ABSTRACT

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This dissertation contributes to the literature on the market design of auctions. I use computational and experimental techniques to make two types of contributions to the literature. First, I provide software that implements a state-of-the-art algorithm for solving multi-unit auctions with asymmetric bidders. This methodological contribution can be used by other economists to solve a variety of auction problems not considered in this dissertation. Second, I undertake the study of one auction environment in particular, utilizing my program to generate hypotheses when bidders participate in a particular sealed-bid, asymmetric multi-unit auction. These hypotheses are then tested in an experimental setting.

COMPUTATIONAL AND EXPERIMENTAL MARKET DESIGN

By

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Dedication

to Lauren and Destro.

Acknowledgements

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I would not have pursued my graduate education without the encouragement of my parents, Barbara and Duncan Higgins. I could not have asked for more love and support. Barbara and Duncan are entirely unique in my experience – they supported and encouraged my pursuits without ever imposing on me their own expectations. This is a nearly impossible trick. Most parents cannot help but encourage in their children actions that they believe will hasten success. Most parents do not recognize that confidence *in* their children is the best expression of love *for* them. If I can do the same with my children, I will feel that I have accomplished something very special.

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Chapter 1: Introduction

This dissertation contributes to the literature on the market design of auctions. I use computational and experimental techniques to make two types of contributions to the literature. First, I provide software that implements a state-of-the-art algorithm for solving multi-unit auctions with asymmetric bidders. This methodological contribution can be used by other economists to solve a variety of auction problems not considered in this dissertation. Second, I undertake the study of one auction environment in particular, utilizing my software to generate hypotheses when bidders participate in a particular sealed-bid, asymmetric multi-unit auction. These hypotheses are then tested in an experimental setting.

Broadly speaking, economists study auctions for three reasons: (1) because a substantial amount of commerce is organized by auctions; (2) because many forms of exchange can be modeled as auctions, especially when strategic behavior is important; and (3) to determine how auctions can be used to achieve a suite of design goals. The first two reasons to study auctions are purely positive, while the third develops techniques that can be used for normative purposes.

The motivation to study auctions from a positive perspective is obvious: billions of dollars of commerce are organized by auctions each year. Commodities such as eggs and tobacco (Sosnick (1963)), as well as differentiated goods such as wine and art (Ashenfelter (1989)), have been traded at auction for many hundreds of years. More recently, internet auctions have become important formats for the exchange of basic consumer goods (eBay, Yahoo!, etc.; Lucking-Reiley (2000)), and for business procurement and supply chain management (Elmaghraby (2000)). Basic market institutions such as the trading floor of a stock or commodity exchange can be usefully modeled as “double-auctions” (Friedman and Rust (1993)). Enduring research agendas have sought to determine why some trades are usually organized as auctions (Bulow and Klemperer (2009) and Wang (1993)), how well auctions aggregate information distributed over many agents (Wilson (1977)), in what circumstances the famed revenue equivalence theorem[1] fails to hold (Maskin and Riley (2000)), or simply how results derived in the study of auctions relates to traditional price theory (Bulow and Roberts (1989)).

The study of auctions from a mechanism design perspective considers how auctions can be used to obtain desirable outcomes such as optimal revenue or economic efficiency (Myerson (1981)), accurate price discovery (Hong and Shum (2004)), minimal winner’s curse (Levin, Kagel and Richard (1996)), etc. Indeed, auctions are not only manipulated to improve outcomes in existing markets, but are used in the *creation* of markets. Auctions of emissions permits, for example, allow economists to harness the power of markets to increase total social welfare by allocating property rights in an efficient way. The issues involved in the manufacture and manipulation of auctions are issues of *market design*, the field of economics concerning “The Economist as Engineer” (Roth (2002)).

Some of the more prominent applications of actively-designed auctions include the use of auctions to sell government debt (Back and Zender (1993)), to distribute property rights for natural resources (Cramton (2009)) or spectrum rights (McMillan (1994)), to encourage environmental conservation (Latacz-Lohmann and Van der

[1] The revenue equivalence theorem posits that in the benchmark model of auctions, the English, Dutch, first-price sealed-bid, and the second-price sealed-bid auctions all yield the same revenue to the seller. See McAfee and McMillan (1987).

Hamsvoort (1997)), and to price externalities (Joskow, Schmalensee and Bailey (1996)).

In each of the applications above, the designer of the auction is responsible for conceptualizing every facet of the market. As Al Roth puts it, "Market design involves a responsibility for detail, a need to deal with all of a market's complications, not just its principal features. Designers therefore cannot work only with the simple conceptual models used for theoretical insights into the general working of markets. Instead, market design calls for an engineering approach."[2] The design of an auction market in practice goes beyond simply choosing from a menu of available auctions. Instead, the design process includes defining the property right or good to be auctioned, specifying the rules of the auction and the terms of payment, and in many cases includes rules governing behavior after the auction.

A great deal of auction theory exists to guide the choices of market designers. However, in many cases theory provides only a rough guide. Practical details of the auction environment often violate one or more assumptions of the theoretical literature. Furthermore, even in those cases when the assumptions of a mathematical model are satisfied, equilibrium theory may offer only qualitative predictions. There are relatively few circumstances in which equilibrium theory can provide useful quantitative predictions of measures of interest, such as expected revenue and surplus. That is, theory often fails to provide a means to evaluate the *economic significance* of the difference between competing auction designs.

Even very small complications to the benchmark models of auction behavior make it impossible to generate quantitative revenue or efficiency comparisons using

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[2] Roth (2002).

existing theory. For example, when bidders in an auction are in some way dissimilar (or *asymmetric*), it is in general not possible to provide a closed-form expression of equilibrium bidding behavior (there are exceptions, but they are special cases). Without an explicit expression of each bidders' strategy, the market designer can often still make comparative static predictions, but certainly cannot quantify expected revenue, efficiency, or the distribution of surplus amongst bidders.

Because of the frequent need to incorporate details that render analytical solutions difficult or impossible to derive, a market designer typically makes use of complementary tools: computational methods can be used to generate predicted bidding functions, and experimental methods can be used to test the predictions.

The first part of this dissertation, comprising chapters 2 and 3, develops a computational technique that can be used to effectively approximate equilibrium bidding strategies in auction games. Bidding strategies are functions that specify an action (a bid) for every situation a bidder might face. Armantier, Florens and Richard (2008) introduced the concept of Constrained Strategic Equilibrium (CSE), a technique that approximates bidding strategies by imposing a parametric form. Chapter 2 reviews the literature on related computational techniques, and demonstrates how the CSE approach differs. Chapter 3 contains details of my implementation of the CSE algorithm. `CSE_SOLVER`, a suite of modular Matlab programs, implements the CSE algorithm and can be used to solve an arbitrary auction problem.

Chapter 3 contains some extensions to the algorithm originally proposed by Armantier, Florens and Richard (1998). First, I demonstrate how bidding strategies can be approximated by monotonically increasing polynomials. I show how *positive polynomials* can be used to approximate monotonic functions.[3] Strategies that increase monotonically are often assumed in theory. The use of a functional form that is totally flexible, and yet is monotonic by construction, provides a computational method to match the theory. Using a function that is monotonic by construction enables the researcher to solve CSE problems with numerical techniques designed for unconstrained optimization. The benefit is potentially faster, more reliable solution of the set of fundamental equations that result from the CSE approach.

I also introduce to the economics literature a technique for distribution estimation that is particularly well-suited to estimating the distributions generated during Monte Carlo simulations. Monte Carlo simulations are used in the CSE algorithm to approximate first order conditions of the constrained equilibrium. The solution of the system of equations generated by the CSE problem relies crucially on the ability to estimate the distribution of winning bids from these Monte Carlo simulations. The density estimation technique discussed in chapter 3, known as Target Distribution Estimation (TDE), allows a researcher to incorporate knowledge about the form of the distribution of the bids, while still maintaining the flexibility of a nonparametric estimator. The CSE procedure typically uses nonparametric density and distribution estimation techniques, as the distribution of the bids is not known *a priori*. However, the distribution of the type draws (values in a buy-auction and costs in sell-auction) is of course known. The distribution of equilibrium bids is a transformation of the

[3] Positive polynomials are polynomials that always take on positive values, despite the fact that no restriction is made on the coefficients. Positive polynomials are constructed using a convolution method explained in chapter 3.

known distribution of types. The TDE makes use of this information and can improve algorithm performance when standard nonparametric techniques fail.

The second part of the dissertation, comprising chapters 4 and 5, use the computational techniques developed in chapters 2 and 3 to examine asymmetric multi-unit auctions. Asymmetric auctions are notoriously difficult to solve – standard numerical techniques used to derive equilibrium bidding functions rely on the specification of differential equations. These differential equations are derived from the first order conditions of each bidder's objective function. When multiple units are auctioned simultaneously, the system of differential equations quickly becomes unmanageable. The `CSE_SOLVER` algorithm is robust enough to solve both benchmark auctions currently found in the literature, and a series of multi-unit auctions specified in chapter 4 of this dissertation. An experimental test of the predictions is presented in chapter 5 of this dissertation. Laboratory experiments are used to evaluate the predictions of the computational models. While individual behavior deviates from equilibrium predictions, aggregate results and comparative static predictions are consonant with the computational results.

Terminology and Notation

Terminology

Bidder Types

In the auction theory literature, it is common to refer to a bidder's preferences, the draws from nature that characterize a bidder, as that bidder's *type*. The term *type* is a generic placeholder for more descriptive, but more context-specific terms. For instance, a bidder's *value* for an item might be the draw from nature that uniquely characterizes the bidder in a *buy-auction,* an auction in which there is a single seller and many buyers, while a bidder's *cost* of producing an item might be the equivalent term in a *sell-auction,* an auction in which there is a single buyer and many sellers. Throughout the dissertation I will use the term *type* when I discuss a generic situation or result, but will also use more specific terminology such as *value* or *cost* when it is appropriate.

Auction Types

As above, I will refer to auctions in which many buyers compete to purchase item(s) from a single monopolistic seller as "buy-auctions" and to auctions in which many sellers compete to provide item(s) to a single monopsonistic buyer as "sell-auctions."

Constrained Strategic Equilibrium (CSE)

I will refer to Constrained Strategic Equilibrium, or CSE, as both an algorithm and an equilibrium concept distinct from Bayesian Nash Equilibrium (BNE). A BNE is defined by a set of strategies such that the strategy of each player is a best response to the strategies of all other players. A CSE is defined by a set of *constrained strategies* such that each player's strategy is a best response to the strategies of all other players. When I refer to solving for a CSE numerically, I will refer to the *CSE algorithm*.

Notation

$X \equiv$ Bidder *type* (draw from nature), a random variable. The i^{th} bidder's type is denoted X_i.

$x \equiv$ The realization of a bidder *type*. The realization of the i^{th} bidder's type is denoted x_i. In Monte Carlo experiments, several realizations are drawn for each bidder's type. The draw of the i^{th} bidder's type in the m^{th} Monte Carlo experiment is denoted x_{im}.

$F(\bullet) \equiv$ Cumulative distribution function of the draw from nature, x.

$b \equiv$ Bid.

$s_i(x_i) \equiv$ Strategy of player *i*; a mapping from type-space to bid-space. $b = s_i(x_i)$.

$a \equiv$ A coefficient of a parametrized bidding strategy.

$G(\bullet) \equiv$ Cumulative distribution function of the critical (marginal) bid.

$g(\bullet) = dG(\bullet)$.

$H(\bullet) \equiv$ Cumulative distribution function of the target distribution (used in a procedure to estimate an unknown distribution).

$h(\bullet) = dH(\bullet)$.

$p(x) \equiv$ A polynomial (I use this notation to represent a generic polynomial, as opposed to a positive or a monotonic polynomial).

$a_{ij} \equiv$ The jth coefficient of the bidding function of the ith representative bidder.

$\mathbf{a}_i \equiv$ The vector of coefficients parameterizing the ith representative bidder's strategy.

$b_{ij} \equiv$ Same as a_{ij} (sometimes necessary to distinguish one type of coefficient from another).

$m_k(x) \equiv$ A k^{th}-stage monotonic polynomial.

$m_k'(x)$ The derivative of a k^{th}-stage monotonic polynomial, which is itself a positive polynomial. A positive polynomial $m_k'(x)$ is expressed in nested form as

$$m_k'(x) = m_{k-1}'(x)\, q_k(x),$$

where

$q_k(x) \equiv \left[1 - 2xa_k + x^2\left(a_k^2 + b_k^2\right)\right]$, and

$m_0'(x) \equiv \lambda$ by definition.

$\lambda \equiv$ The multiplicative parameter of the positive polynomial $m'(x)$.

$d_l^{(j)} \equiv$ Coefficient on x to the power l in the j^{th} step of the positive polynomial.

- Notation is used to express $q_l(x)$ in a more compact and convenient way.

  $q_j = \left[d_0^{(j)} + d_1^{(j)}x + d_2^{(j)}x^2\right]$, therefore, $d_0^{(j)} = 1$, $d_1^{(j)} = -2a_j$, $d_2^{(j)} = \left(a_j^2 + b_j^2\right)$.

$D_l^{(j)} \equiv$ The ***total*** coefficient on x to the power l in the j^{th} step of the positive polynomial. The total coefficient is a function of all of the coefficients d from the j^{th} and all previous steps, $d \equiv \left\{\left\{d_0^{(1)}, d_1^{(1)}, d_2^{(1)}\right\}, ..., \left\{d_0^{(j)}, d_1^{(j)}, d_2^{(j)}\right\}\right\}$.

Chapter 2: A Simulation Approach to Approximating Equilibrium

Introduction

Two things are accomplished in this chapter: I review the literature on computational techniques used to approximate equilibrium bidding functions, and I set up the general problem to be solved using the CSE approach.

The literature review is focused on a technique pioneered by Marshall, et al. (1994), hereafter MMRS. The MMRS-type approach is the dominant approach in the literature on numerical computation of auction equilibrium. Several authors have followed in the footsteps of MMRS, making meaningful improvements to the original algorithm. The basic idea behind the MMRS approach stays the same, however, and there are similarities between the MMRS approach and the constrained strategic equilibrium approach first introduced by Armantier, Florens and Richard (1998), which is implemented in this dissertation.

Both algorithmic approaches use as building blocks the first order conditions of each bidder's objective function, although the expression of these objective functions are different in each algorithm. The objective function of bidder *i* in a first-price, sealed-bid auction in its most general form can be written $U_i\left(x,b\right)\times \Pr_i\left(win\,|\,b\right)$: the utility of winning with a bid of *b*, multiplied by the probability of winning with that same bid. Under the MMRS approach, the expression of $\Pr_i\left(win\,|\,b\right)$ is explicit – the probability of winning with a particular bid is expressed only in terms of primitives of the model, including the known distributions of each bidder's type. The CSE approach is more direct. Rather than expressing the probability of winning in primitive terms, the probability is estimated directly from Monte Carlo experiments. The MMRS approach leads to a more explicit representation of the objective function, and thus the first order condition, which requires unique input "by-hand" in order to make the algorithm appropriate for a particular auction problem. The CSE approach sacrifices some accuracy for greater generality.

Relaxing the Assumptions of the Benchmark Auction Model: Why Computational Approaches are Necessary

Computational techniques are a valuable tool used in the analysis of many real-world auction institutions. Computational techniques are most useful for analyzing auctions when:

1. Equilibrium strategies are known not to exist, but benchmark bid functions would be useful to derive.
2. It is unknown whether or not equilibrium strategies exist.
3. Equilibrium strategies are known to exist, but it is difficult or impossible to derive analytical expressions for the bid functions.

Auctions in which bidders draw their types from more than one probability distribution, commonly referred to as *asymmetric auctions*, are a good example of auctions that often require computational analysis for one of the three reasons above.

The *symmetry* assumption captures the notion that all bidders in an auction share a common uncertainty and outlook. Each bidder knows their own type but not that of their opponents. Bidder *i* views all opposing bidders as having drawn their types from a certain distribution, and believes each other bidder assumes the same, so that the only thing differentiating bidders is the realization of their draw. Formally, suppose that each of *N* bidders draw their preferences for an item independently from a continuous distribution *F*, that each bidder knows their own draw from *F* but not that of any other bidder, that each bidder is risk-neutral, and that these facts are common knowledge among all bidders.[4] This model is known as the symmetric Independent Private Values (symmetric IPV) model of auctions. Assuming all bidders are motivated by profit maximization, it is straightforward to find a single strategy *s* that characterizes the equilibrium behavior of all bidders in a first-price, sealed-bid auction.

Symmetry is the principal assumption of the benchmark IPV model of auctions relaxed in this dissertation. Relaxing the symmetry assumption means allowing each of the *N* bidders in the auction to draw their preferences from an idiosyncratic distribution $F_i,\ i = 1,...,N$.[5]

We study asymmetric auction models because there is often reason to believe that bidders are ex ante heterogeneous, and so will pursue idiosyncratic strategies. The rules of an auction might favor certain classes of bidder, as in FCC spectrum auctions (Ayres and Cramton (1996), McMillan (1994)) and government procurement

[4] *F* is commonly assumed to be twice continuously differentiable and to be defined on a compact support.

[5] Allowing for bidders to draw their preferences from idiosyncratic distributions is the most common way to relax the symmetry assumption. However, it is sometimes assumed that bidders draw their preferences from a single distribution, but that other differences between bidders generate an asymmetry. For instance, bidders might behave differently for some reason, or it might be assumed that bidders have idiosyncratic utility functions, or that bidders are treated differently in the auction because of an observable trait, etc. Such assumptions would also lead to a model that would properly be termed *asymmetric* (Hubbard and Paarsch (2009)).

(McAfee and McMillan (1989), Denes (1997), Hubbard and Paarsch (2009), Krasnokutskaya and Seim (2009), Marion (2007), Marion (2009)). In many more cases there is no favoritism inherent in the auction rules, but there are observable differences between the bidders that partially reveal a bidder's type. Cramton (1995) provides some salient examples in the context of the U.S. Narrowband PCS[6] auction:

> Of the 25 bidders, a few bidders were known to have high values because of their large market share, prior product development, or other advantages. PageNet, for example, had by far the largest market share in paging going into the auction. It also had a well developed product, VoiceNow, that required a substantial slice of narrowband spectrum for nationwide distribution. McCaw was known to have deep pockets, as were some of the other large firms (AirTouch and BellSouth). These differences were known by all. The relevant auction model to analyze was clearly one with asymmetric bidders. (p. 50).

The General (Asymmetric) IPV Model

Before proceeding further, it will be helpful to introduce in detail the fundamental auction problem we are seeking to solve. The fundamental auction problem will be introduced in the context of a single-unit auction. This setting is the easiest to analyze, and all of the literature I subsequently review deals exclusively with this case. The problem will be generalized to the multi-unit case later in the dissertation.

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[6] Auction for nationwide narrowband personal communication services.

Suppose that each of the *N* bidders draw *types*, or values, from a (potentially) idiosyncratic distribution F_i on compact support. That is, we know that $x_i \sim F_i$, and $x_i \in \left[\underline{x}_i, \overline{x}_i\right]$. Bidders are risk neutral and so the utility of winning a buy-auction with bid *b* when a bidder's type is x_i can be expressed as

$$u\left(b; x_i\right) = x_i - b\,.$$

Index the bidders by $1, 2, ..., N$ and, without loss of generality, consider the problem of deciding what to bid from the perspective of bidder 1. Bidder 1 seeks to maximize the expected returns from bidding, i.e. the bidder seeks to maximize

$$\left(x_1 - b\right)\Pr\left(win \mid b\right). \qquad (1)$$

The probability that bidder 1 will win with a bid of *b* is the probability that each of the other $N-1$ bids will be below *b*. Assuming that the $N-1$ other bidders follow strategies $s_2\left(x_2\right), s_3\left(x_3\right), ..., s_N\left(x_N\right)$, the probability can be expressed as

$$\left(x_1 - b\right)\Pr\left(b > s_2\left(X_2\right) \cap b > s_3\left(X_3\right) \cap ... \cap b > s_N\left(X_N\right)\right).$$

Assuming that strategies are monotonically increasing in x, i.e. that each bidder submits bids that are non-decreasing in their value for the item, we know that each of the strategies s_i has an inverse. Denote the inverse function corresponding to s_i by ϕ_i. Then bidder 1's problem can be rewritten in a more useful form as

$$\left(x_1 - b\right)\Pr\left(\phi_2\left(b\right) > X_2 \cap \phi_3\left(b\right) > X_3 \cap ... \cap \phi_N\left(b\right) > X_N\right).$$

Using the inverse strategies allows us to express bidder 1's objective function in a more useful form because we know the distribution of each X_i by assumption. Isolating the X_i in the expression allows us to use the knowledge of the distributions F_i and the independence of each draw x_i to finally write bidder 1's objective function as

$$(x_1 - b) \times F_2(\phi_2(b)) \times F_3(\phi_3(b)) \times \ldots \times F_N(\phi_N(b)).$$

To find the optimal *b*, differentiate with respect to *b* in order to obtain the first order condition

$$(x_1 - b)\left[\sum_{i=2}^{N} f_i \times \phi_i^{'} \times \left(\prod_{j=\{2,\ldots,N\},\, j\neq i} F_j\right)\right] = \prod_{i=1}^{N} F_i. \tag{2}$$

This first order condition gives us a differential equation (in terms of ϕ_i) that characterizes the equilibrium of the auction. The boundary conditions of the ordinary differential equation can be written

$$\phi_i(\underline{b}_i) = \underline{x}_i \tag{3}$$

and

$$\phi_i(\overline{b}_i) = \overline{x}_i, \tag{4}$$

where $\underline{b}_i$ denotes the bid corresponding to the lowest possible type-draw $\underline{x}_i$ and $\overline{b}_i$ denotes the bid corresponding to the highest possible type-draw $\overline{x}_i$.

When the N differential equations (2)-(4) (one-per-bidder) are simultaneously satisfied by a set of optimal strategies $\{s_1(x_1),...,s_N(x_N)\}$, we have a candidate equilibrium.[7]

In order to gain intuition, it is instructive to set up the model in a special case. Suppose there are two bidders, i.e. that N=2. In that case, the objective function of bidder 1 simplifies to

$$\begin{aligned}&(x_1 - b)\Pr\left(b > s_2(X_2)\right) = \\ &(x_1 - b) \times F_2\left(\phi_2(b)\right),\end{aligned} \tag{5}$$

and the resulting first order condition can be written

$$(x_1 - b)\, f_2\left(\phi_2(b)\right)\phi_2^{'}(b) = F_2\left(\phi_2(b)\right). \tag{6}$$

This yields an ordinary differential equation describing how the inverse bid function ϕ_2 varies with b. The exact same procedure, when performed from the perspective of bidder 2, yields

$$(x_2 - b)\, f_1\left(\phi_1(b)\right)\phi_1^{'}(b) = F_1\left(\phi_1(b)\right). \tag{7}$$

Along with a set of boundary conditions, these two equations characterize the equilibrium. To further simplify the example, let $X_1, X_2 \in [0,1]$. This, with one additional assumption discussed below, is the first model specified by MMRS.[8] With the assumption of [0,1] common support, we can write the boundary conditions:

$$\phi_1(0) = \phi_2(0) = 0, \tag{8}$$

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[7] Finding a solution to the system of differential equations does not constitute a *proof* of equilibrium. To prove that the N-tuple $\{s_1(x_1),...,s_N(x_N)\}$ is an equilibrium, we must show that none of the bidders can profitably deviate from the strategy $s_i(x_i)$.

[8] MMRS specify a number of different permutations of their basic model of coalitions. The model here is analogous to their "coalition versus coalition" model.

and

$$\phi_1\left(\bar{b}\right)=\phi_2\left(\bar{b}\right)=1. \tag{9}$$

To see why (8) holds in equilibrium, realize that a bidder with a value draw of 0 will never submit a bid $b>0$, as doing so can only result in a loss. To see why there is a single maximum bid $\bar{b}$ for both bidders, suppose otherwise. Suppose $\phi_1\left(\bar{b}_1\right)=1$ and $\phi_2\left(\bar{b}_2\right)=1$ and, without loss of generality, suppose $\bar{b}_1>\bar{b}_2$. In this case, whenever bidder 1 realized a value of 1, bidder 1 could reduce their bid to $\bar{b}_2$ without reducing their probability of winning the auction. Therefore, $\bar{b}_1=\bar{b}_2=\bar{b}$ and we get (9).[9]

The system of differential equations (6) and (7), with boundary conditions (8) and (9), completely characterize the equilibrium in the MMRS two-bidder case.

In general, the asymmetric auction model (2)-(4) cannot be solved analytically. That is, a closed-form expression cannot be derived for the equilibrium strategies of each of the bidders. In the next section, numerical solution techniques developed by MMRS to solve (6)-(9) (a special case of (2)-(4))will be reviewed in detail. Although many special cases of the asymmetric model have been solved explicitly,[10] the computational approach is necessary for the analysis of the vast majority of asymmetric auction problems.

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[9] Using the assumption of a common support allows me to derive (8) and (9) using the straightforward arguments above. Without the assumption of a common support [0,1], similar boundary conditions can still be established, but the necessary argument is more nuanced. The specific example here contains all the necessary intuition and comes at the cost of very little generality.

[10] Vickrey (1961) famously solved a very special case in which one bidder draws their value from a degenerate distribution. A model with two bidders and uniform distribution of types was solved first by Griesmer, Levitan and Shubik (1967). Their model was later generalized by Kaplan and Zamir (2007).

Computational Approaches: A Review of the Literature

In order to generate predicted bidding functions when analytical solutions to (2)-(4) are unknown, authors have resorted to a number of different computational strategies. I will not attempt to provide a complete catalog of available techniques, but will discuss the major methods that have been employed, and characterize the advantages and disadvantages of each.

The seminal paper in the field of numerical analysis of auctions was by Marshall, et al. (1994). MMRS construct a model that can be re-cast as a special case of (6) - (9) when the distributions F_i are assumed to be uniform. MMRS use this model to study collusion. The MMRS model allows for a sort of "supper-bidder" to form as a result of cooperation among sub-groups of the *N* bidders. These super-bidders act differently than any atomistic bidder would, creating an asymmetry that makes it difficult to derive an analytical solution. The basic technique pioneered by MMRS attempts to numerically solve the differential equations (6) and (7). MMRS specify an approximating function for the (transformed) inverse strategies (in this case the approximating function is a series of piecewise polynomial expansions at regular intervals). Rather than choosing a starting value that satisfies (8) and using a standard shooting algorithm, the major innovation of MMRS is the development of a back-stepping algorithm. To see why this unusual method of solving the differential equations (6) and (7) is necessary, rearrange (6) to isolate the inverse strategy function.

$$\phi_2'(b) = \frac{F_2(\phi_2(b))}{(x_1 - b)\, f_2(\phi_2(b))}. \tag{10}$$

Substitute in the identity[11] $\phi_2^{'}\left(b\right)=\frac{1}{s_2^{'}\left(X_2\right)}$ and invert both sides of (10) to get

$$s_2^{'}\left(X_2\right)=\frac{\left(x_1-b\right)f_2\left(\phi_2\left(b\right)\right)}{F_2\left(\phi_2\left(b\right)\right)}. \tag{11}$$

As X_2 approaches its lower support from the right (so that $\phi_2\left(b\right)$ approaches 0), the denominator of the expression above tends to zero, and so the slope of the bid function s_2 tends to infinity. This makes the differential equation behave poorly near the lower support.[12] Because of this pathology, the differential equation (11) cannot be solved using standard techniques.

MMRS solve this problem by employing a back-solve method. They select a starting point for their algorithm by guessing $\overline{b}$ in (9) and step backwards, tracing out the differential equation until a solution is found such that both (8) and (9) are simultaneously satisfied.

The virtue of the MMRS algorithm is its accuracy. The drawback of this technique for solving auction problems is its lack of generality – the program needs to be modified substantially for use in more complex cases. This is because the technique uses a considerable amount of input, i.e. a significant amount of work is done "by hand" prior to running the algorithm. In order to express the differential equation in terms of the primitives of the model F_i, the system must be explicitly

[11] The derivative of an inverse function is the reciprocal of the derivative of the original function evaluated at the value of the inverse: $df^{-1}\left(x\right)=\frac{1}{f^{'}\left(f^{-1}\left(x\right)\right)}$.

[12] Technically speaking, the differential equation (11) is not Lipschitz.

rewritten for every special case considered. Additionally, MMRS use a transformation of the original differential equations that enhances numerical stability.

Several contributions following in the tradition of MMRS use increasingly sophisticated methods to improve the convergence properties of the original algorithm. Li and Riley (2007) recently offered a substantial generalization of the MMRS algorithm that, among other improvements, uses a more intelligent procedure to select the intervals used in the approximation of ϕ_i. Li and Riley characterize the system of differential equations more generally, allowing for more than two types of bidders and for the fundamental preference parameter to be distributed other than uniform. Riley and Li have made their algorithm, implemented in a software distribution called BIDCOMP^2, available for download.[13]

Gayle and Richard (2008) offer an alternative software package that, like BIDCOMP^2, solves a general system of differential equations that characterize any single-unit asymmetric IPV model.[14,15] The implementation of Gayle and Richard allows for arbitrary combinations of distributions of the preference parameter. For example, a hybrid distribution of types can be specified based on data the user possesses. This distribution can then be fed into the Gayle and Richard solver. The essential idea behind the Gayle and Richard algorithm is unchanged from MMRS and Li and Riley, however. Gayle and Richard seek to solve the same differential equations as MMRS and Li and Riley – the major improvement offered by Gayle and Richard (2008) is in the accuracy with which they approximate the inverse bid

[13] The download page is located at http://www.econ.ucla.edu/riley/bidcomp/, although, as of this writing, several of the files necessary to run key examples are missing due to broken links.
[14] Gayle and Richard's software is available at http://capcp.psu.edu/AsymmetricAuctions/index.html. The software is programmed in FORTRAN90.
[15] Gayle and Richard's software solves an auction problem with a fixed reserve price.

functions. Gayle and Richard use local Taylor-series as the approximating function in their particular application.

Although we have focused on MMRS-type approaches thus far, there are other notable approaches to numerically computing equilibrium.

Athey (1997) takes the approach of discretizing the action space – she restricts bidders to submit one bid from a finite menu of possible bids. This simplification allows a much more straightforward approach to calculating equilibrium – Athey solves a best response (fixed-point) problem at a finite number of points. Athey proceeds by solving a series of individual optimization problems, each determining the point at which a bidder switches from bidding at one discrete point to the next-highest discrete point.

Bajari (2001) compares the performance of three algorithms, two of which are similar to the algorithms we have already reviewed, and one of which is similar in spirit to the algorithm used in this dissertation. Bajari's first algorithm is again essentially that of MMRS – by finding a solution to the one stable boundary condition, the fundamental differential equations can be solved. Bajari reports that this algorithm, though fast and efficient for solving the auction problems posed by MMRS, can be slow to converge in his applied work. Bajari's second algorithm begins with an initial guess of the bidding functions (usually the guess is the zero-profit bid function where each bidder submits a bid equal to their type) and then computes a (potentially infinite) series of best responses. In some sense, this is similar to Athey's algorithm, but the continuous action space means that cycling can easily occur and the algorithm can terminate without giving useful feedback. Bajari's second algorithm can be quick to converge, especially if a good guess of the equilibrium bidding function can be provided, but again, infinite cycling is possible.[16]

Finally, Bajari's third algorithm is similar to the methods introduced by Armantier, Florens and Richard (1998), and so to the methods that I use in this dissertation. Bajari's third algorithm uses global polynomials to approximate the inverse bid functions. Rather than guessing an endpoint value that satisfies a boundary condition and back-solving, Bajari treats the differential equation as an equality constraint (which can be transformed into a system of zeros) that should approximately hold when a high-order polynomial is used to estimate the inverse bid function with sufficient precision. Using a polynomial to approximate the equilibrium inverse bidding function reduces the problem to one of finding coefficients that minimize a system of transformed differential equations (the transformed equations should theoretically equal zero in equilibrium, so minimizing their value provides an effective algorithm). Bajari evaluates the fundamental differential equations at a grid of points covering the range of feasible bids, and uses a nonlinear least squares algorithm to find the polynomial coefficients that minimize the transformed equations.

The Constrained Strategic Equilibrium technique is similar to Bajari's third algorithm. The CSE algorithm was first introduced by Armantier, Florens and Richard (1998), and further developed in Armantier and Richard (2000) and Armantier, Florens and Richard (2008).

[16] I have implemented this algorithm myself and have found that it often cycles when used to solve extremely simple multi-unit auctions, even if reasonable starting points are provided.

Constrained Strategic Equilibrium

The idea underlying the Constrained Strategic Equilibrium technique is both extremely simple and totally general. Although I develop the technique here in the context of auctions, the technique applies to Bayesian games in general.

Before developing the formal framework, it will be helpful to introduce the intuition behind CSE. One way to understand the idea of a CSE is to view it as an extension of Rothkopf's (1969) original "markup" model. Rothkopf's initial insight was to consider what bidders might do if they were constrained to submit bids that were linear functions of their draws. The markup strategy is simple and intuitive: a buyer will submit a bid that is some constant markup[17] of his draw from nature, $a \times x$, the parameter a being a fraction in the range [0,1], and chosen intelligently to maximize expected gains. In the context of a particular common value model, Rothkopf showed that if the bidding functions $s(x;a)$ were constrained to be linear, i.e. of the form $s(x;a) = ax$, and a set of coefficients a were found to define an equilibrium of the restricted game, then these same coefficients would be an equilibrium of the unrestricted game as well. Rothkopf's conclusions are not relevant to the IPV model here, but his simple suggestion to consider linear strategies foreshadows the CSE approach.

Note the similarity between Rothkopf's idea and the discretization of the action space imposed by Athey (1997). Both assumptions simplify the search for an

[17] Rothkopf introduced his model in the context of a procurement auction, i.e. an auction where bidders compete to sell an item to a single buyer. The term "markup" then had the intuitive appeal of referring to price as a markup of each bidder's cost of production. In the context of an auction where a single seller receives bids from many buyers, a more appropriate term might be "mark-down" instead. We will use Rothkopf's original language for consistency.

equilibrium by restricting the action space in some way. Athey discretizes the action space by positing that each bidder chooses from a menu of available bids; Rothkopf allows for bidders to choose instead among a family of linear strategies. While Athey's method requires a point-by-point analysis of the auction problem, Rothkopf's method uses a single coefficient to define the entire strategy profile.

In general, the CSE approach is to use a *set* of coefficients to define the strategy profile. Rather than restrict the strategy space to be parameterized by a single coefficient per bidder (Rothkopf's linear case), Armantier, Florens and Richard (2003) proposed that an arbitrary number of coefficients per bidder could be used to parameterize the strategy profile.

In order to develop the CSE solution procedure, we will set up the same problem reviewed above in (6) through (9) as a constrained strategy problem. We will then generalize the notation and procedure so that the CSE algorithm can handle any auction problem.

Recall from above that there are two bidders, each of whom draws their type from an idiosyncratic distribution F_i. To solve for an equilibrium strategy of bidder 1 with unconstrained strategies, we must find some bid *b* that maximizes bidder 1's objective function for every possible value of X. We did this above by finding

$$b_1{}^* = \arg\max\{(x_1 - b_1)\Pr(win \mid b_1)\}.$$

To solve for an equilibrium of the constrained strategy model, we substitute $s(x_1;\mathbf{a_1})$,[18] a strategy constrained to take a particular functional form, for *b* in (1) to get

$$\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right)\Pr\left(win \mid s\left(x_1;\mathbf{a_1}\right)\right). \tag{12}$$

The $s(x_1;\mathbf{a_1})$ can in principal be any function that can be parameterized by a vector of coefficients $\mathbf{a_1}$, for example an ordinary quadratic polynomial $s(x_1;\mathbf{a_1}) = a_{10} + a_{11}x_1 + a_{12}x_1^2$. Rather than finding the argmax $b_1{*}$ of (1), we search for the optimal coefficients $\mathbf{a}_1{*}$ of (12). That is, we seek to find[19]

$$\mathbf{a_1}{*} = \arg\max\{\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right)\Pr\left(win \mid s\left(x_1;a_1\right)\right)\},$$

for any possible realization x_1.

Let G_1 be the distribution of the highest bid that is not submitted by bidder 1. Then we can immediately rewrite (12) as

$$\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right)G_1\left(s\left(x_1;\mathbf{a_1}\right)\right). \tag{13}$$

Note the difference between how the probability of winning is represented in (5) and how it is represented in (13). Herein lies a key difference between the CSE algorithm implemented in `CSE_SOLVER` and MMRS-type algorithms. To obtain an expression for $\Pr(win \mid b)$ in (5), we noted that bidder 1 wins when $b > s_2(x_2)$. In order to transform the expression $\Pr(win \mid b)$ into a function of the known

[18] Recall that bold notation is used to represent vectors of coefficients (see the Notation section in chapter 1).

[19] Note that the imposition of constrained strategies need not be very restrictive at all. Since $s(x;a)$ can take on any parameterized form, we can approximate the true strategy function $s(x)$ to any degree desirable. In particular, the Weierstrass theorem tells us that we can approximate any continuous function to a desired degree of accuracy using a simple polynomial. See Armantier, Florens and Richard (2003) for a more detailed argument justifying the imposition of parametric form.

distributions F_i, we used the assumed monotonicity of the strategy s_2 to isolate X_2. This allowed us to rewrite $\Pr\left(b > s_2\left(X_2\right)\right)$ as $\Pr\left(\phi_2\left(b\right) > X_2\right)$, and since we know the distribution of X_2, we were finally able write the probability of winning explicitly as $F_2\left(\phi_2\left(b\right)\right)$.

The CSE algorithm takes a more direct approach. Rather than expressing the probability of winning in terms of the distribution of *types*, we express the probability of winning in terms of the distribution of *bids*. Given the distribution and density of types and the relationship $b = s\left(x\right)$, we can derive the density of bids $g\left(b\right)$ using a straightforward transformation of variables

$$g\left(b\right) = f\left(x\right)\left|\frac{dx}{db}\right|.$$

If there exist strategies *s* mapping types to bids, and the types are random variables, then the bids themselves are also random variables.

Rather than expressing the probability of winning in terms of the distribution of types, the CSE algorithm estimates the probability of winning directly in terms of the distribution of bids. Using this definition of G_1 in (13), we can proceed to solve for the equilibrium strategies by maximizing expected profits.

The choice variables of bidder 1 are the parameters $\mathbf{a_1}$ of the strategy s. The first order condition with respect to the j^{th} parameter a_{1j} is written

$$\frac{\partial s}{\partial a_{1j}} \times \left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right) \times \left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) = 0\,, \tag{14}$$

where $\frac{\partial s}{\partial a_{1j}}$ denotes the partial derivative of the parameterized strategy function with respect to the j^{th} parameter and $dG_1 = g_1$. If there are *k* parameters specifying the strategy s, we have *k* first order conditions corresponding to bidder 1's objective function

$$\begin{aligned}
&\frac{\partial s}{\partial a_{11}}\left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) = 0\\
&\frac{\partial s}{\partial a_{12}}\left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) = 0\\
&\ldots\\
&\frac{\partial s}{\partial a_{1k}}\left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) = 0,
\end{aligned} \tag{15}$$

where the only difference between each of the first order conditions above is accounted for by the first term, the partial derivative of the strategy function with respect to the appropriate coefficient.

By an exactly analogous procedure, we also get *k* first order conditions corresponding to the objective function of bidder 2

$$\begin{aligned}
&\frac{\partial s}{\partial a_{21}}\left(g_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\left(x_2 - s\left(x_2;\mathbf{a_2}\right)\right) - G_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\right) = 0\\
&\frac{\partial s}{\partial a_{22}}\left(g_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\left(x_2 - s\left(x_2;\mathbf{a_2}\right)\right) - G_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\right) = 0\\
&\ldots\\
&\frac{\partial s}{\partial a_{2k}}\left(g_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\left(x_2 - s\left(x_2;\mathbf{a_2}\right)\right) - G_2\left(s\left(x_2;\mathbf{a_2}\right)\right)\right) = 0.
\end{aligned} \tag{16}$$

To find the coefficients $(\mathbf{a_1}^*, \mathbf{a_2}^*)$ that completely characterize an equilibrium in constrained strategies, we approximately solve the system of equations (15) and (16). Note that equations (15) and (16) are expressed in terms of a single realization of the pair (X_1, X_2). Of course, we seek to find coefficients $(\mathbf{a_1}^*, \mathbf{a_2}^*)$ that approximately solve (15) and (16) for *arbitrary* realizations of X_1 and X_2. To do this, we use Monte Carlo sampling and a penalty function representation of each first order condition. That is, we use Monte Carlo sampling to find coefficients $(\mathbf{a_1}^*, \mathbf{a_2}^*)$ that minimize a summarization of the first order conditions when X_1 and X_2 take on arbitrary values.

M-good, *N*-bidder Case

Generalizing from the two-bidder case is straightforward. The CSE problem with *N* bidders and *M* identical goods at auction is a system of $N \times M \times k$ equations, where *k* is the degree of the parameterized strategy functions $s(x_1, \mathbf{a_1})$, $s(x_2, \mathbf{a}_2)$, .., $s(x_N, \mathbf{a}_N)$.

$$\begin{pmatrix} \frac{\partial s}{\partial a_{11}}\left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) \\ \dots \\ \frac{\partial s}{\partial a_{1k}}\left(g_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\left(x_1 - s\left(x_1;\mathbf{a_1}\right)\right) - G_1\left(s\left(x_1;\mathbf{a_1}\right)\right)\right) \\ \frac{\partial s}{\partial a_{21}}\left(g_2\left(s\left(x_2;\mathbf{a}_2\right)\right)\left(x_2 - s\left(x_2;\mathbf{a}_2\right)\right) - G_2\left(s\left(x_2;\mathbf{a}_2\right)\right)\right) \\ \dots \\ \frac{\partial s}{\partial a_{Nk}}\left(g_N\left(s\left(x_N;\mathbf{a}_N\right)\right)\left(x_N - s\left(x_N;\mathbf{a}_N\right)\right) - G_N\left(s\left(x_N;\mathbf{a}_N\right)\right)\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \qquad (17)$$

The only difference between the M-unit case and the single-unit case is in the estimation of the distribution and density of the marginal bid, G_i and g_i. In the single-unit case of a buy-auction, $G_i(x)$ represents the probability that the highest bid of all bidders except i is less than x. In the M-unit case, $G_i(x)$ represents the probability that the M^{th}-highest bid of all bidders except i is less than x.

Stages of the CSE Algorithm

To sum up, the stages of the CSE algorithm are:

1. (Strategy Choice): Specify a family of functions to represent each bidder's strategy.
2. (Initialization Stage): Specify starting values $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N\}^0$ for the parameters of each bidder's k-parameter strategy (the superscript "0" above denotes the initial stage).
3. Evaluate the first order condition with parameters $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N\}^0$ for each auction. Note that there are *mc* auctions, where *mc* is the Monte Carlo size.
4. Calculate a single value for each first order condition based on a penalty function Ω:[20]

[20] In practice, the quadratic penalty function is typically used.

$$\begin{pmatrix} foc_{11} \\ \ldots \\ foc_{1k} \\ foc_{21} \\ \ldots \\ foc_{Nk} \end{pmatrix} \approx \begin{pmatrix} \sum_{m=1}^{mc} \Omega\left(\frac{\partial s}{\partial a_{11}} \left(g_1\left(s\left(x_{1m};\mathbf{a_1}\right)\right)\left(x_{1m} - s\left(x_{1m};\mathbf{a_1}\right)\right) - G_1\left(s\left(x_{1m};\mathbf{a_1}\right)\right)\right)\right) \\ \ldots \\ \sum_{m=1}^{mc} \Omega\left(\frac{\partial s}{\partial a_{1k}} \left(g_1\left(s\left(x_{1m};\mathbf{a_1}\right)\right)\left(x_{1m} - s\left(x_{1m};\mathbf{a_1}\right)\right) - G_1\left(s\left(x_{1m};\mathbf{a_1}\right)\right)\right)\right) \\ \sum_{m=1}^{mc} \Omega\left(\frac{\partial s}{\partial a_{21}} \left(g_2\left(s\left(x_{2m};\mathbf{a}_2\right)\right)\left(x_{2m} - s\left(x_{2m};\mathbf{a}_2\right)\right) - G_2\left(s\left(x_{2m};\mathbf{a}_2\right)\right)\right)\right) \\ \ldots \\ \sum_{m=1}^{mc} \Omega\left(\frac{\partial s}{\partial a_{Nk}} \left(g_N\left(s\left(x_{Nm};\mathbf{a}_N\right)\right)\left(x_{Nm} - s\left(x_{Nm};\mathbf{a}_N\right)\right) - G_N\left(s\left(x_{Nm};\mathbf{a}_N\right)\right)\right)\right) \end{pmatrix}$$

5. (Optimization Stage): Use an optimization routine to solve the system of equations above that approximate the parameterized set of first order conditions. Solve in terms of the choice variables $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N\}$.
6. Evaluate the candidate equilibrium characterized by $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_N\}^*$, the result of step 5.
7. If the equilibrium is satisfactory,[21] stop; else increase the degree of the approximating function: $k = k + 1$.
8. Return to step 2.

[21] Methods of evaluating candidate solutions are discussed in more detail in chapter 4.

Chapter 3: Implementation of the CSE Algorithm

Introduction

In this chapter I discuss implementation of the CSE algorithm. I review the key components of a successful implementation, and discuss my specific choices of how to execute the algorithm. Finally, the procedure is benchmarked; it is used to solve auction problems (both symmetric and asymmetric) appearing in the literature.

Recall that there are two defining characteristics of the CSE algorithm:

1. The bid function itself is approximated by a parameterized continuous function. Other state-of-the-art algorithms use point-wise approximations rather than continuous parameterizations of the bid function, and/or approximate the inverse bid function rather than the bid function itself.
2. The conditional probability of winning, as embodied in the distribution and density of the marginal bid, is estimated directly based on Monte Carlo experiments. MMRS-type algorithms represent the probability of winning as a transformation of variables based on the distribution of types.

A successful implementation of the CSE algorithm requires that the conditional probability of winning be estimated accurately, that the parameterization of the bidders' strategies be flexible yet parsimonious, and that a robust optimization routine can be used to find the solution to the necessary first order conditions (a system of zeros).

The algorithm is implemented as a suite of Matlab programs called `CSE_SOLVER`, which can be used to find approximate solutions to the equilibrium bidding functions in any pay-as-bid, sealed-bid auction. I discuss especially the constrained strategies that I implement in `CSE_SOLVER`, and how I estimate the probability of winning for each bidder. I make use of available optimization routines that can be integrated directly into the suite of programs.

The algorithm is programmed in a modular style that allows any auction game to be submitted to the solver without modification of the core script. The user writes a script that carries out any desired number of instances of a given auction[22] and returns a vector of critical bids.[23] `CSE_SOLVER` uses this vector to estimate G, the probability of winning from the perspective of a given bidder, and g, the associated density. The first order conditions are formed based on the user's choice of parameterization of the strategies and the estimates of G and g, and then one of several optimization routines can be used to find the set of coefficients that solves the system of zeros.

The implementation of the CSE algorithm uses two strategies to estimate the probability of winning based on data generated from Monte Carlo experiments. Either kernel methods or orthogonal polynomials are used to estimate G and g in the initial stage and, if necessary, *target distribution* methods are used for increased accuracy in a refinement stage. Kernel and orthogonal polynomial methods are well known, so I will provide only a cursory review of how they are implemented. The target distribution method is much less well known. I provide a full explanation of the target distribution method below.

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[22] This script solves the *winner determination problem*.

[23] The script provided for this purpose is itself easy to manipulate. For most auctions, only a few parameters will need to be changed. If an auction is to be solved that does not conform to the script provided, only the self-contained winner determination problem script needs to be provided by the user.

Parameterized Strategies

Both basic global polynomials and monotonic polynomials are implemented in CSE_SOLVER. The construction of monotonic polynomials is discussed at length below. These strategy functions can be used to approximate with arbitrary precision any continuous function. In theory, each of these functions is capable of approximating an arbitrary function. In practice, the four parameterizations offer advantages and disadvantages.

Polynomial Strategies

$$s(x;a_i) = p_K(x;a_i) \equiv a_{i,K}x^K + a_{i,K-1}x^{K-1} + \ldots + a_{i,2}x^2 + a_{i,1}x + a_{i,0}$$

There are several advantages to using a basic polynomial to approximate bidding functions. Simple global polynomials are extremely flexible. The Weierstrass approximation theorem tells us that any continuous function defined on a bounded interval can be uniformly approximated by a single polynomial. Polynomials are also extremely easy to construct and fast to evaluate using Horner's method. As a purely practical matter, if the estimated bidding functions are to be exported to another software package, say for use in a structural econometric exercise, global polynomials are equally easy to represent in any computer language, including in a spreadsheet.

However, in some cases numerical problems can arise with a single global representation of the bidding strategies. It is well known that polynomials tend to diverge near the endpoints of a closed interval when fit to data. This behavior, known as Runge's phenomenon, can make polynomial strategies ill-behaved at times. In addition, high-order polynomials can fit data *too* well, taking on spurious features of the simulated data (“overfitting”), and often becoming non-monotonic.

Monotonic Polynomial Strategies

$$m_k(x) = \int m_k^{'}(x) = \int m_{k-1}^{'}(x) \times \left[1 - 2xa_k + x^2\left(a_k^2 + b_k^2\right)\right]$$

In auction theory, it is commonly assumed that bidding functions are monotonic. Equilibrium strategies are known to be monotonic in many situations where computational methods are needed to derive explicit bidding functions (Lebrun (1996); Lebrun (1999)). In each of the MMRS-style algorithms reviewed in chapter 1, the assumption of monotonicity is maintained in order to derive an inverse bidding function. As a purely theoretical matter, then, it is desirable to have the option to force the constrained strategy to be monotonic. Moreover, if an equilibrium bidding function either does not exist or cannot be found, the assumption of monotonicity is likely a desirable feature of any approximate solution.

Of course, a regular polynomial construction could be used in conjunction with a constrained optimization technique. That is, we could search for the polynomial functions that most nearly satisfy the system of first order conditions, subject to the constraint that the derivative of the polynomial function be non-negative. However, using polynomials that are monotonic *by construction* allows for the same unconstrained optimization methods to be used without extra feasibility iterations. All control variables (coefficients of the parameterized strategies) will result in a solution to the constrained problem that is, at the very least, a feasible solution to the unconstrained problem, which can save computing time. Should it be impossible to find a solution to the system of first order conditions with a given tolerance, using monotonic polynomials ensures that the closest solution we can find is theoretically feasible, and so provides a good model of behavior.

In addition, the use of monotonic polynomials avoids some of the numerical issues associated with “regular” polynomials. Although high-order monotonic polynomials can still overfit simulated data, the problems of non-monotonicity are of course eliminated, which causes the polynomials to hold shape much better near the extremes of the support.

Finally, “cold starts” to the algorithm are easier to implement with monotonic polynomials. When selecting a starting value for a set of strategy parameters, the user typically needs to provide parameters that are reasonably close to the optimum – a “hot start.” An algorithm that relies on a hot start can be slow to converge with poor starting values. When the user has little information on which to base the choice of the initial parameters, a cold start can be performed instead – several potential starting values can be chosen randomly. When the coefficients of a regular polynomial are randomly chosen, many of the resulting strategies are likely to be non-monotonic, infeasible strategies. This problem is solved without any loss of flexibility by using monotonic polynomials.

The construction of the monotonic polynomials uses a conflation method. Rather than the set of coefficients multiplying each power of the ordinary polynomial, the choice variables of the monotonic polynomial strategy are based on a fully factored expression of the polynomial in terms of its roots. We express the constraints on the fully factored version of the polynomial function, and then expand the expression to its most easily parameterized form.

The construction of the monotonic polynomial strategies proceeds in two stages. First, we construct strictly positive polynomials, i.e. polynomials that never take on negative values. I denote these positive polynomials by $m'(\bullet)$. In the second stage, a monotonically increasing polynomial is generated by integrating the positive polynomial. I denote the monotonic polynomial by $m(\bullet)$. The method derived below is due to Elphinstone (1983) and is implemented as part of the distribution of `CSE_SOLVER`.

Constructing Positive Polynomials

In this section I outline some defining characteristics of positive polynomials. These characteristics are used to generate a procedure for *constructing* positive polynomials. Our goal is to create a formula that can express any positive polynomial (and so is fully general), while never expressing a non-positive polynomial.

Let $a_0, a_1, ..., a_n$ be a series of *n* coefficients. Notice that any and every polynomial in one variable can be expressed as $p(x) = \sum_{j=0}^{n} a_j x^j$. We would like to derive a similarly flexible formula that can represent any positive polynomial, but that is incapable of expressing a non-positive polynomial.

We are interested in creating a polynomial that is positive everywhere in a single variable (a bidder's *type*). Since we are only interested in functions of one variable, it is helpful to think graphically. The most obvious characteristic of the graph of a positive polynomial in one variable is that it never crosses the x-axis. A polynomial that crosses the x-axis is somewhere not strictly positive. This is equivalent to the condition that positive polynomials have no real solutions z to the equation $p(z)=0$. Said another way, positive polynomials have *no real roots*. This is the first important characteristic that we will use in deriving a constructive representation of positive polynomials.

We combine two basic results from the theory of algebra to derive the second important characteristic of positive polynomials. The two basic results are: (1) the fundamental theorem of algebra says that a polynomial of degree *n* has *n* roots[24] and (2) if a polynomial has an odd number of roots, at least one root must be real.[25] Since a positive polynomial has no real roots, we can conclude from (2) alone that a positive polynomial cannot have an odd number of roots. So, taken together, (1) and (2) imply that positive polynomials must be of even degree. That is, the highest power to which the argument *x* is raised must be an even number. To see why, the chain of logic proceeds as follows: (2) implies that any function without a real root must have an even number of roots and (1) implies that a polynomial with an even number of roots must itself be of even degree. As a result, we know that a positive polynomial can be constructed from paired terms. This way, the highest power to which *x* is raised will necessarily be divisible by two. Whereas a standard polynomial in one variable can be constructed by a simple summation formula, $\sum_j a_j x^j$, a positive polynomial should be constructed from terms that appear in pairs, i.e. $\sum_j \varphi_{j1}(x)\varphi_{j2}(x)$ where the pair $\varphi_{j1}, \varphi_{j2}$ are some simple functions that always appear together. We turn next to the exact form of these paired terms.

[24] That is, *n* solutions $z_1, z_2, ..., z_n$ to the equation $p(z_j)=0$. These roots can be real or complex.

[25] This is because complex roots only appear in conjugate pairs (see, for example, Weisstein).

We have thus far established two characteristics shared by all positive polynomials, the *no-real-roots* condition and the *even-degree* condition. To finally construct our first representation of a positive polynomial, realize that every polynomial[26] can be represented in irreducible form by

$$\lambda\left(x-z_1\right)\left(x-z_2\right)...\left(x-z_n\right), \tag{18}$$

where λ is a scalar constant, and again the letter *z* represents the roots of the polynomial. Said another way, every polynomial can be completely factored so that it can be expressed as a constant multiplied by a series of terms $\left(x-z_j\right)$, $j=1,...,n$, where each z_j is a root of the polynomial.

We now collect our three facts. Since a polynomial is everywhere positive if and only if it has only complex roots, and these roots appear only in conjugate pairs, and every polynomial can be represented in irreducible form as in (18), then we can represent every strictly positive polynomial, $m'\left(x\right)$, by a sequence of complex conjugate pairs multiplied together.

$$m'\left(x\right)=\lambda\prod_{j=1}^{k}\left(x-z_j\right)\left(x-\overline{z}_j\right), \tag{19}$$

where λ is a constant[27], z_j is a complex root of $m'\left(x\right)$, and $\overline{z}_j$ is its conjugate. Complex numbers are made up of a real number α, plus an imaginary number **i** multiplied by another real number β. The conjugate is formed by changing the sign of the imaginary part. Plugging this representation into (19) we get

[26] All polynomials can be represented this way, not just positive polynomials. See Binmore and Davies (2001), for example.
[27] The constant λ is the coefficient multiplying the highest-order term of the polynomial.

$$m'(x) = \lambda \prod_{j=1}^{k} \left(x - \left(\alpha_j + i\beta_j \right) \right) \left(x - \left(\alpha_j - i\beta_j \right) \right). \tag{20}$$

Equation (20) can be expanded and simplified to yield

$$m'(x) = \lambda \prod_{j=1}^{k} \left[x^2 - 2x\alpha_j + \alpha_j^{\,2} + \beta_j^{\,2} \right]. \tag{21}$$

Note that the imaginary part of the expression (19) has vanished entirely, which enables us to use (21) as the basis for a numerical representation of any positive polynomial.

Since λ, $\alpha = \{\alpha_1, ..., \alpha_k\}$, and $\beta = \{\beta_1, ..., \beta_k\}$ are simply arbitrary constants, we can make some useful substitutions and represent (21) in an equivalent, but more computationally attractive way. Let $a_j = \dfrac{\alpha_j}{\alpha_j^2 + \beta_j^2}$ and $b_j = \dfrac{\beta_j}{\alpha_j^2 + \beta_j^2}$. Then

$$m'(x) = \lambda \prod_{j=1}^{k} \left[1 - 2xa_j + x^2 \left(a_j^2 + b_j^2 \right) \right] \tag{22}$$

is entirely equivalent to (21). This can be seen easily by substituting the definitions of a_j and b_j into (22).

The equation (22) is superior to (21) because it has a structure that facilitates simple iterative calculations. One of the reasons polynomials are used so often to approximate unknown functions is because polynomials of degree n are related in a straightforward way to polynomials of degree $n+1$. For example, a polynomial $p_2(x) = a_0 + a_1x + a_2x^2$ is equal to $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ when $a_3 = 0$. Because of this, if we find that an unknown function is well-approximated by the quadratic polynomial $p_2(x)$, then we can begin our search for the best-fitting cubic polynomial using the starting values $\{a_0, a_1, a_2, 0\}$. The standard representation of a general polynomial, $p_n(x; a_0, ..., a_n) = \sum_{j=0}^{n} a_j x^j$ has the property that $p_n(x; a_0, ..., a_n) = p_{n+1}(x; a_0, ..., a_n, 0)$; equation (22) also has this property. To calculate a positive polynomial of degree 2, we set $k = 1$. To calculate a positive polynomial of degree 4, we set $k = 2$; the values a_1 and b_1 can be used as starting values.

Constructing Monotonic Polynomials

In this section I show how to group the estimated coefficients of the positive polynomial $m'(x)$ in order to make integration by the power rule straightforward. This enables construction of the monotonic polynomial $m(x) = \int m'(t)\, dt$. The task is conceptually simple, but is difficult to reduce to a reasonable number of steps (so that the number of computer operations does not grow too quickly as we increase k). If the expansion of (22) resulted in an expression like

$$m'_k(x) = D_0 + D_1 x^1 + D_2 x^2 + ... + D_k x^k,$$

such that the coefficient D_τ multiplied x to the power τ, then we could obtain $m_k(x)$ by a simple application of the power rule

$$m_k(x) = \int m'_k(x) = D_0 x + \frac{D_1 x^2}{2} + \frac{D_2 x^3}{3} + ... + \frac{D_{2k} x^{2k+1}}{2k+1}.$$

The expansion of (22) is unfortunately quite a bit messier. As it stands, the terms multiplying each successive power of x are a combination of the parameters $\{\lambda, a, b\}$.We need a way of rewriting (22) which makes grouping terms that multiply the same power of x computationally simple. To collect terms, I introduce some new notation.

Since (22) is calculated iteratively, we can write any given positive polynomial as a product of $k+1$ terms

$$m'_k(x) = m'_0(x)q_1(x)q_2(x)...q_k(x), \tag{23}$$

where $m'_0(x) = \lambda$ is the value of the positive polynomial when $k = 0$, and each term $q_j(x)$ represents the bracketed term in (22), i.e.

$$q_j(x) = \left[1 - 2xa_j + x^2\left(a_j^2 + b_j^2\right)\right]. \tag{24}$$

Let the term multiplying x^0 inside the brackets in (24) be denoted $d_0^{(j)}$, the term multiplying x^1 be denoted $d_1^{(j)}$, and the term multiplying x^2 be denoted $d_2^{(j)}$. Then we can rewrite (22) as

$$m'(x) = \lambda \prod_{j=1}^{k}\left[d_0^{(j)} + d_1^{(j)}x + d_2^{(j)}x^2\right], \tag{25}$$

where

$$\begin{aligned} d_0^{(j)} &= 1 \\ d_1^{(j)} &= -2a_j \quad . \\ d_2^{(j)} &= a_j^2 + b_j^2 \end{aligned} \tag{26}$$

We can now see that for $k = 2$ we have

$$\begin{aligned} m_2^{'}(x) = \lambda[& d_0^{(1)}d_0^{(2)} + d_0^{(1)}d_1^{(2)}x + d_0^{(1)}d_2^{(2)}x^2 + \\ & + d_1^{(1)}d_0^{(2)}x + d_1^{(1)}d_1^{(2)}x^2 + d_1^{(1)}d_2^{(2)}x^3 + \quad . \\ & + d_2^{(1)}d_0^{(2)}x^2 + d_2^{(1)}d_1^{(2)}x^3 + d_2^{(1)}d_2^{(2)}x^4] \end{aligned}$$

Note that there are three separate sets of coefficients multiplying x^2: $d_0^{(1)}d_2^{(2)}$, $d_1^{(1)}d_1^{(2)}$, and $d_2^{(1)}d_0^{(2)}$. We want to collect terms so that we can express $m_2'(x)$, a fourth degree polynomial, in terms of just five coefficients. Notice that the sole coefficient multiplying x to the power τ is the sum of all $d_m d_n$ such that $m+n=\tau$. We can collect terms so that we can express $m_2'(x)$ in the desired form

$$m_2'(x)=\lambda\left[D_0^{(2)}+D_1^{(2)}x+D_2^{(2)}x^2+D_3^{(3)}x^3+D_4^{(4)}\right],$$

where

$$\begin{aligned}
D_0^{(2)} &= d_0^{(1)}d_0^{(2)} \\
D_1^{(2)} &= \left(d_0^{(1)}d_1^{(2)}+d_1^{(1)}d_0^{(2)}\right) \\
D_2^{(2)} &= \left(d_0^{(1)}d_2^{(2)}+d_1^{(1)}d_1^{(2)}+d_2^{(1)}d_0^{(2)}\right). \\
D_3^{(2)} &= \left(d_1^{(1)}d_2^{(2)}+d_2^{(1)}d_1^{(2)}\right) \\
D_4^{(2)} &= d_2^{(1)}d_2^{(2)}
\end{aligned}$$

In this case, we accomplished the grouping of terms by simple inspection. We would like to express the coefficients such that the grouping of terms was "automatic".

By expressing the coefficients in matrix form, we accomplish our goal of grouping terms during the construction of the positive polynomial. Let $D_\tau^{(k)}$ represent the sole coefficient multiplying x to the power τ of a positive polynomial of degree $2k$ (recall that the degree of the positive polynomial is twice k, since the formula (22) always yields polynomials of even degree). Examining the first few values of k reveals a simple pattern.

- $k=1$:

$$D^{(1)} = \begin{pmatrix} D_0^{(1)} \\ D_1^{(1)} \\ D_2^{(1)} \end{pmatrix} = \begin{pmatrix} d_0^{(1)} \\ d_1^{(1)} \\ d_2^{(1)} \end{pmatrix} \text{ by definition.}$$

- $k = 2$:

$$D^{(2)} = \begin{pmatrix} D_0^{(2)} \\ D_1^{(2)} \\ D_2^{(2)} \\ D_3^{(2)} \\ D_4^{(2)} \end{pmatrix} = \begin{pmatrix} d_0^{(2)} & 0 & 0 \\ d_1^{(2)} & d_0^{(2)} & 0 \\ d_2^{(2)} & d_1^{(2)} & d_0^{(2)} \\ 0 & d_2^{(2)} & d_1^{(2)} \\ 0 & 0 & d_2^{(2)} \end{pmatrix} \begin{pmatrix} D_0^{(1)} \\ D_1^{(1)} \\ D_2^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} d_0^{(2)} & 0 & 0 \\ d_1^{(2)} & d_0^{(2)} & 0 \\ d_2^{(2)} & d_1^{(2)} & d_0^{(2)} \\ 0 & d_2^{(2)} & d_1^{(2)} \\ 0 & 0 & d_2^{(2)} \end{pmatrix} \begin{pmatrix} d_0^{(1)} \\ d_1^{(1)} \\ d_2^{(1)} \end{pmatrix} = \begin{pmatrix} d_0^{(2)} d_0^{(1)} \\ d_1^{(2)} d_0^{(1)} + d_0^{(2)} d_1^{(1)} \\ d_2^{(2)} d_0^{(1)} + d_1^{(2)} d_1^{(1)} + d_0^{(2)} d_2^{(1)} \\ d_2^{(2)} d_1^{(1)} + d_1^{(2)} d_2^{(1)} \\ d_2^{(2)} d_2^{(1)} \end{pmatrix}.$$

The general formula for computing the coefficients efficiently is

$$\begin{pmatrix} D_0^{(k)} \\ D_1^{(k)} \\ D_2^{(k)} \\ \dots \\ D_{2k}^{(k)} \end{pmatrix} = \begin{pmatrix} d_0^{(k)} & 0 & \dots & 0 & 0 \\ d_1^{(k)} & d_0^{(k)} & \dots & \dots & \dots \\ d_2^{(k)} & d_1^{(k)} & \dots & \dots & \dots \\ 0 & d_2^{(k)} & \dots & 0 & \dots \\ \dots & 0 & \dots & d_0^{(k)} & 0 \\ \dots & \dots & \dots & d_1^{(k)} & d_0^{(k)} \\ \dots & \dots & \dots & d_2^{(k)} & d_1^{(k)} \\ 0 & 0 & \dots & 0 & d_2^{(k)} \end{pmatrix} \begin{pmatrix} D_0^{(k-1)} \\ D_1^{(k-1)} \\ D_2^{(k-1)} \\ \dots \\ D_{2(k-1)+1}^{(k-1)} \end{pmatrix}. \tag{27}$$

For a polynomial of degree $2k$, the single coefficient on x^0 is constructed by multiplying the "new" coefficient $d_0^{(k)}$ by the single coefficient on x^0 in a polynomial of degree $2(k-1)$. The single coefficient on x^1 is constructed by summing two terms: (1) multiply the coefficient $d_1^{(k)}$ by the single coefficient on x^0 in a polynomial of degree $2(k-1)$; (2) multiply the coefficient $d_0^{(k)}$ by the single coefficient on x^1 in a polynomial of degree $2(k-1)$. The pattern continues. Combinations of the "new" coefficients are multiplied by all existing terms from a lower-degree polynomial to create new terms.

Utilizing this matrix representation, we can calculate the sole coefficients multiplying all powers of x, i.e. powers $0,...,2k$, using just k matrix operations. This representation makes integrating the positive polynomial computationally simple, and allows us to generate any monotonic polynomial using the coefficients from (27) directly in the simple expression

$$m_k(x) = \sum\nolimits_{j=0}^{k} \frac{D_j x^{j+1}}{j+1}. \qquad (28)$$

Estimating G and g

I use three different techniques to estimate the distribution and density of the marginal bid. As suggested by Armantier (2006), I have implemented an estimation strategy based on orthogonal polynomials. Armantier suggests using orthogonal polynomials because of their speed. However, I have found that a strategy combining kernel methods and interpolation to be just as fast, and much more accurate in most cases. Finally, I have implemented a target distribution method which, although slower to calculate than either orthogonal polynomials or kernel methods, can provide smoother estimates. Because the target distribution method sacrifices speed for accuracy, it is most useful as a final refinement to the CSE procedure. In the following sections, I will briefly discuss orthogonal and kernel estimation methods, and discuss the target distribution method in slightly more detail.

Orthogonal Series Method

The unknown density of the critical bid, $g(\bullet)$, can be nonparametrically estimated by

$$\hat{g}(x) = w(x)\sum_{k=0}^{n}\hat{a}_k\varphi_k(x),$$

where $w(x)$ is a weighting function, $\hat{a}_k$ are coefficient weights to be estimated, and $\varphi_k(x)$ are functions chosen to be *mutually orthonormal* with respect to the weighting function $w(x)$. The functions $\varphi_k(x)$ are mutually orthonormal if the following two conditions hold:

(i.) $\int_a^b \varphi_k(x)\varphi_j(x)w(x)dx = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$

(ii.) $\varphi_0(x) = 1$.

There are several families of polynomials that can serve as the φ family of functions. Among them are Legendre, Chebyshev, Laguerre, and Hermite polynomials. Legendre polynomials have the particularly attractive property that the weighting function that causes condition (i.) to hold is simply $w(x) = 1$. The coefficient weights are estimated by

$$\hat{a}_k = \frac{1}{MC}\sum_{i=1}^{MC}\varphi_k(x_i), \text{ for } k = 1, 2, ..., n.$$

The advantage of estimating density functions by orthogonal polynomials is that an estimate of the density at any given point *x* can be calculated without calculating the density at all other points – that is, the estimation is entirely *local*. In addition, orthogonal polynomials can be computed by a recurrence relation, which makes computation numerically attractive. The number of computations needed to arrive at an estimate is relatively small. Additionally, since we need to carryout out *MC* instances of the auction in order to determine the winner, we need to do very few additional calculations in order to estimate $\hat{a}_k$.

The technique is not without its drawbacks. Although the procedure of estimating a density function on a compact interval is simple and computationally attractive, the estimate is not necessarily smooth. As opposed to kernel methods, density estimates using the orthogonal series technique can only be smoothed by increasing the Monte Carlo size or the degree of the approximating polynomial, and the optimal degree of the polynomial approximation is difficult to determine.[28]

Kernel Method

A kernel estimator is of the form

$$\hat{g}(x) = \frac{1}{MC \cdot h} \sum_{i=1}^{MC} K\left(\frac{x - x_i}{h}\right),$$

where *h* is a bandwidth parameter, *K* is the kernel function, and $\{x_1, x_2, ..., x_{MC}\}$ are the observed data. The kernel function can be any function; popular choices are the normal pdf, the triangle pdf, or the epanechnikov pdf. As the

[28] The approximation by orthogonal polynomials converges to the true function as $n \to \infty$. For practical purposes, *n* needs to be large enough such that the difference between successive orthogonal coefficients is small. The value of *n* necessary to provide an accurate estimate of the unknown function is unknown a priori, a fact which favors the more robust kernel method, in my experience.

preceding list makes clear, the kernel function itself is often a probability density function, but this is not required.[29]

Target Distribution Method

The target distribution method is based on a simple premise. Every two continuous distribution functions $G(\bullet)$ and $H(\bullet)$ are related by at least one transformation function $t(\bullet)$ such that $G(x) = H(t(x))$.[30] Suppose *G* is the unknown distribution function we wish to estimate. We can then select H, the *target distribution*, and estimate *G* by $H(\hat{t}(x))$. By estimating the transformation function and fixing the target distribution, we obtain an estimate of the unknown distribution $\hat{G}$ that (i) incorporates any prior information we may possess as to the form of the true distribution that generated our data, (ii) incorporates information on the support of the distribution, and (iii) is sufficiently flexible to provide excellent performance even when the target distribution is chosen poorly.

The target distribution method is nonparametric in the sense that no assumption is made as to the form of *G*. On the other hand, we do place restrictions on *H* in order to ensure that our estimate $\hat{G}$ is itself a distribution function, and that the same procedure yields $\hat{g} = d\hat{G}$, an estimate of the density that has all the properties of a p.d.f.

[29] See Turlach (1993) and Härdle and Linton (1994) for details on properties of the kernel and desirable selection criteria.
[30] See Elphinstone (1983) and citations therein.

Since H is selected to be a continuous distribution, $H\left(t\left(x\right)\right)$ is a distribution so long as $t\left(x\right)$ is monotonically increasing. Therefore, $\frac{d}{dx}H\left(t\left(x\right)\right)=h\left(t\left(x\right)\right)t^{'}\left(x\right)$ is a well-behaved probability density function, again so long as $t\left(x\right)$ is monotonically increasing.

As H is fixed and G is unknown, the computation necessary to carry out the target method reduces to finding an appropriate estimate $\hat{t}\left(x\right)$. The estimate must be: (i) monotonically increasing and (ii) sufficiently flexible to estimate the true $t\left(x\right)$. Using an unrestricted polynomial to estimate $t\left(x\right)$ is tempting, as the Weierstrass theorem tells us that $t\left(x\right)$ can be estimated to any desired degree of accuracy with a polynomial over a closed interval. However, it is also well known that the best approximation of a monotonically increasing function is not necessarily itself a monotonically increasing function. That is, an unrestricted polynomial fits requirement (ii), but not requirement (i). This was Elphinstone's original motivation for the procedure to generate monotonic polynomials by conflation, which I outlined above.

The CSE Solution in Benchmark Cases

First-price auction with uniformly distributed values

Probably the most well known sealed-bid auction model is the first-price symmetric IPV model with uniformly distributed values. Any advanced microeconomics or introductory game theory text will include analysis of this auction model. The unique equilibrium bidding function is well known to be

$$s(x)=\frac{N-1}{N}x\,.$$

The Bayesian Nash Equilibrium is a linear function of the value draw *x*. Thus in this case, the CSE of the auction problem is exactly identical to the unconstrained BNE.

We will solve this auction posed as a CSE problem when $N = 2$. Doing so will provide a great deal of insight into how the algorithm works, and demonstrates the accuracy of the CSE algorithm in a basic case.

Restricting ourselves to a simple linear function, recall that the first order condition of the CSE problem from the perspective of bidder 1 can be written:

$$\frac{d}{da_1}s(x_1;a_1)\Big(\big(x_1-s(x_1;a_1)\big)g\big(s(x_1;a_1)\big)-G\big(s(x_1;a_1)\big)\Big)=0\,. \tag{14}$$

When $s(x_1;a_1)=a_1x_1$, $\frac{d}{da_1}s(x_1;a_1)=x_1$ and the first order condition can be expressed as

$$x_1\Big(\big(x_1-a_1x_1\big)g\big(s(x_1;a_1)\big)-G\big(s(x_1;a_1)\big)\Big)=0\,.$$

In a symmetric equilibrium, $a_1=a_2=a$, and the analytical expressions for *g* and *G* are $\frac{1}{a}$ and x, respectively. Substituting in, we get:

$$x\left(\left(x-ax\right)\frac{1}{a}-x\right)=0\,.$$

Figure 1 illustrates the approach (and the challenge) of trying to find an equilibrium by approximating the first order condition. Figure 1 shows the analytical first order condition for all values of *x* ($x \in [0,1]$), evaluated at candidate solutions for the linear coefficient *a* between 0 and the BNE value of 0.5. The coefficient *a* is represented on the front-facing (left-to-right) axis, and the values *x* are represented on the right-facing (front-to-back) axis. The optimal (BNE) coefficient is located at the far right of edge of the 3D plot. The value of the first order condition is zero for all values of *x* along the right edge of the graph. As *a* decreases (along the front-facing axis, *a* decreases from right-to-left) the first order condition diverges from zero for values of $x > 0$, with the magnitude of the first order condition increasing in *x*.

Figure 1 shows the value of the first order condition for all possible values of *x*. It is the *average* value, however, of the first order condition we are evaluating with the Monte Carlo technique. The average of the first order condition will be 0 at the BNE coefficient a^*, since in equilibrium the value of (14) is 0 for *every x*. The average value of the first order condition is plotted in Figure 2. That is, the front-to-back axis in Figure 1 is collapsed by averaging all values along a "slice" of Figure 1 for a fixed value of *a*. This produces a single line representing the value of the first order condition as *a* increases from left to right. Starting from $a = 1$ (bid = value), the first order condition approaches zero smoothly from below.

The following Mathematica code generates Figure 1and Figure 2.

```
(* FOC for a buy-auction *)
foc[x_,a_]:=(x-a*x)*(x/a)-(x^2)

Plot3D[foc[x,a],Cao and
Tian),{a,0.0000000000001,0.5},AxesLabel→{Style[x,FontSize→18],Style[
a,FontSize→18],Style[foc,FontSize→18]}]

avgfoc[a_]:=Sum[foc[x,a],{x,0,1,0.01}]/101

Plot[avgfoc[a],{a,0.0000000001,1},AxesLabel→{Style[a,FontSize→18],St
yle[foc,FontSize→18]}]
```

Figure 1 – Analytical FOCs of a buy-auction

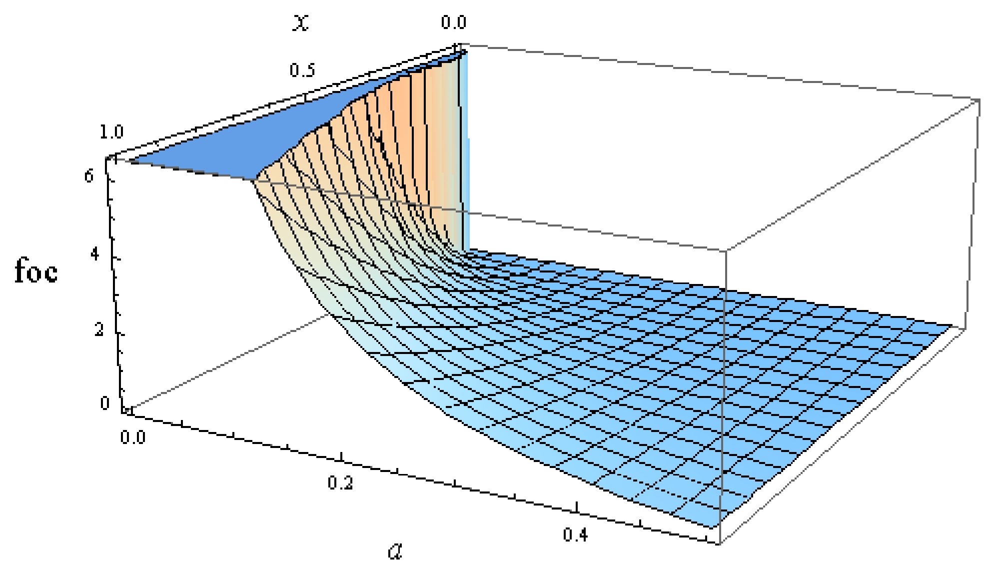
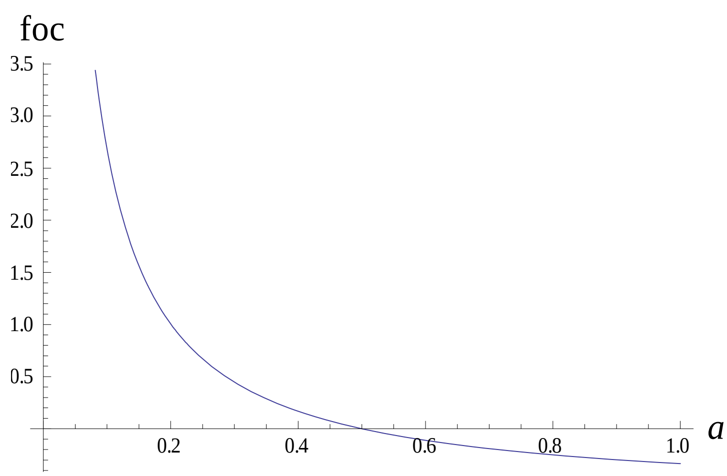


Figure 2 – Average FOCs as *a* varies



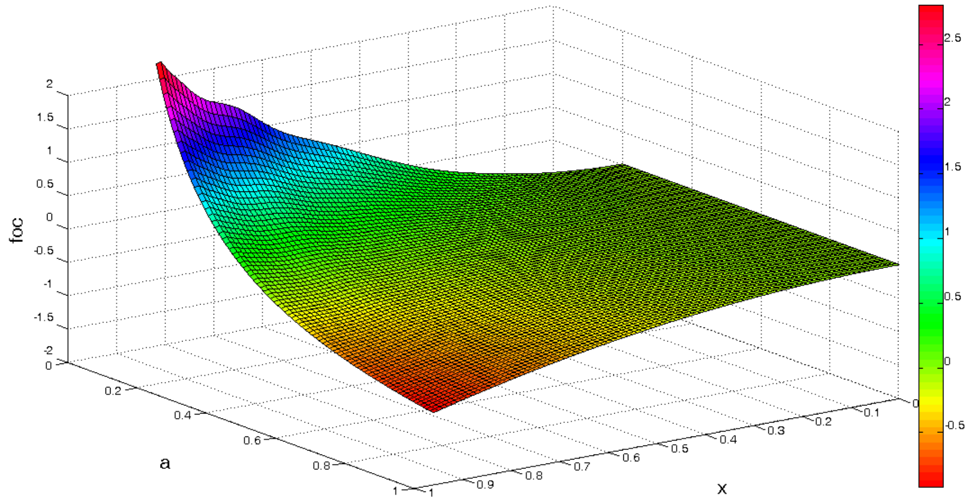
The approximation of the distribution and density functions *G* and *g* introduces Monte Carlo error into this procedure. All the simulation error we get is generated by the difference between the Monte Carlo estimates $\hat{g}$ and $\hat{G}$ and their true values $\frac{1}{a}$ and x. The following MATLAB code generates estimates of *g* and *G* using kernel methods and a Monte Carlo size of 1e4, and plots the value of the first order condition for potential coefficients *a* over the range of *x*.

```
% Set the pseudo-random number stream for replicability.
stream = RandStream('mrg32k3a');
RandStream.setDefaultStream(stream);
stream.Substream = 1;

% Evaluate the FOC at even intervals over the domain of x.
x1 = (0:0.01:1)';
% Evaluate the FOC for all feasible values of the coefficient a.
a = 0.2:0.01:1;
% Draw random numbers for bidder 2. This is the Monte Carlo sample.
x2 = random('unif',0,1,1e4,1);
% Form a grid of the data for 3D plot.
[A1,X1] = meshgrid(a,x1);
[A2,X2] = meshgrid(a,x2);
% Bids of bidder 2.
B2 = A2.*X2;
% Make container arrays for the density and distribution.
PDF = zeros(size(A1));
CDF = zeros(size(A1));
% Fill in the density and distribution.
for i = 1:1:size(X1,2)
    PDF(:,i) = ksdensity(B2(:,i),X1(:,i).*A2(1,i),...
        'support',[0 A2(1,i)],'function','pdf');
    CDF(:,i) = ksdensity(B2(:,i),X1(:,i).*A2(1,i),...
        'support',[0 A2(1,i)],'function','cdf');
end
% Evaluate the FOC.
FOC = X1.*((X1 - A1.*X1).*PDF - CDF);
% Exclude values of x very close to 1 due to numerical instability.
X1(98:101,:) = []; A1(98:101,:) = []; FOC(98:101,:) = [];
% Plot the FOC over all values of (x,a).
surf(X1,A1,FOC)
xlabel('x')
ylabel('a')
zlabel('foc')
colormap hsv
axis([0 1 0 1 -2 2])
```

The result of the code is displayed in Figure 3. The approximation of the first order condition appears to be quite good, although some numerical problems are hidden here by the exclusion of points *x* close to 1 (see Figure 6 and the discussion below). The approximate solution can be seen by tracing the line of light green across a slice of the surface when *a* is about 0.5.

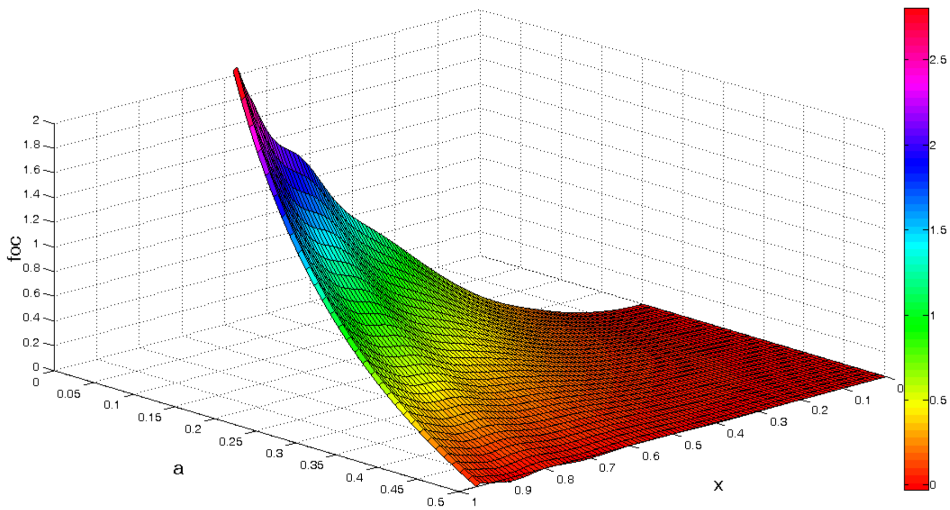
Figure 3 – Simulated FOCs of a buy-auction



In order to see the surface corresponding to $a = 0.5$ (the equivalent of the right edge of Figure 1) we look at Figure 4, which stops at $a = 0.5$ to isolate the slice of Figure 3 along that edge. We can see that at the BNE solution, the value of the simulated first order condition is very near to zero (colored in red).

```
% Plot the FOC over all values of (x,a) s.t. a <= 0.5.
surf(X1(:,1:31),A1(:,1:31),FOC(:,1:31))
xlabel('x'   ,'FontSize',16)
ylabel('a'   ,'FontSize',16)
zlabel('foc','FontSize',16)
colormap hsv
axis([0 1 0 0.5 0 2])
```

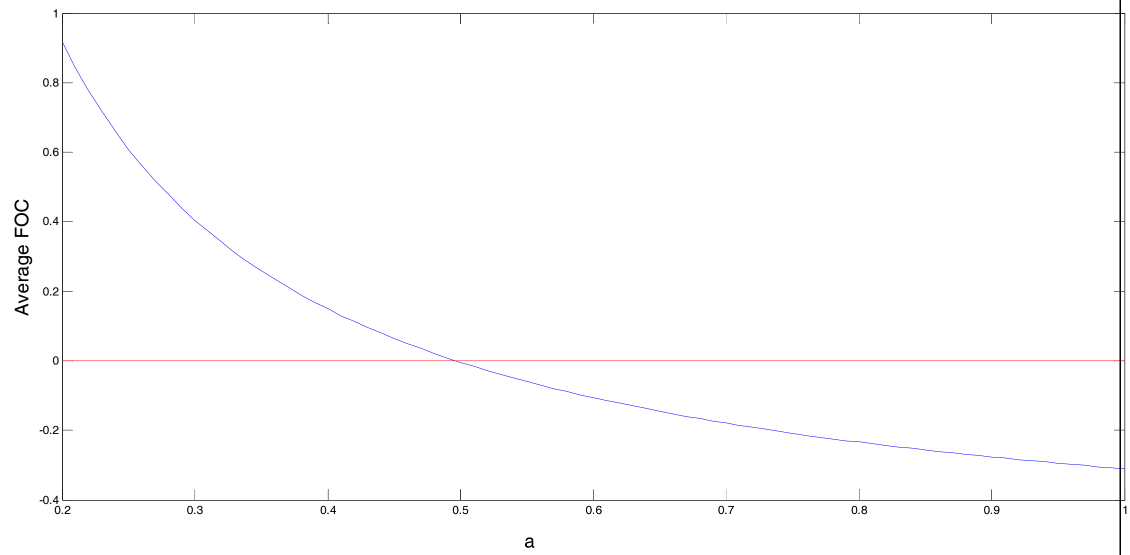
Figure 4 – Simulated FOCs of a buy-auction (focused)



Once again, we collapse the surface by averaging over the x-values for a given value of the coefficient *a*. The results are shown in Figure 5.

```
% Figure 5 -- Average of the x-values at a = 0.5.
avgfoc = mean(FOC,1);
plot(A1(1,:),avgfoc); hold on;
plot(A1(1,:),zeros(size(A1(1,:))),'col','red'); hold off;
xlabel('a','FontSize',16)
ylabel('Average FOC','FontSize',16)
```

Figure 5 – Simulated average FOCs as *a* varies



The constrained equilibrium solution is actually found to be 0.4964 based on a grid search with step size 0.0001. The simulation error of 0.0036 is mostly due to errors in the estimation of the density of the winning bid *g*, and these errors are mostly clustered at the upper end of the range of *x* values. The disturbance for values when *x* is close to 1 can be seen in the front of Figure 4, where there appears to be a wave along the leading edge of the FOC surface.

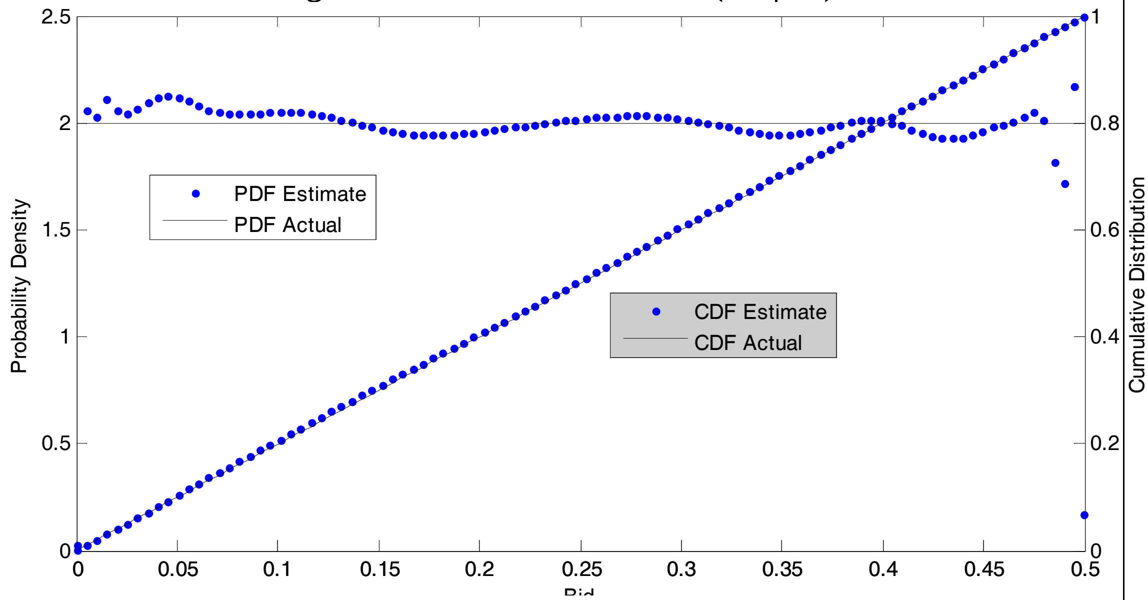
```
% What is the exact minimum? Is it at a = 0.5?
% Do a very fine grid search.
a = 0.45:0.0001:0.55;
x1 = (0:0.01:1)';
[A1,X1] = meshgrid(a,x1);
[A2,X2] = meshgrid(a,x2);
B2 = A2.*X2;
PDF = zeros(size(A1));
CDF = zeros(size(A1));
for i = 1:1:size(X1,2)
    PDF(:,i) = ksdensity(B2(:,i),X1(:,i).*A2(1,i),...
        'support',[0 A2(1,i)],'function','pdf');
    CDF(:,i) = ksdensity(B2(:,i),X1(:,i).*A2(1,i),...
        'support',[0 A2(1,i)],'function','cdf');
end
FOC = X1.*((X1 - A1.*X1).*PDF - CDF);
avgfoc = mean(FOC(1:96,:),1);
minindx = find(foo == min(foo));
```

```
trueMin = A1(1,minindx);
% Result --> 0.4965.
```

Taking a closer look: the following MATLAB code examines in greater detail the slice of the surface in Figure 4 when $a = 0.5$. It is instructive to examine the estimation of g and G at this point, as it illustrates the difficulty of accurately estimating the distribution of a bid on a compact support. The result of this exercise is shown in Figure 6.

```
% Estimate G and g for the BNE coefficient a = 1/2.
% Set coefficient a = 0.5;
a = 0.5;
% Form bids for bidder 2.
b2 = a*x2;
% Estimate density.
[pdf xi] = ksdensity(b2,'support',[0,1*a],'function','pdf');
% Estimate distribution.
[cdf xi] = ksdensity(b2,'support',[0,1*a],'function','cdf');
% Plot both pdf and cdf.
[ax,h1,h2] = plotyy([xi' xi'],[pdf' (ones(size(xi))/(1/2))'],...
    [xi' xi'],[cdf' (xi/(1/2))'])
set(h1(1),'LineStyle','.');set(h1(2),'Color','black');
set(h2(1),'LineStyle','.');set(h2(1),'Color','blue');
set(h2(2),'Color','black');
legend(h1,'PDF Estimate','PDF Actual','Location',[0.2,0.65 0.05
0.01])
legend(h2,'CDF Estimate','CDF Actual','Location',[0.6 0.45 0.05
0.01])
xlabel('Bid')
set(get(ax(1),'Ylabel'),'String','Probability Density')
set(get(ax(2),'Ylabel'),'String','Cumulative Distribution')
title('Kernel Estimates: Pr(win|bid)')
% Note that you may need to re-size the graphics window in order for
text annotations to appear as shown.
```

Figure 6 – Kernel Estimates: Pr(win|bid)

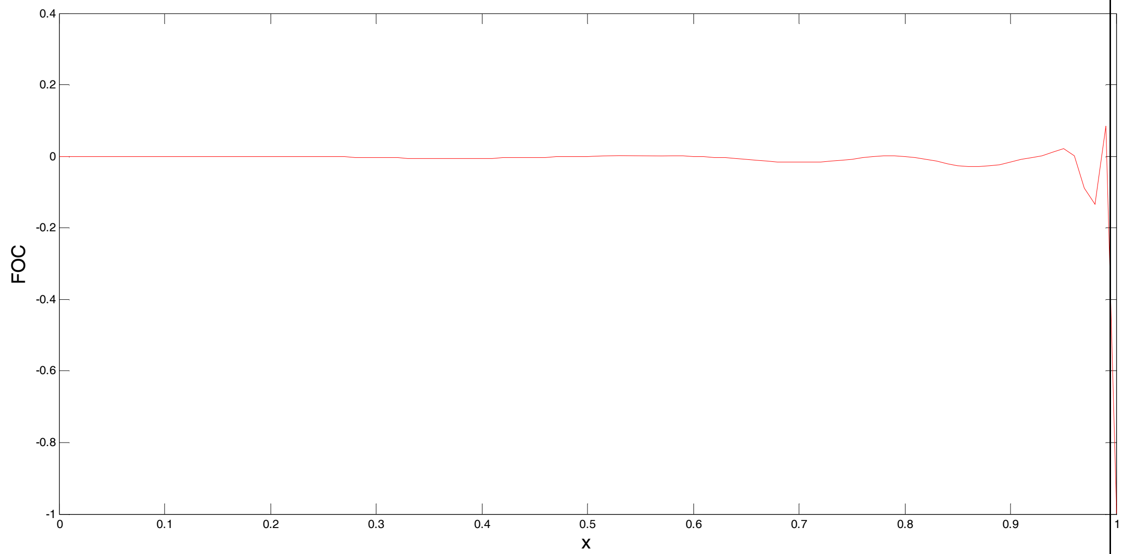


The estimate of *G* is quite good over the entire range of *x*, so much so that estimated points overlay and obscure the true value of the CDF. The estimate of *g* is less precise. Nonparametric methods are in a way “local” – the value of a density estimate at a given point draws on nearby data. Points near either end of a compact support necessarily have fewer data points within a given window, since no data appears above (below) the upper (lower) endpoint of the distribution. I have found instability near the endpoints of the support of bids using both kernel methods and orthogonal polynomials. Using these estimates of the PDF and CDF of bidder 2’s bid, we can evaluate the first order condition at evenly spaced intervals over the domain of *x*.

```
% Evaluate the first order condition at values along x.
x1 = (0:0.01:1)';
b2 = 0.5*x2;
g = ksdensity(b2,0.5*x1,'support',[0 0.5],'function','pdf');
G = ksdensity(b2,0.5*x1,'support',[0 0.5],'function','cdf');
foc = x1.*((x1 - 0.5.*x1).*g - G);
plot(x1,foc,'col','red')
```

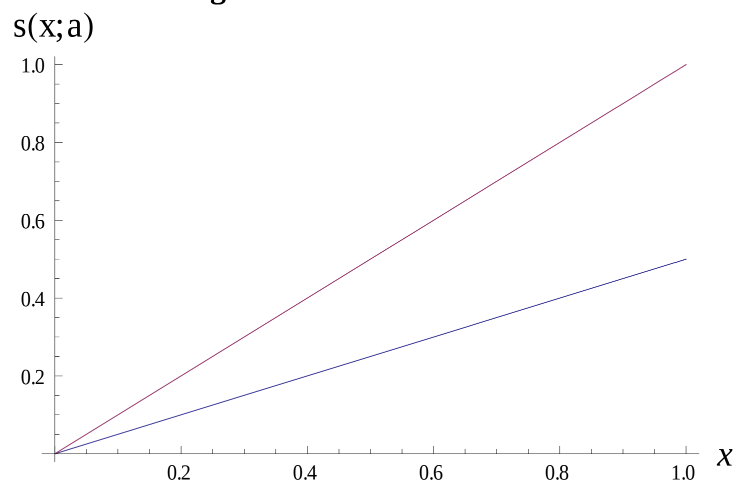
```
xlabel('x','FontSize',16); ylabel('FOC','FontSize',16);
```

Figure 7 – FOC over *x* for *a* = 0.5 (BNE).



The instability of $\hat{g}$ as $x \to 0$ has no visible influence on the estimated first order condition displayed in Figure 7. The estimation error in that region is overwhelmed by the small values of $\left(x_1 - s\left(x_1; a_1\right)\right)$, the gain from winning the auction, in the first order condition (14). As $x \to 0$, notice that $s\left(x_1; a_1\right) \to x_1$ (see the equilibrium bidding function in Figure 8) and so the term multiplying *g* becomes extremely small. On the other hand, the estimation error in $\hat{g}$ as $x \to 1$ has a significant impact on the first order condition, which can be seen clearly in Figure 7. The profit from winning the auction in this region is relatively large, meaning that estimation errors in $\hat{g}$ are amplified. For this reason, we exclude from our calculations values of the first order condition as *x* approaches the upper (lower) end of its support in a buy- (sell-) auction.

Figure 8 – BNE bid function



The MMRS coalition vs. coalition model

The MMRS coalition-vs-coalition model can be estimated by the CSE technique, and the solutions from the two competing algorithms can be compared. The components of the model are as follows. There are 5 total bidders ($n = 5$); two coalitions form, where the first coalition is either of size 4 or 3 (the second coalition is either of size (1 or 2); draws of value are from the uniform [0,1]; a coalition bids according to its highest value (i.e. a coalition of 3 individuals behaves as if its value is the highest among its three individual draws).

We will focus here on a single case of the MMRS model. The MMRS algorithm with coalitions of size 4 and 1 produces the bidding functions represented in Figure 9 (originally appearing as Figure 1A in Marshall, et al. (1994) and reproduced here with permission from Elsevier – see appendix). The same model, submitted to `CSE_SOLVER`, gives the results displayed in Figure 10.

Figure 9 – Bidding functions in MMRS coalition model

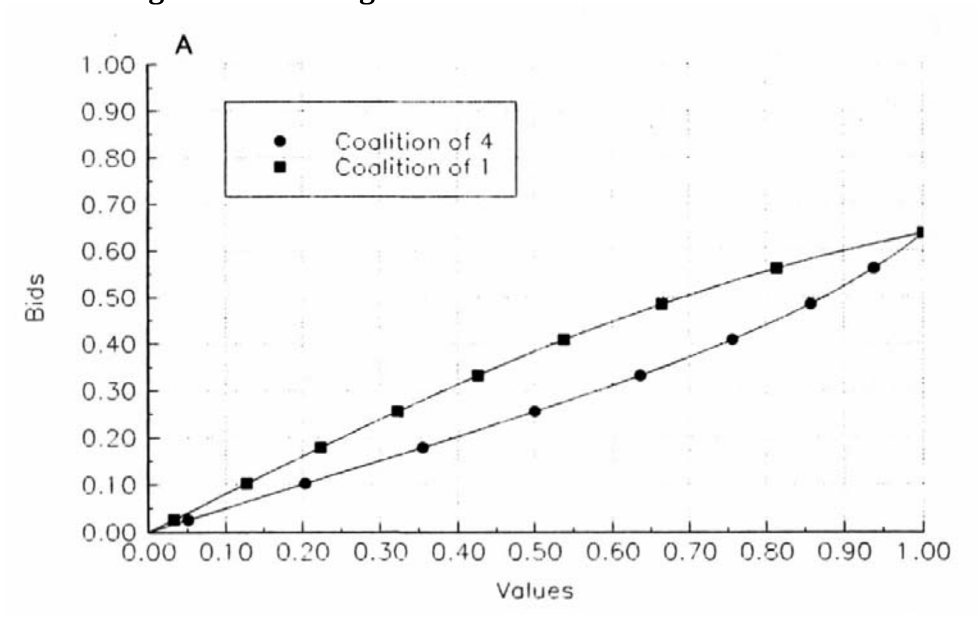
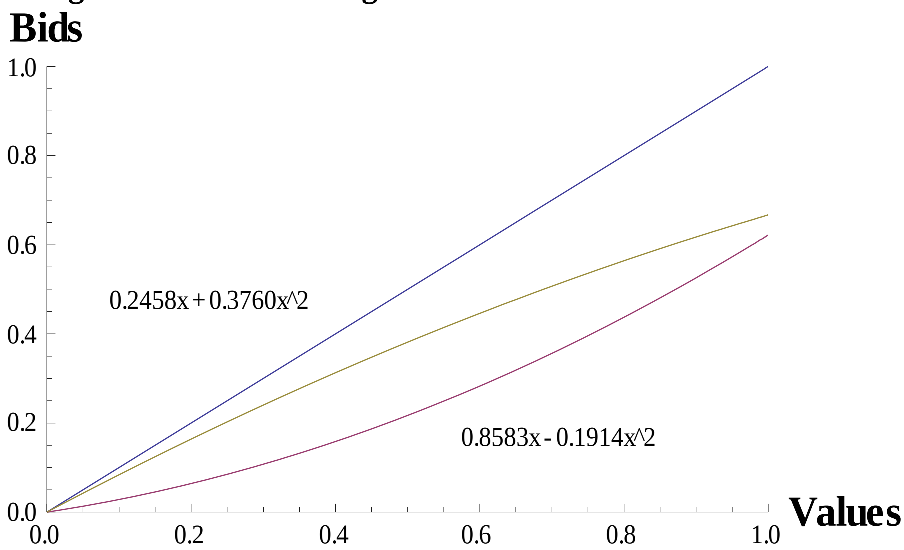
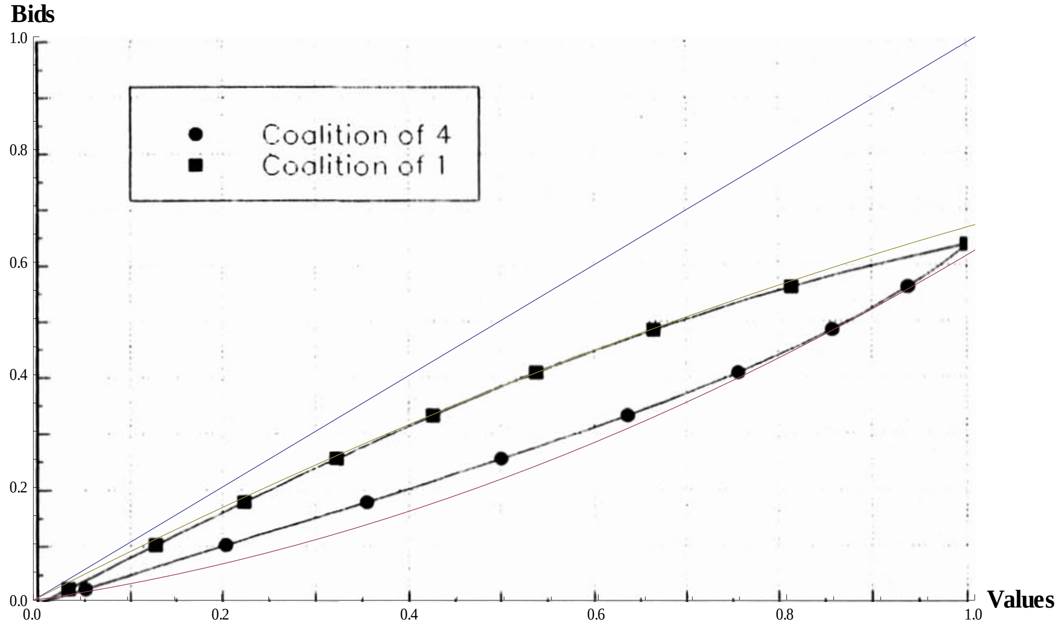


Figure 10 – CSE bidding functions in MMRS coalition model



You can see in Figure 10 that the bidding functions arrived at by the CSE algorithm are nearly equal to those of MMRS. Figure 11 makes the near-equivalence more apparent by overlaying the two plots, showing just how close the two techniques are. The two figures have been overlaid using Mathematica, which allows the plots to be manipulated onto the same scale. I have stretched the CSE plot so that the aspect ratio matches as nearly as possible the exact dimensions of the original plot, which appears as an image underlying the CSE plot.

Figure 11 – MMRS/CSE bidding functions overlaid



The CSE procedure does not quite match the MMRS estimates, which are reported accurate to 6 to 8 digits.[31] The CSE procedure can be improved, however, by imposing the added condition that $s(1;\mathbf{a}_1) = s(1;\mathbf{a}_2)$, the condition that MMRS algorithm relies on for convergence. This condition is imposed simply by restricting one of the four coefficients (the choice of which one is arbitrary) needed to estimate a quadratic CSE. We simply choose one of the coefficients, in this case we fix the coefficient on the second squared-term, and solve the identity:

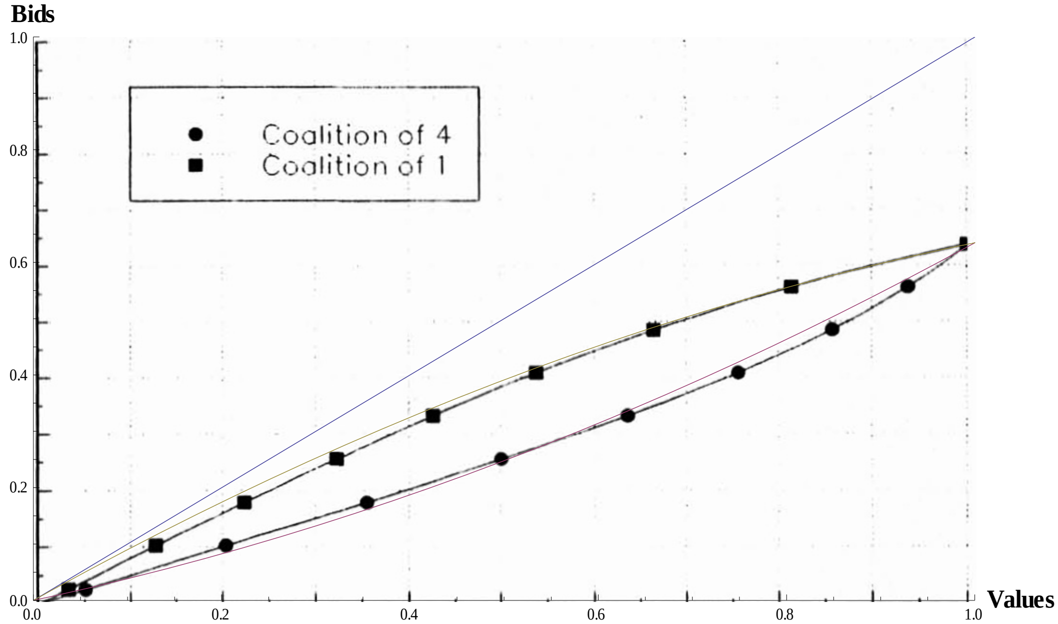
$$s(1;\mathbf{a}_1) = s(1;\mathbf{a}_2)$$
$$\left(a_{11}x + a_{12}x^2\right)\Big|_{x=1} = \left(a_{21}x + a_{22}x^2\right)\Big|_{x=1}$$
$$a_{22} = \left(a_{11} + a_{12}\right) - a_{21}.$$

This restriction means that the CSE bidding functions will be defined by a total of three coefficients $\{a_{11}^*, a_{12}^*, a_{21}^*\}$, and will be arrived at by solving a system of three

[31] The accuracy measure quoted here refers to the estimate of the bid of each coalition when the type draw is 1. This boundary condition defines the entire bid function under the MMRS technique.

equations. The modification needed to implement this procedure in `CSE_SOLVER` is straightforward. The result of running the restricted CSE model is shown in Figure 12. The overlay of the two plots is nearly perfect.[32]

Figure 12 – MMRS/ Restricted CSE bidding functions overlaid



`Notes: The CSE bidding function of the coalition of 4 is given by` **`0.3286x + 0.3053x^2`**`, while the CSE bidding function of the coalition of 1 is given by` **`0.9116x - 0.2778x^2`**`.`

The important question is whether or not the difference between the CSE estimates and the more exact MMRS solutions are economically meaningful. For the coalition model replicated above, MMRS report the auctioneer's expected revenue and bidders' expected per-capita surplus. The exact values reported in MMRS are compared to the CSE results in Table 1 below. The auctioneer's expected revenue and

[32] Using the same measure of accuracy given in MMRS, the restricted CSE method is within 0.0035 of the MMRS estimate of the upper support of the distribution of bids. The estimate of the upper support, denoted by t^*, is given in Table 1 on page 204 of MMRS. I subtract from that value the figure 0.63385650948398442, which is the bid of a bidder with a type draw of 1 using the coefficients given in the notes under Figure 12.

the per-capita surplus of the coalition bidders are estimated identically to at least 3 significant digits. The improvement in per-capita surplus from optimal unilateral deviation is less than 1.5% for both coalitions. The profitability of optimal deviation from the CSE strategies is extremely low.

Table 1 – Revenue and surplus comparison

k1 = 4	Coalition vs. Coalition		
k2 = 1	Auct.	k1	k2
MMRS	0.5057	0.0567	0.0860
CSE	0.5139	0.0565	0.0817
Diff	-0.0082	0.0002	0.0043

Chapter 4: Asymmetric Multi-Unit Auctions with a Quota

Introduction

In this chapter, we use the CSE approach to study asymmetric procurement auctions wherein sellers from two classes draw costs from different distributions. When sellers are asymmetric, a cost-minimizing buyer discriminates among classes of sellers to enhance competition (Myerson (1981)). Establishing quota—a limit on the number of offers that can be accepted from any one class—discriminates simply and effectively. A binding quota increases demand scarcity from the perspective of low-cost sellers, which causes them to lower their offers. The CSE approach is used to solve for the equilibrium strategies of asymmetric auctions with and without a quota, and we find a quota enhances competition and lowers total procurement cost. The quota we impose are similar to mechanisms used widely in practice. In government procurement of construction contracts, for example, “set asides” are used to reserve some contracts for minority-owned and small businesses, effectively placing quota on the number of contracts available for non-minority-owned or large businesses. Because the mechanism is widely used to promote social goals and can also lead to better outcomes for the buyer, our findings have both positive and normative implications.

When sellers are asymmetric, the optimal auction is one that discriminates between sellers (Myerson (1981), Bulow and Roberts (1989)). While the conditions that characterize an optimal auction have been known for some time, implementation remains an open issue. How might a procuring agent, knowing that sellers are observably different in their ability to produce a good, structure an auction that accounts for this asymmetry? Direct implementation of the optimal auction requires that the procuring agent have knowledge of the distribution of seller costs and the ability to discriminate perfectly between sellers. Since neither of these two conditions is likely to be met in practice, we investigate the returns to a simple mechanism that a buyer can easily implement. The mechanism imposes a quota, a limit on the number of winning offers that can come from any single class of sellers, to increase competition within that class. That is, the auctioneer specifies *ex ante* that he will accept no more than x offers from a defined class of sellers.

We find motivation to study simple price discriminating methods in many practical applications. Private firms engage in contract procurement using auctions. Every level of government procures goods from suppliers that are observably heterogeneous in some way. Popular examples include defense-related procurement, procurement of infrastructure contracts, and procurement of fleet vehicles. The federal government also procures environmental services from heterogeneous private landowners using an auction procedure.[33] A mechanism that encourages competition through discrimination could also exist in markets with less structure than a formal auction. Firms, for example, hire from heterogeneous labor pools. Firms cannot perfectly discriminate between workers, and so must pay some workers more than their reservation wage (Ayres and Cramton (1996)).

[33] Although many similar programs exist, the largest single example of what are known as *conservation auctions* is the Conservation Reserve Program implemented by the U.S. Department of Agriculture. See Kirwan, Lubowski and Roberts (2005) for details.

What we refer to as a quota is similar to what some in the auction literature call *set-asides*. When multiple units of a good are being auctioned by the government (whether they be items to be sold or contracts to be purchased), set-asides reserve some number to be won by qualified bidders. Qualified bidders are bidders selected based on observable characteristics, often race or business size, meant in most cases to promote social goals, such as encouraging participation by a minority class of bidders (Denes (1997)). Milgrom (2004) presents a simple example of how set-asides can increase competition in an auction and so enhance the auction outcome from the bid-taker's perspective.[34] The fact that set-asides are used both in the sale of public goods such as spectrum (Ayres and Cramton (1996)), and in government procurement, provides a positive motivation for our study. That is, in addition to or despite possible social goals, governments or firms may use set-asides to reduce procurement costs.

Quotas reduce procurement costs when sellers of several dissimilar classes compete to sell multiple goods.[35] Sellers compete against rivals both similar and dissimilar to themselves. Sellers from these dissimilar classes, having observable characteristics that distinguish them, will offer toward a common margin. This margin is set by a mix of *within-class* and *between-class* competition. When one class of sellers has lower opportunity costs than another, a quota enhances within-class competition. The intuition is straightforward: by limiting the number of winning

[34] Milgrom's example (of a *forward* auction, as opposed to a *reverse*, or procurement, auction) is particularly simple in that the distribution of bidder values does not overlap. Thus the high-value bidders in his example always win all the goods in an auction without set-asides. Our examples are more general, as we allow for cost distributions to overlap.
[35] Similarly, in an auction to *sell* (rather than procure) multiple items, a seller would benefit from quota when buyers of many different classes compete to purchase.

offers, demand from that group declines. The artificial scarcity makes offers more competitive. The tradeoff is that between-class competition is sacrificed: when a low-cost seller is eliminated, high-cost sellers face less competition.

McAfee and McMillan (1989) provide an example in a context of international trade, which we modify slightly for our own purposes. There are six firms, two foreign and low-cost and four domestic and high-cost, competing for two government contracts. Unrestricted competition is characterized by weak competition within the class of low-cost foreign firms. The marginal foreign firm competes with domestic firms to fulfill the second contract, while the stronger foreign firm extracts substantial rents. McAfee and McMillan investigate how price preferences influence the procurement cost of an auction. If a quota were imposed that mandated a maximum of one foreign and one domestic firm to fulfill the government's need, the low-cost foreign firms would be forced to compete directly with each other. Rent that would have been extracted by low-cost foreign firms is reduced while rent accruing to domestic firms increases. The net effect of a quota depends on the net balance of offsetting influences: low-cost foreign firms face tougher competition, while high-cost domestic firms inflate their offers in the absence of direct foreign competition. In this polar example, quota has effectively created two separate auctions, one in which only foreign firms compete, and one in which only domestic firms compete.

The total effect of a quota on procurement cost is the sum of enhanced competition within classes and reduced competition between classes. A quota is thus most beneficial to the buyer when within-class competition is low among low-cost sellers. This happens if demand for the marginal unit typically comes from a high-cost class. In single-price auction without the restriction of a quota, what we will refer to as an “open” auction, sellers from the low-cost class will offer toward the same margin as sellers from the high-cost class and thereby extract substantial rents. Setting a quota effectively reduces the surplus captured by low-cost sellers.

While the idea of discriminating among sellers was laid out in the seminal paper on optimal auctions (Myerson (1981)), the method of discrimination considered here is new in important ways. Most auction papers focusing on implementation of a discriminating policy have examined what are known as *bid preferences*, a discounting of offers for the sole purpose of determining winners. For example, bid preferences in auctions for the procurement of transportation contracts in California take the following form (Krasnokutskaya and Seim (2009); Marion (2009)). A qualified “small bidder” wins a contract so long as its offer is within 5% of the lowest offer by an unqualified seller. Such a preference could be formulated as an actual discount to the qualified offer *for the purposes of evaluation only* (Hubbard and Paarsch (2009)). That is, the bid-taker will rank *discounted* offers, equal to (1-*preference*) × *offer*, from lowest-to-highest, selecting the lowest discounted offer as the winning offer and paying the winning seller their full undiscounted offer. Such bid preference programs are common in government procurement (Hubbard and Paarsch (2009)), and have been applied in high stakes auctions such as the first auctions for spectrum in the United States (Ayres and Cramton (1996)). Bid preferences are also used in, for example, procurement for snow contracts (Flambard and Perrigne (2006)), and have been studied in experimental settings (Corns and Schotter (1999)). A persistent finding is that procurement cost can be reduced by some positive bid preference, so long as the bid preference does not inhibit participation by strong sellers.[36]

Perhaps because most auction research in the area of bid discrimination considers single-unit auctions, set-asides and quotas, which apply only to multi-unit auctions, have received less study. In a multi-unit context quota neatly handles a problem inherent to the preference approach. To implement a price-preference mechanism, the auctioneer must know which class of sellers is low-cost and which are high-cost, and also have a good understanding of cost differences between classes. Such information is not necessary with quota. There are many applications, such as auctions for conservation land, when the auctioneer is less likely to know which class is low-cost and which is high-cost but nevertheless expects costs to differ widely across classes. In this case, providing a bid preference to the wrong party could *increase* procurement cost. On the other hand, a quota can be used by the bid-taker to encourage competition, even if the bid-taker is not able to identify which group of sellers is relatively low-cost, and which group of sellers is relatively high-cost. The bid-taker could always place a binding limit on the number of bids from any one group that can be accepted. Note that to enforce this rule, the bid-taker need not be able to identify which group of sellers is relatively low-cost. The only requirement is that the sellers themselves be aware of this fact.

Our research applies quota in an independent private values (IPV) model of a one-shot, sealed-bid auction. Since our focus is on procurement, we model a pay-as-bid auction as opposed to a uniform price auction. Almost all government

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[36] We do not analyze the effect on participation of imposing a quota. Since there is no participation cost in our model, we assume that each potential bidder will find it in their interest to submit a bid, since the expected profit from doing so is at least weakly positive.

procurement auctions use the pay-as-bid format. Thus our analysis differs from that of Ayres and Cramton (1996), who investigate bid preferences and set-asides in multiple-round, open-bid auctions, and from Denes (1997), who studies multiple auctions over time.

In the next section, we construct a simple two-class model of a procurement auction for multiple goods. We examine a number of special cases to illustrate how large differences in opportunity cost between classes of sellers can lead to low levels of within-class competition. We then estimate bidding functions in these cases using the CSE approach to computation.

Model

Suppose there are two classes of sellers. Type *A* Sellers draw their costs from distribution F^A, while Type *B* sellers draw their costs from distribution F^B. There are *N* total sellers, n^A Type *A* sellers, and likewise n^B Type *B*. The fact that there are n^A Type *A* sellers and n^B Type *B* sellers is common knowledge to all, including the bid-taker. However, it is only required that sellers know the characteristics of F^A and F^B, while the buyer may remain ignorant of these characteristics.

Type *A* sellers draw their costs, c, independently from a distribution F^A on support $\left[\underline{c}^A, \overline{c}^A\right]$ and Type *B* sellers draw their costs from a distribution F^B on support $\left[\underline{c}^B, \overline{c}^B\right]$. The distribution F^B is constructed from distribution F^A by an additive parameter δ. This captures the simplest type of class asymmetry – sellers perceive intra-group cost heterogeneity identically, regardless of their class, and sellers perceive the other group as similar to themselves but with a different average cost (sellers are ex ante high-cost or low-cost).

Consider the problem from the perspective of an arbitrary seller (say seller 1). The buyer will accept the lowest m of the offers submitted by the N sellers, so the probability of a given offer being accepted is the probability that the offer is below the m^{th} lowest of all the other offers. Each seller submits a single offer.

This is the form of the basic model. We will use this model to investigate how procurement cost, and the rent accruing to sellers, increases as within-class competition decreases. To do so, and to provide quantitative predictions for our laboratory experiment, we must derive bidding functions for our model.

Symmetric Sellers

Let $Y_1, Y_2, ..., Y_{N-1}$ represent the cost draws of each of the $N-1$ sellers that are not seller 1, ordered from lowest-to-highest. The unique symmetric equilibrium bidding strategy of the auction when $\delta = 0$ is

$$\beta(x) = E\left[Y_M \mid Y_M > x\right].$$

***PROOF*:** See Weber (1983) or Ortega-Reichert (1968).

The intuition behind this result is simple. A seller facing $N-1$ competitors that are ex ante identical will submit an offer just low enough to be among the m lowest. If the equilibrium is symmetric, i.e. every seller follows the same strategy, only those sellers with the lowest m cost draws will be accepted. Therefore, each seller forms their expectations of what the m^{th} lowest cost draw will be, conditional on it being greater than their own draw (conditional on winning).

With the analytic bid functions given in the proposition, expected procurement cost is easily seen to be $m \bullet E\left[C_m\right]$ where C_m is the m^{th} lowest amongst all N cost draws.[37]

Uniquely in the symmetric case we can derive an analytic solution to the equilibrium bidding function. Since our main interest is in investigating auctions with asymmetric sellers, we will need to introduce the CSE technique for estimating bidding functions.

Asymmetric Sellers

The sharp prediction of behavior and total procurement cost holds only when all sellers are identical ($\delta = 0$). When we consider asymmetric classes of sellers, there is no single strategy that sellers from both groups will follow in equilibrium. Solving for equilibrium bidding functions when sellers are asymmetric is notoriously difficult (Gayle and Richard (2008)). Rather than relax the assumption of asymmetry, we use the CSE algorithm discussed in chapters 2 and 3 to compute equilibrium strategies. Considering constrained strategies often proves advantageous in two ways. First, by constraining the strategy space we can solve auction models that would otherwise prove intractable. Second, strategies that are simple functions of a seller's private information often prove to be more useful predictors of actual behavior than Nash predictions (Kagel and Richard 2001). Since some of the predicted bidding functions presented here will be tested in the lab, it is important to note this as-yet unmentioned benefit of the constrained approach.

[37] This is the m^{th} order statistic from N draws and is distinct from what was denoted by Y_m in the proposition. Y_m denoted the m^{th} order statistic from $N-1$ draws.

The CSE approach has another benefit. When a BNE does *not* exist, we can approximate *likely* outcomes by finding strategies that form a near-equilibrium. That is, we can identify strategies that lead to a situation where the incentives for any one seller to deviate are extremely low. The requirement for a BNE is, of course, that unilateral deviation be unprofitable. A constrained strategic equilibrium is one in which the expected profit from deviation is made arbitrarily small.

By restricting attention to strategies that follow a particular functional form, we are able to estimate bidding strategies numerically. We present here a more detailed explanation of the program only in the linear case, but the extension to polynomial strategies is straightforward, and has been reviewed at length in chapters 2 and 3. We attempt to find the *best* coefficient a, given the assumption that every seller restricts themselves to linear strategies, and that every seller in a group implements the same a. For an arbitrary Type A seller, this amounts to maximizing the offer net of costs, multiplied by the probability that the offer is accepted. This is given by

$$\left(a_A c_i - c_i\right) G\left(a_A c_i\right), \tag{29}$$

where G is the distribution of the *critical offer,* the offer above which no offers will be accepted. Note that the critical offer may be submitted by a member of either group A or group B, so specifying G analytically not a simple matter. To find the optimal a_A, we differentiate to obtain first order condition

$$c_i G\left(a_A c_i\right) - c_i\left(a_A c_i - c_i\right) g\left(a_A c_i\right) = 0, \tag{30}$$

where g is the derivative of G. By doing likewise from the perspective of an arbitrary Type B seller, we obtain a system of zeros that can be solved using readily available numerical recipes.[38,39]

Having found a_A* that most nearly satisfies the first order condition, which we call our optimal constrained strategy of order 1 (*k*=1), it is natural to wonder how good this equilibrium approximation is. For instance, supposing that n^A-1 sellers follow the strategy of bidding $a_A*\times c$ and n^B sellers follow the strategy of bidding $a_B*\times c$, how well could the omitted Type *A* seller do by changing their offer? The concept of equilibrium being built on the idea of unilateral deviation, it is natural to measure the "goodness" of an equilibrium approximation by how well one seller could do by discarding the $a*$-strategy in favor of another; we would like this seller's profit from deviating to be as small as possible. If the benefits to pursuing another strategy are large, we might suppose that sellers would no longer restrict themselves to simple linear strategies. Instead, it seems reasonable to assume that sellers might pursue more complex strategies, should the reward to doing so be substantial. In what follows, we estimate a constrained strategic equilibrium as a sequence of polynomials. We then measure the goodness of a constrained strategic

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38 We use canned routines included in MATLAB's optimization toolbox and KNITRO, a suite of algorithms made freely available for academic and personal use by Ziena (http://www.ziena.com/).

39 We find the vector of polynomials $\left[\beta_K^A(c),\beta_K^B(c)\right]$ that most nearly satisfy (30), subject to theoretical restrictions. We force β_K^A and β_K^B to be monotonic and for $\beta_K^{A,B}\left(\overline{c}^B\right)=\overline{c}^B$. The last restriction means that a bidder receiving the worst possible cost draw will submit a bid exactly equal to their cost draw.

equilibrium of degree K by the increase in expected profit an unconstrained seller can obtain if all other sellers abide by the constrained strategy.[40]

Numerical Results

In all cases we fix Type *A* sellers to draw their costs from the uniform distribution on [0,100]. We generate asymmetry by making a Type *B* distribution that shifts the Type *A* distribution by a constant δ, giving a support on $[\delta, 100+\delta]$.

Symmetric Sellers

We first demonstrate the CSE technique when sellers are symmetric, which allows us to benchmark the CSE against the well-known equilibrium bid strategy derived by Weber (1983) and Ortega-Reichert (1968).[41] We show how successively higher-degree polynomials better approximate the true equilibrium bid function. We also illustrate a measure of the approximation error that we can apply even to the asymmetric case that has no known closed-form solution. We examine the asymmetric cases in the next subsection.

Figure 13 shows the Nash Equilibrium bid function and the linear CSE bids when 10 sellers compete for the right to sell 6 identical goods to the bid-taker. The linear strategy approximation captures the general slope of the Nash Equilibrium bid function quite well. A better approximation is desirable, however. To see why, we calculate the best response of a unilateral deviator. If all but one seller were using the linear CSE strategy, how well could one informed player do by optimizing on his

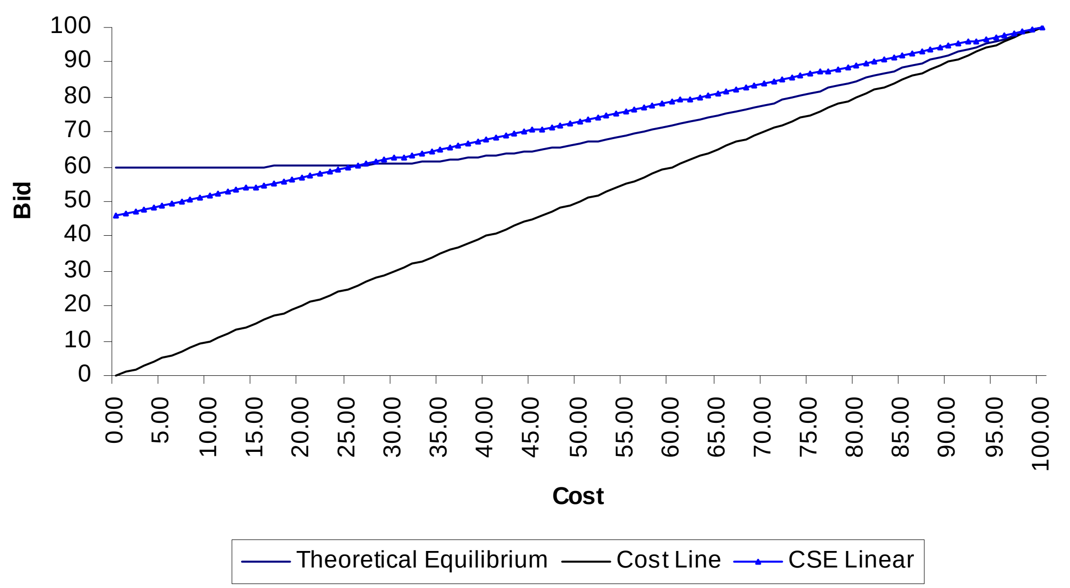
[40] That is, we allow a single bidder to deviate by following the best unilateral strategy, where this new strategy is constrained by a $\bar{K} > K$. $\bar{K}$ is chosen to be sufficiently large that the deviating bidder is essentially unconstrained.
[41] See the theorem by Weber and Ortega-Reichert above.

own? The answer turns out to be that a nearly 16% increase in average profits is possible if a seller optimally deviates. We denote by $Q(k)$ the measure of approximation quality with Monte Carlo size *mc* for any degree *k* of the CSE ($k=1$ for linear strategies, $k=2$ for quadratic strategies, etc.). $Q(k)$ is calculated as

$$Q(k)=\frac{\left[\frac{1}{mc^2}\sum_{i=1}^{mc}\sum_{j=1}^{mc}\Pi\left(b_i^*;s_{-i,j}^{CSE(k)}\right)-\Pi\left(b_i^{CSE(k)};s_{-i,j}^{CSE(k)}\right)\right]}{\overline{\Pi}^{CSE(k)}},$$

where $\Pi(b_i;s_{-i})$ represents the profits accruing to seller i when seller i submits the bid b_i, and all other sellers follow the strategy s_{-i}, b_i^* is the optimal bid for seller i, conditional on all others bidding according to $CSE(k)$, and $\overline{\Pi}^{CSE(k)}$ is the expected profit from following the equilibrium strategies.[42]

Figure 13 – Symmetric Auction with Linear Strategies

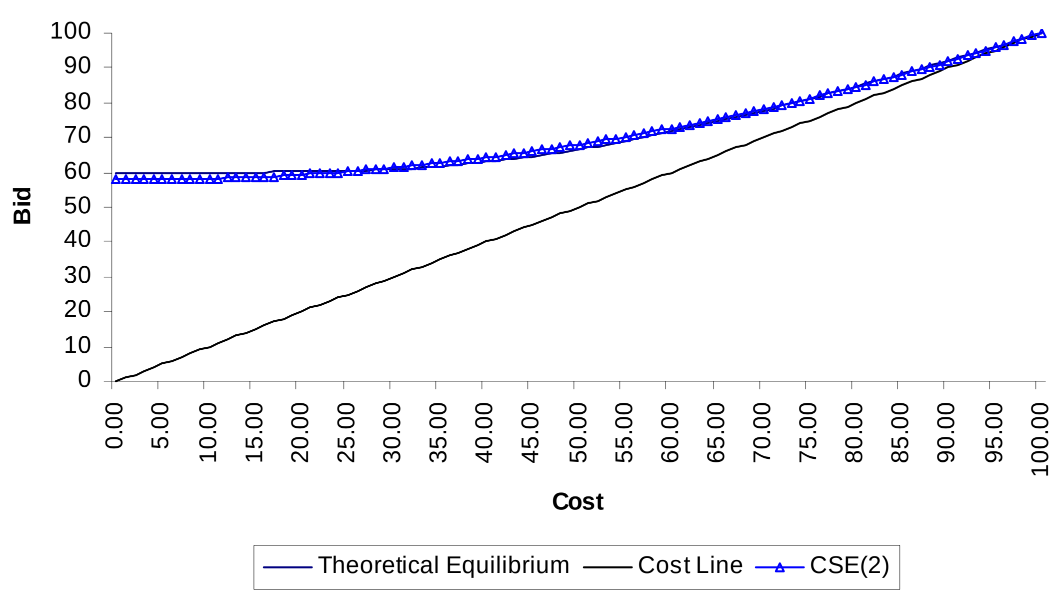


[42] The numerator of this measure of equilibrium stability is referred to as C_4 by Armantier, Florens, and Richard.

An arbitrary seller following the linear CSE strategy will earn an average profit of \$19.15, according to our computations. If this same seller optimally deviates while all his competitors follow the CSE strategy, he can expect to earn an additional \$3.00, which amounts to the 16% increase reported above.

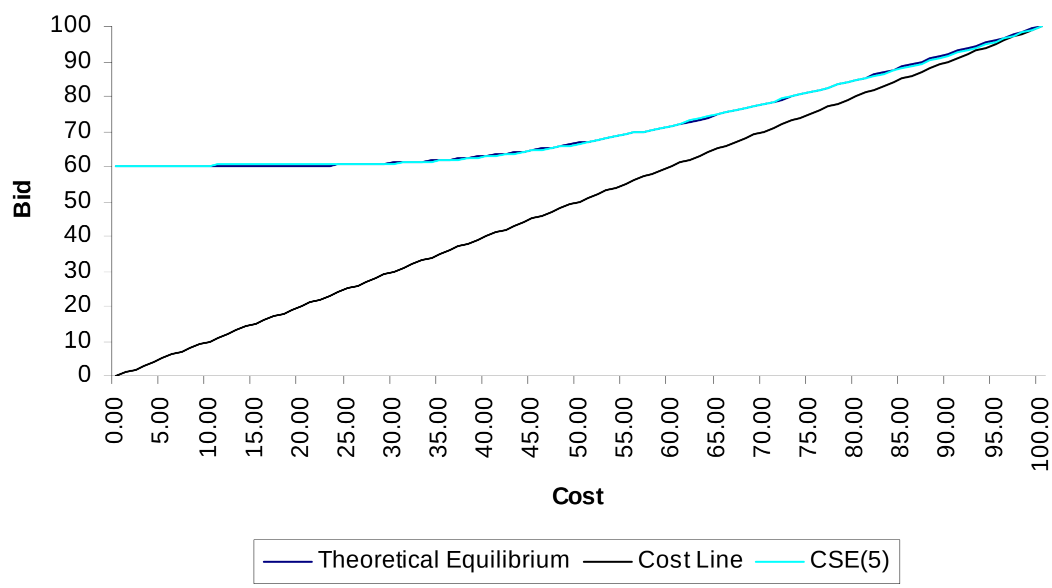
We increase the degree of the polynomial strategy in order to approximate a constrained equilibrium with a lower benefit to unconstrained deviation. Figure 14 displays the results of the quadratic CSE, again against the benchmark of the theoretical equilibrium. A large improvement in the approximation quality is immediately obvious, even from a cursory examination of the figure. This apparent improvement is confirmed by the $Q(2)$ statistic. The advantage to optimal deviation has declined markedly, from 16% in the linear case to 2.5% in the quadratic case. The quadratic strategies approximate true equilibrium more closely, and generate a much lower payoff to deviation from the constrained strategy.

Figure 14 – Symmetric Auction with Quadratic Strategies



Increasing the degree of the polynomial of the constrained strategy makes the quality of the approximation arbitrarily precise. Figure 15 shows the optimal quintic (CSE(5)) bid function as compared to the true Nash Equilibrium bids. In this case, the approximation is so good that the two lines overlay each other almost perfectly, so much so that they are difficult to distinguish. As a reference for the approximation quality possible with the CSE approach, the $Q(5)$ statistic is 0.75%.

Figure 15 – Symmetric Auction with Quintic Strategies



Asymmetric Sellers

Now that we have demonstrated how well the computational technique works, we apply it to several asymmetric auctions. We calculate equilibrium approximations in three cases of asymmetry: low asymmetry ($\delta = 25$), medium asymmetry ($\delta = 50$), and high asymmetry ($\delta = 75$). These cases are selected to demonstrate the decline in competition among low-cost sellers that occurs as the between-class heterogeneity increases (as δ increases). Note that in each case there are five Type *A* sellers and five Type *B* sellers, and the procuring agent wishes to purchase six units. We begin by presenting the results graphically. The bidding functions for low, medium, and high asymmetry cases are presented in Figure 16 through Figure 18. Since there is no theoretical equilibrium for comparison, we plot only the highest-degree CSE bid function computed (CSE(5)). There are several things to notice.

Figure 16 – Low Asymmetry Auction with Quintic Strategies

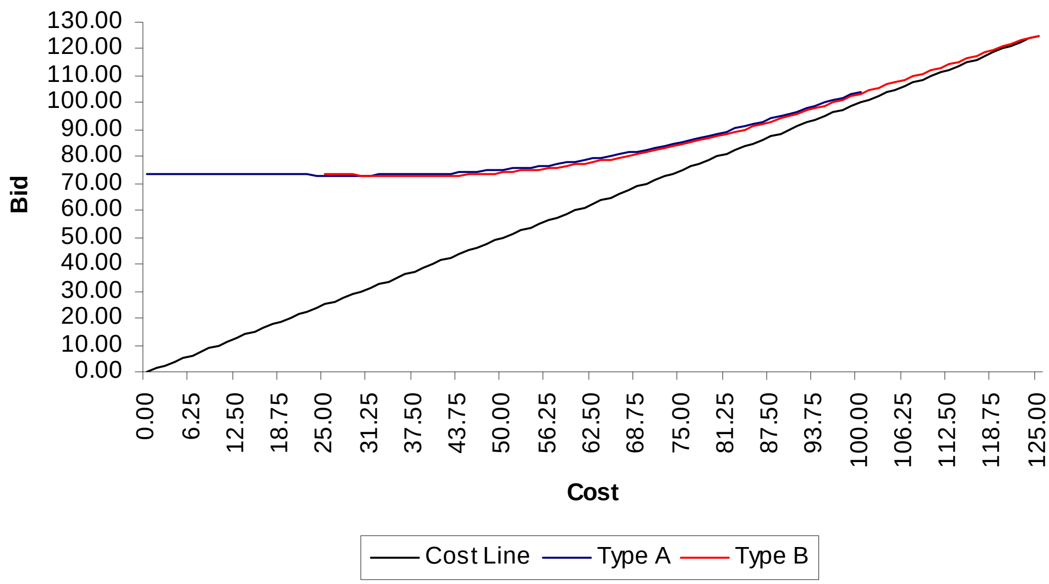


Figure 17 – Medium Asymmetry Auction with Quintic Strategies

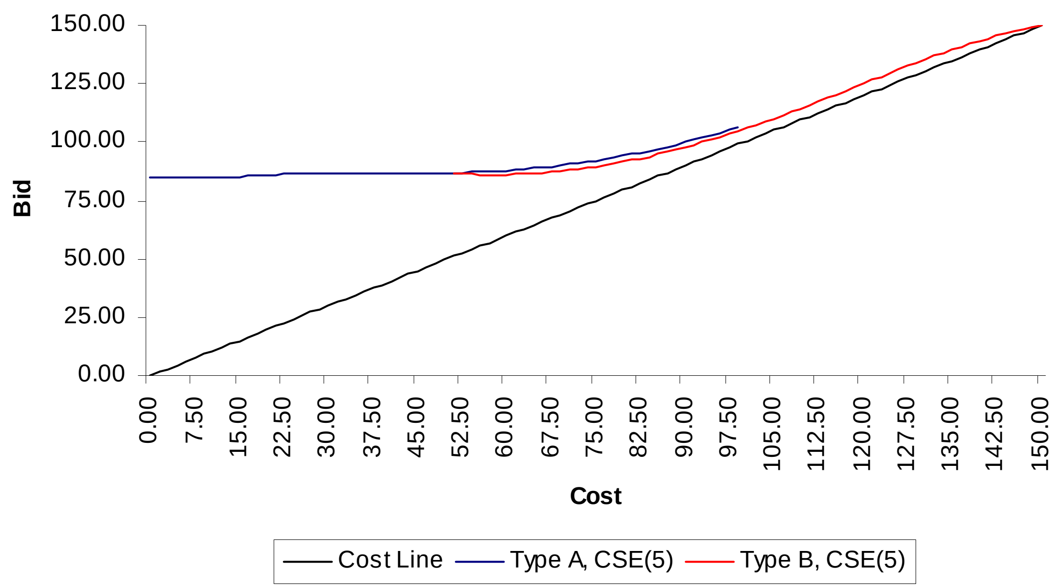
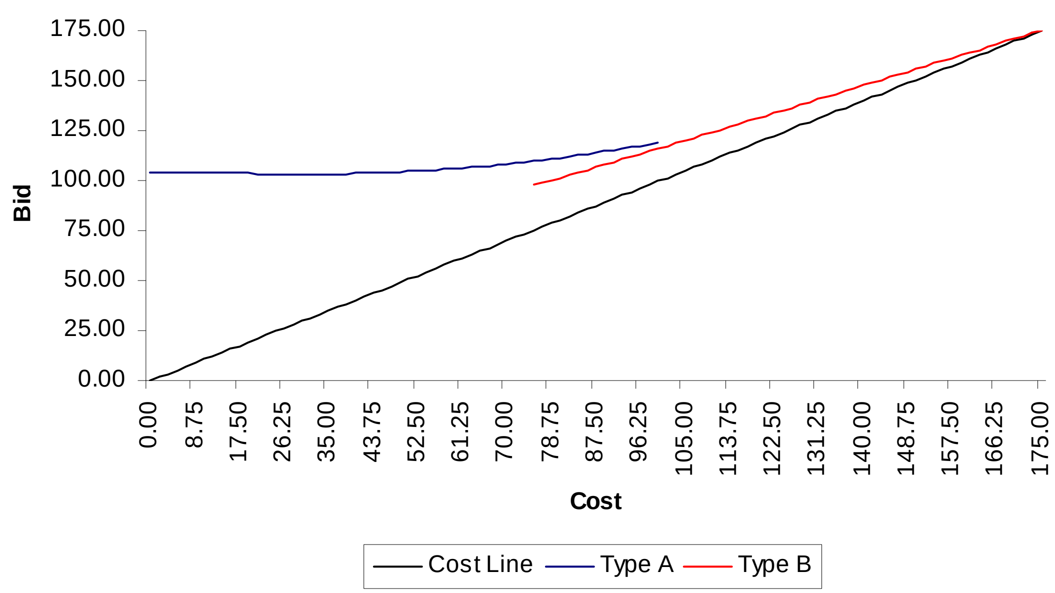


Figure 18 – High Asymmetry Auction with Quintic Strategies



In the low, medium, and high asymmetry cases, Type *B* sellers bid more aggressively than Type *A* sellers with the same cost draw. Since Type *B* sellers compete against a stronger cohort (they bid against four other Type *B* sellers and five Type *A* sellers rather than the reverse), the probability of a Type *B* seller winning is everywhere lower than the probability of a Type *A* seller winning. This result is consistent with theory. See, for example, Krishna (2002), p. 48. The phenomenon has been described as “weakness leads to aggression” (Krishna (2002); p. 47). Because weak sellers face stiffer competition, they face a lower conditional probability of winning. *Ex ante*, a strong seller is more likely to win than a weak seller, both having submitted the same bid. This lower probability of winning induces weak sellers to bid closer to their true cost; hence, “weakness leads to aggression.”

Although Type *B* sellers bid more aggressively in all scenarios, the degree to which they bid more aggressively is increasing in the between-class heterogeneity. The relative aggressiveness of sellers can be seen as the vertical difference between Type *A* and Type *B* bid functions in Figure 16 through Figure 18. The distance between the bid functions increases as Type *B* sellers are progressively made weaker (i.e. their cost distributions are shifted higher), from the low asymmetry case in Figure 16 to the high asymmetry case in Figure 18. The increasing difference in bidding functions between Type *A* and Type *B* sellers is a result of Type *A* sellers increasing their bids in response to the weaker competition provided by Type *B* sellers. The same low-cost seller will increase their bid from about $70 in the low asymmetry case, to just below $90 in the medium asymmetry case, and finally to about $100 in the high asymmetry case. These low-cost sellers extract the most rent from the auction process, and are the ones targeted by the imposition of a quota.

The takeaway is that low-cost bidders are extracting rent because of the observable heterogeneity of the high-cost bidders. The greater is the observable heterogeneity, all else equal, the greater the rent extracted by the low-cost bidders. We have claimed that imposing a quota can reduce procurement cost by substantially reducing the rent accruing to low-cost bidders. We now examine the computed bid functions in auctions with a quota in order to determine if that claim is borne out.

We present the estimated bid functions for auctions with a quota in Figure 19 through Figure 22. The symmetric, low-, medium-, and high-asymmetry cases are presented, just as they were for the open auctions. In each case, the quota is enforced by a simple rule. The bids of no more than four Type *A* sellers and four Type *B* sellers can be accepted. The imposition of a quota has a pronounced impact on bidding behavior. When faced with both within- and between-group competition, low-cost sellers bid much more aggressively. As a direct demonstration of the quota-effect, we present Figure 23, which plots the bid functions before and after the imposition of a quota in the medium asymmetry case (the case will be examined in the laboratory experiment). Note the tradeoff of imposing a quota: Type *A* sellers submit significantly lower bids in an auction with a quota than in an open auction, while Type *B* sellers inflate their bids slightly to reflect their increased chances of winning under the quota regime.

Figure 19 – Symmetric Auction with Quota and Quintic Strategies

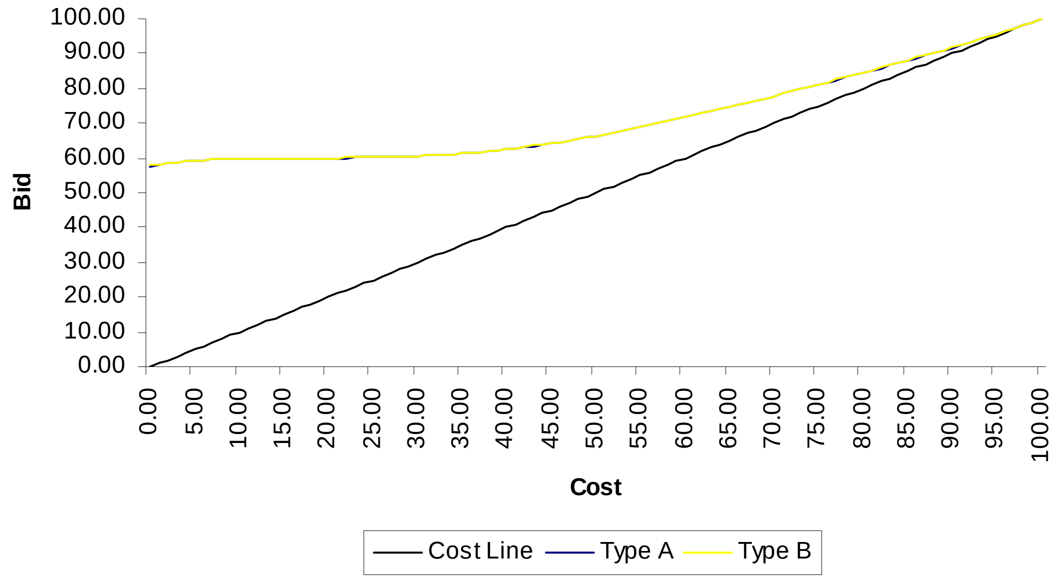


Figure 20 – Low Asymmetry Auction with Quota and Quintic Strategies

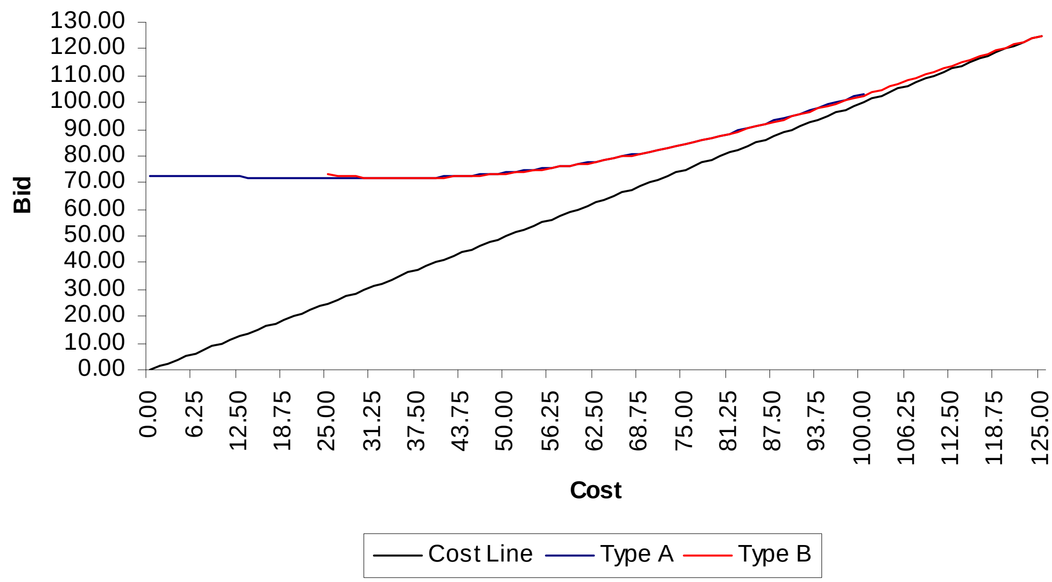


Figure 21 – Medium Asymmetry Auction with Quota and Quintic Strategies

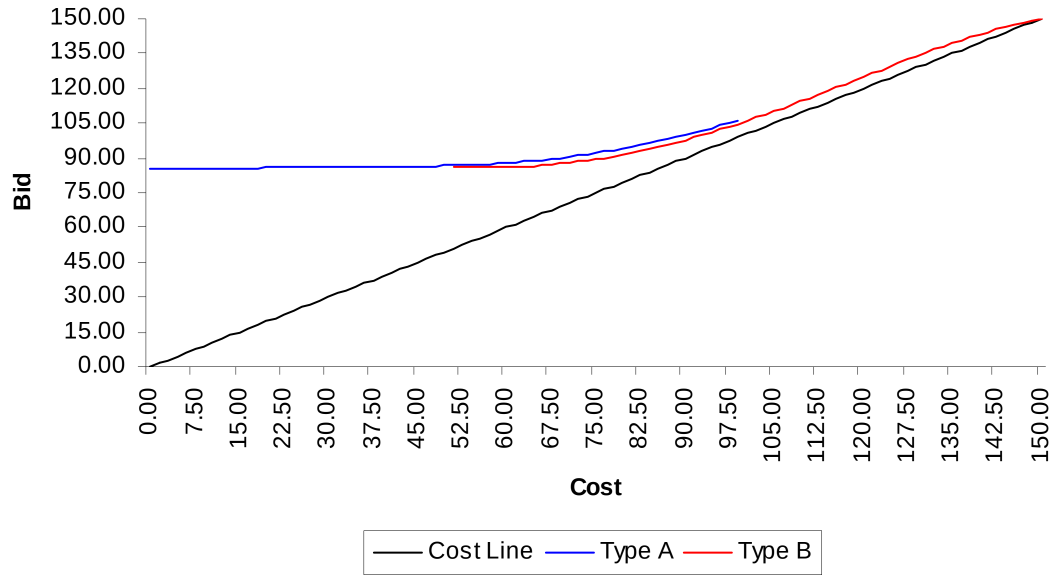


Figure 22 – High Asymmetry Auction with Quota and Quintic Strategies

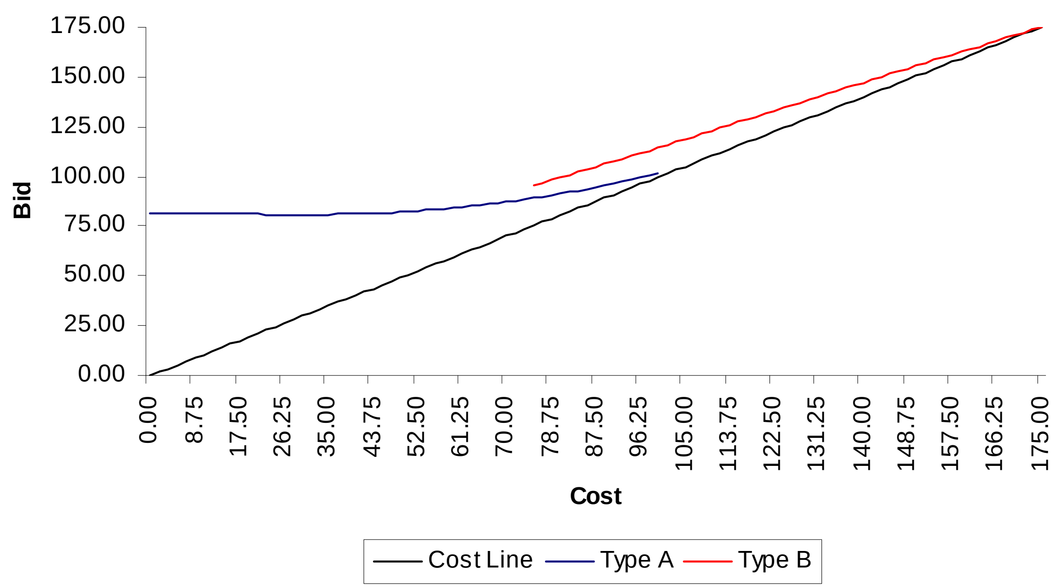
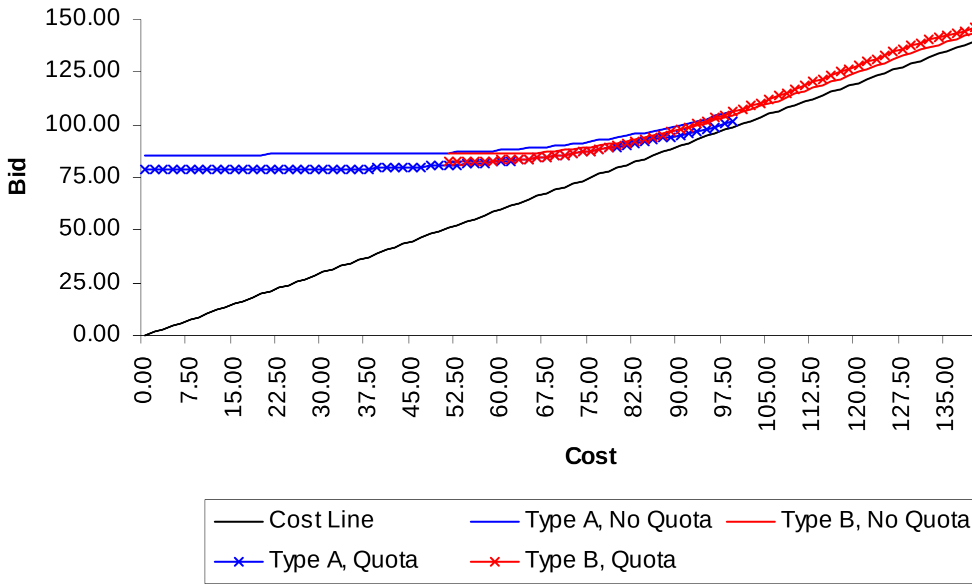


Figure 23 – Medium Asymmetry Auction Comparison



Using these estimated bid functions, we are able to compute the total procurement cost to the bid-taker in constrained equilibrium. Table 2 presents the expected total procurement cost in low-, medium-, and high-asymmetry cases, with and without a quota. The imposition of a quota decreases procurement cost in *all* cases, although when asymmetry is low the difference is negligible. When between-group asymmetry is high, however, the difference becomes more pronounced. We find that a quota can be an effective means to reduce procurement cost in situations of high seller asymmetry, while posing little risk of increasing procurement cost when groups of sellers are similar.

Table 2 – Expected Total Procurement Cost

Symmetric Auction

Open Auction	Quota Auction
381.6442	380.4208
(17.8699)	(18.2493)

Low Asymmetry Auction

Open Auction	Quota Auction
457.2948	452.2005
(16.112)	(17.364)

Medium Asymmetry Auction

Open Auction	Quota Auction
531.4892	509.7331
(13.7573)	(21.2654)

High Asymmetry Auction

Open Auction	Quota Auction
643.5566	564.8096
(15.668)	(24.7912)

Expected Procurement Cost from Monte Carlo Simulations.
Standard deviations in parentheses.

A quota is capable of lowering procurement cost when sellers are asymmetric while not raising procurement cost when sellers are symmetric. This is a consequence of the fact that the returns to competition are increasing at a decreasing rate. In the IPV model of auctions, increasing competition reduces the rent accruing to the winning seller. However, as the number of sellers tends to infinity, the auction becomes perfectly competitive (see McAfee and McMillan (1987), for example); the effect of an extra seller on the behavior of existing sellers becomes negligible. That is, the returns to competition are increasing at a decreasing rate. Imposing a quota in an auction creates artificial scarcity, and thus competition. The greatest increase in competition comes when we move from having one seller in an auction to having two. The marginal increase in competition is still high when we add a third, but declines as we continue to add additional sellers. Likewise, the returns to enforcing a small bit of competition, by creating artificial scarcity, are highest when the amount of competition starts out low, as it does among the Type *A* sellers in the auctions without a quota. In exchange for this extra competition encouraged among Type *A* sellers by the imposition of a quota, we increase the probability that a Type *B* seller will win the auction. This represents a *decrease* in competition facing Type *B* sellers. However, since Type *B* sellers were winning infrequently, i.e. competition was high, the decrease in competitive forces among Type *B* sellers is reduced from a point of relatively high competition.

In the final chapter of the dissertation we turn to experimental evidence, where we examine bidder behavior in laboratory auctions constructed to match the medium-asymmetry case.

Chapter 5: Experimental Evidence

Introduction

We have two motivations for subjecting the quota mechanism to laboratory testing. First, closed-form theoretical predictions are difficult if not impossible to derive, so empirical validation complements the numerical computations presented above. Second, behavior may systematically differ from theory. We show a price discriminating mechanism that is "implementable" in the sense that: (1) the rules can be explained easily to sellers and (2) the results are robust to "misbehavior" by sellers. Human sellers are known to misbehave in that they do not always bid according to standard game theoretic concepts, even in environments where the mathematical theory suggests bidding should be straightforward.[43] Thus, we desire to check that our results are robust to the actions of boundedly rational sellers. We put the auction institutions to their most rigorous test not by simulating particular types of misbehavior unilaterally, but by allowing a group of human sellers to compete for cash in a laboratory setting.

Experimental Procedures

We report on the results from 17 experimental sessions. Ten undergraduate students from the University of Maryland participated in each session. All experiments were computerized, using custom software.[44] In each session, five of the

[43] It is well known, for example, that bidders in second-price sealed-bid auctions frequently fail to submit bids equal to their costs, even though doing so is unambiguously optimal. See Kagel (1995), e.g.

[44] A hearty thanks goes to Daniel Hellerstein for his custom-designed auction software.

subjects were labeled as Type *A* sellers, and five of the subjects were labeled as Type *B* sellers. This information was provided on-screen; subjects did not know the Type-identities of any of their competitors, but did know that there were a total of five Type *A* and five Type *B* sellers. Additionally, subjects knew that Type *A* sellers drew their costs randomly, with each amount between $0.00 and $100.00 being equally likely to occur, while Type *B* sellers drew costs between $50.00 and $150.00.

Subjects entered the lab and were randomly assigned to a role, which corresponded to information displayed for them on their computer terminal when they sat down. Each subject listened as the instructions were read aloud. This way, each subject began with the same set of information, and any questions were answered publicly if part of the instructions were unclear.[45] Subjects were then given time to re-read the instructions on their own before the first auction began.[46] Each subject had an opportunity to practice in their role before participating in an auction for real money.

The treatment in the experiment was whether or not a quota was imposed. The design we employed was a within-design. Each subject participated in both open auctions and auctions with a quota. Because every subject participated in both treatments, we can make both within- and between-comparisons. The order of treatments was varied to control for learning effects, and in some sessions an "A-B-A"-type design was employed to determine if individual bidding behavior within a treatment varied with experience. Because we varied the order in which subjects

[45] Subjects were asked to raise their hand if they had a question. A monitor would listen to the question and the answer would then be given publicly if the question pertained to all subjects.
[46] We include some screenshots from the computerized auction environment in the appendix. This includes a "Welcome" screenshot that began every set of instructions. A full set of screenshots, as well as instructions for all treatments, are available upon request.

faced different auction mechanisms, we are confident that our results are not influenced systematically by seller learning or order effects.[47] During a given experimental treatment (each treatment lasted for at least 10 auctions), subjects maintained their Type-identity. That is, subjects did not change between Type *A* and Type *B* in sequential auctions.[48]

We focus only on the medium asymmetry case in our experimental analysis. While there is probably some merit to confirming that procurement cost is further reduced by a quota as the between-group heterogeneity gets larger, we were concerned about perceived fairness in the experimental auctions.[49] Further, it seems intuitively obvious that the benefit of employing any discriminating mechanism should increase as seller heterogeneity increases. Consequently we didn't believe that it was necessary to test the quota auctions under the high asymmetry condition. On the other hand, we also do not test the quota mechanism when between-group heterogeneity is low. Given that we observe more aggressive bidding in our experimental sessions than we expected, this would have been desirable. We leave this for future research.

Results

We begin our analysis by presenting the data in full. Figure 24 shows a scatter plot of all bids against all costs in the open auction treatment, while

[47] We discard the first 3 rounds of each treatment in our regression analysis, a standard practice in experimental economics, mean to account for an initially steep learning curve. Our results are robust to the inclusion or exclusion of additional rounds. We find no evidence for end-of-round type effects, and so do not exclude any auctions at the end of a treatment sequence.

[48] We did, however, experiment with changing bidder Types over treatments.

[49] When between-group heterogeneity is large, one group necessarily earns far larger profits than the other

Figure 25 shows a scatter plot of all bids against all costs in the quota treatment. The bidding behavior by Type and quartile is also summarized in Table 3. A brief inspection reveals that bidding behavior in the lab does broadly conform to that predicted by our computational results. In the open auction, competitive bids are almost flat, and centered about the predicted equilibrium margin (about $87). When cost draws are above the predicted margin, bids increase close to linearly with costs. On the other hand, in Figure 25 we see that bids from those with the lowest costs are depressed significantly by the imposition of a quota. The average bid for subjects with costs below $80 is reduced by about $10. The predicted impact of imposing a quota seems to be realized. Strong sellers feel increased competition and bid more aggressively as a result. The tradeoff should be that weaker sellers, on average, bid less aggressively. Their increased chance of winning should have caused them to increase their bids. Comparing the mean Type *B* bids from the auctions with and without a quota in Table 3, it would seem instead that Type *B* sellers bid *more* aggressively.[50]

[50] These calculations are fully inclusive, however, and so the means may be unduly influenced by "throw away" bids. Individuals who assessed their chances of winning as being almost zero sometimes submitted excessively high bids.

Table 3 – Bidding Behavior

	Seller Bids							
	Type A				Type B			
Cost	0 - 25	25 - 50	50 - 75	75 - 100	50 - 75	75 - 100	100 - 125	125 - 150
	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)	Mean Bid (SE)
Auction Restrictions	No Limits -- Open Auction							
None	79.13	80.86689	92.04782	103.1273	94.81788	96.44045	129.6281	190.2178
	(1.72)	(1.15)	(2.49)	(2.98)	(6.45)	(0.74)	(5.81)	(18.03)
	Quota Auctions							
Highest bid in each group rejected	65.01	71.01	81.28	93.34	83.16	97.04	128	153.32
	(1.59)	(1.66)	(1.04)	(1.17)	(0.99)	(1.32)	(8.03)	(9.47)

	Seller Profits							
	Type A				Type B			
Cost	0 - 25	25 - 50	50 - 75	75 - 100	50 - 75	75 - 100	100 - 125	125 - 150
	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)	Mean Profit (SE)
Auction Restrictions	No Limits -- Open Auction							
None	62.30483	39.40278	21.63756	6.525411	17.0264	4.443	0.546931	-0.02237
	(1.99)	(1.24)	(1.07)	(0.75)	(0.93)	(0.56)	(0.25)	(0.07)
	Quota Auctions							
Highest bid in each group rejected	50.8	30.13	12.88	1.66	16.62	4.06	0.45	-1.53
	(1.70)	(1.81)	(1.08)	(0.30)	(1.23)	(0.55)	(0.22)	(1.63)

Figure 24 – Scatterplot of Bid Data

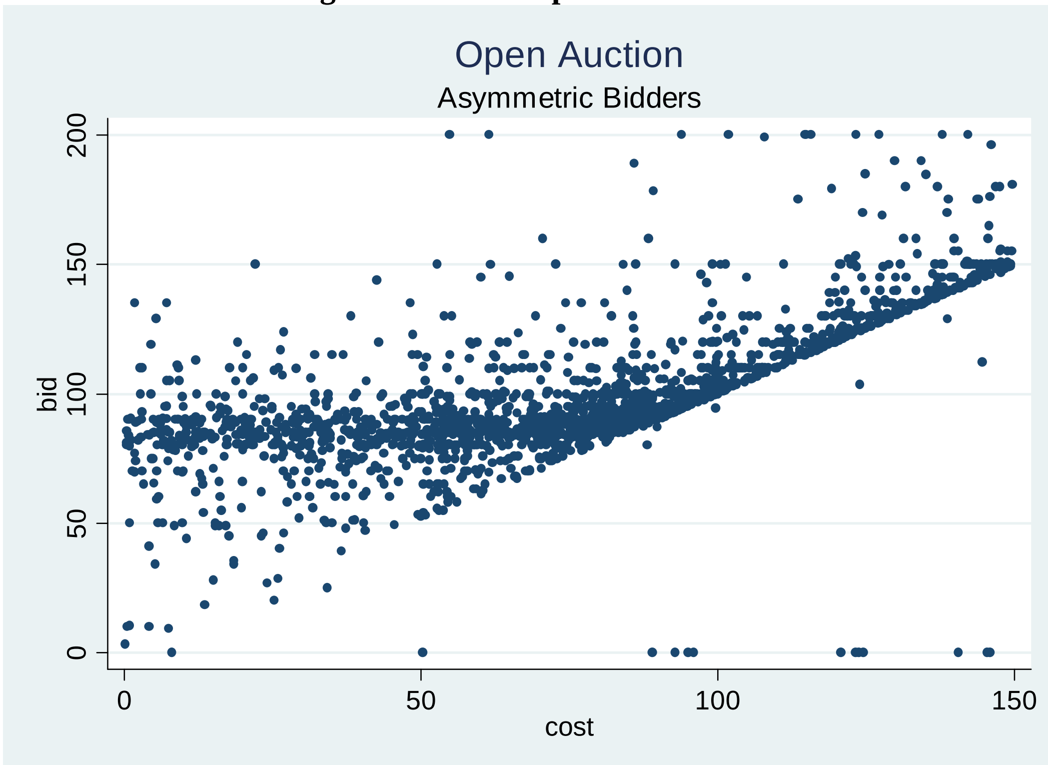
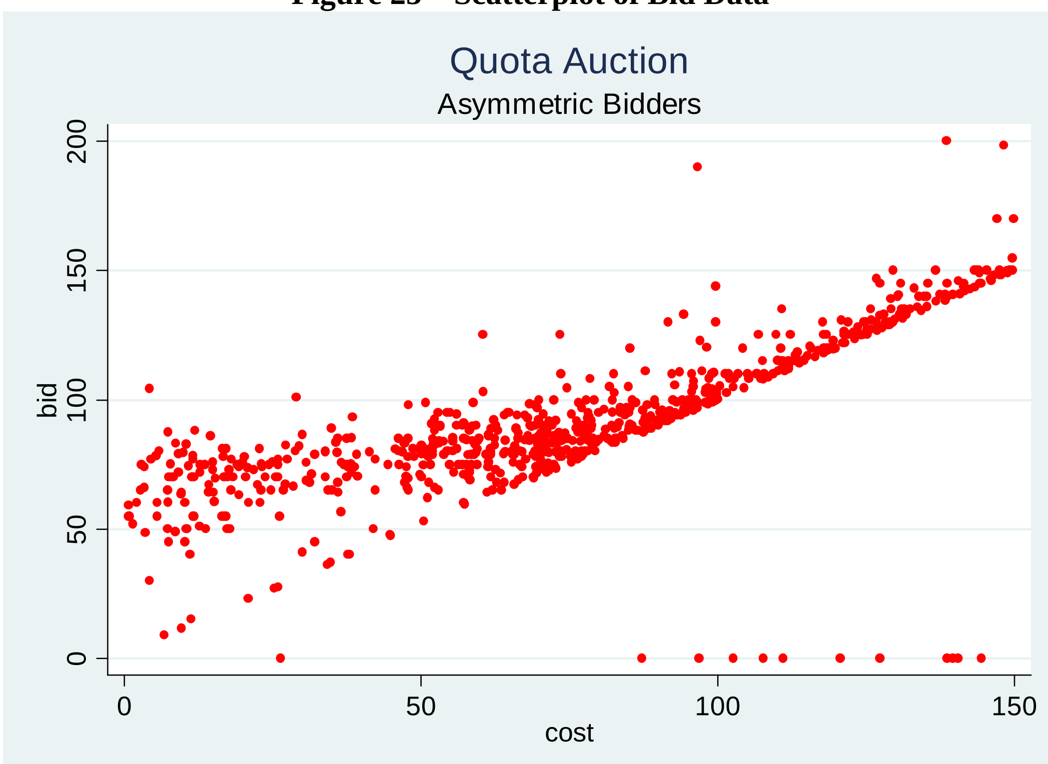


Figure 25 – Scatterplot of Bid Data



Formal analysis of the experimental data follows the same pattern from the presentation of the computational evidence. We first present an econometric model of the bidding functions. We then present the empirical distribution of procurement cost with and without a quota. We calculate a series of counterfactual procurement cost distributions to increase the power of our tests and to check the robustness of our conclusions. We fit a panel model to our data and use it to predict procurement cost over a number of draws – a Monte Carlo experiment. The imposition of a quota has a more pronounced impact than anticipated.

Econometric Model

As in our computational model, we specify our econometric model such that each subject's bid is a polynomial function of their cost draw. The panel nature of the data allows us to control for the effect of subject-specific bidding behavior by including fixed or random effects in our specification. That is, we specify

$$b_{i,t} = \alpha_i + \sum_{k=1}^{K} \beta_k \bullet c_{i,t} + \varepsilon_{i,t},$$

where $b_{i,t}$ $\left(c_{i,t}\right)$ is the bid (cost draw) of subject *i* in auction *t*, $\varepsilon_{i,t}$ is a normally distributed error term, and α_i's and the β's are parameters to be estimated.[51]

We estimated both regressions in which Type *A* and Type *B* bids were pooled, and in which Type *A* and Type *B* bids were modeled separately.[52] That is, we

[51] In all specifications, a Hausman test supports the use of random effects in the sense that the null hypothesis of consistency is not rejected. Therefore, since random effects are more efficient than fixed effects under the null hypothesis, we report only the results of models with random effects.
[52] We ran the regression in which Type *A* and Type *B* bids were modeled separately by including a full set of interaction terms of a dummy variable (D=1 if Type *B*, 0 otherwise) with each polynomial cost term. This is not identical to running two completely separate models in the case of a panel data regression. However the results, both the predictions and the resulting statistics, are very similar.

estimated regressions where only a single bidding function was specified, which would be appropriate if Type *A* and Type *B* bidding behavior was indistinguishable, and regressions where Type *A* and Type *B* bid functions had independent coefficients. For both the pooled and separate regressions, we ran restricted and unrestricted models.[53] We present here only the unrestricted models. The omitted results are qualitatively similar.

We present the results in Table 4 for open auctions, and Table 5 for auctions with a quota. The pooled models (single bid function) fit the data well, but the chi-square statistic testing the joint significance of all interacted variables in regressions (2) and (4) lead us to conclude that Type A and Type B seller behavior should be modeled separately. The predictions from our selected model are shown in Figure 26. In fact, the predictions of the model look extremely similar to what was predicted by theory, as can be seen by comparison with Figure 23. We run a simple Monte Carlo experiment, showing what the distribution of procurement cost would be in repeated experiments, assuming the estimated coefficients of our econometric model are the true coefficients. The results are displayed in Figure 27. The mean procurement cost in an auction without a quota is \$526.56, compared to a mean procurement cost of \$477.53 in an auction with a quota.

[53] We use our econometric model to test the theoretical restrictions placed on our computational model. Specifically, we test whether or not $\beta^B\left(\overline{c}^B\right)=\overline{c}^B$, or whether the bidder with the highest possible cost draw submits a bid equal to their cost draw. With this restriction and the assumption of monotonicity, this implies that no Type B bidder would bid more than $\overline{c}^B$. Additionally we test whether or not we can impose the restriction that $\beta^A\left(\overline{c}^B\right)=\overline{c}^B$. In all cases our data rejects the theoretical restrictions based on a likelihood ratio test. Consequently, the coefficients we derive in our CSE calculations are not directly comparable to the coefficients we estimate econometrically. The predicted bidding behavior, however, is comparable.

Table 4 – Econometric Model of the Open Auction

	(1) Type *A* & *B* Pooled Bid		(2) Fully interacted model Bid	
main				
Cost	0.316*	(2.57)	-0.470*	(-2.02)
Cost^2	-1.620**	(-3.29)	3.267*	(2.34)
Cost^3	3.401***	(4.59)	-7.062*	(-2.29)
Cost^4	-1.596***	(-4.30)	5.625*	(2.48)
Cost*B			5.205*	(2.02)
(Cost^2)*B			-16.20*	(-2.54)
(Cost^3)*B			22.82**	(3.19)
(Cost^4)*B			-12.09***	(-3.63)
B			-0.659	(-1.72)
Constant	0.532***	(48.74)	0.565***	(41.18)
Test of	chi2(4) = 4555.81		chi2(4) = 428.55	
joint			chi2(4) = 20.83	
sigma_u				
Constant	0.0595***	(14.60)	0.0595***	(14.62)
sigma_e				
Constant	0.0589***	(53.33)	0.0584***	(53.33)
Observations	1556		1556	

t statistics in parentheses
* p<0.05, ** p<0.01, *** p<0.001
B=1 if TypeB, 0 otherwise

Table 5 – Econometric Model of the Auction with Quota

	(3)		(4)	
	Type *A* & *B* Pooled		Fully interacted model	
	Bid		Bid	
main				
Cost	0.361*	(1.99)	-0.254	(-0.74)
Cost^2	-1.168	(-1.67)	2.757	(1.39)
Cost^3	2.535*	(2.48)	-6.547	(-1.52)
Cost^4	-1.151*	(-2.31)	5.712	(1.82)
Cost*B			-1.780	(-0.52)
(Cost^2)*B			0.986	(0.12)
(Cost^3)*B			4.754	(0.50)
(Cost^4)*B			-5.476	(-1.23)
B			0.425	(0.83)
Constant	0.440***	(28.88)	0.451***	(23.23)
Test of	chi2(4) = 2947.84		chi2(4) = 491.52	
joint			chi2(4) = 8.58	
sigma_u				
Constant	0.0437***	(8.30)	0.0412***	(8.36)
sigma_e				
Constant	0.0500***	(32.65)	0.0495***	(32.72)
Observations	590		590	

t statistics in parentheses

* p<0.05, ** p<0.01, *** p<0.001

B=1 if TypeB, 0 otherwise

Figure 26 – Predicted Bidding Functions

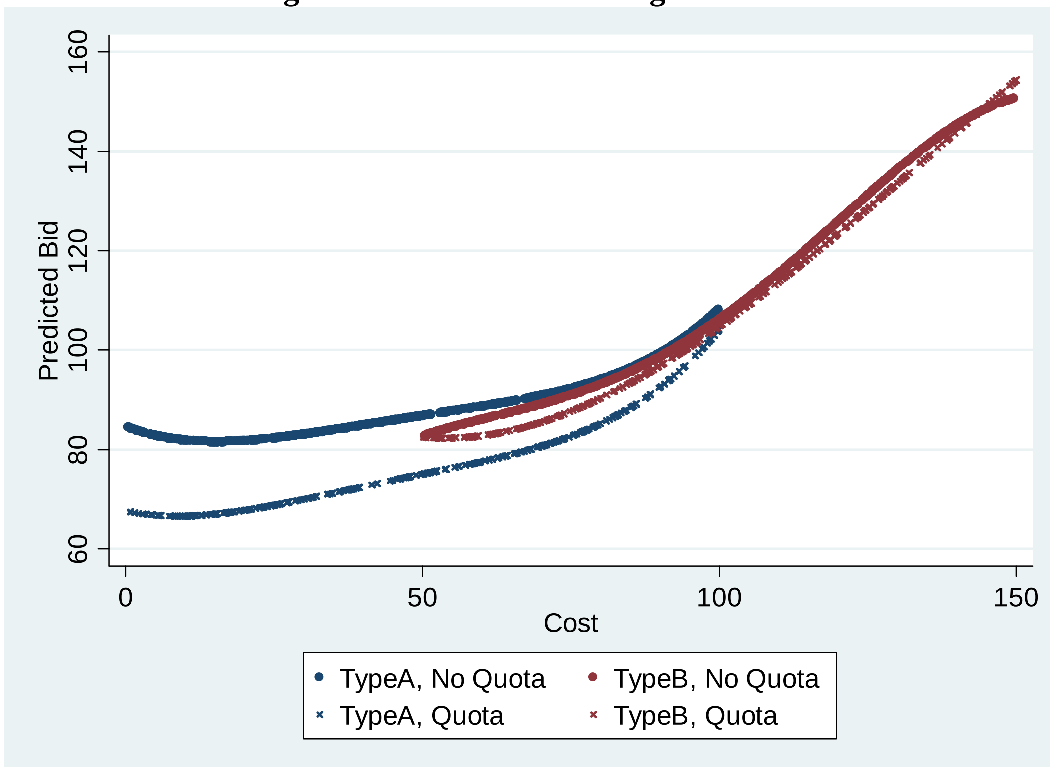
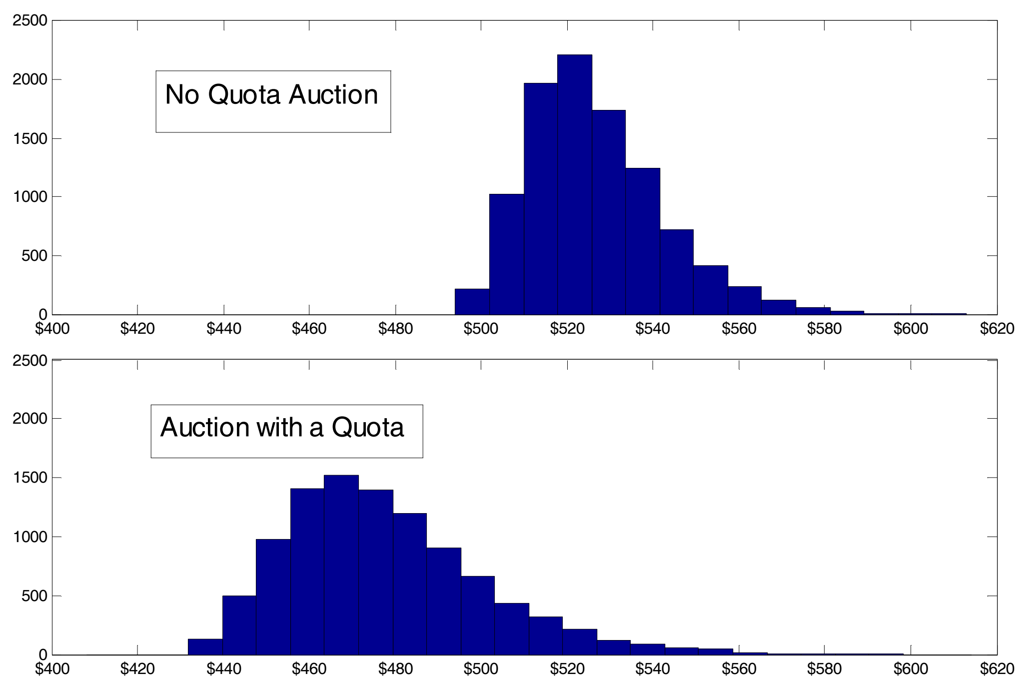


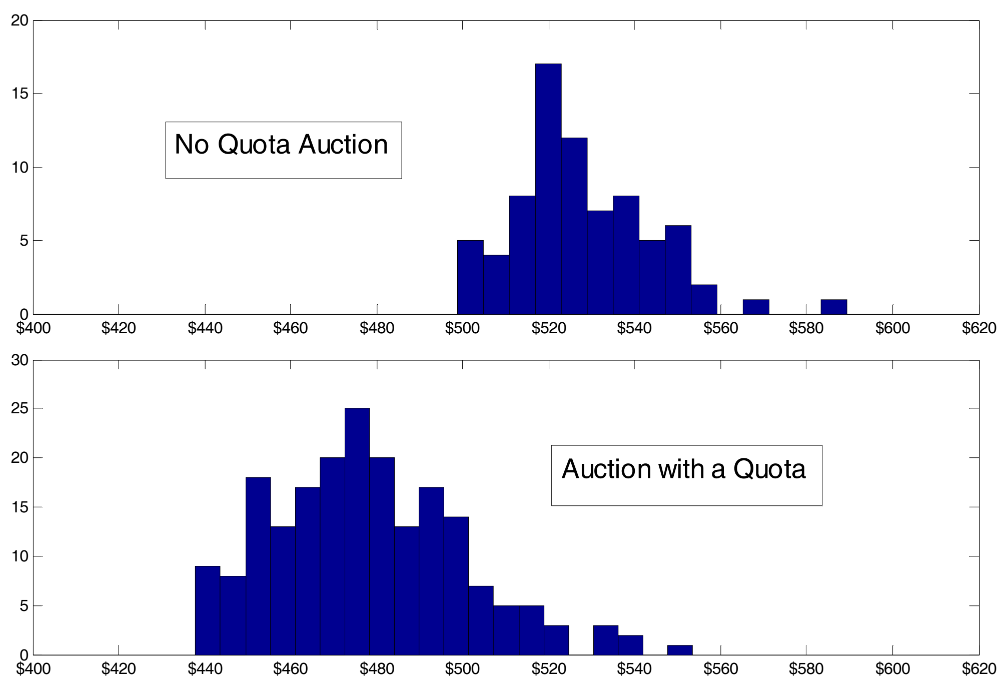
Figure 27 – Distribution of Procurement Cost in Repeated Experiments



We use the same estimated coefficients to run an additional counterfactual experiment. We predict what the procurement cost would have been if the exact cost draws received by sellers in the auctions without a quota were instead obtained in the auctions with a quota, and vice-versa. Simply, we swap the cost draws received by subjects in the quota treatment with the cost draws received by subjects in the open auction treatment. We do this to ensure that our findings are not an artifact of a set of lucky draws that somehow made the quota treatment appear to produce more aggressive bidding and lower procurement costs.

When we apply the estimated coefficients to the swapped cost draws, we obtain results summarized in Figure 28. Nonparametric tests confirm that the data do not come from identical distributions. Procurement cost in auctions with a quota is significantly lower than in auctions without a quota.

Figure 28 – Distribution of Procurement Cost in Repeated Experiments with Actual Cost Draws



We have thus far examined the procurement costs that would be realized, on average, if sellers followed the strategies given by our econometric model. Some amount of "smoothing" takes place when we base our procurement cost estimates on our econometric model, however, since the influence of outliers is minimized. We turn finally to a direct analysis of the observed procurement cost data in each auction of the laboratory experiment.

Procurement Cost

We plot a histogram of the procurement cost under both treatments over all sessions in Figure 29. It is apparent that the mean procurement cost in an auction with a quota is lower than the mean procurement cost in an auction without a quota. A Komogorov-Smirnov test rejects the null hypothesis of identical distributions, supporting the finding that procurement cost is lower in an auction with a quota. The Kolmogorov-Smirnov test statistic[54] is based on the maximum distance between the two empirical CDFs, which we plot in Figure 30. The empirical CDF of procurement cost in an auction without a quota stochastically dominates the empirical CDF of procurement cost in an auction with a quota. A Mann-Whitney rank-sum test bolsters our conclusion (rank-sum = 7054; p-value = 4.6e-9).

[54] The value of the test statistic here is 0.3624, with an associated p-value of 6.2e-7.

Figure 29 – Distribution of Procurement Cost

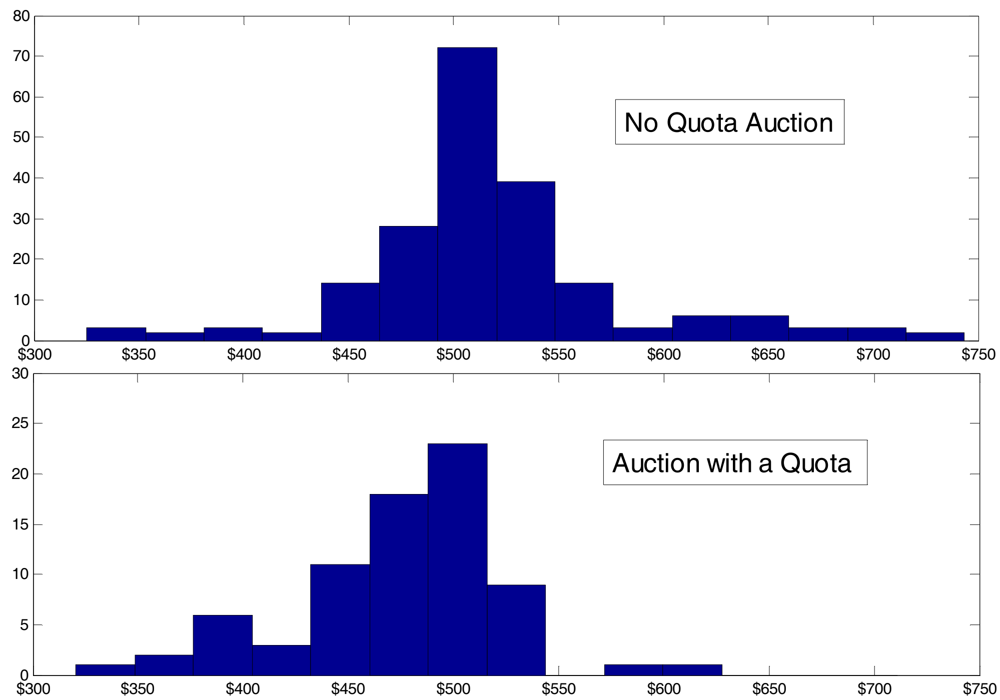
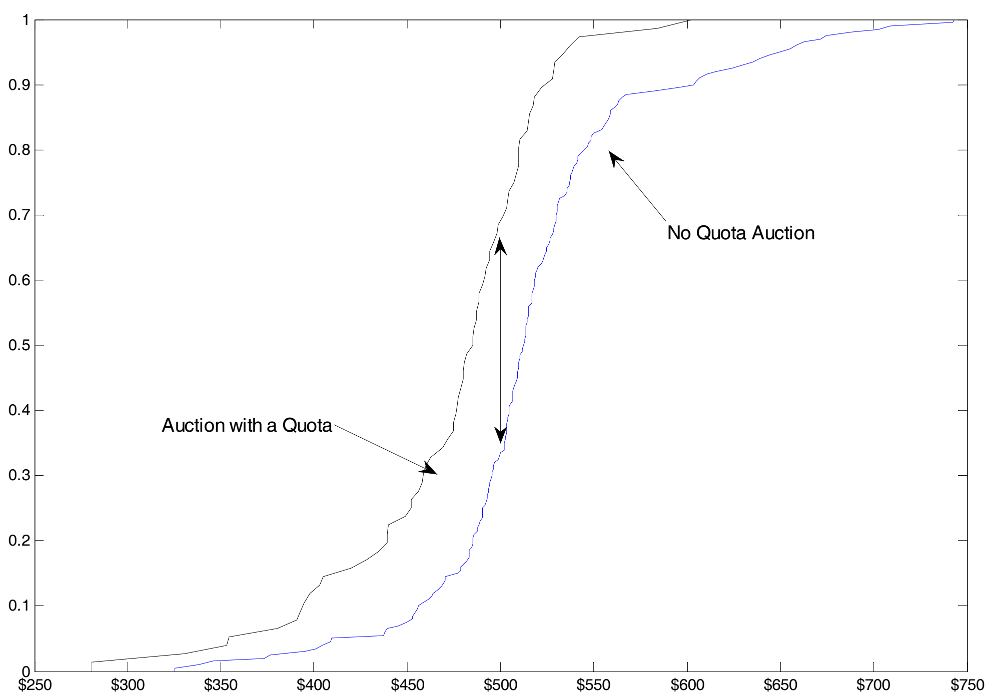


Figure 30 – Empirical CDFs of Procurement Cost



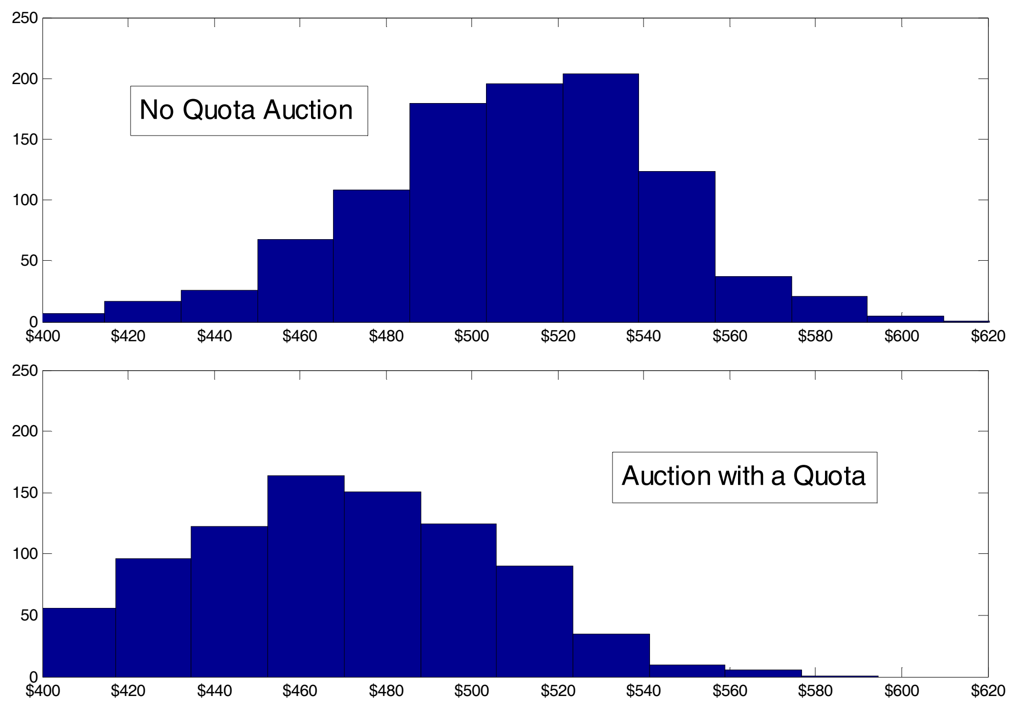
We carry out a recombinant estimation procedure to complement the above tests.[55] In order to make sure that our procurement cost statistics are not unduly influenced by a small number of anomalous bids, we *recombine* bids made in a given auction with bids made in other auctions. In this way, we can create hypothetical auctions – auctions that did not take place but *could have* taken place, thus increasing our sample size. To be clear, we present a simple example. Denote by $b_{t,i}$ the bid placed by seller *i* in auction *t*, $i \in [1,10]$, $t \in [1,T]$. Each auction 1,…,*T* occurred inside the lab, generating payments to subjects. A vector of bids exists for each auction that did take place, for example the first: $\left(b_{1,1}\ b_{1,2}\ b_{1,3}\ b_{1,4}\ b_{1,5}\ b_{1,6}\ b_{1,7}\ b_{1,8}\ b_{1,9}\ b_{1,10}\right)$. An outcome that *could have* occurred but did not is given by $\left(b_{28,1}\ b_{63,2}\ b_{46,3}\ b_{27,4}\ b_{37,5}\ b_{11,6}\ b_{57,7}\ b_{66,8}\ b_{85,9}\ b_{25,10}\right)$, i.e. the bid of the first seller in the 28th auction, the second seller in the 63rd auction, and so on. This vector of bids represents an auction for which we can calculate our outcome of interest, procurement cost.

The assumption underlying the procedure is exchangeability. The manifestation of this assumption in terms of a sealed-bid auction is simply that when a subject makes a bid, there are only two pieces of information relevant to his decision: (1) his cost draw; (2) his knowledge of the distribution of his opponents' costs. If this assumption is reasonable, it allows us to treat the bids submitted in any auction as exchangeable, and thus to create a large set of valid hypothetical auctions that we can evaluate.

[55] See Mullin and Reiley (2006) for further details on the recombinant estimator.

We carry out a recombinant procedure using the algorithm outlined in the appendix. We use this algorithm to generate auctions, which we then solve and compute procurement cost statistics for. The results are summarized by the histogram presented in Figure 31. The conclusions we drew from the raw data are supported by what we find using the recombinant procedure. Mann-Whitney and Kolmogorov-Smirnov tests again reject the null hypothesis of identically distributed procurement costs.

Figure 31 – Procurement Cost under Recombinant Procedure



The key question we have set out to answer is whether or not the changes we observe in seller behavior between auctions without a quota and auctions with a quota add up to a more cost-effective auction from the perspective of the bidtaker. The experimental results do confirm our computational finding that auctions with a quota are more cost-effective. The difference in total procurement cost between the two institutions is more than we expected, however. We estimate that average procurement cost was reduced by 8.7% in the auctions with simple quota as compared to the open auctions (Table 6). Compare this to an expected reduction in procurement cost of 4.1%. Total procurement cost was lower than expected in all experimental auctions. Auctions without a quota were 2.4% less costly than expected, while auctions with a quota were 7.1% less costly than expected.

Table 6 – Experimental / CSE Comparison

Total Procurement Cost

	Experimental		**Simulated**	
	Open	Quota	Open	Quota
Avg cost	$518.57	$473.30	$531.49	$509.73
SD	63.24	54.19	13.76	21.27
N	200	76	10,000	10,000

Chapter 6: Conclusions

In this dissertation I have outlined the CSE procedure proposed by Armantier, Florens and Richard (2008) for approximating the equilibrium bid functions of an auction. I have explored the performance of the procedure in relation to the prevailing computational methods in the literature, and I have programmed an algorithm that allows a CSE solution to be found for a general auction problem. The modular nature of this algorithm allows many auction problems to be solved using a consistent program, avoiding rewriting for special cases. This is the first work of which I am aware to apply a computational equilibrium program to a multi-unit auction case. In those cases when an analytical equilibrium can be found, it has been shown that the computational algorithm yields nearly exact results, even in the multi-unit case.

The CSE algorithm has been used to predict outcomes in a multi-unit, asymmetric procurement auction problem, and these predictions have been compared to the outcomes of a laboratory experiment. The direction of the comparative static predictions given by the CSE algorithm have been confirmed by the laboratory experiment.

The CSE algorithm and computational program given here can be useful for market design problems in many areas of economics.

Appendices

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Appendix B – Instructions for Laboratory Experiment

The following screenshots are samples of the instructions used for the laboratory experiment reported in chapter 5. The instructions had a modular design such that the order of treatment could be varied continuously and randomly. Thus the entire instructions for any one experimental session were composed of instructions similar to those displayed here. Instructions used in a given session and for a given treatment are available upon request.

Welcome!

Today you will be participating in an experiment on economic decision making.

If you follow the instructions and make good decisions, you can earn a considerable amount of money, which will be paid to you after the experiment is over.

Just for participating in this experiment, you will receive a *participation payment* of $5.00. However, you can earn substantially more by actively taking part in the experiment. Your total earnings will be paid to you in cash at the conclusion of the experiment.

We will begin by conducting several auctions. In each auction, you and all other participants are sellers and there is a single buyer. The single buyer is a computer. The buyer will be purchasing tickets, which you will be given, in each auction.

We will be conducting 40 auctions today, one after the other. Each auction will take about 1 minute. They are all separate, independent auctions. Your *bid* and earnings in any one auction will have no influence on your earnings in any other auction.

These instructions will describe how the auction works and prepare you for the auctions. After describing each topic, a list of bullet points will be provided to summarize the topic just introduced. During the experiment you will be able click on the Summary of this auction's rules link in the upper left-hand corner of your web browser to bring up these bullet-point summaries and recall topics quickly.

Auction Instructions

During an auction, each one of you will have a single ticket that you may offer for sale to the buyer.

To offer your ticket for sale, you will submit a *bid*. The lower your *bid* the higher the chance your *bid* will be accepted. If your *bid* is accepted, you will receive your *bid* minus a *cost* that will be posted on your ticket. You will only incur this *cost* if your *bid* is accepted.

The *cost* of your ticket will be randomly determined separately and independently for each bidder in each auction in the experiment.

Both the *cost* and your *bid* will be denominated in a currency called E-Bucks. Each E-Buck will be converted into dollars at the end of the experiment at a rate of .05 real dollars per E-Buck.

<table>
<tr><td rowspan="3">Thus your total earnings for the experiment will be:</td><td colspan="3">(Your $5.00. participation payment)</td></tr>
<tr><td colspan="3">+</td></tr>
<tr><td>(Your net earnings)</td><td>×</td><td>(the .05 E-bucks Conversion Rate)</td></tr>
<tr><td rowspan="3">... and where Your net earnings equals :</td><td colspan="3">(Sum of your accepted bids)</td></tr>
<tr><td colspan="3">-</td></tr>
<tr><td colspan="3">(Sum of costs on your accepted tickets)</td></tr>
</table>

Types of Bidders

There are two types of bidders in the room. Some of you are *Type A* bidders and some of you are *Type B* bidders. You will find out which type you are, and the *cost* of your ticket, when you advance to the bidding screen.

For *Type A* bidders, *cost* may be any amount (rounded to the penny) between and including $0 and $100, with each amount being equally likely.

Imagine a roulette wheel with the stops labeled at $0.00, $0.01, $0.02,..., $99.98, $99.99, $100.00
A hard spin of the wheel would make each of these values equally likely.
Similarly, for *Type B* bidders *cost* may be any amount (rounded to the penny) between and including $50 and $150, with each amount being equally likely.

You may note that the range of *Type B* cost is higher than the range of *Type A*.

Before each auction you will be given a new ticket, with a new randomly chosen *cost*, regardless of whether or not you sold your ticket in the previous auction. In other words, each auction is truly independent. Your actions in any auction will have no bearing on your earnings in any other auction.

More on Bidder Types

Even though you will receive a new ticket before each auction, your bidder *Type* will remain the same.

- If you are a *Type A* bidder, you will always be a *Type A* bidder.
- If you are a *Type B* bidder, you will always be a *Type B* bidder.

Your random *cost* for any round will be known to you and only you. Do not let anyone else see it. You will learn your random *cost* at the beginning of each timed round. You will have up to a minute to submit your *bid* after learning your random *cost*. You can, however, submit a *bid* as soon as you like after learning your random *cost*.

We will review how you submit your *bid* in a few moments.

Although the *cost* of your ticket differs from other bidders in the room, the buyer values all tickets equally. To the buyer, any one ticket is as good as any other ticket.

Review of Types of Bidders

- There are 10 total bidders (you plus 9 others).
- There are two Types of bidders:
    - 5 of you are a *Type A* bidders
    - 5 of you are a *Type B* bidders.
- Each ticket has a *cost* printed on it.

  The *cost*, randomly drawn from the *Type A* or *Type B* interval, will be different for every auction.

Rules of the Game

During each auction you will have a chance to submit any *bid* you choose by clicking in the Your bid textbox on the screen and using your number keys.

Once all bidding tickets are received, they will be compiled and winners will be determined.

The buyer will accept the lowest 6 *bids* in each auction.

Payments will be calculated as follows:

- If your ticket is rejected, you will earn nothing for that auction.
- If your ticket is accepted, you will receive your *bid* minus the *cost* printed on your ticket.
- **Please note carefully:** If your *bid* is less than your *cost*, and your ticket is accepted, you will lose money in that auction. This amount (converted from E-bucks to dollars) will be subtracted from your $5.00 *participation payment*.

Making an offer

We will run one practice auction so that you can familiarize yourself with the bidding screen and the process. You may raise your hand to ask questions at any time during the practice auction.

- To submit a *bid* simply click in the Your bid textbox on the screen and use your number keys to enter your offer.
- The practice auction and all other auctions are timed.

- By the conclusion of the timed round, you must submit a *bid* for your ticket.
- You may submit your offer at any time before the timed round expires, if you wish.
- There is a timer on the offer screen to assist you in budgeting your time. If you have not submitted a *bid* by the time the auction expires, the *bid* displayed on-screen will be submitted for you automatically.

You may not converse with other participants during the entire experiment.

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