

## ABSTRACT

Title of dissertation: ESSAYS ON UNIFORM PRICE AUCTIONS

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When selling divisible goods such as energy contracts or emission allowances, should the entire supply be auctioned all at once or should it be spread over a sequence of auctions? How does the expected revenue in a sequence of uniform price auctions compare to the expected revenue in a single uniform price auction? These are questions that come up when designing high-stake auctions and this dissertation provides answers to them. In uniform price auctions, large bidders have an incentive to reduce demand in order to pay less for their winnings. In a sequence of uniform price auctions, bidders also internalize the effect of their bidding in early auctions on the overall demand reduction in later auctions and discount their bids by the option value of increasing their winnings in later auctions. This dissertation shows that a sequence of two uniform price auctions yields lower expected revenue than a single uniform price auction particularly when competition is not very strong.

It is generally argued that forward trading is socially beneficial. Two of the most common arguments state that forward trading allows efficient risk sharing and improves information sharing. It is also believed that when firms can produce

any level of output, strategic forward trading can enhance competition in the spot market by committing firms to more aggressive strategies. However, firms usually face capacity constraints, which change the incentives for strategic trading ahead of the spot market. This dissertation also studies these incentives through a model where capacity constrained firms engage in forward trading before they participate in the spot market, which is organized as a multi-unit uniform-price auction with uncertain demand. This dissertation shows that when a capacity constrained firm commits itself through forward trading to a more competitive strategy in the spot market, it actually softens competition in the spot market. Hence, its competitor prefers not to follow suit in the forward market and thus behave less competitively in the spot market than otherwise. Moreover, strategic forward trading generally leaves consumers worse off as a consequence of less intense competition in the spot market.

# ESSAYS ON UNIFORM PRICE AUCTIONS

by

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# Chapter 1

## Introduction

In a uniform price auction bidders have an incentive to shade their bids (i.e. reduce demand) in order to lower the price they pay for their purchases<sup>1</sup>. This incentive grows with the quantity demanded and is inversely related to the size of bidders, measured by the maximum quantity they want to buy<sup>2</sup>. When bidders can choose their sizes or choose to behave as if they have different sizes, bidders have more degrees of freedom on determining the optimal bid shading. This dissertation studies two environments where bidders enjoy that extra freedom. In the first case the focus is on a sequence of uniform price auctions, while in the second case the focus is on strategic trading ahead of a uniform price auction.

### 1.1 Sequential Uniform Price Auctions

When designing high-stake auctions, such as auctions for energy contracts or emission allowances, one of the first questions that come up is whether to have a single auction or to spread the supply (or demand in a procurement case) over a sequence of auctions. More often than not the decision has been to have a sequence of

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<sup>1</sup>In a procurement auction, bidders have an incentive to inflate their bids or reduce their supply to increase the price they receive.

<sup>2</sup>In a procurement auction, a bidder's size is measured by the maximum quantity he wants to supply.

auctions. The Regional Greenhouse Gas Initiative (RGGI) which comprises the 10 northeastern states in the U.S. allocates CO<sub>2</sub> emission allowances among electricity generators within the region by means of a sequence of uniform price auctions. The supply of a given vintage of CO<sub>2</sub> emission allowances is spread over four annual auctions and four quarterly auctions.<sup>3</sup> Electricity supply contracts are sold quarterly by Electricité de France, Endesa and Iberdrola (Spain) and were sold by Electrabel (Belgium) through the so called virtual power plant auctions<sup>4</sup>. Gas release programme auctions is the name used for the annual auctions of natural gas contracts used by Ruhrgas, Gas de France (GDF) and Total among others<sup>5</sup>. The New York Independent System Operator allocates installed capacity payments through a sequence of monthly uniform price auctions<sup>6</sup>; and the Colombian system operator will procure forward electricity supply contracts to match the annual forecast electricity demand by means of a sequence of four quarterly auctions<sup>7</sup>.

The seller looks for the auction format that is best suited for achieving her main goals of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. There are several features of the market that should

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<sup>3</sup>See Holt et al. (2007) for more details on the auction design for CO<sub>2</sub> selling emission allowances under the RGGI.

<sup>4</sup>See [www.powerauction.com](http://www.powerauction.com) and Milgrom (2004) for more details on virtual power plant auctions.

<sup>5</sup>See [www.powerauction.com](http://www.powerauction.com) for more details on gas release programme auctions.

<sup>6</sup>See Installed Capacity Manual (2008), NYISO for more details on installed capacity auctions.

<sup>7</sup>See Cramton (2007) and [www.creg.gov.co](http://www.creg.gov.co) for more details on the Colombian electricity market.

be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

When the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction, participation in the auction can be expected to be low, which tends to have a negative effect on expected revenues. For this reason, the seller might prefer a single auction over a sequence of auctions to keep transaction costs low. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. A sequence of sealed-bid auctions is somewhere between a single sealed-bid auction and an ascending auction in terms of the private information revealed through the auctions. Hence, when there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenues. Since the price in an auction might be too high or too low due to some unexpected events, risk averse infra-marginal bidders (i.e. bid-takers) prefer a sequence of auctions over a single auction. If there is a single auction, infra-marginal bidders might end up paying too high or too low a price for all their purchases. But, in a sequence of auctions this risk is reduced since the prices bidders pay for their purchases are determined at several points in time. In the presence of risk averse bidders the seller might also prefer a sequence of sealed-bid auctions, since such auction format might increase the seller's expected revenues not only by increasing

participation of risk averse bidders, particularly bid-takers, but also by encouraging marginal bidders to bid more aggressively due to a weaker winner's curse in a case with affiliated information<sup>8</sup>.

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions<sup>9</sup>. However, there is no theoretical nor empirical research that studies sequences of divisible good auctions. But, in several real-world cases where sequences of auctions are used, such as those mentioned before, the auctioneer sells a divisible good. Moreover, we know from the case of a single auction, that divisible good auctions are not a trivial extension of single object auctions; hence one should not expect the results from sequential single object auctions to extend over to the case of sequential divisible good auctions. Therefore, studying strategic bidding in a sequence of divisible good auctions as well as the efficiency and revenue generation properties of this type of auctions is not only relevant from an academic perspective, but also from a practical standpoint.

Chapter 2 studies a sequence of two uniform price auctions for a divisible good in a pure common value model with symmetric information and aggregate

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<sup>8</sup>In the case of common-values with affiliated signals, the extra information that is revealed through the sequence of auctions reduce the winner's curse and the real risk imposed by aggressive bidding.

<sup>9</sup>See Weber (1983), Milgrom and Weber (1999), Ashenfelter (1989), McAfee and Vincent (1993), Bernhardt and Scoones (1994), Jeitschko (1999), Katzman (1999).

uncertainty. The unique profile of equilibrium bid functions in the second auction is fully characterized, as well as the entire set of equilibrium bid functions in the first auction. Using the characterization of equilibrium bidding, the revenue generation properties of the sequence of two uniform price auctions are compared with those of a single uniform price auction. A sequence of uniform price auctions was chosen over a sequence of pay-as-bid auctions because uniform price auctions are more widely used in energy and emission allowance markets, and there is a growing trend toward the use of this type of auctions in other markets.

Ausubel and Cramton (2002) show bidders in a uniform price auction have an incentive to shade their bids (i.e. reduce demand) in order to lower the price they pay for their purchases. This incentive grows with the quantity demanded and is inversely related to the size of bidders, measured by the maximum quantity they want to buy. In each auction of a sequence of two uniform price auctions bidders have the same incentive to shade their bids, since spreading the supply over two auctions does not change the fact that a bidder behaves like a residual monopsonist. At the first auction of the sequence, bidders know that if they do not buy all the quantity they want in that auction, they still have another opportunity to do so in the second auction. Therefore, bidders discount their first auction bids by the option value of increasing their purchases in the second auction. This is similar to the case of a sequence of single object auctions, where bidders discount their bids in an auction by the option value of participating in later auctions (Milgrom and Weber (1999), Weber (1983), Bernhardt and Scoones (1994) and Jeitschko (1999)).

In a single uniform price auction or in the first auction of the sequence, the

maximum quantity each bidder wants to buy (i.e. his demand) is exogenous. However, in the second auction of the sequence bidders' demands are endogenous, because they depend on the quantities bought in the first auction. Since the bid shading in the second auction depends on bidders' demands in that auction, bidders have an incentive to shape the bid shading in the second auction through their bidding in the first auction. In equilibrium, one bidder holds back in the first auction, by bidding lower prices than his competitors. In that way, this bidder reduces his competition in the second auction by letting the other bidders buy larger quantities in the first auction than otherwise. This feature of equilibrium will be called *dynamic* bid shading to differentiate it from the *static* bid shading described by Ausubel and Cramton (2002). The bidder who benefits the most from this strategic behavior is the largest bidder, because by having a larger demand he can profit the most from the more intense bid shading in the second auction.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding are even stronger when the supply is split evenly between the two auctions of the sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions. These results are in line with the finding that it is better for the seller to use a sealed-

bid auction than a dynamic auction when competition is not very strong (Cramton (1998) and Klemperer (2004)).

This is the first study of a sequence of divisible good auctions. The benefit of modeling sequential divisible good auctions is that it allows for the study of strategic forward looking bidding, which could have not been done by modeling a sequence of single object auctions with either unit or multi-unit demands, or even a sequence of multi-unit auctions with unit demands. Bidders bid in the first auction not only to buy some quantity at that stage, but also to improve their strategic position in the second auction. The improvement in a bidder's strategic position is not a consequence of the bidder strategically revealing information to manipulate his opponents' beliefs, but a consequence of the bidding and the quantity bought in the first auction.

This study relates to a broad literature on how to create and enhance market power<sup>10</sup>. In any market, there are different ways of creating or enhancing market power. For example, firms can create barriers to entry, or create sub-markets either by independently differentiating their products from their competitors' products, or by explicitly coordinating on some type of market segmentation. The underlying idea on the different strategies to create or enhance market power is to profitably differentiate yourself from your potential or actual competitors. This is exactly what happens in a sequence of two uniform price auctions. *Dynamic bid shading* is a strategy that allows bidders to optimally differentiate themselves by splitting up the market into two less competitive markets.

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<sup>10</sup>See Tirole (1988) for a survey on creation or enhancement of market power.



The literature on auctions for split-award contracts studies the case in which a buyer divides the purchases of its input requirements into several (usually two) contracts that are awarded to different suppliers in separate auctions (Anton and Yao (1989, 1992), Perry and Sákovics (2003)). In a sequence of two uniform price auctions, the split or market segmentation, which is endogenous, is not complete (i.e. all bidders buy in both auctions) because of the uncertainty about the residual supply in the second auction. However, as chapter 2 shows for the case of forward trading ahead of a procurement uniform price auction, if bidders' expected profits from the first auction or market are zero, then one bidder, usually the largest one, will wait for the second auction or market even with uncertain residual supply.

This study also relates to a branch of the auction literature that studies auctions with aggregate uncertainty. On one side, Klemperer and Meyer (1989), Holmberg (2004, 2005) and Aromí (2006) study procurement uniform price auctions where firms sell a divisible good and demand is uncertain. These framework is known as the supply function framework since firms compete by submitting supply functions. On the other side, Wang and Zender (2002) study standard divisible goods auctions in a common values model with random noncompetitive demand. The model in this paper is closer to Wang and Zender's (2002) model than to the supply function models, not only because it studies a standard auction where the seller is the auctioneer, but also because it assumes a common values model with random noncompetitive demand.

## 1.2 Forward Trading and Capacity Constraints

It is generally argued that forward trading is socially beneficial. Two of the most common arguments state that forward trading allows efficient risk sharing among agents with different attitudes toward risk and improves information sharing, particularly through price discovery. It is also believed that forward trading enhances competition in the spot market by committing firms to more aggressive strategies. A firm, by selling forward, can become the leader in the spot market (the top seller), thereby improving its strategic position in the market. Still, when firms compete in quantities at the spot market, every firm faces the same incentives, resulting in lower prices and no strategic improvement for any firm. This is Allaz and Vila's (1993) argument. Green (1999) shows when firms compete in supply functions, forward trading might not have any effect on the intensity of competition in the spot market, but in general it will enhance competition. This pro-competitive argument has been used to support forward trading as a market mechanism to mitigate incentives to exercise market power, particularly in electricity markets.

The pro-competitive feature of forward trading has been challenged by recent papers. Mahenc and Salanié (2004) show when, in the spot market, firms producing substitute goods compete in prices instead of in quantities, firms take long positions (buy) in the forward market in equilibrium. This increases the equilibrium spot price compared to the case without forward market. In that paper as in Allaz and Vila's paper, firms use forward trading to credibly signal their commitment to more profitable spot market strategies. However, as Fudenberg and Tirole (1984)

and Bulow et al. (1985) point out, in those cases prices are strategic complements, while quantities are strategic substitutes, which is the reason for the different equilibrium forward positions taken by firms in both papers, and the resulting effect on the intensity of competition. Liski and Montero (2006) show that under repeated interaction it becomes easier for firms to sustain collusive behavior in the presence of forward trading. The reason is that forward markets provide another instrument to punish deviation from collusive behavior, which reduces the gains from defection.

However, all these papers ignore a key point—that firms usually face capacity constraints, which affects their incentives for strategic trading ahead of the spot market. When a capacity constrained firm sells forward, it actually softens competition in the spot market from the perspective of competitors. In the case where there are two firms and one sells its entire capacity forward, its competitor becomes the sole supplier in the spot market, which implies it has the power to set the price.

The following is an example of how forward trading can affect the intensity of competition in the spot market when firms are constrained on the quantity they can offer. The In-City (generation) capacity market in New York is organized as a uniform-price auction, where the market operator (NYISO) procures capacity from the Divested Generation Owners (DGO's). Two of the dominant firms in this market are KeySpan, with almost 2.4 GW of installed capacity and, US Power Gen, with 1.8 GW. Before May 2006, US Power Gen negotiated a three years swap (May 2006 – April 2009) with Morgan Stanley for 1.8 GW, by which it commits to pay (receive from) Morgan Stanley 1.8 million times the difference between the monthly auction price and \$7.57 kw-month, whenever such difference is positive (negative). Morgan

Stanley closed its position by negotiating with KeySpan the exact reverse swap.

The first swap works for US Power Gen as a credible signal that it will bid more aggressively in the monthly auction, since US Power Gen benefits from lower clearing prices in that auction. Also, this financial transaction could be explained on risk hedging grounds. The swap reduces US Power Gen's exposure to the spot price by locking in, at \$7.57 per kw-month, the price it receives for those MWs of capacity its sells in the spot market. On the other side, the outcome of these transactions left KeySpan owning, either directly or financially, 4.2 GW of capacity for three years, which gave it a stronger dominant position in the In-City capacity market, and the incentive to bid higher prices in the monthly auction than otherwise. Moreover, it is difficult to explain this financial transaction on risk hedging grounds, since the swap increases KeySpan's exposure to the uncertain price of the monthly auction, by buying at the fixed price and selling at a variable price (the spot price).

As chapter 3 shows, when capacity constrained firms facing common uncertainty compete in a uniform-price auction with price cap, strategic forward trading does not enhance competition. On the contrary, firms use forward trading to soften competition, which leaves consumers worse off. The intuition of this result is that when a capacity constrained firm commits itself through forward trading to a more competitive strategy in the spot market, its competitor faces a more inelastic residual demand in that market. Hence, its competitor prefers not to follow suit in the forward market and thus behave less competitively in the spot market than it otherwise would, by inflating its bids. Because of capacity constraints a firm's actions in the forward market can change its competitor's strategies in the spot market by

affecting its own marginal revenue in the spot market. This result and its intuition relate to the work of Fudenberg and Tirole (1984) and Bulow et al. (1985) on strategic interactions. Under the assumptions made here, once US Power Gen negotiated the swap with Morgan Stanley, KeySpan would have the incentive to bid higher prices in the monthly auction, than if there were no trading ahead of it, even if KeySpan did not buy the swap from Morgan Stanley.

When studying the effect of forward trading on investment incentives in a model with uncertain demand and Cournot competition in the spot market, Murphy and Smeers (2007) find that in some equilibria of the forward market one of the firms stays out of the market while the other firm trades. These equilibria come up when the capacity constraint of the latter firm binds at every possible realization of demand. Grimm and Zoetl (2007) also study that problem by assuming a sequence of Cournot spot market with certain demand at each market, but varying by market. They also find that when a firm's capacity constraint binds in a particular spot market, this firm is the only one trading forwards which mature at that spot market. These results are in the same line as those on chapter 3. However, when the spot market is organized as a uniform-price auction, as is the case here, they hold even if the capacity constraints only bind for some demand realizations. Also, by modeling the spot market as a uniform-price auction with uncertain demand, the results on this paper are better suited for the understanding of wholesale electricity markets.

The results here are also related to those on demand/supply reduction in uniform-price auctions. As Ausubel and Cramton (2002) show, in uniform-price procurement auctions, bidders have an incentive to reduce supply in order to receive

a higher price for their sales. This incentive grows with the quantity supplied and it is inversely related to the size of the smallest bidder. Large bidders make room for small bidders. When a capacity constrained firm sells forward, it behaves like a smaller bidder in the auction. Therefore, the incentive to inflate bids increases for the other bidders in the auction. Consequently, strategic forward trading can be reinterpreted as a mechanism that allows firms to assign themselves to different markets, in order to strengthen their market power, which leaves firms better off, but at the expense of consumers who end up worse off. As the paper will show, usually the smaller firm decides to trade most of its capacity through the forward market, with the larger firm becoming almost the sole trader on the spot market.

The goal of this chapter is not to challenge the general belief that forward trading is socially beneficial, but yes to challenge the pro-competitive view of forward trading by highlighting the impact of capacity constraints on the incentives for strategic forward trading.

### 1.3 Outline

This dissertation is organized as follows. Chapter 2 analyzes a sequence of two uniform price auctions for divisible goods. Chapter 3 analyzes strategic forward trading when firms are capacity constrained and the spot market is organized a uniform price auction with uncertain demand. Finally, chapter 4 provides the conclusion to the dissertation.

## Chapter 2

### Sequential Uniform Price Auctions

#### 2.1 Introduction

When designing high-stake auctions, such as auctions for energy contracts or emission allowances, one of the first questions that come up is whether to have a single auction or to spread the supply (or demand in a procurement case) over a sequence of auctions. More often than not the decision has been to have a sequence of auctions. The Regional Greenhouse Gas Initiative (RGGI) which comprises the 10 northeastern states in the U.S. allocates CO<sub>2</sub> emission allowances among electricity generators within the region by means of a sequence of uniform price auctions. The supply of a given vintage of CO<sub>2</sub> emission allowances is spread over four annual auctions and four quarterly auctions.<sup>1</sup> Electricity supply contracts are sold quarterly by Electricité de France, Endesa and Iberdrola (Spain) and were sold by Electrabel (Belgium) through the so called virtual power plant auctions<sup>2</sup>. Gas release programme auctions is the name used for the annual auctions of natural gas contracts used by Ruhrgas, Gas de France (GDF) and Total among others<sup>3</sup>. The New York

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<sup>1</sup>See Holt et al. (2007) for more details on the auction design for CO<sub>2</sub> selling emission allowances under the RGGI.

<sup>2</sup>See [www.powerauction.com](http://www.powerauction.com) and Milgrom (2004) for more details on virtual power plant auctions.

<sup>3</sup>See [www.powerauction.com](http://www.powerauction.com) for more details on gas release programme auctions.

Independent System Operator allocates installed capacity payments through a sequence of monthly uniform price auctions<sup>4</sup>; and the Colombian system operator will procure forward electricity supply contracts to match the annual forecast electricity demand by means of a sequence of four quarterly auctions<sup>5</sup>.

The seller looks for the auction format that is best suited for achieving her main goals of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. There are several features of the market that should be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

When the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction, participation in the auction can be expected to be low, which tends to have a negative effect on expected revenues. For this reason, the seller might prefer a single auction over a sequence of auctions to keep transaction costs low. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. A sequence of sealed-bid auctions is somewhere between a single sealed-bid auction and an ascending auction in terms of the private information revealed through the auc-

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<sup>4</sup>See Installed Capacity Manual (2008), NYISO for more details on installed capacity auctions.

<sup>5</sup>See Cramton (2007) and [www.creg.gov.co](http://www.creg.gov.co) for more details on the Colombian electricity market.



tions. Hence, when there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenues. Since the price in an auction might be too high or too low due to some unexpected events, risk averse infra-marginal bidders (i.e. bid-takers) prefer a sequence of auctions over a single auction. If there is a single auction, infra-marginal bidders might end up paying too high or too low a price for all their purchases. But, in a sequence of auctions this risk is reduced since the prices bidders pay for their purchases are determined at several points in time. In the presence of risk averse bidders the seller might also prefer a sequence of sealed-bid auctions, since such auction format might increase the seller's expected revenues not only by increasing participation of risk averse bidders, particularly bid-takers, but also by encouraging marginal bidders to bid more aggressively due to a weaker winner's curse in a case with affiliated information<sup>6</sup>.

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions<sup>7</sup>. However, there is no theoretical

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<sup>6</sup>In the case of common-values with affiliated signals, the extra information that is revealed through the sequence of auctions reduce the winner's curse and the real risk imposed by aggressive bidding.

<sup>7</sup>See Weber (1983), Milgrom and Weber (1999), Ashenfelter (1989), McAfee and Vincent (1993), Bernhardt and Scoones (1994), Jeitschko (1999), Katzman (1999).

nor empirical research that studies sequences of divisible good auctions. But, in several real-world cases where sequences of auctions are used, such as those mentioned before, the auctioneer sells a divisible good. Moreover, we know from the case of a single auction, that divisible good auctions are not a trivial extension of single object auctions; hence one should not expect the results from sequential single object auctions to extend over to the case of sequential divisible good auctions. Therefore, studying strategic bidding in a sequence of divisible good auctions as well as the efficiency and revenue generation properties of this type of auctions is not only relevant from an academic perspective, but also from a practical standpoint.

This chapter studies a sequence of two uniform price auctions for a divisible good in a pure common value model with symmetric information and aggregate uncertainty. The unique profile of equilibrium bid functions in the second auction is fully characterized, as well as the entire set of equilibrium bid functions in the first auction. Using the characterization of equilibrium bidding, the revenue generation properties of the sequence of two uniform price auctions are compared with those of a single uniform price auction. A sequence of uniform price auctions was chosen over a sequence of pay-as-bid auctions because uniform price auctions are more widely used in energy and emission allowance markets, and there is a growing trend toward the use of this type of auctions in other markets.

Ausubel and Cramton (2002) show bidders in a uniform price auction have an incentive to shade their bids (i.e. reduce demand) in order to lower the price they pay for their purchases. This incentive grows with the quantity demanded and is inversely related to the size of bidders, measured by the maximum quantity they

want to buy. In each auction of a sequence of two uniform price auctions bidders have the same incentive to shade their bids, since spreading the supply over two auctions does not change the fact that a bidder behaves like a residual monopsonist. At the first auction of the sequence, bidders know that if they do not buy all the quantity they want in that auction, they still have another opportunity to do so in the second auction. Therefore, bidders discount their first auction bids by the option value of increasing their purchases in the second auction. This is similar to the case of a sequence of single object auctions, where bidders discount their bids in an auction by the option value of participating in later auctions (Milgrom and Weber (1999), Weber (1983), Bernhardt and Scoones (1994) and Jeitschko (1999)).

In a single uniform price auction or in the first auction of the sequence, the maximum quantity each bidder wants to buy (i.e. his demand) is exogenous. However, in the second auction of the sequence bidders' demands are endogenous, because they depend on the quantities bought in the first auction. Since the bid shading in the second auction depends on bidders' demands in that auction, bidders have an incentive to shape the bid shading in the second auction through their bidding in the first auction. In equilibrium, one bidder holds back in the first auction, by bidding lower prices than his competitors. In that way, this bidder reduces his competition in the second auction by letting the other bidders buy larger quantities in the first auction than otherwise. This feature of equilibrium will be called *dynamic* bid shading to differentiate it from the *static* bid shading described by Ausubel and Cramton (2002). The bidder who benefits the most from this strategic behavior is the largest bidder, because by having a larger demand he can profit the

most from the more intense bid shading in the second auction.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding are even stronger when the supply is split evenly between the two auctions of the sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions. These results are in line with the finding that it is better for the seller to use a sealed-bid auction than a dynamic auction when competition is not very strong (Cramton (1998) and Klemperer (2004)).

This is the first study of a sequence of divisible good auctions. The benefit of modeling sequential divisible good auctions is that it allows for the study of strategic forward looking bidding, which could have not been done by modeling a sequence of single object auctions with either unit or multi-unit demands, or even a sequence of multi-unit auctions with unit demands. Bidders bid in the first auction not only to buy some quantity at that stage, but also to improve their strategic position in the second auction. The improvement in a bidder's strategic position is not a consequence of the bidder strategically revealing information to manipulate his opponents' beliefs, but a consequence of the bidding and the quantity bought in the first auction.

When bidders have private information and multi-unit demands or non trivial demands in the case of divisible goods, bidders' beliefs might become asymmetric in any auction after the first one. This asymmetry might be problematic when analyzing sequential auctions. Most of the literature on sequential auctions, which studies sequence of single object auctions, avoids this problem by assuming unit demands, since the winner of an auction does not bid in subsequent auctions<sup>8</sup>. Exceptions to this are Katzman (1999) and Donald, Paarsch and Robert (2006). Katzman (1999) assumes two bidders with demand for two units, and deals with asymmetric bidders' beliefs by studying a sequence of two second price auctions, where the beliefs are irrelevant after the first auction, since the second price auction has a dominant strategy. Donald, Paarsch and Robert (2006) study a sequence of single-unit English auctions with multi-unit demands. They assume that the distribution of valuations is symmetric and remains identical across players, regardless of the number of units they have purchased in previous auctions. Another way of avoiding the problem of asymmetric beliefs is assuming pure common values with symmetric information. This assumption includes two different cases. In one case the value of the good on sale is known by every bidder. In the other case, the value is unknown but every bidder receives the same signal about it.

This chapter relates to a broad literature on how to create and enhance market power<sup>9</sup>. In any market, there are different ways of creating or enhancing market

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<sup>8</sup>See for example Milgrom and Weber (1999), Weber (1983), Bernhardt and Scoones (1994) and Jeitschko (1999).

<sup>9</sup>See Tirole (1988) for a survey on creation or enhancement of market power.

power. For example, firms can create barriers to entry, or create sub-markets either by independently differentiating their products from their competitors' products, or by explicitly coordinating on some type of market segmentation. The underlying idea on the different strategies to create or enhance market power is to profitably differentiate yourself from your potential or actual competitors. This is exactly what happens in a sequence of two uniform price auctions. *Dynamic bid shading* is a strategy that allows bidders to optimally differentiate themselves by splitting up the market into two less competitive markets.

The literature on auctions for split-award contracts studies the case in which a buyer divides the purchases of its input requirements into several (usually two) contracts that are awarded to different suppliers in separate auctions (Anton and Yao (1989, 1992), Perry and Sákovics (2003)). In a sequence of two uniform price auctions, the split or market segmentation, which is endogenous, is not complete (i.e. all bidders buy in both auctions) because of the uncertainty about the residual supply in the second auction. However, as chapter 2 shows for the case of forward trading ahead of a procurement uniform price auction, if bidders' expected profits from the first auction or market are zero, then one bidder, usually the largest one, will wait for the second auction or market even with uncertain residual supply.

This chapter also relates to a branch of the auction literature that studies auctions with aggregate uncertainty. On one side, Klemperer and Meyer (1989), Holmberg (2004, 2005) and Aromí (2006) study procurement uniform price auctions where firms sell a divisible good and demand is uncertain. This framework is known as the supply function framework since firms compete by submitting supply

functions. On the other side, Wang and Zender (2002) study standard divisible goods auctions in a common values model with random noncompetitive demand. The model in this chapter is closer to Wang and Zender's (2002) model than to the supply function models, not only because it studies a standard auction where the seller is the auctioneer, but also because it assumes a common values model with random noncompetitive demand.

The structure of this chapter is as follows. Section 2.2 describes the model of two sequential uniform price auctions. Section 2.3 develops the case with only two bidders, by first characterizing the equilibrium bid functions for the second and first auction, respectively, and then comparing the expected revenue in a sequence of two uniform price auctions with the expected revenue in a single uniform price auction. Section 2.4 does the same as section 2.3 but for the case of three bidders. Section 2.5 concludes.

## 2.2 Model

The seller has a quantity, normalized to one, of a perfectly divisible good for sale. She uses a sequence of two uniform price auctions, selling a quantity  $S_1$  in the first auction and a quantity  $S_2$  in the second auction, with  $S_1 + S_2 = 1$ . The price paid and the quantities bought by each bidder in the first auction is revealed before the second auction takes place. Resale between auctions is not allowed and it also is assumed the discount factor between both auctions is one.

Each bidder has a constant marginal value for the good, up to the maximum

quantity he wants to consume<sup>10</sup>. Moreover, this marginal value,  $v$ , is the same for all bidders and no bidder has private information. This last assumption includes two different cases. In one case, every bidder knows the true value of the good. In the other case, the value is unknown, but every bidder receives the same signal about the value of the good, and winning any quantity in the first auction does not provide any extra information. In this last case,  $v$  can be reinterpreted as the expected value conditional on the signal. For simplicity, it is assumed the seller derives no value for this good<sup>11</sup>.

There are  $N$  strategic bidders, each acting to maximize his expected profits. Each strategic bidder  $l$  wants to consume any quantity,  $q_l$ , up to  $\lambda_l$ , where  $\lambda_l > 0$ . Define  $\tilde{\lambda}$  as the second highest  $\lambda$ , and assume that  $\tilde{\lambda} \leq \frac{S_2}{N}$ . In the seminal analysis of divisible good auctions, Wilson (1979) demonstrated that uniform price auctions have a continuum of equilibria. As it will become clear later, the last assumption is key in reducing the set of possible equilibria up to the point of having a unique profile of equilibrium bid functions on the second auction. A strategy for strategic bidder  $l$  is a pair of piece-wise twice continuously differentiable demand functions, one for each auction,  $(d_{l1}(p), d_{l2}(p))$ , with  $d_{lt} : [0, \infty) \rightarrow [0, \lambda_l]$ .

There is also a continuum of measure 1 of non-strategic bidders, who can consume any quantity up to one. The bid of a non-strategic bidder is just a quantity<sup>12</sup>.

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<sup>10</sup>The constant marginal value assumption is made just for tractability. As it will become clear along this chapter, the results would hold even if the marginal values were decreasing.

<sup>11</sup>The results will not change as long as the seller has a lower value for the good than the bidders.

<sup>12</sup>This can be interpreted as a non-strategic bidder submitting a flat bid at a price of  $v$ , or just submitting a quantity and telling the auctioneer he will buy that quantity at whichever is the



Each one of these bidders has probability  $S_t$  of being assigned to auction  $t$ , with  $t = 1, 2$ . Once a non-strategic bidder is assigned to an auction he can only bid in that particular auction. All non-strategic bidders in auction  $t$  receive the same demand shock  $X_t$ , with  $x_t \in [0, 1]$ . Therefore, a non-strategic bidder in auction  $t$  bids for a quantity  $X_t$ . The demand shocks  $X_1$  and  $X_2$  are *i.i.d.* with  $G(x)$  representing the cumulative distribution function. The aggregate demand from non-strategic bidders in auction  $t$  is given by  $S_t X_t$  with  $S_t x_t \in [0, S_t]$ . Hence, strategic bidders' residual supply at auction  $t$ ,  $Y_t = S_t(1 - X_t)$ , is uncertain with  $F(y_t)$  representing its cumulative distribution function over the interval  $[0, S_t]$ . In a uniform price auction, generally there are multiple equilibria, which complicates the study of a sequence of this type of auctions. As a consequence of the uncertainty on the strategic bidders' residual supply created by the non-strategic bidders' random bidding, most, if not all, of the points on the strategic bidders' equilibrium demand functions will be characterized by equilibrium conditions in greater detail than otherwise<sup>13</sup>.

Since the auctions used by the seller are uniform price auctions, the price paid by bidders at an auction is the clearing price, which is defined as the highest losing bid. This price depends on strategic bidders' residual supply,  $y_t$ , and the demand functions submitted by all strategic bidders,  $p_t = \inf \{p \mid \sum_l d_{lt}(p) \leq y_t\}$ . If  $\sum_l d_{lt}(p_t) = y_t$ , then each strategic bidder  $l$  is assigned a quantity  $q_{lt}(y_t) = d_{lt}(p_t)$ .

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clearing-price.

<sup>13</sup>All the results would hold if instead of assuming the presence of non-strategic bidders it were assumed the supply is uncertain. However, in that case it would be hard to conceptualize the idea that the seller can spread the supply over a sequence of auctions.

If  $\sum_l d_{lt}(p_t) > y_t$ , then the demand curves of some bidders are discontinuous at  $p_t$  and they will be proportionally rationed at such price.

Before the first auction both types of bidders submit their demand functions for that auction to the auctioneer, who aggregates them and find the clearing-price for that auction. Then, after the outcome of the first auction is revealed, both type of bidders submit again their demand functions to the auctioneer, but this time for the second auction. The auctioneer again aggregates the demands and find the clearing-price for the second auction. The main difference between the first and second auction is the maximum quantity each strategic bidder wants to buy in each auction. Since they will likely buy some quantity in the first auction, the maximum quantity a strategic bidder wants to buy in an auction weakly decreases from the first to the second auction.

Given the information structure and the timing of the game, an equilibrium of this model is a profile of strategies, one for each strategic bidder, that defines a subgame perfect equilibrium (SPE) of the entire game. From now on, the word bidder(s) by itself will refer to strategic bidders, while the expression non-strategic bidders will still be used when referring to this other type of bidders.

### 2.3 Two-Bidder Case

This section analyzes the case where there are only two strategic bidders. Given the timing and information structure, the analysis will start focusing on the second auction and once equilibrium bidding in that auction is fully characterized,

the focus will shift to the first auction. As the reader probably already imagines the more interesting findings of this chapter are regarding equilibrium in the first auction and their effects on expected revenue. This is so because the incentives bidders face in the last auction of the sequence are indistinguishable from the incentives they would face in an otherwise identical single uniform price auction.

### 2.3.1 Second Auction

Once the auctioneer has announced the outcome of the first auction, but before the residual supply in the second auction,  $y_2$ , is known, bidders simultaneously choose their demand functions for the second auction. When doing this, bidder  $l$  maximizes his expected profit from the second auction conditional on the quantities purchased by each bidder in the first auction. Define  $q_{l1}(y_1)$  as the quantity bought by bidder  $l$  in the first auction when the residual supply was  $y_1$ . Bidder  $l$ 's optimization problem becomes:

$$\max_{d_{l2}(p_2)} E_2 [(v - p_2) d_{l2}(p_2)] \quad (2.1)$$

$$\text{s.t. } d_{l2}(p_2) \leq \lambda_l - q_{l1}(y_1) \quad (2.2)$$

The most important source of uncertainty in equation (2.1) is non-strategic bidders' demand in the second auction, which translates into uncertainty about the clearing price,  $p_2$ .

As mentioned above, a demand function for bidder  $l$  can be any piece-wise twice continuously differentiable decreasing function mapping from  $\mathfrak{R}_+$  to  $[0, \lambda_l]$ . However, as the next lemma shows, equilibrium demand functions in the second

auction are smooth functions in the interval  $(0, v)$ <sup>14</sup>.

**Lemma 1** *Equilibrium demand functions in the second auction are continuous for every price  $p \in (0, v)$ .*

**Proof.** First, clearly no bidder will bid more than  $v$ , and both bidders will bid  $v$  for their first unit. Now, define  $\underline{d}_{-l_2}(p^*) = \lim_{p \rightarrow p^{*+}} d_{-l_2}(p)$ ,  $\bar{d}_{-l_2}(p^*) = \lim_{p \rightarrow p^{*-}} d_{-l_2}(p)$ , and similarly for the aggregate demand,  $D_2(p)$ . Assume bidder  $-l$ 's demand is discontinuous at  $p^* \in (0, \bar{p}_2)$ . Then  $(\bar{d}_{-l_2}(p^*) - \underline{d}_{-l_2}(p^*)) > 0$ . For any interval  $[p^* - \epsilon, p^*]$  bidder  $l$  must demand additional quantity, otherwise bidder  $-l$  can profitably deviate by withholding demand at  $p^*$ . Define  $p^\epsilon(p^*) = \sup\{p \mid d_{l_2}(p) \geq \underline{d}_{l_2}(p^*) + \epsilon\}$ .

Bidder  $l$  can increase his expected profit by deviating and submitting the following demand function:

$$\tilde{d}_{l_2}^\epsilon(p) = \begin{cases} \underline{d}_{l_2}(p^*) + \epsilon & \text{if } p \in (p^\epsilon(p^*), p^* + \epsilon) \\ d_{l_2}(p) & \text{otherwise} \end{cases} \quad (2.3)$$

The effect of this deviation on expected profits can be split in two parts, an expected loss from higher prices,  $\Omega^\epsilon$ , and an expected gain from larger purchases,  $\Gamma^\epsilon$ .

The expected loss is bounded above by:

$$\Omega^\epsilon < (p^* + \epsilon - p^\epsilon(p^*))(\underline{d}_{l_2}(p^*) + \epsilon)Pr^\epsilon(\Delta p) \quad (2.4)$$

$Pr^\epsilon(\Delta p)$  is the probability that the price changes due to the deviation by bidder  $l$ ; and clearly it converges to zero as  $\epsilon$  does so. Hence, the derivative of the upper bound is zero at  $\epsilon = 0$ .

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<sup>14</sup>The idea for the proofs of the first three lemmas, or part of them, follows Aromí (2006).

Now, the expected gain,  $\Gamma^\epsilon$ , is bounded below by:

$$\Gamma^\epsilon > (v - p^* - \epsilon)\Delta E^\epsilon(q_{l2}) \quad (2.5)$$

$\Delta E^\epsilon(q_{l2})$  is the expected change in quantity bought by bidder  $l$  in the second auction. Clearly, the lower bound of the expected gain is zero at  $\epsilon = 0$ . In the case  $\bar{d}_{l2}(p^*) = \underline{d}_{l2}(p^*)$ :

$$\Delta E^\epsilon(q_{l2}) > (d_{l2}(p^*) - d_{l2}(p^* + \epsilon)) \left[ F(d_{l2}(p^*) + \bar{d}_{-l2}(p^*)) - F(d_{l2}(p^*) + \underline{d}_{-l2}(p^*)) \right] \quad (2.6)$$

However, if  $(\bar{d}_{l2}(p^*) - \underline{d}_{l2}(p^*)) > 0$ , then:

$$\begin{aligned} \Delta E^\epsilon(q_{l2}) &\geq \int_{D_2(p^*+\epsilon)}^{d_{l2}(p^*)+\epsilon+d_{-l2}(p^*+\epsilon)} (y_2 - D_2(p^* + \epsilon)) dF(y_2) \\ &+ \int_{\underline{D}_2(p^*)+\epsilon}^{\bar{D}_2(p^*)} \left[ \frac{\bar{d}_{l2}(p^*) - \underline{d}_{l2}(p^*) - \epsilon}{\bar{D}_2(p^*) - \underline{D}_2(p^*) - \epsilon} (y_2 - \underline{D}_2(p^*) - \epsilon) + \epsilon \right] dF(y_2) \\ &- \frac{\bar{d}_{l2}(p^*) - \underline{d}_{l2}(p^*)}{\bar{D}_2(p^*) - \underline{D}_2(p^*)} \int_{\underline{D}_2(p^*)}^{\bar{D}_2(p^*)} (y_2 - \underline{D}_2(p^*)) dF(y_2) \end{aligned} \quad (2.7)$$

The derivative of the expected change in quantity bought by bidder  $l$  in the second auction evaluated at  $\epsilon = 0$  is positive in both cases. Hence, the upper bound of the expected gain due to the deviation is strictly increasing at  $\epsilon = 0$ .

$$\lim_{\epsilon \rightarrow 0} \frac{\partial \Delta E^\epsilon(q_{l2})}{\partial \epsilon} \Big|_{\epsilon=0} = \int_{\underline{D}_2(p^*)}^{\bar{D}_2(p^*)} \frac{(\bar{d}_{-l2}(p^*) - \underline{d}_{-l2}(p^*)) (\bar{D}_2(p^*) - y_2)}{(\bar{D}_2(p^*) - \underline{D}_2(p^*))^2} dF(y_2) + c > 0 \quad (2.8)$$

where  $c$  is positive a constant. ■

Bidder  $l$  is a residual monopsonist whose residual supply is given by the residual supply both strategic bidders face and the demand from bidder  $-l$ :  $rs_{l2}(p_2) = y_2 - d_{-l2}(p_2)$ . Even knowing the demand from bidder  $-l$ , bidder  $l$ 's residual supply is uncertain due to the uncertainty about  $y_2$ . The goal of bidder  $l$  is to find

the demand function that maximizes his expected profits conditional on bidder  $l$ 's demand function. If bidder  $l$  could find the price-quantity points,  $(p_2, rs_{l2}(p_2))$ <sup>15</sup>, that maximize his ex-post profits for every possible realization of  $y_2$ , and that set of points could be described by a weakly decreasing demand function, then clearly that demand function would maximize his expected profits. Since the uncertainty only affects the location of bidder  $l$ 's residual supply and not its slope, there is always a weakly decreasing demand function that describe the set of ex-post optimal price-quantity points. A more technical proof of the equivalence between the ex-ante and ex-post maximizations can be found in appendix A.

When deciding how much to buy, a monopsonist looks for the quantity such that the marginal addition to his costs equals the marginal addition to his revenue. However, since he pays the same price for all the units he buys, this price is determined by the residual supply he faces, which is his average cost. Hence, a monopsonist pays a price lower than his marginal revenue. Now, a standard result in auctions with uniform pricing rules is that bidders reduce their demands or shade their bids. The reason for this behavior is found on the incentives faced by a monopsonist. The marginal revenue for a bidder is the marginal value he has for the good, and the marginal cost of his purchases is higher than his average cost (i.e. his residual supply) since he pays the same price for all the quantity he buys. Equation (2.9), which is the first order condition for bidder  $l$ , shows that the more inelastic is bidder  $l$ 's residual supply, the more he shades his bids.

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<sup>15</sup>Bidder  $l$  selects a price-quantity point on his residual supply curve for each realization of  $y_2$ . Hence, the price bidder  $l$  selects is the clearing-price.

$$v - p_2 = \frac{d_{l2}(p_2)}{-d'_{-l2}(p_2)} \quad (2.9)$$

In equilibrium, no bidder demands a strictly positive quantity at prices above  $v$ , or bid more than  $v$  for any quantity. Since bidder  $l$  buys  $q_{l1}(y_1)$  in the first auction, when the residual supply in that auction is  $y_1$ , the largest quantity he wants to consume in the second auction is given by  $\lambda_l - q_{l1}(y_1)$ . Define this quantity as  $\mu_l$  and the smallest unsatisfied demand after the first auction as  $\underline{\mu} = \min\{\mu_1, \mu_2\}$ .

The first order conditions for both bidders define a system of differential equations, which defines interior equilibrium bidding in the second auction<sup>16</sup>. However, since the only asymmetry between bidders is in the maximum quantity each bidder wants to buy, represented by the  $\lambda$ s, the system of first order conditions for an interior solution is symmetric, and defines the following differential equation:

$$d'_2(p_2) = -\frac{d_2(p_2)}{v - p_2} \quad (2.10)$$

The differential equation in (2.10) has multiple solutions, one for each possible pair of initial conditions. However, given the assumptions of the model, there is only one pair of initial conditions, and therefore, only one pair of demand functions in the second auction which can be part of an equilibrium. The following two lemmas describe these equilibrium initial conditions.

**Lemma 2** *In equilibrium, bidder  $l$  buys less than  $\lambda_l - q_{l1}(y_1)$  at any price above zero.*

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<sup>16</sup>Interior bidding means  $d_{l2}(p_2) \in (0, \mu_l)$ .

**Proof.** If equilibrium demand functions are strictly decreasing at every price in  $(0, v)$ , then no demand function will reach the quantity  $\lambda_l - q_{l1}(y_1)$  at a strictly positive price. Hence, showing that equilibrium demand functions are strictly decreasing at every price in that interval will prove this lemma.

If bidder  $l$  demands the same positive quantity at every  $p \in [p', p'']$ , there are two possible cases. First, if bidder  $-l$  demands additional quantity for that range of prices, then he can increase his expected profit by withholding demand at prices in  $[p', p'']$ . Second, if no bidder demands additional quantity at that range of prices, bidder  $l$  can withhold demand at every price in  $(p', p'' - \epsilon)$  and increase his expected profit. Define  $p_\epsilon(p') = \inf\{p \mid d_{l2}(p) \leq d_{l2}(p') - \epsilon\}$ .

For example, bidder  $l$  can deviate by submitting:

$$\widehat{d}_{l2}^\epsilon(p) = \begin{cases} d_{l2}(p_\epsilon(p')) & \text{if } p \in (p', p_\epsilon(p')) \\ [d_{l2}(p_\epsilon(p')), d_{l2}(p'')] & \text{if } p = p' \\ d_{l2}(p) & \text{otherwise} \end{cases} \quad (2.11)$$

The effect of this deviation on expected profit can also be split in two parts, an expected loss from lower purchases and an expected gain from lower prices. The expected loss is bounded above by:

$$\Omega^\epsilon < (v - p'')\epsilon (F(d_{l2}(p'') + d_{-l2}(p'')) - F(d_{l2}(p_\epsilon(p')) + d_{-l2}(p_\epsilon(p')))) \quad (2.12)$$

Moreover, the upper bound converge to zero as  $\epsilon$  converges to zero, and its derivative is also zero at  $\epsilon = 0$ . Now, the expected gain is bounded below by:

$$\Gamma^\epsilon > (p'' - p') d_{l2}(p_\epsilon(p')) (F(d_{l2}(p'') + d_{-l2}(p'')) - F(d_{l2}(p_\epsilon(p')) + d_{-l2}(p''))) \quad (2.13)$$



The lower bound also converges to zero as  $\epsilon$  converges to zero, and is strictly increasing in  $\epsilon$  at  $\epsilon = 0$ . Hence, equilibrium demand functions are strictly decreasing at any price in  $(0, v)$ . ■

**Lemma 3** *In the second auction, the equilibrium demand function of the bidder with the smallest unsatisfied demand,  $\lambda_l - q_{l1}(y_1)$ , is continuous at  $p = 0$ .*

**Proof.** Clearly, at a price of zero, every bidder demands the largest quantity he wants to consume. Moreover, at least one of the equilibrium demand functions has to be continuous at  $p = 0$ , otherwise any bidder would have the incentive to increase his demand at a price just above zero.

Now, assume the subscript  $j$  refers to the bidder who wants to consume the smallest quantity after auction one,  $\underline{\mu} = \mu_j$ , and  $i$  refers to the other bidder. Because of symmetric interior equilibrium bidding and the strict monotonicity of equilibrium demand functions, the equilibrium demand function of bidder  $i$  can not be continuous at zero, if that of bidder  $j$  is not. Hence, the equilibrium demand function of bidder  $j$  in the second auction is continuous at  $p = 0$ . ■

The intuition behind the proof of lemma (3) can be explained as follows. After the first auction, the maximum quantities both bidders want to consume might be asymmetric. If that is the case, in equilibrium, the bidder with the largest unsatisfied demand will not demand more than  $\underline{\mu}$  at any positive price, or bid more than zero for any quantity above it. If the residual supply in the second auction happens to be larger than  $2\underline{\mu}$ , then the bidder who has a strictly positive value for a quantity larger than  $\underline{\mu}$  becomes the marginal bidder, the one setting the price. Hence, his

optimal strategy is to bid a price of zero for any quantity above  $\underline{\mu}$ .

These initial conditions together with equation (2.10) define the equilibrium demand functions in the second auction; which once inverted give the following equilibrium bid function:

$$b_{l2}(q_{l2}; \mathbf{q}_1) = \begin{cases} v \left( 1 - \frac{q_{l2}}{\underline{\mu}} \right) & \text{if } q_{l2} < \underline{\mu} \\ 0 & \text{otherwise} \end{cases} \quad (2.14)$$

where  $\mathbf{q}_1 = (q_{11}(y_1), q_{21}(y_1))$ . As discussed before, both bidders bid symmetrically for any quantity up to  $\underline{\mu}$ . The demand reduction or bid shading in the second auction increases with the quantity demanded, but most importantly it increases as  $\underline{\mu}$  decreases. A decrease in the smallest unsatisfied demand in the second auction turns competition in this auction less intense, the smallest bidder becomes smaller. Hence, the residual supply that each bidder faces becomes more inelastic, which increases bid shading. This last feature of equilibrium bidding in the second auction is particularly interesting. In a single auction, the maximum quantity bidders want to buy is exogenous; however, such quantity becomes endogenous through out a sequence of auctions. Therefore, bidders can, and will, affect bid shading in the second auction through their bidding in the first auction.

The second auction equilibrium demand function of each bidder and the equilibrium price in that same auction are easily derived from equation (2.14). Define  $m = \min\{S_2, \lambda_1 + \lambda_2 - y_1\}$ . Then, bidder  $l$ 's equilibrium profit from the second auction, as a function of the residual supply in that auction and the purchases in

the previous auction, can be written as:

$$\pi_{l2}(y_2; \mathbf{q}_1) = \begin{cases} \frac{y_2^2 v}{4 \underline{\mu}} & \text{if } y_2 < 2\underline{\mu} \\ v\mu_l & \text{if } \underline{\mu} = \mu_l \text{ and } 2\underline{\mu} \leq y_2 \leq S_2 \\ v(y_2 - \underline{\mu}) & \text{if } \underline{\mu} < \mu_l \text{ and } 2\underline{\mu} \leq y_2 \leq m \\ v\mu_l & \text{if } \underline{\mu} < \mu_l \text{ and } m \leq y_2 \leq S_2 \end{cases} \quad (2.15)$$

### 2.3.2 First Auction

Now that bidders' equilibrium behavior in the second auction has been derived and understood, it is time to move backward and study equilibrium behavior in the first auction. At this stage, bidders simultaneously and independently choose the demand functions they will submit for the first auction. As in the case of the second auction analyzed before, bidders make their choices without knowing the demand from non-strategic bidders in the first auction, which means bidders do not know the supply left for them in that auction,  $y_1$ .

For a relevant realization of  $y_1$ , an increase in bidder  $l$ 's purchases in the first auction implies a decrease in bidder  $-l$ 's purchases in that same auction<sup>17</sup>. Moreover, since equilibrium bidding in the second auction depends on the smallest unsatisfied demand,  $\underline{\mu}$ , bidder  $l$ 's profit from the last auction in the sequence depends on the demand functions submitted in the first auction. For that reason, when selecting the demand function for the first auction, bidder  $l$  does not look for the demand that maximizes his expected profits from the first auction, but looks for the

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<sup>17</sup>If  $y_1 > \lambda_1 + \lambda_2$  and the increase in bidder  $l$ 's purchases in the first auction is smaller than  $y_1 - \lambda_{-l}$ , then the quantity bought by bidder  $-l$  in the first auction remains unchanged. However, this case is not relevant since both bidders will buy all they want in the first auction.

one that maximizes the expected value of his entire stream of profits. Hence, bidder  $l$ 's optimization problem becomes:

$$\max_{d_{l1}(p_1)} E_1 [(v - p_1) d_{l1}(p_1) + E_2 [\pi_{l2}(\mathbf{q}_1)]] \quad (2.16)$$

$$\text{s.t. } d_{l1}(p_1) \leq \lambda_l \quad (2.17)$$

In order to start characterizing the first auction equilibrium demand functions, the marginal change in bidder  $l$ 's expected profit from the second auction due to a marginal change in his own purchases in the first auction needs to be defined. Since  $d_{l1}(p_1) = y_1 - d_{-l1}(p_1)$  in equilibrium, this change can be expressed in terms of either  $q_{l1}$  or  $q_{-l1}$ <sup>18</sup>. But, as it will become clear later, it is more convenient to express the change in terms of  $q_{-l1}$ . Evidently, the effect of a change in demand reduction depends on whether, after the first auction, bidder  $l$  has the smallest unsatisfied demand or not.

$$E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right] = \begin{cases} - \int_0^{2\mu} \frac{y_2^2}{4} \frac{v}{\underline{\mu}^2} dF(y_2) + \int_{2\mu}^{S_2} v dF(y_2) & \text{if } \underline{\mu} = \mu_l \\ \int_0^{2\mu} \frac{y_2^2}{4} \frac{v}{\underline{\mu}^2} dF(y_2) + \int_{2\mu}^{S_2} v dF(y_2) & \text{if } \underline{\mu} = \mu_{-l} \end{cases} \quad (2.18)$$

If the quantity purchased by bidder  $l$  in the first auction decreases ( $q_{-l1}$  increases), there are two effects on bidder  $l$ 's expected profit from the second auction. On one side, bidder  $l$ 's expected profit from the second auction increases, as the second term in both lines of equation (2.18) shows. By decreasing the quantity he purchases in the first auction, bidder  $l$  increases the maximum quantity he wants to buy in the second auction,  $\mu_l$ . Moreover, the quantity he buys in the second auction actually increases only in the event that the clearing-price is zero, which happens

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<sup>18</sup>Consequently,  $\frac{\partial \pi_{l2}}{\partial q_{l1}} = -\frac{\partial \pi_{l2}}{\partial q_{-l1}}$ .

when  $y_2$  is greater than  $2\underline{\mu}$ . On the other side, bidder  $l$ 's expected profit from the second auction increases or decreases depending on whether bidder  $l$  has the largest unsatisfied demand or not after the first auction. In the case that  $\underline{\mu} = \mu_{-l}$ , the clearing price in the second auction decreases when  $y_2$  is smaller than  $2\underline{\mu}$ , increasing bidder  $l$ 's expected profit from the second auction, as the first term on the bottom line of (2.18) indicates. However, when  $\underline{\mu} = \mu_l$  and  $y_2$  is smaller than  $2\underline{\mu}$ , the effect on bidder  $l$ 's expected profit is the opposite since the clearing price increases.

For ease of notation, equation (2.18) will be rewritten as:

$$E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right] = \begin{cases} \gamma & \text{if } \underline{\mu} = \mu_l \\ \phi & \text{if } \underline{\mu} = \mu_{-l} \end{cases} \quad (2.19)$$

The following lemmas start characterizing the equilibrium demand functions in the first auction, by stating the conditions for them to be smooth and strictly monotonic. Define  $\bar{p}_1 = p_1(0)$ .

**Lemma 4** *Equilibrium demand functions in the first auction are continuous at any price  $p \in (0, \bar{p}_1)$ , as long as  $\underline{D}_1(p) < S_1$ .*

**Proof.** The proof of this lemma is just an extension of the proof of lemma 1. Therefore, instead of writing again the entire proof, only the differences between both cases will be pointed out and their consequences will be developed.

Assume bidder  $-l$ 's demand function is discontinuous at  $p^* \in (0, \bar{p}_1)$ , then  $\bar{d}_{-l1}(p^*) > \underline{d}_{-l1}(p^*)$ . As before, for any interval  $[p^* - \epsilon, p^*]$  bidder  $l$  must demand additional quantity, otherwise bidder  $-l$  can profitably deviate by withholding demand at  $p^*$ . Define  $p^\epsilon(p^*) = \sup\{p \mid d_{l1}(p) \geq \underline{d}_{l1}(p^*) + \epsilon\}$ . Observe that  $p^\epsilon(p^*)$  tends

to  $p^*$  as  $\epsilon$  tends to zero, and it equals  $p^*$  when  $d_{l1}(p)$  is also discontinuous at  $p^*$ . Bidder  $l$  can deviate by submitting a demand function with the same structure as that in equation (2.3). Obviously, this deviation will also yield a loss and a gain in expected profit from the first auction due to higher prices and larger purchases in that auction, respectively.

Assume  $S_1 \geq \bar{D}_1(p^\epsilon(p^*))$ . Then, the upper bound for the expected loss and the lower bound for the expected gain are those on equations (2.4) and (2.5), respectively, with the subscript referring to the auction changed to 1. Also, as it was shown in the proof of lemma 1, this deviation seems to be profitable for bidder  $l$ . However, since the deviation now takes place in the first auction, it also triggers a change in expected profits from the second auction. The change in bidder  $l$ 's expected profits caused by the impact this deviation has in equilibrium bidding in the second auction can be written as:

$$\Delta E_1 [\pi_{l2}] = \int_{D_1(p^* + \epsilon)}^{\bar{D}_1(p^\epsilon(p^*))} E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right] \Delta q_{-l1}(y_1) dF(y_1) \quad (2.20)$$

The derivative of bidder  $l$ 's expected profits from the second auction with respect to  $q_{-l1}$  can take any sign. Hence, bidder  $l$  can suffer an expected loss or an expected gain from the second auction due to his deviation. For ease of notation, the expected loss and gain will be represented by  $\Theta^\epsilon$  and  $\Psi^\epsilon$  respectively. The expected gain is bounded below by zero, by definition, and it is weakly increasing in  $\epsilon$ . Bidder  $l$ 's expected loss is bounded above by:

$$\Theta^\epsilon < \bar{M} (d_{l1}(p^*) - d_{l1}(p^* + \epsilon) + \epsilon) \left[ F(\bar{D}_1(p^\epsilon(p^*))) - F(D_1(p^* + \epsilon)) \right] \quad (2.21)$$

where  $\bar{M}$  is the  $\max_{y_1} E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right]$  when  $y_1 \in [D_1(p^* + \epsilon), \bar{D}_1(p^\epsilon(p^*))]$ . The upper

bound and its derivative with respect to  $\epsilon$  converge to zero as  $\epsilon$  does so.

Now, if  $\bar{D}_1(p^\epsilon(p^*)) > S_1 > \underline{D}_1(p^*)$ , then all the upper and lower bounds still approach zero as  $\epsilon$  does so. Moreover, the signs of their derivatives with respect to  $\epsilon$  remain unchanged. Hence, the deviation by bidder  $l$  is profitable. ■

**Lemma 5** *Equilibrium demand functions in the first auction are strictly decreasing at every price  $p \in (0, \bar{p}_1)$ , as long as  $D_1(p) \leq S_1$ .*

**Proof.** If bidder  $l$  demands the same positive quantity at every  $p \in [p', p'']$ , there are two possible cases. First, if bidder  $-l$  demands additional quantity for that range of prices, then he can increase his expected profit by withholding demand at prices in  $[p', p'']$ . Second, if no bidder demands additional quantity at that range of prices, bidder  $l$  can withhold demand at every price in  $(p', p'' - \epsilon)$  and increase his expected profit. Specifically, bidder  $l$  can deviate by submitting a demand function like the one in (2.11).

If  $S_1 \geq D_1(p')$ , then, after changing the subscript referring to the auction to 1, equations (2.12) and (2.13) represent the lower bound for the expected loss due to smaller purchases and the upper bound for the expected gain due to lower prices, respectively. Moreover, as shown on the proof of lemma 2, such deviation seems to be profitable for bidder  $l$ . However, since the deviation now takes place in the first auction, it also triggers a change in expected profits from the second auction. The change in bidder  $l$ 's expected profits caused by the impact this deviation has in equilibrium bidding in the second auction can be written as:

$$\Delta E_1 [\pi_{l2}] = \int_{D_1(p')-\epsilon}^{D_1(p')} E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right] \Delta q_{-l1}(y_1) dF(y_1) \quad (2.22)$$

In this case  $\Delta q_{-l1}(y_1)$  is positive. As mentioned before, the derivative of bidder  $l$ 's expected profits from the second auction with respect to  $q_{-l1}$  can take any sign. Hence, bidder  $l$  can suffer an expected loss or an expected gain from the second auction due to his deviation. For ease of notation, the expected loss and gain will be represented by  $\Theta^\epsilon$  and  $\Psi^\epsilon$  respectively. In this case, the expected gain is also bounded below by zero, by definition, and it is weakly increasing in  $\epsilon$ .

Now, bidder  $l$ 's expected loss is bounded above by:

$$\Theta^\epsilon < -\underline{M}\epsilon [F(D_1(p')) - F(D_1(p') - \epsilon)] \quad (2.23)$$

where  $\underline{M}$  is the  $\min_{y_1} E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right]$  when  $y_1 \in [D_1(p') - \epsilon, D_1(p')]$ , and it is negative when  $\Delta E_1 [\pi_{l2}]$  is negative. This upper bound and its derivative with respect to  $\epsilon$  converge to zero as  $\epsilon$  does so. Hence, equilibrium demand functions are strictly decreasing at any price in  $(0, \bar{p}_1)$  as long as  $D_1(p') \leq S_1$ . ■

Bidder  $l$  is not only a residual monopsonist in the second auction, but also in the first auction. As a consequence, bidder  $l$  can construct the demand function for the first auction that maximizes the expected value of his stream of profits by finding all the price-quantity points  $(p_1, rs_{l1}(p_1))$  that maximize his ex-post stream of profits for each possible realization of the residual supply in the first auction,  $y_1$ <sup>19</sup>. A detailed mathematical proof of this equivalence can also be found in the appendix. Another implication of bidder  $l$  being a residual monopsonist is that bidder  $l$  has the incentive to shade his bids in the first auction for the same reason as he does in the second auction of the sequence. Since that behavior also comes up

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<sup>19</sup>Ex-post in the first auction means after the realization of the residual supply in the first auction, but before the realization of the residual supply in the second auction.



in single uniform price auction, from now on it will be referred as *static* bid shading or demand reduction.

The first order condition that characterizes bidder  $l$ 's optimal interior bidding is:

$$-d'_{-l1}(p_1)v - (d_{l1}(p_1) - p_1 d'_{-l1}(p_1)) = -d'_{-l1}(p_1)E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right] \quad (2.24)$$

The intuition of equation (2.24) is better understood in terms of the ex-post maximization where bidder  $l$  selects the first auction clearing price,  $p_1$ , that maximizes his stream of ex-post profits conditional on  $y_1$  and bidder  $-l$ 's demand function,  $d_{-l1}(p_1)$ <sup>20</sup>. For a given  $y_1$ , if bidder  $l$  increases  $p_1$ , the quantity he buys in the first auction increases by  $-d'_{-l1}(p_1)$ . Hence, the left hand side of equation (2.24) represents the marginal change in profits from the first auction due to a marginal increase in  $p_1$ . The first term represents the marginal increase in value, while the terms inside the brackets represent the marginal increase in cost. When the clearing price in the first auction increases, bidder  $-l$  buys a smaller quantity in that auction, which affects the demand reduction in the second auction. The right hand side of equation (2.24) represents the expected marginal change in profit from the second auction due to the marginal change in the first auction clearing price. If there were a single auction, or this were the last auction of the sequence, then the last term on the right-hand side would be zero. Hence, when selecting his bid for the first auction, bidder  $l$  balances the marginal change in profit from the first auction with the expected marginal change in profit from the second auction.

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<sup>20</sup>Since in the ex-post maximization bidder  $l$  selects a price-quantity point on his residual supply curve,  $rs_{l1}(p_1)$  it is equivalent to think he selects a clearing price or a quantity.

In a single auction or in the last auction of the sequence, a bidder knows that is the only or the last chance he has to buy the quantity he wants. In the first auction of the sequence, a bidder knows that if he does not buy in the first all the quantity he wants, he still have another opportunity to buy, the second auction. In other words, in a sequence of auctions bidders have the option of buying later. Hence, bidders discount their bids in the first auction by the option value of increasing their purchases in the second auction. The option value for bidder  $l$  is given by the expected marginal change of his profit from the second auction to a change in the quantity he buys in the first auction, which in the two-bidder case analyzed here can be expressed as  $E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{-l1}} \right]$ . In addition, as equation (2.18) shows, the option value of increasing purchases in the second auction is larger for the bidder reaching the second auction with the largest unsatisfied demand than for the other bidder, due to the asymmetric effect on bid shading.

The *F.O.C.s* for both bidders define a system of differential equations, which characterizes interior equilibrium bidding. This system of differential equations does not have explicit solutions. Hence, the next step will be to characterize equilibrium bidding in as great detail as possible. The following proposition states that, as long as the residual supply in the first auction is smaller than  $\min\{S_1, \lambda_1 + \lambda_2\}$ , at the beginning of the second auction the unsatisfied demand of one of the bidders is always smaller than that from his opponent. As the proof of the proposition shows, the cause of the asymmetry can be found on bidders incentive to optimally intensify the demand reduction in the second auction.

**Proposition 1** *In a sequence of two uniform price auctions, bidders always reach the second auction with asymmetric unsatisfied demands. Moreover,  $\mu_j < \mu_i, \forall y_1 < \min\{S_1, \lambda_1 + \lambda_2\}$ , with  $i, j \in \{1, 2\}$  and  $i \neq j$ .*

**Proof.** Define  $\Pi_{11} = \pi_{11} + E_2[\pi_{12}]$ . Assume without loss of generality bidder 2 uses  $d_{21}(p)$ . In equilibrium,  $d_{11}(p_1) = y_1 - d_{21}(p_1)$ . Then, bidder 1's unsatisfied demand after the first auction can be written as  $\mu_1 = \lambda_1 - d_{11}(p_1) = \lambda_1 - y_1 + d_{21}(p_1)$ . Now, define  $\tilde{p}_1$  as the clearing price in the first auction such that  $\lambda_2 - d_{21}(\tilde{p}_1) = \lambda_1 - y_1 + d_{21}(\tilde{p}_1) = \underline{\mu}$ . Then,  $\lim_{p_1 \rightarrow \tilde{p}_1^-} E_2 \left[ \frac{\partial \pi_{12}}{\partial d_{21}} \right] > \lim_{p_1 \rightarrow \tilde{p}_1^+} E_2 \left[ \frac{\partial \pi_{12}}{\partial d_{21}} \right]$ , which implies,  $\lim_{p_1 \rightarrow \tilde{p}_1^-} \frac{\partial \Pi_{11}}{\partial p_1} < \lim_{p_1 \rightarrow \tilde{p}_1^+} \frac{\partial \Pi_{11}}{\partial p_1}$ . Hence, it is never optimal for bidder 1 to select  $\tilde{p}_1$ . Therefore, in equilibrium, bidders can not reach the second auction with identical unsatisfied demands.

By definition, the subscript  $j$  refers to the bidder who reaches the second auction with the smallest unsatisfied demand, while the subscript  $i$  refers to the other bidder,  $\underline{\mu} = \mu_j < \mu_i$ . Bidder  $j$  can be either bidder 1 or 2. Now, it needs to be proved that the bidder labeled as  $j$  is the same bidder for every realization of the residual supply in the first auction that is smaller than  $\min\{S_1, \lambda_1 + \lambda_2\}$ . Continuity of the equilibrium demand functions in the first auction ensure the bidder with the smallest unsatisfied demand after the first auction, bidder  $j$ , is the same bidder (either 1 or 2) for all  $y_1 < \min\{S_1, \lambda_1 + \lambda_2\}$ . If the bidder labeled as  $j$  were a different bidder depending on the realization of  $y_1$ , then there would exist at least one price,  $p_1$ , for which both bidders' unsatisfied demands would be identical. However, that can not happen in equilibrium. ■

A conclusion that can be easily drawn from proposition 1 is that there is no symmetric equilibrium in the first auction when bidders are symmetric. The next proposition generalizes this result by showing that no matter whether the bidders are symmetric or not, equilibrium bidding in the first auction is asymmetric for all  $p \in (\underline{p}_1, \bar{p}_1)$ , where  $\bar{p}_1 = p_1(0)$  and  $\underline{p}_1 = p_1(S_1)$ .

**Proposition 2** *In equilibrium, bidder  $i$  bids less aggressively than bidder  $j$  in the first auction:  $d_{j1}(p) > d_{i1}(p) \forall p \in (\underline{p}_1, \bar{p}_1)$ .*

**Proof.** When  $\lambda_j \geq \lambda_i$ , the result comes trivially from proposition 1. So, the case that needs to be proved is when  $\lambda_j < \lambda_i$ .

First, assume both bidders' equilibrium demand functions in auction one are identical for every strictly positive price. Then,  $\frac{\partial \Pi_{j1}}{\partial p_1}$  and  $\frac{\partial \Pi_{i1}}{\partial p_1}$  equal zero  $\forall p_1 \in (\underline{p}_1, \bar{p}_1)$ . However,

$$\begin{aligned} \frac{\partial \Pi_{j1}}{\partial p_1} &= -d_1(p_1) - d'_1(p_1)(v - p_1) + d'_1(p_1)\gamma \\ &> -d_1(p_1) - d'_1(p_1)(v - p_1) + d'_1(p_1)\phi \\ &= \frac{\partial \Pi_{i1}}{\partial p_1} \end{aligned}$$

where the inequality comes from  $\phi > \gamma$  and  $d'_1(p_1)$  being negative. Therefore, both bidders' equilibrium demand functions in auction one are not identical at every strictly positive price.

The next step is to prove the equilibrium demand functions do not cross each other. First, it will be shown that in equilibrium  $d_{i1}(p)$  is continuous at  $\bar{p}_1$ , but not  $d_{j1}(p)$ . By definition  $\bar{p}_1$  is the  $\inf\{p \mid d_{i1}(p) = 0 \text{ and } d_{j1}(p) = 0\}$ . First, both

demand functions can not be discontinuous at  $\bar{p}_1$  otherwise any bidder can deviate by using a deviation like the one on lemma 4. Second, assume  $d_{j1}(p)$  is continuous at  $\bar{p}_1$ . If  $d_{i1}(p)$  is either continuous or discontinuous at  $\bar{p}_1$ , then  $\lim_{p \rightarrow \bar{p}_1^-} \frac{\partial \Pi_{i1}}{\partial p_1} = -(v - \bar{p}_1 - \phi) \lim_{p \rightarrow \bar{p}_1^-} d'_{j1}(p) - \lim_{p \rightarrow \bar{p}_1^-} d_{i1}(p) = 0$ . Now, if  $d_{i1}(p)$  is continuous (discontinuous) at  $\bar{p}_1$ , then  $(v - \bar{p}_1 - \phi)$  is zero (positive). Hence,  $(v - \bar{p}_1 - \gamma)$  is strictly positive since  $\phi > \gamma$ , which implies  $\lim_{p \rightarrow \bar{p}_1^-} \frac{\partial \Pi_{j1}}{\partial p_1} = -(v - \bar{p}_1 - \gamma) \lim_{p \rightarrow \bar{p}_1^-} d'_{i1}(p) > 0$ . Therefore,  $d_{i1}(p)$  is continuous at  $\bar{p}_1$ , but not  $d_{j1}(p)$ .

Since  $d_{i1}(\bar{p}_1) = 0$  and  $\bar{d}_{j1}(\bar{p}_1) > 0$ , the equilibrium demand function of bidder  $j$  can only cross that of bidder  $i$  from above. Assume the following:

$$\begin{aligned} d_{i1}(p) &< d_{j1}(p) & \text{if} & & p > \tilde{p} \\ d_{i1}(p) &= d_{j1}(p) & \text{if} & & p = \tilde{p} \\ d_{i1}(p) &> d_{j1}(p) & \text{if} & & p < \tilde{p} \end{aligned}$$

Then  $d'_{i1}(\tilde{p}) < d'_{j1}(\tilde{p})$ , and

$$\begin{aligned} \left. \frac{\partial \Pi_{j1}}{\partial p_1} \right|_{\tilde{p}} &= -d_{j1}(\tilde{p}) - d'_{i1}(\tilde{p})(v - \tilde{p} - \gamma) \\ &> -d_{j1}(\tilde{p}) - d'_{i1}(\tilde{p})(v - \tilde{p} - \phi) \\ &> -d_{i1}(\tilde{p}) - d'_{j1}(\tilde{p})(v - \tilde{p} - \phi) \\ &= \left. \frac{\partial \Pi_{i1}}{\partial p_1} \right|_{\tilde{p}} \end{aligned}$$

Hence, the equilibrium demand functions do not cross each other. Consequently, since  $d_{i1}(\bar{p}_1) = 0$  and  $\bar{d}_{j1}(\bar{p}_1) > 0$ , then  $d_{i1}(p) < d_{j1}(p)$  for all  $p \in (\underline{p}_1, \bar{p}_1)$ .

■

Propositions 1 and 2 state a quite interesting feature of equilibrium bidding in a sequence of uniform price auctions which is not found in sequences of single object

auctions. In the first auction of a sequence of two uniform price auctions bidders not only internalize they have another option for buying their desired quantity, but also internalize they can affect the intensity of bid shading in the second auction through their bidding in the first auction. Hence, in the first auction of the sequence bidder  $i$  bids lower prices than bidder  $j$ , allowing bidder  $j$  to buy a larger quantity in that auction than otherwise. This strategy is profitable for bidder  $i$  because even though he buys a lower quantity in the first auction, he then benefits from weaker competition in the second auction, which translates into larger bid shading in the last auction of the sequence<sup>21</sup>. This characteristic of equilibrium bidding will be called *dynamic bid shading* since it is a consequence of the dynamic feature of a sequence of auctions, and also, to differentiate it from the *static bid shading* that comes up even in a single uniform price auction.

The idea behind *dynamic bid shading* relates to a broad literature on how to create or enhance market power. In any market, there are different ways of creating or enhancing market power. For example, firms can create barriers to entry, or create sub-markets either by independently differentiating their products from their competitors' products, or by explicitly coordinating on some kind of market segmentation. The underlying idea on the different strategies to create or enhance market power is to profitably differentiate yourself from your potential or

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<sup>21</sup>Bidder  $i$  not only buys a lower quantity in the first auction, but also pays a lower price in that auction. However, what makes this strategy profitable is the higher expected profit bidder  $i$  can reap from the second auction. Otherwise, there would be asymmetric bid shading in the last auction of the sequence and even in single uniform price auction.

actual competitors. This is exactly what happens in a sequence of two uniform price auctions. *Dynamic bid shading* is a strategy that allows bidders to optimally differentiate themselves by splitting up the market into two less competitive markets. There is no full market segmentation, where bidder  $j$  buys only in the first auction and bidder  $i$  waits for the second auction, because of the uncertainty about the residual supply in the second auction. However, as Herrera-Dappe (2008) shows for the case of forward trading ahead of a uniform price auction, if bidders make no profits from the first auction or market, then bidder  $i$  will wait for the second auction or market even with uncertain residual supply.

If the highest possible residual supply in the first auction,  $S_1$ , is smaller than the maximum quantity both bidders want to consume,  $\lambda_1 + \lambda_2$ , then the system of equations defined by the *F.O.C.s* in (2.24) only characterizes the equilibrium demand functions for prices in the interval  $[\underline{p}_1, \bar{p}_1]$ <sup>22</sup>. Now, it remains to extend both demand functions over  $[0, \underline{p}_1)$  in a way that none of these prices become clearing prices. This can be achieved by using any pair of decreasing twice continuously differentiable functions  $(\tilde{d}_{j1}(p), \tilde{d}_{i1}(p))$  defined over the interval  $[0, \underline{p}_1]$ , that satisfy  $\tilde{d}_{l1}(\underline{p}_1) = d_{l1}(\underline{p}_1)$  for  $l = i, j$  as well as the following inequalities for all  $p \in [0, \underline{p}_1)$ :

$$-(S_1 - \tilde{d}_{j1}(p)) - \tilde{d}'_{j1}(p)(v - p - \phi) > 0 \quad (2.25)$$

and

$$-(S_1 - \tilde{d}_{i1}(p)) - \tilde{d}'_{i1}(p)(v - p - \gamma) > 0 \quad (2.26)$$

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<sup>22</sup>When  $\underline{p}_1$  equals zero, the interval is open at  $\underline{p}_1$ ; because the equilibrium demand functions are not necessarily continuous at zero and  $d_{l1}(0) = \lambda_l$  for all  $l$ .

The left-hand sides on (2.25) and (2.26) are the derivatives of bidder  $i$  and  $j$ 's ex-post stream of profits<sup>23</sup> with respect to the price in the first auction, evaluated using the market-clearing condition and  $y_1 = S_1$ .

According to proposition 2 bidder  $j$  bids more aggressively than bidder  $i$  in any equilibrium of the first auction. However, nothing has been said about the identity of these bidders. If bidders are symmetric, clearly for every pair of equilibrium bid functions there will be two almost identical equilibria, where the only difference between them will be bidders' identity. However, as the maximum quantities bidders want to buy become more asymmetric, bidding lower prices in the first auction becomes less profitable for the smaller bidder. The reason is the smaller bidder's unsatisfied demand after the first auction becomes smaller, leaving him with less quantity to profit from the more intense bid shading in the second auction. Define  $\underline{\lambda}_i$  as the lowest demand of bidder  $i$  for an equilibrium to exist. Clearly,  $\underline{\lambda}_i$  depends on the demand of bidder  $j$ , the marginal value of the good, the split of the supply and the distributions of the second auction residual supply. If bidders demands are such that  $\lambda_2 \in [\underline{\lambda}_1, \lambda_1^{-1}(\underline{\lambda}_2)]$ , then there are two equilibria, one with  $j = 1$  and another with  $j = 2$ . But, when bidders are so asymmetric that  $\lambda_2$  lies outside of that interval, then there is only one equilibrium and the bidder holding back in the first auction (i.e. bidder  $i$ ) is the larger bidder. As Table 2.1 shows, bidders do not have to be too different for only one equilibrium to exist.

The expected marginal change in bidder  $l$ 's profit from the second auction

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<sup>23</sup>Remember ex-post in this case means after the realization of the residual supply in the first auction, but before the realization of the residual supply in the second auction.



Table 2.1: First auction equilibria when  $Y_2 \sim U[0, 1 - S_1]$

$S_1$	$v$	$\underline{\lambda}_i^a$	$\lambda_j$	$\bar{p}_1$	$d_{j1}(\bar{p}_1)$	$d_{i1}(0)$	$\bar{\Delta}^b$	$\underline{\Delta}^c$
0.5	10	0.165	0.18	3.83	0.036	0.111	0.069	0.036
0.5	10	0.092	0.10	2.13	0.020	0.061	0.039	0.020
0.5	10	0.230	0.25	5.32	0.053	0.154	0.096	0.060
0.5	20	0.165	0.18	7.67	0.036	0.111	0.069	0.036
0.5	20	0.092	0.10	4.26	0.020	0.061	0.039	0.020
0.5	20	0.230	0.25	10.65	0.053	0.154	0.096	0.060
0.36	10	0.165	0.18	2.99	0.036	0.111	0.069	0.036
0.36	10	0.092	0.10	1.66	0.020	0.061	0.039	0.020
0.36	10	0.229	0.25	4.16	0.053	0.154	0.096	0.060
0.64	10	0.165	0.18	5.32	0.036	0.111	0.069	0.036
0.64	10	0.092	0.10	2.96	0.020	0.061	0.039	0.020

<sup>a</sup>Lowest  $\lambda_i$  for the equilibrium to exist. <sup>b</sup> $\bar{\Delta} = \max[d_{j1}(p) - d_{j1}(p)]$ .

<sup>c</sup> $\underline{\Delta} = \min[d_{j1}(p) - d_{j1}(p)]$ .

due to a change in the quantity bought by bidder  $-l$  in the first auction,  $\gamma$  or  $\phi$ , depends on the quantity purchased by bidder  $j$  in the first auction,  $q_{j1}$ <sup>24</sup>. The equation  $v - \check{p} - \gamma = 0$  defines a locus of price-quantity points,  $(\check{p}, q_1(\check{p}))$ , where equilibrium bidding has a particular feature. If the equilibrium demand function of bidder  $i$  is perfectly elastic at price  $\check{p}$ , then the optimal quantity demanded by bidder  $j$  at such price will be  $q_1(\check{p})$ . Following Klemperer and Meyer (1989), this locus will be called bidder  $i$ 's Bertrand locus. Similarly, the equation  $v - \hat{p} - \phi = 0$  defines bidder  $j$ 's Bertrand locus of price-quantity points,  $(\hat{p}, q_1(\hat{p}))$ . In this case, if the quantity demanded in equilibrium by bidder  $j$  at price  $\hat{p}$  equals  $q_1(\hat{p})$  and  $d_{i1}(\hat{p}) > 0$ , then the equilibrium demand function of bidder  $j$  will be perfectly elastic at price  $\hat{p}$ .

<sup>24</sup>Remember  $j$  can be either  $l$  or  $-l$ .

Bidder  $j$  equilibrium demand function in the first auction can not go through any point above or to the right of bidder  $j$ 's Bertrand locus; otherwise, bidder  $i$  would be demanding negative quantities or bidder  $j$ 's demand function would be increasing. Since  $\phi > \gamma$ , bidder  $j$ 's Bertrand locus is lower than bidder  $i$ 's Bertrand locus. Hence, bidder  $j$ 's Bertrand locus defines the upper bound of his equilibrium bids in the first auction<sup>25</sup>. The upper bound of bidder  $j$  first auction equilibrium bids becomes<sup>26</sup>:

$$\hat{b}_1^j(q_1) = \begin{cases} v \left( F(2\underline{\lambda}) - \int_0^{2\underline{\lambda}} \frac{y_2^2}{4\underline{\lambda}^2} dF(y_2) \right) & \text{if } q_1 \in [0, \lambda_j - \underline{\lambda}] \\ v \left( F(2(\lambda_j - q_1)) - \int_0^{2(\lambda_j - q_1)} \frac{y_2^2}{4(\lambda_j - q_1)^2} dF(y_2) \right) & \text{if } q_1 \in [\lambda_j - \underline{\lambda}, \lambda_j] \\ 0 & \text{otherwise} \end{cases} \quad (2.27)$$

In addition, since bidder  $j$  buys a larger quantity than bidder  $i$  in the first auction, and the difference is at least  $\lambda_j - \underline{\lambda}$ , the upper bound of bidder  $i$ 's first auction equilibrium bids becomes:  $\hat{b}_1^i(q_1) = \hat{b}_1^j(q_1 + \lambda_j - \underline{\lambda})$ .

The upper bounds of bidders first auction bids take into account the discounting of bidder  $i$ 's option value of increasing his purchases in the second auction,  $\phi(q_1)$ .

However, they do not fully take into account the *dynamic* bid shading that takes

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<sup>25</sup>The way demand functions were extended over the whole domain of prices ensures bids are also bounded above by bidder  $j$ 's Bertrand locus for all  $q_1 \in (d_{j1}(p_1), \lambda_j)$ . However, this is actually irrelevant since those bids are never going to be realized.

<sup>26</sup>Since bidder  $j$  is the bidder with the smallest unsatisfied demand after auction one, if  $\lambda_j > \underline{\lambda}$ , then there is no equilibrium with  $q_{j1} \in [0, \lambda_j - \underline{\lambda}]$ . Hence,  $\hat{b}_1(q_1) = \hat{b}_1(\lambda_j - \underline{\lambda})$  for all  $q_1 \in [0, \lambda_j - \underline{\lambda}]$ .

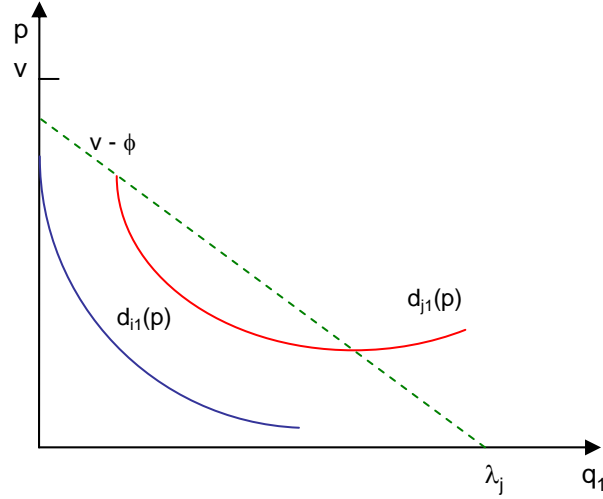


Figure 2.1: Bidder  $j$ 's Bertrand locus<sup>28</sup>

place in the first, and they completely ignore the *static* bid shading in that auction<sup>27</sup>. Since the option value is positive, the terms in brackets in (2.27) are smaller than one; which means the upper bounds of equilibrium bids are smaller than  $v$ . Moreover, when bidder  $j$  buys his maximum demand,  $\lambda_j$ , in the first auction, bidder  $i$ 's option value equals the value of the good; because if he were to buy some quantity in the second auction, he would pay zero for it, since he would become the only bidder submitting a bid in that auction. In addition, the highest equilibrium bid in the first auction is not higher than  $v \left( F(2\lambda) - \int_0^{2\lambda} \frac{y_2^2}{4\lambda^2} dF(y_2) \right)$ .

As mentioned before, the *F.O.C.s* for both bidders define a system of differential equations, which characterizes interior equilibrium bidding. There is not one but multiple pairs of demand functions that solve that system of differential equa-

<sup>27</sup>Only when  $\lambda_j > \underline{\lambda}$  bidder  $i$ 's upper bound takes into account a fraction of the *dynamic* bid shading.

tions. The same problem came up in the second auction with equation (2.10). In that case, the existence of a unique equilibrium was ensured by assuming the smaller bidder will be able to buy, with strictly positive probability, as much as he wanted in the second auction ( $\underline{\lambda} \leq \frac{S_2}{2}$ ). Due to the asymmetric bidding in the first auction, an equivalent assumption in this case would be to assume  $S_1$ , the highest possible residual supply in the first auction, is non smaller than  $2\lambda_j$ .

**Proposition 3** *For any given pair of demands  $(\lambda_i, \lambda_j)$ , there exists a unique profile of equilibrium demand functions in the first auction  $(d_{i1}(p), d_{j1}(p))$ , if  $S_1 \geq 2\lambda_j$  and  $f(u) \geq \int_0^u \frac{y_2^2}{u^3} dF(y_2)$ .*

**Proof.** Assume  $S_1 \geq 2\lambda_j$ . Clearly, at a price of zero every bidder demands the largest quantity he wants to consume. Since both demand functions can not be discontinuous at a price of zero and  $\mu_i < \mu_j$ , then  $\underline{d}_{i1}(0) \equiv \lim_{p \rightarrow 0} d_{i1}(p) < \lambda_i$  and  $\underline{d}_{j1}(0) \equiv \lim_{p \rightarrow 0} d_{j1}(p) = \lambda_j$ . Also,  $d_{j1}(p) < \lambda_j$  for all strictly positive prices. Otherwise, bidder  $j$ 's demand function would cross the upper bound. Hence, the price-quantity points  $(0, \underline{d}_{i1}(0))$  and  $(0, \lambda_j)$  are the bottom conditions for the equilibrium demand functions of bidders  $i$  and  $j$  respectively. Each profile of equilibrium demand functions also has a pair of top conditions  $(\bar{p}, 0)$  and  $(\bar{p}, d_{j1}(\bar{p}))$  defined by the equation  $v - \bar{p} - \phi = 0$ , where  $\bar{p} = \inf\{p | d_{i1}(p) = 0\}$ .

Assume  $(d_{i1}(p), d_{j1}(p))$  is a pair of equilibrium demand functions with top conditions  $(\bar{p}^a, d_{j1}(\bar{p}^a))$ . Also, assume  $(\tilde{d}_{i1}(p), \tilde{d}_{j1}(p))$  is another pair of equilibrium demand functions, but with top conditions  $(\bar{p}^b, \tilde{d}_{j1}(\bar{p}^b))$ , where  $\bar{p}^a > \bar{p}^b$  and  $d_{j1}(\bar{p}^a) < \tilde{d}_{j1}(\bar{p}^b)$ . Since  $\tilde{d}_{j1}(\bar{p}^b) > d_{j1}(\bar{p}^b)$  and  $\gamma(\tilde{d}_{j1}(\bar{p}^b)) > \gamma(d_{j1}(\bar{p}^b))$ , with the last inequality

coming from the assumption that  $f(u) \geq \int_0^u \frac{y_2^2}{u^3} dF(y_2)$ , then  $\tilde{d}'_{i1}(\bar{p}^b) < d'_{i1}(\bar{p}^b)$ .

There exists a price  $\bar{p}^b - \epsilon$  such that  $\tilde{d}_{i1}(\bar{p}^b - \epsilon) = d_{i1}(\bar{p}^b - \epsilon)$ . Moreover, at that price  $\tilde{d}'_{i1}(\bar{p}^b - \epsilon) < d'_{i1}(\bar{p}^b - \epsilon)$ , which implies  $\tilde{d}_{j1}(\bar{p}^b - \epsilon) > d_{j1}(\bar{p}^b - \epsilon)$  since the elasticity of bidder  $i$ 's demand function at a given price increases with the quantity demanded by bidder  $j$ . Therefore,  $\tilde{d}'_{j1}(\bar{p}^b - \epsilon) < d'_{j1}(\bar{p}^b - \epsilon)$ . Hence, since the slopes of the demand functions are monotonic to the top conditions, there is a unique set of bottom conditions for each set of top conditions. Therefore, there is a unique pair of equilibrium demand functions in the first auction. ■

### 2.3.3 Revenue Comparison

When choosing among several auction formats, the seller looks for the auction format that is best suited for achieving her main objectives of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. In this chapter the number of winners is not an issue since the seller is assumed to be unconcerned about the after auction market. Also, efficiency is not an issue for this seller since all the bidders are assumed to have the same value for the good.

When the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction, participation in the auction can be expected to be low, which tends to have a negative effect on expected revenues. For this reason, the seller might prefer a single auction over a sequence of auctions

to keep transaction costs low. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. A sequence of sealed-bid auctions is somewhere between a single sealed-bid auction and an ascending auction, in terms of the private information revealed through the auctions. Hence, when there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenues. Since the price in an auction might be too high or too low due to some unexpected events, risk averse bidders prefer a sequence of auctions over a single auction. If there is a single auction, bidders might end up paying too high or too low a price for all their purchases. But, in a sequence of auctions this risk is reduced since the prices bidders pay for their purchases are determined at several points in time. In the presence of risk averse bidders the seller might also prefer a sequence of sealed-bid auctions, since such auction format might increase the seller's expected revenues not only by increasing participation of risk averse bidders, but also by encouraging them to bid more aggressively due to a weaker winner's curse in a case with affiliated information<sup>29</sup>.

The characterization of equilibrium bidding in the sequence of two uniform price auctions showed that even in an environment without transaction costs, bud-

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<sup>29</sup>In the case of common-values with affiliated signals, the extra information that is revealed through the sequence of auctions reduce the winner's curse and the real risk imposed by aggressive bidding.

get or borrowing constraints, where bidders are risk neutral and the revelation of information is not an issue, a single uniform price auction and a sequence of two uniform price auctions most likely differ in terms of the expected revenues they yield. An ideal scenario for comparing the expected revenues from a single uniform price auction and a sequence of two uniform price auctions would be one with analytical solutions for the equilibrium bid functions. However, as it was discussed before, that is not the case here. Therefore, the approach will be to define an upper bound of the expected revenue in a sequence of two uniform price auctions using the upper bounds of equilibrium bids and then compare it with the expected revenue in a single uniform price auction.

From equation (2.14), the equilibrium price in the second auction as a function of the residual supply in that auction and conditional on the residual supply in the first auction becomes:

$$p_2(y_2 | y_1) = \begin{cases} v \left[ 1 - \frac{y_2}{2(\lambda_j - q_{j1}(y_1))} \right] & \text{if } y_2 < 2(\lambda_j - q_{j1}(y_1)) \\ 0 & \text{otherwise} \end{cases} \quad (2.28)$$

In equilibrium, bidder  $j$  buys more than half the residual supply in the first auction plus  $(\lambda_j - \underline{\lambda})/2$ , as long as the residual supply is smaller than  $\lambda_j + \underline{\lambda}$ . Also, since the equilibrium price in equation (2.28) is decreasing in  $q_{j1}$ , the following is an

upper bound of the equilibrium revenue in a second auction<sup>30</sup>:

$$\widehat{R}_2(y_2 | y_1) = \begin{cases} vS_2 \left[ 1 - \frac{y_2}{\lambda_j + \underline{\lambda} - y_1} \right] & \text{if } y_2 < \lambda_j + \underline{\lambda} - y_1 \\ 0 & \text{otherwise} \end{cases} \quad (2.29)$$

Define  $\Delta^- = \lambda_j - \underline{\lambda}$  and  $\Delta^+ = \lambda_j + \underline{\lambda}$ . Using the upper bounds of individual bids in the first auction, the upper bound of the equilibrium revenue in the first auction becomes:

$$\widehat{R}_1(y_1) = \begin{cases} vS_1 \left( F(2\underline{\lambda}) - \int_0^{2\underline{\lambda}} \frac{y_2^2}{4\underline{\lambda}^2} dF(y_2) \right) & \text{if } y_1 \in [0, \Delta^-) \\ vS_1 \left( F(\lambda_j + \underline{\lambda} - y_1) - \int_0^{\lambda_j + \underline{\lambda} - y_1} \frac{y_2^2}{(\lambda_j + \underline{\lambda} - y_1)^2} dF(y_2) \right) & \text{if } y_1 \in [\Delta^-, \Delta^+) \\ 0 & \text{otherwise} \end{cases} \quad (2.30)$$

If the seller decides to run a single uniform price auction, equilibrium bidding in this auction will be similar to the bidding in the last auction of the sequence. The only difference is that the demand reduction will be driven by the smallest of the highest possible individual demands,  $\underline{\lambda}$ . Hence, the equilibrium revenue in a single auction can be written as:

$$\widehat{R}(y) = \begin{cases} v \left[ 1 - \frac{y}{2\underline{\lambda}} \right] & \text{if } y < 2\underline{\lambda} \\ 0 & \text{otherwise} \end{cases} \quad (2.31)$$

There is not much that can be said regarding the comparison of expected revenues without making any assumption on the distribution of the demand shock

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<sup>30</sup>If  $\lambda_j < \lambda_i$  and  $y_1 > 2\lambda_j$ , then  $q_{j1}(y_1) = \lambda_j$  might be smaller than half the residual supply in the first auction, and  $(\lambda_j - q_{j1}(y_1))$  would be larger than  $(\lambda_j - \frac{y_1}{2})$ . However, for those realizations of  $y_1$ , the equilibrium price in the second auction is zero, since bidder  $i$  is the only strategic bidder submitting a bid in that auction.



received by non-strategic bidders,  $G(x)$ . The following proposition states sufficient conditions for the expected revenue in a sequence of two uniform price auctions to be smaller than that from a single uniform price auction when the demand shocks are uniformly distributed.

**Proposition 4** *When the demand shocks are uniformly distributed, the sequence of two uniform price auctions yields lower expected revenues than a single uniform price auction if:*

$$\frac{S_1+3}{6(1-S_1)\underline{\lambda}} \left( \lambda_j + \underline{\lambda} - \frac{S_1}{2} - \frac{(\lambda_j - \underline{\lambda})^2}{2S_1} \right) < 1 \quad \text{for} \quad \lambda_j - \underline{\lambda} < S_1 < 2\lambda_j$$

$$\frac{S_1+3}{12(1-S_1)S_1\underline{\lambda}} \left( 6\lambda_j\underline{\lambda} - \lambda_j^2 - \underline{\lambda}^2 \right) < 1 \quad \text{for} \quad 2\lambda_j \leq S_1$$

**Proof.** If the demand shocks are uniformly distributed, then the residual supply in any auction is also uniformly distributed. Hence, the expected revenue in a single uniform price auction becomes  $v\underline{\lambda}$ . Adding (2.29) and (2.30), taking expectations and dividing by the expected revenue in a single uniform price auction gives the left-hand side on both inequalities on proposition 4. The right-hand side comes from dividing the expected revenue in a single uniform price auction by itself. ■

Proposition 4 says: (i) When the smallest bidder is the one who bids higher prices in the first auction (i.e.  $\lambda_j = \underline{\lambda}$ ), and he demands less than 18.75% of the aggregate supply, a single uniform price auction yields higher expected revenue than any equilibrium of a sequence of two uniform price auctions. (ii) However, in any other case (i.e.  $\lambda_j = \underline{\lambda} > 0.1875$  or  $\lambda_j > \underline{\lambda}$ ) the upper bound of the expected revenue in a sequence of two uniform price auctions is higher than the expected revenue in a single uniform price auction for at least some values of  $S_1$ . Remember the upper

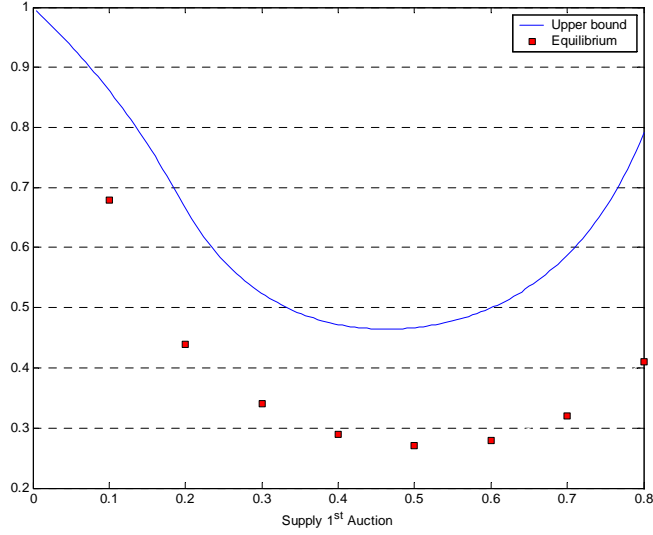


Figure 2.2: Ratio of  $E[Rev_{Seq}]$  to  $E[Rev_{Single}]$  when  $\lambda_i \geq \lambda_j = 0.1$

bound of the expected revenue in a sequence of two uniform price auctions ignores the *static* bid shading that takes place in the first, and it does not fully take into account the *dynamic* bid shading in that auction. Consequently, it does not fully take into account the *static* bid shading in second auction either.

When bidder  $j$  has the smallest demand of both bidders and his demand increases, bidder  $i$ 's option value of increasing his purchases in the second auction decreases. The reason is bid shading in the second auction will be smaller and its response to changes on the quantities purchased in the first auction will also be weaker. As a consequence, the upper bound of the expected revenue in the sequence of uniform price auctions increases more than the expected revenue in a single uniform price auction. The main difference between the cases where  $\lambda_j \geq \lambda_i$  and  $\lambda_j < \lambda_i$ , for a given  $\lambda_j$ , is that in the former case the expected revenue in a single auction is smaller than the upper bound of the expected revenue in a sequence

of auctions when the first auction is small. As  $\lambda_i$  decreases below  $\lambda_j$ , the expected price in a single uniform price auction decreases, since the smallest bidder becomes smaller, and so its expected revenue. The upper bounds of the expected equilibrium prices in the first and second auctions of a sequence also decrease, but their impact on the upper bound of the expected revenue is smaller for small values of  $S_1$ . Hence, in case (ii) the upper bound of expected revenue in a sequence of two uniform price auctions might not convey enough information since it does not fully take into account the bid shading that takes place in the first and second auctions. However, as Figure 2.3 shows, even in case (ii) there are some equilibria of the sequence of auctions that yield lower expected revenue than a single uniform price auction. Moreover, since the uniform distribution satisfies the condition in proposition 3, the equilibria in Figure 3 are not just random equilibria, but the unique equilibria for  $S_1 \geq 0.44$ . The same is true about the equilibria in Figure 2 for values of  $S_1$  greater than or equal to 0.2.

As long as  $\lambda_j < 0.215$ , the upper bound of the expected revenue in a sequence of two uniform price auctions is lower than the expected revenue in a single uniform price auction for some values of  $S_1$ . Moreover, the conditions in proposition 4 tell us that in such case the worst for the seller is to spread the supply fairly evenly over the two auctions in the sequence. In addition, the equilibria depicted on Figure 2.3 shows us that the same is true even when the upper bound of the expected revenue in a sequence is higher than the expected revenue in a single uniform price auction for all relevant supply splits. As the supply in the first auction increases, the expected price in the second auction conditional on the residual supply in the first auction

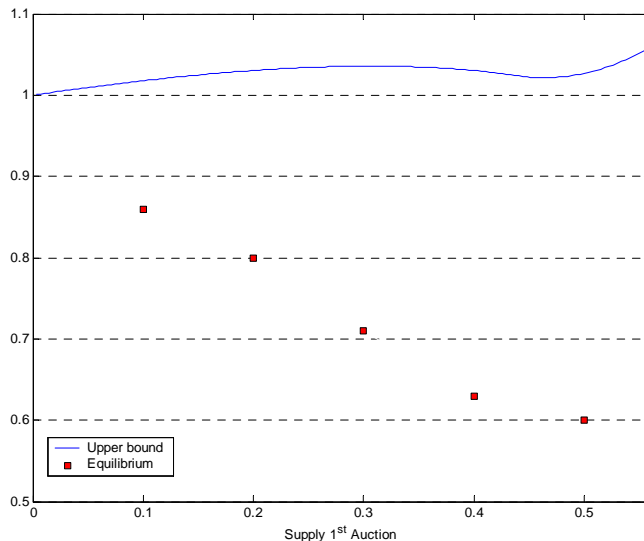


Figure 2.3: Ratio of  $E[Rev_{Seq}]$  to  $E[Rev_{Single}]$  when  $\lambda_i \geq \lambda_j = 0.22$

increases. Consequently, the option value of increasing the quantity purchased in the second auction decreases, which increases the price in the first auction for a given  $y_1$ . At the same time, a larger first auction supply increases the probability of low prices in both auctions at the expense of a reduction in the probability of high prices, also in both auctions. Hence, since  $Y_1$  and  $Y_2$  are identically distributed, and the uniform distribution is symmetric, these effects offset each other when the supply is evenly split between both auctions.

## 2.4 Three-Bidder Case

This section extends the analysis of the previous section to the case of three strategic bidders showing the results obtained when there are only two strategic bidders are not specific to that case. The reason for developing the three-bidder case and not the more general N-bidder case is just clarity of exposition, since the

mathematics in the latter case becomes entangled due to the asymmetries among bidders and the non-existence of symmetric equilibria even in the symmetric case. Moreover, the intuition seems to indicate the incentives and, therefore, the results found in the three-bidder case extend to the more general  $N$ -bidder case.

### 2.4.1 Second Auction

When three strategic bidders participate in a sequence of two uniform price auctions, the equilibrium demand functions in the second auction are continuous for every price  $p \in (0, v)$ . A deviation similar to the one used in lemma 1 for the two-bidder case can be used to rule out discontinuities or elastic segments on the equilibrium demand functions when there are three bidders.

As in the two-bidder case, the only difference among bidders is in the maximum quantity each bidder wants to consume, represented by the  $\lambda$ s. Hence, the system of first order conditions for an interior solution is symmetric, and defines the following differential equation:

$$(n - 1)d'_2(p_2) = -\frac{d_2(p_2)}{v - p_2} \quad (2.32)$$

where  $n$  represents the number of bidders whose demand constraint is not binding at  $p_2$ .

This differential equation also has multiple solutions, one for each possible pair of initial conditions. However, like before, the assumptions of the model guarantee there is only one pair of initial conditions and, therefore, only one pair of second auction demand functions which can be part of an equilibrium. The subscripts  $i, j$

and  $k$  will be used to label bidders according to their unsatisfied demands after the first auction:  $\mu_i \geq \mu_j \geq \mu_k$ . The following lemma describes the equilibrium initial conditions:

**Lemma 6** *In the second auction, in equilibrium:*

- (i) *The demand function of bidder  $k$  might reach his unsatisfied demand at a strictly positive price.*
- (ii) *Bidders  $i$  and  $j$  buy less than their unsatisfied demands at any price above zero.*
- (iii) *If only one bidder has the largest unsatisfied demand ( $\mu_i > \mu_j$ ), then his demand function is discontinuous at  $p = 0$ .*

**Proof.** (i) When there are three bidders participating in the auction, at any given price range the equilibrium demand functions of at least two bidders are strictly decreasing. If at least two of the bidders were demanding the same quantity for some price range, then a deviation like the one in lemma 2 would be profitable. However, since in this particular case, interior bidding is symmetric, equilibrium demand functions are strictly decreasing at least for interior quantities. Nevertheless, it is possible the demand function for bidder  $k$  reaches his unsatisfied demand at a strictly positive price. (ii) The proof of this point is identical to that of lemma 2.

(iii) Clearly, at a price of zero, every bidder demands the largest quantity he wants to consume. Moreover, at least two of the equilibrium demand functions have to be continuous at  $p = 0$ , otherwise some bidder would have the incentive to

increase his demand at a price just above zero.

When  $\mu_i > \mu_j$ , because of symmetric interior equilibrium bidding and the strict monotonicity of at least two equilibrium demand functions, the equilibrium demand function of bidder  $i$  can not be continuous at zero, if that of bidder  $j$  is not. Hence, the equilibrium demand function of bidder  $j$  in the second auction is continuous at  $p = 0$ . Therefore, bidder  $i$  becomes the marginal bidder for every  $y_2 \geq \mu_k + 2\mu_j$ , and his optimal strategy is to bid a price of zero for any quantity above  $\mu_j$ . ■

These initial conditions together with equation (2.32) define the equilibrium demand functions in the second auction; which once inverted give the following bid functions:

$$b_{l2}(q_{l2}; \mathbf{q}_1) = \begin{cases} v \left[ 1 - \frac{q_{l2}^2}{\mu_j \mu_k} \right] & \text{if } q_{l2} \in [0, \mu_k) \\ v \left[ 1 - \frac{q_{l2}}{\mu_j} \right] I(k) & \text{if } q_{l2} \in [\mu_k, \mu_j) \\ 0 & \text{otherwise} \end{cases} \quad (2.33)$$

where  $I(k)$  is an indicator function that equals zero if  $l = k$ , and one otherwise.

As discussed before, all three bidders bid symmetrically for any quantity up to  $\mu_k$ . While bidders  $i$  and  $j$  also bid symmetrically for any quantity in  $[\mu_k, \mu_j]$ . As expected, when the three bidders are active (i.e.  $d_{l2}(p) < \lambda_l$  for all  $l$ ), bidders shade their bids less than when there are only two active bidders. Also, the bid shading or demand reduction in the second auction is determined by  $\mu_k$  and  $\mu_j$ . A decrease in either the smallest or the second smallest unsatisfied demand in the second auction turns competition in that auction less intense. Hence, the residual supply that each bidder faces becomes more inelastic, which increases bid shading. Consequently,

in the three-bidder case, bidders will also affect bid shading in the second auction through their bidding in the first auction.

The second auction equilibrium demand function of each bidder and the equilibrium price in that same auction are easily derived from equation (2.33). Define  $m = \min \{S_2, \sum_l \lambda_l - y_1\}$ . Then, bidder  $l$ 's equilibrium profit from the second auction, as a function of the residual supply in that auction and the purchases in the previous auction, can be written as:

$$\pi_{l2}(y_2; \mathbf{q}_1) = \begin{cases} \frac{y_2^3 v}{27 \mu_j \mu_k} & \text{if } y_2 \in [0, 3\mu_k) \\ \frac{v(y_2 - \mu_k)^2}{4\mu_j} & \text{if } l \neq k \text{ and } y_2 \in [3\mu_k, \mu_k + 2\mu_j) \\ \frac{v(y_2 - \mu_k)\mu_k}{2\mu_j} & \text{if } l = k \text{ and } y_2 \in [3\mu_k, \mu_k + 2\mu_j) \\ v\mu_l & \text{if } l \neq i \text{ and } y_2 \in [\mu_k + 2\mu_j, S_2] \\ v(y_2 - \mu_k - \mu_j) & \text{if } l = i \text{ and } y_2 \in [\mu_k + 2\mu_j, m] \\ v\mu_l & \text{if } l = i \text{ and } y_2 \in (m, S_2] \end{cases} \quad (2.34)$$

## 2.4.2 First Auction

For a relevant realization of  $y_1$ , an increase in a bidder's purchases in the first auction implies a decrease in at least one of the other bidder's purchases in that same auction<sup>31</sup>. Moreover, since equilibrium bidding in the second auction depends on the two smallest unsatisfied demands,  $\mu_k$  and  $\mu_j$ , bidder  $l$ 's profit from the last auction in the sequence depends on the demand functions submitted in the first

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<sup>31</sup>If  $y_1 > \lambda_1 + \lambda_2 + \lambda_3$  and the increase in bidder  $l$ 's purchases in the first auction is smaller than  $y_1 - \sum_{-l} \lambda_{-l}$ , then the quantity bought by the other two bidders in the first auction remains unchanged. However, this case is not relevant since all bidders will buy all they want in the first auction.



auction. For that reason, when selecting the demand function for the first auction, bidder  $l$  does not look for the demand that maximizes his expected profits from the first auction, but looks for the one that maximizes the expected value of his entire stream of profits. Therefore, bidder  $l$  has to take into account not only the effect this bid will have on his profit from the first auction, but also the effect on his expected profit from the second auction through demand reduction.

When there are two bidders and one bidder increases (decreases) the quantity he buys in an auction, there is an identical decrease (increase) in the quantity the other bidder buys in the same auction. When there are more than two bidders, the change in a bidder's purchase also implies a balancing change in the purchases of all other bidders. However, how that change is allocated among the other bidders depends on the elasticity of their demand functions. Also, given the first auction demand functions of all bidders except  $l$ , it is equivalent to think of bidder  $l$  as increasing the quantity he buys in the first auction or increasing the clearing price in that auction. Therefore, the marginal change in bidder  $l$ 's expected profit from the second auction due to a marginal change in the first auction clearing price will be defined in this section. This change depends on bidder  $l$ 's unsatisfied demand after the first auction (i.e. whether  $l$  is  $i$ ,  $j$  or  $k$ ).

$$E_2 \left[ \frac{\partial \pi_{l2}}{\partial p_1} \right] = \mathbf{E}_2 \left[ \frac{\partial \pi_{l2}}{\partial \mathbf{q}_{-11}} \right] \cdot \mathbf{d}'_{-11}(p_1) \quad (2.35)$$

Where  $\mathbf{E}_2 \left[ \frac{\partial \pi_{l2}}{\partial \mathbf{q}_{-11}} \right]$  and  $\mathbf{d}'_{-11}(p_1)$  are both vectors of the corresponding derivatives. For ease of notation, define  $A_{lh} = E_2 \left[ \frac{\partial \pi_{l2}}{\partial q_{h1}} \right]$  with  $l \neq h$ . The expression for

each of the six  $A_{lh}$  are the followings:

$$A_{ij} = \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j^2 \mu_k} dF(y_2) + \int_{3\mu_k}^{\mu_k+2\mu_j} \frac{v(y_2-\mu_k)^2}{4\mu_j^2} dF(y_2) + \int_{\mu_k+2\mu_j}^{S_2} v dF(y_2)$$

$$A_{ik} = \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} dF(y_2) + \int_{3\mu_k}^{\mu_k+2\mu_j} \frac{v(y_2-\mu_k)}{2\mu_j} dF(y_2) + \int_{\mu_k+2\mu_j}^{S_2} v dF(y_2)$$

$$A_{ji} = -A_{ij} + 2 \int_{\mu_k+2\mu_j}^{S_2} v dF(y_2)$$

$$A_{jk} = A_{ji} - \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} dF(y_2) + \int_{3\mu_k}^{\mu_k+2\mu_j} \frac{v(y_2-\mu_k)}{2\mu_j} dF(y_2)$$

$$A_{ki} = - \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j \mu_k^2} dF(y_2) + \int_{3\mu_k}^{\mu_k+2\mu_j} \frac{v(y_2-2\mu_k)}{4\mu_j} dF(y_2) + \int_{\mu_k+2\mu_j}^{S_2} v dF(y_2)$$

$$A_{kj} = A_{ki} + \int_0^{3\mu_k} \frac{y_2^3 v}{27\mu_j^2 \mu_k} dF(y_2) + \int_{3\mu_k}^{\mu_k+2\mu_j} \frac{v(y_2-\mu_k)\mu_k}{2\mu_j^2} dF(y_2)$$

Lemma 4 stated the conditions for the first auction equilibrium demands functions to be continuous in the two-bidder case. The same conditions hold for the three-bidder case. The lemma is restated below and the proof is updated for the three-bidder case. Remember,  $\bar{p}_1 = p_1(0)$ .

**Lemma 7** *Equilibrium demand functions in the first auction are continuous at any price  $p \in (0, \bar{p}_1)$ , as long as  $\underline{D}_1(p) < S_1$ .*

**Proof.** The proof of this lemma is just an extension of the proof of lemma 1. Therefore, instead of writing again the entire proof, only the differences between both cases will be pointed out and their consequences will be developed.

Assume bidder  $-l$ 's demand function is discontinuous at  $p^* \in (0, \bar{p}_1)$ , then  $\bar{d}_{-l}(p^*) > \underline{d}_{-l}(p^*)$ . As before, for any interval  $[p^* - \epsilon, p^*]$  bidder  $l$  must demand additional quantity, otherwise bidder  $-l$  can profitably deviate by withholding demand at  $p^*$ . Define  $p^\epsilon(p^*) = \sup\{p \mid d_{l1}(p) \geq \underline{d}_{l1}(p^*) + \epsilon\}$ . Observe that  $p^\epsilon(p^*)$  tends to  $p^*$  as  $\epsilon$  tends to zero, and it equals  $p^*$  when  $d_{l1}(p)$  is also discontinuous at  $p^*$ . Bidder  $l$  can deviate by submitting a demand function with the same structure as

that in equation (2.3). Obviously, this deviation will also yield a loss and a gain in expected profit from the first auction due to higher prices and larger purchases in that auction, respectively.

Assume  $S_1 \geq \bar{D}_1(p^\epsilon(p^*))$ . Then, the upper bound for the expected loss and the lower bound for the expected gain are those on equations (2.4) and (2.5), respectively, with the subscript referring to the auction changed to 1. Also, as it was shown in the proof of lemma 1, this deviation seems to be profitable for bidder  $l$ . However, since the deviation now takes place in the first auction, it also triggers a change in expected profits from the second auction. The change in bidder  $l$ 's expected profits caused by the impact this deviation has in equilibrium bidding in the second auction can be written as: The change in bidder  $l$ 's expected profits caused by the impact bidder  $l$ 's deviation has in equilibrium bidding in the second auction due to his deviation in the first auction can be written as:

$$\Delta E_1 [\pi_{l2}] = \int_{D_1(p^* + \epsilon)}^{\bar{D}_1(p^\epsilon(p^*))} \mathbf{E}_2 \left[ \frac{\partial \pi_{l2}}{\partial \mathbf{q}_{-11}} \right] \cdot \Delta \mathbf{q}_{-11}(y_1) dF(y_1) \quad (2.36)$$

The derivative of bidder  $l$ 's expected profits from the second auction with respect to  $q_{-11}$  can take any sign. Hence, bidder  $l$  can suffer an expected loss or an expected gain from the second auction due to his deviation. For ease of notation, the expected loss and gain will be represented by  $\Theta^\epsilon$  and  $\Psi^\epsilon$  respectively. The expected gain is bounded below by zero, by definition, and it is weakly increasing in  $\epsilon$ . Bidder  $l$ 's expected loss is bounded above by:

$$\Theta^\epsilon < \bar{M}_l (d_{l1}(p^*) - d_{l1}(p^* + \epsilon) + \epsilon) \left[ F(\bar{D}_1(p^\epsilon(p^*))) - F(D_1(p^* + \epsilon)) \right] \quad (2.37)$$

Since,  $A_{ik} > A_{ij}$ ,  $A_{jk} > A_{ji}$  and  $A_{kj} > A_{ki}$ , then  $\bar{M}_i = \max_{y_1} A_{ik}$ ,  $\bar{M}_j =$

$\max_{y_1} A_{jk}$  and  $\bar{M}_k = \max_{y_1} A_{kj}$ , when  $y_1 \in [D_1(p^* + \epsilon), \bar{D}_1(p^\epsilon(p^*))]$ . The upper bound and its derivative with respect to  $\epsilon$  converge to zero as  $\epsilon$  does so.

Now, if  $\bar{D}_1(p^\epsilon(p^*)) > S_1 > \underline{D}_1(p^*)$ , then all the upper and lower bounds still approach zero as  $\epsilon$  does so. Moreover, the signs of their derivatives with respect to  $\epsilon$  remain unchanged. Hence, equilibrium demand functions are smooth as long as  $\underline{D}_1(p) < S_1$ . ■

One of the features of equilibrium demand functions in the two-bidder case was their strict monotonicity. When three bidders participate in the sequence of two uniform price auctions, it is possible that equilibrium demand functions are constant at some price range. However, at most a single demand function can be constant for a given price range. If the demand functions of two bidders were constant at the same price range, then that of the third bidder should also be constant or he would have the incentive to withhold demand at those prices. Now, if all the demand functions were constant at the same price range, then any of the bidders could deviate by submitting a demand function like the one in (2.11). The proof that such deviation is profitable is the proof of lemma 5 updated to the three-bidder case in the same way as the proof of lemma 4 was updated for lemma 7. In the second auction, interior inelastic segments on the equilibrium demand functions were ruled out because of the symmetric bidding for interior quantities. However, that is not necessarily the case in the first auction.

Bidder  $l$ 's optimal interior bidding in the first auction, conditional on other bidders' demand functions, is characterized by the following equations for all  $y_1 \leq \min\{S_1, \sum_l \lambda_l\}$ :

(i) If  $d_{l1}(p)$  is strictly decreasing:

$$-\sum_{h \neq l} d'_{h1}(p_1)(v - p_1) - d_{l1}(p_1) = -\sum_{h \neq l} d'_{h1}(p_1)A_{lh} \quad (2.38)$$

(ii) If  $d_{l1}(p)$  is constant for  $p \in (p', p'') \subset (\underline{p}_1, \bar{p}_1)$ :

$$\int_{D_1(p')}^{D_1(p)} \left[ -\sum_{h \neq l} d'_{h1}(p_1)(v - p_1 - A_{lh}) - d_{l1}(p_1) \right] dF(y_1) \leq 0 \quad \forall p \in (p', p'') \quad (2.39)$$

$$\int_{D_1(\bar{p}_1)}^{D_1(p_1)} \left[ -\sum_{h \neq l} d'_{h1}(p_1)(v - p_1 - A_{lh}) - d_{l1}(p_1) \right] dF(y_1) = 0 \quad (2.40)$$

Equation (2.38) is the counterpart of equation (2.24) for the three-bidder case.

The left-hand side represents the marginal change in profit from the first auction due to a marginal change in the first auction clearing price. The right hand side represents the expected marginal change in profit from the second auction due to the marginal change in  $p_1$ . For a given  $y_1$  and conditional on the demand functions of all bidders besides  $l$ , an increase in the first auction clearing price decreases the quantity bidder  $l$  buys in that auction, thus increasing his unsatisfied demand in the second auction and altering the demand reduction in that auction. Hence, bidder  $l$ 's option value of increasing his purchase in the second auction becomes:

$$E_2 \left[ \frac{\partial \pi_{l2}}{\partial Q_{-l1}} \right] = \frac{\sum_{h \neq l} d'_{h1}(p_1)A_{lh}}{\sum_{h \neq l} d'_{h1}(p_1)} \quad (2.41)$$

Where  $Q_{-l1}$  represents the quantity bought in the first auction by all bidders besides  $l$ . Equations (2.39) and (2.40) characterize any interior inelastic segments that bidder  $l$ 's equilibrium demand function might have, as long as  $D_1(p) \leq \min\{S_1, \sum_l \lambda_l\}$ .

Equations (2.38) to (2.40) for the three bidders define a system of differential equations, which characterizes interior equilibrium bidding. This system of differen-

tial equations does not have explicit solutions. Hence, as in the two-bidder case, the next step will be to characterize equilibrium bidding in as great detail as possible. When three bidders participate in a sequence of two uniform price auctions, bidders not only shade their bids for the same reason they do it in a single uniform price auction (i.e. *static* bid shading), but they also shade their bids in the first auction to increase the bid shading in the second auction (i.e. *dynamic* bid shading). The following proposition states that as long as the residual supply in the first auction is non greater than  $\min\{S_1, \sum_l \lambda_l\}$ , at the beginning of the second auction the unsatisfied demand of one of the bidders is always smaller than those from his opponents. Like in the two-bidder case, the cause of this asymmetry can be found on the demand reduction that takes place in the second auction and bidders' incentive to increase it.

**Proposition 5** *In a sequence of two uniform price auctions, at least one of the bidders always reach the second auction with a different unsatisfied demand than the others. Moreover,  $\mu_i > \mu_j \geq \mu_k$ ,  $\forall y_1 < \min\{S_1, \sum_l \lambda_l\}$ , with  $i, j, k \in \{1, 2, 3\}$  and  $i \neq j \neq k$ .*

**Proof.** Assume without loss of generality that bidders 2 and 3 submit the functions  $d_{21}(p)$  and  $d_{31}(p)$ , and also  $\lambda_2 - q_{21}(y_1) \leq \lambda_3 - q_{31}(y_1)$ . In equilibrium,  $d_{11}(p_1) = y_1 - d_{21}(p_1) - d_{31}(p_1)$ . Then, bidder 1's unsatisfied demand after the first auction can be written as  $\lambda_1 - d_{11}(p_1) = \lambda_1 - y_1 + d_{21}(p_1) + d_{31}(p_1)$ . (i) If  $\lambda_2 - q_{21}(y_1) < \lambda_3 - q_{31}(y_1)$ , then define  $\tilde{p}_1$  as the clearing price in the first auction such that  $\mu_1 = \mu_2$ . Then,  $\lim_{p_1 \rightarrow \tilde{p}_1^-} A_{ij} > \lim_{p_1 \rightarrow \tilde{p}_1^+} A_{ji}$  and  $\lim_{p_1 \rightarrow \tilde{p}_1^-} A_{ik} > \lim_{p_1 \rightarrow \tilde{p}_1^+} A_{jk}$ , which implies

$\lim_{p_1 \rightarrow \tilde{p}_1^-} \frac{\partial \Pi_{11}}{\partial p_1} < \lim_{p_1 \rightarrow \tilde{p}_1^+} \frac{\partial \Pi_{11}}{\partial p_1}$ . Hence, it is never optimal for bidder 1 to select  $\tilde{p}_1$  when  $d'_{11}(\tilde{p}_1) < 0$ . Moreover, since at most one equilibrium demand function can be constant at a given price range, bidders 1 and 2 reach the second auction with asymmetric unsatisfied demands. (ii) If  $\lambda_2 - q_{21}(y_1) = \lambda_3 - q_{31}(y_1)$ , then for the same reason bidders 1, 2 and 3 can not reach the second auction with symmetric unsatisfied demands.

Now, assume  $\lambda_2 - q_{21}(y_1) < \lambda_3 - q_{31}(y_1)$  and define  $\hat{p}_1$  as the clearing price in the first auction such that  $\mu_1 = \mu_3$ . Then,  $\lim_{p_1 \rightarrow \hat{p}_1^-} A_{ji} = \lim_{p_1 \rightarrow \hat{p}_1^+} A_{ki}$  and  $\lim_{p_1 \rightarrow \hat{p}_1^-} A_{jk} = \lim_{p_1 \rightarrow \hat{p}_1^+} A_{kj}$ , which implies  $\lim_{p_1 \rightarrow \hat{p}_1^-} \frac{\partial \Pi_{11}}{\partial p_1} = \lim_{p_1 \rightarrow \hat{p}_1^+} \frac{\partial \Pi_{11}}{\partial p_1}$ . Hence, in equilibrium, bidders 1 and 3 could reach the second auction with symmetric unsatisfied demands if  $\lambda_2 - q_{21}(y_1) < \lambda_3 - q_{31}(y_1)$ . Actually, whether they are symmetric or not depends on whether  $\lambda_1$  and  $\lambda_3$  are symmetric or not, otherwise a contradiction would arise.

Finally, continuity of equilibrium demand functions in the first auction ensures the ranking of bidders according to their unsatisfied demands after the first auction is the same for all  $y_1 < \min\{S_1, \sum_l \lambda_l\}$ . ■

When bidders are symmetric as well as when  $\lambda_i \leq \lambda_j \leq \lambda_k$ , it is clear from proposition 5 that bidders use bidding in the first auction to optimally shape bid shading in the second auction. In both cases bidder  $i$  demands the smallest quantity at every price, followed by bidder  $j$  and then bidder  $k$ ,  $d_{i1}(p) < d_{j1}(p) \leq d_{k1}(p)$  for all  $p \in (\underline{p}_1, \bar{p}_1)$ . Because of the endogenous asymmetries in the model, it is convenient to focus on partially symmetric equilibria, those where  $\mu_j = \mu_k$ , which only come up when  $\lambda_i \geq \lambda_j = \lambda_k$ . The following proposition shows there is also dynamic bid

shading in partially symmetric equilibria.

**Proposition 6** *In partially symmetric equilibria (i.e.  $\mu_j = \mu_k$ ), bidder  $i$  bids less aggressively than bidders  $j$  and  $k$  in the first auction:  $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$  for all  $p \in (\underline{p}_1, \bar{p}_1)$ :*

**Proof.** First, if  $\mu_j = \mu_k$ , then  $A_{ji} = A_{ki}$  and  $A_{jk} = A_{kj}$ , which implies  $d_{j1}(p) = d_{k1}(p)$ . Hence,  $\mu_j = \mu_k$  only happens if  $\lambda_j = \lambda_k$ . Moreover, if  $\lambda_i$  were smaller than  $\lambda_j$  and  $\lambda_k$ , a contradiction would arise for small realizations of the residual supply,  $y_1$ .

Assume  $d_{i1}(p) = d_{j1}(p) = d_{k1}(p)$  at every strictly positive price. Then,  $\frac{\partial \Pi_{j1}}{\partial p_1}$  and  $\frac{\partial \Pi_{i1}}{\partial p_1}$  equal zero for every price in  $(\underline{p}_1, \bar{p}_1)$ . However,

$$\begin{aligned} \frac{\partial \Pi_{j1}}{\partial p_1} &= -d_1(p_1) - (v - p_1 - A_{ji})d'_1(p_1) - (v - p_1 - A_{jk})d'_1(p_1) \\ &> -d_1(p_1) - (v - p_1 - A_{ij})d'_1(p_1) - (v - p_1 - A_{ik})d'_1(p_1) \\ &= \frac{\partial \Pi_{i1}}{\partial p_1} \end{aligned}$$

where the inequality comes from  $A_{ji} < A_{ij}$  and  $A_{jk} < A_{ik}$ . Hence,  $d_{i1}(p)$  and  $d_{j1}(p)$  are not identical at every strictly positive price.

The next step is to prove bidder  $i$  equilibrium demand functions does not cross those of bidder  $j$  and  $k$ . Define  $\bar{p}_1^i$  is the  $\inf\{p \mid d_{i1}(p) = 0\}$ . First, it will be shown that in equilibrium  $d_{i1}(\bar{p}_1^i) = 0$ , but  $d_{j1}(\bar{p}_1^i)$  and  $d_{k1}(\bar{p}_1^i)$  are strictly positive.

Assume  $d_{j1}(\bar{p}_1^i) = 0$  and  $d_{k1}(\bar{p}_1^i) = 0$ . If  $d_{i1}(p)$  is either continuous or discontinuous at  $\bar{p}_1^i$ , then  $\lim_{p \rightarrow \bar{p}_1^{i-}} \frac{\partial \Pi_{i1}}{\partial p_1} = -2(v - \bar{p}_1^i - A_{ij}) \lim_{p \rightarrow \bar{p}_1^{i-}} d'_{j1}(p) - \lim_{p \rightarrow \bar{p}_1^{i-}} d_{i1}(p) = 0$ . Now, if  $d_{i1}(p)$  is continuous (discontinuous) at  $\bar{p}_1^i$ , then  $(v - \bar{p}_1^i - A_{ij})$  is zero (positive). Hence,  $(v - \bar{p}_1^i - A_{ji})$  and  $(v - \bar{p}_1^i - A_{jk})$  are strictly positive since  $A_{ij} > A_{ji}$  and



$A_{ij} = A_{ik} > A_{jk}$ , which implies  $\lim_{p \rightarrow \bar{p}_1^-} \frac{\partial \Pi_{j1}}{\partial p_1} = -(v - \bar{p}_1^i - A_{ji}) \lim_{p \rightarrow \bar{p}_1^-} d'_{i1}(p) - (v - \bar{p}_1^i - A_{jk}) \lim_{p \rightarrow \bar{p}_1^-} d'_{k1}(p) > 0$ . Therefore,  $d_{i1}(\bar{p}_1^i) = 0$  and  $d_{j1}(\bar{p}_1^i) = d_{k1}(\bar{p}_1^i) > 0$ .

Since  $d_{i1}(\bar{p}_1^i) = 0$  and  $d_{j1}(\bar{p}_1^i) = d_{k1}(\bar{p}_1^i) > 0$ , the equilibrium demand functions of bidder  $j$  and  $k$  can only cross that of bidder  $i$  from above. Assume the following:

$$d_{i1}(p) < d_{j1}(p) \quad \text{if} \quad p > \tilde{p}$$

$$d_{i1}(p) = d_{j1}(p) \quad \text{if} \quad p = \tilde{p}$$

$$d_{i1}(p) > d_{j1}(p) \quad \text{if} \quad p < \tilde{p}$$

Then  $d'_{i1}(\tilde{p}) < d'_{j1}(\tilde{p})$ , and

$$\begin{aligned} \left. \frac{\partial \Pi_{j1}}{\partial p_1} \right|_{\tilde{p}} &= -d_{j1}(\tilde{p}) - d'_{i1}(\tilde{p})(v - \tilde{p} - A_{ji}) - d'_{k1}(\tilde{p})(v - \tilde{p} - A_{jk}) \\ &> -d_{j1}(\tilde{p}) - d'_{i1}(\tilde{p})(v - \tilde{p} - A_{ij}) - d'_{k1}(\tilde{p})(v - \tilde{p} - A_{ik}) \\ &> -d_{i1}(\tilde{p}) - d'_{j1}(\tilde{p})(v - \tilde{p} - A_{ij}) - d'_{k1}(\tilde{p})(v - \tilde{p} - A_{ik}) \\ &= \left. \frac{\partial \Pi_{i1}}{\partial p_1} \right|_{\tilde{p}} \end{aligned}$$

Hence, the equilibrium demand functions do not cross each other. Consequently, since  $d_{i1}(\bar{p}_1^i) = 0$  and  $d_{j1}(\bar{p}_1^i) = d_{k1}(\bar{p}_1^i) > 0$ , then  $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$  for all  $p \in (\underline{p}_1, \bar{p}_1)$ . ■

If the highest possible residual supply in the first auction,  $S_1$ , is smaller than the aggregate quantity bidders want to consume,  $\sum_l \lambda_l$ , then the system of equations defined by the *F.O.C.s* in (2.38) - (2.40) only characterizes the equilibrium demand functions for prices in the interval  $[\underline{p}_1, \bar{p}_1]$ <sup>32</sup>. Now, it remains to extend all demand functions over  $[0, \underline{p}_1)$  in a way that none of these prices become clearing prices. This

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<sup>32</sup>When  $\underline{p}_1$  equals zero, the interval is open at  $\underline{p}_1$ ; because the equilibrium demand functions are not necessarily continuous at zero and  $d_{l1}(0) = \lambda_l$  for all  $l$ .

Table 2.2: First auction partially symmetric equilibria when  $Y_2 \sim U[0, 1 - S_1]$

$S_1$	$v$	$\underline{\lambda}_i^a$	$\lambda_j$	$\lambda_k$	$\bar{p}_1$	$\bar{p}_1^b$	$d_{i1}(0)$	$\bar{\Delta}^c$
0.5	10	0.10	0.10	0.10	5.99	2.94	0.067	0.035
0.5	10	0.15	0.15	0.15	8.99	4.42	0.099	0.052
0.5	20	0.10	0.10	0.10	11.98	5.89	0.067	0.035
0.5	20	0.15	0.15	0.15	17.98	8.83	0.099	0.052
0.3	10	0.10	0.10	0.10	4.28	2.10	0.067	0.035
0.3	10	0.15	0.15	0.15	6.42	3.16	0.099	0.052
0.7	10	0.10	0.10	0.10	9.99	4.90	0.067	0.035

<sup>a</sup>Lowest  $\lambda_i$  for the partially symmetric equilibrium to exist.

<sup>b</sup> $\bar{p}_1^i$  is the  $\inf\{p \mid d_{i1}(p) = 0\}$ , <sup>c</sup> $\bar{\Delta} = \max[d_{j1}(p) - d_{j1}(p)]$ .

can be achieved by using any decreasing twice continuously differentiable functions  $(\tilde{d}_{k1}(p), \tilde{d}_{j1}(p), \tilde{d}_{i1}(p))$  defined over the interval  $[0, \underline{p}_1]$ , that satisfy  $\tilde{d}_{l1}(\underline{p}_1) = d_{l1}(\underline{p}_1)$  as well as the following inequality for all  $p \in [0, \underline{p}_1)$  and for  $l = i, j, k$ :

$$-(S_1 - \sum_{h \neq l} \tilde{d}_{h1}(p)) - \sum_{h \neq l} \tilde{d}'_{h1}(p)(v - p - A_{lh}) > 0 \quad (2.42)$$

The left-hand side is the derivative of bidder  $l$ 's ex-post stream of profits with respect to the price in the first price auction, evaluated using the market-clearing condition and  $y_1 = S_1$ .

In a partially symmetric equilibrium, it is possible to define an upper bound of the the first auction equilibrium bids similar to the one defined in the two-bidder case. When bidders  $j$  and  $k$  reach the second auction with symmetric unsatisfied demands,  $A_{ij} = A_{ik} > A_{jk} = A_{kj} > A_{ji} = A_{ki}$ . The equation  $v - p - A_{ij} = 0$  defines bidders  $j$  and  $k$ 's Bertrand locus when each one of the three bidders bid for positive quantities. Since  $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$ , there is a range of prices at which only  $j$  and  $k$  bid for positive quantities. In that case, bidder  $j$  and  $k$ 's Bertrand

locus is defined by  $v - p - A_{jk} = 0$ . Finally, bidder  $i$ 's Bertrand locus is given by  $v - p - A_{ji} = 0$ . Since  $A_{ij} > A_{jk} > A_{ji}$  and  $d_{i1}(p) < d_{j1}(p) = d_{k1}(p)$ , an upper bound of each bidder's equilibrium bid, when they all bid for positive quantities is given by  $p = v - A_{ij}$ . When bidder  $i$  does not bid for positive quantities, an upper bound of bidders  $j$  and  $k$ 's equilibrium bids is given by  $p = v - A_{jk}$ . Without an analytic solution for the equilibrium it is not possible to pin down the highest price bidder  $i$  bids for a positive quantity,  $\hat{p}_1$ , and  $d_{j1}(\hat{p})$ , which makes impossible to compare the expected revenues from a sequence of two uniform price auction and a single uniform price auction using this upper bound for bidders  $j$  and  $k$ 's bids. Hence, when defining an upper bound for the seller's expected revenue, it will be assumed bidders  $j$  and  $k$  bid according to  $p = v - A_{jk}$  for all  $q_1$ .

$$\hat{b}_1^{jk}(q_1) = \begin{cases} vF(3\lambda_i) & \text{if } q_1 \in [0, \lambda_j - \lambda_i) \\ vF(3(\lambda_j - q_1)) & \text{if } q_1 \in [\lambda_j - \lambda_i, \lambda_j) \\ 0 & \text{otherwise} \end{cases} \quad (2.43)$$

When bidder  $i$  bids for a strictly positive quantity in the first auction, the bid of every bidder is bounded above by  $p = v - A_{ij}$ . Since bidders  $j$  and  $k$  buy larger quantities than bidder  $i$  in the first auction, and the difference is at least  $\lambda_j - \underline{\lambda}$ , the upper bound of bidder  $i$ 's equilibrium bids in the first auction becomes:

$$\hat{b}_1^i(q_1) = \begin{cases} v \left( F(3(\underline{\lambda} - q_1)) - \int_0^{3(\underline{\lambda} - q_1)} \frac{y_2^3}{27(\underline{\lambda} - q_1)^3} dF(y_2) \right) & \text{if } q_1 \in [0, \underline{\lambda}) \\ 0 & \text{otherwise} \end{cases} \quad (2.44)$$

The upper bound of bidder  $i$ 's first auction bids takes into account the discounting of the option value of increasing his purchases in the second auction,  $A_{ij}$ . However, it does not fully take into account the *dynamic* bid shading that takes

place in the first, and it completely ignores the *static* bid shading in that auction<sup>33</sup>. The upper bound of bidders  $j$  and  $k$ 's first auction bids only takes into account their option value of increasing purchases in the second auction when bidder  $i$  does not buy any quantity in the first auction. This option value is larger than the option value when bidder  $i$  buys in the first auction, but smaller than bidder  $i$ 's own option value. In addition, the highest equilibrium bid in the first auction is not higher than  $vF(3\underline{\lambda})$ .

### 2.4.3 Revenue Comparison

In a partially symmetric equilibrium, the price in the second auction as a function of the residual supply in that auction and conditional on the residual supply in the first auction becomes:

$$p_2(y_2 | y_1) = \begin{cases} v \left( 1 - \frac{y_2^2}{9(\lambda_j - q_{j1}(y_1))^2} \right) & \text{if } y_2 < 3(\lambda_j - q_{j1}(y_1)) \\ 0 & \text{otherwise} \end{cases} \quad (2.45)$$

In a partially symmetric equilibrium bidders  $j$  and  $k$  buy the same quantity in the first auction, with each of them buying more than one third of the residual supply in that auction plus  $(\lambda_j - \underline{\lambda})/3$ , as long as the residual supply is smaller than  $2\lambda_j + \underline{\lambda}$ . Also, since the equilibrium price in equation (2.45) is decreasing in  $q_{j1}$ , the

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<sup>33</sup>Only when  $\lambda_j > \underline{\lambda}$ , bidder  $i$ 's upper bound takes into account a fraction of the *dynamic* bid shading.

following is an upper bound of the equilibrium revenue in the second auction<sup>34</sup>:

$$\widehat{R}_2(y_2 | y_1) = \begin{cases} vS_2 \left(1 - \frac{y_2^2}{(2\lambda_j + \underline{\lambda} - y_1)^2}\right) & \text{if } y_2 < 2\lambda_j + \underline{\lambda} - y_1 \\ 0 & \text{otherwise} \end{cases} \quad (2.46)$$

If the seller decides to run a single uniform price auction, equilibrium bidding in this auction will be similar to the bidding in the last auction of the sequence. The only difference is that demand reduction will be driven by the smallest of the highest possible individual demands,  $\underline{\lambda}$ . Hence, the equilibrium revenue in a single auction can be written as:

$$\widehat{R}(y) = \begin{cases} v \left[1 - \frac{y^2}{9\underline{\lambda}^2}\right] & \text{if } y < 3\underline{\lambda} \\ 0 & \text{otherwise} \end{cases} \quad (2.47)$$

**Proposition 7** *When the demand shock is uniformly distributed, partially symmetric equilibria of a sequence of two uniform price auctions yield lower expected revenue than a single uniform price auction, if:*

$$\begin{aligned} \frac{4\lambda_j + 2\underline{\lambda} - S_1}{6\underline{\lambda}} + \frac{3(4\lambda_j - S_1)S_1 - 12(\lambda_j - \underline{\lambda})^2}{8(1 - S_1)\underline{\lambda}} < 1 & \quad \text{for } S_1 \in [2(\lambda_j - \underline{\lambda}), 2\lambda_j - \frac{3}{2}\underline{\lambda}] \\ \frac{(20 + 7S_1)(4\lambda_j + 2\underline{\lambda} - S_1)}{120(1 - S_1)\underline{\lambda}} + \frac{3(56\lambda_j\underline{\lambda} - 24\lambda_j^2 - 31\underline{\lambda}^2)}{80(1 - S_1)\underline{\lambda}} < 1 & \quad \text{for } S_1 \in [2\lambda_j - \frac{3}{2}\lambda_i, 2\lambda_j + \lambda_i] \\ \frac{(\lambda_j + \underline{\lambda})^2}{6S_1\underline{\lambda}} + \frac{48\lambda_j - 15\underline{\lambda}}{16(1 - S_1)} < 1 & \quad \text{for } S_1 \in [2\lambda_j + \underline{\lambda}, \infty) \end{aligned}$$

**Proof.** If the demand shocks are uniformly distributed, then the residual supply in any auction is also uniformly distributed. Hence, the expected revenue in a single uniform price auction becomes  $2v\underline{\lambda}$ . The left-hand side on the three inequalities

<sup>34</sup>If  $\lambda_j < \lambda_i$  and  $y_1 > 3\lambda_j$ , then  $q_{j1}(y_1) = \lambda_j$  might be smaller than one third of the residual supply in the first auction, and  $(\lambda_j - q_{j1}(y_1))$  would be larger than  $(\lambda_j - \frac{y_1}{3})$ . However, for those realizations of  $y_1$ , the equilibrium price in the second auction is zero, since bidder  $i$  is the only strategic bidder submitting a bid in that auction.

in proposition 7 are the upper bound of expected revenue in a sequence of two uniform price auctions as a proportion of the expected revenue in a single uniform price auction. The upper bound of the expected revenue in the first auction of the sequence can be constructed by using (2.43) and (2.44), while the upper bound of the expected revenue in the second auction of the sequence comes from (2.46). ■

Proposition 7 says: (i) When there are two symmetric bidders, each one demanding less than  $1/9$  of the aggregate supply and the third bidder is not smaller than them, a single uniform price auction yields higher expected revenue than any partially symmetric equilibrium of a sequence of two uniform price auctions. (ii) However, if either the two symmetric bidders are the smallest, but they demand more than  $1/9$  of the aggregate supply, or the third bidder is the smallest of all, then the upper bound of the expected revenue of partially symmetric equilibria in a sequence of two uniform price auctions is higher than the expected revenue in a single uniform price auction for at least some values of  $S_1$ . Remember the upper bound of the expected revenue in a sequence of two uniform price auctions ignores the *static* bid shading that takes place in the first, and it does not fully take into account the *dynamic* bid shading in that auction. Consequently, it does not fully take into account the *static* bid shading in the second auction either.

The conditions in proposition 7 are similar to the conditions in proposition 4 for the two-bidder case. When there are three bidders, the seller still suffers the largest loss in expected revenue when spreading the supply evenly over the two auctions in the sequence. However, when a third bidder participates in the auctions, the loss in expected revenue as a consequence of spreading the supply over a sequence of two

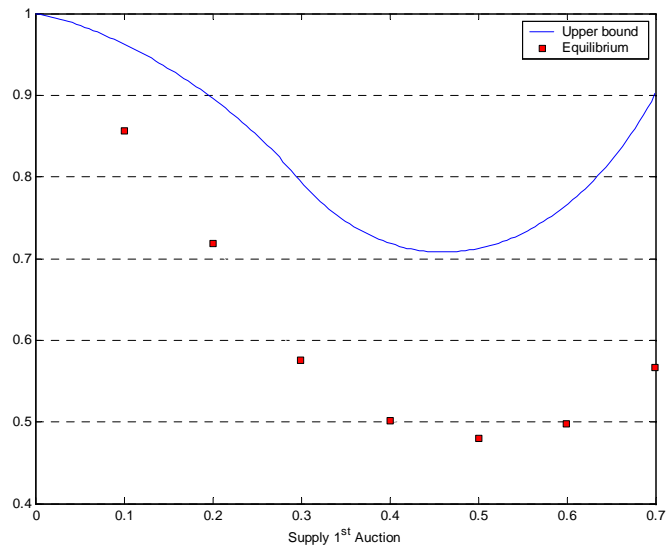


Figure 2.4: Ratio of  $E[Rev_{Seq}]$  to  $E[Rev_{Single}]$  when  $\lambda_i \geq \lambda_j = \lambda_k = 0.1$

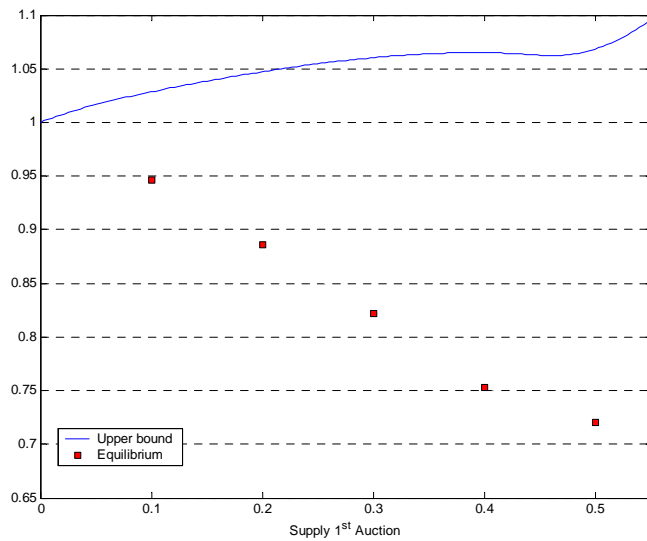


Figure 2.5: Ratio of  $E[Rev_{Seq}]$  to  $E[Rev_{Single}]$  when  $\lambda_i \geq \lambda_j = \lambda_k = 0.15$

uniform price auctions represents a smaller share of the expected revenue the seller can raise by selling the entire supply through a single uniform price auction.

## 2.5 Conclusion

When choosing among several auction formats, the seller looks for the auction format that is best suited for achieving her main objectives of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. One decision that needs to be made by the seller when she has a divisible good for sale is whether to sell the entire supply in one auction or to spread it over several auctions. There are several features of the market that should be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

The seller might prefer a single auction over a sequence of auctions when the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. When there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the market relative to a single sealed-bid auction, possibly increasing expected revenue.



If infra-marginal bidders are risk averse, the seller might also prefer a sequence of sealed-bid auctions, since that auction format reduces bidders' risk which might increase the seller's expected revenue by increasing participation.

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions. However, there is no theoretical nor empirical research that studies sequences of divisible good auctions. This chapter filled that gap in the literature for the case of divisible good auctions with a uniform pricing rule by studying a sequence of two uniform price auctions and comparing its revenue generation properties with those of a single uniform price auction.

In auctions where bidders pay the clearing price for the quantity won, bidders have an incentive to reduce demand (i.e. shade their bids) to pay less for their winnings. This incentive grows with the quantity demanded and is inversely related to bidders' demands. In a sequence of two uniform price auctions, bidders internalize that their bidding in the first auction has an effect on the demand reduction in the later auction. Bidders reduce their demands even more in the first auction with one bidder, usually the largest one, reducing it more than the others and thus strengthening the bid shading or demand reduction in the second auction. Hence, in a sequence of uniform price auctions there is not only *static* demand reduction but also *dynamic* demand reduction.

In any auction within a sequence of single object auctions with the exception

of the last, bids are discounted by the option value of participating in later auctions. In the case of a sequence of two uniform price auctions, bids in the first auction are also discounted respect to what they would be in a single uniform price auction. The discount this time represents the option value of increasing the quantity purchased in the later auction.

In a sequence of two uniform price auctions with non-strategic bidders who bid randomly and strategic bidders with, equilibrium bidding in the second auction was shown to be unique and symmetric for any supply split with  $S_2 \geq N\tilde{\lambda}$ . However, this was not the case in the first auction. Nevertheless, first auction equilibrium bids are bounded above by the value of the good discounted by the option value of increasing the quantity purchased in the second auction<sup>35</sup>. Using this upper bound of equilibrium bids, an upper bound of the expected revenue in a sequence of two uniform price auctions was defined.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding are even stronger when the supply is split evenly between the two auctions of the

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<sup>35</sup>If bidders do not know the actual value of the good and they all receive the same signal about it, then the upper bound is given by the expected value of the good discounted by the option value of increasing the quantity purchased in the second auction.

sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions. These results are in line with the finding that it is better for the seller to use a sealed-bid auction than a dynamic auction when competition is not very strong.

## Chapter 3

### Market Power, Forward Trading and Supply Function Competition

#### 3.1 Introduction

It is generally argued that forward trading is socially beneficial. Two of the most common arguments state that forward trading allows efficient risk sharing among agents with different attitudes toward risk and improves information sharing, particularly through price discovery. It is also believed that forward trading enhances competition in the spot market by committing firms to more aggressive strategies. A firm, by selling forward, can become the leader in the spot market (the top seller), thereby improving its strategic position in the market. Still, when firms compete in quantities at the spot market, every firm faces the same incentives, resulting in lower prices and no strategic improvement for any firm. This is Allaz and Vila's (1993) argument. Green (1999) shows when firms compete in supply functions, forward trading might not have any effect on the intensity of competition in the spot market, but in general it will enhance competition. This pro-competitive argument has been used to support forward trading as a market mechanism to mitigate incentives to exercise market power, particularly in electricity markets.

The pro-competitive feature of forward trading has been challenged by recent papers. Mahenc and Salanié (2004) show when, in the spot market, firms producing substitute goods compete in prices instead of in quantities, firms take long posi-

tions (buy) in the forward market in equilibrium. This increases the equilibrium spot price compared to the case without forward market. In that paper as in Allaz and Vila's paper, firms use forward trading to credibly signal their commitment to more profitable spot market strategies. However, as Fudenberg and Tirole (1984) and Bulow et al. (1985) point out, in those cases prices are strategic complements, while quantities are strategic substitutes, which is the reason for the different equilibrium forward positions taken by firms in both papers, and the resulting effect on the intensity of competition. Liski and Montero (2006) show that under repeated interaction it becomes easier for firms to sustain collusive behavior in the presence of forward trading. The reason is that forward markets provide another instrument to punish deviation from collusive behavior, which reduces the gains from defection.

However, all these papers ignore a key point—that firms usually face capacity constraints, which affects their incentives for strategic trading ahead of the spot market. When a capacity constrained firm sells forward, it actually softens competition in the spot market from the perspective of competitors. In the case where there are two firms and one sells its entire capacity forward, its competitor becomes the sole supplier in the spot market, which implies it has the power to set the price.

The following is an example of how forward trading can affect the intensity of competition in the spot market when firms are constrained on the quantity they can offer. The In-City (generation) capacity market in New York is organized as a uniform-price auction, where the market operator (NYISO) procures capacity from the Divested Generation Owners (DGO's). Two of the dominant firms in this market are KeySpan, with almost 2.4 GW of installed capacity and, US Power Gen, with

1.8 GW. Before May 2006, US Power Gen negotiated a three years swap (May 2006 – April 2009) with Morgan Stanley for 1.8 GW, by which it commits to pay (receive from) Morgan Stanley 1.8 million times the difference between the monthly auction price and \$7.57 kw-month, whenever such difference is positive (negative). Morgan Stanley closed its position by negotiating with KeySpan the exact reverse swap.

The first swap works for US Power Gen as a credible signal that it will bid more aggressively in the monthly auction, since US Power Gen benefits from lower clearing prices in that auction. Also, this financial transaction could be explained on risk hedging grounds. The swap reduces US Power Gen's exposure to the spot price by locking in, at \$7.57 per kw-month, the price it receives for those MWs of capacity it sells in the spot market. On the other side, the outcome of these transactions left KeySpan owning, either directly or financially, 4.2 GW of capacity for three years, which gave it a stronger dominant position in the In-City capacity market, and the incentive to bid higher prices in the monthly auction than otherwise. Moreover, it is difficult to explain this financial transaction on risk hedging grounds, since the swap increases KeySpan's exposure to the uncertain price of the monthly auction, by buying at the fixed price and selling at a variable price (the spot price).

As this chapter shows, when capacity constrained firms facing common uncertainty compete in a uniform-price auction with price cap, strategic forward trading does not enhance competition. On the contrary, firms use forward trading to soften competition, which leaves consumers worse off. The intuition of this result is that when a capacity constrained firm commits itself through forward trading to a more competitive strategy in the spot market, its competitor faces a more inelastic resid-

ual demand in that market. Hence, its competitor prefers not to follow suit in the forward market and thus behave less competitively in the spot market than it otherwise would, by inflating its bids. Because of capacity constraints a firm's actions in the forward market can change its competitor's strategies in the spot market by affecting its own marginal revenue in the spot market. This result and its intuition relate to the work of Fudenberg and Tirole (1984) and Bulow et al. (1985) on strategic interactions. Under the assumptions made here, once US Power Gen negotiated the swap with Morgan Stanley, KeySpan would have the incentive to bid higher prices in the monthly auction, than if there were no trading ahead of it, even if KeySpan did not buy the swap from Morgan Stanley.

When studying the effect of forward trading on investment incentives in a model with uncertain demand and Cournot competition in the spot market, Murphy and Smeers (2007) find that in some equilibria of the forward market one of the firms stays out of the market while the other firm trades. These equilibria come up when the capacity constraint of the latter firm binds at every possible realization of demand. Grimm and Zoetl (2007) also study that problem by assuming a sequence of Cournot spot market with certain demand at each market, but varying by market. They also find that when a firm's capacity constraint binds in a particular spot market, this firm is the only one trading forwards which mature at that spot market. These results are in the same line as those on this chapter. However, when the spot market is organized as a uniform-price auction, as is the case here, they hold even if the capacity constraints only bind for some demand realizations. Also, by modeling the spot market as a uniform-price auction with uncertain demand, the results on

this chapter are better suited for the understanding of wholesale electricity markets.

The results here are also related to those on demand/supply reduction in uniform-price auctions. As Ausubel and Cramton (2002) show, in uniform-price procurement auctions, bidders have an incentive to reduce supply in order to receive a higher price for their sales. This incentive grows with the quantity supplied and it is inversely related to the size of the smallest bidder. Large bidders make room for small bidders. When a capacity constrained firm sells forward, it behaves like a smaller bidder in the auction. Therefore, the incentive to inflate bids increases for the other bidders in the auction. Consequently, strategic forward trading can be reinterpreted as a mechanism that allows firms to assign themselves to different markets, in order to strengthen their market power, which leaves firms better off, but at the expense of consumers who end up worse off. As this chapter will show, usually the smaller firm decides to trade most of its capacity through the forward market, with the larger firm becoming almost the sole trader on the spot market.

The goal of this chapter is not to challenge the general belief that forward trading is socially beneficial, but yes to challenge the pro-competitive view of forward trading by highlighting the impact of capacity constraints on the incentives for strategic forward trading.

The chapter is organized as follows. Section 3.2 analyzes the case where firms are only allowed to sell forwards at date 0. Section 3.3 analyzes the case where firms are also allowed to buy forward and shows that the results do not change. Section 3.4 concludes.



## 3.2 Short Forward Positions

There are two firms which produce and sell an homogeneous good in the spot market (date 1) to satisfy demand from non strategic consumers. At date 0, before the spot market takes place, firms can sell forward contracts (i.e. take short positions) in a competitive forward market, with the good traded in the spot market being the underlying good of the forward contracts. Also, at date 0 competitive risk neutral traders take positions on the forward market<sup>1</sup>. As it is usually assumed, forward contracts mature at the time the spot market meets, date 1. For simplicity, it is assumed the discount factor between forward and spot markets is one. If a firm sells forward at price  $p^h$  and the price in the spot market is  $p$ , the payoff of the forward contract at maturity will be  $(p^h - p)$  per unit. Therefore, forward contracts can be interpreted as specifying the seller receives (pays) the difference between the forward price,  $p^h$ , and the spot price,  $p$ , if such difference is positive (negative). This is just a financial forward, which is settled without physical delivery, but through an equivalent monetary payment<sup>2</sup>. It is assumed along the chapter there is no risk of default from any party involved in a transaction in the forward market. Moreover, no contract can be renegotiated in the spot market.

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<sup>1</sup>It is not necessary that all traders be risk neutral. As long as a large proportion of them are so, the results hold. Also, consumers could be allowed to participate in the forward market without any change on the results.

<sup>2</sup>As Mahenc and Salanié (2004) point out, most actual forward markets function as markets without physical delivery. Nevertheless, the qualitative results would not change if forward contracts were settled through physical delivery.

The demand faced by both firms in the spot market,  $D(p, x)$ , is assumed to be uncertain, with  $x$  being a demand shock which can take any value on the interval  $[0, M]$ .  $F(x)$  is the cumulative distribution function of the demand shock, which is assumed to be strictly increasing, continuous and piece-wise continuously differentiable. The spot market is modeled as a uniform-price auction, where the auctioneer's goal is to ensure enough supply to match demand. A firm's strategy consists of a forward transaction and a piece-wise twice continuously differentiable supply function for the spot market. The realization of the demand uncertainty takes place at date 1, but after firms have chosen their supply functions. Firms are assumed to be capacity constrained, with  $k_l$  representing the installed capacity of firm  $l$ . Each firm's cost function,  $C_l(q_l)$  where  $q_l$  is the quantity produced by firm  $i$ , is assumed to be increasing, piece-wise continuously differentiable and convex.

Firms' cost functions and installed capacities are common knowledge. At date 0, firms simultaneously and independently chose the amount of forward contracts they want to sell. Then, at date 1 given its portfolio of forward contracts and that of its competitor, each firm chooses the supply function it will submit to the auctioneer. This choice is also made simultaneously and independently by both firms. Once the auctioneer has the supply functions from both firms, the demand uncertainty is realized. Given the information structure and the timing of the game, an equilibrium of this model is a profile of strategies, one for each firm, that defines a subgame perfect equilibrium of the entire game. Hence, the first step on the study of firms' incentives to trade forward at date 0 is solving for the spot market equilibrium for every pair of forward transactions,  $\mathbf{h} = (h_1, h_2)$ .

### 3.2.1 Spot Market

At date 1 before the demand shock is realized, firms chose their optimal supply functions taking  $\mathbf{h}$  as given, with that for firm  $l$  ( $l = 1, 2$ ) represented by  $s_l(p; \mathbf{h})$ . In order to perform a meaningful analysis of the forward market, it is necessary to have analytical solutions for the equilibrium spot supply functions. However, as it will become clear later, that might turn out cumbersome. For this reason, the spot market demand will be assumed inelastic,  $D(p, x) = x$ , which will simplify the analysis<sup>3</sup>. To guarantee existence of a relevant equilibrium, it will be assumed there is a price cap,  $\bar{p}$ , in the spot market. Also, proportional rationing will be used when required.

The literature on supply function equilibrium shows that when demand is certain or when it is uncertain but with the highest possible demand,  $M$  in this case, lower than total installed capacity,  $k_1 + k_2$ , there exist multiple equilibria in the spot market (see Klemperer and Meyer (1989)). However, when there is positive probability of both capacity constraints binding, the spot market has a unique equilibrium (see Holmberg (2004) and Aromí (2007)). For this reason, it is assumed that  $M > k_1 + k_2$ <sup>4</sup>.

Firms' supply functions depend on the forward portfolio,  $\mathbf{h}$ . However, for ease

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<sup>3</sup>This might seem a strong assumption. However, for example, wholesale electricity demand can be closely approximated by an inelastic demand. Moreover, the modeling in this chapter fits the functioning of most wholesale electricity markets.

<sup>4</sup>This is also a reasonable assumption in many markets, and particularly in wholesale electricity markets. Another option is to interpret  $D(p, x)$  as the residual demand after subtracting the bids from non-strategic firms.

of notation, such reference will be suppressed hereafter,  $s_l(p) \equiv s_l(p; \mathbf{h})$ . Since the spot market is modeled as a uniform-price auction, the equilibrium spot price for a given profile of supply functions,  $(s_1(p), s_2(p))$ , and quantity demanded,  $x$ , is the lowest price that clears the market:

$$p(x, \mathbf{s}) = \begin{cases} \inf \{p \in [0, \bar{p}] : x \leq s_1(p) + s_2(p)\} & \text{if } x < S(\bar{p}) \\ \bar{p} & \text{otherwise} \end{cases} \quad (3.1)$$

where  $\mathbf{s} = (s_1, s_2)$  and  $S(p)$  is the aggregate supply function.

Remember that the payoff of a forward transaction, when firms have submitted the profile  $\mathbf{s}$  of supply functions and  $x$  is the realization of the spot demand, is just  $(p^h - p(x, \mathbf{s}))$  per unit, with  $p^h$  being the forward price. Hence, if firm  $l$  sold  $h_l$  units in the forward market, its expected profit can be written as:

$$\Pi_l(s_l, s_{-l}; \mathbf{h}) = E \left[ p(x, \mathbf{s}) q_l(x, \mathbf{s}) - C_l(q_l(x, \mathbf{s})) + (p^h - p(x, \mathbf{s})) h_l \right] \quad (3.2)$$

where  $q_l(x, \mathbf{s})$  is the quantity delivered in equilibrium by firm  $i$  for a given realization of the demand and a given pair of supply functions. If there is no excess demand,  $q_l(x, \mathbf{s}) = s_l(p(x, \mathbf{s}))$ , otherwise  $q_l(x, \mathbf{s}) < s_l(p(x, \mathbf{s}))$  due to rationing<sup>5</sup>.

The goal of firm  $l$  when choosing its spot market supply function,  $s_l(p)$ , is to maximize its expected profits, represented by (3.2), subject to its capacity constraint and taking date 0 forward sales as given.

Aromí (2007) characterizes the unique equilibrium when there is no forward trading. The remainder of this section extends his results to the case where firms

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<sup>5</sup>When there is excess demand at the equilibrium price  $p$ , then  $q_l(x, \mathbf{s}) = \underline{s}_l(p) + (x - \underline{S}(p)) \frac{\bar{s}_l(p) - \underline{s}_l(p)}{\bar{S}(p) - \underline{S}(p)}$ , where  $\underline{s}_l(p) \equiv \lim_{\epsilon \rightarrow 0} s_l(p - \epsilon)$ ,  $\bar{s}_l(p) \equiv \lim_{\epsilon \rightarrow 0} s_l(p + \epsilon)$  and the same applies for the aggregate supply.

have previously sold forwards. Let  $c_l(q_l)$  represent the marginal cost function of firm  $l$ , and define  $p_0 = \inf \{p \geq 0 : s_1(p) > 0 \text{ and } s_2(p) > 0\}$ .

**Lemma 8** *When firms have sold forward, the equilibrium supply functions are continuous for every price  $p \in (p_0, \bar{p})$  and  $0 \leq p_0 \leq \max \{c_1(0), c_2(0)\}$ .*

**Proof.** This lemma states that if both firms are offering strictly positive quantity in equilibrium and the spot price is below the price cap, equilibrium supply functions are continuous when firms have sold forwards. This proof follows Aromí's proof for the case when firms did not sell forward.

Assume firm  $-l$  offers  $(\bar{q}_{-l} - \underline{q}_{-l}) > 0$  at a price  $p^* \in (p_0, \bar{p})$ . For any subset  $[p^*, p^* + \varepsilon]$  firm  $l$  must offer additional quantity, otherwise firm  $-l$  can profitably deviate by withholding supply at  $p^*$ . Let's define  $p_l^\varepsilon(p^*) = \inf \{p : s_l(p) \geq \underline{s}_l(p^*) + \varepsilon\}$ , and observe that  $\lim_{\varepsilon \rightarrow 0} p_l^\varepsilon(p^*) = p^*$ .

For example, firm  $l$  can deviate by submitting the following supply function:

$$\widehat{s}_l^\varepsilon(p) = \begin{cases} \underline{s}_l(p^*) + \varepsilon & \text{if } p \in (p^* - \varepsilon, p_l^\varepsilon(p^*)) \\ s_l(p) & \text{otherwise} \end{cases} \quad (3.3)$$

The effect of this deviation on the expected profits can be split in two parts, a loss from lower prices,  $\Omega^\varepsilon$ , and a gain from larger sales,  $\Gamma^\varepsilon$ .

The loss is bounded above by:

$$\Omega^\varepsilon < (p_l^\varepsilon(p^*) - p^* + \varepsilon) (s_l(p_l^\varepsilon(p^*)) - h_l) Pr^\varepsilon(\Delta p) \quad (3.4)$$

$Pr^\varepsilon(\Delta p)$  is the probability that the price changes due to the deviation by firm  $l$ , and clearly it converges to zero as  $\varepsilon$  does so. Moreover, the difference in prices

also converges to zero with  $\varepsilon$ , hence the derivative of the upper bound is zero at  $\varepsilon = 0$ .

Now, the gain  $\Gamma^\varepsilon$ , is bounded below by:

$$\Gamma^\varepsilon > (p^* - \varepsilon - c_l(\underline{s}_l(p^*) + \varepsilon)) \Delta E^\varepsilon(q_l) \quad (3.5)$$

The unit markup is strictly positive at  $\varepsilon = 0$ . In addition, as Aromí shows, the change in expected quantity,  $\Delta E^\varepsilon(q_l)$ , is strictly increasing in  $\varepsilon$  at  $\varepsilon = 0$  and it is independent of forward transactions. Therefore, this deviation is profitable even in the case where firms sell forwards. ■

**Lemma 9** *When firms have sold forward, the equilibrium supply functions are strictly increasing at every  $p \in (p_0, \bar{p})$ .*

**Proof.** This lemma states that if both firms are offering strictly positive quantity in equilibrium and the spot price is below the price cap, equilibrium supply functions are strictly increasing when firms have sold forward. Besides a minor change on the lower bound for the gains in terms of prices to allow firms to sell forward, this proof follows step by step Aromí's proof of its lemma 2.

When firms did not trade ahead of the spot market, if firm  $l$  offers the same quantity for  $p \in [\underline{p}, \bar{p}]$  there are two possible cases. If firm  $-l$  is offering additional units for that range of prices, then firm  $-l$  can increase its expected profits by withholding supply for that range of prices. If no firm offers additional units for that range of prices, firm  $l$  can withhold supply at every  $p \in (\underline{p} - \varepsilon, \bar{p})$ , and increase its expected profits.

For example, firm  $l$  can deviate to:

$$\tilde{s}_l^\varepsilon(p) = \begin{cases} s_l(\underline{p} - \varepsilon) & \text{if } p \in (\underline{p} - \varepsilon, \bar{p}) \\ [s_l(\underline{p} - \varepsilon), s_l(\underline{p})] & \text{if } p = \bar{p} \\ s_l(p) & \text{otherwise} \end{cases} \quad (3.6)$$

The losses in terms of quantities are bounded above by:

$$\Omega^\varepsilon < \underline{p} (s_l(\underline{p}) - s_l(\underline{p} - \varepsilon)) (F(s_l(\underline{p}) + s_{-l}(\underline{p})) - F(s_l(\underline{p} - \varepsilon) + s_{-l}(\underline{p} - \varepsilon))) \quad (3.7)$$

Moreover, the upper bound converge to zero as  $\varepsilon$  converges to zero and its derivative is also zero at  $\varepsilon = 0$ . Now the gains in terms of prices are bounded below by:

$$\Gamma^\varepsilon > (\bar{p} - \underline{p}) (s_l(\underline{p} - \varepsilon) - h_l) (F(s_l(\underline{p}) + s_{-l}(\underline{p})) - F(s_l(\underline{p} - \varepsilon) + s_{-l}(\underline{p} - \varepsilon))) \quad (3.8)$$

The lower bound is strictly increasing in  $\varepsilon$  at  $\varepsilon = 0$ , even after firms sold forward. ■

Since equilibrium supply functions are strictly increasing at every price on the interval  $(p_0, \bar{p})$ , no firm offers in equilibrium its entire installed capacity at a price below the price cap. Aromí showed that when no firm has traded ahead of the spot market, both firms offer all of their installed capacity at the price cap, and the equilibrium supply function of at least one firm is continuous at  $\bar{p}$ . That result still holds when firms have sold forward at date 0. The reason is it is not profitable for firms to reduce the quantity supplied at the price cap below its installed capacity, even when firms have sold forward, since the price can not go higher. Moreover, since

equilibrium supply functions are continuous for prices up to  $\bar{p}$ , if  $\lim_{p \rightarrow \bar{p}} q_l(p) < k_l$  for both firms, at least one of them will find profitable to deviate and sell a larger quantity at prices just below the price cap, no matter whether they sold forward or not.

Firm  $l$ 's residual demand is  $d_l(p; x) \equiv \max\{0, x - s_{-l}(p)\}$ . The collection of price-quantity points that maximize firm  $l$ 's ex-post profits given firm  $-l$ 's supply function form an ex-post optimal supply function. Since the uncertainty in the model, which comes from the additive demand shock, causes firm  $l$ 's residual demand to shift horizontally without affecting its slope, the ex-post optimal supply function is also the ex-ante optimal supply function, the one that maximizes (3.2) given firm  $-l$ 's supply function. This equivalence between ex-ante and ex-post optimal supply functions holds as long as firm are risk-neutral and the uncertainty can be represented by a single random variable that only affects firms' residual demand additively. If firms are risk-averse and the uncertainty in the model enters additively, the ex-ante optimal supply function might not be equivalent to the supply function that maximizes ex-post utility, but it will be equivalent to the one that maximizes ex-post profit.<sup>6</sup>

Therefore, firm  $l$ 's optimization problem can be represented as one where firm  $l$  chooses the clearing price that maximizes its profit for each particular level of

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<sup>6</sup>See Hortaçsu and Puller (2007) for a discussion of the case where firms have private information and are risk-averse.



demand, given its competitor (firm  $-l$ ) supply function.

$$\begin{aligned} \max_{p(x)} & \left[ p(x) (x - s_{-l}(p(x))) + (p^h - p(x)) h_l - C_l(x - s_{-l}(p(x))) \right] \\ & s.t. \ 0 \leq x - s_{-l}(p(x)) \leq k_l \end{aligned} \quad (3.9)$$

Both firms first order conditions for an interior solution define a system of differential equations which characterizes the equilibrium supply functions once  $x - s_{-l}(p(x))$  is replaced by  $s_l(p(x))$ :

$$s'_2(p(x)) = \frac{s_1(p(x)) - h_1}{p(x) - c_1(s_1(p(x)))} \quad (3.10)$$

$$s'_1(p(x)) = \frac{s_2(p(x)) - h_2}{p(x) - c_2(s_2(p(x)))} \quad (3.11)$$

The bottom conditions of the equilibrium supply functions depend on both firms' forward transactions. The following lemma characterizes them.

**Lemma 10** *The equilibrium supply function of firm  $l$  satisfies:  $s_l(c_l(h_l)) = h_l$ ; and  $\forall p < p_0$   $s_l(p) = h_l$  if  $c_l(h_l) \leq p_0$  otherwise  $s_l(p) = s_l(p_0) < h_l$ ,  $l = 1, 2$ .*

**Proof.** Define  $s_l^*(p)$  as firm  $l$ 's equilibrium supply function, and remember that  $p(x)$  is the equilibrium price as a function of demand. Therefore,  $s_l^*(p(x)) = \min \{x - s_{-l}^*(p(x)), k_l\}$  represents the quantity firm  $l$  supplies in equilibrium when demand is  $x$ . Now, if  $s_l^*(p)$  and  $s_{-l}^*(p)$  are continuous and strictly increasing ( $p > p_0$ ) equation (3.10) or (3.11) and  $s_{-l}^*(p)$  define  $s_l^*(p)$ . It is easy to see that when  $x - s_{-l}^*(p(x)) = h_l$ , the only price that satisfy the FOCs is  $p(x) = c_l(h_l)$ .

Now for  $p < p_0$  at least one firm is offering zero quantity in equilibrium. Hence, there are two possible cases  $s_l^*(p_0) = 0$ , which implies  $s_l^*(p) = 0$  for all  $p < p_0$ , and  $s_l^*(p_0) > 0$ , which means firm  $l$ 's residual demand equals the  $\min \{x, k_l\}$

and therefore it is inelastic at every  $p < p_0$ . If  $p_0 < c_l(h_l)$ , then  $s_l^*(p_0) < h_l$ . When  $0 < x < s_l^*(p_0)$ , firm  $l$ 's residual demand is lower than the quantity hedged by its forward sales and it is inelastic, hence  $\frac{\partial \pi_l^h(x)}{\partial p} < 0$  at every  $p < p_0$  and for every  $0 < x < s_l^*(p_0)$ , where  $\pi_l^h(x)$  is firm  $l$ 's ex-post profit. Therefore,  $p(x) = 0 \forall x < s_l^*(p_0)$ ; which means,  $s_l^*(p) = s_l^*(p_0) \forall p < p_0$ .

When  $c_l(h_l) \leq p_0$ , and  $x < h_l$ ,  $\frac{\partial \pi_l^h(x)}{\partial p} < 0$  at every  $p < c_l(h_l)$ , for the same reasons explained above, then  $p(x) = 0$ . However, when  $h_l < x < s_l^*(p_0)$ , firm  $l$ 's residual demand is higher than its contract holdings and also inelastic at any price below  $p_0$ , hence  $\frac{\partial \pi_l^h(x)}{\partial p} > 0$  for all prices in that range, and  $p(x) = p_0$ . Therefore,  $s_l^*(p) = h_l \forall p < p_0$ . ■

Since forwards are assumed to be financial contracts, a firm's residual demand might be lower than its forward portfolio. In that case the firm can be seen not as a seller in the spot market, but as a net buyer. To see this, rewrite (3.9) as  $\pi_i^h = p(s_i(p) - h_i) + C_i(s_i(p)) + p^h h_i$ , where the reference to  $x$  have been suppressed for ease of notation. If  $d_i(p) < h_i$ , then  $s_i(p) < h_i$ , therefore, the first term which is the net revenue from the spot market would be negative. In that event, firm  $i$  does not have any incentive to exercise monopoly power over its residual demand by pushing the equilibrium price as high as it is profitable. On the contrary, it has an incentive to exercise monopsony power by driving down the equilibrium price as much as it is profitable. For example, if  $0 < s_i(p_0) < h_i$ , the optimal strategy for firm  $i$  will be to offer any quantity below  $s_i(p_0)$  at the lowest possible price, which is zero. This is the intuition behind lemma 10.

Lemmas 8 through 10 and equations (3.10) and (3.11) characterize the equilib-

rium profile of supply functions. It can be easily seen from the system of first order conditions and the proof of lemma 10 that such supply functions are actually mutual best responses; hence, there exists at least one equilibrium of the spot market. As Aromí shows for the case when firms have sold no contracts, the monotonicity and continuity of the profile of supply functions with respect to the boundary conditions at the price cap ensure uniqueness of the equilibrium in supply functions. It can also be seen from equations (3.10) and (3.11) that when firms sold in the forward market, the profile of supply functions defined by those equations is also monotonic and continuous with respect to the boundary conditions at  $\bar{p}$ . Hence, the equilibrium defined by lemmas 1 through 3 and equations (3.10) and (3.11) is the unique equilibrium in supply functions when firms have previously sold forwards.

When firm  $l$  sells forward contracts, its marginal net revenue from the spot market decreases, but its marginal cost remains unchanged. Hence, given the strategy of its competitor, if firm  $l$  sold forward contracts, its strategy in the spot market,  $s_l(p)$ , becomes more aggressive (i.e. bids lower prices) than if it did not sell ahead of the spot market. As equation (3.10) or (3.11) show, given the strategy of firm  $-l$ , the higher is  $h_l$ , the lower is the price chosen by firm  $l$  for any realization of  $d_l(x)$ . Hence selling ahead of the spot market shifts firm  $l$ 's supply function outwards. Therefore, a forward sale is just a credible commitment device for a more aggressive selling strategy in the spot market.

When firms face no capacity constraints and there is no price cap, there exist multiple equilibria of the spot market. Also when marginal costs are constant and symmetric, with  $C_l(q_l) = cq_l$ , the supply functions are linear in price in every

equilibria. Moreover, their slopes, which are symmetric, are independent of forward positions and only the intercept of each firm's equilibrium supply function depends on its own forward position<sup>7</sup>. Hence, in this case there is a clear relationship between forward transactions and equilibrium spot supply functions, which is not necessarily true when equilibrium supply functions are non linear in price.

The goal of this chapter is to study how capacity constraints shape firms' incentives for strategic forward trading. Hence, assuming constant and symmetric marginal costs is a sensible choice, since in this way the effect of capacity constraints can be clearly identified. When firms are capacity constrained and there is a price cap<sup>8</sup>, the supply functions in the unique equilibrium are still linear in prices. However, now not only the intercept depends on forward positions, but also the slope of the equilibrium supply functions. The reason is equilibrium supply functions are strictly increasing at every price on the interval  $(p_0, \bar{p})$ , which means no firm offers in equilibrium its total installed capacity at a price below the price cap. Let's define  $ka_l = \max \{0, k_l - h_l\}$  as firm  $l$ 's adjusted capacity and  $ka_m = \min \{ka_1, ka_2\}$ . The following expressions define the equilibrium supply functions:

$$s_l(p) = \begin{cases} \alpha_l + \beta p_0 & p \in [0, p_0] \\ \alpha_l + \beta p & p \in (p_0, \bar{p}) \\ k_l & p = \bar{p} \end{cases} \quad l = 1, 2 \quad (3.12)$$

with

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<sup>7</sup>The difference among all the possible equilibria for a given profile of forwards,  $\mathbf{h}$ , is just the slope of the supply functions.

<sup>8</sup>The role of the price cap is to ensure the existence of a relevant equilibrium, otherwise firms would be offering every unit at a price of infinite.

$$\alpha_l = h_l - \beta c, \quad \beta = \frac{ka_m}{\bar{p} - c} \quad (3.13)$$

$$p_0 = c - \frac{\min\{h_1, h_2\}}{\beta} \quad (3.14)$$

As lemma 10 states, when  $p = c$ , the quantity supplied by firm  $l$  equals its short forward position,  $s_l(c) = h_l$ . Additionally, if  $ka_1 < ka_2$ , then the supply function of firm 1 is continuous at  $\bar{p}$ , while the  $\lim_{p \rightarrow \bar{p}} s_2(p) = k_2 - (ka_2 - ka_1) < k_2$ , which means firm 2 withholds  $(ka_2 - ka_1)$  units.

Firm  $l$  is more aggressive than firm  $-l$  in the spot market, if it offers a larger quantity than firm  $-l$  at every price in  $(p_0, \bar{p})$ . Now, this implies the relatively less aggressive firm is the one that withholds part of its installed capacity in the spot market. In the case of constant symmetric marginal costs, the difference in adjusted capacity not only indicates which firm is the most aggressive one in the spot market, but it also represents the quantity withheld in that market. When firms have symmetric constant marginal costs, but asymmetric installed capacities and they did not sell ahead of the spot market, the optimal strategy for the largest firm is to mimic the other firm at prices on the interval  $[c, \bar{p})$  and then offer its extra capacity at the price cap<sup>9</sup>. Since in equilibrium firms offer any quantity up to its forward holdings at prices below their marginal cost, firm  $l$ 's adjusted capacity represents the portion of firm  $l$ 's installed capacity that is offered at prices above marginal and average cost. Therefore, the firm with the largest adjusted capacity,

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<sup>9</sup>Since the smaller firm has already exhausted its capacity, consumers can only buy from the largest firm. Hence, the optimal price is  $\bar{p}$ .

the less aggressive one, withholds its extra adjusted capacity and offers it at the price cap.

When costs are not symmetric, which firm is relatively more aggressive depends not only on the difference in adjusted capacity, but also on the cost difference. For example, if firms are symmetric in capacity and they have not sold any forward contracts, but their constant marginal costs are different, the firm with the lowest marginal and average cost will be more aggressive in the spot market, even though both firms have exactly the same adjusted capacity. Therefore, difference in adjusted capacity as well as difference in costs are the factors that determine which firm will be relatively more aggressive in the spot market.

### 3.2.2 Forward Market

At date 0 firms compete in the forward market by choosing the amount of forwards they want to sell, while competitive traders take forward positions. The competitive assumption together with the neutrality toward risk by firms and traders implies that (3.2) becomes:

$$\Pi_l(h_l, h_{-l}) = E[p(x, \mathbf{h}) q_l(x, \mathbf{h}) - cq_l(x, \mathbf{h})] \quad (3.15)$$

Equations (3.12) - (3.14) define the equilibrium supply functions in the spot market. Now using them and the demand,  $D(p, x) = x$ , the equilibrium spot price

for a given vector of forward transactions can be written as:

$$p(x; \mathbf{h}) = \begin{cases} 0 & 0 \leq x \leq \underline{S} \\ \frac{x-h_1-h_2}{2\beta} + c & \underline{S} < x < \bar{S} \\ \bar{p} & x \geq \bar{S} \end{cases} \quad (3.16)$$

where,  $\underline{S} \equiv \lim_{p \rightarrow 0} S(p) = h_1 + h_2 + 2\beta(p_0 - c)$  and  $\bar{S} \equiv \lim_{p \rightarrow \bar{p}} S(p) = |ka_1 - ka_2|$ . Both firms' equilibrium supply functions, and therefore the aggregate supply function, are strictly increasing for demand realizations on the interval  $(\underline{S}, \bar{S})$ .

The quantity delivered by each firm in equilibrium in the spot market depends on the demand realization, installed capacity and forward sales. When  $x \in [0, \underline{S}]$ , if  $p_0$  is strictly positive the quantity delivered by firm  $l$  can be  $x$  or zero, depending on whether firm  $l$  supplies a strictly positive quantity at  $p_0$  or not.<sup>10</sup> However, if  $p_0$  equals zero, firm  $l$  delivers  $x \frac{\alpha_l}{\alpha_l + \alpha_{-l}}$  in equilibrium in the spot market<sup>11</sup>. When  $x \in (\underline{S}, \bar{S})$  the quantity delivered by firm  $l$  is given by  $p(x; \mathbf{h})$  and (3.12). If  $x \in [\bar{S}, k_1 + k_2)$ , the firm with the lowest adjusted capacity delivers its entire installed capacity, while the other firm delivers the extra quantity needed to match demand, and when  $x \in (k_1 + k_2, M]$  each firm delivers its entire installed capacity. Hence, firm  $l$ 's spot profit as a function of the demand realization and forward sales is the following:

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<sup>10</sup>Remember that  $p_0 = \inf \{p : s_1(p) > 0 \text{ and } s_2(p) > 0\}$ . Therefore, if  $p_0$  is strictly positive and lower than the marginal cost, at most one firm offers a strictly positive quantity at this price.

<sup>11</sup>This comes from assuming proportional rationing when there is excess supply.

- If  $0 \leq x \leq \underline{S}$

$$\pi_l(x, \mathbf{h}) = \begin{cases} 0 & \text{if } h_l \leq \beta c \text{ and } h_l \leq h_{-l} \\ -cx & \text{if } h_l > h_{-l} \text{ and } h_{-l} \leq \beta c \\ -cx \frac{\alpha_l}{\alpha_1 + \alpha_2} & \text{if } h_l > \beta c \text{ and } h_{-l} > \beta c \end{cases} \quad (3.17)$$

- If  $\underline{S} < x < \bar{S}$

$$\pi_l(x, \mathbf{h}) = \frac{(x - h_{-l})^2 - h_l^2}{4\beta} \quad (3.18)$$

- If  $\bar{S} \leq x < K$

$$\pi_l(x, \mathbf{h}) = \begin{cases} (\bar{p} - c)(x - k_{-l}) & \text{if } ka_l < ka_{-l} \\ (\bar{p} - c)k_l & \text{if } ka_l \geq ka_{-l} \end{cases} \quad (3.19)$$

- If  $K \leq x \leq M$

$$\pi_l(x, \mathbf{h}) = (\bar{p} - c)k_l \quad (3.20)$$

When there are no capacity constraints, the only effect of a forward sale by firm  $l$  is to make its spot market strategy more aggressive by shifting its supply function outward without changing its slope. Hence, if firm  $l$  does not trigger any response from its competitor when selling forward, then firm  $l$  will have no incentive to sell them. Firm  $l$  could have follow the same more aggressive strategy in the spot market without selling forward, but it did not do so, because it would have decreased its expected profits. Firm  $l$ 's more aggressive strategy weakly increases its sales for every demand realization, but it also weakly decreases the equilibrium spot price, with the latter effect being the dominant one. This is the reason why firms would not take short forward positions if there were no capacity constraints.



When firms are limited on the quantity they can produce, the supply reduction or bid inflation in the spot market depends on the smallest adjusted capacity. If firm  $l$  has the smallest adjusted capacity in the spot market, because it either has the smallest installed capacity or sold the most in the forward market, an increase in firm  $l$ 's forward sales decreases the elasticity of its supply function, and therefore, the elasticity of firm  $-l$ 's residual demand. Hence, firm  $-l$  has an incentive to increase the prices at which it offers every single unit. As a result, when firms are capacity constrained, firm  $-l$ 's response might be strong enough to give firm  $l$  the incentive to sell forward. However, if firm  $-l$ 's adjusted capacity is the smallest of both, an increase on the amount sold ahead of the spot market by firm  $l$  which does not alter the ranking of adjusted capacities does not trigger any response from firm  $-l$ .

Expected spot profit is a continuous function of forward transactions, but this function is not differentiable everywhere. The derivative of  $\Pi_l$  with respect to  $h_l$  does not exist at  $h_l = k_l - k_{-l} + \min\{h_{-l}, k_{-l}\}$  (i.e. where  $ka_l = ka_{-l}$ ). Moreover, the left hand side derivative is negative, while the right hand side derivative will never be smaller than the former and it could even be positive. Hence, it is not guaranteed that  $\Pi_l(h_l, h_{-l})$  is quasi-concave in  $h_l$ <sup>12</sup>. As a consequence, existence of pure-strategy equilibria is not guaranteed for every demand distribution. However, as proposition 8 states, only one particular type of equilibrium might exist.

**Proposition 8** *In every possible pure-strategy equilibrium of the forward market, only one firm sells forward, but less than its installed capacity.*

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<sup>12</sup>The best response correspondences might not be closed-graph (i.e. be upper hemi-continuous).

**Proof.** Equilibrium spot profits depend on the demand realization, the forward positions and installed capacities. There are six different cases for the expected profits depending on the pair of forward sales.

**Case (a)**  $ka_l \geq ka_{-l}$ ,  $h_l \leq \beta c$  and  $h_l \leq h_{-l} \Rightarrow \underline{S} = h_{-l} - h_l$ ,  $\bar{S} = 2k_{-l} - h_{-l} + h_l$

$$\frac{\partial \Pi_l^a}{\partial h_l} = -\frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) \quad (3.21)$$

$\frac{\partial \Pi_l^a}{\partial h_l}$  is strictly negative unless  $h_l = 0$ , when it becomes zero.

**Case (b)**  $ka_l \geq ka_{-l}$ ,  $h_l > h_{-l}$  and  $h_{-l} \leq \beta c \Rightarrow \underline{S} = h_l - h_{-l}$ ,  $\bar{S} = 2k_{-l} - h_{-l} + h_l$

$$\frac{\partial \Pi_l^b}{\partial h_l} = -f(\underline{S}) p_0 \underline{S} - \frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) \quad (3.22)$$

$\frac{\partial \Pi_l^b}{\partial h_l}$  is also strictly negative unless  $h_l = h_{-l} = 0$ .

**Case (c)**  $ka_l \geq ka_{-l}$ ,  $h_l > \beta c$  and  $h_{-l} > \beta c \Rightarrow \underline{S} = \alpha_l + \alpha_{-l} = h_l + h_{-l} - 2\beta c$ ,  $\bar{S} = 2k_{-l} - h_{-l} + h_l$

$$\frac{\partial \Pi_l^c}{\partial h_l} = -\frac{c\alpha_{-l}}{(\alpha_1 + \alpha_2)^2} \int_0^{\underline{S}} x dF(x) - \frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) \quad (3.23)$$

In this case  $\frac{\partial \Pi_l^c}{\partial h_l} < 0$ , since  $\alpha_{-l} = h_{-l} - \beta c$  and this is strictly positive by assumption.

**Case (d)**  $ka_l < ka_{-l}$ ,  $h_l \leq \beta c$  and  $h_l \leq h_{-l} \Rightarrow \underline{S} = h_{-l} - h_l$ ,  $\bar{S} = K - (ka_{-l} - ka_l) = 2k_l - h_l + h_{-l}$

$$\frac{\partial \Pi_l^d}{\partial h_l} = -\frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) + \frac{1}{4\beta} \int_{\underline{S}}^{\bar{S}} \frac{(x - h_{-l})^2 - h_l^2}{k_l - h_l} dF(x) \quad (3.24)$$

**Case (e)**  $ka_l < ka_{-l}$ ,  $h_l > h_{-l}$  and  $h_{-l} \leq \beta c \Rightarrow \underline{S} = h_l - h_{-l}$ ,  $\bar{S} = 2k_l - h_l + h_{-l}$

$$\frac{\partial \Pi_l^e}{\partial h_l} = -c \underline{S} f(\underline{S}) - \frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) + \frac{1}{4\beta} \int_{\underline{S}}^{\bar{S}} \frac{(x - h_{-l})^2 - h_l^2}{k_l - h_l} dF(x) \quad (3.25)$$

**Case (f)**  $ka_l < ka_{-l}$ ,  $h_l > \beta c$  and  $h_{-l} > \beta c \Rightarrow \underline{S} = \alpha_l + \alpha_{-l} = h_l + h_{-l} - 2\beta c$ ,

$$\bar{S} = 2k_l - h_l + h_{-l}$$

$$\begin{aligned} \frac{\partial \Pi_l^f}{\partial h_l} &= \frac{-c\bar{p}h_{-l} + c^2k_l}{(\bar{p} - c)\underline{S}^2} \int_0^{\underline{S}} x dF(x) - \frac{h_l}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) \\ &\quad + \frac{1}{4\beta} \int_{\underline{S}}^{\bar{S}} \frac{(x - h_{-l})^2 - h_l^2}{k_l - h_l} dF(x) \end{aligned} \quad (3.26)$$

Define  $\lambda_l(h_{-l}) = k_l - k_{-l} + h_{-l}$  as the value of  $h_l$  such that  $ka_l = ka_{-l}$ . The derivative of  $\Pi_l(h_l, h_{-l})$  with respect to  $h_l$  does not exist at  $h_l = \lambda_l(h_{-l})$ , since  $\lim_{h_l \rightarrow \lambda_l(h_{-l})^-} \frac{\partial \Pi_l(h_l, h_{-l})}{\partial h_l} < \lim_{h_l \rightarrow \lambda_l(h_{-l})^+} \frac{\partial \Pi_l(h_l, h_{-l})}{\partial h_l}$ . When firm  $l$  is the relatively less aggressive firm ( $ka_l > ka_{-l}$ ), the optimal choice for firm  $l$  is to stay out of the forward market at date 0, as (3.21), (3.22) and (3.23) are strictly negative at every  $h_l \in (0, \lambda_l(h_{-l}))$  and zero at  $h_l = 0$ .

Let's assume  $(h_1^*, h_2^*) \gg 0$  is the equilibrium of the forward market. Since  $\frac{\partial \Pi_l(h_l, h_{-l})}{\partial h_l} < 0 \forall h_{-l}$  as long as  $0 < h_l < \lambda_l(h_{-l})$ , if firm 1 sells a strictly positive amount at date 0, it has to be that  $h_1^* > \lambda_1(h_2^*) = k_1 - k_2 + h_2^*$ , which is the same as  $h_2^* < k_2 - k_1 + h_1^*$ . But this contradicts the assumption that  $h_2^* > 0$ , because this assumption implies  $h_2^* > k_2 - k_1 + h_1^*$ . Therefore,  $(h_1^*, h_2^*) \gg 0$  can not be an equilibrium.

Let's assume without loss of generality that  $k_1 > k_2$ . Now,  $\lambda_1(h_2) > 0 \forall h_2$ , which means there is always an  $h_1$  at which firm 1 will be the less aggressive firm. Therefore,  $\frac{\partial \Pi_1(0, h_2)}{\partial h_1} = 0 \forall h_2$ , since when firm 1 does not sell forwards it is always the less aggressive firm ( $ka_1 > ka_2$ ). In addition,  $\lambda_2(0) < 0$ , hence, firm 2 is the most aggressive at  $\mathbf{h} = (0, 0)$  and  $\frac{\partial \Pi_2^d(0, 0)}{\partial h_2} = \frac{1}{4\beta} \int_{\underline{S}}^{\bar{S}} \frac{x^2}{k_2} dF(x) > 0$ . Therefore, no firm selling forwards at date 0 is not an equilibrium. If  $k_1 = k_2$ , both firms will have the

incentive to sell forwards when its competitor does not sell.

If  $h_l$  tends to  $k_l$ , the relevant cases to focus on are: **(c)** when  $h_{-l} = k_{-l}$ , **(e)** when  $h_{-l} = 0$ , and **(f)** when  $0 < h_{-l} < k_{-l}$ . In all the cases  $\lim_{h_l \rightarrow k_l^{(-)}} \frac{\partial \Pi_l(h_l, h_{-l})}{\partial h_l} < 0$ , as long as  $c > 0$ . Also,  $\frac{\partial \Pi_l(h_l, h_{-l})}{\partial h_l} = 0$  when  $h_l > k_l$ , because selling more than its capacity does not have any impact on the spot market, firm  $i$  is already offering every unit at a price of zero and it can not be more aggressive than that. Hence, no firm hedges its entire capacity.

Therefore, in every possible pure-strategy equilibrium of the forward market, only one firm sells forward, but less than its installed capacity. ■

When a capacity constrained firm commits itself through forward trading to a more competitive strategy in the spot market, its competitor faces a more inelastic residual demand in that market. Hence, its competitor prefers not to follow suit in the forward market and thus behave less competitively in the spot market than it otherwise would, by inflating its bids. A firm has an incentive to sell forwards when its competitor does not sell, because the response it triggers in its competitor is strong enough to increase its expected profits. Also, no firm wants to hedge its entire installed capacity, because the negative impact on the spot price would be too large, since its optimal strategy at date 1 would be to offer every single unit at a price of zero. Finally, there can not be an equilibrium where both firms sell strictly positive amounts at date 0, because only one firm at a time can trigger the necessary response on its competitor to turn a forward sale into a profitable action. Because of capacity constraints a firm's actions in the forward market can change its competitor's strategies in the spot market by affecting its own marginal revenue

in the spot market.

Now, by assuming that demand is uniformly distributed on  $[0, M]$  a closed form solution for the equilibrium forward sales,  $h_i^*$ , can be obtained together with conditions that guarantee existence. This allows the study of some features of the equilibria and particularly of the effect of forward transactions on the allocation of total welfare between consumers, which are represented by the auctioneer, and producers. Define  $c = \delta\bar{p}$ , where  $\delta \in (0, 1)$ .

**Proposition 9** *When  $x \sim U[0, M]$ ,  $(h_j^*, h_i^*) = \left(\frac{2(\bar{p}-c)}{2\bar{p}+c}k_j, 0\right)$  is the equilibrium of the forward market if:*

$$\begin{aligned} \frac{(2 + \delta)}{3^{3/2}\delta} &\geq \frac{k_j}{k_i} \\ \frac{(9\delta^3 + 4(1 - \delta)^3)(2 + \delta)}{36\delta^3 - 9\delta^4 + \frac{16}{3}(1 - \delta)^4} &\geq \frac{k_j}{k_i} \\ \frac{(4 - \delta^2)(1 - \delta)\delta}{2(\delta - \delta^2 + 2\delta^3)} &\geq \frac{k_j}{k_i} \end{aligned}$$

Moreover, assume without loss of generality that  $k_1 > k_2$ . If  $k_1 - k_2 > \hat{\gamma}$ , where  $\hat{\gamma}$  is defined by:

$$\hat{\gamma}^2 + \hat{\gamma} \left( \frac{4\delta}{2 + 3\delta} \right) k_2 - \frac{4}{3} \left( \frac{1 - \delta}{2 + 3\delta} \right) k_2^2 = 0$$

then, there is a unique equilibrium with firm 2 selling  $h_2^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_2$ .

**Proof.** Proposition 8 showed the only possible equilibria are those where one firm, which will be called firm  $i$ , sells forward a quantity smaller than its installed capacity ( $0 < h_i^* < k_i$ ), while the other firm, which will be called firm  $j$ , does not sell forward ( $h_j^* = 0$ ). This proof will be divided in two parts. First, it will be shown that

$h_i^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_i$  is the best response to  $h_j^* = 0$ . Then, it will be shown that  $h_j^* = 0$  is the best response to  $h_i^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_i$ .

If  $k_i \leq k_j$  and  $h_j = 0$ ,  $ka_i$  is certainly smaller than  $ka_j$ ; then  $\Pi_i(h_i, 0)$  is a continuously differentiable function for all  $h_i$  in  $(0, k_i)$ . This corresponds to case **(e)** in the proof of proposition 8, hence  $h_i^*$  is defined by:

$$-c\underline{S}f(\underline{S}) - \frac{h_i}{2\beta} \int_{\underline{S}}^{\bar{S}} dF(x) + \frac{1}{4\beta} \int_{\underline{S}}^{\bar{S}} \frac{(x)^2 - h_i^2}{k_i - h_i} dF(x) = 0 \quad (3.27)$$

where the left hand side is  $\frac{\partial \Pi_i^e}{\partial h_i}$  from equation (3.25). If  $x \sim U[0, M]$ , then the *F.O.C.* becomes:

$$-\frac{2\bar{p}+c}{3}h_i + \frac{2}{3}(\bar{p}-c)k_i = 0 \quad (3.28)$$

and,

$$h_i^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_i \quad (3.29)$$

When  $k_i > k_j$  and  $h_j = 0$ ,  $ka_i$  can be either smaller or larger than  $ka_j$ . Therefore,  $\Pi_i(h_i, 0)$  is not continuously differentiable. If  $h_i = 0$ , it is case **(a)** in the proof of proposition 8,  $\Pi_i(h_i, 0) = \Pi_i^a(h_i, 0)$ . When  $h_i \in (0, \lambda_i(0))$ , it is case **(b)**,  $\Pi_i(h_i, 0) = \Pi_i^b(h_i, 0)$ ; while if  $h_i \in (\lambda_i(0), k_i)$ , it is case **(e)**, with  $\Pi_i(h_i, 0) = \Pi_i^e(h_i, 0)$ . Clearly,  $\Pi_i^a(0, 0) > \Pi_i^b(h_i, 0)$ . Therefore, for  $h_i^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_i$  to be firm  $i$ 's best response to  $h_j = 0$ , it has to be that  $\max_{h_i} \Pi_i^e(h_i, 0)$  is not smaller than  $\Pi_i^a(0, 0)$ . Since  $k_i > k_j$ , then:

$$\Pi_i^a(0, 0) = \frac{(\bar{p}-c)}{M} \left[ \left( \frac{k_i^2}{2} + \frac{k_j^2}{6} \right) + (M - k_i - k_j)k_i \right] \quad (3.30)$$

and

$$\Pi_i^e(h_i^*, 0) = -\frac{ch_i^*}{2M} - \frac{(\bar{p}-c)}{M} \left[ \frac{(k_i - h_i^*)}{3} - k_i k_j - (M - k_i - k_j)k_i \right] \quad (3.31)$$

Now, subtracting both expressions, we have:

$$\Pi_i^e(h_i^*, 0) - \Pi_i^a(0, 0) = -\frac{(\bar{p} - c)}{M} \left[ \left( \frac{k_i^2}{2} + \frac{k_j^2}{6} \right) - k_i k_j + \frac{(k_i - h_i^*)}{3} \right] - \frac{c h_i^*}{2M} \quad (3.32)$$

plugging  $h_i^*$ , defining  $k_i = k_j + \gamma$ , and arranging terms:

$$\Pi_i^e(h_i^*, 0) - \Pi_i^a(0, 0) = \frac{2\bar{p} + 3c}{2(2\bar{p} + c)M} \left[ -\gamma^2 - \gamma \left( \frac{4ck_j}{2\bar{p} + 3c} \right) + \frac{4}{3} k_j^2 \left( \frac{\bar{p} - c}{2\bar{p} + 3c} \right) \right] \quad (3.33)$$

Replacing  $c$  by  $\delta\bar{p}$ , (3.33) becomes:

$$\Pi_i^e(h_i^*, 0) - \Pi_i^a(0, 0) = \frac{2 + 3\delta}{2(2 + \delta)M} \left[ -\gamma^2 - \gamma \left( \frac{4\delta k_j}{2 + 3\delta} \right) + \frac{4}{3} k_j^2 \left( \frac{1 - \delta}{2 + 3\delta} \right) \right] \quad (3.34)$$

Define  $\hat{\gamma}$  as the value of  $\gamma$  such that  $\Pi_i^a(0, 0) - \Pi_i^e(h_i^*, 0) = 0$ . Hence,  $h_i^* = \frac{2(\bar{p}-c)}{2\bar{p}+c}k_i$  can be firm  $i$ 's best response to  $h_j = 0$  only if  $k_i - k_j \leq \hat{\gamma}$ . Now, for  $h_i^*$  to be firm  $i$ 's best response to  $h_j = 0$ ,  $h_i^*$  has to be an interior solution,  $h_i^* \in (\lambda_i(0), k_i)$ , where  $\lambda_i(0) = k_i - k_j$ . If  $\gamma$  were equal to  $h_i^*$ , it can be shown that equation (3.33) would be negative. Therefore,  $\hat{\gamma}$  is smaller than  $h_i^*$ ; which means  $h_i^*$  is firm  $i$ 's best response to  $h_j = 0$ , if  $k_i - k_j \leq \hat{\gamma}$ . Consequently, if  $k_1 > k_2$  and  $k_1 - k_2 > \hat{\gamma}$ , there is no equilibrium where the large firm 1 sells forward.

The next step is to find conditions for  $h_j = 0$  to be firm  $j$ 's best response to  $h_i^*$ . When  $ka_i < ka_j$ ,  $h_j = 0$  is the optimal choice for firm  $j$ , since  $\frac{\partial \Pi_j(h_i, h_j)}{\partial h_j} \Big|_{ka_i < ka_j} < 0$  for all  $h_j > 0$  and  $\frac{\partial \Pi_j(h_i, 0)}{\partial h_j} \Big|_{ka_i < ka_j} = 0$ . The expected profit function is not differentiable, but continuous at  $\lambda_j(h_i^*)$ ; and it is also concave for  $h_j$  in  $(0, \lambda_j(h_i^*))$  and  $h_j$  in  $(\lambda_j(h_i^*), k_j)$ . Hence, if the  $\lim_{h_j \rightarrow \lambda_j(h_i^*)^+} \frac{\partial \Pi_j(h_i^*, h_j)}{\partial h_j}$  is non-positive for cases **(d)**, **(e)** and **(f)**, then  $h_j = 0$  is firm  $j$ 's best response to  $h_i^*$ .

$$\lim_{h_j \rightarrow \lambda_j(h_i^*)^+} \frac{\partial \Pi_j^d}{\partial h_j} \Big|_{h_i^*} = \frac{\bar{p} - c}{M} k_j \left( 1 - \frac{(2\bar{p} + c)^2 k_j^2}{27c^2 k_i^2} \right)$$

$$\begin{aligned}\lim_{h_j \rightarrow \lambda_j(h_i^*)^+} \frac{\partial \Pi_j^e}{\partial h_j} \Big|_{h_i^*} &= \left( c + \frac{16(\bar{p} - c)^4 + 54c^3(\bar{p} - c)}{27(2\bar{p} + c)c^2} \right) k_i - \left( c + \frac{4(\bar{p} - c)^3}{9c^2} \right) k_j \\ \lim_{h_j \rightarrow \lambda_j(h_i^*)^+} \frac{\partial \Pi_j^f}{\partial h_j} \Big|_{h_i^*} &= \left( \frac{c(\bar{p} - c)}{2\bar{p} + c} + \frac{3c^3(\bar{p} - c) + 2c^4}{(2\bar{p} + c)(\bar{p} - c)^2} \right) k_i - \left( c + \frac{c^2}{2(\bar{p} - c)} \right) k_j\end{aligned}$$

Define  $c = \delta\bar{p}$ , where  $\delta \in (0, 1)$ . Now, the conditions for  $\lim_{h_j \rightarrow \lambda_j(h_i^*)^+} \frac{\partial \Pi_j^d}{\partial h_j} \Big|_{h_i^*}$  to be non-positive can be expressed as follows:

$$\begin{aligned}\frac{(2 + \delta)}{3^{3/2}\delta} &\geq \frac{k_i}{k_j} \\ \frac{(9\delta^3 + 4(1 - \delta)^3)(2 + \delta)}{36\delta^3 - 9\delta^4 + \frac{16}{3}(1 - \delta)^4} &\geq \frac{k_i}{k_j} \\ \frac{(4 - \delta^2)(1 - \delta)\delta}{2(\delta - \delta^2 + 2\delta^3)} &\geq \frac{k_i}{k_j}\end{aligned}$$

Where the three conditions are for cases **(d)**, **(e)** and **(f)** respectively. For example, if the installed capacities are symmetric, these conditions will be satisfied for any  $\delta$  approximately smaller than 0.48. ■

A very interesting feature of the equilibrium is that when the asymmetry between firms in terms of their installed capacity is larger than  $\hat{\gamma}$ , there is a unique equilibrium of the forward market, where only the smaller firm sells forward. Obviously, having a unique equilibrium is a very interesting feature, but the unique equilibrium in itself is very striking.

In equilibrium, firms split the two markets (forward and spot) between them. When  $|k_1 - k_2| > \hat{\gamma}$ , the small firm trades mainly through the forward market, while the large firm becomes almost the sole seller in the spot market. There is no equilibrium where the large firm sells forward at date 0, because the small firm is relatively so small that its optimal response to the large firm's more aggressive strategy in the spot market is not enough to offset the downward impact of this



latter strategy on the spot price; and on the forward price through the no arbitrage condition. Hence, when seeing in a market that only the small firm takes a hedge against the uncertain price, it would be risky to draw the standard conclusion that this is a sign the smaller firm is more risk averse than the larger one, since as this chapter shows this might happen even when firms are risk neutral.

When a firm takes a short forward position in equilibrium, the size of the forward sale is independent of the other firm's installed capacity. Hence,  $k_{-l}$  only plays a role on determining whether firm  $l$  sells at date 0, but not on how much it sells when it does.

Since demand is assumed to be inelastic, forward trading can not increase or decrease expected welfare, but it can affect the allocation of the gains from trade between consumers and producers. As the following proposition shows, firms are generally better off in aggregate thanks to forward trading. But, the other side of this story is that consumer are worse off by firms' strategic use of forward trading, since it allows firms to step up their exercise of market power. Define  $\bar{k}$  and  $\underline{k}$  as the largest and smallest installed capacity respectively.

**Proposition 10** *When firms are capacity constrained and the small firm takes a short forward position in equilibrium, strategic forward trading reduces expected consumer surplus. However, when the large firm is the one taking the short position in equilibrium, expected consumer surplus decreases if  $\bar{k} - \underline{k} < \tilde{\gamma}$ , but increases if  $\tilde{\gamma} < \bar{k} - \underline{k} \leq \hat{\gamma}$ ; where  $\tilde{\gamma}$  is defined by:*

$$\tilde{\gamma}^2 + (2\underline{k})\tilde{\gamma} - \frac{(1-\delta)^2}{3(1+2\delta)}\underline{k}^2 = 0 \tag{3.35}$$

**Proof.** Let's assume without loss of generality that  $k_1 \geq k_2$ . When  $x \sim U[0, M]$ ,

$\Pi_1^a(0, 0)$  is given by equation (3.30) and

$$\Pi_2^a(0, 0) = \frac{(\bar{p} - c)}{M} \left[ \left( k_1 k_2 - \frac{k_2^2}{3} \right) + (M - k_1 - k_2) k_2 \right] \quad (3.36)$$

There are two possible equilibria,  $\left( \frac{2(\bar{p}-c)}{2\bar{p}+c} k_1, 0 \right)$  and  $\left( 0, \frac{2(\bar{p}-c)}{2\bar{p}+c} k_2 \right)$ . Let's start with the second equilibrium, the one where the small firm 2 takes a short forward position. The expected profits for both firms are the following:

$$\Pi_1^a(0, h_2^*) = \frac{(\bar{p} - c)}{M} \left[ \frac{k_2^2 + h_2^{*2}}{6} - \frac{k_2 h_2^*}{3} + \frac{k_1^2}{2} + (M - k_1 - k_2) k_1 \right] \quad (3.37)$$

$$\Pi_2^e(0, h_2^*) = -\frac{c h_2^{*2}}{2} + \frac{(\bar{p} - c)}{M} \left[ k_1 k_2 - \frac{(k_2 - h_2^*)^2}{3} + (M - k_1 - k_2) k_2 \right] \quad (3.38)$$

Defining  $\Pi_T(0, 0) = \Pi_1^a(0, 0) + \Pi_2^a(0, 0)$  and  $\Pi_T(0, h_2^*) = \Pi_1^a(0, h_2^*) + \Pi_2^e(0, h_2^*)$ ,

replacing  $h_2^*$  and subtracting, we have

$$\Pi_T(0, h_2^*) - \Pi_T(0, 0) = \frac{2}{3} \frac{(\bar{p} - c)^3}{(2\bar{p} + c)^2} \frac{k_2^2}{M} \quad (3.39)$$

which is strictly positive for all  $\bar{p} > c$ . Since the expected gains from trade are constant, the expected consumer surplus decreases when there is strategic forward trading and the small firm takes a short position.

When the large firm is the one selling forward in equilibrium, the expected profits are the following:

$$\Pi_1^e(h_1^*, 0) = -\frac{c h_1^{*2}}{2M} + \frac{(\bar{p} - c)}{M} \left[ k_1 k_2 - \frac{(k_1 - h_1^*)^2}{3} + (M - k_1 - k_2) k_1 \right] \quad (3.40)$$

$$\Pi_2^a(h_1^*, 0) = \frac{(\bar{p} - c)}{M} \left[ \frac{k_1^2 + h_1^{*2}}{6} - \frac{k_1 h_1^*}{3} + \frac{k_2^2}{2} + (M - k_1 - k_2) k_2 \right] \quad (3.41)$$

Defining  $\Pi_T(h_1^*, 0) = \Pi_1^e(h_1^*, 0) + \Pi_2^a(h_1^*, 0)$ , we have

$$\Pi_T(h_1^*, 0) - \Pi_T(0, 0) = \frac{2}{3} \frac{(\bar{p} - c)}{M} \left[ k_2^2 - \frac{3\bar{p}(\bar{p} + 2c)}{(2\bar{p} + c)^2} k_1^2 \right] \quad (3.42)$$

Rearranging and replacing  $k_1$  by  $k_2 + \gamma$ , we obtain

$$\Pi_T(h_1^*, 0) - \Pi_T(0, 0) = \frac{-2\bar{p}(\bar{p} - c)(\bar{p} + 2c)}{(2\bar{p} + c)^2 M} \left[ \gamma^2 + (2k_2)\gamma - \frac{(\bar{p} - c)^2}{3\bar{p}(\bar{p} + 2c)} k_2^2 \right] \quad (3.43)$$

Replacing  $c$  by  $\delta\bar{p}$ , (3.43) becomes:

$$\Pi_T(h_1^*, 0) - \Pi_T(0, 0) = \frac{-2(1 - \delta)(1 + 2\delta)}{(2 + \delta)^2 M} \left[ \gamma^2 + (2k_2)\gamma - \frac{(1 - \delta)^2}{3(1 + 2\delta)} k_2^2 \right] \quad (3.44)$$

Let's define  $\tilde{\gamma}$  as the value of  $\gamma$  such that strategic forward trading does not impact aggregate expected profits. When  $k_1 - k_2 = \tilde{\gamma}$ , the extra expected profits enjoyed by firm 1 exactly offset the loss experienced by firm 2. As it can be seen from equations (3.34) and (3.44),  $\tilde{\gamma}$  is smaller than  $\hat{\gamma}$ . Hence, when the equilibrium where the large firm sells forward does exist, the effect of strategic forward trading on expected aggregate profits and therefore, on expected consumer surplus, depends on the asymmetry between firms. When the difference between  $k_1$  and  $k_2$  is smaller than  $\tilde{\gamma}$ , consumers are worse off, however, when  $k_1 - k_2 \in (\tilde{\gamma}, \hat{\gamma}]$  consumers and the large firm are better off at the expense of the small firm. ■

### 3.3 Long and Short Forward Positions

On the previous section firms were only allowed to take short positions in the forward market. Now, firms are also allowed to take long positions in the forward market if they find such strategy to be optimal. In this section it will be assumed, for tractability reasons, firms not only have symmetric constant marginal costs, but also symmetric installed capacities,  $k_1 = k_2 = k$ . A positive  $h_l$  represents firm  $l$ 's short position in the forward market, while a negative  $h_l$  represents firm  $l$ 's long position.

### 3.3.1 Spot Market

As it was explained before, when firms sell forward, equilibrium supply functions are strictly increasing and continuous for every price on the interval  $(p_0, \bar{p})$ . Moreover, a forward sale by firm  $l$  has the effect of shifting that firm's equilibrium supply function outwards. Also, the amount withheld by the less aggressive firm equals the difference in adjusted capacity. When firms can take long positions on the forward market, the difference in adjusted capacity can be larger than  $k$ , which has an important consequence on the equilibrium in the spot market. For this reason, the cases  $|ka_l - ka_{-l}| < k$  and  $|ka_l - ka_{-l}| \geq k$  will be studied separately.

#### 3.3.1.1 Case a) $|ka_l - ka_{-l}| < k$

If  $0 < ka_l - ka_{-l} < k$ , then firm  $l$  is the least aggressive firm and its withholding will always be smaller than its entire installed capacity. As it will become clear, this assumption guarantees the existence of a unique pure-strategy equilibria, which is still characterized by equations (3.10) and (3.11) and by updated versions of lemmas 1 to 3. In this case, the equilibrium spot price and firm  $l$ 's expected profit are still represented by equations (3.1) and (3.2).

When firm  $l$  takes a long position in the forward market, its marginal net revenue from the spot market decreases, but its marginal cost remains unchanged. Hence, given the strategy of its competitor, if firm  $l$  buys forward contracts, its strategy in the spot market becomes less aggressive (i.e. bids higher prices) than if it did not buy ahead of the spot market. Therefore,  $p_0$ , which was defined as

$\inf \{p : s_1(p) > 0 \text{ and } s_2(p) > 0\}$ , weakly increases with the amount of forwards bought by firms, and can be higher than the marginal cost. Lemmas 4 and 5 are updated version of lemmas 1 and 2, while the top conditions of equilibrium supply functions described before still hold when firms are allowed to buy forward and  $|ka_l - ka_{-l}| < k$ .

**Lemma 11** *When firms have traded forward, the equilibrium supply functions are continuous for every price  $p \in (p_0, \bar{p})$  and  $0 \leq p_0 < \bar{p}$ .*

**Proof.** This lemma states that if both firms are offering strictly positive quantity in equilibrium and the spot price is below the price cap, equilibrium supply functions are continuous when firms have traded forward as long as  $|ka_l - ka_{-l}| < k$ , with  $l = 1, 2$ . The proof of this lemma is identical to the proof of lemma 8, when  $h_l$  is allowed to be negative. ■

**Lemma 12** *When firms have traded forward, the equilibrium supply functions are strictly increasing at every  $p \in (p_0, \bar{p})$ .*

**Proof.** This lemma states that if both firms are offering strictly positive quantity in equilibrium and the spot price is below the price cap, equilibrium supply functions are strictly increasing when firms have traded forward as long as  $|ka_l - ka_{-l}| < k$ , with  $l = 1, 2$ . The proof of this lemma is identical to the proof of lemma 9, when  $h_l$  is allowed to be negative. ■

As it was explained before, a firm's optimization problem can be represented as one where the firm chooses the clearing price for each particular level of demand,

given its competitor supply function. Therefore, firm  $l$ 's optimization problem for a given demand level  $x$  and a given supply function of firm  $-l$  is still represented by equation (3.9), while equations (3.10) and (3.11) characterize the equilibrium supply functions in the spot market for prices on the interval  $(p_0, \bar{p})$ . The following lemma, which is an updated version of lemma 10, characterizes the bottom conditions of equilibrium supply functions when firms are allowed to buy and sell at date 0.

**Lemma 13** *When firms have traded forwards, firm  $l$ 's equilibrium supply function satisfies:  $s_l(p) = \max\{0, h_l\} \forall p < p_0$ , if  $c \leq p_0$ ; otherwise  $s_l(p) = s_l(p_0) < h_l \forall p < p_0$  and  $s_l(c) = h_l$ .*

**Proof.** This proof is similar to the proof of lemma 10. Let's define  $s_l^*(p)$  as firm  $l$ 's equilibrium supply function, and remember that  $p(x)$  is the equilibrium price as a function of demand. Therefore,  $s_l^*(p(x)) = \min\{x - s_{-l}^*(p(x)), k_l\}$  represents the quantity firm  $l$  supplies in equilibrium when demand is  $x$ . Now, if  $s_l^*(p)$  and  $s_{-l}^*(p)$  are continuous and strictly increasing ( $p > p_0$ ) equation (3.10) or (3.11) and  $s_{-l}^*(p)$  define  $s_l^*(p)$ . It is easy to see that when  $x - s_{-l}^*(p(x)) = h_l$ , the only price that satisfy the FOCs is  $p(x) = c$ . However, when at least one firm buys forward,  $p_0$  is strictly higher than  $c$ . Let's assume that firm  $l$  is the one that defines  $p_0$ ,  $s_l^*(p_0) = 0$ . The FOC for firm  $l$  becomes  $s'_{-l}(p_0) = \frac{-h_l}{p_0 - c}$ . If  $h_l < 0$  (i.e. firm  $l$  buys forward) then  $p_0 > c$ , while if  $h_l > 0$ ,  $p_0 < c$ . Hence, since  $s_l^*(p) \geq 0 \forall p$ , then  $s_l(c) = \max\{0, h_l\}$ .

Now for  $p < p_0$  at least one firm is offering zero quantity in equilibrium. Hence, there are two possible cases  $s_l^*(p_0) = 0$ , which implies  $s_l^*(p) = 0$  for all  $p < p_0$ , and  $s_l^*(p_0) > 0$ , which means firm  $l$ 's residual demand equals the  $\min\{x, k_l\}$

and therefore it is inelastic at every  $p < p_0$ . If  $p_0 < c$ , then  $s_l^*(p_0) < h_l$ . When  $0 < x < s_l^*(p_0)$ , firm  $l$ 's residual demand is lower than the quantity hedged by its forward sales and it is inelastic, hence  $\frac{\partial \pi_l^h(x)}{\partial p} < 0$  at every  $p < p_0$  and for every  $0 < x < s_l^*(p_0)$ , where  $\pi_l^h(x)$  is firm  $l$ 's ex-post profit. Therefore,  $p(x) = 0 \forall x < s_l^*(p_0)$ ; which means,  $s_l^*(p) = s_l^*(p_0) \forall p < p_0$ .

When  $c \leq p_0$ , and  $0 \leq x < h_l$ ,  $\frac{\partial \pi_l^h(x)}{\partial p} < 0$  at every  $p < c$ , for the same reasons explained above, then  $p(x) = 0$ . However, when  $\max\{0, h_l\} < x < s_l^*(p_0)$ , firm  $l$ 's residual demand is higher than its contract holdings and also inelastic at any price below  $p_0$ , hence  $\frac{\partial \pi_l^h(x)}{\partial p} > 0$  for all prices in that range, and  $p(x) = p_0$ . Therefore,  $s_l^*(p) = \max\{0, h_l\} \forall p < p_0$ . ■

Now, it becomes clear that as long as  $|ka_l - ka_{-l}| < k$ , buying forwards has exactly the opposite effect on the spot market strategy than selling them. Given the strategy of its competitor, once firm  $l$  bought forward contracts, its strategy in the spot market,  $s_l(p)$ , becomes less aggressive than if it bought none. As equation (3.10) or (3.11) show, given  $s_{-l}(p)$ , the smaller is  $h_l$ , the higher is the price chosen by firm  $l$  for any realization of  $d_l(x)$ . hence buying ahead of the spot market shifts firm  $l$ 's supply function inwards. Therefore, a forward purchase can be seen just as a credible commitment device for a less aggressive strategy in the spot market. As in the case where firms can only sell ahead of the spot market, when  $|ka_l - ka_{-l}| < k$  there exists a unique pair of equilibrium supply functions, which are also represented by equations (3.12) – (3.14).

### 3.3.1.2 Case b) $|ka_l - ka_{-l}| \geq k$

If firm  $-l$  sells forward contracts for an amount equal to or larger than its installed capacity, it is possible that  $ka_l - ka_{-l} \geq k$ . However, in that case firm  $-l$  offers its entire installed capacity at a price of zero in the spot market. This is optimal because for any demand realization firm  $-l$ 's residual demand is smaller than  $h_j$ , and so firm  $-l$  behaves like a net buyer who exercises monopsony power. As a consequence, the optimal spot market strategy for firm  $l$  is to offer the share of its installed capacity that is not hedged by a forward sale at the price cap and the share that is hedged at a price of zero.

If  $h_l$  and  $h_{-l}$  are both smaller than  $k$ , then  $ka_l - ka_{-l} \geq k$  can be true only if firm  $l$  bought forward contracts. Since installed capacities are symmetric, the quantity withheld by the less aggressive firm, in this case firm  $l$ , equals the difference in forward transactions ( $h_{-l} - h_l$ ). However, in the spot market no firm can withhold more than its entire installed capacity. Hence, if the difference in forward positions is greater than or equal to  $k$ , firm  $l$  will not offer any quantity at prices below  $\bar{p}$ , but it will offer its full installed capacity at the price cap.

The optimal response for firm  $-l$  to a strategy like that is to offer the share of its installed capacity that has not been hedged by forward sales at a price below the price cap, but as close as possible to it<sup>13</sup>. However, that price does not exist. Therefore, there is no pure-strategy equilibrium in the spot market when  $h_l$  and  $h_{-l}$  are both smaller than  $k$  and  $|h_{-l} - h_l| \geq k$ . Nevertheless, there is a mixed-

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<sup>13</sup>Remember that demand is uncertain with support  $[0, M]$  and  $M > k_1 + k_2$ . So, whether one or two firms are needed to satisfy demand depends on the particular demand realization.



strategy equilibrium<sup>14</sup>. Before characterizing this equilibrium, some notation needs to be defined. Firm  $l$ 's withholding will be represented by  $w_l$ . So, if  $w_l = k$ , then  $w_{-l} = \min \{k, ka_{-l}\}$ .

**Proposition 11** *If  $w_l = k$ , there exists a unique mixed-strategy equilibrium of the spot market, where firms name prices according to continuous and strictly increasing probability distribution functions,  $G_l(p)$  and  $G_{-l}(p)$ , with support  $[\underline{p}, \bar{p}]$  where  $\underline{p} \geq c$  and  $G_l(p) \leq G_{-l}(p)$ . Moreover, if  $w_l > w_{-l}$ , firm  $l$  names  $\bar{p}$  with strictly positive probability and  $G_l(p) < G_{-l}(p) \forall p \in (\underline{p}, \bar{p})$ .*

**Proof.** As it was explained before, when  $|ka_l - ka_{-l}| \geq k$  and  $\max \{h_l, h_{-l}\} < k$ , there is no pure-strategy equilibrium of the spot market, however, there is a unique equilibrium in mixed-strategy. Let's define  $G_l(p)$  as the equilibrium mixed-strategy of firm  $l$ , and  $[\underline{p}_l, \bar{p}_l]$  as its support, with  $l = 1, 2$ .

The proof has two parts. In the first part some features of this mixed-strategy equilibrium are proved, while in the second part the unique equilibrium is calculated<sup>15</sup>.

#### **Some features of the equilibrium:**

a)  $\underline{p}_1 = \underline{p}_2 = \underline{p} \geq c$ : In equilibrium no firm names a price below its average cost. If a firm does so, there is a positive probability that it will earn negative profits, but it can increase its expected profits by just naming with the same probability any

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<sup>14</sup>See von der Fehr and Harbord (1992, 1993) who study the spot market when firms offer their entire installed capacity at a unique price.

<sup>15</sup>Some ideas for the proof of proposition 11 come from von der Fehr and Harbord (1992) which analyzes a case with discrete demand distribution, asymmetric costs and no forward trading.

price equal to or greater than  $c$ . In addition, if the lowest price named by firm  $-l$  is  $\underline{p}_{-l}$ , firm  $l$  will never name a price  $p_l < \underline{p}_{-l}$ , otherwise it can increase its expected profits by naming a price  $p'_l \in (p_l, \underline{p}_{-l})$ .

b)  $\bar{p}_l = \bar{p}$  for at least one firm: Let's assume  $\bar{p}_{-l} < \bar{p}_l < \bar{p}$ . When firm  $l$  names  $\bar{p}_l$ , it earns a strictly positive profit per unit,  $(\bar{p}_l - c)$ , only after firm  $-l$  has sold its entire installed capacity. Hence, naming a higher price does not have any impact on the expected quantity it sells, but increases the unit markup. Therefore,  $\bar{p}_l = \bar{p}$  for at least one firm.

c) *There is no interior mass point,  $Prob(p_l = p) = 0 \forall p \in (\underline{p}, \bar{p})$* : Let's assume firm  $-l$  names  $p' \in (\underline{p}, \bar{p})$  with positive probability in equilibrium and no firm sold its entire installed capacity at date 0. First, naming  $p'$  with positive probability is not an optimal strategy for firm  $l$ . If it does so, it will be tied with firm  $-l$ , sharing demand proportionally. Let's define  $m_{-l} = \max\{0, h_{-l}\}$  and  $m_l$  in a similar way. Because of the tie, one element of firm  $l$ 's expected profits is:

$$\begin{aligned} Pr(p_{-l} = p') (p' - c) \frac{w_l}{w_l + w_{-l}} \int_{m_{-l}}^{2k} (x - m_{-l}) dF(x) & \text{ if } h_l < 0 \\ Pr(p_{-l} = p') (p' - c) \left[ m_l + \frac{w_l}{w_l + w_{-l}} \right] \int_{m_l}^{2k} x dF(x) & \text{ if } h_{-l} < 0 \end{aligned} \quad (3.45)$$

However, if firm  $l$  names  $p' - \epsilon$ , with  $\epsilon > 0$ , instead of  $p'$ , there will be no tie. Moreover, when firm  $l$  names  $p' - \epsilon$  and firm  $-l$  names  $p'$ , firm  $l$  will sell before firm  $-l$  sells any quantity at a strictly positive price<sup>16</sup>. As  $\epsilon$  converges to 0, the

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<sup>16</sup>Firm  $j$  will sell  $h_j$  at a price of zero if it sold forward at date 0.

corresponding element in firm  $l$ 's expected profits converges to:

$$\begin{aligned}
Pr(p_{-l} = p') (p' - c) \left[ \int_{m_{-l}}^{k+m_{-l}} (x - m_{-l}) dF(x) + \int_{k+m_{-l}}^{2k} k dF(x) \right] & \text{ if } h_l < 0 \\
Pr(p_{-l} = p') (p' - c) \left[ \int_{m_l}^k x dF(x) + k \int_k^{2k} dF(x) \right] & \text{ if } h_{-l} < 0
\end{aligned} \tag{3.46}$$

The terms on (3.46) are larger than those on (3.45). Also, all the other elements on firm  $l$ 's expected profits when it names  $p' - \epsilon$ , can be made arbitrarily close to all those elements when it names  $p'$ , by choosing a small enough  $\epsilon$ . Therefore, naming  $p'$  with positive probability cannot be part of firm  $l$ 's equilibrium strategy when firm  $-l$  already does it.

Second, if firm  $-l$  names  $p'$  with strictly positive probability there exists an  $\underline{\epsilon} > 0$  such that naming any price on the interval  $(p', p' + \underline{\epsilon})$  is not optimal for firm  $l$ . If firm  $l$  names  $p' + \epsilon$ , with  $\epsilon < \underline{\epsilon}$ , (3.45) converges to (3.47) as  $\epsilon$  approaches 0.

$$\begin{aligned}
Pr(p_{-l} = p') (p' - c) \int_k^{2k} (x - k) dF(x) & \text{ if } h_l < 0 \\
Pr(p_{-l} = p') (p' - c) \left[ m_l \int_{m_l}^{k+m_l} dF(x) + \int_{k+m_l}^{2k} (x - k) dF(x) \right] & \text{ if } h_{-l} < 0
\end{aligned} \tag{3.47}$$

For example, when  $h_l$  is negative, firm  $l$  names a price higher than  $p'$  and  $p_{-l} = p'$ , firm  $l$  only sells after its competitor has exhausted its entire installed capacity. Hence, the increase in price has to be large enough to offset the decrease in expected quantity, for naming a price higher than  $p'$  to be an optimal strategy for firm  $l$ .

Therefore, if when firm  $-l$  names  $p'$  with strictly positive probability, firm  $l$  will not name any price on the interval  $(p', p' + \underline{\epsilon})$ , then firm  $-l$  will be better off by naming a price on such interval instead of naming  $p'$  with positive probability,

which contradicts that the later was part of firm  $-l$ 's equilibrium strategy.

*d) The support of  $G_l(p)$ ,  $[\underline{p}, \bar{p}_l]$ , is a convex set for  $l = 1, 2$ :* Let's assume there is a subset  $A \subset [\underline{p}, \bar{p}]$ , such that no firm names any price on  $A$ . Also, define  $p_a = \inf \{p \mid p \in A\}$  and  $p^a = \sup \{p \mid p \in A\}$ . If  $p_a - \epsilon$ , for some  $\epsilon > 0$ , belongs to the support of firm  $l$ 's mixed-strategy, then firm  $l$  can increase its expected profits by choosing  $p_a + \delta \in A$ , instead of  $p_a - \epsilon$ . As  $\epsilon$  converges to zero, the increase in firm  $l$ 's expected profit converges to:

$$\begin{aligned} & \left[ G_{-l}(p_a) \int_k^{2k} (x - k) dF(x) + [1 - G_{-l}(p^a)] \int_{m_{-l}}^{k+m_{-l}} (x - m_{-l}) dF(x) \right] \delta \quad \text{if } h_l < 0 \\ & \left[ G_{-l}(p_a) \int_{k+m_l}^{2k} (x - k) dF(x) + [1 - G_{-l}(p^a)] \int_{m_l}^k x dF(x) \right] \delta \quad \text{if } h_{-l} < 0 \end{aligned} \quad (3.48)$$

Since both expressions on (3.48) are positive, the support of firm  $l$ 's equilibrium mixed-strategy is convex. Now, if firm  $l$  is the only firm with prices on  $A$  as part of its strategy, firm  $l$  can increase its expected profits by choosing  $p^a + \delta$  instead of any price on  $A$ . For example, switching from  $p^a - \epsilon$  to  $p^a + \delta$ , gives increases in expected profits like those on (3.48) as  $\epsilon$  converges to zero. Hence, the support of firm  $-l$ 's equilibrium mixed-strategy is also convex.

**Calculation of the mixed-strategy equilibrium:** Since installed capacities are symmetric and equal to  $k$ , if  $h_l$  and  $h_{-l}$  are both smaller than  $k$ , then  $ka_l - ka_{-l} \geq k$  can be true only if firm  $l$  bought forwards (more than firm  $-l$ ) at date 0. In this case, firm  $l$ 's withholding,  $w_l$ , equals  $k$ , while  $w_{-l} = \min \{k, ka_{-l}\}$ , since firm  $-l$  could have sold forwards.

Now, the expected profit of firm  $l$  when it names  $p$  and firm  $-l$  plays according to the mixed-strategy  $G_{-l}(p) \equiv \Pr(p_{-l} \leq p)$  is:

$$\begin{aligned}
E_{-l}\Pi_l(p, G_{-l}(p)) &= G_{-l}(p)(p-c) \int_k^{2k} (x-k) dF(x) \\
&+ [1-G_{-l}(p)](p-c) \int_{m_{-l}}^{k+m_{-l}} (x-m_{-l}) dF(x) \quad (3.49) \\
&+ k \int_{k+m_{-l}}^{2k} dF(x) \int_p^{\bar{p}} (s-c) dG_{-l}(s) \\
&+ (\bar{p}-c)k \int_{2k}^M dF(x)
\end{aligned}$$

$E_{-l}$  means the expectation is taken with respect to firm  $-l$ 's mixed-strategy. The first term on the right side is the expected profit when firm  $-l$  undercuts firm  $l$ , but the latter is the marginal firm. The second and third term are when firm  $-l$  names higher prices than firm  $l$ . In the second term the marginal firm is  $l$ , while in the third is  $-l$ . Finally, the last term is when demand happens to be larger than the installed capacity of both firms.

In equilibrium the derivative of  $E_{-l}\Pi_l(p, G_{-l}(p))$  with respect to  $p$  equals zero for all  $p \in [\underline{p}, \bar{p}]$ .

$$\frac{\partial E_{-l}\Pi_l(p, G_{-l}(p))}{\partial p} = g_{-l}(p)(p-c)\eta_{-l} - G_{-l}(p)\mu_{-l} + \nu_{-l} \quad (3.50)$$

where  $g_{-l}(p) \equiv G'_{-l}(p)$  and

$$\eta_{-l} = \int_k^{2k} (x-k) dF(x) - \int_{m_{-l}}^{k+m_{-l}} (x-m_{-l}) dF(x) - k \int_{k+m_{-l}}^{2k} dF(x) \quad (3.51)$$

$$\mu_{-l} = \int_{m_{-l}}^{k+m_{-l}} (x-m_{-l}) dF(x) - \int_k^{2k} (x-k) dF(x) \quad (3.52)$$

$$\nu_{-l} = \int_{m_{-l}}^{k+m_{-l}} (x-m_{-l}) dF(x) \quad (3.53)$$

By setting (3.50) equal to zero,  $g_{-l}(p)$  becomes:

$$g_{-l}(p) = \frac{\mu_{-l}G_{-l}(p)}{\eta_{-l}(p-c)} - \frac{\nu_{-l}}{\eta_{-l}(p-c)} \quad (3.54)$$

The expected profit of firm  $-l$  when it names  $p$  and firm  $l$  plays according to the mixed-strategy  $G_l(p)$  is:

$$\begin{aligned}
E_l \Pi_{-l}(G_l(p), p) &= G_l(p)(p-c) \int_{k+m_{-l}}^{2k} (x-k) dF(x) \\
&+ [1-G_l(p)](p-c) \int_{m_{-l}}^k x dF(x) \\
&+ k \int_k^{2k} dF(x) \int_p^{\bar{p}} (s-c) dG_l(s) \\
&+ (\bar{p}-c)k \int_{2k}^M dF(x)
\end{aligned} \tag{3.55}$$

In the same line as before, in equilibrium  $g_l(p)$  becomes:

$$g_l(p) = \frac{\mu_l G_l(p)}{\eta_l (p-c)} - \frac{\nu_l}{\eta_l (p-c)} \tag{3.56}$$

where

$$\eta_l = \int_{k+m_{-l}}^{2k} (x-k) dF(x) - \int_{m_{-l}}^k x dF(x) - k \int_k^{2k} dF(x) \tag{3.57}$$

$$\mu_l = \int_{m_{-l}}^k x dF(x) - \int_{k+m_{-l}}^{2k} (x-k) dF(x) \tag{3.58}$$

$$\nu_l = \int_{m_{-l}}^k x dF(x) \tag{3.59}$$

If  $h_{-l} \leq 0$ ,  $w_l = w_{-l}$ , then (3.54) and (3.56) are the same. However, when  $h_{-l} > 0$ ,  $w_l > w_{-l}$ . In this last case, firm  $l$  names  $\bar{p}$  with strictly positive probability, otherwise we would get a contradiction when solving (3.54) and (3.56). Since in equilibrium each firm has to leave its competitor indifferent among the prices on  $[\underline{p}, \bar{p}]$  and firm  $-l$  by being the smallest of both is more at risk of being undersold, then firm  $l$  has to be less aggressive by stochastically naming higher prices

By solving (3.54) and using the end conditions ( $G_{-l}(\underline{p}) = 0$  and  $G_{-l}(\bar{p}) = 1$ )

together with the fact that  $G_{-l}(p)$  is continuous at  $\bar{p}$ ,  $G_{-l}(p)$  becomes:

$$G_{-l}(p) = \begin{cases} \lambda_{-l} \ln \left( \frac{p-c}{\bar{p}-c} \right) + 1 & \text{if } x \sim U \\ \left( \frac{p-c}{\bar{p}-c} \right)^{\phi_{-l}} \left( 1 + \frac{\lambda_{-l}}{\phi_{-l}} \right) - \frac{\lambda_{-l}}{\phi_{-l}} & \text{otherwise} \end{cases} \quad (3.60)$$

and

$$\underline{p} = \begin{cases} c + (\bar{p} - c) e^{-\frac{1}{\lambda_{-l}}} & \text{if } x \sim U \\ c + (\bar{p} - c) \left( \frac{\lambda_{-l}}{\phi_{-l} + \lambda_{-l}} \right)^{\frac{1}{\phi_{-l}}} & \text{otherwise} \end{cases} \quad (3.61)$$

where  $\lambda_{-l} = \frac{-\nu_{-l}}{\eta_{-l}}$ ,  $\phi_{-l} = \frac{\mu_{-l}}{\eta_{-l}}$ . Then, solving (3.56) and using  $G_l(\underline{p}) = 0$  and (3.61),  $G_l(p)$  becomes:

$$\begin{aligned} G_l(p) &= \begin{cases} \lambda_l \ln \left( \frac{p-c}{\bar{p}-c} \right) + \frac{\lambda_l}{\lambda_{-l}} & \text{if } x \sim U \\ \left( \frac{p-c}{\bar{p}-c} \right)^{\phi_l} \frac{\lambda_l}{\phi_l} \left( \frac{\lambda_{-l}}{\phi_{-l} + \lambda_{-l}} \right)^{\frac{-\phi_l}{\phi_{-l}}} - \frac{\lambda_l}{\phi_l} & \text{otherwise} \end{cases} \\ G_l(\bar{p}) &= 1 \end{aligned} \quad (3.62)$$

where  $\lambda_l$  and  $\phi_l$  are defined in a similar way as  $\lambda_{-l}$  and  $\phi_{-l}$ . The equilibrium mixed-strategies and the lower bound of their support are different when demand is uniformly distributed, because  $\mu_{-l} = \mu_l = 0$ .

From (3.60) and (3.62) it can be seen that  $G_l(p) < G_{-l}(p)$ ,  $\forall p \in [\underline{p}, \bar{p}]$  when  $w_l > w_{-l}$ . However, when  $w_l = w_{-l}$ ,  $G_l(p) = G_{-l}(p)$ , and they are represented by expression (3.60), while  $\underline{p}$  remains the same. ■

Hence, in equilibrium firm  $l$  offers its entire installed capacity at a single price from the interval  $[\underline{p}, \bar{p}]$ , while firm  $-l$  offers  $w_{-l}$  also at a single price from the same interval. In addition, if firm  $-l$  has a short position in the forward market, firm  $-l$  will offer  $h_{-l}$  at a price of zero in the spot market.

### 3.3.2 Forward Market

At date 0 firms and competitive risk-neutral traders take their preferred forward positions. As before, equation (3.15) represents firm  $l$ 's expected profit. Also, expressions 3.16) and (3.17) to (3.20) represent the equilibrium price and net profit in the spot market when  $|ka_l - ka_{-l}| < k$ . When at date 0 no firm sold its installed capacity nor more than that, and one firm bought so much that  $|ka_l - ka_{-l}| \geq k$ , then at date 1 firms play the mixed-strategies described on proposition 11. In those cases, firm  $l$ 's expected profit is given by  $(\bar{p} - c) \left[ \int_{k+m_l}^{2k} (x - k) dF(x) + \int_{2k}^M kdF(x) \right]$ , where  $m_l = \max \{0, h_l\}$ .

When  $0 < ka_l - ka_{-l} < k$ , an increase in firm  $l$ 's long position gives firm  $l$  an incentive to bid higher prices in the spot market. This action increases the price that firm  $l$  receives in the spot market, but at the same time decreases the quantity it sells. Moreover, this less aggressive bidding behavior in the spot market would only be profitable if firm  $-l$  responds by withholding its supply at prices above  $p_0$ <sup>17</sup>. However, firm  $-l$  might withhold some quantity at prices below  $p_0$ , but it does not change its bidding behavior at any price above  $p_0$  and below the price cap, because the slope of equilibrium supply functions at those prices depends on the smallest adjusted capacity, that of firm  $-l$ , which has not changed. Therefore, no firm has a long position in the forward market in equilibrium.

**Proposition 12** *The equilibria of the forward market when firms can buy or sell*

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<sup>17</sup>If this action were optimal without any change on firm  $-l$ 's bidding behavior, then firm  $l$  would have taken it without increasing its long position in the forward market.



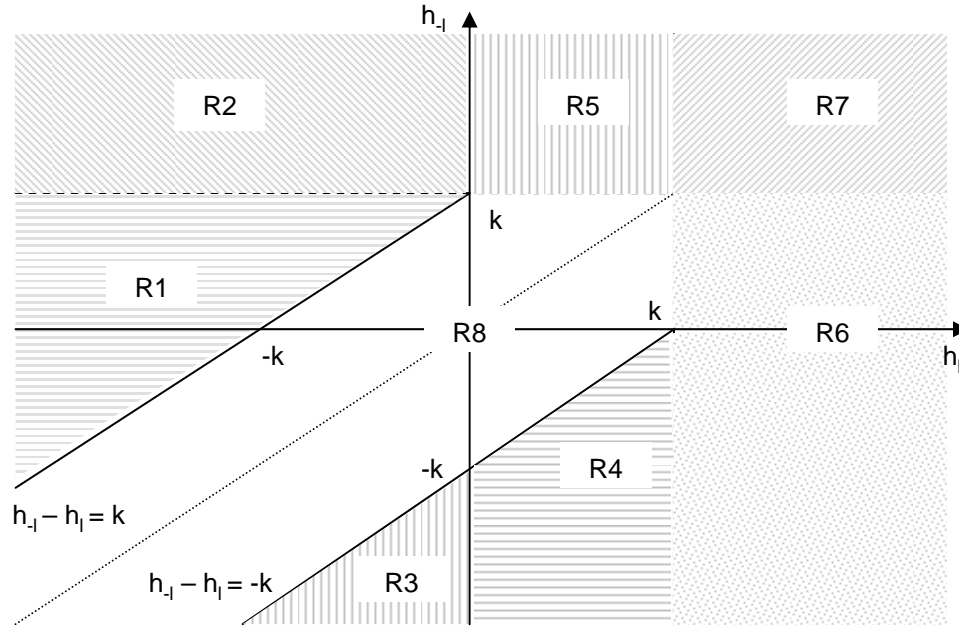


Figure 3.1: Pairs of forward positions organized by firm  $l$ 's expected profit and spot market equilibrium

*forward are the same as those when they can only sell forward.*

**Proof.** The space of all pairs of forward transactions can be divided into eight regions ( $R1, \dots, R8$ ) in terms of the spot market equilibrium and features of expected profit. These regions are represented in Figure 3.1. On regions  $R1$ ,  $R3$  and  $R4$  the equilibrium in the spot market is in mixed-strategies, while on the remaining regions the equilibrium is in pure-strategies, hereafter SFE, for supply function equilibrium.

The expected profit of firm  $l$  is<sup>18</sup>:

$$E\Pi_l(\mathbf{h}) = \begin{cases} (\bar{p} - c) \left[ \int_k^{2k} (x - k) dF(x) + \int_{2k}^M k dF(x) \right] & \text{if } \mathbf{h} \in R1, R2 \text{ or } R3 \\ (\bar{p} - c) \left[ \int_{k+h_l}^{2k} (x - k) dF(x) + \int_{2k}^M k dF(x) \right] & \text{if } \mathbf{h} \in R4 \text{ or } R5 \\ k \int_k^M (p(x) - c) dF(x) & \text{if } \mathbf{h} \in R6 \\ k (\bar{p} - c) \int_{2k}^M dF(x) & \text{if } \mathbf{h} \in R7 \\ \Pi_l^{SFE}(\mathbf{h}) & \text{if } \mathbf{h} \in R8 \end{cases} \quad (3.63)$$

where  $p(x)$  is given by expression (3.16) and  $\Pi_l^{SFE}(\mathbf{h})$  refers to firm  $l$ 's expected profits when the equilibrium in the spot market is a SFE. On the proof of proposition 8 there are six different cases (*a to f*) for  $\Pi_l^{SFE}(\mathbf{h})$ . However, when installed capacities are symmetric there are only four, because cases *b*) and *d*) are not possible<sup>19</sup>.

The  $\lim_{h_l \rightarrow k(-)} \frac{\partial E\Pi_l(\mathbf{h})}{\partial h_l} < 0$ , as long as  $c > 0$  and  $\frac{\partial E\Pi_l(\mathbf{h})}{\partial h_l} = 0$  when  $h_l > k$ . Therefore, firm  $l$  will not sell at date 0 an amount equal to or greater than its entire installed capacity, and because of symmetry the same can be said about firm  $-l$ . Hence, regions *R2*, *R5*, *R6* and *R7* can be ruled out as regions where best responses might intersect.

The next step is to show that for a given  $h_{-l}$ , firm  $l$  is better off by choosing an  $h_l$  in *R8* than any  $h_l$  in *R1*. Firm  $l$ 's expected profit in *R1* is independent

<sup>18</sup> $\Pi_l$  represents firm  $l$ 's expected profit, where the expectation is over the uncertain demand. The  $E$  on  $E\Pi_l$  represents the expectation over firms' spot market strategies. In the case of SFE such expectation is trivial.

<sup>19</sup>For example,  $ka_i > ka_j \Leftrightarrow h_i < h_j$ , therefore, case *b*) does not exist. The opposite inequality rules out *d*).

of  $h_l$ . Also,  $\Pi_l^{SFE}(\mathbf{h})$  converges to firm  $l$ 's expected profit in  $R1$  as  $h_l$  converges to  $(h_{-l} - k)$ . Now,  $\left. \frac{\partial \Pi_l^{SFE}(\mathbf{h})}{\partial h_l} \right|_{h_l < h_{-l}} = -\frac{h_l}{2\beta} \int_{h_{-l}-h_l}^{2k} dF(x)$ , which is strictly positive when  $h_l < 0$ . This derivative does not exist at  $h_l = h_{-l}$ , but when  $h_{-l} < 0$ , the  $\lim_{h_l \rightarrow h_{-l}^{(+)}} \frac{\partial \Pi_l^{SFE}(\mathbf{h})}{\partial h_l} = \frac{1}{4\beta} \int_0^{2k} \frac{x^2 - 2xh_{-l}}{k - h_{-l}} dF(x)$ , which is positive since  $h_{-l} < 0$ . Therefore,  $R1$  and the section of  $R8$  where  $h_l < 0$  and  $h_l < h_{-l}$  can be also ruled out as regions where there might be an equilibrium. By the same reasoning but for firm  $-l$ , it is possible to rule out  $R3$ ,  $R4$  and the section of  $R8$  where  $h_l > h_{-l}$  and  $h_{-l} < 0$ .

Finally, the only region, or subregion, left is  $R8$  where  $\mathbf{h} \geq 0$ . Therefore, if there are equilibria in the forward market, they are the same as those described in proposition 8, when firms were only allowed to sell at date 0. ■

### 3.4 Conclusion

Forward trading allows efficient risk sharing among agents with different attitudes toward risk and improves information sharing, particularly through price discovery. It is also believed that forward trading enhances competition in the spot market. The standard argument claims a firm, by selling forward, can become the leader in the spot market (the top seller), thereby improving its strategic position in the market. Still, every firm faces the same incentives, resulting in lower prices and no strategic improvement for any firm. Due to this effect on competition, forward trading has become a centerpiece of most liberalized electricity markets. However, as this chapter showed, this argument does not hold when firms face capacity con-

straints.

When capacity constrained firms facing common uncertainty compete in a uniform-price auction with price cap, strategic forward trading does not enhance competition. On the contrary, firms use forward trading to soften competition, which leaves consumer worse off. The intuition of this result is that when a capacity constrained firm commits itself through forward trading to a more competitive strategy at the spot market, its competitor faces a more inelastic residual demand in that market. Hence, its competitor prefers not to follow suit in the forward market and thus behaves less competitively at the spot market than it otherwise would, by inflating its bids. Therefore, forward trading allows firms to step up the exercise of market power, which leaves them better off at the expense of consumers.

The results on this chapter generalize to the standard auction case where the auctioneer is the seller and the bidders are the buyers. Bidders in uniform-price auctions have an incentive to reduce demand in order to pay a lower price for their purchases. This incentive grows with the quantity demanded. In a standard auction, when a bidder with demand for a finite quantity buys forward, it behaves like a smaller bidder in the auction. Therefore, the incentive to reduce their bids increases for the other bidders in the auction. Consequently, strategic forward trading intensify demand reduction in standard uniform-price auctions, which reduces seller's expected revenue.

## Chapter 4

### Conclusion

#### 4.1 Sequential Uniform Price Auctions

When choosing among several auction formats, the seller looks for the auction format that is best suited for achieving her main objectives of revenue maximization and efficiency. Sometimes, the seller is also interested in the market that results after the auction, like in spectrum auctions, and prefers an auction that yields a diverse pool of winners even at the expense of revenue maximization and efficiency. One decision that needs to be made by the seller when she has a divisible good for sale is whether to sell the entire supply in one auction or to spread it over several auctions. There are several features of the market that should be considered when deciding between a single auction and a sequence of auctions such as transaction costs, budget or borrowing constraints, private information and bidders's risk aversion.

The seller might prefer a single auction over a sequence of auctions when the transaction costs of bidding in an auction are high relative to the profits bidders can expect to make in that auction. In the event that bidders face budget or borrowing constraints a single auction might limit the quantity they can buy, while in a sequence of auctions bidders have the chance to raise more capital if needed. When there is private information about the value of the good being auctioned, a sequence of sealed-bid auctions improves the discovery of the collective wisdom of the

market relative to a single sealed-bid auction, possibly increasing expected revenue. If infra-marginal bidders are risk averse, the seller might also prefer a sequence of sealed-bid auctions, since that auction format reduces bidders' risk which might increase the seller's expected revenue by increasing participation.

In addition, the effect of strategic bidding on revenue generation and efficiency should be considered when deciding between a single auction and a sequence of auctions. There is an extensive literature that studies equilibrium bidding, revenue generation and efficiency in sequences of single object auctions, such as sequences of first price, second price or even English auctions. However, there is no theoretical nor empirical research that studies sequences of divisible good auctions. Chapter 2 filled that gap in the literature for the case of divisible good auctions with a uniform pricing rule by studying a sequence of two uniform price auctions and comparing its revenue generation properties with those of a single uniform price auction.

In auctions where bidders pay the clearing price for the quantity won, bidders have an incentive to reduce demand (i.e. shade their bids) to pay less for their winnings. This incentive grows with the quantity demanded and is inversely related to bidders' demands. In a sequence of two uniform price auctions, bidders internalize that their bidding in the first auction has an effect on the demand reduction in the later auction. Bidders reduce their demands even more in the first auction with one bidder, usually the largest one, reducing it more than the others and thus strengthening the bid shading or demand reduction in the second auction. Hence, in a sequence of uniform price auctions there is not only *static* demand reduction but also *dynamic* demand reduction.

In any auction within a sequence of single object auctions with the exception of the last, bids are discounted by the option value of participating in later auctions. In the case of a sequence of two uniform price auctions, bids in the first auction are also discounted respect to what they would be in a single uniform price auction. The discount this time represents the option value of increasing the quantity purchased in the later auction.

In a sequence of two uniform price auctions with non-strategic bidders who bid randomly and strategic bidders with, equilibrium bidding in the second auction was shown to be unique and symmetric for any supply split with  $S_2 \geq N\tilde{\lambda}$ . However, this was not the case in the first auction. Nevertheless, first auction equilibrium bids are bounded above by the value of the good discounted by the option value of increasing the quantity purchased in the second auction<sup>1</sup>. Using this upper bound of equilibrium bids, an upper bound of the expected revenue in a sequence of two uniform price auctions was defined.

The *static* and *dynamic* bid shading together with the discounting of the option value of increasing the quantity purchased in the second auction reduce the seller's expected revenue when using a sequence of two uniform price auctions. The *dynamic* bid shading and the option value discounting, which are not present in single uniform price auction, are particularly strong when there are few bidders and at least one of them demands a small share of the supply. These features of equilibrium bidding

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<sup>1</sup>If bidders do not know the actual value of the good and they all receive the same signal about it, then the upper bound is given by the expected value of the good discounted by the option value of increasing the quantity purchased in the second auction.

are even stronger when the supply is split evenly between the two auctions of the sequence. Hence, in those cases it is certainly more profitable for the seller to use a single uniform price auction than a sequence of two uniform price auctions. These results are in line with the finding that it is better for the seller to use a sealed-bid auction than a dynamic auction when competition is not very strong.

## 4.2 Forward Trading and Capacity Constraints

Forward trading allows efficient risk sharing among agents with different attitudes toward risk and improves information sharing, particularly through price discovery. It is also believed that forward trading enhances competition in the spot market. The standard argument claims a firm, by selling forward, can become the leader in the spot market (the top seller), thereby improving its strategic position in the market. Still, every firm faces the same incentives, resulting in lower prices and no strategic improvement for any firm. Due to this effect on competition, forward trading has become a centerpiece of most liberalized electricity markets. However, as chapter 3 showed, this argument does not hold when firms face capacity constraints.

When capacity constrained firms facing common uncertainty compete in a uniform-price auction with price cap, strategic forward trading does not enhance competition. On the contrary, firms use forward trading to soften competition, which leaves consumer worse off. The intuition of this result is that when a capacity constrained firm commits itself through forward trading to a more competitive strategy at the spot market, its competitor faces a more inelastic residual demand in



that market. Hence, its competitor prefers not to follow suit in the forward market and thus behaves less competitively at the spot market than it otherwise would, by inflating its bids. Therefore, forward trading allows firms to step up the exercise of market power, which leaves them better off at the expense of consumers.

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## Appendix A

### Equilibrium Bidding in Sequential Uniform Price Auctions

#### A.1 First Order Conditions

##### A.1.1 Ex-ante Profit Maximization

The expected profit of bidder  $l$  before auction  $t$ , when he bids  $d_{lt}(p)$  and his competitors bid their equilibrium demand functions  $d_{mt}(p)$  with  $m \neq l$ , can be written as:

$$E_t [\Pi_{lt}] = E_t [(v - p_t) d_{lt}(p_t) + I(t) E_{t+1} [\pi_{lt+1}(\mathbf{q}_t)]]$$

where  $I(t)$  is an indicator function which equals one if  $t = 1$  and zero if  $t = 2$ . Remember  $p_t$  is the clearing price in auction  $t$ . Also,  $\pi_{lt+1}(\mathbf{q}_t)$  is the ex-post profit from auction  $t + 1$  when the vector of purchases in auction  $t$  was  $\mathbf{q}_t$ .

Bidder  $l$ 's optimization problem at  $t$  is:

$$\begin{aligned} \max_{d_{lt}(p)} E_t [\Pi_{lt}] \\ \text{s.t. } d_{lt}(p) \leq \lambda_l - q_{l,t-1}(y_{t-1}) \end{aligned}$$

where  $q_{l0} = 0$ .

The most important source of uncertainty in auction  $t$  is the demand from non-strategic bidders in that auction, which translates into uncertainty about the clearing price in that auction,  $p_t$ . Let's define a probability measure over realizations

of the clearing price, from the perspective of bidder  $l$ , conditional on him bidding  $d_{lt}(p)$ , while his competitors bid the demand functions  $\{d_{mt}(p), m \in -l\}$ :

$$H^t(p, d_{lt}(p)) \equiv \Pr [p_t \leq p \mid d_{lt}(p)]$$

By the definition of the clearing price, the event  $p_t \leq p$  is equivalent to  $d_{lt}(p) + \sum_{-l} d_{-lt}(p) \leq y_t$ . This probability distribution can be written as:

$$\begin{aligned} H^t(p, d_{lt}(p)) &= \Pr[d_{lt}(p) + \sum_{-l} d_{-lt}(p) \leq y_t \mid d_{lt}(p)] \\ &= \int I\{d_{lt}(p) + \sum_{-l} d_{-lt}(p) \leq y_t\} dF(y_t) \end{aligned}$$

where  $I\{\cdot\}$  is the indicator function for the enclosed event.

Now, bidder  $l$ 's expected profit maximization can be written as:

$$\max_{d_{lt}(p)} \int_{\underline{p}}^{\bar{p}} [(v - p)d_{lt}(p) + I(t)E_{t+1}[\pi_{lt+1}]] dH^t(p, d_{lt}(p))$$

Integration by parts of the expected profit function yields:

$$c - \int_{\underline{p}}^{\bar{p}} \left( \left( v - p + I(t)E_{t+1} \left[ \frac{\partial \pi_{lt+1}}{\partial d_{lt}} \right] \right) d'_{lt}(p) - d_{lt}(p) \right) H^t(p, d_{lt}(p)) dp$$

where  $c$  is a constant. Now define:

$$M^t(p, d_{lt}, d'_{lt}) \equiv \left( \left( v - p + I(t)E_{t+1} \left[ \frac{\partial \pi_{lt+1}}{\partial d_{lt}} \right] \right) d'_{lt}(p) - d_{lt}(p) \right) H^t(p, d_{lt}(p))$$

Then, the Euler equation becomes:

$$\frac{\partial}{\partial p} M^t_{d'_{lt}} = M^t_{d_{lt}}$$

Evaluating the derivatives,

$$\begin{aligned} M^t_{d'_{lt}} &= \left( v - p + I(t)E_{t+1} \left[ \frac{\partial \pi_{lt+1}}{\partial d_{lt}} \right] \right) H^t(p, d_{lt}(p)) \\ M^t_{d_{lt}} &= \left( I(t)E_{t+1} \left[ \frac{\partial^2 \pi_{lt+1}}{\partial d_{lt}^2} \right] d'_{lt}(p) - 1 \right) H^t(p, d_{lt}(p)) \\ &\quad + \left( \left( v - p + I(t)E_{t+1} \left[ \frac{\partial \pi_{lt+1}}{\partial d_{lt}} \right] \right) d'_{lt}(p) - d_{lt}(p) \right) H^t_{d_{lt}}(p, d_{lt}(p)) \end{aligned}$$

where  $H_{d_{it}}^t(p, d_{it}(p))$  is the derivative of the probability distribution of the clearing price with respect to  $d_{it}$ . Taking the total derivative of  $M_{d_{it}}^t$  with respect to  $p$ :

$$\begin{aligned} \frac{\partial}{\partial p} M_{d_{it}}^t &= \left( I(t) E_{t+1} \left[ \frac{\partial^2 \pi_{it+1}}{\partial d_{it}^2} \right] d_{it}' - 1 \right) H^t(p, d_{it}(p)) \\ &\quad + \left( v - p + I(t) E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{it}} \right] \right) \left( H_p^t(p, d_{it}(p)) + d_{it}'(p) H_{d_{it}}^t(p, d_{it}(p)) \right) \end{aligned}$$

Therefore, the Euler equation becomes:

$$\frac{H_p^t(p, d_{it})}{H_{d_{it}}^t(p, d_{it})} = \frac{-d_{it}(p)}{v - p + I(t) E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{it}} \right]}$$

Now, using the definition of the clearing price, the derivatives of the probability functions become:

$$\begin{aligned} H_p^t(p, d_{it}(p)) &= \frac{\partial}{\partial p} \Pr[d_{it}(p) + \sum_{-l} d_{-it}(p) \leq y_t \mid d_{it}(p)] \\ &= \frac{\partial}{\partial p} [1 - F(d_{it}(p) + \sum_{-l} d_{-it}(p))] \\ &= -f(d_{it}(p) + \sum_{-l} d_{-it}(p)) \sum_{-l} d'_{-it}(p) \end{aligned}$$

and

$$\begin{aligned} H_{d_{it}}^t(p, d_{it}(p)) &= \frac{\partial}{\partial d_{it}} \Pr[d_{it}(p) + \sum_{-l} d_{-it}(p) \leq y_t \mid d_{it}(p)] \\ &= -f(d_{it}(p) + \sum_{-l} d_{-it}(p)) \end{aligned}$$

Hence, the Euler equation becomes:

$$\sum_{-l} d'_{-it}(p) = \frac{-d_{it}(p)}{v - p + I(t) E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{it}} \right]} \quad (\text{A.1})$$

Finally, using  $E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{it}} \right] d_{it}'(p) = \sum_{-l} E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{-it}} \right] d'_{-it}(p)$ , (A.1) becomes:

$$\sum_{-l} \left( v - p - I(t) E_{t+1} \left[ \frac{\partial \pi_{it+1}}{\partial d_{-it}} \right] \right) d'_{-it}(p) = -d_{it}(p) \quad (\text{A.2})$$

### A.1.2 Ex-post Profit Maximization

Assume bidder  $l$  knows the realization of the residual supply at auction  $t$  and his competitors' demand functions at that auction. Also, assume that at stage  $t$ , bidder  $l$  only chooses the price-quantity pair,  $(p_t, q_{lt})$ , that maximizes the sum of his stream of profits and clears the market at  $t$ . Then, his optimization problem in auction  $t$  can be written as:

$$\begin{aligned} \max_{p_t} (v - p_t) \left( y_t - \sum_{-l} d_{-lt}(p_t) \right) + I(t) E_{t+1} [\pi_{lt+1}(\mathbf{q}_t)] \quad (\text{A.3}) \\ \text{s.t. } d_{lt}(p_t) \leq \lambda_l - q_{lt-1}(y_{t-1}) \end{aligned}$$

In this case, the *F.O.C.* for interior bidding becomes:

$$-(y_t - \sum_{-l} d_{-lt}(p_t)) - \sum_{-l} \left( v - p_t - I(t) E_{t+1} \left[ \frac{\partial \pi_{lt+1}(\mathbf{q}_t)}{\partial d_{-lt}} \right] \right) d'_{-lt}(p) = 0 \quad (\text{A.4})$$

Which is identical to the Euler equation in (A.2), once  $(y_t - \sum_{-l} d_{-lt}(p_t))$  has been replaced by  $d_{lt}(p)$ .

## A.2 Second Order Conditions

Since the ex-ante and ex-post profit maximization problems are equivalent under the assumptions made here, only the second order conditions for the latter case will be developed.

$$\begin{aligned} \frac{\partial^2 \Pi_{l1}}{\partial p_1^2} &= \sum_{-l} d'_{-l1}(p_1) + \sum_{-l} (1 + I(t) E_2 \left[ \frac{\partial^2 \pi_{l2}}{\partial d_{-l1} \partial p_1} \right]) d'_{-l1}(p_1) \\ &\quad - \sum_{-l} (v - p_1 - I(t) E_2 \left[ \frac{\partial \pi_{l2}}{\partial d_{-l1}} \right]) d''_{-l1}(p_1) \quad (\text{A.5}) \end{aligned}$$

Evaluating the *F.O.C.* in (A.4) at the equilibrium, and then totally differentiating it with respect to  $p_1$  to obtain an expression for the second term on (A.5) gives:

$$\begin{aligned} \frac{\partial^2 \Pi_{l1}}{\partial p_1^2} &= d'_{l1}(p_1) + \sum_{-l} d'_{-l1}(p_1) \\ &< 0 \end{aligned}$$

Therefore, any solution to equation (A.4) would define a global maximum if bidders had unlimited demands.

Now, when  $t = 2$  and bidder  $l$  wants to consume any quantity up to  $\lambda_l$ , only one demand function that solves (A.4) is a global maximum, and that is the one that also satisfy the end conditions described in lemmas 2 and 3 for the two-bidder case and lemma 6 for the three-bidder case. For every  $y_2 \in [0, N\mu_j]$ , with  $N = 2, 3$ , the demand functions are characterized by (A.4), where the global *S.O.C.s* are satisfied. For  $y_2 \in (N\mu_j, S_2]$  all but one bidder buy all they want to consume, therefore, the best the other bidder can do is to choose a price of zero. Hence, the profiles of second auction bid functions given by equation (2.14) for the two-bidder case and equation (2.33) for the three-bidder case are Nash equilibrium of the second auction.

The sum of ex-post profit from the first auction and expected profit from the second auction,  $\Pi_{l1}$ , is twice continuously differentiable with respect to the first auction clearing price at every price besides  $\tilde{p}_1$ .<sup>1</sup> Hence, when  $t = 1$ , any solution to (A.4) locally maximizes  $\Pi_{l1}$  either on  $(\underline{p}_1, \tilde{p}_1)$  or  $(\tilde{p}_1, \bar{p}_1)$ . Obviously, if the left-hand side and right-hand side derivatives of  $\Pi_{l1}$  with respect to  $p_1$ , evaluated at  $\tilde{p}_1$ , have the same sign, then any solution to (A.4) when  $t = 1$  will globally maximize

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<sup>1</sup>Remember  $\tilde{p}_1$  is defined by  $\lambda_i - d_{i1}(\tilde{p}_1) = \lambda_j - d_{j1}(\tilde{p}_1)$ .

$\Pi_{l_1}$ . If  $S_1 < \sum_l \lambda_l$ , the way demand functions were extended over the interval  $[0, \underline{p}_1)$  guarantees none of these prices will maximize  $\Pi_{l_1}$ . If  $S_1 \geq \sum_l \lambda_l$ , then for some realizations of  $y_1$  all but one bidder buy all the quantity they want to consume, and the best the other bidder can do is to choose a price of zero for those realizations of  $y_1$ . Hence, at least the local *S.O.C.s* are satisfied.

Since the system of differential equations defined by the set of *F.O.C.s* does not have analytical solutions when  $t = 1$ , the only way to find profiles of demand functions which are solutions to that system is through numerical methods. In that case it can be easily checked whether each bidder's demand function is a global maximum conditional on the other bidders' demand functions (i.e. if the profiles are Nash equilibria of the first auction). Tables 2.1 and 2.4.2 present some equilibria and show the set of equilibria is not the empty set.

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