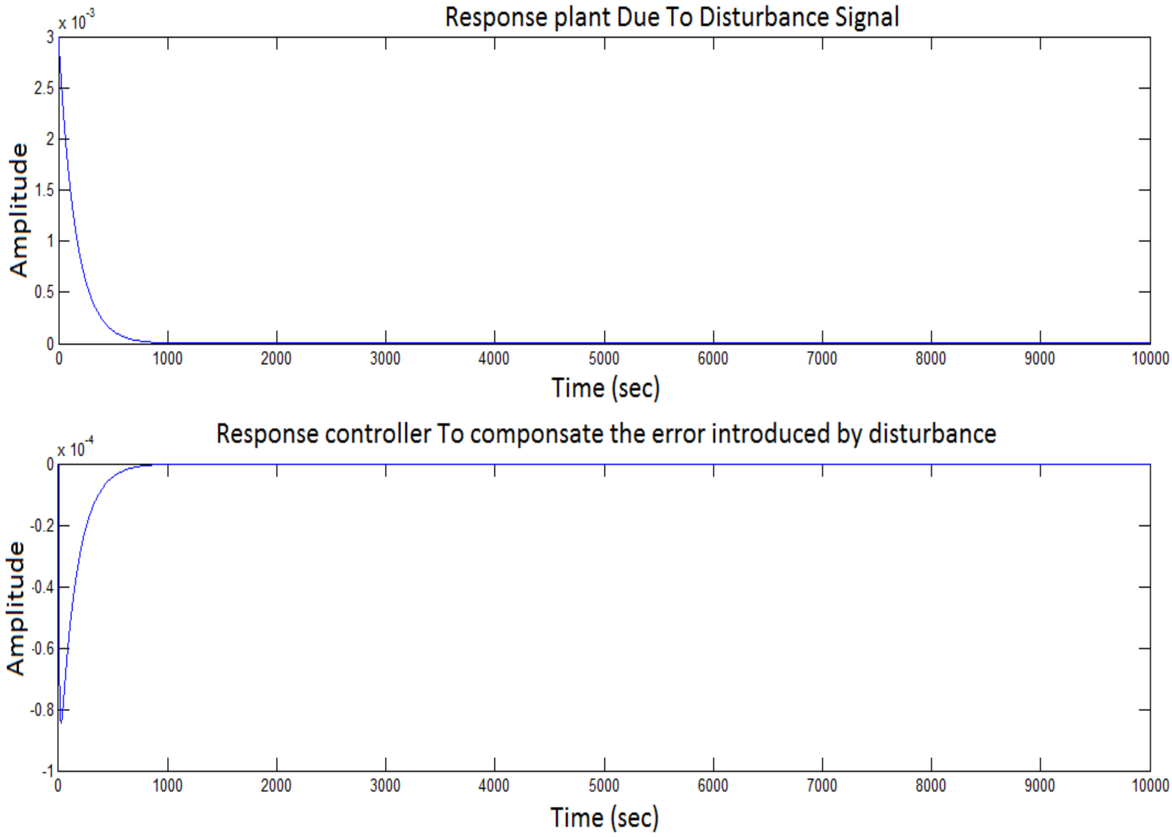


Figure (4.8): Plant and controller response due to introduced disturbance input

For test the efficacity of the system in minimizing and rejecting the disturbance signal we have introduced a noise signal, as result, we see that The curve of controller is compensative to the curve of the plant for reason minimizing the error which mean that our system functionality have satisfaction to exclude the noise disturbance signal.

- **Step Response Closed-Loop System CL= LFT (P, K) and output response to disturbance:**

The plot of the output response to disturbance signal and output response to input when using the close loop feedback is shown in Figure (4.9).

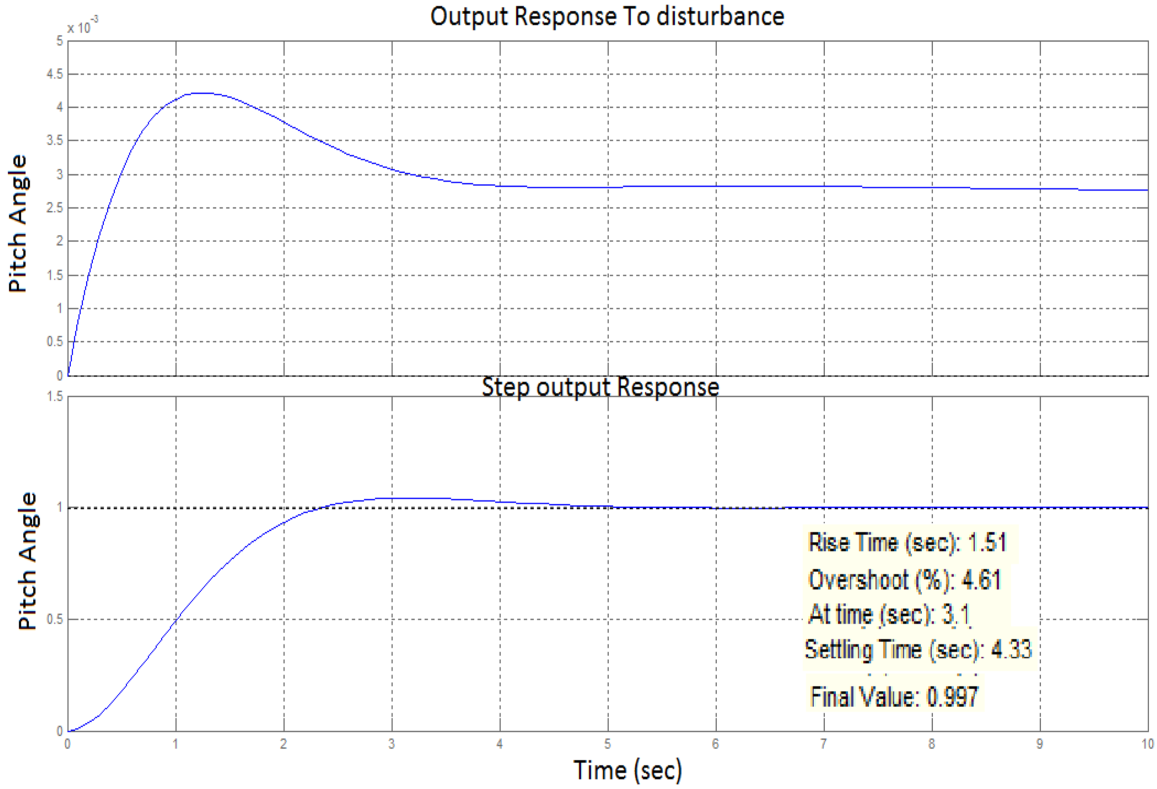


Figure (4.9): Step response close-loop system and output disturbance

We have inject a disturbance signal with step input of 0.3 radian disturbance in angle and we see from the first plot the output response of this disturbance at value 0.003 radian with mean the input signal disturbance minimized 100 time. Moreover, we conclude that our system is stable because it returns to the zero after it is subjected by a temporary disturbance.

From the second plot, we see the powerful of our method since the system go directly to the steady state and all results satisfies the design requirements.

- **Bode Diagram for Close-Loop System: CL = LFT (P, K):**

  From the plot in Figure (4.10), we see the robustness of the system and the smoothness variation of the phase. In addition, if we see the Bode plot, the low frequency gain has increased while keeping the bandwidth frequency the same, this is a good low pass filter we can use it for tracking of signal input. This tells us that steady-state error has reduced while keeping the same rise time. The above step response shows that the steady-state error got eliminated. Now all design requirements are satisfied.

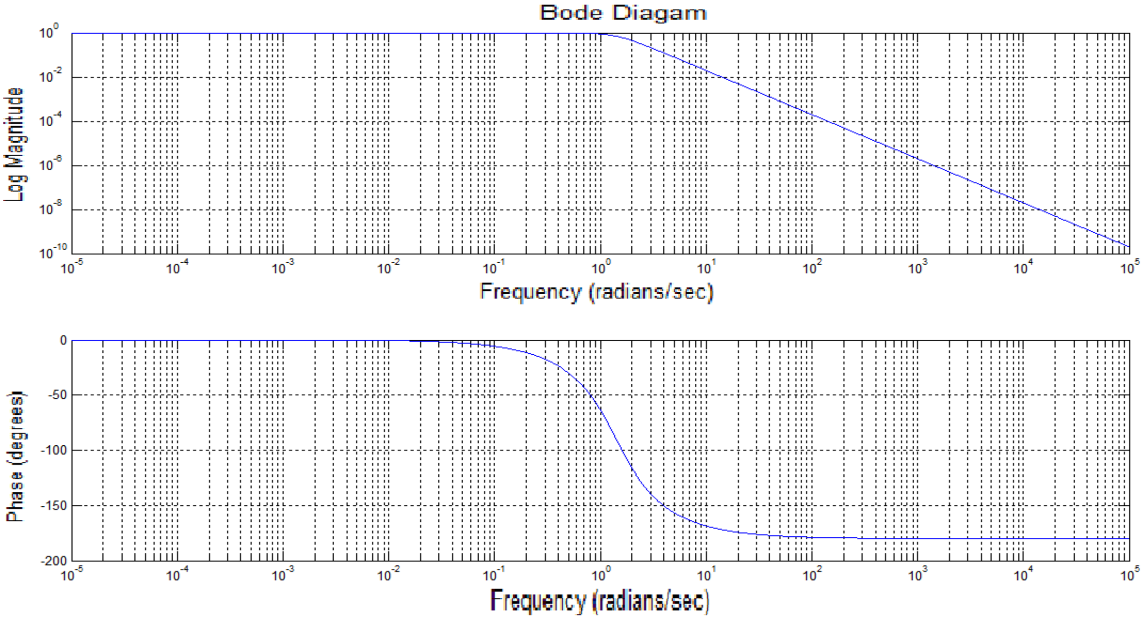


Figure (4.10): Bode diagram for close loop system

- **Rout-Locus Output Plant Response**

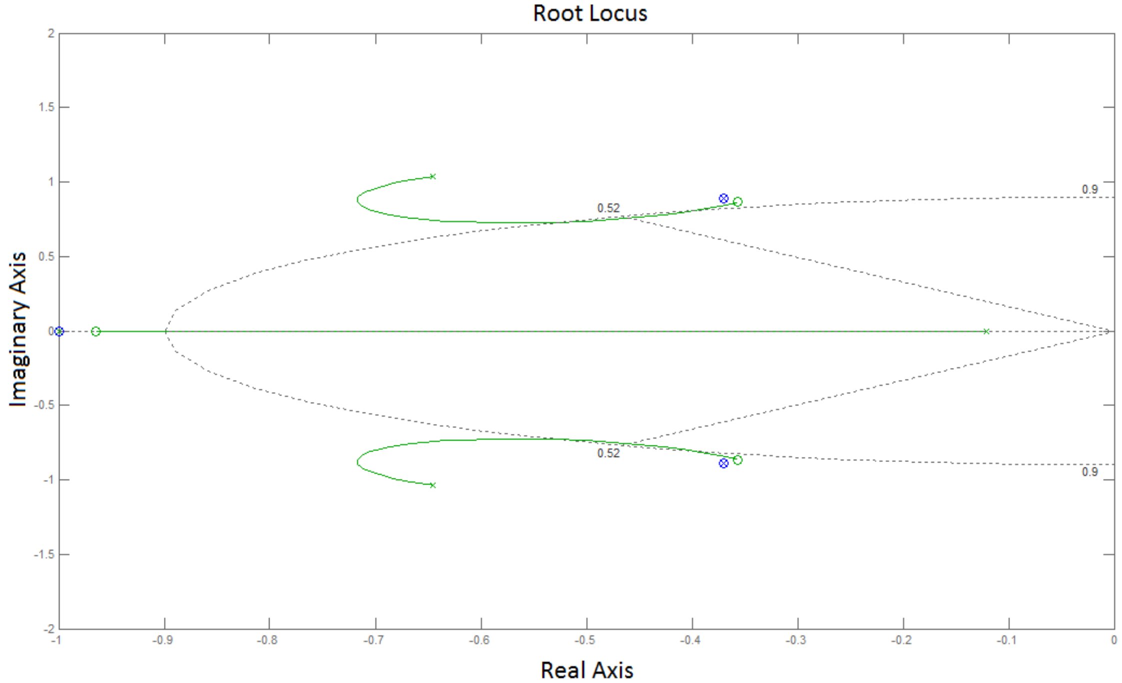


Figure (4.11): Rout-locus output plant response

From the plot of Figure (4.11), we see the powerful of the method used and the robustness of the system and the pole placement of the poles to the region between -1 and 0 to ensure stability of the system and we can define the position of poles inside the design region by fixing the controller gain.

And from all this results we have the minimized the H_2-norm optimal controller $\|T_{zw}\|_2 \leq 1.802e+000$.

Now we want to realize the mixed objective by fixing the values obtained for H_2-norm which means fixing the external characteristic and minimizing l_1-norm which means internal characteristic of the system.

We use convex optimization approach to find the optimal solution l_1 that minimize the cost function.

The state space equations of the controller are:

$$\dot{X}(t) = A_k * X(t) + B_k * u(t)$$
$$Y(t) = C_k * X(t) + D_k * u(t)$$

Where:

$$\text{Ak} = \begin{bmatrix} -0.4226 & -1.1547 & 0 & 0 & 0.0000 \\ 1.1547 & -1.5774 & 0 & 0 & -0.0000 \\ 0.0000 & -0.0000 & -0.3097 & 56.6233 & -0.6448 \\ 0.0000 & -0.0000 & -0.0136 & -0.4327 & -0.0042 \\ 0 & 0 & 0 & 56.7000 & -0.6655 \end{bmatrix}; \quad \text{Bk} = \begin{bmatrix} 0.5373 \\ -0.5373 \\ -0.6409 \\ -0.0039 \\ -0.6655 \end{bmatrix}$$

$$\text{Ck} = \begin{bmatrix} 0.0001 & -0.0001 & 0.0141 & -0.3308 & -0.0166 \end{bmatrix}; \quad \text{Dk} = \begin{bmatrix} 0 \end{bmatrix}$$

We want to minimize $Y(t) = C_k * X(t) + D_k * u(t)$ where this function considered as cost function:

```
[Ak,Bk,Ck,Dk]=unpck(K)
cvx_begin
 variable x(5)
  minimize norm(Ck*x+Dk,1)
 subject to
 Tzw<=1.802;
cvx_end
```

The output program is:

```
Calling SDPT3: 2 variables, 0 equality constraints
------------------------------------------------------------
 num. of constraints =  1
 dim. of socp   var  =  2,   num. of socp blk  =  1
 dim. of linear var  =  1
*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 0|0.000|0.000|4.8e-011|1.1e+000|7.7e+000| 1.414214e+000| 0:0:00| chol  1  1
 3|0.989|0.989|5.2e-007|7.8e-004|7.2e-005| 5.857996e-004| 0:0:00| chol  1  1
 4|0.989|0.989|6.4e-008|8.6e-006|7.9e-007| 6.428092e-006| 0:0:00| chol  1  1
   stop: max(relative gap, infeasibilities) < 1.49e-008
-------------------------------------------------------------------
 number of iterations   =  5
 primal objective value =  8.39797738e-009
 dual   objective value = -2.90430783e-009
 gap := trace(XZ)       = 1.14e-008
 relative gap           = 1.14e-008
 actual relative gap    = 1.13e-008
 rel. primal infeas     = 4.78e-010
 rel. dual   infeas     = 4.81e-011
 Total CPU time (secs)  = 0.3
 CPU time per iteration = 0.1
 termination code       =  0
 DIMACS: 4.8e-010  0.0e+000  4.8e-011  0.0e+000  1.1e-008  1.1e-008
-------------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +8.39798e-009
```

Then the optimal value and the solution l_1 that minimize the cost function. Optimal value (cvx_optval): +8.39798e-009

4.12. L_1 / H_∞ Simulation

4.12.1. Closed-Loop Transfer Function

To solve this problem, a feedback controller will be added to improve the system performance. Figure (4.12) shown below is the block diagram of a typical unity feedback system.

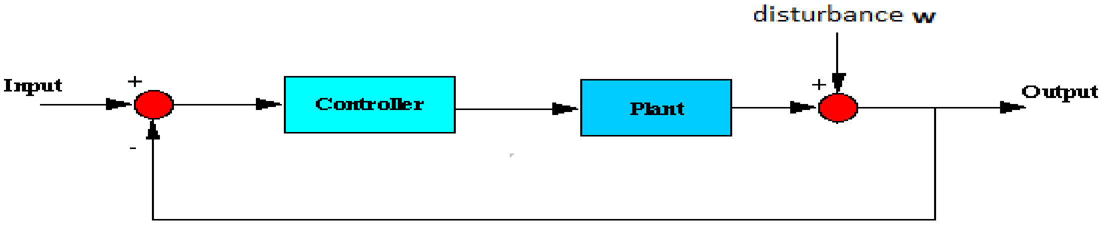


Figure (4.12): Close-loop feedback with controller and plant

A dynamic controller needs to be designed so that the step response satisfies all design requirements.

The state space equations of the controller are:

$$\dot{X}(t) = A_k * X(t) + B_k * u(t)$$
$$Y(t) = C_k * X(t) + D_k * u(t)$$

Where:

$$Ak = \begin{bmatrix} -0.2519 & -0.5547 & 0 & 0 & 0.0000 \\ 0.5547 & -2.7481 & 0 & 0 & 0 \\ -0.0658 & 0.0138 & -0.1611 & 49.7586 & -0.8683 \\ -0.0058 & 0.0012 & -0.0006 & -1.0334 & -0.0238 \\ 0 & 0 & 0 & 56.7000 & -0.6655 \end{bmatrix} ;\ Bk = \begin{bmatrix} -0.7448 \\ 0.7448 \\ -0.9064 \\ -0.0055 \\ -0.9412 \end{bmatrix}$$

$$Ck = \begin{bmatrix} -0.2007 & 0.0420 & 0.4630 & -21.1565 & -0.6931 \end{bmatrix};\ Dk = \begin{bmatrix} 0 \end{bmatrix}$$

From the state space model, we get the transfer function of the controller:

$$K(s) = \frac{0.5298s^4 + 1.93s^3 + 1.93s^2 + 1.485s + 0.2578}{s^5 + 4.86s^4 + 8.921s^3 + 9.201s^2 + 3.296s + 0.3185} \quad (4.9)$$

The transfer functions of the plant with considering the weighted function of signal disturbance:

$$\mathrm{P}(\mathrm{s}) = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921s}$$

Where the weighted function of signal disturbance is:

$$W_o = 1$$

Then the transfer function of the output Z to disturbance w is:

$$T_{zw} = \frac{Z}{w} = \frac{1}{1 + P(s) * K(s)} = [1 + P(s) * K(s)]^{-1}$$

4.12.2. Building the program H_∞-optimal controller

Second, we build the program H_∞-optimal controller, this program can found the optimal external optimization of our system.

1. Use the values of state space system (A, B, C, D) we generate G(s).
2. Define the weighting functions for the system showed in Figure (2.18).
3. Build the system interconnection to obtain the generalized plant P(s).
4. Extact the portioned state-space matrices from the plant P(s) and we got[Ap, Bp, Cp, Dp]
5. Decomposition of Ap, Bp, Cp and Dp using linear fractional transformation to Bp1, Bp2, Cp1, Cp2, Dp11, Dp12, Dp21and Dp22.

$$\left[\begin{array}{c|c} A_p & \mathrm{B}_p \\ \hline C_p & \mathrm{D}_p \end{array}\right] \Leftrightarrow \left[\begin{array}{c|cc} A_p & \mathrm{B}_{p1} & \mathrm{B}_{p2} \\ \hline C_{p1} & \mathrm{D}_{p11} & \mathrm{D}_{p12} \\ C_{p2} & \mathrm{D}_{p21} & \mathrm{D}_{p22} \end{array}\right]$$

6. Test stabilizability of (Ap, Bp2) and (Ap, Bp1).
7. Test detectability of (Cp2, Ap) and (Cp1, Ap).
8. $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$ (meaning that D_{12} is orthogonal to C_1) and no coupling in D_{12}.
9. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (meaning D_{21} is orthogonal to B_1) and no coupling in D_{12}.
10. Compute H_∞ optimal controller using the function `HINFSYN` we extract all characteristics of systems as output of this function, this function has to work as follow:

    [K, CL, GAM, INFO] = HINFSYN (P, NMEAS, NCON)

This function is from robust toolbox; it takes as input the matrix P and number of measurements and gives us all characteristic of the system. After execution of this function, the results obtained are:

- H_∞ optimal controller:

$$K = \begin{bmatrix} -0.2519 & -0.5547 & 0 & 0 & 0.0000 & -0.7448 & 5.0000 & \\ 0.5547 & -2.7481 & 0 & 0 & 0 & 0.7448 & 0 & \\ -0.0658 & 0.0138 & -0.1611 & 49.7586 & -0.8683 & -0.9064 & 0 & \\ -0.0058 & 0.0012 & -0.0006 & -1.0334 & -0.0238 & -0.0055 & 0 & \\ 0 & 0 & 0 & 56.7000 & -0.6655 & -0.9412 & 0 & \\ -0.2007 & 0.0420 & 0.4630 & -21.1565 & -0.6931 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{Inf} & \end{bmatrix}$$

- Gamma and $\min_{K \in S} \left\| T_{zw} \right\|_{\infty}$ value:

```
Test bounds:      0.0000 <  gamma  <=   1000.0000

  gamma    hamx_eig  xinf_eig  hamy_eig   yinf_eig   nrho_xy   p/f
1.000e+003  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0000    p
  500.000  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0000    p
  250.000  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0000    p
  125.000  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0000    p
   62.500  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0001    p
   31.250  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0005    p
   15.625  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0019    p
    7.813  1.2e-001  4.0e-003  1.2e-001   0.0e+000    0.0076    p
    3.906  1.2e-001  4.1e-003  1.2e-001   0.0e+000    0.0314    p
    1.953  1.2e-001  4.3e-003  1.2e-001   0.0e+000    0.1464    p

 Gamma value achieved:     1.9531
norm between 1.5818 and 1.5834
achieved near 0
```

From the results output program, we plot all different diagrams that clarify characteristic of system: the diagrams who clarify the systems characteristics are as follows:

- **Output Plant and Controller Response:**

The plot of the plant and controller response when introducing a disturbance signal is shown in Figure (4.13).

To test the efficacity of the system in minimizing and rejecting the disturbance signal we have introduced a noise signal, as result, we see that The curve of controller is compensative to the curve of the plant for reason minimizing the error which mean that our system functionality have satisfaction to exclude the noise disturbance signal. Moreover, we conclude that our system is stable because it returns to the zero after it is subjected by a temporary disturbance.

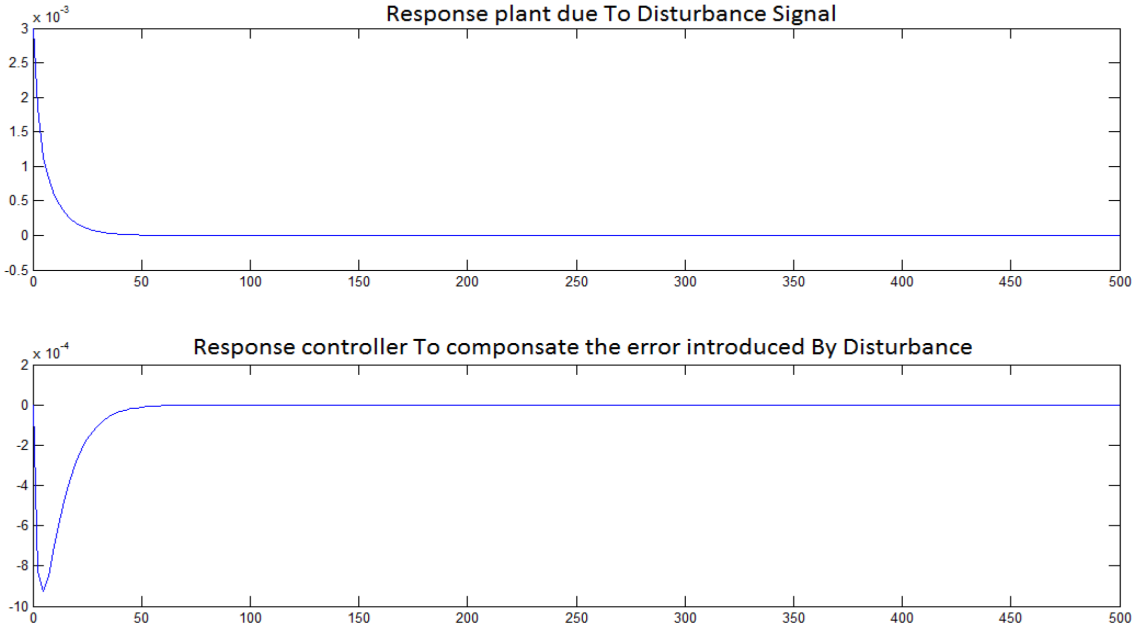


Figure (4.13): Plant and controller response due to introduced disturbance input

- **Rout-Locus Output Plant Response:**

  From the plot of Figure (4.14), we see the powerful of the method used and the robustness of the system and the pole placement of the poles to the region between -1 and 0 to ensure stability of the system and we can define the position of poles inside the design region by fixing the controller gain.

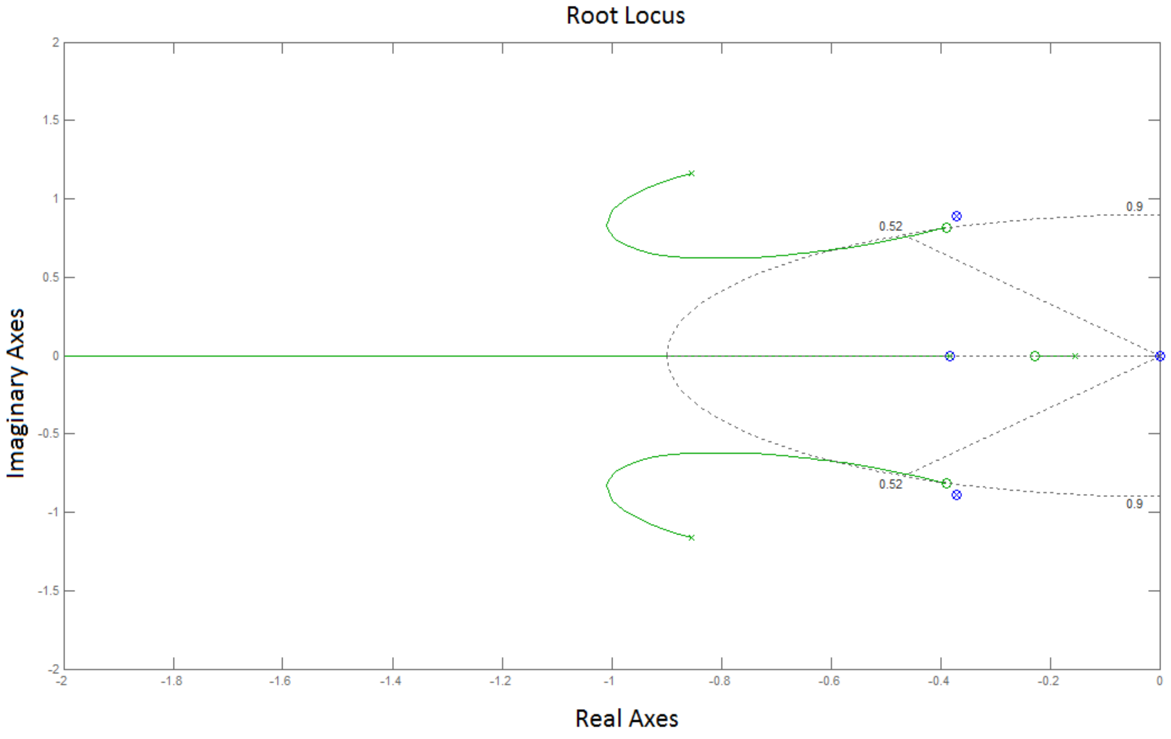


Figure (4.14): Rout-locus output plant response

And from all this results we have the minimized the H_∞-norm optimal controller. $1.5818 \le \min_{K \in S} \left\| T_{zw} \right\|_\infty \le 1.5834$

Now we want to realize the mixed objective by fixing the values obtained for H_∞-norm which means fixing the external characteristic and minimizing l_1-norm which means internal characteristic of the system.

We use convex optimization approach to find the optimal solution l_1 that minimize the cost function.

The state space equations of the controller are:

$$\dot{X}(t) = A_k * X(t) + B_k * u(t)$$
$$Y(t) = C_k * X(t) + D_k * u(t)$$

Where:

$$\text{Ak} = \begin{bmatrix} -0.2519 & -0.5547 & 0 & 0 & 0.0000 \\ 0.5547 & -2.7481 & 0 & 0 & 0 \\ -0.0658 & 0.0138 & -0.1611 & 49.7586 & -0.8683 \\ -0.0058 & 0.0012 & -0.0006 & -1.0334 & -0.0238 \\ 0 & 0 & 0 & 56.7000 & -0.6655 \end{bmatrix}; \ \text{Bk} = \begin{bmatrix} -0.7448 \\ 0.7448 \\ -0.9064 \\ -0.0055 \\ -0.9412 \end{bmatrix}$$

$$\text{Ck} = \begin{bmatrix} -0.2007 & 0.0420 & 0.4630 & -21.1565 & -0.6931 \end{bmatrix}; \ \text{Dk} = \begin{bmatrix} 0 \end{bmatrix}$$

We want to minimize $Y(t) = C_k * X(t) + D_k * u(t)$ where this function considered as cost function:

```
[Ak,Bk,Ck,Dk]=unpck(K)
>> cvx_begin
  variable x(5)
   minimize norm(Ck*x+Dk,1)
   subject to
  >> Tzw<=1.5818;
  >> Tzw>=1.5834;
 cvx_end
```

The output program is:

```
Calling SDPT3: 2 variables, 0 equality constraints
------------------------------------------------------------
 num. of constraints =  1
 dim. of socp   var  =  2,   num. of socp blk  =  1
 dim. of linear var  =  1
*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
-------------------------------------------------------------------
   stop: max(relative gap, infeasibilities) < 1.49e-008
-------------------------------------------------------------------
 number of iterations   =  5
 primal objective value =  8.39797738e-009
 dual   objective value = -2.90430783e-009
 gap := trace(XZ)       = 1.14e-008
 relative gap           = 1.14e-008
 actual relative gap    = 1.13e-008
 rel. primal infeas     = 4.78e-010
 rel. dual   infeas     = 4.81e-011
 norm(X), norm(y), norm(Z) = 1.0e+000, 2.9e-009, 1.0e+000
 norm(A), norm(b), norm(C) = 2.0e+000, 2.0e+000, 2.0e+000
 Total CPU time (secs)  = 0.3
 CPU time per iteration = 0.1
 termination code       =  0
 -------------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +8.39798e-009
```

Then the optimal value and the solution l_1 that minimize the cost function for infinity norm: Optimal value (cvx_optval)= +8.39798e-009

4.13.Comparison between l_1 / H_2 and l_1 / H_∞

The results of l_1 / H_∞ are more flexible compared to the results of l_1 / H_2 because the H_∞ work to the worse case and can accept all solutions.

From another viewpoint, the l_1 / H_2 method has more exactitude results compared to l_1 / H_∞ since it works to the average energy value, which mean solution close to optimal.

CHAPTER 5 CONCLUSIONS, RECOMMENDATIONS AND PERSPECTIVES

5.1. General Conclusion

Resolving the problem of robust controllers design capable of guaranteeing stability in the face of plant uncertainty was addressed in this Thesis. More specifically the problem of designing a linear time invariant stabilizing controllers that minimize the l_1 norm of a certain closed-loop transfer function and maintain the H_2 norm (mixed l_1/H_2), or the H_∞ , norm (mixed l_1/H_∞), of a different transfer function below a pre-specified level was addressed.

Our main research direction was the application of new and computationally tractable analysis and synthesis methods for uncertain systems. In particular, we considered LTI systems, and based on convex optimization approach.

To test the efficacity of the system in minimizing or rejecting the disturbance signal. I have introduced a noise signal, as result, our system functionality has successfully excluded the noise disturbance signal and minimized the input disturbance by 100 times. Moreover, our system is stable because it returned to the zero after it was subjected by a temporary disturbance. This method is powerful since the system goes directly to the steady state and all results satisfy the design requirements. In addition to that, I got a good low pass filter that can be used in many applications for tracking of signal input. The powerful of the method show the robustness of the system and the pole placement to the region between -1 and 0.

From all these results, I have minimized the H_2-norm and H_∞-norm optimal controller by using convex optimization approach, I found the optimal value and the solution l_1 that minimized the cost function.

The main result of this thesis showed that, a two-stage process can synthesize suboptimal controllers, involving a convex optimization problem, which optimizes the internal characteristics of the controller, an H_∞, H_2 optimization, which optimize the external characteristic of the system. Furthermore, this approach also provides a CVX-based parameterization of all suboptimal output feedback controllers, for mixed l_1/H_∞ and l_1/H_2 problems. Results showed that mixed l_1/H_∞ and l_1/H_2 objectives produced robust optimal solutions.

5.2. Recommendations and Future Work

With the fact that systems come more and more complicated, we recommend the development of this method to Mutli-objective, linear time variant (LTV) control synthesizing problems for systems with time delays.

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APPENDICES

This m-file used to compute and simulate the optimal H-2 controller

```
% Author: niazi Abouhwaij
% This m-file was used to compute and simulate the optimal H-2
controller
% for the pich controller process
% 1-input, 1-output plant state-space matrices

A=[-0.313  56.7  0 ; -0.0139   -0.426   0; 0  56.7   0];
B=[0.232 ; 0.0203 ; 0];
C=[0   0  1];
D=[0];
G=pck(A,B,C,D);

% weighting functions
numWo=[1];
denWo=[1];
Wo=nd2sys(numWo,denWo);
numWi=[1];
denWi=[1];
Wi=nd2sys(numWi,denWi);
numWe=[1];
denWe=[1 1];
We=nd2sys(numWe,denWe);
numWu=[30];
denWu=[1];
Wu=nd2sys(numWu,denWu);

% Build the system interconnection to obtain the generalized
plant P(s)
% using the "sysic" command

systemnames='Wo Wi Wu We G ';
inputvar='[ w1; w2 ; u ]';
outputvar='[ Wu; We; Wo - G ]';
input_to_Wo='[ w1 ]';
input_to_Wi='[ w2 ]';
input_to_Wu='[ u ]';
input_to_We='[Wo - G ]';
input_to_G='[ u + Wi ]';
sysoutname='P';

cleanupsysic='yes';
sysic;
% Extract partitioned state-space matrices
[AP,BP,CP,DP]=unpck(P);
```

```
BP1=BP(:,1:2);
BP2=BP(:,3);
CP1=CP(1:2,:);
CP2=CP(3,:);
DP11=DP(1:2,1:2);
DP12=DP(1:2,3);
DP21=DP(3,1:2);
DP22=DP(3,3);

% Stabilizability of (AP,BP2) and (AP,BP1)
% PBH test: [-AP+sI BPi] must not lose rank at closed RHP
eigvals of AP

rhpeigs=[];
[V,EIGS] = eig(AP);
eigs = diag(EIGS);
for i=1:length(AP)
if real(eigs(i)) >= 0
rhpeigs=[rhpeigs;eigs(i)];
end
end
p = length(rhpeigs);
r = length(AP); % Rank of [sI-AP] for s not eigval(AP)
rkb1=[]; % ranks at rhp eigenvalues
rkb2=[]; % ranks at rhp eigenvalue
for j=1:p
rkb1=[rkb1 ; rank([(rhpeigs(j)*eye(length(AP)))-AP BP1])];
rkb2=[rkb2 ; rank([(rhpeigs(j)*eye(length(AP)))-AP BP2])];
end
if min(rkb1) < length(AP)
sprintf('PROBLEM: (AP,BP1) IS NOT STABILIZABLE')
stop
end
if min(rkb2) < length(AP)
sprintf('PROBLEM: (AP,BP2) IS NOT STABILIZABLE')
stop
end
% Detectability of (CP2,AP) and (CP1,AP)
% PHB test: [(-AP+sI)' CP2']' must not lose rank at closed RHP
eigvals of AP
rkc1=[]; % ranks at rhp eigenvalues
rkc2=[]; % ranks at rhp eigenvalues
for j=1:p
rkc1=[rkc1 ; rank([(rhpeigs(j)*eye(length(AP)))-AP; CP1])];

rkc2=[rkc2 ; rank([(rhpeigs(j)*eye(length(AP)))-AP; CP2])];
end
if min(rkc1) < length(AP)
sprintf('PROBLEM: (CP1,AP) IS NOT DETECTABLE')
stop
end
```

```
if min(rkc2) < length(AP)
sprintf('PROBLEM: (CP2,AP) IS NOT DETECTABLE')
stop
end
% There are two more condition that should be checked. The
matrices
% [AP-jwI BP2; CP1 DP12] and [AP-jwI BP1; CP2 DP21] must have
full column
% and row rank respectively for all w.
ww = logspace(-2,4,200);
for i=1:200
rk = rank([sqrt(-1)*ww(i)*eye(length(AP))-AP BP2; CP1 DP12]);
if rk < (length(AP)+length(BP2(1,:)))
sprintf('PROBLEM: FIRST MATRIX IS RANK DEFICIENT AT FREQ.
=%g',ww(i))
stop
end
end
for i=1:200
rk = rank([sqrt(-1)*ww(i)*eye(length(AP))-AP BP1; CP2 DP21]);
if rk < (length(AP)+length(CP2(:,1)))
sprintf('PROBLEM: SECOND MATRIX IS RANK DEFICIENT AT FREQ.
=%g',ww(i))
stop
end
end
% compute the H2 optimal controller
nmeas=1; % number of measurements to controller
ncon=1; % number of control signals from controller
ricmethod=1;
[K,Tzw,Kfi,Gfi,CL,hamx,hamy]=h2syn(P,nmeas,ncon,ricmethod)

% H2-norm achieved
h2norm(Tzw)

% closed-loop sensitivity from output disturbance to process
output.
% i.e. S(s)=(I+GK)^(-1)
S=starp(abv(sbs(0,1),sbs(1,mmult(-1,G,K))),1);
%S=starp(abv(sbs(0,1),sbs(1,mmult(-1,G,K))),1);
% closed-loop system from output disturbance to process inputs
% i.e. SK(s)=-(I-GK)^(-1)K

KS=starp(abv(sbs(1,eye(1)),sbs(1,G)),mmult(K,-1));
%KS=starp(abv(sbs([0],eye(1)),sbs(1,G)),mmult(K,-1));
% simulate the output response to a change in setpoint for
delta
tfinal=2000 ; % final time for simulation
```

```
intstep=0.25; % integration step
step_input=0.01 ; % step input: 0.3 rad disturbance in angle
% simulate plant response first
plant_output_resp=trsp(mmult(S,Wo,0.0001),step_input,tfinal,intstep);
K_output_resp=trsp(mmult(KS,Wo,0.0001),step_input,tfinal,intstep);
% plot the output and actuator responses
figure(1)
%subplot(211)

vplot(plant_output_resp,'b');
grid
title('plant Output Response')
XLABEL('Time (sec)')
YLABEL('pitch angle(rad)')
%subplot(212)
figure(2)
vplot(sel(K_output_resp,1,1));
grid
title('Controller output  Response')
XLABEL('Time (sec)')
YLABEL('pitch angle(rad)')
[Ac,Bc,Cc,Dc]=unpck(CL);
figure(3)
step(Ac,Bc,Cc,Dc);
[Nc Dc]=ss2tf(Ac,Bc,Cc,Dc,1)
Nc=Nc(1,:);
CPlant=tf(Nc,Dc);
grid
figure (4)
 Bode(CPlant);
grid
[Ak,Bk,Ck,Dk]=unpck(K);
[Ng Dg]=ss2tf(Ak,Bk,Ck,Dk);
G=tf(Ng,Dg);
[Ap,Bp,Cp,Dp]=unpck(P);
[N D]=ss2tf(Ap,Bp,Cp,Dp,1);
N=N(3,:);
Plant=tf(N,D);
XLABEL('Time (sec)')
YLABEL('pitch angle(rad)')
 figure(5)
 zeta=0.52;
 Wn=0.9;
 rlocus (Plant,G);

 sgrid (zeta,Wn);
axis ([-1 0 -2 2]);
```

This m-file was used to compute and simulate the optimal H-infinity controller

```
% Author: niazi Abouhwaij
% This m-file was used to compute and simulate the optimal H-
infinity controller
% for the pich controller process
% 1-input, 1-output plant state-space matrices

A=[-0.313  56.7  0 ; -0.0139   -0.426   0; 0  56.7   0];%
B=[0.232 ; 0.0203 ; 0];
C=[0   0  1];
D=[0];
G=pck(A,B,C,D);

% weighting functions
numWo=[1];
denWo=[1];
Wo=nd2sys(numWo,denWo);
numWi=[1];
denWi=[1];
Wi=nd2sys(numWi,denWi);
%Wi=daug(Wi1,Wi1);
numWe=[1];
denWe=[1 1];
We=nd2sys(numWe,denWe);
numWu=[1];
denWu=[1];
Wu=nd2sys(numWu,denWu);

% Build the system interconnection to obtain the generalized plant
P(s)
% using the "sysic" command

systemnames='Wo Wi Wu We G ';
inputvar='[ w1; w2 ; u ]';
outputvar='[ Wu; We; Wo - G ]';
input_to_Wo='[ w1 ]';
input_to_Wi='[ w2 ]';
input_to_Wu='[ u ]';
input_to_We='[Wo - G ]';
input_to_G='[ u + Wi ]';
sysoutname='P';

cleanupsysic='yes';
sysic;
% Extract partitioned state-space matrices

[AP,BP,CP,DP]=unpck(P);
BP1=BP(:,1:2);
BP2=BP(:,3);
CP1=CP(1:2,:);
CP2=CP(3,:);
```

```
DP11=DP(1:2,1:2);
DP12=DP(1:2,3);
DP21=DP(3,1:2);
DP22=DP(3,3);

% Stabilizability of (AP,BP2) and (AP,BP1)
% PBH test: [-AP+sI BPi] must not lose rank at closed RHP
eigvals of AP
rhpeigs=[];
[V,EIGS] = eig(AP);
eigs = diag(EIGS);
for i=1:length(AP)
if real(eigs(i)) >= 0
rhpeigs=[rhpeigs;eigs(i)];
end
end
p = length(rhpeigs);
r = length(AP); % Rank of [sI-AP] for s not eigval(AP)
rkb1=[]; % ranks at rhp eigenvalues
rkb2=[]; % ranks at rhp eigenvalue
for j=1:p
rkb1=[rkb1 ; rank([(rhpeigs(j)*eye(length(AP)))-AP BP1])];
rkb2=[rkb2 ; rank([(rhpeigs(j)*eye(length(AP)))-AP BP2])];
end
%if min(rkb1) < length(AP)
% sprintf('PROBLEM: (AP,BP1) IS NOT STABILIZABLE')
% stop
%end
if min(rkb2) < length(AP)
sprintf('PROBLEM: (AP,BP2) IS NOT STABILIZABLE')
stop
end
% Detectability of (CP2,AP) and (CP1,AP)
% PHB test: [(-AP+sI)' CP2']' must not lose rank at closed RHP
eigvals of AP
rkc1=[]; % ranks at rhp eigenvalues
rkc2=[]; % ranks at rhp eigenvalues
for j=1:p
rkc1=[rkc1 ; rank([(rhpeigs(j)*eye(length(AP)))-AP; CP1])];
rkc2=[rkc2 ; rank([(rhpeigs(j)*eye(length(AP)))-AP; CP2])];
end
if min(rkc1) < length(AP)
sprintf('PROBLEM: (CP1,AP) IS NOT DETECTABLE')
stop
end

if min(rkc2) < length(AP)
sprintf('PROBLEM: (CP2,AP) IS NOT DETECTABLE')
stop
end
```

```
% There are two more condition that should be checked. The matrices
% [AP-jwI BP2; CP1 DP12] and [AP-jwI BP1; CP2 DP21] must have full column
% and row rank respectively for all w.
ww = logspace(-4,2,200);
for i=1:200
rk = rank([sqrt(-1)*ww(i)*eye(length(AP))-AP BP2; CP1 DP12]);
if rk < (length(AP)+length(BP2(1,:)))
sprintf('PROBLEM: FIRST MATRIX IS RANK DEFICIENT AT FREQ. = %g',ww(i))
stop
end
end
for i=1:200
rk = rank([sqrt(-1)*ww(i)*eye(length(AP))-AP BP1; CP2 DP21]);
if rk < (length(AP)+length(CP2(:,1)))
sprintf('PROBLEM: SECOND MATRIX IS RANK DEFICIENT AT FREQ. = %g',ww(i))
stop
end
end

% compute the H-infinity optimal controller
nmeas=1; % number of measurements to controller
ncon=1; % number of control signals from controller
ricmethod=2;
[K,Tzw,CL,X,GAM,hamx,hamy]=hinfsyn(P,nmeas,ncon,0,1000,ricmethod);
% Hinf norm achieved
hinfnorm(Tzw)
% closed-loop sensitivity from output disturbance to process output.
% i.e. S(s)=(I+GK)^(-1)
S=starp(abv(sbs(0,1),sbs(1,mmult(-1,G,K))),1);
% closed-loop system from output disturbance to process inputs
% i.e. SK(s)=-(I-GK)^(-1)K
KS=starp(abv(sbs([0],eye(1)),sbs(1,G)),mmult(K,-1));
% simulate the output response to a change in setpoint for delta
tfinal=10000; % final time for simulation
intstep=5; % integration step
step_input=0.003; % step input: 0.3 rad disturbance in angle

% simulate plant response first
plant_output_resp=trsp(mmult(S,Wo,0.01),step_input,tfinal,intstep);
K_output_resp=trsp(mmult(KS,Wo,0.01),step_input,tfinal,intstep);
% plot the state responses
figure(1)
vplot(plant_output_resp,'b');
title('plant Output Response')
XLABEL('Time()')
YLABEL('Amplitude')
```

```
figure(2)
vplot(sel(K_output_resp,1,1));
grid
title('Controller output  Response')
XLABEL('Time()')
YLABEL('Amplitude')
[Ac,Bc,Cc,Dc]=unpck(CL);
figure(3)
step(Ac,Bc,Cc,Dc);
[Nc Dc]=ss2tf(Ac,Bc,Cc,Dc,1)
Nc=Nc(1,:);
CPlant=tf(Nc,Dc);
grid
figure (4)
 Bode(CPlant);
grid
[Ak,Bk,Ck,Dk]=unpck(K);
[Ng Dg]=ss2tf(Ak,Bk,Ck,Dk);
G=tf(Ng,Dg);
[Ap,Bp,Cp,Dp]=unpck(P);
[N D]=ss2tf(Ap,Bp,Cp,Dp,1);
N=N(3,:);
Plant=tf(N,D);
 figure(5)
 zeta=0.52;
 Wn=0.9;
 rlocus (Plant,G);

 sgrid (zeta,Wn);
axis ([-1 0 -2 2]);
```

Function:HINFSYN H-infinity controller synthesis.

```
%HINFSYN  H-infinity controller synthesis.
% [K,CL,GAM,INFO] = HINFSYN(P,NMEAS,NCON) or
% [K,CL,GAM,INFO] = HINFSYN(P,NMEAS,NCON,KEY1,VALUE1,KEY2,VALUE2,...)
%    computes H-infinity controller K for partitioned plant P via the
%    gamma-iteration, computing the minimal cost GAM in [GMIN,GMAX] for
%    which the closed-loop system CL= LFT(P,K) satisfies
%                 HINFNORM(CL) < GAM.
%    NMEAS is the number of measurement outputs from the plant and NCON is
%    the number of control inputs to the plant; these may be omitted if
%    P=MKTITO(P,NMEAS,NCON) or P=AUGW(SYS,W1,W2,W3).
%
%       KEY      |VALUE    | MEANING
%     --------------------------------------------------------------
%      'GMAX'    | real    | initial upper bound on GAM (Inf default)
%      'GMIN'    | real    | initial lower bound on GAM (0 default)
%      'TOLGAM'  | real    | relative error tolerance for GAM (.01 default)
%      'METHOD'  |         | H-infinity solution method:
%                |'ric'    | - (default) standard 2-Riccati solution, DGKF1989
%                |'lmi'    | - LMI solution, Packard 1992, Gahinet 1994
%                |'maxe'   | - maximum entropy, HINFSYNE, Glover-Doyle 1988
%      'S0'      | real    |  (default=Inf) frequency S0 at which entropy is
%                |         |      evaluated, only applies to METHOD 'maxe'
%      'DISPLAY' |'on/off'| display synthesis information to screen,
%                |         |  (default = 'off')
%        --------------------------------------------------------------
%      outputs:
%        K     -  H-infinity controller
%        CL    -  lft(P,K) (closed-loop system)
%        GAM   -  H-infinity cost
%        INFO  -  Structure array containing possible additional information
%                 depending on 'METHOD':
%                 INFO.AS  - all solutions controller, LTI two-port LFT
%                 INFO.KFI  - full information gain matrix (constant
%                             feedback U2 = KFI*[X; U1] )
%                 INFO.KFC  - full control gain matrix (constant
%                             output-injection; KFC is the dual of KFI)
%                 INFO.GAMFI - H-infinity cost for full information KFI
%                 INFO.GAMFC - H-infinity cost for full control KFC
%     See also: AUGW, H2SYN, LOOPSYN, MKTITO, NCFSYN, LTI/NORM
% OLD HELP
% function [k,g,gfin,ax,ay,hamx,hamy] =
hinfsyn(p,nmeas,ncon,gmin,gmax,tol,ricmethd,epr,epp,quiet)
%  This function computes the H-infinity (sub) optimal n-state
%  controller, using Glover's and Doyle's 1988 result, for a system P.
%  the system P is partitioned:
%                          | a   b1   b2   |
%             p    =       | c1  d11  d12  |
%                          | c2  d21  d22  |
%   where b2 has column size of the number of control inputs (NCON)
%   and c2 has row size of the number of measurements (NMEAS) being
%   provided to the controller.
%   Copyright 1991-2005 MUSYN Inc. and The MathWorks, Inc.
$Revision: 1.1.8.5 $
```

Function:H2SYN H2 controller synthesis

```
%H2SYN  H2 controller synthesis.
%
% [K,CL,GAM,INFO]=H2SYN(P,NMEAS,NCON)
% [K,CL,GAM,INFO]=H2SYN(MKTITO(P,NMEAS,NCON)))
%  Calculates the H2 optimal controller K and the closed loop
%  system CL=LFT(P,K).  NMEAS and NCON are the dimensions of
%  the measurement outputs from P and the controller inputs to P.
%
%     inputs:
%       P    -   LTI two-port plant
%       NMEAS   -   measurement outputs from plant to controller
%       NCON    -   control inputs to plant from controller
%     outputs:
%       K    -  H2 optimal controller
%       CL   -  closed-loop system CL=LFT(P,K)
%       GAM  -  GAM = H2NORM(CL)
%       INFO -  struct array with various information, such as
%               NORMS  -  norms of 4 different quantities, full
%               information control cost (FI), output estimation
%               cost (OEF), direct feedback cost (DFL) and full
%               control cost (FC).  NORMS = [FI OEF DFL FC];
%     KFI    -  full information/state feedback control law
%     GFI    -  full information/state feedback closed-loop
%     HAMX   -  X Hamiltonian matrix
%     HAMY   -  Y Hamiltonian matrix
%
%     Comment:  For discrete plants and for continuous plants
%               zero feedthrough term (D11 = 0),
%               GAM =sqrt(FI^2 + OEF^2+ trace(D11*D11'));
%               otherwise, GAM is infinite
%
%     See also: AUGW, HINFSYN, LTI/NORM, LTRSYN
%   Copyright 1991-2004 MUSYN Inc. and The MathWorks, Inc.
% $Revision: 1.1.6.2 $
```