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# Analytical Solutions for a Large-Scale Long-Lived Rotating Layer of Fluid Heated Underneath

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# ANALYTICAL SOLUTIONS FOR A LARGE-SCALE LONG-LIVED ROTATING LAYER OF FLUID HEATED UNDERNEATH

by

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In this work, the system of equations for a large- scale long lived rotating layer of fluid with the deformable upper free surface and non-deformable lower free surface heated underneath has been reviewed and derived. The quasi-geostrophic approximation, the beta effect and the method of multi-scale expansions have been employed to and as a result, an equation governing the evolution of large-scale perturbations, has been derived. The effect of each term present in the upper surface deformation equation has been analyzed and the analytical solutions have been obtained by virtue of employing auxiliary Riccati equation method. The soliton solutions obtained contributes to the sustenance of the vortex structure of the long-lived rotating layer of fluid due to the existence of two terms namely the nonlinear term or the so-called beta effect and the diffusion term resulted from the presence of heating energy from below.

The solution obtained, has been also applied to the case of long-lived vortex structure of the Great red spot of Jupiter and the results for the large-scale perturbations and averaged dominant terms of non-dimensional components of the velocity fields have been presented. The results show the correlation between the heating of the fluid motion from the lower layers, which is one of the fundamental features of the Great Red spot of Jupiter, and the sustenance of the vortex structure.

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# CHAPTER 1

# INTRODUCTION

Analytical solutions for a rotating layer of fluid heated underneath is of great interest. In the past six decades, there have been many studies devoted to the numerical solutions for a rotating layer of fluid heated underneath. However, very few analytical solutions for the above fundamental phenomenon can be found in the literature and very important features of the above phenomenon such as the deformation of the upper surface due to the onset of convection was not taken into account in the previous analytical solutions.

The first analytical solutions for a rotating layer of fluid heated from below were obtained by Chandrasekhar in 1953 [1]. He obtained the analytical solutions for three cases of boundary conditions: both boundaries free, one boundary free and the other one rigid, both boundaries rigid. In obtaining his solutions, nonlinear terms in the governing equations were neglected (linear theory of stability). Also, the deformation of the upper surface and the beta effect were neglected in his analytical solutions. He showed that the Coriolis force or the angular velocity $\Omega$ has an inhibitive effect on the onset of convection. In other words, an increase in the angular velocity results in the inhibition of the onset of convection [1].

Zhang and Roberts [2] developed the solution obtained by Chandrasekhar. They employed asymptotic analysis and obtained asymptotic solutions for a rotating layer of fluid heated from below. Also, the exact solutions constructed works well in the case that the Prandtl number is sufficiently small. However, nonlinear terms were neglected in the governing equations.

Petviashvili in 1980 [3] also obtained analytical solution for an inviscid layer of fluid subject to the Coriolis force. His solution is the first soliton solution accounts for the sustenance of the vortex structure of a rotating layer of fluid. In his governing equations the effect of convection and viscosity has been disregarded [3-5]

Busse [6,7] has also made significant contributions to the solution of a rotating layer of fluid heated from below, with application to planetary cases, both analytically and numerically. The model that was used by him is known as the deep model characterized by the Taylor-Proudman theorem. Linear analysis of the governing equation associated with weakly nonlinear analysis was carried out by him.

Analytical solutions give us better understanding and help us to explain the physics of the problem in a clear way. Therefore, we have been motivated to obtain analytical solutions for a large-scale long-lived rotating layer of fluid heated underneath with the deformable upper and non-deformable lower surfaces.

In deriving the system of governing equations, the Coriolis effect is considered with the assumption that the rotation period of flow is long compared to the rotation period of the overall system. Therefore, the quasi-geostrophic approximation has been employed together with the Boussinesq approximation and the beta plane approximation or the so-called beta effect. Heating of fluid from below, viscosity of fluid and the deformation of the upper surface have been taken into account in deriving the system of governing equations. However, the effect of the surface tension has been neglected at the deformable upper boundary condition. We have employed the method of multi-scale expansions in order to obtain the equation for the upper surface deformation. In order to solve the equation for the upper surface deformation analytically, the auxiliary equation method [8,9] along with the Riccati expansion method [10,11] have been used. The soliton solutions of the equation for the upper surface deformation, account for the sustenance of Rossby waves or vortices.

One of the applications of the mathematical model in our study is to model large-scale and long-lived vortex structures in planetary atmospheres such as the mysterious problem of long-lived vortex structures of the Great Red Spot of Jupiter which have been observed for more than 300 years. It is proposed that the Great Red Spot of Jupiter is a Rossby solitary vortex and retains its shape for a long time [12,13]. The other important feature of the Great Red Spot of Jupiter is the heating of the atmosphere from the lower layers [14,15], which has been taken into consideration in the mathematical model.

# CHAPTER 2

# MATHEMATICAL MODEL

In this chapter, we describe the mathematical model of a large-scale long-lived rotating layer of fluid heated underneath. Therefore, we start with the basic assumptions made in order to derive the governing equations and boundary conditions. Quasi-geostrophic approximation and multi-scale expansion methods will be applied to the governing equations and boundary conditions in order to derive simpler form of the governing equations and boundary conditions. As a result, we will derive an equation that governs the upper surface deformation.

## 2.1 Basic Assumptions

In our mathematical model, a single rotating layer of fluid, assumed to be horizontally unbounded, is considered. The lower surface of fluid is supposed to be a non-deformable stress-free isothermal plane. However, the upper surface is supposed to be a deformable stress-free isothermal plane. Hence, there are no tangential stresses at the lower and upper boundaries. The lower non-deformable surface implies that there is no vertical velocity at the lower boundary. In contrast, the upper deformable surface implies that the vertical fluid velocity at the upper boundary needs to be taken into consideration. In other words, the convective motion deforms the upper surface. Thus, the vertical displacement of the surface from its unperturbed position should be taken into account at the upper boundary [14,15], whereas the deformation of the upper surface of fluid was neglected in some publications [16-18].

We denote $L$, $H$ and $\Theta$ as characteristic horizontal length scale, the vertical length and the temperature difference between the lower and upper boundaries respectively as shown in Fig. 2.1. In our model, the horizontal dimensions are assumed to be much more than the vertical one. In other words, we assume that $L \gg H$.

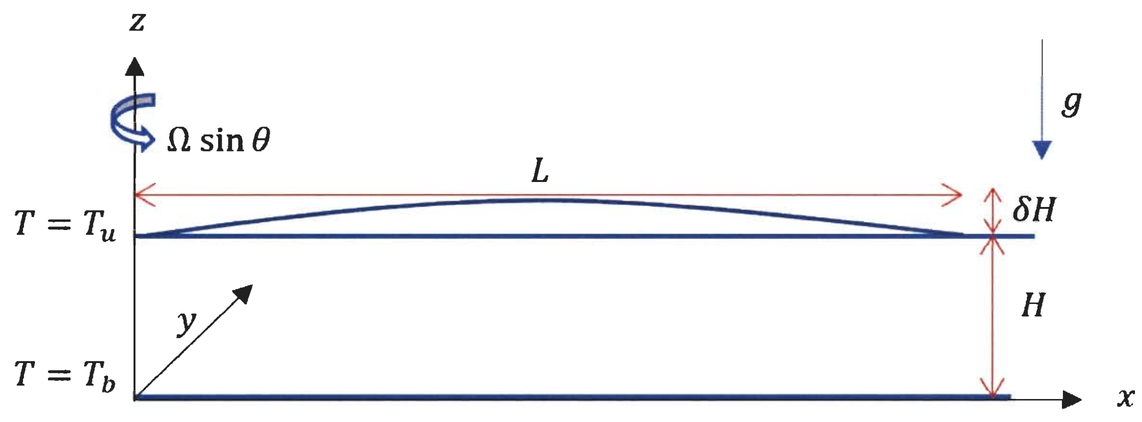



**Figure 2.1 The sketch of a single layer model heated underneath**

The lower non-deformable stress-free surface is at a constant temperature $T_b$. The deformable upper surface is assumed to be at a constant temperature $T_u$. Therefore, $\Theta = T_b - T_u$ by its definition. $\delta H$ is defined as the deviation of the upper surface from the unperturbed state. $\mathbf{\Omega}$ is denoted the angular rotation velocity vector of the overall system. As illustrated in Fig. 2.2, $\mathbf{\Omega}$ is of the form $\mathbf{\Omega} = \Omega \sin\theta\, \boldsymbol{k} + \Omega \cos\theta\, \boldsymbol{j}$ where $\theta$ is latitude, $\Omega \sin\theta$ and $\Omega \cos\theta$ are the vertical and horizontal components of the angular rotation velocity vector $\mathbf{\Omega}$ respectively as shown in Fig. 2.2. Due to the assumption made, which is $L \gg H$, we neglect the horizontal component of the angular rotation velocity vector $\boldsymbol{\Omega}$ [19]. Therefore, the angular rotation velocity vector might be approximated to the form of $\boldsymbol{\Omega} = \Omega \sin\theta\, \boldsymbol{k}$. The thin shell of fluid shown in Fig. 2.2 has the local vertical unit vector $\boldsymbol{k}$ and the velocity component in that direction is $w$. The northward velocity is $v$ and the eastward velocity (into the paper in Fig. 2.2) is $u$. Unit vectors eastward and northward are defined $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively. In our mathematical model, we consider the Cartesian model of a rotating spherical thin fluid layer. It is noted that the lower and upper surfaces of fluid in our model corresponds to the inner and outer surfaces of a spherical thin fluid layer as shown in Fig. 2.2.

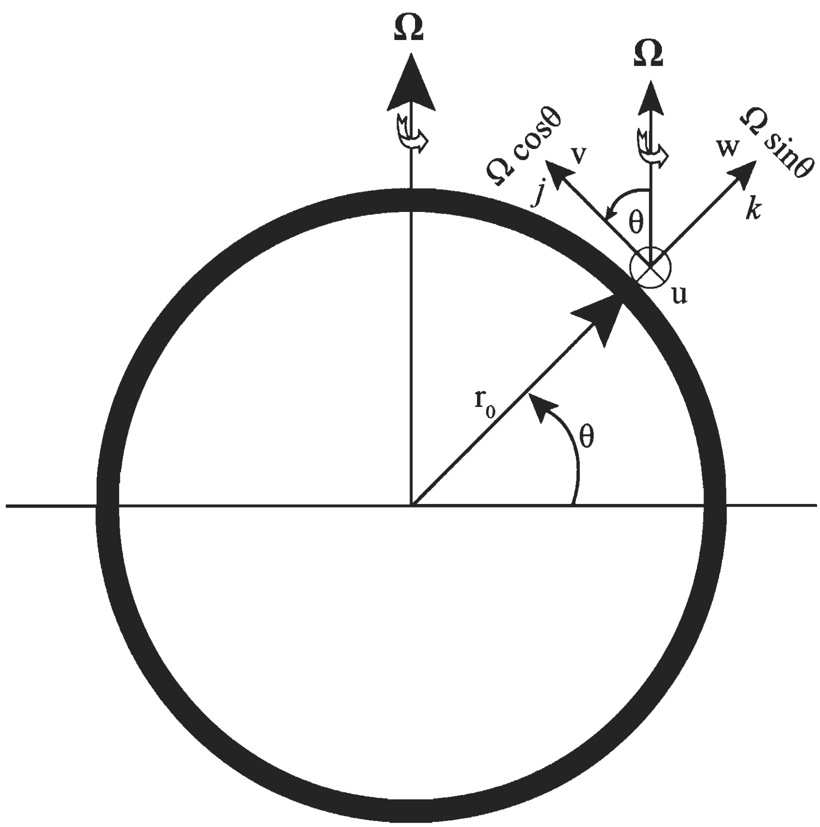



**Figure 2.2 The thin shell of fluid and the local coordinate frame at latitude $\theta$**

By virtue of employing the Taylor series expansion at an arbitrary latitude far from the equator $\theta_0$ [20] and using the so-called β plane approximation in order to model the flow on a sphere, we have:

$$\Omega \sin\theta = \Omega \sin\theta_0 + \Omega\cos\theta_0(\theta - \theta_0) - \frac{\Omega \sin\theta_0}{2}(\theta - \theta_0)^2 + \cdots \tag{2.1}$$

From the geometry shown in Fig. 2.2, $y = r_0(\theta - \theta_0)$ can be inferred and by neglecting the last term in the above equation, Eq. (2.1) might be approximated to

$$\Omega \sin\theta = \Omega \sin\theta_0 + \frac{\beta_0}{2}y, \tag{2.2}$$

where $\beta_0 = 2\frac{\Omega\cos\theta_0}{r_0}$ and $\frac{\beta_0}{2}y \ll \Omega \sin\theta_0$.

Eq. (2.2) might be written in form of

$$\boldsymbol{\Omega} = \Omega \sin\theta\, \boldsymbol{k} = \Omega \sin\theta_0\,(1 + \beta y)\boldsymbol{k}, \tag{2.3}$$

where $\beta = \frac{\beta_0}{2\Omega \sin\theta_0}$ and $\beta y \ll 1$.

β is a constant parameter characterizing the beta effect or known as the latitudinal variation of the local vertical component of the angular rotation velocity vector **Ω.**

It is noted that either Eq. (2.2) or Eq. (2.3) is referred as the β plane approximation [12,14,15,19,21] and is widely used in Geophysical Fluid Dynamics.

The other important assumption we made in our mathematical model is the time scale of the rotation of flow is long compared to the rotation period of the frame from which the flow is observed. This assumption implies that we should use a rapid-rotation approximation in our model. That is the reason we employ the quasi-geostrophic approximation and as a result, the term $2\boldsymbol{\Omega} \times \boldsymbol{V}$, which is denoted the Coriolis acceleration term, becomes a dominant term in the Navier-Stokes equations and is balanced by the pressure gradient force. The above assumption is widely used in study of planetary large-scale Rossby vortices in giant planets. For instance, the characteristic rotation period of JGRS vortex is about one week, whereas Jupiter rotation period is about 10 hours [12]. Therefore, in our model we consider a case in which the Taylor Number (Ta) is much larger than unity by virtue of assuming a rapid-rotation case. We also use the Boussinesq approximation meaning that the density of fluid variations are negligible except in the buoyancy term since the thickness of fluid layer in our model is small compared to its horizontal lengths. Hence, we neglect the density variations across the fluid layers.

The other important features of our model is the heating of fluid motion from below. This fundamental feature leads to the existence of diffusion terms and as a result, our system of equations is different from those ordinarily used in geophysical hydrodynamics literature [19]. In the next section, we introduce the governing equations of the fluid motion heated underneath given our assumptions.

## 2.2 Derivation of the Governing Equations

In this section, we derive the system of governing equations and boundary conditions both in dimensional and non-dimensional forms given the assumptions made in the previous section.

### 2.2.1 Dimensional Form

By applying the Boussinesq approximation, we formulate the system of equations [22] governing the fluid motion heated underneath in the vector forms and then transform it into scalar forms

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla)\boldsymbol{V} + 2\boldsymbol{\Omega} \times \boldsymbol{V} = -\frac{\nabla p}{\rho_0} + \nu \Delta \boldsymbol{V} + \frac{\rho}{\rho_0}\boldsymbol{g}\,, \tag{2.4}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{V}.\nabla T = \kappa \Delta T\,, \tag{2.5}$$

$$\rho = \rho_0[1 - \alpha(T - T_0)]\,, \tag{2.6}$$

$$\nabla.\boldsymbol{V} = 0\,, \tag{2.7}$$

where T and P are defined as the fluid temperature and pressure respectively. $T_0$ and $\rho_0$ are a reference temperature and density respectively. $\nu$ is the fluid kinematic viscosity. $\alpha$ is defined as the thermal expansion coefficient of the fluid. $\boldsymbol{g}$ is the gravity acceleration vector and $\kappa$ is denoted the fluid thermal diffusivity.

$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is referred to laplacian. The velocity vector is defined as $\boldsymbol{V} = (u, v, w)$ where $u$, $v$ and $w$ are the horizontal components and the vertical component of the fluid motion respectively.

Given the linear change in the fluid density with respect to the fluid temperature shown in Eq. (2.6), we substitute it into Eq. (2.4). Therefore, Eq. (2.4) is of the form

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla)\boldsymbol{V} + 2\boldsymbol{\Omega} \times \boldsymbol{V} = -\frac{\nabla p}{\rho_0} + \nu \Delta \boldsymbol{V} + \boldsymbol{g} + g\alpha(T - T_0)\boldsymbol{k}\,, \qquad (2.8)$$

where g is the magnitude of the gravity acceleration vector $\boldsymbol{g}$ and $\boldsymbol{k}$ is the local vertical unit vector shown in Fig 2.2.

Given the assumptions made in the section 2.1, let us now formulate the boundary conditions as follows:

$$at\ z = 0: \quad T - T_0 = T_b\,, \quad p = p_b\,, \quad w = 0 \ \ and \quad w_{zz} = 0 \qquad (2.9)$$

$$at\ z = H + \delta H: \quad T - T_0 = T_u\,, \quad p = p_u\,, \quad w = \frac{d}{dt}(\delta H) \ \ and \ \ w_{zz} = 0, (2.10)$$

where $T_0$ is a reference temperature, $T_b$ and $T_u$ are constant temperatures at the lower and upper boundaries respectively, $p_b$ is an arbitrary constant pressure at the lower boundary, $p_u$ is a constant pressure exerted on the upper free surface. By virtue of neglecting surface tension at the upper boundary and assuming that above our single layer fluid there is a gas with so small density, we may set $p_u = 0$ at the upper boundary condition. $\delta H$ is the deviation of the upper surface from the unperturbed state. $w_{zz}$ is set to zero at the boundaries due to the assumption of stress-free boundary conditions. Therefore, we neglect tangential stresses at the lower and upper surfaces. In other words, it is assumed $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ at both boundaries. By differentiating the continuity equation, namely, Eq.(2.7) with respect to the vertical coordinate $z$ and using the aforementioned assumption, we infer $w_{zz} = 0$ at the lower and upper boundaries.

We now define $p = p_s + p'$, $T - T_0 = T_s + T'$ and $\boldsymbol{V} = \boldsymbol{V}_s + \boldsymbol{V}'$, where $p_s$ and $T_s$ are static pressure and temperature respectively. $\boldsymbol{V}_s$ is the fluid velocity vector when the fluid is at rest and as a result, is equal to zero vector. $\boldsymbol{V}'$ is denoted the convective velocity vector of the fluid or the perturbed velocity vector and represents the perturbed fluid motion. $p'$ and $T'$ are deviations from linear hydrostatic pressure and temperature respectively. $p'$ and $T'$are also known as the perturbed pressure and temperature respectively. In other words, $p'$ and $T'$ represent the perturbations [22]. It is obvious that the stagnant fluid and its corresponding field (0, $p_s$, $T_s$) must satisfy Eqs. (2.4) – (2.7) as follows:

$$\frac{\nabla p_s}{\rho_0} = \boldsymbol{g} + g\alpha T_s \boldsymbol{k}\,, \tag{2.11}$$

$$\Delta T_s = 0\,. \tag{2.12}$$

The solutions to Eqs. (2.11) and (2.12) are as follows:

$$T_s = -\frac{\Theta}{H} z + T_b + T_0\,, \tag{2.13}$$

$$p_s = p_b - \rho_0 g \left[\alpha \frac{\Theta}{2H} z^2 - \alpha(T_b + T_0) z + z\right]. \tag{2.14}$$

where $\Theta = T_b - T_u$ .

For details of obtaining the above solutions, see Appendix A. We could also write the equation (2.13) in the following form:

$$\nabla T_s = -l_1 \boldsymbol{k}\,, \tag{2.15}$$

where $l_1 = \frac{\Theta}{H}$.

$(\boldsymbol{V}, p, T)$ must also satisfy Eqs. (2.4) and (2.5) as follows:

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla)\boldsymbol{V} + 2\boldsymbol{\Omega} \times \boldsymbol{V} = -\frac{\nabla(p_s + p')}{\rho_0} + \nu\Delta\boldsymbol{V} + \boldsymbol{g} + g\alpha(T_s + T')\boldsymbol{k}\,, \tag{2.16}$$

$$\frac{\partial(T_s + T')}{\partial t} + \boldsymbol{V}.\nabla(T_s + T') = \kappa\Delta(T_s + T')\,. \tag{2.17}$$

Substituting Eqs. (2.13) and (2.14) into the above equations we obtain the following system of governing equations:

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla)\boldsymbol{V} + 2\boldsymbol{\Omega} \times \boldsymbol{V} = -\frac{\nabla p'}{\rho_0} + \nu\Delta\boldsymbol{V} + g\alpha T'\boldsymbol{k}\,, \tag{2.18}$$

$$\frac{\partial T'}{\partial t} + \boldsymbol{V}.\nabla T' - l_1 w = \kappa\Delta T', \tag{2.19}$$

$$\nabla.\boldsymbol{V} = 0\,. \tag{2.20}$$

Note that $w$ is the vertical component of the fluid velocity vector $\boldsymbol{V}$ and $l_1$ is a constant defined by Eq. (2.15).

Let us now formulate the boundary conditions for the above system of equations as follows:

$$at\ z = 0: \quad T' = 0, \quad p'_z = 0\,, \quad w = 0 \quad and \quad w_{zz} = 0 \tag{2.21}$$

Following Tikhomolov [14,15] and Gershuni [22] for the upper boundary conditions, the assumption that the upper boundary condition remains at constant temperature $T_u$ and the expression obtained for $T_s$ , we may form the following relation for $T'$:

$$at\ z = H + \delta H : \quad T' + \frac{dT_s}{dz}\delta H = 0 \implies T' = \frac{\Theta}{H}\,\delta H. \tag{2.22}$$

Given the assumption that $p_u = 0$ , we may also form the following relationship for $p'$at the upper boundary:

$$at\ z = H + \delta H : \quad p' = \rho_0 g \delta H. \tag{2.23}$$

As shown in Eq. (2.10), vertical component of the fluid velocity, $w$ at the upper boundary is of the form

$$at\ z = H + \delta H : \quad w = \frac{d}{dt}(\delta H) \quad and \quad w_{zz} = 0 \tag{2.24}$$

### 2.2.2 Non-dimensional Form

By introducing the following non-dimensional variables, the governing equations, namely, Eqs. (2.18) – (2.20) and the boundary conditions, namely, Eqs. (2.21) – (2.24) can be non-dimensionalized.

$$\boldsymbol{V} = \boldsymbol{V}^* \frac{\kappa}{H}, \quad w = w^* \frac{\kappa}{H}, \quad t = t^* \frac{H^2}{\kappa}, \quad h = \frac{\delta H}{H}, \quad p' = p^* \frac{\rho_0 \nu \kappa}{H^2}, \quad T' = T^*\,\Theta$$

$$x = x^* H, \quad y = y^* H, \quad \beta = \frac{\beta^*}{H}, \quad \beta y = \beta^* y^*, \quad \nabla = \frac{\nabla^*}{H}, \quad \Delta = \frac{\Delta^*}{H^2}. \tag{2.25}$$

Substituting Eqs. (2.3) and the above non-dimensional variables into Eq. (2.18) – (2.20), we obtain the non-dimensional form of the governing equations as follows:

$$\frac{\kappa}{\nu}\frac{\partial \boldsymbol{V}^*}{\partial t^*}+\frac{\kappa}{\nu}(\boldsymbol{V}^*.\nabla^*)\boldsymbol{V}^*+2\frac{\Omega\sin\theta_0 H^2}{\nu}(1+\beta^* y^*)\boldsymbol{k}\times\boldsymbol{V}^*$$

$$=-\nabla^* p^*+\Delta^*\boldsymbol{V}^*+\frac{g\alpha\Theta H^3}{\nu\kappa}T^*\boldsymbol{k}, \tag{2.26}$$

$$\frac{\partial T^*}{\partial t^*}+\boldsymbol{V}^*.\nabla^* T^*-w^*=\Delta^* T^*, \tag{2.27}$$

$$\nabla^*.\boldsymbol{V}^*=0. \tag{2.28}$$

Using non-dimensional parameters, we could make Eqs. (2.26) – (2.28) simpler. Given $P=\frac{\nu}{\kappa}$, $Ta^{\frac{1}{2}}=D=2\frac{\Omega\sin\theta_0 H^2}{\nu}$ and $R=\frac{g\alpha\Theta H^3}{\nu\kappa}$, where P is the Prandtl number, $Ta$ is the Taylor number, $R$ is the Rayleigh number and $D$ is defined as the square root of the Taylor number. Substituting these three independent non-dimensional parameters into Eq. (2.26). Therefore, the non-dimensional system of governing equations is of the form

$$\frac{1}{P}\frac{\partial \boldsymbol{V}^*}{\partial t^*}+\frac{1}{P}(\boldsymbol{V}^*.\nabla^*)\boldsymbol{V}^*+D(1+\beta^* y^*)\boldsymbol{k}\times\boldsymbol{V}^*=-\nabla^* p^*+\Delta^*\boldsymbol{V}^*+RT^*\boldsymbol{k}, \tag{2.29}$$

$$\frac{\partial T^*}{\partial t^*}+\boldsymbol{V}^*.\nabla^* T^*-w^*=\Delta^* T^*, \tag{2.30}$$

$$\nabla^*.\boldsymbol{V}^*=0. \tag{2.31}$$

For the sake of clarity, we henceforth omit the star on the non-dimensional variables of Eqs. (2.29) – (2.31). Therefore, we have

$$\frac{1}{PD}\frac{\partial \boldsymbol{V}}{\partial t}+\frac{1}{PD}(\boldsymbol{V}.\nabla)\boldsymbol{V}+(1+\beta y)\boldsymbol{k}\times\boldsymbol{V}=-\frac{1}{D}\nabla p+\frac{1}{D}\Delta\boldsymbol{V}+\frac{RT}{D}\boldsymbol{k}, \tag{2.32}$$

$$\frac{\partial T}{\partial t}+\boldsymbol{V}.\nabla T-w=\Delta T, \tag{2.33}$$

$$\nabla . \mathbf{V} = 0 \,. \tag{2.34}$$

By substituting the non-dimensional variables from equation (2.25) into the boundary conditions and omitting the star on non-dimensional variables, we obtain the non-dimensional form of the boundary conditions, namely, Eqs. (2.21) – (2.24) as follows:

$$at\ z = 0: \quad T = 0, \quad p_z = 0\,, \quad w = 0 \quad and \quad w_{zz} = 0\,, \tag{2.35}$$

$$at\ z = 1 + h: \quad T = h\,, \quad p = qh\,, \quad w = \frac{dh}{dt} \quad and \quad w_{zz} = 0\,, \tag{2.36}$$

where $h = \frac{\delta H}{H}$ and $q = \frac{gH^3}{\nu\kappa}$.

It is worth noting that $h$ is assumed to be function of $x$, $y$ and $t$ and is assumed $h \ll 1$. Let us now transform the above system of governing equations into the scalar form. Hence, we have

$$\frac{1}{PD}\frac{\partial u}{\partial t} + \frac{1}{PD}\left(uu_x + vu_y + wu_z\right) - (1 + \beta y)v = -\frac{p_x}{D} + \frac{1}{D}\Delta u\,, \tag{2.37}$$

$$\frac{1}{PD}\frac{\partial v}{\partial t} + \frac{1}{PD}\left(uv_x + vv_y + wv_z\right) + (1 + \beta y)u = -\frac{p_y}{D} + \frac{1}{D}\Delta v\,, \tag{2.38}$$

$$\frac{1}{PD}\frac{\partial w}{\partial t} + \frac{1}{PD}\left(uw_x + vw_y + ww_z\right) = -\frac{p_z}{D} + \frac{1}{D}\Delta w + \frac{RT}{D}\,, \tag{2.39}$$

$$\frac{\partial T}{\partial t} + uT_x + vT_y + wT_z - w = \Delta T\,, \tag{2.40}$$

$$u_x + v_y + w_z = 0\,. \tag{2.41}$$

In the next step to solve the above system of governing equations, we employ the quasi-geostrophic approximation.

## 2.3 Quasi-Geostrophic Approximation

In the quasi-geostrophic theory, three important assumptions are used, one of which is the time scale of the motion of flow is long compared to the rotation period of the frame from which the flow is observed. The second one is the frictional diffusion time scale of the flow is long compared to the rotation period. The third one is the vertical velocity of the fluid might be neglected [12,14,15,19].

As mentioned in the section (2.1), it is assumed that $Ta \gg 1$ in our model, given $Ta^{\frac{1}{2}} = D$ and $D = 2\frac{\Omega \sin\theta_0 \, H^2}{\nu} = E^{-1}$, where E is a dimensionless number defined as the Ekman number. Therefore, we conclude that $D \gg 1$ and $E \ll 1$.

It is noted that when the Ekman number is much less than unity, we could use the quasi-geostrophic approximation in Eq. (2.29) or Eq. (2.32). Employing the quasi-geostrophic approximation signifies that the Coriolis term $(2\boldsymbol{\Omega} \times \boldsymbol{V})$ is approximately balanced by the pressure gradient [12,14,15,19]. In other words, by applying the quasi-geostrophic approximation, we might neglect the unsteady, convective inertia and diffusion terms in Eq. (2.32). Therefore, Eq. (2.32) is approximated to the form of

$$(1+\beta y)\boldsymbol{k} \times \boldsymbol{V} = -\frac{1}{D}\nabla p + \frac{1}{D}RT\boldsymbol{k}, \tag{2.42}$$

or:

$$-(1+\beta y)v\boldsymbol{i}+(1+\beta y)u\boldsymbol{j}=-\frac{1}{D}\nabla p+\frac{1}{D}RT\boldsymbol{k}\,. \tag{2.43}$$

By virtue of the assumption $\beta y \ll 1$, we have the following results:

$$u^{(0)}=-\frac{1}{D}p_y\,, \tag{2.44}$$

$$v^{(0)}=\frac{1}{D}p_x\,, \tag{2.45}$$

where $p_x=\frac{\partial p}{\partial x}$, $p_y=\frac{\partial p}{\partial y}$, $u^{(0)}$ and $v^{(0)}$ are the non-dimensional geostrophic horizontal velocity components. It is noted that $u^{(0)}$ and $v^{(0)}$ are the dominant terms of $u$ and $v$ respectively.

Similarly, $w^{(0)}$ is defined as the non-dimensional geostrophic vertical velocity component and is assumed to be zero according to the quasi-geostrophic approximation explained above. Thus, it can be written

$$w^{(0)}=0\,. \tag{2.46}$$

The above approximations, namely, Eqs. (2.44) – (2.46) is known as the quasi-geostrophic approximation [12,14,15,19]. Let us now simplify the system of governing equations derived in the previous section, namely, Eqs. (2.37) – (2-41) by employing the above quasi-geostrophic approximation. Therefore, the system of governing equations could be approximated to the form of

$$w_z-\frac{1}{PD^2}\Delta_2 p_t-\frac{1}{PD^3}J(p,\Delta_2 p)-\beta\frac{p_x}{D}+\frac{1}{D^2}{\Delta_2}^2 p+\frac{1}{D^2}\Delta_2 p_{zz}=0\,, \tag{2.47}$$

$$p_z=RT, \tag{2.48}$$

$$T_t + \frac{1}{D} J(p,T) - w = \Delta_2 T + T_{zz} \,, \tag{2.49}$$

where $J(f,g) = f_x g_y - f_y g_x$ , $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_2{}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ .

For details of the derivation for Eqs. (2.47) – (2.49), see Appendix B. The system of Eqs. (2.47) – (2.49) is akin to the one obtained by Tikhomolov in his paper [14]. Boundary conditions for the above system of equations are of the form

$$at\ z = 0: \quad T = 0, \quad p_z = 0\,, \quad w = 0 \quad and \quad w_{zz} = 0\,, \tag{2.50}$$

$$at\ z = 1 + h: \ T = h\,, \ p = qh\,, \ w = \frac{dh}{dt} = h_t + uh_x + vh_y \ and \ \ w_{zz} = 0. \tag{2.51}$$

Approximating $u$ and $v$ by $u^{(0)}$ and $v^{(0)}$ respectively at the upper boundary, yields

$$at\ z = 1 + h: \quad w = h_t + u^0 h_x + v^0 h_y = h_t + \frac{q}{D}\left(-h_y h_x + h_x h_y\right) = h_t\,. \tag{2.52}$$

Therefore, the upper boundary condition (2.51) is of the form

$$at\ z = 1 + h: \quad T = h\,, \quad p = qh\,, \quad w = h_t \quad and \quad w_{zz} = 0. \tag{2.53}$$

In order to solve the system of Eqs. (2.47) – (2.49) together with the boundary conditions, namely, Eqs. (2.40) and (2.53), we employ the method of multi-scale expansions which will be explained in details in the following section.

## 2.4 Method of Multi-Scale Expansions

Following Tikhomolov [15] and Newell & Whitehead [23], we seek the asymptotic expansions for the dynamic fields such as $p, w$ and $T$ in Eqs. (2.47) – (2.49) and the boundary conditions, namely, Eqs. (2.50) and (2.53). Therefore, we define the slow coordinates and time such as $X, Y$ and $\tau$ and expand the above dynamic fields as a power series in $\varepsilon$ where $\varepsilon$ is a small parameter and $\varepsilon^2 \approx R - R_{cr}$ where $R_{cr}$ is the critical Rayleigh number and will be defined later [15], [23]. In other words, the above dynamic fields will be represented as a function of both the slow variables and regular variables. Hence, we begin with the definition of the slow variables as follows:

$$X = \varepsilon x\,, \qquad Y = \varepsilon y\,, \qquad \tau = \varepsilon^2 t\,, \tag{2.54}$$

where $\varepsilon$ is a small parameter and $\varepsilon \ll 1$. Note that the vertical coordinate $z$ remains as before. By virtue of the transformation (2.54), the operators in Eqs (2.47) – (2.49) might be changed to the form of [15], [23]

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\,, \qquad \frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\,, \qquad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau},$$

$$\Delta_2 \to \Delta_2 + 2\varepsilon \left( \frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y} \right) + \varepsilon^2 \Delta_{2X}\,,$$

$$\begin{aligned} \Delta_2{}^2 \to \Delta_2{}^2 &+ 4\varepsilon \left( \frac{\partial^4}{\partial x^3 \partial X} + \frac{\partial^4}{\partial y^3 \partial Y} \right) + 6\varepsilon^2 \left( \frac{\partial^4}{\partial x^2 \partial X^2} + \frac{\partial^4}{\partial y^2 \partial Y^2} \right) + 4\varepsilon^3 \left( \frac{\partial^4}{\partial x \partial X^3} + \frac{\partial^4}{\partial y \partial Y^3} \right) \\ &+ 4\varepsilon \left( \frac{\partial^4}{\partial x \partial y^2 \partial X} + \frac{\partial^4}{\partial x^2 \partial y \partial Y} \right) + 2\varepsilon^2 \left( \frac{\partial^4}{\partial y^2 \partial X^2} + 4 \frac{\partial^4}{\partial x \partial y \partial X \partial Y} + \frac{\partial^4}{\partial x^2 \partial Y^2} \right) \\ &+ 4\varepsilon^3 \left( \frac{\partial^4}{\partial y \partial X^2 \partial Y} + \frac{\partial^4}{\partial x \partial X \partial Y^2} \right) + \varepsilon^4 \Delta_{2X}{}^2, \end{aligned} \tag{2.55}$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$, $\Delta_2{}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ and $\Delta_{2X}{}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$. For details of the above derivation see Appendix C.

Before we expand the dynamic fields as a function of the slow variables and regular variables, we would like to make some assumptions in order to simplify the rest of our derivation. Starting with the deformation of the upper surface $h(x, y, t)$, we follow Tikhomolov [15] and assume that the deformation of the upper surface is very small and is in the order of $\varepsilon^2$. We also assume that the leading order of the deformation is only dependent of the slow variables $(X, Y, \tau)$. Therefore, $h(x, y, t)$ might be written in form of

$$h(x, y, t) = \varepsilon^2\left[\mathcal{H}^{(0)}(X, Y, \tau) + \varepsilon\mathcal{H}^{(1)} + \varepsilon^2\mathcal{H}^{(2)} + \cdots\right], \tag{2.56}$$

where the deformation $\mathcal{H}^{(0)}$ is independent of $x$, $y$ and $t$ while all other terms on the right hand side of Eq. (2.56) might be function of $x$, $y$ and $t$ as well as $X$, $Y$ and $\tau$.

Following the upper boundary condition (2.53) in which $T = h$, we assume the following form for $T$:

$$T = \varepsilon^2\left[\vartheta^{(0)}(X, Y, z, \tau) + \varepsilon\vartheta^{(1)} + \varepsilon^2\vartheta^{(2)} + \cdots\right], \tag{2.57}$$

where $\vartheta^{(0)}$ is independent of $x$, $y$ and $t$ while all other terms on the right hand side of Eq. (2.57) are function of $x$, $y$ and $t$ as well as $X$, $Y$, $z$ and $\tau$.

Given the upper boundary condition (2.53) in which $w = h_t$ and the transformation $\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \varepsilon^2\frac{\partial}{\partial \tau}$, the appropriate form for $w$ is assumed to be

$$w = \varepsilon^4\left[W^{(0)}(X, Y, z, \tau) + \varepsilon W^{(1)} + \varepsilon^2 W^{(2)} + \cdots\right], \tag{2.58}$$

where $W^{(0)}$ is independent of $x$, $y$ and $t$ while all other terms on the right hand side of Eq. (2.58) might be function of $x$, $y$ and $t$ as well as $X$, $Y$, $z$ and $\tau$.

It is worth noting that the above expression for $w$ is in agreement with the assumption we made in the quasi-geostrophic approximation in the previous section which was $w^{(0)} = 0$. The appropriate expression for $p$ is assumed to be of the form

$$p = p^{(0)}(X, Y, \tau) + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \varepsilon^3 p^{(3)} + \cdots, \tag{2.59}$$

where $p^{(0)}$ is independent of $x$, $y$ , $z$ and $t$ while all other terms on the right hand side of Eq. (2.59) might be function of $x$, $y$, $z$ and $t$ as well as $X$, $Y$ and $\tau$.

The appropriate expansion expression for $\beta$ is assumed to be of the form [15]:

$$\beta = \varepsilon^3(B^{(0)} + \varepsilon B^{(1)} + \cdots), \tag{2.60}$$

The appropriate expansion expression for $R$ is assumed to be of the form [15, 23]:

$$R = R_{cr} + O(\varepsilon^2), \quad or \quad R = R_{cr} + \varepsilon^2 R_2, \tag{2.61}$$

where $R_{cr}$ is the critical Rayleigh number or the minimum value of the Rayleigh number for the onset of convection and defined as follows:

$$as\ D \to \infty \qquad R_{cr} \approx 3\left(\frac{\pi^2}{2}\right)^{\frac{2}{3}} D^{\frac{4}{3}} = 8.6956 D^{\frac{4}{3}}, \tag{2.62}$$

where $D = 2\frac{\Omega \sin\theta_0\, H^2}{\nu}$.

It is worth noting that the above asymptotic expression for $R_{cr}$ was obtained by Chandrasekhar [1]. He determined the above expression by the linear stability analysis of the system of governing equations for a rotating layer of fluid heated from below with free boundary conditions. It is noted that the above expression for $R_{cr}$ should be used in the cases in which the value of $D$ is sufficiently large. Therefore, it is in agreement with our assumption, which is $D \gg 1$. Although, the deformation of the upper surface and the beta effect were neglected in his derivation, we can still use it for the expansion of the Rayleigh number in Eq. (2.61) since $R_{cr}$ by its definition is the minimum value of the Rayleigh number for the onset of convection and as a result, the deformation due to convection can be disregarded. For details of the derivation for the above asymptotic expression, see his paper [1].

It is evident that from Eq. (2.62) and based on the definition of the Rayleigh number, a corresponding $q$, where $q = \frac{gH^3}{\nu\kappa}$, for $R_{cr}$ can be defined in form of

$$as\ D \to \infty \qquad q_{cr} = \frac{R_{cr}}{\alpha\Theta} \approx \frac{8.6956 D^{\frac{4}{3}}}{\alpha\Theta}, \tag{2.63}$$

where $\Theta = T_b - T_u$ and $\alpha$ is defined as the thermal expansion coefficient of the fluid.

Employing the transformations (2.54) and (2.55), we substitute the Eqs. (2.56) – (2.61) into the system of Eqs. (2.47) – (2.49) and the boundary conditions (2.50) and (2.53). Therefore, we obtain

$$W_z{}^{(0)} - \frac{1}{PD^2}\Delta_{2X}p_\tau{}^{(0)} - \frac{1}{PD^3}J_X\left(p^{(0)},\Delta_{2X}p^{(0)}\right) - B^{(0)}\frac{p_X{}^{(0)}}{D} + \frac{1}{D^2}\Delta_{2X}{}^2p^{(0)}$$

$$+\frac{1}{D^2}\Delta_{2X}p_{zz}{}^{(2)} = 0\,, \qquad \mathit{at\ the\ fourth\ order\ or\ } \varepsilon^4 \qquad (2.64)$$

$$p_z{}^{(2)} = R_{cr}\vartheta^{(0)}, \qquad \mathit{at\ the\ second\ order\ or\ } \varepsilon^2 \qquad (2.65)$$

$$\vartheta_{zz}{}^{(0)} = 0\,, \qquad \mathit{at\ the\ second\ order\ or\ } \varepsilon^2 \qquad (2.66)$$

where $J_X(f,g) = f_X g_Y - f_Y g_X$ , $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ and $\Delta_{2X}{}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$ .

The boundary conditions of the above system of equations are as follows:

$$\mathit{at}\ z = 0: \quad \vartheta^{(0)} = 0, \quad p_z{}^{(0)} = 0\,, \quad W^{(0)} = 0 \quad \mathit{and} \quad W_{zz}{}^{(0)} = 0\,, \qquad (2.67)$$

$$\mathit{at}\ z = 1 + \varepsilon^2\mathcal{H}^{(0)} : \vartheta^{(0)} = \mathcal{H}^{(0)}, p^{(0)} = \eta\mathcal{H}^{(0)}, W^{(0)} = \mathcal{H}_\tau{}^{(0)}\ \mathit{and}\ W_{zz}{}^{(0)} = 0, (2.68)$$

where $\eta$ is a constant, approximated to $\eta \approx \varepsilon^2 q$ , and $q = \frac{gH^3}{\nu\kappa}$ .

It is worth noting that Eq. (2.64) has been derived at the fourth order of $\varepsilon$. Also Eqs. (2.65) and (2.66) have been derived at the second order of $\varepsilon$.

For details of the derivation for Eqs. (2.64) – (2.66), see Appendix D. Eq. (2.66) together with $\vartheta^{(0)} = 0$ and $\vartheta^{(0)} = \mathcal{H}^{(0)}$ at the lower and upper boundaries respectively result in

$$\vartheta^{(0)} = z\mathcal{H}^{(0)}\,. \qquad (2.69)$$

Substituting the above equation into Eq. (2.65) and differentiating both sides of the equation with respect to $z$ yields

$$p_{zz}{}^{(2)} = R_{cr}\mathcal{H}^{(0)}. \tag{2.70}$$

We now substitute $p_{zz}{}^{(2)}$ from the above equation into Eq. (2.64). Therefore, it yields

$$W_z{}^{(0)} - \frac{1}{PD^2}\Delta_{2X}p_\tau{}^{(0)} - \frac{1}{PD^3}J_X\left(p^{(0)}, \Delta_{2X}p^{(0)}\right) - B^{(0)}\frac{p_X{}^{(0)}}{D} + \frac{1}{D^2}\Delta_{2X}{}^2p^{(0)} + \frac{R_{cr}\Delta_{2X}\mathcal{H}^{(0)}}{D^2} = 0. \tag{2.71}$$

Note that $p^{(0)}$ is independent of the coordinate z. By virtue of Integrating the above equation along the coordinate z, from $z = 0$ to $z = 1 + \varepsilon^2\mathcal{H}^{(0)}$, and taking into account the boundary conditions (2.67) and (2.68), we derive

$$\mathcal{H}_\tau{}^{(0)} - \frac{\eta\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)}{PD^2}\Delta_{2X}\mathcal{H}_\tau{}^{(0)} - \frac{\eta^2\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)}{PD^3}J_X\left(\mathcal{H}^{(0)}, \Delta_{2X}\mathcal{H}^{(0)}\right) - B^{(0)}\eta\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)\frac{\mathcal{H}_X{}^{(0)}}{D} + \frac{\eta\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)}{D^2}\Delta_{2X}{}^2\mathcal{H}^{(0)} + \frac{R_{cr}\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)\Delta_{2X}\mathcal{H}^{(0)}}{D^2} = 0\,, \tag{2.72}$$

where $\eta$ is a constant, approximated to $\eta \approx \varepsilon^2 q$ , and $q = \frac{gH^3}{\nu\kappa}$.

Eq. (2.72) is the equation which governs the deformation of the upper surface. In order to simplify it, we should pay our attention to the assumption we made which is $\varepsilon \ll 1$. This assumption leads us to neglect the term $\varepsilon^2\mathcal{H}^{(0)}$ in the above equation since

$\varepsilon^2\mathcal{H}^{(0)} \approx h \ll 1$. This is a reasonable conclusion; However, we are inclined to keep this term in the $B$ term. The reason that makes us to keep $\varepsilon^2\mathcal{H}^{(0)}$ in the $B^{(0)}$ term, is the existence of $B^{(0)}$ parameter which has sufficiently large value according to the equation (2.60). Therefore, Eq. (2.72) simplifies to

$$\mathcal{H}_\tau{}^{(0)} - \frac{\eta\Delta_{2X}\mathcal{H}_\tau{}^{(0)}}{PD^2} - \frac{\eta^2 J_X\left(\mathcal{H}^{(0)}, \Delta_{2X}\mathcal{H}^{(0)}\right)}{PD^3} - B^{(0)}\eta\left(1 + \varepsilon^2\mathcal{H}^{(0)}\right)\frac{\mathcal{H}_X{}^{(0)}}{D}$$
$$+ \frac{\eta\Delta_{2X}{}^2\mathcal{H}^{(0)}}{D^2} + \frac{R_{cr}\Delta_{2X}\mathcal{H}^{(0)}}{D^2} = 0\,, \qquad (2.73)$$

where $\mathcal{H}^{(0)} \approx \frac{1}{\varepsilon^2}h$ , $B^{(0)} \approx \frac{1}{\varepsilon^3}\beta$ , $\eta \approx \varepsilon^2 q$, $q = \frac{gH^3}{\nu\kappa}$ , $J_X(f, g) = f_X g_Y - f_Y g_X$, $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ and $\Delta_{2X}{}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$ .

For the sake of clarity, we henceforth omit the $(0)$ on the variables of the above equation. Therefore, Eq. (2.73) might be written in the form of

$$\mathcal{H}_\tau - \frac{\eta}{PD^2}\Delta_{2X}\mathcal{H}_\tau - \frac{\eta^2 J_X(\mathcal{H}, \Delta_{2X}\mathcal{H})}{PD^3} - \frac{B\eta}{D}\mathcal{H}_X - \varepsilon^2\frac{B\eta}{D}\mathcal{H}\mathcal{H}_X + \frac{\eta}{D^2}\Delta_{2X}{}^2\mathcal{H}$$
$$+ \frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H} = 0\,. \qquad (2.74)$$

Eq. (2.74), which governs the upper surface deformation, is akin to the equation obtained by Tikhomolov in his paper [14].

It is worth pointing out that the nonlinear term $\mathcal{H}\mathcal{H}_X$ plays an important role in Eq. (2.74). As shown in the next section, by assuming the solution of Eq. (2.74), $\mathcal{H}(X, Y, \tau)$, in the form of $\mathcal{H}(\xi)$ where $\xi = X + Y - \lambda\tau$ , the Jacobian term, namely, $J_X(\mathcal{H}, \Delta_{2X}\mathcal{H})$ vanishes. Therefore, the only nonlinear term remains, will be the term

$\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ or equivalently the term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_\xi$. The presence of this term in the above equation will result in the existence of soliton solution for Eq. (2.74). This term is also important since it is the only term in the equation which accounts for the cyclone-anticyclone asymmetry [12] in Eq. (2.74). We keep in mind that the presence of the term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ in Eq. (2.74) emerges from keeping the term $\varepsilon^2 \mathcal{H}^{(0)}$ in the $B^{(0)}$ term in Eq. (2.72) due to the large value of the parameter $B$. Given the relation $B^{(0)} \approx \frac{1}{\varepsilon^3}\beta$, we could conclude the $\beta$ effect is the main reason for the existence of the nonlinear term, namely, the term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ in Eq. (2.74) and as a result of presence of this term in the above equation, the soliton solution is obtainable. In other words, the $\beta$ effect, namely, the consideration of the sphericity of the rotating layer of fluid, is mainly accountable to the existence of the soliton solution. However, one might ask what if the $\beta$ effect is neglected in the derivation of our governing equation. Nezlin in his book [12], raised this question and discussed about it. In that case, we should have kept the term $\varepsilon^2 \mathcal{H}^{(0)}$ in the Jacobian term of Eq. (2.72). However, it is a far small term in the Jacobian term. As a result, the term $\varepsilon^2 \mathcal{H} \frac{\eta J_X(\mathcal{H}, \Delta_{2X}\mathcal{H})}{PD^3}$, namely, the cubic nonlinearity term accounts for the cyclone-anticyclone asymmetry. However, the method to obtain a soliton solution, makes the Jacobian term vanish as discussed above. Therefore, the analytical approach presented in the next section does not cover the case in which the $\beta$ effect disregarded.

The other important term in Eq. (2.74) is the diffusion term, namely, the term $\frac{R_{cr}}{D^2} \Delta_{2X}\mathcal{H}$. As shown in the next section, in the absence of the term $\frac{R_{cr}}{D^2} \Delta_{2X}\mathcal{H}$, given the presence of all other terms in Eq. (2.74), there is no soliton solution for Eq. (2.74). Note that the existence of convection or equivalently the presence of heating from below in our model yields the presence of the term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$. By virtue of the existence of this term, the dissipative losses in the vortex structure can be compensated, and as a result, the presence of $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$ gives rise to the sustenance of the long-lived large-scale vortex structure [14]. Therefore, we conclude that in the absence of the effect of convection there is no soliton solution for Eq. (2.74).

In the following chapter, we present the analytical solution for the upper surface deformation equation, namely, Eq. (2.74).

# CHAPTER 3

# ANALYTICAL SOLUTIONS

In recent years, a vast variety of analytical methods has been used and developed in order to construct exact solutions for nonlinear partial differential equations. These techniques are such as Riccati expansion method [10], tanh method, extended tanh method, sine cosine method, auxiliary equation method [8,9], F-expansion method, Jacobi elliptic method, Exp-function method and so on.

In order to solve Eq. (2.74) analytically, we employ the so-called auxiliary equation method [8,9] with consideration of the Riccati equation as the auxiliary equation [10,11]. The auxiliary equation method is a straightforward technique proposed to solve nonlinear partial equations. [8,9,11].

## 3.1 Auxiliary Riccati Equation Method

First, auxiliary Riccati equation method [11] is described briefly and then the method is applied Eq. (2.74), and consequently the solutions will be obtained.

Suppose a nonlinear partial differential equation is of the form:

$$P(\mathcal{H},\mathcal{H}_\tau,\mathcal{H}_X,\mathcal{H}_X,\mathcal{H}_{\tau\tau},\mathcal{H}_{XX},\mathcal{H}_{YY}\ldots)=0\,, \tag{3.1}$$

where $\mathcal{H}$ assumed to be dependent on variables $X$, $Y$ and $\tau$ and is expressed in the form of $\mathcal{H}(X, Y, \tau)$.

In order to solve Eq. (3.1), variable transformation in the following form is suggested:

$$\mathcal{H}(X, Y, \tau) = \mathcal{H}(\xi), \qquad \xi = X + Y - \lambda\tau, \tag{3.2}$$

where $\lambda$ is the wave speed.

By using the above transformation, which is so-called travelling wave transformation, Eq. (3.1) will be changed to an ordinary differential equation in the following form:

$$Q(\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{H}''', \dots) = 0. \tag{3.3}$$

The solution of Eq. (3.3), which is so-called travelling wave solution, is assumed to be of the form

$$\mathcal{H}(\xi) = \sum_{i=0}^{n} d_i F^i(\xi), \tag{3.4}$$

where the positive integer $n$ can be determined by balancing the highest order derivative terms and the highest order nonlinear terms in the above equation, $d_i$ are unknown constant coefficients will be determined later and $F(\xi)$ satisfying the following auxiliary Riccati equation:

$$F'(\xi) = q_0 + F^2(\xi), \tag{3.5}$$

where $q_0$ is a real constant coefficient and $F' = \frac{dF}{d\xi}$ .

The general solution of Eq. (3.5) is of the form [12]

$$F(\xi) = \begin{cases} F_1(\xi) = -\sqrt{-q_0}\ \tanh(\sqrt{-q_0}\xi)\ , \\ F_2(\xi) = -\sqrt{-q_0}\ \coth(\sqrt{-q_0}\xi), \end{cases} \quad when\ q_0 < 0\ , \tag{3.6}$$

$$F(\xi) = -\frac{1}{\xi}\ , \quad when\ q_0 = 0\ , \tag{3.7}$$

$$F(\xi) = \begin{cases} F_1(\xi) = \sqrt{q_0}\ \tan(\sqrt{q_0}\xi)\ , \\ F_2(\xi) = -\sqrt{q_0}\ \cot(\sqrt{q_0}\xi), \end{cases} \quad when\ q_0 > 0\ . \tag{3.8}$$

We now return to Eq. (2.74) and apply the auxiliary Riccati equation method.

## 3.2 Applying Auxiliary Riccati Equation Method

For the sake of simplicity, we rearrange Eq. (2.74) in the form of

$$\mathcal{H}_\tau - a\Delta_{2X}\mathcal{H}_\tau - eJ_X(\mathcal{H},\Delta_{2X}\mathcal{H}) - b\mathcal{H}_X - \varepsilon^2 b\mathcal{H}\mathcal{H}_X + Pa{\Delta_{2X}}^2\mathcal{H} + c\Delta_{2X}\mathcal{H} = 0 \tag{3.9}$$

where $a = \frac{\eta}{PD^2}$ , $b = \frac{B\eta}{D}$ , $c = \frac{R_{cr}}{D^2}$ , $e = \frac{\eta^2}{PD^3}$ , $\eta \approx \varepsilon^2 q$, $q = \frac{gH^3}{\nu\kappa}$ and $P$ is the Prandtl number. Note that it is assumed all $a$, $b$, $c$, $e$ and $P$ take positive values in the above equation.

We seek to transform Eq. (3.9) into an ordinary differential equation. Therefore, we employ variable changing proposed by Eq. (3.2) and as a result, Eq. (3.9) transforms into the following equation:

$$(-\lambda - b)\mathcal{H}' + 2a\lambda\mathcal{H}''' - \varepsilon^2 b\mathcal{H}\mathcal{H}' + 4Pa\mathcal{H}'''' + 2c\mathcal{H}'' = 0\ , \tag{3.10}$$

where $\mathcal{H}' = \frac{d\mathcal{H}}{d\xi}$, $\mathcal{H}'' = \frac{d^2\mathcal{H}}{d\xi^2}$ , $\mathcal{H}''' = \frac{d^2\mathcal{H}}{d\xi^2}$ and $\mathcal{H}'''' = \frac{d^4\mathcal{H}}{d\xi^4}$ .

Note that by virtue of the transformation presented in Eq. (3.2), the Jacobian term has vanished in Eq. (3.10). For details of the derivation for Eq. (3.10), see Appendix E.

Integrating Eq. (3.10). with respect to $\xi$ results in

$$(-\lambda - b)\mathcal{H} + 2a\lambda\mathcal{H}'' - \varepsilon^2 b\frac{\mathcal{H}^2}{2} + 4Pa\mathcal{H}''' + 2c\mathcal{H}' = A\,, \tag{3.11}$$

where A is a constant of integration.

In order to determine the constant A, we make the following assumption:

$$as \;\; \xi \to \infty : \;\; \mathcal{H} \to 0\,, \quad \mathcal{H}' \to 0\,, \quad \mathcal{H}'' \to 0 \quad and \quad \mathcal{H}''' \to 0. \tag{3.12}$$

By virtue of the above assumption, Eq. (3.11) is of the form

$$(-\lambda - b)\mathcal{H} + 2a\lambda\mathcal{H}'' - \varepsilon^2 b\frac{\mathcal{H}^2}{2} + 4Pa\mathcal{H}''' + 2c\mathcal{H}' = 0\,. \tag{3.13}$$

According to the auxiliary Riccati equation method explained above, it is assumed that the solution of Eq. (3.13) is of the form

$$\mathcal{H}(\xi) = \sum_{i=0}^{n=3} d_i F^i(\xi) = d_0 + d_1 F(\xi) + \; d_2 F^2(\xi) + d_3 F^3(\xi)\,, \tag{3.14}$$

where $d_0$ , $d_1$ , $d_2$ and $d_3$ are unknown constant coefficients and will be determined later.

The positive integer $n$ in Eq. (3.14) was determined by balancing the highest order derivative terms and the highest order nonlinear terms in Eq. (3.13) as follows [8,10]:

$$\begin{cases} O\left[\dfrac{\partial^r \mathcal{H}}{\partial \xi^r}\right] = n + r \\ O\left[\mathcal{H}^s \dfrac{\partial^r \mathcal{H}}{\partial \xi^r}\right] = sn + r \end{cases} \Rightarrow \begin{cases} O\left[\dfrac{\partial^3 \mathcal{H}}{\partial \xi^3}\right] = n + 3 \\ O[\mathcal{H}^2] = 2n \end{cases} \Rightarrow n + 3 = 2n \Rightarrow n = 3 \quad (3.15)$$

Given Eq. (3.14), we obtain the following expression for $\mathcal{H}^2$:

$$\mathcal{H}^2 = {d_0}^2 + 2d_0 d_1 F + ({d_1}^2 + 2d_0 d_2)F^2 + (2d_0 d_3 + 2d_1 d_2)F^3 + ({d_2}^2 + 2d_1 d_3)F^4 + 2d_2 d_3 F^5 + {d_3}^2 F^6. \quad (3.16)$$

We now differentiate both sides of Eq. (3.14) with respect to $\xi$ and use the Riccati equation, namely, Eq. (3.5). Therefore, we obtain the following expressions for $\mathcal{H}'$:

$$\mathcal{H}' = d_1 q_0 + 2d_2 q_0 F + (d_1 + 3d_3 q_0)F^2 + 2d_2 F^3 + 3d_3 F^4. \quad (3.17)$$

In the same fashion, we obtain the following expression for $\mathcal{H}''$ and $\mathcal{H}'''$ :

$$\mathcal{H}'' = 2d_2 {q_0}^2 + (2d_1 q_0 + 6d_3 {q_0}^2)F + 8d_2 q_0 F^2 + (2d_1 + 18d_3 q_0)F^3 + 6d_2 F^4 + 12d_3 F^5, \quad (3.18)$$

$$\mathcal{H}''' = 2d_1 {q_0}^2 + 6d_3 {q_0}^3 + 16d_2 {q_0}^2 F + (8d_1 q_0 + 60d_3 {q_0}^2)F^2 + 40d_2 q_0 F^3 + (6d_1 + 114d_3 q_0)F^4 + 24d_2 F^5 + 60d_3 F^6. \quad (3.19)$$

Substituting $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}''$, $\mathcal{H}'''$ and $\mathcal{H}^2$ from Eqs. (3.14), (3.16), (3.18), (3.19) and (3.16) respectively into Eq. (3.13), and setting each coefficient of $F^i$ $(0 \le i \le 6)$ to zero, yields seven equations as follows:

$$(-\lambda - b)d_0 + 4a\lambda d_2 {q_0}^2 - \frac{b}{2}\varepsilon^2 {d_0}^2 + 8Pad_1 {q_0}^2 + 24Pad_3 {q_0}^3 + 2cd_1 q_0 = 0, \quad (3.20)$$

$$(-\lambda - b)d_1 + 2a\lambda(2d_1 q_0 + 6d_3 {q_0}^2) - b\varepsilon^2 d_0 d_1 + 64Pad_2 {q_0}^2 + 4cd_2 q_0 = 0, \quad (3.21)$$

$$(-\lambda-b)d_2+16a\lambda d_2q_0-\frac{b}{2}\varepsilon^2({d_1}^2+2d_0d_2)+4Pa(8d_1q_0+60d_3{q_0}^2)+2c(d_1$$
$$+3d_3q_0)=0\,, \qquad (3.22)$$

$$(-\lambda-b)d_3+2a\lambda(2d_1+18d_3q_0)-b\varepsilon^2(d_0d_3+d_1d_2)+160Pad_2q_0+4cd_2$$
$$=0, \qquad (3.23)$$

$$12a\lambda d_2-\frac{b}{2}\varepsilon^2({d_2}^2+2d_1d_3)+4Pa(6d_1+114d_3q_0)+6cd_3=0\,, \qquad (3.24)$$

$$24a\lambda d_3-b\varepsilon^2d_2d_3+96Pad_2=0\,, \qquad (3.25)$$

$$-\frac{b}{2}\varepsilon^2{d_3}^2+240Pad_3=0\,. \qquad (3.26)$$

We solve the above set of equations with the aid of Maple 15. Consequently, 10 sets of solutions have been obtained. In order to see all 10 sets of solutions see the Appendix F. It is worth noting that our interest is only in those sets of solutions in which $q_0<0$. Strictly speaking, we seek hyperbolic travelling solutions since only those solutions are considered soliton solutions. As indicated in Eq. (3.6), the hyperbolic function solutions exist when $q_0<0$. Therefore, the sets of solutions in which $q_0>0$ have been disregarded. Additionally, we also seek those sets of solutions for the above system of equations that do not contradict the assumption made in Eq. (3.12). Therefore, we present here the sets of solutions that satisfy both $q_0<0$ and the assumption indicated in Eq. (3.12). The only set of solutions for the above system of equations that satisfies the two aforementioned requirements is the fifth set of solutions. The fifth set of solutions with the assumption that $\frac{3c}{2P}<1$ , after some algebraic manipulations is as follows:

$$d_0 = \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \qquad d_1 = -60\frac{c}{\varepsilon^2 b}, \qquad d_2 = -\frac{120}{\varepsilon^2 b}\sqrt{2Pac}\,, \qquad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = -\frac{c}{8Pa}, \qquad \lambda = -4\sqrt{\frac{2Pc}{a}}, \qquad ab^2 = 32Pc\left(\frac{3c}{2P} - 1\right)^2. \qquad (3.27)$$

Note that the assumption $\frac{3c}{2P} < 1$ or equivalently $\frac{3R_{cr}}{2D^2} < P$ is a reasonable assumption due to the definition of $R_{cr}$ in the equation (3.59). Note that it is assumed $D \gg 1$.

Given Eqs. (3.6), (3.14) and the above set of solutions, the solutions for $\mathcal{H}$ are of the form

$$\mathcal{H}(\xi) = +\frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}} + \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}tanh\left(\sqrt{\frac{c}{8Pa}}\xi\right) - \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}tanh\left(\sqrt{\frac{c}{8Pa}}\xi\right)^2$$
$$-\frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}tanh\left(\sqrt{\frac{c}{8Pa}}\xi\right)^3, \qquad (3.28)$$

or:

$$\mathcal{H}(\xi) = +\frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}} + \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}coth\left(\sqrt{\frac{c}{8Pa}}\xi\right) - \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}coth\left(\sqrt{\frac{c}{8Pa}}\xi\right)^2$$
$$-\frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}coth\left(\sqrt{\frac{c}{8Pa}}\xi\right)^3. \qquad (3.29)$$

For the sake of brevity, we henceforth stick with the solution (3.28). Given the travelling wave transformation defined in Eq. (3.2), the value for $\lambda$ and the constraint $ab^2 = 32Pc\left(\frac{3c}{2P} - 1\right)^2$ obtained in the fifth set of solutions, the solution (3.28) is of the form

$$\mathcal{H}(X,Y,\tau) = \frac{1}{\varepsilon^2}\frac{15c}{4\left(P-\frac{3}{2}c\right)} + \frac{1}{\varepsilon^2}\frac{15c}{4\left(P-\frac{3}{2}c\right)}tanh\left(\sqrt{\frac{c}{8Pa}}\left(X+Y+4\sqrt{\frac{2Pc}{a}}\tau\right)\right)$$
$$-\frac{1}{\varepsilon^2}\frac{15c}{4\left(P-\frac{3}{2}c\right)}tanh\left(\sqrt{\frac{c}{8Pa}}\left(X+Y+4\sqrt{\frac{2Pc}{a}}\tau\right)\right)^2$$
$$-\frac{1}{\varepsilon^2}\frac{15c}{4\left(P-\frac{3}{2}c\right)}tanh\left(\sqrt{\frac{c}{8Pa}}\left(X+Y+4\sqrt{\frac{2Pc}{a}}\tau\right)\right)^3. \qquad (3.30)$$

As mentioned above it is assumed that $\frac{3c}{2P} < 1$ . Let us now plug the values of $a$, $b$ and $c$, defined in Eq. (3.9), in Eq. (3.30) and the constraint $ab^2 = 32Pc\left(\frac{3c}{2P} - 1\right)^2$. Therefore, the solution (3.30) and the constraint are of the form

$$\mathcal{H}(X,Y,\tau) = \frac{1}{\varepsilon^2}\frac{15R_{cr}}{2(2PD^2-3R_{cr})} + \frac{1}{\varepsilon^2}\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\left(X+Y+4P\sqrt{\frac{2R_{cr}}{\eta}}\tau\right)\right)$$
$$-\frac{1}{\varepsilon^2}\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\left(X+Y+4P\sqrt{\frac{2R_{cr}}{\eta}}\tau\right)\right)^2$$
$$-\frac{1}{\varepsilon^2}\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\left(X+Y+4P\sqrt{\frac{2R_{cr}}{\eta}}\tau\right)\right)^3, \qquad (3.31)$$

with the constraint $B\eta^{\frac{3}{2}} = 4\sqrt{2R_{cr}}\,D\left(P - \frac{3R_{cr}}{2D^2}\right)$.

It is worth noting that the above solution satisfies the assumption made in Eq. (3.12). In order to obtain the expression for $h(x, y, t)$, we infer the relation $h(x, y, t) \approx \varepsilon^2 \mathcal{H}(X, Y, \tau)$ from Eq. (2.56). Given the transformation defined in the Eq. (2.54) and the relationship $B \approx \frac{1}{\varepsilon^3}\beta$ obtained from Eq. (2.60), the solution (3.31) and the constraint are approximated to

$$h(x,y,t) = \frac{15R_{cr}}{2(2PD^2 - 3R_{cr})} + \frac{15R_{cr}}{2(2PD^2 - 3R_{cr})} tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\varepsilon\left(x + y + 4P\varepsilon\sqrt{\frac{2R_{cr}}{\eta}}t\right)\right)$$

$$-\frac{15R_{cr}}{2(2PD^2 - 3R_{cr})} tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\varepsilon\left(x + y + 4P\varepsilon\sqrt{\frac{2R_{cr}}{\eta}}t\right)\right)^2$$

$$-\frac{15R_{cr}}{2(2PD^2 - 3R_{cr})} tanh\left(\sqrt{\frac{R_{cr}}{8\eta}}\varepsilon\left(x + y + 4P\varepsilon\sqrt{\frac{2R_{cr}}{\eta}}t\right)\right)^3, \tag{3.32}$$

with the constraint $\beta\eta^{\frac{3}{2}} = 4\varepsilon^3\sqrt{2R_{cr}}\, D\left(P - \frac{3R_{cr}}{2D^2}\right)$, where $\eta \approx \varepsilon^2 q$ and $q = \frac{gH^3}{\nu\kappa}$.

Let us now substitute $\eta$ with $\varepsilon^2 q$ thus, $\varepsilon$ vanishes from the above solution and the constraint. Therefore, the solution (3.32) is of the form

$$h(x,y,t)=\frac{15R_{cr}}{2(2PD^2-3R_{cr})}+\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8q}}\left(x+y+4P\sqrt{\frac{2R_{cr}}{q}}t\right)\right)$$
$$-\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8q}}\left(x+y+4P\sqrt{\frac{2R_{cr}}{q}}t\right)\right)^2$$
$$-\frac{15R_{cr}}{2(2PD^2-3R_{cr})}tanh\left(\sqrt{\frac{R_{cr}}{8q}}\left(x+y+4P\sqrt{\frac{2R_{cr}}{q}}t\right)\right)^3, \qquad (3.33)$$

with the constraint $\beta q^{\frac{3}{2}}=4\sqrt{2R_{cr}}\,D\left(P-\frac{3R_{cr}}{2D^2}\right)$, where $q=\frac{gH^3}{\nu\kappa}$ and $R_{cr}\approx 8.6956D^{\frac{4}{3}}\ as\ D\to\infty$.

The above solution with the constraint is one of the soliton solutions for Eq. (2.74). One of an important coefficients in the above solution, $\sqrt{\frac{R_{cr}}{q}}$, namely, the so-called pumping rate, arising from the existence of convection, to replenish the dissipative losses in the long-lived large-scale vortex structure [14]. The presence of the constraint in the above solution can be interpreted in a way that the balance between the dissipative losses and the heating energy from which the deformation of the upper free surface arises, requires the coefficient of terms in Eq. (2.74) meets the above constraint in order to have the above solition solution to account for the sustenance of the vortex structure. It is worth noting that the existence of the constraint is a feature inherent in Eq. (2.74) and cannot be avoided.

In our model, it is assumed that $D\gg 1$. Also, one can see the above solution works well if $\frac{PD^2}{R_{cr}}\gg 1$ or $\frac{R_{cr}}{PD^2}\ll 1$ . For sufficiently large values of $D$, $R_{cr}$ might be substituted with the asymptotic expression from Eq. (2.62), namely, $R_{cr} \approx 8.6956D^{\frac{4}{3}}$ . Given $D \gg 1$ , we notice that the relationship $\frac{PD^2}{R_{cr}} \gg 1$ is really satisfied.

One also can see from the constraint that if $\beta = 0$, the constraint will not be satisfied anymore due to the assumption $\frac{R_{cr}}{PD^2} \ll 1$ . Therefore, as discussed in the previous chapter, in the case $\beta = 0$, there is no solution for Eq. (2.74).

By virtue of substituting $R_{cr}$ with the expression $8.6956D^{\frac{4}{3}}$ for sufficiently large value of $D$, the solution (4.33) and the constraint are approximated to

$$h(x,y,t) = \frac{15}{0.46PD^{\frac{2}{3}} - 6} + \frac{15}{0.46PD^{\frac{2}{3}} - 6} tanh\left(1.04257D^{\frac{2}{3}}\sqrt{\frac{1}{q}}\left(x + y + 16.6811PD^{\frac{2}{3}}\sqrt{\frac{1}{q}}t\right)\right)$$

$$- \frac{15}{0.46PD^{\frac{2}{3}} - 6} tanh\left(1.04257D^{\frac{2}{3}}\sqrt{\frac{1}{q}}\left(x + y + 16.6811PD^{\frac{2}{3}}\sqrt{\frac{1}{q}}t\right)\right)^2$$

$$- \frac{15}{0.46PD^{\frac{2}{3}} - 6} tanh\left(1.04257D^{\frac{2}{3}}\sqrt{\frac{1}{q}}\left(x + y + 16.6811PD^{\frac{2}{3}}\sqrt{\frac{1}{q}}t\right)\right)^3, \quad (3.34)$$

with the constraint $\beta q^{\frac{3}{2}} = 16.6811\, D^{\frac{5}{3}}\left(P - 13.0434D^{-\frac{2}{3}}\right)$ , where $q = \frac{gH^3}{\nu\kappa}$ , $D = 2\frac{\Omega \sin\theta_0 H^2}{\nu}$ , $P = \frac{\nu}{\kappa}$, $\beta = \frac{\beta_0 H}{2\Omega \sin\theta_0}$ and $\beta_0 = 2\frac{\Omega \cos\theta_0}{r_0}$ .

It is worth mentioning that the solution (3.34) and the constraint are valid for sufficiently large values of $D$. It can be inferred from the solution (3.34) that if $P^{\frac{3}{2}}D \geq 6245$ the solution (3.34) yields $h \approx 0.1$. As discussed above, we should keep in mind that the solution (3.34) works well for sufficiently large values of $D$. However, it can be also seen from the solution (3.34) that if $D \to \infty$ the perturbation approaches to zero and there is no soliton solution for Eq. (2.74), which is expectable since the larger the value of $D$, the greater the value of $R_{cr}$, and as a result, the onset of convection will be precluded as $D \to \infty$. Therefore, the deformation of the upper surface due to convection, will be obviated in the case $D \to \infty$. The assumption $D \to \infty$ can be as a result of an increase in the angular velocity $\Omega$. Therefore, it can be also inferred that an increase in $\Omega$ gives rise to the inhibition of the onset of convection discussed by Chandrasekhar in his paper [1].

As discussed in chapter 2, the existence of convection in our model yields the presence of the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$ in Eq. (2.74). Therefore, we can see as $D \to \infty$, the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$, can be neglected in Eq. (2.74), which means there is no source of energy to compensate the dissipative losses in the vortex structure. Therefore, in the absence of the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$, the solution (3.34) for the deformation of the upper free surface does not work as a soliton solution for Eq. (2.74) and the constraint also is not met since the constraint is a balance needed between heating energy and dissipative losses in order to have the above soliton solution. In the next section, we use the Jovian data as an example and present the results based on the solution (3.34) and the constraint.

# CHAPTER 4

# RESULTS

The sustenance of vortex structures in the Jupiter is a puzzling phenomenon. It is proposed that the Great Red Spot of Jupiter is a Rossby solitary vortex [12,13]. The hyperbolic traveling solution, which is a soliton solution obtained in the previous chapter, can be applied to the case of the Great Red Spot of Jupiter. Therefore, in this chapter, we employ the Jovian parameters as an example case for our solution, namely, the solution (3.34). The following Jovian data is used for the rest of calculation [14,15,24,25]:

$$\nu \approx 10^8 \frac{cm^2}{s}, \quad \kappa \leq \nu \implies 1 \leq P, \qquad \alpha \approx 0.01 \frac{1}{K}, \quad \Theta \approx 10K$$

$$\Omega \approx 1.76 \times 10^{-4} \frac{rad}{s}, \theta_0 \approx 22 \implies \quad 2\Omega \sin\theta_0 \approx 1.32 \times 10^{-4},$$

$$r_0 \approx 69911km \approx 6.9911 \times 10^9 cm \quad \implies \beta_0 \approx 4.67 \times 10^{-14}\, cm^{-1}s^{-1} \qquad (4.1)$$

Note that $\theta_0 \approx 22$ is the latitude at which the Great Red Spot of Jupiter exists and $\nu$ is the effective turbulent viscosity for the Jovian atmosphere.

As explained in chapter 3, In order to obtain acceptable results from the solution (3.34), the constraint $\beta q^{\frac{3}{2}} = 16.6811\, D^{\frac{5}{3}}\left(P - 13.0434 D^{-\frac{2}{3}}\right)$ has to be satisfied. Additionally $D$ has to adopt sufficiently large value. Therefore, we need to determine the depth of our model, namely, $H$, in a way that meets the foregoing requirements. Given the above Jovian data, it can be inferred that $H \approx 3 \times 10^5\ m$ or $H \approx 300km$. This value for $H$ yields $D \approx 1200$ and also meets the constraints if $P \approx 10$, and $q \approx q_{cr} \approx \frac{8.6956D^{\frac{4}{3}}}{\alpha\Theta}$. Therefore, for the rest of our calculation, $P$ and $H$ are set to 5 and $3 \times 10^5 m$ respectively. It is worth noting that different models adopt different depth for the Great red spot of Jupiter. In some shallow water models, the thickness has been set to be about 25 km [12] and in some other models it is on the order of several hundred kilometers [24]. In fact, the depth of the rotating fluid layer fluctuates. It is important that the value determined for $H$ is much less than the horizontal dimensions of the Great red spot of Jupiter. As a result, the vortex is still considered in the Rossby regime [12] and the quasi-geostrophic approximation used in the section (2.3) is valid.

Substituting the Jovian data in the solution (3.34), the perturbation $h(x, y, t)$ is of the form

$$\begin{aligned} h(x,y,t) &= 0.0595 + 0.0595\, tanh\big(0.1118(x + y + 8.94434t)\big) \\ &\quad - 0.0595\, tanh\big(0.1118(x + y + 8.9443t)\big)^2 \\ &\quad - 0.0595\, tanh\big(0.1118(x + y + 8.9443t)\big)^3. \end{aligned} \tag{4.2}$$

The horizontal size of the Great Red Spot of Jupiter, which is known as $12000km \times 25000km$ [12], needs to be transformed to the non-dimensional size according to Eq. (2.25). Therefore, $-20 \leq x \leq 20$ and $-41.6 \leq y \leq 41.6$. In the same fashion, according to Eq. (2.25), it can be obtained that $t = 1$ corresponds to 1.43 year.

As illustrated in the following figures the perturbation $h$ decays when $t = 10$. In other words, after about 14 years the perturbation decays.

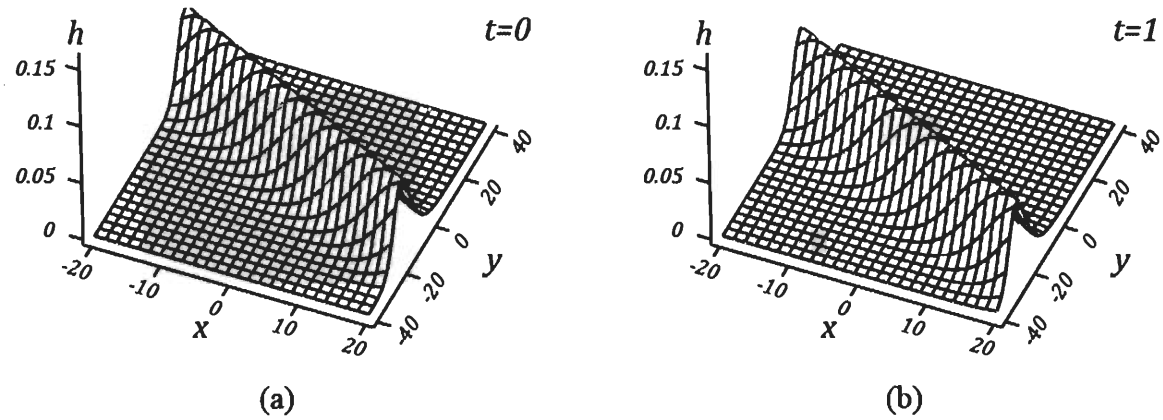



**Figure 4.1 The time evolution of the perturbation $h = (x, y)$**

As the figures 5.1, 5.2 and 5.3 illustrate, the period of time needs to decay the perturbation $h(x, y)$ is much less than the period of time over which the vortex structure of the Great Red Spot of Jupiter has existed. The discrepancy between the time period obtained and the age of the Great Red Spot of Jupiter might be due to neglect of the effects of zonal flows on the Jovian vortex structures in our model since it has been proposed that the zonal flows might also give rise to the uniqueness and localization of the long-lived vortex structure of the Great Red Spot of Jupiter [12,14].

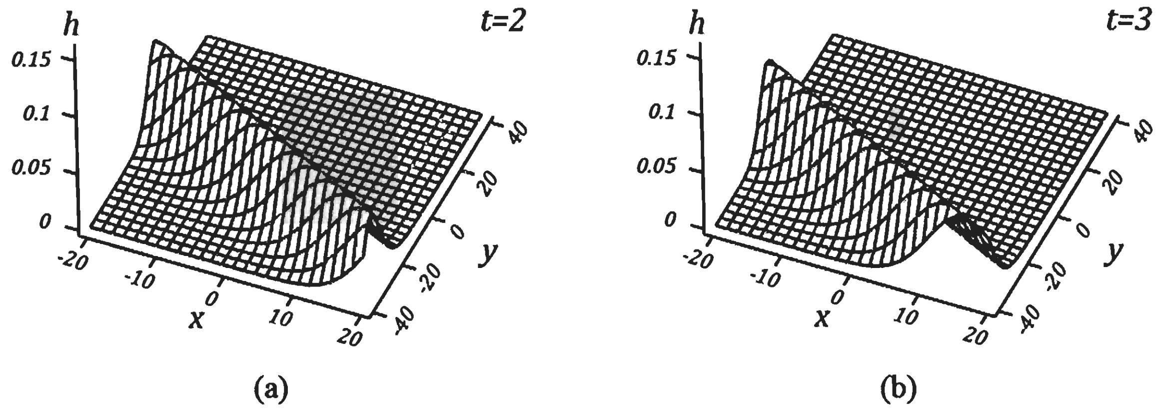



**Figure 4.2 The time evolution and decaying of the perturbation $h = (x, y)$**

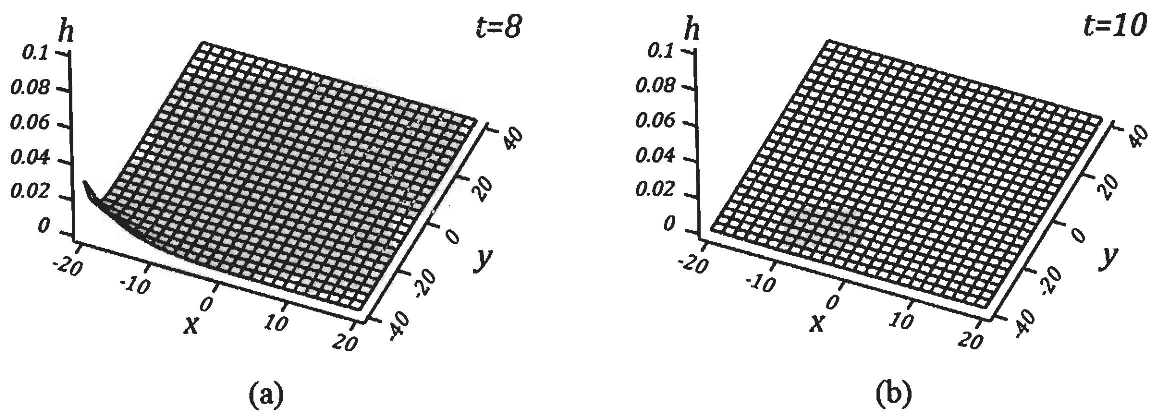



**Figure 4.3 The decaying of the perturbation $h = (x, y)$**

Let us now calculate the dominant terms of non-dimensional components of the velocity field, namely, $u^{(0)}$, $v^{(0)}$ and $w^{(0)}$ based on the quasi-geostrophic approximation made in the section 2.3 for the case of the Great Red spot of Jupiter. Given Eqs. (2.44), (2.45) and (2.46), $w^{(0)}$ is set to zero, $u^{(0)}$ and $v^{(0)}$ can be approximated by substituting $p$ with $p^{(0)}(X, Y, \tau)$ from Eq. (3.56), using the relationships from chapter 2 such as

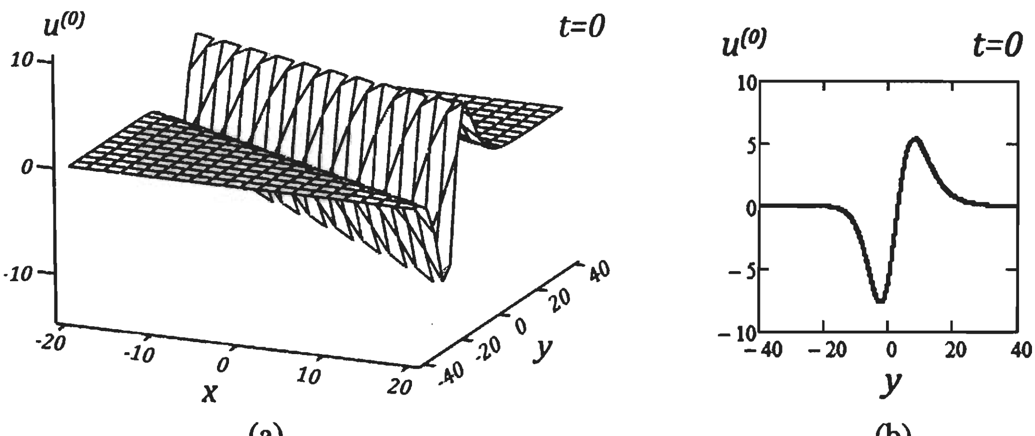
$p^{(0)} = \eta \mathcal{H}^{(0)}$, where $\eta = \varepsilon^2 q$, $\mathcal{H}^{(0)} \approx \frac{1}{\varepsilon^2} h$, $X = \varepsilon x$, $Y = \varepsilon y$, $\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}$ and $\frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}$, we infer the following equations for $u^{(0)}$, $v^{(0)}$ and $w^{(0)}$:

$$u^{(0)} = -\frac{1}{D} p_y \approx -\frac{\varepsilon}{D} \frac{\partial p^{(0)}}{\partial Y} = -\frac{\varepsilon^3}{D} q \frac{\partial \mathcal{H}^{(0)}}{\partial Y} \approx -\frac{q}{D} \frac{\partial h}{\partial y} \,, \tag{4.3}$$

$$v^{(0)} = \frac{1}{D} p_x \approx -\frac{\varepsilon}{D} \frac{\partial p^{(0)}}{\partial X} = \frac{\varepsilon^3}{D} q \frac{\partial \mathcal{H}^{(0)}}{\partial X} \approx -\frac{q}{D} \frac{\partial h}{\partial x} \,, \tag{4.4}$$

$$w^{(0)} = 0 \,. \tag{4.5}$$

It is already discussed that in order to meet the constraint in the case of the Great Red Spot of Jupiter, $q \approx q_{cr} \approx \frac{8.6956 D^{\frac{4}{3}}}{\alpha \Theta}$. Therefore, $q$ needs to be substituted with $q_{cr}$ in Eqs. (4.3) and (4.4). The results are as follows:




(a) (b)

**Figure 4.4 The time evolution of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 0$, (b) a cut at $x = 0$.**

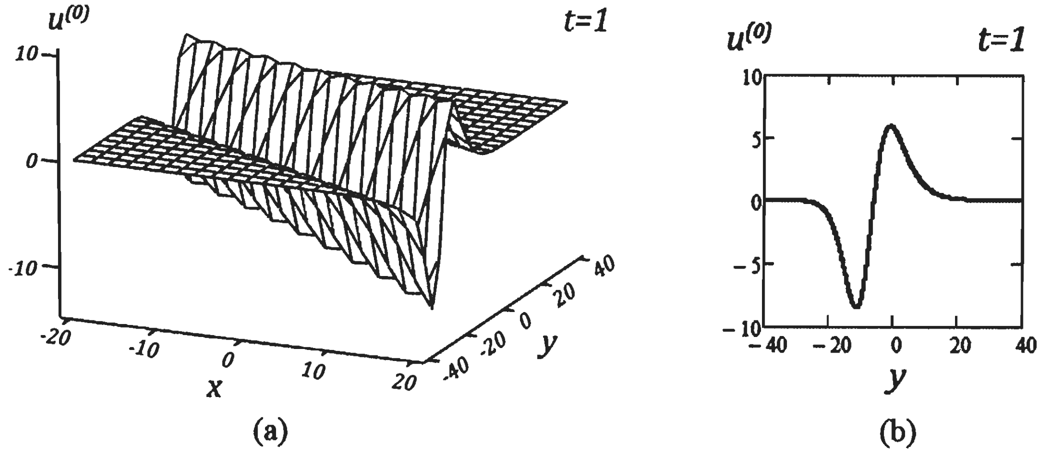



**Figure 4.5** The time evolution of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 1$, (b) a cut at $x = 0$.

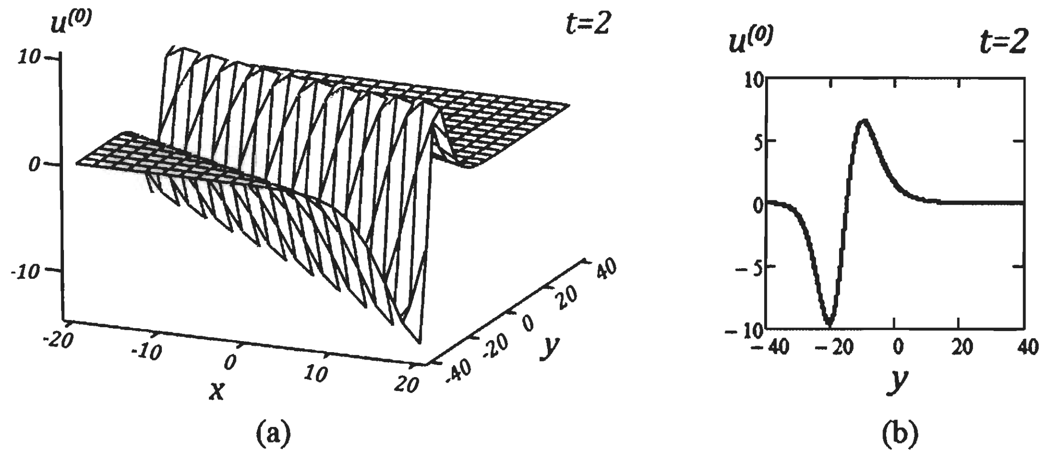



**Figure 4.6** The time evolution of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 2$, (b) a cut at $x = 0$.

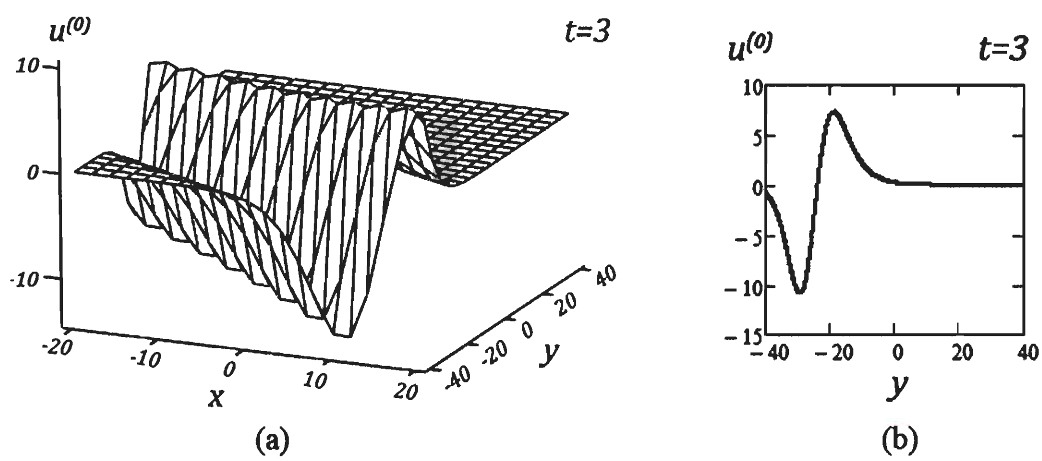
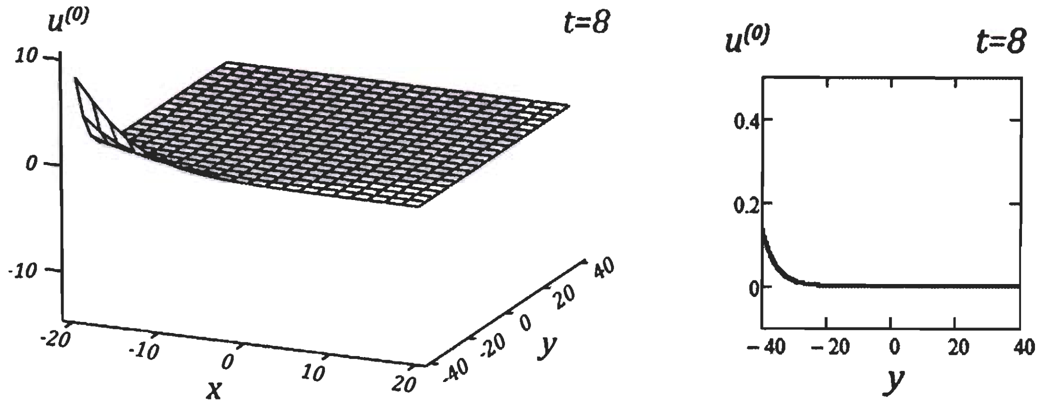
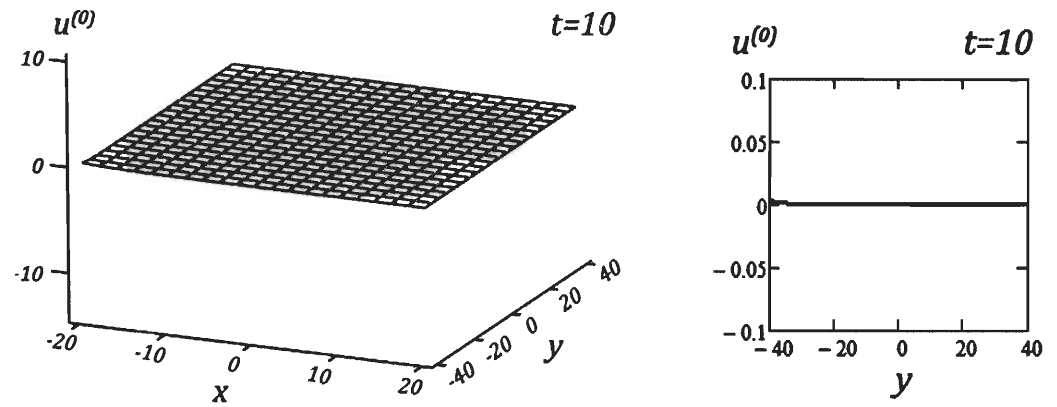



**Figure 4.7** The time evolution of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 3$, (b) a cut at $x = 0$.




(a) (b)

**Figure 4.8** The decaying of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 8$, (b) a cut at $x = 0$.




(a) (b)

**Figure 4.9** The decaying of the non-dimensional mean $u^{(0)}$, (a) the 3D $u^{(0)}$ at $t = 10$, (b) a cut at $x = 0$.

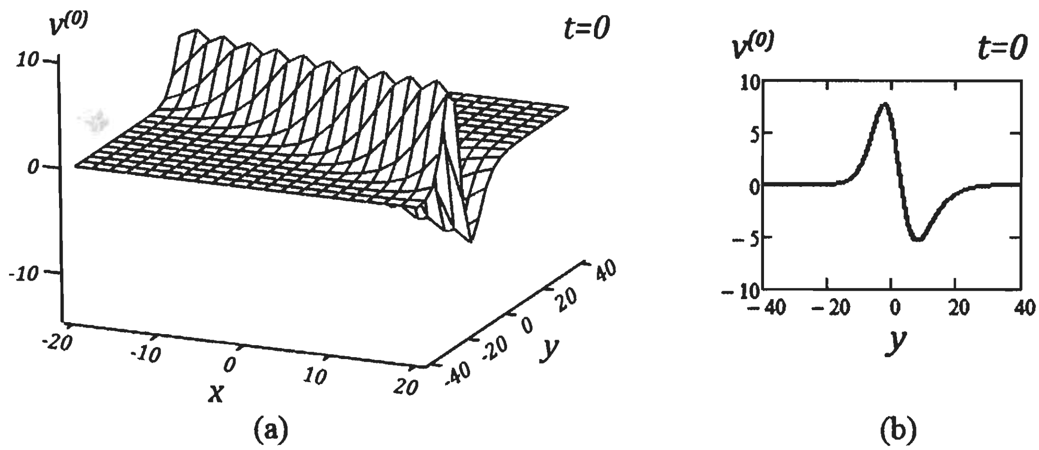



**Figure 4.10** The time evolution of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 0$, (b) a cut at $x = 0$.

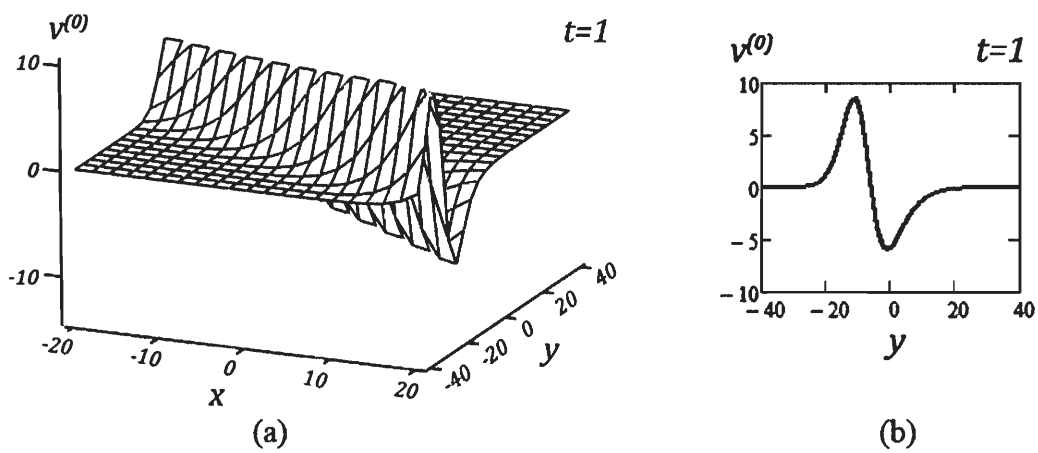



**Figure 4.11** The time evolution of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 1$, (b) a cut at $x = 0$.

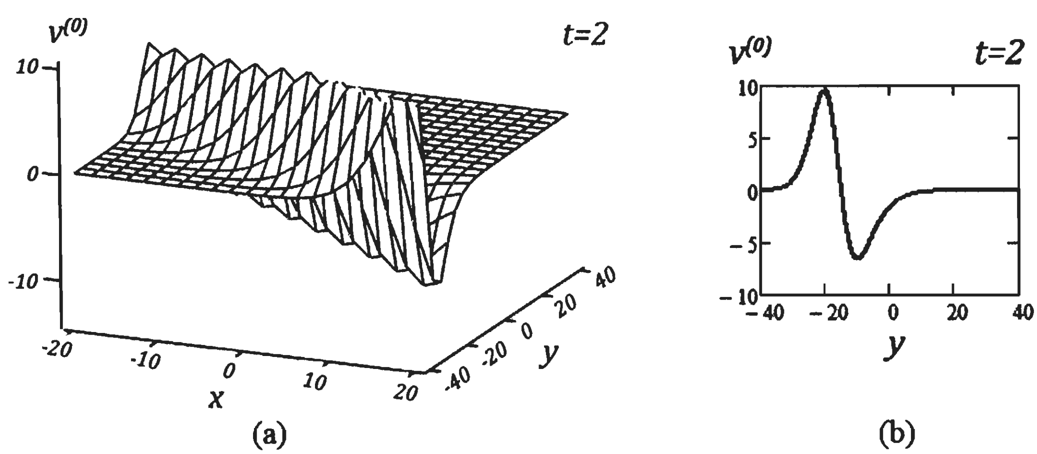



**Figure 4.12** The time evolution of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 2$, (b) a cut at $x = 0$.

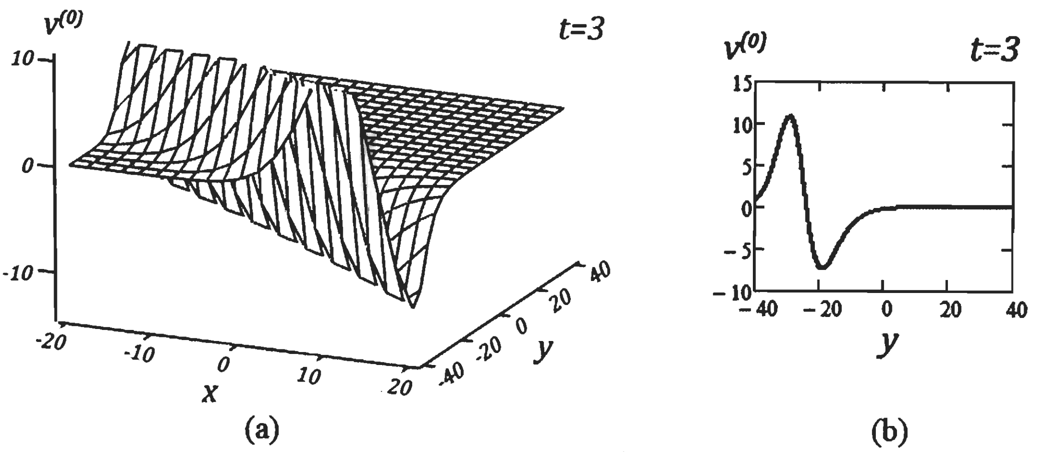



**Figure 4.13** The time evolution of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 3$, (b) a cut at $x = 0$.

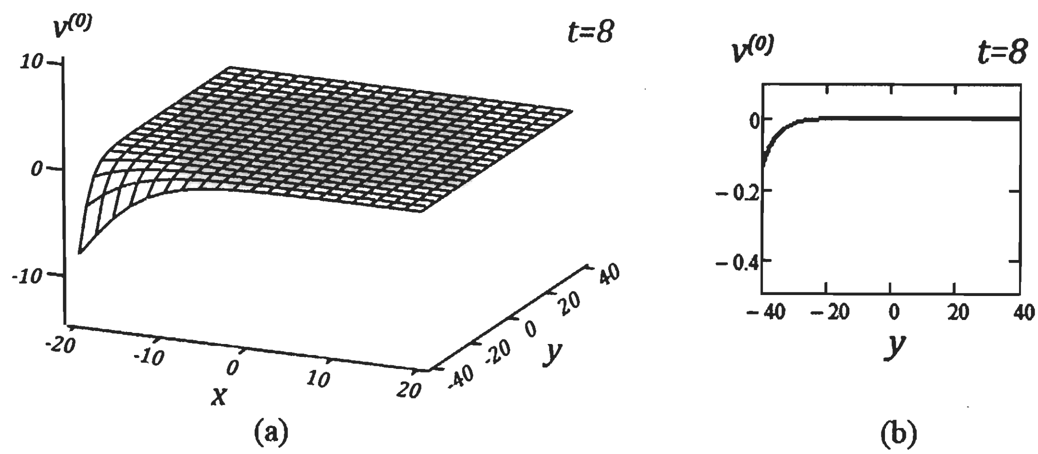



**Figure 4.14** The decaying of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 8$, (b) a cut at $x = 0$.

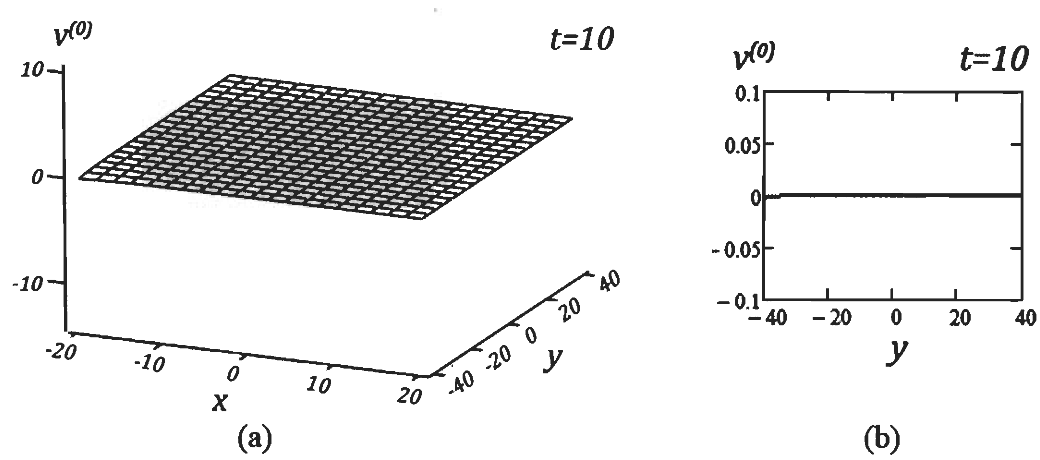



**Figure 4.15** The decaying of the non-dimensional mean $v^{(0)}$, (a) the 3D $v^{(0)}$ at $t = 10$, (b) a cut at $x = 0$.

# CHAPTER 5

# DISCUSSIONS

We have derived an equation which governs the evolution of large-scale perturbations in a large-scale long-lived rotating layer of fluid heated underneath. The soliton solutions, namely, the hyperbolic travelling solutions have been obtained for the deformation equation, one set of which is our interest and accounts for the sustenance of Rossby vortex in our mathematical model. The effect of all terms in the deformation equation, namely, Eq. (2.74) has been examined. There are two nonlinear terms including the $B$ term or $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ and the Jacobian term, namely, $J_X(\mathcal{H}, \Delta_{2X}\mathcal{H})$ in Eq. (2.74). By virtue of the variable transformation employed, the Jacobian term has vanished. Therefore, the only nonlinear term remains in the equation, is the term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$, emerging from the existence of the so-called beta effect, namely, the consideration of the sphericity of the rotating layer of fluid, in the system of governing equations. We have shown that the nonlinear term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ accounts for the existence of soliton solutions for Eq. (2.74) and as a result, give rises to the sustenance of the vortex structure in our model. The term $\varepsilon^2 \frac{B\eta}{D} \mathcal{H}\mathcal{H}_X$ also accounts for the cyclone-anticyclone asymmetry in Eq. (2.74).

It can also be inferred from the solution (3.33) or (3.34) and the constraint, in the case $B$ or $\beta$ is set to zero, or in the absence of the nonlinear term $\varepsilon^2 \frac{B\eta}{D}\mathcal{H}\mathcal{H}_X$ in Eq. (2.74), there is no soliton solution for Eq. (2.74).

In order to obtain soliton solutions in the absence of the nonlinear term $\varepsilon^2 \frac{B\eta}{D}\mathcal{H}\mathcal{H}_X$, the term $\varepsilon^2\mathcal{H}\frac{\eta J_X(\mathcal{H},\Delta_{2X}\mathcal{H})}{PD^3}$, namely, the cubic nonlinearity term would have to be kept in the equation (2.72). However, the variable transformation employed, makes the Jacobian term vanish and as a result, the cubic nonlinearity term disappears as well. Therefore, the soliton solution obtained excludes the case in which the beta effect disregarded. The effect of the cubic nonlinearity term in the absence of the beta effect can be examined in future work by a different analytical method.

It has been shown that the existence of convection or heating energy from below yields the presence of the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$ in Eq. (2.74). The diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$ replenishes the effect of the dissipative losses inherent in the long-lived large-scale vortex structures. It can be inferred from the solution with the constraint obtained that in absence of the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$, given all other terms exist, there is no soliton solution for Eq. (2.74) and as a result the deformation does not exist.

The diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$, emerging from the existence of convection or heating energy from below, contributes to the presence of an important coefficients in the solution, $\sqrt{\frac{R_{cr}}{q}}$, namely, the so-called pumping rate for generation of a large-scale disturbance to replenish the dissipative losses in the vortex structure [14]. In fact, the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$, contributes to the destabilization of Eq. (2.74) or the deformation of the upper free surface and as a result, accounts for the sustenance of the long-lived vortex structure. However, the other diffusion term in the equation (2.74), namely, $\frac{\eta}{D^2}{\Delta_{2X}}^2\mathcal{H}$ and also the dispersive term, namely, $\frac{\eta}{PD^2}\Delta_{2X}\mathcal{H}_\tau$ play a different role and contribute to the stabilization of Eq. (2.74) and as a result, accounts for decaying of the perturbation or deformation. Consequently, in order to have a soliton solution for Eq. (2.74), which accounts for the sustenance of the vortex structure, as one can see from our solution, It is required to maintain a balance between heating energy from below and the dissipative losses. The foregoing balance yields the constraint obtained. In principle, the constraint is an intrinsic feature of Eq. (2.74) and cannot be neglected.

In conclusion, there are two terms, namely, the diffusion term $\frac{R_{cr}}{D^2}\Delta_{2X}\mathcal{H}$ and the nonlinear term $\varepsilon^2\frac{B\eta}{D}\mathcal{H}\mathcal{H}_X$ in Eq. (2.74) that account for the sustenance of the vortex structure and there are the other two terms, namely, the other diffusion term $\frac{\eta}{D^2}{\Delta_{2X}}^2\mathcal{H}$ and the dispersive term $\frac{\eta}{PD^2}\Delta_{2X}\mathcal{H}_\tau$ that account for the decaying of the vortex structure.

The solution with the constrained obtained works well for sufficiently large values of $D$, where $D$ is the reciprocal of the Ekman number $E$. However, one can see from the solution that as $D \to \infty$ the perturbation approaches to zero and there is no soliton solution for Eq. (2.74). In other words, the deformation of the upper surface due to convection, will be precluded in the case $D \to \infty$, which corresponds to the absence of convection already discussed above. It can be also inferred from the solution that an increase in the angular velocity $\Omega$ gives rise to the inhibition of the onset of convection discussed by Chandrasekhar in his paper [1].

Finally, it has been of our interest to apply the solution obtained to the case of the long-lived large-scale vortex structure of the Great Red Spot of Jupiter. The results obtained for deformation of the upper surface $h(x, y)$ is close to the 1D results obtained numerically by Tikhomolv in his paper [14]. The only discrepancy is in amplitude of $h$ due to the use of different data for the non-dimensional parameters. By virtue of obtaining $h$ and the existence of the relationship between $h$ and the averaged dominant terms of non-dimensional components of the velocity fields, namely, $u^{(0)}$, $v^{(0)}$ and $w^{(0)}$,

The discrepancy between the time period obtained and the age of the Great Red Spot of Jupiter might be explained due to the effects of the zonal flows on the Jovian vortex structures, whose effects have not been taken into account in our model. It has been proposed that the presence of zonal flows might have effects on the uniqueness and localization of the vortex structure of the Great Red Spot of Jupiter [12,14]. It is obvious that our model is not yet a complete model in order to fully describe the vortex structure of the Great Red spot of Jupiter. However, our model and the solutions obtained are capable of presenting the correlation between the heating of the fluid motion from the lower layers, which is one of the fundamental features of the Great Red spot of Jupiter [14], and the sustenance of the vortex structure.

# CHAPTER 6

# CONCLUSION

The system of equations for a large- scale long lived rotating layer of fluid with the deformable upper free surface and non-deformable lower free surface has been reviewed and derived. The quasi-geostrophic approximation, the beta effect and the method of multi-scale expansions have been employed to simplify the system of equations and boundary conditions and as a result an equation governing the evolution of large-scale perturbations, has been derived. The effect of each term present in the upper surface deformation equation has been analyzed and the analytical solutions have obtained by virtue of employing auxiliary Riccati equation method [11]. The soliton solutions obtained for the deformation of upper surface, namely, $h(x, y)$ contributes to the sustenance of the vortex structure of the long-lived rotating layer of fluid due to the existence of two terms namely the nonlinear term or the so-called beta effect and the diffusion term resulted from the presence of heating energy from below.

The solution obtained, has been also applied to the case of long-lived vortex structure of the Great red spot of Jupiter and the results for the large-scale perturbations and averaged dominant terms of non-dimensional components of the velocity fields have been presented.

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# APPENDIX A

## The Static Solutions for Governing Equations

Given Eqs. (2.4) - (2.8), we obtained Eqs. (2.11) and (2.12) as follows:

$$\frac{\nabla p_s}{\rho_0} = \boldsymbol{g} + g\alpha T_s \boldsymbol{k}, \tag{A.1}$$

$$\Delta T_s = 0. \tag{A.2}$$

By taking curl of both sides of Eq. (A.1) and using the relation that the curl of the gradient is equal to zero we have

$$\frac{\partial T_s}{\partial x} = \frac{\partial T_s}{\partial y} = 0\,. \tag{A.3}$$

In other words, $T_s$ depends only on the vertical coordinate z. Given Eq. (A.2), we conclude that $T_s$ is of the form

$$T_s = -\,l_1 z + l_2\,, \tag{A.4}$$

where $l_1$ and $l_2$ are constants and can be determined by applying the boundary conditions (2.9) and (2.10) in the absence of perturbations at the boundaries as follows:

$$at\ z = 0: \quad T_s - T_0 = T_b \quad \Longrightarrow \quad l_2 = T_b + T_0\,, \tag{A.5}$$

$$at\ z = H: \quad T_s - T_0 = T_u \quad \Longrightarrow \quad l_1 = \frac{T_b - T_u}{H} = \frac{\Theta}{H}. \tag{A.6}$$

Assuming $l_1$ as a positive value implies that the fluids heated from below and therefore, the static temperature at the lower boundary is higher than the one at the upper boundary. Given Eqs. (A.4), (A.5) and (A.6), $T_s$ is of the form

$$T_s = -\frac{\Theta}{H}z + T_b + T_0\,, \tag{A.7}$$

where $\Theta = T_b - T_u$ .

By substituting $T_s$ from the equation (A.7) into the equation (A.1), we obtain

$$\frac{\nabla p_s}{\rho_0} = g\left[\alpha\left(-\frac{\Theta}{H}z + T_b + T_0\right) - 1\right]\boldsymbol{k}. \tag{A.8}$$

It can be inferred from the above equation that $p_s$ depends only the vertical coordinate z. Therefore, Eq. (A.8) changes to the form of

$$\frac{1}{\rho_0}\frac{dp_s}{dz} = g\left[\alpha\left(-\frac{\Theta}{H}z + T_b + T_0\right) - 1\right]. \tag{A.9}$$

Integrating the above equation from 0 to z, given that $at\ z = 0:\ p_s = p_b$ , we have

$$p_s = p_b - \rho_0 g\left[\alpha\frac{\Theta}{2H}z^2 - \alpha(T_b + T_0)z + z\right]. \tag{A.10}$$

## APPENDIX B

## The Derivation of the Equations (2.47) - (2.49)

We substitute $w$ in Eqs. (2.37) and (2.38) by $w^{(0)}$ from Eq. (3.43). Therefore, we obtain

$$\frac{1}{PD}\frac{\partial u}{\partial t}+\frac{1}{PD}\left(uu_x+vu_y\right)-(1+\beta y)v=-\frac{p_x}{D}+\frac{1}{D}\Delta u\,, \tag{B.1}$$

$$\frac{1}{PD}\frac{\partial v}{\partial t}+\frac{1}{PD}\left(uv_x+vv_y\right)+(1+\beta y)u=-\frac{p_y}{D}+\frac{1}{D}\Delta v\,, \tag{B.2}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

Taking derivative of Eqs. (B.1) and (B.2) with respect to $y$ and $x$ respectively, and subtracting Eq. (B.1) from Eq. (B.2), we obtain

$$\frac{1}{PD}\left(v_{xt}-u_{yt}\right)+\frac{1}{PD}\left[u_x\left(v_x-u_y\right)+u\left(v_x-u_y\right)_x+v_y\left(v_x-u_y\right)+v\left(v_x-u_y\right)_y\right]$$
$$+u_x+v_y+\beta y\left(u_x+v_y\right)+\beta v=\frac{1}{D}\Delta\left(v_x-u_y\right). \tag{B.3}$$

Given Eqs. (2.44) and (2.45), we conclude

$$v^{(0)}{}_x-\;u^{(0)}{}_y=\frac{1}{D}\left(p_{xx}+p_{yy}\right)=\frac{1}{D}\Delta_2 p\,, \tag{B.4}$$

where $\Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Approximating $v_x - u_y$ , $u$ and $v$ in Eq. (B.3) by $v^{(0)}{}_x - u^{(0)}{}_y$ , $u^{(0)}$ and $v^{(0)}$ from Eqs. (B.4), (2.44) and (2.45) respectively, yields

$$\frac{1}{PD^2}\Delta_2 p_t + \frac{1}{PD}\left[(u_x + v_y)\frac{\Delta_2 p}{D} - \frac{p_y \Delta_2 p_x}{D^2} + \frac{p_x \Delta_2 p_y}{D^2}\right] + (1+\beta y)(u_x + v_y) + \beta\frac{p_x}{D}$$

$$= \frac{1}{D^2}\Delta(\Delta_2 p)\,. \qquad (B.5)$$

By virtue of using the continuity equation (2.41) and the assumption $\beta y \ll 1$ from Eq. (2.3), Eq. (B.5) reduces to

$$\frac{1}{PD^2}(\Delta_2 p_t - w_z \Delta_2 p) + \frac{1}{PD}\left[+\frac{p_x \Delta_2 p_y}{D^2} - \frac{p_y \Delta_2 p_x}{D^2}\right] - w_z + \beta\frac{p_x}{D} = \frac{1}{D^2}\Delta(\Delta_2 p)\,. \qquad (B.6)$$

We now use the assumption that $w_z \Delta_2 p \ll \Delta_2 p_t$. Hence, Eq. (B.6) simplifies to

$$\frac{1}{PD^2}\Delta_2 p_t + \frac{1}{PD^3}J(p, \Delta_2 p) - w_z + \beta\frac{p_x}{D} = \frac{1}{D^2}\Delta(\Delta_2 p)\,, \qquad (B.7)$$

where $J(f, g) = f_x g_y - f_y g_x$

Eq. (B.7) might be written in the form of

$$w_z - \frac{1}{PD^2}\Delta_2 p_t - \frac{1}{PD^3}J(p, \Delta_2 p) - \beta\frac{p_x}{D} + \frac{1}{D^2}{\Delta_2}^2 p + \frac{1}{D^2}\Delta_2 p_{zz} = 0\,, \qquad (B.8)$$

where ${\Delta_2}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ .

The Eq. (B.8) is exactly same as Eq. (2.47). Eq. (2.48) is easily derived from Eq. (2.43) by projecting Eq. (2.43) onto the z-axis. Therefore, we derive

$$p_z = RT. \qquad (B.9)$$

It is worth noting that Eq. (B.9) is akin to the hydrostatic equation commonly used in geophysical fluid dynamics [19].

In order to derive Eq. (2.49), $u$ and $v$ in Eq. (2.40) are substituted by $u^{(0)}$ and $v^{(0)}$ from Eqs. (2.44) and (2.45) respectively. Hence, we obtain

$$\frac{\partial T}{\partial t} + \frac{1}{D}\left(p_x\, T_y - p_y\, T_x\right) + wT_z - w = \Delta T. \tag{B.10}$$

We substitute $w$ in the nonlinear term of the above equation by $w^{(0)}$ from Eq. (2.46). However, $w$ in the linear term might be kept. Therefore, Eq. (B.10) is of the form

$$\frac{\partial T}{\partial t} + \frac{1}{D}J(p,T) - w = \Delta T, \tag{B.11}$$

or:

$$T_t + \frac{1}{D}J(p,T) - w = \Delta_2 T + T_{zz}. \tag{B.12}$$

# APPENDIX C

## The Derivation of the Equation (2.55)

Let us rewrite the transformation (2.54), proposed to define the slow variables, as follows:

$$X = \varepsilon x\,, \qquad Y = \varepsilon y\,, \qquad \tau = \varepsilon^2 t\,, \tag{C.1}$$

or:

$$\frac{\partial X}{\partial x} = \varepsilon\,, \qquad \frac{\partial Y}{\partial y} = \varepsilon\,, \qquad \frac{\partial \tau}{\partial \tau} = \varepsilon^2. \tag{C.2}$$

We now define an arbitrary function F, where $F = F(x, y, t, X, Y, \tau)$. Differentiating F with respect to $x$, $y$ and $\tau$ yields

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial X}{\partial x}\frac{\partial F}{\partial X} = \frac{\partial F}{\partial x} + \varepsilon\frac{\partial F}{\partial X} = \left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X}\right)F. \tag{C.3}$$

In the same way, we obtain

$$\frac{\partial F}{\partial y} = \left(\frac{\partial}{\partial y} + \varepsilon\frac{\partial}{\partial Y}\right)F, \tag{C.4}$$

$$\frac{\partial F}{\partial t} = \left(\frac{\partial}{\partial t} + \varepsilon^2\frac{\partial}{\partial \tau}\right)F. \tag{C.5}$$

Let us differentiate both sides of Eq. (C.3) with respect to $x$. Therefore, Eq. yield

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x} + \varepsilon\frac{\partial F}{\partial X}\right) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial X}{\partial x}\frac{\partial^2 F}{\partial X \partial x} + \varepsilon\frac{\partial^2 F}{\partial x \partial X} + \varepsilon\frac{\partial X}{\partial x}\frac{\partial^2 F}{\partial X^2}, \qquad (C.6)$$

or:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial x^2} + 2\varepsilon\frac{\partial^2 F}{\partial x \partial X} + \varepsilon^2\frac{\partial^2 F}{\partial X^2} = \left(\frac{\partial^2}{\partial x^2} + 2\varepsilon\frac{\partial^2}{\partial x \partial X} + \varepsilon^2\frac{\partial^2}{\partial X^2}\right)F. \qquad (C.7)$$

In the same way, we obtain

$$\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2}{\partial y^2} + 2\varepsilon\frac{\partial^2}{\partial y \partial Y} + \varepsilon^2\frac{\partial^2}{\partial Y^2}\right)F. \qquad (C.8)$$

The Eqs. (C.7) and (C.8) result in:

$$\Delta_2 F = \Delta_2 F + 2\varepsilon\left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right)F + \varepsilon^2 \Delta_{2X} F, \qquad (C.9)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

Let us now differentiate both sides of Eq. (C.7) with respect to $x$. Therefore, Eq. (C.7) yields

$$\frac{\partial^3 F}{\partial x^3} = \frac{\partial^3 F}{\partial x^3} + \frac{\partial X}{\partial x}\frac{\partial^3 F}{\partial X \partial x^2} + 2\varepsilon\frac{\partial^3 F}{\partial x^2 \partial X} + 2\varepsilon\frac{\partial X}{\partial x}\frac{\partial^3 F}{\partial X^2 \partial x} + \varepsilon^2\frac{\partial^3 F}{\partial x \partial X^2} + \varepsilon^2\frac{\partial X}{\partial x}\frac{\partial^3 F}{\partial X^3} \qquad (C.10)$$

or:

$$\frac{\partial^3 F}{\partial x^3} = \frac{\partial^3 F}{\partial x^3} + 3\varepsilon\frac{\partial^3 F}{\partial x^2 \partial X} + 3\varepsilon^2\frac{\partial^3 F}{\partial x \partial X^2} + \varepsilon^3\frac{\partial^3 F}{\partial X^3}, \qquad (C.11)$$

or:

$$\frac{\partial^3 F}{\partial x^3} = \left(\frac{\partial^3}{\partial x^3} + 3\varepsilon\frac{\partial^3}{\partial x^2 \partial X} + 3\varepsilon^2\frac{\partial^3}{\partial x \partial X^2} + \varepsilon^3\frac{\partial^3}{\partial X^3}\right)F, \qquad (C.12)$$

In the same way, we obtain

$$\frac{\partial^3 F}{\partial y^3} = \left(\frac{\partial^3}{\partial y^3} + 3\varepsilon\frac{\partial^3}{\partial y^2 \partial Y} + 3\varepsilon^2\frac{\partial^3}{\partial y \partial Y^2} + \varepsilon^3\frac{\partial^3}{\partial Y^3}\right)F\,. \qquad (C.13)$$

By virtue of differentiating both sides of Eq. (C.11) with respect to $x$, we obtain

$$\frac{\partial^4 F}{\partial x^4} = \frac{\partial^4 F}{\partial x^4} + \frac{\partial X}{\partial x}\frac{\partial^4 F}{\partial X \partial x^3} + 3\varepsilon\frac{\partial^4 F}{\partial x^3 \partial X} + 3\varepsilon\frac{\partial X}{\partial x}\frac{\partial^4 F}{\partial X^2 \partial x^2} + 3\varepsilon^2\frac{\partial^4 F}{\partial x^2 \partial X^2} + 3\varepsilon^2\frac{\partial X}{\partial x}\frac{\partial^4 F}{\partial X^3 \partial x}$$
$$+ \varepsilon^3\frac{\partial^4 F}{\partial x \partial X^3} + \varepsilon^3\frac{\partial X}{\partial x}\frac{\partial^4 F}{\partial X^4}, \qquad (C.14)$$

or:

$$\frac{\partial^4 F}{\partial x^4} = \frac{\partial^4 F}{\partial x^4} + 4\varepsilon\frac{\partial^4 F}{\partial x^3 \partial X} + 6\varepsilon^2\frac{\partial^4 F}{\partial x^2 \partial X^2} + 4\varepsilon^3\frac{\partial^4 F}{\partial x \partial X^3} + \varepsilon^4\frac{\partial^4 F}{\partial X^4}, \qquad (C.15)$$

or:

$$\frac{\partial^4 F}{\partial x^4} = \left(\frac{\partial^4}{\partial x^4} + 4\varepsilon\frac{\partial^4}{\partial x^3 \partial X} + 6\varepsilon^2\frac{\partial^4}{\partial x^2 \partial X^2} + 4\varepsilon^3\frac{\partial^4}{\partial x \partial X^3} + \varepsilon^4\frac{\partial^4}{\partial X^4}\right)F. \qquad (C.16)$$

In the same fashion, we obtain

$$\frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^4}{\partial y^4} + 4\varepsilon\frac{\partial^4}{\partial y^3 \partial Y} + 6\varepsilon^2\frac{\partial^4}{\partial y^2 \partial Y^2} + 4\varepsilon^3\frac{\partial^4}{\partial y \partial Y^3} + \varepsilon^4\frac{\partial^4}{\partial Y^4}\right)F\,. \qquad (C.17)$$

Given Eqs. (C.7) and (C.8), the term $\frac{\partial^4 F}{\partial x^2 \partial y^2}$ is of the form

$$\frac{\partial^4 F}{\partial x^2 \partial y^2} = \left(\frac{\partial^2}{\partial x^2} + 2\varepsilon\frac{\partial^2}{\partial x \partial X} + \varepsilon^2\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial^2}{\partial y^2} + 2\varepsilon\frac{\partial^2}{\partial y \partial Y} + \varepsilon^2\frac{\partial^2}{\partial Y^2}\right)F\,, \qquad (C.18)$$

or:

$$\frac{\partial^4 F}{\partial x^2 \partial y^2} = \left(\frac{\partial^4}{\partial x^2 \partial y^2} + 2\varepsilon \frac{\partial^4}{\partial x^2 \partial y \partial Y} + \varepsilon^2 \frac{\partial^4}{\partial x^2 \partial Y^2} + 2\varepsilon \frac{\partial^4}{\partial x \partial X \partial y^2} + 4\varepsilon^2 \frac{\partial^4}{\partial x \partial X \partial y \partial Y} + 2\varepsilon^3 \frac{\partial^4}{\partial x \partial X \partial Y^2} + \varepsilon^2 \frac{\partial^4}{\partial X^2 \partial y^2} + 2\varepsilon^3 \frac{\partial^4}{\partial X^2 \partial y \partial Y} + \varepsilon^4 \frac{\partial^4}{\partial X^2 \partial Y^2}\right) F, \qquad (C.19)$$

Given Eqs. (C.16), (C.17) and (C.18), we obtain the following equation:

$$\Delta_2{}^2 F = \left[\Delta_2{}^2 + 4\varepsilon\left(\frac{\partial^4}{\partial x^3 \partial X} + \frac{\partial^4}{\partial y^3 \partial Y}\right) + 6\varepsilon^2\left(\frac{\partial^4}{\partial x^2 \partial X^2} + \frac{\partial^4}{\partial y^2 \partial Y^2}\right) + 4\varepsilon^3\left(\frac{\partial^4}{\partial x \partial X^3} + \frac{\partial^4}{\partial y \partial Y^3}\right) + 4\varepsilon\left(\frac{\partial^4}{\partial x \partial y^2 \partial X} + \frac{\partial^4}{\partial x^2 \partial y \partial Y}\right) + 2\varepsilon^2\left(\frac{\partial^4}{\partial y^2 \partial X^2} + 4\frac{\partial^4}{\partial x \partial y \partial X \partial Y} + \frac{\partial^4}{\partial x^2 \partial Y^2}\right) + 4\varepsilon^3\left(\frac{\partial^4}{\partial y \partial X^2 \partial Y} + \frac{\partial^4}{\partial x \partial X \partial Y^2}\right) + \varepsilon^4 \Delta_{2X}{}^2\right] F, \qquad (C.20)$$

where $\Delta_2{}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ and $\Delta_{2X}{}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$.

# APPENDIX D

## The Derivation of the Equations (2.64) - (2.66)

Starting with Eq. (2.48) and rewriting it as follows:

$$p_z = RT. \qquad (D.1)$$

We substitute Eqs. (2.57), (2.59) and (2.61), which are corresponding to the temperature, pressure and the Rayleigh number expansions respectively, into the above equation. Therefore, Eq. (D.1) is of the form

$$\frac{\partial}{\partial z}\left(p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \varepsilon^3 p^{(3)} + \cdots\right) = (R_{cr} + \varepsilon^2 R_2)\varepsilon^2\left(\vartheta^{(0)} + \varepsilon\vartheta^{(1)} + \varepsilon^2\vartheta^{(2)} + \cdots\right) \quad (D.2)$$

where $p^{(0)}$ is of the form $p^{(0)}(X, Y, \tau)$.

Eq. (D.2) results in

$$p_z{}^{(0)} = 0, \qquad at\ the\ leading\ order\ or\ \varepsilon^0 \qquad (D.3)$$

$$p_z{}^{(1)} = 0, \qquad at\ the\ first\ order\ or\ \varepsilon^1 \qquad (D.4)$$

$$p_z{}^{(2)} = R_{cr}\vartheta^{(0)} \qquad at\ the\ second\ order\ or\ \varepsilon^2 \qquad (D.5)$$

$$p_z{}^{(3)} = R_{cr}\vartheta^{(1)} \qquad at\ the\ third\ order\ or\ \varepsilon^3 \qquad (D.6)$$

$$p_z{}^{(4)} = R_{cr}\vartheta^{(2)} + R_2\vartheta^{(0)} \qquad at\ the\ fourth\ order\ or\ \varepsilon^4 \qquad (D.7)$$

Eq. (D.5) is exactly same as Eq. (2.65). Eq. (C.4) implies that $p^{(1)}$ is independent of the vertical coordinate $z$ and is of the form $p^{(1)}(X, Y, x, y, \tau, t)$. For the sake of simplicity, we now assume that $p^{(1)}$, same as $p^{(0)}$, is just dependent of the slow variables. Therefore, $p^{(1)}$ might be in the form of $p^{(1)}(X, Y, \tau)$ .

Also, from Eq. (2.57) it is assumed that $\vartheta^{(0)} = \vartheta^{(0)}(X, Y, z, \tau)$. Thus, it can be inferred from Eq. (D.5) that $p^{(2)}$ is of the form $p^{(2)}(X, Y, z, \tau)$.

Given the upper boundary condition at which $p = qh$ and $T = h$, let us substitute the deformation, temperature and pressure expansions presented by Eqs. (2.56), (2.57) and (2.59) respectively into the above boundary conditions. Therefore, it results in

$$at\ z = 1 + \varepsilon^2\mathcal{H}^{(0)}: \quad p^{(0)}(X, Y, \tau) = \eta\mathcal{H}^{(0)}, \qquad \vartheta^{(0)} = \mathcal{H}^{(0)}, \tag{D.8}$$

$$at\ z = 1 + \varepsilon^2\mathcal{H}^{(0)}: \quad p^{(1)}(X, Y, \tau) = \eta\mathcal{H}^{(1)}, \qquad \vartheta^{(1)} = \mathcal{H}^{(1)}, \tag{D.9}$$

$$at\ z = 1 + \varepsilon^2\mathcal{H}^{(0)}: \quad p^{(2)}(X, Y, \tau) = \eta\mathcal{H}^{(2)}, \qquad \vartheta^{(2)} = \mathcal{H}^{(2)}, \tag{D.10}$$

where $\eta$ is a constant, approximated to $\eta \approx \varepsilon^2 q$ , and $q = \frac{gH^3}{\nu\kappa}$ .

It might be inferred from Eqs. (D.8), (D.9) and (D.10) that $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are just functions of the slow variables same as $\vartheta^{(0)}$. Therefore, we can conclude from Eqs. (D.6) and (D.7) that $p^{(3)}$and $p^{(4)}$ are only dependent of the slow variables and might be presented in the form of

$$p^{(3)} = p^{(3)}(X, Y, z, \tau) \quad and \quad p^{(4)} = p^{(4)}(X, Y, z, \tau)\,. \tag{D.11}$$

Consequently, $p$ is of the form

$$p = p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \varepsilon^2 p^{(2)}(X,Y,z,\tau) + \varepsilon^3 p^{(3)}(X,Y,z,\tau) + \varepsilon^4 p^{(4)}(X,Y,z,\tau) + \cdots, \qquad (D.12)$$

Let us now rewrite Eq. (2.47) as follows:

$$w_z - \frac{1}{PD^2}\Delta_2 p_t - \frac{1}{PD^3}J(p,\Delta_2 p) - \beta\frac{p_x}{D} + \frac{1}{D^2}{\Delta_2}^2 p + \frac{1}{D^2}\Delta_2 p_{zz} = 0\,. \qquad (D.13)$$

Employing the transformations presented in Eq. (2.55), we substitute Eqs. (2.58), (3.59) and (2.60), which are corresponding to the vertical velocity $w$, pressure and $\beta$ expansions respectively, into Eq. (D.13). Therefore, each term in Eq.(D.13) is of the form

$$w_z = \frac{\partial}{\partial z}\left(\varepsilon^4 W^{(0)}(X,Y,z,\tau) + \varepsilon^5 W^{(1)} + \varepsilon^6 W^{(2)} + \cdots\right), \qquad (D.14)$$

or:

$$w_z = \frac{\partial}{\partial z} W^{(0)}(X,Y,z,\tau) \qquad \textit{at the fourth order or } \varepsilon^4. \qquad (D.15)$$

It is noted that at the orders lower than the fourth order $w_z$ is set to zero.

Given the expansion (D.12) for $p$ and the transformations presented in Eq. (2.55), the term $\Delta_2 p_t$ is of the form

$$\Delta_2 p_t = \left[\Delta_2 + 2\varepsilon\left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right) + \varepsilon^2 \Delta_{2X}\right]\left(\frac{\partial}{\partial t} + \varepsilon^2\frac{\partial}{\partial \tau}\right)\left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \varepsilon^2 p^{(2)}(X,Y,z,\tau) + \varepsilon^3 p^{(3)} + \cdots\right], \qquad (D.16)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

Eq. (D.16) yields

$$\Delta_2 p_t = \varepsilon^4 \Delta_{2X} \frac{\partial}{\partial \tau} p^{(0)}(X,Y,\tau) + \varepsilon^5 \Delta_{2X} \frac{\partial}{\partial \tau} p^{(1)}(X,Y,\tau) + \cdots, \qquad (D.17)$$

or:

$$\Delta_2 p_t = \varepsilon^4 \Delta_{2X} p_\tau{}^{(0)}(X,Y,\tau) \qquad \text{at the fourth order or } \varepsilon^4. \qquad (D.18)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

Given $J(f,g) = f_x g_y - f_y g_y$ and the expansion (D.12) for $p$, the term $J(p, \Delta_2 p)$ is of the form

$$\begin{aligned} J(p,\Delta_2 p) = \Bigg[ & \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\right) \left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \cdots\right] \left(\Delta_2 + 2\varepsilon \left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right)\right. \\ & \left. + \ \varepsilon^2 \Delta_{2X}\right) \left(\frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\right) \left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \cdots\right] \\ & - \left(\frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\right) \left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \cdots\right] \Bigg[\Delta_2 + 2\varepsilon \left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right) \\ & + \ \varepsilon^2 \Delta_{2X}\Bigg] \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\right) \left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \cdots\right] \Bigg], \end{aligned} \qquad (D.19)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

Eq. (D.19) yields

$$\begin{aligned} J(p,\Delta_2 p) = \varepsilon^4 & \left[\left(\frac{\partial}{\partial X} \Delta_{2X} \frac{\partial}{\partial Y}\right) - \left(\frac{\partial}{\partial Y} \Delta_{2X} \frac{\partial}{\partial X}\right)\right] p^{(0)}(X,Y,\tau) \\ + \varepsilon^5 & \left[\left(\frac{\partial}{\partial X} \Delta_{2X} \frac{\partial}{\partial Y}\right) - \left(\frac{\partial}{\partial Y} \Delta_{2X} \frac{\partial}{\partial X}\right)\right] p^{(1)}(X,Y,\tau) + \cdots, \end{aligned} \qquad (D.20)$$

or:

$$J(p,\Delta_2 p) = \varepsilon^4 J_X\left(p^{(0)}, \Delta_{2X} p^{(0)}\right) \qquad \text{at the fourth order or } \varepsilon^4, \qquad (D.21)$$

where $J_X(f,g) = f_X g_Y - f_Y g_X$ , $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ .

Given the transformations presented in Eq. (2.55) and the expansion (D.12) for pressure, the term ${\Delta_2}^2 p$ is of the form

$$
{\Delta_2}^2 p = \Big[{\Delta_2}^2 + 4\varepsilon\left(\frac{\partial^4}{\partial x^3 \partial X} + \frac{\partial^4}{\partial y^3 \partial Y}\right) + 6\varepsilon^2\left(\frac{\partial^4}{\partial x^2 \partial X^2} + \frac{\partial^4}{\partial y^2 \partial Y^2}\right) + 4\varepsilon^3\left(\frac{\partial^4}{\partial x \partial X^3} + \frac{\partial^4}{\partial y \partial Y^3}\right) + 4\varepsilon\left(\frac{\partial^4}{\partial x \partial y^2 \partial X} + \frac{\partial^4}{\partial x^2 \partial y \partial Y}\right) + 2\varepsilon^2\left(\frac{\partial^4}{\partial y^2 \partial X^2} + 4\frac{\partial^4}{\partial x \partial y \partial X \partial Y} + \frac{\partial^4}{\partial x^2 \partial Y^2}\right) + 4\varepsilon^3\left(\frac{\partial^4}{\partial y \partial X^2 \partial Y} + \frac{\partial^4}{\partial x \partial X \partial Y^2}\right) + \varepsilon^4 {\Delta_{2X}}^2\Big]\left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \varepsilon^2 p^{(2)}(X,Y,z,\tau) + \varepsilon^3 p^{(3)} + \cdots\right], \qquad (D.22)
$$

where ${\Delta_2}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ and ${\Delta_{2X}}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$ .

Eq. (D.22) yields

$$
{\Delta_2}^2 p = \varepsilon^4 {\Delta_{2X}}^2 p^{(0)}(X,Y,\tau) + \varepsilon^5 {\Delta_{2X}}^2 p^{(1)}(X,Y,\tau) + \cdots , \qquad (D.23)
$$

or:

$$
{\Delta_2}^2 p = \varepsilon^4 {\Delta_{2X}}^2 p^{(0)}(X,Y,\tau) \qquad \textit{at the fourth order or } \varepsilon^4, \qquad (D.24)
$$

where ${\Delta_2}^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}$ and ${\Delta_{2X}}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$ .

Given the transformations presented in Eq. (2.55) and the expansion (D.12) for $p$, the term $\Delta_2 p_{zz}$ is of the form

$$
\Delta_2 p_{zz} = \left[\Delta_2 + 2\varepsilon\left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right) + \varepsilon^2 \Delta_{2X}\right]\left(\frac{\partial^2}{\partial z^2}\right)\left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \varepsilon^2 p^{(2)}(X,Y,z,\tau) + \varepsilon^3 p^{(3)}(X,Y,z,\tau) + \varepsilon^4 p^{(4)}(X,Y,z,\tau) + \cdots\right]. \qquad (D.25)
$$

Eq. (D.25) is simplified to

$$\Delta_2 p_{zz} = \varepsilon^4 \Delta_{2X} p_{zz}{}^{(2)}(X,Y,z,\tau) + \varepsilon^5 \Delta_{2X} p_{zz}{}^{(3)}(X,Y,z,\tau) + \cdots, \qquad (D.26)$$

or:

$$\Delta_2 p_{zz} = \varepsilon^4 \Delta_{2X} p_{zz}{}^{(2)}(X,Y,z,\tau) \qquad at\ the\ fourth\ order\ or\ \varepsilon^4, \qquad (D.27)$$

where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

Given the transformations presented in Eq. (2.55), the expansion (D.12) for $p$ and the expansion for $\beta$ from Eq. (2.60), the term $\beta p_x$ is of the form

$$\beta p_x = \left[\varepsilon^3 (B^{(0)} + \varepsilon B^{(1)} + \cdots)\right]\left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\right)\left[p^{(0)}(X,Y,\tau) + \varepsilon p^{(1)}(X,Y,\tau) + \cdots\right] \quad (D.28)$$

Eq. (D.28) yields

$$\beta p_x = \varepsilon^4 B^{(0)} p_X{}^{(0)}(X,Y,\tau) + \varepsilon^5\left[B^{(1)} p_X{}^{(0)}(X,Y,\tau) + B^{(0)} p_X{}^{(1)}(X,Y,\tau)\right] + \cdots. \quad (D.29)$$

or:

$$\beta p_x = \varepsilon^4 B^{(0)} p_X{}^{(0)}(X,Y,\tau) \qquad at\ the\ fourth\ order\ or\ \varepsilon^4, \qquad (D.30)$$

where $p_x = \frac{\partial p}{\partial x}$ and $p_X{}^{(0)} = \frac{\partial p^{(0)}}{\partial X}$.

We now substitute Eqs. (D.15), (D.18), (D.21), (D.24), (D.27) and (D.30) into Eq. (D.13). Therefore, Eq. (D.13) is of the form

$$W_z{}^{(0)} - \frac{1}{PD^2}\Delta_{2X} p_\tau{}^{(0)} - \frac{1}{PD^3} J_X\left(p^{(0)}, \Delta_{2X} p^{(0)}\right) - B^{(0)} \frac{p_X{}^{(0)}}{D} + \frac{1}{D^2}\Delta_{2X}{}^2 p^{(0)}$$

$$+ \frac{1}{D^2}\Delta_{2X} p_{zz}{}^{(2)} = 0, \qquad at\ the\ fourth\ order\ or\ \varepsilon^4 \qquad (D.31)$$

Eq. (D.31) is exactly same as Eq. (2.64). In order to derive the last governing equation, Eq. (2.66), let us rewrite Eq. (2.49) as below:

$$T_t + \frac{1}{D} J(p, T) - w = \Delta_2 T + T_{zz} \,. \tag{D.32}$$

It is worth noting that from Eqs. (D.9) and (D.10) might be inferred

$$\vartheta^{(1)} = \vartheta^{(1)}(X, Y, z, \tau) \quad and \quad \vartheta^{(2)} = \vartheta^{(2)}(X, Y, z, \tau) \,. \tag{D.33}$$

In other words, $\vartheta^{(1)}$ and $\vartheta^{(1)}$ are only dependent of the slow variables. Given the expansion equation for $T$, Eq. (2.57), it might be written in the form of

$$T = \varepsilon^2 \left[ \vartheta^{(0)}(X, Y, z, \tau) + \varepsilon \vartheta^{(1)}(X, Y, z, \tau) + \varepsilon^2 \vartheta^{(2)}(X, Y, z, \tau) + \cdots \right], \tag{D.34}$$

Given the transformations presented in Eq. (2.55), We substitute $T$, $p$ and $w$ from Eqs. (D.34), (D.12) and (2.58) respectively, into Eq. (D.32). Therefore, Eq. (D.32) is of the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau}\right) \left[\varepsilon^2 \vartheta^{(0)}(X, Y, z, \tau) + \varepsilon^3 \vartheta^{(1)}(X, Y, z, \tau) + \cdots\right] \\
&\quad + \frac{1}{D} \Bigg[ \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\right) \left[p^{(0)}(X, Y, \tau) + \varepsilon p^{(1)}(X, Y, \tau) + \cdots\right] \left(\frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\right) \left[\varepsilon^2 \vartheta^{(0)}(X, Y, z, \tau)\right. \\
&\quad \left. + \varepsilon^3 \vartheta^{(1)}(X, Y, z, \tau) + \cdots\right] \\
&\quad - \left(\frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\right) \left[p^{(0)}(X, Y, \tau) + \varepsilon p^{(1)}(X, Y, \tau) + \cdots\right] \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X}\right) \left[\varepsilon^2 \vartheta^{(0)}(X, Y, z, \tau)\right. \\
&\quad \left. + \varepsilon^3 \vartheta^{(1)}(X, Y, z, \tau) + \cdots\right] \Bigg] - \left[\varepsilon^4 W^{(0)}(X, Y, z, \tau) + \varepsilon^5 W^{(1)} + \cdots\right] \\
&\quad = \left[\Delta_2 + 2\varepsilon \left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right) + \varepsilon^2 \Delta_{2X}\right] \left[\varepsilon^2 \vartheta^{(0)}(X, Y, z, \tau) + \varepsilon^3 \vartheta^{(1)}(X, Y, z, \tau) + \cdots\right] \\
&\quad + \left(\frac{\partial^2}{\partial z^2}\right) \left[\varepsilon^2 \vartheta^{(0)}(X, Y, z, \tau) + \varepsilon^3 \vartheta^{(1)}(X, Y, z, \tau) + \cdots\right],
\end{aligned} \tag{D.35}$$

Eq. (D.35) yields

$$\varepsilon^2\, \vartheta_{zz}{}^{(0)}(X,Y,z,\tau) + \cdots = 0\,, \qquad (D.36)$$

or:

$$\vartheta_{zz}{}^{(0)} = 0\,, \qquad \textit{at the second order or } \varepsilon^2 \qquad (D.37)$$

Eqs. (D.31), (D.5) and (D.37) are exactly same as Eqs. (2.64), (2.65) and (2.66) respectively.

# APPENDIX E

## The Derivation of the Equation (3.10)

Let us rewrite Eq. (3.9) as follows:

$$\mathcal{H}_\tau - a\Delta_{2X}\mathcal{H}_\tau - eJ_X(\mathcal{H}, \Delta_{2X}\mathcal{H}) - b\mathcal{H}_X - \varepsilon^2 b\mathcal{H}\mathcal{H}_X + Pa{\Delta_{2X}}^2\mathcal{H} + c\Delta_{2X}\mathcal{H} = 0, (E.1)$$

where $a = \frac{\eta}{PD^2}$ , $b = \frac{B\eta}{D}$ , $c = \frac{R_{cr}}{D^2}$ , $e = \frac{\eta^2}{PD^3}$ , $\eta \approx \varepsilon^2 q$, $q = \frac{gH^3}{\nu\kappa}$ , $J_X(f, g) = f_X g_Y - f_Y g_X$, $\Delta_{2X} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ and ${\Delta_{2X}}^2 = \frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2}$ .

Using the transformation (3.2) results in

$$\mathcal{H}_\tau = \frac{\partial\mathcal{H}}{\partial\xi}\frac{\partial\xi}{\partial\tau} = \frac{d\mathcal{H}}{d\xi}(-\lambda) = -\lambda\mathcal{H}', \qquad (E.2)$$

$$\mathcal{H}_X = \frac{\partial\mathcal{H}}{\partial\xi}\frac{\partial\xi}{\partial X} = \frac{d\mathcal{H}}{d\xi} = \mathcal{H}', \qquad (E.3)$$

$$\mathcal{H}_{XX} = \frac{\partial\mathcal{H}_X}{\partial X} = \frac{\partial\mathcal{H}_X}{\partial\xi}\frac{\partial\xi}{\partial X} = \frac{d(\mathcal{H}')}{d\xi} = \mathcal{H}'', \qquad (E.4)$$

$$\mathcal{H}_{XX\tau} = \frac{\partial(\mathcal{H}_{XX})}{\partial\tau} = \frac{\partial(\mathcal{H}_{XX})}{\partial\xi}\frac{\partial\xi}{\partial\tau} = \frac{d(\mathcal{H}'')}{d\xi}(-\lambda) = -\lambda\mathcal{H}'''. \qquad (E.5)$$

In the same fashion, the following relations could be easily derived.

$$\mathcal{H}_{XXXX} = \mathcal{H}_{XXYY} = \mathcal{H}_{YYYY} = \mathcal{H}'''', \quad \mathcal{H}_Y = \mathcal{H}', \quad \mathcal{H}_{YY} = \mathcal{H}'',$$

$$\mathcal{H}_{YY\tau} = -\lambda\mathcal{H}''', \quad \mathcal{H}_{YYX} = \mathcal{H}_{XXY} = \mathcal{H}_{XXX} = \mathcal{H}_{YYY} = \mathcal{H}''' . \qquad (E.6)$$

Given the above relationships, $\Delta_{2X}\mathcal{H}, J_X(\mathcal{H}, \Delta_{2X}\mathcal{H})$, $\Delta_{2X}\mathcal{H}_\tau$ and ${\Delta_{2X}}^2\mathcal{H}$ are of the fom

$$\Delta_{2X}\mathcal{H} = \mathcal{H}_{XX} + \mathcal{H}_{YY} = 2\mathcal{H}'', \tag{E.7}$$

$$J_X(\mathcal{H}, \Delta_{2X}\mathcal{H}) = \mathcal{H}_X(\mathcal{H}_{XXY} + \mathcal{H}_{YYY}) - \mathcal{H}_Y(\mathcal{H}_{XXX} + \mathcal{H}_{YYX}) = 0\,, \tag{E.8}$$

$$\Delta_{2X}\mathcal{H}_\tau = \mathcal{H}_{XX\tau} + \mathcal{H}_{YY\tau} = -2\lambda\mathcal{H}''', \tag{E.9}$$

$${\Delta_{2X}}^2\mathcal{H} = \mathcal{H}_{XXXX} + \mathcal{H}_{YYYY} + 2\mathcal{H}_{XXYY} = 4\mathcal{H}''''. \tag{E.10}$$

Substituting Eqs. (E.2) - (E.10) into Eq. (E.1), yields

$$-\lambda\mathcal{H}' - a(-2\lambda\mathcal{H}''' \ ) - b\mathcal{H}' - \varepsilon^2 b\mathcal{H}\mathcal{H}' + 4Pa\mathcal{H}'''' + 2c\mathcal{H}'' = 0\,, \tag{E.11}$$

or:

$$(-\lambda - b)\mathcal{H}' + 2a\lambda\mathcal{H}''' - \varepsilon^2 b\mathcal{H}\mathcal{H}' + 4Pa\mathcal{H}'''' + 2c\mathcal{H}'' = 0\,. \tag{E.12}$$

# APPENDIX F

## Ten Sets of Solutions for the System of Equations (3.20) - (3.26)

We solve the above system of equations with the aid of Maple 15. The obtained 10 sets of solutions are as follows:

**The first set of solution**:

$$d_0 = -\frac{1}{\varepsilon^2}, \qquad d_1 = -\frac{180}{19}\frac{c}{\varepsilon^2 b}, \qquad d_2 = 0, \qquad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = \frac{1}{152}\frac{c}{Pa}, \qquad \lambda = 0, \qquad 1 + \frac{1800}{6859}\frac{c^3}{Pab^2} = 0. \qquad (F.1)$$

The above set of solutions is neglected since $q_0 > 0$. Also, the last equation cannot be satisfied given the assumption that $a$, $b$, $c$ and $P$ take only positive values.

**The second set of solutions**:

$$d_0 = -\frac{1}{\varepsilon^2}, \qquad d_1 = -\frac{540}{19}\frac{c}{\varepsilon^2 b}, \qquad d_2 = 0, \qquad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = -\frac{11}{152}\frac{c}{Pa}, \qquad \lambda = 0, \qquad 1 - \frac{11(1800)}{6859}\frac{c^3}{Pab^2} = 0. \qquad (F.2)$$

In the above set of solutions, which is the steady state set of solutions, $q_0 < 0$; However, the above set of solutions does not satisfy the assumption made in Eq. (3.12). In other words, the above set of solutions contradicts the following condition:

$$as \ \ \xi \to \infty : \qquad \mathcal{H} \to 0. \qquad (F.3)$$

Therefore, the above steady state set of solutions is neglected.

**The third set of solutions** (with the assumption that $\frac{c}{P} < 1$ ):

$$d_0 = -\frac{7c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \quad d_1 = 60\frac{c}{\varepsilon^2 b}, \quad d_2 = -\frac{120}{\varepsilon^2 b}\sqrt{2Pac}, \quad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = \frac{c}{8Pa}, \quad \lambda = -4\sqrt{\frac{2Pc}{a}}, \quad ab^2 = 32Pc\left(\frac{c}{P} - 1\right)^2. \qquad (F.4)$$

The above set of solutions is neglected since $q_0 > 0$.

**The fourth set of solutions**:

$$d_0 = -\frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \quad d_1 = 60\frac{c}{\varepsilon^2 b}, \quad d_2 = -\frac{120}{\varepsilon^2 b}\sqrt{2Pac}, \quad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = \frac{c}{8Pa}, \quad \lambda = -4\sqrt{\frac{2Pc}{a}}, \quad ab^2 = 32Pc\left(\frac{c}{P} + 1\right)^2. \qquad (F.5)$$

The above set of solutions is neglected since $q_0 > 0$ .

**The fifth set of solutions** (with the assumption that $\frac{3c}{2P} < 1$ ):

$$d_0 = \frac{15c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \quad d_1 = -60\frac{c}{\varepsilon^2 b}, \quad d_2 = -\frac{120}{\varepsilon^2 b}\sqrt{2Pac}, \quad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = -\frac{c}{8Pa}, \quad \lambda = -4\sqrt{\frac{2Pc}{a}}, \quad ab^2 = 32Pc\left(\frac{3c}{2P} - 1\right)^2. \qquad (F.6)$$

The above set of solutions is a desirable set of solutions since $q_0 < 0$ and it also satisfies the condition stated in Eq. (F.3).

**The sixth set of solutions**:

$$d_0 = \frac{3c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \qquad d_1 = -60\frac{c}{\varepsilon^2 b}, \qquad d_2 = -\frac{120}{\varepsilon^2 b}\sqrt{2Pac}\,, \qquad d_3 = 480P\frac{a}{\varepsilon^2 b},$$

$$q_0 = -\frac{c}{8Pa}, \qquad \lambda = -4\sqrt{\frac{2Pc}{a}}, \qquad ab^2 = 32Pc\left(\frac{3c}{2P}+1\right)^2. \qquad (F.7)$$

The above set of solutions is neglected since it does not satisfy the condition stated in Eq. (F.3).

**The seventh set of solutions** (with the assumption that $\frac{5c}{47P} < 1$ ):

$$d_0 = \frac{105\sqrt{47}}{47^2}\frac{c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \qquad d_1 = \frac{180}{47}\frac{c}{\varepsilon^2 b}, \qquad d_2 = -\frac{360\sqrt{47}}{47}\frac{\sqrt{2Pac}}{\varepsilon^2 b},$$

$$d_3 = 480P\frac{a}{\varepsilon^2 b}, \qquad q_0 = -\frac{c}{376Pa},$$

$$\lambda = -\frac{12\sqrt{47}}{47}\sqrt{\frac{2Pc}{a}}, \qquad ab^2 = \frac{288}{47}Pc\left(\frac{5c}{47P}-1\right)^2. \qquad (F.8)$$

The above set of solutions is neglected since it does not satisfy the condition stated in Eq. (F.3).

**The eighth set of solutions:**

$$d_0 = -\frac{15\sqrt{47}}{47^2}\frac{c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}\,, \qquad d_1 = \frac{180}{47}\frac{c}{\varepsilon^2 b}\,, \qquad d_2 = -\frac{360\sqrt{47}}{47}\frac{\sqrt{2Pac}}{\varepsilon^2 b}\,,$$

$$d_3 = 480P\frac{a}{\varepsilon^2 b}\,, \qquad q_0 = -\frac{c}{376Pa}\,,$$

$$\lambda = -\frac{12\sqrt{47}}{47}\sqrt{\frac{2Pc}{a}}\,, \qquad ab^2 = \frac{288}{47}Pc\left(\frac{5c}{47P}+1\right)^2. \qquad (F.9)$$

The above set of solutions is neglected since it does not satisfy the condition stated in Eq. (F.3).

**The ninth set of solutions** (with the assumption that $\frac{45c}{584P} < 1$ ):

$$d_0 = \frac{150\sqrt{73}}{73^2}\frac{c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}\,, \qquad d_1 = \frac{300}{73}\frac{c}{\varepsilon^2 b}\,, \qquad d_2 = \frac{480\sqrt{73}}{73}\frac{\sqrt{2Pac}}{\varepsilon^2 b}\,,$$

$$d_3 = 480P\frac{a}{\varepsilon^2 b}\,, \qquad q_0 = -\frac{c}{584Pa}\,,$$

$$\lambda = -\frac{16\sqrt{73}}{73}\sqrt{\frac{2Pc}{a}}\,, \qquad ab^2 = \frac{512}{73}Pc\left(\frac{45c}{584P}-1\right)^2. \qquad (F.10)$$

The above set of solutions is neglected since it does not satisfy the condition stated in Eq. (F.3).

**The tenth set of solutions:**

$$d_0 = -\frac{30\sqrt{73}}{73^2}\frac{c}{\varepsilon^2 b}\sqrt{\frac{2c}{Pa}}, \qquad d_1 = \frac{300}{73}\frac{c}{\varepsilon^2 b}, \qquad d_2 = \frac{480\sqrt{73}}{73}\frac{\sqrt{2Pac}}{\varepsilon^2 b},$$

$$d_3 = 480P\frac{a}{\varepsilon^2 b}, \qquad q_0 = -\frac{c}{584Pa},$$

$$\lambda = -\frac{16\sqrt{73}}{73}\sqrt{\frac{2Pc}{a}}, \qquad ab^2 = \frac{512}{73}Pc\left(\frac{45c}{584P}+1\right)^2. \qquad (F.11)$$

The above set of solutions is neglected since it does not satisfy the condition stated in Eq. (F.3).