Evaluation of adjoint representation Wilson line 4-point correlators in the MV model

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Abstract

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The Color Glass Condensate, a QCD effective theory, is presented. It describes the small momentum fraction (x) gluons in a high energy hadron or nucleus, in the saturation regime. Due to the high density of small-x gluons, they can be modelled as a classical color field with large-x partons acting as sources for this field. It is explained how the multiple scatterings of partons off of a color field can be summed using Wilson lines. The main point of this work is to derive an analytical expression for the adjoint representation Wilson line 4-point correlators. These correlators arise when computing the scattering amplitudes for processes which involve two gluons propagating through the color field together. The computation is done by assuming the Gaussian distribution of color sources, which was argued in the MV model.

Keywords: color glass condensate, gluon saturation, Wilson line, adjoint representation, MV model

Tiivistelmä

Demirci, Sami
Adjungoidussa esityksessä olevien Wilsonin viivojen 4-pistekorrelaattoreiden laskeminen MV mallissa
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Työssä esitellään Värilasikondensaatiksi kutsuttu QCD:n efektiivinen teoria. Se kuvaa suurienergisen hadronin tai ytimen pienen liikemääräfraktion (x) gluoneita saturaatioalueessa. Pienen (x):n gluonien suuren tiheyden vuoksi niitä voidaan kuvailla klassisella värikentällä, jonka lähteenä toimivat suuren (x):n partonit. Työssä selitetään kuinka partonien lukuisat siroamiset värikentästä voidaan summata käyttäen Wilsonin viivoja. Työn päätavoite on johtaa analyyttinen lauseke adjungoidun esityksen Wilsonin viivojen 4-pistekorrelaattoreille. Näitä korrelaattoreita ilmaantuu, kun lasketaan sironta-amplitudeja prosesseille, joissa kaksi gluonia etenee yhdessä värikentän lävitse. Lasku tehdään olettaen Gaussinen jakauma värilähteille, joka perusteltiin MV mallissa.

Avainsanat: värilasikondensaatti, gluonisaturaatio, Wilsonin viiva, adjungoitu esitys, MV malli

Foreword

It may sound like a cliché, but I have always had the desire to know how the world works. When I was young I was in awe of how devices built by people work, how every part, every cog and gear, does it job in perfect harmony. First I wanted to become a car mechanic, and later I wanted to be an aeroplane mechanic. After some iterations and getting acquainted with natural sciences, I decided that I wanted to understand the universe in the most fundamental level possible. Thus I wanted to become a physicist, and now I am dealing with matters of quantum mechanics. So in the end, it seems like I became a mechanic after all.

I would like to thank my supervisor professor Tuomas Lappi for giving me such an interesting topic, for his guidance and for always having time to discuss my questions and problems. I would also like to thank my dear friends and colleagues in and around FYS4 for striking up interesting conversations and for creating a lovely working environment.

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Sami Demirci

Contents

1 Introduction

The theory of strong interactions, quantum chromodynamics (QCD), has taken its place as the theory describing the interactions between the constituents of regular hadronic matter. These constituents are called quarks and gluons. The validity of QCD in the high energy regime has been studied extensively in collider experiments, for instance at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC).

The need for colliders is due to the asymptotic freedom of QCD, the coupling between quarks and gluons becomes weak at high momentum transfers permitting the usage of weak coupling methods, for instance perturbation theory. This is important, because the presence of strong coupling makes theoretical considerations tremendously more difficult. Even though we study the weakly coupled regime of QCD at high energies, some difficulties and subtleties arise when going to higher and higher collision energies. One of these is the rapid rise of parton densities of small momentum fraction (small-x) partons as implied by measurements of parton distribution functions, for instance at Hadron-Electron Ring Accelerator (HERA) [1]. Due to this, the high energy hadron can not be viewed as a bunch of individual partons, but as a dense system of partons all interacting with each other. This is when the density of the partons compensates for the small coupling constant, and multiple scattering processes become important.

To address these high density effects arising when probing the small-x constituents of hadrons, the QCD effective theory of Color Glass Condensate (CGC) was formulated. The CGC was built on the ideas devised in the McLerran-Venugopalan (MV) model, which was formulated in references [2–4]. The basic idea is that the small-x parton densities become so large that the partons can be modelled as a classical color field. The large-x partons, for instance the valence partons, act as sources for these small-x partons. [5]

In this thesis we will be considering a proton-hadron collision. We consider a process in which the scattering is initiated by a gluon from the proton. This gluon proceeds to scatter with the dense small-x parton system of the nucleus.

The gluon splits into two gluons during the scattering process and these produced gluon hadronize after the scattering producing hadron showers. The scattering off of the nucleus can not be described by ordinary perturbation theory, for we have to take into account the multiple scatterings of the same gluon. Assuming an eikonal scattering, the multiple scatterings of the gluons can be incorporated into path-ordered exponentials of the color field of the nucleus, which are called Wilson lines. Eikonal scattering means that the scattering particle travels in a straight line and its trajectory is not altered by the scattering.

Every scattering gluon contributes one adjoint representation Wilson line to the scattering amplitude. The squared scattering amplitude of the gluon splitting process considered in the thesis contains 2-, 3- and 4-point functions of these adjoint Wilson lines. We will concentrate on the 4-point function as the lower-point functions can be found as limits of the 4-point function. Additionally the 2- and 3-point functions are easy to compute using the same method we will use for the 4-point function. The usual way of computing the 4-point function of the adjoint Wilson lines is to express it as a linear combination of 8-point functions of fundamental Wilson lines. The linear combination easily becomes cumbersome to compute and thus one usually resorts to using the large-N approximation scheme. To avoid all these difficulties, we will compute the adjoint 4-point function directly in the adjoint representation.

The Wilson lines depend on the configuration of the color field of the nucleus. This configuration can not be known and thus we have to average over all the possible configurations with some probability weight. The color field can be connected to the densities of the color sources in the nucleus and thus we can express the Wilson lines as their functionals. Then we do not average over the color field, but over the source densities. The MV model argues that the probability weight of these densities should be Gaussian. This Gaussian approximation allows us to evaluate the adjoint 4-point function of the Wilson lines analytically, as we will do in the section 3.

How we can make sure that we really probe the small-x partons of the nucleus so that it is justifiable to use the CGC framework? We want to probe the large-x partons of the projectile proton and the small-x partons of the nucleus. This lets us to model the scattering using CGC, but still allows us to use collinear factorization for the projectile due to the diluteness of the proton at large-x. As an estimate, we can relate the momentum fractions of the interacting partons to final state variables

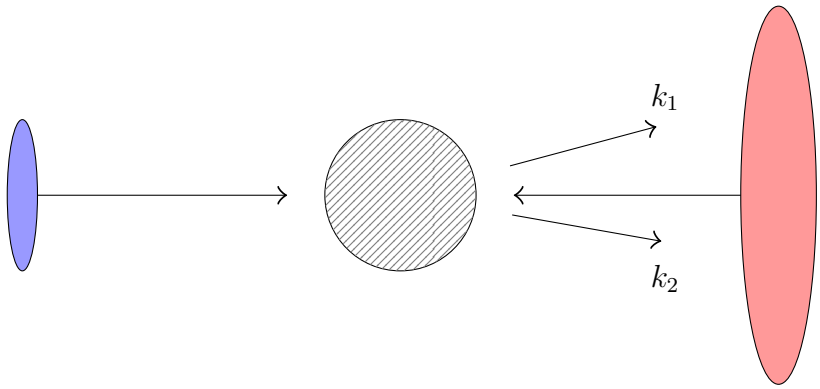


Figure 1. Production of two partons at forward rapidities in a p-A collision. The proton is on the left, the nucleus is on the right and the blob in the middle describes the interactions. k_1 and k_2 are the momenta of the produced partons.

by assuming a $(2 \to 2)$ process. Define x_p to be the momentum fraction of the parton coming from the proton with respect to that particular proton, and define x_A to be the momentum fraction of the parton coming from the nucleus with respect to the nucleon from which it came from. Additionally, let us consider the kinematics in the CM frame of the proton-nucleon system. With these assumptions, we find

$$x_p = \frac{k_{1T}}{\sqrt{s}} e^{y_1} + \frac{k_{2T}}{\sqrt{s}} e^{y_2}, \quad x_A = \frac{k_{1T}}{\sqrt{s}} e^{-y_1} + \frac{k_{2T}}{\sqrt{s}} e^{-y_2}, \tag{1}$$

where $\sqrt{s}$ is the energy per interacting nucleon pair in the CM frame, k_{iT} are the transverse momenta of the produced particles, and

$$y_i \equiv -\ln\left(\tan\left(\frac{\theta_i}{2}\right)\right) \tag{2}$$

are their pseudorapidities. Here θ_i is the angle between the direction of the produced particle and the positive direction of the collision axis. We have defined the proton to be travelling to the positive direction. The figure 1 illustrates the situation. [6]

From (1) we can see that when the rapidities of the produced particles are positive and large, x_A is small and x_p is large. Thus in particle production at forward rapidities, i.e. in the fragmentation region of the proton, the probed parton densities can be expected to be such, that we can use the CGC formalism. [6]

The CGC describes the high energy hadronic matter, and thus provides a description of the initial stages of ultrarelativistic heavy ion collisions. By studying multiparticle correlations in p-A collisions with CGC, we can study the effects of

gluon saturation to observables measured in collisions at LHC and RHIC. By studying p-A collisions, and not A-A collisions, allows us to study correlations arising due to the initial state of the colliding particles, without the interference of correlations arising due to the final state, the quark-gluon plasma (QGP). CGC also provides a possible initial condition for the later stages of heavy ion collisions.

2 Color glass condensate and Wilson lines

2.1 The light-cone coordinates

We will almost exclusively work in light-cone coordinates. These coordinates are extremely useful when discussing particles travelling at or near the speed of light. The (+,−,transverse)-components of the light-cone coordinates for any 4-vector x are defined as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad (\mathbf{x}_\perp)^i = x^i, \quad i = 1,2. \tag{3}$$

The metric tensor $g^{\mu\nu}$ for the light-cone coordinates is not diagonal and its non-zero elements are

$$g^{+-} = g^{-+} = 1, \quad g^{11} = g^{22} = -1. \tag{4}$$

For comparison, the matrix representation of the usual Minkowski metric tensor in regular spacetime coordinates $\eta^{\mu\nu}$ and in light-cone coordinates $g^{\mu\nu}$ read

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{5}$$

The light-cone metric implies that

$$x^+ = x_-, \quad x^- = x_+. \tag{6}$$

The dot product of two vectors x,y in the light-cone coordinates gives

$$x \cdot y = x^\mu y_\mu = x^+ y^- + x^- y^+ - \mathbf{x}_\perp \cdot \mathbf{y}_\perp \tag{7}$$

and thus

$$x^2 = 2x^+ x^- - \mathbf{x}_\perp^2. \tag{8}$$

One should note that some texts use different normalization for the (+,−)-components of the light-cone coordinates (3). [7]

From this point forward, we will consider a collision of a light projectile, for instance a proton, travelling to the positive z-direction and a heavy nucleus travelling to the negative z-direction. Both projectiles are assumed to be highly energetic in our reference frame and thus their light-cone momentums have only one large component. More explicitly the momentum of the proton is

$$p_p \approx (p_p^+,0,0,0) \tag{9}$$

and the momentum of the nucleus is

$$p_N \approx (0,p_N^-,0,0). \tag{10}$$

The figures 2 and 3 illustrate the two coordinate systems for the case of a proton-nucleus collision.

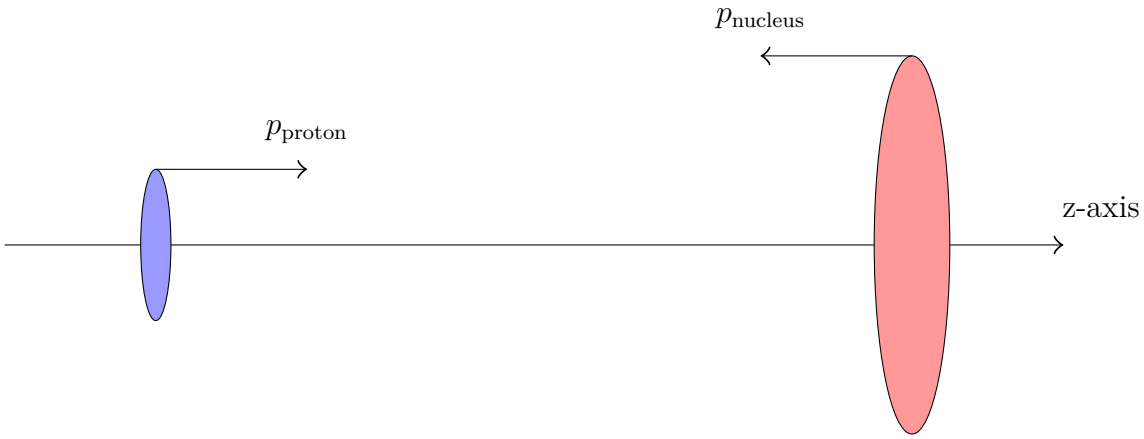


Figure 2. Proton-nucleus collision in regular spacetime coordinates (t,x,y,z).

2.2 Parton densities of a proton at high energies

The parton densities of the proton can be measured in a variety of collision experiments. Deep inelastic lepton-proton scattering is an example of one of such experiments. This type of scattering is preferred because the amount of produced particles is small so it is easier control in comparison to, for instance, a proton-proton scattering. Such experiments were done at the HERA particle accelerator by colliding electrons and positrons with protons in 1992-2007 [1].

The color glass condensate approach was motivated by the observation that the gluon densities in a proton rises rapidly as we consider partons with smaller and

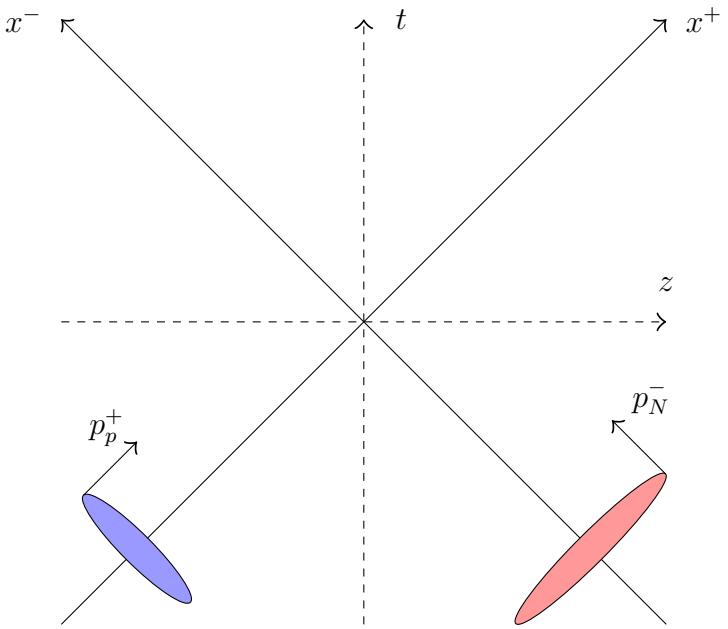


Figure 3. Proton-nucleus collision in light-cone coordinates (x^+,x^-,x,y).

smaller momentum fraction (x) [8]. The parton distribution functions (PDF) of a proton at probe virtualities $Q^2 = 1.9$ GeV^2 and $Q^2 = 10$ GeV^2 can be seen in figures 4. The xu_v and xd_v are the valence up- and down-quark PDFs respectively, the xS is the sea quark PDF and the xg is the gluon PDF. From the figures 4 we see that the gluon density of a proton grows much more rapidly at small-x than other parton densities. This trend becomes apparent when the energy (or resolution) of the probe is high enough so it can see the small-x partons inside the proton.

The proton consists of valence quarks. In addition, it consists of gluon and (anti)quark fluctuations. These partons can fluctuate into states of even greater number of partons. All these fluctuations are present in the proton during an interaction with a probe, but only fluctuations with longer lifetimes than the time resolution of the probe, can actually be seen by the probe. At higher energy i.e. smaller x, the probe sees more of these fluctuations because their lifetimes are increased due to time dilation, and because the time scale of the interaction is shorter. This results in larger parton densities seen in the process. At small-x the gluon density is much higher than other parton densities. The large difference between the gluon and sea quark densities is due to the fact that the sea quark quark-antiquark pairs are produced from gluons in the vertex $g \to q\bar{q}$. This implies that the sea quark production is suppressed by a factor of strong coupling constant g_s in comparison

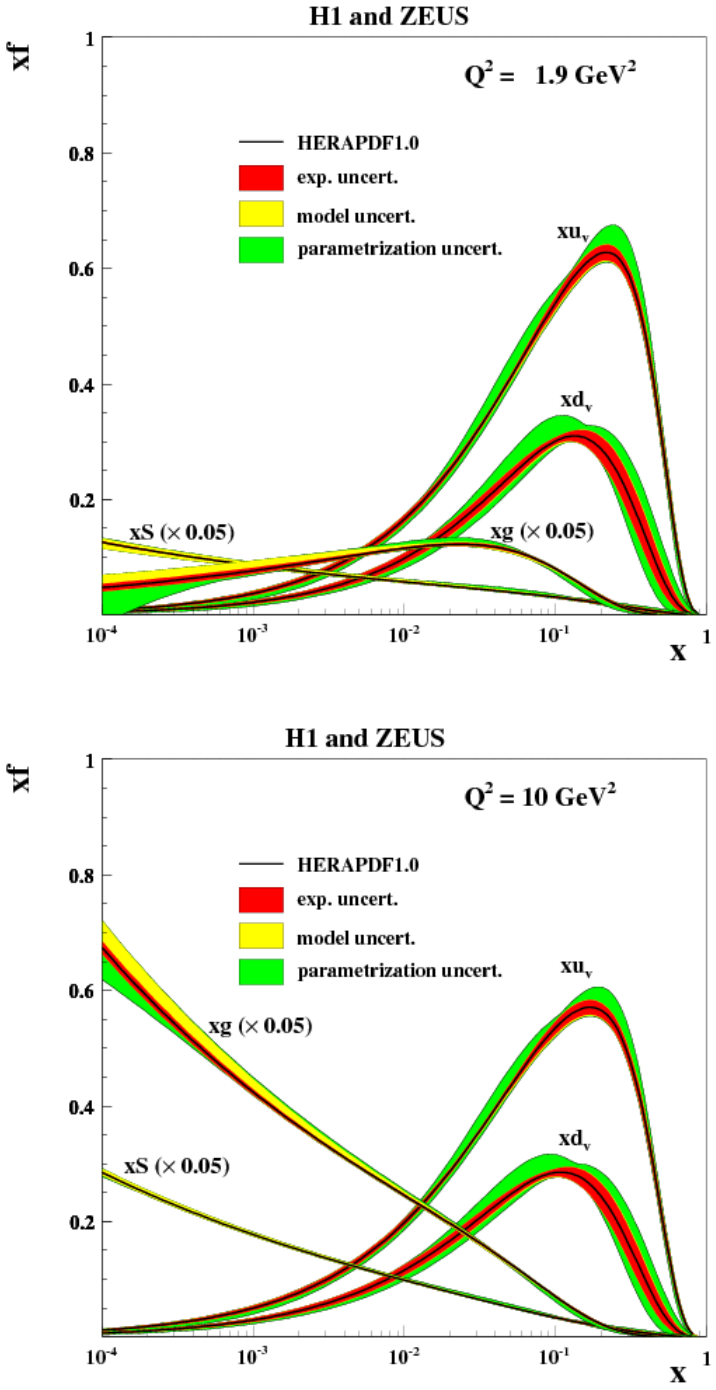


Figure 4. The parton distribution functions of a proton when probed with virtualities (Q^2) of 1.9 (top) and 10 GeV^2 (bottom). Figures from ref. [1].

to the production of gluons. [5]

Hadronic total cross sections increase with increasing center of mass energy $\sqrt{s}$. The Froissart bound follows from the unitarity of the scattering matrix (the S-matrix) and gives the upper bound for the increase of the total cross section with increasing energy at high energies. The bound states that the total cross section can not grow faster than some constant times $\ln^2(s)$ as s increases. The rising parton densities in a hadron at high energy result in rising cross sections. [9]

The linear QCD evolution equation Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [10] predicts the increase of gluon densities as with decreasing x, but the prediction violates the Froissart bound at small-x. The problematically rapid growth of the gluon density is cured by taking the non-linear effects into account, which result in a non-linear evolution equation called the Balitsky-Kovchegov (BK) equation. This equation predicts the rapid rise of the gluon density at small-x, but with it saturating at some sufficiently small x due to the non-linear terms in the equation. This solves the problem of the Froissart bound violation. [5]

When small-x gluon densities rise, recombination ($gg \to g$) becomes as important as gluon splitting ($g \to gg$). This leads to a saturation of gluon densities. The transverse size of a gluon is proportional to the inverse of its transverse momentum $x_T \propto k_T^{-1}$. Thus it is intuitive that the small transverse momentum i.e. large transverse size gluons start to overlap and that their density can not grow indefinitely, but they start to fuse together. [9]

The saturation (momentum) scale Q_s separates the saturated and unsaturated gluons. The gluon saturation effects are important for gluons whose transverse momentum is smaller than the saturation scale i.e. $k_T^2 < Q_s^2$. The saturation scale increases as s increases i.e. x decreases. If the saturation effects are not taken into account, the cross sections of hadronic collisions may increase faster than is allowed by the Froissart bound and thus violate the unitarity of the S-matrix. This implies that the saturation effects are indeed an important property of high energy hadrons. [9]

The QCD effective theory Color Glass Condensate (CGC) was formulated to address the gluon saturation effects in hadrons. The saturated gluons form a dense system which results in nonlinear effects that are not important in an unsaturated or dilute system. [5]

The figure 5 illustrates the landscape of the different regions of high energy QCD.

On the left we see the nonperturbative region i.e. the region where the momentum scale of the probe is so small that the coupling constant in the interaction is large, of the order 1. In the dilute (non-saturated) region, the linear DGLAP and BFKL equations describe the evolution of the parton distributions in increasing probe virtuality Q^2 and decreasing momentum fraction x respectively. When evolving the gluon distribution towards decreasing x, we have to replace the BFKL equation with the non-linear BK and JIMWLK (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner) equations as the parton saturation effects become important. The BK equation is only valid in the large-N limit whereas the JIMWLK equation is its finite-N generalization. The saturation scale Q_s separates the dilute and saturated region. [11]

2.3 The McLerran-Venugopalan model

Let us consider a high energy nucleus in an infinite momentum frame (IMF) moving in the negative z-direction. In this frame, the momentum of the nucleus has only the p_A^--component. We will be focusing on the dense system of small-x gluons in the nucleus. Now we shall discuss the classical field description of the small-x gluon field of the high energy nucleus as given by the McLerran-Venugopalan -model.

A small-x gluon carries a small fraction of the longitudinal nuclear momentum and has a large longitudinal extent. Because of this, the gluon interacts with all the nucleons in the nucleus on the longitudinal direction. One might think that because the nucleus and the nucleons as a whole are colorless (or "white"), the gluon would not interact with the colored constituents of the nucleus. However, the small-x gluons have some transverse momentum, and if this transverse momentum is much larger than the QCD scale Λ_{QCD}, the transverse extent is much smaller than the transverse extent of the nucleus. This results in the possibility of the gluon seeing only parts of individual nucleons in the nucleus. For instance, the gluon could see only one quark in a nucleon and thus interact with it. Thus the small-x gluon with high enough transverse momentum can, and most presumably will, only interact with parts of the nucleus. The nucleons are taken to be independent and thus the interactions are taken to be random i.e. the gluon's interaction with a nucleon does not interfere with the gluon's interactions with other nucleons. [10]

In the IMF frame, the high energy nucleus is highly Lorentz-contracted in the longitudinal direction. Thus the nucleus can be approximated as a two dimensional

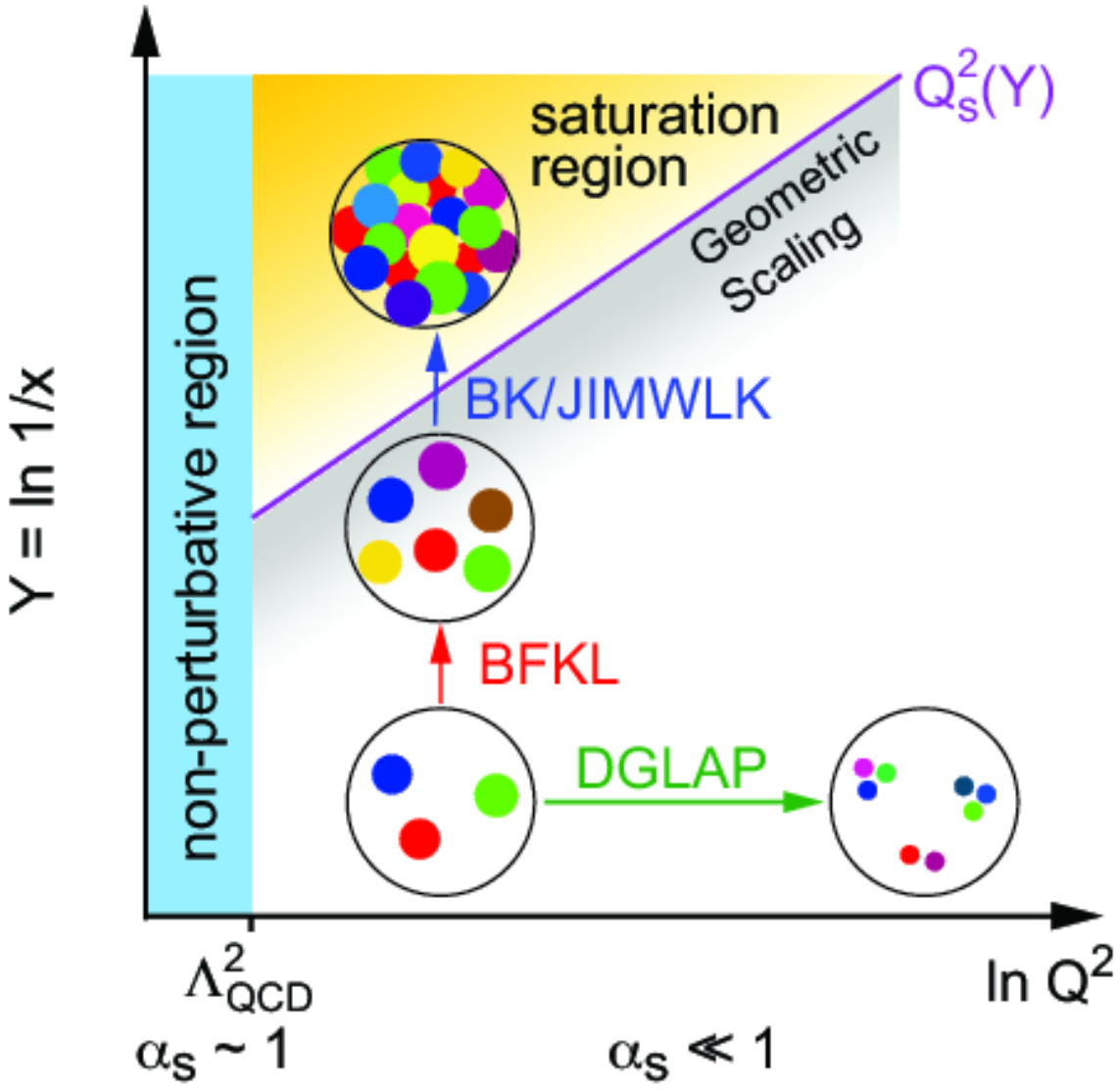


Figure 5. The "map" of high energy QCD. The figure illustrates the partonic content of a hadron or a nucleus, and the validity regions of evolution equations describing them. Q is the virtuality of the probe and x is the momentum fraction of the partons. Figure from ref. [11].

object in the transverse plane. It can be argued that the gluon density in the transverse plane of a nucleus is

$$\rho_{\text{gluon}} = \frac{xG_A}{S_\perp}, \tag{11}$$

where xG_A is the gluon distribution of the nucleus and $S_\perp$ is the transverse area of the nucleus. For a dilute (non-saturated, relatively non-dense) nucleus, we can assume the nucleon-nucleon effects are small and thus

$$xG_A = AxG_N \sim A, \tag{12}$$

where G_N is the gluon distribution of a single nucleon and A is the the number of nucleons in the nucleus or the mass number. The radius of a nucleus R_A is often approximated as

$$R_A = r_0 A^{1/3}, \tag{13}$$

where r_0 is just a constant that varies a bit from nucleus to nucleus, but this is not important for us. Equation (13) implies that the transverse size of the nucleus scales as

$$S_\perp \sim R_A^2 \sim A^{2/3}, \tag{14}$$

and thus from the equation (15) we get

$$\rho_{\text{gluon}} = \frac{xG_A}{S_\perp} \sim A^{1/3}. \tag{15}$$

Thus the gluon density ρ_{gluon} is greatly enhanced in a large nucleus consisting of multiple nucleons compared to just one nucleon discussed earlier. [10]

The high gluon density implies high occupation numbers of gluons which in turn implies that the small-x gluon field should be treated classically. The strongest possible gluon field in QCD at small coupling g is of the order $\sim \frac{1}{g}$. The highly saturated gluon system is expected to reach this and thus the gluon field is expected to be of that order i.e. $A^\mu \sim \frac{1}{g}$. [10]

2.4 Separation of partons: sources and fields

The small-x partons are much more delocalized in the longitudinal direction than the larger-x partons. The Lorentz-contraction of these larger-x partons is larger than that affecting the small-x partons. Because of this, the small-x partons see the larger-x partons as a thin sheet of classical color sources. The separation of the small- and larger-x partons is assumed to be purely kinetic in the framework of the MV-model. [12]

The lifetimes of the small-x partons are short compared to the lifetimes of the larger-x partons. Thus the large-x partons seem like light-cone-time-independent objects to the small-x partons. Because of this, the large-x partons are referred to as static color sources for the smaller-x partons. These large-x color sources are assumed to be unaffected by the momentum kicks received from the emission of small momentum (small-x) partons. This is referred to as the eikonal approximation. Under this approximation, the light-cone current J^μ generated by the color sources has only one non-zero component, the $(-)$-component

$$J_a^\mu = \delta^{\mu-}\rho_a(x_T,x^+), \tag{16}$$

where $\delta^{\mu-}$ picks the one non-zero component of the current. Here $\rho^a(x_T,x^+)$ denotes the color charge density of the color sources in the nucleus. The staticity of the color sources can be seen in the light-cone time x^- indepence of the color charge density $\rho^a(x_T,x^+)$. [5, 12]

The small-x gluon field can be solved, at lowest order, from the Yang-Mills equation with the color source J^μ seen in the equation (16). The equation reads

$$[D_\mu,F^{\mu\nu}] = J^\nu, \tag{17}$$

where D_μ is the covariant derivative defined as

$$D_\mu \equiv \partial_\mu - igA_\mu^a t^a, \tag{18}$$

and the field tensor is defined as

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \tag{19}$$

Here we have used the notation

$$A_\mu \equiv A_\mu^a t^a \quad \text{and} \quad F_{\mu\nu} \equiv F_{\mu\nu}^a t^a. \tag{20}$$

A convenient choice for the gauge is the Lorenz gauge

$$\partial_\mu A^\mu = 0. \tag{21}$$

In this gauge, the gluon field A can be solved from the Yang-Mills equation (17) yielding

$$A^+ = A^i = 0, \quad A^- = -\frac{1}{\nabla_\perp^2}\rho(x^+,x_T), \tag{22}$$

where the notation means that A^- is a solution to the equation

$$\nabla_\perp^2 A^- = -\rho(x^+,x_T). \tag{23}$$

The solution (22) gives a connection between the gluon fields A and the source densities ρ. [5, 13]

The equation can be solved using the Fourier transform. The solution for the field A^- reads

$$A^- = -\int d^2z_T G_0(x_T - z_T)\rho_a(x^+,z_T), \tag{24}$$

where

$$G_0(x_T - z_T) = -\int \frac{d^2k_T}{(2\pi)^2}\frac{e^{ik_T\cdot(x_T-z_T)}}{k_T^2} \tag{25}$$

is the Green's function for the two-dimensional Laplacian. [13]

2.5 CGC target averages of operators

In the Color Glass Condensate formalism we deal with the nuclear target gluon field (A) dependent operators. These operators are usually colorless combinations of Wilson lines and they are computed in some particular configuration of the target field. In practise, the field configuration can not be known, and thus we have to average over all possible configurations with some probability distribution $W[A]$. Taking the average is sensible because the configuration is unknown, it does not change during one scattering event but is different from scattering to scattering. As we discussed earlier, the target field A can be related to the density of color sources ρ. Thus we can replace the average over the field configurations with the average over the source density average with a weight $W[\rho]$.

In the presence of one nuclear source, the operator average of an nuclear source ρ_A dependent operator $O[\rho_A]$ is explicitly written as a functional integral

$$\left\langle O \right\rangle \equiv \int [D\rho_A] W_A[\rho_A] O[\rho_A], \tag{26}$$

where $W_A[\rho_A]$ is the probability weight of the sources as discussed before and $D\rho_A$ is the integration measure. This means integrating over all different possible target source densities of the nucleus while weighting them with the probability of finding the field in that particular configuration. After expressing all the desired operators, for some process in terms of the source densities, the observable quantities are extracted by performing the averaging procedure. Thus we still need to define the weight $W[\rho]$ to be able to calculate any observable quantities related to these operators. [13]

The MV-model provides the required probability distribution for a large nucleus and argues that it should be a Gaussian in the source density ρ. Written explicitly, the weight $W[\rho_A]$ reads

$$W_A[\rho_A] = \exp\left[-\int dx^+ d^2\mathbf{x}_\perp \frac{\rho_{A,a}(x^+,\mathbf{x}_\perp)\rho_{A,a}(x^+,\mathbf{x}_\perp)}{2\mu_A^2(x^+)} \right], \tag{27}$$

where $\mu_A^2(x^+)$ describes the color source density of the nucleus. The Gaussianity of the probability weight can be justified by noticing that a highly Lorentz-contracted nucleus has a very high density of color charges. Additionally these color charges originating from different nucleons are taken to be uncorrelated. Because of there are many uncorrelated (independent) color charges at every impact parameter (fixed transverse coordinate, whole longitudinal extent) of the nucleus, the so called central limit theorem states that, in such a system, the distribution of the color charges should be nearly Gaussian. [5, 13]

2.6 Wilson lines

The scattering of a particle off the strong color field of the nucleus can not be studied using ordinary perturbation theory. This is due to the high density gluon fields compensating for the small coupling constant. Thus we need to take into account the multiple scatterings of the particles scattering off the color field. In the eikonal approximation, these scatterings are summed by Wilson lines defined as

$$\begin{aligned} U(\mathbf{x}_\perp) &= \mathcal{P}_+ \exp\left[ig \int_{-\infty}^{\infty} dz^+ A_A^-(z^+,\mathbf{x}_\perp)\cdot T \right] \\ &= \mathcal{P}_+ \exp\left[-ig \int_{-\infty}^{\infty} dz^+ d^2\mathbf{z}_\perp G_0(\mathbf{x}_\perp - \mathbf{z}_\perp)\rho_a(z^+,\mathbf{z}_\perp)T^a \right], \end{aligned} \tag{28}$$

where the equivalence of the two exponentials follows from equation (24). Here $\mathcal{P}_+$ denotes path-ordering with respect to the (+)-light-cone coordinate z^+ and the T^a is a generator of $SU(3)$ (or more generally, $SU(N)$) in some representation. If the scattering particle is a quark, the generator is in the fundamental representation and if the particle is a gluon, the generator is in the adjoint representation. We will denote the adjoint representation Wilson line as U and a fundamental representation Wilson line as V. [6, 14]

The path-ordered exponential (28) is defined as

$$U(\mathbf{x}_\perp) = \sum_{n=0}^{\infty}(-ig)^n \int \prod_{i=1}^{n} d^2\mathbf{z}_{i\perp} G_0(\mathbf{x}_\perp - \mathbf{z}_{i\perp}) \int_{-\infty}^{\infty} dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ \ldots \int_{-\infty}^{z_{n-1}^+} dz_n^+ \\ \times \rho_{a_1}(z_1^+,\mathbf{z}_{1\perp})\rho_{a_2}(z_2^+,\mathbf{z}_{2\perp}) \ldots \rho_{a_n}(z_n^+,\mathbf{z}_{n\perp}) T^{a_1} T^{a_2} \ldots T^{a_n}, \tag{29}$$

which is almost like the Taylor series of an exponential, but the factorial is missing and has been incorporated into the bounds of the integrations over the (+)-coordinate. Setting all the integral bounds from $-\infty$ to ∞ would introduce the factorial, but we would have to introduce path-ordering for the integrand [15]. [14]

2.7 Wilson lines in scattering processes

The main point of this thesis is to compute the four point correlators of adjoint representation Wilson lines. These quantities arise when we have a process where two gluons propagate through a dense gluon field. An example of such a process can be found in the paper [6] where the scattering amplitude $\mathcal{M}$ for the partonic process $gA \to ggX$ is computed. We shall take this computation as an example, which illustrates how Wilson lines are incorporated to the calculations of scattering amplitudes.

In the lowest order, the process $gA \to ggX$ has two contributing Feynman graphs. These graphs correspond to the gluon splitting before and after the interaction with the color field of the nucleus. These graphs are seen in the figure 6. The scattering amplitude for this process can be written by applying Feynman rules to these two graphs. Every gluon interacting with the color field introduces an adjoint Wilson line. Thus the contribution corresponding to the graph on the left contains one, and the graph on the right contains two adjoint Wilson lines. [6]

The scattering amplitude of the partonic process is the sum of the algebraic expressions corresponding to the graphs 6. The scattering amplitude can be written

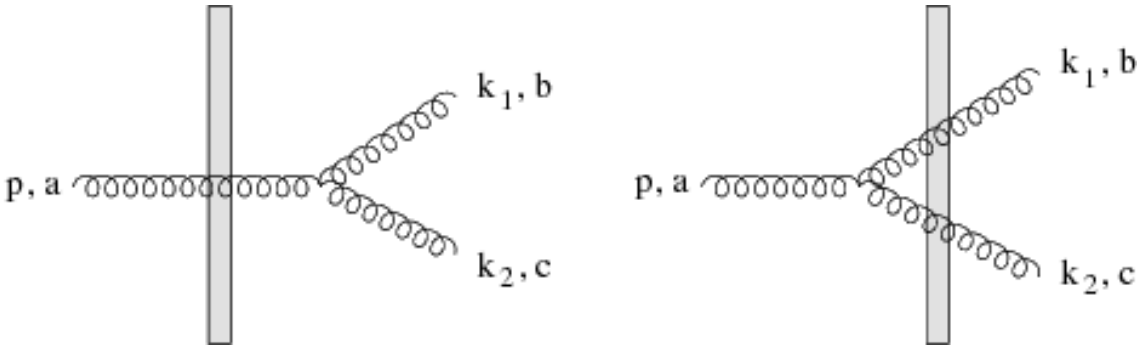


Figure 6. The Feynman graphs for gluon splitting in the presence of a color field of a nucleus. The left figure corresponds to the case where the gluon splits after its interaction with the color field, and the right figure corresponds to the gluon splitting before the interaction with both produced gluon interacting with the field. The figures are from ref. [6].

as

$$\mathcal{M}_{gA\to ggX} = \mathcal{M}_1 + \mathcal{M}_2, \tag{30}$$

where $\mathcal{M}_1$ denotes the contribution of the graph on the left and $\mathcal{M}_2$ denotes the contribution of the graph on the right. The scattering amplitude squared is

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_2\mathcal{M}_1^*. \tag{31}$$

Both parts of the amplitude contain Fourier transforms of adjoint Wilson lines in position space. The Wilson line and color structure dependences of the parts of the amplitude are

$$\mathcal{M}_1 \propto f^{dbc}U_{da}(x_T), \quad \mathcal{M}_2 \propto f^{aef}U_{be}(x_T)U_{cf}(y_T), \tag{32}$$

where the antisymmetric structure constants f come from the Feynman rule for the 3-gluon vertex, d is an internal color index and x_T,y_T are transverse coordinates of the gluons propagating through the color field. Of course the amplitude also depends on the Lorentz part, i.e. the part with the free propagators and polarizations tensors, but they are not interesting for us right now. [6]

Due to the dependences seen in (32), the amplitude squared contains four different, colorless combinations of adjoint representation Wilson lines. The quantity of interest, the Wilson line combination of four lines arises from the term $|\mathcal{M}_2|^2$. Explicitly written, this quantity reads

$$|\mathcal{M}_2|^2 \propto f^{aef}f^{ae'f'}U_{be}(x_T)U_{cf}(y)U_{be'}(\bar{x}_T)U_{cf'}(\bar{y}). \tag{33}$$

The two and three point operators of Wilson lines can be found by taking appropriate limits of transverse coordinates of the four point operator seen here, according to [6].

To calculate any observable quantities, we have to take the target average of the Wilson line operators, as discussed before. This averaging procedure only affects quantities that depend on the field or color charge densities of the target nucleus, and thus we can calculate the averages before doing the Fourier transforms of the Wilson line operators. This is we have to be able to calculate quantities like

$$f^{aef}f^{ae'f'}\left\langle U_{be}(x_T)U_{cf}(y_T)U_{be'}(\bar{x}_T)U_{cf'}(\bar{y}_T)\right\rangle \tag{34}$$

Next we shall discuss how to evaluate an arbitrary four point correlator of adjoint Wilson lines arising in scattering process calculations.

3 Wilson line correlators

3.1 Color algebra

In this subsection I shall present the group theoretic definitions and identities required for the evaluation of fundamental and adjoint representation Wilson line correlators. All the considerations are made for the $SU(N)$ group with arbitrary $N \in \mathbb{N}$ even though for us, the case $N = 3$ is the most interesting. This is done because keeping the arbitrary N in calculations has its uses. We can for example simplify our final results, or make some intermediate steps easier, by taking advantage of the large N limit used for instance in the paper [16].

3.1.1 Color algebra needed for the fundamental representation calculation

Let us define t^a to be the generators of the Lie group $SU(N)$ in the fundamental representation. The generators are represented by $N^2 - 1$ linearly independent traceless and Hermitian $N \times N$ matrices. Thus the index a runs from 1 to $N^2 - 1$ and

$$\mathrm{Tr}(t^a) = 0 \tag{35}$$

for any $a \in \{1,2,...,N^2 - 1\}$. The conventional choice for the normalization of the generators is

$$\mathrm{Tr}(t^a t^b) = \frac{\delta^{ab}}{2} \tag{36}$$

and their commutation relation is

$$[t^a,t^b] \equiv t^a t^b - t^b t^a = i f^{abc} t^c, \tag{37}$$

where f is the totally antisymmetric structure constant. Summation over repeated indices is left implicit. [15]

An important identity for the generators of the fundamental representation is the Fierz identity

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}, \tag{38}$$

where the generators have been written using the index notation for matrices. Using this, we can easily derive the expression for the sum of the squares of fundamental representation generators

$$[t^a t^a]_{ij} = t^a_{ik} t^a_{kj} = \frac{1}{2}\delta_{ij}\underbrace{\delta_{kk}}_{=N} - \frac{1}{2N}\underbrace{\delta_{ik}\delta_{kj}}_{=\delta_{ij}} = \frac{N^2-1}{2N}\delta_{ij} \equiv C_F \delta_{ij} \tag{39}$$

or when written in the matrix form

$$t^a t^a = C_F \mathbb{I}_N. \tag{40}$$

Here we have defined the quadratic Casimir

$$C_F \equiv \frac{N^2-1}{2N}, \tag{41}$$

where the subscript F refers to the fundamental representation. From equation (40) we can easily see that the sum of the squares commutes with all other matrices. We will make use of this property when dealing with tadpole contributions to the Wilson line correlators. [16]

As we will later see, the only identity we need for the evaluation of fundamental representation Wilson line correlators is the Fierz identity (38). This is a result of the simplicity of the fundamental Wilson line singlet states. [16]

3.1.2 Color algebra needed for the adjoint representation calculation

Evaluating adjoint representation Wilson line correlators is not quite as straightforward as the evaluation of fundamental representation ones. This is mainly due to the more complicated structure of the singlet states and the more complex expression for the Fierz type identity for the adjoint representation generators. Because of these complications, we need many more color algebraic identities to be able to evaluate the adjoint representation correlators without reverting the analysis to the fundamental case. This could be done, but it would render the final result much more complicated because of the redundant degrees of freedom we would have to introduce to the calculations. These degrees of freedom would of course vanish in the final result, but they still render the final result unnecessarily complicated. I will give an example of this in the coming sections of this thesis.

The generators of the adjoint representation of $SU(N)$ are defined element wise using the antisymmetric structure constants f seen in the equation (37). The explicit

expression for the adjoint generators T reads

$$T^a_{bc} \equiv -if^{abc}, \tag{42}$$

where the subscript indices b,c are the matrix indices of the generator T^a and all the indices a,b,$c \in \{1,2,...,N^2-1\}$ which is evident from the equation (37) . Thus the adjoint representation generators are antisymmetric $(N^2-1)\times(N^2-1)$ matrices. [15] The adjoint representation generators obey the commutation relation

$$[T^a,T^b] \equiv T^aT^b - T^bT^a = if^{abc}T^c. \tag{43}$$

Now with our choice for the normalization of the fundamental representation generators, we have

$$\mathrm{Tr}(T^aT^b) = f^{cda}f^{cdb} = N\delta^{ab} \equiv C_A\delta^{ab}, \tag{44}$$

where we have defined the quadratic Casimir

$$C_A \equiv N, \tag{45}$$

where the subscript A refers to the adjoint representation. [17] Using equation (44) we can easily get the relation

$$[T^aT^a]_{bc} = T^a_{bd}T^a_{dc} = -f^{abd}f^{adc} = f^{adb}f^{adc} = C_A\delta^{bc} \tag{46}$$

or when written in matrix form

$$T^aT^a = C_A\mathbb{I}_N. \tag{47}$$

The anticommutator of fundamental representation generators reads

$$\{t^a,t^b\} \equiv t^at^b + t^bt^a = \frac{1}{N}\delta^{ab}\mathbb{I}_N + d^{abc}t^c, \tag{48}$$

where d is the totally symmetric structure constant. For these we have an identity

$$d^{cda}d^{cdb} = \frac{N^2-4}{N}\delta^{ab} \tag{49}$$

[17]. It sometimes turns out to be useful useful to define symmetric $(N^2-1)\times(N^2-1)$ matrices D as

$$D^a_{bc} = d^{abc} \tag{50}$$

[18].

Using equations (37) and (48), we can find explicit expressions for the elements of the antisymmetric and symmetric structure constants with respect to the fundamental representation generators. From equation (37) we get

$$[t^a,t^b] = if^{abd}t^d \implies [t^a,t^b]t^c = if^{abd}t^dt^c \implies -i[t^a,t^b]t^c = f^{abd}t^dt^c.$$

Now by taking the trace and using equation (36), we get

$$f^{abc} = -2i\,\mathrm{Tr}([t^a,t^b]t^c). \tag{51}$$

From equation (48) we get

$$\{t^a,t^b\} = \frac{1}{N}\delta^{ab}\mathbb{I}_N + d^{abd}t^d \implies \{t^a,t^b\}t^c = \frac{1}{N}\delta^{ab}t^c + d^{abd}t^dt^c$$

and now, by taking the trace and using the equations (36) and (35), we get

$$d^{abc} = 2\,\mathrm{Tr}(\{t^a,t^b\}t^c). \tag{52}$$

The contractions of the structure constants in equations (44) and (49) can be calculated by writing the structure constants as traces of fundamental representation generators by using the equations (51) and (52) and then contracting the generators with the same index by using the Fierz identity given in equation (38).

Using the fundamental generator representations (51) and (52) of the structure constants, we can show that they are "traceless" in a sense that

$$d^{aab} = 0 \tag{53}$$

and

$$f^{aab} = 0. \tag{54}$$

Equation (54) follows from the equation (51) as follows

$$f^{aab} = -2i\,\mathrm{Tr}(\underbrace{[t^a,t^a]}_{=0}t^b) = 0$$

and equation (53) follows from (52) and (40)

$$d^{aab} = 2\,\mathrm{Tr}(\underbrace{\{t^a,t^a\}}_{=2C_F\mathbb{I}_N}t^b) = 4C_F\,\underbrace{\mathrm{Tr}(t^b)}_{=0} = 0.$$

Remembering the (anti)symmetricity of the (anti)symmetric structure constants under pairwise exchange of two indices, we conclude that, assuming summation over

a repeated index, the structure constant evaluates to zero whenever it has two of the same index. This is true for the antisymmetric structure constants even without the summation, but requires the summation for the symmetric structure constants.

Furthermore we will need the Jacobi and Jacobi-like identities for the structure constants which read

$$f^{abe}f^{ecd} + f^{cbe}f^{aed} + f^{dbe}f^{ace} = 0, \tag{55}$$

$$f^{abe}d^{ecd} + f^{cbe}d^{aed} + f^{dbe}d^{ace} = 0 \tag{56}$$

and

$$f^{abe}f^{cde} = \frac{2}{N}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}) + d^{ace}d^{bde} - d^{bce}d^{ade} \tag{57}$$

[19]. We will also need lesser known identity

$$\begin{aligned} d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma} = &\left[\frac{2(24-10N^2+N^4)}{N(48-29N^2+3N^4)}\right][\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4} + \delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}] \\ &+\left[\frac{2(N^2-12)(4-N^2)}{N(48-29N^2+3N^4)}\right]\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\ &+\left[\frac{24-15N^2+N^4}{48-29N^2+3N^4}\right][d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma} + d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}]. \end{aligned} \tag{58}$$

which is valid for $N = 2,3$. I have included the proof of identity (58) in the appendix A. Lastly, we will need trace identities for the matrices (42) and (50) up to traces of four matrices. I will cite them here, without proof, as they were presented in the

appendix A of the paper [20]. The identities read

$$\mathrm{Tr}(T^a) = \mathrm{Tr}(D^a) = \mathrm{Tr}(T^a D^b) = 0 \tag{59a}$$

$$\mathrm{Tr}(T^a T^b) = N\delta^{ab} \tag{59b}$$

$$\mathrm{Tr}(D^a D^b) = \frac{N^2-4}{N}\delta^{ab} \tag{59c}$$

$$\mathrm{Tr}(T^a T^b T^c) = i\frac{N}{2} f^{abc} \tag{59d}$$

$$\mathrm{Tr}(T^a T^b D^c) = \frac{N}{2} d^{abc} \tag{59e}$$

$$\mathrm{Tr}(D^a D^b T^c) = i\frac{N^2-4}{2N} f^{abc} \tag{59f}$$

$$\mathrm{Tr}(D^a D^b D^c) = \frac{N^2-12}{2N} d^{abc} \tag{59g}$$

$$\mathrm{Tr}(T^a T^b T^c T^d) = \delta^{ad}\delta^{bc} + \frac{1}{2}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd}) + \frac{N}{4}(f^{ade}f^{bce} + d^{ade}d^{bce}) \tag{59h}$$

$$\mathrm{Tr}(T^a T^b T^c D^d) = i\frac{N}{4}(d^{ade}f^{bce} - f^{ade}d^{bce}) \tag{59i}$$

$$\mathrm{Tr}(T^a T^b D^c D^d) = \frac{1}{2}(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd}) + \frac{N^2-8}{4N} f^{ade}f^{bce} + \frac{N}{4} d^{ade}d^{bce} \tag{59j}$$

$$\mathrm{Tr}(T^a D^b T^c D^d) = -\frac{1}{2}(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd}) + \frac{N}{4}(f^{ade}f^{bce} + d^{ade}d^{bce}) \tag{59k}$$

$$\mathrm{Tr}(T^a D^b D^c D^d) = i\frac{2}{N} f^{ade}d^{bce} + i\frac{N^2-8}{4N} f^{abe}d^{cde} + i\frac{N}{2} d^{abe}f^{cde} \tag{59l}$$

$$\begin{aligned}\mathrm{Tr}(D^a D^b D^c D^d) = {} & \frac{N^2-4}{N^2}\delta^{ad}\delta^{bc} + \frac{1}{2}\delta^{ac}\delta^{bd} + \frac{N^2-8}{2N^2}\delta^{ab}\delta^{cd} + \frac{N}{4} f^{ade}f^{bce} \\ & + \frac{N^2-16}{4N} d^{ade}d^{bce} - \frac{4}{N} d^{abe}d^{cde}.\end{aligned} \tag{59m}$$

Now we have all the color algebraic machinery we need for the evaluation of fundamental and adjoint representation Wilson line correlators.

3.2 Evaluation of Wilson line correlators

Next I shall explain how one can evaluate the Wilson line correlators analytically in the MV-model. First I will discuss the Wilson line dipole since it is the most simple physically relevant correlator and its evaluation is the most straightforward. It is useful to discuss the dipole first for it gives some insight to the evaluation process without introducing transitions between singlet states, which makes the evaluation of higher point correlators more difficult. This is due to the dipole only having one possible singlet configuration.

After evaluating the dipole, I will discuss the main point of this thesis, the evaluation of the four point correlator of adjoint representation Wilson lines. This can be done by reverting the treatment to the fundamental representation case, but this method renders the result unnecessarily complicated. I will discuss this method mainly to explain the evaluation of fundamental representation correlators and to demonstrate how the fundamental representation method differs from the adjoint representation case. The main result of this thesis is the evaluation of the four point adjoint representation Wilson line correlator directly in the adjoint representation. In the following calculations of correlators, we will be using the method and notation presented in [16] with some required modifications. Finally I will briefly discuss the calculation of matrix exponentials that appear in the final result of the evaluation of the Wilson line correlators.

3.2.1 The Wilson line dipole

We wish to calculate the fundamental representation Wilson line dipole of the form

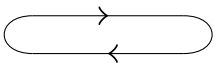
$$C_{dip,F}(\mathbf{x}_1,\mathbf{x}_2) = \Big\langle V(\mathbf{x}_1)_{\beta_1\alpha_1} V^*(\mathbf{x}_2)_{\beta_2\alpha_2} \Big\rangle, \tag{60}$$

where all the $\mathbf{x}_i \equiv \mathbf{x}_{i,\perp}$. I will suppress the symbol $\perp$ from this point forward. We are interested in calculating correlators that correspond to color singlet states. The only color singlet for the dipole (60) is [14]

$$\mathcal{N}^{(a)} = C_{dip,F}(\mathbf{x}_1,\mathbf{x}_2)\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2} = \Big\langle \operatorname{Tr}(V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)) \Big\rangle. \tag{61}$$

To ease the $SU(N)$ algebraic considerations, we introduce the graphical representation of Wilson lines presented in the paper [16]. Using this representation, we can express the sole singlet configuration (61) of the dipole as

$$\mathcal{N}^{(a)} = \Big\langle \operatorname{Tr}(V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)) \Big\rangle \hat{=}$$



, (62)

where the line with an arrow pointing to the right represents a Wilson line V, the line with an arrow pointing to the left represents a Hermitian conjugate of a Wilson line $V^\dagger$ and the endpoints represent how the Wilson lines are multiplied. The horizontal direction corresponds to the x^+ coordinate which is increasing to the right. Every loop represents a trace and thus because of the cyclicity of the trace, one can start reading the graph from any point.

The two point function of the source ρ_A given by (26) with the Gaussian weight (27) is [16]

$$\Big\langle \rho_{A,a}(x^+,\mathbf{x}_\perp)\rho_{A,b}(y^+,\mathbf{y}_\perp)\Big\rangle = \delta^{ab}\delta(x^+-y^+)\delta(\mathbf{x}_\perp-\mathbf{y}_\perp)\mu_A^2(x^+). \tag{63}$$

It is important to notice that the correlator (63) is local in the +-component of the spatial coordinate which can be thought of as the temporal coordinate for the eikonal Wilson line. The locality of the transverse components is not as important but the δ^{ab}-term plays a crucial role in the upcoming calculation. Because of the Gaussian form of the weight $W_A[\rho_A]$ (27), we can evaluate higher point functions of ρ_a's by connecting different ρ_a's pairwise by the use of Wick's theorem. This means that, for instance, the four point correlator of sources would be

$$\Big\langle \rho_a\rho_b\rho_c\rho_d\Big\rangle = \Big\langle \rho_a\rho_b\Big\rangle\Big\langle \rho_c\rho_d\Big\rangle + \Big\langle \rho_a\rho_c\Big\rangle\Big\langle \rho_b\rho_d\Big\rangle + \Big\langle \rho_a\rho_d\Big\rangle\Big\langle \rho_b\rho_c\Big\rangle.$$

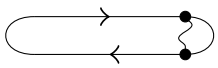
Additionally because of the Gaussian weight, averages over an odd power of sources goes to zero

$$\Big\langle \prod_{odd}\rho_i\Big\rangle = 0.$$

By writing the Wilson lines in the dipole (61) as a power series, we notice that the averaging procedure $\Big\langle \dots \Big\rangle$ only affects the densities ρ. By using Wick's theorem, we can write the higher point correlation functions of ρ's as product of two point functions (63). These two point functions force the indices of two different generators of $SU(N)$ to be the same. Thus by writing the higher point correlator of ρ's as product of two point functions, we get contributions corresponding to contracting the generators in every possible way. Because of the locality of the two point function in x^+, only contractions that happen at the same time contribute.

The contractions of generators can happen between different Wilson lines (ladder type contributions) or within a same Wilson line (tadpole type contribution). We can also express the contractions of generators within the formalism of the graphical representation of paper [16]. The ladder type contribution for the dipole is

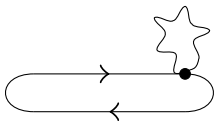
$$\Big\langle \mathrm{Tr}(V(\mathbf{x}_1)t^at^aV^\dagger(\mathbf{x}_2))\Big\rangle \hat{=}$$



(64)

and the tadpole type contribution is

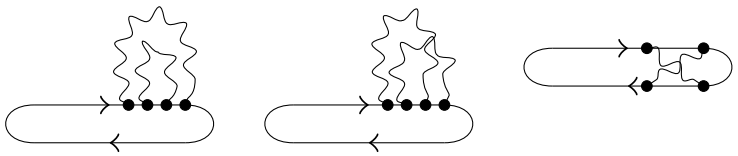
$$\Big\langle \mathrm{Tr}(V(\mathbf{x}_1)t^at^aV^\dagger(\mathbf{x}_2))\Big\rangle \hat{=}$$



, (65)

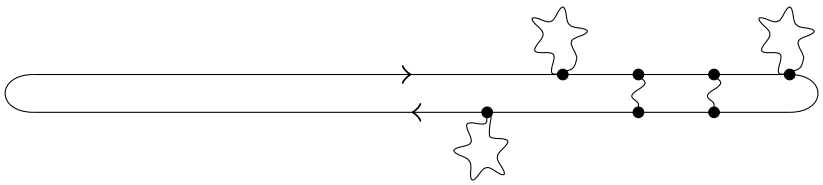
where the black dots represent generators of $SU(N)$ and the wavy line indicates which two generators have the same index, that is, which two generators do we contract. Both endpoints of the wavy line have a generator, so the tadpole type contribution has two generators at the same point. Note that for the dipole, these two contributions have the same color algebraic structure. However, this is not generally true if we have more Wilson lines, that is, when we are discussing higher-than-2-point functions of Wilson lines. The main difference of these different types of contributions is their origin. As we discussed earlier, the tadpole generators originate from the same Wilson line and the ladder generators from different lines, and thus it can be expected them to have different contributions to the correlator that arise from the remaining part of the Wilson line associated to that particular contraction. Furthermore, when we draw the black dots in the right edge of the graph, we mean that the color generator contractions are due to the contraction of two of the ρ's with the largest values of the (+)-coordinate. Thus, the color structure (64) corresponds to the contraction of the largest-(+) ρ of the upper Wilson line and the largest-(+) ρ of the lower Wilson line. In turn, the color structure (65) corresponds to the contraction of the largest and second largest-(+) ρ's of the upper Wilson line.

As stated before, the path-ordering of Wilson lines dictates the possible contractions. Due to locality in x^+, not all contractions contribute. Examples of contractions that do not contribute include



(66)

and an example of a contributing graph would be



. (67)

In the end, the tadpole contributions $\propto t^a t^a$ factor out because of their commutativity, which is evident from the equation (40). Because the contractions of ρ_a's have to be local in light-cone-time in order of them to contribute, it is natural to do the contractions in the order with increasing (or decreasing) (+)-coordinate. This is precisely how we will be doing the contractions in the following calculations.

The tadpole contributions factorize out from the expression of two explicitly written Wilson lines and effectively form one-point functions of Wilson lines [13]. This result can be assumed to generalize to higher-point correlators: we get one one-point correlator for every Wilson line in the correlator. The dipole can then be written as [13]

$$\begin{aligned}&\Big\langle V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)\Big\rangle \\ &= \underbrace{\Big\langle V(\mathbf{x}_1)\Big\rangle\Big\langle V(\mathbf{x}_2)\Big\rangle}_{\propto\mathbb{I}} \times \underbrace{\Big\langle V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)\Big\rangle}_{\text{Only ladder contributions}} .\end{aligned} \tag{68}$$

Thus

$$\begin{aligned}&\Big\langle \operatorname{Tr}(V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2))\Big\rangle \\ &= \Big\langle V(\mathbf{x}_1)\Big\rangle\Big\langle V(\mathbf{x}_2)\Big\rangle \times \underbrace{\Big\langle \operatorname{Tr}(V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2))\Big\rangle}_{\text{Only ladder contributions}}.\end{aligned} \tag{69}$$

As we saw in equations (68) and (69), the sum of all contributions $\mathcal{M}_D$ factorizes into a product of all ladder type contributions $\mathcal{N}_D$ and all tadpole type contributions $\mathcal{T}_D$

$$\mathcal{M}_D = \mathcal{N}_D\mathcal{T}_D. \tag{70}$$

Let us define an object that is associated with the tadpole contractions that reads

$$L(x,x) \equiv g^2 \int_{\mathbf{z}_\perp} G_0(\mathbf{x}_\perp - \mathbf{z}_\perp)^2, \tag{71}$$

where the bounds of the integral will be discussed later in the thesis due to infrared divergences the propagator integrals possess. The tadpole contributions exponentiate and evaluate to

$$\mathcal{T}_D \equiv \exp\Big[-\frac{1}{2}C_F\mu_A^2[L(x_1,x_1) + L(x_2,x_2)]\Big], \tag{72}$$

where the factor $\frac{1}{2}$ arises because the endpoints of the tadpole contractions are path-ordered due to them originating from the same Wilson line. This means that every tadpole type contraction brings in a delta function which eliminates one integral i.e. we get [14]

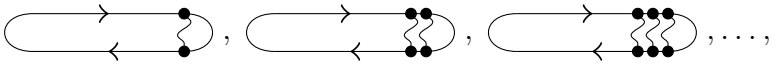
$$\int_{-\infty}^{z_1^+} dz_2^+ \delta(z_1^+ - z_2^+) = \frac{1}{2}, \tag{73}$$

due to path-ordering. We denote the integrated density by

$$\mu_A^2 \equiv \int_{-\infty}^{\infty} dz^+ \mu_A^2(z^+), \tag{74}$$

where we use the same symbol but without the argument. [16]

For the dipole, the only ladder type contributions are



where the contributions are proportional to

$$L(x,y) \equiv g^2 \int_{\mathbf{z}_\perp} G_0(\mathbf{x}_\perp - \mathbf{z}_\perp) G_0(\mathbf{y}_\perp - \mathbf{z}_\perp), \tag{75}$$

which is a generalization of the equation (71). Summing over all the possible ladder contributions we get

$$\mathcal{N}_D = \sum_{n=0}^{\infty} \int_{z_1^+ < \ldots < z_n^+} \mathcal{N}_n(z_1^+, \ldots, z_n^+), \tag{76}$$

where $\mathcal{N}_n$ is the contribution with n ladder type contractions in it. Writing $\mathcal{N}_n$ explicitly, we get

$$\mathcal{N}_D = \sum_{n=0}^{\infty} \int_{z_1^+ < \ldots < z_n^+} \Big[\prod_{i=1}^{n} \mu_A^2(z_i^+) \Big] \Big[C_F L(x_1,x_2) \Big]^n. \tag{77}$$

Now we can remove the path-ordering by introducing a factor $\frac{1}{n!}$ and thus

$$\mathcal{N}_D = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{z_1^+, \ldots, z_n^+} \Big[\prod_{i=1}^{n} \mu_A^2(z_i^+) \Big] \Big[C_F L(x_1,x_2) \Big]^n. \tag{78}$$

In equation (78) we identify the power series of the exponential and employ the notation (74) and thus get

$$\mathcal{N}_D = \exp \Big[\mu_A^2 C_F L(x_1,x_2) \Big]. \tag{79}$$

Substituting equations (72) and (79) to the equation (70), we get

$$\mathcal{M}_D = \exp \Big[\mu_A^2 C_F L(x_1,x_2) - \frac{1}{2} C_F \mu_A^2 [L(x_1,x_1) + L(x_2,x_2)] \Big]. \tag{80}$$

We can define Γ-functions as

$$\Gamma(x-y) \equiv L(x,x) + L(y,y) - 2L(x,y), \tag{81}$$

where we have used the translational invariance of the transverse coordinates i.e. the functions can only depend on the relative difference of the transverse coordinates. Using this, we can write equation (80) as

$$\mathcal{M}_D = \exp \Big[-\frac{1}{2} C_F \mu_A^2 \Gamma(x_1 - x_2) \Big]. \tag{82}$$

After contracting all the generators in the Wilson lines i.e. when $V \to \mathbb{I}_N$, we are left with factors

$$\mathrm{Tr}(\mathbb{I}_N) = N \tag{83}$$

to every contributing graph. This can be thought to be the initial condition for the dipole as it corresponds to the case where the quark does not scatter at all i.e. $V = \mathbb{I}$. Because of the contribution (83) we get the final result

$$C_{dip,F}(\mathbf{x}_1,\mathbf{x}_2) = N \exp\Big[-\frac{1}{2} C_F \mu_A^2 \Gamma(x_1 - x_2)\Big]. \tag{84}$$

The result (84) corresponds to the result given in the paper [13], where the differing factor of N is explained by the tracing procedure. [16]

The adjoint representation Wilson line dipole is

$$\mathcal{N}_A^{(a)} = \Big\langle \mathrm{Tr}(U(\mathbf{x}_1)U^\dagger(\mathbf{x}_2))\Big\rangle. \tag{85}$$

The structure of the adjoint representation Wilson line is the same as with the fundamental representation one. Only the representation of the group generators are changed. Thus, because all the generator contractions for the dipole are done between adjacent generators, only the quadratic Casimir and the initial condition (the trace) give a differing contribution. The Casimir factor is replaced as $C_F \to C_A$ and the initial condition changes as $N \to N^2 - 1$ and thus we get the result

$$C_{dip,F}(\mathbf{x}_1,\mathbf{x}_2) = (N^2 - 1) \exp\Big[-\frac{1}{2} C_A \mu_A^2 \Gamma(x_1 - x_2)\Big]. \tag{86}$$

The propagator integral (L-function) (75) is an infrared divergent quantity. The origin of the divergences is the fact that the color neutrality of the target nucleus is not taken into account in the weight (27). The L-function can be regularized by introducing a upper cutoff Λ_{QCD} to the $\mathbf{z}_\perp$-integral. Physically, the addition of the cutoff corresponds to requiring color neutralization at the distance scale of $1/\Lambda_{QCD}$. The introduction of the cutoff renders the L-function quadratically divergent with respect to the cutoff scale. However, the L-functions combine to form Γ-functions that are only logarithmically divergent. The physical reason for the softening of the singularity is the color singlet structure of the Wilson lines in question. The color singlets are almost free of long-ranged color interactions, but they still introduce the logarithmic divergence. In this thesis we will only consider color singlet states so we can always expect the L-functions to combine into Γ-functions. [14, 16]

3.2.2 Fundamental representation 8-point correlators

The goal of this thesis is the evaluation of adjoint representation Wilson line correlators directly in the adjoint representation. In this section I will show how one can straightforwardly express the adjoint four point correlators as a linear combinations of fundamental representation eight point correlators. I will also show how one can evaluate the fundamental eight point correlators.

Let us take a concrete example. One possible adjoint representation four point correlator is

$$C_{4,A}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \Big\langle \operatorname{Tr}\Big[U(\mathbf{x}_1)U^\dagger(\mathbf{x}_2)\Big] \operatorname{Tr}\Big[U(\mathbf{x}_3)U^\dagger(\mathbf{x}_4)\Big]\Big\rangle. \tag{87}$$

The adjoint representation Wilson line can be expressed elementwise as fundamental representation Wilson lines by the identity [21]

$$U_{ab}(\mathbf{x}) = 2\operatorname{Tr}[t^a V^\dagger(\mathbf{x}) t^b V(\mathbf{x})], \tag{88}$$

where a,b are the matrix indices of the adjoint Wilson line U, the t's are the fundamental representation generators and the V's are the fundamental Wilson lines. Note that the fundamental Wilson lines depend on the same transverse coordinate. Now, making the replacement

$$U_{ab}(\mathbf{x}) \to 2\operatorname{Tr}[t^a V^\dagger(\bar{\mathbf{x}}) t^b V(\mathbf{x})], \tag{89}$$

and contracting all generators using the Fierz identity (38), we get an fundamental Wilson line 8-point correlator expression for the adjoint 4-point correlator. The exact equality is recovered by taking the limit $\bar{\mathbf{x}} \to \mathbf{x}$.

To demonstrate the process, let us consider the correlator in equation (87). First, we express the adjoint Wilson lines in the correlator in index form as

$$C_{4,A}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \Big\langle U_{ab}(\mathbf{x}_1)U^\dagger_{ba}(\mathbf{x}_2)U_{cd}(\mathbf{x}_3)U^\dagger_{dc}(\mathbf{x}_4)\Big\rangle = \Big\langle U_{ab}(\mathbf{x}_1)U_{ab}(\mathbf{x}_2)U_{cd}(\mathbf{x}_3)U_{cd}(\mathbf{x}_4)\Big\rangle,$$

where we have used the fact that the adjoint Wilson lines are real i.e.

$$U^* = U \iff U^\dagger_{ab} = U_{ba}. \tag{90}$$

Next we replace the adjoint Wilson line elements with fundamental Wilson line traces using (89). Doing this, we get

$$\begin{aligned} C_{4,A}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) =& 2^4\Big\langle \operatorname{Tr}[t^a V^\dagger(\bar{\mathbf{x}}_1) t^b V(\mathbf{x}_1)] \operatorname{Tr}[t^a V^\dagger(\bar{\mathbf{x}}_2) t^b V(\mathbf{x}_2)] \\ &\times \operatorname{Tr}[t^c V^\dagger(\bar{\mathbf{x}}_3) t^d V(\mathbf{x}_3)] \operatorname{Tr}[t^c V^\dagger(\bar{\mathbf{x}}_4) t^d V(\mathbf{x}_4)]\Big\rangle\Big|_{\bar{\mathbf{x}}_i=\mathbf{x}_i}. \end{aligned}$$

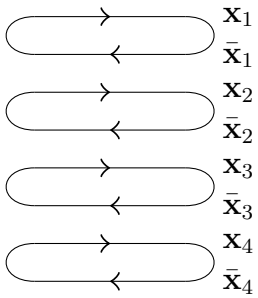
Now we shall employ the Fierz identity (38) to contract the generators with the same index. Let us leave the limit $\bar{\mathbf{x}}_i = \mathbf{x}_i$ implicit for simplicity. The "Fierzing" procedure yields

$$
\begin{aligned}
C_{4,A}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = {} & \operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& -\frac{1}{N}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& -\frac{1}{N}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& +\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& -\frac{1}{N}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& +\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& +\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& -\frac{1}{N^3}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& -\frac{1}{N}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& -\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& +\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& -\frac{1}{N^3}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_2)V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& +\frac{1}{N^2}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& -\frac{1}{N^3}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)] \\
& -\frac{1}{N^3}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_4)V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_3)] \\
& +\frac{1}{N^4}\operatorname{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)]\operatorname{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\operatorname{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)]\operatorname{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)],
\end{aligned}
\tag{91}
$$

where we could represent the traces graphically using the same graphical representation we used for the dipole. For instance, the last product of traces in equation

(91) would get the following graphical representation:

$$\mathrm{Tr}[V(\mathbf{x}_1)V^\dagger(\bar{\mathbf{x}}_1)]\,\mathrm{Tr}[V(\mathbf{x}_2)V^\dagger(\bar{\mathbf{x}}_2)]\,\mathrm{Tr}[V(\mathbf{x}_3)V^\dagger(\bar{\mathbf{x}}_3)]\,\mathrm{Tr}[V(\mathbf{x}_4)V^\dagger(\bar{\mathbf{x}}_4)]$$

$\hat{=}$ 

. (92)

Now we have expressed the adjoint Wilson line 4-point correlator (87) as a linear combination of fundamental Wilson line 8-point correlators (91). Thus, by calculating the fundamental 8-point correlators, we can evaluate the adjoint 4-point correlator by taking the limit introduced in (89).

Let us now evaluate the 8-point correlators using the procedure presented in the paper [16] and we shall largely use the notation presented in the paper. The paper discusses the evaluation of the fundamental Wilson line 4-point correlators but the process generalizes straightforwardly for 8-point correlators. In matter of fact, the main ideas from the dipole calculation generalize to the calculation of any 2n-point fundamental correlator. First, we need a basis for possible singlet states for the 8 Wilson lines which equates to finding all the possible contractions for

$$\left\langle V_{\beta_1\alpha_1}(\mathbf{x}_1)V^*_{\beta_2\alpha_2}(\mathbf{x}_2)V_{\beta_3\alpha_3}(\mathbf{x}_3)V^*_{\beta_4\alpha_4}(\mathbf{x}_4)V_{\beta_5\alpha_5}(\mathbf{x}_5)V^*_{\beta_6\alpha_6}(\mathbf{x}_6)V_{\beta_7\alpha_7}(\mathbf{x}_7)V^*_{\beta_8\alpha_8}(\mathbf{x}_8)\right\rangle \quad (93)$$

so that the states preserve the color flow i.e. a Wilson line always follows a Hermitian conjugate of a Wilson line and vice versa. The following calculation needs us to choose a basis with only either the α's or only the β's contracted. Because of the path ordering, we can always do the group generator contractions starting from α's or from β's i.e. starting from the $x^+ = -\infty$ side or from the $x^+ = \infty$ side, which leaves the topology of the other side unchanged. Let us now do the contractions starting from the beta-side so that our singlet base consist of every possible color flow preserving beta-index contractions. Let us also choose $\alpha_i = \alpha_{i+1}, i = 1,3,5,7$ for notational clarity. This choice can be changed in the end of the calculation and it will only affect the initial conditions which are easy to change.

The 8-point correlators are extremely cumbersome to write down algebraically or graphically, so to lighten up the notation, we shall use the notation that I introduced

in my research report. The notation is closely related to the notation used when describing cyclic permutations. Using the cyclic notation, we can write

$$\mathrm{Tr}[V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)]\,\mathrm{Tr}[V(\mathbf{x}_3)V^\dagger(\mathbf{x}_4)]\,\mathrm{Tr}[V(\mathbf{x}_5)V^\dagger(\mathbf{x}_6)]\,\mathrm{Tr}[V(\mathbf{x}_7)V^\dagger(\mathbf{x}_8)]\hat{=}(1,2)(3,4)(5,6)(7,8), \tag{94}$$

where every bracket implies a trace and the number represents a Wilson line with the same index in its parameter. For the fundamental Wilson lines we will use the convention that every odd numbered parameter represents a Wilson line and every even numbered parameter represents a Hermitian conjugate of a Wilson line. In the graphical representation the i:th line from the top will always represent a Wilson line (conjugate) having the parameter x_i.

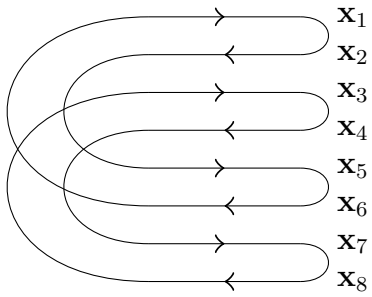
Using the cyclic notation we can write our choice for the 8-point correlator (93) basis as

$$\begin{aligned}
\mathcal{N}^{(1)} &= (1,2,5,6)(3,4,7,8) & \mathcal{N}^{(2)} &= (1,2,3,4)(5,6,7,8)\\
\mathcal{N}^{(3)} &= (1,2,3,4,7,8,5,6) & \mathcal{N}^{(4)} &= (1,2,5,6,7,8,3,4)\\
\mathcal{N}^{(5)} &= (1,2,5,6,3,4,7,8) & \mathcal{N}^{(6)} &= (1,2)(3,4,7,8,5,6)\\
\mathcal{N}^{(7)} &= (1,2,3,4)(5,6)(7,8) & \mathcal{N}^{(8)} &= (1,2,5,6,7,8)(3,4)\\
\mathcal{N}^{(9)} &= (1,2,5,6,3,4)(7,8) & \mathcal{N}^{(10)} &= (1,2,7,8,3,4)(5,6)\\
\mathcal{N}^{(11)} &= (1,2)(3,4)(5,6,7,8) & \mathcal{N}^{(12)} &= (1,2,7,8,5,6,3,4)\\
\mathcal{N}^{(13)} &= (1,2,3,4,5,6)(7,8) & \mathcal{N}^{(14)} &= (1,2)(3,4,5,6,7,8)\\
\mathcal{N}^{(15)} &= (1,2,5,6)(3,4)(7,8) & \mathcal{N}^{(16)} &= (1,2,7,8,5,6)(3,4)\\
\mathcal{N}^{(17)} &= (1,2)(3,4,7,8)(5,6) & \mathcal{N}^{(18)} &= (1,2,7,8,3,4,5,6)\\
\mathcal{N}^{(19)} &= (1,2,3,4,5,6,7,8) & \mathcal{N}^{(20)} &= (1,2,3,4,7,8)(5,6)\\
\mathcal{N}^{(21)} &= (1,2)(3,4,5,6)(7,8) & \mathcal{N}^{(22)} &= (1,2)(3,4)(5,6)(7,8)\\
\mathcal{N}^{(23)} &= (1,2,7,8)(3,4,5,6) & \mathcal{N}^{(24)} &= (1,2,7,8)(3,4)(5,6).
\end{aligned} \tag{95}$$

Let us have one more example to summarize all the different representations we have for the Wilson line singlet states i.e. the correlators. The first basis state in

equation (95) reads

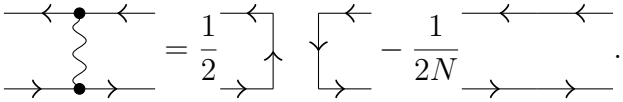
$$\begin{aligned}\mathcal{N}^{(1)} &= (1,2,5,6)(3,4,7,8)\\ &= \mathrm{Tr}[V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)V(\mathbf{x}_5)V^\dagger(\mathbf{x}_6)]\,\mathrm{Tr}[V(\mathbf{x}_3)V^\dagger(\mathbf{x}_4)V(\mathbf{x}_7)V^\dagger(\mathbf{x}_8)]\\ &= \end{aligned}$$



(96)

We will suppress the transverse component parameters x_i and keep the ordering of the Wilson lines i.e. the horizontal lines unchanged. We will let the connections in the end points change and thus this convention still allows us to represent every possible topology without any restrictions.

Now we can begin the calculation which is mostly a generalization of the calculation involving the dipole. The graphical representation for the Fierz identity (38) is [16]



(97)

Just like in the dipole case, the ladder and tadpole type contributions decouple. The first term in (97) implies that the generator contractions may result in state transitions between the singlet states (95). This was not possible for the dipole due to it having only one possible singlet state. The factorization of all contraction contributions $\mathcal{M}_{F,8}$ is completely analogous to the dipole factorization (70) and it reads

$$\mathcal{M}_{F,8} = \mathcal{N}_{F,8}\mathcal{T}_{F,8}, \tag{98}$$

where, again, $\mathcal{N}_{F,8}$ includes all contractions between different Wilson lines and $\mathcal{T}_{F,8}$ contains all the tadpole contractions. The evaluation of the tadpole contributions yields

$$\mathcal{T}_{F,8} = \exp\Big[-\frac{1}{2}C_F\mu_A^2\sum_{i=1}^{8}L(x_i,x_i)\Big]. \tag{99}$$

Let us now denote $\mathcal{N}_{F,8,n}(z_1^+,\ldots,z_n^+)$ as the part of the ladder type contributions $\mathcal{N}_{F,8}$ that includes precisely n ladder type contractions. Thus the whole ladder type

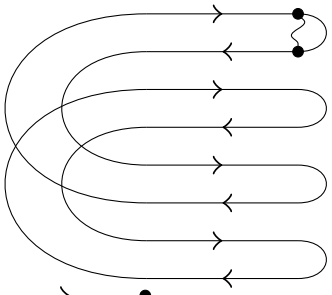
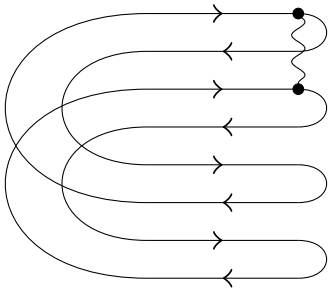
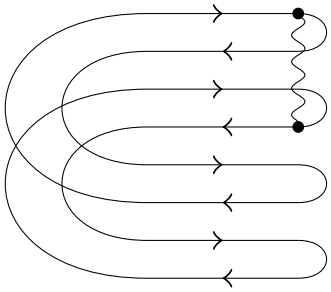
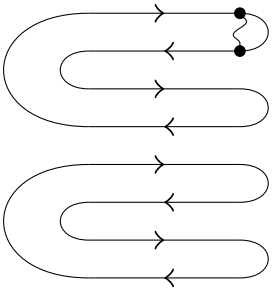
contribution can be written as

$$\mathcal{N}_{F,8} = \sum_{n=0}^{\infty} \int_{z_1^+ < \ldots < z_n^+} \mathcal{N}_{F,8,n}(z_1^+, \ldots, z_n^+). \tag{100}$$

The singlet states in (95) span a basis and thus we can write every $\mathcal{N}_{F,8,n}(z_1^+, \ldots, z_n^+)$ as their linear combination

$$\mathcal{N}_{F,8,n}(z_1^+, \ldots, z_n^+) = \sum_{i=1}^{24} a_{i,n} \mathcal{N}^{(i)}. \tag{101}$$

When we have written the contribution $\mathcal{N}_{F,8,n-1}$ as a linear combination of the basis states, we can get the contribution $\mathcal{N}_{F,8,n}$ recursively by adding a gluon link (a ladder) in every possible way to the contribution $\mathcal{N}_{F,8,n-1}$. Thus we get a recursion relation

$\mathcal{N}_{F,8,n} = \mu_A^2(z_n^+) a_{1,n-1} \Big[L(\mathbf{x}_1,\mathbf{x}_2)$  $- L(\mathbf{x}_1,\mathbf{x}_3)$  $+ L(\mathbf{x}_1,\mathbf{x}_4)$  $\ldots \Big] + \mu_A^2(z_n^+) a_{2,n-1} \Big[L(\mathbf{x}_1,\mathbf{x}_2)$  $+ \ldots \Big]$

$+ \mu_A^2(z_n^+) a_{3,n-1} \Big[\ldots \Big] + \ldots$

(102)

In equation (102) the $\mu_A^2(z_n^+)$,L's and $\pm$ in front of the L's come from the components of the Wilson lines that are associated with the generator contraction in question.

The $\pm$ in front of the L's come from the imaginary unit in the Wilson line which results in a $-$-sign to the conjugated Wilson line. Thus we get a factor $(\pm i)^2 = -1$ when we have contraction between two Wilson lines or two conjugated Wilson lines. We get a factor $-i \cdot i = +1$ when the contraction is done between a Wilson line and a conjugated Wilson line.

Next we shall employ the graphical Fierz identity (97) to express the equation (102) in the form (120). After we have done this, we can replace all the linearly independent basis topologies (95) with linearly independent $\mathbb{R}^{24}$ vectors. The obvious choice is to replace the basis topologies with the standard basis vectors as

$$\mathcal{N}^{(i)} = e_i, \quad i \in 1,2,\ldots,23,24, \tag{103}$$

where for instance

$$e_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}. \tag{104}$$

Doing the replacement, we can rewrite the recursion relation (120) as

$$\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{24,n} \end{bmatrix} = \mu_A^2(z_n^+) M \begin{bmatrix} a_{1,n-1} \\ a_{2,n-1} \\ \vdots \\ a_{24,n-1} \end{bmatrix}, \tag{105}$$

where M is the z^+-evolution matrix describing the evolution of the Wilson line states as we add gluon contractions to them. It is straightforward to find the elements of the matrix M by comparing equation (105) to the fully contracted form of equation (102). The recursion relation (105) is can be easily solved yielding

$$\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{24,n} \end{bmatrix} = \left[\prod_{i=1}^{n} \mu_A^2(z_i^+) \right] M^n \begin{bmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{24,0} \end{bmatrix}, \tag{106}$$

where the values of $a_{i,0}$ are extracted from the expression of the correlator we are calculating. We shall discuss this more when we get the final result. The equation (106) does not take into account the initial conditions of fully contracted Wilson line correlators. Noticing that the $a_{i,n}$ corresponds to the correlator $\mathcal{N}^{(i)}$, we can

add the initial conditions as

$$\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix}\begin{bmatrix}a_{1,n}\\ a_{2,n}\\ \vdots\\ a_{24,n}\end{bmatrix} = \left[\prod_{i=1}^{n}\mu_A^2(z_i^+)\right]\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix} M^n \begin{bmatrix}a_{1,0}\\ a_{2,0}\\ \vdots\\ a_{24,0}\end{bmatrix}. \tag{107}$$

To get the final expression, we still have to sum over all possible number of ladders, introduce the path-ordered integration over the +-coordinate originating from the definition of the Wilson line and multiply with the tadpole contributions. This yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}\underbrace{\left[\int_{z_1^+<\ldots<z_n^+}\prod_{i=1}^{n}\mu_A^2(z_i^+)\right]}_{\text{Remove path-ordering}\Longrightarrow\frac{1}{n!}}\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix} M^n \begin{bmatrix}a_{1,0}\\ a_{2,0}\\ \vdots\\ a_{24,0}\end{bmatrix}\mathcal{T}_{F,8}\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\left[\int_{z_1^+,\ldots,z_n^+}\prod_{i=1}^{n}\mu_A^2(z_i^+)\right]}_{=\mu_A^{2n}}\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix} M^n \begin{bmatrix}a_{1,0}\\ a_{2,0}\\ \vdots\\ a_{24,0}\end{bmatrix}\mathcal{T}_{F,8}\\
&=\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix}\underbrace{\sum_{n=0}^{\infty}\frac{1}{n!}\mu_A^{2n}M^n}_{\exp(\mu_A^2 M)}\begin{bmatrix}a_{1,0}\\ a_{2,0}\\ \vdots\\ a_{24,0}\end{bmatrix}\mathcal{T}_{F,8}\\
&=\begin{bmatrix}\mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)}\end{bmatrix}\exp(\mu_A^2 M)\begin{bmatrix}a_{1,0}\\ a_{2,0}\\ \vdots\\ a_{24,0}\end{bmatrix}\mathcal{T}_{F,8}.
\end{aligned} \tag{108}$$

Now the full expression for any fundamental 8-point correlator $C_{F,8}$ can be found

from

$$C_{F,8} = \sum_{i=1}^{24} c_i \mathcal{N}^{(i)}$$
$$= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(24)} \end{bmatrix} \exp(\mu_A^2 M) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{24} \end{bmatrix} \mathcal{T}_{F,8}. \tag{109}$$

Let us now discuss this final result (109). The initial conditions of the basis states $\mathcal{N}^{(i)}$ depend on how we choose the close the $z^+ = \infty$ side of the basis topologies i.e. the right side of the topologies in the graphical representation. In the fundamental representation, the initial condition is found by replacing every Wilson line with an identity matrix i.e. every loop (or trace) in the topology gives a power of $\text{Tr}[\mathbb{I}] = N$. For example an one loop topology has an initial condition of N and a three loop topology has an initial condition of N^3. The coefficients c_i are the factors multiplying the correlator we wish to evaluate. For instance the correlator

$$N^2 \Big\langle \text{Tr}[V(\mathbf{x}_1)V^\dagger(\mathbf{x}_2)V(\mathbf{x}_3)V^\dagger(\mathbf{x}_4)] \,\text{Tr}[V(\mathbf{x}_5)V^\dagger(\mathbf{x}_6)V(\mathbf{x}_7)V^\dagger(\mathbf{x}_8)] \Big\rangle \tag{110}$$

would have the coefficients

$$c_2 = N^2, \quad c_j = 0, \quad \forall j \neq 2. \tag{111}$$

The L-functions in the tadpole and ladder contributions combine to form Γ-functions and the remaining L-functions can be written as Γ-functions by adding suitable zeroes to the matrix elements. The analytical evaluation of the matrix exponential is cumbersome, but can be done using diagonalization or Jordan form. We shall discuss this more in the section 3.3. The exact expression for the transition matrix M can be found in the appendix B. The transition matrix elements were found using the Mathematica code that I wrote as part of my research training. I have included the code in the appendix C.

Now we know how to calculate the adjoint 4-point correlator (87) by using the relation (91) and by taking the limit discussed earlier in conjuction with the replacement (89). Note that the equation (91) includes fundamental 8-point correlators with 4 different topologies for the connections in the $z^+ = \infty$ side of the basis topologies. Thus our result for the adjoint 4-point correlator will be a sum of 4 different contractions of the transition matrix (as seen in (109)) with 4 different initial conditions for the basis states.

3.2.3 Adjoint representation 4-point correlators

Now we get to the main point of this thesis: how to evaluate an adjoint representation Wilson line 4-point correlator in the MV-model? We shall use the same method that we used in the fundamental case earlier. Now the basis states and state transitions are much more complicated, but the procedure is mostly analogous.

According to [14], the singlet state base for four adjoint representation Wilson lines

$$S_{A,4} = \langle U(x_1)_{\beta_1\alpha_1} U(x_2)_{\beta_2\alpha_2} U(x_3)_{\beta_3\alpha_3} U(x_4)_{\beta_4\alpha_4} \rangle \tag{112}$$

can be chosen to be

$$\begin{aligned}
\mathcal{N}^{(1)} &= S_{A,4}\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4} \\
\mathcal{N}^{(2)} &= S_{A,4}\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4} \\
\mathcal{N}^{(3)} &= S_{A,4}\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\
\mathcal{N}^{(4)} &= S_{A,4}d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma} \\
\mathcal{N}^{(5)} &= S_{A,4}d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma} \\
\mathcal{N}^{(6)} &= S_{A,4}d^{\alpha_1\alpha_2\gamma}f^{\alpha_3\alpha_4\gamma} \\
\mathcal{N}^{(7)} &= S_{A,4}d^{\alpha_1\alpha_3\gamma}f^{\alpha_2\alpha_4\gamma} \\
\mathcal{N}^{(8)} &= S_{A,4}d^{\alpha_1\alpha_4\gamma}f^{\alpha_2\alpha_3\gamma}
\end{aligned} \tag{113}$$

where the indices are matrix indices of the adjoint representation Wilson lines U. This basis is only valid for $SU(3)$ for the symmetric structure constants d are zero for $SU(2)$ and the result of appendix A implies that the set of states (113) is not a complete base for $SU(N), \quad N > 3$. This omits the possibility of employing the large N limit but still leaves the result valid for the QCD-case.

Again, the calculation does not depend on the choice of the contractions of the $z^+ = -\infty$ i.e. the beta-index side. The choice only affects the initial conditions as it did in the fundamental case. For notational clarity, let us consider the case where $\beta_1 = \beta_2$ and $\beta_3 = \beta_4$.

Let us use the same graphical representation for the Wilson lines and the generator contractions as before. Note that in the adjoint case the Wilson line "directions" can change as long as the arrows are mirrored in the whole loop. This means that

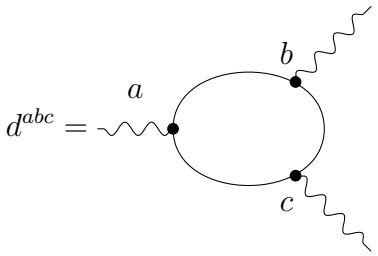
$$\begin{aligned}
\mathcal{N}^{(1)} &= \mathrm{Tr}[U(\mathbf{x}_1)U^\dagger(\mathbf{x}_2)]\,\mathrm{Tr}[U(\mathbf{x}_3)U^\dagger(\mathbf{x}_4)] \\
&= \mathrm{Tr}[U^\dagger(\mathbf{x}_1)U(\mathbf{x}_2)]\,\mathrm{Tr}[U^\dagger(\mathbf{x}_3)U(\mathbf{x}_4)].
\end{aligned} \tag{114}$$

This follows from $U \in \mathbb{R}$ and from the invariance of the trace under transposing

$$\begin{aligned}\mathrm{Tr}[U(\mathbf{x}_1)U^\dagger(\mathbf{x}_2)] &= U_{ab}(\mathbf{x}_1)U^\dagger_{ba}(\mathbf{x}_2) \\ &= U^\dagger_{ba}(\mathbf{x}_1)U_{ab}(\mathbf{x}_2) = \mathrm{Tr}[U^\dagger(\mathbf{x}_1)U(\mathbf{x}_2)]\end{aligned}$$

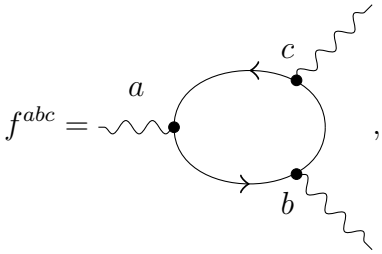
and it is valid for traces of multiple adjoint Wilson lines i.e. the order of the lines gets mirrored and the lines are turned into their Hermitean conjugates and the conjugates are turned into non-conjugated Wilson lines.

Calculations using the graphical representation get cumbersome rather quickly in the adjoint representation as we will shortly see. However, it is useful for getting the initial algebraic expression for the generator contractions. This is why it is useful to introduce graphical representations for the structure constants f and d. The symmetric structure constant d can be represented as



(115)

and the antisymmetric structure constant f can be represented as



, (116)

where the arrow denotes the ordering of the indices in f. The antisymmetric structure constant is invariant under cyclic permutation and thus we can start reading the indices from any point as long as we are faithful to the ordering. Changing the direction of the arrow in the circular path introduces an overall $-$-sign. The symmetric structure constant has no ordering due to its symmetricity under any permutation. The graphical representation of f differs from regular Feynman 3-gluon vertex because the Feynman rule is symmetric under permutations whereas f is not, thus the ordering is required.

The structure of the adjoint Wilson line is the same as the fundamental one. The two-point correlator of the sources ρ's are the same. With these remarks, we

expect the adjoint Wilson line contractions to factorize into the ladder and tadpole contributions

$$\mathcal{M}_A = \mathcal{N}_A \mathcal{T}_A. \tag{117}$$

The tadpole contributions yield

$$\mathcal{T}_A = \exp\Big[-\frac{1}{2} C_A \mu_A^2 \sum_{i=1}^{4} L(x_i,x_i)\Big]. \tag{118}$$

We will again denote the n-ladder contribution of $\mathcal{N}_A$ as $\mathcal{N}_{A,n}(z_1^+,\ldots,z_n^+)$. The whole ladder type contribution is then

$$\mathcal{N}_A = \sum_{n=0}^{\infty} \int_{z_1^+<\ldots<z_n^+} \mathcal{N}_{A,n}(z_1^+,\ldots,z_n^+). \tag{119}$$

The singlet states (113) form a basis and thus we can write

$$\mathcal{N}_{A,n}(z_1^+,\ldots,z_n^+) = \sum_{i=1}^{8} A_{i,n} \mathcal{N}^{(i)}. \tag{120}$$

The $\mathcal{N}_{A,n}$ can be found from $\mathcal{N}_{A,n-1}$ by adding one ladder type contraction in every way to the basis states

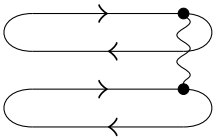
$$\begin{aligned}
\mathcal{N}_{A,n} = {} & \mu_A^2(z_n^+) A_{1,n-1} \Big[L(\mathbf{x}_1,\mathbf{x}_2) \quad - L(\mathbf{x}_1,\mathbf{x}_3) \quad + \ldots \Big] \\
& + \mu_A^2(z_n^+) A_{2,n-1} \Big[L(\mathbf{x}_1,\mathbf{x}_2) \quad + L(\mathbf{x}_1,\mathbf{x}_3) \quad + \ldots \Big] \\
& + \mu_A^2(z_n^+) A_{3,n-1} \Big[\ldots \Big] \\
& + \mu_A^2(z_n^+) A_{4,n-1} \Big[L(\mathbf{x}_1,\mathbf{x}_2) \quad + \ldots \Big] \\
& + \ldots .
\end{aligned} \tag{121}$$

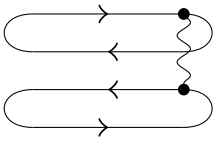
As in the fundamental case, the sign in front of the L-functions depends on the orientation of the graphically represented adjoint Wilson lines in the endpoints of the contraction. We have to fix the orientation for the graph we are calculating, but the specific choice does not alter the end result. This is due to a $-$-sign that is introduced to every contraction endpoint on every flipped Wilson line. Thus the flipping of an Wilson line introduces the same amount of $-$-signs in front of the L-functions as to the generator contractions that are written as antisymmetric structure constants f.

Let us have some examples on how to calculate the contractions in the equation (121) and how the Wilson line orientation invariance arises. Let us first define

$$\tilde{S}_{A,4} = \langle U(x_1)_{\beta_1\alpha_1} U(x_2)_{\beta_2\alpha_2} U(x_3)_{\beta_3\alpha_3} U(x_4)_{\beta_4\alpha_4} \rangle \delta^{\beta_1\beta_2}\delta^{\beta_3\beta_4} \tag{122}$$

Take for example the second contraction term in the recursion relation (121)

$-L(\mathbf{x}_1,\mathbf{x}_3)$  $= -L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}T^{\gamma}_{\alpha_1\alpha_2}T^{\gamma}_{\alpha_3\alpha_4}$

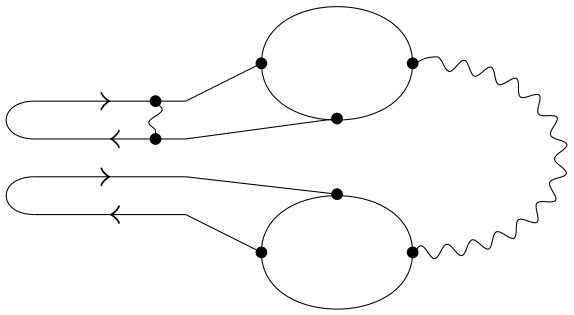
$= L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}T^{\gamma}_{\alpha_1\alpha_2}T^{\gamma}_{\alpha_4\alpha_3} = L(\mathbf{x}_1,\mathbf{x}_3)$  , (123)

where we see the invariance under conjugation of adjoint Wilson lines. Now using the identities color algebra discussed in subsection 3.1.2 we get

$$\begin{aligned}
&L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}T^{\gamma}_{\alpha_1\alpha_2}T^{\gamma}_{\alpha_4\alpha_3} = -L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}f^{\gamma\alpha_1\alpha_2}f^{\gamma\alpha_4\alpha_3} \\
&= L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}f^{\alpha_1\alpha_2\gamma}f^{\alpha_3\alpha_4\gamma} \\
&= L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}\left[\frac{2}{N}\left(\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4} - \delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3}\right) + d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma} - d^{\alpha_2\alpha_3\gamma}d^{\alpha_1\alpha_4\gamma}\right] \\
&= L(\mathbf{x}_1,\mathbf{x}_3)\tilde{S}_{A,4}\left[\frac{2}{N}\left(\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4} - \delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3}\right) + d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}\right. \\
&- \left[\frac{2(24-10N^2+N^4)}{N(48-29N^2+3N^4)}\right][\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4} + \delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}] \\
&- \left[\frac{2(N^2-12)(4-N^2)}{N(48-29N^2+3N^4)}\right]\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\
&\left. - \left[\frac{24-15N^2+N^4}{48-29N^2+3N^4}\right][d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma} + d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}]\right],
\end{aligned} \tag{124}$$

where we have written the contraction as a linear combination of the basis states (113).

Let us take an example involving a contraction involving structure constants. One contributing contraction is



$$= \tilde{S}_{A,4} T^{\eta}_{\alpha_1 a} T^{\eta}_{b\alpha_2} d^{ab\gamma} d^{\alpha_3\alpha_4\gamma} \tag{125}$$

$$\begin{aligned}
&= \tilde{S}_{A,4} T^{\eta}_{\alpha_1 a} T^{\eta}_{b\alpha_2} D^{\gamma}_{ab} d^{\alpha_3\alpha_4\gamma} = \tilde{S}_{A,4} T^{\alpha_1}_{\eta a} T^{\alpha_2}_{b\eta} D^{\gamma}_{ab} d^{\alpha_3\alpha_4\gamma} \\
&= \tilde{S}_{A,4} d^{\alpha_3\alpha_4\gamma} \operatorname{Tr}\left[T^{\alpha_2} T^{\alpha_1} D^{\gamma}\right] \\
&= \tilde{S}_{A,4} d^{\alpha_3\alpha_4\gamma} \frac{N}{2} d^{\alpha_2\alpha_1\gamma} = \tilde{S}_{A,4} \frac{N}{2} d^{\alpha_1\alpha_2\gamma} d^{\alpha_3\alpha_4\gamma},
\end{aligned}$$

where we used the trace identity listed in (59). All the remaining contractions in equation (121) can be expressed using the chosen singlet base (113) by using the color algebra introduced in subsections 3.1.1 and 3.1.2.

After doing all the contractions on the RHS of the recursion relation (121) and expressing them as a linear combination of the basis states (113), we can express the basis states as vectors as we did in the fundamental case. Now our basis is 8-dimensional, so we replace the states with $\mathbb{R}^8$ standard basis vectors e_i as

$$\mathcal{N}^{(i)} = e_i, \quad i \in 1,2,\ldots,8. \tag{126}$$

Doing the replacements, we can write the recursion relation as

$$\begin{bmatrix} A_{1,n} \\ A_{2,n} \\ \vdots \\ A_{8,n} \end{bmatrix} = \mu_A^2(z_n^+) W \begin{bmatrix} A_{1,n-1} \\ A_{2,n-1} \\ \vdots \\ A_{8,n-1} \end{bmatrix}, \tag{127}$$

where W is the z^+-evolution matrix. After the same treatment we did in the fun-

damental case, we arrive to a similar looking result

$$\begin{aligned} C_{A,4} &= \sum_{i=1}^{8} c_i \mathcal{N}^{(i)} \\ &= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(8)} \end{bmatrix} \exp(\mu_A^2 W) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} \mathcal{T}_{A,4}, \end{aligned} \tag{128}$$

where, again, the tadpole contributions combine with the L-functions on the diagonal of the W-matrix to form Γ-functions. The rest of the L-functions form Γ-functions despite the W-matrix elements are much more complicated than the M-matrix elements we discussed in the fundamental case.

The W-matrix elements are listed in the appendix B. An important thing to note is that the W-matrix is block diagonal with 5- and 3-dimensional blocks

$$W = \begin{bmatrix} W_{5\times5} & 0 \\ 0 & W_{3\times3} \end{bmatrix}. \tag{129}$$

This ultimately means that the first 5 and the last 3 basis states do not mix through the z^+-evolution represented by the W-matrix. Additionally because the correlators arising in physical calculations are usually color states corresponding to the 5 first basis states (113) (i.e. $c_6 = c_7 = c_8 = 0$), we can reduce the equation (128) to a simpler form.

The simplification follows straightforwardly from the equation (128). Let us now show this. Let us first define

$$W_1 = \begin{bmatrix} W_{5\times5} & 0 \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 \\ 0 & W_{3\times3} \end{bmatrix} \tag{130}$$

for which

$$W = W_1 + W_2 \tag{131}$$

and

$$[W_1, W_2] = 0. \tag{132}$$

Due to the commutativity (132) it holds that

$$\exp(W) = \exp(W_1 + W_2) = \exp(W_1)\exp(W_2), \tag{133}$$

and thus assuming $c_6 = c_7 = c_8 = 0$, we get from the equation (128)

$$\begin{aligned}
C_{A,4} &= \sum_{i=1}^{8} c_i \mathcal{N}^{(i)} = \sum_{i=1}^{5} c_i \mathcal{N}^{(i)} \\
&= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(8)} \end{bmatrix} \exp(\mu_A^2 W) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} \mathcal{T}_{A,4} \\
&= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(8)} \end{bmatrix} \exp(\mu_A^2 W_1) \underbrace{\exp(\mu_A^2 W_2)}_{\to \exp(0)=1} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} \mathcal{T}_{A,4} \\
&= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(8)} \end{bmatrix} \exp(\mu_A^2 W_1) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} \mathcal{T}_{A,4} \\
&= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(5)} \end{bmatrix} \exp(\mu_A^2 W_{5\times 5}) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_5 \end{bmatrix} \mathcal{T}_{A,4},
\end{aligned} \tag{134}$$

where we have used the fact that the upper block $W_{5\times 5}$ only operates on the first 5 coefficients c_i and the first 5 initial conditions $\mathcal{N}^{(i)}$. Additionally we have used the fact that the lower block only operates on the last 3 c_i's which are zero and thus every power of W_2 acting on the c_i-coefficient vector gives a zero and therefore the exponent with W_2 evaluates to 1. Because $W_{5\times 5}$ does not operate on the 3 last initial conditions $\mathcal{N}^{(i)}$, the last 3 basis states are redundant and thus we can make the replacement we did in the last line of (128). The final reduced expression for the adjoint 4-point correlator reads

$$\begin{aligned}
C_{A,4} &= \sum_{i=1}^{5} c_i \mathcal{N}^{(i)} \\
&= \begin{bmatrix} \mathcal{N}_{\text{ini}}^{(1)} & \mathcal{N}_{\text{ini}}^{(2)} & \cdots & \mathcal{N}_{\text{ini}}^{(5)} \end{bmatrix} \exp(\mu_A^2 W_{5\times 5}) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_5 \end{bmatrix} \mathcal{T}_{A,4}.
\end{aligned} \tag{135}$$

The initial conditions are found the same way as in the fundamental representation case by setting every Wilson line to identity $U \to I$. The initial conditions depend on how we contract the β-indices of the singlet states (112). For instance, if we wish to calculate

$$\mathrm{Tr}[U(\mathbf{x}_1)U^\dagger(\mathbf{x}_2)]\,\mathrm{Tr}[U(\mathbf{x}_3)U^\dagger(\mathbf{x}_4)],$$

we would have to contract the singlets with $\delta^{\beta_1\beta_2}\delta^{\beta_3\beta_4}$. With this β-contraction choice, the initial condition for the first basis state would be

$$\mathcal{N}_{\mathrm{ini}}^{(1)} = \mathrm{Tr}[\mathbb{I}]\,\mathrm{Tr}[\mathbb{I}] = (N^2-1)^2, \tag{136}$$

where (N^2-1) is the dimension of the adjoint representation. The initial condition for the 4th basis state would be

$$\mathcal{N}_{\mathrm{ini}}^{(4)} = d^{\alpha_1\alpha_1\gamma}d^{\alpha_3\alpha_3\gamma} = 0 \tag{137}$$

and for the 5th state we would get

$$\mathcal{N}_{\mathrm{ini}}^{(5)} = d^{\alpha_1\alpha_3\gamma}d^{\alpha_1\alpha_3\gamma} = \left(\frac{N^2-4}{N}\right)\delta^{\gamma\gamma} = \left(\frac{N^2-4}{N}\right)(N^2-1). \tag{138}$$

It is important to note that the result (135) is only valid for $N = 3$ i.e. the QCD-case. The $N = 2$ case might work with some modification to the result. In this case the symmetric structure constants vanish ($d = 0$) and thus the antisymmetric structure constants f can be written as linear combinations of the Kronecker deltas δ which can be seen from the equation (57). This implies that the singlet basis for $N = 2$ is not the same as for $N = 3$ and must be constructed separately if one wishes to calculate correlators for $N = 2$. For values $N > 3$ this calculation will not work because of the result derived in the appendix A. The result implies that the basis (113) does not span the whole basis for the possible singlet state configurations and must be supplemented with some additional states.

3.3 Analytical evaluation of matrix exponentials

The matrix exponentials arising in the calculation of the Wilson line correlators can be evaluated in a closed form, but this is generally a cumbersome process, especially for large matrices. Nevertheless, we shall discuss how one could evaluate these exponents, if one would like to do so. This is straightforward if we can exploit the

diagonalizability of the exponentiated matrix. For non-diagonalizable matrices, we need to utilize the Jordan canonical form of the matrix. We shall discuss both of these cases in this section.

3.3.1 Diagonalizable matrix

Take a diagonalizable matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ with $n \in \mathbb{N}$. For this a diagonalizable matrix A, one can find a basis change matrix P so that

$$B = P^{-1}AP, \tag{139}$$

where the matrix B is diagonal. Inverting (139), we get

$$A = PBP^{-1}, \tag{140}$$

Furthermore, we can easily see that

$$A^n = (PBP^{-1})^n = PB^nP^{-1}. \tag{141}$$

Now we are ready to tackle the challenge that is $\exp(A)$. The matrix exponential is defined as a series, and thus

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = P\Big[\sum_{k=0}^{\infty} \frac{B^k}{k!}\Big]P^{-1}, \tag{142}$$

where the equation (141) was used to get the second equality. Due to the diagonality of the matrix B, we get

$$B^k = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & B_{nn} \end{bmatrix}^k = \begin{bmatrix} B_{11}^k & 0 & \cdots & 0 \\ 0 & B_{22}^k & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & B_{nn}^k \end{bmatrix}, \tag{143}$$

and thus

$$e^A = P\Big[\sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} B_{11}^k & 0 & \cdots & 0 \\ 0 & B_{22}^k & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & B_{nn}^k \end{bmatrix}\Big]P^{-1} = P \begin{bmatrix} e^{B_{11}} & 0 & \cdots & 0 \\ 0 & e^{B_{22}} & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & e^{B_{nn}} \end{bmatrix} P^{-1} \tag{144}$$

The equation (144) gives us the analytic expression for the exponentiated matrix, as we desired.

3.3.2 Non-diagonalizable matrix

Let us have an non-diagonalizable matrix $M \in \mathcal{M}_{n\times n}(\mathbb{C})$ with $n \in \mathbb{N}$. One way to tackle $\exp(M)$ is to employ the so called Jordan form of the matrix M. The Jordan form coincides with the diagonalized form of the matrix when considering a diagonalizable matrix, and thus is a more general object. It can be thought as writing the matrix in a basis in which it is closest to a diagonal form. [22]

Let us first introduce the definition of a nilpotent matrix. Let $N \in \mathcal{M}_{n\times n}(\mathbb{C})$ be a nilpotent matrix. Then there exists a $q \in \mathbb{N}$ so that

$$N^q = 0. \tag{145}$$

We will need the definition of the nilpotent matrix in the coming discussion. [22]

Even though the matrix M can not be diagonalized, we can "block-diagonalize" it. The Jordan form of the matrix M reads

$$J = S^{-1}MS = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & J_i \end{bmatrix}, \tag{146}$$

where the $J_1, J_2, \ldots J_i$ are Jordan blocks. The Jordan blocks are of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{bmatrix}. \tag{147}$$

The Jordan blocks are not necessarily of the same size, but the are always square blocks. Every block J_i has one eigenvector with an eigenvalue λ_i. All the diagonal elements are the same λ_i and the elements just above the diagonal are ones. All the other elements are zeroes. [22]

Let us now discuss how we find the basis change matrix S in equation (146). The matrix M does not have enough eigenvectors to form a basis, and thus they have to be supplemented with so called generalized eigenvectors. These vectors belong to strings initiated by the proper eigenvectors of M. After finding the eigenvalues of the matrix M using the usual procedure, we can find the proper eigenvector x_1 corresponding to the eigenvalue λ_1 as

$$Mx_1 = \lambda_1 x_1. \tag{148}$$

The string of vectors $(x_1,x_2,\ldots,x_j)$ initiated by the eigenvector x_1 can be found from the expression

$$Mx_j = \lambda x_j + x_{j-1}. \tag{149}$$

These eigenvectors and generalized eigenvectors form the columns of the basis change matrices S. Explicitly written, the matrix S reads

$$S = \begin{bmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & \ldots & z_1 & z_2 & z_3 \end{bmatrix}, \tag{150}$$

where x_a,y_b,z_c represent strings of vectors corresponding to eigenvalues of the matrix M. [22]

Let us now discuss how to exponentiate the matrix M using the Jordan form as discussed in [23]. Inverting the equation (146) we get

$$M = SJS^{-1}. \tag{151}$$

Let us again consider the Jordan form J (146). Let us divide the matrix J into two parts

$$J = D + N, \tag{152}$$

where the D contains all the diagonal elements of J and N contains all the ones that lie on the superdiagonal of J. Let us denote the blocks of D and N as D_i and N_i respectively and let us keep the sizes of the blocks the same as for J so that

$$J_i = D_i + N_i. \tag{153}$$

The blocks D_i are diagonal

$$D_i = \lambda_i \mathbb{I} \tag{154}$$

and thus

$$D_i N_i = N_i D_i. \tag{155}$$

From this it follows

$$
\begin{aligned}
DN = &\begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & D_i \end{bmatrix} \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & N_i \end{bmatrix} = \begin{bmatrix} D_1N_1 & 0 & \cdots & 0 \\ 0 & D_2N_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & D_iN_i \end{bmatrix} \\
= &\begin{bmatrix} N_1D_1 & 0 & \cdots & 0 \\ 0 & N_2D_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & N_iD_i \end{bmatrix} = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & N_i \end{bmatrix} \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & D_i \end{bmatrix} \\
= &ND
\end{aligned}
\tag{156}
$$

that is

$$[D,N] = 0. \tag{157}$$

Using the Jordan form (151) and the equation (152) we get

$$e^M = e^{SJS^{-1}} = Se^JS^{-1} = Se^{D+N}S^{-1} = Se^De^NS^{-1}, \tag{158}$$

where D is a diagonal matrix and N is a strictly upper triangular matrix and is thus nilpotent. The exponent of D is easily computed as it is a trivially diagonalizable matrix and the series expansion of the $\exp(N)$ terminates at some point, so we have to sum a finite amount of powers of D. Thus we have found a way to express the exponent of a non-diagonalizable matrix in an analytical way.

Computing the matrix exponential in this way can easily become cumbersome. Luckily, there exist computing systems with pre-existing commands that provides us with the Jordan form and the basis change matrices of a given matrix. This makes computing the exponential much easier. One of such systems is Wolfram Mathematica and its command named *JordanDecomposition*. The remaining task is then to compute the nilpotent part and to take the product of the evaluated matrices. This can still be a cumbersome task for large matrices with horrible eigenvalues and (generalized) eigenvectors, but might be worthwhile to do for smaller matrices.

4 Conclusion

The main point of this thesis was getting the expression for the 4-point correlator of adjoint Wilson lines from which the correlator can be evaluated. The most general expression can be found in the equation (128). Due to the block-diagonality ((5×5) and (3×3)) of the adjoint transition matrix ((8×8)) found in the appendix B, the general expression decouples, and can be simplified to yield the expression (135). This is an useful simplification, because scattering processes usually only contain the states corresponding to the first 5 singlet states in our chosen basis (113). The (5×5)-block contains all the information about the evolution of these 5 states. Thus we only need to calculate the exponential of a (5×5)-matrix and not the exponential of the whole (8×8)-matrix.

The evaluation of the adjoint 4-point correlator can be related to the evaluation of a fundamental 8-point correlator as discussed before. In that case we would have to compute the exponential of a (24×24)-matrix, which would be computationally much more expensive than the exponential of the (5×5)-matrix. Additionally we would potentially have to contract the matrix exponential with multiple (initial condition, coefficient)-pairs. Thus it is analytically much neater to do the whole computation in the adjoint representation as done in this thesis.

The result of this work can be used to study multi-particle correlations within the Color Glass Condensate framework, without having to rely on the large-N limit when computing the adjoint Wilson line 4-point correlators, potentially giving better results. The large-N limit is used extensively in literature by first expressing the adjoint Wilson line correlators in terms of fundamental Wilson line correlators, and then by neglecting all but the leading-N terms.

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A Representing $d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma}$ as a linear combination of chosen basis singlet states

We start out by writing $d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma}$ as a general element of space spanned by the basis singlet states (113). Doing this we get

$$\begin{aligned} d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma} =& A\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4} + B\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4} + C\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\ &+ Dd^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma} + Ed^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma} \\ &+ Fd^{\alpha_1\alpha_2\gamma}f^{\alpha_3\alpha_4\gamma} + Gd^{\alpha_1\alpha_3\gamma}f^{\alpha_2\alpha_4\gamma} + Hd^{\alpha_1\alpha_4\gamma}f^{\alpha_2\alpha_3\gamma}, \end{aligned} \tag{159}$$

where A,B,C,D,E,F,G,$H \in \mathbb{C}$ are the coefficients of the states. Contracting the equation from the both sides (159) with $\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}$, we get

$$\begin{aligned} d^{\alpha_1\alpha_4\gamma}d^{\alpha_1\alpha_4\gamma} =& (\frac{N^2-4}{N})(N^2-1) \\ =& A\delta^{\alpha_1\alpha_1}\delta^{\alpha_4\alpha_4} + B\delta^{\alpha_1\alpha_4}\delta^{\alpha_1\alpha_4} + C\delta^{\alpha_1\alpha_4}\delta^{\alpha_1\alpha_4} \\ &+ Dd^{\alpha_1\alpha_1\gamma}d^{\alpha_4\alpha_4\gamma} + Ed^{\alpha_1\alpha_4\gamma}d^{\alpha_1\alpha_4\gamma} + Fd^{\alpha_1\alpha_1\gamma}f^{\alpha_4\alpha_4\gamma} \\ &+ Gd^{\alpha_1\alpha_4\gamma}f^{\alpha_1\alpha_4\gamma} + Hd^{\alpha_1\alpha_4\gamma}f^{\alpha_1\alpha_4\gamma} \\ =& A(N^2-1)^2 + B(N^2-1) + C(N^2-1) + D\cdot 0 \\ &+ E(\frac{N^2-4}{N})(N^2-1) + F\cdot 0 + G\cdot 0 + H\cdot 0 \end{aligned}$$

$$\implies (\frac{N^2-4}{N})(N^2-1) = A(N^2-1)^2 + B(N^2-1) + C(N^2-1) + E(\frac{N^2-4}{N})(N^2-1).$$

Contractions with rest of the basis states are done in the same fashion. We get

a system of equations

$$\begin{aligned}
\frac{4}{N}-5N+N^3 =&A(N^2-1)^2+B(N^2-1)\\
&+C(N^2-1)+E(\frac{4}{N}-5N+N^3),\\
\frac{4}{N}-5N+N^3 =&A(N^2-1)+B(N^2-1)^2\\
&+C(N^2-1)+D(\frac{4}{N}-5N+N^3),\\
0 =&A(N^2-1)+B(N^2-1)+C(N^2-1)^2\\
&+D(\frac{4}{N}-5N+N^3)+E(\frac{4}{N}-5N+N^3),\\
32-\frac{24}{N^2}-\frac{17N^2}{2}+\frac{N^4}{2} =&B(\frac{4}{N}-5N+N^3)+C(\frac{4}{N}-5N+N^3)\\
&+D(24-\frac{16}{N^2}-9N^2+N^4)+E(32-\frac{24}{N^2}-\frac{17N^2}{2}+\frac{N^4}{2}),\\
32-\frac{24}{N^2}-\frac{17N^2}{2}+\frac{N^4}{2} =&A(\frac{4}{N}-5N+N^3)+C(\frac{4}{N}-5N+N^3)\\
&+D(32-\frac{24}{N^2}-\frac{17N^2}{2}+\frac{N^4}{2})+E(24-\frac{16}{N^2}-9N^2+N^4),\\
0 =&F(4-5N^2+N^4)+G(-2+\frac{5N^2}{2}-\frac{N^4}{2})-H(-2+\frac{5N^2}{2}-\frac{N^4}{2}),\\
0 =&F(-2+\frac{5N^2}{2}-\frac{N^4}{2})+G(4-5N^2+N^4)+H(-2+\frac{5N^2}{2}-\frac{N^4}{2}),\\
0 =&-F(-2+\frac{5N^2}{2}-\frac{N^4}{2})+G(-2+\frac{5N^2}{2}-\frac{N^4}{2})+H(4-5N^2+N^4)
\end{aligned} \tag{160}$$

for contractions with $\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}$, $\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}$, $\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3}$, $d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma}$, $d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}$, $d^{\alpha_1\alpha_2\gamma}f^{\alpha_3\alpha_4\gamma}$, $d^{\alpha_1\alpha_3\gamma}f^{\alpha_2\alpha_4\gamma}$ and $d^{\alpha_1\alpha_4\gamma}f^{\alpha_2\alpha_3\gamma}$ respectively. Solving this system of equations for the coefficients gives

$$\begin{aligned}
A=B&=\frac{2(24-10N^2+N^4)}{N(48-29N^2+3N^4)},\\
C&=\frac{2(N^2-12)(4-N^2)}{N(48-29N^2+3N^4)},\\
D=E&=\frac{24-15N^2+N^4}{48-29N^2+3N^4},\\
F=G&=H=0.
\end{aligned} \tag{161}$$

Substituting the coefficients to the equation (159), we get the desired identity

$$\begin{aligned} d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma} =& \left[\frac{2(24-10N^2+N^4)}{N(48-29N^2+3N^4)}\right][\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}+\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}] \\ &+\left[\frac{2(N^2-12)(4-N^2)}{N(48-29N^2+3N^4)}\right]\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\ &+\left[\frac{24-15N^2+N^4}{48-29N^2+3N^4}\right][d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma}+d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}]. \end{aligned} \tag{162}$$

We can check the correctness of identity (162) by evaluating it for $N = 2$,3 and comparing it to known results. This is easy for $N = 2$, for we know that the symmetric structure constants d are zero in this case. Thus in the case of $N = 2$ equation (162) reduces to

$$\begin{aligned} 0 =& \left[\frac{2(24-10\cdot 2^2+2^4)}{2(48-29\cdot 2^2+3\cdot 2^4)}\right][\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}+\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}] \\ &+\left[\frac{2(2^2-12)(4-2^2)}{2(48-29\cdot 2^2+3\cdot 2^4)}\right]\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\ \iff & 0=0. \end{aligned}$$

$\implies$ Equation (162) is trivially valid for $SU(2)$.

Next we shall consider the less trivial $SU(3)$ case. Now the symmetric structure constants are not generally zero and the equation (162) gives

$$\begin{aligned} d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma} =& \frac{1}{3}[\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}+\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}] \\ &+\frac{1}{3}\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} \\ &-[d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma}+d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}]. \end{aligned}$$

This can be written more neatly as

$$d^{\alpha_1\alpha_2\gamma}d^{\alpha_3\alpha_4\gamma}+d^{\alpha_1\alpha_3\gamma}d^{\alpha_2\alpha_4\gamma}+d^{\alpha_1\alpha_4\gamma}d^{\alpha_2\alpha_3\gamma}=\frac{1}{3}[\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}+\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}+\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3}]. \tag{163}$$

The equation (163) can be found in literature, for instance in the paper [19]. From this we can conclude that the identity (162) is also valid for the case of $SU(3)$.

We can also try and check whether or not the identity (162) is valid for an arbitrary N. If the identity would be valid for all N, the squares (or contractions with

itself) of the LHS and the RHS should be equivalent. This condition is necessary, but not sufficient for the validity. Doing the contractions of (162) we get the equation

$$\frac{\left(N^2-4\right)^2\left(N^2-1\right)}{N^2} = \frac{(N-2)(N-1)(N+1)(N+2)\left(N^6-23N^4+164N^2-192\right)}{N^2\left(3N^4-29N^2+48\right)}, \tag{164}$$

which is valid for $N = 3$, but not for $N = 4$. From this we can conclude that the identity (162) is not valid for $N > 3$ and thus the singlet base (113) does not span the whole singlet space when considering the $N > 3$ case. Thus if we would want to calculate the adjoint representation Wilson line four point function so that we would be able to take the large-N limit in the end, we would have to know how the base of the singlet states behave with increasing N. However, the $N = 3$ case is sufficient for us at this time.

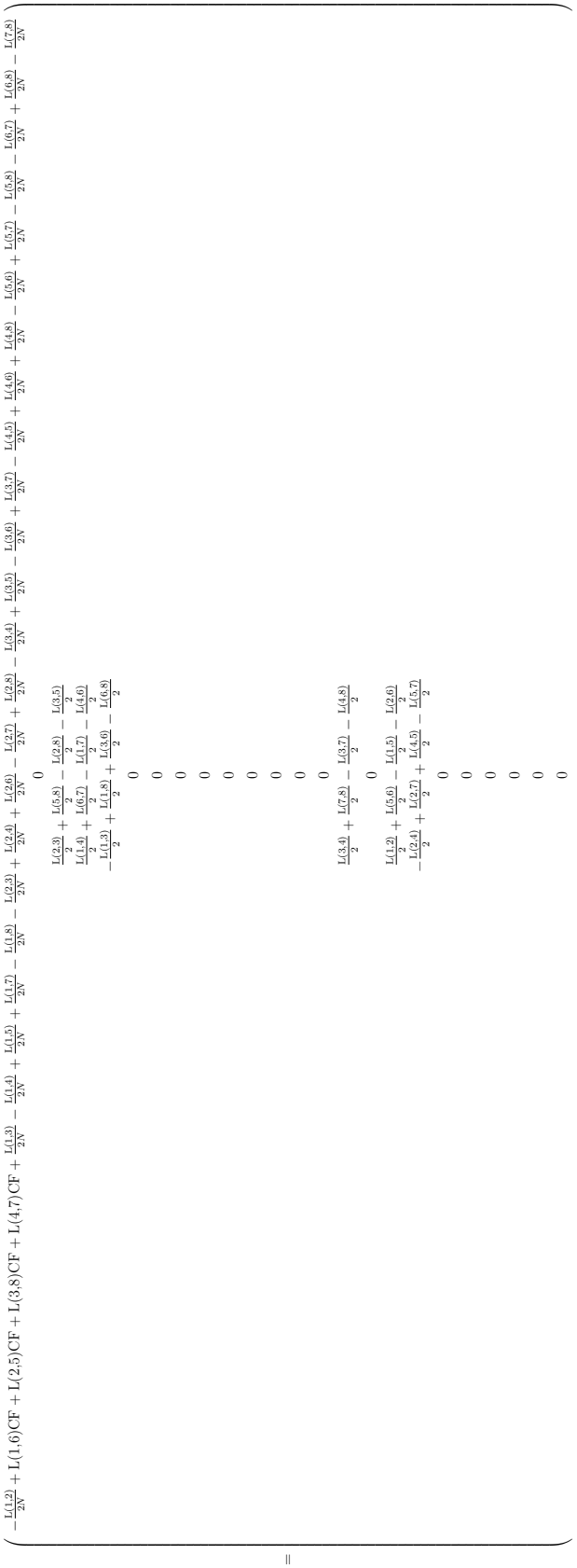
B Transition matrix elements

B.1 Fundamental representation matrix

In this appendix I present the elements for the (24×24) fundamental transition matrix with the base (95). We will use the notation $L(i,j) \equiv L(x_i,x_j)$. The matrix is valid for an arbitrary N. For presentation purposes, let us define the matrix as

$$\begin{bmatrix} x_1 & x_2 & \dots & x_{24} \end{bmatrix}, \tag{165}$$

where x_i are column vectors. The column vectors read



$$x_1 = \begin{pmatrix} -\frac{L(1,2)}{2N} + L(1,6)CF + L(2,5)CF + L(3,8)CF + L(4,7)CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\ 0 \\ \frac{L(2,3)}{2} + \frac{L(5,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,5)}{2} \\ \frac{L(1,4)}{2} + \frac{L(6,7)}{2} - \frac{L(1,7)}{2} - \frac{L(4,6)}{2} \\ -\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(3,6)}{2} - \frac{L(6,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(3,4)}{2} + \frac{L(7,8)}{2} - \frac{L(3,7)}{2} - \frac{L(4,8)}{2} \\ 0 \\ \frac{L(1,2)}{2} + \frac{L(5,6)}{2} - \frac{L(1,5)}{2} - \frac{L(2,6)}{2} \\ -\frac{L(2,4)}{2} + \frac{L(2,7)}{2} + \frac{L(4,5)}{2} - \frac{L(5,7)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{166}$$

$$x_2 = \begin{pmatrix} -\frac{L(1,2)}{2N} + L(1,4)CF + L(2,3)CF + L(5,8)CF + L(6,7)CF + \frac{L(1,3)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\ 0 \\ \frac{L(1,6)}{2} + \frac{L(4,7)}{2} - \frac{L(1,7)}{2} - \frac{L(4,6)}{2} \\ \frac{L(2,5)}{2} + \frac{L(3,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,5)}{2} \\ 0 \\ 0 \\ \frac{L(5,6)}{2} + \frac{L(7,8)}{2} - \frac{L(5,7)}{2} - \frac{L(6,8)}{2} \\ 0 \\ 0 \\ 0 \\ \frac{L(1,2)}{2} + \frac{L(3,4)}{2} - \frac{L(1,3)}{2} - \frac{L(2,4)}{2} \\ -\frac{L(2,6)}{2} + \frac{L(2,7)}{2} + \frac{L(3,6)}{2} - \frac{L(3,7)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,5)}{2} + \frac{L(1,8)}{2} + \frac{L(4,5)}{2} - \frac{L(4,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{167}$$

$$
x_3 = \begin{pmatrix}
-\frac{L(1,2)}{2N} + L(1,6)CF + L(2,3)CF + L(4,7)CF + L(5,8)CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
\frac{L(2,5)}{2} + \frac{L(3,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,5)}{2} \\
\frac{L(1,4)}{2} + \frac{L(6,7)}{2} - \frac{L(1,7)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(3,6)}{2} - \frac{L(1,3)}{2} - \frac{L(2,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(4,5)}{2} + \frac{L(7,8)}{2} - \frac{L(4,8)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
-\frac{L(2,4)}{2} + \frac{L(2,7)}{2} + \frac{L(3,4)}{2} - \frac{L(3,7)}{2} \\
0 \\
0 \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,8)}{2} + \frac{L(5,6)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \tag{168}
$$

$$
x_4 = \begin{pmatrix}
-\frac{L(1,2)}{2N} + L(1,4)\mathrm{CF} + L(2,5)\mathrm{CF} + L(3,8)\mathrm{CF} + L(6,7)\mathrm{CF} + \frac{L(1,3)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
\frac{L(1,6)}{2} + \frac{L(4,7)}{2} - \frac{L(1,7)}{2} - \frac{L(4,6)}{2} \\
\frac{L(2,3)}{2} + \frac{L(5,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,5)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(3,4)}{2} - \frac{L(4,8)}{2} \\
\frac{L(3,6)}{2} + \frac{L(7,8)}{2} - \frac{L(3,7)}{2} - \frac{L(6,8)}{2} \\
-\frac{L(2,6)}{2} + \frac{L(2,7)}{2} + \frac{L(5,6)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(4,5)}{2} - \frac{L(1,5)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \tag{169}
$$

$$
x_5 = \begin{pmatrix}
-\frac{L(1,2)}{2N} + L(1,8)CF + L(2,5)CF + L(3,6)CF + L(4,7)CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
-\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(3,8)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(5,8)}{2} - \frac{L(1,5)}{2} - \frac{L(2,8)}{2} \\
\frac{L(3,4)}{2} + \frac{L(6,7)}{2} - \frac{L(3,7)}{2} - \frac{L(4,6)}{2} \\
\frac{L(1,4)}{2} + \frac{L(7,8)}{2} - \frac{L(1,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(2,3)}{2} + \frac{L(5,6)}{2} - \frac{L(2,6)}{2} - \frac{L(3,5)}{2} \\
0 \\
0 \\
-\frac{L(2,4)}{2} + \frac{L(2,7)}{2} + \frac{L(4,5)}{2} - \frac{L(5,7)}{2} \\
0
\end{pmatrix} \tag{170}
$$

$$
x_6 = \begin{pmatrix}
\frac{L(1,3)}{2N} + L(1,2)CF + L(3,6)CF + L(4,7)CF + L(5,8)CF - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
-\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(2,3)}{2} - \frac{L(2,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(3,4)}{2} + \frac{L(6,7)}{2} - \frac{L(3,7)}{2} - \frac{L(4,6)}{2} \\
\frac{L(4,4)}{2} + \frac{L(2,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(3,5)}{2} + \frac{L(3,8)}{2} + \frac{L(5,6)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
\frac{L(4,5)}{2} + \frac{L(7,8)}{2} - \frac{L(4,8)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
0
\end{pmatrix} \tag{171}
$$

$$
x_7 = \begin{pmatrix}
0 \\
-\frac{L(5,7)}{2} + \frac{L(5,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,2)}{2N} + L(1,4)\mathrm{CF} + L(2,3)\mathrm{CF} + L(5,6)\mathrm{CF} + L(7,8)\mathrm{CF} + \frac{L(1,3)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} \\
0 \\
\frac{L(2,5)}{2} + \frac{L(3,6)}{2} - \frac{L(2,6)}{2} - \frac{L(3,5)}{2} \\
\frac{L(2,7)}{2} + \frac{L(3,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,7)}{2} \\
0 \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(4,5)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,7)}{2} + \frac{L(1,8)}{2} + \frac{L(4,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
\frac{L(1,2)}{2} + \frac{L(3,4)}{2} - \frac{L(1,3)}{2} - \frac{L(2,4)}{2} \\
0 \\
0
\end{pmatrix} \tag{172}
$$

$$
x_8 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(3,8)}{2} - \frac{L(4,8)}{2} \\
\frac{L(3,6)}{2} + \frac{L(4,7)}{2} - \frac{L(3,7)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
-\frac{L(1,2)}{2N} + \mathrm{L(1,8)CF} + \mathrm{L(2,5)CF} + \mathrm{L(3,4)CF} + \mathrm{L(6,7)CF} + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(5,8)}{2} - \frac{L(1,5)}{2} - \frac{L(2,8)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,6)}{2} + \frac{L(7,8)}{2} - \frac{L(1,7)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
\frac{L(2,3)}{2} + \frac{L(4,5)}{2} - \frac{L(2,4)}{2} - \frac{L(3,5)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(2,6)}{2} + \frac{L(2,7)}{2} + \frac{L(5,6)}{2} - \frac{L(5,7)}{2}
\end{pmatrix} \tag{173}
$$

$$
x_9 = \begin{pmatrix}
0 \\
0 \\
0 \\
-\frac{L(3,7)}{2} + \frac{L(3,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\
-\frac{L(1,7)}{2} + \frac{L(1,8)}{2} + \frac{L(4,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
\frac{L(2,3)}{2} + \frac{L(5,6)}{2} - \frac{L(2,6)}{2} - \frac{L(3,5)}{2} \\
0 \\
-\frac{L(1,2)}{2^N} + L(1,4)CF + L(2,5)CF + L(3,6)CF + L(7,8)CF + \frac{L(1,3)}{2^N} + \frac{L(1,5)}{2^N} - \frac{L(4,6)}{2^N} + \frac{L(4,7)}{2^N} - \frac{L(1,8)}{2^N} - \frac{L(2,3)}{2^N} + \frac{L(2,4)}{2^N} + \frac{L(2,6)}{2^N} - \frac{L(2,7)}{2^N} + \frac{L(2,8)}{2^N} - \frac{L(3,4)}{2^N} + \frac{L(3,5)}{2^N} + \frac{L(3,7)}{2^N} - \frac{L(3,8)}{2^N} - \frac{L(4,5)}{2^N} + \frac{L(4,6)}{2^N} - \frac{L(4,7)}{2^N} + \frac{L(4,8)}{2^N} - \frac{L(5,6)}{2^N} + \frac{L(5,7)}{2^N} - \frac{L(5,8)}{2^N} - \frac{L(6,7)}{2^N} + \frac{L(6,8)}{2^N} \\
0 \\
0 \\
\frac{L(2,7)}{2} + \frac{L(5,8)}{2} - \frac{L(2,8)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(3,4)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(4,5)}{2} - \frac{L(1,5)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0
\end{pmatrix} \tag{174}
$$

$$
x_{10} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\frac{L(2,5)}{2} + \frac{L(6,7)}{2} - \frac{L(2,6)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
\frac{L(2,3)}{2} + \frac{L(7,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,7)}{2} \\
0 \\
0 \\
-\frac{L(1,2)}{2N} + L(1,4)\mathrm{CF} + L(2,7)\mathrm{CF} + L(3,8)\mathrm{CF} + L(5,6)\mathrm{CF} + \frac{L(1,3)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
-\frac{L(3,5)}{2} + \frac{L(3,6)}{2} + \frac{L(5,8)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(4,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,4)}{2} \\
-\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(4,5)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(3,4)}{2} - \frac{L(4,8)}{2}
\end{bmatrix}
\tag{175}
$$

$$
x_{11} = \begin{pmatrix}
\frac{L(1,3)}{2N} + L(1:2)CF + L(3:4)CF + L(5:8)CF + L(6;7)CF - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} - \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(2,3)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0 \\
\frac{L(3,6)}{2} + \frac{L(4,7)}{2} - \frac{L(3,7)}{2} - \frac{L(4,6)}{2} \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,8)}{2} + \frac{L(2,5)}{2} - \frac{L(2,8)}{2} \\
0 \\
0 \\
0 \\
-\frac{L(3,5)}{2} + \frac{L(3,8)}{2} + \frac{L(4,5)}{2} - \frac{L(4,8)}{2} \\
0 \\
\frac{L(1,6)}{2} + \frac{L(2,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
\frac{L(5,6)}{2} + \frac{L(7,8)}{2} - \frac{L(5,7)}{2} - \frac{L(6,8)}{2} \\
0 \\
0
\end{pmatrix}
\tag{176}
$$

$$
x_{12} = \begin{pmatrix}
-\frac{L(1,2)}{2N} + L(1,4)CF + L(2,7)CF + L(3,6)CF + L(5,8)CF + \frac{L(1,3)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
\frac{L(2,3)}{2} + \frac{L(6,7)}{2} - \frac{L(2,6)}{2} - \frac{L(3,7)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(4,7)}{2} - \frac{L(4,7)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
\frac{L(2,5)}{2} + \frac{L(7,8)}{2} - \frac{L(2,8)}{2} - \frac{L(5,7)}{2} \\
-\frac{L(3,5)}{2} + \frac{L(3,8)}{2} + \frac{L(5,6)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(3,4)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,8)}{2} + \frac{L(4,5)}{2} - \frac{L(4,8)}{2} \\
0
\end{pmatrix}
\tag{177}
$$

$$
x_{13} = \begin{pmatrix}
-\frac{L(1,2)}{2V} + L(1,6)CF + L(2,3)CF + L(4,5)CF + L(7,8)CF + \frac{L(1,3)}{2V} - \frac{L(1,4)}{2V} + \frac{L(1,5)}{2V} + \frac{L(1,7)}{2V} - \frac{L(1,8)}{2V} + \frac{L(2,4)}{2V} - \frac{L(2,5)}{2V} + \frac{L(2,6)}{2V} - \frac{L(2,7)}{2V} + \frac{L(2,8)}{2V} - \frac{L(3,4)}{2V} + \frac{L(3,5)}{2V} - \frac{L(3,6)}{2V} + \frac{L(3,7)}{2V} - \frac{L(3,8)}{2V} + \frac{L(4,6)}{2V} - \frac{L(4,7)}{2V} + \frac{L(4,8)}{2V} - \frac{L(5,6)}{2V} + \frac{L(5,7)}{2V} - \frac{L(5,8)}{2V} - \frac{L(6,7)}{2V} + \frac{L(6,8)}{2V} \\
0 \\
0 \\
\frac{L(4,7)}{2} + \frac{L(5,8)}{2} - \frac{L(4,8)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,4)}{2} + \frac{L(5,6)}{2} - \frac{L(1,5)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(2,4)}{2} + \frac{L(2,5)}{2} + \frac{L(3,4)}{2} - \frac{L(3,5)}{2} \\
0 \\
0 \\
\frac{L(2,7)}{2} + \frac{L(3,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,7)}{2} \\
-\frac{L(1,7)}{2} + \frac{L(1,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\
0 \\
\frac{L(1,2)}{2} + \frac{L(3,6)}{2} - \frac{L(1,3)}{2} - \frac{L(2,6)}{2} \\
0 \\
0 \\
0
\end{pmatrix} \tag{178}
$$

$$
x_{14} = \begin{bmatrix}
\frac{L(1,3)}{2N} + L(1,2)CF + L(3,8)CF + L(4,5)CF + L(6,7)CF - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
0 \\
0 \\
\frac{L(1,4)}{2} + \frac{L(2,5)}{2} - \frac{L(1,5)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(3,4)}{2} + \frac{L(5,8)}{2} - \frac{L(3,5)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
-\frac{L(4,6)}{2} + \frac{L(4,7)}{2} + \frac{L(5,6)}{2} - \frac{L(5,7)}{2} \\
\frac{L(1,6)}{2} + \frac{L(2,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,6)}{2} \\
-\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(2,3)}{2} - \frac{L(2,8)}{2} \\
0 \\
\frac{L(3,6)}{2} + \frac{L(7,8)}{2} - \frac{L(3,7)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0
\end{bmatrix} \tag{179}
$$

$$
x_{15} = \begin{pmatrix}
-\frac{L(3,7)}{2} + \frac{L(3,8)}{2} + \frac{L(4,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,7)}{2} + \frac{L(1,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\
-\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(3,6)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
0 \\
\frac{L(2,3)}{2} + \frac{L(4,5)}{2} - \frac{L(2,4)}{2} - \frac{L(3,5)}{2} \\
0 \\
-\frac{L(1,2)}{2N} + L(1,6)CF + L(2,5)CF + L(3,4)CF + L(7,8)CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(4,5)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} \\
\frac{L(2,7)}{2} + \frac{L(5,8)}{2} - \frac{L(2,8)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(5,6)}{2} - \frac{L(1,5)}{2} - \frac{L(2,6)}{2} \\
0 \\
0
\end{pmatrix}
\tag{180}
$$

$$
x_{16} = \begin{pmatrix}
0 \\
0 \\
\frac{L(2,3)}{2} + \frac{L(4,7)}{2} - \frac{L(2,4)}{2} - \frac{L(3,7)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(6,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,6)}{2} \\
-\frac{L(1,3)}{2} + \frac{L(4,4)}{2} + \frac{L(3,6)}{2} - \frac{L(4,6)}{2} \\
0 \\
0 \\
\frac{L(2,5)}{2} + \frac{L(7,8)}{2} - \frac{L(2,8)}{2} - \frac{L(5,7)}{2} \\
-\frac{L(1,2)}{2N} + \mathrm{L(1,6)CF} + \mathrm{L(2,7)CF} + \mathrm{L(3,4)CF} + \mathrm{L(5,8)CF} + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
-\frac{L(0,5)}{2} + \frac{L(3,8)}{2} + \frac{L(4,5)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,8)}{2} + \frac{L(5,6)}{2} - \frac{L(6,8)}{2}
\end{pmatrix}
\tag{181}
$$

$$
x_{17} = \begin{pmatrix}
-\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(2,5)}{2} - \frac{L(2,6)}{2} \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(3,5)}{2} + \frac{L(3,6)}{2} + \frac{L(5,8)}{2} - \frac{L(6,8)}{2} \\
0 \\
0 \\
0 \\
\frac{L(1,4)}{2} + \frac{L(2,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,4)}{2} \\
0 \\
0 \\
0 \\
\frac{L(4,5)}{2} + \frac{L(6,7)}{2} - \frac{L(4,6)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
\frac{L(1,3)}{2N} + \mathrm{L(1,2)CF} + \mathrm{L(3,8)CF} + \mathrm{L(4,7)CF} + \mathrm{L(5,6)CF} - \frac{L(1,4)}{2N} + \frac{L(4,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
-\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(2,3)}{2} - \frac{L(2,8)}{2} \\
0 \\
0 \\
\frac{L(3,4)}{2} + \frac{L(7,8)}{2} - \frac{L(3,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
0
\end{pmatrix}
\tag{182}
$$

$$x_{18} = \begin{pmatrix} -\frac{L(2,4)}{2} + \frac{L(2,5)}{2} + \frac{L(4,7)}{2} - \frac{L(5,7)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(1,4)}{2} + \frac{L(5,6)}{2} - \frac{L(1,5)}{2} - \frac{L(4,6)}{2} \\ 0 \\ 0 \\ \frac{L(2,3)}{2} + \frac{L(7,8)}{2} - \frac{L(2,8)}{2} - \frac{L(3,7)}{2} \\ \frac{L(1,2)}{2} + \frac{L(6,7)}{2} - \frac{L(1,7)}{2} - \frac{L(2,6)}{2} \\ 0 \\ \frac{L(3,4)}{2} + \frac{L(5,8)}{2} - \frac{L(3,5)}{2} - \frac{L(4,8)}{2} \\ 0 \\ -\frac{L(1,2)}{2^N} + L(1,6)\mathrm{CF} + L(2,7)\mathrm{CF} + L(3,8)\mathrm{CF} + L(4,5)\mathrm{CF} + \frac{L(1,3)}{2^N} - \frac{L(1,4)}{2^N} + \frac{L(1,5)}{2^N} + \frac{L(1,7)}{2^N} - \frac{L(1,8)}{2^N} - \frac{L(2,3)}{2^N} + \frac{L(2,4)}{2^N} - \frac{L(2,5)}{2^N} + \frac{L(2,6)}{2^N} + \frac{L(2,8)}{2^N} - \frac{L(3,4)}{2^N} + \frac{L(3,5)}{2^N} - \frac{L(3,6)}{2^N} + \frac{L(3,7)}{2^N} + \frac{L(4,6)}{2^N} - \frac{L(4,7)}{2^N} + \frac{L(4,8)}{2^N} - \frac{L(5,6)}{2^N} + \frac{L(5,7)}{2^N} - \frac{L(5,8)}{2^N} - \frac{L(6,7)}{2^N} + \frac{L(6,8)}{2^N} - \frac{L(7,8)}{2^N} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,3)}{2} + \frac{L(1,8)}{2} + \frac{L(3,6)}{2} - \frac{L(6,8)}{2} \\ 0 \end{pmatrix} \tag{183}$$

$$x_{19} = \begin{pmatrix} 0 \\ \frac{L(1,4)}{2} + \frac{L(5,8)}{2} - \frac{L(1,5)}{2} - \frac{L(4,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(2,4)}{2} + \frac{L(2,5)}{2} + \frac{L(3,4)}{2} - \frac{L(3,5)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(1,6)}{2} + \frac{L(7,8)}{2} - \frac{L(1,7)}{2} - \frac{L(6,8)}{2} \\ \frac{L(1,2)}{2} + \frac{L(3,8)}{2} - \frac{L(1,3)}{2} - \frac{L(2,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,2)}{2N} + L(1,8)CF + L(2,3)CF + L(4,5)CF + L(6,7)CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\ -\frac{L(4,6)}{2} + \frac{L(4,7)}{2} + \frac{L(5,6)}{2} - \frac{L(5,7)}{2} \\ 0 \\ 0 \\ -\frac{L(2,6)}{2} + \frac{L(2,7)}{2} + \frac{L(3,6)}{2} - \frac{L(3,7)}{2} \\ 0 \end{pmatrix} \tag{184}$$

$$
x_{20} = \begin{pmatrix}
0 \\
0 \\
-\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(5,8)}{2} - \frac{L(6,8)}{2} \\
0 \\
\frac{L(2,5)}{2} + \frac{L(3,6)}{2} - \frac{L(2,6)}{2} - \frac{L(3,5)}{2} \\
0 \\
\frac{L(1,4)}{2} + \frac{L(7,8)}{2} - \frac{L(1,7)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,2)}{2} + \frac{L(3,8)}{2} - \frac{L(1,3)}{2} - \frac{L(2,8)}{2} \\
0 \\
\frac{L(4,5)}{2} + \frac{L(6,7)}{2} - \frac{L(4,6)}{2} - \frac{L(5,7)}{2} \\
-\frac{L(1,2)}{2N} + \mathrm{L(1,8)CF} + \mathrm{L(2,3)CF} + \mathrm{L(4,7)CF} + \mathrm{L(5,6)CF} + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N} \\
0 \\
0 \\
0 \\
-\frac{L(2,4)}{2} + \frac{L(2,7)}{2} + \frac{L(3,4)}{2} - \frac{L(3,7)}{2}
\end{pmatrix} \tag{185}
$$

$$
x_{21} = \begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(4,7)}{2} + \frac{L(5,8)}{2} - \frac{L(4,8)}{2} - \frac{L(5,7)}{2} \\ 0 \\ 0 \\ \frac{L(1,4)}{2} + \frac{L(2,5)}{2} - \frac{L(1,5)}{2} - \frac{L(2,4)}{2} \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(2,3)}{2} - \frac{L(2,6)}{2} \\ -\frac{L(3,7)}{2} + \frac{L(3,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(1,3)}{2N} + \mathrm{L}(1,2)\mathrm{CF} + \mathrm{L}(3,6)\mathrm{CF} + \mathrm{L}(4,5)\mathrm{CF} + \mathrm{L}(7,8)\mathrm{CF} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N}
\end{pmatrix} \tag{186}
$$

$$
x_{x2} = \begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(2,3)}{2} - \frac{L(2,4)}{2} \\ 0 \\ 0 \\ 0 \\ -\frac{L(5,7)}{2} + \frac{L(5,8)}{2} + \frac{L(6,7)}{2} - \frac{L(6,8)}{2} \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(2,5)}{2} - \frac{L(2,6)}{2} \\ 0 \\ -\frac{L(3,7)}{2} + \frac{L(3,8)}{2} + \frac{L(4,7)}{2} - \frac{L(4,8)}{2} \\ 0 \\ 0 \\ 0 \\ -\frac{L(3,5)}{2} + \frac{L(3,6)}{2} + \frac{L(4,5)}{2} - \frac{L(4,6)}{2} \\ \frac{L(4,3)}{2N} + L(1,2)CF + L(3,4)CF + L(5,6)CF + L(7,8)CF - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(1,8)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} - \frac{L(2,7)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N}
\end{bmatrix}
\tag{187}
$$

$$
x_{23} = \left(\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
-\frac{L(2,4)}{2} + \frac{L(2,5)}{2} + \frac{L(4,7)}{2} - \frac{L(5,7)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{L(1,4)}{2} + \frac{L(5,8)}{2} - \frac{L(1,5)}{2} - \frac{L(4,8)}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
-\frac{L(1,3)}{2} + \frac{L(1,6)}{2} + \frac{L(3,8)}{2} - \frac{L(6,8)}{2} \\
\frac{L(2,3)}{2} + \frac{L(6,7)}{2} - \frac{L(2,6)}{2} - \frac{L(3,7)}{2} \\
0 \\
\frac{L(1,2)}{2} + \frac{L(7,8)}{2} - \frac{L(4,7)}{2} - \frac{L(2,8)}{2} \\
0 \\
-\frac{L(1,2)}{2N} + L(1,8)\mathrm{CF} + L(2,7)\mathrm{CF} + L(3,6)\mathrm{CF} + L(4,5)\mathrm{CF} + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} + \frac{L(2,8)}{2N} - \frac{L(3,4)}{2N} + \frac{L(3,5)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} - \frac{L(5,6)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N}
\end{array}\right) \tag{188}
$$

$$
x_{24} = \begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L(2,5)}{2} + \frac{L(6,7)}{2} - \frac{L(2,6)}{2} - \frac{L(5,7)}{2} \\ 0 \\ -\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(3,8)}{2} - \frac{L(4,8)}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{L(1,5)}{2} + \frac{L(1,6)}{2} + \frac{L(5,8)}{2} - \frac{L(6,8)}{2} \\ 0 \\ 0 \\ 0 \\ \frac{L(2,3)}{2} + \frac{L(4,7)}{2} - \frac{L(2,4)}{2} - \frac{L(3,7)}{2} \\ 0 \\ \frac{L(1,2)}{2} + \frac{L(7,8)}{2} - \frac{L(1,7)}{2} - \frac{L(2,8)}{2} \\ -\frac{L(3,5)}{2} + \frac{L(3,6)}{2} + \frac{L(4,5)}{2} - \frac{L(4,6)}{2} \\ -\frac{L(1,2)}{2N} + \mathrm{L(1,8)CF} + \mathrm{L(2,7)CF} + \mathrm{L(3,4)CF} + \mathrm{L(5,6)CF} + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} + \frac{L(1,5)}{2N} - \frac{L(1,6)}{2N} + \frac{L(1,7)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} - \frac{L(2,5)}{2N} + \frac{L(2,6)}{2N} + \frac{L(2,8)}{2N} + \frac{L(3,5)}{2N} - \frac{L(3,6)}{2N} + \frac{L(3,7)}{2N} - \frac{L(3,8)}{2N} - \frac{L(4,5)}{2N} + \frac{L(4,6)}{2N} - \frac{L(4,7)}{2N} + \frac{L(4,8)}{2N} + \frac{L(5,7)}{2N} - \frac{L(5,8)}{2N} - \frac{L(6,7)}{2N} + \frac{L(6,8)}{2N} - \frac{L(7,8)}{2N}
\end{pmatrix} \tag{189}
$$

B.2 Adjoint representation matrix

In this appendix I present the elements for the (8×8) adjoint transition matrix with the base (113). We will use the notation $L(i,j) \equiv L(x_i,x_j)$. The matrix is valid for $N = 3$. For presentation purposes, let us define the matrix as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{bmatrix}, \tag{190}$$

where x_i are column vectors. The column vectors read

$$x_1 = \begin{bmatrix} \frac{-\mathrm{L}(1,3)+\mathrm{L}(1,4)+\mathrm{L}(2,3)-\mathrm{L}(2,4)}{N} + (\mathrm{L}(1,2)+\mathrm{L}(3,4))N + \frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))N\left(N^2-9\right)}{3N^4-29N^2+48} \\ \frac{2(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))\left(N^2-8\right)\left(2N^2-3\right)}{N(3N^4-29N^2+48)} \\ -\frac{2(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))N\left(2N^2-13\right)}{3N^4-29N^2+48} \\ -\frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))\left(N^4-15N^2+24\right)}{3N^4-29N^2+48} \\ \frac{2(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))\left(N^4-7N^2+12\right)}{3N^4-29N^2+48} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{191}$$

$$x_2 = \begin{bmatrix} \frac{2(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))\left(N^2-8\right)\left(2N^2-3\right)}{N(3N^4-29N^2+48)} \\ \frac{-\mathrm{L}(1,2)+\mathrm{L}(1,4)+\mathrm{L}(2,3)-\mathrm{L}(3,4)}{N} + (\mathrm{L}(1,3)+\mathrm{L}(2,4))N + \frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))N\left(N^2-9\right)}{3N^4-29N^2+48} \\ -\frac{2(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))N\left(2N^2-13\right)}{3N^4-29N^2+48} \\ \frac{2(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))\left(N^4-7N^2+12\right)}{3N^4-29N^2+48} \\ -\frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))\left(N^4-15N^2+24\right)}{3N^4-29N^2+48} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{192}$$

$$x_3 = \begin{bmatrix} \frac{2(\mathrm{L}(1,2)-\mathrm{L}(1,3)-\mathrm{L}(2,4)+\mathrm{L}(3,4))}{N} \\ -\frac{2(\mathrm{L}(1,2)-\mathrm{L}(1,3)-\mathrm{L}(2,4)+\mathrm{L}(3,4))}{N} \\ (\mathrm{L}(1,4)+\mathrm{L}(2,3))N \\ \mathrm{L}(1,2)-\mathrm{L}(1,3)-\mathrm{L}(2,4)+\mathrm{L}(3,4) \\ -\mathrm{L}(1,2)+\mathrm{L}(1,3)+\mathrm{L}(2,4)-\mathrm{L}(3,4) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{193}$$

$$x_4 = \begin{bmatrix} -\frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))(N-2)(N+2)\left(N^2-8\right)\left(N^2-6\right)}{6N^6-58N^4+96N^2} \\ \frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))(N-2)(N+2)\left(5N^4-44N^2+48\right)}{6N^6-58N^4+96N^2} \\ -\frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))\left(5N^4-58N^2+152\right)}{6N^4-58N^2+96} \\ \frac{\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4)}{N}+\frac{N\left(N^2-9\right)(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))}{9N^4-87N^2+144}+\frac{1}{6}(3\mathrm{L}(1,2)+\mathrm{L}(1,3)+2\mathrm{L}(1,4)+2\mathrm{L}(2,3)+\mathrm{L}(2,4)+3\mathrm{L}(3,4))N \\ \frac{(\mathrm{L}(1,3)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(2,4))\left(N^2-8\right)\left(N^4-7N^2+12\right)}{6N^5-58N^3+96N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{194}$$

$$x_5 = \begin{bmatrix} \frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))(N-2)(N+2)\left(5N^4-44N^2+48\right)}{6N^6-58N^4+96N^2} \\ -\frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))(N-2)(N+2)\left(N^2-8\right)\left(N^2-6\right)}{6N^6-58N^4+96N^2} \\ -\frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))\left(5N^4-58N^2+152\right)}{6N^4-58N^2+96} \\ \frac{(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))\left(N^2-8\right)\left(N^4-7N^2+12\right)}{6N^5-58N^3+96N} \\ \frac{\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4)}{N}+\frac{N\left(N^2-9\right)(\mathrm{L}(1,2)-\mathrm{L}(1,4)-\mathrm{L}(2,3)+\mathrm{L}(3,4))}{9N^4-87N^2+144}+\frac{1}{6}(\mathrm{L}(1,2)+3\mathrm{L}(1,3)+2\mathrm{L}(1,4)+2\mathrm{L}(2,3)+3\mathrm{L}(2,4)+\mathrm{L}(3,4))N \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{195}$$

$$x_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}(2\mathrm{L}(1,2)+\mathrm{L}(1,3)+\mathrm{L}(1,4)+\mathrm{L}(2,3)+\mathrm{L}(2,4)+2\mathrm{L}(3,4))N \\ \frac{1}{4}(-\mathrm{L}(1,3)+\mathrm{L}(1,4)+\mathrm{L}(2,3)-\mathrm{L}(2,4))N \\ \frac{1}{4}(-\mathrm{L}(1,3)+\mathrm{L}(1,4)+\mathrm{L}(2,3)-\mathrm{L}(2,4))N \end{bmatrix}, \tag{196}$$

$$x_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}(-\mathrm{L}(1,2) + \mathrm{L}(1,4) + \mathrm{L}(2,3) - \mathrm{L}(3,4))N \\ \frac{1}{4}(\mathrm{L}(1,2) + 2\mathrm{L}(1,3) + \mathrm{L}(1,4) + \mathrm{L}(2,3) + 2\mathrm{L}(2,4) + \mathrm{L}(3,4))N \\ \frac{1}{4}(\mathrm{L}(1,2) - \mathrm{L}(1,4) - \mathrm{L}(2,3) + \mathrm{L}(3,4))N \end{bmatrix} \tag{197}$$

and

$$x_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}(\mathrm{L}(1,2) - \mathrm{L}(1,3) - \mathrm{L}(2,4) + \mathrm{L}(3,4))N \\ \frac{1}{4}(\mathrm{L}(1,2) - \mathrm{L}(1,3) - \mathrm{L}(2,4) + \mathrm{L}(3,4))N \\ \frac{1}{4}(\mathrm{L}(1,2) + \mathrm{L}(1,3) + 2\mathrm{L}(1,4) + 2\mathrm{L}(2,3) + \mathrm{L}(2,4) + \mathrm{L}(3,4))N \end{bmatrix}. \tag{198}$$

C Mathematica code for finding the fundamental representation transition matrix

I have included the Mathematica code that I wrote as a part of my research training in this appendix. This is done because the research report containing the code was not properly published due to it being more of an internal report. The code uses the cyclic notation that was also used in this thesis.

Slightly modified version of my research report's appendix: This section contains a Wolfram Alpha code that calculates the recursion matrix for any 2n-point correlator. It requires the user to input the basis topologies with one side chosen to be constant. The basis topologies are given in a form that resembles the permutation cycle notation. For example the basis topology $\mathrm{Tr}(U(x_1)U^\dagger(x_2))\,\mathrm{Tr}(U(x_3)U^\dagger(x_4))$ is written as (1,2)(3,4) and $\mathrm{Tr}(U(x_1)U^\dagger(x_2)U(x_3)U^\dagger(x_4))$ is written as (1,2,3,4). The numbers represent the number of the line. Every bracket represent a loop and the numbers in the brackets tell the ordering of the lines. The lines are connected to the next line in the loop and the last line is connected to the first line in the loop. The odd numbered lines go from left to right ($= U$) and the even numbered lines go from right to left ($= U^\dagger$). The basis topologies are given to the script by adding them to the list e[i]. The topologies of the fundamental four-point function are given to the script as an example, as list elements e[1] and e[2]. A 2n-point basis topologies $\mathcal{N}^{(i)}$ are given to the script as list elements e[i]. User can add as many e[i]'s as there are basis topologies. User should make sure that one side of the topology is kept constant and that the arrows on the lines are followed carefully. After giving the script the required basis topologies, evaluating the notebook gives the corresponding recursion matrix.

```
(*Author: Sami P. Demirci*)
(*Basis topologies. Odd numbers represent Wilson lines and even
  numbers represent daggered Wilson lines. The coordinates have been
  chosen so that Wilson lines go from left to right and the daggered
  Wilson lines go in the opposite direction. Numbers in parentheses
  represent closed loops and their connections. For example (1,4,3,2)
  means that the Wilson line loop connections are 1 → 4 → 3 →
   2 → 1 → ... Furthermore (1,2)(3,4) means that there are two separate closed
       loops with following connections: 1 → 2 → 1 ... and 3 → 4 → 3 ...*)

e[1] := "(1,2)(3,4)"

e[2] := "(1,2,3,4)"

countTheLines[] := (
  loops1 = StringSplit[e[1], {")(", "(", ")", ","}];
  n1 = 1;
  While[n1 < Length[loops1] + 1,
   loops1 = ReplacePart[loops1, n1 → ToExpression[Extract[loops1, n1]]];
   n1++;
  ];
  loops1 = Sort[loops1, Greater];
  NW := Extract[loops1, 1];
  numberOfBasisTopologies := (NW / 2)!;
 )

countTheLines[];

(*Permutes a loop. For example permutation["1,2,3,4"] = 2,3,4,1.*)

permutation[loop_] := (
  listOfLines5 = StringSplit[loop, ","];
  x5 = Extract[listOfLines5, 1];
  listOfLines5 = Append[Drop[listOfLines5, 1], x5];
  numberOfLines5 = Length[listOfLines5];
  n5 = 1;
  out5 = "";
  While[n5 < numberOfLines5 + 1,
   out5 = StringJoin[out5, ",", Extract[listOfLines5, n5]];
   n5++;
  ];
  out5 = StringTrim[out5, ","];
  out5
 )
```

```
(*Arranges a loop so that the line with
 the smallest number is in front of the expression.*)

arrangeLoop[loop_] := (
  listOfLines7 = StringSplit[loop, ","];
  numberOfLines7 = Length[listOfLines7];
  x7 = Extract[listOfLines7, 1];
  n7 = 1;
  While[n7 < numberOfLines7 + 1,
   y7 = Extract[listOfLines7, n7];
   If[ToExpression[y7] < ToExpression[x7]
    , listOfLines7 = Append[Extract[TakeDrop[listOfLines7, 1], 2], x7];
    x7 = Extract[listOfLines7, 1];
    n7 = 1
    , n7++
   ];
  ];
  m7 = 1;
  out7 = "";
  While[m7 < numberOfLines7 + 1,
   out7 = StringJoin[out7, ",", Extract[listOfLines7, m7]];
   m7++;
  ];
  out7 = StringTrim[out7, ","];
  out7
 )

(*Arranges all loops so that the line with the smallest number in the loop
 is in front of the expression. Arranges the order of the loops depending
 on the number of the first number in the loop from smallest to largest.*)

arrangeAllLoops[top_] := (
  listOfLoops8 = StringSplit[top, {")(", "(", ")"}];
  numberOfLoops8 = Length[listOfLoops8];
  n8 = 1;
  While[n8 < numberOfLoops8 + 1,
   listOfLoops8 = ReplacePart[listOfLoops8,
     n8 → StringSplit[arrangeLoop[Extract[listOfLoops8, n8]], ","]];
   n8++;
  ];
  listOfLines8 = Range[numberOfLoops8];
  m8 = 1;
  While[m8 < numberOfLoops8 + 1,
   listOfLines8 = ReplacePart[listOfLines8,
```

```
    m8 → ToExpression[Extract[Extract[listOfLoops8, m8], 1]]];
   m8++;
  ];
  listOfLines8 = Reverse[Sort[listOfLines8, Greater]];
  k8 = 1;
  While[k8 < numberOfLoops8 + 1,
   listOfLines8 =
    ReplacePart[listOfLines8, k8 → ToString[Extract[listOfLines8, k8]]];
   k8++;
  ];
  out8 = Range[numberOfLoops8];
  n8 = 1;
  While[n8 < numberOfLoops8 + 1,
   m8 = 1;
   While[m8 < numberOfLoops8 + 1,
    If[Extract[listOfLines8, n8] == Extract[Extract[listOfLoops8, m8], 1],
     out8 = ReplacePart[out8, n8 → Extract[listOfLoops8, m8]];
     m8 = numberOfLoops8 + 1, m8++];
   ];
   n8++;
  ];
  n8 = 1;
  outer8 = "";
  While[n8 < numberOfLoops8 + 1,
   m8 = 1;
   loop8 = "";
   While[m8 < Length[Extract[out8, n8]] + 1,
    loop8 = StringJoin[loop8, ",", Extract[Extract[out8, n8], m8]];
    m8++;
   ];
   outer8 = StringJoin[outer8, "(", StringTrim[loop8, ","], ")"];
   n8++;
  ];
  outer8
 )

(*Checks which side of the topology is kept constant.*)
```

```
whichSideIsConstant[] := (
  rhsIsConstant = True;
  i20 = 1;
  x20 = Extract[StringSplit[arrangeAllLoops[e[1]], {")(", "(", ")", ","}], 2];
  While[i20 < numberOfBasisTopologies + 1,
   If[
    x20 == Extract[StringSplit[arrangeAllLoops[e[i20]], {")(", "(", ")", ","}], 2],
    Null,
    rhsIsConstant = False];
   i20++;
  ];
  keepRHSConstant := rhsIsConstant;
 )

whichSideIsConstant[];

(*Adds a link to a topology.*)
```

```
addALink[top_, numberOfLine_] := (
  If[keepRHSConstant,
   If[OddQ[numberOfLine],
    StringReplace[top, {StringJoin[",", ToString[numberOfLine], ","] →
        StringJoin[",", "t", ",", ToString[numberOfLine], ","],
       StringJoin["(", ToString[numberOfLine], ","] →
        StringJoin["(", "t", ",", ToString[numberOfLine], ","],
       StringJoin[",", ToString[numberOfLine], ")"] →
        StringJoin[",", "t", ",", ToString[numberOfLine], ")"]}],
    StringReplace[top, {StringJoin[",", ToString[numberOfLine], ","] →
        StringJoin[",", ToString[numberOfLine], ",t,"],
       StringJoin["(", ToString[numberOfLine], ","] →
        StringJoin["(", ToString[numberOfLine], ",t,"],
       StringJoin[",", ToString[numberOfLine], ")"] →
        StringJoin[",", ToString[numberOfLine], ",t)"]}]],
   If[OddQ[numberOfLine],
    StringReplace[top, {StringJoin[",", ToString[numberOfLine], ","] →
        StringJoin[",", ToString[numberOfLine], ",t,"],
       StringJoin["(", ToString[numberOfLine], ","] →
        StringJoin["(", ToString[numberOfLine], ",t,"],
       StringJoin[",", ToString[numberOfLine], ")"] →
        StringJoin[",", ToString[numberOfLine], ",t)"]}],
    StringReplace[top, {StringJoin[",", ToString[numberOfLine], ","] →
        StringJoin[",t,", ToString[numberOfLine], ","],
       StringJoin["(", ToString[numberOfLine], ","] →
        StringJoin["(t,", ToString[numberOfLine], ","],
       StringJoin[",", ToString[numberOfLine], ")"] →
        StringJoin[",t,", ToString[numberOfLine], ")"]}]
   ]]
 )

(*Checks if the topology with two links is a
 term that gets a CF factor when the links are removed.*)
```

```
isThisACFTerm[top_] := (
  loopList6 = StringSplit[top, {")(", "(", ")"}];
  numberOfLoops6 = Length[loopList6];
  out6A = False;
  out6B = False;
  If[StringContainsQ[top, "t,t"], out6A = True, out6A = False];
  top6 = "";
  n6 = 1;
  While[n6 < numberOfLoops6 + 1,
   top6 = StringJoin[top6, "(", permutation[Extract[loopList6, n6]], ")"];
   n6++;
  ];
  If[StringContainsQ[top6, "t,t"], out6B = True, out6B = False];
  out6 = out6A || out6B;
  out6
 )

(*Permutes a loop until there's a link in front.*)

arrangeLoopTFirst[loop_] := (
  T = False;
  loop9 = loop;
  While[T == False,
   If[StringTake[loop9, 1] == "t", T = True, loop9 = permutation[loop9]];
  ];
  loop9
 )

(*Removes the link of a loop with only one link.*)

fierz1TLoop[loop_] := (
  loop10 = arrangeLoopTFirst[loop];
  loopList10 = StringSplit[loop10, "t"];
  loop10A = Extract[loopList10, 1];
  loop10A = StringTrim[loop10A, ","];
  loop10B = Extract[loopList10, 2];
  loop10B = StringTrim[loop10B, ","];
  {loop10A, loop10B}
 )

(*Removes the links of a loop with two links.*)
```

```
fierz2TLoops[loop1_, loop2_] := (
  x11 = StringDrop[arrangeLoopTFirst[loop1], 2];
  y11 = StringDrop[arrangeLoopTFirst[loop2], 2];
  StringJoin[x11, ",", y11]
 )

(*Removes the links of a topology.*)

fierz[top_] := (
  loopList12 = StringSplit[top, {")(", "(", ")"}];
  numberOfLoops12 = Length[loopList12];
  numberOfTLoops12 = 0;
  out12 = "";
  i12 = 1;
  j12 = 1;
  While[i12 < numberOfLoops12 + 1,
   x12 = Extract[loopList12, i12];
   T12 = StringContainsQ[x12, "t"];
   If[T12, numberOfTLoops12++;
    k12[j12] = x12;
    j12++, out12 = StringJoin[out12, "(", x12, ")"]];
   i12++;
  ];
  If[numberOfTLoops12 == 1,
   y12 = fierz1TLoop[k12[1]];
   out12 = StringJoin[out12, "(", Extract[y12, 1], ")(", Extract[y12, 2], ")"],
   out12 = StringJoin[out12, "(", fierz2TLoops[k12[1], k12[2]], ")"]
  ];
  out12
 )

(*Takes a topology as an input and
 gives the corresponding standard basis vector.*)
```

```
topologyToVector[top_] := (
  top13 = arrangeAllLoops[top];
  out13 = Range[numberOfBasisTopologies];
  n13 = 1;
  While[n13 < numberOfBasisTopologies + 1,
   x13 = arrangeAllLoops[e[n13]];
   T13 = (x13 == top13);
   If[T13, out13 = ReplacePart[out13, n13 → 1],
    out13 = ReplacePart[out13, n13 → 0]];
   n13++;
  ];
  out13
 )

(*Adds a term to the recursion matrix.*)

addToAMatrix[matrix_, rowNumber_, columnNumber_, addThis_] := (
  row14 = Extract[matrix, rowNumber];
  row14 =
   ReplacePart[row14, columnNumber → (Extract[row14, columnNumber] + addThis)];
  matrix14 = ReplacePart[matrix, rowNumber → row14];
  matrix14
 )

(*Handles the correct adding of terms
 into the matrix. Takes a vector as an input.*)

addAVectorToAMatrix[matrix_, vector_, columnNumber_] := (
  matrix15 = matrix;
  n15 = 1;
  While[n15 < numberOfBasisTopologies + 1,
   matrix15 = addToAMatrix[matrix15, n15, columnNumber, Extract[vector, n15]];
   n15++;
  ];
  matrix15
 )

(*Main program*)
```

```
r = 1;
n = 1;
rMatrix = ConstantArray[0, {numberOfBasisTopologies, numberOfBasisTopologies}];
While[r < numberOfBasisTopologies + 1,
  n = 1;
  While[n < NW + 1,
   m = n + 1;
   While[m < NW + 1,
    topo = e[r];
    topo = addALink[topo, n];
    topo = addALink[topo, m];
    T = isThisACFTerm[topo];
    If[T, topo = CF * topologyToVector[e[r]], topo =
      (1 / 2) * topologyToVector[fierz[topo]] - (1 / (2 N)) * topologyToVector[e[r]]];
    rMatrix = addAVectorToAMatrix[rMatrix, (-1)^(n + m + 1) *
       StringJoin["L(", ToString[n], ",", ToString[m], ")"] * topo, r];
    m++;
   ];
   n++;
  ];
  r++;
 ];
MatrixForm[rMatrix]
```

$$\begin{pmatrix} L(1,2)\,CF + L(3,4)\,CF + \frac{L(1,3)}{2N} - \frac{L(1,4)}{2N} - \frac{L(2,3)}{2N} + \frac{L(2,4)}{2N} & \frac{L(1,2)}{2} - \frac{L(1,3)}{2} - \frac{L(2,4)}{2} + \frac{L(3,4)}{2} \\ -\frac{L(1,3)}{2} + \frac{L(1,4)}{2} + \frac{L(2,3)}{2} - \frac{L(2,4)}{2} & L(1,4)\,CF + L(2,3)\,CF - \frac{L(1,2)}{2N} + \frac{L(1,3)}{2N} + \frac{L(2}{2} \end{pmatrix}$$