ABSTRACT

Title of dissertation: TOPICS IN MARKET DESIGN

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Dissertation directed by: Professor Lawrence Ausubel
Department of Economics

My dissertation consists of two papers covering distinct topics within Microeconomic Theory. The first chapter is drawn from Matching Theory. One of the oldest but least understood matching problems is Gale and Shapley's (1962) "roommates problem": is there a stable way to assign $2N$ students into N roommate pairs? Unlike the classic marriage problem or college admissions problem, there need not exist a stable solution to the roommates problem. However, the traditional notion of stability ignores the key physical constraint that roommates require a room, and it is therefore too restrictive. Recognition of the scarcity of rooms motivates replacing stability with Pareto optimality as the relevant solution concept. This paper proves that a Pareto optimal assignment always exists in the roommates problem, and it provides an efficient algorithm for finding a Pareto improvement starting from any *status quo*. In this way, the paper reframes a classic matching problem, which previously had no general solution, to become both solvable and economically more meaningful.

The second chapter focuses on the role networks play in market and social

organization. In network theory, externalities play a critical role in determining which networks are optimal. Adding links can create positive externalities, as they potentially make distant vertices closer. On the other hand, links can result in negative externalities if they increase congestion or add competition. This paper will completely characterize the set of optimal and equilibrium networks for a natural class of negative externalities models where an agent's payoff is a function of the degree of her neighbors. These results are in sharp contrast to the optimal and equilibrium networks for the standard class of positive externalities models where payoff is a function of the distance two agents are apart. This highlights the role externalities play in optimal and equilibrium network structure.

TOPICS IN MARKET DESIGN

by

Thayer Morrill

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Advisory Committee:
Professor Lawrence Ausubel, Chair/Advisor
Professor Peter Cramton
Professor Emel Filiz Ozbay
Professor Daniel Vincent
Professor Subramanian Raghavan

Dedication

I would like to dedicate this work to my wife, Melinda Sandler Morrill. I can not imagine going through the process without her. What I can imagine is how much worse the end product would have been.

Acknowledgments

My second year at the University of Maryland was a pivotal year for me. I was lucky enough to have Lawrence Ausubel, Peter Cramton, and Daniel Vincent as instructors. I may not have learned all the economics I know from the three of them, but it is not much of an overstatement to say I learned all the economics I enjoy from them. My committee has been extremely supportive throughout my graduate school experience, and for that, and for all I learned from them, I am extremely grateful.

I would like to particularly thank my adviser Lawrence Ausubel. He is my economic role model even if he more than once wrote empirical papers. I am forever grateful for his mentorship and support.

My wife, Melinda Sandler Morrill deserves special acknowledgment. She has read every word in this dissertation many times over. All of these ideas where shaped while talking to her. If any of the thoughts in these papers are expressed clearly, she is the one who deserves the credit.

I would like to thank Daniel Aromi, Jacques Cremer, Jonah Gelbach, Emel Filiz Ozbay, Erkut Ozbay, and John Shea for their helpful suggestions. I received very useful comments on my job market paper during many seminars, thank you to those participants.

Finally, I would like to thank my dog and dissertation buddy, Jerry Sandler Morrill. Almost everything you see here was written to the sounds of Jerry snoring.

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Chapter 1

Introduction

Matching Theory, along with Auction Theory, are two of the great success stories of microeconomic theory. One of the original matching theory papers was Gale and Shapley's 1962 paper *College Admission and the Stability of Marriage*. This paper consists of a simple but interesting problem, a clever algorithm, and the discovery that the solution space has a particularly desirable attribute. Theorists continued to study the problem simply because it was interesting and the results they discovered were so elegant. Eventually, about twenty years after the problems were introduced, economists started to recognize that the knowledge gained from solving these problems could be applied to designing markets, specifically the doctor-hospital resident market. In this way, matching theory has followed much of mathematics. Academics studied the problems for years because the properties discovered are beautiful, and eventually they recognized ways in which this elegant theory can be applied.

The progress made in matching theory is almost completely contained in the subfield of two-sided Matching Theory. In two-sided matching theory, we explore how to best pair two types of distinct agents. This is commonly called the Marriage Problem. Just as natural, and in fact more general, is the problem of how to best pair two agents of the same type. This question, called the Roommates Problem, has received almost no attention from the literature. This is because the very paper that introduced it, Gale and Shapley (1962), proved that there does not need to exist a stable solution to the Roommates Problem while there always exists a stable solution to the Marriage Problem.[1] Economists accepted that an equilibrium need not exist to the Roommate Problem and turned their attention elsewhere.

The second chapter in my dissertation seeks to rectify this situation. An equilibrium does always exist in the Roommates Problem, but economists have simply been looking in the wrong place. I introduce an alternative solution to the classic formulation of the Roommates Problem. I prove that an equilibrium always exists under this alternate notion of stability, and I discuss several algorithms for making an equilibrium assignment and improving one that is in disequilibrium.

This is an original and significant contribution to the field of economics. I have reframed a classic problem, which previously had no general solution, in a way that is economically more meaningful and now solvable. This is particularly important in the context of market design. As with the Marriage Problem, the mechanisms we study for the Roommates Problem have the potential to be applied to help mitigate market failure in some specific markets.

The third chapter in my dissertation looks at a different subfield of microeconomic theory, Network Theory. Classical economic theory often assumes a continuum of agents in markets. There are situations where this is both convenient and appropriate; however, the world is fundamentally a discrete place. There is no

[1] A matching is stable if no two unassigned agents prefer each other to their current assignment.

market with a continuum of agents, and there are many situations where the specific combinatorial characteristics of the network are critical to the economic characteristics of the market. For example, a car manufacturer does not buy brakes from a market. Rather it has long term relationships with a few manufacturers. Moreover, there are relatively few manufacturers of both cars and brakes. As such, the specific relationships that the manufacturers do and do not have are critical to understanding the market. It is precisely in this situation where a graph, and not a continuum, is the appropriate way to model the market.

While Network Theory has received a lot of attention in recent years, there has been little attention paid to how the choice of utility function affects which networks will emerge. This is the focus of the third chapter in my dissertation. I am able to completely characterize one of the most natural classes of utility functions: when the payoff an agent receives is a nonincreasing function of the degree of her neighbors. This is meant to model congestion or competition. An agent receives a payoff for the relationships she has, but these relationships are less beneficial the more people she has to compete with. The classic example in the literature is a model of co-authors. The benefit an academic receives from having a co-author is decreasing in the number of co-authors that person has as she will have less time to devote to the project.

This characterization of degree based utility models is important for its own sake, but especially as a contrast to the known results for distance based utility functions. A utility function is distance based if the payoff two agents receive from being connected is a nonincreasing function of the distance they are apart in the network. The networks I find to be optimal and stable for a degree based function are strikingly different from the optimal networks for a distance based utility function. The fact that both models are intuitive yet lead to dramatically different optimal networks means that when a researcher is attempting to model an economic situation with a network based model, she needs to be particularly careful how she chooses the underlying utility function.

Chapter 2

The Roommates Problem Revisited

One of the oldest but least understood matching problems is Gale and Shapley's (1962) "roommates problem": is there a stable way to assign $2N$ students into N roommate pairs? Unlike the classic marriage problem or college admissions problem, there need not exist a stable solution to the roommates problem. However, the traditional notion of stability ignores the key physical constraint that roommates require a room, and it is therefore too restrictive. Recognition of the scarcity of rooms motivates replacing stability with Pareto optimality as the relevant solution concept. This paper proves that a Pareto optimal assignment always exists in the roommates problem, and it provides an efficient algorithm for finding a Pareto improvement starting from any *status quo*. In this way, the paper reframes a classic matching problem, which previously had no general solution, to become both solvable and economically more meaningful.

2.1 Introduction

Economics is often defined as the study of how to efficiently allocate scarce resources. As such, assignment problems are at the heart of economics. Two-sided matching theory asks how to best match agents of two distinct types. Examples include students and schools, residents and hospitals, or kidneys and people in need of a transplant. A different but related question asks how to best pair two agents of the same type. Examples of these one-sided matches include roommates at a university, lab partners in a science class, and partners in a police force. Two-sided matching theory has been well studied by economists who have created an elegant and applicable theory. One-sided matching theory has been comparatively neglected[1].

This neglect is likely due to the very paper that introduced it. In their classic 1962 article *College Admissions and the Stability of Marriage,* Gale and Shapley introduce both the marriage problem and the roommates problem. While Gale and Shapley prove a stable match always exists in a two-sided market, they introduce the roommates problem to demonstrate that a stable pairing need not exist in a one-sided market. Since a stable match need not exist, economists have been stymied in their attempts to find and analyze solutions to this important assignment problem. Unfortunately, this has led many economists to turn their attention elsewhere, and as a result, the economics literature on this classic problem is sparse.

This paper starts by questioning if stability is the correct equilibrium concept. Gale and Shapley define a set of marriages as unstable if either there exist a man and woman who are not married but prefer each other to their current spouse or there exists someone who would prefer to be single than married to their current partner. Stability in the roommates problem is borrowed from the marriage model. A pairing is unstable if two students prefer to live with each other rather than their current assignment.[2] Stability fits the marriage model so well that no other solution

[1]Roth and Sotomayor (1990) is an excellent introduction to the two-sided matching literature. Gusfield and Irving (1989) is also a nice introduction. Interestingly, although the economics literature on the roommates problem is very small, there is a comparatively large computer science literature on it. Roth and Sotomayor, two economists, mention the roommates problem only as an example. In contrast, Gusfield and Irving, two computer scientists, devote nearly a quarter of the book to the roommates problem. Finding a traditionally-stable roommate pairing (if one exists) is considered a "hard" algorithmic question. The bulk of their presentation is a polynomial-time algorithm for finding a traditionally-stable pairing when one exists. Tan (1991) establishes a necessary and sufficient condition for the existence of a stable pairing. Chung (2000) extends Tan's result to a sufficient condition for the existence of a stable pairing when preferences are weak.

[2]I am interested in the case where each student is required to have a roommate. Con-

concept has been needed or suggested. The same is not true of the roommates problem. Roommates face an additional constraint that married couples do not; roommates must have a room in which to live. A student may prefer another to her assigned roommate; however, she needs a room in which to live and presumably does not have the right to evict her current roommate. Therefore, the traditional notion of stability is too restrictive.

I will present Gale and Shapley's original example to highlight this point.

Example (Gale and Shapley, 1962): *A Stable Assignment Need Not Exist*

Suppose there are four students: α, β, γ *and* δ*.* α*'s top choice is* β*,* β*'s top choice is* γ*,* γ*'s top choice is* α*, and all three rank* δ *last. Gale and Shapley define an assignment to be unstable if two students are not currently roommates but prefer each other to their current assignment. Under this definition, there does not exist a stable assignment since whoever is assigned to* δ *prefers the other two students to* δ *and is the top choice of one of these students. In the words of Gale and Shapley:*

> "...whoever has to room with δ will want to move out, and one of the other two will be willing to take him in."

While one of the other two may be willing to take him in, it is quite a different matter whether this student is *able* to take him in. In order to take him in, either his current roommate must voluntarily leave, be evicted, or an additional room must be available. With a scarcity of rooms and with no student willing to change his

sequently, I do not include in my definition of stability the additional requirement that each student prefers her assignment to being unassigned.

assignment to δ, the original assignment is an equilibrium after all.

If an agent can dissolve her partnership unilaterally, then stability is the natural equilibrium concept. If she finds someone she prefers who also prefers her, then both parties will dissolve their current partnership and pair together. As a result, the original assignment is not an equilibrium. However, if a partnership requires *bilateral* agreement to dissolve, then two agents wanting to change their assignment is not enough to disturb the original pairing. If bilateral agreement is required, an assignment will only be changed if all involved parties agree. Since an agent will only agree if the new assignment makes her better off, any deviation from the original set of assignments must be a *Pareto improvement*. Therefore, when bilateral agreement is required to dissolve a partnership, Pareto optimality, not stability, is the proper equilibrium concept. If an assignment is Pareto optimal, then there is no reassignment that all parties will consent to; therefore, the original assignment will not be disturbed.

Most of matching theory studies assignments that can be unilaterally dissolved. Assignments which can only be dissolved with bilateral agreement are an important but little studied second category. As argued above for the roommates problem, an essential but scarce input creates the need for bilateral agreement. Additional examples include police officers who require a police car to do patrol and partners in a science class who must work at a common laboratory. The same requirement can be created by a legally binding contract that can only be modified by mutual consent. For example, many professional athletes have no-trade clauses in their contract which they may waive at their discretion. In the presence of such clauses, the assignment of an athlete to a team can only be disturbed when all relevant parties approve the trade.

This paper focuses on the roommates problem as reconsidered using the equilibrium concept of Pareto optimality. I will show there always exists an efficient assignment. Therefore, unlike the case where stability is applied, an equilibrium always exists in the roommates problem. Moreover, I show an inefficient assignment can always be Pareto improved to an efficient one. These results motivate several questions. If an assignment has not been made, how should we make it? If an assignment has been made, how can we determine if the assignment is efficient? If an assignment is inefficient, how can we Pareto improve it? These questions are the focus of this paper. In particular, the last two turn out to be complicated. To answer them I introduce an algorithm, The Roommate Swap, which identifies whether an assignment is inefficient and finds a Pareto improvement when it is.

Much of the analysis in this paper relies on tools from graph theory. Networks are a natural way of representing assignment problems, particularly one like the roommates problem where two agents are paired. In particular, my algorithm relies heavily on applying Edmund's Blossom algorithm[3] to the graph theoretic representation of the roommates problem.

The paper is organized as follows. Section 2 formally introduces the problem and proves existence. Section 3 details the Roommate Swap algorithm. Section 4 examines the strategic implications of several assignment mechanisms. Section 5 looks at extensions and modeling issues, and section 6 concludes. The appendix

[3] Edmunds (1965).

provides several technical proofs and a discussion of the computational complexity of the Roommate Swap algorithm.

2.2 The Roommates Problem Revisited

We wish to assign $2N$ students to M rooms. Students have preferences over all other students that are strict, complete, and transitive. All rooms are identical and students have no preference as to which room they are assigned.

An assignment is a function that pairs students. Every student is assigned to exactly one other student, and assignments are symmetric.

Definition 1. *Let S be a set of students with $|S| = 2N$. A function $\mu : S \to S$ is an* ***assignment*** *of S if:*

1. $\mu(s) \neq s$.
2. $\mu(s_1) = \mu(s_2) \Rightarrow s_1 = s_2$.
3. $\mu(\mu(s)) = s$.

The traditional equilibrium concept is based on the notion of a blocking pair.

Definition 2. *Two students s and t are a* ***blocking pair*** *to an assignment μ if $\mu(s) \neq t$ but $s \succ_t \mu(t)$ and $t \succ_s \mu(s)$. An assignment is* ***stable*** *if there does not exist a blocking pair*[4]*.*

[4]The traditional definition of stability also includes the constraint that the person prefers her assignment to being unassigned. In my model every student must be assigned to some room, so I omit this additional constraint.

As argued in the introduction, this is not the proper equilibrium concept for the roommates problem. A roommate assignment is an equilibrium if it is Pareto optimal.

Definition 3. *An assignment μ is* ***inefficient*** *if there exists a different assignment μ' such that for every student s, $\mu'(s) \succeq_s \mu(s)$. An assignment is* ***Pareto optimal (efficient)*** *if it is not inefficient.*

Since preferences are strict, if μ' Pareto improves μ, then at least four students must strictly prefer μ' to μ. As the following result proves, the set of stable assignments is a subset of the set of efficient assignments.

Proposition 1. *If an assignment is stable, then it is Pareto efficient.*

Proof. I will prove the contrapositive. If an assignment μ is inefficient, then there exists an assignment μ' that Pareto improves μ. Let s be any student such that $\mu(s) \neq \mu'(s)$. Since μ' is a Pareto improvement of μ, both $\mu'(s) \succ_s \mu(s)$ and $s \succ_{\mu'(s)} \mu(\mu'(s))$. Therefore, s and $\mu'(s)$ form a blocking pair to μ. □

Note that the reverse direction need not hold. An assignment can be Pareto efficient but not be stable. In Gale and Shapley's original example, no assignment is stable yet every assignment is Pareto efficient.

With the following assumptions, the general case of 2N students and M rooms reduces to the more familiar case of 2N students and N rooms:

Assumption 1. *Each student prefers having a room to not having a room.*

Assumption 2. *Each student would rather have a room to herself than to be assigned a roommate.*

Assumption 3. *At most two students can be assigned to a room.*

Note that if $N > M$, some students will not be assigned a room. Such a student cannot be involved in a Pareto improving switch by Assumption 1. Similarly, if $N < M$, a number of students will not be assigned a roommate. Assumption 2 implies such a student will never be involved in a Pareto improving switch. Therefore, the only set of students relevant for this problem are those who have been assigned a roommate. By Assumption 3, this set has exactly twice as many students as rooms. Without loss of generality, for the rest of the paper I will assume there are $2N$ students and N rooms.

Gale and Shapley show that an assignment without a blocking pair need not exist. However, an efficient assignment always exists.

Proposition 2. *An efficient roommate assignment always exists.*

Proof. (Random serial dictatorship[5]) Assign every student a priority (randomly or otherwise). Assign the student with highest priority her most preferred roommate and remove them both from consideration. From students who remain, assign the student with highest priority her most preferred roommate among those students that are unassigned. Remove these two from consideration and repeat until no

[5] Abdulkadiroglu and Sonmez (1998) is a very nice paper on the Random Serial Dictatorship mechanism. They analyze it in the context of a housing allocation problem where n students are to be assigned to n rooms, but it is rather interesting how robust the Random Serial Dictatorship is. The same mechanism can be used to make a Pareto efficient assignment of a student and a room, two students to be roommates, three or more students to be roommates, students to be roommates and the room they will live in, etc.

students remain. This assignment is Pareto efficient. To see this, note that if a student is involved in a Pareto improvement, then necessarily her roommate must be involved as well. The student with highest priority, s_1, receives her top choice, s_2, so neither she nor her choice can be involved in a Pareto improvement. Let s_3 be the student who chooses second. Since neither s_1 or s_2 are involved in any Pareto improvements, if s_3 is part of a Pareto improvement she must be reassigned to a student among $S \setminus \{s_1, s_2\}$. However, s_3 already receives her top choice among this set. Therefore, s_3 (and consequently the student she chooses) is not part of any Pareto improvement. Similarly, the student who chooses third is not part of any Pareto improvement, and so on. □

The following is a stronger statement and implies Proposition 1. It is stated to motivate the Roommate Swap algorithm.

Proposition 3. *If an assignment μ is inefficient, there exists an efficient assignment μ' which Pareto improves μ.*

The proof is straightforward but is included as it motivates the need for the Roommate Swap Algorithm.

Proof. Let μ be an assignment and $PI(\mu)$ be the set of strict Pareto improvements of μ. Transitivity of preference implies $\forall \mu' \in PI(\mu), PI(\mu') \subseteq PI(\mu)$. Since $\mu' \in PI(\mu) \setminus PI(\mu'), PI(\mu') \subset PI(\mu)$. Since there are only a finite number of possible assignments, the following chain must converge to the empty set:

$$PI(\mu) \supset PI(\mu_1) \supset PI(\mu_2) \supset \ldots, \text{ where } \mu_i \in PI(\mu_{i-1})$$

In particular, there must exist an j such that $PI(\mu_j) = \emptyset$. μ_j is an efficient assignment which Pareto improves μ. $\square$

Put simply, if μ is not efficient, there exists a Pareto improvement μ_1. μ_1 is either efficient or can be Pareto improved to μ_2, etc. We must eventually reach an efficient assignment, and since preferences are transitive, this assignment must Pareto improve μ.

Propositions 2 and 3 motivate two distinct but related problems. The first problem is how to make an efficient assignment when no assignment has yet been made. The second is how to Pareto improve an inefficient assignment to an efficient one. Although these two problems are very similar, it is surprising how different these processes end up being. The serial dictatorship used in Proposition 2 to show existence provides a linear-time procedure for finding an efficient assignment. In contrast to the ease of finding an efficient assignment, it is rather difficult to even determine if any given assignment is efficient let alone how to improve it. Preferences between students need not interact when assigning students, but they interact directly when determining if one assignment Pareto improves another. This makes it significantly more complicated to determine if an assignment is efficient than it is to simply find an efficient assignment.

At this point the reader may object as there is an obvious and trivial algorithm to determine if an assignment is efficient. Namely, one could simply look at each possible reassignment and determine if it Pareto improves the original. If no assignment Pareto improves the original, then the original is efficient. Unfortunately,

Students	Number of Possible Assignments
2	1
4	3
6	15
8	105
10	945
12	10,395
14	135,135
16	2,027,025
18	34,459,425
20	654,729,075
30	6,190,283,353,629,370
2N	$\frac{(2N)!}{2^N(N!)}$

this algorithm is of no practical use as the growth of the number of assignments relative to students being assigned is factorial. Specifically, given 2N students there exists $\frac{(2N)!}{2^N(N!)} = (2N-1)(2N-3)(2N-5)\cdots(3)(1)$ many ways of assigning them to be roommates.[6] Even for small N, this is prohibitively large. For example, there exists on order of 6 quadrillion (6×10^{15}) many ways to assign 30 students to be roommates. Therefore, a more sophisticated process is required.

[6]A short proof appears in the Appendix.

2.3 The Roommate Swap Algorithm

This section demonstrates an $O(n^2)$ algorithm for determining if an assignment is efficient. Moreover, when an assignment is inefficient I provide an $O(n^3)$ algorithm, The Roommate Swap, for finding a Pareto improvement.[7] Much of the analysis uses tools from graph theory, so it is necessary to present some definitions and results. This document is intended to be self-contained, but I refer the reader to *Introduction to Graph Theory*, second edition, by Douglas West for a more detailed analysis of graph theory.

A graph consists of vertices and edges between them. For my purposes, all edges are undirected.

1. Two vertices are **adjacent** if there is an edge between them.

2. The **degree** of a vertex v, denoted $d(v)$, is the number of vertices it is adjacent to.

3. A **path** is a sequence of vertices $\{v_1, v_2, \ldots, v_k\}$ such that no vertex appears twice and any two consecutive vertices are adjacent.

4. A **cycle** is a sequence of vertices $\{v_1, v_2, \ldots, v_k\}$ such that no vertex appears twice, any two consecutive vertices are adjacent, and v_1 and v_k are adjacent.

5. Two vertices are **connected** if there is a path between them. Since our graphs are undirected, connected is a reciprocal relationship. A graph is connected if all vertices are connected.

[7]A discussion on the computational complexity of the algorithm appears in the appendix.

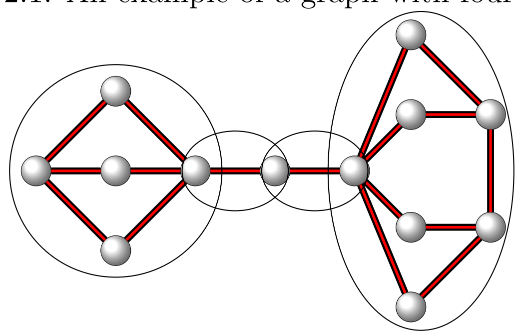
6. A vertex is **incident** to an edge if it is one of the edge's endpoints. $G \setminus v$ is the graph that results from deleting the vertex v and all edges incident to v.

7. A vertex v is a **cut-vertex** if G is connected, but $G \setminus v$ is not.

8. A **block** is a maximal subgraph containing no cut vertex.

Note that the subgraph consisting of two vertices and an edge between them contains no cut-vertex, so any edge is either a block or a subset of a block. I will refer to any block containing only two vertices as a trivial block. Since every vertex in our graph has at least one edge incident to it, this is the smallest block possible. Figure 2.1 shows an example where the blocks have been circled.

Figure 2.1: An example of a graph with four blocks.



Definition 4. *Given an assignment μ, a set of students X is* ***closed under roommates*** *if $s \in X$ implies $\mu(s) \in X$.*

Given a set of preferences $\succ$ and assignment μ, I will induce a graph, $G^{\mu}_{\succ}$, as follows:

- Each vertex corresponds to a student. Label the vertices s_1 through s_n. When referring to the graph, I will use the term vertex and student interchangeably.

- A *solid* edge is drawn between roommates. By definition, each vertex is incident to exactly one solid edge.

- Draw a *dashed* edge between any two students that form a blocking pair to μ. That is to say, if s_i prefers s_j to her current roommate and vice versa.

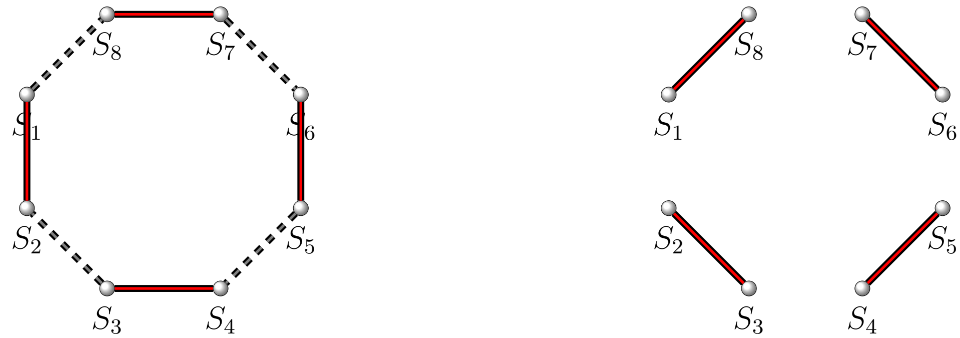
When the preferences and assignment are clear from the context, I will just refer to the graph as G. I will call a path that alternates between dashed and solid edges (or vice versa) an **alternating path**. Similarly, a cycle that alternates between dashed and solid edges is an **alternating cycle**.

Lemma 1. *An assignment μ is efficient under preferences $\succ$ if and only if $G^{\mu}_{\succ}$ contains no alternating cycle. Moreover, if μ' Pareto improves μ and s is a student such that $\mu(s) \neq \mu'(s)$, then s is contained in an alternating cycle in $G^{\mu}_{\succ}$.*

The intuition for sufficiency is captured in Figure 2.2. In an alternating cycle, we can simply "swap" roommates. We eliminate the solid edges, make the dashed edges in the cycle solid, and leave everyone outside the cycle unchanged. This is a well-defined reassignment that Pareto improves the initial assignment.

Proof. Suppose $G^{\mu}_{\succ}$ contains an alternating cycle C. An alternating cycle is closed under roommates as each vertex is incident to a solid edge in the cycle. This implies $V(G) \setminus C$ is closed under roommates as well (V(G) means the vertex set of G). We will construct a Pareto improvement μ'. For every $v \in V(G) \setminus C$ let $\mu'(v) = \mu(v)$. This is well defined since $V(G) \setminus C$ is closed under roommates. For every $v \in C$, let $\mu'(v)$ be the vertex it shares a dashed edge with in the cycle C. This is well defined

Figure 2.2: An alternating cycle with its corresponding Pareto improvement.



as each vertex is incident to exactly one dashed edge in the cycle and sharing a dashed edge is a reciprocal relationship. A dashed edge indicates that both vertices prefer each other to their original assignment. Therefore, μ' Pareto improves μ.

Suppose that μ' is a Pareto improvement of μ. Let G' be the subgraph consisting of all solid edges in $G^{\mu}_{\succ}$ and only the dashed edges between vertices not paired by μ that are paired by μ' (since μ' is a Pareto improvement, there must be a dashed edge between such vertices). Note that any vertex v in G' either has degree[8] 1 (if $\mu(v) = \mu'(v)$) or degree 2 (if $\mu(v) \neq \mu'(v)$). Moreover, for any vertex v, if $d(v) = 2$, then $d(\mu(v)) = d(\mu'(v)) = 2$. Choose any vertex t such that $d(t) = 2$. t is connected via a solid edge to $\mu(t)$. Since $d(t) = 2$, $d(\mu(t)) = 2$ and so $\mu(t)$ must be connected via a dashed edge to $\mu'(\mu(t))$. $\mu'(\mu(t))$ must be connected via a solid edge to $\mu(\mu'(\mu(t)))$ which must be connected to a dashed edge via $\mu'(\mu(\mu'(\mu(t))))$, and so on. Eventually this process must cycle as there is only a finite number of vertices. However, a cycle to any vertex $s \neq t$ would mean the degree of s is at least three which is not possible. Therefore, the process must cycle back to our first vertex t.

[8]The degree of a vertex v, denoted $d(v)$, is the number of edges v is incident to.

Moreover, it must cycle via a dashed edge as we have already exhausted t's solid edge. By construction, this is an alternating cycle. □

Lemma 2. *Let t be any student.*

1. *t and $\mu(t)$ are contained in a unique block, B_t.*

2. *If t is part of an alternating-cycle C, then $C \subseteq B_t$.*

3. *If t is involved in a Pareto improvement, then B_t is non-trivial. That is to say if there exists an assignment μ' such that μ' Pareto improves μ and $\mu'(t) \neq \mu(t)$, then $|B_t| > 2$.*

Proof.

1. Since there is an edge between t and $\mu(t)$, they are in at least one block together. Since the intersection of two blocks contains at most one student,[9] t and her roommate must be in exactly one block together. Call this block B_t.

2. A cycle contains no cut-vertex, so it must be a subset of a block. An alternating-cycle containing t must contain $\mu(t)$ since t lies on a solid edge in the alternating-cycle. Since B_t is the unique block containing t and $\mu(t)$, the cycle must be contained in B_t.

3. If t is involved in a Pareto improvement, then by Lemma 1 t is contained in an alternating-cycle. By (2) this alternating-cycle is contained in B_t, so B_t must

[9]See West pg. 156. The intuition is that if if two blocks B_1 and B_2 share two vertices, then after cutting a vertex, at least one of the two must remain. Call this vertex v. v is connected to all remaining vertices as it is in a block with each of them. But if every vertex has a path to v, then all vertices are connected. Therefore $B_1 \cup B_2$ has no cut-vertex contradicting the maximality of a block.

contain more than just t and $\mu(t)$.

□

Lemma 2, part (2) says that if a student t is part of a Pareto improvement (and consequently an alternating-cycle), then she must be reassigned to a member of B_t. Therefore, no edge between t and a vertex outside of B_t can be part of an alternating-cycle. Let G' be the graph obtained by deleting all edges between t and any vertex not in B_t. Then G contains an alternating-cycle if and only if G' contains an alternating-cycle. This motivates the following procedure.

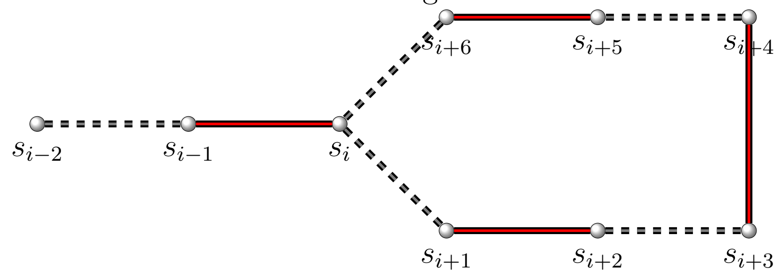
Pruning a Graph

1. Start with a graph G.
2. Determine the set of blocks $B_1, B_2, \ldots, B_m$.
3. For each student-roommate pair s and $\mu(s)$, locate the unique block that both are in. Remove *all* edges from either s or $\mu(s)$ to any student outside this block.

A key point is that if a student s was in a block B with $\mu(s) \notin B$, then after pruning the graph, s is no longer in B. By iterating the pruning process we end up with a graph in which all blocks are closed under roommates. Note that these blocks may be trivial, but by Lemma 2, the students in such a block are not involved in any Pareto improvements.

Proposition 4. *Any non-trivial block closed under roommates contains an alternating cycle.*

Figure 2.3: A “Blossom”.



The algorithm in this proof was inspired by Edmunds' Blossom Algorithm from graph theory[10] and Gale's Top-Trading Cycles Algorithm.[11]

Proof. Look at any non-trivial block B closed under roommates. Every vertex v in B must be incident to a dashed edge. Otherwise v is only connected (by a solid edge) to $\mu(v)$ which would mean removing $\mu(v)$ disconnects v from the rest of the block. This is not possible since a block contains no cut-vertices. Start with any vertex s. First take a dashed edge to a new vertex s_1 then continue on a solid edge to $s_2 = \mu(s_1)$. Continue alternating between dashed and solid edges until we cycle. We must eventually cycle since there is a finite number of vertices.

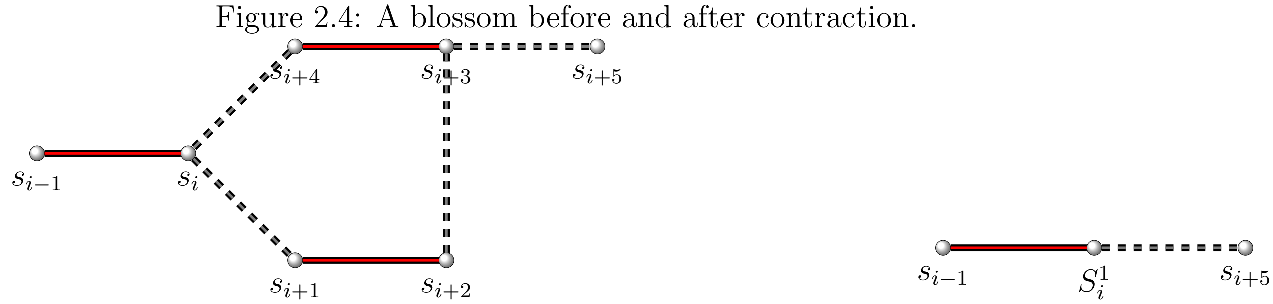
If our cycle is even (a cycle is even if it contains an even number of vertices), then we are done. By construction, an even cycle alternates between dashed and solid edges and is therefore an alternating cycle. Therefore, assume our cycle is odd, $\{s_i, s_{i+1}, s_{i+2}, \ldots, s_{i+2m}\}$. By construction, any odd cycle looks like Figure 2.3, except possibly of different length. Edmunds refers to this as a blossom. The vertices $\{s_1, s_2, \ldots, s_i\}$ are the stem, s_i is the base of the blossom, and s_i must connect to s_{i+1} and s_{i+2m} via dashed edges.

[10]Edmunds (1965). A discussion of the Blossom algorithm appears in West, page 142.
[11]Shapley and Scarf (1974).

There must by a dashed edge from one of $s_{i+1}, s_{i+2}, \ldots, s_{i+2m}$ to a vertex outside the cycle. Otherwise s_i would be a cut-vertex as deleting it would disconnect $s_{i+1}, s_{i+2}, \ldots, s_{i+2m}$ from the rest of the graph. What we will do is contract the odd cycle into a single super-vertex S_i^1. The superscript indicates the number of contractions we performed to result in S_i. See Figure 2.4 for an example. Make any edge that was previously between a vertex in the cycle and a vertex t outside the cycle now between S_i^1 and t. Note that there is a solid edge between s_{i-1} and S_i^1 and all other edges incident to S_i^1 must be dashed as for any $s_j \in \{s_{i+1}, s_{i+2}, \ldots, s_{i+2m}\}$, $\mu(s_j) \in \{s_{i+1}, s_{i+2}, \ldots, s_{i+2m}\}$.

Now continue with one of the unexplored dashed edges incident to S_j^1. Again, we must eventually cycle. If the cycle is even, stop. If the cycle is odd, contract the blossom and continue. There must always be an unexplored dashed edge out of an odd cycle (or else the base vertex would be a cut vertex), so after any odd cycle we will be able to continue. Since we have a finite number of vertices and edges and each contraction reduces the number of vertices, we cannot continue indefinitely. The algorithm only stops with an even cycle, and since the algorithm must eventually terminate, we must eventually reach an even cycle.

Figure 2.4: A blossom before and after contraction.



Any alternating cycle containing super-vertices can be expanded to an alternating cycle containing no super-vertices. No matter how we enter the blossom, either the edge to the left or to the right is solid. We can follow the cycle in the direction of the solid edge all the way to base vertex. This is an alternating path to the base, the cycle connects to the base with a dashed edge, and then continues along the stem starting with a solid edge. So indeed, this expands an alternating path through a super-vertex to an alternating path through the cycle that was contracted. If our super-vertex is the result of multiple contractions, then our base vertex is now a super-vertex but otherwise nothing changes. Moreover, the base is a super-vertex containing fewer contractions, so we can proceed by induction on the number of contractions to get the desired result.

□

Proposition 4 implies a simple procedure for determining whether an assignment is efficient.

Determining if an assignment μ is efficient given preferences $\succ$.

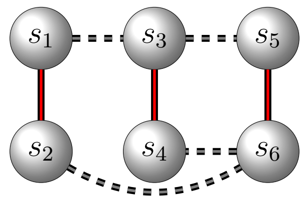
1. Induce graph $G^{\mu}_{\succ}$.

2. Iteratively prune $G^{\mu}_{\succ}$ until all blocks are closed under roommates.

3. If all blocks are trivial, then our assignment is efficient. If there exists a non-trivial block, then by Proposition 4 and Lemma 1, our assignment is inefficient.

The algorithm in the proof finds an alternating cycle when one exists. Once we have located an alternating cycle then just as we did in Figure 2.2 on page

Figure 2.5: A non-trivial block closed under roommates, but s_1 and s_2 are not contained in any alternating-cycle.



19, we "swap" roommates to get a Pareto improvement, . For this reason I call the algorithm the Roommate Swap. Note that we have now answered the two key questions from the previous section. The Roommate Swap identifies if a given assignment is efficient. Moreover, when an assignment is inefficient, it finds a Pareto improvement.

The Roommate Swap determines if a given assignment is efficient. However, a particular student likely does not care whether the assignment can be Pareto improved. Rather, she would like to know if *she* can be part of a Pareto improvement. Unfortunately, Proposition 4 does not generalize to the statement if a student t is contained in a non-trivial block closed under roommates, then t is involved in a Pareto improvement. Figure 2.5 is a non-trivial block that is closed under roommates, but s_1 and s_2 are not part of any Pareto improvements.

The Roommate Swap does not determine if a particular student can improve her assignment. However, it is not biased. If we randomly choose the vertex we start with, and when we have a choice, we randomly choose which edge to continue on, then the Roommate Swap will find any Pareto improvement with probability that is uniformly bounded away from zero. Therefore, if the Roommate Swap is run repeatedly, it will determine if an individual student is involved in a Pareto improvement with probability one.

2.4 Strategic Implications

This paper has focused on two problems: finding an efficient assignment and finding an efficient Pareto improvement of an inefficient assignment. Continuing the pattern from previous sections, finding a strategy-proof mechanism for making an assignment is easier than finding a strategy-proof mechanism for improving an assignment. In fact, we will find that there does not exist a mechanism for selecting a Pareto improvement of a given assignment that makes truthful revelation of preferences a dominant strategy. These results follow very closely the results for two-sided matching theory presented in Roth and Sotomayor (1990).

Following the matching literature, I will use dominant strategy as my equilibrium concept.

Definition 5. *A* ***dominant strategy*** *is a strategy that is a best response to all possible strategies of the other agents. An assignment mechanism is* ***strategy proof*** *if it is a dominant strategy for each agent to reveal her preferences truthfully.*

There does exist a strategy-proof mechanism for making an efficient assignment. In fact, we have already seen this mechanism several times.

Observation 1. *The serial dictatorship is strategy proof.*

In the serial dictatorship, a student's preferences are irrelevant unless she is the one choosing her roommate. Since she gets her top choice, she does best when she submits her true preferences regardless of the preferences submitted by other students.

Finding an incentive compatible, efficient assignment mechanism is very closely related to Social Choice theory and Arrow's Impossibility Theorem. The Gibbard-Satterthwaite Theorem says that if arbitrary preferences are possible, then the unique incentive-compatible, Pareto optimal mechanism is the dictatorship mechanism. Unfortunately this cannot be directly applied as we are restricting the domain of allowable preferences. A students is only allowed to have preferences over her own assignment, and therefore, she is forced to be indifferent between many assignments. For example, a student does not have a single most-preferred assignment, but rather, she is indifferent among all assignments that match her to her most-preferred roommate. A dictator mechanism would not be Pareto optimal as, among her top choices, the dictator would select a Pareto optimal assignment only by chance. The serial dictatorship is the generalization of the dictatorship mechanism that has the properties of incentive compatibility and Pareto optimality. Due to the corresponding uniqueness results for the dictatorship mechanism, it seems likely that the serial dictatorship is the unique incentive-compatible mechanism for selecting an efficient assignment.

Lemma 3. *There does not exist a strategy-proof mechanism for selecting a Pareto improvement of an inefficient assignment.*

Lemma 3 is proved in the appendix. This is quite a general result, but it is rather easy to proof. A strategy-proof mechanism must be able to handle any initial assignment and any profile of preferences. Following the path of Roth (1982), I demonstrate a case that no mechanism is able to handle.

2.5 Extensions and Modeling Issues

2.5.1 Extensions to the Model

Not surprisingly, the existence of an efficient solution is quite general. For example, if students have preferences over both their roommate and the room they are assigned, then Propositions 2 and 3 still hold. In fact, the same proofs are still valid. Similarly, if more than two students are assigned to be roommates, the same existence results hold.

This paper has focused on one-sided matches, but there are many interesting examples of two-sided matches with a physical constraint. Whenever a two-sided match requires bilateral approval to dissolve, then any Pareto optimal assignment will be an equilibrium. For example, an airline matches a pilot with a navigator in order to fly an airplane. The presence of a physical constraint, the airplane, means a blocking pair is not enough to disturb an assignment.

The extra structure inherent in a two-sided match makes it easier to find a Pareto improvement to a two-sided match than a one-sided match. Here we can use a slight variation of the Top Trading Cycles algorithm[12] to determine if an assignment is Pareto optimal and to Pareto improve the assignment when it is not. For a given pilot p, define a navigator n to be *achievable for p* if n weakly prefers p

[12] See Shapley and Scarf, 1974.

to her current assignment. Have each pilot point to her most-preferred, achievable navigator. Note that a pilot always has a navigator to point to as her current assignment is achievable. Have each navigator point to their current assignment. There must exist a cycle since there are only a finite number of agents and each agent is pointing to someone. If the cycle is trivial (the pilot is pointing to the navigator she is currently assigned to), then neither the pilot nor the navigator can be involved in a Pareto improvement and we can remove them from consideration. If the cycle is non-trivial, then it represents a strict Pareto improvement for all agents in the cycle. Future drafts of this paper will contain a more detailed discussion of two-sided matches with a physical constraint.

When students have preferences over both their roommate and the room they are assigned, there still exists an efficient assignment. However, the Roommate Swap does not readily generalize to this case. The notion of a "swap" completely characterizes a Pareto improvement when only one other factor is involved in a pairing; however, with multiple dimensions a Pareto improvement can be much more complicated.

However, there is one very specific but important case where the Roommate Swap can be readily generalized. If students have lexicographical preferences over their roommate and room, then we will be able to find a Pareto improvement for any inefficient assignment. If the students care about the room first and the roommate second, then we can run the Top Trading Cycles algorithm to find a Pareto improvement when one exists. If a student cares about her roommate first and her room second, then we can first run the Roommate Swap and next run the Top

Trading Cycles algorithm. There are a variety of ways we can aggregate individual preferences over rooms to a single roommate-pairing preference over rooms that will result in an efficient allocation. Thus, starting with an arbitrary assignment and lexicographical preferences over roommates and rooms, we can determine if an assignment is efficient, and if not, Pareto improve it to an efficient one.

2.5.2 Alternative Equilibrium Concepts

This paper has focused on pairing two agents as this is the classic framing of the roommates problem. While I believe efficiency, not stability, is the correct equilibrium concept for this classic problem, the more agents that are assigned to be together, the less compelling efficiency becomes as an equilibrium. If six people are assigned to an office, it is likely that a person can switch desks with a student in another office without requiring unanimous approval from her current officemates.

We can formulate an alternative equilibrium that has more appeal in this case. Instead of assigning six people to be officemates, we make six keys to each office and give each person a key to one office. Since the rooms are homogeneous, this is just an indirect way of assigning officemates. If we allow students to trade keys, then an assignment is an equilibrium if no two students wish to trade offices. Note that we are honoring the physical constraint; no student is being evicted. Moreover, this does not allow a student to block another student from switching her assignment.

Similar to stability, there need not exist an equilibrium if students can trade rooms. Suppose there are four students, a, b, c, and d, with preferences as follows:

$$
\begin{aligned}
a &: b \text{ is most preferred} \\
b &: c \text{ is most preferred} \\
c &: d \text{ is most preferred} \\
d &: a \succ c \succ b
\end{aligned}
$$

If a is assigned to b and c is assigned to d, then b and d will trade places. If a is assigned to d and b is assigned to c, then a and c will trade places. Finally, if a is assigned to c and b is assigned to d, then a and d will trade places. Since these are the only possible assignments, there is no equilibrium.

In general, an argument can be made for either equilibrium. The new equilibrium might be a reasonable model for condominiums or rooms in a group house. If a person decides to sell her condominium, she does not need the approval of the other condominium owners in the building. Note however that there also exists building cooperatives. Here a sale does require the approval of the board, so in this context, Pareto optimality is a more natural equilibrium concept. Similarly, depending on the lease a person signs, subletting a room in an apartment or group house may or may not require the approval of the landlord or other tenants. Therefore, whether or not Pareto optimality is the best equilibrium concept depends on the particular lease signed.

2.6 Conclusion

The roommates problem is one of three assignment problems introduced by Gale and Shapley in their classic 1962 paper *College Admissions and the Stability of Marriage*. This is the paper that created the field of matching theory, and the reason why the roommates problem was included is that it is such a natural assignment problem. While their other two assignment problems, the marriage problem and the college admission problem, have been studied extensively, little progress has been made on the roommates problem. This paper hopes to make several contributions to the matching theory literature on the roommates problem. First, identifying Pareto optimality instead of stability as the proper equilibrium makes the roommates problem economically more meaningful. With this improved equilibrium concept, I have shown that an equilibrium always exists. Most importantly, I demonstrate how to improve an inefficient assignment to an efficient one if we are not in equilibrium. For such a natural assignment problem as the roommates problem, this is likely to have real world applications. Therefore, this paper reframes a classic matching problem, which previously had no general solution, in a way that is both solvable and economically more meaningful.

2.7 Appendix

Lemma 4. *There are* $\frac{(2N)!}{2^N(N!)} = (2N-1)(2N-3)(2N-5)\cdots(3)(1)$ *many ways to assign 2N students to be roommates.*

Proof. The proof is by induction. When $N = 1$, the result is trivial as there is only one way to assign two students to be roommates. Assume $N > 1$ and by induction there are $(2N-3)(2N-5)\cdots(3)(1)$ many ways to assign 2(N-1) many students to be roommates. Select a student s. There are 2N-1 possible roommates for s, and by assumption, for any roommate we pick, there are $(2N-3)(2N-5)\cdots(3)(1)$ many ways to assign the remaining 2N-2 many students. Therefore, there is a total of $[2N-1] \times [(2N-3)(2N-5)\cdots(3)(1)]$ many ways of assigning roommates. □

Lemma 3 *There does not exist a strategy-proof mechanism for selecting a Pareto improvement of an inefficient assignment.*

Proof. Suppose there are four students, a, b, c, and d, and an initial assignment, μ_1 pairing a with b and c with d. Moreover, suppose the student's preferences are as follows.

$$a : c \succ d \succ b$$

$$b : c \succ d \succ a$$

$$c : b \succ a \succ d$$

$$d : b \succ a \succ c$$

With four students, there are three possible assignment. Note that an assignment is completely determined by who a (or any other student) is assigned to. Let μ_2 denote the assignment where a is paired with c and μ_3 denote the assignment where a is paired with d. In our original assignment μ_1, each person is paired with their least preferred roommate, so μ_1 is Pareto dominated by both of the other assignments. Suppose for contradiction that their exists a strategy-proof mechanism M for selecting an efficient, Pareto improving assignment. Note that if a submits the preferences $c \succ b \succ d$ and all other students submit true preferences, then μ_2 is the only assignment that Pareto improves μ_1 (relative to the submitted preferences). In such a case, M must select μ_2. Similarly, if b submits the preferences $c \succ a \succ d$ and all other students submit true preferences, then M must select μ_3 as it is now the only Pareto improving assignment. When all students submit true preferences, M must select either μ_2 or μ_3. If M selects μ_2, then b can do better by deviating and submitting the preferences $c \succ a \succ d$. If M selects μ_3, then a can do better by submitting preferences $c \succ b \succ d$. Either way, M is not strategy proof which is a contradiction. □

2.7.1 Computational Complexity

The purpose of this section is to demonstrate that the Roommate Swap is a polynomial time algorithm and therefore implementable. I demonstrate that it is at worst an $O(N^3)$ algorithm where N is the number of students.

Each iteration of the algorithm involves the following steps, performed in sequence:

1. *Induce the graph.* This is at worst $O(N^2)$ as a graph is defined by its edges and there are at most $\frac{N(N-1)}{2}$ many edges.

2. *Iteratively prune the graph until all blocks are closed under roommates.* West (2001), pg. 157, details an $O(N)$ algorithm for determining blocks. We need to iterate at most N times as each iteration eliminates at least one student from each block or stops looking at a block if it is already closed under roommates.

Therefore iteratively pruning the graph is at worst $O(N^2)$.

3. *Find an alternating-cycle.* This process is $O(N)$. At each step we either travel to a previously unvisited vertex, which we can do at most N times, or contract a minimum of two vertices, which we can do at most $\frac{N}{2}$ times. So the algorithm must conclude in at most $N + \frac{N}{2}$ steps. As it takes at most N steps to expand a cycle containing super-vertices to a proper cycle, finding an alternating-cycle concludes in $O(N)$ time.

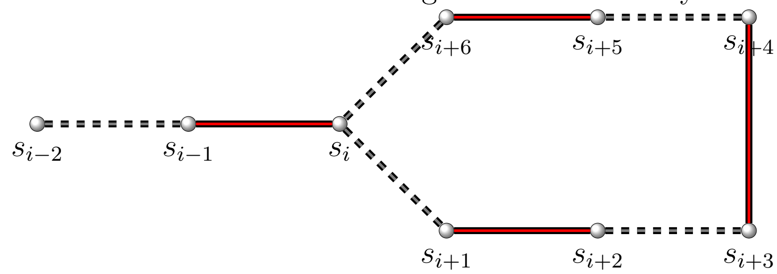
   Therefore each iteration is $O(N^2)$.

Observation 2. *In each iteration of the Roommate Swap, at least one student is reassigned her top achievable student.*

Proof. The search process ends with an alternating-cycle that may or may not contain super-vertices. Dashed edges from standard vertices are chosen to be the vertex's most preferred student among those who prefer her to their current assignment. Therefore, if the alternating cycle contains no super-vertices, then half the students receive their top achievable match. A grey edge from a super-vertex is not necessarily the student's most preferred achievable student. However, if the alternating-cycle contains a super-vertex and we need to expand our contractions, then there must be a *last* odd-cycle that needs to be expanded.

Figure 6 shows a last cycle with six vertices, but the analysis is the same for fewer or greater vertices. Our alternating path must go through s_i and either s_{i+1} or s_{i+6}. None of these edges involve super-vertices (this is our last expansion) so

Figure 2.6: An odd cycle.



by construction, s_{i+1} is s_i's top achievable student and s_i is s_{i+6}'s top achievable choice. Either way, at least one student receives her top achievable choice.

□

The significance of this is that once a student has been assigned her top achievable choice, neither she nor her roommate can ever be involved in another Pareto improvement. Therefore we can eliminate them both from consideration. Since we eliminate at least two students after every iteration, there can be at most $\frac{N}{2}$ iterations.

The algorithm performs $O(N)$ many iterations of an $O(N^2)$ process. Therefore it is, at worst, $O(N^3)$.

Chapter 3

Externalities in Networks

In network theory, externalities play a critical role in determining which networks are optimal. Adding links can create positive externalities, as they potentially make distant vertices closer. On the other hand, links can result in negative externalities if they increase congestion or add competition. This paper introduces two new equilibrium concepts and will completely characterize the set of optimal and equilibrium networks for a natural class of negative externalities models where an agent's payoff is a function of the degree of her neighbors. These results are in sharp contrast to the optimal and equilibrium networks for the standard class of positive externalities models where payoff is a function of the distance two agents are apart.

3.1 Introduction

Networks have long been studied in sociology, computer science, physics and mathematics. However, economists have only recently begun to focus on networks. This is surprising as graphs are a natural model of many economic situations. Most people find a job through a network of friends and associates.[1] Similarly, a car manufacturer does not buy its brakes from a marketplace but rather has long term relationships with its suppliers.

Fortunately, networks have started to receive the attention they deserve.[2] Whether out of convenience or necessity, virtually all networking papers employ a reduced form utility function. However, there has been very little attention paid

[1]See Granovetter (1973, 1995), Rees (1966), and Montgomery (1991).

[2]Jackson (2003) is an excellent survey.

to how the choice of utility function affects structural predictions. In particular, there is an interesting feature embedded in a utility function–whether or not additional links cause positive or negative externalities to uninvolved vertices. This paper seeks to show that the choice of underlying utility function and the treatment of externalities is of critical importance to which networks are optimal.

Jackson and Wolinsky's seminal paper *A Strategic Model of Social and Economic Networks* [1996, JET], hereafter **JW**, introduced two reduced form models and solved for the socially optimal network in each. These models can be described by the utility an individual gets from a graph.

JW's Connections Model

$$u_i(G) = \sum_{j \neq i} \delta^{d(i,j)} - c * d_{v_i}$$

where $0 \leq \delta \leq 1$, $d(i,j)$ is the length of the shortest path between v_i and v_j, and d_{v_i} is the number of agents v_i has an edge with.

JW's Co-author Model

$$u_i(G) = \sum_{j: e_{i,j} \in E(G)} [\frac{1}{d_{v_i}} + \frac{1}{d_{v_j}} + \frac{1}{d_{v_i} d_{v_j}}]$$

The Connections Model has a very nice solution space. If we measure aggregate utility as the sum of individual utilities, then the socially optimal graph is either empty, a star, or complete. A star is a network with one central agent involved in every connection[3]. Moreover, for appropriate link costs, these graphs will be pairwise stable. The Co-author Model has a much less interesting solution space.

[3]See Figure 3.1 on page 45 for an example.

For an even number of participants, 2N, the socially optimal graph is simply N pairs. It is possibly for this reason that the Connections Model (or derivations of it) appears much more frequently in the subsequent literature.

The star turns out to be optimal for a wide class of models. Bloch and Jackson (2007) define a utility function to be *distance based* if there exist c and f such that

$$u_i(G) = \sum_{j \neq i} f(d(i,j)) - c * d_{v_i}$$

where $d(i,j)$ is the number of links in the shortest path between vertices i and j, f is a nonincreasing function, and c is a cost per link. They demonstrate that the unique non-trivial efficient network is the star network. The star formation appears frequently in the literature under a variety of different utility functions. Given their result, it is not surprising that the common ingredient among these models is they are distance based.

The fact that the star is optimal for such a wide class of utility functions makes a compelling case for it as a real world model. However, there are at least two points of concern. First, in the star every vertex's payoff is strictly increasing in the size of the network, moreover at an increasing rate. Thus, not only do these models predict the optimal network to be a star but the largest star possible. However, we expect real-world networks to lose their value and start to break down as they get too large. Moreover, the entire world organized as a star is clearly not ideal for any situation. Another concern is that, although star networks are widely predicted by economic theory, they are not commonly observed in practice. We may observe star-like networks such as airlines' hub-and-spoke systems, but I know of no true star networks.

A distance based model is an environment where all externalities must be positive. If two agents form a link, then they weakly decrease the distance all other agents are apart and weakly increase the number of other agents they are connected to. There are at least two important considerations that a model with only positive externalities does not capture. First, in most networks we expect there to be congestion issues. This is especially true if we are discussing a computer network, but congestion also occurs in most economic networks. For example, **JW's** co-authors model is meant to capture that as the number of co-authors you have increases, you have less time to devote to each one. We would expect the same thing to occur in a network of friends, a communications network, a network of business associates, etc. A star should be especially prone to congestion issues as there is a clear bottleneck, the central node.

A second consideration is that most economic networks involve competition among agents. In a network of buyers and sellers, an additional link literally means increased competition. In a network of gamblers exchanging private information about a horse race, the value of the information decreases with each additional person who learns the inside tip. When an MBA student talks about networking, they are referring to contacts to help them get a job. One can imagine it would be very helpful to have a friend forward your resume to her boss. However, the value of this is substantially decreased if she also forwards the resume of twenty other friends.[4]

[4]Calvo-Armengol and Jackson (2004) captures this effect. In their model, when an

A natural way of modeling a network where an agent's payoff is adversely affected by competition or congestion is through a degree based utility model. I will define a utility function to be *degree-based* if there exists a ϕ such that

$$u_v(G) = \sum_{w \in N(v)} \phi(d_w)$$

where ϕ is non-increasing and d_w is the number of direct relationships w has[5]. In this environment, externalities can only be negative. A new connection in a network weakly increases the degree of each of an agent's neighbors, so if the agent is not directly involved with the new link, her payoff must weakly decrease. This paper will completely characterize the socially optimal and equilibrium set of networks for degree based utility functions.

First, I will show that the regular network is socially optimal for any degree based utility function[6]. Next, I will introduce two new equilibrium concepts which extend the traditional notion of pairwise stability. Under pairwise stability, an agent is able to biilaterally add an edge or unilaterally drop an edge. Strong pairwise stability, a natural extension of pairwise stability, allows an agent to both drop an edge and add another concurrently. Under strong pairwise stability with transfers, an agent is also able to transfer utility to her immediate neighbors. I will be able to completely characterize the set of strongly-pairwise-stable networks for any degree based utility function. Finally, with only a mild assumption on the consistency of

employed agent hears about a job, she randomly picks an unemployed acquaintance to pass the job to. In this model, edges have negative externalities as an acquaintance adding a connection decreases the chances you will be randomly chosen in the event you are unemployed.

[5]The notation $w \in N(v)$ will be explained more fully in the section on Graph Theory terminology, but it means any vertex w that has a direct edge with v.

[6]A network is regular if all agents have the same number of connections

externalities, I will be able to show that a network is strongly pairwise stable with transfers if and only if it is the socially optimal regular graph.

These results for degree based utility functions are not only interesting in their own right but especially as a contrast to the positive externalities environment. If we use as our metric maximum degree minus minimum degree, then the star and a regular graph are as different as two graphs can possibly be. It is interesting that two very natural classes of utility functions can lead to such strikingly different optimal and equilibrium networks. As such, this should be taken as a note of caution for researchers using a reduced form utility function to model a social network. The choice of utility function and more broadly the treatment of externalities in these networks are of critical importance to the predictions of the model.

As one of the first papers in networks and specifically one that characterized the solution space of two reduced form models, **JW** has greatly influenced the models in subsequent papers. The solution space for the connections model, the star, is quite interesting whereas the solution space for the co-authors model is trivial. Possibly for this reason, most subsequent papers are based roughly on the connections model. As a result they are positive externalities models. This is unfortunate as most situations of interest to economists involve competition and thus exhibit negative externalities. The final contribution of this paper is to propose and solve a negative externalities model that both has a more interesting solution space than the co-authors model and is a more natural counterpoint to the connections model.

This paper is closely related to Bloch and Jackson (2007). The two papers, in conjunction, are able to completely characterize the two most intuitive, general classes of network models. The focus of Bloch and Jackson is on network formation. They introduce several games in which players make decisions about both link formation and transfers to other agents in the network. Their paper highlights the importance of externalities and demonstrates a reasonable way in which agents might overcome them. The equilibrium concept I introduce, strong pairwise stability with transfers, is a core concept which complements the network formation games introduced in their paper.

This paper is most closely related to Jackson and Wolinsky (1996). They present and solve two reduced form models. Their first model, the connections model, is a distance based model, while their second, the co-authors model, is essentially a degree based model. Therefore, the results presented here in conjunction with the results in Bloch and Jackson (2007) can be viewed as a generalization of **JW**.

There are at least two other papers, Currarini (2002) and Goyal and Joshi (2002), that look at externalities in networks; however, both papers take a substantially different modeling approach than this paper. Currarini (2002) focuses on the partition of vertices into connected components. In Currarini's model the value of a network depends only on this partition. Externalities are defined by whether the value of a network increases or decreases when the partition becomes finer. The network matters in that it determines the partition, but in Currarini's framework the role of network architecture is minimized. Goyal and Joshi's (2002) approach is more similar to this paper but still substantially different. They examine two interesting models where agents have varying degrees in equilibrium. The focus of their paper is how differing degrees affect agent payoff. Externalities, in the form of whether or not agents are strategic substitutes or compliments, end up being crucial to solving their models, but their two games are not an attempt to actually model externalities. In fact, strategic complementarities are defined specifically in terms of the particular payoff function they use and their concept is not readily generalizable. In contrast, this paper's primary aim is to model network externalities in as general a way as possible without diminishing the role of network architecture.

The remainder of this article is organized as follows. Section 2 provides a brief overview of graph theory terminology. Section 3 introduces degree based utility functions and completely characterizes the socially optimal and equilibrium networks for any degree based utility function. Section 4 introduces and solves a specific reduced form utility function. Section 5 concludes, and the Appendix provides complete proofs.

3.2 Graph Theory Terminology

This paper uses no theoretical results from graph theory, but I will borrow from its terminology to facilitate discussion. I will represent a network as a graph where vertices represent agents and an edge represents a relationship between the two agents. All edges are undirected.

The following are some definitions with corresponding notation:

- E(G) is the set of edges in a graph G. $e_{u,v} \in E(G)$ is an edge between vertices u and v.

- V(G) is the set of vertices in G.

- u and v are *adjacent* if $e_{u,v} \in E(G)$. $u \leftrightarrow v$ indicates u and v are adjacent.

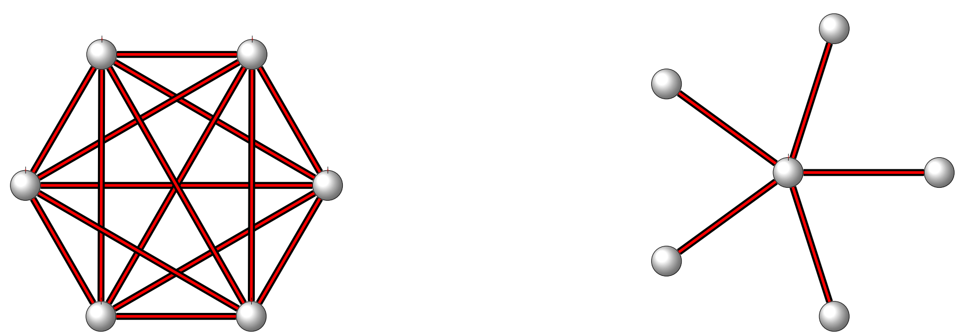
- The *neighborhood* of v is the set of vertices adjacent to v. Symbolically, $N(v) = \{u \in V(G) | e_{u,v} \in E(G)\}$.

- The *degree* of v is the number of vertices v is adjacent to. Symbolically, $d_v = |N(v)|$.

- G is *complete* if all vertices are adjacent.

- A *star* is a graph with one center vertex, in which all remaining vertices are adjacent only to the center.

Figure 3.1: A complete graph and a star.



- A *path* between u and v is a set of edges $\{e_{v_1,v_2}, e_{v_2,v_3}, e_{v_3,v_4}, \ldots, e_{v_{n-1},v_n}\}$ such that $v_1 = u$ and $v_n = v$.

- u and v are *connected* if there exists a path between them. The *components* of a graph are its maximal connected subgraphs. Note that the connection relation is transitive, symmetric, and reflexive, so it is an equivalence relation. The equivalence classes of the connection relation are the connected components.

This paper assumes the number of vertices are fixed. Therefore, a graph is completely characterized by its set of edges. As a result, I will slightly abuse notation and use G and E(G) interchangeably. For example, $G \cup e_{u,v}$ represents the new graph created by adding an edge between u and v.

3.3 Degree Based Utility Functions

In this section I will introduce a class of models which is a natural counterpoint to the more widely studied distance based models. A utility function is *degree-based* if there exists a ϕ such that

$$u_v(G) = \sum_{w \in N(v)} \phi(d_w)$$

. A particularly desirable feature of degree based utility models is that they are interesting even when links are costless to form. For any positive externalities model, if links are costless then the complete network must be socially optimal as adding links can not harm the agents but may help them. As a result, positive externalities models are only interesting to study when links are costly. However, there are situations where incurring a cost for a link does not have a clear interpretation. For example, it is not clear how an agent incurs a cost that is completely separate from the benefit of having a friendship. Time spent maintaining a business relationship or even expenses incurred "wining and dining" a potential partner might reasonably be considered costs, but it is difficult to apply this to a model of personal relationships.

After all, having someone to spend time with and someone to go to costly activities such as dinners or baseball games are some of the benefits, not costs, of having a friend. Therefore, it is particularly attractive that we do not need separate costs for links in order to make a degree-based utility function nontrivial.

The star does not perform as well with a rival utility function. The perimeter vertices only get utility from their immediate neighbor, the center of the star. However, the center is connected to so many vertices (the maximum possible) that its value has decreased. Moreover, we do not expect the star to be an equilibrium of a network game as two perimeter vertices would do better by dropping their connection to the center agent and forming a link to each other. In this environment, a more symmetric graph does better socially and is more likely to be pairwise stable. A regular graph, where all agents have the same number of connections, is the natural place to look. Unfortunately, regular graphs do not always exist.[7] However, a regular graph always exists when there is an even number of vertices. I will define a new class of graphs which exist regardless of the parity of the number of agents.

Definition 6. *Let* $\overline{d} = \max\{d(v) : v \in V(G)\}$ *and* $\underline{d} = \min\{d(v) : v \in V(G)\}$. *Then:*

1. *A graph is* ***nearly-regular*** *if* $(\overline{d} - \underline{d}) \leq 1$.

2. *A graph is* ***nearly-n-regular*** *if* $(n-1) \leq \underline{d} \leq \overline{d} \leq n$.

[7] For example, if we have an odd number of vertices, we can not have a $(2a+1)$-regular graph. Since every edge contributes two to the sum of all vertex degrees, the total sum of degrees must be even. An odd-regular graph with an odd number of vertices would have an odd total degree sum which is not possible.

The next proposition completely characterizes the set of socially optimal networks when there is an even number of vertices. I will be able to simplify this characterization once I impose a mild assumption on the consistency of externalities.

Proposition 5. *Suppose there is an even number of agents. A network G is socially optimal if and only if for every vertex v, $d_v \in \arg\max x\phi(x)$. In particular, for any $n \in \arg\max x\phi(x)$, all n-regular graphs are socially optimal.*

Proof. Each agent receives a payoff *from* her neighbors and contributes utility *to* her neighbors. As an accounting identity, the sum of what every agent receives must equal the sum of what every agent contributes. In particular

$$U(G) = \sum_{i=1}^{N} u_i(G) = \sum_{i=1}^{N} \sum_{v_j \in N(v_i)} \phi(d_{v_j}) = \sum_{i=1}^{N} d_{v_i}\phi(d_{v_i}) \tag{3.1}$$

Let $n \in \arg\max x\phi(x)$. Since we have an even number of vertices, an n-regular graph exists[8]. Pick any n-regular graph H. By Equation 3.1, H must be socially optimal. Moreover, if J is a network with a vertex v such that $d_v \notin \arg\max x\phi(x)$, then $U(J) < U(H)$.

$\square$

[8]To see this, label the vertices 0 through $|V(G)| - 1$. If n is even, then connect vertex i to vertices $i \pm jmod(|V(G)|)$, for $1 \leq j \leq \frac{n}{2}$. If n is odd, then connect vertex i to vertex $i + \frac{|V(G)|}{2} mod(|V(G)|)$ and to vertices $i \pm jmod(|V(G)|)$, for $1 \leq j \leq \frac{n-1}{2}$

3.3.1 Pairwise Stability

Pairwise stability is the standard equilibrium concept in Network Theory. Intuitively, it says no agent wishes to unilaterally drop one of her connections, and no two agents wish to bilaterally add a connection. More formally:

Definition 7. *A graph G is pairwise stable if:*

1. *If $e_{i,v} \in E(G)$, then $u_i(G) > u_i(G - e_{i,v})$ and $u_j(G) > u_j(G - e_{i,v})$.*

2. *If $e_{i,v} \notin E(G)$, then either $u_i(G) > u_i(G + e_{i,v})$ or $u_j(G) > u_j(G + e_{i,v})$.*

So far in this section we have assumed that links are costless. However, we must add link costs in order to make pairwise stability interesting. Otherwise, as long as ϕ is strictly positive, the only pairwise stable graph is the complete graph. For the remainder of this subsection, I will assume $u_v(G) = \sum_{w \in N(v)} \phi(d_w) - c \cdot d_v$. Moreover, to avoid any nuisances, I will assume $c \neq \phi(n)$ for any $n \in \mathbb{N}$. This assumption is without loss of generality since one can perturb c by an arbitrarily small ϵ.

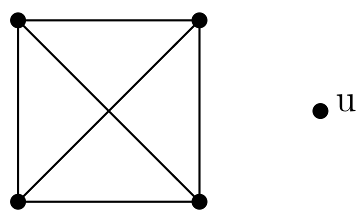
This next structure appears several times so I will explicitly define it.

Definition 8. *G is a* ***maximal nearly-n-regular graph*** *if it is nearly n regular and there does not exist a nearly-n-regular graph G' such that $E(G) \subset E(G')$.*

Proposition 6. *Let $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$. If utility is rival, then all maximal nearly-n-regular graphs are pairwise stable.*

Proof. Let G be any maximal nearly-n-regular graph. Look at any vertices u and v such that $e_{u,v} \notin E(G)$. The maximality of G implies at least one u or v must have

Figure 3.2: An undesirable pairwise stable graph



degree n. Without loss of generality, assume $d_v = n$. Then $u_i(G + e_{u,v}) - u_i(G) = \phi(n+1) - c < 0$ by the maximality of n. Similarly, look at any vertices u and v such that $e_{u,v} \in E(G)$. $u_i(G) - u_i(G - e_{u,v}) \geq \phi(n) - c \geq 0$. Therefore G is pairwise stable. □

Pairwise stability is a fairly weak concept. Pairwise stability only allows a vertex to unilaterally drop a connection or to bilaterally add a connection, but not both. For example, if $\max\{x \in \mathbb{N} | \phi(x) \geq c\} = 3$, then the graph in Figure 3.2 is pairwise stable since u will be unwilling to add an edge with any vertex that already has degree 3. However, this is unsatisfying as any of the vertices in the 4-clique would be happy to *exchange* one of their edges for an edge with u. Similarly, as long as the vertex is willing to drop one of her edges, u would be happy to form an edge with any vertex in the 4-clique. This leads to a new solution concept.

Definition 9. *A graph G is* ***Strongly Pairwise Stable*** *if*

1. *G is pairwise stable*

2. *There does not exist a* $u, v, w \in V(g)$ *such that* $u_u(G') > u_u(G)$ *and* $u_v(G') > u_v(G)$ *where* $G' = G + e_{u,v} - e_{v,w}$.

With this stronger solution concept, we are able to completely characterize the set of strongly pairwise stable graphs.

Proposition 7. *Let* $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$ *and suppose* ϕ *is strictly decreasing. Then G is strongly pairwise stable if and only if G is a maximal nearly-n-regular graph.*

Proof. It is straightforward to verify that a maximal nearly-n-regular graph is strongly pairwise stable. To prove the other direction, look at any strongly pairwise stable graph G. First note that $\max\{d_v | v \in V(G)\} = n$ since otherwise G would not be pairwise stable.[9] Suppose for contradiction that G is not nearly-regular. Then there exists a u and v such that $d_v - d_u \geq 2$. Let $w \in N(v) \setminus N(u) \neq \emptyset$ and let $G' = G + \{e_{u,w}\} - \{e_{v,w}\}$. Note that $u_w(G') > u_w(G)$ since $d_u + 1 < d_v$. Moreover, $u_u(G') > u_u(G)$ since $d_w \leq n$ and $\phi(n) > c$. Therefore G is not strongly pairwise stable, a contradiction. Since G has max degree n and is nearly-regular, it must be nearly-n-regular. If G is not maximal, then there are two non-adjacent vertices of degree n-1 and therefore G is not pairwise stable. □

3.3.2 Equilibrium with Transfers

In this section I will introduce a new core concept which generalizes pairwise stability to allow agents to make transfers. This complements the games introduced in Block and Jackson (2007). They define several new network games which extend

[9]If the maximum degree is greater then n, some vertex would want to drop an edge. If the max degree is less then n, any unconnected vertices would be better off adding an edge.

the traditional network games to allow players to make financial transfers. In their paper they make a distinction between who an agent is able to make a transfer to and on what an agent is able to condition this transfer payment. In the direct transfer game, an agent can only make a transfer for a link she is directly involved with. In the indirect transfer games she can only make demands on her own relationships, but she is free to subsidize *any* relationship. In their standard game, a transfer is conditional only on a link forming, but in their game with contingent transfers, an agent can condition a payment on the entire network structure.

Transfers are a natural concept in a social network. We often see situations where maintaining the relationship is more important to one of the agents than the other. In the business world, one person may be in a position of greater power or may simply have more connections than the other. In the academic world, the marginal value of publishing an article should decrease with the total number of publications an author has. Therefore, even if two co-authors are equally talented, the relationship should be relatively more valuable to the person with fewer publications. In such a situation, it is natural for a transfer to occur. The person to whom the relationship is more valuable will have the incentive to put forth more effort into the project or else might agree to do the less desirable aspects of the work.

I propose a new core concept which generalizes pairwise stability to allow agents to make transfers to the people they have a relationship with. I only allow direct transfers as I find indirect transfers much less natural. A transfer between two agents with a relationship can be non-pecuniary; however, there is less flexibility when the transfer is between two agents who are not linked. It is hard to interpret an indirect transfer as anything other that a monetary payment. In some situations this may be perfectly natural, but especially with social networks, direct monetary transfers occur infrequently. An academic can do many things to either ease the burden on her coauthor or make the relationship more valuable, but it would be strange for her to pay an academic to *not* collaborate with any of her coauthors. Despite this restriction, we will still be able to reach some powerful conclusions.

More formally, the game consists of a graph G and a matrix of transfers T. In the transfer matrix, t_{ij} represents the transfer from agent i to agent j. An agent can only make a transfer to someone she has a direct relationship with. Therefore, $t_{ij} > 0$ only if $e_{ij} \in G$. To avoid ambiguity, I will require that $t_{ij} = -t_{ji}$ as we care only about the net transfer.

Therefore, given a network G and transfers T, agent i receives a payoff of

$$\pi_i(G,T) = \sum_{v_j \in N(v_i)} [\phi(d_{v_j}) + t_{ji}] \tag{3.2}$$

Individually, an agent should be able to drop any of her edges and change the transfers she offers. Two agents should also be able to form a link if they so desire. Anytime an agent changes her edges or transfers, she alters the payoff of her neighbors and, therefore, potentially jeopardizes these relationships. However, if two agents are able to move bilaterally to establish a mutually beneficial relationship and to adjust their transfers so that *all* of their neighbors are better off, then they do not jeopardize these relationships and surely the network is in disequilibrium.

Definition 10. *Given a network G with transfers T, agent v_i* ***blocks*** $< G, T >$ *if* *there exists an agent* v_j[10]*, subsets* $A \subseteq N(v_i)$*,* $B \subseteq N(v_j)$*, and transfers* T' *on the set* $\{v_i, v_j, N(v_i) \setminus A, N(v_j) \setminus B\}$ *such that:*

$$\pi_x(G', T + T') > \pi_x(G, T)$$

for every $x \in \{v_i, v_j, N(v_i) \setminus A, N(v_j) \setminus B\}$ *where* $G' = G \cup e_{i,j} \setminus \{e_{i,k} : k \in A\} \setminus \{e_{j,k} : k \in B\}$*.*

This definition is purely a generalization of strong pairwise stability. An agent can drop any edge, add an edge (as long as the other agent wishes to as well), or do both simultaneously. If the agent is also able to compensate all the agents she remains in a relationship with so that they are made better off by the change, then surely we are in disequilibrium.

Definition 11. *A network* G *is* ***strongly pairwise stable with transfers*** *if there exists transfers* T *such that no agent blocks* $< G, T >$*. In such a case, we say the transfers* T ***support*** G*.*

When it is clear from context that it is a network with transfers, I will just say strongly pairwise stable instead of strongly pairwise stable with transfers. With a mild regularity assumption on externalities, I will prove that the only network that is strongly pairwise stable with transfers is the socially optimal, nearly-regular network.

Whenever an edge is added to an network, there is a social trade off. There is a direct benefit to the two agents forming the relationship but at a cost of a decreased

[10] We allow $v_j = \emptyset$ in which case we interpret $G \cup e_{i,j} = G$.

payoff to all the agents they already share an edge with. This trade-off is captured by the function:

$$to(x) = \phi(x+1) - x(\phi(x) - \phi(x+1)), 1 \leq x \leq |V(G)| - 2$$

When an agent with degree x forms a new edge, she is contributing $\phi(x+1)$ to the new neighbor but at a cost of $\phi(x) - \phi(x+1)$ to the x many agents she was previously connected to. This motivates a regularity condition on externalities.

Definition 12. *The externalities in a network are* ***consistent*** *if* $to(x)$ *is decreasing.*

Definition 13. *Externalities are* ***weakly consistent*** *if any of the following conditions hold:*

1. $to(x) > 0$.
2. $to(x) < 0$.
3. *There exists a integer* M *such that* $to(x) \geq 0$ *for every* $x \leq M$ *and* $to(x) < 0$ *for every* $x > M$.

For each respective case, we define the ***threshold*** *of* $to(x)$ *to be:*

1. $|V(G)| - 1$
2. *1*
3. M

$to(x)$ is decreasing if $\phi(x) - \phi(x+1)$ is increasing. Therefore, if ϕ is concave, then externalities are consistent. However, this condition is much weaker than concavity. Of course, if externalities are consistent then they are weakly consistent.

I will assume that if $to(x)$ is weakly consistent with threshold M with $1 < M < |V(G)| - 1$, then $to(M) = 0$. This does not change the results in any significant way, but it does make the proofs cleaner. Moreover, if $to(M) \neq 0$, then we can replace $\phi(x)$ by $\phi'(x)$ where $\phi'(x) = \phi(x) - to(M)$. Now:

$$\begin{aligned} to_{\phi'} &= \phi'(x+1) - x(\phi'(x) - \phi'(x+1)) \\ &= \phi(x+1) - to(M) - x(\phi(x) - to(M) - \phi(x+1) + to(M)) \\ &= to_{\phi}(x) - to(M) \end{aligned}$$

and therefore $to_{\phi'}(M) = 0$.

With this mild assumption on externalities, we can completely classify the set of socially optimal networks. In a surprising result, once agents are able to make transfers to their neighbors, these networks will also be the only strongly pairwise stable networks.

Proposition 8. *Suppose externalities are weakly consistent.*

1. *When $1 < M < |V(G)| - 1$, then G is socially optimal if and only if G is nearly-(M+1)-regular.*

2. *When $M = |V(G)| - 1$, then the complete network is the unique socially optimal network.*

3. *When $M = 1$, then the unique socially optimal network is the trivial network*[11].

[11]The trivial network consists of $\frac{|V(G)|}{2}$ many pairs if $|V(G)|$ is even and $\frac{|V(G)-3|}{2}$ many pairs plus the three remaining vertices connected as a path if $|V(G)|$ is odd. See Figure 3.3 on page 66 for an example.

I will prove Proposition 8.1 here. The remaining cases, which are more technical, are proved in the Appendix.

Proof. We know from the previous section that G is optimal if and only if every vertex has degree from the $argmax x\phi(x)$. Note that

$$\begin{aligned} to(x) &= \phi(x+1) - x(\phi(x) - \phi(x+1)) \\ &= (x+1)\phi(x+1) - x\phi(x) \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{2\leq i\leq x} to(i-1) &= \sum_{2\leq i\leq x} [i\phi(i) - (i-1)\phi(i-1)] \\ &= x\phi(x) - \phi(1) \end{aligned}$$

since $\sum_{2\leq i\leq x}[i\phi(i) - (i-1)\phi(i-1)]$ is a telescoping series. In particular,

$$x\phi(x) = [\sum_{2\leq i\leq x} to(i-1)] + \phi(1) \tag{3.3}$$

Since, $to(x) \geq 0$ for every $x \leq M$ and $to(x) < 0$ for every $x > M$, $x\phi(x)$ is maximized at $x = M+1$. Note that

$$\begin{aligned} (M+1)\phi(M+1) &= [\sum_{2\leq i\leq (M+1)} to(i-1)] + \phi(1) \\ &= to(M) + [\sum_{2\leq i\leq (M)} to(i-1)] + \phi(1) \\ &= 0 + [\sum_{2\leq i\leq (M)} to(i-1)] + \phi(1) \\ &= M\phi(M) \end{aligned}$$

Where the second to last equality follows from our assumption that $to(M) = 0$. Since $to(x) \neq 0$ for every $x \neq M$, $argmax(to(x)) = \{M, M+1\}$. As mentioned previously, a nearly regular graph always exists. Therefore, a network is optimal if and only if it is nearly-(M+1)-regular.

□

I will now prove the main theorem of this section.

Proposition 9. *Suppose externalities are weakly consistent.*

1. *When* $1 < M < |V(G)| - 1$*,* G *is strongly pairwise stable with transfers if an only if* G *is nearly-(M+1)-regular.*

2. *When* $M = |V(G)| - 1$*, the complete network is the unique network that is strongly pairwise stable with transfers.*

3. *When* $M = 1$*, the unique network that is strongly pairwise stable with transfers is the trivial network.*

This proposition is proved with a sequence of lemmas. Complete proofs are given in the Appendix, but I will list the lemmas and give the intuition to the proof here.

Lemma 5. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with degree less than* M*, then* u *is adjacent to every vertex with degree less than or equal to* M*.*

Intuition. If there are two non-adjacent agents with degree less than the threshold, then adding an edge is socially beneficial as the trade-off for both agents is positive. Moreover, all the social gains are realized by the two agents. Since it is socially beneficial, they benefit enough to be able to compensate all of their neighbors and still improve their payoff.

Lemma 6. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with degree greater than* $M+1$*, then* u *is not adjacent to any vertex with degree greater than or equal to* $M+1$*.*

Intuition. It is socially beneficial for the two agents with degree greater than the threshold to cut their relationship. Their neighbors, who receive all the benefit from the two agents cutting an edge, are willing and able to increase their transfers by enough to make it in u or v's best interest to cut the edge. We have to be a little more careful than in Lemma 5 since there may be transfers between the two agents which would affect their willingness to drop the edge.

Lemma 7. *Suppose* T *supports a network* G*. Then for every two agents* i *and* j *such that* $e_{i,v} \in G$*,*

$$t_{ij} \leq \phi(d_j) - (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$$

Proof. v_i has $d_i - 1$ many neighbors who would be willing to pay up to $\phi(d_i - 1) - \phi(d_i)$ for i to sever her relationship with j. i receives a benefit of $\phi(d_j) - t_i, j$ from her relationship with j, so if $\phi(d_j) - t_{ij} < (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$ then i and all her remaining neighbors do strictly better if i drops her relationship with j and accepts a transfer of $\phi(d_i - 1) - \phi(d_i) - \epsilon$ from each of her remaining neighbors. □

Lemma 8. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. If there exists a vertex* u *with* $d_u > M+1$*, then all of* u*'s neighbors are adjacent.*

Intuition. If there is an agent u such that $d_u > M + 1$, $v, w \in N(u)$, but $e_{vw} \notin G$, then v and w are better off dropping their edges with u and creating an edge between themselves. They do not need to compensate their neighbors for such a move as their degree has not changed. We know from Lemma 6 that d_v and d_w are both less than $M + 1$, so, both v and w are made better off as they each have degree less than u. The only thing we have to be careful about is that u may be subsidizing her relationship with either or both of the agents, and when they sever the relationship with u they forgo the subsidy. Lemma 7 sets a bound on the transfer from u to v or w that ensures it will always be in v and w's best interest to forgo the transfer and establish a relationship with each other.

Lemma 9. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be strongly pairwise stable. No vertex in* G *has degree greater than* $M + 1$*.*

Intuition. This is a pigeonhole argument. Suppose for contradiction there is a vertex u with $d_u > M + 1$. By Lemma 6, every neighbor of u must have degree less than or equal to M. By Lemma 8, all neighbors of u must be adjacent. However, there are at least $M + 1$ neighbors of u. All are adjacent to the other neighbors of u (there are at least M other neighbors of u) plus u itself. Therefore, all neighbors of u must have degree at least $M + 1$, a contradiction.

Lemma 10. *Suppose* $1 < M < |V(g)| - 1$ *and let* G *be socially optimal. No vertex in* G *has degree less than* M*.*

Intuition. If there exists a vertex u with $d_u < M$, we know from Lemma 5 that u must be adjacent to all vertices with degree less than or equal to M. We can establish that there must be two vertices, v and w that are adjacent to each other, but neither of which are adjacent to u. Either of these agents would be better off dropping the edge with the other and instead establishing an edge with u. Moreover, this will be socially beneficial as both v and w must have degree $M+1$ and socially we are indifferent whether or not they are adjacent. However, since $d_u < M$, we strictly prefer that u create a new relationship. Since this switch is socially beneficial and all the gains are realized by u and the vertex that switches, they will be able to compensate u's neighbors for their decreased payoff. Again, we have to be careful about transfers between v and w, but only one can be receiving a positive transfer from the other.

Lemma 11. *Suppose* $1 < M < |V(g)| - 1$. *If G is nearly-(M+1)-regular, then G is strongly pairwise stable with transfers.*

Proof. Let G be any nearly-(M+1)-regular graph. Define a set of transfers T by:

$$t_{uv} = \begin{cases} 0 & d_u = d_v \\ \phi(M) - \phi(M+1) & d_u = (M+1), d_v = M \\ \phi(M+1) - \phi(M) & d_u = M, d_v = M+1 \end{cases}$$

Every agent with degree M receives a total payoff of $M\phi(M)$ and every agent with degree $M+1$ receives a payoff of $(M+1)\phi(M+1)$. Since $to(M) = 0$, $M\phi(M) = (M+1)\phi(M+1)$.

A nearly-(M+1)-regular graph is optimal, so adding an edge cannot increase social payoff. Since all the benefits are captured by the two agents adding an edge and all costs are incurred by their neighbors, it is not possible for the two agents adding the edge to make all their neighbors better off. Similarly, an agent u has no wish to delete one of her edges. u's remaining neighbors receive all the benefit, while u incurs all the costs. Since the costs are greater than or equal to the benefits (the original graph was socially optimal), u's remaining neighbors will not be able to compensate u so that all are better off. Finally, two vertices u and v can not do better by each dropping an edge and creating an edge with each other. The new relationship is worth at most $\phi(M)$ which is exactly what they received from their previous relationship. □

Lemma 9 and Lemma 10 establishes the nearly-(M+1)-regularity is necessary for strong pairwise stability with transfers. Lemma 11 establishes that being nearly-(M+1)-regular is sufficient as well. This is a surprising and powerful result. In a network of relationships, an agent should be able to sever any ties it chooses and establish new ties when it is mutually desirable. Moreover, there should always be informal ways an agent can exert effort that is costly for herself but makes the relationship more beneficial for a partner. My result establishes that if this is case, then the only network which will be an equilibrium is the socially optimal network.

3.4 A Reduced Form Utility Model

A particular degree based utility function of interest is:

$$u_i(G) = w_i + \sum_{i \leftrightarrow j} \gamma^{d_{v_j}} w_{i,j} - \sum_{i \leftrightarrow j} c_{i,j}, \text{ where } 0 \leq \gamma \leq 1. \tag{3.4}$$

Recall JW's Connections Model is:

$$u_i(G) = w_i + \sum_{j \neq i} \delta^{t_{i,j}} w_{i,j} - \sum_{i \leftrightarrow j} c_{i,j}$$

where $t_{i,j}$ is the length of the shortest path between v_i and v_j.

As mentioned before, the Connections Model has only positive externalities. The co-authors model is the negative externalities model JW examine, but the utility function presented in Equation 3.4 is a more natural negative externalities counterpoint to the connections model. A vertex only gets utility from its neighbors, and this utility is a decreasing function of each neighbor's degree. This also fits **JW**'s motivation for the co-authors model. The benefit to working with a colleague is decreasing in the number of co-authors she has as she will have less time to devote to your project.

With our results from Section 3, we can quickly solve for the symmetric version of Equation 3.4. Let

$$u_i(G) = \sum_{i \leftrightarrow j} \gamma^{d_{v_j}} \tag{3.5}$$

where $0 < \gamma < 1$. Further, suppose $\gamma = \frac{\tau}{\tau+1}$ for some integer τ.

By assumption:

$$\begin{aligned} to(x) &= (x+1)\gamma^{x+1} - x\gamma^x \\ &= \gamma^x((x+1)\gamma - x) \\ &= \gamma^x(x(\gamma - 1) + 1) \end{aligned}$$

which is a decreasing function of x. Moreover

$$to(\tau) = \gamma^\tau((\tau+1)\gamma - \tau)$$

$$
\begin{aligned}
&= \gamma^{\tau}((\tau+1)\frac{\tau}{\tau+1} - \tau) \\
&= \gamma^{\tau}((\tau - \tau) \\
&= 0
\end{aligned}
$$

Therefore all the assumptions of Proposition 3 on page 58 are met.

Proposition 10. *Suppose* $u_i(G) = \sum_{i \leftrightarrow j} \gamma^{d_{v_j}}$. *Then*

1. *G is socially optimal if and only if G is nearly-*$(\tau+1)$*-regular.*

2. *G is strongly pairwise stable with transfers if and only if G is nearly-*$(\tau+1)$*-regular.*

3.5 Conclusion

Distance based and degree based models are the two most intuitive models of an agent's payoff from a network. While much attention has been paid to distance based models, very little has been paid to degree based models. This paper completely characterizes the set of optimal and stable networks for this natural class of utility functions. The predicted networks are interesting in their own right but especially so when taken in contrast to the optimal networks for distance based models. It is striking that two intuitive models can lead to such dramatically different predictions. In particular, this paper, taken in conjunction with the results from Bloch and Jackson (2007), provides a generalization and simplification of results in the classic networks paper by Jackson and Wolinsky (1996).

3.6 Appendix - Proofs

The trivial network consists of $\frac{|V(G)|}{2}$ many pairs if $|V(G)|$ is even and $\frac{|V(G)-3|}{2}$ many pairs plus the three remaining vertices connected as a path if $|V(G)|$ is odd.

Proposition 8. *Suppose externalities are weakly consistent.*

1. *When* $1 < M < |V(G)| - 1$, *then* G *is socially optimal if an only if* G *is nearly-(M+1)-regular.*

2. *When* $M = |V(G)| - 1$, *then the complete network is the unique socially optimal network.*

3. *When* $M = 1$, *then the unique socially optimal network is the trivial network.*

Proof. Proposition 8.2 Let G be any graph that contains two non-adjacent vertices u and v, and suppose the threshold is $|V(G)|-1$. By the definition of the threshold, $to(x) > 0$. Therefore,

$$\begin{aligned} U(G \cup e_{u,v}) - U(G) &= (d_u+1)\phi(d_u+1) - d_u\phi(d_u) + (d_v+1)\phi(d_v+1) - d_v\phi(d_v) \\ &= to(d_u) + to(d_v) \\ &> 0 \end{aligned}$$

and G can not be optimal. Therefore, the only optimal network is one where all vertices are adjacent.

Proposition 8.3 We know from Equation 3.1 on page 48 that

$$U(G) = \sum_{i=1}^{N} d_{v_i}\phi(d_{v_i})$$

Figure 3.3: Trivial graph for an even and odd number of vertices.

and from Equation 3.3 on page 57 that

$$x\phi(x) = [\sum_{2 \leq i \leq x} to(i-1)] + \phi(1)$$

Since $to(x) < 0$ for every $x > 0$, $x\phi(x)$ is strictly decreasing for $x \geq 1$. Therefore, if a 1-regular graph exists, it must be optimal. A 1-regular graph exists when there is an even number of vertices (the trivial graph), but does not when the number of vertices is odd. Since $2\phi(2) > 0$, the trivial graph must be optimal when there is an odd number of agents.

□

Lemma 5. *Suppose* $1 < M < |V(g)| - 1$ *and let G be strongly pairwise stable. If there exists a vertex u with degree less than M, then u is adjacent to every vertex with degree less than or equal to M.*

Proof. Suppose for contradiction that G is supported by transfers T and has two non-adjacent vertices u and v with $d_u < M$ and $d_v \leq M$. Let

$$t'_{vx} = \begin{cases} \phi(d_v) - \phi(d_v + 1) + \epsilon & x \in N(v) \\ \phi(d_u + 1) - \phi(d_v + 1) & x = u \end{cases}$$

$$t'_{ux} = \begin{cases} \phi(d_u) - \phi(d_u + 1) + \epsilon & x \in N(u) \\ \phi(d_v + 1) - \phi(d_u + 1) & x = v \end{cases}$$

Then

$$\begin{aligned}
\Delta\pi_u(G \cup e_{i,j}, T+T') &= \phi(d_v+1) + t'_{vu} - \sum_{w \in N_G(u)} t'_{uw} \\
&= \phi(d_v+1) + [\phi(d_u+1) - \phi(d_v+1)] + d_u(\phi(d_u+1) - \phi(d_u) - \epsilon) \\
&= \phi(d_u+1) - d_u(\phi(d_u) - \phi(d_u+1)) - d_u * \epsilon) \\
&= to(d_u) - d_u * \epsilon \\
&> 0
\end{aligned}$$

for ϵ sufficiently small.

For $x \in N_G(u)$

$$\begin{aligned}
\Delta\pi_x(G \cup e_{i,j}, T+T') &= \phi(d_u+1) - \phi(d_u) + t'_{ux} \\
&= \phi(d_u+1) - \phi(d_u) + \phi(d_u) - \phi(d_u+1) + \epsilon \\
&= \epsilon \\
&> 0
\end{aligned}$$

Similarly, $\pi_v(G \cup e_{i,j}, T+T') - \pi_v(G,T) > 0$ and $\pi_x(G \cup e_{i,j}, T+T') - \pi_u(G,T) > 0$ for every $x \in N_G(v)$. Therefore, u and v block $< G, T >$, contradicting the assumption that T supports G. □

Lemma 6. *Suppose $1 < M < |V(g)| - 1$ and let G be strongly pairwise stable. If there exists a vertex u with degree greater than $M+1$, then u is not adjacent to any vertex with degree greater than or equal to $M+1$.*

Proof. Suppose for contradiction that G is supported by transfers T and has two adjacent vertices u and v with $d_u > M+1$ and $d_v \geq M+1$. Let $r_{xu} = \phi(d_u -$

$1) - \phi(d_u) - \epsilon$ for every $x \in N(v) \setminus u$. In other words, each neighbor of u *gives* u $\phi(d_u - 1) - \phi(d_u) - \epsilon$. Similarly, let $s_{vx} = \phi(d_v) - \phi(d_v - 1) + \epsilon$ for every $x \in N(u) \setminus v$.

Then if u drops its edge with v in order to receive transfers R, it loses the benefit from v, $\phi(d_v)$, no longer makes the transfer $t_{u,v}$, and gains the transfers R from each of its remaining neighbors. Specifically,

$$\Delta\pi_u(G \setminus e_{uv}, T + R) = -\phi(d_v) + t_{uv} + (d_u - 1)(\phi(d_u - 1) - \phi(d_u) - \epsilon)$$

Similarly,

$$\Delta\pi_v(G \setminus e_{uv}, T + S) = -\phi(d_u) + t_{vu} + (d_v - 1)(\phi(d_v - 1) - \phi(d_v) - \epsilon)$$

Adding these two equations yields

$$\begin{aligned} \Delta\pi_u(G \setminus e_{uv}, T + R) + \Delta\pi_v(G \setminus e_{uv}, T + S) &= \\ -to(d_v - 1) - to(d_u - 1) + t_{uv} + t_{vu} - \epsilon(d_u + d_v - 2) &= \\ -to(d_v - 1) - to(d_u - 1) - \epsilon(d_u + d_v - 2) &> 0 \end{aligned}$$

for sufficiently small ϵ. The first equality comes from rearranging terms. The second equality follows since $t_{uv} = -t_{vu}$ by definition. The final inequality follows since $d_u - 1 > M$ and $d_v - 1 \geq M$, so by the definition of the threshold, $\phi(d_u - 1) < 0$ and $\phi(d_v - 1) \leq 0$. But, since $\Delta\pi_u(G \setminus e_{uv}, T + R) + \Delta\pi_v(G \setminus e_{uv}, T + S) > 0$, either $\Delta\pi_u(G \setminus e_{uv}, T + R) > 0$ or $\Delta\pi_v(G \setminus e_{uv}, T + S) > 0$.

By construction, $\pi_x(G \setminus e_{uv}, T + R) - \pi_x(G, T) > 0$ for every $x \in N(u) \setminus v$ and $\pi_x(G \setminus e_{uv}, T + S) - \pi_x(G, T) > 0$ for every $x \in N(v) \setminus u$. Since

$$[\pi_u(G \setminus e_{uv}, T + R) - \pi_u(G, T)] + [\pi_v(G \setminus e_{uv}, T + S) - \pi_v(G, T)] > 0$$

at least one of $[\pi_u(G\setminus e_{uv}, T+R) - \pi_u(G,T) > 0$ or $\pi_v(G\setminus e_{uv}, T+S) - \pi_v(G,T) > 0$. Whichever one is greater than zero blocks G, a contradiction. □

In equilibrium, there is a limit to how much an agent is willing to transfer another agent.

Lemma 7. *Suppose T supports a network G. Then for every two agents i and j such that $e_{i,v} \in G$,*

$$t_{ij} \leq \phi(d_j) - (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$$

.

Proof. v_i has $d_i - 1$ many neighbors who would be willing to pay up to $\phi(d_i - 1) - \phi(d_i)$ for i to sever her relationship with j. i receives a benefit of $\phi(d_j) - t_i, j$ from her relationship with j, so if $\phi(d_j) - t_{ij} < (d_i - 1)(\phi(d_i - 1) - \phi(d_i))$ then i and all her remaining neighbors do strictly better if i drops her relationship with j and accepts a transfer of $\phi(d_i - 1) - \phi(d_i) - \epsilon$ from each of her remaining neighbors. □

Lemma 8. *Suppose $1 < M < |V(g)| - 1$ and let G be strongly pairwise stable. If there exists a vertex u with $d_u > M + 1$, then all of u's neighbors are adjacent.*

Proof. Suppose not, and let u be such that $d_u > M + 1$, $v, w \in N(u)$, but $e_{vw} \notin G$. We know from Lemma 6 that $d_v, d_w \leq M$. Since v and w are not adjacent, we know from Lemma 5 that neither v nor w has degree less than M. Therefore, $d_v = d_w = M$.

From Lemma 7

$$t_{uv} \leq \phi(d_v) - (d_u - 1)(\phi(d_u - 1) - \phi(d_u))$$

$$= \phi(d_v) - \phi(d_u) + \phi(d_u) - (d_u - 1)(\phi(d_u - 1) - \phi(d_u))$$

$$= \phi(M) - \phi(d_u) + to(d_u - 1)$$

Similarly, $t_{uw} \leq \phi(M) - \phi(d_u) + to(d_u - 1)$. Therefore

$$\begin{aligned}
\Delta\pi_v(G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T) &= \phi(d_w) - \phi(d_u) - t_{uv} \\
&= \phi(M) - \phi(d_u) - t_{uv} \\
&\geq \phi(M) - \phi(d_u) - (\phi(M) - \phi(d_u) + to(d_u - 1)) \\
&= -to(d_u - 1) \\
&> 0
\end{aligned}$$

where the last inequality follows from $d_u > M + 1$ and therefore, $to(d_u - 1) < 0$.

Similarly, $\Delta\pi_w(G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T) > 0$. Note that since the degree of v and w has not changed, all vertices in $N(v) \cup N(w) \setminus u$ are indifferent between $< G + e_{vw} \setminus \{e_{uv}, e_{uw}\}, T >$ and $< G, T >$. Therefore agents v and w block $< G, T >$ contradicting the stability of G.

□

Lemma 9. *Suppose $1 < M < |V(g)| - 1$ and let G be strongly pairwise stable. No vertex in G has degree greater than $M + 1$.*

Proof. This is a pigeonhole argument. Suppose for contradiction there is a vertex u with $d_u > M + 1$. By Lemma 6, every neighbor of u must have degree less than or equal to M. By Lemma 8, all neighbors of u must be adjacent. However, there are at least $M + 1$ neighbors of u. All are adjacent to the other neighbors of u (there are at least M other neighbors of u) plus u itself. Therefore, all neighbors of u must have degree at least $M+1$, a contradiction. □

Lemma 10. *Suppose $1 < M < |V(g)| - 1$ and let G be socially optimal. No vertex in G has degree less than M.*

Proof. Suppose for contradiction there exists a vertex u with $d_u < M$. Since $M < |V(G)| - 1$, there exists a v not adjacent to u. By Lemma 5, $d_v > M$. Therefore, by Lemma 9, $d_v = M+1$. Since v has $M+1$ neighbors and u has less than M neighbors, there must exist a w which is adjacent to v but not adjacent to u. Repeating the logic above, $d_w = M+1$. We will demonstrate that u, w, and all of their neighbors can be made better off if u adds an edge with w and w drops it's edge with v.

From Lemma 7

$$\begin{aligned} t_{wv} &\leq \phi(d_v) - (d_w - 1)(\phi(d_w - 1) - \phi(d_w)) \\ &= \phi(M+1) - (M)(\phi(M) - \phi(M+1)) \\ &= to(M) \\ &= 0 \end{aligned}$$

Similarly $t_{vw} \leq 0$, therefore $t_{wv} = t_{vw} = 0$. Let $G' = G \cup e_{uw} \setminus e_{vw}$. Then

$$\begin{aligned} \Delta\pi_w(G', T) &= \phi(d_u + 1) - \phi(d_v) - t_v w \\ &= \phi(d_u + 1) - \phi(d_v) \\ \Delta\pi_u(G', T) &= \phi(d_w) \\ \Delta\pi_x(G', T) &= \phi(d_u + 1) - \phi(d_u) \text{ for every } x \in N(u) \\ \Delta\pi_x(G', T) &= 0 \text{ for every } x \in N(w) \end{aligned}$$

Let

$$t'_{ux} = \begin{cases} \phi(d_u) - \phi(d_u+1) + \epsilon & x \in N(u) \\ \phi(d_v) - \phi(d_u+1) + \omega & x = w \end{cases}$$

$$t'_{wx} = \delta \text{ for every } x \in N(w) \setminus v.$$

Now

$$\begin{aligned}
\Delta\pi_w(G', T+T') &= \omega - d_w * \delta \\
\Delta\pi_u(G', T+T') &= \phi(d_u+1) - d_u(\phi(d_u) - \phi(d_u+1)) - d_u * \epsilon - \omega \\
&= to(d_u) - d_u * \epsilon - \omega \\
\Delta\pi_x(G', T+T') &= \epsilon \text{ for every } x \in N(u) \\
\Delta\pi_x(G', T+T') &= \delta \text{ for every } x \in N(w)
\end{aligned}$$

Since $d_u < M$, $to(d_u) > 0$, and therefore, u, w and all of their neighbors can be made better off in $G \cup e_{uw} \setminus e_{vw}$. This contradicts the strong pairwise stability of G.

□

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