Pro Gradu thesis:

Gauge/Gravity Dualities

Timo Alho

UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF PHYSICS

Abstract

In this thesis we review gauge/gravity dualities. Some details on the background topics of supersymmetric gauge theories, string theories and differential geometry of anti de Sitter spacetimes are given. We will then proceed to motivate the Maldacena duality, in its original form as a duality between type IIB string theory and the $\mathcal{N} = 4$ super Yang-Mills theory, by considering D3 branes and interactions of open strings on them. After introducing the Maldacena duality with some level of technical detail, we finally review more general gauge/gravity dualities and some results pertaining to QCD.

Preface

I would like to thank my supervisor Dr. Kimmo Tuominen for giving me the opportunity to write this thesis on a very interesting subject on the forefront of theoretical physics, and also for providing indispensable guidance and advice in learning this difficult subject. I would also like to thank Dr. Jan Rak for providing my first experiences with research in physics.

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1 Introduction

"Non-Euclidean calculus and quantum physics are enough to stretch any brain, and when one mixes them with folklore, and tries to trace a strange background of multi-dimensional reality behind the ghoulish hints of the Gothic tales and the wild whispers of the chimney-corner, one can hardly expect to be wholly free from mental tension."

–H.P. Lovecraft, "Dreams in the Witch House"–

The AdS/CFT duality is an interesting application combining non-Euclidean calculus, quantum field theory and extra dimensions with string theory. It is currently the only known plausible realization of the idea that QCD, which in some ways shows string-like behaviour in its bound states such as mesons, might actually be describable as a theory of strings. Indeed, among the early suggestions for an explanation of the strong interactions, bosonic string theory was a strong candidate, until it was realized that it inevitably contained a spin-2 boson, the graviton (this discovery, of course, led to the use of string theory as a theory of gravitation, and later a proponent for a theory of everything in the form of superstring theory). Once QCD was invented, Gerard t'Hooft showed that in the limit where the number of colors, N, is large, non-planar diagrams in the perturbative expansion of QCD are suppressed and the theory takes on a stringy character. Nonetheless, no concrete connection between a string theory and QCD was found until 1998, when Maldacena together with Witten, Gubser, Klebanov and Polyakov proposed a duality between type IIB superstring theory on a five dimensional anti de Sitter (AdS_5, for short) spacetime background and a conformal supersymmetric Yang-Mills theory living on the boundary of that space[1] [1, 2, 3].

The Maldacena conjecture added the vital ingredient of an extra fifth dimension (actually the string theory has five further dimensions, which are compactified on a sphere). The remarkable feature that a four dimensional theory encodes the full dynamics of a five dimensional theory was a realization of yet another idea which had been present for some time: quantum gravitational holography. The idea of holography in quantum gravity stems from the Bekenstein-Hawking conjecture for black hole entropy, which states that the entropy of a black hole is proportional to the surface area of its event horizon. If entropy is to originate in the microscopic realm as (the logarithm of) the possible number of microstates corresponding to a macrostate, then for a black hole the number of microstates is only proportional to the area of the event horizon. Then the physics inside the three dimensional black hole seems to be somehow "holographically" encoded on its two dimensional horizon, and therefore any theory of quantum gravity should be holographic, in the sense that it can be written as a theory with one dimension less and without gravity.

One of the main incentives leading to the AdS/CFT duality has been the relation between string theory branes and gauge theories. Since the second superstring revolution in the mid-90's, it has been known that there is an intimate connection between gauge field theories and extended objects in string theory known as Dp-branes.

A D-brane in string theory is a subspace on which open strings can end. The endpoints of open strings can have either Neumann or Dirichlet boundary conditions, independently for each direction in space. Since Dirichlet boundary conditions on a given coordinate of the the string fix the location of its end point on that coordinate, choosing Dirichlet boundary conditions for $D-p-1$ coordinates (where D is the dimension of the ambient space) defines a $p+1$ dimensional hyperplane on which the ends of the strings are still free to move. The two ends of the strings may be bound on different D-branes.

[1] Henceforth in this document we will use the term "Maldacena duality" to refer specifically to the duality between type IIB string theory on an $AdS_5 \times S^5$ background and the $\mathcal{N}=4$ super Yang-Mills theory. AdS/CFT duality may refer to any duality between a conformal field theory and the corresponding string theory. For more general scenarios we will use either the term gauge/gravity dualities or gauge/string dualities. AdS/QCD will refer to any approximate duality proposed between gravity or string theories and QCD.

In supergravity, p-branes are essentially black hole solutions that extend in p spatial dimensions (and their spacetime dimension is therefore $p+1$). Since supergravity is the low-energy effective theory of string theory, it is generally expected that the p-brane solutions are also solutions of the full string theory.

It is now known that the D-brane and p-branes are in fact the same object, which also promotes the D-branes from an artificial wall set by the boundary conditions to a dynamical object which is perturbed by vibrations of the strings attached to it. These objects are called Dp-branes.

A field theory on the stack of branes is generated by N Dp-branes stacked on top of each other. In this case, an open string which has both of its ends on the brane stack has two indices from $1 \ldots N$, denoting to which brane in the stack its ends are bound to. Therefore the string states are $N \times N$ matrices, which can be shown to form an adjoint representation of $U(N)$. The low-energy effective theory for these open strings on the stack, with type IIB string theory in the bulk, is actually the $\mathcal{N} = 4$ super Yang-Mills field theory (a $U(1)$ factor related to the center-of-mass dynamics of the brane decouples for most purposes, leaving an $SU(N)$ symmetry). Since the geometry near the brane is that of an anti de Sitter space, low-energy means short strings that do not extend far from the brane and the full symmetries of the SYM and type IIB string theory (compactified on an 5-sphere) match, it was conjectured by Maldacena that type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory are exactly dual at all energy scales.

Since the AdS/CFT duality involves several seemingly unrelated concepts, we will begin this thesis with two rather disconnected chapters. In the first one, we will quickly introduce the supersymmetry algebra, the $\mathcal{N} = 1$ supersymmetric gauge theory, and derive the $\mathcal{N} = 4$ theory by dimensional reduction from ten to four dimensions. In the second chapter, we will delve into the geometry and conformal infinity of AdS_n. Then in the chapter about string theory and branes, we will begin to draw the various ingredients together, and after that we will finally get to the Maldacena conjecture. The final chapter concentrates on various applications and variations of the duality.

2 Supersymmetric gauge theories

In this chapter we will give a short review of supersymmetric gauge theories. We will also derive the four-dimensional $\mathcal{N} = 4$ super Yang-Mills (abbreviated SYM) action as the dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ SYM action.

2.1 The Coleman-Mandula -theorem and the supersymmetry algebra

The Coleman-Mandula -theorem [4, 5] states, in essence (there are some technical assumptions required), that the Lie algebra of the symmetry group of a consistent non-trivial relativistic quantum field theory with a mass gap is a direct product of the Poincaré algebra and the algebra of some internal symmetry group. Therefore continuous internal symmetries cannot mix the various irreducible representations of the Poincaré group and hence particles of different spin cannot mix.

A loophole in the Coleman-Mandula theorem is the assumption that the symmetry group is generated by a Lie algebra. We can avoid this restriction by generalizing the group generators to form a graded Lie superalgebra. A graded Lie superalgebra has a gradation and a generalized Lie bracket which respects the gradation, i.e. if g_i and g_j are elements of the algebra with grades i, j respectively, then the grade of $[g_i, g_j]$ is $i + j$. The bracket is then antisymmetric (corresponding to a commutator) or symmetric (corresponding to an anticommutator) depending on the grade,

$$[g_i, g_j] = -(-1)^{ij}[g_j, g_i]. \tag{2.1}$$

The gradation is naturally $\mathbb{Z}_2$, since only the parity of the grade matters for the bracket. Note that the term Lie superalgebra is actually a misnomer: a Lie superalgebra is not a Lie algebra, because the generalized bracket does not respect the antisymmetry of the Lie bracket.

By the Haag-Lopuszanski-Sohnius theorem [5, 6], the only Lie superalgebra consistent with a relativistic field theory is the supersymmetry algebra, which generalizes the Poincaré algebra to include the supersymmetry generators Q^I and $\bar{Q}^I$, which are now the odd part of the algebra and transform as Weyl spinors, with the generalized commutation relations [5, 7, 8]

$$\begin{aligned}
\left\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\right\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} && (2.2)\\
\left\{Q^I_\alpha, Q^J_\beta\right\} &= \epsilon_{\alpha\beta} Z^{IJ} && (2.3)\\
\left\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta}\left(Z^{IJ}\right)^* && (2.4)\\
\left[P_\mu, Q^I\right] &= \left[P_\mu, \bar{Q}^I\right] = 0 && (2.5)\\
\left[M_{\mu\nu}, Q^I_\alpha\right] &= -i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta && (2.6)\\
\left[M_{\mu\nu}, \bar{Q}^{\dot\alpha I}\right] &= -i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta I} && (2.7)
\end{aligned}$$

where $\{\ ,\ \}$ denotes the anticommutator (which is the generalized Lie bracket when both of the elements are odd), σ^μ are the Pauli spin matrices, P_μ are the translation generators, $i\sigma_{\mu\nu} = \frac{i}{4}\left(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu\right)$ are the Lorentz transform generators in the Weyl spinor representation and $Q^I, \bar{Q}^J$ are supercharges, which extend the Poincaré algebra. The Poincaré generators obey the standard commutation relations amongst themselves. The indices $I = 1, \cdots, \mathcal{N}$ label independent copies of the supersymmetry algebra. The $\mathcal{N} > 1$ algebras are called extended supersymmetry algebras. The Z^{IJ} are central charges, which are antisymmetric under the exchange of I and J and commute with all elements of the algebra.

Eq. (2.5) and (2.6, 2.7) are determined by the Jacobi identities, or may alternatively be derived from the transformation properties of the supercharges, which are Weyl spinors. For example, if the supercharges did not commute with the translation generators, they would depend on the spacetime coordinates and we would have a local symmetry, which leads to supergravity. The grading requires the right side of Eq. (2.2 - 2.4) to be bosonic, and the rest of the commutation relations exclude the possibility of anything proportional to $M_{\mu\nu}$. In the case of a single supersymmetry the transformation properties then fix the anticommutators except for a normalization, which we choose as shown here. In the presence of several supercharges Eq. (2.2) gains a factor Δ^{IJ}, which can be chosen to be diagonal. Eq. (2.3, 2.4) are the most general choice allowed by the other commutation relations. Note that when $\mathcal{N}=1$ the antisymmetry of the central charges requires them to vanish.

2.2 R-symmetry

The $\mathcal{N}=1$ supersymmetry algebra is invariant under a global $U(1)$ phase transformation of the supercharges and the extended supersymmetry algebras are invariant under an $SU(\mathcal{N})$ transformation mixing the supercharges. These are called R-symmetries.

2.3 $\mathcal{N}=1$ Super Yang-Mills

In order to construct supersymmetric field theories, we must find such multiplets of fields and such transformation laws that represent the supersymmetry algebra on these fields. General methods to do this can be found in the literature [5, 7], but we will only consider here the one representation we are interested in. This is the supersymmetric generalization of a pure Yang-Mills gauge theory, called a super Yang-Mills theory. The Lagrangian in the $\mathcal{N}=1$ case is [9]

$$\mathcal{L} = \mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda\right\}, \tag{2.8}$$

where D_μ is the gauge covariant derivative, $F_{\mu\nu} = [D_\mu, D_\nu] = \partial A_\mu - \partial A_\nu + ig\,[A_\mu, A_\nu]$ is the field strength tensor, A_μ is the gauge field, λ is a spinor in the adjoint representation of the gauge group and Γ^μ are the Dirac matrices. The exact nature of λ depends on the spacetime dimension D [10]: in $D=3,4$ λ is a Majorana spinor, in $D=6$ it is a Weyl spinor and in $D=10$ it is a Majorana-Weyl spinor. In ten dimensions this Lagrangian transforms as a total derivative under the supersymmetry transformation [9]

$$A_\mu \to A_\mu - i\bar{\zeta}\Gamma_\mu\lambda, \qquad \lambda \to \lambda + \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\zeta, \tag{2.9}$$

where ζ is an infinitesimal spinor transformation parameter and $\Gamma^{\mu\nu} = \frac{1}{2}\left[\Gamma^\mu, \Gamma^\nu\right]$.

2.4 $\mathcal{N}=4$ Super Yang-Mills

The supersymmetry theory relevant to the AdS/CFT duality is the $\mathcal{N}=4$ super Yang-Mills in four dimensions. The four-dimensional $\mathcal{N}=4$ SYM can be derived by dimensionally reducing the $\mathcal{N}=1$ SYM Lagrangian (Eq. (2.8)) from ten to four dimensions [10]. Let Greek indices with an overbar run over all ten dimensions, Latin indices over the six dimensions to be compactified and bare Greek indices over the remaining four. Then the gauge kinetic term becomes

$$\begin{aligned}
F_{\bar{\mu}\bar{\nu}}F^{\bar{\mu}\bar{\nu}} &\to (\partial_{\bar{\mu}}A_{\bar{\nu}} - \partial_{\bar{\nu}}A_{\bar{\mu}} + ig[A_{\bar{\mu}}, A_{\bar{\nu}}])(\partial^{\bar{\mu}}A^{\bar{\nu}} - \partial^{\bar{\nu}}A^{\bar{\mu}} + ig[A^{\bar{\mu}}, A^{\bar{\nu}}]) \\
&= F_{\mu\nu}F^{\mu\nu} + \partial_\mu A_a\partial^\mu A^a + \partial_\nu A_b\partial^\nu A^b + ig[A_\mu, A_a]\partial^\mu A^a - ig[A_a, A_\nu]\partial^\nu A^a \\
&\quad + ig\partial_\mu A_a[A^\mu, A^a] - ig\partial_\nu A_a[A^a, A^\nu] \\
&\quad + i^2g^2[A_a, A_\nu][A^a, A^\nu] + i^2g^2[A_\mu, A_a][A^\mu, A^a] + i^2g^2[A_a, A_b][A^a, A^b] \\
&= F_{\mu\nu}F^{\mu\nu} + 2D_\mu X_i D^\mu X^i - g^2[X_i, X_j][X^i, X^j], \quad X^i \equiv A^i.
\end{aligned} \tag{2.10}$$

The compactified components A^a of the gauge field have become six scalars X^i, all transforming in the adjoint representation, and an interaction term for them.

Compactifying the spinor term gives

$$\bar{\lambda}\Gamma^{\bar{\mu}} D_{\bar{\mu}}\lambda = \bar{\lambda}\left(\Gamma^{\mu}\partial_{\mu}\lambda + ig\Gamma^{\mu}[A_{\mu},\lambda] + ig\Gamma^{i}[X_i,\lambda]\right) = \bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda + ig\bar{\lambda}\Gamma^{i}[X_i,\lambda]. \tag{2.11}$$

This equation contains a ten-dimensional spinor λ and the ten-dimensional Dirac matrices Γ^{μ}, Γ^{i}. In order to bring this to the form of a four-dimensional field theory we must break down the spinor and the gamma matrices to components corresponding to four-dimensional representations of the Poincaré group. From the fact that the gamma matrices form a ten-dimensional Clifford algebra, we can deduce some properties required of the decomposition:

$$\begin{aligned} & \{\Gamma^{\bar{\mu}}, \Gamma^{\bar{\nu}}\} && = 2\eta^{\bar{\mu}\bar{\nu}}\mathbb{I}_{10} \\ \Rightarrow\ & \{\Gamma^{\mu}, \Gamma^{\nu}\} && = 2\eta^{\mu\nu}\mathbb{I}_{10} \\ & \left\{\Gamma^{i}, \Gamma^{j}\right\} && = 2\delta^{ij}\mathbb{I}_{10} \\ & \left\{\Gamma^{\mu}, \Gamma^{i}\right\} && = 0. \end{aligned} \tag{2.12}$$

In addition we have the Weyl condition

$$(\mathbb{I}_{10} + \Gamma^{11})\lambda = 0, \tag{2.13}$$

where $\Gamma^{11} = \prod_{\bar{\mu}=1}^{10} \Gamma^{\bar{\mu}}$, for the ten-dimensional spinor. One way to do the decomposition is as follows: first write the ten-dimensional spinor as

$$\lambda = \lambda \otimes \xi, \tag{2.14}$$

where $\lambda = \lambda_{\mu A}$ has one four-dimensional spacetime index and another internal four-dimensional index, which we will denote by capital Latin alphabet. The spinor $\xi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ can be chosen constant by virtue of Eq. (2.13). Then define the ten-dimensional gamma matrices:

$$\begin{aligned} \Gamma^{\mu} &= \gamma^{\mu} \otimes \mathbb{I}_4 \otimes \sigma_3, \\ \Gamma^{3+i} &= \mathbb{I}_4 \otimes \alpha^{i} \otimes \sigma_1, \\ \Gamma^{6+i} &= \gamma^{5} \otimes \beta^{i} \otimes \sigma_3, \quad \text{where } i \in 1,2,3 \end{aligned} \tag{2.15}$$

with γ^{μ} being the four-dimensional gamma matrices and the α and β matrices satisfying the algebra

$$\left\{\alpha^{i}, \alpha^{j}\right\} = \left\{\beta^{i}, \beta^{j}\right\} = 2\delta^{ij}\mathbb{I}_4 \tag{2.16}$$

and σ_n being the Pauli spin matrices. The first factor of the tensor products operates on the four-dimensional spacetime index of the spinor, the second factor on the internal index and the final one on the ξ factor.

It is easy to check that the gamma matrices defined this way satisfy the ten dimensional Clifford algebra, that the Γ^{μ} matrices satisfy the four-dimensional Clifford algebra on the spacetime index of the spinor λ and that the Weyl condition Eq. (2.13) is fulfilled. In addition, the Γ^{k} matrices with $k \in \{4...9\}$ form the Clifford algebra of $SO(6)$. Inserting this decomposition into Eq. (2.11) and carrying out the ξ products gives

$$\begin{aligned} \bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda + \bar{\lambda}\Gamma^{i}[X_i,\lambda] &= \sum_{A=1}^{4}\bar{\lambda}_A\gamma^{\mu}D_{\mu}\lambda_A + g\sum_{i=1}^{3}\sum_{A,B=1}^{4}\alpha^{i}_{AB}\bar{\lambda}_A[X_i,\lambda_B] \\ &\quad + ig\sum_{i=4}^{6}\sum_{A,B=1}^{4}\beta^{i}_{AB}\bar{\lambda}_A\gamma^{5}[X_i,\lambda_B]. \end{aligned} \tag{2.17}$$

The single Majorana fermion of ten-dimensional $\mathcal{N}=1$ SYM therefore becomes four Majorana fermions of $\mathcal{N}=4$ SYM.

The full $\mathcal{N}=4$ super Yang-Mills Lagrangian is then

$$\begin{aligned}\mathcal{L} &= \operatorname{tr}\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu X_i D^\mu X^i - \frac{g^2}{2}[X_i, X_j][X^i, X^j] - i\sum_{A=1}^{4}\bar{\lambda}_A\gamma^\mu D_\mu\lambda_A \\ &\quad -ig\sum_{i=1}^{3}\sum_{A,B=1}^{4}\alpha^i_{AB}\bar{\lambda}_A[X_i,\lambda_B] + g\sum_{i=4}^{6}\sum_{A,B=1}^{4}\beta^i_{AB}\bar{\lambda}_A\gamma^5[X_i,\lambda_B]\}\end{aligned} \tag{2.18}$$

2.5 Symmetries of the $\mathcal{N}=4$ SYM

The most obvious symmetry of the $\mathcal{N}=4$ SYM is its Lorentz invariance, which forms the group $SO(1,3)$. In addition, this theory is exceptional in that it enjoys an unbroken conformal symmetry in the quantum level [9]. From the discussion of section 3.4, taking $n=5$ we see that the four dimensional conformal group is actually isomorphic to the group $SO(2,4)$. In addition, $\mathcal{N}=4$ SYM has the R-symmetry $SU(4)$, which mixes the four supercharges and is isomorphic to $SO(6)$, giving the group $SO(2,4)\times SO(6)$. Since spinors are involved, the covering groups of these need to be considered, so that we have $SU(2,2)\times SU(4)$. Finally the supercharges transform under this group in such a way as to form the supergroup $SU(2,2|4)$, which has $(2,2)$ bosonic generators and 4 fermionic generators, and which is the full symmetry group of the theory.

2.6 The 't Hooft coupling

In a $SU(N)$ gauge theory, when N is considered a free parameter of the theory, the effective perturbative coupling is not actually the Yang-Mills -coupling g_{YM}, but it is the coupling $g^2_{YM}N$. This is because any diagram calculation will involve a sum over Feynman diagrams for each of the N species of particles in the fundamental representation of the gauge group separately for each vertex. Therefore the final diagram picks up a factor of $g^2_{YM}N$ from each vertex. Because of this, the large N limit in itself is not interesting, but we must take the 't Hooft coupling $\lambda = g^2_{YM}N$ as a constant while letting N run to infinity. [11]

3 Anti de Sitter space

We will now review the properties of the anti de Sitter spacetime (AdS), which will be the background of the Maldacena duality. First we introduce AdS_n as a submanifold of the pseudo-Euclidean manifold $\mathbb{R}^{2,n-1}$ and derive its metric. Then we will show that it is a vacuum solution of the Einstein equations with a non-zero cosmological constant, proving that it is a maximally symmetric constant negative curvature space in the process. After constructing a few different coordinate parametrizations for the AdS_n manifold, we will study the conformal infinity of the space.We will mostly follow the presentation in [12].

3.1 AdS_n as an embedded manifold

Geometrically we can define an n-dimensional AdS_n space as the set

$$\mathrm{AdS}_n = \{ y \in \mathbb{R}^{2,n-1} : y^2 = y^c y^d \eta_{cd} = -b^2, b \in \mathbb{R} \}, \tag{3.1}$$

endowed with a metric inherited from the metric $\eta_{ab} = \mathrm{diag}(-1, +1, +1, ..., +1, -1)$ of the ambient pseudo-Euclidean $\mathbb{R}^{2,n-1}$ space[2]. In this section Latin indices run from 0 to n, and b is the "radius" of the AdS_n manifold.

In order to find an explicit form for the AdS_n metric, introduce the stereographic coordinates $(x^\mu) = (x^1, ..., x^n)$ such that

$$\begin{aligned} y^0 &= \rho \frac{1+x^2}{1-x^2} \\ y^\mu &= \rho \frac{2x^\mu}{1-x^2} \end{aligned} \tag{3.2}$$

where

$$x^2 \equiv x^\mu x^\nu \eta_{\mu\nu},$$

$\eta_{\mu\nu} = \mathrm{diag}(+1, +1, ..., +1, -1)$, Greek indices run over the AdS_n components from 1 to n, and they are raised and lowered with $\eta_{\mu\nu}$. It is easy to see that $y^2 = -\rho^2$, so indeed when $\rho^2 = b^2$ the x^μ describe the AdS manifold.

Now the standard formula for the induced metric $g_{\mu\nu}$ of an embedded manifold gives the AdS_n metric:

$$g_{\mu\nu} = \frac{\partial y}{\partial x^\mu} \cdot \frac{\partial y}{\partial x^\nu} = \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu} \eta_{ab}. \tag{3.3}$$

We need the partial derivatives

$$\begin{aligned} \frac{\partial x^2}{\partial x^\mu} &= \frac{\partial (x^\alpha x^\beta)}{\partial x^\mu} \eta_{\alpha\beta} = 2x_\mu, \\ \frac{\partial y^0}{\partial x^\mu} &= b \frac{2x_\mu (1-x^2) + 2x_\mu (1+x^2)}{(1-x^2)^2} = b \frac{4x_\mu}{(1-x^2)^2} \quad \text{and} \\ \frac{\partial y^\nu}{\partial x^\mu} &= \frac{2\delta_\mu^\nu (1-x^2) + 4x_\mu x_\nu}{(1-x^2)^2} \end{aligned}$$

[2]Our choice of sign here differs from that in [12], where the metric in the embedding space is chosen to be mostly minus, giving a positive "radius squared".

from which we may construct the metric. With a slight abuse of notation in contracting the y^α with $\eta_{\mu\nu}$, we have

$$
\begin{aligned}
g_{\mu\nu} &= \frac{\partial y^a}{\partial x^\mu}\frac{\partial y^b}{\partial x^\nu}\eta_{ab} = -\frac{\partial y^0}{\partial x^\mu}\frac{\partial y^0}{\partial x^\nu} + \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\eta_{\alpha\beta} \\
&= b^2\frac{-16x_\mu x_\nu + 4(1-x^2)^2\eta_{\mu\nu} + 16x_\mu x_\nu(1-x^2) + 16x_\mu x_\nu x^2}{(1-x^2)^4} \\
&= 4b^2\frac{\eta_{\mu\nu}}{(1-x^2)^2}. \qquad (3.4)
\end{aligned}
$$

The AdS_n metric is conformally flat, that is, it factorizes to the form $g_{\mu\nu}(x) = e^{\phi(x)}\eta_{\mu\nu}$, with $\phi(x) = \log 4b^2 - 2\log(1-x^2)$ in our case. Note that inverting this metric is simple, $g^{\mu\nu} = e^{-\phi(x)}\eta^{\mu\nu}$.

We will now show that the AdS_n metric solves the vacuum Einstein equations in $D = n$ dimensions with a positive cosmological constant. In this case the Einstein equations simplify to

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}\Lambda g_{\mu\nu} \qquad \Big| \cdot g^{\mu\nu} \\
R(1-\frac{1}{2}D) &= \frac{1}{2}D\Lambda \\
R_{\mu\nu} = \frac{1}{2}\Lambda g_{\mu\nu}(1+\frac{D}{2-D}) &= \frac{\Lambda}{2-D}g_{\mu\nu}. \qquad (3.5)
\end{aligned}
$$

We need to work out the Ricci tensor. During the following calculation, we raise and lower Greek indices by $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$. Contractions are of course carried out with $g_{\mu\nu}$, which amounts to inserting the additional $e^{\pm\phi(x)}$ factor. For the Riemann tensor we need the Christoffel symbols:

$$
\begin{aligned}
\Gamma^\mu_{\nu\rho} &= \frac{1}{2}e^{-\phi}\eta^{\mu\lambda}\left(\partial_\nu e^\phi \eta_{\rho\lambda} + \partial_\rho e^\phi \eta_{\nu\lambda} - \partial_\lambda e^\phi \eta_{\nu\rho}\right) \\
&= \frac{1}{2}\left(\delta^\mu_\rho\partial_\nu\phi + \delta^\mu_\nu\partial_\rho\phi - \eta_{\nu\rho}\partial^\mu\phi\right). \qquad (3.6)
\end{aligned}
$$

From them we form the derivatives and products

$$
\begin{aligned}
\partial_\rho\Gamma^\mu_{\nu\sigma} &= \frac{1}{2}\left(\delta^\mu_\sigma\partial_\rho\partial_\nu\phi + \delta^\mu_\nu\partial_\rho\partial_\sigma\phi - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi\right) \\
\Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} &= \frac{1}{4}(2\delta^\mu_\rho\partial_\sigma\phi\partial_\nu\phi - \eta_{\sigma\rho}\partial^\mu\phi\partial_\nu\phi + \delta^\mu_\sigma\partial_\rho\phi\partial_\nu\phi + \delta^\mu_\nu\partial_\rho\phi\partial_\sigma\phi \\
&\quad -\eta_{\nu\rho}\partial^\mu\phi\partial_\sigma\phi - \delta^\mu_\rho\eta_{\nu\sigma}\partial^\lambda\phi\partial_\lambda\phi), \qquad (3.7)
\end{aligned}
$$

and then we are ready to write down the Riemann tensor:

$$
\begin{aligned}
R^\mu{}_{\nu\rho\sigma} &= \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho} \\
&= \frac{1}{2}\left(\delta^\mu_\sigma\partial_\rho\partial_\nu\phi - \delta^\mu_\rho\partial_\sigma\partial_\nu\phi - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi + \eta_{\nu\rho}\partial_\sigma\partial^\mu\phi\right) \\
&\quad +\frac{1}{4}(\delta^\mu_\rho\partial_\sigma\phi\partial_\nu\phi - \delta^\mu_\sigma\partial_\rho\phi\partial_\nu\phi + (\eta_{\nu\sigma}\partial_\rho\phi - \eta_{\nu\rho}\partial_\sigma\phi)\partial^\mu\phi \\
&\quad +(\eta_{\nu\rho}\delta^\mu_\sigma - \eta_{\nu\sigma}\delta^\mu_\rho)\partial^\lambda\phi\partial_\lambda\phi). \qquad (3.8)
\end{aligned}
$$

Contracting μ and ρ, we have

$$
\begin{aligned}
R_{\nu\sigma} &= R^\mu{}_{\nu\mu\sigma} = \frac{1}{2}(\partial_\sigma\partial_\nu\phi - D\partial_\sigma\partial_\nu\phi - \eta_{\nu\sigma}\partial_\mu\partial^\mu\phi + \partial_\sigma\partial_\nu\phi) \\
&\quad +\frac{1}{4}(D\partial_\sigma\phi\partial_\nu\phi - \partial_\sigma\phi\partial_\nu\phi + \eta_{\nu\sigma}\partial_\mu\phi\partial^\mu\phi - \partial_\sigma\phi\partial_\nu\phi + \eta_{\nu\sigma}\partial^\lambda\phi\partial_\lambda\phi - \eta_{\nu\sigma}D\partial^\lambda\phi\partial_\lambda\phi) \\
&= \left(1-\frac{D}{2}\right)\left(\partial_\sigma\partial_\nu\phi - \frac{1}{2}\partial_\sigma\phi\partial_\nu\phi\right) + \frac{1}{2}\eta_{\nu\sigma}\left(\left[1-\frac{D}{2}\right](\partial\phi)^2 - \partial^2\phi\right), \qquad (3.9)
\end{aligned}
$$

where $(\partial\phi)^2 = \partial^\mu\phi\partial_\mu\phi$ and $\partial^2\phi = \partial^\mu\partial_\mu\phi$. Plugging in the derivatives

$$\begin{aligned}\partial_\mu\partial_\nu\phi &= 4\frac{\eta_{\mu\nu}}{1-x^2} + 8\frac{x_\mu x_\nu}{(1-x^2)^2} \\ \partial_\mu\phi\partial_\nu\phi &= 16\frac{x_\mu x_\nu}{(1-x^2)^2} \end{aligned} \tag{3.10}$$

gives the final result

$$\begin{aligned} R_{\mu\nu} &= -4\frac{D-1}{(1-x^2)^2}\eta_{\mu\nu} = -\frac{D-1}{b^2}g_{\mu\nu} = \frac{\Lambda}{2-D}g_{\mu\nu}, \quad \text{when} \\ \Lambda &= \frac{(D-1)(D-2)}{b^2}, \end{aligned} \tag{3.11}$$

so indeed AdS_n is a solution of the Einstein equations with a positive (in our choice of signs) cosmological constant. In addition, it is of constant negative curvature, which can be easily seen by calculating the curvature scalar

$$R = R_{\mu\nu}g^{\mu\nu} = -\frac{D(D-1)}{b^2}. \tag{3.12}$$

Combining Eq. (3.10), Eq. (3.8) and Eq. (3.12), we further see that the maximum symmetry condition

$$R^\mu{}_{\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\nu\sigma}\delta^\mu_\rho - g_{\nu\rho}\delta^\mu_\sigma), \tag{3.13}$$

which guarantees that the spacetime has the full possible set of $D(D+1)/2$ Killing vectors [13], is satisfied:

$$R^\mu{}_{\nu\rho\sigma} = \frac{4}{(1-x^2)^2}(\eta_{\rho\nu}\delta^\mu_\sigma - \eta_{\sigma\nu}\delta^\mu_\rho) = \frac{R}{D(D-1)}\left(\frac{4b^2}{(1-x^2)^2}\eta_{\sigma\nu}\delta^\mu_\rho - \frac{4b^2}{(1-x^2)^2}\eta_{\rho\nu}\delta^\mu_\sigma\right). \tag{3.14}$$

If we can now find a group of symmetries on AdS_n with dimension $n(n+1)/2$, we know based on the above that there are no more symmetries. Since the group $SO(2,n-1)$ preserves the metric on $\mathbb{R}^{2,n-1}$, it also maps the embedded AdS_n to itself. With $y \in \mathrm{AdS}_n$ a vector of the embedding space, $y \in \mathbb{R}^{2,n-1}, y^2 = -b^2$, consider the transformation by an element $\Lambda \in SO(2,n-1)$ such that $y \mapsto y'$. Let a^i, b^i be in the tangent space $T_y\mathrm{AdS}_n$ of the point $y \in \mathrm{AdS}_n$ and a'^i, b'^i in $T_{y'}\mathrm{AdS}_n$. Then

$$a' \cdot b' = \Lambda^i{}_k a^k \Lambda^j{}_l b^l \eta_{ij} = a^k b^l \eta_{kl} = a \cdot b, \tag{3.15}$$

since $\Lambda^T\eta\Lambda = \eta$ by the definition of $SO(2,n-1)$. Therefore $SO(2,n-1)$ induces the corresponding symmetry on AdS_n. Also, $\dim SO(2,n-1) = n(n+1)/2$ so this is the full symmetry group of AdS_n.

3.2 Alternative coordinate systems

While AdS_n is conveniently defined in stereographic coordinates, some other coordinate systems will turn out useful later.

Let us first define the so called "light cone coordinates" for the embedding space [3]:

$$u = y^0 + y^n, \quad v = y^0 - y^n \tag{3.16}$$

$$\begin{aligned} \vec{y} &= (y^1, y^2, ..., y^{n-1}), \\ \vec{y}^2 &= y^\alpha y^\beta \eta^s_{\alpha\beta}, \quad \eta^s_{\alpha\beta} = \mathrm{diag}(1,1,...1,s), \\ y^2 &= \vec{y}^2 - uv. \end{aligned} \tag{3.17}$$

[3] This differs from the definition in [12] by including the sign s. This way we avoid using complex coordinates, which produces problems and indeed a potentially imaginary norm for some vectors in the presentation given by [12].

Here s defines the signature of the embedding manifold, $s=-1$ being the case described above. When $s=1$ we have a "Euclidean" version of the AdS_n manifold, which is actually the n dimensional hyperbolic space. The above definition indeed describes the embedding space: the map $f:(u,v,\vec{y})\rightarrow \mathbb{R}^{2,n-1}$, with the usual metric in $\mathbb{R}^{2,n-1}$,

$$f(u,v,y^1,y^2,...,y^{n-1})=\left(\frac{1}{2}(u+v),\frac{1}{2}(u-v),y^1,y^2,...,y^{n-1}\right) \tag{3.18}$$

is an isometry to the embedding space. The case with $s=1$ gives the alternative case where the isometry defined by Eq. (3.18) goes to $\mathbb{R}^{1,n}$ and the subspace produced by the equation $y^2=-b^2$ gives a version of AdS_n with Euclidean signature.

From the light cone coordinates we further define

$$\begin{aligned} \xi^\alpha &\equiv \frac{y^\alpha}{u}, \quad \alpha=1,...,n-1 \\ \vec{\xi} &\equiv (\xi^1,...,\xi^{n-1}), \quad \vec{\xi}^2=\frac{\vec{y}^2}{u^2} \end{aligned} \tag{3.19}$$

so that the AdS_n equation becomes

$$y^2 = u^2\vec{\xi}^2-uv=-b^2, \quad \text{or} \tag{3.20}$$

$$v = u\vec{\xi}^2+\frac{b^2}{u}. \tag{3.21}$$

Now we can pick the set $(u,\vec{\xi})$ as coordinates in the AdS_n. To calculate the metric in this case, use the dot product derived from Eq. (3.17)[4],

$$y_1\cdot y_2=\vec{y}_1\cdot\vec{y}_2-\frac{1}{2}u_1v_2-\frac{1}{2}u_2v_1, \quad \vec{y}_1\cdot\vec{y}_2=y_1^\alpha y_2^\beta \overset{s}{\eta}_{\alpha\beta}, \tag{3.22}$$

directly on the formula for the induced metric of an embedded manifold. Denoting the light-cone components by $y=(u,v,y^\alpha)$ we have

$$\begin{aligned} ds^2 &= \frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial u}du^2+\sum_{\alpha=1}^{n-1}\frac{\partial y}{\partial\xi^\alpha}\cdot\frac{\partial y}{\partial\xi^\alpha}(d\xi^\alpha)^2+ \\ &\quad 2\sum_{\alpha=1}^{n-1}\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial\xi^\alpha}dud\xi^\alpha+\sum_{\alpha\neq\beta}\frac{\partial y}{\partial\xi^\alpha}\cdot\frac{\partial y}{\partial\xi^\beta}d\xi^\alpha d\xi^\beta \\ &= \left[\sum_{\beta=1}^{n-1}\overset{s}{\eta}_{\beta\beta}\left(\frac{\partial y^\beta}{\partial u}\right)^2-\frac{\partial u}{\partial u}\frac{\partial v}{\partial u}\right]du^2+\sum_{\alpha=1}^{n-1}\overset{s}{\eta}_{\alpha\alpha}\left(\frac{\partial y^\alpha}{\partial\xi^\alpha}\right)^2(d\xi^\alpha)^2 \\ &= \frac{b^2}{u^2}du^2+u^2d\vec{\xi}^2, \\ d\vec{\xi}^2 &\equiv \sum_{\alpha=1}^{n-2}(d\xi^\alpha)^2+s(d\xi^{n-1})^2. \end{aligned} \tag{3.23}$$

This metric further simplifies by setting $b=1$ and changing variables by $u\rightarrow 1/\xi^0$, which leads to

$$ds^2=\left(-\frac{d\xi^0}{(\xi^0)^2}\right)^2(\xi^0)^2+\frac{d\vec{\xi}^2}{(\xi^0)^2}=\frac{1}{(\xi^0)^2}\left((d\xi^0)^2+d\vec{\xi}^2\right). \tag{3.24}$$

[4] Here subscript indices denote different vectors, not covariant components.

3.3 The boundary of AdS_n

A projective boundary can be defined on the Anti de Sitter space. This is in some sense the limiting geometry in the limit of a very large $\vec{y}$. Define the rescaled light cone coordinates for the embedding space

$$\tilde{u}S = u, \quad \tilde{y}^\alpha S = y^\alpha, \quad \tilde{v}S = v, \tag{3.25}$$

With these, the AdS_n equation is

$$\vec{\tilde{y}}^2 - \tilde{u}\tilde{v} = -\frac{b^2}{S^2}. \tag{3.26}$$

Now consider points very far away (large $\vec{y}$), but with the scale S correspondingly large so that the coordinates with a tilde remain finite. As $S \to \infty$, the right hand side of Eq. (3.26) goes to zero, and therefore the boundary is defined by

$$\vec{\tilde{y}}^2 - \tilde{u}\tilde{v} = 0. \tag{3.27}$$

We still have one degree of freedom too much, since the overall scale of the coordinates cannot matter now that we essentially scaled out an infinity from them. To remedy this, we define the points in the boundary $\partial\mathrm{AdS}_n$ of AdS_n as the set of possible directions to which the points may escape to infinity, or equivalently as equivalence classes of the set $\{(u, v, \vec{y}) : \vec{y}^2 - uv = 0\}$ under rescaling

$$(u, v, \vec{y}) \sim (tu, tv, t\vec{y}) \quad \text{for all } t \in (0, \infty). \tag{3.28}$$

To eliminate the redundancy caused by the equivalence, we may choose a representative point from each equivalence class.

As a first exercise in dealing with the AdS boundary, we show that topologically it is $S^1 \times S^{n-2}$ in the case of Minkowski signature $s = -1$, and $\mathbb{R} \times S^{n-2}$ in the Euclidean case $s = 1$:

$$\begin{aligned} t^2\vec{y}^2 - t^2 uv &= t^2\sum_{\alpha=1}^{n-2}(y^\alpha)^2 + st^2(y^{n-1})^2 + t^2(y^0)^2 - t^2(y^n)^2 = 0 \\ \Rightarrow t^2\sum_{\alpha=1}^{n-2}(y^\alpha)^2 + t^2(y^0)^2 &= t^2(-s(y^{n-1})^2 + (y^n)^2). \end{aligned} \tag{3.29}$$

Now fix the scale such that $s(y^{n-1})^2 + (y^n)^2 = 1$. This leads to the equations

$$\begin{aligned} \sum_{\alpha=1}^{n-2}(y^\alpha)^2 + (y^0)^2 &= 1 \\ -s(y^{n-1})^2 + (y^n)^2 &= 1, \end{aligned} \tag{3.30}$$

which is the equation of a sphere for the two subspaces when $s = -1$. When $s = 1$, the latter equation becomes that of a hyperbola, which is homeomorphic to $\mathbb{R} \times \mathbb{Z}_2$. In the latter case, we could also have moved the y^{n-1} term to the left hand side in Eq. (3.29) and fixed the scale such that

$$\begin{aligned} t^2\sum_{\alpha}(y^\alpha)^2 + t^2(y^0)^2 + t^2(y^{n-1})^2 &= 1 \\ \Rightarrow y^n &= \pm 1 \end{aligned} \tag{3.31}$$

in order to reach the conclusion that the topology is equivalent to $S^{n-1} \times \mathbb{Z}_2$, that is, two disconnected $n-1$ dimensional spheres. The meaning of this second result is not completely clear, since we have been unable to find an explicit conformal transformation which would map $S^{n-2} \times \mathbb{R} \times \mathbb{Z}_2$ to

$S^{n-1} \times \mathbb{Z}_2$. The fact that in the two-dimensional case an inversion maps $S^1 \times \mathbb{R}$ to $S^1 \times S^1$ does cast some doubt on the above result. The error could be that the scale fixing in Eq. (3.31) somehow implicitly scales different components of the vector differently.

Another way to choose the scale is setting $v \to 1$. Since this only works when $v \neq 0$, it gives a coordinate patch on the $v \neq 0$ part of the boundary manifold. In this case, Eq. (3.27) gives $u = \vec{y}^2$, so we may choose $\vec{y}$ as the coordinate set. By symmetry, another choice is $u \to 1$ for $u \neq 0$, so that $v = \vec{\tilde{y}}^2$ and $\vec{\tilde{y}}$ gives coordinates on the patch $u \neq 0$. Obviously these two scalings give different coordinates for the same point. The coordinate transformation between them is

$$\vec{\tilde{y}} = \frac{\vec{y}}{\vec{y}^2} \tag{3.32}$$

Since the points on the boundary are defined only up to an arbitrary scale, we cannot inherit a full metric from the bulk AdS_n. Nonetheless, the angles between vectors are well defined because normalization eliminates scale, and we can therefore inherit a conformal structure from AdS_n. Finding out this structure is easiest in the $(tu, \vec{\xi})$ coordinate set. Notice that $\vec{\xi}$ does not carry the scale t since it is defined as the ratio of u and $\vec{y}$. The boundary condition in these coordinates leads to $\xi^0 = (tu)^{-1} = \frac{\vec{\xi}^2}{tv}$, and therefore the inherited metric on the boundary is

$$(ds^2)_{\text{boundary}} = \frac{t^2 v^2}{\vec{\xi}^4} d\vec{\xi} \tag{3.33}$$

which indeed explicitly depends on the choice of scale. Now the cosine of the angle between two vectors $\vec{a}, \vec{b}$ in the tangent space of the boundary at $(tu, tv, \vec{\xi})$ is

$$\cos\theta_{\vec{a}\vec{b}} = \frac{\frac{t^2v^2}{\vec{\xi}^4}\sum_i a^i b^i}{\sqrt{\frac{t^4v^4}{\vec{\xi}^8}\sum_j (a^j)^2 \sum_k (a^k)^2}} = \frac{\sum_i a^i b^i}{\sqrt{\sum_j (a^j)^2 \sum_k (a^k)^2}}, \tag{3.34}$$

which is independent of scale. We may therefore take the inner product on the tangent space of the boundary to be the same as that between the $\vec{\xi}$ components in the bulk AdS_n, which is in practice either the Minkowskian or the Euclidean inner product. Since $\vec{y} = u\vec{\xi}$, it is easy to see that using $\vec{y}$ with the same inner product is equivalent.

In light of Eq. (3.34) we see that the AdS_n boundary is locally the flat $n-1$ dimensional conformal space. The change of coordinates in Eq. (3.32) is a conformal transformation, and we should expect to see the full conformal group to manifest itself on the boundary. It is indeed generated by the $SO(2, n-1)$ symmetry of AdS_n.

3.4 The conformal group on the boundary

The goal of this section is to demonstrate that $SO(2, n-1)$ transformations acting on AdS_n act as the conformal group on its boundary. We begin with a small overview of the conformal group.

In this section we denote vectors on the boundary $\partial\mathrm{AdS}_n$ by Latin characters from the end or middle of the alphabet, and vectors in the tangent space of $\partial\mathrm{AdS}_n$ by Latin characters from the beginning of the alphabet. Even powers of a vector denote half the power of the dot product of a vector with itself.

Let $x \in \partial\mathrm{AdS}_n$. Then the conformal group consists of dilations:

$$x \mapsto x' = \lambda x, \lambda \in \mathbb{R} \tag{3.35}$$

and special conformal transformations, which are compositions of an inversion, translation and another inversion, $x \mapsto x'$ such that:

$$\begin{aligned} x &\mapsto x'' = \frac{x}{x^2} + p, \quad p \in \partial\mathrm{AdS}_n \\ x'' &\mapsto x' = \frac{x''}{x''^2} = \frac{\frac{x}{x^2} + p}{(\frac{x}{x^2} + p)^2} = \frac{x + x^2 p}{1 + 2x \cdot p + x^2 p^2}. \end{aligned} \tag{3.36}$$

It is often convenient to leave the last inversion implicit:

$$\frac{x'}{x'^2} = \frac{x}{x^2} + p. \tag{3.37}$$

It is easy to see that the transformation (3.35) is conformal. To see that the special conformal transformations are indeed conformal, start by implicitly differentiating Eq. (3.37) with respect to x^α, which yields in components

$$\frac{\frac{\partial x'^\mu}{\partial x^\alpha} x'^2 - 2\left(\frac{\partial x'^\nu}{\partial x^\alpha} x'_\nu\right) x'^\mu}{x'^4} = \frac{\delta^\mu_\alpha x^2 - 2x_\alpha x^\mu}{x^4}. \tag{3.38}$$

Contracting α in this equation by a vector $a \in T_x\partial\mathrm{AdS}_n$, and using the transformation law

$$a'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} a^\alpha$$

for tangent vectors under a coordinate transformation, yields

$$\frac{a'^\mu x'^2 - 2(a' \cdot x')x'^\mu}{x'^4} = \frac{a^\mu x^2 - 2(a \cdot x)x^\mu}{x^4}. \tag{3.39}$$

Now contract the equation on both sides with the corresponding equation for another tangent vector b to get

$$\frac{a' \cdot b'}{x'^4} = \frac{a \cdot b}{x^4}. \tag{3.40}$$

This easily leads to

$$\frac{a \cdot b}{\sqrt{a^2 b^2}} = \frac{a' \cdot b'}{\sqrt{a'^2 b'^2}}, \tag{3.41}$$

which is a statement of conformal invariance under the transformation.

The infinitesimal versions of the transformations are

$$x \mapsto x' = x(1 + \lambda) \tag{3.42}$$

$$x \mapsto x' = x(1 - 2p \cdot x) + x^2 p. \tag{3.43}$$

Let us examine the effects that a Lorentz transformation $\Lambda^\mu{}_\nu \in SO(2, n-1)$ acting on $\mathbb{R}^{2,n-1}$ has on the embedded AdS_n and its boundary. An infinitesimal Lorentz transformation can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu, \tag{3.44}$$

where $\omega^{\mu\nu}$ is an antisymmetric matrix [5]. Applying the coordinate transformation from the basis y^α to $(u, v, \vec{y})$,

$$\Lambda^\mu{}_\nu{}' = \frac{\partial y'^\mu}{\partial y^\alpha} \frac{\partial y^\beta}{\partial y'^\nu} \Lambda^\alpha{}_\beta, \tag{3.45}$$

[5] Notice that the ω in Eq. (3.44) has one covariant and one contravariant index, so it is not, in general, antisymmetric.

gives an infinitesimal transformation of the form

$$\Lambda^{\mu}{}_{\nu}{}' = \delta^{\mu}_{\nu} + \begin{pmatrix} a & 0 & \vec{\alpha}^T \\ 0 & -a & \vec{\beta}^T \\ \frac{1}{2}\vec{\beta} & \frac{1}{2}\vec{\alpha} & \omega_{n-1} \end{pmatrix}, \tag{3.46}$$

acting on the coordinates $(u, v, \vec{y})$, where $\vec{\alpha}$ and $\vec{\beta}$ are column vectors and ω_{n-1} is the lower-right $(n-1) \times (n-1)$ submatrix of $\omega^{\mu}{}_{\nu}$.

This transformation acts on the point $(u, v, \vec{y})$ in the AdS_n space such that

$$\begin{pmatrix} u \\ v \\ \vec{y} \end{pmatrix} \mapsto \begin{pmatrix} u(1+a) + \vec{\alpha} \cdot \vec{y} \\ v(1-a) + \vec{\beta} \cdot \vec{y} \\ \frac{1}{2}(u\vec{\beta} + v\vec{\alpha}) + (1 + \omega_{n-1})\vec{y} \end{pmatrix}. \tag{3.47}$$

The effect on points on the boundary of AdS_n are given by the same formula, but whatever convention we use for picking representatives from the projective equivalence classes of the boundary points may not be respected. If, for example, we hold the convention that the point with $v = 1$ is chosen as the representative (in which case, since $u = \vec{y}^2$, we can parametrize the whole boundary except for the point $v = 0$ with the vector $\vec{y}$), the transformed point may have $v \neq 1$. We therefore must explicitly enforce the convention by dividing by the transformed $v = v(1-a) + \vec{\beta} \cdot \vec{y}$, which yields

$$\vec{y}' = \frac{\frac{1}{2}(u\vec{\beta} + v\vec{\alpha}) + (1 + \omega_{n-1})\vec{y}}{v(1-a) + \vec{\beta} \cdot \vec{y}} = \frac{\frac{1}{2}(\vec{y}^2\vec{\beta} + \vec{\alpha}) + (1 + \omega_{n-1})\vec{y}}{1 - a + \vec{\beta} \cdot \vec{y}} \tag{3.48}$$

and further, using the fact that all the transformation parameters are infinitesimal

$$\vec{y}' = \vec{y}(1 + a - \vec{\beta} \cdot \vec{y}) + \omega_{n-1}\vec{y} + \frac{1}{2}(\vec{y}^2\vec{\beta} + \vec{\alpha}). \tag{3.49}$$

With Eq. (3.49) we can analyze the effect of the transformation on the boundary. First of all, if only $\vec{\alpha}$ is non-zero, we have

$$\vec{y} \mapsto \vec{y} + \frac{1}{2}\vec{\alpha}, \tag{3.50}$$

which is a translation. If only $\omega_{n-1} \neq 0$,

$$\vec{y} \mapsto \vec{y} + \omega_{n-1}\vec{y} \tag{3.51}$$

which is an infinitesimal, potentially Lorentz[6], rotation. If only $a \neq 0$,

$$\vec{y} \mapsto \vec{y}(1+a), \tag{3.52}$$

so that we have an infinitesimal dilation. Finally, if only $\vec{\beta}$ is non-zero, we have the transformation

$$\vec{y} \mapsto \vec{y}(1 - \vec{\beta} \cdot \vec{y}) + \frac{1}{2}\vec{y}^2\vec{\beta}, \tag{3.53}$$

which is the same as Eq. (3.43) with the identification $2p = \vec{\beta}$, i.e. a special conformal transformation.

Therefore we see that the Lorentz symmetries of the embedding space generate the conformal group on the boundary of the AdS_n[7]

[6] The sign ultimately sneaks in from the signature of the embedding $\mathbb{R}^{2,n-1}$ space through the $\omega^{\mu}{}_{\nu}$ matrix

[7] Formally we would still need to prove that the infinitesimal generators commute as required by the Lie algebra of the conformal group.

4 Superstring theory, branes and Yang-Mills theories

The twist of the Maldacena duality is intimately related to the connections between branes in superstring theory and Yang-Mills theories living on those branes. In this section we will look into superstrings, supergravity and branes, as a motivation for the next chapter, where we get to the Maldacena duality itself.

4.1 Superstring theory

We will review string theory here only extremely briefly, omitting all details. The fundamental idea is to replace point particles with one-dimensional strings. Since the equation of motion for a free classical point particle is the geodesic equation, which minimizes the path length of its world line (or in spacetimes with Lorentzian signature, maximizes proper time), a natural generalization of the action for a string is the one that minimizes the area of the world sheet swept out by the string moving through spacetime. If the string has world-volume coordinates $X^\mu(x^\alpha)$, where indices μ, ν etc are indices on the D-dimensional spacetime through which the string is moving, and $x^\alpha \in \{\sigma, \tau\}$ are coordinates on the world sheet, with $x^1 \equiv \sigma$ spacelike and $x^2 \equiv \tau$ timelike, and indices α, β refer to the world sheet coordinates, then its world sheet area is

$$A = \int \sqrt{|\det \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}|} d^2x \tag{4.1}$$

where $g_{\mu\nu}$ is the spacetime metric and the determinant is taken over the α, β indices. The square root makes this action difficult to quantize, but an action which is classically equivalent to minimizing the area is

$$S = -\frac{T}{2} \int h^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\nu \sqrt{|\det h|} g^{\mu\nu} d^2x \tag{4.2}$$

where T is the string tension, which is the only free parameter of the theory, and $h_{\alpha\beta}$ is an auxiliary field which may be interpreted as a metric on the world sheet. Using the equations of motion one can eliminate the $h_{\alpha\beta}$ field and show that Eq. (4.1) and Eq. (4.2) are indeed equivalent. Quantizing the action Eq. (4.2) by Fourier-decomposing the solutions of the wave equation resulting from minimizing the action, promoting the Fourier coefficients to operators and imposing canonical commutation relations (which then lead to commutation relations for the Fourier coefficient operators) on the world sheet coordinates X_μ and their canonical momentum conjugates $P^\mu = -g^{\mu\nu} \frac{\partial \mathcal{L}}{\partial \dot{X}^\nu} = T\dot{X}^\mu$ leads to the theory of the bosonic string. For the Lorentz-symmetry anomalies resulting from the quantization to cancel, it turns out that $D = 26$ is required.

Unfortunately the bosonic theory has two problems which make it unsatisfactory: it has no fermionic states, and the ground state of the string turns out to have negative mass squared, which spoils causality. Both of these problems can be resolved by adding a fermionic degree of freedom on the world sheet. The simplest action is

$$S = \frac{1}{2\pi\alpha'} \int (h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \sqrt{|\det h|} d^2x, \tag{4.3}$$

where the ρ^α matrices satisfy the Dirac-Clifford algebra in two dimensions, and $\alpha' = \frac{1}{2\pi T}$ is the string parameter. This action is globally supersymmetric, but usually local supersymmetry is required on the world-sheet. This can be achieved by adding some extra terms to the action, which can then be gauged away, resulting ultimately in the addition of some gauge-choice conditions. In this superstring theory, the cancellation of Lorentz-anomalies on a flat background requires $D = 10$. Negative mass squared, or tachyonic, string states appear also in superstring theory, but there are five consistent versions of the theory where the tachyons may be projected away by the so-called

GSO projection (This projection also generates space-time supersymmetry from the world-sheet supersymmetry). They are known as type I, type IIA, type IIB and heterotic string theories, with the heterotic string theory coming in two flavours, one with gauge group $SO(32)$ and the other with $E_8 \times E_8$.

In a string theory, we have the choice of using either open or closed strings, or both. Open strings alone are not consistent, since there is always an interaction where both ends of two open strings interact, forming a closed string. The string theory where we have both open and closed unoriented strings is called Type I superstring theory.

For closed strings, the wave solutions of the string equations of motion contain excitations that move around the string in anticlockwise and clockwise directions, called left- and right-movers. In a theory with only closed oriented strings, it turns out we can choose what is essentially the chirality of the massless ground states of the left- and right-movers independently, and that the interactions do not mix the choice of chiralities. Choosing both to be the same gives type IIB superstring theory, and choosing them to be opposite gives type IIA superstring theory. In addition, the boundary conditions for fermionic closed strings may be periodic or anti-periodic, and may also be chosen for the left- and right movers independently. In a consistent string theory, both boundary conditions must be present. Periodic boundary conditions are known as Ramond (or R) boundary conditions and anti-symmetric boundary conditions are known as Neveu-Schwarz (or NS) boundary conditions. Since we have separate boundary conditions for left- and right movers, the possible choices of boundary conditions give so called RR, R-NS, NS-R and NS-NS sectors of the theory.

The one remaining consistent superstring theory is called heterotic. In heterotic string theory, the right moving degrees of freedom come from a type II superstring theory (in this case we do not need to distinguish between IIA and IIB, since we are dealing only with right movers) and the left movers come from bosonic string theory. Of the 26 bosonic dimensions, 16 are considered as compactified on a 16-dimensional torus so that the dimension of the whole theory is 10. [7]

4.2 Symmetries of superstring theory

In addition to the symmetries of its background, type IIB superstring theory also has two spacetime supersymmetries, so it is an $\mathcal{N} = 2$ supersymmetric theory. In ten dimensions this means that the total number of supersymmetry generators is $2^{D/2} = 32$. In type I superstring theory the open string boundary conditions break half of the supersymmetries, leaving an $\mathcal{N} = 1$ supersymmetric theory with 16 generators. The type IIB theory does not admit any gauge symmetries per se (although compactifying it on various manifolds can generate such symmetries), whereas type I theory classically allows any Lie group as a gauge group, by introducing group theory indices called Chan-Paton factors on the endpoints of the string. Quantum mechanically, $SO(32)$ and $E_8 \times E_8$ are singled out as the only consistent gauge groups. [14]

4.3 Branes

Superstring theory has two superficially distinct concepts of branes. The first one is related to boundary conditions of open strings [15]:

1. Neumann boundary conditions: $\partial_\sigma X^\mu = 0$, where σ is the spacelike world-sheet coordinate, on the endpoints of the string. This is a Lorentz-invariant condition, which physically requires that no momentum must flow out of the ends of the string and that the endpoints must move at the speed of light [14].

2. Dirichlet boundary conditions: The endpoint coordinates X^μ are constant. This boundary condition is clearly not Lorentz -invariant, and is often ignored in pure string theory.

The Dirichlet and Neumann boundary conditions can also be mixed. We can take Neumann boundary conditions for $p+1$ spacetime coordinates, and Dirichlet boundary conditions for the remaining $D-p-1$ coordinates. The endpoints are then free to move in the $p+1$ directions with Neumann boundary conditions, and are fixed in the remaining directions (the string itself is still of course free to move in all of spacetime).

The $p+1$ dimensional volume in which the string endpoints are constrained to lie on is called a D-brane, D for Dirichlet and brane as a generalization of the word membrane. At first sight it seems that the choice of Dirichlet boundary conditions introduces an unphysical and arbitrary object, and that we should simply stick to Neumann boundary conditions. It was however shown in [16] (where they also coin the term D-brane) that the D-brane actually arises from the toroidal compactification of an open string theory. They showed that the compactified theory is a theory of closed strings in the bulk and open strings with endpoints constrained on the brane. The closed strings interact with the brane, so it is indeed a dynamical object, instead of an arbitrary choice of boundary conditions.

The other kind of brane in string theory is known as the p-brane. These are essentially solutions to the low-energy supergravity approximation of string theory which have a singularity extended in p space dimensions. A 0-brane is a conventional black hole. A 1-brane is the string, the 2-brane is an object with a singularity that is a 2 dimensional membrane, and so on. A p-brane may have a charge related to a $p+1$ -form gauge potential of the supergravity approximation. If the charge is equal to the mass of the brane, it is called an extremal brane, and in this case the event horizon surrounding the singularity becomes degenerate with the singularity itself. [15]

It turns out that p-branes which have a charge from the Ramond-Ramond sector of string theory are actually the same objects as D-branes. They are therefore usually termed Dp-branes, to emphasize the equivalence. [17, 18, 19].

We will now calculate explicitly the Dp-brane solution from the supergravity approximation of type IIB superstring theory.

4.4 Finding the Dp-brane solution

We start from the action of type IIB supergravity theory. This is given by [12]:

$$S = -s\frac{1}{16\pi G_1 0}\int d^{10}x\sqrt{|g|}\left(e^{-2\phi}(R+4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}\sum_n \frac{1}{n!}F_n^2 + \dots\right), \tag{4.4}$$

where ϕ is the dilaton field, F_n are the RR sector field strengths, s is the signature (-1 for Minkowski). The dots represent omitted NS-NS sector field strength tensors and fermionic fields, which we will not be concerned with. This truncation is consistent in the sense that solutions of the equations of motion for this action are also solutions to the equations of motion for the full action [20]. For the IIB string theory, there are only field strengths with n odd. In order to make calculations easier, we carry out a conformal transformation to the Einstein frame, which may be defined by requiring that the metric contains a pure Einstein-Hilbert -term $\sqrt{|g|}R$ (the issue of the physical significance of such transformations is discussed for example in [21]). The conformal transformation which takes Eq. (4.4) to the Einstein frame is

$$g_{\mu\nu} \to e^{-\frac{1}{2}\phi}g_{\mu\nu}. \tag{4.5}$$

The Einstein frame action is then [12]

$$S = -\frac{s}{2\kappa_D^2}\int d^Dx\sqrt{g}\{R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\sum_n\frac{1}{n!}e^{a_n\phi}F_n^2\}. \tag{4.6}$$

We will show that this action has a classical solution of the form expected for a Dp -brane, and specifically study the D3 -brane metric in more detail. It is generally expected that the these solutions extend to a solution of the full string theory, with order α' corrections [20].

4.4.1 Equations of Motion

To obtain the classical equations of motion, we use the Palatini method, which essentially amounts to treating the metric and the connection as independent variables. The Palatini method works in the Einstein frame, since the Einstein-Hilbert -term appears in the action without directly coupling to any fields. This saves us the trouble of breaking down the Ricci scalar in Eq. (4.6) to its representation in terms of the metric, which would involve a large amount of terms.

In terms of the metric and the connection, Eq. (4.6) becomes

$$\begin{aligned} S &= -\frac{s}{2\kappa_D^2}\int d^Dx\sqrt{g}\{g^{\mu\nu}\left(\partial_\rho\Gamma^\rho_{\mu\nu}-\partial_\nu\Gamma^\rho_{\mu\rho}+\Gamma^\rho_{\lambda\rho}\Gamma^\lambda_{\mu\nu}-\Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\rho}\right)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &- \frac{1}{2n!}e^{a_n\phi}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_n}F_{\mu_1\mu_2\cdots\mu_n}F_{\nu_1\nu_2\cdots\nu_n}\}, \end{aligned} \tag{4.7}$$

where we have, for simplicity, considered the case where only one F_n is non-zero. Varying this with respect to the metric gives

$$\begin{aligned} \frac{\delta S}{\delta g^{\alpha\beta}} &= -\frac{s}{2\kappa_D^2}\int d^Dx\{\left(-\frac{1}{2}g_{\alpha\beta}g^{\mu\nu}+\delta^\mu_\alpha\delta^\nu_\beta\right)R^\rho{}_{\mu\rho\nu}-\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi \\ &- \frac{1}{2n!}e^{a_n\phi}(F_\alpha{}^{\nu_2\nu_3\cdots\nu_n}F_{\beta\nu_2\nu_3\cdots\nu_n}+F^{\nu_1}{}_\alpha{}^{\nu_3\cdots\nu_n}F_{\nu_1\beta\nu_3\cdots\nu_n}+\cdots \\ &+ F^{\nu_1\nu_2\cdots\nu_{n-1}}{}_\alpha F_{\nu_1\nu_2\cdots\nu_{n-1}\beta})+\frac{1}{2n!}e^{a_n\phi}\frac{1}{2}g_{\alpha\beta}F_n^2+\frac{1}{4}g_{\alpha\beta}\partial^\mu\phi\partial_\mu\phi\}\sqrt{g}\delta g^{\alpha\beta}. \end{aligned} \tag{4.8}$$

The terms involving the components of F_n can be combined to give $nF_\alpha{}^{\nu_2\nu_3\cdots\nu_n}F_{\beta\nu_2\nu_3\cdots\nu_n}$ using the total antisymmetry of the two-form F_n. The classical solution is obtained by setting the term in curly braces to zero. Contracting the α and β indices on the resulting equation, we find the Ricci scalar to be

$$R=\frac{1}{2}\partial^\rho\phi\partial_\rho\phi+\frac{1}{2n!}e^{a_n\phi}\frac{2n-D}{2-D}F_n^2. \tag{4.9}$$

Substituting this back to the equation (and renaming indices for convenience) gives the final form of the first equation of motion:

$$R^\mu{}_\nu=\frac{1}{2}\partial^\mu\phi\partial_\nu\phi+\frac{1}{2n!}e^{a_n\phi}(nF^{\mu\mu_2\mu_3\cdots\mu_n}F_{\nu\mu_2\mu_3\cdots\mu_n}-\delta^\mu_\nu\frac{n-1}{D-2}F_n^2). \tag{4.10}$$

The second equation of motion is obtained by varying Eq. (4.7) with respect to ϕ. This leads to the equation

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi)=\frac{a_n}{2n!}e^{a_n\phi}F_n^2. \tag{4.11}$$

For the F_n variation, we must remember that since it is a field-strength n-form, the dynamical variable is not F_n itself but an $n-1$ -form tensor potential $A_{\mu_2\mu_3\cdots\mu_n}$ such that $F=dA$, or in components $F_{\mu_1\mu_2\cdots\mu_n}=\frac{1}{(n-1)!}\partial_{\mu_1}A_{\mu_2\mu_3\cdots\mu_n}$. This gives

$$\partial_\mu(\sqrt{g}e^{a_n\phi}F^{\mu\nu_1\nu_2\cdots\nu_n})=0. \tag{4.12}$$

Finally the variation with respect to the Christoffel symbols gives no new information, but only restores the standard relation between the metric tensor and the metric connection. For a derivation, see for example [22].

4.4.2 The brane solution

Now that we have the equations of motion, we may start looking for solutions. The types of solutions we want are such that the geometry has a Poincaré symmetry in $p+1$ -directions corresponding to a brane with p spacelike dimensions and a timelike dimension, a radial direction, and a spherical symmetry in the remaining $D-p-1$ dimensions. A diagonal metric satisfying these symmetries is

$$ds^2 = sB^2dt^2 + C^2\sum_{i=1}^{p}(dx^i)^2 + F^2dr^2 + G^2r^2d\Omega_{d-1}^2, \tag{4.13}$$

with all coefficients depending only on r, where $d\Omega_{d-1}^2$ is the metric of the unit sphere S^{d-1}, with the metric matrix $\gamma = \mathrm{diag}(1, \sin\phi_1, \sin\phi_1\sin\phi_2, \cdots, \sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2})$. The coordinates are then x^i and t for the brane and y^a for the directions transverse to the brane. We also use the set $(r, \phi_1, \cdots, \phi_{d-1})$ with $r^2 = \sum_{a=1}^{d}(y^a)^2$ and ϕ_k being the angular coordinates on the surface of the $(d-1)$-dimensional sphere. The total dimension of the space is then $D = p+1+d$, and s determines the sign of the timelike dimension and therefore whether we are dealing with a Lorentzian or a Euclidean manifold. We also require that the metric tends to flat as $r\to\infty$, or equivalently that all the coefficients tend to 1 on that limit.

In order to find a solution, we take the ansatz where all the coefficients of a metric are of the form Eq. (4.13) and all the fields are functions of the radial coordinate r only (the geometric variables of course may also depend on the angular coordinates on the unit sphere), and in addition the electric field tensor F_n is of the form

$$F_{ti_1i_2\cdots i_pr} = \epsilon_{i_1i_2\cdots i_p}k(r), \tag{4.14}$$

and therefore obviously $n = p+2$. The indices t and r here mean that the specific index must be the one referring to the t or r coordinate respectively. Indices i_k mean that the corresponding index must be one of those referring to the on-brane coordinates x^i, and ϵ is the covariant fully antisymmetric rank-p tensor. Due to the antisymmetry of F_n it is fully defined by Eq. (4.14).

We can now write the equation of motion for F_n, Eq. (4.12), in a more explicit form. Raising all the indices of F with the inverse metric gives

$$\begin{aligned} F^{ti_1i_2\cdots i_pr} &= g^{t\mu_0}g^{i_1\mu_2}\cdots g^{i_p\mu_p}g^{r\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}} = g^{tt}g^{i_1j_1}\cdots g^{i_pj_p}g^{rr}\epsilon_{j_1j_2\cdots j_p}k(r) \\ &= \frac{s}{B^2C^{2p}F^2}\tilde{\epsilon}^{i_1i_2\cdots i_p}k(r), \end{aligned} \tag{4.15}$$

where the indices j_k go through the brane coordinates x^i, and $\tilde{\epsilon}^{i_1i_2\cdots i_p}$ denotes an object which is numerically exactly the same as $\epsilon_{i_1i_2\cdots i_p}$, but with upper indices for notational clarity [8]. Note that it is then not a rank-p contravariant tensor (which is why we write it with a tilde).

With the observation that

$$\sqrt{g} = BC^pF(Gr)^{d-1}\sqrt{|\gamma|}, \tag{4.16}$$

where $|\gamma|$ is the determinant of the metric on the unit sphere, the equation of motion for F_n gives

$$\partial_r\left(\frac{s}{BC^pF}(Gr)^{d-1}e^{a\phi}k(r)\right) = 0, \tag{4.17}$$

which leads to

$$k(r) = se^{-a\phi}BC^pF\frac{Q}{(Gr)^{d-1}} \tag{4.18}$$

where Q is a constant of integration.

[8] We do not apply the Einstein summation convention to pairs of indices that are both superscript or both subscript.

We will later need the results

$$F_n^2 = n! F_{t12\cdots pr} F^{t12\cdots pr} = n! s e^{-2a\phi} \frac{Q^2}{(Gr)^{2(d-1)}} \tag{4.19}$$

and

$$F^{\mu\xi_2\xi_3\cdots\xi_n} F_{\nu\xi_2\xi_3\cdots\xi_n} = (n-1)! s e^{-2a\phi} \delta^\mu_\nu \frac{Q^2}{(Gr)^{2(d-1)}}, \tag{4.20}$$

when $\mu, \nu \in \{t, 1, 2, \cdots, p, r\}$ and else 0.

In order to work out the equations of motion for the metric, we need to compute the Ricci tensor. We will do this in the vielbein formalism. Since the metric is of the diagonal form

$$ds^2 = s(A_0)^2 (dz^0)^2 + \sum_{\mu=1}^{D-1} (A_\mu)^2 (dz^\mu)^2 \tag{4.21}$$

where $\{z_\mu\} = \{t, x^i, r, \phi^k\}$, we can write the vielbein as

$$e^a_\mu = A_\mu, \tag{4.22}$$

with $A_\mu = (B, \underbrace{C, \cdots, C}_{p \text{ components}}, F, Gr, Gr\sin\phi_1, Gr\sin\phi_1\sin\phi_2, \cdots, Gr\sin\phi_1\sin\phi_2\cdots\sin\phi_{d-2})$. We denote flat indices by Latin letters or Greek letters with an overbar. Then the metric tensor is

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu = \eta_{\mu\nu} A_\mu A_\nu, \text{ no summation over } \mu,\ \nu \text{ on the r.h.s.} \tag{4.23}$$

We denote the inverse vielbein by $E^\mu_a = A^\mu = \frac{1}{A_\mu}$.

Now the spin connection can be derived from the requirements that it is compatible with the metric and torsion free [23]:

$$\begin{aligned} \omega^a{}_b &= -\omega^b{}_a \quad &(4.24)\\ T^a &= de^a + \omega^a{}_b \wedge e^b = 0. \quad &(4.25) \end{aligned}$$

The wedge product of two 1-forms is $a \wedge b = a_\mu dx^\mu \wedge b_\nu dx^\nu = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu) dx^\mu \wedge dx^\nu$, where the dx^μ are basis 1-forms. The wedge product of basis 1-forms dx^μ, dx^ν is a basis 2-form $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. The exterior derivative of a 1-form is $da = \partial_\mu a_\nu dx^\mu \wedge dx^\nu$. Using Eq. (4.25), we have

$$0 = \partial_\mu e^a_\nu dx^\mu \wedge dx^\nu + \frac{1}{2}(\omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu) dx^\mu \wedge dx^\nu. \tag{4.26}$$

Since the vielbein is diagonal, only the $\nu = a$ term of the first sum survives. Picking a term with a fixed a and a given component of the 2-form gives (note that due to the antisymmetry of the basis 2-forms, two terms of the μ, ν sum always contribute to the single 2-form component, canceling the 1/2 in front of the wedge product)

$$\begin{aligned} & \partial_\mu A_a + \omega^a{}_{a\mu} A_a - \omega^a{}_{\bar{\mu}a} A_\mu = 0 \\ \Rightarrow\quad & \omega^a{}_{\bar{\mu}a} = \frac{1}{A_\mu} \partial_\mu A_a. \end{aligned} \tag{4.27}$$

The index μ appearing on the left with an overbar and on the right without an overbar means that the same index must appear on both sides of the equation, but on the left it is a flat index and on the right it is a curved index (which is made flat by the factor $\frac{1}{A_\mu}$, which is the inverse vielbein $E^\mu_{\bar{\mu}}$).

We will follow this convention throughout this section. Via the metricity condition (we have to set this explicitly, since we already used it in deriving Eq. (4.27))

$$\omega^{\bar{\mu}}{}_{aa} = -\frac{1}{A_\mu}\partial_\mu A_a. \tag{4.28}$$

Since the vielbein only depends on r and the angular coordinates on the sphere, the non-zero components are

$$\begin{aligned} \omega^{a}{}_{\bar{r}a} &= -\omega^{\bar{r}}{}_{aa} = \frac{1}{F}\partial_r A_a \text{ and} \\ \omega^{\bar{\phi}_l}{}_{\phi_k\phi_l} &= -\omega^{\bar{\phi}_k}{}_{\phi_l\phi_l} = \frac{1}{A_{\phi_k}}\partial_{\phi_k}A_{\phi_l} = \frac{\cos\phi_k}{\sin\phi_k}\frac{A_{\phi_l}}{A_{\phi_k}} \text{ when } l > k,\ 0 \text{ otherwise.} \end{aligned} \tag{4.29}$$

The curvature 2-form is

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_{b\mu\nu}dx^\mu \wedge dx^\nu. \tag{4.30}$$

In order to extract the Ricci tensor from here, we need to make the μ index flat with the vielbein. We also flatten the index ν (In the following, the μ sum is denoted explicitly since it involves an extra subscript index, whereas the Einstein summation convention still applies to the c sum. We will follow a similar convention systematically.):

$$R_{a\bar{\nu}} = R^{\bar{\mu}}{}_{a\bar{\mu}\bar{\nu}} = \sum_\mu \frac{1}{A_\mu A_\nu}(\partial_\mu\omega^\mu{}_{a\nu} - \partial_\nu\omega^\mu{}_{a\mu} + \omega^\mu{}_{c\mu}\omega^c{}_{a\nu} - \omega^\mu{}_{c\nu}\omega^c{}_{a\mu}). \tag{4.31}$$

It is easy to see here that the Ricci tensor is diagonal, so we only need to consider the $\bar{\nu} = a$ terms. In addition, the $\nu \neq r, \phi_k$ terms are independent of the $\omega^{\phi_l}{}_{\phi_k\phi_l}$ components of the spin connection, and we have

$$\begin{aligned} R_{\bar{\nu}\bar{\nu}} &= \frac{1}{A_\nu}\left(-\frac{1}{F}\partial_r\frac{1}{F}\partial_r A_\nu + \sum_{\mu\neq r}\frac{1}{A_\mu}\frac{1}{F}\partial_r A_\mu\left(-\frac{1}{F}\partial_r A_\nu\right) + \frac{1}{F^2}(\partial_r A_\nu)^2\right) \\ &= \frac{1}{F^2}\left(-\frac{F}{A_\nu}\partial_r\frac{A_\nu}{F}\partial_r\log A_\nu - \sum_{\mu\neq r}(\partial_r\log A_\mu)(\partial_r\log A_\nu) + (\partial_r\log A_\nu)^2\right) \\ &= -\frac{1}{F^2}\left((\log A_\nu)'' + (\log(F^{-1}\prod_{\mu\neq r}A_\mu))'(\log A_\nu)'\right) \\ &= -\frac{1}{F^2}\left((\log A_\nu)'' + (\log(fr^{d-1}))'(\log A_\nu)'\right), \end{aligned} \tag{4.32}$$

with prime denoting a partial derivative with respect to r, and f defined as

$$fr^{d-1} \equiv BC^pF^{-1}(Gr)^{d-1}. \tag{4.33}$$

Note how the ϕ_k dependencies cancel in the logarithmic derivatives.

When $\nu = r$, different terms contribute and the result is

$$\begin{aligned} R_{\bar{r}\bar{r}} = &-\frac{1}{F^2}\left((\log(Ffr^{d-1}))'' + ((\log B)')^2 - (\log F)'(\log(Ffr^{d-1}))'\right. \\ &\left.+p((\log C)')^2 + (d-1)((\log Gr)')^2\right). \end{aligned} \tag{4.34}$$

The $\nu = \phi_l$ components gain terms of the same form as Eq. (4.32), and additionally the ϕ_k derivatives now contribute the extra terms

$$
\begin{aligned}
&\sum_k \frac{1}{A_{\phi_l} A_{\phi_k}} \left(\partial_{\phi_k} \omega^{\phi_k}{}_{\phi_l \phi_l} - \partial_{\phi_l} \omega^{\phi_k}{}_{\phi_l \phi_k} + \omega^{\phi_k}{}_{\phi_c \phi_k} \omega^{\phi_c}{}_{\phi_l \phi_l} - \omega^{\phi_k}{}_{\phi_c \phi_l} \omega^{\phi_c}{}_{\phi_l \phi_k} \right) \\
= &\sum_{k=l+1}^{d-1} \frac{1}{A_{\phi_l} A_{\phi_k}} \left(\partial_{\phi_k} \frac{\cos\phi_k}{\sin\phi_k} \frac{A_{\phi_l}}{A_{\phi_k}} - \partial_{\phi_l} \frac{\cos\phi_l}{\sin\phi_l} \frac{A_{\phi_k}}{A_{\phi_l}} + \sum_{c=1}^{l-1} \frac{\cos^2\phi_c}{\sin^2\phi_c} \frac{A_{\phi_k} A_{\phi_l}}{A_{\phi_c}^2} \right) + \sum_{c=1}^{l-1} \frac{1}{A_{\phi_l}^2} \frac{\cos^2\phi_c}{\sin^2\phi_c} \frac{A_{\phi_l}^2}{A_{\phi_c}^2} \\
= &\frac{1}{G^2 r^2} \left[\sum_{k=1}^{l-1} \frac{1 - (d-2-k)\frac{\cos^2\phi_k}{\sin^2\phi_k}}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{k-1}} + \frac{d-1-l}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{l-1}} \right]. \qquad (4.35)
\end{aligned}
$$

The term in brackets equals $d-2$. This is easily seen for $l = 1$ (the sine product is 1 for l = 1, and the sum is empty), and if the claim holds for the l:th term, then

$$
\begin{aligned}
&\sum_{k=1}^{l-1} \frac{1 - (d-2-k)\frac{\cos^2\phi_k}{\sin^2\phi_k}}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{k-1}} = d - 2 - \frac{d-1-l}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{l-1}} \\
\Rightarrow &\sum_{k=1}^{(l+1)-1} \frac{1 - (d-2-k)\frac{\cos^2\phi_k}{\sin^2\phi_k}}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{k-1}} + \frac{d-1-(l+1)}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{(l+1)-1}} \\
= &\sum_{k=1}^{l-1} \frac{1 - (d-2-k)\frac{\cos^2\phi_k}{\sin^2\phi_k}}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{k-1}} + \frac{1 - (d-2-l)\frac{\cos^2\phi_l}{\sin^2\phi_l}}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_{l-1}} + \frac{d-2-l}{\sin^2\phi_1 \sin^2\phi_2 \cdots \sin^2\phi_l} \\
= &\; d-2 \qquad (4.36)
\end{aligned}
$$

and therefore it holds for all $l \geq 1$.

Combining all this and raising the first index gives the final form of the components of the Ricci tensor.

$$
\begin{aligned}
R^{\bar{t}}{}_{\bar{t}} &= -\frac{s}{F^2} \left((\log B)'' + (\log(f r^{d-1}))' (\log B)' \right) \\
R^{\bar{i}}{}_{\bar{i}} &= -\frac{1}{F^2} \left((\log C)'' + (\log(f r^{d-1}))' (\log C)' \right) \\
R^{\bar{r}}{}_{\bar{r}} &= -\frac{1}{F^2} \Big((\log(F f r^{d-1}))'' + ((\log B)')^2 - (\log F)' (\log(F f r^{d-1}))' \\
&\quad + p((\log C)')^2 + (d-1)((\log Gr)')^2 \Big) \\
R^{\bar{\alpha}}{}_{\bar{\alpha}} &= -\frac{1}{F^2} \left((\log Gr)'' + (\log(f r^{d-1}))' (\log Gr)' - (d-2) \frac{F^2}{G^2 r^2} \right)
\end{aligned}
\qquad (4.37)
$$

Converting the indices in Eq. (4.9) to flat gives

$$
R^a{}_a = e^a_\mu E^\mu_a R^\mu{}_\mu = \delta^a_\mu R^\mu{}_\mu, \qquad (4.38)
$$

since the vielbein is diagonal. Then, using Eq. (4.9), Eq. (4.37), Eq. (4.19) and Eq. (4.20) we get the equations

$$
\begin{aligned}
-\frac{s}{F^2}\left((\log B)'' + (\log(fr^{d-1}))'(\log B)'\right) &= s(d-2)\frac{K^2}{F^2}, && (4.39)\\
-\frac{1}{F^2}\left((\log C)'' + (\log(fr^{d-1}))'(\log C)'\right) &= s(d-2)\frac{K^2}{F^2}, && (4.40)\\
-\frac{1}{F^2}\Big((\log(Ffr^{d-1}))'' + ((\log B)')^2 - (\log F)'(\log(Ffr^{d-1}))' & \\
+p((\log C)')^2 + (d-1)((\log Gr)')^2\Big) &= s(d-2)\frac{K^2}{F^2} \\
&\quad +\frac{1}{2F}(\phi')^2 && (4.41)\\
-\frac{1}{F^2}\left((\log Gr)'' + (\log(fr^{d-1}))'(\log Gr)' - (d-2)\frac{F^2}{G^2r^2}\right) &= s(p+1)\frac{K^2}{F^2}, && (4.42)\\
\phi'' + \phi'(\log(fr^{d-1}))' &= a_n s(D-2)K^2 && (4.43)
\end{aligned}
$$

where

$$K^2 = \frac{1}{2(D-2)}e^{-a_n\phi}\frac{Q^2}{(Gr)^{2(d-1)}}F^2. \tag{4.44}$$

We will not attempt to find a general solution to these equations, but will instead specifically look for a brane solution of the desired form. We start with the ansatz

$$\log\left(\frac{B^s}{C}\right) = c_B\log f,\ \log\left(\frac{F}{G}\right) = c_F\log f, \tag{4.45}$$

where c_B and c_F are to be determined. This, combined with Eq. (4.39) amd Eq. (4.40), leads to

$$
\begin{aligned}
&\left(\log\frac{B^s}{C}\right)'' + \left(\log\frac{B^s}{C}\right)'(\log(fr^{d-1}))' = c_B\left((\log f)'' + (\log f)\left[(\log f)' + \frac{d-1}{r}\right]\right)\\
=&\quad c_B\left(\frac{f''}{f} + \frac{f'(d-1)}{fr}\right) = 0\\
\Rightarrow&\quad f'' + f'\frac{d-1}{r} = 0. && (4.46)
\end{aligned}
$$

This has the family of solutions

$$f = \frac{C_1}{r^{d-2}} + C_2, \tag{4.47}$$

where C_1 and C_2 are constants of integration. Since we demand that $f\to 1$ as $r\to\infty$, we pick $C_2 = 1$, and write the solution as

$$f = 1 - \left(\frac{r_0}{r}\right)^{d-2}. \tag{4.48}$$

Let us further write

$$g \equiv C^{p+1}G^{d-2} = f\frac{CF}{BG} = f^{1-(c_B-c_F)}, \tag{4.49}$$

and then, using Eq. (4.40) and Eq. (4.42)

$$
\begin{aligned}
&(\log C^{p+1})'' + (\log C^{p+1})'(\log(fr^{d-1}))'\\
&+(\log(Gr)^{d-2})'' + (\log(Gr)^{d-2})'(\log frd-1)' - (d-2)^2\frac{F^2}{G^2r^2}\\
=&\quad (\log g)'' + (\log g)'(\log fr^{d-1})' + \frac{d-2}{r}\left[(\log f)' + \frac{d-2}{r}\left(1-\frac{F^2}{G^2}\right)\right] = 0. && (4.50)
\end{aligned}
$$

A solution is given by $g \equiv 1$ or equivalently $c_B - c_F = 1$, which leads to $\left(1 - \left(\frac{r_0}{r}\right)^{d-2}\right)^{2c_F+1} = 1$ and therefore $c_F = -1/2$, $c_B = 1/2$.

Using this on Eq. (4.45) allows us to re-write the F^2/G^2 term in Eq. (4.42) in terms of f, which gives

$$(\log G)'' + (\log G)'(\log f r^{d-1})' = s(p+1)K^2. \tag{4.51}$$

Adding this to Eq. (4.43), multiplied with factors to cancel the right hand side, leads to

$$(a_n \log G - \frac{p+1}{D-2}\phi)'' + (a_n \log G - \frac{p+1}{D-2}\phi)'(logfr^{d-1})' = 0$$

and therefore

$$\phi = \log G^{\frac{D-2}{p+1}a_n}. \tag{4.52}$$

Now we can reduce Eq. (4.51) to a function of G only, which gives us a differential equation for G:

$$\frac{G''}{G} - \frac{(G')^2}{G^2} + \frac{G'}{Gr}\left[\frac{d-1-\left(\frac{r_0}{r}\right)^{d-2}}{1-\left(\frac{r_0}{r}\right)^{d-2}}\right] = s\frac{(p+1)Q^2}{2(D-2)r^{2(d-1)}\left(1-\left(\frac{r_0}{r}\right)^{d-2}\right)}G^{-\frac{2\Delta}{p+1}}, \tag{4.53}$$

where $\Delta = (d-2)(p+1) - \frac{1}{2}(D-2)a_n^2$. We take the ansatz

$$G = \left(1 + \left(\frac{h}{r}\right)^{d-2}\right)^A = H^A, H \equiv 1 + \left(\frac{h}{r}\right)^{d-2} \tag{4.54}$$

with A and h to be determined. After some computation this gives

$$A = \frac{p+1}{\Delta} \tag{4.55}$$

$$h^{2(d-2)} + r_0^{d-2}h^{d-2} = -s\frac{\Delta Q^2}{2(d-2)^2(D-2)} \tag{4.56}$$

and therefore

$$G = H^{\frac{p+1}{\Delta}} = \left(1 + \left(\frac{h}{r}\right)^{d-2}\right)^{\frac{p+1}{\Delta}}. \tag{4.57}$$

With the relations we have established, we now know all other components of the metric except C. This can be solved by adding Eq. (4.40) and Eq. (4.51), multiplied with factors such that the right hand side is eliminated, to yield (as one solution, which we choose here)

$$CG^{\frac{d-2}{p+1}} = 1. \tag{4.58}$$

Then the final metric is

$$ds^2 = H^{-2\frac{d-2}{\Delta}}\left(sfdt^2 + \sum_{i=1}^{p}(dx^i)^2\right) + H^{2\frac{p+1}{\Delta}}(f^{-1}dr^2 + r^2 d\Omega_{d-1}^2). \tag{4.59}$$

4.4.3 The extremal D3 -brane

The charge Q, which appeared as a constant of integration in Eq. (4.18), is related to the number N of coincident branes by [12]

$$Q = N g_s \frac{(2\pi\ell_s)^4}{\Omega_5} = 16 N g_s \ell_s^4 \pi, \tag{4.60}$$

where ℓ_s is the string length and Ω_5 is the volume of the 5-d unit sphere.

The $D = 10$, $p = 3$ case is the D3 -brane solution. The compactification sphere then has dimension $d = 6$. Setting $r_0 = 0$ is the extremal solution, and the metric becomes

$$\begin{aligned}
a_5 &= 0, \quad f = 1, \quad \Delta = (p+1)(d-2) = 16, \\
h^{d-2} &= \left(\frac{Q^2}{16}\right)^{1/2} = \frac{Q}{4}, \\
H &= 1 + \frac{Q}{4r^4} \\
ds^2 &= \left(1 + \frac{Q}{4r^4}\right)^{-1/2} (sdt^2 + \sum_{i=1}^{3}(dx^i)^2) + \left(1 + \frac{Q}{4r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2)
\end{aligned} \tag{4.61}$$

Let us then define the variable $U = r/\ell_s^2$, and go to the limit where $r \to 0$ and $\ell_s^2 \to 0$ in such a way that U becomes the meaningful variable. Then Eq. (4.61) goes to

$$\begin{aligned}
ds^2 &= \left(\frac{4Ng_s\pi}{\ell_s^4 U^4}\right)^{-1/2} (sdt^2 + \sum_{i=1}^{3}(dx^i)^2) + \left(\frac{4Ng_s\pi}{\ell_s^4 U^4}\right)^{1/2} (\ell_s^4 dU^2 + U^2 \ell_s^4 d\Omega_5^2) \\
&= \ell_s^2 \left(\frac{U^2}{L^2} d\vec{x}_4^2 + \frac{L^2}{U^2} dU^2 + L^2 d\Omega_5^2\right),
\end{aligned} \tag{4.62}$$

with $L^2 = \sqrt{4Ng_s\pi}$ and $d\vec{x}_4^2$ is the Euclidean or Minkowski metric, as determined by s. Rescaling to remove the singular ℓ_s^2 and multiplying by L^2 brings the metric to a form which is clearly a direct product of a 5-sphere and the AdS_5 metric Eq. (3.23), both with radius L^2:

$$ds^2 = \frac{L^4}{U^2} dU^2 + U^2 d\vec{x}_4^2 + L^4 d\Omega_5^2. \tag{4.63}$$

4.4.4 The non-extremal D3 brane

If we take $r_0 \neq 0$, we have a non-extremal D3 brane. We can then go to the near-horizon limit and let $r_0 \to 0$ in such a way that r_0/r remains meaningful. Writing this in terms of U and a new parameter U_0 replacing r_0, the metric becomes

$$ds^2 = \frac{L^4}{U^2}(1 - \frac{U_0^4}{U^4})^{-1} dU^2 + U^2 \left[s\left(1 - \frac{U_0^4}{U^4}\right) dt^2 + d\vec{x}_3^2\right] + L^4 d\Omega_5^2. \tag{4.64}$$

This is the metric of a black hole in $\mathrm{AdS}_5 \times S^5$, with the event horizon at $U = U_0$. This event horizon originates of course as the event horizon of the black p-brane.

4.5 Branes and gauge theory

On a stack of N branes on top of each other, an open string going from one brane back to itself or another brane on the stack has two indices from $1 \ldots N$ determining the starting and ending branes. Such indices related to the ends of open strings are called Chan-Paton factors. The quantum states corresponding to such open strings must carry the corresponding indices, and therefore become $N \times N$ matrices. It turns out that these are $U(N)$ matrices, corresponding to a local gauge symmetry. The $U(1)$ part factorizes out for most purposes, leaving an $SU(N)$ symmetry. [14]

The $SU(N)$ degrees of freedom on an extremal D3 brane generate, in the low energy approximation, an $\mathcal{N} = 4$ super Yang-Mills field theory. While an exact proof is beyond the scope of this discussion, it is already suggestive that if we take the $\mathcal{N} = 1$ supersymmetric gauge theory, which must exist in the bulk as a part of the supergravity theory, and dimensionally reduce it to take place on the brane, we end up with the $\mathcal{N} = 4$ SYM, as seen in section 2.4.

It is also interesting to consider what are the consequences of letting the brane be non-extremal. In such a case, according to the Bekenstein-Hawking conjecture, there must be a temperature related to the horizon. As a matter of fact, this temperature indeed becomes the temperature of the field theory. We will return to this several times in later sections.

5 The Maldacena duality

In the previous section we saw that the low energy theory describing the interactions of the ends of open strings bound on a stack of N D3 branes is the $\mathcal{N}=4$ super Yang-Mills theory with the gauge group $SU(N)$. Since the energy of a string is coarsely proportional to string length, we see that low-energy means short strings that do not venture far from the brane. On the other hand, we saw that spacetime geometry near the brane is that of an $\mathrm{AdS}_5 \times S^5$. So it seems that as long as the geometry is anti de Sitter, the open strings bound to the branes form a SYM theory. What if the space is made to have an AdS_5 geometry throughout, not just near the brane? This idea, combined with the fact that type IIB string theory on an $\mathrm{AdS}_5 \times S^5$, with the S^5 compactified, has a set of symmetries that is isomorphic to those of the $\mathcal{N}=4$ SYM theory, leads to the Maldacena conjecture [1, 2, 3]:

Type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ background with N units of five-form flux through the sphere S^5, is exactly equivalent, or dual, to the $\mathcal{N}=4$ super Yang-Mills theory on a four dimensional flat Minkowski space, with the gauge group $SU(N)$. The identification between the two theories is defined by the equation

$$\langle \exp \int \phi_0 \mathcal{O} \rangle_{SYM} = Z[\phi(\vec{x},0) = \phi_0(\vec{x})]_{\mathrm{String}}, \tag{5.1}$$

where $\mathcal{O}$ is an operator of the SYM, the expectation value on the left is with respect to the SYM action, z is the radial coordinate such that the boundary is at $z=0$, $\vec{x}$ represents the rest of the coordinates and therefore also parametrizes the boundary (for example in the fashion described in section 3.3), $\phi(\vec{x},z)$ is a field of the string theory that is dual to the operator $\mathcal{O}$, $\phi_0(\vec{x})$ is an arbitrary prescribed field on the boundary and $Z[\phi(\vec{x},0) = \phi_0(\vec{x})]_{\mathrm{String}}$ is the partition function of the string theory, with the constraint that the field ϕ must tend to ϕ_0 on the boundary. The relation between the couplings of the theories is

$$g_s = g_{YM}^2, \quad L^4 = 4\pi g_s N \alpha'^2, \tag{5.2}$$

where g_s is the string coupling constant, g_{YM} is the Yang-Mills coupling, N is the rank of the gauge group $SU(N)$ or the quantized amount of 5-form flux on the S^5, $\alpha' = \frac{1}{2\pi T} = l^2/2$ where l is the zero-mode length of the string and T is the string tension, and finally L^2 is the radius of both the S^5 and the AdS_5.

This prescription allows us to find the expectation values of the operator $\mathcal{O}$ from the string theory partition function by taking functional derivatives with respect to ϕ_0 on both sides of Eq. (5.1). The ϕ_0 field acts as a source term on field theory side, such that each functional derivative brings down a factor of $\mathcal{O}$. Setting the ϕ_0 field to zero in the end then recovers the operator expectation value:

$$\begin{aligned} \langle \mathcal{O}(\vec{x}_1)\dots\mathcal{O}(\vec{x}_n)\rangle &= \left[\frac{\delta}{\delta\phi_0(\vec{x}_1)}\cdots\frac{\delta}{\delta\phi_0(\vec{x}_n)}\int \mathrm{d}X \exp(\int \phi_0\mathcal{O})\exp(i\int \mathcal{L}_{\mathrm{SYM}}(X))\right]\Big|_{\phi_0=0} \\ &= \left[\frac{\delta}{\delta\phi_0(\vec{x}_1)}\cdots\frac{\delta}{\delta\phi_0(\vec{x}_n)}\int_G \mathrm{d}Y \exp(S_{\mathrm{String}}(Y))\right]\Big|_{\phi_0=0}, \end{aligned} \tag{5.3}$$

where $G = \{\phi(\vec{x},z) | \lim_{z\to 0}\phi(\vec{x},z) = \phi_0(\vec{x})\}$ is the set of functions over which the integrations range and $\mathrm{d}X$ and $\mathrm{d}Y$ stand for functional integration over all fields of the SYM and string theories, respectively. Notice that the functional derivative on the right is with respect to a variable that defines the integration domain, which may not be so easy to define generally. Practical calculations involve the classical approximation, though, and in this case the functional integral reduces to picking out the solution that minimizes the action, so one may simply solve the classical problem for a general ϕ_0 and then take the functional derivative as usual. Also one or both of the integrals may be divergent in the general case, but this is to be expected since a QFT usually requires renormalization. If the conjecture holds true, then renormalization on one side should somehow define the renormalization on the other side. This is addressed in [24].

Notice that in this formulation, the D3 brane that inspired the conjecture has been entirely left out. The SYM theory lives on the boundary of the AdS_5 space, which contains no branes at all. Only the geometry near the brane has been taken as an ingredient of the duality.

The part left to define in order to make concrete calculations is the correspondence between operators on the SYM side and fields on the string side (or AdS side, which we will use as a more generic term once we start approximating the string theory with various low-energy descriptions). Right away we see that the fields and their corresponding operators must have conjugate quantum numbers in order to form singlets for the functional integral. Especially the product of the scaling dimensions of the boundary field ϕ_0 and the boundary operator $\mathcal{O}$ must be $d-1$, so that the action is invariant with respect to scaling.

5.1 Comparing symmetries

The first evidence for the Maldacena conjecture comes from the symmetries of the two theories. First of all, both have the same conformal symmetry, since the SYM lives natively on four dimensional Minkowski space, and is a conformal theory, whereas the boundary of AdS_5 has only a conformal structure (as opposed to having a complete metric structure) as we have shown in section 3.4, and therefore any theory defined on the boundary must have four dimensional conformal symmetry. The string theory also has extra dimensions that form an S^5, which gives it an $SO(6)$ symmetry. The corresponding symmetry on the SYM side is the $SU(4) \sim SO(6)$ R-symmetry rotating the supercharges. Finally both theories have 16 unbroken supersymmetries (the background metric breaks half of the 32 supersymmetries of the string theory). These groups combine to a $SU(2,2|4)$ supergroup on both sides.

5.2 The large N and large 't Hooft coupling limits

From Eq. (5.2) we see that the 't Hooft coupling is

$$\lambda = g_s N. \tag{5.4}$$

Now the limit with λ constant and $N \to \infty$ corresponds to $g_s \to 0$. Therefore on this limit only the first term in a loop expansion of string theory survives and we have a classical string theory. On the other side we have the large N limit of the full quantum SYM theory. On this limit the duality is between a classical string theory and a fully quantum mechanical field theory!

We also see from Eq. (5.2) that

$$\lambda = \frac{L^4}{4\pi\alpha'^2}, \tag{5.5}$$

and hence that the strong 't Hooft coupling limit $\lambda \to \infty$ with the radius L kept constant corresponds to the small α' limit on the string theory side. Since the masses of the string theory non-ground state excitations are proportional to α', this is the massless limit of string theory, which is the corresponding supergravity theory.

Combining both of these limits gives us a method to calculate observables in the large N and strong coupling regime of the $\mathcal{N} = 4$ SYM theory by solving problems in classical supergravity. Since the strong coupling limit of field theories has previously been accessible only to numerical lattice calculations, this is a very powerful result.

Fig. 1 is a schematical depiction of various limits of the string theory with respect to the variables of its dual field theory.

We can also conjecture the duality on any of these limits individually. One can then consider if any of the following conjectures are true:

1. $\mathcal{N} = 4$ SYM is dual to Type IIB string theory on $\mathrm{AdS}_5 \times S^5$.
2. The large 't Hooft coupling limit of $\mathcal{N} = 4$ SYM is dual to Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$.
3. The large N limit of $\mathcal{N} = 4$ SYM is dual to classical Type IIB string theory on $\mathrm{AdS}_5 \times S^5$.
4. The simultaneous large N and large 't Hooft coupling limit of $\mathcal{N} = 4$ SYM is dual to classical Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$.

As we have established, conjecture 1 implies all the rest and conjectures 2 and 3 together imply 4. Except for symmetry arguments, most direct evidence in favour of the duality is actually evidence only for conjecture 4. This is because that is the only limit where explicit calculations have been made.

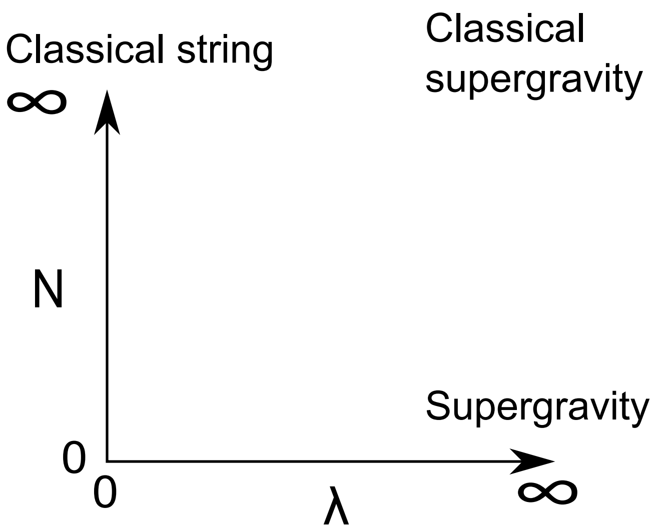

Figure 1: A schematic on the various limits of the string theory dual to the $\mathcal{N} = 4$ super Yang-Mills with respect to SYM parameters.

5.3 The classical supergravity limit

On the supergravity limit, the action on the right hand side of Eq. (5.1) is that of type IIB supergravity. If we go at the same time to the classical limit, the functional integral picks out the classical solution only, which is the one that minimizes the action. We have then

$$\langle \exp(\int \mathcal{O}\phi_0) \rangle = \exp i \int \mathcal{L}_{\mathrm{Sugra}}(\phi(\phi_0)), \tag{5.6}$$

where $\phi(\phi_0)$ is the classical supergravity solution expressed as a function of the boundary field ϕ_0.

In general, finding the classical solution with arbitrary boundary conditions is still a formidable task. We will instead attempt at first to find a solution to the simpler problem of free fields in the AdS_5 background with the given boundary conditions. Since the equation of motion for a free field is linear, it can be written in the general form $D_x\phi(x) = 0$, where D_x is a linear differential operator acting on the coordinates $x = (\vec{x}, z)$ and $\phi(x)$ is the field. If we then have a function $K(\vec{y}, \vec{x}, z)$ such that

$$\begin{aligned} D_x K(\vec{y}, \vec{x}, z) &= 0, \quad \text{when } z \neq 0, \\ \lim_{z\to 0} K(\vec{y}, \vec{x}, z) &= \delta(\vec{x} - \vec{y}), \end{aligned}$$

we can construct a solution with the boundary values $\phi_0(\vec{y})$ by decomposing it as

$$\phi(x) = \int \mathrm{d}\vec{y}\ \phi_0(\vec{y}) K(\vec{y}, \vec{x}, z) \tag{5.7}$$

so that

$$\lim_{z\to 0} \phi(\vec{x}, z) = \int \mathrm{d}\vec{y}\ \phi_0(\vec{y}) \lim_{z\to 0} K(\vec{y}, \vec{x}, z) = \int \mathrm{d}\vec{y}\ \phi_0(\vec{y}) \delta(\vec{y} - \vec{x}) = \phi_0(\vec{x}) \tag{5.8}$$

and

$$D_x \phi(\vec{x}, z) = D_x \int \mathrm{d}\vec{y}\ \phi_0(\vec{y}) K(\vec{y}, \vec{x}, z) = \int \mathrm{d}\vec{y}\ \phi_0(\vec{y}) D_x K(\vec{y}, \vec{x}, z) = 0. \tag{5.9}$$

This is of course just the familiar method of Greens functions, except now with the twist that the inhomogenic term is defined on the boundary instead of the bulk space.

5.3.1 Massless scalar fields

The simplest case is that of a free massless scalar field. In a background with the metric $g_{\mu\nu}$, the equation of motion is

$$D_\mu D^\mu \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \partial^\mu \phi(x) \right) = 0. \tag{5.10}$$

If we pick the Poincaré coordinates as defined in Eq. (3.24), denoting $\xi^0 = z, \vec{\xi} = \vec{x}$, the metric tensor becomes $g_{\mu\nu} = \frac{1}{z^2}\eta_{\mu\nu}$, $g = \frac{1}{z^{2d}}$, $g^{\mu\nu} = z^2 \eta^{\mu\nu}$. The easiest way to derive the propagator is to use symmetries. The conformal Minkowski space on the boundary of AdS_5 is compactified by a single point P at infinity, so we may first look for the propagator which produces a delta function on that point. The propagator cannot depend on the $\vec{x}$ coordinate due to translation invariance, so Eq. (5.10) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{z^{d-2}} \frac{\mathrm{d}}{\mathrm{d}z} K(z) = 0. \tag{5.11}$$

The ansatz $K(z) = cz^\alpha$ then leads to $\alpha(\alpha + 1 - d) = 0$. The solution $\alpha = 0$ is constant and cannot have a delta function singularity at infinity. The propagator for this special case is then

$$K(\infty, \vec{x}, z) = cz^{d-1}, \tag{5.12}$$

where c is a normalization constant. The conformal invariance of the boundary theory allows us to find the propagator with a delta function at the origin. The mapping

$$z \mapsto \frac{z}{x^2}, \quad \vec{x} \mapsto \frac{\vec{x}}{x^2} \tag{5.13}$$

where $x^2 = z^2 + \vec{x}^2$, induces the mapping

$$\vec{x} \mapsto \frac{\vec{x}}{(z^2 + \vec{x}^2)^2} \tag{5.14}$$

on the boundary, which is a conformal transformation and also maps the point at infinity to the origin (here we enforced the condition that $u = 1/z = 1$ to fix the coordinates on the boundary, as in section 3.3). It is also an isometry of AdS_d. Let $a, b \in T_x\mathrm{AdS}_d$ and $x' = x/x^2$. Then the tangent vectors transform as

$$a'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} a^{\nu} = \frac{a^{\mu} x^2 - 2x \cdot a x^{\mu}}{x^4} \tag{5.15}$$

and, denoting $a^{\mu} b^{\nu} \eta_{\mu\nu} = a \cdot b$, their inner product becomes

$$g'_{\mu\nu} a'^{\mu} b'^{\nu} = \frac{1}{(z/x^2)^2} \left(\frac{a \cdot b \; x^4}{x^8} \right) = \frac{1}{z^2} a \cdot b = g_{\mu\nu} a^{\mu} b^{\nu}. \tag{5.16}$$

In the embedding space, this corresponds to the reflection $y^0 \mapsto y^0, y^{\alpha} \mapsto y^{\alpha}, y^n \mapsto -y^n$, as can be seen by solving for the y coordinates in terms of the set $(z, \vec{x})$ and the AdS_d condition Eq. (3.21).

Applying the transformation to Eq. (5.12) gives

$$K(0, \vec{x}, z) = c \frac{z^{d-1}}{(z^2 + \vec{x}^2)^{d-1}} \tag{5.17}$$

That this satisfies the Laplace equation in the bulk can be verified by direct calculation. The delta function property can be seen as follows: when $\vec{x} \neq 0$, the propagator clearly converges to zero as $z \to 0$. On the other hand, the integral over the coordinates $\vec{x}$ is independent of z, since

$$\int \mathrm{d}^{d-1}\vec{x} \; \frac{z^{d-1}}{(z^2 + \vec{x}^2)^{d-1}} = \int \mathrm{d}^{d-1}\vec{x}' z^{d-1} \; \frac{z^{d-1}}{(z^2 + z^2\vec{x}'^2)^{d-1}} = \int \mathrm{d}^{d-1}\vec{x}' \; \frac{1}{(1 + \vec{x}'^2)^{d-1}} = \frac{1}{c}, \tag{5.18}$$

where the last equality defines the normalization constant c introduced above. Therefore the integral over any set including the point $\vec{x} = 0$ goes to $1/c$ as z goes to zero, so the propagator indeed becomes a delta function at this limit. Now an application of translation invariance on the boundary gives the full propagator:

$$K(\vec{y}, \vec{x}, z) = c \frac{z^{d-1}}{(z^2 + (\vec{x} - \vec{y})^2)^{d-1}} \tag{5.19}$$

and the classical solution is

$$\phi(x) = c \int \mathrm{d}\vec{y} \phi_0(\vec{y}) \frac{z^{d-1}}{(z^2 + (\vec{x} - \vec{y})^2)^{d-1}}. \tag{5.20}$$

We now have enough data to form the n-point functions of the operator corresponding to the scalar field. The scalar action is

$$I(\phi_0) = -\frac{1}{2} \int \mathrm{d}^d x \; \sqrt{g} \partial_{\mu} \phi \partial^{\mu} \phi = -\frac{1}{2} \int \mathrm{d}^d x \; \left\{ \partial_{\mu} \left(\sqrt{g} \phi \partial^{\mu} \phi \right) - \phi \partial_{\mu} (\sqrt{g} \partial^{\mu} \phi) \right\}. \tag{5.21}$$

The last term vanishes by the equation of motion. The total derivative term becomes a surface integral, which is non-zero at the boundary $z \to 0$. Setting $z = \epsilon$ for now to suppress divergences, we have

$$I(\phi_0) = -\frac{1}{2} \int_{z=\epsilon} \mathrm{d}\vec{x} \; z^{-d+2} \phi(\vec{x}, z) \partial_z \phi(\vec{x}, z). \tag{5.22}$$

Evaluating the derivative of ϕ and using the fact that z is small, so that we may pick only the terms proportional to the highest power of z, gives

$$\lim_{z \to 0} \partial_z \phi(\vec{x}, z) = c(d-1) z^{d-2} \int \mathrm{d}^{d-1}\vec{y} \; \phi_0(\vec{y}) \frac{1}{(\vec{x} - \vec{y})^{2d-2}}. \tag{5.23}$$

The $\phi(\vec{x}, z)$ term in the action becomes simply $\phi_0(\vec{x})$ on the limit $z \to 0$, and the action becomes

$$I(\phi_0) = -\frac{c(d-1)}{2} \int \mathrm{d}^{d-1}\vec{x} \; \mathrm{d}^{d-1}\vec{y} \; \frac{\phi_0(\vec{x}) \phi_0(\vec{y})}{(\vec{x} - \vec{y})^{2(d-1)}}, \tag{5.24}$$

which is non-singular, so we have taken the limit $z \to 0$. Inserting this into the master formula then gives

$$\begin{aligned}
\langle \mathcal{O}(\vec{x}')\mathcal{O}(\vec{y}')\rangle &= \left[\frac{\delta}{\delta\phi_0(\vec{x}')}\frac{\delta}{\delta\phi_0(\vec{x}')}\exp\left(\frac{c(d-1)}{2}\int \mathrm{d}^{d-1}\vec{x}\ \mathrm{d}^{d-1}\vec{y}\ \frac{\phi_0(\vec{x})\phi_0(\vec{y})}{(\vec{x}-\vec{y})^{2(d-1)}}\right)\right]\Bigg|_{\phi_0=0} \\
&= \frac{c(d-1)}{(\vec{x}'-\vec{y}')^{2(d-1)}}.
\end{aligned} \tag{5.25}$$

This is exactly the result for the 2-point function of a conformal operator of scaling dimension $d-1$, which is fixed in the field theory side by conformal invariance [11]. As expected, the propagator derived by a classical calculation is indeed characteristic of a quantum theory.

The 3-point function is zero, but the 4-point function is

$$\begin{aligned}
\langle \mathcal{O}(\vec{x}')\mathcal{O}(\vec{y}')\mathcal{O}(\vec{z}')\mathcal{O}(\vec{w}')\rangle &= c^2(d-1)^2\left(\frac{1}{(\vec{z}'-\vec{x}')^{2(d-1)}}\frac{1}{(\vec{y}'-\vec{w}')^{2(d-1)}}\right. \\
&\quad +\frac{1}{(\vec{y}'-\vec{x}')^{2(d-1)}}\frac{1}{(\vec{z}'-\vec{w}')^{2(d-1)}} \\
&\quad \left.+\frac{1}{(\vec{z}'-\vec{y}')^{2(d-1)}}\frac{1}{(\vec{x}'-\vec{w}')^{2(d-1)}}\right),
\end{aligned} \tag{5.26}$$

which is simply the 4-point function of a free field theory, as expected.

5.3.2 Massless abelian gauge fields

We will now compute the propagator for a massless abelian gauge field A_μ with the equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0, \tag{5.27}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.28}$$

as usual. In this section Greek indices go through the full set $\{z, \vec{x}\}$, with 0 being the z –component, and Latin indices run over components of $\vec{x}$. The gauge field actually defines a 1-form $A = A_\mu dx^\mu$ in d dimensions. The boundary limit of the 1-form must be a 1-form in $d-1$ dimensions, which means that the radial component proportional to dz must vanish. We are then looking for a set of gauge propagators $K^i(\vec{y}, \vec{x}, z)$ such that

$$\lim_{z\to 0} K^i(\vec{y}, \vec{x}, z) = \delta(\vec{x}-\vec{y})dx^i, \tag{5.29}$$

that is, the propagator must tend to a delta function on the ith component and to zero on all other components. We use again the trick of constructing first the propagator where the delta is at infinity, and argue that it must again be independent of other components than z:

$$K^i(\infty, \vec{x}, z) = f(z)\mathrm{d}x^i. \tag{5.30}$$

Notice that the index of K denotes the component of the boundary field that we want to be non-zero. This function must satisfy Eq. (5.27). We calculate

$$\begin{aligned}
F_{0i} &= \partial_z f(z) = f'(z), \\
\sqrt{g}F^{0i} &= z^{4-d}f'(z), \\
\partial_z(\sqrt{g}F^{0i}) &= (4-d)z^{3-d}f'(z) + z^{4-d}f''(z) = 0.
\end{aligned} \tag{5.31}$$

Clearly if $f(z) = z^{\alpha}$, then both terms in Eq. (5.31) are of the same power, which gives a constraint on α. We have

$$\begin{aligned} \alpha(\alpha+3-d)z^{2-d+\alpha} &= 0 \Rightarrow \alpha = d-3 \\ K_i(\infty, \vec{x}, z) &= z^{d-3}\mathrm{d}x^i. \end{aligned} \tag{5.32}$$

Application of the same inversion as in the previous section gives the propagator to the origin of the boundary:

$$\begin{aligned} x^\mu &\mapsto \frac{x^\mu}{z^2+\vec{x}^2} \\ \Rightarrow K_i(0,\vec{x},z) &= \left(\frac{x^i}{z^2+\vec{x}^2}\right)^{d-3} \mathrm{d}\left(\frac{x^i}{z^2+\vec{x}^2}\right) \\ &= \frac{z^{d-3}x^i}{(z^2+\vec{x}^2)^{d-3}}\mathrm{d}\left(\frac{1}{z^2+\vec{x}^2}\right) + \frac{z^{d-3}}{(z^2+\vec{x}^2)^{d-2}}\mathrm{d}x^i \\ &= \mathrm{d}\left(\frac{z^{d-3}x^i}{(z^2+\vec{x}^2)^{d-2}}\right) - \frac{\mathrm{d}(z^{d-3}x^i)}{(z^2+\vec{x}^2)^{d-2}} \\ &\quad -\frac{z^{d-3}x^i}{(z^2+\vec{x}^2)}\mathrm{d}\left(\frac{1}{(z^2+\vec{x}^2)^{d-3}}\right) + \frac{z^{d-3}}{(z^2+\vec{x}^2)^{d-2}}\mathrm{d}x^i \\ &= \mathrm{d}\left(\frac{z^{d-3}x^i}{(z^2+\vec{x}^2)^{d-2}}\right) - \frac{\mathrm{d}(z^{d-3}x^i)}{(z^2+\vec{x}^2)^{d-2}} \\ &\quad -\frac{d-3}{d-2}z^{d-3}x^i\mathrm{d}\left(\frac{1}{(z^2+\vec{x}^2)^{d-2}}\right) + \frac{z^{d-3}}{(z^2+\vec{x}^2)^{d-2}}\mathrm{d}x^i \\ &= \frac{1}{d-2}\mathrm{d}\left(\frac{z^{d-3}x^i}{(z^2+\vec{x}^2)^{d-2}}\right) - \frac{1}{d-2}\frac{\mathrm{d}(z^{d-3}x^i)}{(z^2+\vec{x}^2)^{d-2}} + \frac{z^{d-3}}{(z^2+\vec{x}^2)^{d-2}}\mathrm{d}x^i \\ &= \frac{1}{d-2}\mathrm{d}\left(\frac{z^{d-3}x^i}{(z^2+\vec{x}^2)^{d-2}}\right) + \frac{d-3}{d-2}\frac{1}{(z^2+\vec{x}^2)^{d-2}}\left(-z^{d-4}x^i\mathrm{d}z + z^{d-3}\mathrm{d}x^i\right). \end{aligned}$$

The first term on the final line is a pure gauge term, which we transform away. Using translation invariance and dropping the constant factor, the final propagator is

$$K_i(\vec{y},\vec{x},z) = \frac{z^{d-3}\mathrm{d}x^i - z^{d-4}(x^i-y^i)\mathrm{d}z}{(z^2+(\vec{x}-\vec{y})^2)^{d-2}} \tag{5.33}$$

and a general 1-form gauge field $A_0(\vec{x}) = a_i(\vec{x})\mathrm{d}x^i$ on the boundary is then generated by

$$A_0(\vec{x}) = \int \mathrm{d}\vec{y}\; a_i(\vec{y})\frac{z^{d-3}\mathrm{d}x^i - z^{d-4}(x^i-y^i)\mathrm{d}z}{(z^2+(\vec{x}-\vec{y})^2)^{d-2}}. \tag{5.34}$$

The action is

$$S = \int_{\mathrm{AdS}_d} F\wedge *F = \int_{\mathrm{AdS}_d} \mathrm{d}A\wedge *F = \int_{\mathrm{AdS}_d} \mathrm{d}(A\wedge *F) - \int_{\mathrm{AdS}_d} F\wedge \mathrm{d}*F = \int_{\partial\mathrm{AdS}_d} A\wedge *F, \tag{5.35}$$

where we used the equation of motion in the form language, $\mathrm{d}*F = 0$. The unit normal to the boundary in our choice of coordinates is $n^\mu = (-z, 0, \dots, 0)$. Since, on the boundary, the 1-form A does not have a component in the z –direction, and the unit normal has a non–zero component only in the z –direction, a little index gymnastics reveals that the action becomes

$$S = \int \mathrm{d}\vec{x}\; A^i F_{0i} n^0. \tag{5.36}$$

The field strength tensor is

$$\begin{aligned}
F_{0i} &= \int \mathrm{d}\vec{y} \left[a_i(\vec{y}) \frac{(d-3)z^{d-4}}{(z^2+(\vec{x}-\vec{y})^2)^{d-2}} - a_i(\vec{y}) \frac{2(d-2)z^{d-2}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1}} \right. \\
&\quad \left. + a_i(\vec{y}) \frac{z^{d-4}}{(z^2+(\vec{x}-\vec{y})^2)^{d-2}} - a_j(\vec{y}) \frac{2(d-2)(x^j-y^j)(x_i-y_i)z^{d-4}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1}} \right] \\
&= \int \mathrm{d}\vec{y} \left[a_i(\vec{y}) \frac{(d-2)z^{d-4}}{(z^2+(\vec{x}-\vec{y})^2)^{d-2}} - a_i(\vec{y}) \frac{2(d-2)z^{d-2}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1}} \right. \\
&\quad \left. - a_j(\vec{y}) \frac{2(d-2)(x^j-y^j)(x_i-y_i)z^{d-4}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1}} \right]. \qquad (5.37)
\end{aligned}$$

Inserting this, the square root of the metric determinant and the gauge field A with the $g^{\mu\nu}$that raises its index, and taking $z \to 0$ gives

$$S \propto \int \mathrm{d}\vec{x}\ \mathrm{d}\vec{y} \sum_{i,j} a_i(\vec{x}) a_j(\vec{y}) \left[\frac{\delta^{ij}}{(\vec{x}-\vec{y})^{2d-4}} - \frac{2(x^i-y^i)(x^j-y^j)}{(\vec{x}-\vec{y})^{2d-2}} \right]. \qquad (5.38)$$

Both the a_i and a_j have a lower index even though they are summed over because we used the $g^{\mu\nu}$ to cancel a z^2.

Exponentiating and taking two functional derivatives gives the two-point function for the corresponding operator $J^i(\vec{x})$, which is a conserved current:

$$\langle J^i(\vec{x}) J^j(\vec{y}) \rangle = \frac{\delta^{ij}}{(\vec{x}-\vec{y})^{2d-4}} - \frac{2(x^i-y^i)(x^j-y^j)}{(\vec{x}-\vec{y})^{2d-2}}. \qquad (5.39)$$

The current conservation is easily demonstrated:

$$\begin{aligned}
\partial_i \langle J^i(\vec{x}) J^j(\vec{y}) \rangle &= -\frac{2(d-2)(x^j-y^j)}{(\vec{x}-\vec{y})^{2d-2}} \\
&\quad - \frac{2d(x^j-y^j)(\vec{x}-\vec{y})^2 - 4(d-1)(x^i-y^i)(x^i-y^i)(x^j-y^j)}{(\vec{x}-\vec{y})^{2d-4}} = 0. \qquad (5.40)
\end{aligned}$$

Since we are still dealing with a free theory, the higher n-point functions are naturally zero for odd n and the usual combinations of the 2-point function for even n.

5.3.3 Massive fields

In the case of massive fields, the conformal invariance of the boundary becomes apparent. In deriving the propagator, the equation of motion now has a mass term on the right side, so that the analog of Eq. (5.11) is solved by

$$K(\infty, \vec{x}, z) = c z^{d-1+\lambda_\pm}, \qquad (5.41)$$

where $\lambda_\pm$ are solutions of $\lambda(d-1+\lambda) = m^2$,

$$\lambda_+ = \frac{1}{2}(-(d-1) + \sqrt{(d-1)^2+4m^2}), \lambda_- = \frac{1}{2}(-(d-1) - \sqrt{(d-1)^2+4m^2}). \qquad (5.42)$$

Out of these only the solution corresponding to λ_+ tends to infinity at $z \to \infty$. The same procedures as in the massless case then lead to the propagator

$$K(\vec{y}, \vec{x}, z) = c \frac{z^{d-1+\lambda_+}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1+\lambda_+}}. \qquad (5.43)$$

Unfortunately, this does not tend to a delta function on the boundary. On the other hand, if the numerator were $z^{d-1+2\lambda_+}$, it would. If we extract this term, we find that

$$\lim_{z\to 0} z^{-\lambda_+} c \int \mathrm{d}\vec{y} \frac{z^{d-1+2\lambda_+}}{(z^2+(\vec{x}-\vec{y})^2)^{d-1+\lambda_+}} \phi_0(\vec{y}) = \lim_{z\to 0} z^{-\lambda_+} \phi_0(\vec{x}). \tag{5.44}$$

The vanishing $z^{-\lambda_+}$ term is a consequence of the divergence of the metric at the boundary. A conformal rescaling while approaching the boundary allows us to recover a finite value for the field ϕ_0, but since the rescaling can be chosen in any way that gives a first order zero on the boundary, this means that we cannot actually specify the boundary field as a function. The field ϕ_0 must instead be specified as a conformal density of weight $-\lambda_+$. In order for the action to be invariant, the operator $\mathcal{O}$ coupling ϕ_0 must have conformal weight $d-1+\lambda_+$.

This is explicitly verified by calculating the 2-point function for $\mathcal{O}$, which proceeds in exactly the same way as in section 5.3.1. The result is

$$\langle \mathcal{O}(\vec{x}')\mathcal{O}(\vec{y}')\rangle = \frac{1}{(\vec{x}'-\vec{y}')^{2(d-1+\lambda_+)}}, \tag{5.45}$$

which is exactly the 2-point function for a conformal operator of weight λ_+, which is fixed by conformal invariance on the field theory side.

Similarily, the conformal weight of an operator coupling to a p-form field can be shown to be $d-1+\lambda_+-p$.

5.4 Interactions and Witten diagrams

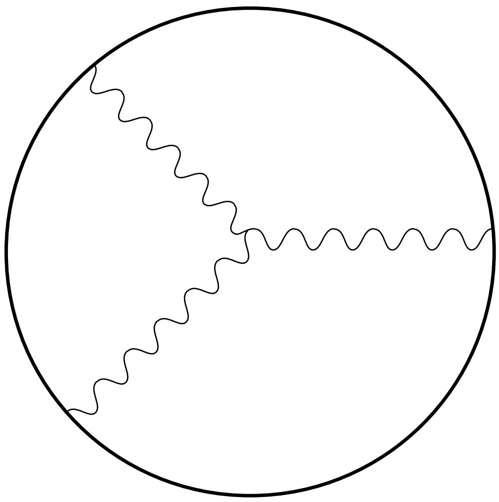

Figure 2: A Witten diagram representing the 3-point function of an abelian gauge field. The circle represents the conformal one-point compactified boundary of AdS, and the interior of the circle is the bulk space.

So far our calculations have only dealt with free fields of the bulk theory, which of course yield free fields on the boundary. The more interesting case of interactions can be addressed in the very same way as in an ordinary field theory, by expanding the exponential of the interaction part of the action in a power series. This then generates vertex rules and propagators analogously to Feynman graphs, with the major exception that external particles propagate to and from the boundary of the AdS spacetime. Bulk-to-bulk propagators and all vertices are those of the bulk theory. The resulting expansions may be graphically depicted as Witten diagrams, see Fig. 2. Quantum corrections to the classical supergravity approximation may be calculated by adding loops to the diagrams. This generates $\frac{1}{N}$ corrections to the boundary theory.

5.5 Finite temperature and the AdS black hole

The most obvious modification to the duality is to consider different backgrounds in the bulk. A simple alternative choice is to take a space which is asymptotically AdS_5 which conserves the conformal symmetry of the boundary, but that has a black hole in the bulk. The simplest black hole configuration has the metric

$$ds^2 = \frac{L^2}{z^2}\left[-(1-\frac{z^4}{z_0^4})dt^2 + d\vec{x}^2 + \frac{dz^2}{1-\frac{z^4}{z_0^4}}\right]. \tag{5.46}$$

It turns out that this metric induces a finite temperature on the field theory side. The temperature is defined by going to periodic Euclidean time and determining the period β such that the metric is free of singularities. Such a period is $\beta = \pi z_0$. By standard finite temperature field theory arguments, this corresponds to the temperature $T = 1/\beta = \frac{1}{\pi z_0}$ [25].

The coordinate transform $z \mapsto L^2/r$ leads to the other commonly used form of the metric

$$ds^2 = \frac{r^2}{L^2}\left(-\left(1-\frac{r_0^4}{r^4}\right)dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}\left(1-\frac{r_0^4}{r^4}\right)^{-1} dr^2, \tag{5.47}$$

where the horizon radius $r_0 = \frac{L^2}{z_0}$. The relationship between temperature and horizon radius in these coordinates is then $T = \frac{r_0}{\pi L^2}$. This form of the metric is the same as the near-brane form of the brane solution we developed in section 4.4.

Most of the interesting results in AdS/CFT are actually calculations in this finite temperature formalism.

6 Applications of the duality

In this section we will review, with varying degrees of detail, some applications of the AdS/CFT duality and related dualities. This list is simply a choice of a few topics that are interesting and suitable for inclusion here, and is far from complete.

Many of the applications do not actually directly use the prescription given in the previous section, but instead rely on general arguments about the behaviour of fields in the bulk coupling to operators on the boundary.

6.1 The stress-energy tensor

The field coupling to the stress-energy tensor of the SYM is the metric tensor of the AdS -space. While the stress-energy tensor could be evaluated directly by the master formula Eq. (5.1), it is easier to derive it from the more general holographic renormalization formalism presented in [24].

We compute the stress-energy tensor of the finite temperature $\mathcal{N} = 4$ SYM. The field dual to the stress-energy tensor in the supergravity side of the duality is the metric tensor. The first step is to convert the metric to the Graham-Fefferman coordinate system

$$ds^2 = \frac{1}{z^2}(dr^2 + g_{\mu\nu}dx^\mu dx^\nu) \tag{6.1}$$

which isolates the singularity on the boundary, and also shows explicitly the representative metric $g_{\mu\nu}$ (of the conformal class of metrics on the boundary) induced by this specific form of the metric in the bulk [26]. The metric Eq. (5.46) can be brought to this form by the transformation [27]

$$z = \frac{w}{\sqrt{1 + \frac{w^4}{4z_0^4}}}. \tag{6.2}$$

Since we are interested in the finite temperature case, we take the signature to be Euclidean, and have

$$ds^2 = \frac{L^2}{w^2}\left(dz^2 + \frac{(1-\frac{w^4}{4z_0^4})^2}{1+\frac{w^4}{4z_0^4}}dt^2 + (1+\frac{w^4}{4z_0^4})d\vec{x}^2\right). \tag{6.3}$$

The expectation value of the boundary stress-energy tensor is then given by [24, 28]

$$\langle T_{ij}\rangle = \frac{L^3}{4\pi G_5}\left[g_{(4)ij} - \frac{1}{8}g_{(0)ij}\left((\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g_{(2)}^2\right) - \frac{1}{2}(g_{(2)}^2)_{ij} + \frac{1}{4}g_{(4)ij}\mathrm{Tr}\ g_{(2)}\right], \tag{6.4}$$

where $g_{(n)}$ is the term with n derivatives in the Taylor expansion of the boundary metric $g_{\mu\nu}$, and G_5 is the five-dimensional Newton's constant. Expanding the non-radial part of Eq. (6.3) gives

$$g_{\mu\nu}dx^\mu dx^\nu = (1 - \frac{3w^4}{4z_0^4})dt^2 + (1+\frac{w^4}{4z_0^4})d\vec{x}^2 + O(\frac{w^8}{z_0^8}). \tag{6.5}$$

The ten dimensional coupling constant in terms of the string theory parameters in $\mathrm{AdS}_5\times S^5$ is [29]

$$16\pi G_{10} = (2\pi)^7\alpha'^4 g_s^2 = \frac{8L^8\pi^5}{N^2}. \tag{6.6}$$

We can relate this to the five-dimensional Newton's constant by writing the ten dimensional gravitational action and using the fact that the S^5 part factorizes:

$$\frac{1}{16\pi G_{10}}\int \mathrm{d}^{10}x\sqrt{G} = \frac{1}{16\pi G_{10}}\int \mathrm{d}^5x\sqrt{g}\int \mathrm{d}^5y\sqrt{\gamma} = \frac{L^5\pi^3}{16\pi G_{10}}\int \mathrm{d}^5x\sqrt{g} = \frac{1}{16\pi G_5}\int \mathrm{d}^5x\sqrt{g}. \tag{6.7}$$

This gives the factor in front of Eq. (6.4) as

$$\frac{L^3}{G_5} = \frac{2N^2}{\pi}. \tag{6.8}$$

Working out the factors for the spatial components gives

$$T_{xx} = \frac{L^3}{4\pi G_5}\frac{1}{4z_0^4} = \frac{N^2}{2\pi^2}\frac{\pi^4 T^4}{4} = \frac{N^2\pi^2 T^4}{8}. \tag{6.9}$$

The time-like component has an extra factor of three, and the stress-energy tensor becomes

$$\langle T_{\mu\nu}\rangle = \begin{pmatrix} -\frac{3N^2\pi^2T^4}{8} & 0 & 0 & 0 \\ 0 & \frac{N^2\pi^2T^4}{8} & 0 & 0 \\ 0 & 0 & \frac{N^2\pi^2T^4}{8} & 0 \\ 0 & 0 & 0 & \frac{N^2\pi^2T^4}{8} \end{pmatrix}. \tag{6.10}$$

This result is consistent with the fact that the boundary theory is conformal: the stress-energy tensor is traceless.

The numerical factor in the components of the tensor is interesting. In the weak coupling limit, the ideal gas approximation to the pressure is

$$T_{xx} = p(T) = (g_B + \frac{7}{8}g_F)\frac{\pi^2}{90}T^4 = (8+7)(N^2-1)\frac{\pi^2}{90}T^4 = \frac{\pi^2(N^2-1)T^4}{6}, \tag{6.11}$$

where g_B and g_F are the number of bosonic and fermionic degrees of freedom, respectively. There are six scalars and two vectors in the bosonic sector, and four fermions with four antifermions, all multiplied by the number of degrees of freedom N^2-1 coming from the adjoint representation of the gauge group. In the large-N limit this expression has a factor of $3/4$ difference to the gravitational result. This shows that the strong coupling (to which the classical approximation of the AdS_5 supergravity is dual to) correction to the stress-energy tensor is simply $3/4$. The same factor also relates the energy density T_{tt} in the weak and strong coupling limits, which obviously must hold already in order to ensure the tracelessness of the energy momentum tensor.

6.2 Entropy

According to the Bekenstein-Hawking -conjecture, the entropy of a black hole is $\frac{A}{4G_N}$, where A is the area of the event horizon [30]. In context of the AdS/CFT duality, the entropy thus associated to the finite-temperature AdS_5 black hole should then be same as that of the field theory, since if they indeed are dual they surely must have the same number of fundamental degrees of freedom. The entropy becomes [27]

$$S_{\mathrm{BH}} = \frac{A}{4G_5} = \frac{z_0^{-3}V_3N^2}{4\cdot\frac{\pi}{2}} = \frac{\pi^2}{2}N^2V_3T^3. \tag{6.12}$$

The entropy of a free, weakly coupled $\mathcal{N}=4$ SYM plasma is

$$S_{\mathrm{YM}} = \frac{2\pi^2}{3}N^2V_3T^3, \tag{6.13}$$

which again differs by the same factor $3/4$ as the pressure and energy density.

6.3 Shear viscosity

The shear viscosity of hot SYM plasma can be related hydrodynamically to the stress-energy tensor in the local rest frame. We consider the rest frame as the one where the three-momentum density locally disappears, $T_{i0} = 0$. Then the stress-energy tensor can be written as[9] [32, 31]

$$\begin{aligned} T_{ij} &= -\delta_{ij} p + \eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial^k u_k \right) + \zeta \delta_{ij} \partial^k u_k, \\ T_{00} &= -\epsilon = -3p. \end{aligned} \tag{6.14}$$

Here ζ is the bulk viscosity, η is the shear viscosity, p is the pressure and u_i is the local flow three-velocity. In a conformal field theory, the stress-energy tensor must be traceless, tr $T_{\mu\nu} = \epsilon - 3p - 3\zeta\partial_k u_k = 0$, so $\zeta = 0$.

The shear viscosity η is given by the Kubo formula [32, 31]:

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int \mathrm{d}t\mathrm{d}x e^{i\omega t} \langle [T_{xy}(t,x),\ T_{xy}(0,0)] \rangle = \lim_{\omega\to 0} \frac{1}{2\omega i} \left[G_A(\omega) - G_R(\omega) \right]. \tag{6.15}$$

On the other hand, the absorption cross-section of gravitons on the supergravity (large 't Hooft coupling) limit by a black three-brane is known to be related to the stress energy tensor on the brane. The connection is [33, 34]

$$\sigma(\omega) = \frac{8\pi G_{10}}{\omega} \int \mathrm{d}t\mathrm{d}x e^{i\omega t} \langle [T_{xy}(t,x),\ T_{xy}(0,0)] \rangle. \tag{6.16}$$

Therefore the shear viscosity is $\eta = \frac{1}{16\pi G_{10}} \sigma(0)$ (the value of $\sigma(0)$ is of course defined as a limit). The graviton absorption cross section on the $\omega \to 0$ classical gravity limit is simply the area of the black hole horizon [31],

$$\sigma(0) = \frac{\pi^3 L^8}{z_0^3}. \tag{6.17}$$

Using Eq. (6.6) we can then work out

$$L^8 = \frac{\kappa^2 N^2}{4\pi^5}, \tag{6.18}$$

where $\kappa^2 = 8\pi G_{10}$, and

$$\eta = \frac{\pi^3 L^8}{2\kappa^2 z_0^3} = \frac{\pi N^2 T^3}{8}. \tag{6.19}$$

Dividing this by the entropy density $s = S/V_3 = \frac{\pi^2 N^2 T^3}{2}$ gives the famous result

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{6.20}$$

A more general method for deriving this result and many other coefficients from the background metric of a gravitational dual of a theory is presented in [35] and [36, 37], where it is also shown that the ratio $\eta/s = 1/4\pi$ holds for all for all theories with a gravity dual in the large N, large λ limit, and more generally it is conjectured, based on string theoretical arguments, that it is a lower limit for all relativistic quantum field theories with a zero chemical potential [36]. Experiments at RHIC suggest that the ratio is quite close to this in the strongly coupled quark gluon plasma generated at the collisions, and it has even been suggested that the RHIC experiments may violate this conjectured bound, although the results are not yet conclusive [38]. In addition, the $\mathcal{N} = 2$ $Sp(2)$ gauge theory

[9] There is a difference in sign here compared to [31], because of our opposing sign convention for the metric, which also inverts the sign of the stress-energy tensor.

in four dimensions, which has an AdS_5 dual, has a negative curvature squared correction to the supergravity result, thus violating the bound [39]. In [40] it is shown that maintaining causality in a similar Gauss-Bonnet theory requires a new bound $\eta/s \geq \frac{16}{25}\frac{1}{4\pi}$. It is also suggested that some more subtle effect may cause the theory to break causality already when $\eta/s < \frac{1}{4\pi}$, thus reinstantiating the original viscosity bound.

6.4 AdS/QCD

In addition to the somewhat rigorous $\mathrm{AdS}_5 \times S^5/\ \mathcal{N} = 4$ SYM -duality, various phenomenological models proposing a holographic gravity dual for realistic QCD have been put forward. In order to approach QCD from a gravity dual viewpoint, one must break much of the symmetries of the original Maldacena scenario. This means breaking supersymmetry, conformality, introducing fields in the fundamental representation of the gauge group in addition to the adjoint fields in the $\mathcal{N} = 4$ SYM and introducing confinement. [41]

An alternative approach is the so-called 'bottom-up' approach where one picks a convenient five dimensional background metric and chooses a field content such that, when interpreted as a gravity dual to a four dimensional gauge theory by applying the master formula Eq. (5.1), some desired features of QCD are reproduced. [41]

6.4.1 Adding quarks

In the standard AdS/CFT scenario, all the particles are in the adjoint representation of the gauge group. This is understood in the brane interpretation of the duality, where the particle content of the SYM theory consists of various ways that open strings can end on the stack on N branes. Since each end of the string can end on any of the N branes, each particle has two indices $1..N$, and therefore lives in the adjoint representation.

In order to remove the other index, so that we can put a particle in the fundamental representation, we have to put one end of the string to somewhere not on the D3 brane. A number of technical requirements are also involved, so that the simplest method turns out to be adding a D7 brane, which we will call a flavor brane, in the $\mathrm{AdS}_5 \times S^5$ background, where the open strings can end. The D7 brane extends in the dimensions that the D3 brane does and in four more, such that the radial dimension is a part of the brane and that the S^5 part is broken to an S^4 (the slice enveloped in the D7 brane) and an S^2 (the part which is not contained in the D7). Adding more than one, say N_f, D7 branes, gives the string an index $1..N_F$ which corresponds to having several quark flavors. In order to keep strings extending from the stack of D7 branes back to itself from forming a $SU(N_f)$ gauge theory, those modes must be decoupled. The coupling constant for them is $\lambda' = \lambda(2\pi l_s)^4 N_f/N$, which goes to zero when $N >> N_f$. Here l_s is the string length. [42]

Decoupling the D7 branes in this way is equivalent to ignoring the effects of the brane on itself and the surrounding geometry, so this is called the probe brane approximation. On the field theory side this corresponds to ignoring quark loops, or the so called quenched approximation. Steps have been taken to move beyond the probe approximation. [41]

We can give mass to the quarks by letting the D7 and D3 branes have some separation in the two dimensions orthogonal to both, since then there is a minimum length for the string between the flavor branes and the D3 gauge branes, so that the rest-energy of a quark will be the energy of the ground state of a string stretched between the branes.

The particle content of the theory will that of the $\mathcal{N} = 4$ SYM plus N_f fundamental $\mathcal{N} = 2$ hypermultiplets. Therefore the total supersymmetry of the theory is broken to $\mathcal{N} = 2$. [42]

6.4.2 Confinement

Since conformal theories cannot contain an energy scale, they cannot have a mass gap and therefore are not confining. In the context of the AdS/CFT duality this can also be calculated directly by considering the energy of two strings hung from the boundary towards the bulk AdS space. It turns out that the minimum of energy is reached when the two strings connect in the bulk, and the energy of that configuration relative to one where the strings stretch separately to infinity (one has to consider the difference here, since either energy alone is infinite) is

$$E = -\frac{4\pi^2(2g_{YM}^2 N)^{1/2}}{\Gamma(1/4)^4 L} \tag{6.21}$$

where L is the separation of the string endpoints on the boundary (quarks in field theory terms). The inverse proportionality, which was actually guaranteed by conformal symmetry, shows that the quarks tend to separate further, and are therefore indeed never confined. As the quark separation increases, the string dips deeper in to the bulk. [43]

Since the conformal symmetry of the boundary is generated from the AdS geometry, it is the geometry that needs to be modified to confine quarks. By inserting a block into the space at some finite z, we can make the energy increase as the separation increases. Consider, for simplicity, a hard wall at $z = z_0$ (i.e. some feature of the geometry such that the string configuration minimizing the world sheet area never dips below z_0). Once the quark separation becomes large enough that the string touches the wall it no longer has any choice but to extend along it, and since the geometry along slices of constant z is flat (as can be seen from the metric Eq. (3.24)), the energy as a function of quark separation must increase proportionally to the length L' of the section of string lying on the wall. [43, 41]

Of course the hard wall scenario is only the simplest choice, and there are several physically more plausible methods that generate confinement, since the only essential requirement is that the geometry is modified such that at quark separation beyond some limit the energy increases as the separation increases.

When finite temperature is applied by including a black hole in the bulk, deconfinement appears naturally once temperature becomes high enough that the black hole horizon crosses the wall that is causing the confinement. Once the wall is behind the horizon, the ends of the two strings stretching from the brane are sucked in to the black hole before they can reach the wall, and cannot therefore join, making the theory deconfined.

6.4.3 The bottom-up approach

Another way to approach the problem of finding a holographic dual to QCD, instead of trying to deform the AdS/CFT scenario, is to start with known features of QCD and attempt to construct a five-dimensional theory that reproduces those features in its dual. This tends to produce a more phenomenological result than the stringy approach, but on the other hand, some surprisingly good fits to QCD parameters have been found. For illustration, we will very briefly describe one such model, taken from [44].

The background is taken to be the standard AdS_5 space:

$$ds^2 = r^2 d\vec{x}^2 + \frac{dr^2}{r^2}, \tag{6.22}$$

where $\vec{x}$ is the four dimensional coordinate, and r is the fifth coordinate, which is interpreted as an energy scale. This metric is conformally invariant, and in order to break the invariance a hard wall is imposed at $r = r_0$.

The field content in the field theory side is taken to be the left- and right handed currents $\bar{q}_L\gamma^\mu t^a q_L$ and $\bar{q}_R\gamma^\mu t^a q_R$ corresponding to the chiral flavor symmetry and the chiral order parameter $\bar{q}_R^\alpha q_L^\beta$. The AdS duals acting as sources for these operators are the fields $A^a_{L\mu}$, $A^a_{R\mu}$ and $(2/z)X^{\alpha\beta}$, respectively. The A fields are vector gauge fields transforming in the fundamental representation of the chiral symmetry group, and $X^{\alpha\beta}$ transforms under the adjoint representation of $SU(N_f)_L$ on one of its indices and under the adjoint of $SU(N_f)_R$ on the other index. The action of the theory is

$$S = \int_{r_0}^{\infty} \mathrm{d}^5 x\sqrt{-g}Tr\left\{|DX|^2 + 3|X|^2 - \frac{1}{4g_5}(F_L^2 + F_R^2)\right\}, \tag{6.23}$$

where $D_\mu X = \partial_\mu X - iA_{L\mu}X + iXA_{R\mu}$ is the covariant derivative, F_L and F_R are the field strength tensors of A_L and A_R, respectively, and g_5 is the five dimensional gauge coupling, which is considered a free parameter of the model.

Applying the AdS/CFT style recipe then gives the expectation values of the operators and any desired combinations of them, which allows extraction of several QCD observables. The model overall has three free parameters, the hard wall position z_0, the expectation value of the quark condensate and the quark mass (the latter two parameters come from determining the boundary conditions of the X field). The gauge coupling g_5 is determined by matching the value of the vector current.

Any three observables can be made to match by fitting the free parameters to them. Any further observables will then be predictions of the model. The first column of table 1 shows the results of fitting the pion and rho masses and the pion decay width, and then calculating four more observables. The second column shows the results of finding the overall best fit for the seven observables.

Observable	Measured [MeV]	AdS A [MeV]	AdS B [MeV]
m_π	139.6±0.0004[9]	139.6*	141
m_ρ	775.8±0.5	775.8*	832
m_{a_1}	1230±40	1363	1220
f_π	92.4±0.35	92.4*	84.0
$F_\rho^{1/2}$	345±8	329	353
$F_{a_1}^{1/2}$	433±13	486	440
$g_{\phi\pi\pi}$	6.03±0.07	4.48	5.29

Table 1: Several meson variables from the AdS/QCD model described in the text. AdS A is the best fit to the starred observables, whereas AdS B is the best fit to all seven.[44]

The fit results are surprisingly good considering the simple starting point of the model and the fact that three QCD operators were rather arbitrarily chosen while the rest were completely ignored. This suggests that the idea of QCD being dual to a five dimensional theory in a deformed anti De Sitter spacetime indeed captures some essential features of QCD.

[9] The peculiar expression 139.6 ± 0.0004 comes from the results column of the table in the original text [44]. The error limit which is much smaller than the least significant digit of the measured value apparently serves the purpose of denoting that the measured value is coarsely 139.6, but that it is actually known to an error margin of 0.0004.

7 Discussion

The calculations and symmetry comparisons presented in section 5 give some basic justification for the Maldacena duality. Due to these and many more successful tests, the duality in its original form, as a duality between the $\mathcal{N} = 4$ SYM and type IIB string theory on an $AdS_5 \times S^5$ background, is nowadays considered by experts as almost surely true, even though an exact proof is still lacking. Current research focuses on deriving further results from the duality, and especially on searching for less symmetric gravity duals which would be closer to QCD.

The primary application of the duality has been in understanding strongly coupled gauge theories. Several phenomenological models have produced surprisingly accurate values for QCD parameters. Another success for the duality has been the derivation of hydrodynamic transport coefficients from the finite temperature formalism, and even some contact with experiment has been made, as values of shear viscosity measured in the RHIC collider have been found to be near the AdS/CFT -derived lower limit.

The generally accepted interpretation for gauge/gravity dualities at the moment is that they are merely convenient mathematical tools for calculating variables in the strongly coupled non-perturbative regime of gauge theories. This is a very natural interpretation when gauge/gravity dualities are used to calculate observables in QCD, since in this case there is no exact duality, and therefore one cannot make very fundamental interpretations based on these approximate relations.

On a more speculative note, consider the consequences if a precise duality would be found and proven between a string theory and QCD. The interpretation that quarks and gluons are actually endpoints of open strings on a brane or a system of branes (or on the conformal infinity of an anti de Sitter spacetime) on which we would then live on would be indistinguishable from the four dimensional field theory point particle interpretation in the context of QCD alone. In this case one of these interpretations might eventually become favoured on basis of cosmological results or perhaps by a direct observation of extra dimensions in a collider. In general as long as there is no duality between the whole Standard Model plus gravity and a string theory (which is unlikely, since no known gauge/string duality has gravity on the field theory side), it is possible to experimentally distinguish a situation where reality is described by QCD plus the rest of the Standard Model, and the duality between QCD and a string theory is just a fortunate mathematical curiosity, from the situation where the observable universe is indeed a system of branes embedded in a higher dimensional spacetime. The latter scenario could even give a whole new meaning to the term "brane cosmology": experimentally probing other branes in the bulk spacetime, for example by observing gravitons (in practice, gravitational waves) arriving or scattering from distant branes or other objects in the bulk! Of course all this is wildly speculative.

We do have reason to look forward to the near future too, though. Several predictions have been made, based on various AdS/QCD models, of effects that might be observable in the LHC. Many of these predictions compare AdS/QCD as a calculational technique to perturbative QCD or lattice QCD, but a few of them are based on general arguments applying to any theories which have a gravity dual. Therefore LHC results contradicting predictions of the latter type have a chance to prove that QCD indeed has no exact gravity dual. On the other hand, observation of large extra dimensions and gravitational phenomenon in the TeV -scale could lend considerable support to stringy brane scenarios.

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A Notation

A.1 Differential geometry

We work with the mostly plus metric, that is, $\eta_{\mu\nu} = (-1, +1, \ldots, +1)$, and contract the the first and third indices of the Riemann tensor to form the Ricci tensor, that is, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. From section 3.2 onward, somewhat unconventionally $\eta_{\mu\nu} = (s, +1, \ldots, +1)$ where $s = \pm 1$, so we can mostly handle a Euclidean and a Minkowski space at the same time.

A.2 Spinors and supersymmetry

When working with spinors in the Weyl representation, undotted Greek indices transform in the $(\frac{1}{2}, 0)$ representation of the Poincaré group, while the dotted indices transform in the conjugate representation $(0, \frac{1}{2})$. Undotted indices are raised by the matrix

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.1}$$

while dotted indices are lowered by

$$\epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

These matrices may be derived from the charge conjugation properties of Weyl spinors [7]. The lowering of undotted indices and raising of dotted indices follow from these, since raising and then lowering an index must amount to the identity transformation.

Capital Latin indices, usually I and J, label the independent supersymmetry generators, and $\mathcal{N}$ denotes the number of these generators.