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Concepts of variable in middle-grades mathematics textbooks during four eras of mathematics education in the United States

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Concepts of Variable in Middle-Grades Mathematics Textbooks during Four Eras of Mathematics Education in the United States

by

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Dedication

I dedicate this dissertation to all my loved ones who encouraged and supported me during these years of schooling. I also dedicate this work to my Lord and Savior Jesus Christ for granting me the strength and wisdom that enabled me to come this far in my education, and for continually reminding me of His promises for my life!

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Concepts of Variable in Middle-Grades Mathematics Textbooks during Four Eras of Mathematics Education in the United States

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Abstract

This study used content analysis to investigate the development of the concept of variables in middle grades mathematics textbooks during four eras of mathematics education in the United States (*New Math*, *Back to Basics*, *Problem Solving*, and the *NCTM Standards era*: 1957 – 2009). It also examined the nature of support that the curricula provide for teachers to enact variables ideas in the classroom. Findings revealed that each of the middle grades mathematics curricula examined used variables, but in varied proportions and levels of complexity. Formal definitions for variables were found in 11 of the 12 students' editions examined. The characteristics of the definitions for variables found in the different curricula were, however, different from one another.

The uses of variables as *placeholders* and as *labels* dominated the uses of variables in the mathematics curricula. The least used category of variables was as an *abstract symbol*. When examined in terms of the content areas, the use of variables as *placeholders* dominated *Number and Operations*, and *Algebra* contents. In *Geometry*, *Measurement*, and *Data Analysis and Probability* content areas, the use of variables as *labels* was predominant.

Overall, the data did not reveal any systematic or drastic change in the treatment of variable ideas during the 50 year period within which this study is situated. There was however, a steady increase in the use of variables as *varying quantities* across grade levels, and the four eras of mathematics education in the United States. There were also some noticeable changes in the treatment of variable ideas found in *Math Connects* curriculum when compared to the treatment in the other three curricula.

The data collected also supported the evidence of guidance provided to teachers in the respective curricula to enact variable ideas in the classroom. However, the amount of guidance identified was limited in the majority of the curricula. Limitations of the study, implications for curriculum and teacher development, as well as recommendations for future research are also presented.

Chapter 1

Background of the Study

The concept of variable is an important tool in the teaching and learning of school mathematics and other related school subjects (e.g., physics, chemistry, economics, and the like). The evidence of the importance of this concept is seen in the many uses of variables in school mathematics today, and also in the statements made by mathematicians, mathematics educators, and researchers, among others about variables. Eisenberg (1991) stated, for example, that understanding the use of variables is the basis of all abstractions in mathematics. To Leitzel (1989), this concept is so important that not understanding it may block students' success in algebra. Sir Percy Nunn (a well known mathematician and a onetime president of The Mathematical Association) stated in 1919 that the discovery of the concept of variable is probably the most important event in the history of humanity and the sovereignty of its use will remain as one of the most important successes of the history of humanity. Skemp (1971) points out that "it is largely through the use of symbols that we achieve voluntary control over our thoughts" (Skemp, 1971, p. 82). And, Philipp (1992) cited Rajaratnam (1957) who stated in the late 1950s that the concept of variables is so important that its discovery was a milestone in the history of mathematics.

School algebra, for example, relies heavily on the use of variable for its presentation. Compared, for example, with the goal of arithmetic which is to find numerical answers, the main focus of school algebra is to find general methods and rules, and use algebraic symbols and language to express these rules in a general form (Booth, 1988). The superiority of the use of variable over plain language in achieving this goal of school algebra, and in other content areas in mathematics (e.g., number theory, geometry), is evident when one tries to describe rules and undertake simple procedures in school mathematics.

For instance, in number theory, one can describe the product of two fractions in plain English as ‘the fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators’. This same procedure can be described in algebraic language using variables as $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, where b and d are non-zero real numbers. Similarly, in geometry, the length of the hypotenuse of a right-angled triangle can be described as the ‘square root of the sum of the squares of the lengths of the legs of the triangle’. This procedure can be described in algebraic language using variables as $c^2 = a^2 + b^2$ where a, b represent the lengths of the legs, and c represents the length of the hypotenuse of the right triangle. In each of the above descriptions, many people would agree that the use of variables simplifies the generalization more succinctly than using plain English.

In spite of these superior uses of variables to express our thoughts easily and concisely, working with variables has been found to be difficult for many students as well as many seasoned mathematics teachers (Clement, 1982; Macgregor & Stacey, 1997; Schoenfeld & Arcavi, 1988; Ursini &Trigueros, 1997). Ursini and Trigueros (1997) revealed, for example, that even after several algebra courses, first year university students still have difficulties in understanding the principal uses of variables, and that their understanding of each use of variable concept remains at an action level where they produce mechanistic answers to routine questions. Booth (1988) found that students perceived $5y$ to mean 5 yachts, or 5 yogurts, when it should have been conceived as 5 times the number represented by the variable y. Wagner (1981) reported that high school students identified different letters as always representing different numbers in equations. That is, they treated n and w in the equations $7n + 22 = 109$ and $7w + 22 = 109$ as representing two different numbers, and thought that there was a linear ordering correspondence between the letters in the alphabet and the number system. Specifically, in their view, an n would represent a smaller number than w.

These difficulties arise, in part, because of the multifaceted nature of variables in school mathematics. Usiskin (1988) reported that many students think that all variables are letters that stand for numbers. Yet the roles of a variable are not always to represent numbers. In geometry, variables often represent points, as seen by the use of the variables *A, B,* and *C* when we write "if *AB* = *BC,* then triangle ABC is isosceles." In logic, the variables *p* and *q* often stand for propositions; in analysis, the variable *f* often stands for a function; in linear algebra the variable *A* may stand for a matrix, or the variable *v* for a vector, and in higher algebra, the variable * may represent an operation. The last of these demonstrates that variables need not necessarily be represented by letters (Usiskin, 1988).

Consequently, these many uses of variables make it difficult for both students and teachers to understand and develop competency in their use (Clement, 1982; Kieran, 1992; Küchemann, 1978, 1981; Macgregor & Stacey, 1997; Usiskin, 1988). This assertion is supported by Schoenfeld and Arcavi's (1999) claim that while variables are formal tools in the service of generalization, their multiple uses make them difficult to understand and are the cause of many students' difficulties when studying mathematics. Consistent with Schoenfeld and Arcavi's assertion, Chiarugi, Fracassina, Furinghetti and Paola (1995) observed that many difficulties students encounter in the study of algebra can be traced to their inadequate construction of the concept of variable, and that the many different uses of variables in different contexts make them difficult to understand.

Among others, the many uses of variables, the assertions about the complex nature of variables, and the numerous research findings that point to learners' difficulty with variables continue to incite researchers to investigate middle school students', high school students', college students', as well as teachers' understandings and misconceptions of the concept of variables (Boz, 2007; Clement et al., 1981; Küchemann, 1978; Philipp, 1992; White & Mitchelmore, 1996) in order to address these difficulties. However, very few studies have examined the extent to which this concept is introduced and/or presented in the school mathematics curriculum.

In fact, only one study (AAAS Project 2061 conducted in 2000) attempted to rate the development of variables and other topics (functions, and operations) described by the AAAS Project as core content that should be present in high school mathematics curriculum. This Project examined 13 high school algebra textbooks that were published between 1992 and 1998 in relation to some specific benchmarks of the *Curriculum and Evaluation Standards for School Mathematics* by the National Council of Teachers of Mathematics (NCTM) (1989) to determine the extent to which the selected high school mathematics textbooks can be judged as unsatisfactory, satisfactory, or excellent in their presentation of the selected topics. Similar topics – number concepts, number skills, geometry concepts, geometry skills, algebra graph concepts, and algebra equations were also examined in middle grades mathematics textbooks against specific benchmarks of the *Curriculum and Evaluation Standards for School Mathematics*. Thus, no study was found in the literature that examined, in detail, how the concept of variable is introduced and developed in the middle school mathematics curricula, or examined the development of variable ideas over time in school mathematics curriculum.

Curriculum materials (which are defined largely by the textbooks students use) have been found to be the major determinant of what is taught and learned in most classrooms in the United States, and in classrooms around the world (Grouws & Smith, 2000; Jones & Tarr, 2007). Robitaille and Travers (1992) noted that textbooks are present not only in classrooms but are also frequently used by teachers and students, and influence the instructional decisions that teachers make on a daily basis. Begle's (1973) data from the National Longitudinal Study of Mathematics Achievement provided evidence that students learn what is in the textbook, and desist from learning topics not covered in the textbook. Over 75 percent of the teachers in a survey sponsored by the National Advisory Committee on Mathematics Education reported using a single textbook predominantly in the classroom and 53 percent of the teachers reported that they followed the texts closely (Porter, 1981). Recent studies have also revealed that most middle-grades (grades 6-8) mathematics teachers use textbooks most of the time for their instruction (Grouws & Smith, 2000; Weiss, Banilower, McMahon, & Smith, 2001).

Thus, because textbooks as major curriculum material, have a marked influence on the teaching and learning of mathematics, it might be a worthwhile endeavor to investigate these materials that many students and teachers use, and their prospects in impacting students' opportunities to learn and acquire facility with the use of vital concepts such as variables in school mathematics. Results from such studies could provide pointers to possible links between some well-documented difficulties that students have with variables and their treatment in the curriculum. The findings could also serve as an important prerequisite step towards addressing how the concept of variable is approached in the curriculum, and its subsequent presentation in the actual classroom setting.

The purpose of the present study, therefore, was to explore the development of the concept of variable in popular middle-grades mathematics textbooks from four eras of mathematics education in the United States: *The New Math*, *Back to Basics*, *Problem Solving*, and the *NCTM Standards era* (Fey & Graeber, 2003; Payne, 2003). Thus, building on the work of other researchers (Jones & Tarr, 2007; Küchemann, 1978, 1981; Philipp, 1992; Usiskin, 1988, 1989), I developed a framework and used it to examine the extent to which popular middle-grades mathematics textbooks present the concept of variable for students to learn with relational understanding, and for teachers to implement as intended. In addition, I explored the extent to which the development and the presentation of this concept in the middle-grades mathematics textbooks have changed, or have kept the status quo over the past five decades. In the following section, I present a brief historical account on the origin and development of the concept of variables to help readers understand the nature and the complexity of this important concept in school mathematics.

Brief Historical Development of the Concept of Variable

Boyer (1991) and Kieran (1992) provided historical accounts of the development of the concept of variable in relation to the growth of algebra over time. According to these researchers, the history of algebra can be divided into three periods, with each period being defined by the concept of variable prevalent within it. The first period, which lasted for more than 3,000 years, was described as the period of rhetorical algebra. This was the era when mathematical problems were expressed and solved rhetorically – that is, exclusively using words, and variables or symbols were not used at all.

Boyer (1991) reported that Diophantus' use of a symbol to represent an unknown quantity in mathematics problems signaled the beginning of the second period in the development of algebra. Specifically, Boyer wrote that during the middle of the third century, Diophantus (ca. AD 250) developed "syncopated arithmetic" using strings of numbers and a symbol called the 'arithmetos' (which resembled the last letter of the Greek word) to designate the unknown.

Later, Viete (1540-1603) was credited for using consonants to indicate given or known magnitudes and vowels to indicate true unknowns. This represents the third period in the development of algebra, which began in the late sixteenth century. That is, for the first time in the history of mathematics, letters were used as symbols to represent sets of numbers. Viete's consonants represented sets of arbitrary constants, or parameters that identified particular characteristics of relationships between the true varying quantities, which were represented by vowels. Descartes (1596-1650), in the early seventeenth century, established the tradition of using letters near the beginning of the alphabet to represent parameters and those near the end of the alphabet to represent varying quantities.

The extant literature revealed that, in spite of the fact that the origin of analytic geometry lies solely with Descartes, it was Fermat (1601-1665) who wrote in 1629 that, "Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved" (Boyer, 1991, p. 346). Thus, Fermat, along with Descartes and Viete, used variables to represent entire sets of numbers and illustrated the relationships between these sets with sets of points on coordinate graphs.

Sfard (1992) and colleagues noted that the general acceptance among the mathematics community of the concept of variable as a varying quantity and as a generalizer of a set of numbers progressed slowly in the 200 years following Viete's momentous contribution (Sfard, 1992; Sierpinska, 1992). There is agreement among mathematicians, however, that the development of the concept of variable beyond its use as a single, specific unknown was a fundamental reason for the phenomenal advancement of mathematics in the last few centuries as compared to its relatively slow progress during the prior 4,000 years.

As the concept of variable developed historically, however, new uses of the variable did not replace its uses during prior times, but rather expanded it into a complex repertoire. Evidence of this complexity is seen in the descriptions by contemporary researchers of the many ways in which variables are currently used in school mathematics: as *labels*, *objects*, *unknowns*, *varying quantities*, *constants*, *parameters*, *generalized numbers*, *placeholders*, *arguments*, and *abstract symbols* (Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1988; Wagner & Parker, 1993). Although experts categorize the use of variables differently, there is agreement among them that any particular use is determined by the mathematical context in which it is used. Thus, the various uses of variables are often barely distinguishable from one another – they are similar in some ways, opposing in other ways, overlapping, and often difficult to articulate clearly. For example, $5x$ might be considered a label for the name of an object, an unknown in a word problem, or a generalized number in a pattern, depending on the context in which it is used.

Statement of the Problem

It is commonly agreed that the concept of variable is central to the teaching and learning of school mathematics (Philipp, 1992; Sasman et al., 1997; Usiskin, 1988, 1989; Wagner, 1999). According to Eisenberg (1991), understanding the use of variables is the basis of all abstractions in mathematics. Percy Nunn argued that the discovery of this concept is probably the most important event in the history of humanity and the sovereignty of its use will remain as one of the most important successes of the history of humanity. And, Philipp (1992) cited Rajaratnam (1957) to have said that this concept is so important that its discovery was a milestone in the history of mathematics.

Despite this importance of the role of variables in school mathematics and other school subjects, a mounting body of research indicates that students of all ages and grade levels, as well as teachers, have problems working with variables. Several studies have documented difficulties experienced by students as they attempted to model word problems involving two variables used as varying quantities (Clement et al., 1981; Mestre & Gerace, 1986; Lochhead & Mestre, 1988; Macgregor & Stacey, 1993; White & Mitchelmore, 1996), or as they attempt to interpret the meaning of variables used in a mathematics problem (Boz, 2002; Collis 1975; Rosnick 1982; White & Mitchelmore, 1996), or define variables they used in solving problems (Graham & Thomas, 2000; Rosnick, 1982; Schoenfeld & Arcavi, 1988), to mention but a few.

While many studies have investigated students' as well as teachers' understanding and misconceptions of variables (Clement et al., 1981; Küchemann, 1978; Philipp, 1992; White & Mitchelmore, 1996), very few studies have examined the extent to which this concept is introduced and/or developed in school mathematics curriculum, and the trend of its development over time in mathematics textbooks. Specifically, no study was found in the research literature that examined, in detail, how the concept of variables is introduced and developed in middle-grades mathematics curricula – a stage described by researchers as crucial in students' transition from arithmetic to algebra (e.g., Herscovics & Linchevski, 1994; Ketterlin-Geller, Jungjohann, Chard, & Baker, 2007; Kieran, 2004; Kieran & Chalouh, 1992).

By not taking for granted the fact that the many uses of variables could be the culprit in causing students' difficulties with this concept, it might also be possible that the problems students encounter in working with variables originate from how this concept is presented in the curriculum – which is defined largely by the textbooks students use. Some researchers assert that while students' fluidity with the use of variables is essential to their success in school mathematics and beyond, this concept is overlooked or treated as a simple concept in some mathematics textbooks (e.g., Kieran, 1981; Schoenfeld & Arcavi, 1988). Graham and Thomas (2000), for example, made the claim that, "the idea of a variable is in fact a key concept in algebra – although many elementary texts do not explain or even mention it" (p. 265).

Reys et al. (2003) however, observed that, "if mathematics content is not included in curriculum materials, then teachers are unlikely to present the content. Likewise, the instructional approaches suggested by the materials often influence teachers' pedagogical strategies. Indeed, the implemented curriculum often closely mirrors the content and pedagogical approach presented in textbooks" (Reys, et al., 2003, p. 75).

Thus, given the complex nature of variables and the findings from the extant literature that students of all ages and grade levels (Clement et al., 1981; Küchemann, 1978; Philipp, 1992; White & Mitchelmore, 1996), as well as teachers possess limited understanding of variable (Al-Ghafri, Jones, & Hirst, 2002; Asquith, Stephens, Knuth, & Alibali, 2007; Boz, 2002, 2007; Mohr, 2008), it is hard to believe that a significant improvement in the teaching and learning of this concept will occur without some curriculum support for it. In other words, it is logical to say that if teachers are not equipped with curriculum materials that design rich opportunities for students to work with variables, in addition to being provided with the necessary guidance on how to enact these opportunities in the classroom, it is unlikely that teachers themselves will modify curricular tasks and implement them in ways that will offer students the appropriate opportunities to develop proficiency in the use of variables.

The preceding argument finds further support in the recommendation by the National Research Council (NRC, 2004) that "a curriculum should include enough support for teachers to enact it as intended. Such support should allow teachers to educate themselves about mathematics content, students' mathematical thinking, and relevant classroom issues…. It might help … teachers to analyze common student errors in order to think about next steps for those who make them" (NRC, 2004, p. 76).

The premise of the present study, therefore, stemmed from the fact that instruction that enhances students' understanding of variables and that can be implemented successfully in many mathematics classrooms is unlikely to happen unless significant attention and supports are woven into the curriculum materials for students as well as for teachers. Consequently, there is the need to examine middle-grades mathematics textbooks for the extent to which they support students' development of, and meaningful engagement with variables in accordance with their various uses in school mathematics.

I do not mean by this, however, that if we equip teachers and students with the right materials on the meaning and uses of variable, this will automatically lead to improved mathematics instruction and students' understanding of it. Nevertheless, there is substantial evidence to support the argument that teachers will draw heavily on the tasks from the textbooks in their possession and students will learn from these materials (Begle, 1973; English, 1980; Osborn, 1985; Porter, 1981; Young & Reigluth, 1988). The need, therefore, to examine the extent to which middle-grades mathematics textbooks provide such opportunity for students and teachers to work with variables has prompted this study.

Purpose of the Study

As stated earlier, the purpose of this study is to examine the extent to which popular middle-grades mathematics textbooks in the USA provide opportunities for students to understand the concept of variables, as well as how these materials provide support for teachers to teach about variables in their classrooms. In addition, this study explores the extent to which the development and presentation of variables in middle-grades mathematics textbooks have changed (if they have), during the last five decades, categorized into four recent eras of mathematics education in the United States (Fey & Graeber, 2003; Payne, 2003). I achieved these objectives by examining four popular middle-grades (grades 6-8) mathematics textbook series published for use with 'average-students', and their corresponding teachers' editions selected from four eras of mathematics education (*New Math*, *Back to Basics*, *Problem Solving*, and *NCTM Standards*) in the United States.

Research Questions

This study investigated the extent to which selected middle-grades mathematics textbooks develop the concept of variables for students to learn with an in-depth understanding, and for teachers to implement successfully in the classrooms. Specifically, the study addressed the following six research questions:

1. How do middle-grades mathematics textbooks develop the concept of variable (i.e., in terms of whether and how they introduce, define and/or explain it, and at which grade level(s) in the middle-grade mathematics textbooks do these occur)?
2. To what extent do middle-grades mathematics textbooks present activities and tasks that address each of the uses of variables (e.g., *labels*, *unknowns*, *generalized numbers*) as described by researchers in the mathematics education community, and in which order do these uses occur in the mathematics textbooks?
3. Which use(s) of variable is/are prevalent within which content areas (i.e., geometry, number theory, algebra, etc.) in middle-grades mathematics textbooks?
4. How does the development and/or presentation of the concept of variable differ across different grade levels of the same textbook series (e.g., in terms of the compositions of the various uses of variable)?

5. To what extent has the development and/or the presentation of variables in middle-grades mathematics textbooks changed during the past five decades (i.e., by comparing the development among the textbooks from the four major eras of mathematics reform in the USA)?
6. To what extent do the teacher's editions of middle grades mathematics textbooks provide guidance to teachers on the treatment of variables (i.e., in terms of alerting teachers to the various uses of variables, to students' misconceptions, and to students' difficulties with variables)?

Significance of the Study

The extant literature informs us that students and teachers make extensive use of the textbooks at their disposal, and that the instructional approaches suggested by the textbooks often influence teachers' pedagogical strategies (Grouws & Smith 2000; NRC, 2004; Reys et al., 2003; Weiss, Banilower, McMahon, & Smith, 2001). Consequently, it is imperative that the textbooks that are intended for use in the classroom be appropriate for both students and teachers. Thus, the importance of examining textbooks for the extent to which they develop vital concepts (such as variables) in school mathematics cannot be overemphasized, especially, when these concepts have been documented as being difficult for students to understand.

In this respect, it is expected that the findings from this study will inform curriculum developers and evaluators of curriculum in their future efforts to improve classroom materials on variables for teachers' and students' use. Specifically, if the results show, for example, that the content related to the uses of variables in middle-grades mathematics textbooks depicts a dominance of students' engagement with this concept at a low level (e.g., based on Küchemanns' hierarchy of students' understanding of variables described later in this study), then a call could be made for changes in the future editions of these mathematics textbooks to increase students' meaningful engagement with variables that will meet their grade level expectations. Similarly, if the results indicate that some textbooks offer appropriate opportunities for students' engagement with variables, then such curricula could serve as a model for future editions, with an eye to improve what is already supportive of students' learning (Stein et al., 1996). Thus, it is anticipated that the results of the present study will offer relevant information for middle-grades mathematics curriculum developers and for writers of middle-grades mathematics textbooks in their future efforts to revise and improve curriculum materials on variables.

Furthermore, it is worth stating here that, although this study does not directly investigate students' understanding of variables, it may contribute to that domain of research by documenting the opportunities mathematics textbooks offer students to acquire competency in using variables. The findings can also shed light on some possible sources of students' difficulties and misconceptions of variable that might originate from how the curriculum presents it. In addition, it is expected that the methodology used in this study will provide some guidelines for future researchers who want to engage in similar research studies.

In summing up the significance of the study, I wish to reiterate here that although the present study does not specifically address the problem of how variable is presented in the actual classroom setting, it makes an important prerequisite step towards this direction by addressing how variable is approached in the intended curriculum – which is central to classroom practices. The results of this curriculum analysis can also point to possible links between some well-documented difficulties that students have with variables and the treatment of this topic in the curriculum, which can in turn have important implications for future revision efforts.

Limitations of the Study

The materials that were examined in this study included only popular middle-grades mathematics textbooks used during four recent eras of mathematics education in the United States. As a result, the findings may not be generalizable beyond those materials (or similar materials), and the time periods within which this study is situated. Also, the number of textbooks that were analyzed in this study was relatively small (i.e., 24 mathematics textbooks). Against this background, it is prudent to acknowledge the fact that the findings might, perhaps, be altered if a larger sample is used. Consequently, any attempt to extend the results beyond these textbooks has to be done with caution.

Second, the present study did not interview individuals who were involved in the development of the selected middle-grades mathematics textbooks, or teachers and students who enact and learn from these curricula materials about their perspectives on the presentation and the uses of variable in the textbooks. This researcher is, thus, aware that obtaining such a data could add some valuable insight to the results of the study, and hence, considers this as a limitation to the study.

Finally, threats to reliability and validity of execution of coding instructions as well as practice of coding to establish a desirable reliability existed. The possibility of coder fatigue existed as the amount of documents to be examined is large (i.e., about 24 textbooks). Coding schedules however, took into consideration the length of units that were coded, and reliability was enhanced by the use of code-recode strategies and measures of inter-coder agreement (reliability and validity issues are treated more fully in Chapter 3).

Definition of Terms

For the purpose of this study, the following terms and definitions were used:

Conception: A conception is the whole cluster of internal representations and associations evoked by a concept; a concept being a mathematical object, process, or idea in its formally defined and objective form generally accepted by mathematicians (Sfard, 1991). A conception therefore refers to the individual and subjective ways that a person visualizes, knows, and uses a concept.

Concept of Variable: The concept of variable is a fundamental mathematical structure usually defined by the uses of variable in algebra (Küchemann, 1978; Usiskin, 1988; Wagner & Parker, 1993). Although algebraic thinking does not always require using variables, the use of variables is the most familiar part of working algebraically. It is what sets algebra apart from arithmetic. The study of advanced mathematics depends on a robust understanding of the many facets of the concept of variable.

Content Analysis: Content analysis is a research tool/technique used to determine the presence of certain words or concepts within texts or sets of texts. Researchers quantify and analyze the presence, meanings and relationships of such words and concepts, then make inferences about the messages within the texts, the writer(s), the audience, and even the culture and time of which these are a part. Texts can be defined broadly as books, book chapters, essays, interviews, discussions, newspaper headlines and articles, historical documents, speeches, conversations, advertising, theater, informal conversation, or any occurrence of communicative language. To conduct a content analysis on any such text, the text is coded, or broken down, into manageable categories on a variety of levels–word, word sense, phrase, sentence, or theme and then examined using one of content analysis' basic methods: conceptual analysis or relational analysis.

Mathematics Curriculum Materials: The National Research Council ([NRC], 2004) defines school mathematics curriculum as "a set of materials for use at each grade level, a set of teacher guides and accompanying classroom assessments, a listing of prescribed or preferred classroom manipulatives or technologies, materials for parents, homework booklets, and so forth" (p. 38). Robitaille et al. (1993) distinguished between three main aspects of the curriculum: the *intended curriculum*, the *implemented curriculum* and the *achieved curriculum*. The intended curriculum refers to the aims and objectives of a given program as specified by those developing the program and the materials to support its introduction and use. The implemented curriculum concerns what happens in practice in the classroom, the teaching approaches, the learning activities and materials teachers draw on using the program. The attained curriculum relates to the outcome of the program: the knowledge, skills, understanding and the attitude displayed by the students who experienced the program. The intended curriculum is the focus of this study.

Middle-Grades: Middle-grades or middle school is any school intermediate between elementary school and senior high school. It usually includes the seventh and the eighth grade, and sometimes the sixth or the ninth grade. In this study, the middle-grades is defined to include the sixth, the seventh and the eighth grade (6-8). This definition is consistent with those used by the NTCM (1989, 2000), the NRC (2004), and the AAAS (2000) in their classification of grade bands in the United States education system.

Popular Textbooks: In this study, popular textbook series is defined as the mathematics textbook series having the largest market share during a given era. I used Weiss (1978, 1987) and Weiss et al. (2001) to determine which textbook series were the most popular during the *Back to Basics*, *Problem Solving*, and the *Standards* eras. In the absence of market share data, popular mathematics textbook series were determined by a "professional consensus" of mathematics educators who were familiar with the middle-grades mathematics curriculum during the past 60 years (Jones & Tarr, 2007).

Variable Task: Any tasks in the curriculum that employ variable ideas for its presentation.

Variable Ideas: These refer to any of the seven uses of variables (*labels*, *unknowns*, *varying quantities*, *constants*, *generalized numbers*, *placeholders*, and *abstract symbols*) in a variable task found in the mathematics textbooks.

Summary

In this chapter, I provided a background to the study and a brief account on the development of the concept of variables. This was followed by the statement of the problem, the purpose of the study, the research questions, the significance of the study, and the limitations of the study. Chapter 2 presents the conceptual frameworks on variable, a review of the literature on students' and teachers' understanding of variables, a review of the use of mathematics textbooks in the classroom, and a review of related content analysis in mathematics education. Chapter 3 contains the methodology and the design of the study. Chapter 4 presents the results of the study. The discussion of the findings, the conclusions and the implications for further research and curriculum development are presented in Chapter 5.

Chapter 2

Conceptual Framework and Review of the Related Literature

In this chapter, I present the research that constitutes the conceptual basis of my study and explain how I use this research to conceptualize variables. Specifically, I provide brief discussions of the many ways in which experts currently categorize the uses of variable to help illustrate the nature of this concept. I also discuss four hierarchies of students' understanding of variables based on Küchemann's (1978; 1981) work, and the results of other researchers (Filloy & Rojano, 1989; Kieran, 1988; Herscovics & Linchevski, 1994) that supported this hierarchy. Findings from various studies that examined students' and teachers' understanding of variables are also presented. Further in the chapter, I discuss some recent similar content analyses that have been conducted in mathematics education. Thus, my discussions towards the end of the chapter provide information about students' understanding of variables, the difficulties students encounter in developing proficiency in the use of variables, as well as establish a link between the present study and the body of research that it fits into. These discussions also inform the development of an analytical framework used to examine the tasks containing variable ideas in the selected middle-grades mathematics textbooks in this study.

Present Conceptions of Variables

Diophantus' use of symbols during the middle of the third century to represent an unknown quantity in mathematics problems signaled the beginning of the use of variables in algebra. Reports from the extant literature indicated that by the 16th century, many more new uses of variables emerged within the mathematics community. As the concept of variable developed historically, however, new uses of variable did not replace its uses during prior times, but rather expanded its use into a complex repertoire. Some of these uses are often barely distinguishable from one another, similar in some ways, opposing in other ways, overlapping, and often difficult to articulate clearly. For example, $5x$ might be considered a *label* for the name of an object, an *unknown* in a word problem, or a *generalized number* in a pattern, depending on the context within which it is used. Below, I describe five of the most common present works on the conceptions of variable.

Küchemann's (1978) Categorization of Variables

Using Collis' (1975) work on children's interpretation of letters in generalized arithmetic, Küchemann (1978) developed an algebra test based on the categorization of the different roles of the letters, and used it to examine students' understanding of variable in a landmark study conducted with 3,000 British students (Küchemann, 1978; 1981). Using students' responses to the test, Küchemann identified six ways in which students interpreted and used variables: the variable *evaluated;* the variable *ignored;* the variable as an *object;* the variable as a specific *unknown;* the variable as a *generalized number;* and the variable as a *varying quantity.* The instrument used by Küchemann in this study was constructed based on Piaget's stages of concrete and formal operational thoughts. As a result, Küchemann classified the first three uses of variables (the variable *evaluated,* the variable *ignored,* and the variable as an *object)* as concrete operational, and the last three (specific *unknown, generalized number,* and *varying quantity*) as formal operational (Küchemann, 1978; 1981). Table 1 presents the six uses of variables with their definitions and specific examples as documented by Küchemann in his study.

Table 1

Küchemann's Categorization of Variables

Role of literal symbol	Definition	Example
Letter evaluated	Numerical value that can be determined by trial and error	If $m = 3n + 1$ and $n = 4$, then $m = ?$
Letter ignored	No need to handle the expression containing the variable	if $a + b = 43$, then $a + b + 2 = ?$
Letter as object	Shorthand for an object rather than for a characteristic of it	$P = 4s$ in finding the perimeter of a square
Letter as specific unknown	Specific, abeit unknown, number that can be operated on without evaluating	if $r = s + t$ and $r + s + t = 30$, then $r = ?$
Letter as generalized number	Multiple values can be taken	What can you say about c if $c + d = 10$, and $c < d$?
Letter as variable	Relationship between letter as their value systematically changes	Which is the larger $2n$ or $n + 2$?

Note. Data in Column 1 and 3 are from "Children's understanding of numerical variables" by D. Küchemann, (1978), *Mathematics in School*, 7(4), p. 24.

Based on these categories, Küchemann (1978, 1981) articulated four hierarchies of students' use and understanding of variables. These hierarchies, which are explained later in this chapter, will be employed in Chapter 5 in discussing the findings, and the implications of the results of this study.

Usiskin's (1988) Conception of Variable in Algebra

Usiskin devoted a considerable amount of his research efforts in the late 1980s to explore the uses of variables in school algebra. In one of his papers entitled *conceptions of school algebra and uses of variables*, Usiskin (1988) provided detailed explanations of variables as conceived in algebra. Usiskin (1988) documented four possible conceptions of variables related to algebra: *generalized arithmetic, a study of procedures for solving certain kinds of problems, the study of relationships among quantities*, and *the study of* *structures*.

Usiskin (1988) contended that, if algebra is viewed as generalized arithmetic, then variables are pattern generalizers that allow students to analyze operations like multiplication, addition or division. Usiskin referred to the commutative property of addition described in the equation $a + b = b + a$ as an example of this conception of variables. In this conception, Usiskin explained, variables do not have specific values, but rather, they allow users to analyze operations like addition and multiplication. The key instructions for students in this conception of algebra, according to Usiskin, are on *translating* and *generalizing* known relationships among numbers, and not on finding specific unknown values.

A second conception of variables explained by Usiskin is linked to the view of algebra as procedures for solving problems. In this conception, variable is viewed as *unknown* value that can be found or solved for in an equation. This use of variable mainly focuses on equations and their solutions. An example of such use of variables is seen in the equation $27 = 4x + 3$. In this equation, the variable x represents the unknown that can be found. In other words, x is simply taking the place of a specific number that can be found by solving the equation. Thus, if we consider algebra as the study of procedures for solving certain kinds of problems, then variables can be viewed as an unknown or a placeholder. The key instructions for students in this conception of algebra are on *solving* and *finding*, because there is a feel for knowing the *unknowns* in such situations.

Usiskin identified a third conception of variable with the use of algebra as the study of relationship between or among quantities. Here, the variable is understood as *argument* (i.e., “standing for a domain value of a function” or *parameter* “standing for a number on which other numbers depend" (p. 10)). According to Usiskin, variables really do vary in this conception, and one can look at how the changes in one variable affect the others. An example of such use of variables is seen in the formula $A = lw$ to represent the area of a rectangle with length (l) and width (w). Thus, if we consider algebra as the study of relationships between or among quantities, then variables are either *arguments or parameters*.

The final conception of variable described by Usiskin is when algebra is viewed as a structure. According to Usiskin, if algebra is considered as study of structures, then variable is thought of as an arbitrary symbol. Here, the variable is little more than an arbitrary mark on paper that allows for algebraic manipulations. The key instructions for students in this conception of algebra are on *manipulations* and *justifications*. Usiskin (1988) noted that the first three uses of variables relate more to school algebra than does the last.

Usiskin (1988) pointed out that the multiple ways in which variables can be used in school mathematics are the main culprit in causing students difficulties when working with variables. In a related study, Usiskin (1999b) documented numerous problems that arise when teachers try to teach an oversimplified conception of variable. He emphasized the importance of allowing students the opportunity to explore the different uses of variables with the aim of helping them to develop the understanding that the uses of variables are inextricably related to the conceptions that are held of algebra, and how algebra is used.

The flow chart in Figure 1 illustrates with examples, each of the four conceptions of algebra and the corresponding uses of variables under each conception as documented by Usiskin (1988).

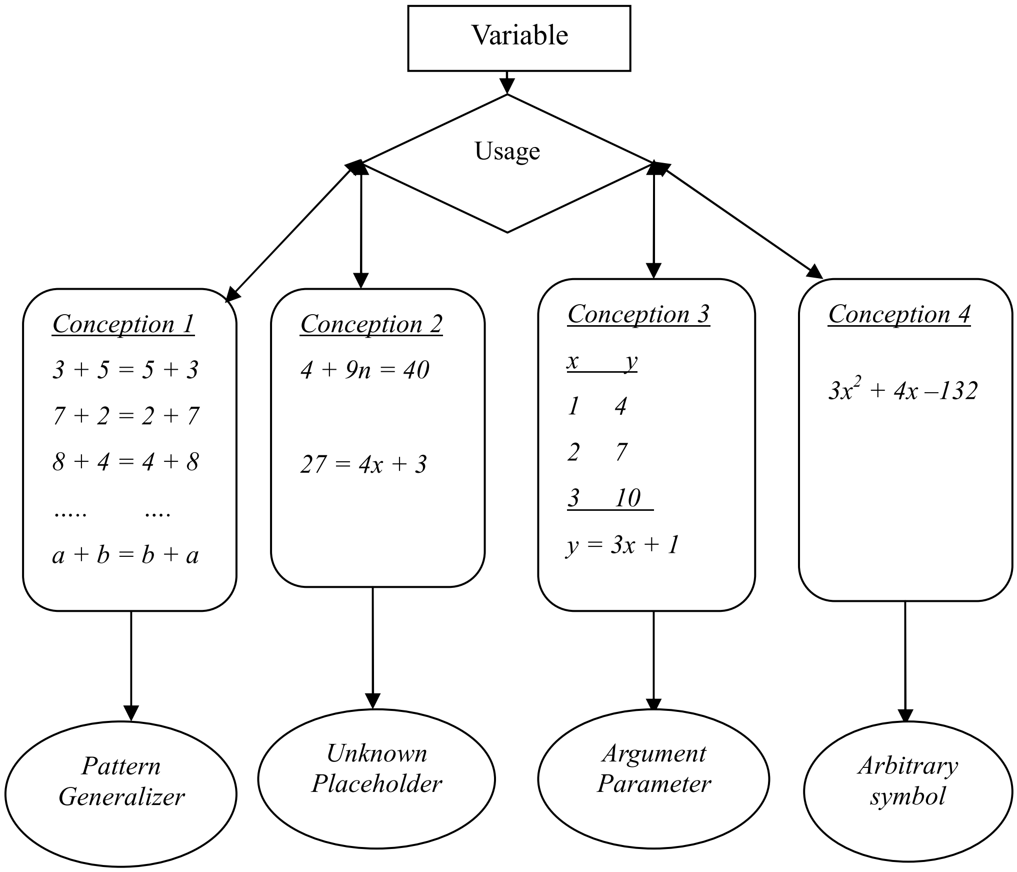


Figure 1: A flow chart depicting the uses of variables in the four conceptions of algebra (Usiskin, 1988).

Philipp's (1992) Categorization of Literal Symbols

Philipp (1992) categorized the uses of variables as labels, constants, parameters, unknowns, generalized numbers, varying quantities, and abstract symbols. Philipp (1992) stated that this categorization of variables includes only the uses of letters related to the concept of variable. Philipp (1992) used several examples to illustrate the various uses of letters in school mathematics (see Table 2). He explained further that the view of variable as a varying quantity implies that students have the understanding that variable may represent many values, including an unlimited number of values.

Table 2

Philipp's Categorization of Literal Symbols

Uses of literal symbols	*Examples*
Labels	f, y in $3f = 1y$ (3 feet in 1 yard)
Constants	π, e, c
Unknowns	x in $5x - 9 = 11$
Generalized numbers	a, b in $a + b = b + a$
Varying quantities	x, y in $y = 9x - 2$
Parameters	m, b in $y = mx + b$
Abstract symbols	e, x in $e * x = x$

Note. Adapted From "The Many Uses of Algebraic Variables" by R. A. Philipp, (1992), *Mathematics Teacher*, 85 (7), p. 560.

Philipp illustrated the uses of variable as constants, parameters, and varying quantities with the exponential equation, $A = Pe^{kt}$ used to model the amount of money (A) that one would have if the initial deposit (P) is compounded continuously at an interest rate k, for certain time t. In this example, the *e* is a constant, P and k are parameters that will be fixed for a particular situation, and A and t are varying quantities that relate to each other.

Philipp explained unknowns to be the values of the literal symbols to be found in solving equations, such as the value(s) of the variable x that satisfies the equation $x^2 - 9 = 26$. Generalized numbers are represented by literal symbols in which all values will make the statement to be true, as with the identity ($a + b = b + a$). Abstract symbols are used in mathematical systems, such as e and $x$in $e * x = x$, where e is an identity for the operation.

Schoenfeld and Arcavi's (1988) Contribution to the Concept of Variable

Schoenfield and Arcavi (1988) discussed the richness and the multifaceted nature of the concept of variable. From a large list of descriptors of variable, these researchers interviewed mathematicians, mathematics educators, and computer scientists for their understandings of the concept of variable by asking them to choose one word from the list of words: "symbol, placeholder, pronoun, parameter, argument, pointer, name, identifier, empty space, void, reference, instance" (p. 151) that best describes variables. These researchers found that even the experts described this fundamental concept in different ways. Furthermore, they examined different literature for their definitions and explanations of variable.

Schoenfield and Arcavi documented ten different definitions of variable found in textbooks and in the research literature. Among these definitions are the following:

A variable is a letter that represents a number;

A variable is a symbol in a mathematical formula representing a variable, placeholder;

A quantity which may assume unlimited number of values is called a variable;

A variable is a quantity of force which throughout a mathematical calculation or investigation is assumed to vary or capable of varying in value;

A variable is a general purpose term in mathematics for an entity which takes various values in any particular context;

The domain of the variable may be limited to a particular set of numbers or *algebraic quantities;*

A variable is a named entity possessing a value that may change during execution of the program;

A variable is associated with a specific memory location and the variable's value is the content of that memory location.

These definitions are similar in some ways but different in other aspects, which typically show the complexity of such a fundamental, but very important concept in school mathematics and its related subjects (e.g., physical sciences, computer science and in commerce). For instance, whereas the definitions a) "a variable is a letter that represents a number", and b) "a symbol in a mathematical formula representing a variable, placeholder" are very similar because they both refer to a *variable* as representing a particular value or number, they are both different from the definitions of *variable* as c) "a quantity which may assume unlimited number of values is called a variable", and d) "a quantity of force which throughout a mathematical calculation or investigation is assumed to vary or capable of varying in value". Furthermore, all four definitions of variables in school mathematics are different from the definition of *variable* that a computer scientist will provide - "a variable is a named entity possessing a value that may change during execution of the program. A variable is associated with a specific memory location and the variable's value is the content of that memory location".

Wagner's (1999) Contribution to the Concept of Variable

Wagner (1999) added an interesting dimension to the discussion of *variable* by explaining ways in which variables are similar and different from *numbers* and *words*.

Wagner explained that, on the one hand, variables are like words in that they act as placeholders. They are often chosen to abbreviate words and they can mean different things in different contexts. On the other hand, variables are different from words because, although they can mean different things in different contexts, they must maintain the same meaning throughout the same context, whereas words can change meanings even within the same sentence. Another distinction between variables and words that Wagner provided was that, while words are made up of letters, there are specific rules about the ways they are used together as opposed to the use of letters for variables.

Wagner cautioned that in order to address these complexities, it is imperative that students be given opportunities to explore the different uses of variables, and to reflect upon their conceptions of variables. Thus, it is important that, when teaching algebra (and for that matter, school mathematics), teachers are aware of the different uses and meaning of variables, and explicitly discuss them with students. Such discussions should be related to the contexts in which the variables are explored. In addition, it is important that teachers themselves be cognizant of the different ways these symbols are used and recognize the particular characteristics they exhibit in various contexts (Wagner, 1999).

Kucheman's (1978) Hierarchy of Students' Understanding of Variables

As explained earlier, Küchemann found in his landmark study (Küchemann, 1978, 1981) that students understand and use variables in six different ways: as *object*, as letters *evaluated*, as letters *ignored;* as *unknown;* as *generalized numbers;* and as a *varying quantity*. Using Piaget's stages of concrete and formal operational thoughts, Küchemann (1978, 1981) articulated four hierarchies of students' understanding and use of variable based on these six uses of variables. For the purpose of this study, I refer to these four hierarchies as *Level 1* and *Level 2*, *Level 3*, and *Level 4* of students' understanding of variables respectively, and explain each of them in detail in the paragraphs that follow.

Level 1 and *Level 2*: Küchemann explained that the variable is *evaluated* when students avoid working with the unknown by simply giving it an arbitrary value or by attempting to find its value through trial and error. For instance, in the equation $x + 5 = 8$, the value of x can be found by trial and error. Thus, the x need never be operated with as an unknown number. On the other hand, the variable is *ignored* when students do not use it in the process of transforming an expression or in solving an equation. The student may acknowledge that the variable exists in the problem, but can often avoid working with the unknown by performing instead some sort of matching procedure. The following question illustrates this use of variable: given that $a + b = 43$, what is $a + b + 2 =$? In this problem, the value of the (?) can be determined without actually giving meaning to the individual variables involved in the equation. That is, a correct solution can be found by simply matching the $a + b$ in both equations and focusing on the $+2$. The 2 is then added to the 43 to obtain the desired result.

The use of variable as an *object* represents Küchemann's third category of students' understanding of variable at *Level 1* and *Level 2*. This use occurs when the meaning of the variable is taken by the student as a *label* of some *thing*, such as when the perimeter of an equilateral triangle with sides of length s is represented by $3s$. In this case, the "s" does not have to be considered as unknown length, but can instead be interpreted as a label for each side of the triangle. Students' manipulations of algebraic symbols following syntactic rules of algebra are often exemplary of the use of variable as an object (Küchemann, 1981).

Küchemann (1978) consequently classified the above three categories of use of variable (denoted by *Level 1* and *Level 2*) as indicating elementary levels of understanding in the hierarchy of understanding of variables. According to Küchemann, the primary difference between *Level 1* and *Level 2* is that students with *Level 2* understanding are able to solve more complex problems than those at *Level 1*. Specifically, students classified at *Level 2* use mathematical syntax and make decisions as to which operations or methods are needed to solve the problems, yet still, they use variables by evaluating them, ignoring them, or treating them as objects. Studies by Filloy and Rojano (1989) and other researchers (Kieran, 1988; Herscovics & Linchevski, 1994) supported Küchemann's findings that students can successfully avoid working with the variable as a specific unknown by ignoring it or evaluating it. Filloy and Rojano (1989) stated, in addition, that ignoring the variable may be a more elementary way of using the variable than evaluating it.

Level 3: By contrast to *Level 1* and *Level 2*, Küchemann's *Level 3* understanding of variable indicates that students are able to use a variable as a specific unknown. That is, when one solves an equation such as $3x + 5 = 4 - 2x$, one is seeking the *specific unknown*, represented by the variable, that will make the equation a true statement. Students at this level of understanding of variables can solve problems which are more abstract than those requiring only *Level 2* understanding. Thus, when variables are used by students as specific unknowns, Küchemann considered the students to be at a higher level of understanding the concept of variable. Küchemann explained that the variable is used to represent a specific, but unknown, number, when its value remains unknown throughout the problem. For example, if: $e + f = 8$, $e + f + g = ?$, then $? = 8 + g$. Here, g is a number which cannot be evaluated, but which, nonetheless, must be used to complete the equation. In this case, the values of e, f, and g remain unknown even after the student finished solving the problem.

In addition, students at Level 3 are able to accept the lack of closure - this means that they can describe the total cost of two items, one of which costs \$3 and the other x dollars, as $3 + x$ and also understand that $3 + x$ represents a single number as well as the operation of addition. Level 3 understanding of variable appears to be a bridge between arithmetic and algebraic thinking. Herscovics and Linchevski (1994) concurred with Küchemann that the ability to work with the variable as a true unknown represents a higher cognitive level than that of simply ignoring or evaluating it.

Level 4: According to Küchemann, the two most advanced uses of variable are those of *generalized numbers* and true *varying quantity*. These indicate Küchemann's *Level 4* understanding of variables. Variables are employed as *generalized numbers* when they are used to represent entire sets of numbers, such as the commutative property of addition for Real numbers: $a + b = b + a$.

On the other hand, when the variable is used as a true variable, or varying quantity, it offers students the chance to see the changing nature of the variable as well as a systematic relationship between two sets of numbers, as in functions. For example, when students are asked to tell the greater of $2n$ and $n + 2$, it is necessary for them to see the relationship between the two numbers as being dependent on the varying values of n. Another illustration of the use of variable at this level is seen in the functional relationship between x and y in $y = mx + b$. Here, the two variables, x and y, are *dependently varying* with each other, while the other two variables, m and b, are not varying, but rather are parameters that may be used as specific unknowns in this case.

Thus, in addition to being able to treat variables as generalized numbers, students at Level 4 (Küchemann's highest level of understanding) can treat variables as varying quantities and interpret functional relationships between them. They can also solve problems which have a more complex structure and require more difficult problem solving methods than those which would designate Level 3 understanding. Gray, Loud and Sokolowski (2005) pointed out that it is these advanced uses of variables that are the most crucial foundations for understanding major concepts of calculus, but at the same time, the most difficult for students to understand.

Students' Misconceptions and Inappropriate Use of Variables

Middle and High School Students' Inappropriate Use of Variable

Just as there are many different uses of variable, research studies that examined students' understanding of the concept of variable pointed to many incorrect ways in which students deal with this concept. Data from Küchemann's study, for example, provided evidence of students' inappropriate use of variables (Küchemann, 1981). Küchemann discovered that the participants in his study exhibited certain common error patterns in his first three categories of use of variables: the variable *evaluated*, the variable *ignored*, and the variable used as an *object*. Küchemann illustrates students' inappropriate evaluation of the variable with the following question, "What can you say about r if $r = s$ and $r + s + t = 30$?" Many subjects responded that $r = 10$. According to Küchemann, these participants appear unable to work with the *r* as the sum of the two unknowns, s and t. They inappropriately evaluated the variables as each representing the number 10. Chalouh and Herscovics (1988) describe this as a cognitive obstacle and stated that some students are unable to work with algebraic expressions because of the lack of a numerical referent. That is, some students appear unable to perform meaningful arithmetic operations on a variable unless they perceive it as representing a number. In Küchemann's description, these students may need to evaluate the variable before they are able to work with it.

In a similar study, Wagner's (1981) work with high school students revealed that many of them evaluate variables incorrectly in equations. Specifically, Wagner's participants identified different letters as always representing different numbers in equations. For example, they treated the n and w in the equations $7n + 22 = 109$ and $7w + 22 = 109$, as representing two different numbers. Likewise, they treated different variables in the same equation as if they could not represent the same number. These participants, in addition, thought that there was a linear ordering correspondence between the letters in the alphabet and the number system. In their view, for example, an n would represent a smaller number than, say, w.

The research literature also provided substantial evidence of secondary school students' inappropriate use of variable as an object or label (Booth, 1988; Küchemann, 1981). Booth's (1988) study provided evidence supporting Kuchmann's findings that students inappropriately use variables as objects or labels. Booth reported that subjects in his Strategies and Errors in Secondary Mathematics (SESM) project described variables in expressions as designating objects which began with that letter. Specifically, Booth's participants mistook $5y$ to mean 5 yachts, or 5 yogurts, when it should have been conceived as 5 times the number represented by the variable y. Consistent with this finding, several studies documented difficulties experienced by students as they attempt to model word problems involving two variables which are used as varying quantities (Clement et al., 1981; Lochhead & Mestre, 1988; Macgregor & Stacey, 1993; Mestre, 1982; Mestre & Gerace, 1986; Mestre & Lochhead, 1983; White & Mitchelmore, 1996).

College Students' Inappropriate Use of Variable

The difficulties students encounter when working with variables do not end at the pre-tertiary education levels, but in many instances, persist during their undergraduate studies. Rosnick (1982) interviewed ten college students and reported that their use of variables in solving word problems was "imprecise, inconsistent, paradoxical, and over-associative" (p. 24). He consequently hypothesized that his subjects' difficulties with the word problems in his study may have been an indication of a larger and more pervasive difficulty of their not being able to understand the symbols needed to solve the problems (Rosnick, 1982). White and Mitchelmore (1996) found that many first-year college students' inabilities to solve calculus problems were rooted in their lack of sufficiently well-developed concepts of variable.

Studies by Clement, Lochhead and Monk (1981), and Mestre and Lochhead (1983) also provide evidence that undergraduate students exhibit error patterns which are indicative of inappropriate use of variables as objects in their famous *Student-Professor Problem*. Specifically, Clement and his colleagues asked 150 undergraduate engineering students enrolled in a Calculus I course to write an equation to represent the statement, "There are six times as many students as professors at this university." These researchers reported that thirty seven percent of the participants responded incorrectly, with two-thirds of the incorrect responses being $6S = P$ instead of $S = 6P$. Analysis of the responses to this problem, clarified by interviews, indicated that subjects did not misunderstand the problem, but rather, they used the variables as objects, instead of a representative of the number of students. In that case, there are “six times as many students as professors” became $SSSSSS = P$, which led them to the equation $6S = P$.

Results of other studies, using variations of the *Student-Professor Problem*, indicate that students often reverse the relationship between the two variables as they attempt to model the relationship symbolically (Kaput, 1987). Clement and his colleagues (1981) identified two causes for the variable reversal exhibited by the undergraduate students in their study. The first was a static comparison of variables which are being inappropriately used as objects, and the second was word order matching. In a related study, Kaput (1987) speculated that the inappropriate use of the variable as an object in the *Student-Professor Problem* occurs because natural language use and algebraic representation are “mutually opposing” (Kaput, 1987, p.22).

Kaput asserted that a mental model for this problem is difficult to translate into symbols because it may be too close to the student’ own experience. Kaput hypothesizes that “many errors are based on the automatic ‘default’ natural language encoding processes” (1987, p. 186). In Kaput’s study with first-year algebra students, more subjects made several of these reversal errors when they were presented with situations very similar to their experiences (e.g., 5 pennies for every nickel) than with unfamiliar situations (e.g., 5 assemblers for each solderer in a factory).

Rosnick (1982) contended that the inability to define and use variables as specific unknowns in word problems appears to be a characteristic of students who have not yet progressed to the highest levels of understanding the concept of variable. To gain insight into undergraduate students' incorrect uses of variables as referents in word problems, Rosnick (1982) administered a written diagnostic test to 245 college calculus students. One type of incorrect usage of variable identified by Rosnick was that of "imprecise definition." Eighty-seven percent of his subjects either gave no definitions for the variables they used, or defined variables nominally, not quantitatively. That is, they used x as an object and wrote, "$x =$ books," instead of, "$x =$ the number of books." Rosnick stated that expert problem solvers may not always write down the definitions for the variables they use, but they could define them properly when asked to do so.

Rosnick explored this possibility by interviewing ten subjects and asking them to define their variables. According to Rosnick, all ten of these subjects defined the variables they used to solve word problems imprecisely. Nine of the ten subjects interviewed also exhibited an inconsistent use of the variable. Rosnick observed that this problem occurred when the subjects unconsciously shifted from one meaning of the variable to another related, but different meaning. He noted further that, even when these subjects were able to correctly solve a problem, they often did not recognize that fact because they did not have a clear understanding of the variable as a specific unknown - a representative of a particular quantity. This difficulty was also observed in those subjects Rosnick identified as exhibiting a "paradoxical" use of variables, which occurred when the subject used the same variable for two contradictory quantitative values. One of Rosnick's subjects had difficulty solving a pecan and cashew problem because he wrote two equations, one in which p stood for the number of pounds of pecans, and another in which p stood for the number of pecans.

Other results from studies that focused on students' use of variables in functions and their graphs (Harel & Dubinsky, 1992; Herscovics, 1989; Leinhardt, Zaslavsky & Stein, 1990; White & Mitchelmore, 1996; Romberg, Fennema & Carpenter, 1993) also revealed that subjects' were handicapped in their understanding of functional relationships by their narrow views of variables as replacements for single numbers, rather than as signifiers of continuous and related sets of numbers. White and Mitchelmore (1996) documented college students' inability to use variables to solve calculus related rates problems. In this context, variables are used as varying quantities which model relationships between dependent quantities. White and Mitchelmore contended that these relationships in calculus are much more complex than those usually seen in algebra courses. Thus, whereas the variables in algebraic functions describe how one quantity changes with respect to a change in another quantity, the variables in many calculus functions describe how the rate of change of one quantity changes with respect to the rate of change of another quantity.

White and Mitchelmore found that "a major source of student difficulties in applying calculus lies in an underdeveloped concept of a variable" (White & Mitchelmore, 1996, p. 91). They identified a large number of errors they call "manipulation focus" (White & Mitchelmore, 1996, p. 91). According to these researchers, students exhibit a manipulation focus when they simply manipulate variables as empty symbols, rather than treating them as representatives of varying quantities that are related. They cited three specific manifestations of manipulation focus in their study.

The first was the failure to distinguish the difference between a general relationship and a specific value. For example, in a problem in which subjects were to find the relationship between the rate of change of the radius and the volume of a sphere, the subjects did not view information such as $V = 64$ as a specific case. Instead, they tried to take the derivative of this value for the volume.

A second example of a manipulation focus was that students searched for symbols to which some known procedure can be applied, without regard for the meaning of the symbols. For example, again in related rates problems, subjects used a simple given relationship between two variables to substitute for one of the variables long before it was appropriate to do so.

A third example of a manipulation focus was that students only remember procedures in terms of the variables used when they were first learned. That is, despite the nature of the question, if an x and a y were somehow involved in the problem, the subjects only sought to find the derivative dy/dx. White and Mitchelmore called this last type of error, "the y, x syndrome" (White & Mitchelmore, 1996, p.91). These researchers stated that students who exhibit the manipulation focus have a concept of variable limited to the algebraic symbols alone, without regard for their meaning in the context of a problem. In effect, students are using variables as empty objects to be manipulated, rather than as representatives of the varying quantities which the functions in typical calculus problems model. Other similar studies also investigated undergraduate students' understanding of variables in inequalities. And unlike the use of the variable in equations, the use of the variable in inequalities requires that students understand and use the variable as a representative of an entire set of numbers.

In a nut shell, Usiskin stated that, the multiple ways in which variables can be used is often the main culprit in causing difficulties that many students encounter when working with variables. Usiskin (1999), thus, points to the problems caused when teachers try to teach an oversimplified conception of variable. He outlines the importance of allowing students the opportunity to explore the different uses of variables to help them develop an in-depth understanding of this fundamental but important concept in mathematics. To this end, I argue that, because students and teachers draw heavy on the content and pedagogical approaches presented in their textbooks, the ways in which variable ideas are presented in the textbooks could go a long way in providing opportunities for students to explore the different uses of variables in school mathematics.

Teachers' Understanding of Variable

Teachers' knowledge of mathematics is central to their teaching – their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing of topics, among others, are influenced extensively by their knowledge (Ball et al., 2005). Research results over the past 15 years have, however, revealed that many teachers' knowledge of fundamental concepts in mathematics is dismayingly thin (e. g., Ball, 1990; Ball & Cohen, 1996; Ma, 1999). Basic topics such as fraction, proportional reasoning, simple probability, equality symbol and variables have been documented as difficult for teachers to understand and teach efficiently (Asquith et al., 2007; Ball, 1990; Borko et al. 1992; Boz, 2002; Pittman, Koellner & Brendefur, 2007).

Boz (2002), for instance, examined the relationships between prospective teachers' subject matter knowledge and pedagogical content knowledge of variables. Participants in his study were presented with a scenario in which they had to respond to students' questions that may be asked when they teach about variables. Specifically, this study asked teacher candidates the following two questions: a) “Teacher, why does $2a + 5b$ not equal $7ab$?” and b) “While solving equations, why does x change its sign when it is brought to the other side?”

Boz reported that 80% of respondents did not provide a valid explanation to part a) of this question, and that, about half of these participants treat letters as objects. The other half did not even provide any sort of valid explanation. For the part b), 67 out of 149 (representing 45%) of the participants did not give a valid mathematical explanation as to why x changes sign when it is brought to the other side of the equality sign. Thus, these prospective teachers know that a literal symbol changes its sign when it is brought to the other side of an equation. However, they seem to forget or don’t know why this rule works, and this seems to affect their explanations. Boz concluded that these prospective teachers seemed to know the “knowing that” but not the “knowing why” about variables. He explained that, the “knowing that” of variable includes symbolization, manipulation and interpretation of the various uses of variables in different mathematical situations. The “knowing why” of variables, on the other hand, includes understanding why rules in manipulation of literal symbols work and anticipation of the consequences of using these rules.

In a more recent study, Boz (2007) investigated 184 prospective mathematics teachers’ understanding and use of letter variables and found that the teachers were comfortable using letters in familiar contexts where the variables assume the role of *unknowns* but had difficulties in using letters as *generalized numbers* or *arguments* in functional relationships. In addition, these teachers were unable to give an appropriate interpretation of the variable within the context of the problem it was used.

Mohr (2008) conducted a study in which he asked pre-service elementary teachers to complete the statement: *A variable is*………….. Mohr reported that approximately 77% of the participants stated that a variable is an *unknown*, or a number to be solved for in an equation. Only 14.5% of the participants referred to a variable as a *quantity that can change*, or a *quantity that can take on more than one value*.

Lockhead (1980) conducted a study involving 202 university faculty and asked them the following question: "Write one sentence in English that gives the same information as the following equation: $A = 7S$. S is the number of solderers in a factory. Now please clarify the answer you gave above. (p. 31)". A correct interpretation of this would be that the number of assemblers in the factory is seven times the number of solderers, since the number of solderers, represented by S, must be multiplied by seven in order to equal the number of assemblers, represented by A. Lockhead (1980) reported that thirty-five percent of the college faculty responded incorrectly.

Asquith et al., (2007) reported results from a study that focused on teachers' knowledge of students' understanding of core algebraic concepts. Specifically, this study examined middle school mathematics teachers' knowledge of students' understanding of the equal sign and variable, and students' success in applying their understanding of these concepts. The researchers found that teachers' predictions of students' understanding of variable aligned to a large extent with students' actual responses to the corresponding items. However, these teachers rarely identified misconceptions about either variable or the equal sign as students' obstacle to solving problems that required application of these concepts.

Al-Ghafri, Jones and Hirst (2002) investigated 251 trainee-teachers' explanations of students' errors in algebra and their suggested ways for addressing such errors. Findings from their study indicated that only a small proportion of trainee-teachers understand the sort of characteristics that determine the complexity of an algebra problem such as the number of variables in the problem, the nature of the elements involved and, more importantly, students' interpretations of the letter variables.

Taken as a whole, these results indicate that teachers have limited understanding of variables, as well as narrow knowledge of students' misconceptions and difficulties with variables. On the contrary, teachers' knowledge of variable should be broad, and should include, among others, the principal uses of variables, awareness of different roles of variables, flexibility, versatility and connectedness among the different roles and uses, and students' difficulties with variables. Even (1993) argues that teachers should acquire the "basic repertoire" which gives insights into, and a deeper understanding of general and more complicated knowledge.

With respect to variables, the general and more complicated knowledge is the teachers' ability to integrate all the uses of variables into one concept, being able to shift from one use to the other flexibly, and being aware of students' difficulties and misconceptions with this concept. Thus, teachers' knowledge and understanding of variables should involve being able to ascribe different roles to symbols to represent problem situations and to manipulate them correctly. It should also involve, as Ursini and Trigerous (1997) put it, being able to "distinguish between the different roles and shift from one to another in a flexible way, integrating them as components of the same mathematical object." (p. 254).

Recognizing that teachers do not always possess the knowledge necessary to teach some of these fundamental concepts in mathematics, many curriculum developers include in their teacher's edition, notes on how and why certain contents are important and presented in ways they are. The National Research Council (NRC, 2004) recommends that "a curriculum should include enough support for teachers to enact it as intended. Such support should allow teachers to educate themselves about mathematics content, students' mathematical thinking, and relevant classroom issues…. It might help … teachers to analyze common student errors in order to think about next steps for those who make them" (NRC 2004, p.76). To this end, there are some K-12 curriculum materials (called *educative curriculum materials*) that are designed to not only promote students' learning but also to promote teachers' learning (Davis & Krajcik, 2005; Schneider & Krajcik, 2002).

Ball contended that teachers' opportunities to learn must equip them with the mathematical knowledge and skill that will enable them to teach mathematics effectively. One way to achieve this (in the case of the concept of variables) could be to design opportunities in the instructors' textbooks for teachers to delve more deeply into the concept of variable. The overriding purpose of such supports would be to provide teachers with ample opportunities to learn about the various uses of variables in school mathematics, how the uses are related, and to learn about students' misconception and inappropriate use of variable as well as how students can learn about variables more appropriately, among others.

In sum, providing teachers with curriculum materials that not only design opportunities for students to engage meaningfully with variables but also offer teachers the guidance necessary to teach them successfully in their classrooms could help teachers develop a better understanding of variables, and place them in a better position to teach them appropriately for students to understand. These fore-going discussions and others have prompted the need to include the examination of teachers' textbooks in this study.

Use of Mathematics Textbooks in the Classroom

Textbooks are necessary and effective elements for teaching and learning in every classroom in the world. Down (1988) stated that textbooks for better or worse dominate what students learn. They set the curriculum, and often the facts learned in most subjects. For many students, textbooks are their first and sometimes the only early exposure to books and reading. The public regards textbooks as authoritative, accurate, and necessary. And teachers rely on them to organize lessons and structure subject matter.

Robitaille and Travers (1992) found that textbooks are present not only in classrooms, they are also frequently used by teachers and students, and influence the instructional decisions that teachers make on daily basis (Tyson-Bernstein & Woodward, 1991; Robitaille & Travers, 1992). With respect to textbook use in mathematics classrooms, Grouws and Smith (2000) reported that most middle-grades mathematics teachers use most of the textbook most of the time. These researchers observed that the mathematics teachers of 75% of the eighth grade students involved in the 1996 National Assessment of Educational Progress (NAEP) reported using their textbooks on a daily basis.

Begle's (1973) data from the National Longitudinal Study of Mathematics Achievement cites evidence that students learn what is in the text and do not learn topics not covered in the textbook. Over 75 percent of the teachers in a survey sponsored by the National Advisory Committee on Mathematics Education reported using a single textbook predominantly in the classroom and 53 percent of the teachers reported that they followed the texts closely (Porter, 1981). A similar report was provided by Weiss et al. (2001) that more than 60% of middle-grades mathematics teachers “cover” at least three fourths of their textbook each year. Research on students’ use of mathematics textbooks in the 2000 NAEP was not any different from that of teachers’ use. Specifically, Braswell et al. (2001) documented that 72% of participating eighth grade students in the 2000 NAEP reported doing mathematics problems from a textbook every day.

All these reports seem to support the notion that textbooks have a marked influence on what is taught and learnt in many mathematics classrooms. Consequently, it would be a worthwhile endeavor to investigate curricular materials that many mathematics teachers and students use to determine the prospects of such resources in impacting students’ opportunities to learn and acquire facility with variables (an essential concept in school mathematics).

Textbook Content Analysis in Mathematics Education

In the following pages, I describe a number of similar curriculum analyses (sometimes referred to as *content analysis* in the research literature) conducted in mathematics education in recent years. This will support and strengthen the need for the present study and will also fit it into the existing body of research in this area.

Content analysis is a research technique used to determine the presence of certain words, symbols or concepts within texts or sets of texts. In content analysis, researchers quantify and analyze the presence, meanings and relationships of such words and concepts, then make inferences about the messages within the texts. Texts can be defined broadly as books, book chapters, essays, interviews, discussions, newspaper headlines and articles, historical documents, speeches, conversations, advertising, theater, informal conversation, or any occurrence of communicative language.

The U.S. Department of Education (1999), for example, conducted content analysis for promising and exemplary programs in terms of quality of program, usefulness to others, educational significance, and evidence of effectiveness and success. Specifically, this study examined the materials based on whether the programs' learning goals were challenging, clear, and appropriate. The researchers also looked for the content alignment with its learning goals, its accuracy and appropriateness for the intended audience; that is, whether the instructional design was engaging and motivating for the intended student population.

Furthermore, the materials were examined to see if the system of assessment was appropriate and was designed to guide teachers' instructional decision making. Other questions addressed in this study were: whether the program can be successfully implemented, adopted, or adapted in multiple educational settings; whether the programs' learning goals reflect the vision promoted in national standards in mathematics; whether the programs' learning goals address important individual and societal needs; and whether the program makes a measurable difference in student learning. There were 61 programs voluntarily submitted by the developers or publishers to be included in this study. The researchers found five of these programs to be *exemplary* because they provided convincing evidence of their effectiveness in multiple sites with multiple populations. Another five of the programs were designated *promising* based on preliminary evidence of effectiveness in one or more sites.

Clopton and colleagues (1999) used the Mathematically Correct and the San

Diego Standards to rate 10 curricular programs for grades 2, 5, and 7 based on mathematical depth, clarity of objectives, clarity of explanations, concepts, procedures, and definitions of terms. This review includes a summary of the structure of the program, evaluations of a selected set of content areas, and evaluations of the programs' quality. In terms of student work, the researchers focused on the quality and sufficiency of student work and its range, depth, and scope. The team also established criteria at each of three grade levels (2, 5 and 7) that represented significant progress in mathematics achievement that will produce competent eighth-graders ready for a rigorous algebra program. The ratings in these areas were made on a scale from 1 (poor) to 5 (outstanding) with the overall evaluation being assigned a letter grade.

Among other findings, Clopton and colleagues reported that there exist programs that are reasonably well on target to achieve the desired level of students' mathematics development. On the other hand, there also exist programs that are not even close to achieving these objectives. The majority of programs reviewed fall in the middle of this range (i.e., they provide some support for modest levels of achievement, but would need supplementation to varying degrees to approximate the desired level of achievement). Furthermore, Clopton *et al.* observed that the programs reviewed vary considerably in style and in the demands they place on teachers. In some cases, the achievements, even to the levels indicated, would require considerable expertise and professional judgment on the part of the teacher. These researchers, thus, envision many of these programs being implemented ineffectively, and advised that this be an important factor when schools are considering curriculum materials for adoption.

The American Association for the Advancement of Science (AAAS) conducted a similar study under Project 2061 (AAAS, 1999). This project developed a methodology and used it to review selected middle-grades mathematics curricula and some high school algebra materials in reference to the *Curriculum and Evaluation Standards for School Mathematics* from the National Council of Teachers of Mathematics (NCTM, 1989). The team of researchers analyzed the curriculum materials for alignment between instruction and the selected learning goals. This involves, among others, estimating the degree to which the materials (including their accompanying teacher's guides) reflect what is known generally about students' learning and effective teaching, and the degree to which the materials support students' learning of the specific knowledge and skills for which a content match has been found (NRC, 2004).

The report explained that the focus on examining the learning goals was to look for evidence that the materials have a sense of purpose; build on students' ideas about mathematics; engage students in mathematics; develop mathematical ideas; promote students' thinking about mathematics; assess students' progress in mathematics; and enhance the mathematics learning environment. With respect to algebra in the high school mathematics curriculum, the review encompassed ideas from functions, operations, and variables. Similar topics examined in the middle-grades include number concepts, number skills, geometry concepts, geometry skills, algebra graph concepts, and algebra equations.

Thirteen contemporary mathematics textbook series written specifically for middle grades students between 1994 and 1999 were evaluated according to a set of benchmarks related to the aforementioned core content that should be present in mathematics instruction. Each textbook series was rated as having most, partial, or minimal content according to each benchmark. The research team found that only four of the textbook series addressed four or more benchmarks in depth, and no series sufficiently addressed all of the benchmarks. In terms of quality, none of the popular textbooks were among the best rated.

A study by Adams et al. (2000) used the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000) to compare Singapore's middle school mathematics curriculum with the USA *Connected Mathematics Program* and *Mathematics in Context* curricula. The research teams' method of content analysis was based on 72 questions that compared the curricula against the 10 overarching standards (number, algebra, geometry, measurement, data and probability, problem solving, reasoning and proof, communication, connection, and representation), and 13 questions that examined six principles (equity, curriculum, teaching, learning, assessment, and technology).

The two USA programs were chosen, in part, because evidence by AAAS's Project 2061 suggested that they were among the top of the 13 National Science Foundation-sponsored projects studied. An interesting and valued feature in the Adams report was that these programs were compared with the "traditional" Singapore mathematics textbooks "under the authority of a traditional teacher" (Adams et al., 2000, p.1). Adams and colleagues explained that they selected the "traditional approach" program from Singapore as a measure of comparison based on the performance of students from both countries on the TIMSS 1995 assessment. Specifically, while the performance of students from the United States on TIMSS "dropped from mediocre at the elementary level through lackluster at the middle school level and down to truly distressing at the high school level", the performance of students from Singapore was at the very top (Adams et al., 2000, p.1). When these NSF programs were compared to the NCTM (PSSM), they showed strong alignment, but when contrasted with the Singapore curriculum, they revealed delays in the introduction of basic materials.

Valverde, Bianchi, Wolfe, Schmidt and Houang (2002) analyzed the content of the textbooks in their sample according to the characteristics of lessons. These characteristics included the primary nature of lessons (concrete and pictorial vs. textual and symbolic), components of the lesson, and student performance expectations. To measure textbook lessons along these dimensions, the researchers divided lessons into blocks, "classified according to whether they constituted narrative or graphical elements; exercise or question sets; worked examples; or activities" (p. 141). The research team found that "textbooks across all populations were mostly made up of exercises and question sets" (p. 143). They also reported that the amount of narrative and worked examples increased over the three grade levels, whereas the number of activities decreased. The analysis further revealed that, "the most common expectation for student performance was that they read and understand, recognize or recall or that they use individual mathematical notations, facts or objects. This is followed . . . by the use of routine mathematical procedures" (p. 128).

Jones and Tarr (2007) analyzed probability content in selected popular middle-grades mathematics textbooks (6, 7 and 8) from a historical perspective, selecting two series, one *popular* and the other *alternative*, from four recent eras of mathematics education in the United States (New Math, Back to Basics, Problem Solving, and the Standards) to determine the extent to which probability content was treated, the types of instructional devices used and the cognitive level of the probability tasks in these textbooks. Jones and Tarr found that Standards era textbook series devoted significantly more attention to probability than other series, with more than half of all tasks analyzed located in Standards era textbooks. In addition, a little over 85% of tasks for six series required low levels of cognitive demand, whereas the majority of tasks in the alternative series from the Standards era required high levels of cognitive demand.

Stylianides (2005) developed an analytic framework and used it to investigate the opportunities that are designed in middle-grades mathematics curriculum for students to engage in reasoning and proof. The framework he developed was used to analyze the algebra, geometry, and number theory units of the *Connected Mathematics* textbooks. Another major objective of this study was to examine the nature of support the curriculum materials provide for teachers to enact the proof and reasoning opportunities in the classroom. Stylianides found that about 40% of the tasks are designed to engage students in reasoning and proving, with many more opportunities designed for students to use inductive rather than deductive modes of reasoning. He also found that the reasoning and proving opportunities were distributed unevenly across grade levels and content areas, and very limited guidance was provided to teachers to support their use of proof tasks.

Taken as a whole, the findings from these studies point to the importance of how the results of content analysis of mathematics education programs or particular topics and concepts in mathematics can provide valuable insights on existing mathematics education programs and materials. This information could help in the design of future mathematics education programs, to enhance classroom instruction and promote students' learning.

Mathematics Education Eras in the United States

Mathematics education in the United State has undergone a number of reforms over the years in attempts to improve or amend existing practices in school mathematics that are judged unsatisfactory at a given point in time. Below, I describe four of the most recent eras of mathematics education in the USA within which this study is situated.

New Math

Recent reforms in mathematics education in the USA can be traced to 1957, when the USA responded to the U. S. S. R.'s technological advancement shown by the launching of the Sputnik satellite (DeVault & Weaver, 1970; Osbourne & Crosswhite, 1970). This led to a focus on school mathematics programs to develop "high quality mathematics for college-capable students, particularly those heading for technical or scientific careers" (National Advisory Committee on Mathematics Education [NACOME], 1975). These curricula were to allow the use of an approach to mathematics that will enable children to learn basic language and structure of mathematics as soon as possible.

This mathematics program, known as "new math", according to researchers (Howson, Keitel & Kilpatrick, 1981), took set-theoretic foundations of mathematics and used them to create curricula dominated by attention to formal structure, properties, deductive proof, and building numeric systems relying heavily on the ideas of set, relation and function with the major aim of linking school mathematics with university or higher mathematics. Howson, Keitel and Kilpatrick (1981) pointed out that the need to bring the content of school mathematics more in line with university mathematics became necessary because the century preceding World War II saw the discovery of more mathematics than ever existed in the history of man.

A few years after its implementation, however, there were complaints from parents about the content of the new curricula. This complaint was also fueled by poor students' performance, and students' lack of practical understanding of the mathematics that was taught. A growing concern also emerged from the public and elementary school teachers that students were unable to compute accurately (Payne, 2003). These issues led the NACOME in 1975 to re-examine the new math curriculum and make recommendations for revision (NACOME, 1975).

The committee recommended, among other things, that "a) the logical structure of mathematics be maintained, b) concrete experiences be an integral part, c) applications be included to a wide realm, d) symbols and formalities be fostered, e) by eighth grade, a calculator be made available, f) metric system be used, and g) statistical ideas be included" (p. 139). Despite these recommendations and efforts to improve the system, there were concerns that the "new math" was leaving too many students behind unprepared to solve everyday word problems and to apply school mathematics concepts beyond the classroom. This led to the emergence of a different program, known as the *back to basics* program, described below.

Back to Basics

By the 1960s, the *new math* materials were being produced commercially, but as explained in the preceding paragraphs, problems arose in terms of teacher training and implementation of the program. Specifically, the new math curricular materials were being used by teachers who were not well trained in its concepts, resulting in students learning abstract concepts without properly obtaining the necessary basic skills needed

(Walmsley, 2001). Elementary teachers complained that the introduction of abstract concepts brought about confusion, and that many students were not even able to complete simple addition tasks correctly (Walmsley, 2001).

Critics also saw that the new math materials were based on theory and not on empirical evidence (Hayden, 1981; Park, 2001). This growing concern about the new math program blossomed into a full-fledged reactionary movement in the 1970s that focused students on the fundamentals of mathematics. Porter et al. (1991) described this movement as “guaranteeing basic skills became the agenda; easy content for all students.” (p. 12).

Consequently, at the beginning of the mid to the late 1970s, a new movement known as “back to basics movement” emerged in response to addressing some of the weaknesses identified in the new math program. The growth of this program (back to basics) was also fueled by the complaints from the business community that the existing curriculum was producing graduates who lacked simple mathematics skills needed in business (Confrey, 2007). This program, in which the “basics” were primarily defined as computational skills (Usiskin, 1985), was accompanied heavily by an “accountability” movement that looked at results or students’ performance. As the new math program, the back to basics movement failed to achieve its objectives, and was short-lived as a result.

Problem Solving

In the 1980s, two national reports were released advocating a new look at the school mathematics program in order to “prevent the eroding educational foundations and threats to the future of the American society” (Gardiner, 1983, p. 5). These reports were *An Agenda for Action* published by the NCTM, and *A Nation at Risk* published by the

National Committee on Excellence in Education. The NCTM formed a *Commission on Standards for School Mathematics*, charging the commission to:

> Create a coherent vision of what it means to be mathematically literate, both in a world that relies on calculators and computers to carry out mathematical procedures and in a world where mathematics is rapidly growing and extensively being applied in diverse fields, and create a set of standards to guide the revision of the school mathematics curriculum and its associated evaluation towards this vision (NCTM, 1989, p. 1).

Specifically, after a decade of focused attention on procedures and algorithms (back to basics), the NCTM (1980) and other organizations (College Board, 1983; National Academy of Sciences and National Academy of Engineering, 1982; National Commission on Excellence in Education, 1983; National Science Foundation and Department of Education, 1980) issued reports calling for focus on *problem solving* in mathematics classrooms during the 1980s. Usiskin (1985) summarized these recommendations as follows: "Taken as a body, reports from inside and outside mathematics education agree almost unanimously that … emphasis should be shifted from rote manipulation to problem solving" (p. 15).

NCTM Standards

In 1989, the NCTM published *Curriculum and Evaluation Standards for School Mathematics*, calling for reform of mathematics education on a wide scale. In this document, the Council provided recommendations for mathematical content that ought to receive increased or decreased attention in the classroom and outlined important mathematical processes, such as problem solving and communication that should be encouraged and fostered as students do mathematics. This document, along with their subsequent publications: *Professional Standards for Teaching Mathematics* (NCTM, 1991) and *Assessment Standards for School Mathematics* (NCTM, 1995) provided classroom teachers and mathematics educators with a conceptual anchor for reforming their practice (Jones & Tarr, 2007).

In particular, several internal factors and international influences in the 1990s lead the NCTM to refine the existing mathematics curriculum. Scholars concerned with the results of the United States in international comparisons, for example, began to raise concerns that United States practices were insufficiently competitive in mathematics and science. In fact, this concern began with the publication of *The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective* (McKnight et al., 1987) which was based on the results from the Second International Mathematics and Science Study (SIMMS, 1982) and continued with Harold Stevenson, an international scholar on Asian instruction, who argued that the NCTM Standards were too vague, that the grade bands were too obscure and broad, combined too many vision statements, and had too much focus on pedagogy with content, and lacked clear measurable criteria (Stevenson, n.d.).

This argument was supported by the results of the Third International Mathematics and Science Study (TIMSS) and video study by Stigler and colleagues (1997) of selected curricula and classroom practices of some better performing countries including Japan and Germany. The need for revising the mathematics curriculum found further support from students' poor performance on internal assessments such as the NAEP, SAT, and decreases in the number of students entering scientific and mathematical fields.

In an attempt to focus the reform of mathematics education into the new millennium, the NCTM published *Principles and Standards for School Mathematics* (NCTM, 2000). This document represented further refinements of the earlier Standards documents in an integrated format, and provided more detailed narrative of the recommendations of the Council. These standards were intended, among other things, to ensure quality, identify explicit goals, and promote change. They were purposed to create mathematically literate workers, encourage lifelong learning, provide opportunities for all, to use technology broadly, to integrate mathematical content, to address equity and support an informed electorate (Confrey, 2007).

The *Standards* were also structured by grade bands, and each addressed standards of problem solving, communication, reasoning and proof, connections, and representation. Issues of pedagogy were integrated into issues of content, emphasizing the importance of active participation in learning by students. The *Standards* also drew heavily on research on students' thinking, students' misconceptions and how students learned particular ideas as they encounter challenging tasks (Confrey, 2007). The documents downplayed heavy reliance on memorization and procedural understanding based on numerous studies documenting disintegration of students' apparent knowledge when asked for reasons and explanations, and stressed conceptual understanding (Erlwanger, 1973; Ginsburg, 1991; Kamii, 1985). The NCTM is still revising the current curriculum by identifying *Curriculum Focal Points for PreK - 8 Mathematics* (NCTM, 2006) to meet the mathematics education needs in the United States.

Impetus for the Present Study

Thus far in this chapter, I have discussed the conceptual framework of variables as put forward by Küchemann (1978, 1981), Usiskin (1988), Philipp (1992), Schoenfeld and Arcavi (1988), and Wagner (1999). The discussion also included various studies that examined students' and teachers' understanding of variables. Research studies that examined textbook use in mathematics education, as well as content analyses in mathematics education have been discussed. I also described in this chapter four eras of mathematics education in the United States within which this study is situated, and the factors that contributed to the birth of each of the eras. In the following pages, I make connections between the foregoing discussions and the present study by explaining the rationales behind investigating each of the research questions of interest to the study.

To begin with, the extant literature reveals many different definitions of variable (Schoenfield & Arcavi, 1988). Consequently, it might be important to determine which definition(s) of variable is/are found in middle-grades mathematics textbooks. In addition, it might be helpful to substantiate (or do otherwise) the contention by researchers (e.g., Graham & Thomas, 2000; Kieran, 1981) that in spite of the importance of the concept of variable, many elementary textbooks relegate its treatment to the background - they do not explain or even mention it. As a result, one would like to know if popular middle-grades mathematics textbooks define and introduce the concept of variable, which definition(s) they offer, and if the definition(s) are similar across middle-grades mathematics textbook series from different eras of mathematics education in the United States. This could inform the mathematics education community about the kind of definition(s) of variable that students are most likely to provide if they are asked to define this term, or interpret the role of variables within the context of a given mathematical task. It could also shed light on the findings from the research literature that point to "imprecise definition" as one type of incorrect usage of variable by students (Rosnick, 1982).

Second, the NCTM recommends in the *Principles and Standards for School Mathematics* (2000) that "students at the middle grade level should develop an initial conceptual understanding of different uses of variables" (NCTM, 2000, p. 221). This recommendation is consistent with the calls by earlier researchers (Usiskin, 1988; Wagner, 1999) that students be given ample opportunity to explore the various uses of variables, and not an over-simplified version of it. In spite of these recommendations, findings from the literature seemed to indicate that the majority of students and some teachers think of variables as representing a single or unique number (Kieran, 1992; Mohr, 2008). And also that, students are more likely to conceive and use variables as *labels* or *unknowns* than they do as *varying quantity*, *generalized numbers* or *abstract symbol*, among others (e.g., Boz, 2007; Clement et al., 1981; Mohr, 2008). It might, thus, be possible that the frequency of use of each of the categories of variable in mathematics textbooks, and the order in which the categories occur in the textbooks may explain why students understand, and use one conception of variable more frequently than the others. This establishes the need to examine how middle-grades mathematics textbooks develop the concept of variable, and the order of this development.

Third, although the concept of variable identifies traditionally with algebra, recent recommendations in mathematics education call for connections among the content areas (NCTM, 2000). As a result, it might be informative to know how variable is used in algebra, and in the other content areas in school mathematics (i.e., geometry, number and operations, measurement, and data analysis and probability), as well as the uses that are prevalent under each of the content areas in middle-grades mathematics textbooks. Thus, a major objective of the present study is to examine how the uses of variable play out in different content areas in middle-grades mathematics textbooks.

Fourth, it might be possible that different levels of emphases are placed on the use of variables in general, and on the use of the various categories of variables in mathematics textbooks for different grade levels (i.e., grade 6, 7 and 8). I speculate that the extent of inclusion of variable ideas in middle-grades mathematics textbooks will increase in number, as well as in sophistication as one progresses from the 6th to the 7th, to the 8th grade. This speculation, however, needs to be substantiated.

Fifth, due to the growing emphases on algebra as a gateway to school mathematics, and the increased attention to connecting the content areas (*Connection Standards*) in recent recommendations of professional organizations such as the NCTM (1989, 2000, 2006), one might expect to observe changes in the treatment and presentation of algebra, and for that matter, changes in the treatment of variable (which is central to the study of algebra) in school mathematics textbooks over time. Unfortunately, there has not been any systematic study on the treatment of this concept in mathematics textbooks over time. Consequently, it is important to reveal the degree to which middle-grades mathematics textbooks have or have not maintained the status quo in terms of the treatment of variable during the past five decades.

The last, but not the least, is the need to examine the support provided to teachers on the treatment of variables. The research literature suggests that many teachers have limited understanding of variables (Al-Ghafri et al., 2002; Asquith et al., 2007; Boz, 2002, 2007). Consequently, it is logical to believe that even if there is a plethora of opportunities designed in the curriculum materials for students to engage with variable in different contexts and levels, it is likely that teachers will need explicit guidance in the teachers' edition in order to implement variable tasks in ways that are consistent with the curriculum developers' intents. Thus, it seems not to be enough to examine only the opportunity for students to engage with variables, but equally important is the need to examine the guidance offered to teachers on how to enact these opportunities in the classroom. In this respect, guidance to teachers could include explicit notes on the various uses of variables, alerting teachers to the various definitions of variable in mathematics textbooks and in the research literature, enlightening teachers on students' common misconceptions and possible students' confusions with variables, and the likes. Collectively, these rationales provided the impetus for the conduct of the study.

Summary

In this chapter, I presented five conceptual frameworks on variable and reviewed the related literature on students' and teachers' understanding of variables. This was followed by the literature on the extent of use of mathematics textbooks in the classroom, research on related content analysis in mathematics education, and the descriptions of four eras of mathematics education in the United States. I ended the chapter by connecting the conceptual framework, the reviewed literature and the present study. Chapter 3 contains the method, the design, and a description of the data analysis techniques. Chapter 4 presents the results and Chapter 5 discusses the results of the study, implications of the results, and the recommendations for future research.

Chapter 3

Research Methodology and Design

In this chapter, I present the method and the design of the study. My discussion is organized into four sections. First, I identify the six research questions that the study addresses. Then I provide the criteria for the four eras of mathematics education programs in the United States within which this study is situated. Next, I describe the sample of mathematics textbooks for the study, and provide rationales for selecting each of the mathematics textbook series that is intended for examination in the study. Last, I present the research design and the analytical framework that was used to examine the variable ideas in the selected mathematics textbooks. The analytical framework is also illustrated with some specific examples from the UCSMP *Pre-Transition Mathematics* (Grade 6) and *Transition Mathematics* (Grade 7) textbooks that I used in my pilot study to test the applicability of the framework.

Research Questions

The present study investigated the extent to which selected middle-grades mathematics textbooks develop the concept of variables for students to learn with an in-depth understanding and for teachers to implement successfully in the classrooms. I accomplished this by addressing the following six research questions:

1. How do middle-grades mathematics textbooks develop the concept of variable (i.e., in terms of whether and how they introduce, define and/or explain it, and

at which grade level(s) in the middle-grade mathematics textbooks do these occur)?

2. To what extent do middle-grades mathematics textbooks present activities and tasks that address each of the uses of variables (e.g., *labels*, *unknowns*, *generalized numbers*) as described by researchers in the mathematics education community, and in which order do these uses occur in the mathematics textbooks?
3. Which use(s) of variable is/are prevalent within which content areas (i.e., geometry, number theory, algebra, etc.) in middle-grades mathematics textbooks?
4. How does the development and/or presentation of the concept of variable differ across different grade levels of the same textbook series (e.g., in terms of their compositions of the various categories of uses of variable)?
5. To what extent has the development and/or the presentation of variables in middle-grades mathematics textbooks changed within the past five decades (i.e., by comparing the development among the textbooks from the four major eras of mathematics reform in the USA)?
6. To what extent do the teacher's editions of middle-grades mathematics textbooks provide guidance to teachers on the treatment of variables (i.e., in terms of alerting teachers to the various uses of variables, to students' misconceptions, and to students' difficulties with variables)?

Specific Timelines for Mathematics Education Reform Eras

This study is situated within four recent eras of mathematics education in the

United States. I refer to them as the *New Math*, *Back to Basics*, *Problem Solving*, and the *NCTM Standards*, and explained each of them previously in Chapter 2. The organization of these time periods is similar to those described by some earlier researchers (Carnine, 1995; Jones & Tarr, 2007; Nibbelink, Stockdale, Hoover & Mangru, 1987). The designation of timelines for mathematics education eras in this study is more in line with the classification used by Jones and Tarr (2007) in their analyses of probability content in popular middle-grades mathematics textbooks. These researchers referred to the period 1957 to 1972 as the *New Math*, the period 1973 to 1983 as the *Back to Basics*, the 1984 to 1993 as the *Problem Solving*, and the 1994 to 2004 as the *NCTM Standards*.

I wish to state here that it is difficult to determine the precise beginning and the end of each era, and a significant event that marks the start of a new era of mathematics education in the United States. For example, Jones and Tarr (2007) noted that the publication of the *Curriculum and Evaluation Standards for School Mathematics* in 1989 does not necessarily immediately impact the textbooks that were published that year or the following year. Nevertheless, for the purpose of this study, I acknowledge the need to specify timelines for each era. Hereafter, I refer to the years 1957 to 1972 as the *New Math* era, the period 1973 to 1983 as the *Back to Basics* era, 1984 to 1993 as the *Problem Solving* era, and 1994 to 2009 as the *NCTM Standards* era. For each of these eras, I selected one popular students' edition mathematics textbook series for grades 6, 7 and 8 that is intended for the average-student, and the corresponding teachers' guides for examination with respect to their treatment of the concept of variable. Table 3 displays the years that depict the beginning and the terminal points of each era.

Table 3

Operational Time Frames for Recent Eras in Mathematics Education in the USA

Name of Era	Year
New Math	1957-1972
Back to Basics	1973-1983
Problem Solving	1984-1993
NCTM Standards	1994-2009

Note. Adapted from "Probability in middle grades mathematics textbooks, 1957-2004" by D. L. Jones (2004), Doctoral dissertation, University of Missouri.

Sample of the Study

The sample for this study was popular middle-grades (6th, 7th and 8th grade) mathematics textbooks used during the past five decades categorized into four eras of mathematics education in the USA. A total of 24 popular middle-grades mathematics textbooks for the average student were examined in this study. This composition is made up of 12 student editions and their corresponding 12 teachers guides selected from the four recent eras of mathematics education in the U.S.

Why Choose the Middle Grades?

The middle grades is chosen for this study, in part, because it is regarded as a crucial stage for students' transition from arithmetic to algebra where students are more likely to be exposed to the uses of letter variables to a greater extent than their prior arithmetic experience might have offered them at the elementary school (Herscovics & Linchevski, 1994; Ketterlin-Geller, Jungjohann, Chard & Baker, 2007; Kieran, 2004; Kieran & Chalouh, 1992). Second, the extant literature reveals a lack of detailed knowledge on how the concept of variable is treated in middle-grades mathematics textbooks. Specifically, no study was found in the research literature that examined the treatment of this concept at the middle-grades from a curricular perspective.

Why Choose Popular Textbooks for Average Students?

Popular middle-grades mathematics textbooks for "average students" were chosen because the goal was to investigate the materials on the treatment of *variables* that were available to the majority of the students during each of the identified eras of mathematics education in the United States in order to gain a broader perspective on the intended opportunities students have to engage with this vital concept in school mathematics. The *average students'* textbook is defined as the mathematics textbook that is neither for remedial nor accelerated students. This definition also excludes textbooks that are primarily focused on algebra, and are geared toward mathematically promising students or towards remedial students in the middle-grades.

Why Examine both the Students' and Teachers' Editions?

Both the students' and teachers' textbooks were examined because the existing literature indicates that both groups have limited understanding of variables (e.g., Asquith et al., 2007; Boz, 2002; Clement et al., 1981; Kaput, 1987; Mohr, 2008, Rosnick, 1982; White & Mitchelmore, 1996). Consequently, it is not sufficient to examine only the opportunity for students to engage with variables, but equally important is the need to examine the guidance provided for teachers on how to enact these opportunities in the classroom since the curriculum is not self enacted. Thus, the examination of both the teachers' and students' editions provides a more comprehensive insight on the treatment of this topic in the curriculum than would be gained by investigating either one of the curricula alone.

Why Examine all the Content Areas in the Selected Mathematics Textbooks?

All the content areas in the selected mathematics textbooks were examined for variable ideas. Although the concept of variable is traditionally identified with algebra, the uses of variables are not restricted to algebra. That is, variable ideas are also used in geometry, number and operations, measurement, and data analysis and probability content areas. In addition, recent recommendations in the mathematics education community (NCTM, 2000) call for connection among the content areas. For these reasons, it is appropriate to gain an understanding of how variable is used in algebra, as well as, how it is used in the other content areas in school mathematics.

Specific Textbook Selection Criteria

To be included in this study, the middle-grades mathematics textbooks had to be classified as popular or have large market share in U.S. middle schools during one of the four eras of mathematics education in the United States (*New math*, *Back to Basics*, *Problem Solving*, and *NCTM Standards*). I used the textbook market share data from Weiss (1978, 1987) and Weiss et al. (2001) to select textbooks from each of the four eras of mathematics education in the USA for examination. In the absence of market share data, I determined popular textbook series based on the consensus from experts of mathematics educators who were familiar with the middle-grades curriculum during the past 50 years and affiliated with the Center for the Study of Mathematics Curriculum (Jones & Tarr, 2007). Furthermore, each of the selected mathematics curricula had to have three textbooks, one each for grades 6, 7 and 8, and be intended for use by the average level students. To further help with the identification and selection of mathematics textbooks for this study, I drew on the criteria used by Jones and Tarr (2007) in their selection of popular middle-grades mathematics textbooks (6, 7 and 8) for their analyses of probability content from a historical perspective.

Selected Middle-Grades Mathematics Textbooks

Using the above criteria, the following mathematics textbooks were selected for examination in this study:

Table 4

Eras in Mathematics Education, Timelines and Textbooks Selected

Name of Era	Year	Textbook	Publisher
New Math	1957-1972	Modern School Mathematics	Houghton Mifflin
Back to Basics	1973-1983	Holt School Mathematics	Holt, Rinehart and Winston
Problem Solving	1984-1993	Mathematics Today	Harcourt Brace Jovanovich
NCTM Standards	1994-2009	Math Connects	Glencoe/McGraw-Hill

Table 4 summarizes information on each era and the respective mathematics curriculum selected for that period. The *Modern School Mathematics* textbook series selected for the New Math era was written by Dolciani, Beckenbach, Wooten, Chinn and Markert (1967a, 1967b), and Duncan, Capps, Dolciani, Quast and Zweng (1967c) and published by Houghton Mifflin. The *Holt School Mathematics* textbook series selected for the Back to Basics era was written by Nichols, Anderson, Dwight, Flournoy, Kalin, Schluep and Simon (1974a, 1974b, 1974c), and published by Holt, Rinehart and Winston. The *Mathematics Today* textbook series selected for the Problem Solving era was written by Abbott and Wells (1985a, 1985b, 1985c), and published by Harcourt Brace Jovanovich. And the *Math Connects* textbook series selected for the NTCM Standards era was written by Day, Frey and Howard, and McGraw-Hill authors (2008), and published by Glencoe/McGraw-Hill. I wish to state here that the *Math Connects* textbook series was formally known as *Mathematics: Applications and Connections* (Collins et al., 1998a, 1998b, 1998c) and was published by Glencoe/McGraw-Hill.

It is also worth reporting here that there is no textbook market share data for the New Math era. Because of the lack of this data, the selection of the *Modern School Mathematics* textbook series for the New Math era was based on a "professional consensus" of mathematics educators familiar with the middle-grades curriculum during the past 50 years, and affiliated with the Center for the Study of Mathematics Curriculum. According to Jones and Tarr (2007), the following professors: Glenda Lappen, Betty Phillips, and Sharon Senk (all of Michigan State University); Zalman Usiskin (University of Chicago); Denisse Thompson (University of South Florida); Douglas Grouws, Barbara Reys, and Robert Reys (all of University of Missouri); and Christian Hirsch (Western Michigan University), all of whom are professors of mathematics education reported that the *Modern School Mathematics* textbook was one of the most (if not *the* most) popular textbook series for middle-grades students during the New Math era.

The selections of the *Holt School Mathematics* textbook series for the Back to Basics era, and the *Mathematics Today* for the Problem Solving era were based on data from Weiss (1978, 1987). Also, because textbook market share data were not available for 2009, I used the Weiss et al. (2001) data to identify popular textbook series in use by 2001 for the Standards era. These data identified *Mathematics: Applications and Connections* (Collins et al., 1998a, 1998b, 1998c) as the most widely used mathematics textbook for the Standards era. However, in this study, I used the current replacement of the *Mathematics: Applications and Connections,* the *Math Connects* (2008) textbook series, as the most popular textbook series for the Standards era.

Each of the selected textbook series was designed for grades 6, 7, and 8 as specified in the textbook selection criteria. The *Modern School Mathematics* textbook series were labeled 6, 7 and 8 respectively. The *Holt School Mathematics* textbook series were labeled Levels 64 – 76 (grade 6), Levels 77 – 91 (grade 7) and Levels 92 – 106 (grade 8). The *Mathematics Today* textbook series were labeled Brown (grade 6), Silver (grade 7), and Gold (grade 8). And the *Math Connects* textbook series are labeled as Course 1 (grade 6), Course 2 (grade 7), and Course 3 (grade 8).

Research Design

In this study, I used content analysis (Krippendorff, 2004; Magid, McKnight, McKnight, & Murphy, 2000) and mixed methods research design (Johnson & Onwuegbuzie, 2004; Tashakkori & Teddlie, 1998) to collect and analyze both qualitative and quantitative data in order to address the six research questions.

Content Analysis

There are many definitions of content analysis, but on a basic level, content analysis can be said to be the analysis of the content of some type of message or symbol. Stone, Dunphy, Smith and Ogilvie (1966) defined content analysis as "….any research technique for making inferences by systematically and objectively identifying specific characteristics within text" (p. 5). Similarly, Krippendorff (2004) defined content analysis as: "a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the contexts of their use" (p.18). Along the same lines as Krippendorff, Weber (1990) described content analysis as "a research method that uses a set of procedures to make valid inferences from text" (p. 9).

Magid, McKnight, McKnight and Murphy (2000) identified five main steps in the process of developing a content analysis: 1) identify documents that are relevant to the research purpose; 2) select a sample document to analyze; 3) develop a category-coding procedure; 4) conduct the content analysis; and 5) interpret the results.

That is, to conduct content analysis, the researcher must first identify documents that are relevant to the research purpose, or identify documents to be examined based on the research questions. After these documents have been identified, the researcher must decide on an appropriate sample size to be used. Krippendorff (1980) recommends that the sample size be large enough to provide sufficient information, yet small enough for the analysis to be feasible. Often times, the sample size is also dictated by trade-off between available resources (time, money, and people) and the desire for generalizability: a larger sample size may involve more resources, but be more generalizable.

Once a sample has been selected, the next task is devising a system of encoding information from the sample. In qualitative approaches, this is usually done by some sort of categorizing. When a categorizing approach is used, the first step is to determine what units (words, sentences, syntactical units, referential units, pages, math problems and the likes) will be placed in which categories. A coding process can also be designed to include matrices, charts, tables, or other ways to display whatever information the researcher will be analyzing (Magid, McKnight, McKnight & Murphy, 2000).

The next task is to determine categories. According to Holsti (1969), categories should "reflect the purposes of research, be exhaustive, be mutually exclusive, independent, and be derived from a single classification principle" (p. 95). Although it is possible to use categories that are not mutually exclusive, Weber (1985) noted that this can lead to problems in statistical analysis. Weber added that, if the categories are not exhaustive, an “other” category can be added so that all units are coded.

The execution of the content analysis can be completed by hand or by computer. Using either procedure, often times the most well thought out research plan may be disrupted by unforeseen problems. Some of these problems may arise from inadequacies in the available data or unexpected costs related to the research process. According to Krippendorff, (1980), the solution to these problems is not to discontinue the content analysis, but to go back and modify the design, keeping the research objective in mind.

The final stage of content analysis is to interpret the results. In doing so, it is highly recommended that one takes into account the contexts within which the data were collected, and the purpose of the study. Regardless of the approach used or the purpose for the content analysis, it is also prudent to discuss what practical or theoretical implications can be drawn from the findings, any major shortcomings or limitations of the methodology used, and directions or suggestions for future research.

In the conduct of this study, I followed the five steps of content analysis outlined above to examine the extent to which popular middle-grades mathematics textbooks develop and present the concept of variable. I developed a coding and categorization scheme and used it to examine variable ideas in 24 middle-grades mathematics textbooks made up of 12 student editions and their corresponding teacher editions chosen from four eras of mathematics education in the United States. The coding and categorization scheme were based on the various frameworks on variables from the extant literature (Küchemann, 1978; Philipp, 1992; Schoenfeld & Arcavi, 1988, Usiskin, 1988).

Mixed Methods

The present study is framed within a mixed methods paradigm of research; gathering and analyzing both qualitative and quantitative data concurrently (Johnson & Onwuegbuzie, 2004; Tashakkori & Teddlie, 1998). Johnson and Onwuegbuzie (2004) defined mixed methods research as “the class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study” (p. 17). That is, this research perspective legitimizes the use of multiple approaches in answering research questions, rather than restricting or constraining researchers’ choices: it is inclusive, pluralistic, and complementary, and it suggests that researchers take an eclectic approach to method selection and collecting multiple data in such a way that the resulting mixture or combination is likely to produce complementary strengths (Brewer & Hunter, 1989).

The qualitative portion of the present study addressed three of the research questions (i.e., research questions 1, 5 and 6) by examining the qualitative differences in the introduction, definition and/or explanation of terms in the narratives, assignments, work examples, and notes/alerts for teachers, among others, on variable idea sets in the selected middle-grades mathematics textbooks using constant comparison and cross-case analytical methods (Denzin, 1989; Glaser & Strauss, 1967; Yin, 1994, 2003).

The quantitative portion of the study addressed three of the research questions (i.e., research questions 2, 3, and 4). Data collection techniques for these questions included tallies for existence and frequency of pages, lessons and contents related to the various uses of variable, and report this information using percentages and graphs.

Reliability Measures

A key ingredient in content analysis is the reliability of the coding procedure. That is, in content analysis, it is imperative that categories be designed clearly, and instructions made specific enough so that when the coding procedure is applied, it remains consistent both over time by the same coder and among different observers. Krippendorff (1980) defined three types of reliability: stability, reproducibility, and accuracy. Stability means that the contents are coded the same by the same coder at different times. The problem that may arise here may be from ambiguities in the text, ambiguities in the coding rules, simple recoding errors, and cognitive changes within the coder (Weber, 1985). For instance, by the time a coder has finished a lengthy coding process, he or she may have made subtle changes in his or her judgments about the placements within different categories, and this may affect the stability of the coding. In order to guard against this, I recoded an earlier-coded section near or at the end of the coding process. This helped me determine how consistent my coding was, and/or if there had been any changes.

The possibility of coder fatigue may also affect the stability of the coding as the amount of documents to be examined was relatively large (i.e., 24 textbooks for this study). In order to reduce coder fatigue and enhance reliability, my coding schedules took into account the length of units that was coded at any given time, and ensured that only a reasonable number of units was coded during any given period of coding.

The second type of reliability is reproducibility. This is often called *inter-coder reliability*. Reproducibility deals with the extent to which the coding of materials is consistent among two or more different coders. According to Krippendorff (1980), this gives one of the surest measures of whether coding categories are clear and consistent or not. In order to establish inter-coder reliability, I employed the assistance of two fellow doctoral candidates (one in mathematics education and the other in applied mathematics) who were familiar with my work to perform check-coding (Miles & Huberman, 1994) (See Table 7 for Cohen Kappa estimates).

Accuracy, the third type of reliability, measures the extent to which a process conforms to a known standard (Krippendorff, 1980). Accuracy is usually established by comparing the performance of one coder with what is known to be a correct performance (Krippendorff, 1980). This is the strongest measure of reliability, but perhaps, the most difficult to achieve since a known standard with regards to the research objectives may not exist. Krippendorff made the point that in the more problematic parts of content analysis, such as the interpretation and transcription of complex textual matter, suitable accuracy standards are not easy to find. That is, because interpretation can be compared only with other interpretations, attempts to measure accuracy presuppose the privileging of some interpretations over others, and this puts any claims regarding precision or accuracy on epistemologically shaky grounds (Krippendorff, 1980). Thus the use of accuracy is limited to coder training and other areas where objective standards are readily available. With respect to the present study, there was no known objective standard for examining the concept of variable from mathematics curricula perspective. The lack of known standards made it difficult to document accuracy measures in this study.

Analytical Framework: An Overview

I developed an analytical framework and used it to examine variable ideas in the selected middle-grades mathematics textbooks. This analytical framework was based on Usikin's (1988) conceptions of variables, Schoenfeld and Arcavi's (1988) framework on variable, Philipps' (1992) categorization of literal symbols, and Küchemann's (1978, 1981) work on students' understanding of variables. I used the framework to examine the uses of variables within the narratives, worked examples, assignments, exercises or question sets, activities and projects, chapter reviews, chapter summaries and chapter test blocks in the selected mathematics textbooks.

I organized the analytical framework into three parts: The first is intended for examining variable ideas within the narratives and worked examples in the students' textbooks. The second is for examining the uses of variables within the exercises or question sets, activities and projects, chapter review, chapter summaries and chapter test in the students' textbooks. The final framework is designed to identify the support provided for teachers (within the narratives, notes, highlights, alerts, and the likes) on teaching about variables in the teachers' edition textbooks.

I employed a method similar to that of Valverde et al. (2002) to divide the data on variables in the selected textbooks according to whether they were located within narratives, worked examples, exercises or question sets, activities and projects, chapter review, summaries and chapter tests. Valverde et al. (2002) defined *narrative elements* as sentences and paragraphs that are used in the text to explain concepts and topics through descriptions and discussions. They may tell stories, and state facts and principles through narration. They defined *activity elements* as segments of the textbook that contain instructions and suggestions for students' activities. Often, these activities contain instructions for the conduct of some sort of 'hands-on' experience. In mathematics, this might include collecting and using data.

Valverde et al. (2002) referred to *exercise and question sets* as those tasks that provide students with instructions and opportunities to practice and acquire particular skills. In this way, *Exercise and question sets* are similar to an activity block in requiring that students engage in performances that are different from reading and understanding. They are also similar to narrative blocks, in the sense that the exercises provide all that is necessary for the pedagogical experience. Another block type, *worked examples*, refers to material that details the execution of a particular algorithm or solution strategy through an illustration with detailed annotations and description (pp. 141-142).

Table 5 summaries each of the frameworks and the type of data that each was used to collect.

Table 5

Overview of the Analytical Framework

Framework	Block to Examine	Examples of Tasks to Examine
Framework I	Narratives and Worked Examples	Introduction, definition, explanation of terms related to variable and types of activities on variable, use of technology to explore variable, and context characteristics of use of variables.
Framework II	Exercises, Projects, Assignments, Activities, Chapter Review, and Chapter Test	Uses of variables as: labels, constants, placeholders, generalized numbers, continuous unknowns, varying quantities, and abstract symbols.

Framework III	Support for Teachers	Information to teachers on definitions of variable, notes on uses of variables, alerts to students' misconceptions and difficulties with variables, suggestions to address students' problems with variables

I document the purpose(s) of the uses of variables within each of the afore-mentioned blocks as to whether the variables are being defined, explained, used as objects, labels, unknowns, varying quantities, constants, parameters, generalized numbers, placeholders, arguments, and abstract symbols (Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1988; Wagner & Parker, 1993).

In the section that follows, I provide more detailed description of the type of information that was gathered, how the information was collected, how it was coded, and the process I followed to record it in formats that facilitated easy analysis. In addition, I illustrate segments of the analytical framework with sample data from the University of Chicago School Mathematics Project (UCSMP) middle-school mathematics textbooks that I used for a pilot study to test the applicability of the framework. The UCSMP mathematics textbooks are labeled *Pre-Transition Mathematics* (sixth grade), *Transition Mathematics* (seventh grade), and *Algebra* (eighth grade) and was published by Wright Group/McGraw Hill (2007/2008). For the pilot, I examined the *Pre-Transition Mathematics* (sixth grade) and the third edition of the *Transition Mathematics* (seventh grade) textbooks only.

Analytical Framework I: Variable Ideas within the Narratives, Assignments and Worked Examples in the Students' Textbooks

To examine variable ideas within the narratives, assignments and worked examples in the selected students' textbooks, I identified and reported: a) how early variable ideas were introduced in each textbook; b) whether definition(s) and/or explanation of variable was provided; c) which type(s) of definitions was provided (refer to the various definitions of variable in Chapter 2); d) whether terms such as unknowns, placeholders, constants, and the like, were used and explained in the textbooks; e) whether different types of activities on variable were provided to offer students the opportunity to explore the various uses of variables (such as their uses as *labels*, *objects*, *unknowns*, *varying quantities*, *constants*, *parameters*, *generalized numbers*, *placeholders*, *arguments*, and *abstract symbols*) in the textbooks; f) whether there exist activities that required the use of technology to explore variable concepts; g) whether variable ideas were explored within different representations such as in tables, graphs, equations, expressions, etc; h) whether variable ideas were used to model real-world phenomena or were used to just facilitate symbolic manipulations ; and i) identified the grade level(s) textbook in which each of the aforementioned occurred.

Below (Figure 2) is an example that illustrates how the concept of variable was introduced in the UCSMP *Transition Mathematics* (Grade 7) students' textbook:

In this chapter, you will see some uses of letters denoting variables. They may describe patterns, appear in formulas, or be unknown quantities. The word *variable* was introduced in the late 1600s by the German mathematician Gottfried Leibniz. However, the *idea* of variables is much older. For centuries, societies have used letters and other symbols to represent numbers. Diophantus, a Greek mathematician who wrote in the years 250–275 CE, used a Greek letter with an accent to represent an unknown quantity. Seventh-century manuscripts from India used both Sanskrit letters and the names of colors in the same way. In about 1300, Zhu-Shijie, a Chinese mathematician, used different placements of numeric symbols to represent unknowns.

Many of the symbols we use today come from more recent contributions. In 1591, the French mathematician François Viète used letters to represent unknowns and constants in equations. Viète's work quickly led to the invention of a great deal more mathematics. Within 100 years, the ideas behind almost all of elementary algebra and calculus had been discovered, and these letters became known as variables because their values could change. The power of variables is their ability to describe patterns and solve problems.

Here is an equation written in symbols similar to those used in Europe in 1631. Can you figure out how we would write this equation today?

aaaa − 4bba == +3.ccc

$$a^4 - 4b^2a = 3c^3$$

Figure 2. Introduction of variables, *Transition Mathematics* p. 68-69. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

This background provides a brief history on the origin of variables and some of its different uses. This introduction occurred very early in the students' textbook (*Transition*

Mathematics, p. 68-69) and was followed by the following explanations and definition of variable on the subsequent page (p. 70):

The definition of variable in Figure 3 is one of the many definitions of variable identified in the literature (Schoenfield & Arcavi, 1988). This definition helps students understand that a variable can stand for a number, a set of numbers, or objects that may not necessarily be numbers. It also informs students that variables may be denoted by other symbols than letters – “letters are often used for variables”. A similar definition of variable was provided on page 138 of the *Pre-Transition Mathematics*, the sixth grade student textbook of the UCSMP middle-grades textbook series.

What Are Patterns?

A **pattern** is a general idea for which there may be many examples. A single example of a pattern is called an **instance.** Here are three instances of a pattern using negative powers of 10.

$$24 \cdot 10^{-2} = 24 \cdot 0.01$$

$$5.2 \cdot 10^{-2} = 5.2 \cdot 0.01$$

$$75 \cdot 10^{-2} = 75 \cdot 0.01$$

We can describe this pattern in words:

Multiplying a number by 10^{-2}
gives the same result as
multiplying the number by 0.01.

But there is a simpler way to describe this pattern by using symbols.

$$n \cdot 10^{-2} = n \cdot 0.01$$

What Are Variables?

The letter n used above is a *variable*. A **variable** is a symbol that can stand for any one of a set of numbers or other objects. Letters are often used for variables. Here n can stand for any number. For example, when n is 12, we substitute the number 12 for every n to get the instance $12 \cdot 10^{-2} = 12 \cdot 0.01$.

Figure 3. Definition of variables, *Transition Mathematics* p. 70. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Prior to the above introduction and definition of variable in *Transition Mathematics*, there was very minimal sighting of variables being used in this textbook. After the introduction, however, there were very frequent sightings of letter variables used to translate words into algebraic expressions, to evaluate algebraic expressions, to develop expressions and formulas, to develop the Pythagorean Theorem, to develop formulas in spreadsheets, in open sentences, and in inequalities and their graphs, among others. The following example (Figure 4) illustrates how the use of variables as a *generalized number* was presented in *Transition Mathematics*:

Example 2

Give three instances of the pattern $a + b = b + a$.

Solution

Think of two numbers. For example, let 2.75 equal a and let 10.49 equal b.
The pattern $a + b = b + a$ becomes:

$$\underline{\ ?\ } + \underline{\ ?\ } = \underline{\ ?\ } + \underline{\ ?\ }$$

Repeat this process two more times so that a and b are always different numbers.

$$\underline{\ ?\ } + \underline{\ ?\ } = \underline{\ ?\ } + \underline{\ ?\ }$$

and

$$\underline{\ ?\ } + \underline{\ ?\ } = \underline{\ ?\ } + \underline{\ ?\ }$$

Because any letter may be used as a variable, there are many possible descriptions of a pattern. The sentence $a + b = b + a$ describes the same pattern as the sentence $y + z = z + y$.

Patterns with words can also be described with variables.

Figure 4. Variables as generalized numbers, *Transition Mathematics* p. 70 example 2. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Some interesting characteristics of this example are that, it uses decimal numbers, integers as well as fractions to illustrate variable as *generalized numbers*. These uses of different sets of number categories help students realize that variables are not only used to illustrate whole number concepts, but can stand in for fractions as well as decimal numbers. In addition, the *generalization* was illustrated using different sets of letters ($a + b = b + a$ and $y + z = z + y$), again indicating to students that no particular sets of letters are reserved for any particular use of variables, but that one can choose flexibly among letters to illustrate variable ideas.

Analytical Framework II: Variables Ideas within the Exercises, Question sets, Activities, Chapter Review, Chapter Summary, and Chapter Test in the Student Textbook

Table 6 presents specific categorizations (with examples) that I employed to classify the uses of variables within the exercises or question sets, activities, chapter reviews, chapter summaries, and chapter tests in the students' textbook in this study.

Table 6

Categorization of Variables by their Use in this Study

Role of Variables	Definition	Example
Labels	Shorthand for the name of object	f, y in $3f = 1y$ (3 feet in 1 yard)
Constants	Quantity with a fixed value in a specified context	π, e, c
Place-holders/ Specific Unknowns	Specific unknown number in equations to be solved	$x + 5 = 8$
Continuous Unknowns	Unknown *values* to be found in equations, expressions or inequalities	$x^2 - 3x = 28, \quad 2x - 4 < 7$
Generalized Numbers	Stands for a set of numbers which gives true statement	$a + b = b + a$
Varying quantity/ Parameters	Functional relationships between variables parameters are constant that varies in the general form of the relationships	x and y in $y = -2x + 6$ m, b in $y = mx + b$
Abstract symbols	Literal symbols without number referent	e, x in $e * x = x$

In the following pages, I describe each of the categories in detail, and provide examples from the UCSMP *Pre-Transition Mathematics* and *Transition Mathematics* that illustrate the particular use of variables.

Labels or Objects: Here, letter variables stand as shorthand for objects rather than for a characteristic of them. For example, the s in the formula $P = 4s$ used to find the perimeter of a square, stands for the sides of the square, rather than the characteristics of it. Other examples of the use of variable as a label are the use of $f, y\ in\ 3f = 1y$ to denote 3 feet in 1 yard, the use of m to denote meter, *L* to denote liter, *t* to denote ton etc. Figure 5 shows an example from the UCSMP *Pre-Transition Mathematics* textbook that illustrates the use of variable as a *label*.

9. Use the fact that 476 ÷ 65 = 7 R21 to find
a. 477 ÷ 65. **b.** 480 ÷ 65. **c.** 550 ÷ 65.

Figure 5. Variables used as a label, *Pre-Transition Mathematics*, p. 420 Q. 9. Taken from *The UCSMP: Pre-transition Mathematics* (2009), by McConnell, Feldman, et al., published by Wright Group/McGraw Hill.

Here, the variable *R* is used as a label to denote remainder. In other words, *R* is used as shorthand for "reminder" and therefore does not have a numerical referent. In this equation, there is no feel to solve for the variable *R*, but rather to view it as a label or an abbreviation for the word *remainder*.

Constants: Constants are letters that represent quantities with a fixed value, either in a specified context or in general form. For instance, the symbol π that denotes the ratio of the circumference of a circle to its diameter, and is approximately 3.1416 is a constant. Some other examples of *constants* include physical constants, such as density of materials (p), and gravity (g) which is defined as $g = 9.8\ m/sec^2$ for an object close to the Earth's surface, and $g = 2.3\ m/sec^2$ for an object on the moon. It is worth mentioning here that, even though gravity varies inversely to the square of the distance from the center of the Earth to the object, for most contexts it stands for a constant value.

Specific Unknowns/Placeholders: This is the use of variable to represent specific unknown numbers in equations that can be solved. The use of variables as specific unknowns involves what Küchemann may refer to as letters evaluated or ignored. Thus, many of the equations involving variable of this type can be solved simply by assigning the variable an arbitrary value, or by finding the value of the variable through trial and error. For example, in the equation $x + 5 = 8$, the value of x can be found by trial and error. The x need never be operated with as an unknown number. Thus, here the variable represents a replacement for a single number, rather than as a signifier of a continuous and related set of numbers. In such use of variables, one may acknowledge the existence of the variable in the problem, but can often avoid working with it by performing instead some sort of matching procedure to find its value. Figures 6 and 7 show examples of the use of variable that will be classified in this study as *Specific Unknowns* or *Placeholder*:

19. In the diagram below, the values of a, b, c, d, e, and f are -6, -2, 1, 2, 4, and 5, though not necessarily in that order. If the sum of the numbers in each circle equals 0, what is the value of each variable?

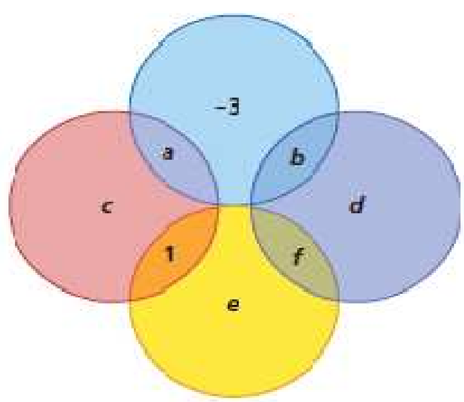


Figure 6. Variables used as a placeholder, *Pre-Transition Mathematics,* p. 254. Taken from *The UCSMP: Pre-transition Mathematics* (2009), by McConnell, Feldman, et al., published by Wright Group/McGraw Hill.

9. Jasmin and Janelle are driving to Cincinnati. They think they can average 60 miles per hour for the 310 miles. At this rate, how long will it take them to get to Cincinnati?

   a. Let *t* be the time (in hours) it will take them. Write an equation involving *t* that can answer the question.

   b. Solve your equation. Check the solution in the original equation.

   c. Answer the question with a sentence.

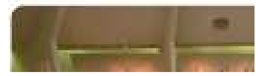


Figure 7. Variables used as a placeholder, *Pre-Transition Mathematics,* p. 450. Taken from *The UCSMP: Pre-transition Mathematics* (2009), by McConnell, Feldman, et al., published by Wright Group/McGraw Hill.

In each of these questions (Figure 6 & 7), there is a feel to solve the situation for the unknown or substituting the given values for the variables in the expressions to obtain a single numeric answer for the variable. For example, one may model question 9 in Figure 7 as $60t = 310$ and solve for the time (*t*) using the method of trial and error, or by dividing both sides of the equation by 60. Tasks of these types in the sampled textbooks were coded as the use of variables as unknown or placeholder.

Continuous Unknowns: There are three scenarios in the present study that represent the use of variables as continuous unknowns. The first is the use of variables in equations whose solutions yield more than one value for the variable (e.g., the solution to the quadratic equation $x^2 - 3x = 28$, and of other polynomial equations with higher degrees). The second is solution to inequalities. Solutions to inequalities may, in many cases, result in a set of values for the variable as opposed to a single value as described in the use of variables as *placeholders*. This type of task provides opportunities for students to recognize that variables can assume more than one value, assume an unlimited number of values or can be used as a signifier of continuous and related sets of numbers. Some illustrations of such uses of variable are the role of *x* in $2x - 4 < 7$.

The third use of variable in the category of continuous unknown is in situations where the final answer to a mathematics expression, equation or inequality is not a single numeric answer, but rather an expression involving literal symbols such as “$a + 3$” representing the final answer to a mathematics problem. This type of answer to mathematics problems is referred to as *lack of closure* in the literature, and has been reported in the literature as being difficult for the majority of students to fathom as a “final answer” because it contradicts their prior arithmetic experiences or solutions that always result in a numerical value (Collis, 1975). Küchemann provided the following example to explain that when the variable is used to represent an unknown number, its value remains unknown throughout the problem: if $e + f = 8$, and $e + f + g =?$, then $? = 8 + g$. Notice here that g is a number which cannot be evaluated, but which, nonetheless, must be used to complete the equation. In this case, the values of e, f, and g remain unknown even after students finish solving the problem. Figure 8 depicts an example of a task taken from the UCSMP *Pre-Transition Mathematics* (Grade 6) student textbook that illustrates the use of variable as *Continuous Unknowns.* Specifically, the solution to part (a) of this task is $\frac{21}{k}$; a solution that will be described as *lack of closure* by Collis (1975) and colleagues. Furthermore, there exists more than one value for the variable k that answers the parts (*b*) and (*c*) of this question. All these attributes of this question classify the use of variable in it as continuous unknowns.

21. Consider the addition $\frac{1}{k} + \frac{2}{k} + \frac{8}{k} + \frac{3}{k} + \frac{7}{k}$.

a. Find the sum.

b. Find a value for k that gives a sum that is a proper fraction.

c. Find a value for k that gives a sum that is an improper fraction.

d. Find a value for k that gives a sum of 1.

Figure 8. Variable as continuous unknown, *Pre-Transition Mathematics* p. 189 Q. 21. Taken from *The UCSMP: Pre-transition Mathematics* (2009), by McConnell, Feldman, et al., published by Wright Group/McGraw Hill.

Generalized Numbers: Variables are employed as generalized numbers when they are used in expressions to represent patterns or sequences. An example of such use of variables is n in $2n + 1$ to generalize an expression for odd numbers, or the use of m in representing the general form of the sequence $1, 4, 9, 16 \ldots \ldots m^2$. Within the context of generalizing patterns, however, a variable can be used in different ways. Wagner and Parker (1993) used the phrase, *generalized number*, to indicate a variable that describes a set of numbers in mathematical identities, or a variable that represents quantities that give a true statement for any of a set of values in a specified context. For instance, the use of $a + b = b + a$ to generalize the commutative property of addition will denote the use of variables as *generalized numbers* in this study. Figure 9 shows an example of such use of variable in a task taken from the UCSMP *Transition Mathematics* (Grade 7) textbook that was classified as *generalized numbers*:

30. Pick a positive integer n. The next greater positive integer is $n + 1$. The next greater is $n + 2$ and so on.
    a. Add any five consecutive integers. Explain why the sum is divisible by 5.
    b. If you add six consecutive integers, is the sum always divisible by 6?
    c. Explore the idea of Parts a and b and make a generalization.

Figure 9. Variable as generalized numbers, *Transition Mathematics* p. 492 Q. 30. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Usiskin (1988) considers the use of variable in mathematical modeling ($T = -0.4Y + 1020$) as pattern generalizers because this activity allows students to use the key instructions of this conception of variable (i.e., to *translate* and *generalize*) to translate information in verbal form into symbolic form in order to solve a situation. In the present study, if the overriding purpose of a task involving the use of variables is for students to model a mathematical situation (i.e., to translate mathematical situations in verbal form into mathematical symbols), the use of variables in that task was classified as pattern generalizers. Figure 10 shows an example of such use of variables.

10. At the botanical gardens, Latayna bought a annuals at \$2.49 each and p perennials at \$5.99 each. Write an expression that represents how much Latanya spent.

Figure 10. Variables as generalized numbers, *Transition Mathematics* p. 645. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Varying Quantities and Parameters: When variables are used as varying quantities, they offer students the opportunity to see their changing nature. In this way, students have the chance to explore relationship between two or more variables as their values systematically change. Thus, varying quantities have a functional relationship between variables (such as those relationships that can be found between x and y in $y = mx + b$). Also in this category are formulas such as $A = lw$ and $A = \frac{1}{2}ab$ used to find the area of a rectangle, triangle and the like. Parameters are constants in the functional relationship equations that vary in other equations or expressions of the same general form. They usually identify particular characteristics of relationships between the varying quantities. An example of parameters is the role of m (slope) and b (y-intercept) in $y = mx + b$. Figure 11 shows an example of task from the UCSMP *Transition Mathematics* (Grade 7) that was classified the use of variables as *varying quantities*:

16. Use the formula $t = T - \frac{2f}{1{,}000}$. This formula gives the estimated air temperature t (in degrees Fahrenheit) at an altitude of f feet when the ground temperature is T. Suppose it is 75°F at ground level.

a. Find the temperature at an altitude of 2,000 feet.

b. Find the temperature at an altitude of 5,000 feet.

Figure 11. Variables as varying quantities, *Transition Mathematics* p. 95, Q. 16. Taken

from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

The use of variables in the formula in Figure 11 to relate the estimated air temperature with altitude and ground temperature offers students the opportunity to explore the effect of a change in one variable (in this case, in the altitude) on another variable (the estimated air temperature).

Abstract Symbols: In this conception, variables are used as arbitrary symbols without number referent. That is, variables are used as abstract symbols or arbitrary marks in the structure such as the use of symbols in proofs. For example, in the questions: given that $a * b = 2a + b$, prove that $a * (b + c) = (a * b) * c$, or factor $x^2 + 2x - 3$, the variables are used as abstract symbols. Here, the variable is little more than an arbitrary mark on paper that allows for algebraic manipulations. The key instructions for students in this conception of algebra are on *manipulations* and *justifications*. Thus, when students are asked to factor expressions such as $x^2 + 2x - 3$, we are more interested in their ability to logically manipulate the expression involving the variable than we are in their ability to solve for the variable - which in this case is meaningless. An example of such use of variable taken from UCSMP *Transition Mathematics* is shown in Figure 12: The essence of this task is not on solving for the variables (x and y) but rather, to assess students facility with logical manipulation of the expression (i.e., the correct removal of the parenthesis, multiplication of a fraction and a whole number and the appropriate grouping of the like terms, which in this case, are delineated by the variables involved).

28. Rewrite the expression without using parentheses.
$\frac{3}{5}(2x + 3y) - \frac{2}{5}(3x - 2y)$ **(Lesson 7-3)**

Figure 12. Variables as abstract symbols, *Transition Mathematics* p. 588 Q. 28. Reprinted

Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

It is worth stating here that my categorization of the uses of variable for this study integrates the roles of variables from Philipp's (1992), Küchemann's (1978, 1981), and Usiskin's (1988) frameworks on the uses of variables. The main difference between each of these researchers' (Philipp, 1999; Küchemann, 1979, 1981; Usiskin, 1988) categorization and mine is that my framework integrates the various components of the conceptions of variables into one composite whole. For example, Usiskin's (1988) framework only describes the uses of variables as generalized numbers, unknown, argument or parameter, and arbitrary symbols. One might argue that Usiskin's list of the uses of variables is incomplete, in that, the other uses of variable such as placeholders, constants, and labels were not included in his description of variables.

Similar argument can be made about the descriptions provided by other researchers (e.g., Küchemann's description did not include the conception of variables as abstract symbols). Thus, in the present study, the seven roles of variables described in Table 4 emerged from the integration of the various frameworks on variables described in Chapter 2 of this study. These seven roles of variables were employed to examine the variable ideas in the mathematics textbooks selected for examination in this study.

I wish to clarify here further that I consider *variable ideas* to be those uses of variables represented by *letter variables* only. I am, however, aware that not all variables are represented by letters (e.g., a symbol in an open sentence such as 3 + □ = 5), and that not every use of letters in mathematics problems represent variable ideas (e.g., some metric systems (cm, kg, km), and lettering or numbering (a, b, c), and the like). As a result, these uses were not considered in this study as variable ideas and were not included in the data collection and analysis.

Analytical Framework III: Identifying Support for Teachers in the Teachers' Editions

I made the argument earlier that it is not sufficient to examine only the opportunity for students to work with variables but equally important is the need to examine the guidance offered to teachers on how to enact these opportunities in the classroom since the curriculum is not self enacted. The research literature (e.g., Asquith et al., 2007; Boz, 2002; Mohr, 2008), and my personal experiences working with pre-service and in-service teachers in teacher education and professional development suggest that many teachers have limited understanding of variables. Consequently, it is logical to believe that even if there is a plethora of opportunities designed in the curriculum for students to engage meaningfully with variable, it is likely that teachers will need explicit guidance in their teachers' textbooks in order to implement the tasks in ways that are consistent with the curriculum developers' intents.

In this respect, I examined the teachers' editions of the selected mathematics textbooks with particular attention to: a) whether teachers have been alerted to the various definitions of variable in textbooks and in the literature; b) whether explicit notes on the various uses of variables have been provided in the texts in ways that can help teachers educate themselves on variables; c) whether teachers' attention has been drawn to students' common misconception of variables, to students' confusions and difficulties when using variables; and d) whether suggestions have been offered on how to address common students' misconceptions and difficulties with variables. These constituted the framework for analyzing the teachers' editions. The following excerpts taken from the UCSMP *Transition Mathematics* (Figures 13-16) show some guidance provided to teachers on the treatment of variable ideas.

Algebra is generally thought of as using letters to represent numbers. In this sense, the study of algebra begins here. However, in earlier grades and in Chapter 1, students have been introduced to many algebraic concepts. They have had to fill in blanks in sentences such as $8 + \underline{\ ?\ } = 13$; this is simply a different form of $8 + x = 13$. They have seen formulas such as $A = \ell w$ for the area of a rectangle or $A = s^2$ for the area of a square. They have probably seen patterns described with variables, such as $a + b = b + a$ for the Commutative Property of Addition. We assume no previous exposure to variables, but if students have worked with them, you can build on that experience.

This chapter discusses four of the main uses of variables. In Lesson 2-1, variables are introduced as ***pattern generalizers***. Lessons 2-2 and 2-3 continue this theme by working on the translation of patterns

(continued on next page)

Figure 13. Notes to teachers on how to introduce variables, *Transition Mathematics* p. 68. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

The notes presented in Figure 13 inform teachers about some experiences students might have had in learning mathematics from their earlier grades that may be similar to the use of variables that teachers can build on to introduce letter variable ideas to them. At the same time, teachers were cautioned in the note to not assume any students' previous exposure to the concept of variables. The note went on to enlighten teachers on four main uses of variables: pattern generalizers and unknown, among others.

Background

This lesson contains an important transition from arithmetic to algebra: ***variables generalize arithmetic.***

Some books introduce variables first as unknowns by solving equations such as $4x = 36$. But that does not convey the important idea that a variable can stand for any one of many values, and it carries the notion that variables are somehow "mysterious." This book begins with the following aspects of variables.

What are variables? Letters are usually used for variables that stand for numbers. In later courses, symbols such as # or * are sometimes used when variables stand for an operation.

Describing number patterns with variables. To describe the pattern that a (nonzero) number divided by itself is 1, students could use $\frac{a}{b} = 1$ or $\frac{n}{n} = 1$. But $\frac{a}{b} = 1$ does not signify that the numerator and

(continued on next page)

Figure 14. Alerts to teachers on variables, *Transition Mathematics* p. 70. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

This alert informs teachers that some mathematics textbooks do not offer students the opportunity to see that variables can sometimes assume more than one value or unlimited number of values. It also provides a rationale to teachers as to why variable ideas were presented the way they were treated in the chapter. Furthermore, it educates teachers that variables need not necessarily be represented by letters but symbols such as # and * can be used to represent variables in more advanced mathematics. Similar alerts, notes and cautions are contained in Figures 15 and 16 to guide teachers on the teaching of variable ideas.

ENGLISH LEARNERS
Vocabulary Development

Some students may have difficulty understanding that a *variable* can represent one or more numerical values. Encourage students to rewrite variable expressions using shapes, such as $4 + \square$ or $5 \cdot \square$, where the square or rectangle is filled in with a value. Once students understand that they can fill in the square or rectangle with a number, it may be easier for them to understand the variable expressions $4 + a$ and $5 \cdot b$.

Accommodating the Learner

Different variables are used to represent different values. If a pattern involves the same number more than once, then the same variable should be used. For example, $n - n = 0$ can be used for $5 - 5 = 0$ and $30 - 30 = 0$.

Have students create equations that could fit the following patterns: $a \cdot b = b \cdot a$, $c + 2 > c$, and $5 \cdot n = 3 \cdot n + 2 \cdot n$.
Answers vary.

Figure 15. Alerts to teachers on students' difficulties, *Transition Mathematics* p. 72. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Background

In patterns and formulas, variables have the feeling of being *knowns* and so formulas and patterns are not particularly threatening to most students. But variables as *unknowns* can be threatening, and this is the major reason we have put this lesson toward the end of this chapter.

What is an open sentence? In the sentence $a < b$, a is the subject, $<$ is the verb, and b is the direct object (or predicate). A sentence is *open* if its truth-value is not known. We suggest that you not work on ways to solve sentences at this time. Many later lessons are devoted to sentence solving.

How formulas lead to open sentences. A formula consists of a statement in which one variable is expressed in terms of others. Thus a formula has at least two variables. One way to obtain an open sentence is to assign a value to all of the variables but one. Thus, formulas naturally lead to open sentences.

Figure 16. Notes to teachers on the approach to the treatment of variables in the textbook, *Transition Mathematics* p. 110. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright

Group/McGraw Hill.

Validating the Framework and Reliability Measures

The framework went through the first stage of refinement during the testing of its applicability using UCSMP *Pre-Transition Mathematics* (sixth grade) and *Transition Mathematics* (seventh grade) textbooks in a pilot study. I refined it further by making an initial test and validation of the categories created with some sample coding and informal analysis of exercises from the selected textbooks in this study. Using the method of constant comparison (Glaser & Strauss, 1967), I compared the characteristics of each of the existing categories to see whether the features of a new task indicate a match or mismatch that could lead to the generation of new categories and/or adjustment of the characteristics of an initial category in order to include the new feature(s). In this way, the description of the framework was refined and systematically validated through further data collection and analysis.

Reliability

To check the reliability of my coding, I enlisted the assistance of two of my colleagues (referred to here as *Coder A* and *Coder B*) to perform check-coding (Miles & Huberman, 1994) to help me determine whether the coding categories were well defined, or if they needed further refinement.

Coder A: This coder was a Mathematics Education doctoral candidate who was finalizing her dissertation during the same period that I was undertaking this study. As a result, this colleague and I read each other's work from the beginning of developing our respective proposals. Consequently, by the time of coding the selected textbooks, she was familiar with my work and the framework, and for that matter, needed no further training in order to undertake the actual coding. I met with *Coder A* on 12 different occasions, to code a textbook at a time, for a total of 12 student edition textbooks. We worked for approximately 2 hours every time we met to code one textbook. At the beginning of each coding session, one of us decided on 100 pages at random to be coded from a textbook (say, pages 105-205 in Math Connects, grade 6), then we spent an hour coding those pages individually, and spent the remaining hour discussing our agreements/disagreements, and then came to a consensus on those we had coded differently. We first coded all the four 6^{th} grade textbooks (one from each of the four curricula), and then moved on to code the four 7^{th} grade textbooks and then the 8^{th} grade textbooks in that order.

Coder B: This coder was a second year doctoral student in applied mathematics. *Coder B* was not familiar with my work as *Coder A* was. As a result, I provided him with training on the framework and on the coding procedure before we began coding the actual textbooks. First, I gave him a copy of my proposal to read (and asked him to pay particular attention to the framework). We met two weeks afterwards to discuss any questions or concerns he might have regarding the framework. At this meeting, we also pilot coded 200 pages of the UCSMP *Transition Mathematics* textbook to ensure that the instructions on the coding and the framework were clear. We spent three hours on this discussion/training session. We began coding the actual textbooks one week following the training session.

Contrary to *Coder A* (who coded 100 pages from each of the 12 students textbooks), *Coder B* only helped with the coding from four textbooks, *one textbook each selected at random from* the respective curricula. Specifically, I met with *Coder B* four times for approximately 2 hours each time to code one textbook. At the beginning of every coding session, one of us decided on 100 pages at random to be coded, then we spent one hour coding those pages individually, and spent the other hour discussing our agreements and/or disagreements. *Coder B* and I coded 100 pages each from the following textbooks: Modern School Mathematics, Grade 7; Holt School Mathematics Grade 6; Mathematics Today, Grade 8; and Math Connects Grade 7. The Cohen Kappa estimates with *Coder A* and *Coder B* are reported in Table 7.

Table 7

Inter-Coder Reliability Estimates for Variable Pages and Variable Categories

Criterion	*Agreement with Coder A*		*Agreement with Coder B*	
	Initial	Final	Initial	Final
Variable Pages	99%	100%	98%	100%
Variable Category	86%	100%	88%	100%

Based on Cohen's guidelines for interpreting reliability estimates: $K < 0$ – no agreement; $0.00 \leq K \leq 0.20$ – slight agreement; $0.21 \leq K \leq 0.40$ – fair agreement; $0.41 \leq K \leq 0.60$ – moderate agreement; $0.61 \leq K \leq 0.80$ – substantial agreement; and $0.81 \leq K \leq 1.00$ – almost perfect agreement, the data in Table 7 indicates an almost perfect agreement with *Coder A* and *Coder B* for both the Variable Page and Variable Category criteria.

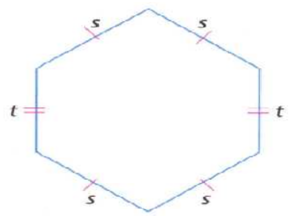
Data Reduction, Unit of Analysis and Coding

I consider variable tasks to be the unit of analysis. Specifically, any task that contains letter variables and fits the descriptions provided in the analytical framework was considered a variable task. To identify these tasks, I examined each instructional page of the selected textbooks, beginning from the first page in the first chapter to the last page of the last chapter of the selected mathematics textbooks for variable ideas.

I define instructional pages as all pages included in the chapters of the textbook. This definition is based on Flanders (1994) definition that, "all pages in each book from the opening of the first chapter through the last chapter were counted as instructional pages. This counting excluded supplementary exercises at the end of books, glossaries, appendices, answer pages, and indices" (1994b, pp. 267-268). Thus, in each textbook, grade level, and era, I collected data on: a) the number of instructional pages, b) the number of instructional pages on which variables were found in each textbook, and categorized into content areas in the textbook, c) the proportion of the instructional pages in which variables were found, d) the proportion of variable tasks on the pages that contained variables, e) the frequency counts of the uses of variables as *labels*, *unknowns*, *varying quantities*, *constants*, *generalized numbers*, *placeholders*, *arguments*, and *abstract symbols*, f) the proportion of the use of variables as *labels*, *unknowns*, *varying quantities*, *constants*, *parameters*, *generalized numbers*, *placeholders*, *arguments*, and *abstract symbols*, and (g) the frequency counts of highlights, notes, alerts and the like on variables for teachers.

In order to calculate the proportion of instructional pages on which variable ideas were found, for example, I counted the number of instructional pages containing variable ideas and divided it by the total number of instructional pages in the textbook. The percent of variable tasks on a variable page was obtained by dividing the number of tasks employing variable ideas on an instructional page by the total number of tasks on that page. For example, Figure 17a contains a total of six tasks. In this counting process, questions *8-11*, and questions *12-15* were counted as single tasks because the respective questions within the tasks used the same values for the same variables to evaluate similar variable expressions (see questions *8-11* in Figure 17a). Examination of each of these six tasks in Figure 17a indicates that all of the tasks employed the concept of variables. Hence, the percentage of variable tasks on this page is calculated to be 100%.

In 6 and 7, the sides of the hexagon below are of lengths s or t.



6. a. Write an expression for the perimeter of the hexagon.
   b. What is the perimeter of the hexagon if s is 25 inches and t is 14 inches?
7. Suppose one of the sides of length t is removed. Use s and t to represent the total length of the sides of the resulting figure.

In 8–11, evaluate the expression when $d = 8$.

8. $d + d$
9. $100 - 4d$
10. $2 + 5d$
11. $\sqrt{\frac{d}{2}}$

In 12–15, find the value of the expression when $m = 2$ and $x = 11$.

12. $3m + 6x$
13. $3mx$
14. $2.5x + m^2$
15. πx^2

APPLYING THE MATHEMATICS

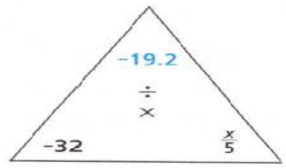
16. Let A be an age from 4 to 9 years. The average height of a boy of age A is $30.4 + 2.4A$ inches.
    a. What is the average height for 6-year-old boys?
    b. What is the average height for 8-year-old boys?
    c. For each additional year of age, how much does the average height change?
17. Suppose x is 6,543 and y is 789.
    a. When evaluating $yx - xy$, Carlos wrote: $yx - xy = 7{,}896{,}543 - 6{,}543{,}789 = 1{,}352{,}754$. His answer is incorrect. Why?
    b. Evaluate $xy - yx$.
    c. Will the answer to Part b change if the values of x and y are changed? Why or why not?

Figure 17a. Variable tasks on variable page, *Transition Mathematics* p. 86. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Similarly, the percentage of variable tasks in the page contained in Figure 17b is 50% because only four (i.e., questions 23, 25, 27, and 28) of the eight questions (tasks) on this page require explicit use(s) of variables for their solutions. Although the remaining four questions in Figure 17b (i.e., questions 22, 24, 26, and 29) can also be solved by employing variable ideas, the use of variables is not a necessary requirement in order to achieve their solutions. Consequently, they were not counted as variable tasks in this study.

22. A car travels 300 miles using 17 gallons of gas. At the same rate of fuel efficiency, how far can it go on 10 gallons? **(Lessons 9-5, 9-2)**

23. Write the related facts for this fact triangle and solve for x. **(Lesson 9-4)**



24. The town of St. Trout, Minnesota, placed an 8-foot-tall ice sculpture in the center of the town's skating rink. On Sunday at 8 A.M., the temperature increased and the sculpture began to melt. By noon, the sculpture was 6 feet 8 inches tall. What was the change per hour in the height of the sculpture? **(Lesson 9-4)**

25. **Multiple Choice** Which of the following is true? **(Lesson 9-3)**

A $\frac{a}{b} \div \frac{c}{d} = \frac{ab}{cd}$ **B** $\frac{a}{b} \div \frac{c}{d} = \frac{b}{a} \cdot \frac{c}{d}$

C $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$ **D** $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{(b + d)}$

26. Suppose a car-rental agency charges \$35 per day or any part of a day to rent an economy car, and there is a 7% sales tax. How much does it cost to rent a car for 100 hours? **(Lessons 9-2, 9-1, 3-7)**

27. Many stars are now known to have planets orbiting them. Suppose $\frac{1}{L}$ of the planets have life on them. And suppose $\frac{1}{H}$ of the planets with life on them have intelligent life. What fraction of the planets have intelligent life? **(Lesson 8-4)**

28. Rewrite the expression without using parentheses. $\frac{3}{5}(2x + 3y) - \frac{2}{5}(3x - 2y)$ **(Lesson 7-3)**

EXPLORATION

29. In aerospace travel, the Mach number is the ratio of the speed of an object to the speed of sound in a gas. Do research to find the four types of flight conditions based on the value of the Mach number.

Figure 17b. Variable tasks on variable page, *Transition Mathematics* p. 588. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

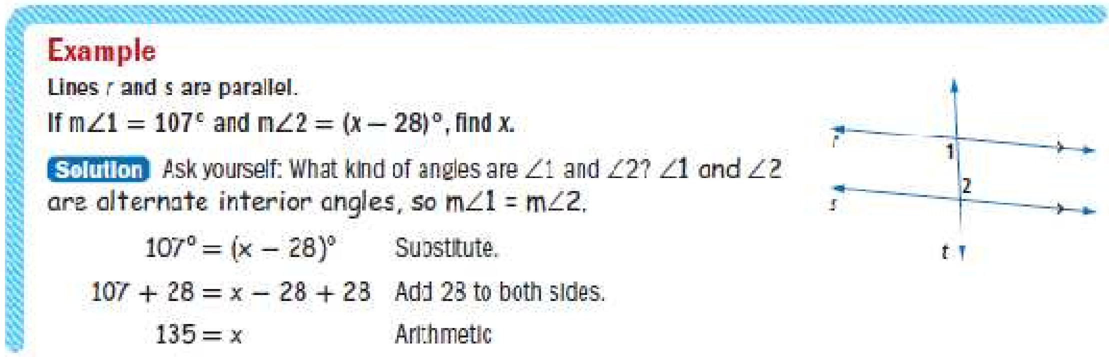
Contrary to the counting of the tasks in questions *8-11* and questions *12-15* in Figure 17a, I counted the tasks in Figure 18 as different tasks in the curricula because, although these tasks may essentially be designed to provide students with only one sort of learning opportunity on variables (i.e., variables used as *varying quantities*), they may include multiple occasions for students to experience this kind of opportunity. And, given that students will not solve all tasks in the curriculum they use, having multiple occasions of the same learning opportunity in the curriculum seems to be a potentially good indicator of the likelihood that teachers will choose to implement these tasks in their classrooms and students will learn from them (Stylianides, 2005).

40. A general rule for finding a man's shoe size in the United States is to multiply the length of his foot in inches by 3 and then subtract 22. A formula describing this rule is $S = 3\ell - 22$, where S is U.S. men's shoe size and ℓ is the length of a man's foot in inches. Nate's foot is 11 inches long. Find his shoe size.

41. The formula $C = 0.6n + 4$ estimates the temperature in degrees Celsius when n is the number of cricket chirps in 15 seconds. If a cricket chirps 25 times in 15 seconds, what is an estimate for the temperature?

Figure 18. Variables used as varying quantity, *Transition Mathematics* p. 127 Q. 40-41. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

If a task contained more than one conception of variable, that task was multiple coded as containing those uses of variables contained in it. An example of this situation is the task in Figure 19 that incorporates two conceptions of variables: one as a label (i.e., r, s and m to denote lines and angles in the task) and the other as specific unknown (i.e., x) to be determined. Consequently, the use of variables in this task was double coded as a *label* and as a *specific unknown*.


Example

Lines *r* and *s* are parallel.

If $m\angle 1 = 107°$ and $m\angle 2 = (x - 28)°$, find x.

Solution Ask yourself: What kind of angles are $\angle 1$ and $\angle 2$? $\angle 1$ and $\angle 2$ are alternate interior angles, so $m\angle 1 = m\angle 2$.

$107° = (x - 28)°$	Substitute.
$107 + 28 = x - 28 + 28$	Add 28 to both sides.
$135 = x$	Arithmetic

Figure 19. Variables used as labels and placeholder, *Transition Mathematics* p. 396. Taken from *The UCSMP: Transition Mathematics*, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

Coding Scheme

The following codes were used to record variable ideas that were sighted in the sample mathematics textbooks.

Table 8

Coding Scheme for Variable Ideas

Code	Definition
Defn	definition of variable, vocabularies or terms related to variables
Nar	narrative portion of textbook containing variable ideas
Wexa	worked examples portion of textbook containing variable ideas
Cr	chapter review portion of textbook containing variable ideas
Ct	chapter test portion of textbook containing variable ideas
Rls	variable ideas in real life situation
Tech	use of technology to explore variable ideas
Lab	label
Const	Constants
Gen#	generalize numbers
Pchold	specific unknowns
Cont Unk	continuous unknown
VarQty	varying quantity
Abst	abstract symbol
Para	parameter or argument
Itsm	informing teachers of students' misconceptions
Sasm	suggestions for addressing students' misconception
Alt	alerts to teachers on treatment of variable ideas
Nt	notes to teachers on variable ideas

Coding the Data

I reduced the data in terms of numbers, ratios, percentages and proportion: e.g., I considered the number of tasks that have specific characteristics out of the total number of tasks examined in each categorization. For the initial sighting and recording of variable ideas in the textbooks, I created a matrix with the following entries: *chapter/section*, *page(s)* and *description*. Once a variable idea is sighted in the textbook, it was recorded in this matrix accordingly, noting the chapter/section, page number(s) and a brief description of the variable using the codes in Table 8. For example, the description "lab in nar p29" in the matrix will stand for the sighting of a "variable used as a label in the narrative portion of textbook on page 29". These were further classified into categories and subcategories based on the analytical framework.

Data Analysis

The data collected in this study were analyzed both qualitatively and quantitatively. The qualitative data analyses include the use of constant comparison (Glaser & Strauss, 1967; Goetz & LeCompte, 1981; Merriam, 1988). Merriam (1988) defines constant comparative method as a "method that involves comparing one segment of data with another to determine similarities and differences. Data are grouped together on a similar dimension" (p. 18). Goetz and LeCompte (1981) state that "as events are constantly compared with previous events, new topological dimension, as well as new relationships, may be discovered" (p.58). These dimensions become categories and are given names, with an overriding objective to locate patterns in the data and arrange them in relationship to each other. Tesch (1990) views comparison as the main intellectual activity that underlies all analysis in qualitative research. Tesch made this point when she stated that:

> The main intellectual tool is comparison. The method of comparing and contrasting is used for practically all intellectual tasks during analysis: forming categories, establishing the boundaries of the categories, assigning the segments to categories, summarizing the content of each category, finding negative evidence, etc. The goal is to discern conceptual similarities, to refine the discriminative power of categories, and to discover patterns. (Tesch, 1990, p.96).

I used constant comparison to compare the development and presentation of variable ideas in the selected middle-grades mathematics textbooks among different grade levels within a single era, among the same grade level across textbooks of different eras, and among textbook series from different eras. To analyze the definitions of variables in the textbooks using constant comparative method, for example, I employed the 10 definitions of variables documented by Schoenfeld and Arcavi (1988) as a "partial framework" (Glaser & Strauss, 1967, p. 45) to compare, discover the attributes, and categorize the definitions of variables found in the selected middle-grades textbooks.

The quantitative analyses involved calculation of descriptive statistics: means, standard deviations, percentages, and the use of visual displays (e.g., bar graphs) to address some of the research questions. In the following pages, I provide brief descriptions of specific techniques that were used to analyze the data for each of the research questions.

To address the first question on how middle-grades mathematics textbooks develop the concept of variable, I identified and examined the introductions, definition(s), explanation of terms, and worked examples related to variable ideas in the narratives of the selected mathematics textbooks in the respective eras. I also report on the similarities and differences in the approaches to the introduction and presentation of variable ideas across the textbooks from the four eras (i.e., cross-case analysis and constant comparison of the presentation of variable ideas across eras). In addition, I examined and report on the context characteristics of variable ideas (i.e., whether variables were used purely at the symbolic level to enhance students' manipulation skills, used in tabular and diagrammatic representations, or used in modeling real-life situations) in the different mathematics curricula.

To address the second question on the extent to which middle-grades mathematics textbooks present activities and tasks that address each of the uses of variables, I document the presence and frequencies of the various uses of variables (i.e., *labels*, *unknowns*, *varying quantities*, *constants*, *generalized numbers*, *placeholders*, and *abstract symbols*) in the students' editions of the selected mathematics textbooks, and the order in which they appeared. I then used graphical displays and descriptive statistics to determine the extent to which quantitative differences exist in the proportions of use of variables in the entire sample of textbooks.

The third question on the uses of variable that are prevalent within the respective content areas was answered by documenting the presence and the frequency counts of the various uses of variables within each of the content areas in the middle grades mathematics textbooks. I then compared the proportions of the various uses of variables under each content area, and across the five content areas, for the same curriculum, and for the entire sample of textbooks using graphical displays and descriptive statistics.

I addressed the fourth research question on the differences in the treatment of variables in different grade level textbooks by identifying and reporting the grade level(s) textbooks in which introduction, definitions and explanations related to variables occurred. I also compared the proportion of variable related pages, and the composition of various categories of use of variable among the grade level textbooks for each textbook series using graphical displays and descriptive statistics.

The fifth research question on the changes in the development of variables in middle-grades mathematics textbooks during the past five decades was addressed using cross-case analysis (Denzin, 1989; Yin, 1994, 2003), with the different cases being the presentation of variable ideas from the textbook series from the four different mathematics textbook series, to determine the qualitative differences and similarities in the presentation and development of variable ideas among the mathematics textbooks across the four eras. Specific comparisons included the nature of introduction and definitions, the amount of related content on variable in the textbooks of each era, the context characteristics of variable ideas in the textbook from each era, the composition of various categories of uses of variable in the textbook series of each era, and the like. I also examined possible linear trends in the development of variable ideas in terms of the proportion of variable pages, and the composition of the use of variable categories over time among the mathematics curricula from the four eras of mathematics education.

Finally, the question on the guidance and support available to teachers in the teacher's editions textbooks to enact variables in middle grades mathematics classroom was answered by reporting the existence, the nature and quantity of such supports, including notes that enlighten teachers on the definitions of variables, on the various uses of variables, alerts for teachers on students' misconceptions on variables, to students' difficulties with variables, among others.

Summary

In this chapter, I described the research methods and design I used in this study. I also provided the timeline for each of the four eras of mathematics education within which the study is situated. In addition, I described the criteria for selecting the textbooks used in this study, and provided the rationale for selecting each of them. Further in the chapter, I described the research techniques (content analysis) and research approach (mixed methods) that were used to examine the variable ideas in the textbooks. I also described the data collection tool, coding scheme, reliability and validity measures, and tested the applicability of the data collection protocol through a pilot study using UCSMP middle grades mathematics textbooks. I ended the chapter with a description of how the data collected were analyzed to answer the research questions. Chapter 4 presents the results of the study. The discussion of the findings, the conclusions and the implications for further research and curriculum development are presented in Chapter 5.

Chapter 4

The Results of the Study

In this chapter, I present the results of the study. I organized the results by information on variable ideas in the different mathematics curriculum examined, followed by the answers to each of the six research questions investigated in this study. I achieved these in the following manner: First, I provide a brief report on the treatment of variables in *Modern School Mathematics* (selected for the New Math era), *Holt School Mathematics* (selected for the Back to Basics era), *Mathematics Today* (selected for the Problem Solving era), and *Math Connects* (selected for the NCTM Standards era).

I then present the result to *Research Question 1* by addressing how the middle-grades mathematics textbooks develop the concept of variables. This is followed by the result to *Research Question 2* on the extent to which the middle-grades mathematics textbooks present activities and tasks that address each of the uses of variables (e.g., *labels*, *unknowns*, *generalized numbers*) and the order in which these uses occurred in the mathematics textbooks. Thereafter, I report the results on the uses of variable that are prevalent within geometry, measurement, number and operations, algebra and data analysis and probability in sampled mathematics textbooks (*Research Question 3*).

Next, I present the results on how the development and/or presentation of the concept of variable differ across different grade levels of the same math curriculum (*Research Question 4*), and the extent to which the development and/or the presentation of variables in middle-grades mathematics textbooks in the United States have changed during the past five decades (*Research Question 5*). I conclude the section with the results to *Research Question 6* which examined the support provided to teachers (in the teachers' guides) to enact variable ideas in the classroom.

Treatment of Variable Ideas by Mathematics Curricula

I examined 12 student editions middle grades mathematics textbooks (made up of three textbooks from each of the four eras of mathematics education in the United States) on their treatment of the concept of variables. I also examined the corresponding teachers' textbooks on the guidance they offer teachers to treat variable ideas in the classroom. Table 9 presents information on the number of instructional pages, the number of pages that contain variable ideas, the percentage of pages that contain variable ideas, the first, and the last page of variable ideas in each of the 12 student edition textbooks.

Table 9

Instructional Pages, Variable Pages, and Percentage of Pages of Variable Ideas

Textbooks	Grade	No. of Inst. Pages	No. of Var. Pages	Percent of Var. Pages	First Var. Page	Last Var. Page
Modern School Mathematics	6	341	234	69	2	341
	7	523	288	55	8	523
	8	492	323	66	7	491
Holt School Mathematics	6	354	111	31	26	354
	7	386	173	45	2	383
	8	386	206	53	18	386
Mathematics Today	6	392	86	22	40	389
	7	440	264	60	12	439
	8	438	248	57	5	437
Math Connects						
	6	669	304	45	22	669
	7	665	308	46	35	665
	8	665	411	62	29	655

The proportion of pages that contain variable ideas (reported in percent) was obtained by dividing the number of instructional pages containing variable ideas by the total number of instructional pages in the textbook. The information presented in the table reveals that variable ideas were found in each of the 12 students' edition mathematics textbooks examined in this study.

A cursory examination of the table indicates that, overall, the *Modern School Mathematics* curriculum selected from the New Math era has the highest proportions of variables pages. The *Holt School Mathematics* program from the Back to Basics era, records the least proportions of variable pages in the mathematics curricula examined. The proportions of pages containing variable ideas ranged from 22% to 69% with the mean of 51% and standard deviation of 10 for the entire sample of textbooks examined.

The first page of variable ideas occurred as early as on the 2nd instructional page of some of the textbooks examined (e.g. *Modern School Mathematics* grade 6, and *Holt School Mathematics* grade 7) while the last page of variable ideas occurred as late as on the last instructional page of the majority of the textbooks examined. In the following pages, I provide a more detailed description on the treatment of variable ideas in each of the middle grades mathematics curricula investigated.

New Math Era: *Modern School Mathematics Curriculum*

As the name indicates, the content of the *Modern School Mathematics* (MSM) curriculum was *modern* both in terms of the topics covered and the style and language of presentation. The New Math reform recommendations in the late 1950s seemed to have a direct influence on the presentation of topics covered in this mathematics curriculum. The structure of number systems was emphasized using the ideas of set, relations, deductive proofs, and functions. A major aim of these emphases was to link school mathematics with university or higher mathematics. Table 10 presents information on topics covered in the *Modern School Mathematics* textbooks for the middle grades.

Table 10

Chapter Titles for Modern School Mathematics Textbook Series

Grade	Ch	Title	Ch	Title
	1	Sets, Numbers & Numerals	7	Fractional Numbers
	2	Add, Sub., Mult. & Divide	8	Mult. & Division: Fractions
	3	Geometry & Graphing	9	Geometry & Graphing
Grade 6	4	Multiplication & Division	10	Decimals, Per Cent
	5	Statements	11	Integers
	6	Number Theory		
	1	Sets & Numbers	8	Line & Angle Relationships
	2	Properties: Addition & Subtraction	9	Coordinate System on a line
	3	Properties: Multiplication &	10	Fractions & Rational Numbers
Grade 7	4	Division	11	Decimals Rational Numbers
	5	Numbers & Numerals	12	Measurement & Geometry
	6	Algorithms & Arithmetic	13	Percentage & Statistics
	7	Set & Geometry	14	The Set of Integers
		Number Theory		
	1	Rational Numbers	10	Open Number Sentences
	2	Add. & Subt.: Rational Numbers	11	Solving Open No. Sentences
	3	Multi. & Div: Rational Numbers	12	Using Equations
Grade 8	4	Geometric Figures in the Plane	13	Square Roots: Similar Figures
	5	Congruence & Measurement	14	Pyramids & Prisms
	6	Exponents & Scientific Notation	15	Cones, Cylinders, & Spheres
	7	The Metric System	16	Relations, Functions, Graphs
	8	Precision & Accuracy	17	Probability
	9	Decimal Numerals & Real Number		

Treatment of Variable Ideas in the Narrative Portion

Some of the chapters in the *Modern School Mathematics* curriculum that employed variable ideas include geometry and graphing, statements, properties (addition and subtraction), coordinate system on a line, properties (multiplication and division), open number sentences, solving open number sentences, multiplication and division of rational numbers, and using equations.

Information presented in Table 9 indicates that the *Modern School Mathematics* curriculum employed variable ideas frequently in developing the topics covered in the middle grades textbooks. Although, there was presence of the use of variables in each of the seven categories of variables (labels, placeholders, constants, continuous unknowns, generalized numbers, varying quantity, and abstract symbol) in the narrative portion of this curriculum, the most frequently employed category was variable as *generalized numbers*. Each grade level textbook begins with number and operations concepts (called rational number concepts in this curriculum), and employed variables predominantly in the narratives to generalize arithmetic patterns, sequences and to prove theorems.

Information contained in Figure 20 illustrates a typical page that contains variable ideas in the narrative portion of the *Modern School Mathematics* textbooks. There were very frequent uses of variable ideas similar to the information in Figure 20 in the *Modern School Mathematics* textbooks.

Consider the following illustration:
If a is represented by $2 + 3$ and b is represented by $4 + 1$, then you can see that $a = b$ because $2 + 3 = 4 + 1$.
Now, let c be represented by 8. Add c to both a and b. You have
$a + c = b + c$ because $(2 + 3) + 8 = (4 + 1) + 8$.
This illustrates that if $a = b$, then $a + c = b + c$.
Is the converse true? That is, if you are given $a + c = b + c$, can you show that $a = b$? Yes, you can by arguing as follows. By adding $-c$ to both members, you get (by the addition property of equality),
$(a + c) + (-c) = (b + c) + (-c)$.
By using other properties, you can show:

$$a + [c + (-c)] = b + [c + (-c)]$$
$$a + 0 = b + 0$$
$$a = b$$

Therefore you can combine the addition property of equality and its converse in the statement: $a = b$ and $a + c = b + c$ are equivalent equations.
(Modern School Mathematics Grade 8, p. 284)

Figure 20. Use of variables as generalized numbers in the *MSM* textbook series.

Throughout the curriculum, the ideas of sets were used in conjunction with other topics, including the presentation of variable ideas. This makes the nature of variable ideas in this curriculum relatively complex than might be found in typical middle grades mathematics textbooks. For example, there were several instances in this curriculum where notations such as $\{x: x \in S \cup T\}$ were used in the context of solving simple linear equations in which the variable functions as a placeholder.

In the *Modern School Mathematics* curriculum for the middle grades, variables were first used to denote sets and to label points at the beginning of each of the three textbooks. Writing of expressions using variables and evaluation of variable expressions came much later (towards the middle chapters of the respective grade level textbooks). The idea of variables as varying quantities was developed through functions, relations, and in the use of formulas (e.g., area, volume, and circumference), among others.

Many of the tasks employing variable ideas in this curriculum were proofs, generalizations and justifications. Variables ideas were rarely employed to solve problems relating to real-world situations in this curriculum. Table 11 presents information on the number of instructional pages, number of pages containing variables, percentage of pages containing variables, the first, and the last page containing variable ideas in the respective textbooks of the *Modern School Mathematics* curriculum.

Table 11

Instructional pages, Variable pages, and Percentage of Variable Pages in Modern School Mathematics Curriculum

Era of Math Education	Grade	Number of Instructional Pages	No. & % of Var. Pages	First Page Of Var. idea	Last Page Of Var. idea
	6	341	234 (69%)	2	341
New	7	523	288 (55%)	8	523
Math	8	492	323 (66%)	7	491

As seen in the table, the 6th grade *Modern School Mathematics* textbook consisted of 341 instructional pages. Two hundred and thirty-four (234) pages of the 341 pages contained variable ideas. This represents 69% of the instructional pages in the 6th grade mathematics textbook that contain variable ideas of some sort. Variables appeared as early as on the 2nd instructional page in the 6th grade textbook and as late as the last instructional page (341) in the textbook. Figure 21 provides a global view of the location of variable ideas and the percent of variable tasks on a variable page in the 6th grade textbook of the *Modern School Mathematics* curriculum.

The percent of variable tasks on a variable page (shaded in the display) was obtained by dividing the number of tasks employing variable ideas on an instructional page by the total number of tasks on that page (rounded to the nearest tens). The horizontal axis of the Figure names the respective page numbers whilst the vertical axis represents the percentage of variable tasks (in an increment of 10). As illustrated in the figure, every chapter in the 6th grade textbook of this curriculum employed variable ideas. However, some chapters employed variable ideas at a higher percentage than others. For example, one can see that chapter 3 on *Geometry and Graphing* employed variable ideas at a higher percentage than Chapter 10 on *Decimals, Per Cent* did.

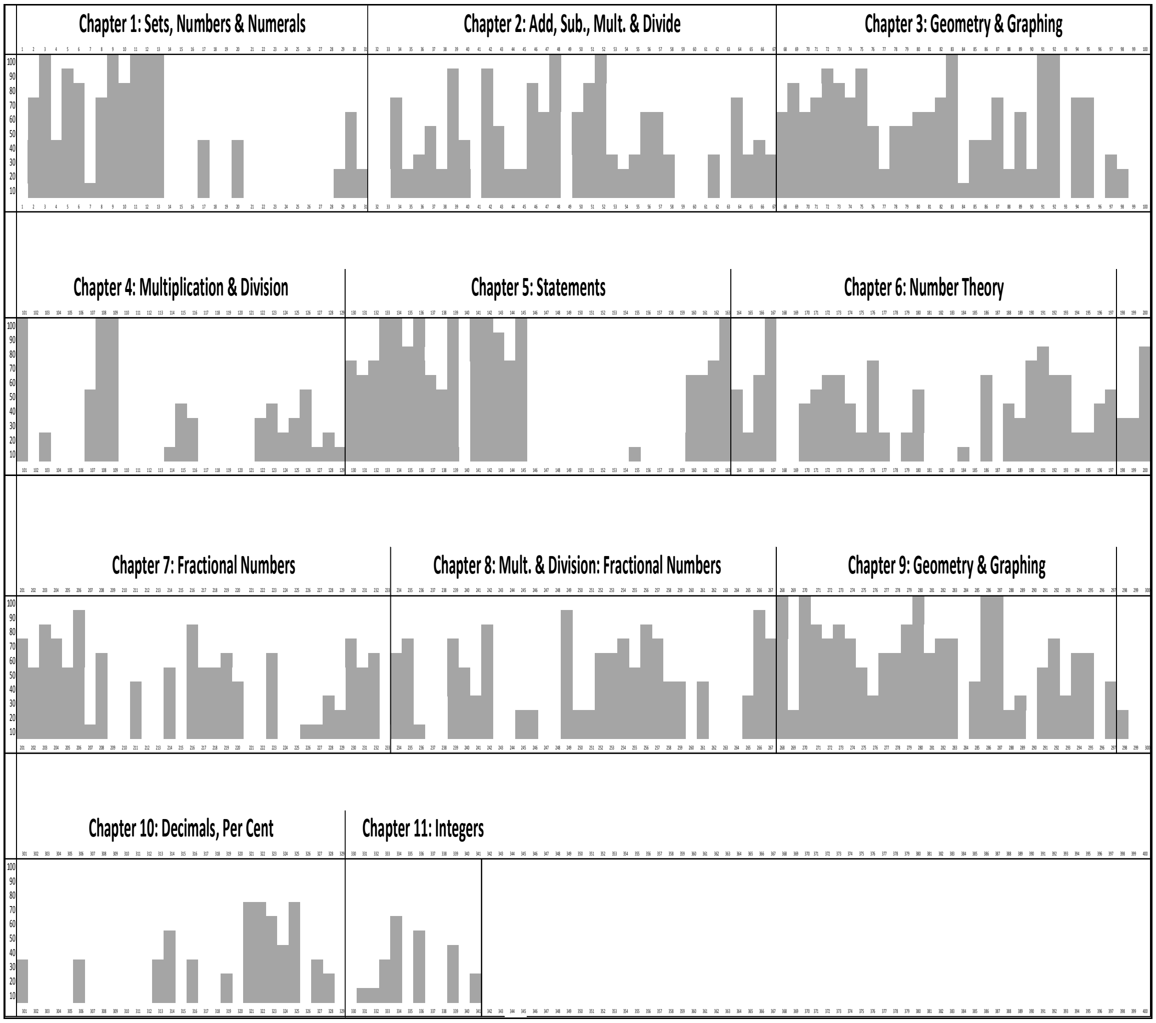


Figure 21. A visual display on the locations and the percent of variable tasks on a variable page in the 6th grade textbook of the Modern School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

The 7th grade textbook of the *Modern School Mathematics* curriculum consists of 523 instructional pages of which 288 contain variable ideas. This constitutes 55% of the instructional pages in the 7th grade textbook with variables ideas sighting. The first instructional page on which variable was found in this textbook was page 8 and the last instructional page containing variable ideas was page 523.

Figure 22 provides a global view of the location of variables and the percentage of variable tasks on variable pages in the 7th grade textbook of the *Modern School Mathematics* curriculum. Again, as the figure reveals every chapter in the 7th grade textbook employed variable ideas. However, there were vast differences in the extent to which different chapters employed variable ideas in the 7th grade textbook. Specifically, whereas chapter 8 on *Line and Angle Relationships* employed variable ideas on almost every page (see Figure 22), variable ideas were found on only one page in chapter 4 on *Numbers and Numerals*.

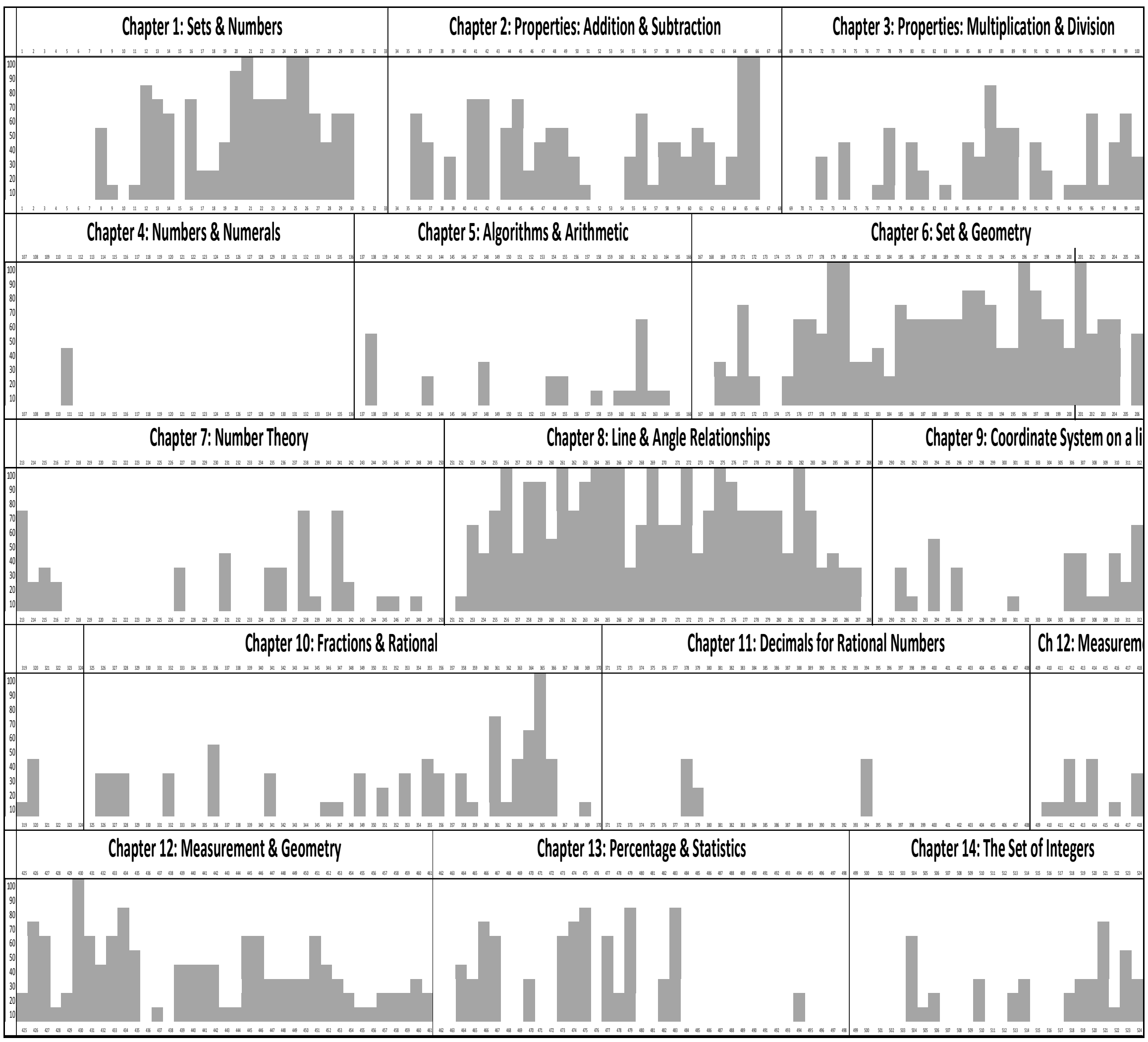


Figure 22. A visual display on the locations and the percent of variable tasks on a variable page in the 7th grade textbook of Modern School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

Similarly, the 8th grade textbook consisted of 492 instructional pages of which 323 (representing 66%) contain variables ideas. The first sighting of variable was on page 7, and the last sighting of variables was on page 491 in the textbook. Figure 23 displays the locations and the percentage of variable tasks on variable pages in the 8th grade textbook of the *Modern School Mathematics* curriculum.

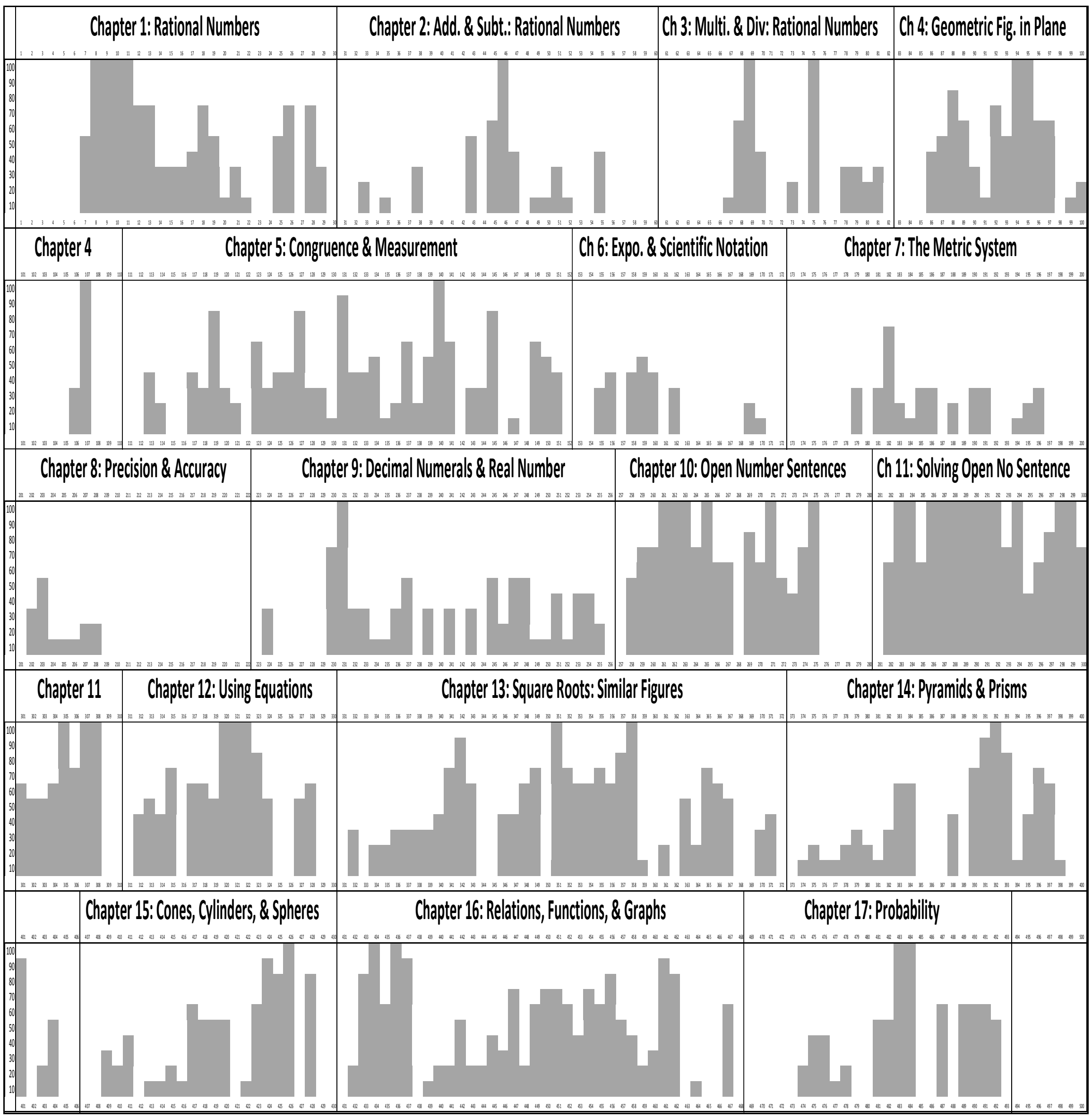


Figure 23. A visual display on the locations and the percent of variable tasks on a variable page in the 8th grade textbook of Modern School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

As in the case of the 6th and the 7th grade textbooks, every chapter in the 8th grade textbook employed variable ideas but in different proportions. In this textbook, chapter 11on *Solving Open Number Sentences* records the highest proportion in the use of variables, whilst chapter 8 on *Precision and Accuracy* has the least proportion in use of variables.

Definitions for Variable in Modern School Mathematics Textbooks

Definition(s) for variables were found in each of the middle grades textbooks of the *Modern School Mathematics* curriculum. Figure 24 presents the definition of variable, and the context within which the definition was presented in the 6th grade textbook of the *Modern School Mathematics* curriculum.

	Symbol	Name	Use
SETS	{ }	Braces	To show a set of elements
	$\cup \cap$	Union, inter	To show an operation with sets
	$\subset \quad \supset$	Subset superset	To state a relation between sets
NUMBERS	A letter like **n**	Placeholder	To hold the place for a numeral as in $4 + n = 9$
	$+ - * \div$	Addition (plus) Subtraction (minus), Divide Multiplication (times)	To show an operation with numbers
	$> \ < \ =$	Greater than, less than equals	To state a relation between numbers

Figure 24. Definition of variable (*Modern School Mathematics*, Grade 6, p. 11).

As revealed in the definition, there was no mention of the word *variable* in the definition/explanation of variables in the 6th grade textbook. Rather, a synonym (placeholder) was used to refer to the variable idea. And, although 69% of the instructional pages in the 6th grade textbook contain variable ideas of some sort, there was no mention of the word "variable" in the entire 6th grade textbook. The information in

Figure 24 also illustrates the extent to which *set ideas* were embedded in the presentation of every topic in the *Modern School Mathematics* curriculum (see also the definition of variable in the 7th grade textbook).

The 7th grade textbook defined variables differently from the definition found in the 6th grade textbook. The definition in the 7th grade textbook was presented in a subtitle called *Commutative Property of Addition; Variables*. The textbook defined variable in the *Commutative Property of Addition; Variables* as follows:

> "$3+6$"and "$6+3$" both name the number 9, you have $3+6=6+3$. Of course, the statement "$3+6$"and "$6+3$" is just a special case of the general law: Changing the order of the addends in a sum of whole numbers does not change the sum. Do you know how to state this general law in symbol? You cannot use numerals like "1", "2", "3" which name specific whole numbers. Instead, when you wish to refer to members of a set, *in general*, you commonly use letters like a, b, c, x, y, or even special characters like □, o, ?and ∇. Such a letter or other symbol used to denote any of the elements of a given set is called a **variable.** The members of the given set are the **values** of the variable. For example, the following statement contains two variables a and b, each denoting the members of the set W of whole numbers. Remember that $W=\{0,1,2,3\ \ldots\ldots\}$. Commutative Property of Addition: If a and b denote whole numbers, then $a+b=b+a$.
>
> (Modern School Mathematics Grade 7, p. 134)

Figure 25. Definition of variable (*Modern School Mathematics*, Grade 7, p. 134).

A Similar definition for variable was found in the 8th grade textbook of the Modern School Mathematics curriculum. An important characteristic that is consistent in the definition for variable found in the 7th and 8th grade textbooks of this curriculum was that variable was defined in terms of its use as *generalized numbers*. Consequently, each of the textbooks employed variable ideas predominantly in the narrative section to develop arithmetic and algebraic properties, generalizations, proofs, relations and functions.

It is also worth pointing out that, in all the textbooks of the *Modern School Mathematics* curriculum for the middle grades, variables were used prior to defining what variables are or explaining how they are used. Even in the sixth grade textbook where the definition for variable occurred relatively early (i.e., on page 11), variable ideas were found on page 2 through page 10 (used as *labels* and *placeholders*) before the definition occurred on page 11. The treatment of variables in the other two textbooks (grades 7 and 8) followed a similar pattern of development. Apart from the definitions and the accompanying explanations offered for variables in these textbooks, there were no introductions, historical or background information on variables found in the *Modern School Mathematics* textbook for the middle grades.

Distribution by Percent of Variable Ideas in Different Grade Levels

Table 12 presents the percentages of the uses of variable categories in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test in the students' textbooks for the respective *Modern School Mathematics* textbooks. As can be seen in the table, the uses of variable as labels, and as placeholders constitute a significant portion of the uses of variables in these blocks of the curriculum. The constitution of these two categories (label and placeholder) was higher in the 6^{th} and the 7^{th} grade level textbooks than it was in the 8^{th} grade textbook. Specifically, the uses of variables in the two categories constitute 75% in the 6^{th} grade and 76% in the 7^{th} grade, but only 55% in the 8^{th} grade textbook.

The finding of the large proportion of the use of variables as labels and placeholders in the exercises or question sets, activities, etc., was contrary to the predominant use of variable as generalized numbers in the narrative portions of this curriculum. Thus, whereas variables were used predominantly in the narrative as generalized numbers, in the exercises or question sets, activities and projects, chapter review, chapter summaries and chapter test portions of the *Modern School Mathematics* curriculum, variable ideas were employed mainly as placeholders, and as labels.

Another important observation on the treatment of variable ideas in this curriculum is that variables were used in all the seven categories of use of variables in each of the three textbooks of the *Modern School Mathematics* curriculum. It is also worth noting that, with the exception of the *Constant* category, the percentage of the use of variable categories in this curriculum was more evenly distributed in the 8th grade textbook than it was in the 6th and 7th textbooks. Table 12 reports the percentages of the uses of variables in the seven categories of use of variables in the *Modern School Mathematics* curriculum for the middle grades.

Table 12

Distribution by Percent of Variable Category in Different Grade Levels in Modern School Mathematics Curriculum

Grades Level	Variable (sub)Category						
	Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Grade 6 (*n* = 2148)	22	2	57	5	1	11	2
Grade 7 (*n* = 1044)	54	1	22	9	10	2	2
Grade 8 (*n* = 1429)	31	1	24	13	11	16	4

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk. = Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

Distribution by Percent of Variable Ideas in Different Content Areas

With respect to the use of variable ideas in different content areas in the *Modern School Mathematics* textbooks, an examination of Table 13 reveals that a little over 60% of the uses of variables in the *number and operations* content area classifies into the use of variables as placeholders. The next highest category of use of variable in this content area was as a label (20%). The remaining 18% of the usage of variable in the *number and operations* content area was distributed among *continuous unknown*, *generalized numbers, abstract symbol* and *varying quantity* with the latter having the least percentage of use (i.e., 2%).

Similarly, the use of variables as placeholders dominated *algebra* content (35%) in the *Modern School Mathematics* curriculum. This was followed by the use of variable as varying quantities (23%), and then as continuous unknowns (22%). The use of variables as *labels* was highest in *measurement*, *geometry*, and *data analysis and probability* content areas. In particular, the use of variables as a label constitutes 71% of all the uses in *geometry*, 63% in *measurement* and 49% in *data analysis and probability* in the *Modern School Mathematic* textbooks for middle grades. It is also worth pointing out that the percentage of the use of variable as *continuous unknowns* was the largest in *algebra* compared to its use in the other content areas (see Table 13).

Table 13

Distribution by Percent of Variable Category in Different Content Areas in Modern School Mathematics Curriculum

Content Area		Variable (sub)Category						
		Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Number & Op.	(*n* = 1867)	20	0	62	7	6	2	3
Algebra	(*n* = 1045)	6	0	35	22	11	23	3
Geometry	(*n* = 1065)	71	1	10	1	1	15	1
Measurement	(*n* = 491)	63	11	4	2	0	20	0
Data & Prob.	(*n* = 117)	49	2	27	3	11	8	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk. = Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

There was limited use of synonyms for variables in this curriculum. As stated earlier, the 6th grade textbook used the word "placeholder" to refer to variables throughout the entire textbook. This word (placeholder) was, however, not used in the 7th and 8th grade textbooks. There was no use of technology to explore variable ideas in the *Modern School Mathematics* curriculum for the middle grades.

Support for Teachers on Variable Ideas

The *Modern School Mathematics* curriculum, in the teachers' textbooks, provided support to teachers to guide them in implementing variable ideas. Specifically, there were evidences of guidance provided to teachers to enhance their content knowledge, to offer them teaching suggestions, and to alert them to students' common misconceptions and difficulties with variables. For example, throughout the teachers' edition textbooks, this curriculum presented proofs of generalizations involving the use of variables found in the students' textbooks to educate teachers on the justification of these generalizations.

The following excerpt in Figure 26 taken from the 8th grade *Modern School Mathematics* teachers' edition textbook shows how variables were used in the teachers' edition to prove the uniqueness of additive inverse of a rational number.

The additive inverse of a rational number is unique, that is, there is only *one* such inverse. You can see formally that this is true by assuming that there are two rational numbers, b and c such that:

$$a + b = 0 \text{ and } a + c = 0$$

If so, then:

$$a + b = a + c$$
$$-a + (a + b) = -a + (a + c)$$
$$(-a + a) + b = (-a + a) + c$$
$$0 + b = 0 + c$$
$$a + b = a + c$$
$$b = c$$

That is, the two inverses are the same. The importance of knowing that the additive inverse is unique lies in the fact that this uniqueness lets you conclude that if $a + b = 0$, then $b = -a$ and $a = -b$".

(Modern School Mathematics, Grade 8, p. 14)

Figure 26. Educating teachers on variable ideas (*MSM* Textbook, Grade 8, p. 14).

Teaching suggestions on variable ideas were also found in the *Modern School Mathematics* curriculum to guide teachers to enact tasks employing variable ideas. An example of this is the following suggestion on why $-x$ does not always denote a negative number. This was found in the 7th grade teachers' edition textbook:

> "Since no symbol is introduced in this book for the opposite of a number, the property of subtraction of integers is not presented entirely in symbol. It is suggested that if you wish to introduce such a symbol to the class you use a lowered negative sign as in $-x$ rather than a raised sign as in ^{-}x. This is because the raised sign is already established as part of a symbol for a negative number, and it is undesirable for students to get the impression that "opposite of" always refers to a negative number. Great problems arise in algebra if students assume that $-x$ or ^{-}x always denotes a negative number. If the symbol $-b$ is introduced and defined to represent the opposite of b, then the property of subtraction of integers can be stated entirely in symbols in the form $a - b = a + (-b)$. Students should be warned in any event that a and b represent integers in $a - b$ here, and not whole numbers. That is the variables a and b can represent either positive or negative numbers or zero" (Modern School Mathematics, grade 7, p. 57).

There were substantial number of instances in the *Modern School Mathematics* textbooks where contents related to variable ideas were presented to enhance teachers' knowledge as well as provide guidance to them on the enactment of variable ideas as illustrated in the preceding examples. There were, however, very few instances where teachers' attentions were drawn to students' common misconceptions and difficulties with variables in this curriculum. Thus, the majority of the guidance provided for teachers in this curriculum was directed towards educating teachers and offering them teaching suggestions, than helping them to identify and address students' misconceptions and difficulties with variables.

Back to Basics Era: *Holt School Mathematics Curriculum*

The authors of the *Holt School Mathematics* (HSM) program indicated in their curriculum that they believed that every student, when given the suitable learning materials and the proper motivation, can learn mathematics. This belief played an instrumental role in the presentation of the content in this mathematics curriculum for the middle grades. For example, in order for deficiency in reading not to stand in students' way of learning mathematics, the authors used simple language and presented many concepts by means of illustrations. The *Holt School Mathematics* program also employed a spiral development of concepts: introducing concepts and skills at elementary level and developing them more and more completely as students move through the grades.

Table 14 reports the chapter titles that are covered in the *Holt School Mathematics* textbooks for grades 6, 7 and 8. The recommendations of the Back to Basics reform in the 1970s also seemed to influence the presentation of the contents in the *Holt School Mathematics* curriculum for the middle grades: this mathematics program placed heavy emphasis on basic skills development, and on the treatment of the metric system. Serious emphasis was also devoted to estimation skills, decimal computation, and percent with a decrease in emphasis on sets and numeration systems other than base ten.

Table 14

Chapter Titles for Holt School Mathematics in the Back to Basics Era

Grade	Ch	Title	Ch	Title
	1	Numeration	8	Fractions & Ratio
	2	Addition and Subtraction	9	Decimals
	3	Multiplication	10	Measurement
Grade 6	4	Division	11	Percent
	5	Geometry	12	Geometry
	6	Number Theory	13	Graphs & Integers
	7	Fractions		
	1	Sets & Whole Numbers	9	Percent & Its Uses
	2	Operations	10	Measurement: Metric System
	3	Problem Solving	11	Geometry
	4	Geometry	12	Add-Subtract Integers
Grade 7	5	Number Theory	13	Multiply-Divide Integers
	6	Fractions	14	Coordinate Geometry
	7	Working with Fractions	15	Probability & Graphing
	8	Decimals		
	1	Numeration & Computation	9	Rational Numbers
	2	The Decimal System	10	Percent
	3	Decimals & Measurement	11	Real Numbers
Grade 8	4	Geometry	12	Geometry
	5	Integers	13	The Metric System
	6	Equations & Inequalities	14	Coordinate Geometry
	7	Problem Solving	15	Probability & Statistics
	8	Number Theory		

Treatment of Variable Ideas in the Narrative Portion

Variable ideas were found in the respective textbooks of the HSM curriculum. Principal topics that incorporated variable ideas in the HSM were: addition and subtraction, geometry, metric system, coordinate geometry, percent, and equations and inequalities. This curriculum addresses the various uses of variables (except the use of variable as an *abstract symbol*) through generalizations, equation solving, inequalities, formulas, graphs, and labeling points. Solution methods to many different types of equations in which variables function as placeholders or specific unknowns were discussed. Related inequalities were developed following the study of equations. Each of the early chapters initially includes the use of equations, and then proceeds to inequalities, gradually developing more complex types of equations, inequalities, and their graphs.

Few activities were presented to help students write equations and formulas from verbal statements, and then use them to solve proportions, percent and commission problems. That is, some equations and formulas using variables were developed within real-world contexts and for applications purposes. In the later chapters, students explore the use of variables in equations to describe relationships between variables (e.g.,$y = x - 2$) and learn how to write functional equations to represent various situations (e.g., function machines). The main idea of how quantities change was treated implicitly through working with formulas (area, volume, and perimeter formulas) in this curriculum. Coordinate Geometry and Pythagorean Theorem were also used to develop relationships between variables. There was a substantial usage of special characters such as □, o,? and ∇ instead of letter variables in this curriculum to denote the *unknown*. In all, variables were used mainly as *placeholders* in the HSM curriculum for the middle grades.

Table 15 presents information on the number of instructional pages, number of pages containing variables, percentage of pages containing variables, the first, and the last page containing variable ideas in the HSM curriculum. As can be seen in the table, the 6th grade textbook consisted of 354 instructional pages, of which 111 pages contain variable ideas. This number constitutes 31% of the instructional pages in the 6th grade textbook that contain variable ideas. Variables were first found on page 26, and last on the 354th instructional page in the 6th grade textbook.

Table 15

Instructional pages, Variable pages, and Percentage of Variable Pages in *Holt School Mathematics Curriculum*

Era of Math Education	Grade	No. of Instructional Pages	No. & % of Var. Pages	First Page of Var. idea	Last Page of Var. idea
	6	354	111 (31%)	26	354
Back to	7	386	173 (45%)	2	383
Basics	8	386	206 (53%)	18	386

Figure 27 provides a global view of the location of variables, and the percentage of variable tasks on variable pages in the 6th grade textbook of the HSM textbook.

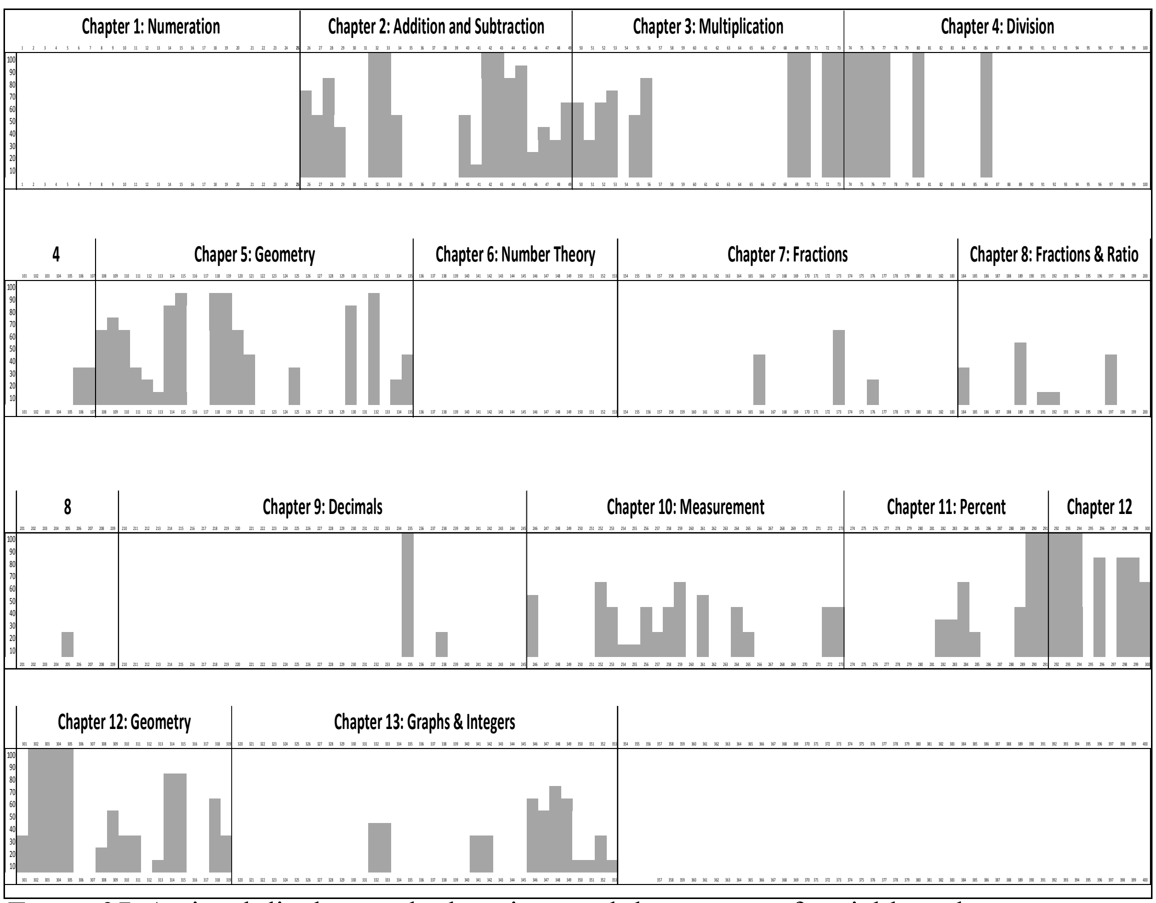


Figure 27. A visual display on the locations and the percent of variable tasks on a variable page in the 6th grade textbook of Holt School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

An examination of the figure reveals that, with the exception of chapter 1 on *Numeration*, and chapter 6 on *Number Theory*, every chapter in the 6th grade HSM textbook employed variable ideas in the presentation of its content. Among the chapters employing variable ideas, chapter 2 on *Addition and Subtraction* had the highest proportion of variable ideas, whilst chapter 7 on *Fractions* had the lowest proportion of variable ideas.

There were 32 more instructional pages in the 7th grade textbook than in the 6th grade textbook of the HSM textbook series. Thus, the 7th grade textbook had a total of 386 instructional pages, of which 173 contained variable ideas. This represents 45% of the instructional pages in the 7th grade textbook of the HSM curriculum with variables ideas sighting. The first instructional page containing variable was page 2, and the last page containing variables was 383. Figure 28 provides a global view of the location of variables, and the percentage of variable tasks on variable pages in the 7th grade textbook of the HSM curriculum.

An inspection of the figure reveals that, with the exception of chapter 8 on *Decimals*, every chapter in the 7th grade HSM textbook employed variable ideas in the presentation of its content. The figure reveals also that, among the chapters employing variable ideas, chapter 14 on *Coordinate Geometry* has the highest proportion of variable ideas, and chapter 7 on *Working with Fractions* has the lowest proportion of variables.

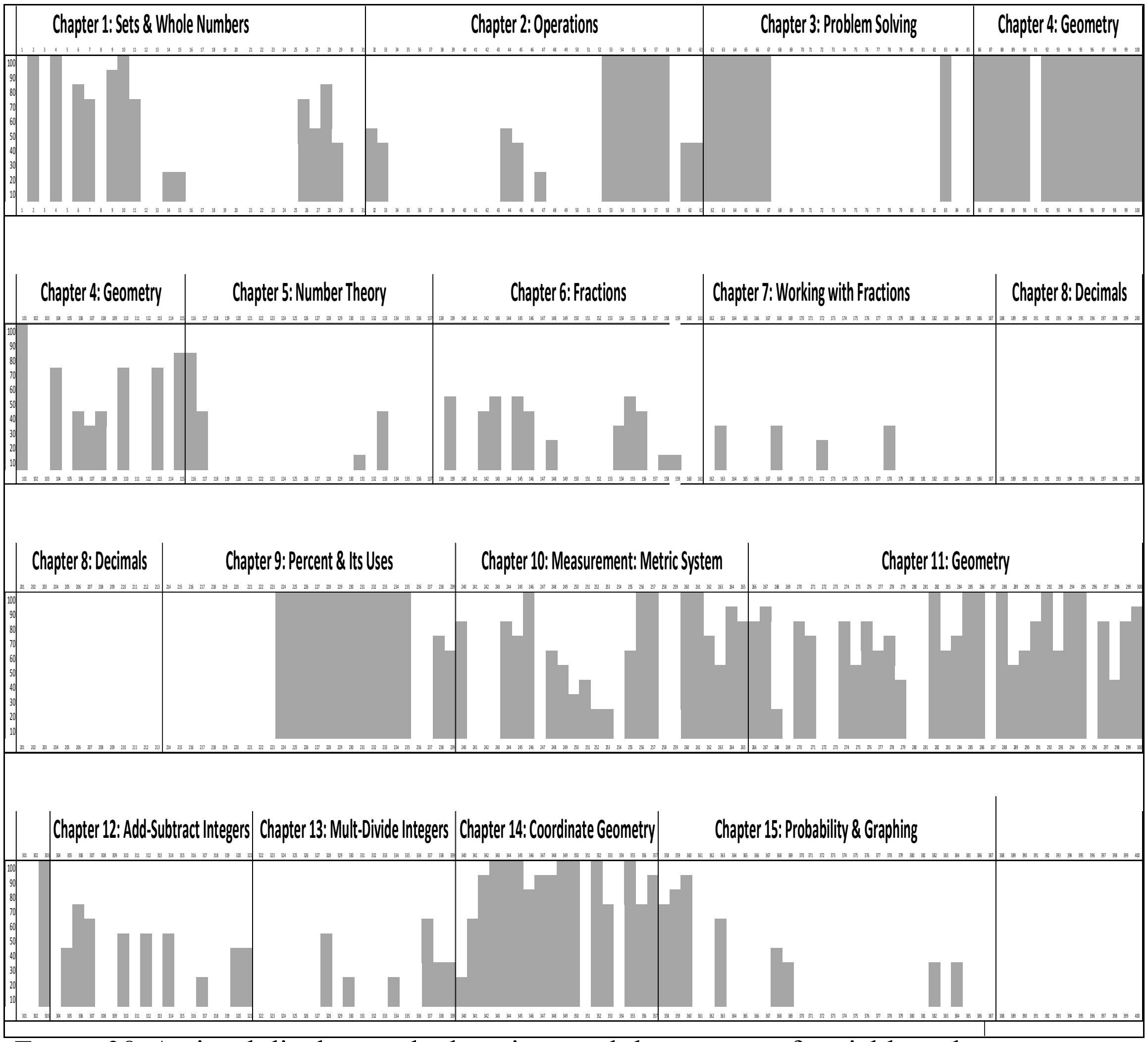


Figure 28. A visual display on the locations and the percent of variable tasks on a variable page in the 7th grade textbook of Holt School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

Similarly, the 8th grade textbook consisted of 386 instructional pages of which 206 pages (representing 53%) contained variable ideas. The first sighting of variable was on page 18, and the last variable idea was sighted on page 386 in this textbook. The location of variables and the percentage of variable tasks on variable pages in the 8th grade textbook of the HSM textbooks are displayed in Figure 29.

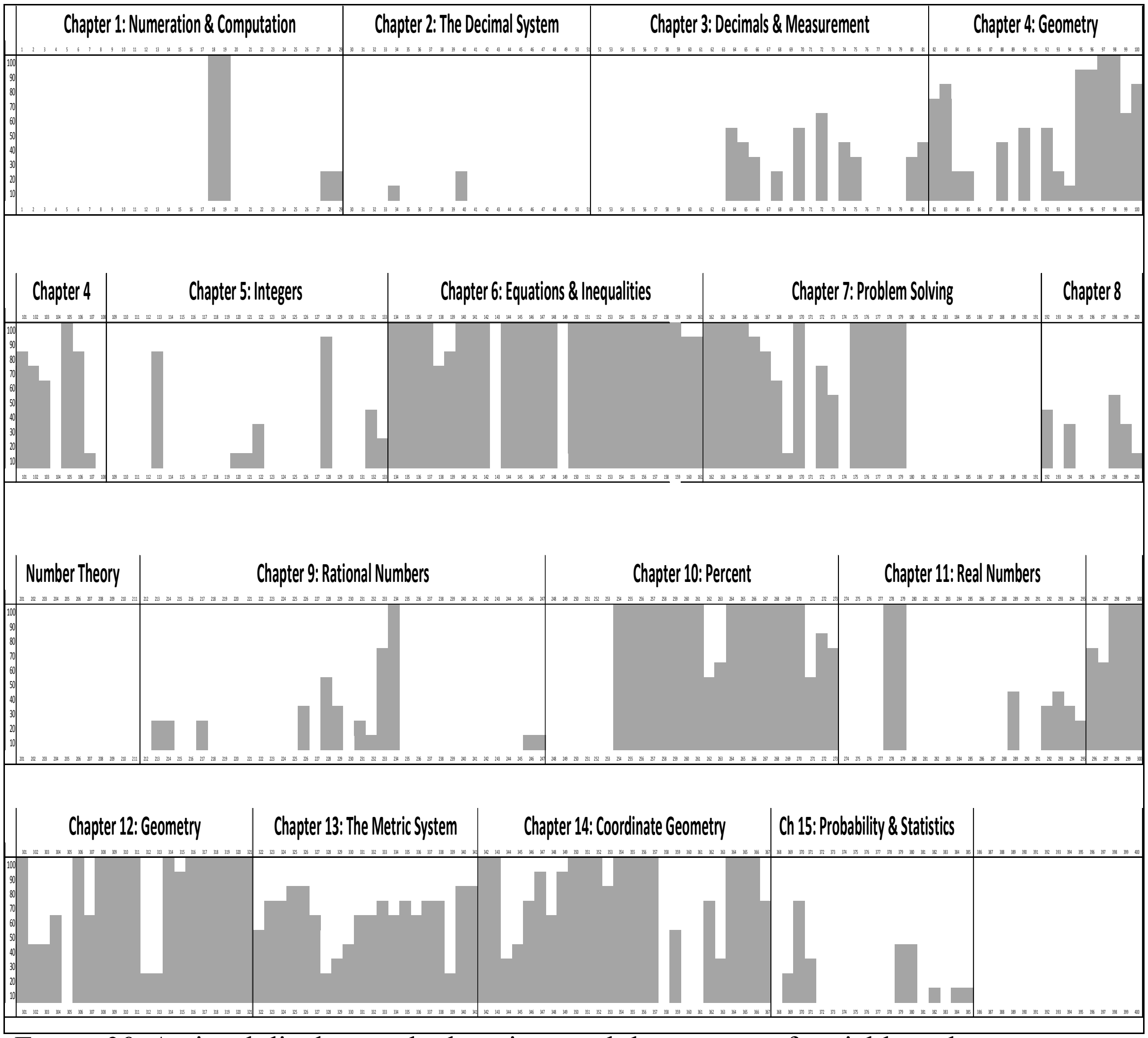


Figure 29. A visual display on the locations and the percent of variable tasks on a variable page in the 8th grade textbook of Holt School Mathematics curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

Contrary to the presentation of variable ideas by the chapters in the 6th and 7th grade textbooks, every chapter in the 8th grade textbook employed variable ideas to present its content. The chapter employing the highest proportion of variable ideas in the 8th grade textbook was chapter 6 on *Equations and Inequalities*, and the one with the lowest proportion of variable ideas was chapter 2 on *The Decimal System*.

Definitions for Variable in Holt School Mathematics Textbooks

Definition(s) for variables were found in each of the three textbooks of the HSM curriculum. The characteristics of the definitions found in this curriculum identified variable with its use as a *placeholder*. That is, variable was defined as a *placeholder* for a number that can be found or identified by solving an equation. Figure 30 contains the three definitions of variable found in the different textbooks of the HSM curriculum:

Consider the equation $n + 4 = 9$
$\lfloor$ **a variable**

If the replacements are — the solution is

$1, 2, 3, 4, 5, \ldots \ldots \ldots \ldots . .9, 10 \rightarrow 5$

$1, 3, 5, 7, 9 \rightarrow 5$

$2, 4, 6, 8, 10 \rightarrow$ no solution

(Holt School Mathematics, Grade 6, p. 42)

The equation $x + 7 = 12$
$\lfloor$ **a variable**

The solution is 5, because $5 + 7 = 12$ is true

(Holt School Mathematics, Grade 7, p. 53)

A mathematical sentence with the symbol $+$ is an equation. An equation may be true, false, or open.

$4 + 5 = 9$ is a true statement

$13 - 9 = 5$ is a false statement

$x + 7 = 10$ is an open sentence
$\lfloor$ **called a variable**

(Holt School Mathematics, Grade 8, p. 134)

Figure 30. Definitions for variables in the *Holt School Mathematics* curriculum.

It can easily be seen that each of these definitions was provided within the context of equation solving, and hence, characterizes variables as a *placeholder* for a number that can be found by solving the given equation. In all the three textbooks, variables were used for labeling points and as placeholders before formal definitions for variables were presented. Apart from these definitions and the accompanying explanations (as shown in Figure 30), there were no introductions, historical or background information related to the origin of variables, and the likes about variables in any of the three HSM textbooks for the middle grades. Words such as placeholders and unknowns that are sometimes used to refer to variables were not used or found in the HSM curriculum. On the other hand, there were substantial usage of symbols such as □, o, and ∇ instead of letters to denote variable ideas in all the three grade level textbooks of the HSM textbooks.

Distribution by Percent of Variable Ideas in Different Grade Levels

Table 16 reports information on the distribution (in percents) of the uses of variable categories in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test in the students' textbooks for different grade levels in the HSM textbooks. A cursory examination of the table reveals that, with the exception of the use of variable as an *abstract symbol*, the HSM curriculum employed variables in all of the other six categories of use of variables, but in different proportions. For the most part, variables were used as *labels*, and as *placeholders* in all the three textbooks. In fact, the use of variables in these two categories constituted at least 66% of the uses of variables in all three textbooks. Specifically, the use of variables in the two categories constituted 80% in the 6th grade textbook, 66% in the 7th grade textbook, and 70% in the 8th grade textbook. The next highest category of use of variables was the *varying quantity* category (this constitutes 9%, 12% and 15%, respectively for the 6th, 7th, and 8th grade textbooks).

Thus, the HSM curriculum in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test portions provided far more opportunities for students to engage with variables in these two categories (i.e., as labels and placeholders) than it does for students to learn about variables as *generalized numbers*, *continuous unknowns*, *varying quantities*, *constants*, and *abstract symbols*. Another important observation is that this curriculum did not use variables as an *abstract* *symbol* in the presentation of its contents.

Table 16

Distribution by Percent of Variable Category in Different Grade Levels in Holt School Mathematics Curriculum

Grades Level		Variable (sub)Category					
	Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst*
Grade 6 (*n* = 665)	33	5	47	1	5	9	0
Grade 7 (*n* = 953)	33	7	33	12	3	12	0
Grade 8 (*n* = 1560)	23	4	47	8	3	15	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

Distribution by Percent of Variable Ideas in Different Content Areas

With respect to the use of variable ideas in different content areas in the HSM textbooks, the information contained in Table 17 reveals that, for the *number and operations* content area, more than three-fourths of all the uses of variable identifies with the use of variables as *placeholders* (77%). The next highest categories of use of variable in the *number and operations* content area were as *label* (9%), and as *generalized number* (9%). Similarly, the use of variables as placeholders dominates *algebra* related content (51%) in this curriculum. This was followed by the use of variable in the category of *continuous unknown* (19%), and that of *labels* (15%). In *measurement*, *geometry*, and *data analysis and probability*, the use of variables as *labels* dominated, constituting 47% of the uses in *geometry*, 95% in *measurement,* and 73% in *data analysis and probability.* The *varying quantity* and *constant* categories were employed in relatively higher proportions within *geometry* than they were in the other content areas in this curriculum.

Table 17

Distribution by Percent of Variable Category in Different Content Areas in Holt School Mathematics Curriculum

Content Area		Variable (sub)Category					
		Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*
Number & Op.	(*n* = 1034)	9	0	77	3	9	2
Algebra	(*n* = 1149)	15	0	51	19	7	8
Geometry	(*n* = 1189)	47	16	9	0	0	28
Measurement	(*n* = 484)	95	1	2	0	0	2
Data & Prob.	(*n* = 75)	73	0	0	27	0	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

The data collected reveals no evidence of the use of technology or manipulative in developing variable ideas in the HSM textbooks for the middle grades. In addition, research issues on students' difficulty, misconceptions, and the likes on variables were not mentioned or discussed in this curriculum.

Support for Teachers on Variable Ideas

The *Holt School Mathematics* curriculum (in the teachers' textbooks) provided support to guide teachers as they enact variable ideas in the classroom. First, this curriculum provided an expanded version of the definition for variable in the teachers' textbook:

> a variable is a symbol, usually a letter, used in an equation in place of a number or numbers which make a statement true. Note that the solution set for all equations here contain only one number. You might ask students to define the following terms in their own words, after discussing the developmental items: variable,

> solution, open equation, replacement set, and solution set (Holt School Mathematics, Grade 7, p. 53).

When compared to the definitions found in the students' textbooks (see Figure 30), this definition provides teachers with a much richer conception of variables by indicating to them, for example, that variable can be used to represent *numbers* (not just a single number), and could be denoted by other characters than letters (implication of "usually a letter" in the definition). The same paragraph of the definition also offered some teaching suggestions to teachers on variables: "You might ask students to define the following terms in their own words, after discussing the developmental items: variable, solution, open equation, replacement set, and solution set". Similar teaching suggestions on variables were identified throughout the teachers' textbooks of this curriculum, e.g., "point out that if a variable is used twice in an equation, each must be replaced by the same number to make a true statement" (Holt School Mathematics, Grade 8, p. 134).

Furthermore, there were instances in the curriculum where teachers were alerted (implicitly) to some students' common misconceptions in working with variables: "Note that this lesson introduces the term $2x$. When a number is multiplied by a variable, we do not use the multiplication sign. A multiplication sign is only used to show the multiplication of two numbers such as $10 \cdot 6$. You might ask your students what would be wrong with putting in the multiplication symbol in that case?" (Holt School Mathematics, Grade 8, p. 136). Even though teachers were not informed explicitly that this is a major misconception that the majority of students may carry about variables, it is worth pointing out that this situation has been identified as a misconception that a majority of novice students to algebra harbor about variables and multiplication. The origin of this misconception is usually traced back to students' prior arithmetic experience (Kieran, 1981).

Another teaching instruction on variables found in this mathematics curriculum is the following: "you may want the students to get in the habit of writing $a = 5$, $b = 6$, and $c =$?, before they substitute the values in $c^2 = a^2 + b^2$. If they do this, students are less likely to confuse the values" (Holt School Mathematics, Grade 8, p. 260). In another development of the same textbook, teachers were advised to "ask students to describe $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ in words before they attempt $\frac{2}{5} + \frac{1}{5} =$ "(Holt School Mathematics, Grade 8, p. 220). Providing students with such opportunities will promote their thinking about the role of the variables in equations, and hence help them understand the meaning of variables they use in formulas and in other instances better. There were limited instances where support was provided to enhance teachers' content knowledge on variable ideas in the teachers' textbooks. This might be due to the relative ease of the content covered in this curriculum (Back to Basics era), leading authors to assume that teachers understand the content and issues related to variables.

Problem Solving Era: *Mathematics Today Curriculum*

The *Mathematics Today* (MT) curriculum was selected for the Problem Solving Era. This curriculum was one of the comprehensive mathematics programs for grades K-8 used during the 1980s. The middle grades series covered basic number concepts and skills, as well as practical math skills such as time, measurement, estimation, probability and statistics. Throughout every chapter in every grade, a special emphasis is placed on *problem-solving* strategies and *applications*. In addition, every chapter in each textbook (6, 7 and 8^{th} grades) contains a unit entitled *Calculator* or *Computer* in which students are introduced to computer programs, or the use of calculators in school mathematics.

The inclusion of these units was consistent with the reform recommendations of the Problem Solving era, which among others, was to "create a coherent vision of what it means to be mathematically literate, both in a world that relies on calculators and computers to carry out mathematical procedures and in a world where mathematics is rapidly growing and extensively being applied in diverse fields" (NCTM, 1989, p. 1). Table 18 presents topics that are covered in the *Mathematics Today* curriculum.

Table 18

Chapter Titles in Mathematics Today selected for Problem Solving Era

Grade	Ch	Title	Ch	Title
Grade 6	1	Numeration	8	Fractions: Multiplic. & Division
	2	Addition and Subtraction	9	Decimals: Add. & Subtraction
	3	Multiplication	10	Decimals: Multiplic. & Division
	4	Division	11	Measurement
	5	Graphing	12	Geometry
	6	Number Theory and Fractions	13	Ratio & Percent
	7	Fractions: Addition & Subtraction	14	Integers
Grade 7	1	Whole Numbers: Add & Subtract	9	Ratio & Proportion
	2	Whole Numbers: Multi. & Divide	10	Percent
	3	Equations	11	Measurement
	4	Decimals: Addition & Subtraction	12	Geometry
	5	Decimals: Multiplication & Division	13	Perimeter, Area, and Volume
	6	Number Theory	14	Probability and Statistics
	7	Fractions: Addition & Subtraction	15	Integers
	8	Fractions: Multiplication & Division		
Grade 8	1	Whole Numbers and Operations	9	Measurement
	2	Decimals	10	Perimeter, Area, and Volume
	3	Number Theory	11	Integers
	4	Fractions	12	Rational Numbers
	5	Solving Equations	13	Probability and Statistics
	6	Geometry	14	Real No. & Coordinate Plane
	7	Ratio	15	Right Triangles, Similarity Trig
	8	Percent		

Treatment of Variable Ideas in the Narrative Portion

Variable ideas were found in each of the three textbooks of the *Mathematics Today* curriculum. The principal chapters that employed variable ideas in this curriculum include measurement, geometry, equations, perimeter, area, volume, and solving equations. In the 6th grade textbook, variables were used predominantly in the narrative section as *labels*. In fact, this use alone accounts for approximately 80% of all the uses of variable in the narrative portion of the 6th grade textbook in this mathematics curriculum. In the 7th and 8th grade textbooks, variable ideas were used in relatively more proportions to *generalize arithmetic* patterns (e.g., in describing the commutative, distributive and associative properties of numbers and in describing multiplication and division rules of fractions, among others), and as a *placeholder* in solving proportion and percentage situations.

Placeholder ideas were also developed through evaluating variable expressions by substituting given numeric values for the variables in a particular variable expression. This was further explored through solution to one-step and two-step equations. Variables were also used as *varying quantities* in developing formulas for finding area, circumference, surface area, perimeter, among others in this curriculum. The concept of varying quantities was further explored through the calculations of interest rates $(I = prt)$, distance, time, rate situations $(d = rt)$, and in solving and graphing functions $(x + y = 6)$.

Table 19 presents information on the number of instructional pages, number of pages containing variables, percentage of pages containing variables, the first, and the last page containing variable ideas in the *Mathematics Today* textbooks. As displayed in the table, the 6th grade textbook consisted of 392 instructional pages. Eighty-six pages of the 392 contained variable ideas. This represents 22% of the instructional pages on which variables were sighted. The 40th instructional page was the first page on which variable ideas were sighted, and the 389th instructional page was the last page that variables were found in this textbook.

Table 19

Instructional pages, Variable pages, and Percentage of Variable Pages in Mathematics Today Curriculum

Era of Math Education	Grade	No. of Instructional Pages	No. & % of Var. Pages	First Page Of Var. idea	Last Page Of Var. idea
	6	392	86 (22%)	40	389
Problem	7	440	264 (60%)	12	439
Solving	8	438	248 (57%)	5	437

The location of variables and the percent of variable tasks in the 6th grade textbook are shown in Figure 31. An immediate observation that can be made from Figure 31 is that five of the 14 chapters in the 6th grade MT textbook did not use variables in the presentation of their contents. In addition, 6 of the 9 chapters that used variable ideas did so sparingly. Thus, the majority of the chapters in the 6th grade MT textbook did not employ variable ideas at all, or did so thinly (see Figure 31). In this textbook, chapter 11 on *Measurement* records the most use of variables.

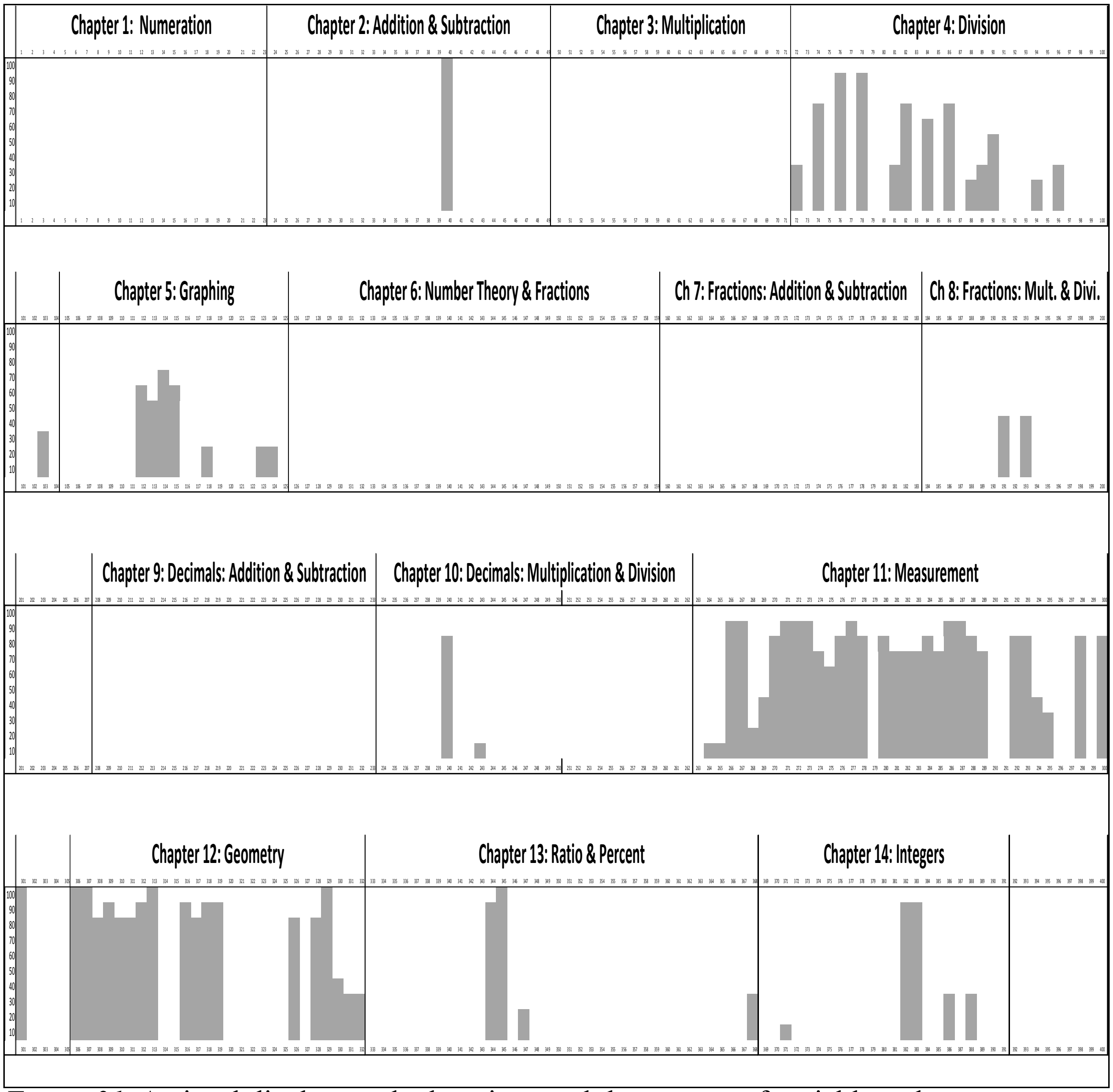


Figure 31. A visual display on the locations and the percent of variable tasks on a variable page in the 6th grade textbook of Mathematics Today curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

The 7th grade textbook of the MT consisted of 440 instructional pages of which 264 contained variable ideas. This represents 60% in the 7th grade MT textbook with variable ideas sighting. The first page on which variables were found was page 12 and the last page containing variable ideas was page 439. Figure 32 provides a global view of the location and the percent of variable tasks on variable pages in the 7th grade textbook.

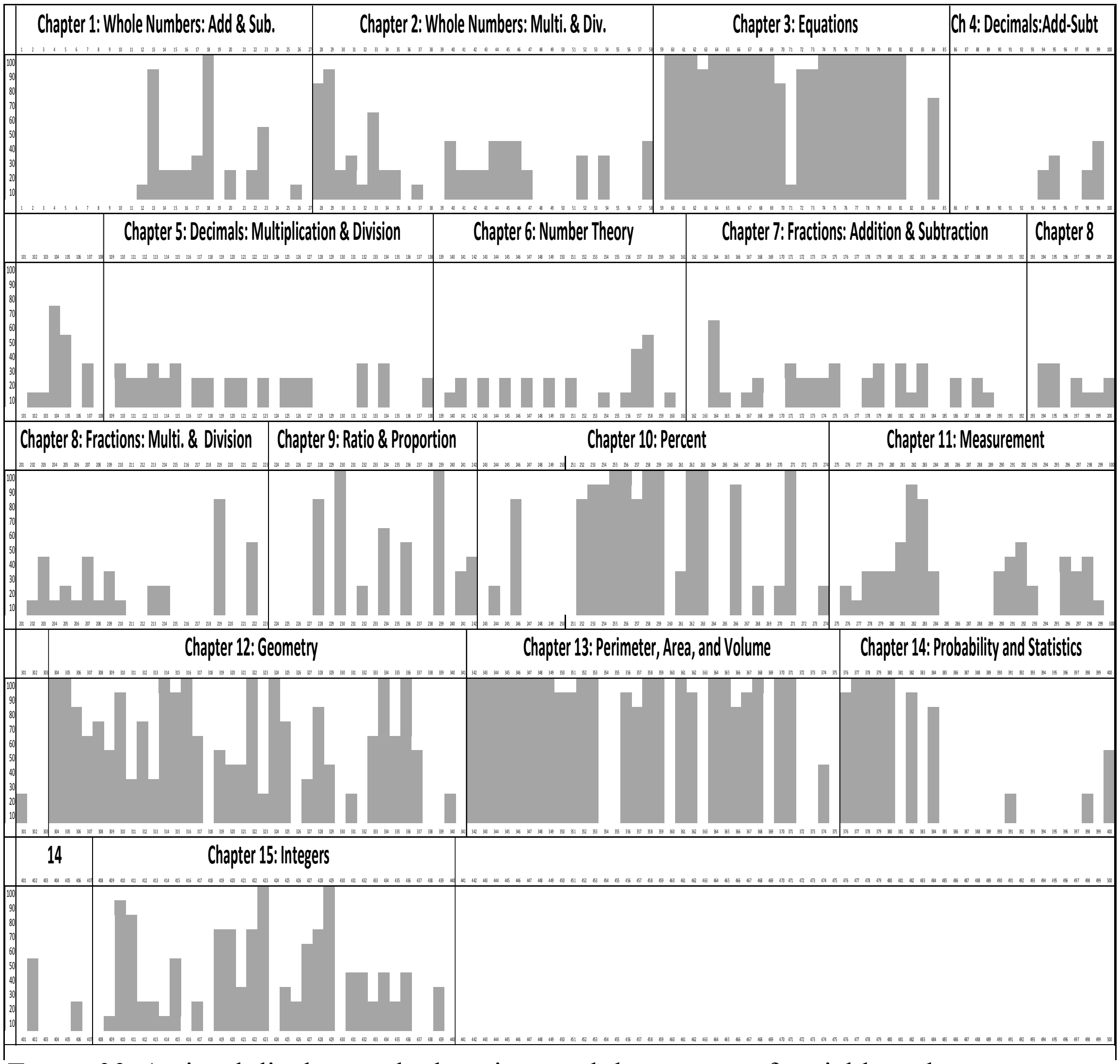


Figure 32. A visual display on the locations and the percent of variable tasks on a variable page in the 7th grade textbook of Mathematics Today curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

Contrary to the sighting of variable ideas in the 6th grade textbook, variable ideas were found in every chapter of the 7th grade textbook, and at a relatively higher proportion of use than was observed in the 6th grade textbook. Chapter 3 on *Equations* employed variable ideas the most, while chapter 4 on *Decimals: Addition and Subtraction* employed variable ideas the least in the 7th grade textbook.

Similarly, the 8th grade textbook consisted of 438 instructional pages of which 248 (representing 57%) contain variable ideas. The first sighting of variable was on page 5 and the last page on which variables were sighted was page 437. The treatment of variable ideas by the chapters in the 8th grade textbook was similar to that of the treatment found in the 7th grade textbook of this mathematics curriculum.

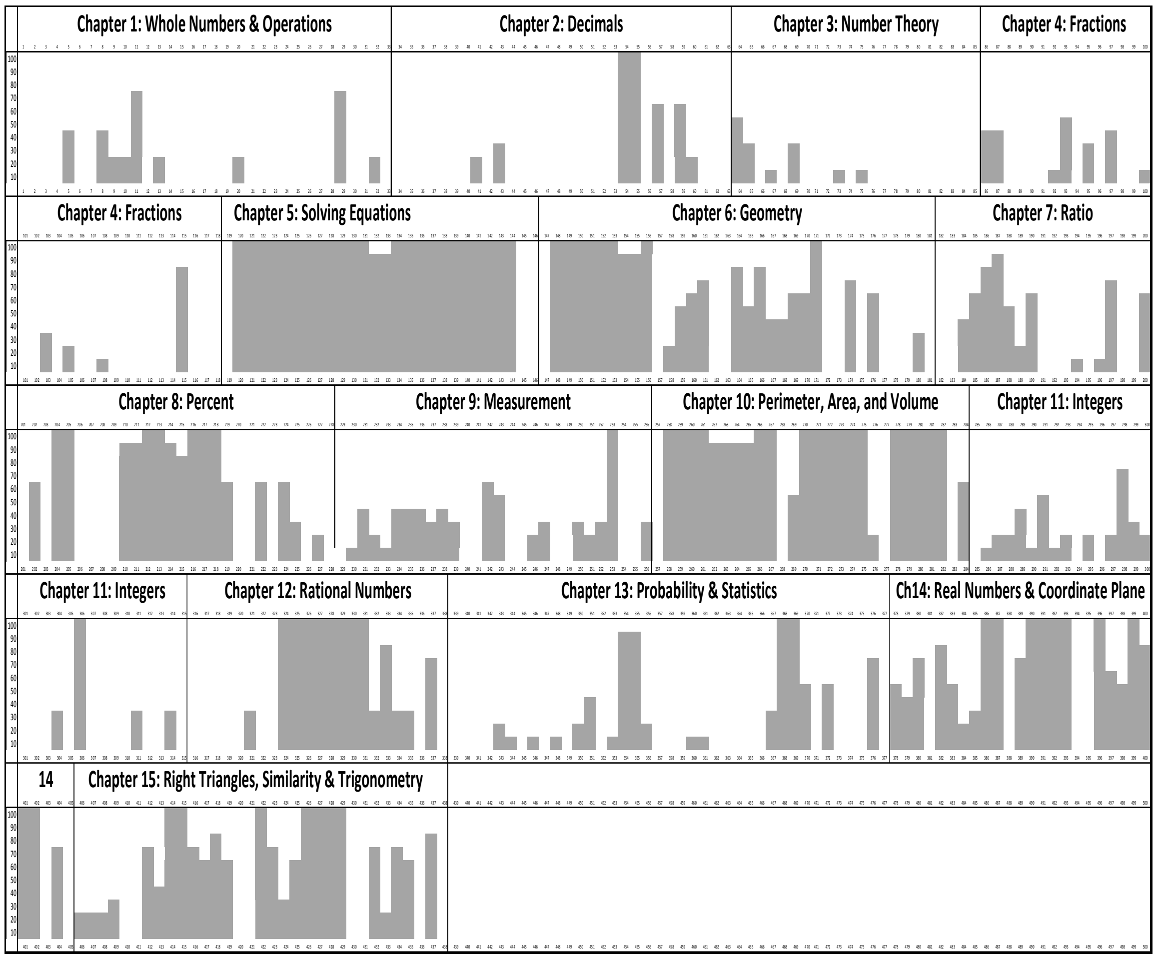


Figure 33. A visual display on the locations and the percent of variable tasks on a variable page in the 8th grade textbook of Mathematics Today curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

As evidenced in Figure 33, every chapter in the 8th grade textbook employed variable ideas to present its content, but in varying proportions. Chapter 5 on *Solving Equations* had the highest proportion of use of variables, whilst chapter 3 on *Number Theory* had the least proportion of use of variables in the 8th grade textbook.

Definitions for Variable in Mathematics Today Textbooks

Only two of the three textbooks of the MT series offered formal definitions for variables (i.e., the 7th and the 8th grade textbooks). Specifically, there was no formal definition for variables found in the 6th grade textbook of the MT curriculum. The characteristics of the definitions of variables found in the MT curriculum (i.e., in the 7th and 8th grade textbooks) depict variables as *continuous unknown* quantities. Figure 34 contains the definitions found in the 7th and the 8th grade textbooks of the MT curriculum:

> Rebecca is three years older than Christopher. You can write mathematical **expressions** to stand for their ages. Let a letter stand for Christopher's age. Use $x \rightarrow x$ then $x + 3$ stands for Rebecca's age $\rightarrow x + 3$, the letter x is a **variable**. To **evaluate** an expression, you substitute a number for the variable.
>
> (Mathematics Today, Grade 7, p. 60)
>
> *Using Coefficients and Variables*: Mathematical expressions can be composed of numbers, letters, or a combination of numbers and letters. Expressions such as $3, x, xy, x \div 2$ and ab are **terms**, since they are numbers, variables, or products and quotients of numbers and variables. If a term consists of a number and a letter, the number portion of the term is called the **coefficient.** The letter portion of the term is called the **variable**
>
> $9x \qquad f \quad -3g \quad + \quad 7$
>
> The coefficient of the variable f is 1.
> The coefficient of the variable g is 3
>
> (Mathematics Today, Grade 8, p. 143)

Figure 34. Definitions for variables in the MT curriculum.

Apart from the definitions and explanations provided (as presented in Figure 34), there were no introduction, historical or background information on variables found in any of the three middle grades textbooks of the MT curriculum. Words such as placeholders and unknowns that are sometimes used to refer to variable were not used or found in the MT curriculum for the middle grades. In both the 7th and the 8th grade textbooks where formal definitions of variables were offered, variables were used (e.g., as placeholders) prior to being defined.

Distribution by Percent of Variable Ideas in Different Grade Levels

Table 20 presents the distribution, in percent, of variable ideas for different grade level textbooks of the MT curriculum. The data in the table indicate that variables were used predominantly in three categories in this curriculum (i.e., as *placeholders*, as *labels* and as *varying quantities*). The usage of variable ideas in these three categories constitutes a combined percentage of 97%, 89% and 85%, respectively in the 6th, 7th, and 8th grade textbooks. There was limited usage of variables as continuous unknowns, constants, and generalized numbers in the MT curriculum. Second, one can observe that while there was very high use of variables as *labels* in the 6th grade textbook, this percentage reduced significantly for a trade off in the use of variables as *placeholders* in the 7th and the 8th grade textbooks of the MT textbooks (see Table 20). Another important observation is that the MT textbooks did not use variable as an *abstract symbol*.

Table 20

Distribution by Percent of Variable Category in Different Grade Levels in Mathematics Today Curriculum

Grades Level	Variable (sub)Category						
	Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Grade 6 (*n* = 677)	76	2	1	0	0	21	0
Grade 7 (*n* = 2402)	25	6	49	2	4	15	0
Grade 8 (*n* = 2233)	29	5	38	5	6	18	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

Distribution by Percent of Variable Ideas in Different Content Areas

Table 21 reports the distribution, in percents, of variable ideas in different content areas of the MT curriculum. Data in the table reveal that about 73% of the uses of variables in the *number and operations* content area in this curriculum identifies as a *placeholder*. The second most frequently used category was the use of variables as a *label* (18%), with the remaining five categories of use of variables accounting for a combined percentage of only 9% in the *number and operations* content area (i.e., 7% for generalized numbers, and 2% for varying quantity).

Similarly, the use of variables as *placeholders* dominates *algebra* related contents (59%). This was followed by the use of variables as *varying quantities* (14%), with the *label* category constituting 11%, *continuous unknowns* and *generalized numbers* being both 8%. In all *measurement*, *geometry*, and *data analysis and probability* related content areas, the use of variables as *labels* dominated, constituting at least 40% in *geometry*, 77% in *measurement*, and 81% in data *analysis and probability* content areas in the middle grade textbooks (see Table 21).

Table 21

Distribution by Percent of Variable Category in Different Content Areas in Mathematics Today Curriculum

Content Area		Variable (sub)Category						
		Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Number & Op.	(*n* = 1227)	18	0	73	0	7	2	0
Algebra	(*n* = 1943)	11	0	59	8	8	14	0
Geometry	(*n* = 1512)	40	16	9	0	0	34	0
Measurement	(*n* = 714)	77	2	0	0	0	20	0
Data & Prob.	(*n* = 239)	81	0	15	0	3	2	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

Even though every chapter in this mathematics curriculum contains a unit on

Calculator or *Computer* in which students were introduced to computer programs or to the use of calculators to perform some sort of mathematical computations, these activities did not incorporate variable ideas. In addition, research issues on students' difficulties, misconceptions and the likes about the concept of variables were not mentioned or discussed in this curriculum.

Support for Teachers on Variable Ideas

There were few instances in the teachers' textbooks of the MT curriculum where support was provided for teachers to enact variable ideas. Among them, the authors of the MT curriculum informed teachers to emphasize to students that "whenever the same variable occurs more than once in an equation, the same value must be used. For instance, if $a = 7$ and $b = 4$, the left side of $a + b = b + a$ becomes $7 + 4$; the right side becomes $4 + 7$" (Mathematics Today, Grade 7, p. 70). In another paragraph of the same textbook, the authors enlightened teachers on the possible extensions of the distributive property: "the distributive property states that for any whole numbers $a, b \ and \ c$, $a \times (b + c) = (a \times b) + (a \times c)$. Although it deals explicitly with cases in which there are two addends $(b + c)$, it can be extended to cases in which there are more than two addends" (Mathematics Today, Grade 7, p. 30).

A Teaching suggestions on the multiplication of a number and a variable were also found in the MT curriculum. In the 8th grade textbook, the authors suggested to teachers to "remind students that two letters side-by-side or letter preceded by a number are multiplication expressions: ab means a times b; $3x$ means 3 times x" (Mathematics Today, Grade 8, p. 143).

There were, however, limited instances of support provided in this curriculum to educate teachers on variable ideas, to provide teachers with teaching suggestions, and to alert teachers to students' common misconceptions and difficulties with variables in the MT curriculum. Seemingly, it may be assumed by the authors of these textbooks that teachers are proficient with variables and understand the issues related to variables and the implications for their students.

NCTM Standards Era: *Math Connects Curriculum*

The authors of the *Math Connects* (MC) curriculum reported that the curriculum is organized around the new NCTM Focal Points, and is designed to meet most State Standards. The writers maintained that they examined the content needed to be successful in *Geometry* and *Algebra* and back mapped the development of mathematical content, concepts, and procedures to Pre-K to ensure a solid foundation and seamless transition from grade level to grade level.

In addition, the authors described the MC curriculum as focusing on three key areas of vocabulary to build mathematical literacy, intervention options aligned to Response to Intervention (RtI), and a comprehensive assessment system of diagnostic, formative, and summative assessments. This curriculum includes formal definitions of mathematical concepts, worked-out examples, real-life problems, and research-based strategies in the treatment of the contents. All the preceding areas of emphasis are in line with the recent NCTM (2006) recommendations: e.g., the *Standards* also drew heavily on research on students' thinking, students' misconceptions, and how students learned particular ideas as they encounter challenging tasks (Confrey, 2007). Table 22 presents the chapter titles that are covered in the MC curriculum for grades 6, 7 and 8.

Table 22

Chapter Titles for Math Connects Curriculum for the NCTM Standards Era

Grade	Ch	Title	Ch	Title
	1	Algebra: Number Patterns & Functn	7	Percent & Probability
	2	Statistics & Graphs	8	System of Measurement
Grade 6	3	Operations with Decimals	9	Geometry: Angles & Polygons
	4	Fractions and Decimals	10	Measurement: Peri., Area, &Vo
	5	Operations with Fractions	11	Integers & Transformations
	6	Ratio, Proportions & Functions	12	Algebra: Properties & Equation
	1	Introduction to Algebra & Functions	7	Applying Percents
	2	Integers	8	Statistics: Analyzing Data
Grade 7	3	Algebra: Linear Equatns & Functns	9	Probability
	4	Fractions, Decimals, & Percents	10	Geometry: Polygons
	5	Applying Fractions	11	Measurement: 2D, 3D Figure
	6	Ratios and Proportions	12	Geometry & Measurement
	1	Algebra: Integers	7	Measurement: Area & Vol.
	2	Algebra: Rational Numbers	8	Algebra: Equatn & Inequalities
Grade 8	3	Real Numbers & Pythagorean Theo.	9	Algebra: Linear Functions
	4	Proportions & Similarity	10	Algebra: Nonlinear Fun, Poly
	5	Percent	11	Statistics
	6	Geometry & Spatial Reasoning	12	Probability

Treatment of Variable Ideas in the Narrative

Each of the three textbooks (grades 6, 7 and 8) used variable ideas in the presentation of its contents. In this curriculum for the middle grades, variable ideas were explored mainly through Algebra: Number Patterns and Functions, Ratios, Proportions and Functions, Geometry: Angles and Polygons, Measurement: Perimeter, Area, and Volume, Integers and Transformations, Algebra: Properties and Equations, Algebra: Linear Equations and Functions, Algebra: Equations and Inequalities, Algebra: Nonlinear Functions and Polynomials, among others.

Typically, after the definition and the accompanying explanation for *variable* were offered, students were presented with numerous variable expressions with specific values for the variables to evaluate the given expressions. Solving of equations came after the evaluations of variable expressions when students were presented with equations involving variables (e.g., $2n = 10$) that can be solved in one-step. In many of these instances, students were asked to solve for the unknown value *using mental math*, or to substitute numeric values in equations with the unknown quantities in order to solve them.

The MC curriculum also used variables to generalize arithmetic patterns and properties such as the commutative, associative, and distributive properties of addition and multiplication. More work with variables is discussed in two-step equation solving situations, in inequalities, and in functional relations such as $y = mx + b$. The concept of symbolic equations was treated by translating verbal descriptions to symbolic expressions using variables, and graphing equations. Students were also provided with opportunities that required them to use variable equations to model real world problems, and write function rules. Much of the emphasis, though, was on solving equations, after using variables to represent the situations.

There were instances in the MC curriculum where cups and counters (manipulative materials) were used in activities to represent variables in developing meaning for inverse operations in solving equations. In these activities (as shown in Figure 35), the cups were used to stand for the *unknown* while the counters represent the *numbers*.

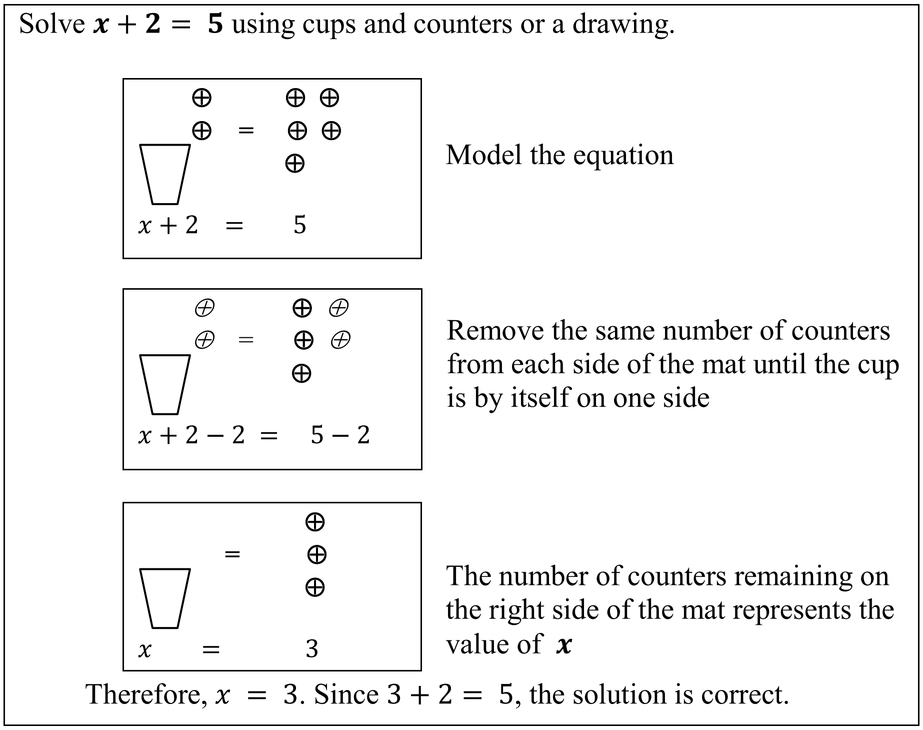


Figure 35. Solving equations using models (Math Connects, Grade 7, p. 134).

Technology, in the form of graphing calculators, was also employed to create tables of values to explore the relationships between two variables. Some of these relationships were illustrated in graphs. Most linear equations or functions were presented as relationships between the variables *x* and *y*. Few parabolic, cubic, and exponential functions were discussed using variables. Some of the chapters used variables to solve proportional reasoning situations, percentage problems, formulas for calculating areas, volume, circumference of geometric figures, and in input-output charts, among others. There were also few instances in this curriculum where systems of equations and inequalities involving two variables were discussed.

The context characteristics of the tasks employing variable ideas in MC curriculum were somehow unique: many of the activities in the MC textbooks that employed variables were presented within real-world contexts. In the 6th grade textbook, for example, students were asked to create a real-world problem in which they would solve the equation $a + 12 = 30$ (Math Connects, Grade 6, p. 60, question 38). Another typical assignment that was frequently found in this curriculum asked students to "write about a real-world situation that can be represented by equations" such as $y = 5x$ and the "explain what the variable represents in the situation" (Math Connects, Grade 6, p. 353, question 24). Table 23 summarizes information on the number of instructional pages, number of pages containing variables, percentage of pages containing variables, the first, and the last page that variable ideas were sighted in the *Math Connects* textbooks for the middle grades.

Table 23

Instructional pages, Variable pages, and Percentage of Variable Pages in Math Connects Curriculum

Era of Math Education	Grade	No. of Instructional Pages	No. & % of Var. Pages	First Page Of Var. idea	Last Page Of Var. idea
	6	669	304 (45%)	22	669
NCTM	7	665	308 (46%)	35	665
Standards	8	665	411 (62%)	29	655

As can be observed from the table, the 6th grade textbook consisted of 669 instructional pages, of which 304 contained variable ideas. This number of variable pages represents 45% of the instructional pages containing variable ideas of some sort. Variable ideas first appeared on the 22nd instructional page, and were last found on the 669th page. Figure 36 provides a global view of the locations and the percent of variable tasks on variable pages in the 6th grade textbook of the MC curriculum. The information presented in Figure 36 indicates that every chapter in the 6th grade textbook employed variable ideas in the presentation of its content, but at varied proportions. Chapter 12 on *Algebra:*

Properties and Equations employed variable ideas the most, while chapter 2 on *Statistics and Graphs* employed variable ideas the least.

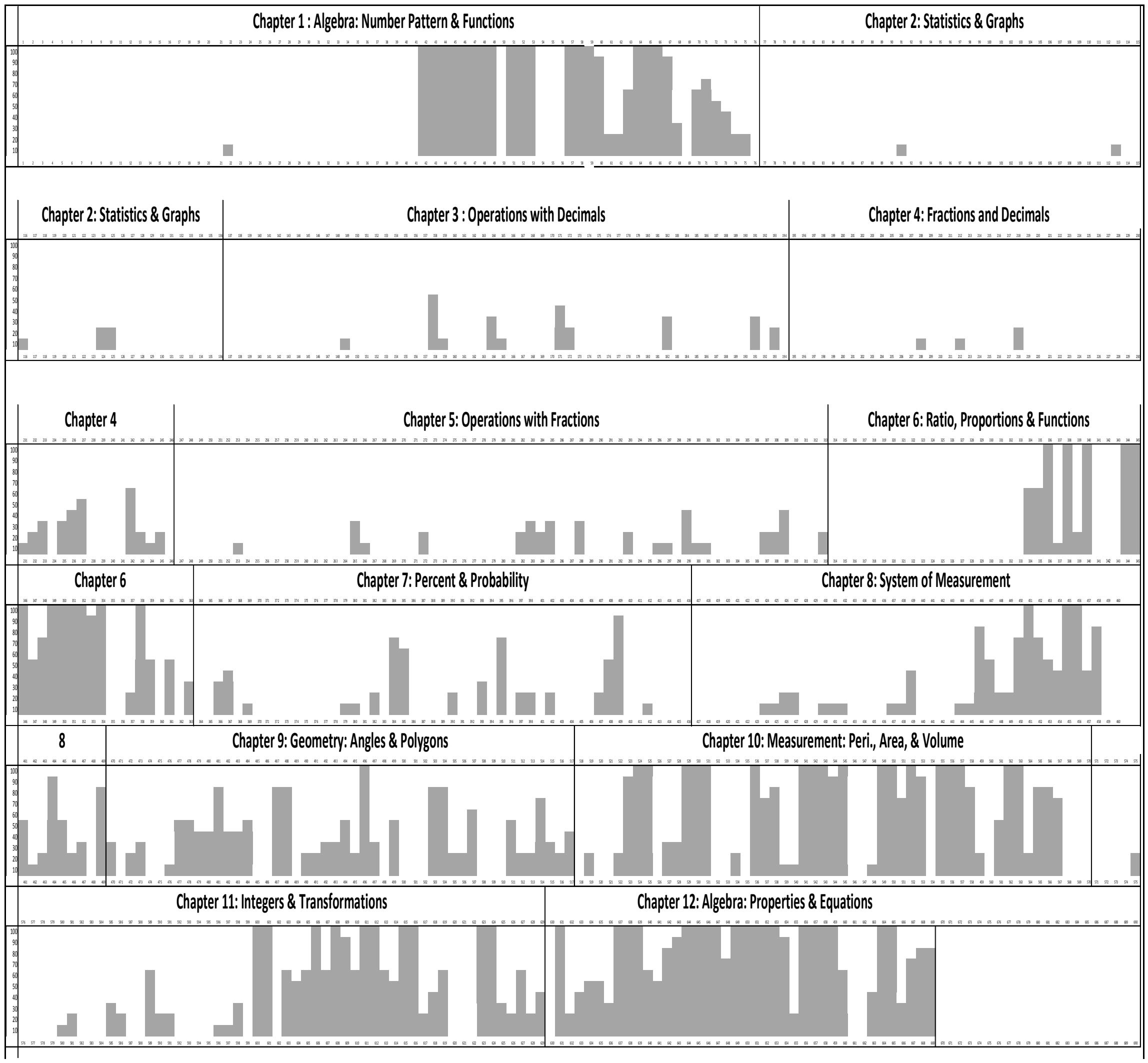


Figure 36. A visual display on the locations and the percent of variable tasks on a variable page in the 6th grade textbook of Math Connects curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

The 7th grade textbook had slightly fewer instructional pages (665) than the 6th grade textbook (669), but contained more variable pages (308). This number (308) represents 46% of the instructional pages in the 7th grade textbook that contained variable ideas. The first instructional page employing variable ideas was 35, and the last page containing variable ideas was the last instructional page in the textbook (page 665).

Figure 37 provides a global view of the locations and the percent of variable tasks on variable pages in the 7th grade textbook of the MC curriculum.

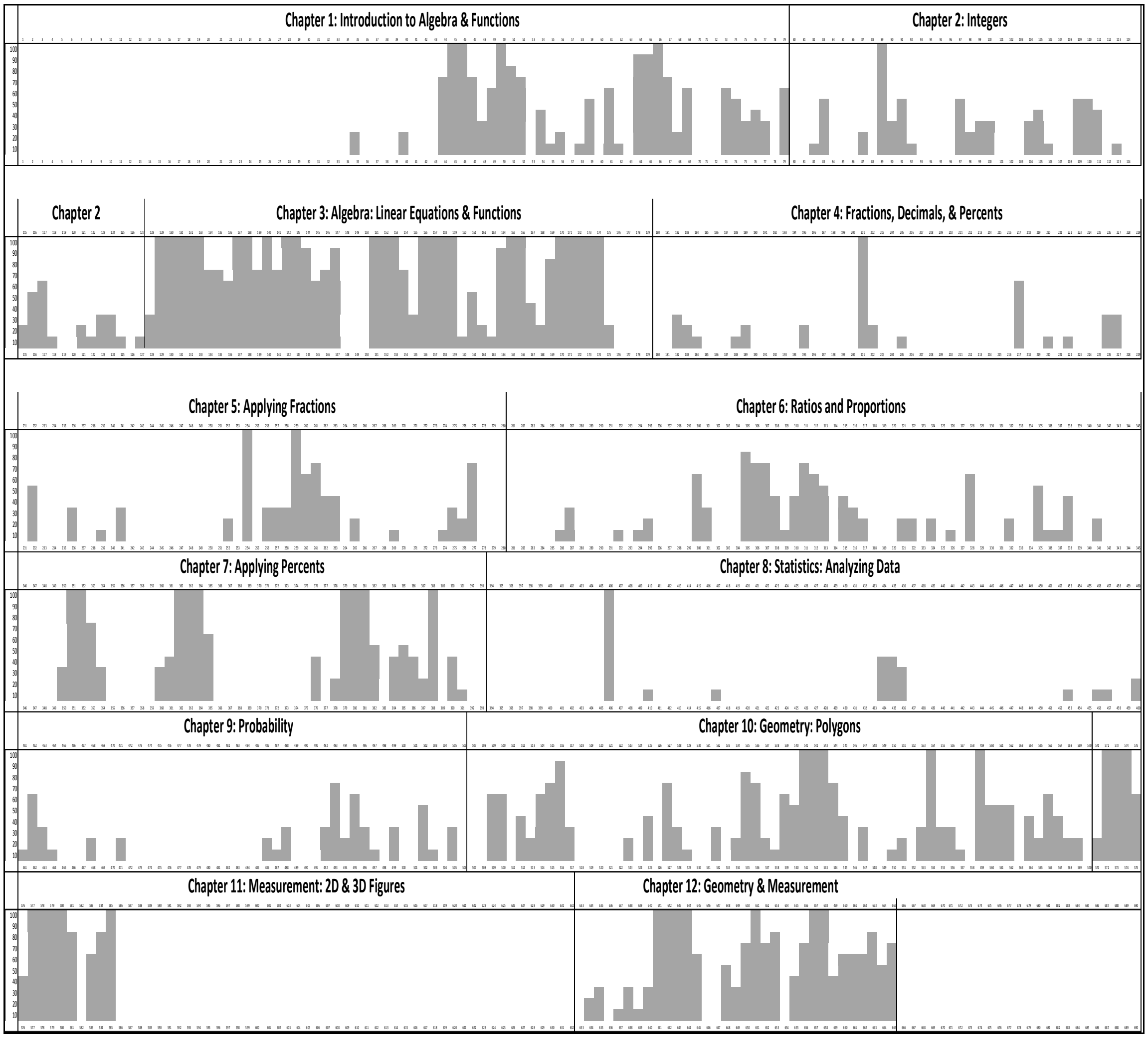


Figure 37. A visual display on the locations and the percent of variable tasks on a variable page in the 7th grade textbook of Math Connects curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

Information displayed in the figure shows that variable ideas were used in every chapter of the 7th grade textbook of the MC curriculum as well. However, not every chapter employed variable ideas in the same proportion. For example, chapter 3 on *Algebra: Linear Equations and Functions* employed variable ideas to a far greater extent than chapter 8 on *Statistics: Analyzing Data* does. One can also see from the display that, in general, *algebra* and *geometry* employed variables in higher proportions than *measurement* and *data analysis and probability* related content areas (see Figure 37).

Similarly, the 8th grade textbook consisted of 665 instructional pages of which 411 (representing 52%) contained variable ideas. The first sighting of variable ideas was on page 29 and the last page containing variable idea was page 655. The locations and the percent of variable tasks on variable pages in the 8th grade textbook of the MC curriculum are displayed in Figure 38.

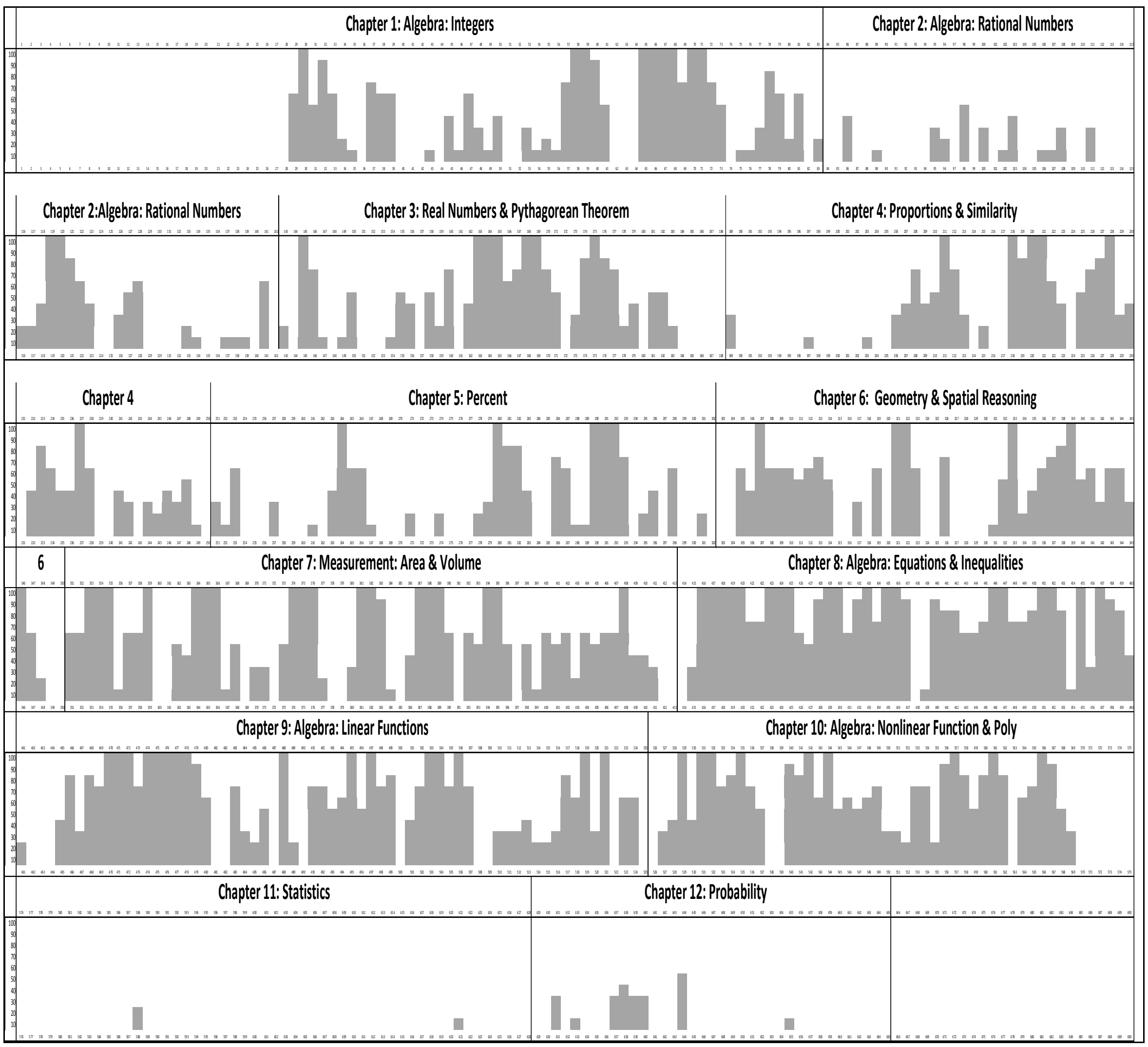


Figure 38. A visual display on the locations and the percent of variable tasks on a variable page in the 8th grade textbook of Math Connects curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

As in all the other grade level textbooks of the MC curriculum, variable ideas were explored in all the chapters of the 8th grade textbook but at a varied proportion. Chapter 8 on *Algebra: Equations and Inequalities* employed variable ideas the most, while chapter 11 on *Statistics* employed variable ideas the least (see Figure 38).

Definitions for Variable in Math Connects Curriculum

Formal definitions for variables were found in each of the three textbooks of the MC curriculum. As can be observed from the definitions contained in Figure 39, the MC textbooks for the middle grades characterized variables as a *placeholder* for a number. This characterization is true in the definition found in all three textbooks of the MC curriculum.

A variable is a symbol, usually a letter, used to represent a number. The expression $2 + n$ represents the sum of two and some number. Algebraic expressions are combinations of variables, numbers and at least one operation. Any letter can be used as a variable. The letter x is often used as a variable. It is also common to use the first letter of the value you are representing.

(Math Connects, Grade 6, p. 42)

A variable is a symbol that represents an unknown quantity. The branch of mathematics that involves expressions with variables is called **algebra**. The expression $n + 2$ is called an algebraic expression because it contains a variable, a number and at least one operation.

(Math Connects, Grade 7, p. 44)

A variable is a symbol, usually a letter, used to represent a number. The branch of mathematics that involves expressions with variables is called **algebra**. The expression $2 \times n$ is called an algebraic expression because it contains a variable, a number and at least one operation. To evaluate or find the value of an algebraic expression, first replace the variable or variables with the known value to produce a numerical expression.

(Math Connects, Grade 8, p. 29)

Figure 39. Definitions for Variables in MC Textbooks.

The 8th grade textbook offered a formal definition for variable before using variable ideas to treat topics covered in the textbook. In the other two grade level textbooks (the 6th and the 7th), variables were used before formal definitions were provided. Apart from the definitions and the accompanying explanations offered for variables in these textbooks, there were no introductions, historical or background information on variables presented in the MC curriculum.

Also, aside from the formal definitions provided, this curriculum explained that "choosing a variable to represent an unknown quantity is called defining a variable" (Math Connects, Grade 7, p. 50). After providing this explanation in the textbooks, students were asked frequently to *define* a variable and then use it to solve given tasks. Towards the end of the 8th grade textbook, this mathematics curriculum distinguished between dependent and independent variables.

Distribution by Percent of Variable Ideas in Different Grade Levels

As Table 24 reveals, variables were used in all seven categories of use of variables (including the use as an *abstract symbol*) in the MC curriculum. The use of variables as *placeholders* and *labels* surpassed all the other uses of variables by far in the 6th and 7th grade textbooks of the MC curriculum. However, the use of variable as *varying quantity* was the highest in the 8th grade textbook. One can also see a steady increase in the use of variables as *varying quantity* across the grade levels (i.e., 18%, 23%, and 28% for 6th, 7th, and the 8th grade textbooks respectively), while the opposite was true in the use of variables as *labels* (i.e., a steady decrease in the use of variables as *labels* across the grade levels).

Table 24

Distribution by Percent of Variable Category in Different Grade Levels in Math Connects Curriculum

Grades Level	Variable (sub)Category						
	Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Grade 6 (*n* = 1502)	26	2	34	8	11	18	1
Grade 7 (*n* = 1437)	21	1	42	2	10	23	1
Grade 8 (*n* = 2224)	13	4	27	12	7	28	9

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

Another observation from the information contained in Table 24 is the relative even distribution in the proportion of use of variable categories in the 8th grade textbook compared to the distribution in the 6th and the 7th grade textbooks.

Distribution by Percent of Variable Ideas in Different Content Areas

With respect to the use of variables in different content areas of the MC textbooks, an examination of Table 25 reveals that about one-half (49%) of the uses of variables in the *number and operations* content area classifies as *label*s. This was followed by the use of variables as a *placeholder* (37%). The use of variables as a *placeholder* dominated *algebra* contents (40%). This was followed by the use of variables as *varying quantity* (22%), *continuous unknown* (13%) and *generalized number* (13%) in that order.

The use of variables as *varying quantities* dominated *geometry* contents, making up 43% of all the uses of variables in *geometry* contents in the MC textbooks. The second most frequently used category of variable in *geometry* was as *label* (25%), with *placeholder* (21%) category being the third. Eighty-eight percent (88%) of all the uses of variables in *measurement* was as *labels*. *Data Analysis and Probability* related content is also dominated by the use of variables as *labels* in this curriculum (53%).

Table 25

Distribution by Percent of Variable Category in Different Content Areas in Math Connects Curriculum

Content Area		Variable (sub)Category						
		Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
Number & Op.	(*n* = 644)	49	0	37	3	6	3	2
Algebra	(*n* = 3045)	5	0	40	13	13	22	7
Geometry	(*n* = 1182)	25	10	21	0	1	43	0
Measurement	(*n* = 160)	88	0	8	0	2	2	0
Data & Prob.	(*n* = 133)	53	0	14	0	6	27	0

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

This curriculum also used technology in the form of graphing calculators to explore the relationships between two variables as their values change. It also showed evidences of incorporating research findings (e.g., students were asked to interpret the role of variables they used in solving problems) related to students' understanding and misconceptions of variables in its presentation of variable ideas.

Support for Teachers on Variable Ideas

The corresponding teachers' textbooks of the MC curriculum were examined to determine the extent to which they provide support for teachers to teach variable ideas in middle grades classrooms. This curriculum contains three major sections (Tips for New Teachers, Focus on Mathematical Content, and Teach) under which guidance was provided for teachers to implement the content. The guidance for enacting variable ideas was found under these headings as well. For instance, under *Tips for New Teachers* in the 8th grade textbook, the following guidance was provided: "some students may have difficulty understanding why $2x + 3x$ is not $5x^2$. Guide these students through the process of applying the distributive property to show why the simplified form of $2x + 3x$ must be $5x$" (Math Connects, Grade 8, p. 418).

The following information was identified under *Focus on Mathematical Content* in the 6th grade textbook: "Like terms have the same variables. Show students that $2xy$ and $5yx$ are like terms because they contain the same variables" (Math Connects, Grade 6, p. 637). Under the same heading in the 8th grade textbook, teachers were advised to "point out to students that $-b$ does not necessarily mean that the value of $-b$ is negative. For example, if $b = -4x$, then $-b = 4x$" (Math Connects, Grade 8, p. 47).

The 6th grade textbook contains the following assignment: "look up the word *variable* in a dictionary. What definition of the word matches the use in this lesson? If classmates use different dictionaries, compare the meanings among the dictionaries" (Math Connects, Grade 6, p. 42b). The completion of this exercise could help teachers (as well as students) develop a better understanding of what variables mean based on different definitions, and how variables are used elsewhere. In another paragraph of the same textbook, teachers were asked to "review the definitions of *variable* and expressions with students. Ask them to give examples of each" (Math Connects, Grade 6, p. 47).

There was a lot more support provided to teachers on variable ideas in the MC curriculum. Below are samples of some more supports found in the 6th and 8th grade textbooks of this curriculum: "The value of a variable never changes within an expression" (Math Connects, Grade 6, p. 344), "when students use the formula for the surface area of a rectangular prism ($S = 2lw + 2lh + 2wh$), encourage them to first list the length, width, and height of the prism, then write out the formula, and then substitute the values into the formula. This will help them to make sure they substitute values correctly" (Math Connects, Grade 6, p. 556).

The 8th grade textbook alerted teachers to "point out that the order may be misleading in some verbal expressions. Remind students that $n - 6$ can be written as $-6 + n$. Although the 6 appears first in the phrase, *6 less than a number*, the reference to 6 need not occur first in the algebraic version" (Math Connects, Grade 8, p. 61). Another important teaching suggestion in the 8th grade textbook was the following question: "what does $8w$ represent in the equation?" (Math Connects, Grade 8, p. 47). The answer provided in the textbook was "8 dollars times the number of weeks she has to save" (Math Connects, Grade 8, p. 47). This situation was identified in the research literature as one of the common misconceptions that students have on the role of variables. The literature reported that the majority of students will interpret $8w$ as, say 8 watches (Booth, 1988; Küchemann, 1981).

The Answers to the Research Questions

<u>Question 1: How do middle-grades mathematics textbooks develop the concept of variable (i.e., in terms of whether and how they introduce, define and/or explain it, and at which grade level(s) in the middle-grade mathematics textbooks do these occur)?</u>

Variable ideas were found in each of the four middle grades mathematics curricula examined in this study. In the majority of the curricula (with the exception of the *Math Connects* curriculum), there was no formal introduction of the concept of variables:

variables were introduced simply by using them, and then offering definition and explanations on their uses afterwards. In some of the textbooks, variable ideas were encountered as early as the 2nd instructional page in the textbooks (*Modern School Mathematics* grade 6, and *Holt School Mathematics* grade 7) and as late as the last page in almost all of the textbooks examined. In nearly all the textbooks, variable ideas were used primarily to solve equations, inequalities, formulas, and to label lines, points, and angles, among others.

Formal definitions for variables were found in 11 of the 12 students' textbooks examined. The exception was the 6th grade textbook of the *Mathematics Today* curriculum (selected for the Problem Solving Era), which did not provide formal definitions for variables. The length of the definitions (and the accompanying explanations, where provided), ranged from few words: "in the equation $x + 7 = 12$, x is **a variable,** the solution is 5, because $5 + 7 = 12$ is true" (*Holt School Mathematics Grade* 7, p. 53) to paragraphs "A variable is a symbol, usually a letter, used to represent a number. The expression $2 + n$ represents the sum of two and some number. Algebraic expressions are combinations of variables, numbers and at least one operation. Any letter can be used as a variable. The letter x is often used as a variable. It is also common to use the first letter of the value you are representing" (Math Connects, Grade 6, p. 42).

The characteristics of the definitions of variables found in different mathematics curricula were, however, different from one another. In particular, the four mathematics curricula defined variables in three different ways. Specifically, the *Modern School Mathematics* from the New Math Era defined variables in terms of their use as *generalized numbers*. In contrast, the *Holt School Mathematics* from the Back to Basics

Eras and the *Math Connects* from NCTM Standards Era defined variables as *placeholders.* The definition found in the *Mathematics Today* curriculum characterized variables as *continuous unknowns*.

Apart from the 8th grade textbook of the *Math Connects* curriculum that defined variables prior to using them, all the other textbooks examined in this study used variable ideas prior to providing a formal definition for variables. None of the textbooks provided historical information on the origination of variables. Table 26 reports the presence of definitions, and explanation on variable ideas in the sampled textbooks. In the majority of the textbooks, the definitions of variables were followed by brief explanations of how variables are used in mathematics.

Table 26

Presence of Definitions and Explanation of Variable ideas in Middle Grade Mathematics Textbooks

Math Ed Era	Definition			Explanation		
	6th	7th	8th	6th	7th	8th
New Math	√	√	√		√	√
Back to Basic	√	√	√	√	√	√
Problem Solving		√	√		√	√
NCTM Standards	√	√	√	√	√	√

There were few instances of the use of words such as placeholders or unknowns to refer to *variables* in the treatment of variable ideas in the selected textbooks. To be more precise, the 6th grade textbook of the *Modern School Mathematics* curriculum used the word *placeholders* instead of the word *variable* to refer to variable ideas throughout the presentation of its contents, and the *Math Connects* textbooks used the terms *placeholder*, and *unknown quantity* in a few instances to refer to variable ideas in the curriculum.

The proportion of variable pages in the selected textbooks ranged from 22% to 69% with the mean of 51% and standard deviation 11% (see Table 9). Only one mathematics curriculum (*Math Connects*) used technology to explore variable ideas. For the most part, the intended treatment of variables in almost all the curricula examined seemed to be aimed at enhancing students' symbolic manipulative skills rather than to help them use variable ideas to solve real-world situations.

Question 2: To what extent do middle-grades mathematics textbooks present activities and tasks that address each of the uses of variables (e.g., labels, unknowns, generalized numbers) as described by researchers in the mathematics education community, and in which order do these uses occur in the mathematics textbooks?

Differences were observed in the narrative sections of different mathematics curricula on the extent to which they employed variable ideas. In particular, the *Modern School Mathematics* curriculum used variables predominantly in the narratives to generalize arithmetic patterns, and to prove mathematical relations. The *Holt School Mathematics*, the *Mathematics Today*, and the *Math Connects* curricula on the other hand, used variables predominantly as placeholders to solve equations, and as labels to name lines, points, and angles. In addition, the *Math Connects* curriculum emphasized more connections between variables and functions than was observed in the other three mathematics curricula.

Table 27 displays information on the opportunities that exist in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test sections of the respective curricula for middle grades students to engage with the various uses of variables.

Table 27

Distribution by Percent of Variable Category in Middle Grades Mathematics Curricula

Textbook Series		Variable (sub) Category						
		Lab.	*Const.*	*Pchold.*	*Cont. Unk.*	*Gen#*	*VarQty*	*Abst.*
New Math Era								
Modern Sch. Math	(*n* = 4621)	31	2	38	9	6	12	2
Back to Basics Era								
Holt Sch. Math	(*n* = 3178)	28	5	43	8	3	13	0
Problem Solving Era								
Math Today	(*n* = 5312)	33	5	38	3	4	17	0
NCTM Standard Era								
Math Connects	(*n* = 5163)	19	3	33	8	9	24	4

Note. Lab. = Label; Const. = Constants; Pchold. = Placeholders; Cont. Unk.= Continuous Unknowns; Gen# = Generalized Numbers; VarQty = Varying Quantity; Abst. = Abstract Symbols.

An examination of the table reveals that, across all the four eras of mathematics education, the sets of middle grades mathematics textbooks examined provide opportunities for students to engage with variables as labels, placeholders, varying quantities, generalized numbers, continuous unknowns, and constants, but in varying proportions. In addition, it can be observed that only two of the four mathematics curricula (*Modern School Mathematics* and *Math Connects*) provided opportunity for middle grades students to engage with variable as an *abstract symbol*.

Figure 40 used the information contained in Table 27 to visually display the proportions of use of variable categories in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test sections of the respective curricula for each of the middle grades mathematics curricula. As evidenced in the display, the use of variable as *placeholder* was the highest of all the uses of variables in the middle grade mathematics curricula examined. The second overall highest category of use of variables was *labels*. This was consistently the second highest across all the curricula examined, except in the *Math Connects* textbook where the use of variables as *varying quantities* surpassed the use of variables as label. The third highest category of use of variables in the middle grades mathematics textbooks was as *varying quantity*. The least used category of variable in the textbooks was as an *abstract symbol*.

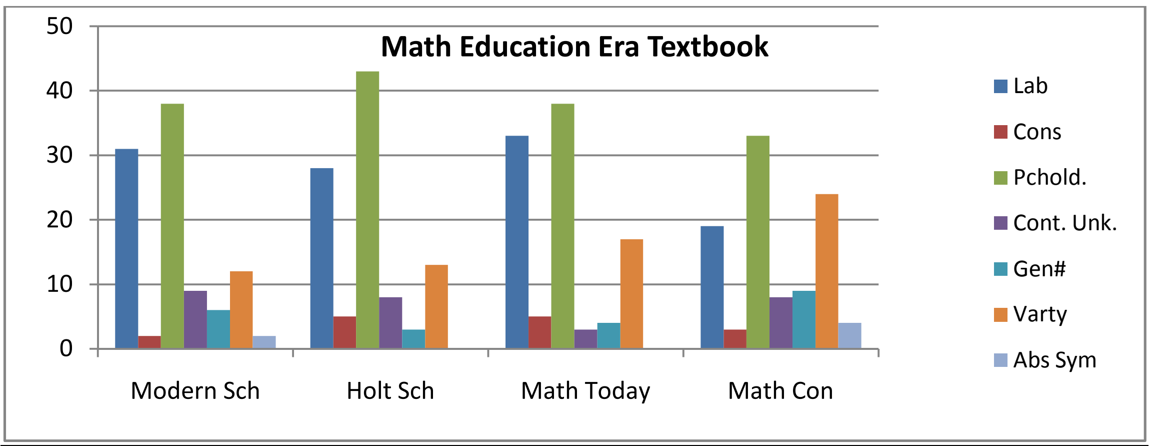


Figure 40. Percent of variable category in middle grades mathematics curriculum. Lab = Label; Cons = Constants; Pchold. = Placeholders; Cont. Unk. = Continuous Unknowns; Gen# = Generalized Numbers; Varty = Varying Quantity; Abs Sym = Abstract Symbols.

Thus, the sample of textbooks examined in this study treated variables predominantly as placeholders, and as labels. In other words, variables were used principally as *placeholders* to solve equations, and as *labels* to name lines, points, and angles in these curricula. There was less emphasis on the connections between variables and functions or between algebraic equations and functions in the set of textbooks examined, but a strong emphasis on the relation between variables and equation solving. The order of occurrence of variables in the selected textbooks was, for the most part, as label, placeholder, continuous unknowns, generalized numbers, and varying quantity and constants. More specifically, the use of variables as continuous unknowns, varying quantities and constants occurred mostly towards the final quarter of the instructional pages in the majority of the textbooks examined.

Question 3: Which use(s) of variable is/are prevalent within which content areas (i.e., geometry, number theory, algebra etc.) in middle-grades mathematics textbooks?

Across all the textbooks examined, *algebra* content employed variable ideas the most, while *data analysis and probability* content recorded the least usage of variable ideas. Figure 41 provides a visual display of the range of opportunities that exist in the different mathematics curricula for students to engage with the different uses of variables within different content areas in the middle grades mathematics curricula.

An examination of the graphs (Figure 41) reveals that the *number and operations* content area predominantly used variables as placeholders. In fact, in three out of the four mathematics curricula examined (the exception being the *Math Connects* curriculum), the proportion of use of variables as a placeholder alone constitutes more than 60% within the *number and operations* content area: it was 62%, 77%, and 73% in the Modern School Mathematics, Holt School Mathematics and Mathematics Today curricula respectively. The second most frequently used category of variable within the *number and operations* content area was label. This was followed by the use of variables as varying quantity (see Figure 41).

Similar patterns of use of variables occurred within the *algebra* content area. The proportion of use of variables as placeholder was consistently the highest within the *algebra* content area across all the four mathematics curricula examined. The next highest categories of use of variables in *algebra* were as a varying quantity. This was followed by the use of variables as a continuous unknown. Contrary to the vast disparity in the distribution of the uses of variables in the other content areas, one can observe a relatively smoother distribution in the proportions of uses of variable within *algebra* and *geometry* (see Figure 41).

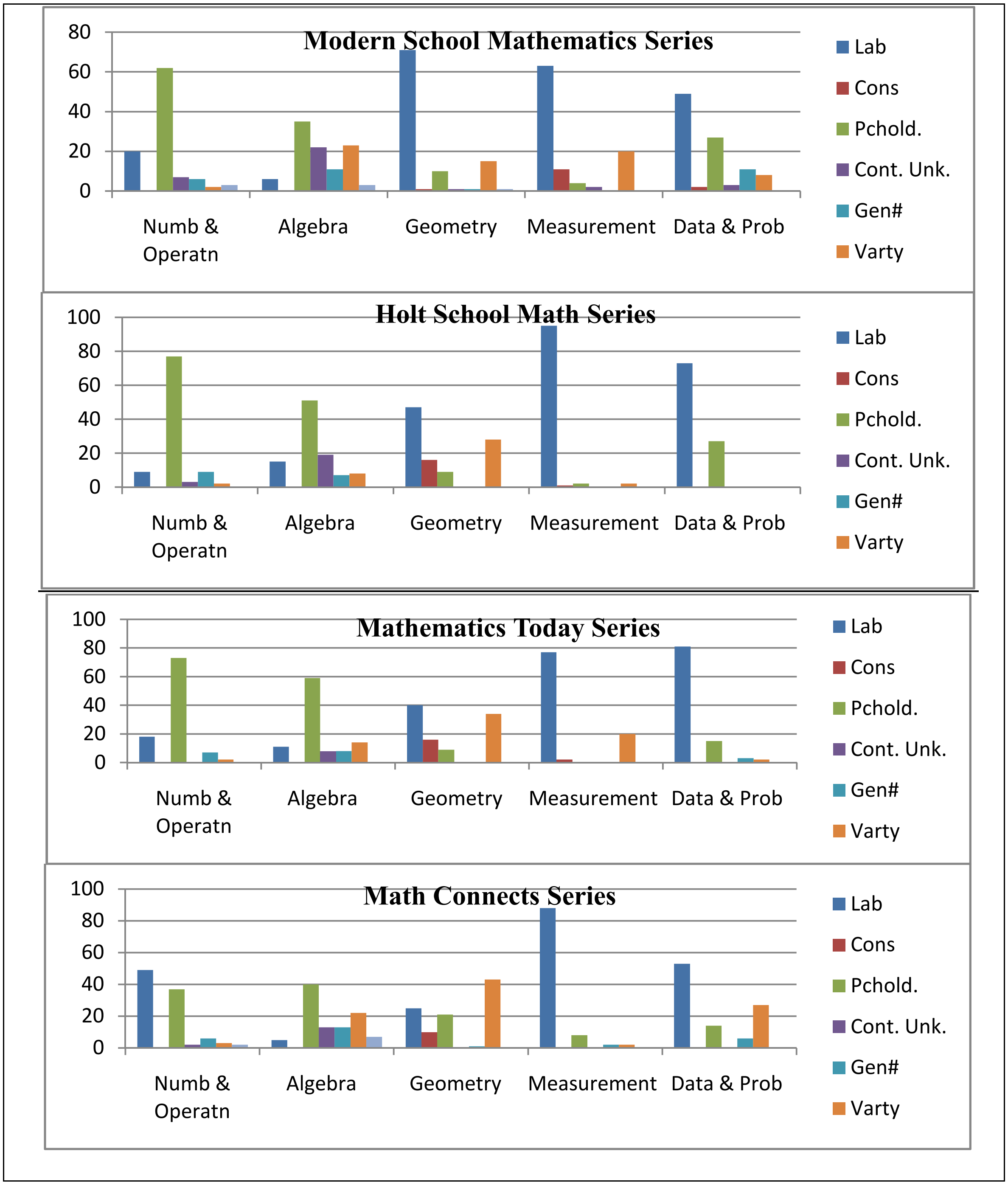


Figure 41. The uses of variables within different content areas in different curricula. Lab = Label; Cons = Constants; Pchold. = Placeholders; Cont. Unk. = Continuous Unknowns; Gen# = Generalized Numbers; Varty = Varying Quantity; Abs Sym = Abstract Symbols.

The principal use of variables in *geometry* was as *label*. This was followed by the use of variable as *varying quantities*. It is also worth pointing out that variable ideas were used at relatively higher percentage as *constants* in developing *geometric* concepts than employed in the other content areas (as constants). Similarly, across all the set of textbooks examined, the use of variables as *label* dominated *measurement*, as well as the *data analysis and probability* content areas. In fact, the use of the label category alone constituted at least 60% in *measurement*, and 40% in *data analysis and probability* areas in the middle grades mathematics textbooks.

<u>Question 4: How does the development and/or presentation of the concept of variable differ across different grade levels of the same textbook series (e.g., in terms of the compositions of the various uses of variable)?</u>

For the most part, the definitions for variables, and the styles of presentation of variable ideas within a mathematics curriculum seemed to be similar irrespective of the grade level of the textbook. This might be due to the fact that the development of the topics in the majority of the curricula examined was spiral in nature. That is, the topics that were covered in the sixth grade textbook were basically re-treated in the 7th and 8th grade textbooks of the respective curricula, but at a relatively higher level of complexity.

On the contrary, the visual displays on the locations, and the percent of variable tasks on variable pages indicated that the 6th grade textbooks employed variable ideas to a much lesser extent than did the 7th and the 8th grade textbooks in each curriculum. For example, there were several chapters, as well as pages in the 6th grade mathematics textbooks of the respective curricula that did not employ variable ideas at all. This was not the case in the 7th and 8th grade textbooks. Furthermore, when examined in terms of the locations and the percent of variable tasks on variable pages, the 7th and 8th grade textbooks across all the four mathematics curricula examined employed variable ideas at a relatively higher percentage than did the 6th grade textbooks. A side-by-side visual display on the locations and the percent of variable tasks on variable pages in the 6th, 7th and the 8th grade textbooks from *Mathematics Today* curriculum in Figure 42 attests to these observations.

However, when decomposed into the percentage of variable pages only (without taking into account the percent of variable tasks on variable pages), there exists no clear trend in the extent of use of variable ideas by grade levels. For example, the information presented earlier in Table 9 at the beginning of this chapter indicates that the percentages of variable pages were 69%, 55%, and 66% for the 6th, 7th and 8th grade textbooks, respectively, in the *Modern School Mathematics* curriculum, and 22%, 60% and 57% for the 6th, 7th and 8th grade textbooks respectively in the *Mathematics Today* curriculum.

These percentages reveal no consistent increasing or decreasing trend in the extent of use of variables by grade levels. In contrast, the percentages of variable pages were 31%, 45% and 53% for the 6th, 7th and 8th grade textbooks, respectively, in the *Holt School Mathematics* curriculum, and 45%, 46% and 62% for the 6th, 7th and 8th grade textbooks, respectively, in the *Math Connects* curriculum. These percentages reveal a consistent increase in the proportions of variable pages as one move to a higher grade level textbook of the same mathematics curriculum (depicting evidence of a linear trend in the proportions of variable pages by grade level).

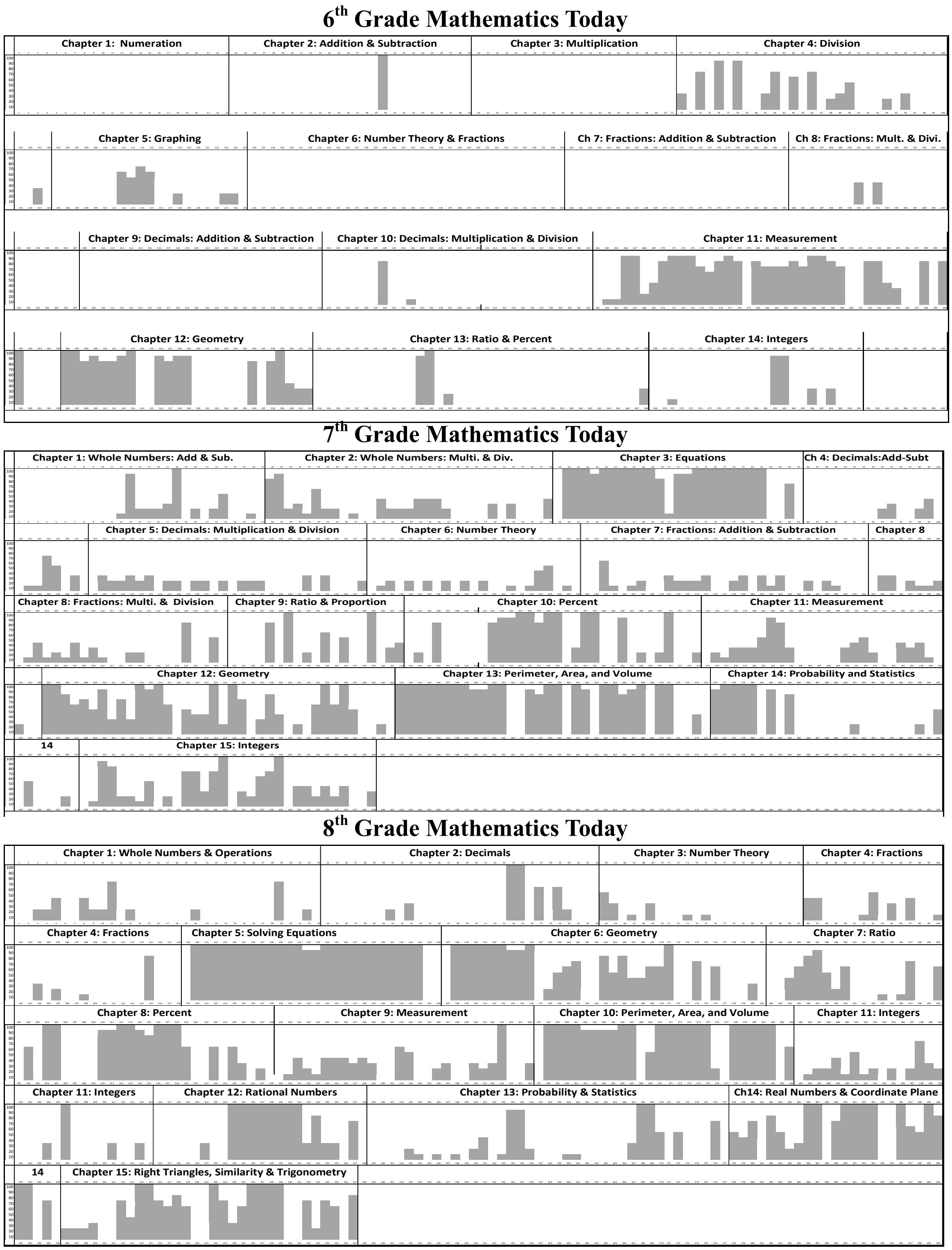


Figure 42. A visual display on the locations and the percent of variable tasks on variable pages in the 6-8 grade textbooks of the Mathematics Today curriculum. The horizontal axis names the page numbers and the vertical axis denotes the percentage of variable tasks.

In terms of the compositions of the various uses of variables, the data revealed a wider disparity in the distribution of the proportions of use of variable categories in the 6th and 7th grade textbooks than in the 8th grade textbooks. In other words, the distribution of the proportions of use of variable categories seemed to be skewed in favor of *labels* and *placeholders* in the 6th and the 7th grade textbooks than in the 8th grade textbooks.

Question 5: To what extent has the development and/or the presentation of variables in middle-grades mathematics textbooks changed during the past five decades (i.e., by comparing the development among the textbooks from the four major eras of mathematics reform in the USA)?

Trends in the use of variable categories: When examined as a whole, the data did not reveal a systematic or drastic change in the treatment of variable ideas over the past 50 years in the middle grades mathematics textbooks. For example, in terms of the overall proportions of the use of variable categories, the data indicated that middle grades mathematics curricula consistently employed variables predominantly as *placeholders* and *labels*. There was, however, a change to this pattern of use of variables by the treatment of variable ideas in the *Math Connects* textbooks (selected for the NCTM Standards era: 1994-2009). Specifically, the use of variables as *varying quantities* replaced the use of variable as *labels* as the second highest use of variables in the *Math Connects* curriculum.

One can also observe from Figure 43 that there is a steady increase in the proportion of use of variables as *varying quantities* across the four curricula over the 50 year period within which this study is situated. No similar pattern of change in the proportion of use of variables was evidenced in the other categories of use of variables within the same period (see Figure 43).

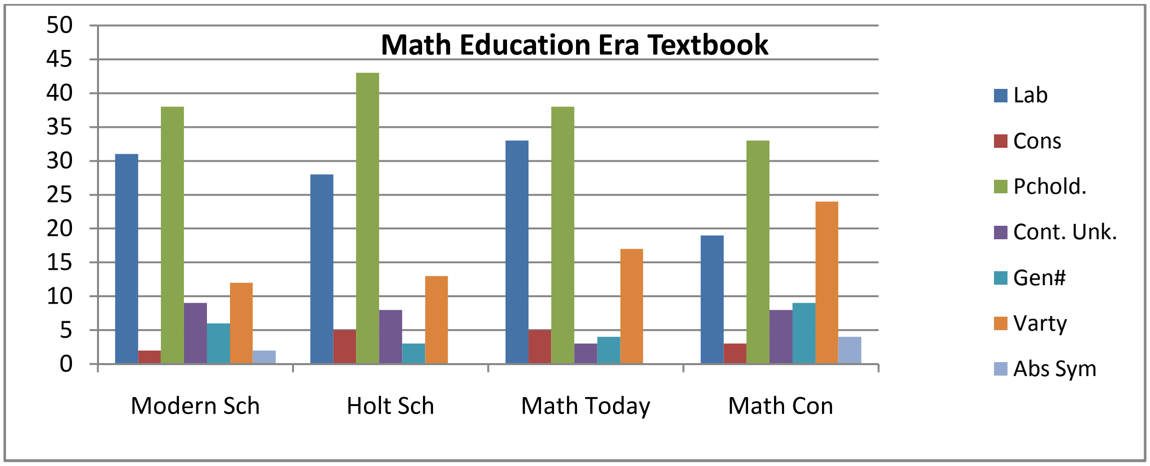


Figure 43. Percent of use of variable categories in middle grades curricula by era. Lab = Label; Cons = Constants; Pchold. = Placeholders; Cont. Unk. = Continuous Unknowns; Gen# = Generalized Numbers; Varty = Varying Quantity; Abs Sym = Abstract Symbols.

Changes in the definition of variable: In terms of definition of variable, the information obtained revealed no systematic pattern of change in the definitions. Thus, even though the characteristics of the definitions of variable were different in the mathematics curricula selected from different mathematics education eras, there was no specific pattern to the changes observed: the definition of variables changed from variables being conceived as *generalized numbers* in the textbooks selected from the New Math era, to the conceptualization of variables as *placeholders* in the textbooks selected from Back to Basics era, to the characterization of variables as *continuous unknown quantities* in the textbooks selected from Problem Solving era, and back to variables being viewed as *placeholders* in the textbooks selected from NCTM Standards Era.

Trends in the proportions of pages containing variables: The overall proportions of pages containing variable ideas in the textbooks from the respective curricula did not follow any systematic pattern; they were highest in the New Math era, became relatively low in the Back to Basics and Problem Solving eras, and then got relatively high again in the textbooks selected from the NCTM Standards Era.

Trends in the use of variables by content area: When analyzed by content areas, one observes a sharp decrease in the use of variables within the *number and operations* content area across the four eras of mathematics education (i.e., within the past 50 years). On the contrary, there was a sharp increase in the use of variables in the *algebra* content area during the same period. There were no consistent changes in the uses of variables in *geometry*, *measurement*, and *data analysis and probability* content areas in the same period based on the information gathered from the set of curricula examined (Figure 44).

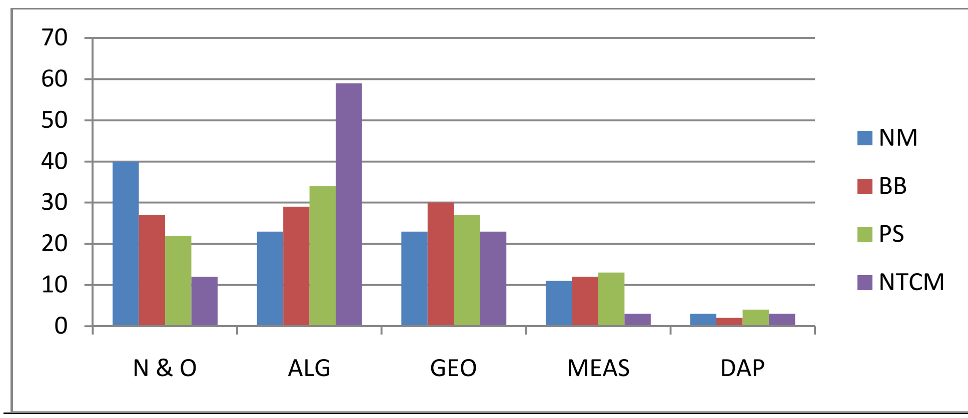


Figure 44. Proportions of uses of variables in different content areas by eras. NM = New Math; BB = Back to Basics; PS = Problem Solving; NCTM = National Council of Teachers of Mathematics.

Changes in the treatment of variables in the Math Connects curriculum: There were some noticeable differences in the treatment of variable ideas in the *Math Connects* curriculum (the textbook series selected for the NCTM Standards Era) that are worth commenting on. First, the context characteristics of the majority of tasks employing variable ideas in the *Math Connects* curriculum relate to real-world experiences of the learners. In fact, many of the worked examples employing variable ideas in this curriculum were entitled “real-world example”. This was generally not the case in the treatment of variable ideas in the other three mathematics curricula. The use of variables to model real-world situations helps students realize that variables can be employed meaningfully to solve problems in their lives, more so than just seeing variables as a tool to manipulate mathematical structures (that may not be immediately meaningful in their lives).

Second, it was only in the *Math Connects* curriculum that technology was employed to explore variable ideas. Even though calculator and computer applications were found in every chapter of the *Mathematics Today* textbooks, for example, there was no evidence of their use to develop variable ideas in that curriculum.

Furthermore, the *Math Connects* curriculum used manipulative materials (cups and counters, see Figure 35) to represent variables in developing meaning for inverse operations in equation solving situations. These activities afford students the opportunity to experience, in more concrete terms, some of the abstract nature of working with variables. In addition, the *Math Connects* curriculum was the only mathematics program that distinguished between *independent* and *dependent* variable in the entire sample of textbooks examined. This distinction could help students develop a better understanding of the role of variables in exploring relationships between two quantities (i.e., in the *varying quantity* category of use of variables).

There was also evidence of the discussion of research findings regarding students' misconceptions about variables in the *Math Connects* curriculum. For example, it was suggested to teachers (in the teachers' guide) to ask students about the possible meaning of $8w$? A specific example is the following: "what does $8w$ represent in the equation? (Math Connects, Grade 8, p. 47). The answer provided was "8 dollars times the number of weeks she has to save". This situation was identified in the research literature as one of the common misconception that the majority of students have in interpreting the role of variables. For example, Booth (1988) reported that the majority of the participants in her study mistook $5y$ to mean 5 yachts, or 5 yogurts, when it should have been conceived as 5 times the number represented by the variable y.

Influences of Reform recommendations: Apparently, reform recommendations in the respective eras seemed to have a direct influence on the materials covered, which in turn influenced the treatment of variable ideas in the respective curriculum. For example, the presentation of variable ideas in the *Modern School Mathematics* curriculum (for the New Math Era) was at a relatively higher level of abstraction than was the treatment in the other three curricula examined. A possible explanation of this could be traced to some of the reform recommendations of the New Math era, which among others was; to introduce mathematical structures to students as early as possible, and to link school mathematics to college mathematics. Similarly, the NCTM stated in the *Principles and Standards* that "children learn through exploring their world; thus, interests and everyday activities are natural vehicles for developing mathematical thinking" (NCTM 2000, pp. 73-74). Consequently, more learning tasks (including the exploration of variable ideas) were presented through "real-world examples" in the *Math Connects* curriculum than was found in the other curricula examined.

<u>Question 6: To what extent do the teacher's editions of middle grades mathematics textbooks provide guidance to teachers on the treatment of variables (i.e., in terms of alerting teachers to the various uses of variables, to students' misconceptions, and to students' difficulties with variables)?</u>

The corresponding teachers' editions of the respective curricula were examined to determine the nature of support they provide for teachers to enact variable ideas in middle grades classrooms. The data collected support the evidence of guidance provided to teachers (in terms of enhancing teachers' content knowledge, offering teaching suggestions, and alerting teachers to students' common misconceptions and difficulties with variables) to teach variable ideas in the respective curricula.

For example, the authors of the *Mathematics Today* curriculum informed teachers to emphasize to students that "whenever the same variable occurs more than once in an equation, the same value must be used. For instance if $a = 7$ and $b = 4$, the left side of $a + b = b + a$ becomes $7 + 4$; the right side becomes, $4 + 7$" (Mathematics Today, Grade 7, p. 70). In another paragraph of the 8th grade textbook, the authors suggested to teachers to "remind students that two letters side-by-side or letter preceded by a number are multiplication expressions: ab means a times b; $3x$ means 3 times x" (Mathematics Today, Grade 8, p. 143). This particular teaching suggestion was found in the teachers' textbook of all the curricula examined.

The Math Connects authors, among others, asked teachers to "point out to students that $-b$ does not necessarily mean that the value of $-b$ is negative. For example, if $b = -4x$, then $-b = 4x$"(Math Connects, Grade 8, p. 47). In addition, teachers were alerted to the following "some students may have difficulty understanding why $2x + 3x$ is not $5x^2$. Guide these students through the process of applying the distributive property to show why the simplified form of $2x + 3x$ must be $5x$" (Math Connects, Grade 8, p. 418). The *Modern School Mathematics* curriculum provided more support to educate teachers on variable ideas than did the other three curricula. One such instance is presented below in Figure 45:

The additive inverse of a rational number is unique, that is, there is only *one* such inverse. You can see formally that this is true by assuming that there are two rational numbers, b and c such that:

$$a + b = 0 \text{ and } a + c = 0$$

If so, then:

$$\begin{aligned} a + b &= a + c \\ -a + (a + b) &= -a + (a + c) \\ (-a + a) + b &= (-a + a) + c \\ 0 + b &= 0 + c \\ a + b &= a + c \\ b &= c \end{aligned}$$

That is, the two inverses are the same. The importance of knowing that the additive inverse is unique lies in the fact that this uniqueness lets you conclude that if $a + b = 0$, then $b = -a$ and $a = -b$" (Modern School Mathematics, Grade 8, p. 14).

Figure 45. Educating teachers on variable ideas (MSM Textbook, Grade 8, p. 14*).*

Overall, the *Modern School Mathematics* program and the *Math Connects* curriculum record the most support available to teachers to enact variable ideas, whereas the *Holt School Mathematics* and *Mathematics Today* curricula record the least support provided. On the whole, there were more supports provided on teaching suggestions for variables than on alerting teachers to students' misconceptions, and on enhancing teachers' knowledge about variable ideas in the curricula examined.

Summary

This chapter presents the results of the study. First, I provided a brief report on the treatment of variables in each of the mathematics curricula selected for the respective eras: *Modern School Mathematics* (New Math), *Holt School Mathematics* (Back to Basics), *Mathematics Today* (Problem Solving), and *Math Connects* (NCTM Standards). This was followed by the results to the six research questions examined in this study. In the chapter that follows (Chapter 5), I present the summary of the results, the discussion of the findings, the conclusions and the implications for curriculum development and for future research.

Chapter 5

Summary, Discussion, and Conclusions

This study investigated the extent to which popular middle grades mathematics textbooks selected from four eras of mathematics education in the United Stated (Th*e New Math*, *Back to Basics*, *Problem Solving*, and the *NCTM Standards era*) (Fey & Graeber, 2003; Payne, 2003) treat the concept of variable, and the extent to which the treatment has changed over the past 50 years. In addition, the study examined the nature and the amount of support the selected mathematics curricula provided for teachers to enact variable ideas in the classroom. In this chapter, I present a summary of the study and discuss the findings in relation to the research questions and the related literature. Limitations of this study, implications for curriculum and teacher development, as well as recommendations for future research are also presented.

Summary of the Results

Data from the study indicate that each of the middle grades mathematics curricula examined employed variable ideas, but in different proportions and levels of complexity. The percentage of pages containing variable ideas ranged from 22% to 69% with a mean of 50% and standard deviation of 11 in the four mathematics curricula. Overall, the *Modern School Mathematics* curriculum selected from the New Math era recorded the highest percentage of pages containing variables, while the *Holt School Mathematics* curriculum from the Back to Basics era had the least percentage of pages containing variables.

The first page of variable ideas occurred within the first quarter of the instructional pages in all the textbooks examined. In some of the textbooks, variable ideas occurred as early as on the 2nd instructional page. The last page of variable ideas occurred in the fourth quarter of the instructional pages in all the textbooks, and as late as on the last instructional page of the majority of the textbooks examined. On the whole, *algebra* and *geometry* contents employed variable ideas the most, whereas measurement and data analysis and probability contents employed variable ideas the least.

Formal definitions for variable were found in 11 of the 12 students' textbooks examined. The characteristics of the definitions of variables in the textbooks changed from the conceptualization of variables as generalized numbers (in the *Modern School Mathematics* curriculum) to the characterization of variables as placeholders (in the *Holt School Mathematics* and *Math Connects* curricula), and to variables being viewed as continuous unknowns (in the *Mathematics Today* curriculum). With the exception of the 8th grade *Math Connects* textbook that defined variables prior to using them, all the other textbooks examined in this study used variables prior to providing a formal definition for what variables are.

In addition to the definitions offered, a few of the textbooks provided explanations on how variables are used. These explanations were given in the contexts of equation solving and/or writing mathematical expressions. However, none of the curricula examined provided historical information on variables. There were rare cases of uses of words such as *placeholders* or *unknowns* to refer to variables in the mathematics curricula. There was, however, substantial usage of special characters such as □, o, and ∇ instead of letters to denote variable ideas in all the *Holt School Mathematics* textbooks for the middle grades.

Differences in the treatment of variable ideas were also observed in the narrative sections of textbooks from different mathematics curricula. For instance, whereas the *Modern School Mathematics* curriculum used variables predominantly in the narrative to generalize arithmetic patterns and prove mathematical relations, the *Holt School Mathematics*, the *Mathematics Today*, and the *Math Connects* curricula treated variables predominantly as placeholders to solve equations, and as labels to name lines, points, and angles, among others. Much more emphasis on the connections between variables and functions were also observed in the *Math Connects* curriculum than was discovered in the other three mathematics curricula.

The use of variables as *placeholder* and *label* dominated the uses of variables in the exercises, question sets, activities and projects, chapter review, chapter summaries and chapter test sections of all the textbooks examined. The use of variables in these two categories constitutes a combined percentage of approximately 70% of all the uses of variables in the curricula examined. The constitution of these two categories (label and placeholder) was higher in the 6th and the 7th grade textbooks than it was in the 8th grade textbooks. Another category of use of variable that stood out among the others was the use of variables as *varying quantities*. Variables were used as *varying quantities* in the textbooks primarily to develop formulas, to represent equations of lines, and to solve systems of inequalities. The least proportion of use of variable categories in the curricula was as an *abstract symbol*.

When examined in terms of content areas, the use of variables as *placeholder* dominated *number and operations*, as well as *algebra* contents. In *geometry*, *measurement*, and *data analysis and probability* content areas, the use of variables as *label* was predominant. In addition, there was a high use of variables as *varying quantities* in *geometry* than in the other content areas.

Overall, the set of middle-grades mathematics textbooks examined provide more opportunities for students to engage with variables as *labels* and *placeholders* than they provide for students to learn about the other uses of variables (i.e., as varying quantities, generalized numbers, continuous unknowns, constants and abstract symbols).The opportunities were even more limited in the case of the use of variables as *abstract symbol*. In fact, only two mathematics curricula (*Modern School Mathematics* and *Math Connects*) provided opportunity for middle grades students to learn about the uses of variables as *abstract symbol*. The order of occurrence of the uses of variable categories was largely as: labels, placeholders, continuous unknowns, generalized numbers, and varying quantity. The use of variables as varying quantity and continuous unknowns occurred mainly towards the third quarter of the instructional pages in the textbooks.

For the most part, the definitions for variables, and the styles of presentation of variable ideas in the respective mathematics curricula seemed similar irrespective of the grade level textbooks. This might be due to the use of spiral development of topics in the respective curricula. There was less use of variable ideas in the 6^{th} grade textbooks than in the 7^{th} and 8^{th} grade textbooks. There were no major observed differences between the use of variables in the 7^{th} and the 8^{th} grades textbooks.

The data gathered from the analysis did not reveal a systematic or drastic change in the treatment of variable ideas during the past 50 years in middle grades mathematics curricula. The use of variables as *placeholders* and *labels* consistently dominate the uses of variables in middle grades mathematics curriculum. There was, however, a steady increase in the use of variables as *varying quantities* over the 50 year period.

In terms of changes in the definition of variable, the information gathered revealed that, even though the characteristics of the definitions of variable were different in the mathematics curricula selected from different mathematics education eras, there was no specific pattern to the changes observed. Similarly, the overall proportions of pages containing variable ideas in the textbooks from the respective curricula did not follow any systematic pattern: they were highest in the New Math eras, relatively low in the Back to Basics and Problem Solving eras, and then high again in the NCTM Standards Era.

When analyzed by content areas, one sees a sharp decrease in the use of variables within the *number and operations* content area during the 50 year period within which this study is situated. On the contrary, the data revealed a sharp increase in the use of variables in *algebra* contents over the same period. There were no consistent changes in the uses of variables in *geometry*, *measurement*, and *data analysis and probability* content areas during the same period.

There were some noticeable changes in the treatment of variable ideas in *Math Connects* (the textbook series selected for the NCTM Standards Era). Namely, there was a relative increase in the use of variable ideas to model "real-world problems" rather than to just facilitate symbolic manipulation skills in mathematics, technology was used to explore variable ideas, concrete materials (manipulative materials: cups and counters) were used to represent variables in developing meaning for inverse operations in equation solving situations, there was a distinction between *independent* and *dependent* variables, and there were discussions of research findings relating to students' understanding and misconceptions about variables.

The data collected also support evidence of guidance provided to teachers (in terms of enhancing their content knowledge, offering them teaching suggestions, and alerting teachers to students' common misconceptions and difficulties with variables) to enact variable ideas in the respective curricula. However, this guidance, in my opinion was inadequate, and very limited in some of the curricula. The *Modern School Mathematics* program and the *Math Connects* curriculum record the most support available to teachers to enact variable ideas, whereas the *Holt School Mathematics* and *Mathematics Today* curricula record the least support provided.

Discussion

The results of this study indicated that each of the middle grades mathematics curricula examined employed variable ideas, but in different proportions and levels of complexity. Overall, the *Modern School Mathematics* curriculum from the New Math era records the highest proportion of variable pages, while the *Holt School Mathematics* series from the Back to Basics era had the least proportion of variable pages in the sample. The nature of variable ideas found in the *Modern School Mathematics* curriculum also seemed to be relatively more complex than those found in the other three curricula. This might be due to the influences of the reform recommendations of the New Math Era, which among others was to; create curricula dominated by attention to formal structure, properties, deductive proof, and building numeric systems relying heavily on the ideas of set, relation and function with the major aim of linking school mathematics with university or higher mathematics. The different levels of use of variables in different mathematics curricula suggest that middle grades students learning from different mathematics curricula may have different opportunities to engage with variables.

Formal definitions for variable were found in almost all the textbooks examined (in 11 out of the 12 textbooks examined). The characteristics of the definitions were, however, different. Specifically, while two of the curricula (*Holt School Mathematics* and *Math Connects* curricula) defined variables as a placeholder to represent a number in equations, the *Modern School Mathematics* curriculum defined variables as a generalized number, and the *Mathematics Today* curriculum defined variables as a continuous unknown (that can assume infinitely many values). Thus, the four middle grades mathematics curricula examined employed three different definitions of variables. This finding is consistent with that of Schoenfield and Arcavi (1988) that different textbooks, different researchers, and even different experts describe this fundamental concept in different ways. Accordingly, many mathematics education researchers have not accepted a common definition of the concept of variable.

Given the lack of consensus on the definition of variables, curriculum designers find themselves in the position to choose among the available definitions of variable to use in their materials. The type of definition they choose influences the treatment of variable ideas in their curriculum, and hence determines the learning opportunities available to students using these curricula. Evidence of this was found in the treatment of variable ideas by the different mathematics curricula examined in this study; the *Modern School Mathematics* curriculum, for example, defined variables in connection with their use as generalized numbers, and subsequently, used variables predominantly in the narratives to generalize arithmetic patterns and prove mathematical relations. Similarly, the *Holt School Mathematics* curriculum defined variables as a placeholder, and then used variables preponderantly as *placeholders* in solving equations.

Thus, the differences observed in the treatment of variables in the narrative portions of different curricula can be traced, to some extent, to the definition of variable employed by the respective curricula. In other words, the way mathematics curricula defined variables influences the way they treat variable ideas. Consequently, if different middle grades mathematics curricula (chosen from the same era, for example), could define variables differently, one could expect differences in their approaches to the treatment of variables, and hence, different learning experiences for different middle grades students learning from different mathematics curricula from the same era.

Another important finding regarding the definitions of variable is that of their location in the various textbooks. In particular, apart from the 8th grade textbook of the *Math Connects* curriculum that defined variables prior to using them, all the other 10 textbooks that defined variables used variables prior to defining what they are or how they are used. A question that arises from this finding is, whether it was better for textbooks to define variables prior to using them, or if it is appropriate to use variables before defining what they are? The answer to this question may depend on a number of factors. Among them are: whether and how variable was defined and used in the previous grade level textbooks? The kind of exposure students had with variables in their previous learning, and the nature of the use of variable prior to being defined in the present textbook? It can be argued that, if variables were defined and explained in the previous grade level textbooks used by the students, then it may not hurt the students' understanding if they encounter variable ideas in their present textbook before definitions are offered.

The extant literature reveals a variety of ways in which variables are used in school mathematics. Among others, variables are used for making general statements, characterizing general procedures, investigating the generality of mathematical issues, and handling finitely or infinitely many cases at once (Schoenfeld & Arcavi, 1988). The idea of variables is also used in labeling points or vertices, sides, and angles of figures, and the likes. The NCTM recommends (2000) that "students at the middle grade level should develop an initial conceptual understanding of different uses of variables" (NCTM, 2000, p. 221).

Although the set of middle grades mathematics curricula examined employed all the above uses and conceptions of variables in school mathematics as recommended by the NCTM and others (NCTM, 2000; Usiskin, 1999b; Wagner, 1999), the results indicated that the curricula provided far more opportunities for students to engage with variables as *labels* and *placeholders*, than they provided for students to engage with the other uses of variables. To be more specific, the use of variables in these two categories (placeholder and label) constituted approximately 70% in the curricula examined.

It was also observed that the use of variables as placeholders and labels occurred at the beginning of the textbooks whilst the use of variables of other types (e.g., as varying quantity and continuous unknown) generally occurred towards the third quarter of the textbooks examined. An important implication of these findings is that students who are taught mathematics with these curricula will more likely develop the conceptions of variables as labels and placeholders than they will with the other uses of variables in school mathematics due to their relative proportions of use as well as order of occurrence in the curriculum. Also supporting the above claim are research findings suggesting that the majority of teachers do not often cover topics located near the end of the textbook, due in part, to time limitations (e.g., Jones & Tarr, 2004; Valverde et al., 2002). Given that the use of variables as continuous unknowns and varying quantity are mainly located in topics near the end of the textbook, this may further limit students' opportunity to engage with the uses of variables of these types. These observations may also explain the findings by researchers that, in spite of the many ways in which variables are used in school mathematics, most students think of variables as representing a single, unique number - a placeholder (Kieran, 1992; Kuchemann, 1981; Mohr, 2008).

Opportunities for students to engage with variables as abstract symbols were even more limited in the textbooks examined. Specifically, only two curricula (*Modern School Mathematics* and *Math Connects*) provided opportunities for middle grades students to engage with the use of variables as an *abstract symbol*. This finding supports Usiskin's (1988) claim that the least used category of variables in school mathematics is that of abstract symbol. In fact, the abstract symbol category was employed sparingly, even in those curricula where it was used. This might be due to the fact that many of the concepts that employ variables as an *abstract symbol* are not of major focus in the middle grades mathematics curriculum.

Over 30 years ago, Küchemann (1978, 1981) articulated four hierarchies of students' understanding and use of variables. In this hierarchy, Küchemann placed the use of variables as labels and placeholders in level 1 and level 2 (the lowest levels of his classification). A variable that varies (*varying quantity*) is considered to be of a higher level of formality than the variable as generalized number or continuous unknown, which are again more formal than a variable as a placeholder. At the top end is the use of variable as an arbitrary symbol (Herscovics & Linchevski, 1994; Filloy & Rojano, 1989). Researchers observed that the advanced uses of variables are the most crucial foundations for success in college mathematics (Gray, Loud, & Sokolowski, 2005), including understanding of major concepts of calculus.

The results of this study, however, indicated that the uses of variables that are prevalent in the set of middle grades mathematics curricula examined are those at the lowest level of Küchemanns' hierarchy (labels and placeholders). It can therefore be argued that, if these curricula are good representation of the materials that the majority of middle grades students learn from (popular middle grades textbooks) during the respective eras, then the conception of variables that the majority of these students will develop will be those of low levels of variables, which according to Gray, Loud and Sokolowski (2005) will not be very useful to them when learning advanced mathematics, unless the curricula they use in the future grades provide more opportunities for them to engage with advanced uses of variables. The data, however, showed a steady increase in the use of variables as a varying quantity across grade levels and mathematics education eras, while the use of variables as labels decreased during the same period. This might be due to the move to have more algebra in the 8^{th} grade (NTCM, 2006). There were no specific trends in the uses of variables in the other categories in the middle grades mathematics textbooks.

Another important question that was asked in this study was on the uses of variable that were prevalent within the various content areas. The results of the study showed that variables were employed predominantly as *placeholders* within Algebra and

Number and Operations content areas, while the use of variables as *labels* dominated Geometry, Measurement, and Data Analysis and Probability content areas. One may be interested in finding out why that was the case. Furthermore, algebra contents employed variables ideas at a much higher percentage than did the other content areas. In addition to the higher percentage of use, the use of variable ideas in the seven categories was more evenly distributed in algebra than it was in the other content areas. These high and even distributions of variable ideas in algebra compared to their use in the other content areas might be due to the fact that the concept of variable identifies traditionally with algebra than the other content areas.

Kieran (1981) observed that despite the importance of the concept of variable, many mathematics curricula continue to discuss the concept of variables like a simple term. This observation was true in the treatment of variable ideas found in the textbooks examined. For example, even though almost all the textbooks defined variables, very limited explanation and/or other information that will help students understand variables were provided in the curriculum. Moreover, none of the curricula develop the concepts of variables in ways that students can clearly understand the richness of this concept without being overwhelmed by its complex nature. Specifically, explicit activities were not found in the textbooks that were purposely designed to help learners acknowledge and distinguish among the various uses of variable.

This state of affairs might be due to the fact that many mathematics curriculum designers do not consider the concept of variables as a *topic* to be treated on its own in the curriculum. Instead, they view the concept of variables as a *tool* to use in developing other topics that are treated in the curriculum. As such, they do not afford this concept the due treatment that it might deserve in the curriculum. However, based on the frequency of use of variables in the middle grades mathematics curriculum (e.g., see the visual displays on the locations and the percent of variable tasks on a variable page in Chapter 4) and the multifaceted nature of this concept (as explained in Chapter 2), it might be a welcomed development for mathematics curriculum designers to begin treating *variables* as a topic in the middle grades mathematics curriculum. I therefore concur with Wagner (1981) that, if we want students to gain appreciation of the power of variables and yet not be overwhelmed by its complex nature, then textbook developers need to explicitly and carefully introduce students to the various uses of variables in school mathematics.

A common misconception about variables reported repeatedly in the literature was that of students' interpretation of the role of variables. The literature reports that in many situations, students interpret the role of variables in functions and equations as *objects* or *labels* when they were not (Booth, 1988; Clement et al., 1981; Küchemann, 1981). Booth (1988) for example, reported that the majority of high school students mistook $5y$ to mean 5 yachts, or 5 yogurts, when it should have been conceived as 5 times the number represented by the variable y. This misconception was also linked to the occurrence of the *reversal error* difficulties reported by Clement et al. (1981) and other researchers (Lochhead & Mestre, 1988; Macgregor & Stacey, 1993; Mestre, 1982; Mestre & Gerace, 1986; Mestre & Lochhead, 1983; White & Mitchelmore, 1996) in the *Student-Professor Problem* as students attempt to model word problems involving two variables which are used as varying quantities.

A surprising finding regarding this misconception was that only one curriculum (*Math Connects*) addressed it by providing explicit opportunities for students to reflect on the meaning of variables in similar situations. Given the frequency with which this misconception has been reported in the research literature, I was expecting the majority of the textbooks, especially those that were designed after these studies were published to have addressed it in the student editions, or at least caution teachers in their teachers' guides to discuss this situation with students. This was however not the case as only the *Math Connects* curriculum showed evidence of addressing this misconception. I speculate that either many curriculum designers are not familiar with this issue, or assumed that teachers are aware of it, and hence, will modify the tasks in the curriculum to discuss this with their students. But, given that teachers themselves may hold similar misconceptions on the interpretation of the role of a variable, it is very unlikely for them to review this with students without curricula support for it.

Another important finding that is worth discussing is the use of technology to develop the concept of variables. The results indicated that only one curriculum (and in limited instances), employed technology to develop the concept of variables. Recent studies in algebra have, however, shown that technology (calculators and computers applications) lead to new possibilities in studying the relationships between two sets of numbers. Thus, the advent of technology has been found to provide different possibilities for students' understanding of the overall uses of variables, especially, the varying nature of variables.

For example, technology can be used with equations of the form $x + y = k$ to illustrate changes in quantities. Spreadsheet applications can also be employed to describe how various quantities change as one variable increases or decreases. Given that the use of technology in school mathematics was not a major focus in some of the earlier eras of mathematics education investigated in this study (the *New Math* and *Back to Basics* eras), it was understandable not to find the curricula from those eras using technology to develop variables. It is my hope, however, that many contemporary middle grades mathematics textbooks use technology to help students develop the meaning of variables.

The results of this study also seemed to indicate that reform recommendations influence the design of textbooks (at least in terms of the treatment of variable ideas) to some extent. As stated earlier, the recommendations in the New Math era seemed to have greatly influenced the treatment of topics covered in the *Modern School Mathematics* curriculum, including the treatment of variables. For example, the concept of *Sets* was used in conjunction with the treatment of every topic, including the definitions of variables found in the *Modern School Mathematics* textbooks. This was in line with the efforts during the New Math era to build numeric systems relying heavily on the ideas of *sets*, relations and functions with the major aim of linking school mathematics with university or higher mathematics.

The Back to Basics era recommended a shift in emphasis to the learning of the basics. Consequently, the curriculum selected for that era (*Holt School Mathematics*) treated variables at the most basic level, when compared to the treatment of variable ideas in all the other curricula examined in this study. Specifically, variable ideas in the *Holt School Mathematics* were employed mainly in equation solving situations, where they function as *placeholders*.

Similarly, in conjunction with the NCTM (2000) reform recommendations that advocate the use of manipulative materials to build mathematical understanding, and called for increased emphasis of studying the mathematical aspects of real-world phenomena, the *Math Connects* curriculum employed variables to model real world situations, used manipulative materials to develop variable ideas, explored variable ideas through the use of technology, and addressed research findings regarding students' misconceptions of variables. These observations seemed to establish possible links between reform recommendations and the treatment of topics in the mathematics curricula from the respective eras, and in particular, the treatment of variable ideas. More studies are needed to confirm this claim.

The data collected also supported evidence of guidance provided to teachers to enact variable ideas in the respective curricula. However, in my opinion, the nature and the amount of support identified were inadequate, and very limited in some of the mathematics curricula. Overall, the *Modern School Mathematics* program and the *Math Connects* curriculum record the most support provided to teachers to enact variable ideas, whereas the *Holt School Mathematics* and *Mathematics Today* curricula record the least support provided. A possible explanation of this finding could stem from the fact that, since the majority of the content in the *Modern School Mathematics* curriculum was difficult (which was one of the major reasons cited for the downfall of this program), it was necessary that a substantial amount of support be provided to teachers in order for them to teach the curriculum effectively. In a similar line of reasoning, one could argue that since the current NCTM recommendations call for emphasis on teacher support, as stated in the *Principles and Standards*:

> Teachers need several different kinds of mathematical knowledge – knowledge about the whole domain; deep, flexible knowledge about curriculum goals and

> about the important ideas that are central to their grade level; knowledge about the challenges students are likely to encounter in learning these ideas......... This kind of knowledge is beyond what most teachers experience in standard pre-service mathematics courses in the United States. …Teachers must have frequent and ample opportunities and resources to enhance and refresh their knowledge" (NCTM, 2000, p. 17),

it was reasonable to see more evidence of support provided for teachers in the *Math Connects* curriculum than was found in those curricula published 20 years ago.

On the contrary, the contents of the *Mathematics Today* (selected from the Problem Solving era) and the *Holt School Mathematics* curricula (selected from the Back to Basics era with the "basics" primarily defined as computational skills) were relatively easy, compared to those in the *Modern School Mathematics* and the *Math Connects* curricula. Consequently, teachers' knowledge of the content might have been assumed by the authors of the *Holt School Mathematics* and *Mathematics Today* curricula, thereby accounting for the limited support offered to teachers in these curricula to enact the curriculum.

Furthermore, in the majority of the instances where guidance was offered, teachers were mainly provided with teaching suggestions without explaining to them why the suggestions were important. Supports aimed at *educating* teachers on variable ideas were even more limited within the curricula. It might be the case that the authors of these curricula assumed that teachers are proficient with the uses of variables themselves, and understand the learning implications for their students. The literature on teachers' knowledge of variables, however, points to the contrary; that teachers' conception of variable is weak (e.g., Boz, 2002, 2007; Mohr, 2008), and that they rarely identify students' misconceptions on the use of variables (e.g., Asquith et al., 2007).

As Ball (2003) espoused, teachers' opportunities to learn must equip them with the mathematical knowledge and skill that will enable them to teach effectively. The National Research Council (NRC, 2004) recommends that "a curriculum should include enough support for teachers to enact it as intended. Such support should allow teachers to educate themselves about mathematics content, students' mathematical thinking, and relevant classroom issues…. It might help … teachers to analyze common student errors in order to think about next steps for those who make them" (NRC, 2004, p.76).

One way to achieve this (in the case of the concept of variables) could be to design ample opportunities in the instructors' textbooks for teachers to delve more deeply into the concept of variable. The overriding purpose of such supports would be to provide teachers with ample opportunities to learn about the various uses of variables in school mathematics, how the uses are related, and about students' misconception in the use of variables, among others.

Taken as a whole, the results of this study provided some encouraging news, as well as raised some concerns. Among the good news were the findings that each of the curricula examined provided opportunities for students to engage with variables at the middle grades. Support for the enactment of variable ideas was also found in the curricula to guide teachers implement variable ideas in the classroom. Also, when compared to the others, the curriculum selected for the present NCTM era (*Math Connects*) showed a substantial improvement in the treatment of variable ideas, in terms of its relative use of variable ideas to model real-world phenomena, its use of technology to explore variable ideas, its use of models (manipulative materials) to represent variables ideas in equation solving, and its discussions of research issues related to students' understanding and misconceptions about variables. Thus, students' engagement with variables will be enhanced if the majority of the contemporary textbooks that are currently in use by students employ a similar or better treatment of the concept of variables as those found in the *Math Connects* curriculum.

On the contrary, the results also showed that different middle grades mathematics curricula defined variables differently. In addition, almost all the curricula examined treated variables as *placeholders* and *labels* (which are considered to be low levels of the use of variables) instead of their uses as continuous unknowns, generalized numbers or varying quantities that could better prepare students for further studies in mathematics. One could argue, for example, that the discovery of the steady increase in the proportion of use of variables as *varying quantity* in the data as one move up the grade levels is an indication that students may have ample opportunities in their future grades to experience variables at those advanced levels. There could also be some textbooks that were used during the respective eras that were not part of the sample of curricula investigated that might have dealt with variables in a much better manner than those curricula examined in this study.

If the preceding arguments do not hold, and the treatment of variable ideas found in the curricula examined in this study mirrors that of the majority of mathematics curricula that middle grades students learn from, then students' conception of variables will be, at best, at the lower level, and this may have serious repercussion for their advance studies in mathematics. Lastly, there was no obvious structure or organization to the treatment of variable ideas by any of the four mathematics curricula examined in ways that will help students clearly understand the richness of the concept of variables without being overwhelmed by its complex nature, and this to me, is not desirable.

Significance of the Study

It is expected that the findings from this study will inform curriculum developers and evaluators of middle grades mathematics curriculum in their future efforts to improve classroom materials on variables for students to use as well as for teachers to implement in their classrooms. For example, among others, the results of this study showed that the opportunities that exist in the middle grades mathematics textbooks for students to use variables depict a dominance of students' engagement with variable at a low level – *placeholders* and *labels* (Küchemann, 1981). Consequently, curriculum developers might consider changes to the treatment of variable ideas in future editions of middle grades mathematics textbooks to increase students' opportunities to engage with the other uses of variables that will be beneficial to them in their further studies in mathematics.

Secondly, even though this study does not directly investigate students' understanding of variables, it contributes to that body of research by documenting the opportunities that mathematics textbooks offer students to acquire competency with the use of variables. That is, this study makes an important prerequisite step towards addressing students' difficulties with variables by examining how variable ideas are presented in the intended curriculum – which is central to classroom practices.

The findings from this study also shed light on some possible links between students' understanding of variable and its presentation in the curriculum. For example, the relatively large emphasis on the use of variables as *placeholders* in the majority of the curricula examined could explain the phenomenon reported repeatedly in the research literature that, in spite of the many uses of variables in school mathematics, the majority of students, as well as some teachers think of variables as representing a single or unique number – a placeholder (Kieran, 1992; Mohr, 2008).

In addition, it is expected that the methodology used in this study will provide some guidelines for future researchers who may use content analysis to carry out research in mathematics education. In order to encourage the use of this methodology (which is rarely used) in mathematics education, the NRC in one of its well referenced publications, *On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Evaluations* (2004), devoted an entire chapter to discuss content analysis in mathematics education, and encouraged mathematics education researchers to engage in such endeavors. The methodology employed in this study contributes to the knowledge base on the use of content analysis in mathematics education research, and hence can serve as a model for future researchers who will engage in similar studies.

The study also contributes to the research literature on teacher development, and the support that is available in the curriculum for teachers to implement the intended curriculum, by documenting the nature and amount of support that is available in the textbook for teachers to teach fundamental topics and concepts such as variables. Researchers (e.g., Ball et al., 2005; Ma, 1999) have long observed that many teachers do not possess firm grasp of fundamental concepts in mathematics. In addition, many recent recommendations in mathematics education require teachers to teach mathematics in ways that they have not experienced as students. These changes will require support for them, and this study contributes indirectly to that body of research by reporting on the type and nature of support that are available in the textbook for teachers to teach about variables in their classroom.

Finally, the analysis of the treatment of variables in popular middle grades mathematics textbooks selected to span a 50 year period of mathematics education in the United States provides some historical report on the changes that this concept has undergone in school mathematics curriculum.

Implications for Mathematics Education

Implications for Curriculum Development and Teacher Education

The results of the study indicate that none of the curricula examined developed the concepts of variables in ways that will help students clearly understand the richness of the concept of variables without being overwhelmed by its complex nature. Thus, explicit activities were not found in the textbooks that are specifically targeted at helping learners acknowledge the various uses of variable. As a result, textbook developers might consider designing activities in their curricula that will explicitly and carefully introduce students to the various uses of variables in school mathematics. For example, when variables such as p are used to label a point on a line, at some point in the curriculum, it might be helpful to point out to students that, here the variable is used as a label, and later on when they encounter the use of variables in developing the associative property of addition $[(a+b)+c=a+(b+c)]$, students can be informed on the role of the variable (as *pattern generalizer*) and so on.

In addition, it might be helpful for curriculum developers to familiarize themselves with the research findings on students' (as well as teachers') difficulties and misconceptions of fundamental topics in school mathematics, and then develop specific activities in the curriculum to address these difficulties. As Wagner (1981) pointed out, some students seem to think that changing the variable in an equation may result in a different solution. Wagner found the majority of participants in his study thinking that the value represented by n and w in $7n + 22 = 109$ and $7w + 22 = 109$ could never be the same. In their view, an n would represent a smaller number than, say, w. Textbook developers can explicitly check this misconception by asking students to solve a given equation, then write the same equation beside it with a different variable for students to discuss.

Also, given that the results from this study indicated that the support provided to teachers to enact variable ideas is limited in some curricula, teacher education programs could make the discussing of variable ideas part of their curriculum. A properly organized activity could provide opportunities for teachers to examine the various conceptions of variables, and to learn about students' difficulties and misconceptions on the uses of variables in their education.

Recommendations for Future Research

As acknowledged earlier, the use of a larger sample of textbooks could provide a more accurate picture on the treatment of variable ideas in middle grades textbooks that will allow for generalization of the results. Consequently, it might be a worthwhile endeavor in the future to conduct a study on the treatment of variable ideas using a larger sample of middle grades mathematics textbooks. This larger sample could be defined to include *popular* and *alternate* middle grades mathematics textbooks. Future research could also examine multiple mathematics curricula from the same era, or a larger number of contemporary textbooks (including reform and commercially based textbooks) on their approaches to the concept of variables, in order to determine the current state of affairs on the treatment of this concept in middle grades mathematics curricula in the United States. In addition, future research may also consider examining the nature of the treatment of the concept of variables in mathematics curricula from other countries to determine what might be learned from international perspective on this topic.

Also, since this study was conducted on the *intended* curriculum, there are likely to be fewer opportunities for students to learn about variables in the *implemented* or *enacted* curriculum (Tarr, Chavez, Reys, & Reys, 2006) than those opportunities identified and reported in this study. Future research is needed to investigate the actual enactment of variable ideas in the classroom. Furthermore, I am not aware of any research that links the nature or extent of the treatment of variable ideas in textbooks to students' ability to work with variables. I recommend that future research examine how the opportunities in textbooks on variables relate to actions in the classroom, and subsequently to students' performance on variable ideas. Such research could help the mathematics education community learn what features of curricula and instructional practices related to variables help students to develop understanding of this concept. Researchers may also consider the extent to which different learning styles, the use of manipulative materials, developmental appropriateness of students, among others influence students' engagement with the concept of variables in the classroom.

Lastly, future research is needed to examine how and when to introduce letter variables into the school mathematics curriculum. It might also be useful to investigate the extent to which textbook publishers themselves are aware of the myriad ways in which variables are used, and the extent to which they recognize the particular characteristics variables exhibit in various contexts, as well.

Limitations of the Study

As explained earlier, the materials that were examined in this study included only popular middle-grades mathematics textbooks used during four eras of mathematics education in the United States. As a result, the findings may not be generalizable beyond those materials (or similar materials) that were examined, or beyond the time frame within which this study is situated. Also, the number of textbooks that were analyzed was relatively small (i.e., 12 students' textbooks and their corresponding 12 teachers' guides). Against this background, it is prudent to acknowledge the fact that the findings might, perhaps, be altered if a larger sample was used. Consequently, any attempt to extend the results beyond these textbooks has to be done cautiously.

Second, the fifth research question that examined the change in the trend of presentation of variable ideas over time could have been addressed more appropriately if the same mathematics textbook series from the same publisher that was used during each of the four eras of mathematics education was selected. This could have made it possible to determine the extent to which the same publisher changes the approaches to the treatment of variable over time, in accordance with the respective reform recommendations. However, no such single mathematics textbook series met all the textbook selection criteria outlined in this study.

In addition, this study examined the intended curriculum. It could be possible that what is in the intended curriculum is not exactly what is enacted in the actual classroom with students. That is, it is possible that teachers may not teach all the lessons in a textbook related to variable ideas or assign all the problems that contain variable ideas.

So, the actual opportunities that students have to engage with variables are likely to be less than those reported in this study.

It is also worth pointing out that although a greater portion of the findings reported in this study were presented in percentages or proportions (instead of frequencies, for instance), the goal was not to set a predetermined proportions to achieve, but instead to consider the appropriateness of the treatment of variables based on the best available information in the curricula, and to present this information in the format that readers can easily make sense of it. In particular, it is important to state that, even though, the percentages of the uses of some of the variable categories reported here may seem small relative to others, the frequency of their occurrences in the curriculum may be sufficient enough to provide learners with the desired learning experiences that will commensurate with their learning expectations.

Furthermore, the present study did not interview individuals who were involved in the development of the selected middle-grades mathematics curricula, or teachers and students who enact and learn from these curricula materials about their perspectives on the presentation and the uses of variable in the textbooks. This researcher is, thus, aware that obtaining such a data could add some valuable insight to the results of the study, and hence, considered this as a limitation to the study.

Also, it is important to state here that when interpreting the results of this study, it is essential to be aware of the fact that other factors such as issues related to adoption, psychological readiness of learners, and federal and local school policies, among others may influence the treatment and inclusion of topics (including variables) in mathematics textbooks and not just what is deemed appropriate for a particular group of students controls the materials presented in the textbook.

Finally, threats to reliability and validity of execution of coding instructions as well as practice of coding to establish a desirable reliability may have occurred. For example, the possibility of coder fatigue may have occurred as the amount of documents examined was large (i.e., about 24 textbooks). Coding schedules however, took into consideration the length of units that were coded, and reliability was enhanced by the use of code-recode strategies and measures of inter-coder agreement.

Final Conclusions

Meaningful and long-lasting improvements in students' learning will require changes in many areas of our education system. At the center of this system is the curriculum, which is defined largely by the textbooks students and teachers use. A careful examination of the content depth, instructional strategies, and the treatment of fundamental concepts in these textbooks is required to judge whether there are potentials for students to actually learn important mathematics. This study contributes to these efforts by examining the opportunities that exist in popular middle grades mathematics textbooks for students to engage with variables. Clearly, the research report presented in this study is not exhaustive. However, I believe that my analysis of popular middle grades mathematics textbooks selected to span a 50 year period of mathematics education in the United States provides some important insight into the state of the treatment of variable ideas in middle grades mathematics textbooks. Future research can expand on the work presented here to help the mathematics education community better understand issues related to variable ideas in school mathematics

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