Lowest order perturbative quantum field theory calculations of bound state decay widths

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Abstract

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The bound state decay widths of the processes $J/\psi \to l^+l^-$, $\eta_c \to gg$, O-Ps $\to \gamma\gamma\gamma$, $J/\psi \to ggg$ and $J/\psi \to \gamma gg$ are derived in the lowest order perturbative theory of QED and QCD. In these calculations, three different methods for processing the invariant amplitude are presented. The derived decay width results agree with the established ones in the literature. Moreover, experimentally verifiable ratios between calculated decay widths are presented. The theoretical predictions are $R^{J/\psi}_{1\text{th}} = \Gamma(J/\psi \to \gamma gg)/\Gamma(J/\psi \to ggg) \approx 0.104$ and $R^{J/\psi}_{2\text{th}} = \Gamma(J/\psi \to l^+l^-)/\Gamma(J/\psi \to ggg) \approx 0.060$, which agree relatively well with experimental results. Finally it is argued that the development can be generalized for other heavy mesons with the same quantum numbers.

Keywords: bound state, particle decays, perturbative QFT, J/ψ meson

Tiivistelmä

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Alimman kertaluvun häiriöteoreettisia kvanttikenttäteorialaskuja sidottujen tilojen hajoamisleveyksille
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Työssä lasketaan sidottujen tilojen hajoamisleveydet prosesseille $J/\psi \to l^+l^-$, $\eta_c \to gg$, O-Ps $\to \gamma\gamma\gamma$, $J/\psi \to ggg$ ja $J/\psi \to \gamma gg$. Prosessit lasketaan perturbatiivisen QED:n ja QCD:n alimmassa kertaluvussa. Laskuissa käydään läpi kolme erilaista tekniikkaa invariantin amplitudin käsittelemiseksi. Johdetut hajoamisleveystulokset käyvät yksiin kirjallisuudesta löytyvien tulosten kanssa. Lisäksi tarkastellaan kokeellisesti varmistettavissa olevia hajoamisleveyssuhteita. Teoreettiset ennusteet ovat $R_{\text{1th}}^{J/\psi} = \Gamma(J/\psi \to \gamma gg)/\Gamma(J/\psi \to ggg) \approx 0.104$ sekä $R_{\text{2th}}^{J/\psi} = \Gamma(J/\psi \to l^+l^-)/\Gamma(J/\psi \to ggg) \approx 0.060$, jotka ovat melko lähellä kokeellisia arvoja. Lopuksi argumentoidaan, että tulokset voidaan yleistää koskemaan muita raskaita mesoneita joilla on samat kvanttiluvut.

Avainsanat: sidottu tila, hiukkasen hajoaminen, perturbatiivinen QFT, J/ψ mesoni

Foreword

Being at such an early stage of my own academic career, there is not much that I can say about the life of a theoretical physicist. However, I have the feeling that the first ten years, or so, of my life on this journey will be filled with meticuluos familiarization with already proven results and theories. To see the forest from all the trees; so to speak. Only after adequate control of the knowledge accumulated by scientists before me, I can dream of inventing something new.
As such, the work at hand is more or less general knowledge on the field of particle physics. A friend of mine once asked from me, why do I need to recalculate a collision process which has already been done back in the 1970's? My response to his question was, why do the first graders need to learn how to calculate 1+1=2? After all a lot of people have done that calculation before them. Why would they need to know how to do that? With this the motivation for this master's thesis should become obvious.
Lastly I would like to thank professor Kari J. Eskola for his guidance during this work. We had many intriguing conversations on the topic going down to the finest details at times. I would also like to thank my friends and my colleagues of FYS4 for enduring my rants when I wasn't able to solve some particular calculation. And finally, I would like to thank Veera for her patience. She has been the one who has had to tolerate my long working hours the most.

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Topi Löytäinen

Contents

1 Introduction

A typical book on quantum field theory (QFT) usually starts with a quick revision of classical field theory or assumes it to be known [1]–[4]. With such an in-depth topic, a certain level of pre-existing knowledge has to be assumed. Therefore, in this thesis it is assumed that the reader is somewhat familiar with the phenomena of particle physics and the mathematical machinery needed for its description. However, it is our desire to present a detailed account of the most important calculations such that the reader should be able to follow the development nearly without the need of a pen and a paper.

The main interest in this thesis is in the lowest order (LO) perturbative quantum field theory (QFT) calculations of bound state decays. The decays of the heavy mesons J/ψ and η_c are taken as examples for decay processes that are described by quantum chromodynamics (QCD). Moreover, the decay of the ortho-positronium (O-Ps) is taken as an example of a process that can be described completely within the framework of quantum electrodynamics (QED). In particular, we shall explicitly derive the decay widths of the following decay processess: $J/\psi \to l^+l^-$, $\eta_c \to gg$, O-Ps $\to \gamma\gamma\gamma$, $J/\psi \to ggg$ and $J/\psi \to \gamma gg$.

We are interested in the decays of those states where the orbital angular momentum is zero i.e. $L = 0$. This is to say that we consider bound state particles with spin 1 or 0. Moreover, it should be noted that both J/ψ and η_c are bound states of the charm c and anticharm $\bar{c}$ quarks. The difference between these two particles is that J/ψ is a $J^{PC} = 1^{--}$ particle and η_c is a $J^{PC} = 0^{-+}$ particle. Similarly with J/ψ, the O-Ps is a $J^{PC} = 1^{--}$ particle. However, the O-Ps is a bound state of an electron e^- and a positron e^+.

A particular emphasis is on the decay of J/ψ as one of our earliest motivations for this topic came from the desire to understand the diffractive J/ψ-production in deep inelastic scattering [5]. The decay width calculations allow us to derive ratios between different decay channels. For example we derive the known result of the ratio between $J/\psi \to \gamma gg$ and $J/\psi \to ggg$:

$$\frac{\Gamma(J/\psi \to \gamma gg)}{\Gamma(J/\psi \to ggg)} = \frac{36}{5}\frac{\alpha}{\alpha_s}Q_c^2, \tag{1.1}$$

where Q_c is the fractional electric charge of the charm quark, α is the QED coupling constant and α_s is the QCD coupling constant [6]. As one can see, the equation 1.1 allows us to figure out the ratio between the QED and QCD coupling constants at the given energy scale.

Because of the wide variety of different notational conventions, it is sensible to explicitly state our notational choices. Throughout this thesis we work in a 4D Minkowski space with the metric chosen to be $g_{\mu\nu} = (+, -, -, -)$. Moreover, we shall adopt the Dirac-Pauli representation. The reader may check appendix E for the explicit form of this representation. When Greek letters $\mu,\nu,\xi,\ldots$ are used as indices, they represent Lorentz indices going over the values 0,1,2,3. Similarly when Latin letters $i,j,k,\ldots$ are used as indices, they go over the values 1,2,3. Moreover, in general any three-vector is written with a bold faced letter or a vector sign on top of it, e.g. $\boldsymbol{k}$ or $\vec{k}$. The choice depends on the context. The four-vector counterpart is then simply written as k. Lastly, throughout this thesis we shall also work with natural units e.g. $\hbar = c = 1$. Any other notational convention should be obvious from the context.

Before presenting the decay width calculations, we shall introduce a method for solving a two-body bound state decay process to LO analytically. This is an important result from which we take full advantage of. Our development for the invariant amplitude of bound state decays rests on the idea that we can first neglect the annihilation of the bound state to calculate the binding, and then neglect the binding to calculate the annihilation [1, p.233]. Then the desired invariant amplitude of the bound state decay is obtained simply by multiplying these two together. We shall also take the extreme nonrelativistic (NR) limit, where the momentum of the heavy constituent particles is set to zero. This is a rather justifiable approximation when working in the center-of-mass (CMS) frame of the decaying particle. In addition, the development holds at the heavy-quark mass energy scales where α and α_s are small enough, such that perturbation theory can be applied. After this we shall consider the processes of interest.

We shall calculate the decay width of the process $J/\psi \to l^+l^-$ in two different ways. First by explicitly forming the matrix structure from the quark and antiquark spinors and calculating through. Then secondly, and maybe more elegantly, by taking advantage of the helicity states of the J/ψ. There are two main reasons for this development. First of all, any mathematical tools that allow us to circumvent some of the tediousness of a brute calculation are valuable in themselves. Secondly, the development should give a more in depth understanding of the physical process at hand.

With this so called helicity technique at our disposal, we shall briefly revisit the calculation of the decay width of $\eta_c \to gg$ as given in [7]. This is of interest to us because η_c is in the spin singlet configuration and expressing this state in the helicity basis gives us the whole picture of how the addition of angular momenta of spin-$\frac{1}{2}$ particles works for this technique. In addition, this allows us to point out the importance of how one chooses the set of independent basis spinors for the Dirac equation (DE) solutions ψ. This choice is also intimately related to the Clebsch-Gordan coeffiecients.

A bound system of an electron and its antiparticle positron is called positronium and when it is in the spin triplet configuration it is called O-Ps. This decay width calculation is well known in literature but has still evoked interest in the 21st century [1], [8]–[10]. We are interested in this calculation since it may be used as a stepping stone to the calculations of $\Gamma(J/\psi \to ggg)$ and $\Gamma(J/\psi \to \gamma gg)$. After having done the positronium calculation, the J/ψ decay width calculations essentially boil down to figuring out the correct color factors.

Some of the calculational details have been placed to the appendices in order to make the main text more concise. A reader interested in these details will find them from there. The main text references them appropriately. Moreover, as we shall extensively use the Feynman rules to derive the invariant amplitude, a quick recap of QED and QCD Feynman rules has been gathered to appendix D. Before the conclusions we shall briefly discuss how these calculations relate to experimental results and how they can be generalized to cover other heavy mesons with the same quantum numbers. Next, let us turn our attention to two-body bound states.

2 Two-body bound state decays in general

The method of solving NR two-body bound state decays is common knowledge in the field of particle physics. In this section we shall briefly go through the main points and results of this development. A reader more interested in the details may refer to literature for further information [1], [2], [7], [11]–[18].

Let us then consider an unstable particle α of mass M, which can decay to n other particles as $\alpha \rightarrow 1 + 2 + \cdots + n$. The decay event can result in multiple different final states $|f_i\rangle$. Let us choose a particular decay process with a final state $|f\rangle$ that we are interested in. We may then define a quantity called the decay width Γ_f as

$$\Gamma_f \equiv \frac{N_f}{N_\alpha \Delta T}, \tag{2.1}$$

where N_f is the number of those decay events where the final state is $|f\rangle$, N_α is the number of decaying particles and ΔT is the time interval in which we observe the decay events. We can clearly see from equation 2.1 that the decay width is proportional to the inverse of the average lifetime of the particle.

Then, by taking e.g. the plane wave approximation method and "putting the particles in a box" we may derive the decay width to the channel $f = 1...n$ to be

$$\Gamma_f = \frac{1}{2E_\alpha} \int \left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^{(4)} \left(p_\alpha - \sum_{j=1}^{n} p_j \right) |\mathcal{M}_{fi}(\alpha \rightarrow 1...n)|^2, \tag{2.2}$$

where E_α, E_i are the energies and p_α,p_i the four-momenta of the particles in question [18]. The delta function $\delta^{(4)}$ takes care of energy conservation and the invariant amplitude $\mathcal{M}_{fi}$ contains the physics of the decay. We can also see that since the energy E_α depends on the frame, the decay width also depends on the frame. Usually the frame is chosen to be the CMS frame of the decaying particle since in it we have the proper time of the system. Then the problem of finding the decay width is a two fold process: determine the invariant amplitude $\mathcal{M}$ for the process and compute the phase space integral.

Next we need to figure out the form of the invariant amplitude. Let us start by considering the wavefunction $\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2,t)$ of the bound state. If the interaction potential is time independent, we may write the wavefunction as follows:

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2,t) = \psi(\boldsymbol{r}_1,\boldsymbol{r}_2)e^{-iEt/\hbar}. \tag{2.3}$$

In this thesis we shall limit ourselves to time independent potentials of QED and QCD. We may then focus only on the spatial part $\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)$. For a purely QED process, such as the O-Ps decay, we know the interaction potential $V(\boldsymbol{r}_1,\boldsymbol{r}_2)$ to be the Coulombic potential. However, for the decays of J/ψ and η_c the strong interaction potential is unknown. So how can we solve a decay process of a bound state whose interaction potential is unknown?

The answer is that, from first principles, we cannot. However, we can formally write the decay width to be proportional to the square of the wavefunction at zero e.g. $|\psi(\boldsymbol{r}=0)|^2$. We can then experimentally determine this value. However, ratios between different decay channels can be determined exactly as the wavefunction terms will cancel each other out. Let us first derive the general form for the invariant amplitude of a two-body bound state decay process and then solve for the exact QED bound state decay solution.

Consider a system of two point like particles a_1 and a_2 as shown below in figure 2.1 whose positions are described by the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. These particles form the decaying particle α which we are interested in. Moreover, we are interested in NR systems which consist of two particles of the same mass m. And by neglecting any binding energies, it also follows that $M = 2m$. With these we may write the CMS coordinate $\boldsymbol{R}$ and relative coordinate $\boldsymbol{r}$ as

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{2.4}$$

Abiding to this notation we can then write the conjugate momenta as

$$\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \qquad \boldsymbol{k} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2). \tag{2.5}$$

In the center-of-mass frame the total momentum $\boldsymbol{K}$ is zero and thus $\boldsymbol{k}_1 = -\boldsymbol{k}_2 = \boldsymbol{k}$. With these results it proves out easier to work in the momentum space. So we take the Fourier transform of the normalized wavefunction $\psi(\boldsymbol{r})$ of the relative motion:

$$\tilde{\psi}(\boldsymbol{k}) = \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\psi(\boldsymbol{r}), \qquad \int \frac{d^3k}{(2\pi)^3}|\tilde{\psi}(\boldsymbol{k})|^2 = 1. \tag{2.6}$$

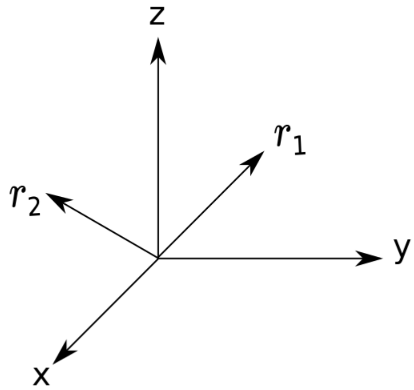


Figure 2.1. Two point like particles a_1 and a_2 of the mass m in a rectangular coordinate system described by the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively.

Then by separating the NR wavefunction of the bound state $|\Psi_{BS}\rangle$ to the wavefunction for the relative motion and the free motion of the CMS of the bound state, it follows that $|\Psi_{BS}\rangle$ can be written as

$$|\Psi_{BS}\rangle = \sqrt{\frac{2}{M}} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(\boldsymbol{k}) \, |\boldsymbol{k}_1\boldsymbol{s}_1, \boldsymbol{k}_2\boldsymbol{s}_2\rangle \,, \tag{2.7}$$

where the front factor $\sqrt{2}/\sqrt{M}$ follows from the $2E_{\bar{K}}$ particles in a box relativistic normalization convention and $s_{1,2}$ denote the spins of a_1,a_2 [2, p.149]. We have now written the bound state wavefunction in terms of certain factors multiplying the free particle solution $|\boldsymbol{k}_1,\boldsymbol{s}_1\rangle$ and $|\boldsymbol{k}_2,\boldsymbol{s}_2\rangle$. Then it naturally follows that the invariant amplitude for the decay from the bound state to some final state $1...n$ may be written as

$$\mathcal{M}(\alpha \to 1...n) = \sqrt{\frac{2}{M}} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(\boldsymbol{k}) \mathcal{M}(a_1a_2 \to 1...n). \tag{2.8}$$

Moreover, by expanding $\mathcal{M}(a_1a_2 \to 1...n)$ around $\boldsymbol{k} = 0$, the momentum integral over $\tilde{\psi}(\boldsymbol{k})$ gives us the position-space wavefunction evaluated at zero $\psi(\boldsymbol{r} = 0)$. Thus, our final result is:

$$|\mathcal{M}_{fi}(\alpha \to 1...n)|^2 \approx \frac{2}{M} |\psi(\boldsymbol{r} = 0)|^2 |\mathcal{M}_{fi}(a_1a_2 \to 1...n)\Big|^2_{\boldsymbol{k}=0}. \tag{2.9}$$

We may then use the Feynman rules to calculate the free particle invariant amplitude on the right hand side (r.h.s.) of equation 2.9. As mentioned earlier, we cannot define $|\psi(\boldsymbol{r} = 0)|^2$ from first principles for the decays of J/ψ or η_c. However, for the O-Ps decay, which we go through in section 5, we know how to do this. Since O-Ps is

a hydrogen like system for which $L = 0$, the wavefunction can be found by repeating the development given in many textbooks, e.g. [11], [12].
Let us go through the main points of this development. We assume that the binding potential is time independent and has no angular dependence. Then by working in spherical coordinates, we may separate the wavefunction solution to a radial part and an angular part. The solutions to the angular part give us the well known sperical harmonics. After factoring out the asymptotic behaviour as $\boldsymbol{r} \to \infty$, the radial part is solved with a power series ansatz. This gives us a recursion formula which must terminate at some point. With this we can define the principal quantum number n and solve for the normalized eigenfunction as $n = 1$ and $l = m = 0$. This gives us the LO position-space wavefunction:

$$\psi(\boldsymbol{r}) \equiv \psi_{100}(r,\theta,\phi) = R_{10}(r) Y_0^0(\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \tag{2.10}$$

where $a = 2/(m\alpha)$ is now twice the Bohr radius [11, p.138]. This follows from the fact that as we are solving the O-Ps system in the CMS coordinates, the reduced mass of the system is half of the electron mass m. Moreover, α is the QED coupling constant. This concludes our formulation of the NR bound state system and with it we may start to consider our first decay width $\Gamma(J/\psi \to l^+ l^-)$.

3 Decay width of $J/\psi \to l^+l^-$

3.1 Explicit spin matrix approach

In this section we shall work through the calculation with conventional techniques involving explicit spinor and spin matrix manipulations. Furthermore, we shall proceed in the manner outlined in section 2. Then, in the LO this decay process can be described by the Feynman diagram given in figure 3.1. As discussed in section 1, we are interested in those particles for which $L = 0$. Since we know that J/ψ is a $J^{PC} = 1^{--}$ particle, it follows that $J = S = 1$. J/ψ must also be in the color singlet state, $Q\bar{Q} \widehat{=} \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ involving all quark colors symmetrically. Since the decay to the lepton channel is a QED process, where only the quark lines involve color, we can straightforwardly write the color factor F_c as $F_c = \frac{3}{\sqrt{3}} = \sqrt{3}$. The free particle process $c\bar{c} \to l^-l^+$ invariant amplitude may be written as

$$\begin{aligned} -i\mathcal{M}^{C=0}_{S=1}(c\bar{c} \to l^-l^+) &= F_c(\bar{u}_3(-ie\gamma_\mu)v_4)\left(\frac{-ig^{\mu\nu}}{q^2}\right)(\bar{v}_2(ieQ_c\gamma_\nu)u_1) \\ &= -F_c\frac{ie^2Q_c}{q^2}(\bar{u}_3\gamma_\mu v_4)(\bar{v}_2\gamma^\mu u_1), \end{aligned} \tag{3.1}$$

where $u_1 \equiv u(p_1,s_1)$ and similarly for other Dirac spinors. Moreover, Q_c stands for the fractional charge of the charm quark and $q = p_1 + p_2 = p_3 + p_4$.

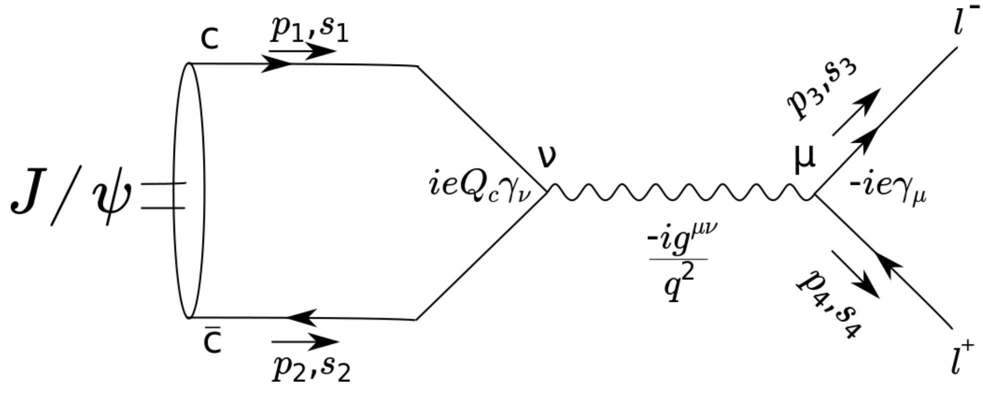


Figure 3.1. The LO Feynman diagram for J/ψ decay into a lepton pair.

We shall use a notation where spin $s = \frac{1}{2}$ is denoted as $s = \uparrow$ and $s = -\frac{1}{2}$ similarly as $s = \downarrow$. With this convention the standard spinors $u(p,s)$ and $v(p,s)$ at the NR limit $\boldsymbol{p} \to 0$ can be written as

$$u(p,s) = N(\boldsymbol{p}) \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} \xrightarrow{\boldsymbol{p}\to 0} \sqrt{2m} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} ; \begin{cases} \chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } s = \uparrow \\ \chi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } s = \downarrow \end{cases}$$
$$v(p,s) = N(\boldsymbol{p}) \begin{pmatrix} \frac{-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi_{\tilde{s}} \\ \chi_{\tilde{s}} \end{pmatrix} \xrightarrow{\boldsymbol{p}\to 0} \sqrt{2m} \begin{pmatrix} 0 \\ \chi_{\tilde{s}} \end{pmatrix} ; \begin{cases} \chi_{\tilde{s}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } s = \uparrow \\ \chi_{\tilde{s}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } s = \downarrow . \end{cases} \tag{3.2}$$

Note that there is some freedom in choosing the basis vectors. Here we have adapted the definition as given by Halzen and Martin [17, p.104]. In addition, it should be noted that the factor $\sqrt{2m}$ results from the normalization convention of $N(\boldsymbol{p}) = N(-\boldsymbol{p}) = \sqrt{E_p + m}$, where m is now the mass of the charm quark.
Let us consider the latter term $(\bar{v}_2\gamma^\mu u_1)$ of equation 3.1. In the sum over the index μ the first term $\mu = 0$ disappears:

$$(\bar{v}_2\gamma^0 u_1) = v_2^\dagger\gamma^0\gamma^0 u_1 = v_2^\dagger u_1 = 2m \begin{pmatrix} 0 & \chi_{\tilde{s}_2}^\dagger \end{pmatrix} \begin{pmatrix} \chi_{s_1} \\ 0 \end{pmatrix} = 0. \tag{3.3}$$

While for $k = 1,2,3$ we get

$$(\bar{v}_2\gamma^k u_1) = 2m \begin{pmatrix} 0 & \chi_{\tilde{s}_2}^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \begin{pmatrix} \chi_{s_1} \\ 0 \end{pmatrix} = 2m\chi_{\tilde{s}_2}^\dagger\sigma^k\chi_{s_1} = 2m(\sigma^k)_{\tilde{s}_2 s_1}, \tag{3.4}$$

where the last equality is just a notation for the matrix product. Notice that this does not specify the spin states s_2 and s_1 and thus holds in general. The invariant amplitude then becomes:

$$-i\mathcal{M} = -F_c\frac{2mie^2Q_c}{q^2}(\bar{u}_3\gamma_k v_4)(\sigma^k)_{\tilde{s}_2 s_1}, \tag{3.5}$$

where we have lightened up the notation by dropping the sub- and superscripts of $\mathcal{M}$. We know that the J/ψ will initially be in the triplet configuration $S = 1$, thus we have to sum over all the possible spin states and take the average of them. The square of the total invariant amplitude $\mathcal{M}_{tot}$ for the free particle annihilation, can then be written with $\mathcal{M}_{s_1 s_2}$ as

$$\overline{|\mathcal{M}_{tot}|}^2 = \frac{1}{3}\sum_{s_3 s_4}\left(|\mathcal{M}_{\uparrow\uparrow}|^2 + |\mathcal{M}_{\downarrow\downarrow}|^2 + \frac{1}{2}|(-i\mathcal{M}_{\uparrow\downarrow}) - (-i\mathcal{M}_{\downarrow\uparrow})|^2\right), \tag{3.6}$$

where the front factor 1/3 comes from averaging over the J/ψ spin states. Note especially that now the initial state $|SM_z\rangle = |10\rangle$ is given by $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ contrary to what we are used to with the Clebsch-Gordan coefficients. This follows from the adopted symmetric sign convention of equation 3.2. For more details see appendix A.1. Then the rest of the calculation is straightforward. Let us inspect the first element in equation 3.6 more closely:

$$|\mathcal{M}_{\uparrow\uparrow}|^2 = \frac{12m^2e^4Q_c^2}{q^4}(\bar{u}_3\gamma_k v_4)(\sigma^k)_{\uparrow\uparrow}(\bar{v}_4\gamma_l u_3)(\sigma^l)^*_{\uparrow\uparrow}, \tag{3.7}$$

where we have now inserted $\uparrow\uparrow$ in place of $s_2 s_1$ and $F_c^2 = 3$. It should be noted that all of the quantities in brackets in equation 3.7 are just numbers and thus can be moved freely around. Then the matrix product $(\bar{u}_3\gamma_k v_4)(\bar{v}_4\gamma_l u_3)$, which is just a number, can be written with the summation convention as

$$(\bar{u}_3\gamma_k v_4)(\bar{v}_4\gamma_l u_3) = (\bar{u}_3)_i(\gamma_k)_{ij}(v_4)_j(\bar{v}_4)_k(\gamma_l)_{kn}(u_3)_n. \tag{3.8}$$

Note that $(u_3)_n$ is just the component n of the 4-vector u_3 and thus can be moved first. Then all the spinors are next to each other and we may use the well known projection operators

$$\sum_{s=\uparrow,\downarrow} u(p,s)\bar{u}(p,s) = \not{p} + m \quad \text{and} \quad \sum_{s=\uparrow,\downarrow} v(p,s)\bar{v}(p,s) = \not{p} - m. \tag{3.9}$$

Then we are left with a matrix product which can be identified as a trace over the matrix product. Furthermore, at this energy scale we may neglect the masses of the leptons i.e. $M \gg m_l$. Here M is the mass of the J/ψ and m_l is the mass of the lepton. Keeping in mind all this, we may process the squared invariant amplitude further:

$$\begin{aligned}\sum_{s_3 s_4}|\mathcal{M}_{\uparrow\uparrow}|^2 &= \frac{12m^2e^4Q_c^2}{q^4}\underbrace{\sum_{s_3 s_4}(u_3\bar{u}_3\gamma_k v_4\bar{v}_4\gamma_l)}_{Trace}(\sigma^k)_{\uparrow\uparrow}(\sigma^l)^*_{\uparrow\uparrow} \\ &= \frac{12m^2e^4Q_c^2}{q^4}\mathrm{Tr}(\not{p}_3\gamma_k\not{p}_4\gamma_l)(\sigma^k)_{\uparrow\uparrow}(\sigma^l)^*_{\uparrow\uparrow}.\end{aligned} \tag{3.10}$$

For the trace in equation 3.10 we may use the well known trace identity

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{3.11}$$

and write:

$$
\begin{aligned}
\sum_{s_3 s_4} |\mathcal{M}_{\uparrow\uparrow}|^2 &= \frac{12 m^2 e^4 Q_c^2}{q^4} \cdot 4 \cdot p_3^\alpha p_4^\beta \left(g_{\alpha k} g_{\beta l} - g_{\alpha\beta} g_{kl} + g_{\alpha l} g_{\beta k} \right) (\sigma^k)_{\uparrow\uparrow} (\sigma^l)^*_{\uparrow\uparrow} \\
&= \frac{48 m^2 e^4 Q_c^2}{q^4} (p_{3k} p_{4l} - (p_3 \cdot p_4) g_{kl} + p_{3l} p_{4k}) (\sigma^k)_{\uparrow\uparrow} (\sigma^l)^*_{\uparrow\uparrow}.
\end{aligned} \tag{3.12}
$$

Notice that we still have the summation over the Lorentz indices k and l. This is because the zeroth index goes always to zero in the NR-limit. Moreover, we are considering the invariant amplitude for which $s_2 s_1 = \uparrow\uparrow$. So we need to consider the values of $(\sigma^k)_{\uparrow\uparrow}$ and $(\sigma^l)^*_{\uparrow\uparrow}$. Essentially we need to multiply the Pauli spin matrices with the appropriate two-spinors. For example when $k = 1$ we get from equation 3.4:

$$
(\sigma^1)_{\uparrow\uparrow} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1. \tag{3.13}
$$

Carrying this over for $k = 2$ and $k = 3$, we can denote the values of $(\sigma^k)_{\uparrow\uparrow}$ and $(\sigma^l)^*_{\uparrow\uparrow}$ with 4-vectors as below:

$$
(\sigma^k)_{\uparrow\uparrow} = (0,1,i,0) \quad \& \quad (\sigma^l)^*_{\uparrow\uparrow} = (0,1,-i,0). \tag{3.14}
$$

Let us then consider the product between the momentum 4-vectors and these 4-vectors. Notice in particular that with our notational convention the subscript denotes the particle in question and the superscript the element of the momentum 4-vector e.g. p_3^1 is the underlined element in $p_3^\mu = (p_3^0, \underline{p_3^1}, p_3^2, p_3^3)^{\mathrm{T}}$. In addition, keeping in mind the minus sign coming from the metric:

$$
\begin{aligned}
(p_{3k} p_{4l} - (p_3 \cdot p_4) g_{kl} + p_{3l} p_{4k}) (\sigma^k)_{\uparrow\uparrow} (\sigma^l)^*_{\uparrow\uparrow} &= (-p_3^1 - i p_3^2)(-p_4^1 + i p_4^2) \\
&\quad - (p_3 \cdot p_4)(-1 - 1) + (-p_3^1 + p_3^2)(-p_4^1 - i p_4^2) \\
&= 2(p_3 \cdot p_4) + 2(p_3^1 p_4^1 + p_3^2 p_4^2).
\end{aligned} \tag{3.15}
$$

We may then continue to consider the other $\mathcal{M}$'s. Again it is beneficial to explicitly denote the values of $(\sigma^k)_{s_2 s_1}$ and $(\sigma^l)^*_{s_2 s_1}$ for the other $s_2 s_1$ states:

$$
\begin{aligned}
(\sigma^k)_{\downarrow\downarrow} &= (0,1,-i,0) \quad \& \quad (\sigma^l)^*_{\downarrow\downarrow} = (0,1,i,0) \\
(\sigma^k)_{\uparrow\downarrow} &= (0,0,0,-1) \quad \& \quad (\sigma^l)^*_{\uparrow\downarrow} = (0,0,0,-1) \\
(\sigma^k)_{\downarrow\uparrow} &= (0,0,0,1) \quad \& \quad (\sigma^l)^*_{\downarrow\uparrow} = (0,0,0,1).
\end{aligned} \tag{3.16}
$$

We can immediately see that for $|\mathcal{M}_{\downarrow\downarrow}|^2$ the expression is the same as for $|\mathcal{M}_{\uparrow\uparrow}|^2$,

$$
\sum_{s_3 s_4} |\mathcal{M}_{\downarrow\downarrow}|^2 = \ldots = \frac{48 m^2 e^4 Q_c^2}{q^4} (2(p_3 \cdot p_4) + 2(p_3^1 p_4^1 + p_3^2 p_4^2)). \tag{3.17}
$$

Let us then denote the state $|SM_z\rangle = |10\rangle$ in equation 3.6 by I:

$$
\begin{aligned}
I :=& \frac{1}{2}\sum_{s_3 s_4} |(-i\mathcal{M}_{\uparrow\downarrow}) - (-i\mathcal{M}_{\downarrow\uparrow})|^2 \\
=& \frac{1}{2}\sum_{s_3 s_4} |F_c \frac{2mie^2 Q_c}{q^2}(\bar{u}_3 \gamma_k v_4)[(\sigma^k)_{\downarrow\uparrow} - (\sigma^k)_{\uparrow\downarrow}]|^2.
\end{aligned}
\tag{3.18}
$$

Notice in particular how the labeling of the spins works. By looking at equation 3.16, we see that only the term $k = 3$ contributes to I. Moreover, by taking advantage of the trace identity of equation 3.11 and by remembering the minus coming from the metric, we can immediately write down the form

$$
I = \frac{48m^2 e^4 Q_c^2}{q^4}(2(p_3 \cdot p_4) + 4p_3^3 p_4^3). \tag{3.19}
$$

Now combining equation 3.19 with 3.17, the result which we have for $|\mathcal{M}_{\uparrow\uparrow}|^2$ and by remembering the 1/3 front factor we arrive at

$$
\overline{|\mathcal{M}_{tot}|}^2 = \frac{1}{3}\frac{48m^2 e^4 Q_c^2}{q^4}(6(p_3 \cdot p_4) + 4(p_3^1 p_4^1 + p_3^2 p_4^2 + p_3^3 p_4^3)). \tag{3.20}
$$

Next we will clean up this result with dynamical variables. The Mandelstam variable s is defined as

$$
s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{3.21}
$$

which we can open up,

$$
s = p_3^2 + 2p_3 \cdot p_4 + p_4^2, \tag{3.22}
$$

and use the result $p_i^2 = m_i^2 \approx 0$. Thus we have $s = 2p_3 \cdot p_4$. Moreover, in the CMS frame we have $\boldsymbol{p}_3 = -\boldsymbol{p}_4$ and the last term of equation 3.20 reduces to $-4|\boldsymbol{p}_3|^2$. Furthermore, in the CMS-frame the energy is evenly distributed for the outgoing leptons i.e. $E_3 = E_4$ and because $\sqrt{s} = E_{tot} = E_3 + E_4$, we can write $|\boldsymbol{p}_3|^2$ with the energy-momentum relation $E_i^2 \approx |\boldsymbol{p}_i|^2$ as $|\boldsymbol{p}_3|^2 = s/4$. Thus we have the results

$$
6(p_3 \cdot p_4) = 3s \quad \text{and} \quad 4(p_3^1 p_4^1 + p_3^2 p_4^2 + p_3^3 p_4^3) = -s. \tag{3.23}
$$

Moreover, in the NR-limit we have $M^2 = q^2 = s$ and $m = M/2$. Inserting these into equation 3.20 we get

$$
\begin{aligned}
\overline{|\mathcal{M}_{tot}|}^2 &= \frac{16(M/2)^2 e^4 Q_c^2}{s^2}(3s - s) = \frac{4se^4 Q_c^2}{s^2} 2s = 8e^4 Q_c^2 \\
&= 8 \cdot 16\pi^2 \alpha^2 Q_c^2,
\end{aligned}
\tag{3.24}
$$

where in the last equality we used the result $e^4 = 16\pi^2\alpha^2$. Now by applying equation 2.9 we get

$$\overline{|\mathcal{M}_{tot}(J/\psi \to l^+l^-)|}^2 = \frac{2}{M}|\psi(\boldsymbol{r}=0)|^2 8 \cdot 16\pi^2\alpha^2 Q_c^2. \tag{3.25}$$

Then from equation 2.2 we have

$$\begin{aligned}\Gamma(J/\psi \to l^+l^-) = \frac{1}{2M}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}&(2\pi)^4\delta^{(4)}(p_{J/\psi} - p_3 - p_4)\\ &\overline{|\mathcal{M}_{tot}(J/\psi \to l^+l^-)|}^2.\end{aligned} \tag{3.26}$$

By neglecting the running of α, we notice that the invariant amplitude is just a constant. The integral of equation 3.26 can be worked down to the following form:

$$\Gamma(J/\psi \to l^+l^-) = \frac{\sqrt{\lambda(M^2,0,0)}}{64\pi^2 M^3}\int d\Omega \overline{|\mathcal{M}_{tot}(J/\psi \to l^+l^-)|}^2, \tag{3.27}$$

where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$ and $d\Omega$ is the infinitesimal solid angle element over which we are integrating. For details, see the appendix A.2. Since $\overline{|\mathcal{M}_{tot}(J/\psi \to l^+l^-)|}^2$ is just a constant, the solid angle integral over $d\Omega$ gives 4π. By substituting these we get

$$\begin{aligned}\Gamma(J/\psi \to l^+l^-) &= \frac{M^2}{64\pi^2 M^3}\cdot 4\pi \cdot \frac{2}{M}|\psi(\boldsymbol{r}=0)|^2 \cdot 8 \cdot 16\pi^2\alpha^2 Q_c^2\\ &= 16\pi\alpha^2 Q_c^2 \frac{|\psi(\boldsymbol{r}=0)|^2}{M^2}.\end{aligned} \tag{3.28}$$

This is the well known Van Royen-Weisskopf formula [19].

3.2 Helicity basis approach

In this section we will repeat the previous calculation in the helicity basis. This approach is rather ingenious and saves us from the explicit spin matrix calculations of the previous section. Before this, however, we shall change our choice of the basis spinors in equation 3.2. The only change we need to make, is to flip the sign of the spin down basis spinor of the antiparticle. That is, in the NR-limit we define the helicity basis vectors as follows:

$$
\begin{aligned}
u(p,h) &= N(\boldsymbol{p}) \begin{pmatrix} \phi_h \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\phi_h \end{pmatrix} \xrightarrow{\boldsymbol{p}\to 0} \sqrt{2m}\begin{pmatrix} \phi_h \\ 0 \end{pmatrix}; \begin{cases} \phi_h = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } h = \uparrow \\ \phi_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } h = \downarrow \end{cases} \\
v(p,h) &= N(\boldsymbol{p}) \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi_h \\ \chi_h \end{pmatrix} \xrightarrow{\boldsymbol{p}\to 0} \sqrt{2m}\begin{pmatrix} 0 \\ \chi_h \end{pmatrix}; \begin{cases} \chi_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } h = \uparrow \\ \chi_h = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ for } h = \downarrow . \end{cases}
\end{aligned}
\tag{3.29}
$$

where $N(\boldsymbol{p})$ is as in equation 3.2 and $h = \uparrow, \downarrow$ refer to the positive and negative helicity, correspondingly. The above sign convention is chosen for example in reference [20, p.44-45]. And we can immediately see why this recovers the standard Clebsch-Gordan coefficient results. The reader is encouraged to see equation A.9 to verify this. Then, as given in reference [21], the Dirac spinors can be written in terms of the J/ψ initial state polarization vectors ϵ as

$$
\begin{aligned}
&u(\uparrow)\bar{v}(\uparrow) = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_J + M}{2}\right)\not{\epsilon}(\uparrow); \quad u(\downarrow)\bar{v}(\downarrow) = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_J + M}{2}\right)\not{\epsilon}(\downarrow); \\
&\frac{1}{\sqrt{2}}\left[u(\uparrow)\bar{v}(\downarrow) + u(\downarrow)\bar{v}(\uparrow)\right] = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_J + M}{2}\right)\not{\epsilon}(0),
\end{aligned}
\tag{3.30}
$$

where $\uparrow$, $\downarrow$ and 0 for ϵ represent the helicity states 1, -1 and 0 respectively. Moreover, we have set $p_J = p_c/2 = p_{\bar{c}}/2$ where p_J is the 4-momentum of J/ψ and $p_{c/\bar{c}}$ is the quark 4-momentum. In addition, M denotes the mass of the J/ψ. Note that we have adopted the notational convention of reference [21], where the polarization vector $\epsilon^\mu(p_J,h)$ is denoted by $\epsilon^\mu(h)$. The subtleties of how these identities come about are not very important for the decay width calculation. Thus the finer details of this method have been relegated to appendix A.3.

The invariant amplitude is still given by equation 3.1. Now we shall denote the lepton vertex $(\bar{u}_3\gamma_\mu v_4)$ with L_μ. With these we may write the invariant amplitude as follows:

$$
-i\mathcal{M}_h = -F_c\frac{ie^2Q_c}{q^2}L_\mu(\bar{v}_2\gamma^\mu u_1), \tag{3.31}
$$

where the subscript h denotes the helicity state of the J/ψ. Then similarly as what we did in equation 3.8, we may identify the trace as

$$
(\bar{v}_2)_i(\gamma^\mu)_{ij}(u_1)_j = (u_1)_j(\bar{v}_2)_i(\gamma^\mu)_{ij} = \mathrm{Tr}(u_1\bar{v}_2\gamma^\mu) \tag{3.32}
$$

and use the results of equation 3.30 to process equation 3.31 further:

$$
\begin{aligned}
-i\mathcal{M}_h &= -F_c \frac{ie^2Q_c}{q^2} L_\mu \mathrm{Tr}(u_1\bar{v}_2\gamma^\mu) = -F_c \frac{ie^2Q_c}{q^2} L_\mu \mathrm{Tr}\left(-\frac{1}{\sqrt{2}}\left(\frac{\not{p}_J + M}{2}\right)\not{\epsilon}(h)\gamma^\mu\right) \\
&= \frac{F_c}{2\sqrt{2}} \frac{ie^2Q_c}{q^2} L_\mu \underbrace{\mathrm{Tr}(M\not{\epsilon}(h)\gamma^\mu)}_{=M\epsilon_\nu(h)4g^{\nu\mu}} = \frac{F_c}{2\sqrt{2}} \frac{ie^2Q_c}{q^2} L_\mu \cdot 4M\epsilon^\mu(h) \\
&= F_c \frac{\sqrt{2}ie^2Q_c}{q^2} M L_\mu \epsilon^\mu(h),
\end{aligned}
\tag{3.33}
$$

where in moving from the first line to the second line, we have used the fact that a trace over an odd number of Dirac gamma matrices vanishes.

Then the summing over of the initial states of J/ψ can be written in terms of a sum over the helicity eigenvalues $0, \pm 1$. Again by averaging over the initial states and summing over the final state helicities, the total squared invariant amplitude for the process $c\bar{c} \to l^-l^+$ can be written as

$$
\overline{|\mathcal{M}_{tot}(c\bar{c} \to l^-l^+)|}^2 = \frac{1}{3} \sum_{h=0,\pm 1} \sum_{s_3 s_4} F_c^2 \frac{2e^4Q_c^2}{q^4} M^2 L_\mu L_\nu^* \epsilon^\mu(h)\epsilon^{\nu^*}(h). \tag{3.34}
$$

We may then write the lepton vertex part as a trace similarly as in equation 3.10,

$$
\sum_{s_3 s_4} L_\mu L_\nu^* = \mathrm{Tr}(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu), \tag{3.35}
$$

where we have again neglected the masses of the leptons. By substituting the result of equation 3.35 into equation 3.34, we get

$$
\overline{|\mathcal{M}_{tot}|}^2 = \frac{2e^4Q_c^2M^2}{q^4} \sum_{h=0,\pm 1} \epsilon^\mu(h)\epsilon^{\nu^*}(h)\mathrm{Tr}(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu). \tag{3.36}
$$

We can further simplify this by noting that $M^2 = q^2 = p_J^2$ and that the sum over the helicity states of the polarization vectors can be written as

$$
\sum_{h=0,\pm 1} \epsilon^\mu(h)\epsilon^{\nu^*}(h) = -g^{\mu\nu} + \frac{p_J^\mu p_J^\nu}{p_J^2}. \tag{3.37}
$$

Let us then substitute equation 3.37 into equation 3.36,

$$
\begin{aligned}
\overline{|\mathcal{M}_{tot}|}^2 &= \frac{2e^4Q_c^2}{q^2}\left(\left(-g^{\mu\nu} + \frac{p_J^\mu p_J^\nu}{p_J^2}\right)\mathrm{Tr}(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu)\right) \\
&= \frac{2e^4Q_c^2}{q^2}\left(-\mathrm{Tr}(\not{p}_3\underbrace{\gamma^\nu\not{p}_4\gamma_\nu}_{=-2\not{p}_4}) + \frac{1}{p_J^2}\mathrm{Tr}(\not{p}_3\not{p}_J\not{p}_4\not{p}_J)\right).
\end{aligned}
\tag{3.38}
$$

Next we will use the results $p_J = p_4 + p_3$, $\not{p}_{3/4}^2 = p_{3/4}^2 = m_{3/4}^2 \approx 0$ and $2\mathrm{Tr}(\not{p}_3\not{p}_4) = 8(p_3 \cdot p_4) = 4s = 4q^2$:

$$\begin{aligned}\overline{|\mathcal{M}_{tot}|}^2 &= \frac{2e^4Q_c^2}{q^2}\left(2\mathrm{Tr}(\not{p}_3\not{p}_4) + \frac{1}{p_J^2}\mathrm{Tr}(\not{p}_3(\not{p}_3 + \not{p}_4)\not{p}_4(\not{p}_3 + \not{p}_4))\right) \\ &= \frac{2e^4Q_c^2}{q^2}\left(4q^2 + \frac{1}{p_J^2}\mathrm{Tr}((\underbrace{(m_3^2\not{p}_4 + \not{p}_3 m_4^2)(\not{p}_3 + \not{p}_4))}_{\approx 0}\right).\end{aligned} \tag{3.39}$$

Thus only the first term contributes and we are left with

$$\overline{|\mathcal{M}_{tot}|}^2 = 8e^4Q_c^2 = 8 \cdot 16\pi^2\alpha^2Q_c^2 \tag{3.40}$$

which is what we had in equation 3.24. Then the rest of the calculation proceeds as in the previous section.

4 Decay width of $\eta_c \to gg$

A calculation for the decay width of $\eta_c \to gg$ using the explicit spin matrix approach, as presented in section 3.1, is done in detail in reference [7]. However, here we shall derive this decay width by using the helicity basis approach as it offers a slightly easier method for the calculation. Because of this we will only outline this calculation here and further details can be found from [7], [15] and appendix B.
With the J/ψ case we went through all the possible ways of combining spin-$\frac{1}{2}$ particles to a $J = S = 1$ and $L = 0$ particle state. Since η_c is a $J^{PC} = 0^{-+}$ particle with $J = L = S = 0$, this gives us the remaining spin singlet state. By now the reader should be sufficiently familiar with our method of calculating the decay width. The three LO Feynman diagrams for the free-particle process are given in figures 4.1, 4.2 and 4.3 and the invariant amplitude is the sum of these three graphs:

$$\begin{aligned}
-i\mathcal{M}^{C=0}_{S=1}(c\bar{c} \to gg) = {} & \bar{v}_2 \left(-ig_s\gamma_\nu t^b_{jk}\right) \epsilon^{\nu*}_2 \frac{i(\not{p}_1 - \not{k}_1 + m)}{(p_1 - k_1)^2 - m^2} \epsilon^{\mu*}_1 \left(-ig_s\gamma_\mu t^a_{ki}\right) u_1 \\
& + \bar{v}_2 \left(-ig_s\gamma_\mu t^a_{jk}\right) \epsilon^{\mu*}_1 \frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2} \epsilon^{\nu*}_2 \left(-ig_s\gamma_\nu t^b_{ki}\right) u_1 \\
& + \bar{v}_2 \left(-ig_s t^d_{ji}\gamma_\delta\right) u_1 \frac{-ig^{\delta\rho}\delta^{cd}}{(p_1 + p_2)^2} \epsilon^{\mu*}_1 g_s f^{abc} \Big[g_{\mu\rho}(-2k_1 - k_2)_\nu \\
& \qquad + g_{\rho\nu}(k_1 + 2k_2)_\mu + g_{\nu\mu}(k_1 - k_2)_\rho\Big] \epsilon^{\nu*}_2,
\end{aligned} \tag{4.1}$$

where g_s is the strong coupling constant, t's are the generators of SU(3) and f^{abc} are the structure constants of SU(3).

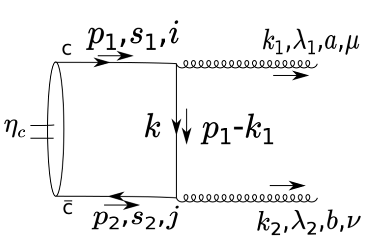


Figure 4.1.
The t-channel graph.

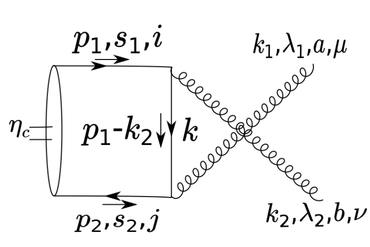


Figure 4.2.
The u-channel graph.

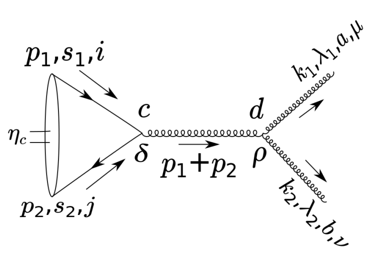


Figure 4.3.
The s-channel graph.

Notice that we have written the gluon propagator in the Feynman gauge. Let us first inspect the color factors of these terms. Just like the J/ψ, η_c is in the color singlet configuration with $i = j$. Then when we sum over the color indices we get the following color factors:

$$\begin{aligned} &\text{t-channel:} \quad \frac{1}{\sqrt{3}}(t^b t^a)_{ii} = \frac{1}{\sqrt{3}}\mathrm{Tr}(t^b t^a) = \frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}. \\ &\text{u-channel:} \quad \frac{1}{\sqrt{3}}(t^a t^b)_{ii} = \frac{1}{\sqrt{3}}\mathrm{Tr}(t^a t^b) = \frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}. \\ &\text{s-channel:} \quad \frac{1}{\sqrt{3}}(t^c f^{abc})_{ii} = \frac{1}{\sqrt{3}}\underbrace{\mathrm{Tr}(t^c)}_{=0} f^{abc} = 0, \end{aligned} \tag{4.2}$$

where T(F) = 1/2 is the normalization for the fundamental representation of SU(3). We can see that the color factor is the same for the channels t and u. Moreover, the color factor of the s-channel vanishes and thus the graph 4.3 with a color-singlet virtual gluon does not contribute at all. We may then simplify equation 4.1 further

$$\begin{aligned} -i\mathcal{M}^{C=0}_{S=1} = -ig_s^2\left(\frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}\right)\Bigg[&\frac{\bar{v}_2\not{\epsilon}_2^*(\not{p}_1 - \not{k}_1 + m)\not{\epsilon}_1^* u_1}{-2p_1\cdot k_1} \\ &+ \frac{\bar{v}_2\not{\epsilon}_1^*(\not{p}_1 - \not{k}_2 + m)\not{\epsilon}_2^* u_1}{-2p_1\cdot k_2}\Bigg]. \end{aligned} \tag{4.3}$$

Next we shall move into the rest frame of η_c and work in the Coulomb gauge. These give us the following constraints:

$$p \equiv p_1 = p_2 = (m,\vec{0}), \quad k_1 = (m,\vec{k}), \quad k_2 = (m,-\vec{k}), \quad k_i\cdot\epsilon_j^* = 0, \tag{4.4}$$

where we have denoted the 3-vector with an arrow and the indices i,j get values 1,2. Next we shall apply the Dirac equation,

$$(\not{p} - m)u(p) = 0, \tag{4.5}$$

to simplify the invariant amplitude further. From the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ it follows that

$$\not{p}_1\not{\epsilon}_i^* = -\not{\epsilon}_i^*\not{p}_1 + 2\underbrace{(p_1\cdot\epsilon_i^*)}_{=0} = -\not{\epsilon}_i^*\not{p}_1, \tag{4.6}$$

where the inner product goes to zero since we are working in the NR-limit where $\vec{p}_1 = 0$ with the Coulomb gauge condition where $\epsilon_i^0 = 0$. Thus we have shown that in this case $\not{p}_1$ and the polarization vectors $\not{\epsilon}_i$ anticommute. This same anticommutation relation holds also for the term $\not{k}_i\not{\epsilon}_i$ by the virtue of the Lorentz gauge. Furthermore, it is evident that $m\mathbb{1}_4$ commutes with $\not{\epsilon}_i^*$. As we have terms such as $(\not{p}_1 + m)\not{\epsilon}_i^* u_1$ appearing in equation 4.3, we can use equation 4.5 to show that

$$(\not{p}_1 + m)\not{\epsilon}_i^* u_1 = \not{\epsilon}_i^* \underbrace{(-\not{p}_1 + m)u_1}_{=0} = 0. \tag{4.7}$$

Then by anticommuting $\not{k}_1$ with $\not{\epsilon}_i^*$ we are left with terms such as $\bar{v}_2\not{\epsilon}_i^*\not{\epsilon}_j^*\not{k}_1 u_1$ in the numerator. Then from equation 4.4, it follows that

$$-2p_1 \cdot k_i = -2m^2. \tag{4.8}$$

With these we arrive to the following form of the invariant amplitude:

$$-i\mathcal{M}_{S=1}^{C=0} = -ig_s^2 \left(\frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}\right) \frac{1}{-2m^2} \bar{v}_2[\not{\epsilon}_2^*\not{\epsilon}_1^*\not{k}_1 + \not{\epsilon}_1^*\not{\epsilon}_2^*\not{k}_2]u_1. \tag{4.9}$$

It is at this point where we would like to apply our helicity basis approach. It can be shown, as developed in appendix B.1, that the spin singlet state of the spinors may be presented as

$$\begin{aligned} &(1) \quad \frac{1}{\sqrt{2}}[v(\uparrow)\bar{u}(\downarrow) - v(\downarrow)\bar{u}(\uparrow)] = \frac{1}{\sqrt{2}}\gamma^5 \left(\frac{\not{p}_\eta + M_\eta}{2}\right) \\ &(2) \quad \frac{1}{\sqrt{2}}[u(\uparrow)\bar{v}(\downarrow) - u(\downarrow)\bar{v}(\uparrow)] = \frac{1}{\sqrt{2}} \left(\frac{\not{p}_\eta + M_\eta}{2}\right)\gamma^5, \end{aligned} \tag{4.10}$$

where M_η is the mass of η_c, (1) holds for a final state and (2) holds for an initial state. Then the total invariant amplitude $\mathcal{M}_{tot}$ in the spin singlet state with $\mathcal{M}_{s_1 s_2}$ is written as

$$-i\mathcal{M}_{tot} = \frac{1}{\sqrt{2}}\left[(-i\mathcal{M}_{\uparrow\downarrow}) - (-i\mathcal{M}_{\downarrow\uparrow})\right]. \tag{4.11}$$

Next we want to apply the results of equation 4.10 to equation 4.9. How this comes about is shown below for one term:

$$(\bar{v}_2)_i[\not{\epsilon}_2^*\not{\epsilon}_1^*\not{k}_1]_{ij}(u_1)_j = (u_1)_j(\bar{v}_2)_i[\not{\epsilon}_2^*\not{\epsilon}_1^*\not{k}_1]_{ij} = \mathrm{Tr}(u_1\bar{v}_2\not{\epsilon}_2^*\not{\epsilon}_1^*\not{k}_1). \tag{4.12}$$

We may then use the result of equation 4.10 and take out the constant $1/(2\sqrt{2})$. Moreover, the term with M_η vanishes since the trace of odd number of gamma matrices vanishes. Then in combining equations 4.11, 4.10 and 4.9 together we get

$$-i\mathcal{M}_{tot} = -ig_s^2 \left(\frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}\right) \frac{1}{-2m^2}\frac{1}{2\sqrt{2}}\left[\mathrm{Tr}(\gamma^5\not{\epsilon}_2^*\not{\epsilon}_1^*\not{k}_1\not{p}_\eta) + \mathrm{Tr}(\gamma^5\not{\epsilon}_1^*\not{\epsilon}_2^*\not{k}_2\not{k}_\eta)\right]. \tag{4.13}$$

In order to evaluate this we need the following trace identity:

$$\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = -4i\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma, \tag{4.14}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

This identity is derived in appendix B.2. Let us simplify the notation slightly by denoting the front factor by C,

$$C = -ig_s^2 \left(\frac{\mathrm{T(F)}\delta^{ab}}{\sqrt{3}}\right) \frac{1}{-2m^2} \frac{1}{2\sqrt{2}}. \tag{4.16}$$

In using these results we take advantage of the fact that $p_\eta = (2m, \vec{0})$, where m is the mass of a charm quark. Thus equation 4.13 becomes

$$\begin{aligned} -i\mathcal{M}_{tot} &= C[(-4i\epsilon^{\mu\nu\rho 0}\epsilon^*_{2\mu}\epsilon^*_{1\nu}k_{1\rho}2m) + (-4i\epsilon^{\mu'\nu'\rho'0}\epsilon^*_{1\mu'}\epsilon^*_{2\nu'}k_{2\rho'}2m)] \\ &= 8imC[\underbrace{(\epsilon^{0\mu\nu\rho}\epsilon^*_{2\mu}\epsilon^*_{1\nu}k_{1\rho})}_{=-(\boldsymbol{\epsilon}^*_2\times\boldsymbol{\epsilon}^*_1)\cdot\boldsymbol{k}_1} + \underbrace{(\epsilon^{0\mu'\nu'\rho'}\epsilon^*_{1\mu'}\epsilon^*_{2\nu'}k_{2\rho'})}_{=-(\boldsymbol{\epsilon}^*_1\times\boldsymbol{\epsilon}^*_2)\cdot\boldsymbol{k}_2}], \end{aligned} \tag{4.17}$$

where we have accounted for the minus sign coming from the exchange $\mu\nu\rho 0 \to 0\mu\nu\rho$ and identified $\epsilon^{0\mu\nu\rho}$ with the Levi-Civita symbol ϵ^{ijk}. Notice in particular that since the zeroth term is taken away by p_η the remaining indices are only spatial indices and thus allows us to write this scalar triple product result.

We can use the fact that $\boldsymbol{k}_2 - \boldsymbol{k}_1 = -2\boldsymbol{k} = -2\hat{e}_k m$, where $\hat{e}_k$ is a unit vector into the direction of $\boldsymbol{k}$. Moreover, we know that for any two three vectors $\vec{a}$ and $\vec{b}$, it holds that $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$. Then the total invariant amplitude simplifies further:

$$-i\mathcal{M}_{tot} = -8imC[(\boldsymbol{\epsilon}^*_1 \times \boldsymbol{\epsilon}^*_2) \cdot (\mathbf{k}_2 - \mathbf{k}_1))] = -16im^2C(\boldsymbol{\epsilon}^*_1 \times \boldsymbol{\epsilon}^*_2) \cdot \hat{e}_k. \tag{4.18}$$

Next we can take advantage of the fact that the scalar triple product $(\boldsymbol{\epsilon}^*_1 \times \boldsymbol{\epsilon}^*_2) \cdot \hat{e}_k$ is invariant in rotations. This intuitive result can be understood through the geometric interpretation of the scalar triple product. Geometrically, for three 3-vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, the product $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ is the volume of a parallelepiped defined by three vectors. As such, it is then clear that the value of $|(\boldsymbol{\epsilon}^*_1 \times \boldsymbol{\epsilon}^*_2) \cdot \hat{e}_k|^2$ cannot change in rotations. Thus we may choose $\hat{e}_k$ to be e.g. along the z-axis. It then follows that the triple product $(\boldsymbol{\epsilon}^*_1 \times \boldsymbol{\epsilon}^*_2) \cdot \hat{e}_k$ has only two possible values of $\pm i$ and gets them when the gluons are either left-handed (L) or right-handed (R). This can be seen by explicitly calculating all the possible cases with the polarization vectors as given in equation A.3. We may then finally form the squared invariant amplitude for the bound state as given by equation 2.9. We will sum over the color indices a and b and over the final state spins $\lambda = \mathrm{L,R}$:

$$\overline{|\mathcal{M}_{tot}(\eta_c \to gg)|}^2 = \frac{2}{M_\eta}|\psi(\mathbf{r}=0)|^2 g_s^4 \left(\frac{16m^2}{4m^2\sqrt{2}}\right)^2 \left(\frac{\mathrm{T(F)}}{\sqrt{3}}\right)^2 \sum_{a,b}\delta^{ab}\delta^{ab} \Big(\underbrace{|-i|^2}_{\lambda=\mathrm{R}} + \underbrace{|+i|^2}_{\lambda=\mathrm{L}}\Big), \tag{4.19}$$

where we can now use the fact that $\sum_{a,b}\delta^{ab}\delta^{ab} = \mathrm{Tr}(\mathbb{1}_8) = 8$ and simplify our result down to

$$\overline{|\mathcal{M}_{tot}(\eta_c \to gg)|}^2 = \frac{2}{M_\eta}|\psi(\mathbf{r}=0)|^2 \left(\frac{2}{3}\right) 16 g_s^4. \tag{4.20}$$

Again the squared invariant amplitude has no angular dependence and we may use our result in equation 3.27. For the strong coupling constant we may use the result $g_s^2 = 4\pi\alpha_s$. Moreover, now the integration over the angles gives us 2π instead of 4π. This is because we have two identical final state particles so that the phase space integral constains one extra 1/2. With these steps our final result becomes

$$\Gamma(\eta_c \to gg) = \left(\frac{2}{3}\right)\frac{16\pi\alpha^2}{M_\eta^2}|\psi(\mathbf{r}=0)|^2. \tag{4.21}$$

This is the result derived by Silverman and Yao in [15].

5 Decay width of O-Ps→ $\gamma\gamma\gamma$

Ortho-positronium (O-Ps) is a $L = 0$, $J = S = 1$ bound state of an electron e^- and a positron e^+. In spectroscopic notation this state is denoted by $^3\mathrm{S}_1$. The decay of this bound state into three photons is a pure QED process and has many similarities to the process $J/\psi \to ggg$. However, the process $J/\psi \to ggg$ cannot be described within the framework of QED but we need QCD for its description. And QCD, being a non-abelian gauge theory, is in general more difficult than QED. However, as we shall see, most of the QCD complications do not contribute for the J/ψ calculation. Our discussion relies heavily on the ideas presented in reference [1, p.233] and as such, we will follow its notational convention. It should be pointed out that, even though this process was calculated in the 1940's for the first time [8], it has evoked some interest in the 21st century [9], [22], [23]. Let us then begin by forming the invariant amplitude.

5.1 Deriving the invariant amplitude

The Feynman graphs of this process to LO are given in figures 5.1-5.6. Notice in particular our convention for indexing the emitted photons and the fact that these different graphs can be categorized as different permutations of (123). We denote the positron e^+ spinor with v_2 and the electron e^- spinor by u_1. When working in the extreme NR limit we can take the 4-momentum of e^- and e^+ to be $p_1 = p_2 \equiv p$ where $p = (m,\vec{0})$. Here m is the mass of e^-.

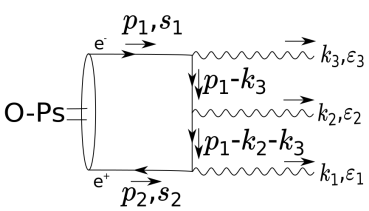


Figure 5.1.
Permutation (123).

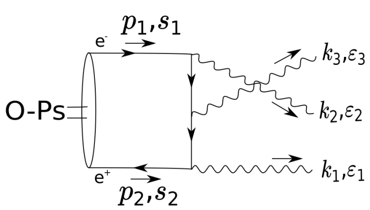


Figure 5.2.
Permutation (132).

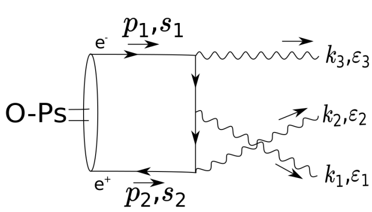


Figure 5.3.
Permutation (213).

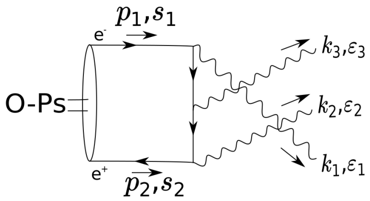


Figure 5.4.
Permutation (231).

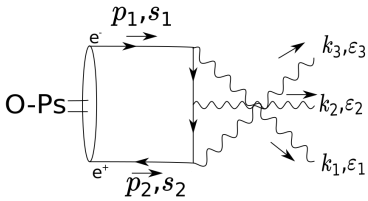


Figure 5.5.
Permutation (321).

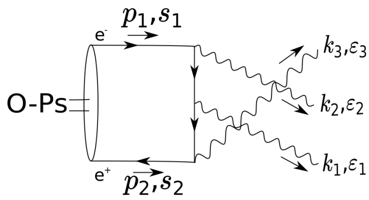


Figure 5.6.
Permutation (312).

The invariant amplitude can then be written as

$$\begin{aligned}-i\mathcal{M}_{S=1}^{C=0}(e^-e^+ \to \gamma\gamma\gamma) = \sum_{\text{permut}} \bar{v}_2(-ie\gamma^\mu)\epsilon_{1\mu}\left(\frac{i(\not{p} - \not{k}_2 - \not{k}_3 + m)}{(p-k_2-k_3)^2 - m^2}\right) \\ (-ie\gamma^\nu)\epsilon_{2\nu}\left(\frac{i(\not{p} - \not{k}_3 + m)}{(p-k_3)^2 - m^2}\right)(-ie\gamma^\sigma)\epsilon_{3\sigma}u_1,\end{aligned} \tag{5.1}$$

where the permutation sum contains all orderings of the photons as given in figures 5.1-5.6. Notice that here we have chosen the polarization vectors ϵ_i to be real and therefore we do not need to write in the complex conjugate * as would be given by the Feynman rules. In the rest frame of the O-Ps we have the following conservation laws $k_1+k_2+k_3 = 2p = 2m$ and $\vec{k}_1+\vec{k}_2+\vec{k}_3 = 0$. It then follows that $p-k_2-k_3 = k_1-p$ which we can use to simplify equation 5.1:

$$-i\mathcal{M}_{S=1} = \sum_{\text{permut}} -ie^3\bar{v}_2\not{\epsilon}_1\left(\frac{\not{k}_1 - \not{p} + m}{(k_1-p)^2 - m^2}\right)\not{\epsilon}_2\left(\frac{\not{p} - \not{k}_3 + m}{(p-k_3)^2 - m^2}\right)\not{\epsilon}_3 u_1. \tag{5.2}$$

Next we shall peel down the 4×4 matrix structure of the elements between the spinors u and v down to a 2×2 matrix structure. This is justified when we are working in the limit where the momentum of the spinors u and v goes to zero. This can be seen by looking at the spinor solutions of equation 3.29. Holding on to the 4×4 matrix structure would just account to carrying extra zeros around. This is to say that from the matrix

$$\not{\epsilon}_1\left(\frac{\not{k}_1 - \not{p} + m}{(k_1-p)^2 - m^2}\right)\not{\epsilon}_2\left(\frac{\not{p} - \not{k}_3 + m}{(p-k_3)^2 - m^2}\right)\not{\epsilon}_3, \tag{5.3}$$

we pick up the relevant parts to a 2×2 matrix a_{123}. By looking at the solutions of equation 3.29, it is evident that the 2×2 matrix is acquired by multiplying with a 2×4 matrix from the left and by a 4×2-matrix from the right:

$$a_{123} = \begin{pmatrix} 0_2 & \mathbb{1}_2 \end{pmatrix}\not{\epsilon}_1\left(\frac{\not{k}_1 - \not{p} + m}{(k_1-p)^2 - m^2}\right)\not{\epsilon}_2\left(\frac{\not{p} - \not{k}_3 + m}{(p-k_3)^2 - m^2}\right)\not{\epsilon}_3\begin{pmatrix} \mathbb{1}_2 \\ 0_2 \end{pmatrix}. \tag{5.4}$$

Then in equation 5.2 the spinor $\bar{v}_2$ is replaced by $-\chi_s^\dagger$ and u_1 by ϕ_s, where χ_s and ϕ_s are as in equation 3.29. Then equation 5.2 can be written as follows:

$$-i\mathcal{M}_{s=1} = i2me^3 \sum_{\text{permut}} \chi_s^\dagger a_{123} \phi_s, \tag{5.5}$$

where the permutation sum is now over the indices 123. Next we need to process the from of a_{123} further. We shall adopt the following notational convention:

$$\omega_i = k_i^0; \quad \hat{k}_i = \frac{\vec{k}_i}{\omega_i}; \quad \hat{\delta}_i = \hat{k}_i \times \hat{\epsilon}_i, \tag{5.6}$$

where the hat over ϵ_i emphasises that the photon polarization vector is a unit vector. Moreover, it follows from the Coulomb gauge that we can choose the first component of the photon polarization vector to be zero, i.e. $\epsilon_i^0 \equiv 0$, from which it immediately follows that $\hat{\epsilon}_i \cdot \hat{k}_i = 0$. And since ϵ_i is normalized to unity we can see that $\hat{\delta}_i$ is also a unit vector. Before evaluating a_{123} further, let us explicitly state the matrix structure of the terms appearing in equation 5.4:

$$\begin{aligned} \not{\epsilon}_i &= \begin{pmatrix} 0_2 & -\hat{\epsilon}_i \cdot \vec{\sigma} \\ \hat{\epsilon}_i \cdot \vec{\sigma} & 0_2 \end{pmatrix}; \quad \not{k}_i = \begin{pmatrix} \omega_i \mathbb{1}_2 & -\vec{k}_i \cdot \vec{\sigma} \\ \vec{k}_i \cdot \vec{\sigma} & -\omega_i \mathbb{1}_2 \end{pmatrix}; \\ -\not{p} + m &= 2m \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix}; \quad \not{p} + m = 2m \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \end{aligned} \tag{5.7}$$

and note that

$$(k_i - p)^2 - m^2 \equiv (p - k_i)^2 - m^2 = p^2 - 2p \cdot k_i + k_i^2 - m^2 = -2\omega_i m. \tag{5.8}$$

We can then break a_{123} down into $I_1 \not{\epsilon}_2 I_2$ and process further:

$$\begin{aligned} I_1 &= \begin{pmatrix} 0_2 & \mathbb{1}_2 \end{pmatrix} \not{\epsilon}_1 \left(\frac{\not{k}_1 - \not{p} + m}{(k_1 - p)^2 - m^2} \right) = \begin{pmatrix} \hat{\epsilon}_1 \cdot \vec{\sigma} & 0 \end{pmatrix} \frac{1}{-2\omega_1 m} \left(\not{k}_1 - \not{p} + m \right) \\ &= \frac{1}{-2\omega_1 m} \begin{pmatrix} \omega_1 \hat{\epsilon}_1 \cdot \vec{\sigma} & i\omega_1 \hat{\delta}_1 \cdot \vec{\sigma} \end{pmatrix} = -\frac{1}{2m} \begin{pmatrix} \hat{\epsilon}_1 \cdot \vec{\sigma} & i\hat{\delta}_1 \cdot \vec{\sigma} \end{pmatrix}, \end{aligned} \tag{5.9}$$

where we have used the results

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = (\vec{a} \cdot \vec{b})\mathbb{1}_2 + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}; \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}; \quad \omega_i \hat{\delta}_i = \vec{k}_i \times \hat{\epsilon}_i. \tag{5.10}$$

We may similarly develop the remaining terms:

$$\begin{aligned} I_2 &= \left(\frac{\not{p} - \not{k}_3 + m}{(p - k_3)^2 - m^2} \right) \not{\epsilon}_3 \begin{pmatrix} \mathbb{1}_2 \\ 0_2 \end{pmatrix} = \frac{1}{-2\omega_3 m} (\not{p} + m - \not{k}_3) \begin{pmatrix} 0 \\ \hat{\epsilon}_3 \cdot \vec{\sigma} \end{pmatrix} \\ &= -\frac{1}{-2\omega_3 m} \begin{pmatrix} i\omega_3 \hat{\delta}_3 \cdot \vec{\sigma} \\ \omega_3 \hat{\epsilon}_3 \cdot \vec{\sigma} \end{pmatrix} = -\frac{1}{2m} \begin{pmatrix} i\hat{\delta}_3 \cdot \vec{\sigma} \\ \hat{\epsilon}_3 \cdot \vec{\sigma} \end{pmatrix}. \end{aligned} \tag{5.11}$$

We can then multiply everything together to get

$$a_{123} = -\frac{1}{4m^2}\Big((\vec{\sigma}\cdot\hat{\epsilon}_1)(\vec{\sigma}\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\epsilon}_3) + (\vec{\sigma}\cdot\hat{\delta}_1)(\vec{\sigma}\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\delta}_3)\Big)\,. \tag{5.12}$$

This is now the (123) permutation of the Feynman graphs. The full invariant amplitude contains five more terms such as this. Essentially nothing but the labeling of the indices changes for the remaining five terms. However, let us still show this explicitly for the permutation a_{132}. By looking at figure 5.2 and comparing it with figure 5.1, we can see that the indices 2 and 3 are swapped. The first electron propagator stays the same, the photon polarization vectors ϵ_2 and ϵ_3 change places and in the second electron propagator we get k_2 instead of k_3. So by looking at equations 5.2, 5.3 and 5.4 we can see that a_{132} has the following form:

$$a_{132} = \begin{pmatrix} 0_2 & \mathbb{1}_2 \end{pmatrix} \not{\epsilon}_1 \left(\frac{\not{k}_1 - \not{p} + m}{(k_1-p)^2 - m^2}\right) \not{\epsilon}_3 \left(\frac{\not{p} - \not{k}_2 + m}{(p-k_2)^2 - m^2}\right) \not{\epsilon}_2 \begin{pmatrix} \mathbb{1}_2 \\ 0_2 \end{pmatrix}. \tag{5.13}$$

We can then use the results of equations 5.7-5.11 with the appropriate swapping of the indices 3 and 2. With these replacements we can multiply $a_{132} = I_1\not{\epsilon}_3 I_2$ to get

$$\begin{aligned} a_{132} &= -\frac{1}{2m}\begin{pmatrix} \hat{\epsilon}_1\cdot\vec{\sigma} & i\hat{\delta}_1\cdot\vec{\sigma} \end{pmatrix} \begin{pmatrix} 0_2 & -\hat{\epsilon}_3\cdot\vec{\sigma} \\ \hat{\epsilon}_3\cdot\vec{\sigma} & 0_2 \end{pmatrix} - \frac{1}{2m}\begin{pmatrix} i\hat{\delta}_2\cdot\vec{\sigma} \\ \vec{\epsilon}_2\cdot\vec{\sigma} \end{pmatrix} \\ &= \frac{1}{4m^2}\begin{pmatrix} i(\hat{\delta}_1\cdot\vec{\sigma})(\hat{\epsilon}_3\cdot\vec{\sigma}) & -(\hat{\epsilon}_1\cdot\vec{\sigma})(\hat{\epsilon}_3\cdot\vec{\sigma}) \end{pmatrix} \begin{pmatrix} i\hat{\delta}_2\cdot\vec{\sigma} \\ \vec{\epsilon}_2\cdot\vec{\sigma} \end{pmatrix} \\ &= -\frac{1}{4m^2}\Big((\hat{\epsilon}_1\cdot\vec{\sigma})(\hat{\epsilon}_3\cdot\vec{\sigma})(\hat{\epsilon}_2\cdot\vec{\sigma}) + (\hat{\delta}_1\cdot\vec{\sigma})(\hat{\epsilon}_3\cdot\vec{\sigma})(\hat{\delta}_2\cdot\vec{\sigma})\Big)\,, \end{aligned} \tag{5.14}$$

which has the same form as equation 5.12. It should then be obvious that the remaining terms go identically. In particular, this allows us to see what the permutation sum in equation 5.5 actually means.

Applying the results of equation 5.10 to the permutation sum would leave us with terms proportional to the unit matrix $\mathbb{1}_2$ and to the Pauli spin matrices σ^i. Taking for example a_{123}:

$$\begin{aligned} a_{123} = -\frac{1}{4m^2}\Big[&(\hat{\epsilon}_1\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\epsilon}_3) + i\Big([(\hat{\epsilon}_1\times\hat{\epsilon}_2)\cdot\hat{\epsilon}_3]\mathbb{1}_2 + i[(\hat{\epsilon}_1\times\hat{\epsilon}_2)\times\hat{\epsilon}_3]\cdot\vec{\sigma}\Big) \\ &+ (\hat{\delta}_1\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\delta}_3) + i\Big([(\hat{\delta}_1\times\hat{\epsilon}_2)\cdot\hat{\delta}_3]\mathbb{1}_2 + i[(\hat{\delta}_1\times\hat{\epsilon}_2)\times\hat{\delta}_3]\cdot\vec{\sigma}\Big)\Big]. \end{aligned} \tag{5.15}$$

Keeping this in mind, we can start to inspect the trace of the permutation sum over the 2×2 matrices a_{123}. First we note that for any square matrices A and B and a constant $c \in \mathbb{C}$ it follows that $\mathrm{Tr}(c(A+B)) = c\mathrm{Tr}(A) + c\mathrm{Tr}(B)$, i.e. trace is a linear mapping. We can then write

$$\sum_{\text{permut}} \mathrm{Tr}(a_{123}) = -\frac{1}{4m^2} \sum_{\text{permut}} \mathrm{Tr}\left((\vec{\sigma}\cdot\hat{\epsilon}_1)(\vec{\sigma}\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\epsilon}_3) + (\vec{\sigma}\cdot\hat{\delta}_1)(\vec{\sigma}\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\delta}_3)\right). \tag{5.16}$$

Then by using the result $\mathrm{Tr}(\sigma^i\sigma^j\sigma^k) = 2i\epsilon^{ijk}$, which is justified in appendix C and where ϵ^{ijk} is the Levi-Civita symbol, we can write

$$\begin{aligned} \sum_{\text{permut}} \mathrm{Tr}(a_{123}) &= \frac{-1}{4m^2} \sum_{\text{permut}} \mathrm{Tr}(\sigma^i\hat{\epsilon}_1^i\sigma^j\hat{\epsilon}_2^j\sigma^k\hat{\epsilon}_3^k) + \mathrm{Tr}(\sigma^i\hat{\delta}_1^i\sigma^j\hat{\epsilon}_2^j\sigma^k\hat{\delta}_3^k) \\ &= \frac{-1}{4m^2} \sum_{\text{permut}} \left(\hat{\epsilon}_1^i\hat{\epsilon}_2^j\hat{\epsilon}_3^k \underbrace{\mathrm{Tr}(\sigma^i\sigma^j\sigma^k)}_{=2i\epsilon^{ijk}} + \hat{\delta}_1^i\hat{\epsilon}_2^k\hat{\delta}_3^k \underbrace{\mathrm{Tr}(\sigma^i\sigma^j\sigma^k)}_{=2i\epsilon^{ijk}}\right) \\ &= \frac{-i}{2m^2} \sum_{\text{permut}} \left(\hat{\epsilon}_1\cdot(\hat{\epsilon}_2\times\hat{\epsilon}_3) + \hat{\delta}_1\cdot(\hat{\epsilon}_2\times\hat{\delta}_3)\right). \end{aligned} \tag{5.17}$$

With $\hat{\epsilon}_1\cdot(\hat{\epsilon}_2\times\hat{\epsilon}_3) = -\hat{\epsilon}_1\cdot(\hat{\epsilon}_3\times\hat{\epsilon}_2)$, we notice that the epsilon triple product terms are completely cancelled. The odd permutations cancel out the even permutations. Similarly with $\hat{\delta}_1\cdot(\hat{\epsilon}_2\times\hat{\delta}_3) = -\hat{\delta}_3\cdot(\hat{\epsilon}_2\times\hat{\delta}_1)$ we notice that the terms where the middle polarization vector is the same are cancelled out completely. This could also be though through the graphs, where figure 5.1 would get cancelled out by figure 5.5, and so forth. With this we see that the trace over the permutations really goes to zero, $\sum_{\text{permut}} \mathrm{Tr}(a_{123}) = 0$.

Then as pointed above, the form of the a_{123} matrices is $a\mathbb{1}_2 + b\sigma^i$, where $a,b \in \mathbb{C}$. With this we have

$$\sum_{\text{permut}} \mathrm{Tr}(a_{123}) = \mathrm{Tr}(a\mathbb{1}_2 + b\sigma_i) = a\mathrm{Tr}(\mathbb{1}_2) + b\mathrm{Tr}(\sigma_i) = 0, \tag{5.18}$$

which forces $a = 0$. We may then get rid of all the terms proportional to the unit matrix. Then equation 5.15 reduces to

$$\begin{aligned} a_{123} = \frac{1}{4m^2}\Big[&(\hat{\epsilon}_1\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\epsilon}_3) - [(\hat{\epsilon}_1\times\hat{\epsilon}_2)\times\hat{\epsilon}_3]\cdot\vec{\sigma} \\ &+ (\hat{\delta}_1\cdot\hat{\epsilon}_2)(\vec{\sigma}\cdot\hat{\delta}_3) - [(\hat{\delta}_1\times\hat{\epsilon}_2)\times\hat{\delta}_3]\cdot\vec{\sigma}\Big]. \end{aligned} \tag{5.19}$$

This obviously holds also for the other permutations of a_{123}. Then from the form of equation 5.19, it is evident that the permutation sum over a_{123} has the form $\vec{\sigma}\cdot\vec{V}$.

We may thus write as follows:

$$\sum_{\text{permut}} a_{123} = \frac{-1}{4m^2}\vec{\sigma} \cdot \underbrace{\sum_{\text{permut}} \left(\hat{\epsilon}_3(\hat{\epsilon}_1 \cdot \hat{\epsilon}_2) - (\hat{\epsilon}_1 \times \hat{\epsilon}_2) \times \hat{\epsilon}_3 + \hat{\delta}_3(\hat{\delta}_1 \cdot \hat{\epsilon}_2) - (\hat{\delta}_1 \times \hat{\epsilon}_2) \times \hat{\delta}_3 \right)}_{=\vec{V}} . \tag{5.20}$$

We may then inspect the form of $\vec{V}$ further by using the triple product identity:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}). \tag{5.21}$$

With it, we can open up the triple products and write them in terms of the dot products. Some of the terms cancel out with each other and some of them can be added together because of the commutativity of the dot product e.g. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. By meticulously keeping track of the terms we arrive at the result

$$\sum_{\text{permut}} a_{123} = \vec{\sigma} \cdot \vec{V}, \tag{5.22}$$

where

$$\vec{V} = -\frac{1}{2m^2} \sum_{\text{cyclic}} \left(\hat{\epsilon}_1(\hat{\epsilon}_2 \cdot \hat{\epsilon}_3 - \hat{\delta}_2 \cdot \hat{\delta}_3) + \hat{\delta}_1(\hat{\epsilon}_2 \cdot \hat{\delta}_3 + \hat{\epsilon}_3 \cdot \hat{\delta}_2) \right) . \tag{5.23}$$

Now the sum contains only the cyclic permutations, i.e. (123), (231) and (312).
Up to this point we have hold on to the notation that the polarization vector is denoted with a hat on top of the epsilon. Now, for notational ease, we shall drop the hat and denote the three vectors with bold letters, e.g. $\boldsymbol{\epsilon}$. It is then obvious from the context which vectors are unit vectors.
It is beneficial to define complex vectors $\boldsymbol{\epsilon}_i^{\pm} = \boldsymbol{\epsilon}_i \pm i\boldsymbol{\delta}_i$. We immediately see that the following holds:

$$\begin{aligned}
&(1) \quad (\boldsymbol{\epsilon}^{\pm})^2 = (\boldsymbol{\epsilon}_i \pm i\boldsymbol{\delta}_i) \cdot (\boldsymbol{\epsilon}_i \pm i\boldsymbol{\delta}_i) = \underbrace{\boldsymbol{\epsilon}_i^2}_{=1} + 2i \underbrace{\boldsymbol{\epsilon}_i \cdot \boldsymbol{\delta}_i}_{=0} - \underbrace{\boldsymbol{\delta}_i^2}_{=1} = 0 \\
&(2) \quad \boldsymbol{\epsilon}_i^{+} \cdot \boldsymbol{\epsilon}_i^{-} = (\boldsymbol{\epsilon}_i + i\boldsymbol{\delta}_i) \cdot (\boldsymbol{\epsilon}_i - i\boldsymbol{\delta}_i) = \boldsymbol{\epsilon}_i^2 + \boldsymbol{\delta}_i^2 = 2 \\
&(3) \quad \boldsymbol{\epsilon}_i^{\pm} \cdot \boldsymbol{\epsilon}_j^{\pm} = (\boldsymbol{\epsilon}_i \pm i\boldsymbol{\delta}_i) \cdot (\boldsymbol{\epsilon}_j \pm i\boldsymbol{\delta}_j) = \boldsymbol{\epsilon}_i \cdot \boldsymbol{\epsilon}_j - \boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j \pm i(\boldsymbol{\epsilon}_i \cdot \boldsymbol{\delta}_j + \boldsymbol{\epsilon}_j \cdot \boldsymbol{\delta}_i).
\end{aligned} \tag{5.24}$$

Then from

$$\begin{aligned}
\boldsymbol{\epsilon}_1^{+}(\boldsymbol{\epsilon}_2^{-} \cdot \boldsymbol{\epsilon}_3^{-}) + \boldsymbol{\epsilon}_1^{-}(\boldsymbol{\epsilon}_2^{+} \cdot \boldsymbol{\epsilon}_3^{+}) =& (\boldsymbol{\epsilon}_1 + i\boldsymbol{\delta}_1)(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_3 - \boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_3 - i(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\delta}_3 + \boldsymbol{\epsilon}_3 \cdot \boldsymbol{\delta}_2) \\
&+ (\boldsymbol{\epsilon}_1 - i\boldsymbol{\delta}_1)(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_3 - \boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_3 + i(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\delta}_3 + \boldsymbol{\epsilon}_3 \cdot \boldsymbol{\delta}_2) \\
=& 2\boldsymbol{\epsilon}_1(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\epsilon}_3 - \boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_3) + 2\boldsymbol{\delta}_1(\boldsymbol{\epsilon}_2 \cdot \boldsymbol{\delta}_3 + \boldsymbol{\epsilon}_3 \cdot \boldsymbol{\delta}_2),
\end{aligned} \tag{5.25}$$

it follows that equation 5.23 can be written as

$$\boldsymbol{V} = -\frac{1}{4m^2}\sum_{\text{cyclic}}\left(\boldsymbol{\epsilon}_1^+(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-)+\boldsymbol{\epsilon}_1^-(\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^+)\right). \tag{5.26}$$

We remind the reader that up to this point we have only dealt with the 2×2-matrix $\sum_{\text{permut}} a_{123}$ inside equation 5.5. Since we know that the structure is of the form $\boldsymbol{\sigma}\cdot\boldsymbol{V}$, we may write

$$\boldsymbol{\sigma}\cdot\boldsymbol{V} = \sigma^1V^1+\sigma^2V^2+\sigma^3V^3 = \begin{pmatrix} V^3 & V^1-iV^2 \\ V^1+iV^2 & -V^3\end{pmatrix}, \tag{5.27}$$

where the components of $\boldsymbol{V} = (V^1,V^2,V^3)$ can be read from equation 5.26. With this development we may finally start to consider what happens to the square of the total invariant amplitude $\mathcal{M}^{tot}_{S=1}$.

5.2 Squaring the invariant amplitude

Since O-PS is in the triplet configuration, we get the same three terms as we had with the J/ψ in section 3. Moreover, the indices denoting the helicities are in the order s_1s_2, e.g. $\mathcal{M}_{S=1;s_1s_2}$. With these we get the following:

$$\begin{aligned}
-i\mathcal{M}_{S=1;\uparrow\uparrow} &= i2me^3\chi_\uparrow^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{V}\phi_\uparrow = ie^3\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix} V^3 & V^1-iV^2 \\ V^1+iV^2 & -V^3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= i2me^3(V^1+iV^2),\\
-i\mathcal{M}_{S=1;\downarrow\downarrow} &= i2me^3\chi_\downarrow^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{V}\phi_\downarrow = ie^3\begin{pmatrix}-1 & 0\end{pmatrix}\begin{pmatrix} V^3 & V^1-iV^2 \\ V^1+iV^2 & -V^3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\
&= i2me^3(-V^1+iV^2),\\
-i\mathcal{M}_{S=1;\uparrow\downarrow} &= i2me^3\begin{pmatrix}-1 & 0\end{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{V}\begin{pmatrix}1\\0\end{pmatrix} = -i2me^3V^3,\\
-i\mathcal{M}_{S=1;\downarrow\uparrow} &= i2me^3\begin{pmatrix}0 & 1\end{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{V}\begin{pmatrix}0\\1\end{pmatrix} = -i2me^3V^3,\\
\frac{1}{\sqrt{2}}\Big((-i\mathcal{M}_{S=1;\uparrow\downarrow})+(-i\mathcal{M}_{S=1;\downarrow\uparrow})\Big) &= -\frac{i2me^3}{\sqrt{2}}2V^3 = -i2me^3\sqrt{2}V^3.
\end{aligned} \tag{5.28}$$

Then in forming the squared invariant amplitude, we will sum over the photon polarization states and average over the initial positronium spin states. This averaging gives us a factor of 1/3 in front. This is to say that we are interested in the unpolarized decay width of the O-Ps. By looking at the results of 5.28 we can readily see the form of the squared invariant amplitude $|\overline{\mathcal{M}^{tot}_{S=1}}|^2$:

$$\begin{aligned}|\overline{\mathcal{M}^{tot}_{S=1}}|^2 &= \frac{1}{3}\left(|\mathcal{M}_{S=1,\uparrow\uparrow}|^2 + |\mathcal{M}_{S=1,\downarrow\downarrow}|^2 + \frac{1}{2}|(-i\mathcal{M}_{S=1;\uparrow\downarrow}) + (-i\mathcal{M}_{S=1;\downarrow\uparrow})|^2\right) \\ &= 4m^2\frac{1}{3}\sum_{\gamma\ \text{polar}}\left(2e^6V_1^2 + 2e^6V_2^2 + 2e^6V_3^2\right),\end{aligned} \tag{5.29}$$

and since $\boldsymbol{V}$ has a three-vector structure we may write

$$|\overline{\mathcal{M}^{tot}_{s=1}}|^2 = 4m^2e^6\frac{2}{3}\sum_{\gamma\ \text{polar}}\mathbf{V}^2, \tag{5.30}$$

where $\mathbf{V}^2 = \boldsymbol{V}\cdot\boldsymbol{V}$. It should also be stressed out that we now have two different sums over which we are calculating. The cyclic permutation sum within $\boldsymbol{V}$ and the sum over the photon polarizations in the squared total invariant amplitude. Next we need to figure out the form of $\mathbf{V}^2$ from equation 5.26,

$$\boldsymbol{V}^2 = \left(\frac{-1}{4m^2}\right)^2\sum_{\text{cyclic}}\left(\boldsymbol{\epsilon}_1^+(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-) + \boldsymbol{\epsilon}_1^-(\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^+)\right)\cdot\sum_{\text{cyclic}}\left(\boldsymbol{\epsilon}_1^+(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-) + \boldsymbol{\epsilon}_1^-(\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^+)\right). \tag{5.31}$$

In each sum we have six terms, so altogether by opening up $\boldsymbol{V}^2$ we get 36 terms. However, by remembering that $(\boldsymbol{\epsilon}^\pm)^2 = 0$ we immediately see that six of these terms are zero. And since $\boldsymbol{\epsilon}_i^+$ is the complex conjugate of $\boldsymbol{\epsilon}_i^-$, we can denote $(\boldsymbol{\epsilon}_i^-\cdot\boldsymbol{\epsilon}_j^-)(\boldsymbol{\epsilon}_i^+\cdot\boldsymbol{\epsilon}_j^+) = |\boldsymbol{\epsilon}_i^+\cdot\boldsymbol{\epsilon}_j^+|^2$. Then by meticulously keeping track of the vectors we get

$$\begin{aligned}\mathbf{V}^2 = \frac{2}{16m^4}\sum_{\text{cyclic}}\Big(&2|\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^-|^2 + (\boldsymbol{\epsilon}_1^+\cdot\boldsymbol{\epsilon}_2^+)(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^-\cdot\boldsymbol{\epsilon}_1^-) + (\boldsymbol{\epsilon}_1^-\cdot\boldsymbol{\epsilon}_2^-)(\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^+)(\boldsymbol{\epsilon}_3^+\cdot\boldsymbol{\epsilon}_1^+) \\ &+ (\boldsymbol{\epsilon}_1^+\cdot\boldsymbol{\epsilon}_2^-)(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^+\cdot\boldsymbol{\epsilon}_1^+) + (\boldsymbol{\epsilon}_1^-\cdot\boldsymbol{\epsilon}_2^+)(\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^+)(\boldsymbol{\epsilon}_3^-\cdot\boldsymbol{\epsilon}_1^-)\Big).\end{aligned} \tag{5.32}$$

Notice in particular that the last two terms appearing on the first line are complex conjugates of each other. The same holds for the terms on the second line. Since $2\text{Re}(c) = (c + c^*)$ for all $c \in \mathbb{C}$, we can write the equation 5.32 in the form

$$\begin{aligned}\mathbf{V}^2 = \frac{1}{4m^4}\sum_{\text{cyclic}}\Big(&|\boldsymbol{\epsilon}_2^+\cdot\boldsymbol{\epsilon}_3^-|^2 + \text{Re}[(\boldsymbol{\epsilon}_1^+\cdot\boldsymbol{\epsilon}_2^+)(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^-\cdot\boldsymbol{\epsilon}_1^-)] \\ &+ \text{Re}[(\boldsymbol{\epsilon}_1^+\cdot\boldsymbol{\epsilon}_2^-)(\boldsymbol{\epsilon}_2^-\cdot\boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^+\cdot\boldsymbol{\epsilon}_1^+)]\Big).\end{aligned} \tag{5.33}$$

Up to this point the development in regards to the real polarization vectors has been completely general. Next we have to choose their form. From conservation of momentum, it is clear that the spatial momentum vectors $\boldsymbol{k}_1$,$\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ define a reaction plane. The polarization vectors can then be chosen to be into the direction of the normal $\boldsymbol{n}$ to this plane or to lie in the reaction plane but so that they are perpendicular to the momentum vectors. This is illustrated in figure 5.7.

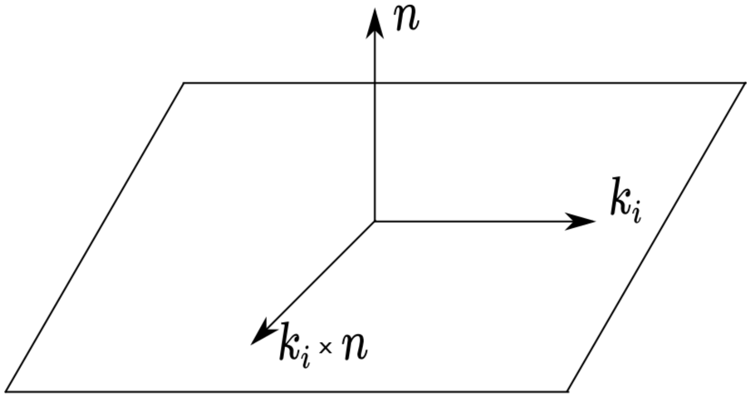


Figure 5.7. The reaction plane defined by the momentum vectors $\boldsymbol{k}_i$. Then related to each momentum vector $\boldsymbol{k}_i$ there are two possible polarization vectors $\boldsymbol{n}$ and $\boldsymbol{k}_i \times \boldsymbol{n}$.

Note that we may choose these vectors to be unit vectors, i.e. all of the vectors $\boldsymbol{\epsilon}_i, \boldsymbol{n}, \hat{\boldsymbol{k}}_i$ and $\boldsymbol{\delta}_i$ are unit vectors. We can then label these vectors as given below:

$$\begin{aligned} &(a) \quad \boldsymbol{\epsilon}_i = \boldsymbol{n}; \quad \boldsymbol{\delta}_i = \hat{\boldsymbol{k}}_i \times \boldsymbol{n}; \quad \boldsymbol{\epsilon}_i^+ = \boldsymbol{\alpha}_i = \boldsymbol{n} + i\hat{\boldsymbol{k}}_i \times \boldsymbol{n}; \quad \boldsymbol{\epsilon}_i^- = \boldsymbol{\alpha}_i^* = \boldsymbol{n} - i\hat{\boldsymbol{k}}_i \times \boldsymbol{n}_i \\ &(b) \quad \boldsymbol{\epsilon}_i = \hat{\boldsymbol{k}}_i \times \boldsymbol{n}; \quad \boldsymbol{\delta}_i = -\boldsymbol{n}; \quad \boldsymbol{\epsilon}_i^+ = -i\boldsymbol{\alpha}_i; \quad \boldsymbol{\epsilon}_i^- = i\boldsymbol{\alpha}_i^*, \end{aligned} \tag{5.34}$$

where * denotes complex conjugate. Note that these choices are consistent with our earlier definitions of $\boldsymbol{\epsilon}_i^\pm$ and $\boldsymbol{\delta}_i$. Notice also that since $\boldsymbol{\epsilon}_i$ can be defined in two different ways, it automatically follows that $\boldsymbol{\delta}$ can also be defined in two different ways.

Next we will sum over the photon polarizations. This sum consists of the permutations of the two choices of equation 5.34 for the (123). Again for notational convenience, let us denote the polarization vectors with $\boldsymbol{\alpha}_i$ as given in equation 5.34. Then, if we choose all the polarization vectors to be according to the option (a), we denote this by $(123) \to (aaa)$ and so forth. The end result is that only the absolute value square term will contribute. This is easy to understand by looking at equation 5.34

as the only difference between these two choices is the sign and the imaginary unit. However, let us see how the last two terms will disappear:

Permutation	$(\boldsymbol{\epsilon}_1^+ \cdot \boldsymbol{\epsilon}_2^+)(\boldsymbol{\epsilon}_2^- \cdot \boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^- \cdot \boldsymbol{\epsilon}_1^-)$
$(123) \rightarrow (aaa)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (baa)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (aba)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (aab)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (abb)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (bab)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (bba)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$
$(123) \rightarrow (bbb)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3^* \cdot \boldsymbol{\alpha}_1^*)$

Permutation	$(\boldsymbol{\epsilon}_1^+ \cdot \boldsymbol{\epsilon}_2^-)(\boldsymbol{\epsilon}_2^- \cdot \boldsymbol{\epsilon}_3^-)(\boldsymbol{\epsilon}_3^+ \cdot \boldsymbol{\epsilon}_1^+)$
$(123) \rightarrow (aaa)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (baa)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (aba)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (aab)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (abb)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (bab)$	$-2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (bba)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$
$(123) \rightarrow (bbb)$	$+2(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2^*)(\boldsymbol{\alpha}_2^* \cdot \boldsymbol{\alpha}_3^*)(\boldsymbol{\alpha}_3 \cdot \boldsymbol{\alpha}_1)$

We can then clearly see that summing all these together we are left with zero. Similarly, for the first term we have 8 possible choices and thus equation 5.33 reduces down to

$$\sum_{\gamma\ \text{polar}} \boldsymbol{V}^2 = \frac{1}{8m^4} \sum_{\text{cyclic}} 2|\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_3|^2 \cdot 8 = \frac{2}{m^4} \sum_{\text{cyclic}} |\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_3|^2, \tag{5.35}$$

which we can substitute into equation 5.30 to get

$$|\overline{\mathcal{M}_{s=1}^{tot}}(e^- e^+ \rightarrow \gamma\gamma\gamma)|^2 = 4m^2 e^6 \frac{4}{3m^4} \sum_{\text{cyclic}} |\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_3|^2. \tag{5.36}$$

We can still process the cyclic sum further by using 5.34. Take for example $\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_3$:

$$\begin{aligned}\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_3 &= (\boldsymbol{n} + i\hat{\boldsymbol{k}}_2 \times \boldsymbol{n}) \cdot (\boldsymbol{n} + i\hat{\boldsymbol{k}}_3 \times \boldsymbol{n}) \\ &= \boldsymbol{n}^2 + i\underbrace{\boldsymbol{n} \cdot (\hat{\boldsymbol{k}}_3 \times \boldsymbol{n})}_{=0} + i\underbrace{(\hat{\boldsymbol{k}}_2 \times \boldsymbol{n}) \cdot \boldsymbol{n}}_{=0} - (\hat{\boldsymbol{k}}_2 \times \boldsymbol{n}) \cdot (\hat{\boldsymbol{k}}_3 \times \boldsymbol{n}) \\ &= 1 - \hat{\boldsymbol{k}}_2 \cdot \hat{\boldsymbol{k}}_3 = 1 - \cos\theta_{23},\end{aligned} \tag{5.37}$$

where θ_{23} is the angle between the vectors $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$. In equation 5.37 we have used the cyclicity of the triple product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, the fact that $\boldsymbol{a} \times \boldsymbol{a} = 0$ for any vector $\boldsymbol{a}$ and the identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}). \tag{5.38}$$

With this we get our final form for the squared invariant amplitude to be

$$\begin{aligned}|\overline{\mathcal{M}^{tot}_{S=1}}(e^- e^+ \to \gamma\gamma\gamma)|^2 &= 4m^2 e^6 \frac{4}{3m^4} \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2 \\ &= 4m^2 e^6 \frac{4}{3m^4} \left[(1 - \cos\theta_{23})^2 + (1 - \cos\theta_{31})^2 + (1 - \cos\theta_{12})^2\right].\end{aligned} \tag{5.39}$$

5.3 Phase space integration

We have now derived the final form for the squared invariant amplitude for the free-particle process. Then according to equation 2.9 we have to multiply this result by $(2/M)|\psi(\mathbf{r} = 0)|^2$ to get the squared invariant amplitude for the O-Ps decay. Moreover, we need to divide the phase space integral by $3! = 6$ because we have three identical particles in the final state. Substituting all this into equation 2.2 we get

$$\begin{aligned}\Gamma(\text{O-Ps} \to \gamma\gamma\gamma) = \frac{1}{6}\frac{1}{2M} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^3 2\omega_1 (2\pi)^3 2\omega_2 (2\pi)^3 2\omega_3} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2 - k_3) \\ \frac{2}{M}|\psi(\mathbf{r} = 0)|^2 4m^2 e^6 \frac{4}{3m^4} \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2,\end{aligned} \tag{5.40}$$

where M is the positronium mass. Then denoting $\psi(\mathbf{r} = 0) = \psi(0)$ and using $M = 2m$ and $e^2 = 4\pi\alpha$, equation 5.40 simplifies to

$$\Gamma_{\text{O-Ps}} = \frac{4\alpha^3}{9\pi^2 m^4}|\psi(0)|^2 \int \frac{d^3k_1 d^3k_2 d^3k_3}{8\omega_1\omega_2\omega_3} \delta^{(4)}(p - k_1 - k_2 - k_3) \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2. \tag{5.41}$$

From equation 2.10 we already know the value of the wavefunction at the origin,

$$|\psi(0)|^2 = \frac{1}{\pi a^3} = \frac{m^3\alpha^3}{8\pi}. \tag{5.42}$$

However, let us keep the explicit form of the square of the wavefunction still in our expression for the decay width. This is because we can use the O-Ps decay width calculation in the J/ψ decay width calculations. Then just before the final answer of the O-Ps decay width, we will substitute the result of equation 5.42.

Our next task is then to do the phase space integration. Let us denote

$$I = \int \frac{d^3k_1 d^3k_2 d^3k_3}{8\omega_1\omega_2\omega_3}\delta^{(4)}(p - k_1 - k_2 - k_3) \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2. \tag{5.43}$$

We may use the following identity

$$\int \frac{d^3k_3}{2\omega_3} = \int d^4k_3 \theta(\omega_3)\delta(k_3^2) \tag{5.44}$$

to write I as

$$I = \int \frac{d^3k_1 d^3k_2}{4\omega_1\omega_2\omega_3} d^4k_3 \delta^{(4)}(p - k_1 - k_2 - k_3)\theta(\omega_3)\delta(k_3^2) \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2. \tag{5.45}$$

Then using the fact that each term in the cyclic sum contributes equally we may write the cyclic sum as $3(1 - \cos\theta_{12})^2$ and do the k_3 integral to get

$$I = 3\int \frac{d^3k_1 d^3k_2}{4\omega_1\omega_2}\theta(\omega_3)\delta(k_3^2)(1 - \cos\theta_{12})^2\Big|_{k_3 = p - k_1 - k_2}. \tag{5.46}$$

Inside the remaining delta function we now have $(p - k_1 - k_2)^2$ which can be opened as follows:

$$\begin{aligned} k_3^2 = (p - k_1 - k_2)^2 &= p^2 + k_1^2 + k_2^2 - 2p\cdot k_1 - 2p\cdot k_2 + 2k_1\cdot k_2 \\ &= 4m^2 - 4m\omega_1 - 4m\omega_2 + 2\omega_1\omega_2(1 - \cos\theta_{12}). \end{aligned} \tag{5.47}$$

Similarly we have $\omega_3 = 2m - \omega_1 - \omega_2$ inside the theta function. Next we shall move into spherical coordinates. Since the integrand has no azimuthal angle dependence, we may trivially integrate over it. Then the polar angle has to be restricted such that momentum is conserved. Thus we get

$$\begin{aligned}
I &= 3\int \frac{4\pi\omega_1^2 d\omega_1 2\pi\omega_2^2 d\omega_2 (d\cos\theta_{12})}{4\omega_1\omega_2}\ \theta(2m-\omega_1-\omega_2) \\
&\quad \delta\left(4m^2 - 4m\omega_1 - 4m\omega_2 + 2\omega_1\omega_2(1-\cos\theta_{12})\right)(1-\cos\theta_{12})^2 \\
&= 3\pi^2 \int d\omega_1 d\omega_2 (d\cos\theta_{12})\ 2\omega_1\omega_2\ \theta(2m-\omega_1-\omega_2) \\
&\quad \frac{1}{2\omega_1\omega_2}\delta\left(\frac{2m^2}{\omega_1\omega_2} - \frac{2m}{\omega_2} - \frac{2m}{\omega_1} + 1 - \cos\theta_{12}\right)(1-\cos\theta_{12})^2 \\
&= 3\pi^2 \int d\omega_1 d\omega_2 (d\cos\theta_{12})\ \theta(2m-\omega_1-\omega_2) \\
&\quad \delta\left(\frac{2m^2}{\omega_1\omega_2} - \frac{2m}{\omega_2} - \frac{2m}{\omega_1} + 1 - \cos\theta_{12}\right)(1-\cos\theta_{12})^2.
\end{aligned} \tag{5.48}$$

The theta function ensures the conservation of energy and gives us the restriction

$$2m > \omega_1 + \omega_2. \tag{5.49}$$

Moreover, since the photons are real, we have $\omega_1, \omega_2 > 0$. Also we see that the delta function makes the $d\cos\theta_{12}$ integration trivial and imposes the restriction

$$\cos\theta_{12} = \frac{2m}{\omega_1\omega_2} - \frac{2m}{\omega_2} - \frac{2m}{\omega_1} + 1. \tag{5.50}$$

Since $-1 \leq \cos\theta \leq 1$ we get

$$\begin{aligned}
1 \geq \frac{2m}{\omega_1\omega_2} - \frac{2m}{\omega_2} - \frac{2m}{\omega_1} + 1 \quad &\& \quad -1 \leq \frac{2m}{\omega_1\omega_2} - \frac{2m}{\omega_2} - \frac{2m}{\omega_1} + 1 \\
\Rightarrow \quad \frac{1}{\omega_2} + \frac{1}{\omega_1} \geq \frac{m}{\omega_1\omega_2} \quad &\& \quad 0 \leq 2m^2 - 2\omega_1 m - 2\omega_2 m + 2\omega_1\omega_2 \\
\Rightarrow \quad \omega_1 + \omega_2 \geq m \quad &\& \quad 0 \leq (m-\omega_1)(m-\omega_2).
\end{aligned} \tag{5.51}$$

Now if both ω_1 and ω_2 would be larger than m then both of the above delta function requirements would be fulfilled. However, equation 5.49 forbids this case. Therefore we are left with the conditions

$$0 \leq \omega_1, \omega_2 \leq m \quad \& \quad m - \omega_1 \leq \omega_2. \tag{5.52}$$

With the above results, we can write equation 5.48 as

$$I = 3\pi^2 \int_0^m d\omega_1 \int_{m-\omega_1}^m d\omega_2 (1-\cos\theta_{12})^2, \tag{5.53}$$

where

$$1 - \cos\theta_{12} = 2m\frac{\omega_1 + \omega_2 - m}{\omega_1\omega_2}. \tag{5.54}$$

With a change of variables $\omega_1/m = x$ and $\omega_2/m = y$, we may write the integral as

$$I = 3\pi^2 4m^2 \int\limits_0^1 dx \int\limits_{1-x}^1 dy \frac{(x+y-1)^2}{x^2 y^2}. \tag{5.55}$$

The first integral with respect to y is a trivial one, giving

$$I = 12\pi^2 m^2 \int\limits_0^1 \frac{dx}{x^2} \left[x(1-x) + x + 2(1-x)\ln(1-x) \right]. \tag{5.56}$$

In order to solve this we make the power series substitution for the logarithm,

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}. \tag{5.57}$$

By inserting this power series into the equation 5.56, we can calculate further as shown below:

$$\begin{aligned} I =& 12\pi^2 m^2 \int\limits_0^1 dx \left[\frac{x-x^2}{x^2} + \frac{x}{x^2} - 2\frac{(1-x)}{x^2} \sum_{k=1}^{\infty} \frac{x^k}{k} \right] \\ =& 12\pi^2 m^2 \int\limits_0^1 dx \left[\frac{2-x}{x} - 2\sum_{k=1}^{\infty} \frac{(1-x)x^{k-2}}{k} \right]. \end{aligned} \tag{5.58}$$

The first few terms of the sum are

$$\frac{1-x}{x} + \frac{1-x}{2} + \frac{(1-x)x}{3} + \cdots \tag{5.59}$$

Thus we may cancel out the first term from the sum and integrate over:

$$\begin{aligned} I &= 12\pi^2 m^2 \left([x]_0^1 - 2\left[\sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{1}{k-1} x^{k-1} - \frac{1}{k} x^k \right) \right]_0^1 \right) \\ &= 12\pi^2 m^2 \left(1 - 2\sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{1}{k-1} - \frac{1}{k} \right) \right) \\ &= 12\pi^2 m^2 \left(1 - 2\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} - \frac{1}{k^2} \right) \right), \end{aligned} \tag{5.60}$$

where in the last line we have used partial fractions decomposition to write the sum in the given form. We notice that the first two terms in the sum form a telescoping sum and the last term is of the form of Riemann zeta function evaluated at two. The telescoping sum gives 1, and for the last term we may use the well known result

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1. \tag{5.61}$$

Plugging all this into equation 5.60 we get:

$$I = 12\pi^2 m^2 \left(1 - 2\left(1 - \left(\frac{\pi^2}{6} - 1\right)\right)\right) = 12\pi^2 m^2 \left(-3 + \frac{\pi^2}{3}\right) \\ = 4\pi^2 m^2 (\pi^2 - 9), \tag{5.62}$$

Substituting this into equation 5.41 we get

$$\Gamma_{\text{O-Ps}} = \frac{16\alpha^3}{9m^2} |\psi(0)|^2 (\pi^2 - 9). \tag{5.63}$$

We can then still substitute the value of equation 5.42 for the square of the wave-function to get

$$\Gamma_{\text{O-Ps}} = \frac{2m\alpha^6}{9\pi}(\pi^2 - 9). \tag{5.64}$$

Note that we can do this because we know the form of the QED potential energy. For the QCD calculations we cannot do this substitution. Equation 5.63 is the result we shall be refering to in the following J/ψ decay width calculations.

6 Decay width of $J/\psi \to ggg$

6.1 Graph contributions

The last section was more or less a prelude to this particular decay process. We will see that the calculation reduces to the one we had before, but this is by no means a trivial result. We begin by considering which of the LO Feynman diagrams contribute to this calculation. Since we have now stated multiple times that the O-Ps result can be used here, the reader might guess that the graphs which contribute to this calculation are the ones in figures 6.1-6.6.

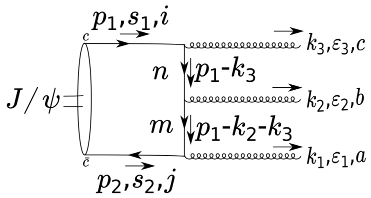


Figure 6.1.
Permutation (123).

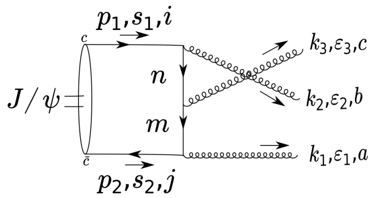


Figure 6.2.
Permutation (132).

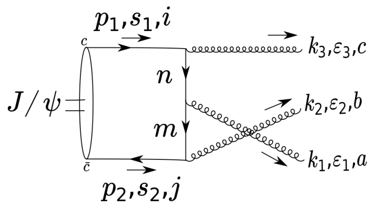


Figure 6.3.
Permutation (213).

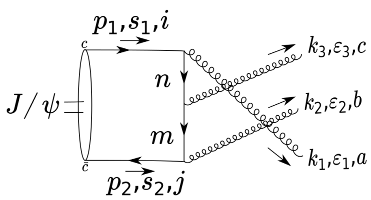


Figure 6.4.
Permutation (231).

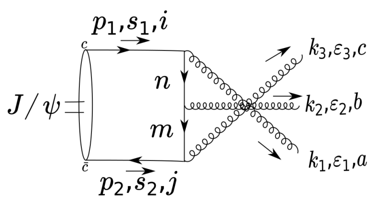


Figure 6.5.
Permutation (321).

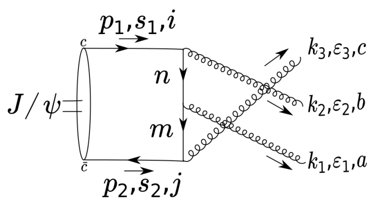


Figure 6.6.
Permutation (312).

These graphs are essentially the same which we had with the O-Ps decay calculation. The only difference is that the photon lines are replaced by gluon lines and now the outgoing gluons may also carry color. However, in QCD gluons have also three- and four-point self-couplings which gives us additional 10 graphs to consider. The s-channel graphs are presented in figures 6.7-6.10

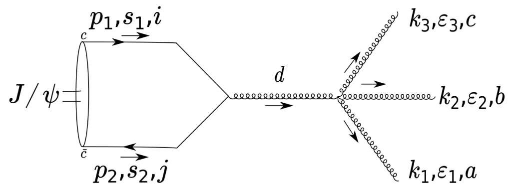


Figure 6.7. First s-channel.

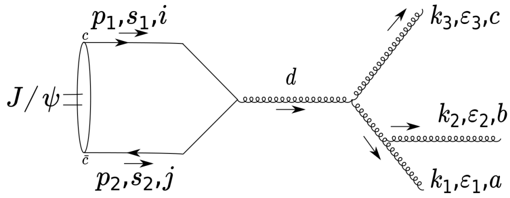


Figure 6.8. Second s-channel.

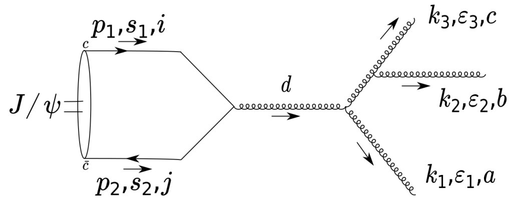


Figure 6.9. Third s-channel.

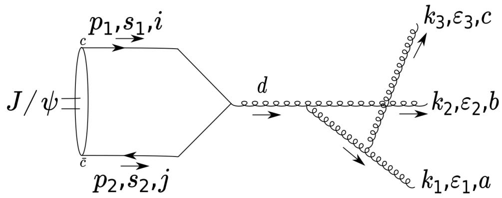


Figure 6.10. Fourth s-channel.

However, the graphs 6.7-6.10 do not contribute because at the vertex where the quarks annihilate the Feynman rules give us $-ig_s(t^d)_{ij}\gamma^\mu$. Since the J/ψ is in a color singlet state, we have $i = j$, which gives us $\mathrm{Tr}(t^d) = 0$ when we sum over the gluon colors. Thus regardless what the rest of the invariant amplitude would look like, this vertex negates contributions from these graphs. Thus, from the conservation of color, it follows that we cannot have a color singlet gluon propagator.

Finally we may also draw six diagrams where the J/ψ would first decay into one real and one virtual gluon and then the virtual one would emit the third gluon. These are given in figures 6.11-6.16.

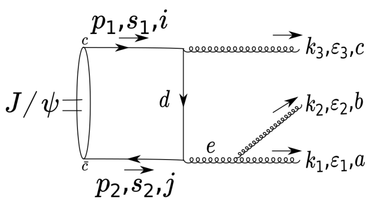


Figure 6.11.
Permutation (12)3.

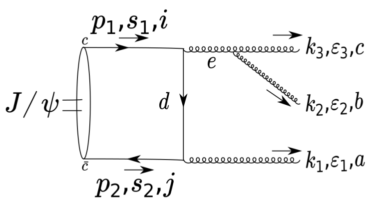


Figure 6.12.
Permutation 1(23).

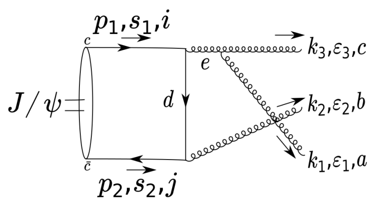


Figure 6.13.
Permutation 2(13).

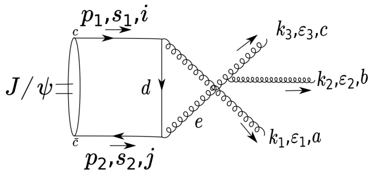


Figure 6.14.
Permutation (23)1.

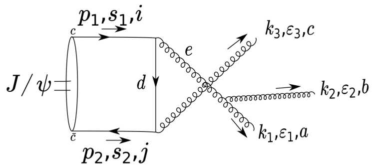


Figure 6.15.
Permutation 3(12).

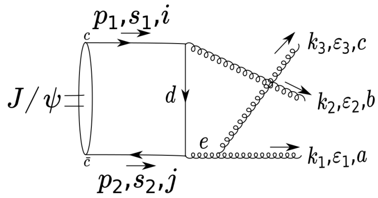


Figure 6.16.
Permutation (13)2.

Note that we have chosen to write parentheses around those two gluons which come from the same quark-gluon vertex. These six graphs in figures 6.11-6.16 are slightly trickier than the s-channel graphs. With a pure color argument, like the one before, we cannot make these graphs disappear. Neither does any argument about spin seem to help us here either. The end result, however, is that the contributions from the (12)3 permutation are exactly cancelled by the 3(12) permutation. Similarly the permutation 1(23) is cancelled by (23)1 and the permutation 2(13) is cancelled by (13)2. Let us see how this result is acquired.
Looking at figure 6.11 we can write down the invariant amplitude as

$$\begin{aligned}-i\mathcal{M}_{(12)3} =&\bar{v}_2(-ig_s(t^e)_{jd}\gamma_{\mu_4})\left(\frac{-i}{(k_1+k_2)^2}\right)\Big(-g_s f^{eab}[g^{\mu_4\mu_1}(k_1+k_2+k_1)^{\mu_2}\\ &+g^{\mu_1\mu_2}(-k_1+k_2)^{\mu_4}+g^{\mu_2\mu_4}(-k_2-(k_1+k_2))^{\mu_1}]\Big)\epsilon_{1\mu_1}\epsilon_{2\mu_2}\\ &\left(\frac{i(\not{p}_1-\not{k}_3+m)}{(p_1-k_3)^2-m^2}\right)(-ig_s(t^c)_{di})\not{\epsilon}_3 u_1.\end{aligned} \tag{6.1}$$

This can be simplified to

$$\begin{aligned}-i\mathcal{M}_{(12)3} =&\bar{v}_2 g_s^3(t^e)_{jd}(t^c)_{di}f^{eab}\frac{\gamma_{\mu_4}}{(k_1+k_2)^2}[g^{\mu_4\mu_1}(2k_1+k_2)^{\mu_2}+g^{\mu_1\mu_2}(k_2-k_1)^{\mu_4}\\ &-g^{\mu_2\mu_4}(2k_2+k_1)^{\mu_1}]\epsilon_{1\mu_1}\epsilon_{2\mu_2}\left(\frac{\not{p}_1-\not{k}_3+m}{(p_1-k_3)^2-m^2}\right)\not{\epsilon}_3 u_1.\end{aligned} \tag{6.2}$$

We can again use the fact that J/ψ is in the color singlet state, sum over the quark colors and identify the color factor in front:

$$C^{abc}=\frac{1}{\sqrt{3}}(t^e)_{id}(t^c)_{di}f^{eab}=\frac{1}{\sqrt{3}}\underbrace{\mathrm{Tr}(t^e t^c)}_{=\mathrm{T(F)}\delta^{ec}}f^{eab}=\frac{\mathrm{T(F)}}{\sqrt{3}}f^{abc}, \tag{6.3}$$

where $\mathrm{T(F)}=1/2$. Then in the Coulomb gauge where $\epsilon_i\cdot k_i=0$, we can write equation 6.2 as

$$\begin{aligned}-i\mathcal{M}_{(12)3} =& g_s^3 C^{abc}\frac{1}{(k_1+k_2)^2}\bar{v}_2[2\not{\epsilon}_1(k_1\cdot\epsilon_2)+(\epsilon_1\cdot\epsilon_2)(\not{k}_2-\not{k}_1)-2\not{\epsilon}_2(k_2\cdot\epsilon_1)]\\ &\left(\frac{\not{p}_1-\not{k}_3+m}{(p_1-k_3)^2-m^2}\right)\not{\epsilon}_3 u_1,\end{aligned} \tag{6.4}$$

where we can denote

$$\not{\kappa}_{12}\equiv[2\not{\epsilon}_1(k_1\cdot\epsilon_2)+(\epsilon_1\cdot\epsilon_2)(\not{k}_2-\not{k}_1)-2\not{\epsilon}_2(k_2\cdot\epsilon_1)], \tag{6.5}$$

to get

$$-i\mathcal{M}_{(12)3}=g_s^3C^{abc}\frac{1}{(k_1+k_2)^2}\bar{v}_2\not{\kappa}_{12}\left(\frac{\not{p}_1-\not{k}_3+m}{(p_1-k_3)^2-m^2}\right)\not{\epsilon}_3u_1. \tag{6.6}$$

We do not need the explicit form of $\not{\kappa}_{12}$ and we can simply denote $\kappa_{12}=(\kappa_{12}^0,\vec{\kappa}_{12})$. Then similarly what we did with the O-Ps calculation, we can simplify the inner 4×4 matrix structure down to a 2×2 matrix structure. We denote this inner 2×2 matrix structure similarly as before with $a_{(12)3}$. Moreover, we notice that the result of equation 5.11 can be used here:

$$\begin{aligned}a_{(12)3}&=\begin{pmatrix}0_2 & \mathbb{1}_2\end{pmatrix}\not{\kappa}_{12}\left(\frac{\not{p}_1-\not{k}_3+m}{(p_1-k_3)^2-m^2}\right)\not{\epsilon}_3\begin{pmatrix}\mathbb{1}_2\\0_2\end{pmatrix}\\ &=\begin{pmatrix}\vec{\kappa}_{12}\cdot\vec{\sigma} & -\kappa_{12}^0\mathbb{1}_2\end{pmatrix}\frac{-1}{2m}\begin{pmatrix}i\hat{\delta}_3\cdot\vec{\sigma}\\ \vec{\epsilon}_3\cdot\vec{\sigma}\end{pmatrix}\\ &=-\frac{1}{2m}\left(i(\vec{\kappa}_{12}\cdot\hat{\delta}_3)\mathbb{1}_2-(\vec{\kappa}_{12}\times\hat{\delta}_3)\cdot\vec{\sigma}-\kappa_{12}^0(\vec{\epsilon}_3\cdot\vec{\sigma})\right),\end{aligned} \tag{6.7}$$

where we have again used $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})=(\vec{a}\cdot\vec{b})\mathbb{1}_2+i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$. Next we multiply $a_{(12)3}$ with $\chi_s^\dagger$ from the left and with ϕ_s from the right. Any term proportional to the unit matrix $\mathbb{1}_2$, will give us a term with the form $C\chi_s^\dagger\phi_s$, where C is some constant. Keeping in mind equation 3.29 we can calculate:

$$\begin{aligned}&\chi_\uparrow^\dagger\phi_\uparrow=\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=0;\quad \chi_\downarrow^\dagger\phi_\downarrow=\begin{pmatrix}-1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=0;\\ &\chi_\downarrow^\dagger\phi_\uparrow+\chi_\uparrow^\dagger\phi_\downarrow=\begin{pmatrix}-1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=0,\end{aligned} \tag{6.8}$$

which is then equivalent to saying that any term proportional to the unit matrix in the expression for the 2×2 matrix element will disappear from the final invariant amplitude. Thus the relevant part of $a_{(12)3}$ can be written in the form

$$a_{(12)3}=\frac{1}{2m}\left((\vec{\kappa}_{12}\times\hat{\delta}_3)\cdot\vec{\sigma}+\kappa_{12}^0(\vec{\epsilon}_3\cdot\vec{\sigma})\right). \tag{6.9}$$

We may then repeat this above calculation for the graph in figure 6.15. The invariant amplitude can be written as

$$
\begin{aligned}
-i\mathcal{M}_{3(12)} =&\bar{v}_2(-ig_s(t^c)_{jd})\not{\epsilon}_3\left(\frac{i(\not{p}_1-\not{k}_1-\not{k}_2+m)}{(p_1-k_1-k_2)^2-m^2}\right)(-ig_s(t^e)_{di}\gamma_{\mu_4})\left(\frac{-i}{(k_1+k_2)^2}\right)\\
&\Big(-g_sf^{eab}[g^{\mu_4\mu_1}((k_1+k_2)+k_1)^{\mu_2}+g^{\mu_1\mu_2}(-k_1+k_2)^{\mu_4}\\
&+g^{\mu_2\mu_4}(-k_2-k_1-k_2))^{\mu_1}]\Big)\epsilon_{1\mu_1}\epsilon_{2\mu_2}u_1.
\end{aligned}
\tag{6.10}
$$

Again, we notice that the color factor in front is

$$
C^{abc}=\frac{1}{\sqrt{3}}(t^c)_{id}(t^e)_{di}f^{eab}=\frac{\mathrm{T(F)}}{\sqrt{3}}f^{abc}. \tag{6.11}
$$

Then with identical steps as before, the invariant amplitude can be taken to the form

$$
-i\mathcal{M}_{3(12)}=g_s^3C^{abc}\frac{1}{(k_1+k_2)^2}\bar{v}_2\not{\epsilon}_3\left(\frac{\not{k}_3-\not{p}+m}{(k_3-p)+m^2}\right)\not{\kappa}_{12}u_1. \tag{6.12}
$$

Immediately we notice that the front factor $g_s^3C^{abc}/(k_1+k_2)^2$ is the same as with the permutation (12)3. We can then identify the 2×2 matrix $a_{3(12)}$ within our expression of the invariant amplitude. Note that now we may use the result of equation 5.9,

$$
\begin{aligned}
a_{3(12)}&=\begin{pmatrix}0_2 & \mathbb{1}_2\end{pmatrix}\not{\epsilon}_3\left(\frac{\not{k}_3-\not{p}+m}{(k_3-p)+m^2}\right)\not{\kappa}_{12}\begin{pmatrix}\mathbb{1}_2\\0_2\end{pmatrix}\\
&=\frac{-1}{2m}\begin{pmatrix}\vec{\epsilon}_3\cdot\vec{\sigma} & i\hat{\delta}_3\cdot\vec{\sigma}\end{pmatrix}\begin{pmatrix}\kappa_{12}^0\mathbb{1}_2\\ \vec{\kappa}_{12}\cdot\vec{\sigma}\end{pmatrix}\\
&=-\frac{1}{2m}\left((\vec{\epsilon}_3\cdot\vec{\sigma})\kappa_{12}^0+i(\hat{\delta}_3\cdot\vec{\kappa}_{12})\mathbb{1}_2-(\hat{\delta}_3\times\vec{\kappa}_{12})\cdot\vec{\sigma}\right),
\end{aligned}
\tag{6.13}
$$

We can again get rid of the term proportional to $\mathbb{1}_2$ and write the relevant part as

$$
a_{3(12)}=-\frac{1}{2m}\left((\vec{\kappa}_{12}\times\hat{\delta}_3)\cdot\vec{\sigma}+\kappa_{12}^0(\vec{\epsilon}_3\cdot\vec{\sigma})\right). \tag{6.14}
$$

We can see that this is exactly the same as equation 6.9 but with a minus sign in front. Moreover, the front factors are also the same. Thus regardless of the spin configuration, these two terms will cancel each other out. This calculation is identical for the pairs 1(23) $\leftrightarrow$ (23)1 and 2(13) $\leftrightarrow$ (13)2 and as such we will not repeat them here.

With this we are then left with only the six Feynman diagrams in figures 6.1-6.6 to consider. If there would be no color factor to consider, this calculation would be identical to the one we had with the O-Ps. Then by looking at figure 6.1 and keeping in mind the O-Ps calculation, the invariant amplitude can then be written as

$$-i\mathcal{M}_{S=1}^{C=0}(c\bar{c} \to ggg) = \sum_{\text{permut}} \bar{v}_2(-ig_s\gamma^\mu t^a_{jm})\epsilon_{1\mu}\left(\frac{i(\not{p}-\not{k}_2-\not{k}_3+m)}{(p-k_2-k_3)^2-m^2}\right)$$
$$(-ig_s\gamma^\nu t^b_{mn})\epsilon_{2\nu}\left(\frac{i(\not{p}-\not{k}_3+m)}{(p-k_3)^2-m^2}\right)(-ig_s\gamma^\sigma t^c_{ni})\epsilon_{3\sigma}u_1, \tag{6.15}$$

It suffices to consider what is the color factor in front of each permutation. Hanging on to our earlier notation, the color factors for each permutation are as follows:

$$\begin{aligned}(123) &\leftrightarrow (t^at^bt^c)_{ji}; \quad (132) \leftrightarrow (t^at^ct^b)_{ji}; \quad (213) \leftrightarrow (t^bt^at^c)_{ji};\\ (231) &\leftrightarrow (t^bt^ct^a)_{ji}; \quad (321) \leftrightarrow (t^ct^bt^a)_{ji}; \quad (312) \leftrightarrow (t^ct^at^b)_{ji}.\end{aligned} \tag{6.16}$$

Then using the fact that J/ψ is in a color singlet state we have $i = j$, summing over $i = 1,2,3$, remembering the $1/\sqrt{3}$ front factor and using the result

$$\mathrm{Tr}(t^at^bt^c) = \frac{\mathrm{T(F)}}{2}(d^{abc} + if^{abc}), \tag{6.17}$$

where d^{abc} are the completely symmetric structure constants and f^{abc} are the completely antisymmetric structure constants, we get the following color factors:

$$\begin{aligned}(123) &\leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{abc}+if^{abc}); \quad (132) \leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{acb}+if^{acb});\\ (213) &\leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{bac}+if^{bac}); \quad (231) \leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{bca}+if^{bca});\\ (321) &\leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{cba}+if^{cba}); \quad (312) \leftrightarrow \frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}(d^{cab}+if^{cab}).\end{aligned} \tag{6.18}$$

The completely symmetric structure constants can be permuted to give d^{abc} for each permutation. Similarly the antisymmetric structure constants can be permuted to give f^{abc} but now the sign depends on whether we are considering an odd or an even permutation. We may then borrow the result of equation 5.5 to write

$$-i\mathcal{M}_{S=1} = i2mg_s^3\frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}\chi_s^\dagger\left(d^{abc}\sum_{\text{permut}} a_{123} + i\sum_{\text{permut}} f^{abc}a_{123}\right)\phi_s. \tag{6.19}$$

Note in particular that d^{abc} can be taken as a common factor and f^{abc} has to be inside the permutation sum. The end result is that the permutation sum with f^{abc} will not contribute to the invariant amplitude. This can indeed be anticipated since $\sum_{\text{permut}} a_{123}$ is fully symmetric in exchanges of final state particles while f^{abc} is fully antisymmetric. In order to recover a fully symmetric $\mathcal{M}$, the f^{abc} terms should vanish. Let us see how this comes about.

6.2 Processing the invariant amplitude

It is fairly easy to see how we end up with the equation 6.19 for the invariant amplitude. We may neglect all the constants and the 2-spinors since they are the same for every term within the sum. We will use our previous result from equation 5.12 and denote the 3-vectors with bold letters. Moreover, denoting this sum by S and by opening it up:

$$\begin{aligned}S = \sum_{\text{permut}} f^{abc} a_{123} = \frac{f^{abc}}{4m^2}\Big[&- (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3) - (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3) \\ &+ (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2) \\ &+ (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3) \\ &- (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1) - (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1) \\ &+ (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1) \\ &- (\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2) - (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2)\Big].\end{aligned} \tag{6.20}$$

Note how the even permutations of (123) have a minus sign in front and the odd permutations have a plus sign. We can then group these terms with $(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_i)$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_i)$. From this we notice that there are commutators of the form $[(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_i),(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_j)]$. For these terms, we can use the following result:

$$\begin{aligned}[(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_i),(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_j)] &= [\sigma^k\epsilon_i^k,\sigma^m\epsilon_j^m] = \epsilon_i^k\epsilon_j^m \underbrace{[\sigma^k,\sigma^m]}_{=2i\epsilon^{kml}\sigma^l} = 2i\boldsymbol{\epsilon}_i\cdot(\boldsymbol{\epsilon}_j\times\boldsymbol{\sigma}) \\ &= 2i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_i\times\boldsymbol{\epsilon}_j).\end{aligned} \tag{6.21}$$

For the terms with $\boldsymbol{\delta}$ we can use the familiar identity of $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbb{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$. With these we get

$$\begin{aligned}S = \frac{f^{abc}}{4m^2}\Big[&(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1)(2i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\epsilon}_2)) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1)[(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\delta}_2)\mathbb{1}_2 + i(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_2)\cdot\boldsymbol{\sigma} \\ &- (\boldsymbol{\epsilon}_2\cdot\boldsymbol{\delta}_3)\mathbb{1}_2 - i(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_3)\cdot\boldsymbol{\sigma}] \\ &(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2)(2i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_1\times\boldsymbol{\epsilon}_3)) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2)[(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\delta}_3)\mathbb{1}_2 + i(\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_3)\cdot\boldsymbol{\sigma} \\ &- (\boldsymbol{\epsilon}_3\cdot\boldsymbol{\delta}_1)\mathbb{1}_2 - i(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_1)\cdot\boldsymbol{\sigma}] \\ &(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_3)(2i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\epsilon}_1)) + (\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3)[(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\delta}_1)\mathbb{1}_2 + i(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_1)\cdot\boldsymbol{\sigma} \\ &- (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\delta}_2)\mathbb{1}_2 - i(\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_2)\cdot\boldsymbol{\sigma}]\Big].\end{aligned} \tag{6.22}$$

We notice that we have terms of the form $(\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_i)(\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_j\times\boldsymbol{\epsilon}_k))$. If we open up these we can get rid of the terms proportional to $\boldsymbol{\sigma}$ by using the Jacobi identity for the cross product,

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=0. \tag{6.23}$$

Then for the rest, we can open up them further and regroup them accordingly:

$$\begin{aligned}S=\frac{f^{abc}}{4m^2}\Big[&2i\boldsymbol{\epsilon}_1\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\epsilon}_2)\mathbb{1}_2+2i\boldsymbol{\epsilon}_2\cdot(\boldsymbol{\epsilon}_1\times\boldsymbol{\epsilon}_3)\mathbb{1}_2+2i\boldsymbol{\epsilon}_3\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\epsilon}_1)\mathbb{1}_2\\&+(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\delta}_2-\boldsymbol{\epsilon}_2\cdot\boldsymbol{\delta}_3)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_1)+(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\delta}_3-\boldsymbol{\epsilon}_3\cdot\boldsymbol{\delta}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_2)\\&+(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\delta}_1-\boldsymbol{\epsilon}_1\cdot\boldsymbol{\delta}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{\delta}_3)\\&+i[\boldsymbol{\delta}_1\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_2-\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_3)+\boldsymbol{\delta}_2\cdot(\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_3-\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_1)\\&+\boldsymbol{\delta}_3\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_1-\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_2)]\mathbb{1}_2\\&+\boldsymbol{\sigma}\cdot[\boldsymbol{\delta}_1\times(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_3-\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_2)+\boldsymbol{\delta}_2\times(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_1-\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_3)\\&+\boldsymbol{\delta}_3\times(\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_2-\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_1)]\Big].\end{aligned} \tag{6.24}$$

To simplify this, we can use the cyclicity of the triple product e.g. $\boldsymbol{\delta}_1\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_2)$ is the same as $-\boldsymbol{\delta}_2\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_1)$. Furthermore, by using the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ to the last two lines, we cancel out the second and the third lines. Then we are left with

$$\begin{aligned}S=\frac{f^{abc}}{4m^2}\Big[&2i\boldsymbol{\epsilon}_1\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\epsilon}_2)\mathbb{1}_2+2i\boldsymbol{\epsilon}_2\cdot(\boldsymbol{\epsilon}_1\times\boldsymbol{\epsilon}_3)\mathbb{1}_2+2i\boldsymbol{\epsilon}_3\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\epsilon}_1)\mathbb{1}_2\\&+2i\boldsymbol{\delta}_1\cdot(\boldsymbol{\epsilon}_3\times\boldsymbol{\delta}_2)\mathbb{1}_2+2i\boldsymbol{\delta}_2\cdot(\boldsymbol{\epsilon}_1\times\boldsymbol{\delta}_3)\mathbb{1}_2+2i\boldsymbol{\delta}_3\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_1)\mathbb{1}_2\Big].\end{aligned} \tag{6.25}$$

By using the anticommutativity of the cross product e.g. $\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$, it is evident that S can be written as

$$S=-\frac{if^{abc}}{2m^2}\sum_{\text{cyclic}}\left(\boldsymbol{\epsilon}_1\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\epsilon}_3)+\boldsymbol{\delta}_1\cdot(\boldsymbol{\epsilon}_2\times\boldsymbol{\delta}_3)\right)\mathbb{1}_2. \tag{6.26}$$

We can see that this is proportional to the unit matrix $\mathbb{1}_2$. Thus regardless of the constants in front of it, we can use our earlier result of equation 6.8, to see that all such contributions will go to zero. This is how we are left with the result of equation 6.27,

$$-i\mathcal{M}_{S=1}=i2mg_s^3\frac{1}{\sqrt{3}}\frac{\mathrm{T(F)}}{2}d^{abc}\chi_s^\dagger\vec{\sigma}\cdot\vec{V}\phi_s. \tag{6.27}$$

Comparing this with equations 5.5 and 5.22 we see that the only difference, in addition to the obvious replacement $e\to g_s$, is that now we have the extra color factor in

front. Once we figure out what happens to this color factor, we can immediately use the result that we know for the decay width of the O-Ps.
As we can see, this is a rather easy task to complete. Squaring equation 6.27 and summing over the colors a,b,c gives the color factor C to be

$$C = \sum_{abc} \left(\frac{1}{\sqrt{3}} \frac{\mathrm{T(F)}}{2} d^{abc} \right)^2 = \frac{5}{9} \frac{1}{16} \underbrace{\sum_a \delta^{aa}}_{\mathrm{Tr}(\mathbb{1}_8)} = \frac{5}{18}, \tag{6.28}$$

where we have used the result

$$d^{acd} d^{bcd} = \frac{N^2 - 4}{N} \delta^{ab} = \frac{5}{3} \delta^{ab}. \tag{6.29}$$

Therefore we may borrow the result of equation 5.39 to state our final result for the squared invariant amplitude:

$$|\overline{\mathcal{M}^{tot}_{S=1}}(c\bar{c} \to ggg)|^2 = \frac{5}{18} 4m^2 g_s^6 \frac{4}{3m^4} \sum_{\text{cyclic}} (1 - \cos\theta_{23})^2, \tag{6.30}$$

where m is now the mass of the c-quark. Furthermore, the phase space integration is also identical to the one we had with the O-Ps. We may then use equation 5.63 to write our final answer,

$$\Gamma(J/\psi \to ggg) = \frac{5}{18} \frac{2^6}{9} \frac{\pi^2 - 9}{M^2} \alpha_s^3 |\psi(0)|^2 = \frac{160}{81} \frac{\pi^2 - 9}{M^2} \alpha_s^3 |\psi(0)|^2, \tag{6.31}$$

where M is the mass of J/ψ. This is in accordance with the standard literature in the field [24], [25].

7 Decay width of $J/\psi \to \gamma gg$

We have now calculated a bound state decay into three final state particles for O-Ps and the heavy meson J/ψ. In these calculations the final state particles have been identical. Moreover, we found the O-Ps decay calculation particularly helpful when calculating the decay of J/ψ into three gluons. We would like to use that result also in the calculation of $\Gamma(J/\psi \to \gamma gg)$. For this, we need to identify the same kind of a permutation sum here.
In the last section we considered in the figures 6.7-6.10 the s-channel decay processes of two quarks into three gluons. It was shown that this type of an annihilation is impossible since the J/ψ is in a color singlet state. Similarly here, we cannot have a graph with a color singlet gluon propagator. With these arguments we can draw the contributing LO Feynman diagrams as presented in figures 7.1-7.6.

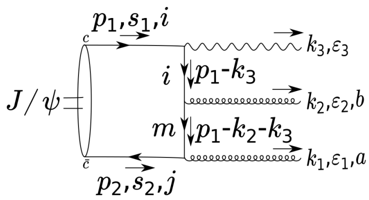


Figure 7.1.
Permutation (123).

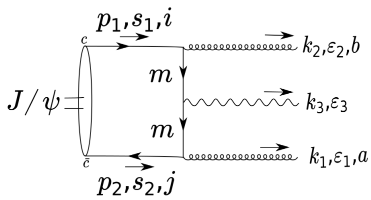


Figure 7.2.
Permutation (132).

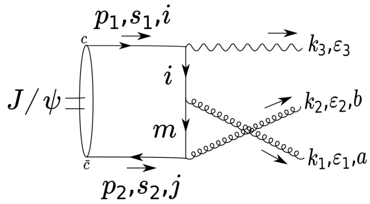


Figure 7.3.
Permutation (213).

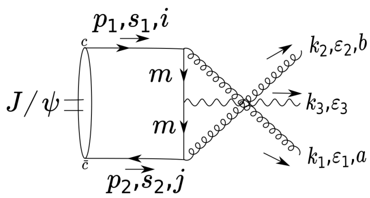


Figure 7.4.
Permutation (231).

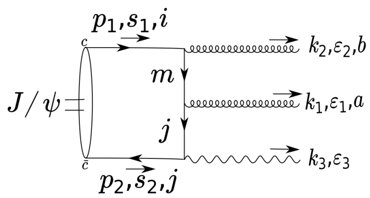


Figure 7.5.
Permutation (312).

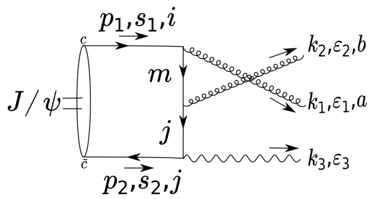


Figure 7.6.
Permutation (321).

The reader might notice that these graphs are somewhat different to what we had in the three gluon case. This follows from the fact that now we are able to distinguish

between some of the final state particles. Note in particular that the photon has the same momentum and polarization vector in all graphs. However, by looking at the graphs we can see that the invariant amplitude can be written in terms of the permutation sum that we have now worked already twice with.
By looking at figure 7.1 we can immediately write the invariant amplitude as

$$\begin{aligned}-i\mathcal{M}^{C=0}_{S=1}(c\bar{c}\to\gamma gg) = \sum_{\text{permut}} \bar{v}_2(-ig_s(t^a)_{jm}\gamma^\mu)\epsilon_{1\mu}\left(\frac{i(\not{p}-\not{k}_2-\not{k}_3+m)}{(p-k_2-k_3)^2-m^2}\right)\\ (-ig_s(t^b)_{mi}\gamma^\nu)\epsilon_{2\nu}\left(\frac{i(\not{p}-\not{k}_3+m)}{(p-k_3)^2-m^2}\right)(-ieQ_c\gamma^\sigma)\epsilon_{3\sigma}u_1.\end{aligned} \tag{7.1}$$

There are few details that are worth mentioning. One quark-gluon vertex is now replaced by a quark-photon vertex. This is equivalent to replacing one g_s by eQ_c, where Q_c is the fractional charge of the charm quark as in section 3. Then by keeping in mind equation 5.2 we can simplify the invariant amplitude to be

$$\begin{aligned}-i\mathcal{M}_{s=1}(c\bar{c}\to\gamma gg) = \sum_{\text{permut}} -ig_s^2eQ_c(t^at^b)_{ji}\bar{v}_2\not{\epsilon}_1\left(\frac{\not{k}_1-\not{p}+m}{(k_1-p)^2-m^2}\right)\not{\epsilon}_2\\ \left(\frac{\not{p}-\not{k}_3+m}{(p-k_3)^2-m^2}\right)\not{\epsilon}_3u_1.\end{aligned} \tag{7.2}$$

Note in particular that the order of t^b and t^a depends on the permutation. However, this does not matter for the resulting color factor. Let us explicitly show this. The color factors can be written in a similar manner as in equation 6.16. This gives us

$$\begin{aligned}&(123)\leftrightarrow(t^at^b)_{ji}; \quad (132)\leftrightarrow(t^at^b)_{ji}; \quad (213)\leftrightarrow(t^bt^a)_{ji};\\ &(231)\leftrightarrow(t^bt^a)_{ji}; \quad (321)\leftrightarrow(t^bt^a)_{ji}; \quad (312)\leftrightarrow(t^at^b)_{ji}.\end{aligned} \tag{7.3}$$

Since J/ψ is in the color singlet state we have $i=j$ and thus we will have a trace over the two matrices. And since $\mathrm{Tr}(t^at^b)=\mathrm{Tr}(t^bt^a)$ we get the color factor C as follows:

$$C = \frac{1}{\sqrt{3}}(t^at^b)_{ii} = \frac{1}{\sqrt{3}}\underbrace{\mathrm{Tr}(t^at^b)}_{=\mathrm{T(F)}\delta^{ab}} = \frac{1}{\sqrt{3}}\mathrm{T(F)}\delta^{ab}, \tag{7.4}$$

where again $\mathrm{T(F)}=1/2$. We can then carry all constants out of the permutation sum and write

$$\begin{aligned}-i\mathcal{M}^{C=0}_{s=1}(c\bar{c}\to\gamma gg) = -ig_s^2eQ_c\frac{1}{\sqrt{3}}\frac{1}{2}\delta^{ab}\sum_{\text{permut}}\bar{v}_2\not{\epsilon}_1\left(\frac{i(\not{p}-\not{k}_2-\not{k}_3+m)}{(p-k_2-k_3)^2-m^2}\right)\\ \not{\epsilon}_2\left(\frac{i(\not{p}-\not{k}_3+m)}{(p-k_3)^2-m^2}\right)\not{\epsilon}_3u_1.\end{aligned} \tag{7.5}$$

In comparing this with equations 5.2 and 5.3, we can see that there is again the same kind of matrix structure. The permutation sum can then be developed in an identical manner as before. We still have to account for the contribution from the color factor. Continuing to sum over a and b:

$$C^2 = \frac{1}{3}\frac{1}{4}\sum_{ab}\delta^{ba}\delta^{ab} = \frac{1}{3}\frac{1}{4}\mathrm{Tr}(\mathbb{1}_8) = \frac{2}{3}. \tag{7.6}$$

With one final modification the rest of the calculation proceeds as with the O-Ps calculation. In the O-Ps calculation we had three identical particles in the final state. Now there are only two. Therefore the 1/3! factor reduces to 1/2!. This is equivalent to multiplying our earlier result by three. Then by using equation 5.64 we get:

$$\Gamma(J/\psi \to \gamma gg) = \frac{2}{3}\cdot 3 \cdot \frac{2^6}{9}\frac{\pi^2-9}{M^2}Q_c^2\alpha\alpha_s^2|\psi(0)|^2, \tag{7.7}$$

which simplifies to:

$$\Gamma(J/\psi \to \gamma gg) = \frac{128}{9}Q_c^2\alpha\alpha_s^2\frac{\pi^2-9}{M^2}|\psi(0)|^2, \tag{7.8}$$

where α is the QED coupling constant, α_s is the QCD coupling constant and M is the mass of J/ψ. This agrees with the established result in literature [26].

8 Discussion

8.1 Helicity basis approach

A decent amount of work in this thesis was spent in learning the helicity basis approach for processing the invariant amplitude in sections 3.2 and 4. For clarity, let us explicitly state how the product of the Dirac spinors can be expressed in terms of the polarization vectors in a certain helicity eigenstate:

$J = 1$ vector meson:

$$u(\uparrow)\bar{v}(\uparrow) = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_{bs}+M}{2}\right)\not{\epsilon}(\uparrow); \quad u(\downarrow)\bar{v}(\downarrow) = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_{bs}+M}{2}\right)\not{\epsilon}(\downarrow);$$
$$\frac{1}{\sqrt{2}}\left[u(\uparrow)\bar{v}(\downarrow) + u(\downarrow)\bar{v}(\uparrow)\right] = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_{bs}+M}{2}\right)\not{\epsilon}(0);$$

$J = 0$ pseudoscalar meson:

$$\frac{1}{\sqrt{2}}[u(\uparrow)\bar{v}(\downarrow) - u(\downarrow)\bar{v}(\uparrow)] = \frac{1}{\sqrt{2}}\left(\frac{\not{p}_{bs}+M}{2}\right)\gamma^5, \tag{8.1}$$

where u and v are the Dirac spinor helicity-base solutions of the quarks that make up the bound state. Furthermore, p_{bs} is the momentum of the bound state, M is the mass of that particle and the polarization vector $\epsilon(h)$ describes the helicity state of the bound state. In particular, notice that the identities of equation 8.1 hold for a meson in an initial state.

In this thesis we have addressed the question of how the process of a two-body bound state decay can be described. A natural sequel to this would be to ask how these bound states are formed in particle collisions. As mentioned in the introduction, one of our earliest motivations for this topic came from the desire to understand the diffractive J/ψ-production in deep inelastic scattering. The helicity basis approach seems to be a particularly useful technique for such considerations. For further details we refer the reader to [5], [21]. Let us then turn our attention to how the calculations for the J/ψ decays compare with experimental results.

8.2 Comparison with experimental results

For the sake of clarity let us present our results for the decay widths on one single page. Collecting the results from equations 3.28, 4.21, 5.64, 6.31 and 7.8 we have:

$$\Gamma(J/\psi \to l^+l^-) = 16\pi\alpha^2 Q_c^2 \frac{|\psi(0)|^2}{M_J^2}, \qquad [19] \tag{8.2}$$

$$\Gamma(\eta_c \to gg) = \left(\frac{2}{3}\right)\frac{16\pi\alpha^2}{M_\eta^2}|\psi(0)|^2, \qquad [15] \tag{8.3}$$

$$\Gamma(\text{O-Ps} \to \gamma\gamma\gamma) = \frac{2m_e\alpha^6}{9\pi}(\pi^2-9), \qquad [8] \tag{8.4}$$

$$\Gamma(J/\psi \to ggg) = \frac{160}{81}\frac{\pi^2-9}{M_J^2}\alpha_s^3|\psi(0)|^2, \qquad [27] \tag{8.5}$$

$$\Gamma(J/\psi \to \gamma gg) = \frac{128}{9}Q_c^2\alpha\alpha_s^2\frac{\pi^2-9}{M_J^2}|\psi(0)|^2, \qquad [26] \tag{8.6}$$

where M_J is the mass of J/ψ, M_η is the mass of η_c, m_e is the mass of an electron, α is the QED coupling constant and α_s is the QCD coupling constant. Furthermore, Q_c is the fractional charge of the charm quark and $\psi(0)$ is the value of the position-space wavefunction at origin. Note in particular that for the O-Ps process we know the explicit form of this wavefunction and can thus substitute a value for it.

These results are in precise agreement with theoretical ones found in the literature. The unfortunate fact is that we still have the square of the wavefunction evaluated at zero appearing in our expressions of the decay widths. However, we can partly get around this dubiety by forming ratios out of the calculated decay widths. Take for example the following:

$$R_{1\text{th}}^{J/\psi} = \frac{\Gamma(J/\psi \to \gamma gg)}{\Gamma(J/\psi \to ggg)} = \frac{36}{5}\frac{\alpha}{\alpha_s}Q_c^2, \tag{8.7}$$

$$R_{2\text{th}}^{J/\psi} = \frac{\Gamma(J/\psi \to e^+e^-)}{\Gamma(J/\psi \to ggg)} = \frac{81\pi}{10(\pi^2-9)}\frac{\alpha^2}{\alpha_s^3}Q_c^2, \tag{8.8}$$

where th labels the fact that these are theoretical predictions in the LO. In order to acquire a value for these ratios, we need to figure out the values of α and α_s at the energy scale of the J/ψ which in our case is the mass $M_J \approx 3$ GeV [28]. The value of α is rather stable up to the mass of the Z-boson $M_Z \approx 91$ GeV but it clearly changes when the energy scale changes [29], [30]. This running of the coupling constant is presented in figure 8.1 [31]. We can then read off the value of the QED coupling constant from the graph to be $\alpha \approx 1/134$.

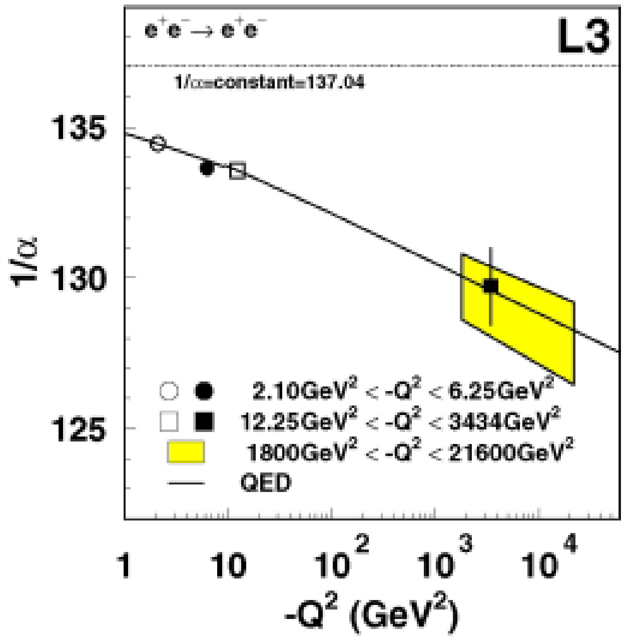


Figure 8.1. Evolution of the QED coupling α as a function of Q^2. Here Q is the energy scale of the process. The figure is from [31].

For the QCD coupling constant, we can read off the value of α_s from figure 8.2 to be around 0.23 when the energy scale is around $Q \approx 3$ GeV [32]. Substituting these values for $R_{1\text{th}}^{J/\psi}$ and $R_{2\text{th}}^{J/\psi}$ in equations 8.7 and 8.8 we get:

$$R_{1\text{th}}^{J/\psi} = \frac{\Gamma(J/\psi \to \gamma gg)}{\Gamma(J/\psi \to ggg)} = \frac{36}{5}\frac{(1/134)}{0.23}\left(\frac{2}{3}\right)^2 \approx 0.104, \tag{8.9}$$

$$R_{2\text{th}}^{J/\psi} = \frac{\Gamma(J/\psi \to l^+l^-)}{\Gamma(J/\psi \to ggg)} = \frac{81\pi}{10(\pi^2-9)}\frac{(1/134)^2}{(0.23)^3}\left(\frac{2}{3}\right)^2 \approx 0.060. \tag{8.10}$$

We can form these same ratios from the experimental data provided by [28]. These values are found to be:

$$R_{1\text{exp}}^{J/\psi} = \frac{\Gamma(J/\psi \to \gamma gg)/\Gamma_{\text{tot}}}{\Gamma(J/\psi \to ggg)/\Gamma_{\text{tot}}} = \frac{8.8\%}{64.1\%} \approx 0.137, \tag{8.11}$$

$$R_{2\text{exp}}^{J/\psi} = \frac{\Gamma(J/\psi \to e^+e^-)/\Gamma_{\text{tot}}}{\Gamma(J/\psi \to ggg)/\Gamma_{\text{tot}}} = \frac{5.971\%}{64.1\%} \approx 0.093, \tag{8.12}$$

where exp labels the fact that these are experimental results. We can see that $R_{1\text{th}}^{J/\psi}$ agrees rather well with $R_{1\text{exp}}^{J/\psi}$ taking into account that our theoretical prediction is only the LO approximation. However, when the ratio depends more strongly on the powers of the coupling constants, we can see that the values start to differ more. This is to say that the discrepancy between the values of $R_{2\text{th}}^{J/\psi}$ and $R_{2\text{exp}}^{J/\psi}$ becomes larger.

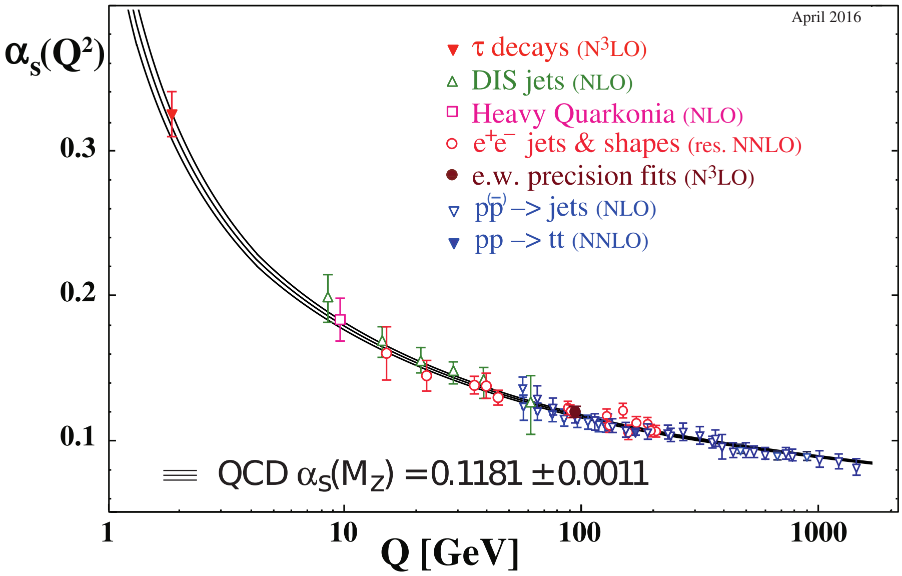


Figure 8.2. Summary of the measurements of $\alpha_s(Q^2)$ as a function of the energy scale Q. The respective order of QCD perturbation theory used in the extraction of α_s is indicated in brackets. The figure is from the reference [32].

Nevertheless, taking into account all the approximations we have made to make the problem easier to solve, it is quite imposing how well our theoretical predictions go together with experimental values. The accuracy of the theoretical predictions could be improved by taking into account higher order terms of the coupling constants. For example for $J/\psi \to e^+e^-$ the first QCD corrections were calculated in [33]. Moreover, by using the Dirac spinor solutions for non-zero momentum and not neglecting the masses of the outgoing leptons, would give us a more precise prediction for the decay width. Nevertheless, these LO calculations are good for order of magnitude approximations.

8.3 Generalization of the results

It should be noted that these calculations can be easily modified to accommodate other $J^{PC} = 0^{-+}$ and $J^{PC} = 1^{--}$ heavy mesons. There are only two parameters that we have to adjust: the fractional charge of the constituent quark and the mass of the decaying meson. This same point is argued in the reference [24]. Then in equations 8.2, 8.3, 8.5 and 8.6 one could simply replace the mass and Q_c to get the decay width for any decaying meson that has the same forementioned properties.

When forming the ratios, we need to account for the running of the coupling constants. Let us take the 1S state of the $\Upsilon(b\bar{b})$-meson as an example which has a mass of $M_\Upsilon \approx 9.4$ GeV [34]–[36]. Reading from figure 8.2 we get the strong coupling constant to be $\alpha_S \approx 0.18$. Similarly, based on figure 8.1, we will adjust the value of α to be 1/132. Moreover, since Υ is a $b\bar{b}$ bound state we have $Q_c = 1/3$. The theoretical predictions are then:

$$R^{\Upsilon}_{1\text{th}} = \frac{\Gamma(\Upsilon \to \gamma gg)}{\Gamma(\Upsilon \to ggg)} = \frac{36}{5}\frac{(1/132)}{0.18}\left(\frac{1}{3}\right)^2 \approx 0.034, \tag{8.13}$$

$$R^{\Upsilon}_{2\text{th}} = \frac{\Gamma(\Upsilon \to e^+e^-)}{\Gamma(J/\psi \to ggg)} = \frac{81\pi}{10(\pi^2-9)}\frac{(1/132)^2}{(0.18)^3}\left(\frac{1}{3}\right)^2 \approx 0.032. \tag{8.14}$$

We can form these same ratios from the experimental data provided by [34]. These values are found to be:

$$R^{\Upsilon}_{1\text{exp}} = \frac{\Gamma(\Upsilon \to \gamma gg)/\Gamma_{\text{tot}}}{\Gamma(\Upsilon \to ggg)/\Gamma_{\text{tot}}} = \frac{2.2\%}{81.7\%} \approx 0.027, \tag{8.15}$$

$$R^{\Upsilon}_{2\text{exp}} = \frac{\Gamma(\Upsilon \to e^+e^-)/\Gamma_{\text{tot}}}{\Gamma(\Upsilon \to ggg)/\Gamma_{\text{tot}}} = \frac{2.38\%}{81.8\%} \approx 0.029. \tag{8.16}$$

Again the theoretical predictions compare quite nicely with the experimental values. In addition, it seems that the theoretical predictions for the Υ ratios are in better agreement with the experimental ratios than the J/ψ counterparts. This suggests that the higher order effects would be relatively decreasing towards higher energy scales. Furthermore, this hints towards our claim that the decay width of any meson fulfilling our assumptions, can be acquired in an identical manner. As a final note it should be pointed out that the Υ-decays are in practise used to measure the running of the α_s. These measurements effectively contribute to the accuracy of figure 8.2. For further details see for example the discussion in [37].

9 Conclusion

The decay of a two-body bound state to some particular final state is by no means a trivial problem. A set of simplifying approximations render the calculation possible and rather straightforward. We have assumed that the bound state decay can be described as a free-particle annihilation process where the momentum of the annihilating particles is set to zero. The invariant amplitude of this free particle process is then weighted by the value of the wavefunction describing the system. The fact that we evaluate this wavefunction at zero has a rather clear intuitive interpretation: when the particles are at the same place, they annihilate.

We then used these principles to calculate an O-Ps decay width and four different heavy meson decay widths. In essence the problem was in figuring out how the invariant amplitude could be processed in the most efficient way. The explicit spin matrix approach is simple in the sense that it requires little insight to the problem at hand. One just calculates all the different matrices whilst meticulously keeping track of the elements.

This in part emphasises the detail of how the Dirac spinor solutions are chosen. The choice of equation 3.2 might be the one that most of people would choose first. However, this does not allow us to use the Clebsch-Gordan coefficients. By changing our definition to the one presented in equation 3.29 we recovered the forementioned results. This choice is then important when explicitly calculating the spin matrix elements.

In the helicity basis approach, the computation of the explicit matrix elements is replaced by the calculation of traces over gamma matrices. However, to be able to use this approach we had to rigorously establish the fact that the assumed identities hold. It was then here, on the grounds of completeness, that the η_c decay calculation was presented. With it we have the complete picture how spin 1/2 particles can be added together and presented in the helicity basis. The result of this development is summarized in equation 8.1.

After this we turned our attention towards the decay of O-Ps into three photons. By peeling down the 4×4 matrix structure to a 2×2, we avoided the explicit calculation of every 4×4 matrix product. In effect this introduced a third method for processing the invariant amplitude. Furthermore, the O-Ps decay calculation also highlights the difference between QED and QCD. We do not know the explicit form of the QCD potential, and thus in the final form of the decay width we are left with $|\psi(0)|^2$. This is not the case for a pure QED process.

The decay widths of J/ψ into three gluons or one photon and two gluons were then relatively easy to calculate. We only had to take into account how the color factor contributes for the calculation. Manipulating the color factors appropriately and summing over the colors we were left with a number multiplying the invariant amplitude. The form of the invariant amplitude was then practically identical to the analogous invariant amplitude of the O-Ps decay. We could then use the O-Ps result to get the J/ψ decay widths.

It was then put forth that the helicity basis approach could be useful in the consideration of heavy quarkonium production in particle collisions. However, any of the three techniques could be used, with the right assumptions, in such calculations. In addition to developing further on the results of this thesis, it would salient to take into account Feynman graphs of higher order in the coupling constant.

In conclusion, this thesis covers a certain range of heavy-quark bound state decays and does it with a set of rather simple assumptions. Moreover, it is obvious that the development can be generalized to encompass any $J^{PC} = 1^{--}$ or $J^{PC} = 0^{-+}$ heavy meson decay. An example of the $\Upsilon(1S)$ decay width ratios was offered to support this view. In considering the decay width ratios it is important to keep in mind the running of the coupling constants as the energy scale is increased. Theoretically one would get a better hold of this only by doing the required next-to-LO calculations. Taking into account the fact that our calculations are only to the LO, the theoretical predictions for the ratios of the decay widths agree relatively well with the experimental results.

A Calculations for section 3

A.1 Addition of fermion spins; photon polarization vectors

We wish to show that with our choice of equation 3.2 for the basis vectors in section 3.1, the addition of angular momenta for the state $|SM_z\rangle = |10\rangle$ of two Dirac spinors u and v is given by $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ where $|\uparrow\downarrow\rangle = |s_1 s_2\rangle$. One way of doing this is to consider the annihilation process of a lepton-pair into a virtual photon $l^+l^- \to \gamma^*$. Before this, we need to establish the form of the photon polarization vector, which we need in calculating this process in the LO by using the Feynman rules.

In section A.3 we shall see that the following development holds also for a spin-1 particle with mass M. Let us then first consider a massless free photon. From QED we know that the wavefunction A^μ of a free photon satisfies in the Lorentz gauge the equation of motion

$$\Box A^\mu = 0, \tag{A.1}$$

where $\Box$ is the d'Alembertian i.e. $\Box = \partial_\nu \partial^\nu$. It is clear that equation A.1 admits solutions of the form $A^\mu = \epsilon^\mu(q)e^{-iq\cdot x}$, where $\epsilon^\mu(q)$ is the polarization vector of the photon and q is the momentum on the photon. The Lorentz gauge $\partial_\mu A^\mu = 0$ imposes the condition

$$q_\mu \epsilon^\mu = 0. \tag{A.2}$$

So essentially there are three independent components of the polarization vector ϵ^μ. In the Coulomb gauge we choose the first component of ϵ^μ to be zero [17, p.134]. As discussed in section A.3, the condition A.2 holds also for a massive (virtual) photon. Then for a virtual photon travelling along the z-axis with the momentum vector $q = (q^0,0,0,q^3)$, the normalized polarization vectors can be then chosen as

$$\begin{cases} \epsilon^\mu_{h=\pm 1} = \mp\frac{1}{\sqrt{2}}(0,1,\pm i,0), \\ \epsilon^\mu_{h=0} = \frac{1}{\sqrt{q^2}}(q^3,0,0,q^0), \end{cases} \tag{A.3}$$

where h tells us the helicity eigenvalue of the photon [17, p.139]. We are particularly interested in the $h = 0$ polarization vector as this is the one which would couple to the fermion-antifermion state $|SM_z\rangle = |10\rangle$.
Let us still explicitly show that the given vectors in equation A.3 are indeed the eigenvectors of the helicity operator. Helicity is defined as the projection of spin to the direction of the momentum and its operator $\hat{h}$ is given by

$$\hat{h} = \hat{e}_p \cdot \hat{\boldsymbol{S}}, \tag{A.4}$$

where $\hat{e}_p$ is the unit vector to the direction of the momentum and $\hat{\boldsymbol{S}}$ is the spin vector operator. In our case, when the photon is travelling along the z-axis, the helicity operator reduces to $\hat{S}_z$. Then taking the spatial components of the polarization vectors, we can check that they fulfil the desired eigenvalue equation. Note that now $\hat{S}_z$ is the spin vector operator for a spin-1 particle, whose matrix representation is given e.g. in [14]. Thus,

$$\begin{aligned}
\hat{S}_z\epsilon_{h=1} &\Rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \Rightarrow \hat{S}_z\epsilon_{h=1} = h\epsilon_{h=1}, \\
\hat{S}_z\epsilon_{h=-1} &\Rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \Rightarrow \hat{S}_z\epsilon_{h=-1} = h\epsilon_{h=-1}, \\
\hat{S}_z\epsilon_{h=0} &\Rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{q^2}} \begin{pmatrix} 0 \\ 0 \\ q^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \hat{S}_z\epsilon_{h=1} = h\epsilon_{h=0}.
\end{aligned} \tag{A.5}$$

We have now established the form of the photon polarization vectors. Now from the Feynman rules we may derive the scattering matrix structure for the process $l^+l^- \to \gamma$ to be

$$\bar{v}_2 \not{\epsilon}_{h=0} u_1. \tag{A.6}$$

As shown in the main text, the $\mu = 0$ case disappears in the NR-limit and by sticking to our notation we may write this element as

$$\bar{v}_2 \not{\epsilon}_{h=0} u_1 = v_2^\dagger \gamma^0 (-\epsilon^3_{h=0} \gamma^3) u_1 = -\frac{2mq^0}{\sqrt{q^2}} \sigma^3_{s_2 s_1}. \tag{A.7}$$

Now for the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ we get the element

$$\frac{1}{\sqrt{2}} \frac{-2mq^0}{\sqrt{q^2}} (\sigma^3_{\uparrow\downarrow} - \sigma^3_{\downarrow\uparrow}), \tag{A.8}$$

and by inspecting the term $(\sigma^3_{\uparrow\downarrow} - \sigma^3_{\downarrow\uparrow})$ more closely:

$$\begin{aligned}(\sigma^3_{\uparrow\downarrow} - \sigma^3_{\downarrow\uparrow}) &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ &= (1-(-1)) = 2 \neq 0,\end{aligned} \tag{A.9}$$

which confirms that leptons in this state annihilate to a γ^*. We now see that if we would have followed the standard Clebsch-Gordan coefficient notation and had a plus sign in between the states, we would have gotten zero. That is to say that the lepton pair in the $(1/\sqrt{2})(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ could not annihilate into a photon. That clearly cannot be the case, since a virtual photon does have the longitudinal polarization state too. This is the reason why the invariant amplitude sign convention has to be chosen as it is chosen in the main text. When we define the spinor basis vectors differently, we regain the traditional Clebsch-Gordan coeffiecient results.

A.2 Phase space integral

The following development is done in detail in reference [18]. However, it is beneficial to go through the main points. The delta function in equation 3.26 can be divided into one-dimensional and a three-dimensional delta function as $\delta^{(4)}(p_{J/\psi} - p_3 - p_4) = \delta(M - E_3 - E_4)\delta^{(3)}(\boldsymbol{p}_3 + \boldsymbol{p}_4)$. Notice that we are still working in the CMS-frame. Then d^3p_4 integration can be done with $\delta^{(3)}(\boldsymbol{p}_3 + \boldsymbol{p}_4)$ which just leaves us the integrand evaluated at $\boldsymbol{p}_3 = -\boldsymbol{p}_4$. Then by moving into spherical coordinates we may write the remaining d^3p_3 as $d\Omega dp_3 p_3^2$. Then by using the delta function result

$$\delta(g(x)) = \frac{\delta(x - x_0)}{|g'(x_0)|}, \tag{A.10}$$

where x_0 is a zero of the function $g(x)$ and g' is the derivative of $g(x)$ with respect to x, we may write the remaining delta function as

$$\delta(M - E_3 - E_4) = \frac{E_3E_4}{p_3M}\bigg|_{\boldsymbol{p}_3=\boldsymbol{p}_z}\delta(p_3 - p_z), \tag{A.11}$$

where $\boldsymbol{p}_z$ is the zero of the expression $M - E_3 - E_4$. It is found to be

$$\boldsymbol{p}_z = \frac{\sqrt{\lambda(M^2,m_3^2,m_4^2)}}{2M}, \tag{A.12}$$

where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$ as given in the main text. Then the remaining integral over dp_3 just leaves us with the integrand evaluated at $\boldsymbol{p}_3 = \boldsymbol{p}_z$

and we get the result

$$\Gamma(J/\psi \to l^+l^-) = \frac{\sqrt{\lambda(M^2,m_3^2,m_4^2)}}{64\pi^2 M^3} \int d\Omega \overline{|\mathcal{M}_{tot}(J/\psi \to l^+l^-)|}^2. \tag{A.13}$$

Since we neglect the masses of the leptons this is equivalent to equation 3.27.

A.3 Helicity basis approach

This approach for the decay width rests heavily on understanding the relationship between the Dirac spinors and the bound state polarization vectors. Before tackling this, it is beneficial to investigate the equation of motion for a massive vector particle. The free particle wave equation for a spin-1 particle of mass M can be written as

$$(g^{\nu\lambda}(\Box + M^2) - \partial^\nu\partial^\lambda)B_\lambda = 0, \tag{A.14}$$

where B_λ is the wavefunction that describes our particle [17, p.138]. Multiplying equation A.14 by ∂_ν from the left,

$$(\partial^\lambda(\Box + M^2) - \partial_\nu\partial^\nu\partial^\lambda)B_\lambda \Rightarrow M^2\partial^\lambda B_\lambda = 0, \tag{A.15}$$

which forces the constraint $\partial^\lambda B_\lambda = 0$ regardless of the gauge and reduces A.14 to the same form as the Klein-Gordon equation (KGE),

$$(\Box + M^2)B_\mu = 0. \tag{A.16}$$

We know that the solutions of equation A.16 are of the form $B_\mu = \epsilon_\mu e^{-iq\cdot x}$. The condition $\partial^\lambda B_\lambda = 0$, then gives us the restriction that $q^\mu\epsilon_\mu = 0$. This is also what we had with equation A.2. We have already shown that polarization vectors of the form of equation A.3 are the eigenvectors of the eigenvalue equation $\hat{h}\epsilon_h = h\epsilon_h$. However, it is still worth noting that the vectors of equation A.3 are normalized as follows:

$$\begin{aligned} \epsilon_{h=0}\cdot\epsilon^*_{h=0} &= \underbrace{\epsilon^0_{h=0}\epsilon^{0*}_{h=0}}_{=\frac{|\boldsymbol{q}|^2}{M^2}} - \underbrace{\epsilon^1_{h=0}\epsilon^{1*}_{h=0}}_{=0} - \underbrace{\epsilon^2_{h=0}\epsilon^{2*}_{h=0}}_{=0} - \underbrace{\epsilon^3_{h=0}\epsilon^{3*}_{h=0}}_{=\frac{E^2}{M^2}} \\ &= \frac{|\boldsymbol{q}|^2 - E^2}{M^2} = \frac{-M^2}{M^2} = -1, \\ \epsilon_{h=\pm1}\cdot\epsilon^*_{h=\pm1} &= \underbrace{\epsilon^0_{h=\pm1}\epsilon^{0*}_{h=\pm1}}_{=0} - \underbrace{\epsilon^1_{h=\pm1}\epsilon^{1*}_{h=\pm1}}_{=\frac{1}{2}} - \underbrace{\epsilon^2_{h=\pm1}\epsilon^{2*}_{h=\pm1}}_{=\frac{1}{2}} - \underbrace{\epsilon^3_{h=\pm1}\epsilon^{3*}_{h=\pm1}}_{=0} \\ &= -1, \end{aligned} \tag{A.17}$$

where we have now identified $\sqrt{q^2} = M$, $q^0 = E$ and $q^3 = |\boldsymbol{q}|$ since $\boldsymbol{q}||z$-axis. Before continuing to derive the results of equation 3.30, let us quickly show why equation 3.37 holds. We will make an ansatz of the form (here $q = p_J$)

$$\sum_h \epsilon^\mu(h)\epsilon^{\nu^*}(h) = Ag^{\mu\nu} + Bp_J^\mu p_J^\nu, \tag{A.18}$$

where A and B are some numbers and $p_J = (p_J^0,0,0,p_J^3)$ is the momentum vector of the particle in question. Then by multiplying equation A.18 by $p_{J\mu}$ from the left:

$$\sum_h \underbrace{(p_J \cdot \epsilon(h))}_{=0}\epsilon^{\nu^*}(h) = Ap_J^\nu + Bp_J^2 p_J^\nu \Rightarrow B = -\frac{A}{p_J^2}, \tag{A.19}$$

where we have used the condition $p_J^\mu \epsilon_\mu = 0$. Then using this result and multiplying equation A.18 by $g_{\mu\nu}$ from the right:

$$\sum_h \underbrace{\epsilon(h) \cdot \epsilon^*(h)}_{=-1} = A(\underbrace{g_{\mu\nu}g^{\mu\nu}}_{=4} - \underbrace{\frac{p_J \cdot p_J}{p_J^2}}_{=1}) \Rightarrow A = -1, \tag{A.20}$$

where we have used the results of A.17. Then substituting these back into equation A.18 we find

$$\sum_h \epsilon^\mu(h)\epsilon^{\nu^*}(h) = -g^{\mu\nu} + \frac{p_J^\mu p_J^\nu}{p_J^2}. \tag{A.21}$$

We may now show how the results of equation 3.30 arise. We shall first explicitly derive the matrix form for the final state particle identities as given below:

$$\begin{aligned} &v(\uparrow)\bar{u}(\uparrow) = -\frac{1}{\sqrt{2}}\not{\epsilon}^*(\uparrow)\left(\frac{\not{p}_J + M}{2}\right); \quad v(\downarrow)\bar{u}(\downarrow) = -\frac{1}{\sqrt{2}}\not{\epsilon}^*(\downarrow)\left(\frac{\not{p}_J + M}{2}\right); \\ &\frac{1}{\sqrt{2}}[v(\uparrow)\bar{u}(\downarrow) + v(\downarrow)\bar{u}(\uparrow)] = -\frac{1}{\sqrt{2}}\not{\epsilon}^*(0)\left(\frac{\not{p}_J + M}{2}\right). \end{aligned} \tag{A.22}$$

From the results of equation A.22, it is then a relatively easy task to derive the results of equation 3.30. We will do this first in the special case where momentum and spin lie on the z-axis, and then rotate this to an arbitrary direction. Up to this point we have been working in the NR-limit where momentum of the charm quarks goes to zero. However, we may show the results of equation 3.30 to hold for any momentum of the vector meson as long as the condition $\boldsymbol{p}_c = \boldsymbol{p}_{\bar{c}} = \boldsymbol{p}_J/2$ holds.
We shall be working with the spinors as given in equation 3.29. Note that $\boldsymbol{\sigma}$ is a vector that contains the Pauli spin matrices e.g. $\boldsymbol{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$. Moreover, since $p_J = 2p$ and $M = 2m$ where p is the momentum of the charm quark and m is the mass of the charm quark. Finally the energy E_p is given as $E_p^2 = |\vec{p}|^2 + m^2$.

Our method of proving the wanted identities relies on forming the explicit form of the matrices of $v(h)\bar{u}(h)$ and showing that the r.h.s. of equation A.22 is the same. Note that for notational convenience, we have truncated the momentum p from our spinors. First we will show the identity for $v(\uparrow)\bar{u}(\uparrow)$:

$$\begin{aligned}
v(\uparrow)\bar{u}(\uparrow) &= (E_p+m)\begin{pmatrix} \frac{\sigma_z p_z}{E_p+m}\chi_\uparrow \\ \chi_\uparrow \end{pmatrix} \begin{pmatrix} \phi_\uparrow^\dagger & \phi_\uparrow^\dagger \left(\frac{\sigma_z p_z}{E_p+m}\right)^\dagger \end{pmatrix}\gamma^0 \\
&= \begin{pmatrix} \sigma_z p_z \chi_\uparrow \phi_\uparrow^\dagger & -\sigma_z p_z \chi_\uparrow \phi_\uparrow^\dagger \left(\frac{\sigma_z p_z}{E_p+m}\right)^\dagger \\ (E_p+m)\chi_\uparrow\phi_\uparrow^\dagger & -\chi_\uparrow\phi_\uparrow^\dagger (\sigma_z p_z)^\dagger \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -p_z & 0 & (E_p-m) & 0 \\ 0 & 0 & 0 & 0 \\ (E_p+m) & 0 & -p_z & 0 \end{pmatrix}.
\end{aligned} \tag{A.23}$$

Then let us open $\frac{-1}{\sqrt{2}}\not{\epsilon}^*(\uparrow)\frac{\not{p}_J+M}{2}$ with $p=p_J/2$ and $m=M/2$,

$$\begin{aligned}
\frac{-1}{\sqrt{2}}\not{\epsilon}^*(\uparrow)\frac{\not{p}_J+M}{2} &= \frac{1}{2}(-\gamma^1+i\gamma^2)(\not{p}+m) \\
&= \frac{1}{2}\begin{pmatrix} 0 & -\sigma_x+i\sigma_y \\ \sigma_x-i\sigma_y & 0 \end{pmatrix} \begin{pmatrix} (E_p+m)\mathbb{1}_2 & -p_z\sigma_z \\ p_z\sigma_z & (-E_p+m)\mathbb{1}_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -p_z & 0 & (E_p-m) & 0 \\ 0 & 0 & 0 & 0 \\ (E_p+m) & 0 & -p_z & 0 \end{pmatrix},
\end{aligned} \tag{A.24}$$

which is the same as equation A.23. Therefore the equality

$$v(\uparrow)\bar{u}(\uparrow) = \frac{-1}{\sqrt{2}}\not{\epsilon}^*(\uparrow)\frac{\not{p}_J+M}{2} \tag{A.25}$$

holds. Now continuing for $v(\downarrow)\bar{u}(\downarrow)$:

$$\begin{aligned}
v(\downarrow)\bar{u}(\downarrow) &= (E_p+m)\begin{pmatrix}\frac{\sigma_z p_z}{E_p+m}\chi_\downarrow \\ \chi_\downarrow\end{pmatrix}\begin{pmatrix}\phi_\downarrow^\dagger & \phi_\downarrow^\dagger\left(\frac{\sigma_z p_z}{E_p+m}\right)^\dagger\end{pmatrix}\gamma^0 \\
&= \begin{pmatrix}\sigma_z p_z \chi_\downarrow \phi_\downarrow^\dagger & -\sigma_z p_z \chi_\downarrow \phi_\downarrow^\dagger\left(\frac{\sigma_z p_z}{E_p+m}\right)^\dagger \\ (E_p+m)\chi_\downarrow\phi_\downarrow^\dagger & -\chi_\downarrow\phi_\downarrow^\dagger(\sigma_z p_z)^\dagger\end{pmatrix} \\
&= \begin{pmatrix}0 & -p_z & 0 & (-E_p+m) \\ 0 & 0 & 0 & 0 \\ 0 & -(E_p+m) & 0 & -p_z \\ 0 & 0 & 0 & 0\end{pmatrix}.
\end{aligned} \tag{A.26}$$

Continuing for $\frac{-1}{\sqrt{2}}\not{\epsilon}^*(\downarrow)\frac{\not{p}_J+M}{2}$:

$$\begin{aligned}
\frac{-1}{\sqrt{2}}\not{\epsilon}^*(\downarrow)\frac{\not{p}_J+M}{2} &= \frac{-1}{2}(-\gamma^1-i\gamma^2)(\not{p}+m) \\
&= \frac{1}{2}\begin{pmatrix}0 & \sigma_x+i\sigma_y \\ -\sigma_x-i\sigma_y & 0\end{pmatrix}\begin{pmatrix}(E_p+m)\mathbb{1}_2 & -p_z\sigma_z \\ p_z\sigma_z & -(E_p-m)\mathbb{1}_2\end{pmatrix} \\
&= \begin{pmatrix}0 & -p_z & 0 & (-E_p+m) \\ 0 & 0 & 0 & 0 \\ 0 & -(E_p+m) & 0 & -p_z \\ 0 & 0 & 0 & 0\end{pmatrix},
\end{aligned} \tag{A.27}$$

which is the same as equation A.26, thus the equality

$$v(\downarrow)\bar{u}(\downarrow) = \frac{-1}{\sqrt{2}}\not{\epsilon}^*(\downarrow)\frac{\not{p}_J+M}{2}. \tag{A.28}$$

Finally continuing for $\frac{1}{\sqrt{2}}(v(\uparrow)\bar{u}(\downarrow)+v(\downarrow)\bar{u}(\uparrow))$. We have now twice calculated these type of matrices, so we may be a bit more straightforward with our calculations. First we have

$$v(\uparrow)\bar{u}(\downarrow) = \begin{pmatrix}\sigma_z\cdot p_z\chi_\uparrow\phi_\downarrow^\dagger & -\sigma_z\cdot p_z\chi_\uparrow\phi_\downarrow^\dagger\left(\frac{\sigma_z\cdot p_z}{E_p+m}\right)^\dagger \\ (E_p+m)\chi_\uparrow\phi_\downarrow^\dagger & -\chi_\uparrow\phi_\downarrow^\dagger(\sigma_z\cdot p_z)^\dagger\end{pmatrix}, \tag{A.29}$$

and similarly

$$v(\downarrow)\bar{u}(\uparrow) = \begin{pmatrix}\sigma_z\cdot p_z\chi_\downarrow\phi_\uparrow^\dagger & -\sigma_z\cdot p_z\chi_\downarrow\phi_\uparrow^\dagger\left(\frac{\sigma_z\cdot p_z}{E_p+m}\right)^\dagger \\ (E_p+m)\chi_\downarrow\phi_\uparrow^\dagger & -\chi_\downarrow\phi_\uparrow^\dagger(\sigma_z\cdot p_z)^\dagger\end{pmatrix}, \tag{A.30}$$

and adding these two together we get:

$$(v(\uparrow)\bar{u}(\downarrow)+v(\downarrow)\bar{u}(\uparrow)) = \begin{pmatrix} -p_z & 0 & (E_p-m) & 0 \\ 0 & -p_z & 0 & (-E_p+m) \\ -(E_p+m) & 0 & p_z & 0 \\ 0 & (E_p+m) & 0 & p_z \end{pmatrix}, \tag{A.31}$$

where we have chosen not to write in the $1/\sqrt{2}$ factor. Next we continue to inspect the r.h.s. of equation A.22. Keeping in mind the form of the polarization vector as given in equation A.3 and the normalization of the polarization vector as shown in equation A.17, we can write down the matrix form:

$$\begin{aligned} -\not{\epsilon}^*(0)\frac{\not{p}_J+M}{2} &= \frac{-1}{m}(p_z\gamma^0-E_p\gamma^3)\begin{pmatrix}(E_p+m)\mathbb{1}_2 & -p_z\sigma_z \\ p_z\sigma_z & -(E_p-m)\mathbb{1}_2\end{pmatrix} \\ &= \frac{-1}{m}\begin{pmatrix}p_z\mathbb{1}_2 & -E_p\sigma_z \\ E_p\sigma_z & -p_z\mathbb{1}_2\end{pmatrix}\begin{pmatrix}(E_p+m)\mathbb{1}_2 & -p_z\sigma_z \\ p_z\sigma_z & -(E_p-m)\mathbb{1}_2\end{pmatrix} \\ &= \begin{pmatrix}-p_z\mathbb{1}_2 & (E_p-m)\sigma_z \\ -(E_p+m)\sigma_z & p_z\mathbb{1}_2\end{pmatrix}, \end{aligned} \tag{A.32}$$

which is the same as in equation A.31. Therefore the identity

$$\frac{1}{\sqrt{2}}(v(\uparrow)\bar{u}(\downarrow)+v(\downarrow)\bar{u}(\uparrow)) = -\frac{1}{\sqrt{2}}\not{\epsilon}^*(0)\frac{\not{p}_J+M}{2} \tag{A.33}$$

holds.

Next we would like to rotate the vectors ϵ, u, v and p, and show that our results hold in any arbitrary direction. To do this we first have to consider what happens to the solutions of the Dirac equation in a proper Lorentz transformation (LT). The Dirac equation can be written as

$$(i\not{\partial}-m)\psi(x)=0, \tag{A.34}$$

which has solutions of the form $\psi(x)=u(p)e^{-ip\cdot x}$ where $u(p)=(u_1(p),u_2(p),u_3(p),u_4(p))^T$. The contravariant 4-vector x^ν transforms as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{A.35}$$

where $\Lambda^\mu{}_\nu$ are the componenets of the LT. It then follows that a covariant 4-vector x_α transforms as

$$x'_\mu = \Lambda_\mu{}^\alpha x_\alpha, \tag{A.36}$$

where $\Lambda_\mu{}^\alpha$ are the components of the inverse LT,

$$(\lambda^{-1})^\mu{}_\nu = g^{\mu\alpha}\Lambda^\beta{}_\alpha g_{\beta\nu} = \Lambda_\nu{}^\mu. \tag{A.37}$$

Then for a LT of the form $x' = \Lambda x$, it follows that there must exist a relation

$$\psi'(x') = S\psi(x), \tag{A.38}$$

where S is some matrix that accomplished the desired transformation. In addition to knowing how the spinor transforms, we need to know how the conjugate spinor $\bar{\psi}$ transforms. The following development derives that result. See reference [17, p.113] for further details. We may write the Dirac equation in two different inertial frames:

$$i\gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} - m\psi(x) = 0 \quad \text{and} \quad i\gamma^\mu \frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0, \tag{A.39}$$

Since from equation A.36 we know that covariant vectors transform with the inverse LT we may open up the primed DE:

$$\begin{aligned}
& i\gamma^\mu \frac{\partial\psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0 \\
\Rightarrow \quad & i\gamma^\mu \frac{\partial}{\partial x'^\mu} S\psi(x) - mS\psi(x) = 0 \\
\Rightarrow \quad & i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu} S\psi(x) - mS\psi(x) = 0 \\
\Rightarrow \quad & i\gamma^\mu S(\Lambda^{-1})^\nu{}_\mu \partial_\nu\psi(x) - Sm\psi(x) = 0 \quad || \to S^{-1} \\
\Rightarrow \quad & i\,\underbrace{S^{-1}\gamma^\mu S(\Lambda^{-1})^\nu{}_\mu}_{=\gamma^\nu} \partial_\nu\psi(x) - m\psi(x) = 0,
\end{aligned} \tag{A.40}$$

where the underbraced equality must hold in order for the DE to be Lorentz covariant i.e. the equations of motion in different frames need to be equivalent. We then have the constraint

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu, \tag{A.41}$$

which also gives

$$S\gamma^\nu S^{-1} = (\Lambda^{-1})^\nu{}_\mu \gamma^\mu. \tag{A.42}$$

Then for a proper infinitesimal LT of the form $\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \epsilon^\nu{}_\mu$, the S-matrix can be shown to be

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \quad \text{and} \quad S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}, \tag{A.43}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{A.44}$$

and each $\epsilon^{\mu\nu}$ is some real number and as discussed below $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Clearly also $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$. Let us then show these results. We demand the invariance of the scalar products,

$$x \cdot y = x' \cdot y' \Leftrightarrow g_{\mu\nu} x^\mu y^\nu = g_{\rho\alpha} \Lambda^\rho{}_\mu x^\mu \Lambda^\alpha{}_\nu y^\nu \Leftrightarrow g_{\mu\nu} = g_{\rho\alpha} \Lambda^\rho{}_\mu \Lambda^\alpha{}_\nu. \tag{A.45}$$

By substituting in our infinitesimal LT and ignoring higher order terms, we get

$$\begin{aligned} g_{\mu\nu} &= g_{\rho\alpha} \Lambda^\rho{}_\mu \Lambda^\alpha{}_\nu = g_{\rho\alpha} (\delta^\rho{}_\mu + \epsilon^\rho{}_\mu)(\delta^\alpha{}_\nu + \epsilon^\alpha{}_\nu) \\ \Rightarrow \quad g_{\mu\nu} &= g_{\mu\nu} + g_{\mu\alpha} \epsilon^\alpha{}_\nu + g_{\rho\nu} \epsilon^\rho{}_\mu \\ \Rightarrow \quad g_{\mu\alpha} \epsilon^\alpha{}_\nu &= -g_{\rho\nu} \epsilon^\rho{}_\mu \quad \Rightarrow \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \end{aligned} \tag{A.46}$$

We can also see that S^{-1} is as given in equation A.43. Again ignoring the higher order terms:

$$SS^{-1} = \left(1 - \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu}\right)\left(1 + \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu}\right) = 1 + \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu} - \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu} = 1. \tag{A.47}$$

Then in substituting these into equation A.41, we can see that the S is truly the desired matrix:

$$\begin{aligned} \left(1 + \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu}\right) \gamma^\alpha \left(1 - \frac{i}{4} \sigma_{\mu\nu} \epsilon^{\mu\nu}\right) &= (\delta^\alpha{}_\beta + \epsilon^\alpha{}_\beta) \gamma^\beta \\ \Rightarrow \quad \gamma^\alpha - \frac{i}{4} \epsilon^{\mu\nu} [\gamma^\alpha, \sigma_{\mu\nu}] &= \gamma^\alpha + \epsilon^\alpha{}_\beta \gamma^\beta \\ \Rightarrow \quad \frac{1}{8} \epsilon^{\mu\nu} [\gamma^\alpha, \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] &= \epsilon^\alpha{}_\beta \gamma^\beta. \end{aligned} \tag{A.48}$$

In order to process this commutator further we need some commutator identities and gamma matrix results:

$$\begin{aligned} &(1) \quad [a, b + c] = ab + ac - ba - ca = [a, b] + [a, c], \\ &(2) \quad [a, bc] = abc - bca = abc - bca + bac - bac = b[a, c] + [a, b]c, \\ &(3) \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}_4 \Rightarrow [\gamma^\mu, \gamma^\nu] = 2(g^{\mu\nu} - \gamma^\nu \gamma^\mu). \end{aligned} \tag{A.49}$$

Let us then process the commutator further:

$$\begin{aligned} [\gamma^\alpha, \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu] &= [\gamma^\alpha, \gamma_\mu \gamma_\nu] - [\gamma^\alpha, \gamma_\nu \gamma_\mu] \\ &= \gamma_\mu [\gamma^\alpha, \gamma_\nu] + [\gamma^\alpha, \gamma_\mu] \gamma_\nu - \gamma_\nu [\gamma^\alpha, \gamma_\mu] - [\gamma^\alpha, \gamma_\nu] \gamma_\mu \\ &= -2\gamma_\mu \gamma_\nu \gamma^\alpha - 2\gamma_\mu \gamma^\alpha \gamma_\nu + 2\gamma_\nu \gamma_\mu \gamma^\alpha + 2\gamma_\nu \gamma^\alpha \gamma_\mu \\ &= -4\gamma_\mu \delta^\alpha{}_\nu + 4\gamma_\nu \delta^\alpha{}_\mu. \end{aligned} \tag{A.50}$$

Substituting this back into the l.h.s. of A.48:

$$\frac{1}{8}\epsilon^{\mu\nu}(-4\gamma_\mu\delta^\alpha_{\ \nu} + 4\gamma_\nu\delta^\alpha_{\ \mu}) = \frac{1}{2}(-\epsilon^{\mu\alpha}\gamma_\mu + \epsilon^{\alpha\nu}\gamma_\nu) = \epsilon^{\alpha\nu}\gamma_\nu = \epsilon^\alpha_{\ \nu}\gamma^\nu. \tag{A.51}$$

We can see that with equation A.51 we have recovered the r.h.s. of equation A.48. Thus we have shown that for a proper LT, the matrix S is given by equation A.43. We have gone through all this to be able to show that the inverse of S is given by

$$S^{-1} = \gamma^0 S^\dagger \gamma^0 \Leftrightarrow S^\dagger\gamma^0 = \gamma^0 S^{-1}. \tag{A.52}$$

Keeping in mind that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ we may inspect $S^\dagger$:

$$\begin{aligned} S^\dagger &= \left(1 - \frac{i}{4}\sigma^{\mu\nu}\epsilon_{\mu\nu}\right)^\dagger = 1 + \frac{i}{4}\epsilon^\dagger_{\mu\nu}\sigma^{\mu\nu\dagger} = 1 + \frac{i}{4}\epsilon_{\mu\nu}\left(\frac{-i}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger})\right) \\ &= 1 + \frac{i}{4}\epsilon_{\mu\nu}\left(\frac{-i}{2}(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0)\right) \quad ||\gamma^0 \rightarrow\leftarrow \gamma^0 \\ \Rightarrow \quad \gamma^0 S^\dagger\gamma^0 &= 1 + \frac{i}{4}\epsilon_{\mu\nu}\underbrace{\left(\frac{-i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\right)}_{=\sigma^{\mu\nu}} \equiv S^{-1}. \end{aligned} \tag{A.53}$$

With these results we may finally inspect how the spinors transform in LTs. From equation A.38 and the solutions of the DE, it follows that $Su = u'$, where u is some Dirac spinor and u' is the transformed one. Then from:

$$\bar{\psi}' = \psi'^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0 = \psi^\dagger\gamma^0 S^{-1} \equiv \bar{\psi}S^{-1}, \tag{A.54}$$

it follows that $\bar{u}' = \bar{u}S^{-1}$. This is to say that the Dirac spinors transform with S multiplied from the left and the conjugate spinors transform with S^{-1} multiplied from the right.

We may then finally rotate the result of equation A.25 to hold in any direction. Multiplying from the right with S and with S^{-1} from the left:

$$\begin{aligned} \underbrace{Sv(\uparrow)}_{=v'(\uparrow)}\underbrace{\bar{u}(\uparrow)S^{-1}}_{\bar{u}'(\uparrow)} &= \frac{-1}{2\sqrt{2}}\left(S\epsilon^*_\mu(\uparrow)\gamma^\mu p_{J\nu}\gamma^\nu S^{-1} + S\epsilon^*_\mu(\uparrow)\gamma^\mu M S^{-1}\right) \\ &= \frac{-1}{2\sqrt{2}}\left(\epsilon^*_\mu(\uparrow)S\gamma^\mu\underbrace{S^{-1}S}_{=\mathbb{1}}\gamma^\nu S^{-1}p_{J\nu} + \epsilon^*_\mu(\uparrow)S\gamma^\mu S^{-1}M\right). \end{aligned} \tag{A.55}$$

Then we use equations A.41 and A.42 to write the inverse LT on every term where it is possible

$$v'(\uparrow)\bar{u}'(\uparrow) = \frac{-1}{2\sqrt{2}}\left(\epsilon^*_\mu(\uparrow)\left(\Lambda^{-1}\right)^\mu_{\ \alpha}\gamma^\alpha\left(\Lambda^{-1}\right)^\nu_{\ \beta}\gamma^\beta p_{J\nu} + \epsilon^*_\mu(\uparrow)\left(\Lambda^{-1}\right)^\mu_{\ \alpha}\gamma^\alpha M\right) \tag{A.56}$$

and by using A.36 it clearly follows that:

$$v'(\uparrow)\bar{u}'(\uparrow) = \frac{-1}{\sqrt{2}}\not{\epsilon}'^{*}(\uparrow)\frac{\not{p}'_J + M}{2}. \tag{A.57}$$

We have now generalized the result of equation A.25 to hold for p_J into an arbitrary direction. We can also see that the same developmet holds for the other results of A.22. The only thing left to do is to show that 3.30 follows from A.22. Let us do this next. We can start by taking the hermitean conjugate of equation A.57 and dropping the primes for clarity:

$$\begin{aligned}
& (v(\uparrow)\bar{u}(\uparrow))^\dagger = \frac{-1}{2\sqrt{2}}(\not{\epsilon}^{*}(\uparrow)(\not{p}_J + M))^\dagger \\
\Rightarrow\quad & \gamma^0 u(\uparrow)v^\dagger(\uparrow) = \frac{-1}{2\sqrt{2}}\left(p^\dagger_{J\mu}\gamma^{\mu\dagger}\epsilon^{*\dagger}_\nu(\uparrow)\gamma^{\nu\dagger} + M\epsilon^{*\dagger}_\nu\gamma^{\nu\dagger}\right) \\
\Rightarrow\quad & \gamma^0 u(\uparrow)v^\dagger(\uparrow) = \frac{-1}{2\sqrt{2}}\left(p_{J\mu}\gamma^0\gamma^\mu\gamma^0\epsilon_\nu(\uparrow)\gamma^0\gamma^\nu\gamma^0 + M\epsilon_\nu(\uparrow)\gamma^0\gamma^\nu\gamma^0\right).
\end{aligned} \tag{A.58}$$

Then by multiplying with γ^0 from the left and the right, we recover the form

$$u(\uparrow)\bar{v}(\uparrow) = -\frac{1}{\sqrt{2}}\left(\frac{\not{p}_J + M}{2}\right)\not{\epsilon}(\uparrow), \tag{A.59}$$

as given in equation 3.30. We can also see that the other results of 3.30 come about identically.

B Calculations for section 4

B.1 Helicity basis identity

Our aim in this section is to derive the results given in equation 4.10. We will work with the solutions of the Dirac equation as is given in the equation 3.29. By looking at equations A.29, A.30 and A.31 we can immediately see the form of $[v(\uparrow)\bar{u}(\downarrow) - v(\downarrow)\bar{u}(\uparrow))]$ in the case where $\vec{p}_\eta||$z-axis:

$$[v(\uparrow)\bar{u}(\downarrow) - v(\downarrow)\bar{u}(\uparrow)] = \begin{pmatrix} p_z & 0 & (-E_p+m) & 0 \\ 0 & -p_z & 0 & (-E_p+m) \\ (E_p+m) & 0 & -p_z & 0 \\ 0 & (E_p+m) & 0 & p_z \end{pmatrix}, \quad \text{(B.1)}$$

with p_z, E_p and m being charm quark quantities. Then on the r.h.s. of equation 4.10 we get

$$\begin{aligned} \gamma^5 \frac{\not{p}_\eta + M}{2} &= \gamma^5(p_0\gamma^0 - p_z\gamma^3 + m\mathbb{1}_4) = \gamma^5 \begin{pmatrix} (E_p+m)\mathbb{1}_2 & -p_z\sigma_z \\ p_z\sigma_z & -(E_p-m)\mathbb{1}_2 \end{pmatrix} \\ &= \begin{pmatrix} p_z\sigma_z & -(E_p-m)\mathbb{1}_2 \\ (E_p+m)\mathbb{1}_2 & -p_z\sigma_z \end{pmatrix}. \end{aligned} \quad \text{(B.2)}$$

We can clearly see that equations B.1 and B.2 are equal and thus the identity

$$\frac{1}{\sqrt{2}}[v(\uparrow)\bar{u}(\downarrow) - v(\downarrow)\bar{u}(\uparrow)] = \frac{1}{\sqrt{2}}\gamma^5 \frac{\not{p}_\eta + M}{2} \quad \text{(B.3)}$$

holds. Of course this proves the result only in the special case that the pseudoscalar meson is moving into the z-axis direction. However, as was developed in the section A.3, this result can also be rotated into an arbitrary direction. Since the proof is essentially identical, we shall not repeat it here. However, we can see that by taking the hermitean conjugate of equation B.3, we get the latter result of equation 4.10:

$$\frac{1}{\sqrt{2}}[\gamma^0 u(\downarrow)v^\dagger(\uparrow) - \gamma^0 u(\uparrow)v^\dagger(\downarrow)] = \frac{1}{2\sqrt{2}}(p_{\eta\mu}\gamma^{\mu\dagger} + M)\gamma^{5\dagger}, \quad \text{(B.4)}$$

and by multiplying with γ^0 from left and right, we recover:

$$\frac{1}{\sqrt{2}}[u(\uparrow)\bar{v}(\downarrow) - u(\downarrow)\bar{v}(\uparrow)] = \frac{1}{\sqrt{2}}\frac{\not{p}_\eta + M}{2}\gamma^5, \tag{B.5}$$

where we have used the results $\gamma^{5\dagger} = \gamma^5$, $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $\{\gamma^5, \gamma^\mu\} = 0$.

B.2 Trace identities

Let us justify the identity of equation 4.14. We know that $\mathrm{Tr}(\gamma^5) = 0$ and $(\gamma^\mu)^2 = \pm\mathbb{1}_4$. First we need to consider the trace $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu)$. Assuming $\mu = \nu$ we have

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^5 \underbrace{\gamma^\mu\gamma^\mu}_{=\pm\mathbb{1}_4}) = \underbrace{\mathrm{Tr}(\pm\gamma^5)}_{=0} = 0. \tag{B.6}$$

We may then assume that $\mu \neq \nu$ and get

$$\gamma^5\gamma^\mu\gamma^\nu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu = \pm i\gamma^\alpha\gamma^\beta, \tag{B.7}$$

since the indices μ and ν have to be one of the values 0,1,2,3 and by the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 0$, whenever $\mu \neq \nu$. Notice that it especially holds that $\alpha \neq \beta$. We then can use the anticommutation of the gammas and the cyclic property of trace to get

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \pm i\mathrm{Tr}(\gamma^\alpha\gamma^\beta) \Rightarrow \mathrm{Tr}(\gamma^\alpha\gamma^\beta) \overset{\text{cycl.}}{=} \mathrm{Tr}(\gamma^\beta\gamma^\alpha) \overset{\text{anticom.}}{=} -\mathrm{Tr}(\gamma^\alpha\gamma^\beta). \tag{B.8}$$

Thus $\mathrm{Tr}(\gamma^\alpha\gamma^\beta) = 0$ and we have

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0. \tag{B.9}$$

With this done we can consider the trace of γ^5 with four other gamma matrices, $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$. Clearly it must hold that $\mu \neq \nu \neq \rho \neq \sigma$. This is to say that each index has a different value. Since if they would not be different we would recover the result of equation B.9. So we have that (μ, ν, ρ, σ) is a permutation of (0,1,2,3). We will first choose $(\mu, \nu, \rho, \sigma) = (0,1,2,3)$:

$$\mathrm{Tr}(\gamma^5 \underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{=-i\gamma^5}) = -i\mathrm{Tr}\Big(\underbrace{(\gamma^5)^2}_{=\mathbb{1}_4}\Big) = -4i. \tag{B.10}$$

Now if we would have some other permutation of the indices (0,1,2,3) we would need to anticommute the gamma matrices to their right places in order to recover the result of equation B.10. Thus with an even number of exchanges we get the same result, $-4i$, and with an odd number we get $+4i$. This is how the result of equation 4.14 comes about.

C Calculations for section 5

C.1 Proof of some identities

We have been extensively using trace and vector identities. Even though some of these results should be familiar to the reader from elementary physics courses, let us still justify some of them. Let us first go through the result $\mathrm{Tr}(\sigma^i\sigma^j\sigma^k) = 2i\epsilon^{ijk}$. Assume we know that

$$\mathrm{Tr}(\mathbb{1}_2) = 2; \quad \mathrm{Tr}(\sigma_i) = 0; \quad \sigma^i\sigma^j = \delta_{ij}\mathbb{1}_2 + i\epsilon^{ijk}\sigma^k; \quad \mathrm{Tr}(\sigma^i\sigma^j) = 2\delta_{ij}. \tag{C.1}$$

With these results we can turn to inspect $\mathrm{Tr}(\sigma^i\sigma^j\sigma^k)$:

$$\begin{aligned}\mathrm{Tr}(\sigma^i\sigma^j\sigma^k) &= \mathrm{Tr}([\delta_{ij}\mathbb{1}_2 + i\epsilon^{ijl}\sigma^l]\sigma^k) = \mathrm{Tr}(\delta_{ij}\sigma^k + i\epsilon^{ijl}\sigma^l\sigma^k)\\ &= \delta_{ij}\underbrace{\mathrm{Tr}(\sigma^k)}_{=0} + i\epsilon^{ijl}\underbrace{\mathrm{Tr}(\sigma^k\sigma^l)}_{2\delta_{kl}}\\ &= 2i\epsilon^{ijk}.\end{aligned} \tag{C.2}$$

The first identity of equation 5.10 follows from

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sigma^i a^i\sigma^j b^j = a^i b^j\sigma^i\sigma^j = a^i b^j(\delta_{ij}\mathbb{1}_2 + i\epsilon^{ijk}\sigma^j) = (\vec{a}\cdot\vec{b})\mathbb{1}_2 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}. \tag{C.3}$$

Then let us inspect the term $\boldsymbol{\epsilon}_3 \times (\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2)$:

$$\begin{aligned}\boldsymbol{\epsilon}_3 \times (\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2) &= \epsilon^{ijk}\epsilon_3^j\epsilon^{klm}\epsilon_1^l\epsilon_2^m = \epsilon^{kij}\epsilon^{klm}\epsilon_3^j\epsilon_1^l\epsilon_2^m\\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_3^j\epsilon_1^l\epsilon_2^m = \epsilon_1^i\epsilon_3^m\epsilon_2^m - \epsilon_2^i\epsilon_3^l\epsilon_1^l\\ &= \boldsymbol{\epsilon}_1(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_2) - \boldsymbol{\epsilon}_2(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_1).\end{aligned} \tag{C.4}$$

Let us also justify the identity of equation 5.38:

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) &= \epsilon^{ijk}a^j b^k\epsilon^{ilm}c^l d^m = \epsilon^{ijk}\epsilon^{ilm}a^j b^k c^l d^m\\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})a^j b^k c^l d^m\\ &= (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}).\end{aligned} \tag{C.5}$$

Finally, let us justify how the three-dimensional integral can be turned to a four-dimensional one as shown below:

$$\int \frac{d^3k_3}{2\omega_3} = \int d^4k_3\theta(\omega_3)\delta(k_3^2), \tag{C.6}$$

where $\theta(\omega_3)$ is the Heaviside step function. First pointing out that for a delta function with a function $g(x)$ as its argument, we can write

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{C.7}$$

where x_i are the zeros of $g(x)$ and g' denotes the derivative of g with respect to x. Starting with the r.h.s. of equation C.6, using equation C.7 and $k_3^2 = \omega_3^2 - \boldsymbol{k}_3^2$:

$$\begin{aligned}
\int d^4k_3 \theta(\omega_3)\delta(k_3^2) &= \int d\omega_3 d^3k_3 \; \theta(\omega_3)\delta(\omega_3^2 - \boldsymbol{k}_3^2) \\
&= \int d^3k_3 \int d\omega_3 \; \theta(\omega_3)\frac{1}{2\omega_3}\left(\delta(\omega_3 + |\boldsymbol{k}_3|) + \delta(\omega_3 - |\boldsymbol{k}_3|)\right) \\
&= \int d^3k_3 \left(\frac{1}{2\omega_3} \underbrace{\theta(\omega_3)\Big|_{\omega_3 = -|\boldsymbol{k}_3|}}_{=0} + \frac{1}{2\omega_3} \underbrace{\theta(\omega_3)\Big|_{\omega_3 = |\boldsymbol{k}_3|}}_{=1} \right) \\
&= \int \frac{d^3k_3}{2\omega_3}.
\end{aligned} \tag{C.8}$$

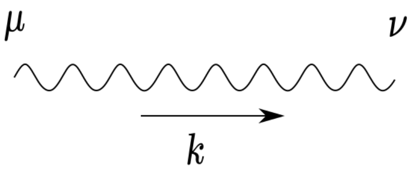
D Feynman rules

There are multiple different sources that offer a comprehensive listing of the Feynman rules. See for example the following [2], [4], [7], [18]. However, for purposes of completeness, let us explicitly state them here.

D.1 QED

Propagators

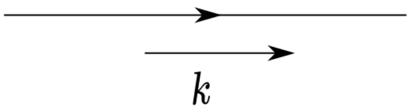
Photon:



$$= -\frac{i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\lambda)\frac{k^\mu k^\nu}{k^2}\right)$$

In the Landau gauge $\lambda = 0$.
In the Feynman gauge $\lambda = 1$.

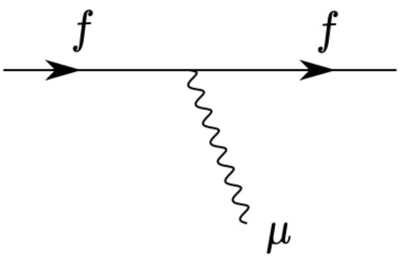
Lepton or a quark:



$$= -\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

Vertex

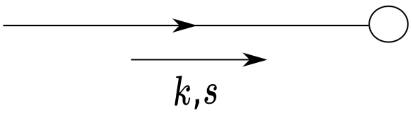
f denotes the flavour.
$Q_f = -1$ for e,μ, τ
$Q_f = +\frac{2}{3}$ for u,c,t
$Q_f = -\frac{1}{3}$ for d,s,b



$$= ieQ_f\gamma^\mu$$

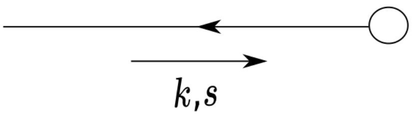
External particles

Incoming lepton or a quark.



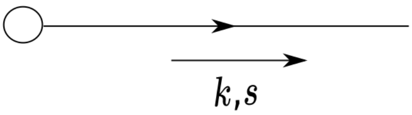
$= u(k,s)$

Incoming antilepton or an antiquark:



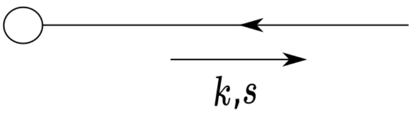
$= \bar{v}(k,s)$

Outgoing lepton or a quark:



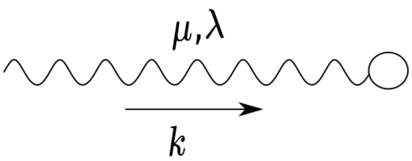
$= \bar{u}(k,s)$

Outgoing antilepton or an antiquark:



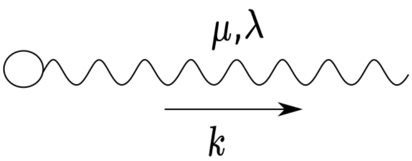
$= v(k,s)$

Incoming photon



$= \epsilon_\mu(k,\lambda)$

Outgoing photon.

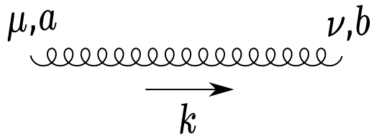


$= \epsilon^*_\mu(k,\lambda)$

D.2 QCD

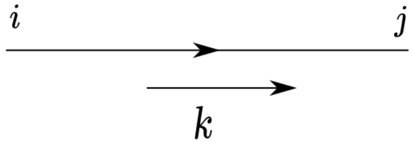
Propagators

Gluon:



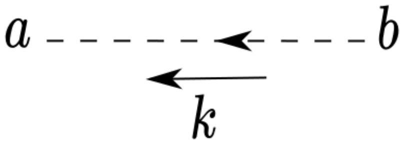
$$= -\frac{i\delta^{ab}}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\lambda)\frac{k^\mu k^\nu}{k^2}\right)$$

Quark:



$$= \frac{i\delta_{ij}(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

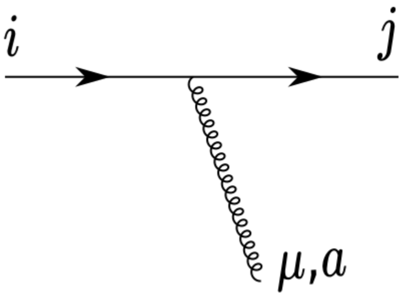
Ghost:



$$= \frac{i\delta^{ab}}{k^2 + i\epsilon}$$

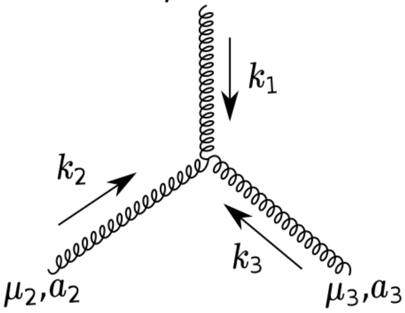
Vertices

Quark-gluon coupling:



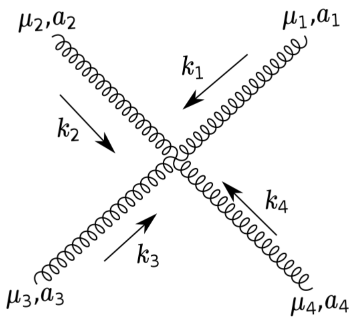
$$= -ig_s(t^a)_{ji}\gamma^\mu$$

3-gluon self-coupling:



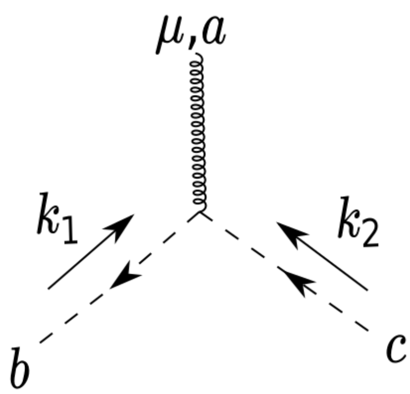
$$\begin{aligned} = -g_s f^{a_1 a_2 a_3}\Big[&g^{\mu_1\mu_2}(k_1 - k_2)^{\mu_3} \\ &+ g^{\mu_2\mu_3}(k_2 - k_3)^{\mu_1} \\ &+ g^{\mu_3\mu_1}(k_3 - k_1)^{\mu_2}\Big]\end{aligned}$$

4-gluon self-coupling:



$$= -ig_s^2\Big[f^{ea_1a_2}f^{ea_3a_4}(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}) + f^{ea_1a_3}f^{ea_4a_2}(g_{\mu_1\mu_4}g_{\mu_3\mu_2} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}) + f^{ea_1a_4}f^{ea_2a_3}(g_{\mu_1\mu_2}g_{\mu_4\mu_3} - g_{\mu_1\mu_3}g_{\mu_4\mu_2})\Big]$$

Ghost-gluon coupling:



$$= -g_s f^{abc} k_1^{\mu}$$

External lines

These are the same as in QED but the photon line is replaced by the gluon line as given above. Color is treated in the vertices.

E Pauli matrices and Dirac matrices

For purposes of completeness, we shall give the explicit form of the Pauli and Dirac matrices. For references, see for example [2], [3], [11].

Pauli matrices

The Pauli matrices σ^i satisfy the following equation:

$$\sigma^i \sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k, \tag{E.1}$$

where δ_{ij} is the Kronecker delta function and ϵ_{ijk} is the Levi-Civita symbol. From it, we can derive the commutation and anticommutation relations

$$[\sigma^i,\sigma^j] = 2i\epsilon_{ijk}\sigma^k; \quad \{\sigma^i,\sigma^j\} = 2\delta_{ij}\mathbb{1}_2. \tag{E.2}$$

The explicit form for the Pauli matrices can be derived to be

$$\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{E.3}$$

With these we may also define the so called Pauli vector $\vec{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$.

Dirac matrices

The Dirac matrices γ^μ satisfy the following anticommutation relation given by the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4. \tag{E.4}$$

In the Dirac representation, the explicit form is

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{E.5}$$

where σ^i are the Pauli spin matrices. Moreover, we may define an additional matrix γ^5 which in the Dirac basis can be explicitly written as follows:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{E.6}$$

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