Strategic Choices in Realistic Settings

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Abstract

In this thesis, we study Bayesian games with two players and two actions (2 by 2 games) in realistic settings where private information is correlated or players have scarcity of attention. The contribution of this thesis is to shed further light on strategic interactions in realistic settings.

Chapter 1 gives an introduction of the research and contributions of this thesis. In Chapter 2, we study how the correlation of private information affects rational agents' choice in a symmetric game of strategic substitutes. The game we study is a static 2 by 2 entry game. Private information is assumed to be jointly normally distributed. The game can, for some parameter values, be solved by a cutoff strategy: that is enter if the private payoff shock is above some cutoff value and do not enter otherwise. Chapter 2 shows that there is a restriction on the value of correlation coefficient such that the game can be solved by the use of cutoff strategies. In this strategic-substitutes game, there are two possibilities. When the game can be solved by cutoff strategies, either, the game exhibits a unique (symmetric) equilibrium for any value of correlation coefficient; or, there is a threshold value for the correlation coefficient such that there is a unique (symmetric) equilibrium if the correlation coefficient is below the threshold, while if the correlation coefficient is above the threshold value, there are three equilibria: a symmetric equilibrium and two asymmetric equilibria. To understand how parameter changes affect players' equilibrium behaviour, a comparative statics analysis on symmetric equilibrium is conducted. It is found that increasing monopoly profit or duopoly profit encourages players to enter the market, while increasing information correlation or jointly increasing the variances of players' prior distribution will make players more likely to choose entry if the equilibrium cutoff strategies are below the unconditional mean, and less likely to choose entry if the current equilibrium cutoff strategies are above the unconditional mean.

In Chapter 3, we study a 2 by 2 entry game of strategic complements in which players' private information is correlated. As in Chapter 2, the game is symmetric and private information is modelled by a joint normal distribution. We use a cutoff strategy as defined in Chapter 2 to solve the game. Given other parameters, there exists a critical value of the correlation coefficient. For correlation coefficient below this critical value, cutoff strategies cannot be used to solve the game. We explore the number of equilibria and comparative static properties of the solution with respect to the correla-

tion coefficient and the variance of the prior distribution. As the correlation coefficient changes from the lowest feasible (such that cutoff strategies are applicable) value to one, the sequence of the number of equilibrium will be 3 to 2 to 1, or 3 to 1. Alternatively, under some parameter specifications, the game exhibits a unique equilibrium for all feasible value of the correlation coefficient. The comparative statics of equilibrium strategies depends on the sign of the equilibrium cutoff strategies and the equilibrium's stability.

We provide a necessary and sufficient condition for the existence of a unique equilibrium. This necessary and sufficient condition nests the sufficient condition for uniqueness given by Morris and Shin (2005). Finally, if the correlation coefficient is negative for the strategic-complements games or positive for the strategic-substitutes games, there exists a critical value of variance such that for a variance below this threshold, the game cannot be solved in cutoff strategies. This implies that Harsanyi's (1973) purification rationale, supposing the perturbed games are solved by cutoff strategies and the uncertainty of perturbed games vanishes as the variances of the perturbation-error distribution converge to zero, cannot be applied for a strategic-substitutes (strategic-complements) game with dependent perturbation errors that follow a joint normal distribution if the correlation coefficient is positive (negative). However, if the correlation coefficient is positive for the strategic-complements games or negative for the strategic-substitutes games, the purification rationale is still applicable even with dependent perturbation errors. There are Bayesian games that converge to the underlying complete information game as the perturbation errors degenerate to zero, and every pure strategy Bayesian Nash equilibrium of the perturbed games will converge to the corresponding Nash equilibrium of the complete information game in the limit.

In Chapter 4, we study how scarcity of attention affects strategic choice behaviour in a 2 by 2 incomplete information strategic-substitutes entry game. Scarcity of attention is a common psychological characteristic (Kahneman 1973) and it is modelled by the rational inattention approach introduced by Sims (1998). In our game, players acquire information about their own private payoff shocks (which here follows a high-low binary distribution) at a cost.

We find that, given the opponent's strategy, as the unit cost of information acquisition increases a player's best response will switch from acquiring information to simply comparing the ex-ante expected payoff of each action (using the player's prior). By studying symmetric Bayesian games, we find that scarcity of attention can generate multiple equilibria in games that ordinarily have a unique equilibrium. These multiple equilibria are generated by the information cost. In any Bayesian game where there are multiple equilibria, there always exists one pair of asymmetric equilibria in which at least one player plays the game without acquiring information. The number of equilibria differs with the value of the unit information cost. There can be 1, 5 or 3 equilibria. Increasing the unit information cost could encourage or discourage a player from choosing entry. It depends on whether the prior probability of a high payoff shock is greater or less than some threshold value. We compare the rational inattention Bayesian game with a Bayesian quantal response equilibrium game where the observation errors are additive and follow a Type I extreme value distribution. A necessary and sufficient condition is established such that both the rational inattention Bayesian game and quantal response game have a common equilibrium.

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Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification.

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Chapter 1

Strategic Choices in Realistic Settings: Introduction

In this thesis, we study how Bayesian players strategically interact in real settings. The real settings refer to correlated private information and scarcity of attention. We introduce the two features into a conventional 2×2 Bayesian game and study how these features affect a player's strategic choice behaviour. The contributions of this thesis shed further lights on the essence of strategic interactions in the real world and provide us with fresh insight into modelling strategic interactions in a realistic setting.

1.1 Introduction

A 2-player entry game is one of the standard Bayesian games that has been widely studied in economics. Its basic form can be expressed as shown in Table 1[1]:

		Player i^*: inactive (0)	Player i^*: active (1)
Player i	inactive (0)	i: 0; i^*: 0	i: 0; i^*: $M+\varepsilon^*$
Player i	active (1)	i: $M+\varepsilon$; i^*: 0	i: $D+\varepsilon$; i^*: $D+\varepsilon^*$

[1] In this thesis, we use $*$ to denote any variable of the opponent. Particularly, $i = (i^*)^*$.

Table 1: The strategic form of a conventional incomplete information entry game

After observing their respective private payoff shocks ε and ε^*, the players make their individual action decisions simultaneously. ε and ε^* have a joint distribution $F(\varepsilon,\varepsilon^*)$. Player *i*'s payoff of choosing action 1 is additively composed by a deterministic payoff (*M* or *D*) and the private payoff shock ε, and player *i*'s payoff of choosing action 0 is 0. Both players collect their payoffs at the end of the game.

The entry game that we have described here makes some strong simplifying assumptions, which this thesis seeks to relax. Specifically, in almost all existing literature related to such a game, either theoretical or empirical, ε and ε^* are assumed independent and players can always pay full attention to their observations. In this thesis, we modify the standard Bayesian game by introducing two 'grains of sands'—private information correlation and scarcity of attention. Private information correlation modifies the information environment and scarcity of attention introduces humanity into rational agents. In this thesis, our main objectives are as follows: (i) to examine rational agents' strategic choice behaviour in a realistic information environment, which is modelled by correlated private information (Chapters 2 and 3), and (ii) to study how rational agents with scarcity of attention make strategic choices in the standard independent-type private information environment (Chapter 4).

1.2 Correlated Private Information (Chapters 2 and 3)

1.2.1 Chapter 2

In Chapter 2, we develop and present a simple model of duopoly entry with correlated private information based on the 2-player static version of the dynamic entry game studied by Pesendorfer and Schmidt-Dengler (2008). In a 2-player static entry game, two identical competing firms simultaneously decide whether to enter a market after observing their respective private payoff shocks. However, unlike conventional entry games, these shocks are not idiosyncratic in this game. This is because there are common and idiosyncratic components of each payoff shock, and each firm only observes its own aggregate shock without knowing its components. An example of this situation is two firms that produce the same good competing for the same market. Each firm expects the private payoff shocks of entry to be correlated with each other, especially when these shocks depend on some common factors of the market.

In Pesendorfer and Schmidt-Dengler's (2008) dynamic entry game where payoff shocks are independent, cutoff strategies are always assumed to be used.[2] We show that in the correlated-information static game, there is a restriction on the value of information correlation. If the value of the given correlation coefficient does not satisfy this restriction, a cutoff strategy cannot be used to solve the static game. This is determined by the normality of the joint prior distribution and the definition of the cutoff strategy. The intuition is that if the correlation coefficient is greater than this critical value, the expected payoff function is no longer monotonic with respect to a player's own private payoff shock, given any strategy of the opponent. For some strategies of the opponent, there are multiple (three) best responses. One of the three best responses will make a player choose entry if the payoff shock is below the best response cutoff value, which contradicts the definition of the cutoff strategy.

Pesendorfer and Schmidt-Dengler's (2008) numerical analysis of a dynamic duopoly entry game with idiosyncratic normally distributed payoff shocks have at least five equilibria. The game assumes that players adopt a cutoff strategy. We analytically prove that for a static correlated-information duopoly entry game, in which payoff shocks are not only normally distributed but also linked by the correlation coefficient, there are at most three equilibria, and hence, for the static version of Pesendorfer and Schmidt-Dengler's (2008) numerical example also (assuming a discount factor equal to zero), in which payoff shocks are independent and normally distributed, there should be at most three equilibria. Furthermore, we provide the comparative statics of the number of equilibria with respect to the correlation coefficient.[3] It is found that for higher degree of information correlation, it is more likely that asymmetric equilibria, i.e. one firm on average prefers entry more than the opponent does, arise. The intuition is that the uncertainty between the players' private payoff shocks is measured by the

[2] Cutoff strategies are defined as when a player's private payoff shock is above a threshold value, they choose entry, or vice versa.

[3] In an independent econometrics paper, Xu (2014) finds a sufficient condition to ensure a unique equilibrium for a 2-player static entry game with flexible payoff specification and positively correlated players' types in order to identify such type of games. We find that his sufficient condition to ensure a unique equilibrium by using a joint normal distribution as the prior distribution is equivalent to the necessary and sufficient condition to ensure that the best response function is a contraction function; this is provided by our Proposition 2. For details, please see Section 2.3. By far, Xu (2014) is the only literature that is the most relevant to this chapter.

correlation coefficient, and this uncertainty determines whether a player's conditional density function of the opponent's payoff shock, given the player's own private payoff shock, can approximately or imprecisely reflect the opponent's private information (private payoff shock).[4] [5] If the uncertainty between the players' private payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private payoff shock given the player's own shock . If the conditional density function can approximately reflect the opponent's private information, multiple equilibria arise. If the conditional density function is imprecise to reflect the opponent 's private information, the game exhibits a unique equilibrium. In the strategic substitutes game, high (low) value of the information correlation coefficient usually represents low (high) uncertainty between the players' private information. However, it is also possible that given certain parameter specifications, the uncertainty between players' private information is high for high values of the correlation coefficient, and so the conditional density function imprecisely reflects the opponent's private information. Accordingly, the game exhibits a unique equilibrium in this situation. These intuitions are established when the best response functions are not contraction function. If the best response functions are contraction function, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem, and hence the game exhibits a unique equilibrium.

Introducing information correlation into an entry game is motivated by the following considerations. First, it is consistent with the widely observed fact that private information, in any form, is hard to be independent. Two entities' private information often depends on some fundamental factors such as the custom of a society or climate. Therefore, each entity has some power of learning or inferring the other entity's private information through the fundamentals. Second, to understand how information correlation affects rational agents' behaviour itself is an interesting topic and few studies about this topic are conducted from the perspective of game theory. Third, and perhaps most importantly, solving the game analytically to understand the inherent structure of

[4]Suppose two random variables ε and ε^*. If ε and ε^* are subject to the relation $\varepsilon^* = a\varepsilon + b + \eta$, where a and b are real numbers and η is a random variable, then we say there exists uncertainty between ε and ε^* and this uncertainty is given by η. Suppose the correlation coefficient between ε and ε^* is given by ρ. Because the relation $\varepsilon^* = a\varepsilon + b + \eta$ holds as long as the correlation coefficient $\rho \in (-1, 1)$, we can consider using ρ to measure the uncertainty between players' private information.

[5]To understand how the density function can reflect the opponent's private information given a player's own private information, please refer to Section 2.4.

the game, i.e. how many equilibria exist and what they are, is technically challenging and is crucial for some further technical application, e.g. identification and estimation of games. This study is also the first step to solve a 2-player correlated-information dynamic entry game. Furthermore, it is a technical preparation to analytically understand a dynamic entry game.

The simplicity of the proposed model makes it well suited for exploring how parameter changes, especially the information correlation coefficient, impact players' equilibrium behaviour. To show this, we calculate the comparative statics of monopoly profit, duopoly profit, variances of prior distribution and the information correlation coefficient, on firms' equilibrium entry threshold in the symmetric equilibrium. We find that increasing the monopoly profit or duopoly profit can always encourage players to adopt lower cutoff strategies and so they are more likely to choose entry. If the present cutoff strategies are positive, then increasing the information correlation or jointly increasing the variances of type distribution will make both players adopt higher cutoff strategies, i.e. they become less likely to choose entry. In contrast, if the present cutoff strategies are negative, then increasing the information correlation or the variances of prior distribution will encourage players to adopt lower cutoff strategies, i.e. they become more likely to choose entry. If the present cutoff strategies equal zero, changing the correlation coefficient or jointly varying the variances of prior distribution does not affect the cutoff strategies.

The intuition is that, if we increase monopoly profit or duopoly profit, the payoff of entry increases and it thereby encourages a player to choose entry. If we change the correlation coefficient, the mean of the conditional distribution of the opponent's payoff shock given a player's own payoff shock changes and the mean has a dominant impact on the player's belief towards the opponent's strategy given their own strategy. The change of mean depends on the sign of the symmetric equilibrium strategies. If we jointly change the variances of the prior distribution, only the variance of the conditional distribution of the opponent's payoff shock given the player's own shock changes. Increasing the variances will assign higher likelihood on low and high payoff shocks in the conditional distribution of the opponent's payoff shock, or vice versa, and the sign of a symmetric equilibrium strategy determines whether this strategy is located in the high payoff shock area or the low payoff shock area in the distribution. The different location determines the different impacts of changing variances on a player's belief.

1.2.2 Chapter 3

In Chapter 3, we develop and exhibit a simple model of duopoly entry with correlated private information in a 2-player static strategic complements game. In this game, $D > M$ and is represented as shown in Table 1. The game is symmetrically specified. In the game, after observing their respective private payoff shocks, the two player firms simultaneously decide whether to enter a market. The private payoff shocks are statistically correlated, and the correlation coefficient of players' joint type distribution measures the degree of information correlation. An example of this situation is two firms that produce complementary inputs entering a local market. One firm expects its private payoff shocks of entry to be correlated with the other firm's because the shocks depend on some common factors of the market. Each firm observes its own aggregated shock without knowing its components, for example, the common factors and idiosyncratic noises if they additively form the aggregated shock.

The game is solved by a cutoff strategy, which is defined as if a player's private payoff shock is above a threshold value, they choose entry, or vice versa. By solving the game, we find a critical value of the correlation coefficient. For correlation coefficients below this critical value, a cutoff strategy cannot be used to solve the game. This result is determined by the normality of the joint prior distribution and the definition of the cutoff strategy. The intuition is that if the correlation coefficient is smaller than this critical value, the expected payoff function is no longer monotonic with respect to the player's own strategies, given any strategy of the opponent. For some strategies of the opponent, there are multiple (three) best responses. One of the three best responses will make a player choose entry if the payoff shock is below the best response cutoff value, which contradicts the definition of cutoff strategies.

Under some parameter specifications, the game exhibits a unique equilibrium. Under other parameter specifications, there may exist two or three equilibria and the number of equilibria changes in the following order as the correlation coefficient increases from the lowest feasible value to 1: $3 \to 2 \to 1$ or $3 \to 1$. The intuition is that the uncertainty between players' private payoff shocks is measured by the correlation coefficient, and the uncertainty between players' private information determines whether a player's conditional density of the opponent's payoff shocks given the player's own payoff shock can approximately or imprecisely reflect the opponent's private information.[6] [7] If the uncertainty between players' private payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private information. If the density function can approximately reflect the other players' private information, then players can obtain enough information to help them match their action strategies, hence leading to multiple equilibria. Otherwise, players cannot obtain enough information to help them match their action strategies and hence the game exhibits a unique equilibrium. In the strategic complements game, high (low) value of correlation coefficient usually represents high (low) uncertainty between players' private information. However, it is also possible that for certain parameter specifications, the game has a unique equilibrium for all feasible values of the correlation coefficient. This is because due to the concerned payoff specification, the two players' ex ante expectations of the opponent's behaviour are unique, irrespective of what payoff shocks will be drawn. Ex ante in this chapter means the expectation is formed before the payoff shocks are drawn, and hence, the expectation is taken for all possible values of payoff shocks. Therefore, we call it ex ante expectation. The expectations are that both players are more likely to choose being inactive, active or not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice. Accordingly, the game exhibits a unique equilibrium to echo the respective expectations.[8]

[6]As in Chapter 2, the uncertainty between players' payoff shocks (random variables) indicate that for two random variables ε and ε^*, the relation $\varepsilon^* = a\varepsilon + b + \eta$ holds, where a and b are two real numbers and η is a random variable that is used to reflect the uncertainty between players' private information. Still, we can consider to use the correlation coefficient between the two random variables to measure the uncertainty between them.

[7]To understand how the density function can reflect the opponent's private information given a player's own private information, please refer to Section 2.4 of Chapter 2

[8]Specifically, if the expectation is that both players are more likely to choose being inactive or that both players are more likely to choose being active, the expectation is dominant in a player's decision making and the uncertainty between players' private information takes a minor role in his decision making. However, if the expectation is that players are not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice, only when the uncertainty between players' private information is high, the expectation is dominant in a player's decision making. These intuitions are established when the best response functions are not contraction function. If the best response functions are contraction function, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem, and hence the game exhibits a unique equilibrium.

The equilibrium strategies are represented by the cutoff strategies, which take all real numbers. The comparative statics of the correlation coefficient or variances of prior distribution on players' equilibrium strategies depend on the sign of the equilibrium strategy and the stability of the equilibrium.[9] For a stable equilibrium, increasing the payoff of entry will make a player more likely choose entry. If the given equilibrium cutoff strategies are negative, increasing the information correlation or jointly increasing the variances of the joint prior distribution will make players less likely choose entry. If the given equilibrium cutoff strategies are positive, increasing information correlation or jointly increasing the variances of the joint prior distribution will make players more likely choose entry. If the given equilibrium cutoff strategies equal zero, changing the information correlation or variances of the joint prior distribution does not have any impact on the equilibrium strategies. For unstable equilibrium, increasing the payoff of entry will make a player less likely choose entry, which contradicts our common sense. Because we use a cutoff strategy to solve the game, if the payoff of entry increases, then given the opponent's strategy, a player will more likely choose entry. Because the game exhibits positive externalities in payoffs, the opponent will also be more likely to choose entry as the best response to the player's change of strategies more favouring entry. Given this best response dynamics, no strategy will converge to an equilibrium in which increasing the payoff of entry makes a player less likely to choose entry. This situation satisfies the Lyapunovian instability of an equilibrium and hence such an equilibrium is unstable.

In this symmetric game, the variances of players' prior distribution are assumed to be identical. There is an equivalence relationship between how the number of equilibria varies with the variances and with the correlation coefficient. We find that under certain parameter specifications, the game exhibits a unique equilibrium. Under other parameter specifications, the number of equilibria changes in the following order as variances increase from the lowest feasible value to $+\infty$: $3 \to 2 \to 1$ or $3 \to 1$. The intuition is that the uncertainty of a player's private payoff shock is determined by the variance of the player's prior distribution, and the uncertainty of both players' private payoff shocks determines whether the conditional density of the opponent's payoff shocks given the player's own payoff shock can approximately or imprecisely reflect the opponent's private information.[10] If the uncertainty of both players' private

[9] The stability concept adopted in this chapter is Lyapunov stability.

[10] In this chapter, no matter how the variances change, they are always assumed to be identical. There-

payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private information. Still, multiple equilibria arise when the density function can approximately reflect the opponent's private information given the player's own private information. Otherwise, the game exhibits a unique equilibrium when the density function is imprecise to reflect the opponent's private information given the player's own private information. The low (high) value of the variance usually represents low (high) uncertainty of a player's private information in the strategic complements game. However, it is also possible that for certain parameter specifications, the game has a unique equilibrium for all feasible values of variances. Similar to the corresponding case for the uncertainty between players' private information, this is also because due to the concerned payoff specification, the two players' ex ante expectations of the opponent's behaviour are unique, irrespective of what payoff shocks will be drawn, and the expectations are that both players are more likely to choose being inactive, active or not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice.[11]

The comparative statics of the number of equilibria with respect to variances is also the necessary and sufficient condition to differentiate unique equilibrium and multiple equilibria. Morris and Shin (2005) study an identically specified game and provide a sufficient condition for unique equilibrium. They focus on how introducing strategic uncertainty can reduce the number of equilibria of a complete information game. The complete information game is symmetric and strategic complements. They also use the cutoff strategy defined in this chapter to solve the game. They argue that when the strategic uncertainty (belief) is sufficiently invariant with respect to all possible strategies, there is a unique equilibrium. Based on this insight, they obtain a sufficient condition to ensure the game exhibits a unique equilibrium. We find that their sufficient condition is essentially the necessary and sufficient to ensure that the best response functions are contraction functions. If both players' best response functions are contractions, then the game is dominance solvable and hence there exists a unique equilibrium. Therefore, we nest Morris and Shin's (2005) result.

fore, we consider the uncertainty of both players' private information (payoff shocks).

[11]Specifically, if the expectation is that both players are more likely to choose being inactive or that both players are more likely to choose being active, the expectation is dominant in a player's decision making and the uncertainty of the player's private information takes a minor role in his decision making. However, if the expectation is that players are not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice, only when the uncertainty of each player's private information is high, the expectation is dominant in a player's decision making.

The incomplete information entry game can be viewed as a perturbed game of a complete information entry game. According to Harsanyi (1973)'s purification rationale, if the perturbation errors on each player's payoff are independent, a Bayesian Nash equilibrium exists that will converge to the mixed strategy equilibrium as perturbation errors tend to zero. In our game, we specify that the variances of the perturbation-error distribution converge to zero, as the process that uncertainty of perturbed games vanishes. We find that, for the strategic complements complete information games if the perturbation errors are negatively correlated, or for the strategic substitutes complete information games if the perturbation errors are positively correlated, there does not exist a Bayesian game that can be solved by the cutoff strategy as perturbation errors tend to zero. Hence, Harsanyi's purification rationale cannot be applied to this situation. The intuition is that by assuming the variances of both players' type distributions are identical, for negative information correlation in the strategic complements game or the positive correlation in the strategic substitutes game, there exists a critical value of variances, below which the expected payoff function is not monotonic with respect to a player's own private payoff shock, and it is possible that given some of the opponent's strategies, the player can have multiple (three) best responses; around one of the best responses, a payoff shock that is below the best response cutoff value can make the player choose entry, which contradicts the definition of the cutoff strategy. Therefore, for negative information correlation in the strategic complements game or positive information correlation in the strategic substitutes game, only if the variances are above the cutoff value, the game can be solved by cutoff strategies.

However, if the information correlation is positive for the strategic complements games or negative for the strategic substitutes games, the purification rationale is still applicable. We find that in these situations, the Bayesian games that are supposed to converge to the complete information game as the perturbation errors that degenerate to zero exist, and during the process, the pure-strategy Bayesian Nash equilibrium will converge to the corresponding Nash equilibrium of the underlying complete information game. Therefore, we extend Harsanyi's purification rationale to dependent perturbation-error situations.

1.3 Scarcity of Attention (Chapter 4)

In Chapter 4, we study how scarcity of attention affects players' strategic behaviour in an incomplete information environment with strategic substitutes. The interaction paradigm and sequence of actions still follow Table 1. In this game, we assume $M > D$ and payoff shocks ε (and ε^*) follows a bivariate distribution: $\varepsilon \in \{u, d\}$, where $Pr(u) = p \in (0,1)$ and $u > d$. Further, to ensure that the underlying game is dominance solvable, we assume $M + d < 0$ and $D + u > 0$.[12] The only difference between this game and the conventional Bayesian games is that both players cannot pay full attention to their observations of private payoff shocks. However, both players can acquire information about their private payoff shocks at a cost, and the information acquisition process in this game is modelled by the rational inattention approach. Psychologists have found that scarcity of attention can account for randomized choice, and in economics literature, Woodford (2008, 2009) and Matějka and Mckay (2015) independently develop the randomized choice theory in the rational inattention framework. The objective of Chapter 4 is to investigate how agents with scarcity of attention strategically interact in an incomplete information environment.

The analysis in this chapter is in line with the literature of entry games. Entry games have been widely studied in industrial organization literature, and it is the most typical form for modelling strategic substitutes behaviour. However, no literature exists that study how psychological factors affect firms' competition. There is a void related to this topic in the industrial organization literature, which this chapter is initially motivated to fill.

Assuming both players' information costs are identical, we first find that there exists a critical value of information cost.[13] If the given information costs of both players are below this value, the game is a Bayesian game. If the given information costs of both players are above or equal to this value, the game becomes a complete information game, in which the players make their best responses without acquiring information, given any strategy of the opponent.

Next, by studying symmetric games, we find that scarcity of attention can generate

[12]The underlying games are the Bayesian games in which players have full attention to their observation.

[13]In this chapter, for simplicity, we refer to information cost to indicate the unit information cost.

multiple equilibria, and different values of information costs lead to different numbers of equilibria.[14] A general rule is that jointly increasing both players' information costs first increases and then decreases the number of equilibria. Specifically, in the symmetric rational inattention Bayesian games, we find that given other primitives, by jointly increasing the information costs from 0 to $+\infty$, the number of equilibria appears in the following sequence: $1 \to 3 \to 5 \to 3$ if multiple equilibria can arise. Alternatively, there always exists a unique equilibrium for any value of information cost. In addition, we find that in any multiplicity situation, there always exists one pair of asymmetric equilibria in which at least one player plays without acquiring information and relies on his prior knowledge. These results about the game's equilibria are mainly caused by the concavity–convexity property of the part of the second iteration of the best response functions in which both players play the game by acquiring information. Furthermore, the concavity–convexity property is ultimately induced by the structure of entropy functions.[15]

For comparative statics of equilibrium strategies, we find that in the symmetric equilibrium and outer asymmetric equilibrium, any improvement in players' expected payoff of entry can increase the probability of entry.[16] If we jointly increase both players' information costs, its impact depends on the relative magnitude between the prior probability of high payoff shock and a threshold value. If the prior probability of high payoff shock is higher (or lower) than the threshold value, increasing the information cost will increase (or decrease) the probability of entry. If the prior probability of high payoff shock equals the threshold value, increasing the information cost does not have any impact on the probability of entry. There is no conclusive result about comparative statics of inner asymmetric equilibrium without particular parameter specification. Finally, in any equilibrium, if we change only one player's information cost, its impacts on both players' equilibrium strategies are not clear without particular parameter specification, but its impact on a player's strategy is found to be always opposite to that on the opponent's strategy.

[14] A symmetric game is defined as a game in which the parameter specifications of both players are identical, particularly the information costs of both players.

[15] The concavity–convexity property means that as the value of a player's strategy increases, the part of the second iteration of the best response functions in which both players acquire information first exhibits concavity and then exhibits convexity. For details, please refer to Section 4.7 of Chapter 4.

[16] In this game, there are three types of equilibrium: symmetric equilibrium, outer asymmetric equilibrium and inner asymmetric equilibrium. They are named according to their location at the best response functions.

We also study how information cost affects a player's expected payoff. A player's information cost does not have any impact on the player's expected payoff, but the opponent's information cost can affect the player's expected payoff through the player's belief towards the opponent's behaviour. Except particular parameter specification, there is no conclusive result about at what value of the opponent's information cost, the player's expected payoff reaches its highest value.

Finally, we study a game in which the players observe their private payoff shocks with an additive noise that follows Type I extreme value distribution. The solution concept is therefore (Bayesian) Quantal Response Equilibria. The similar-looking strategic choice models motivate us to further consider under what conditions the two games can be identical. It is found that there exists a specific set of parameter specification under which both games have a common equilibrium $(\frac{1}{2}, \frac{1}{2})$. Except this situation, the two games will not coincide.

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Chapter 2

Strategic Entry with Correlated Private Information

This chapter studies how correlation of players' private information affects their strategic behaviour. We introduce information correlation into a static 2-player strategic substitutes entry game. The degree of information correlation is measured by the correlation coefficient of a symmetric joint normal distribution, which is used to model players' prior distribution. It is found that a cutoff strategy cannot be used for all values of correlation coefficient to solve the game and there exists a threshold correlation coefficient value to differentiate the unique-equilibrium and the multiple (three)-equilibria situations given other parameters. Finally, by comparative statics analysis of symmetric equilibrium strategies, we find that increasing the payoff of entry encourages players to adopt a lower entry threshold, while increasing the information correlation or jointly increasing the variances of prior distribution increases the positive entry threshold and lowers the negative entry threshold.

2.1 Introduction

This chapter develops and presents a simple model of duopoly entry with correlated private information based on a 2-player static version of the dynamic entry game studied by Pesendorfer and Schmidt-Dengler (2008). In a 2-player static entry game, two identical competing firms simultaneously decide whether to enter a market after observing their respective private payoff shocks. However, unlike conventional entry games, these shocks are not idiosyncratic in this game. This is because there are common and idiosyncratic components of each payoff shock, and each firm only observes its own aggregate shock without knowing its components. An example of this situation is two firms that produce the same good competing for the same market. Each firm expects the private payoff shocks of entry to be correlated with each other, especially when these shocks depend on certain common factors of the market.

In Pesendorfer and Schmidt-Dengler's (2008) dynamic entry game where payoff shocks are independent, cutoff strategies are always assumed to be used.[1] We show that in the correlated-information static game, there is a restriction on the value of information correlation. If the value of the given correlation coefficient does not satisfy this restriction, a cutoff strategy cannot be used to solve the static game. This is determined by the normality of the joint prior distribution and the definition of the cutoff strategy. The intuition is that if the correlation coefficient is greater than this critical value, the expected payoff function is no longer monotonic with respect to a player's own private payoff shock, given any strategy of the opponent. For some strategies of the opponent, there are multiple (three) best responses. One of the three best responses will make a player choose entry if the payoff shock is below the best response cutoff value, which contradicts the definition of the cutoff strategy.

Pesendorfer and Schmidt-Dengler's (2008) numerical analysis of a dynamic duopoly entry game with idiosyncratic normally distributed payoff shocks have at least five equilibria. The game assumes that players adopt a cutoff strategy. We analytically prove that for a static correlated-information duopoly entry game, in which payoff shocks are not only normally distributed but also linked by the correlation coefficient, there are at most three equilibria, and hence, for the static version of Pesendorfer and Schmidt-Dengler's (2008) numerical example also (assuming a discount factor equal to zero), in which payoff shocks are independent and normally distributed, there should be at most three equilibria. Furthermore, we provide the comparative statics of the number of equilibria with respect to the correlation coefficient.[2] It is found that for

[1] Cutoff strategies are defined as when a player's private payoff shock is above a threshold value, they choose entry, or vice versa.

[2] In an independent econometrics paper, Xu (2014) finds a sufficient condition to ensure a unique equilibrium for a 2-player static entry game with flexible payoff specification and positively correlated players' types in order to identify such type of games. We find that his sufficient condition to ensure a unique equilibrium by using a joint normal distribution as the prior distribution is equivalent to the necessary and sufficient condition to ensure that the best response function is a contraction function; this is provided by our Proposition 2. For details, please see Section 2.3. By far, Xu (2014) is the only literature that is the most relevant to this chapter.

higher degree of information correlation, it is more likely that asymmetric equilibria, i.e. one firm on average prefers entry more than the opponent does, arise. The intuition is that the uncertainty between the players' private payoff shocks is measured by the correlation coefficient, and this uncertainty determines whether a player's conditional density function of the opponent's payoff shock, given the player's own private payoff shock, can approximately or imprecisely reflect the opponent's private information (private payoff shock).[3] [4] If the uncertainty between the players' private payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private payoff shock given the player's own shock . If the conditional density function can approximately reflect the opponent's private information, multiple equilibria arise. If the conditional density function is imprecise to reflect the opponent 's private information, the game exhibits a unique equilibrium. In the strategic substitutes game, high (low) value of the information correlation coefficient usually represents low (high) uncertainty between the players' private information. However, it is also possible that given certain parameter specifications, the uncertainty between players' private information is high for high values of the correlation coefficient, and so the conditional density function imprecisely reflects the opponent's private information. Accordingly, the game exhibits a unique equilibrium in this situation. These intuitions are established when the best response functions are not contraction function. If the best response functions are contraction function, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem, and hence the game exhibits a unique equilibrium.

Introducing information correlation into an entry game is motivated by the following considerations. First, it is consistent with the widely observed fact that private information, in any form, is hard to be independent. Two entities' private information often depends on some fundamental factors such as the custom of a society or climate. Therefore, each entity has some power of learning or inferring the other entity's private information through the fundamentals. Second, to understand how information corre-

[3]Suppose two random variables ε and ε^*. If ε and ε^* are subject to the relation $\varepsilon^* = a\varepsilon + b + \eta$, where a and b are real numbers and η is a random variable, then we say there exists uncertainty between ε and ε^* and this uncertainty is given by η. Suppose the correlation coefficient between ε and ε^* is given by ρ. Because the relation $\varepsilon^* = a\varepsilon + b + \eta$ holds as long as the correlation coefficient $\rho \in (-1,1)$, we can consider using ρ to measure the uncertainty between players' private information.

[4]To understand how the density function can reflect the opponent's private information given a player's own private information, please refer to Section 2.4.

lation affects rational agents' behaviour itself is an interesting topic and few studies about this topic are conducted from the perspective of game theory. Third, and perhaps most importantly, solving the game analytically to understand the inherent structure of the game, i.e. how many equilibria exist and what they are, is technically challenging and is crucial for some further technical application, e.g. identification and estimation of games. This study is also the first step to solve a 2-player correlated-information dynamic entry game. Furthermore, it is a technical preparation to analytically understand a dynamic entry game.

The simplicity of the proposed model makes it well suited for exploring how parameter changes, especially the information correlation coefficient, impact players' equilibrium behaviour. To show this, we calculate the comparative statics of monopoly profit, duopoly profit, variances of prior distribution and the information correlation coefficient, on firms' equilibrium entry threshold in the symmetric equilibrium. We find that increasing the monopoly profit or duopoly profit can always encourage players to adopt lower cutoff strategies and so they are more likely to choose entry. If the present cutoff strategies are positive, then increasing the information correlation or jointly increasing the variances of type distribution will make both players adopt higher cutoff strategies, i.e. they become less likely to choose entry. In contrast, if the present cutoff strategies are negative, then increasing the information correlation or the variances of prior distribution will encourage players to adopt lower cutoff strategies, i.e. they become more likely to choose entry. If the present cutoff strategies equal zero, changing the correlation coefficient or jointly varying the variances of prior distribution does not affect the cutoff strategies.

The intuition is that, if we increase monopoly profit or duopoly profit, the payoff of entry increases and it thereby encourages a player to choose entry. If we change the correlation coefficient, the mean of the conditional distribution of the opponent's payoff shock given a player's own payoff shock changes and the mean has a dominant impact on the player's belief towards the opponent's strategy given their own strategy. The change of mean depends on the sign of the symmetric equilibrium strategies. If we jointly change the variances of the prior distribution, only the variance of the conditional distribution of the opponent's payoff shock given the player's own shock changes. Increasing the variances will assign higher likelihood on low and high payoff shocks in the conditional distribution of the opponent's payoff shock, or vice versa, and the sign of a symmetric equilibrium strategy determines whether this strategy is located in the high payoff shock area or the low payoff shock area in the distribution. The different location determines the different impacts of changing variances on a player's belief.

This chapter proceeds as follows: Section 2.2 presents the model. Section 2.3 exhibits the best response function. Section 2.4 describes how the conditional density function of the opponent's payoff shock given a player's own payoff shock can be used to reflect the opponent's private information. Section 2.5 studies the comparative statics of the number of equilibria with respect to the correlation coefficient and the stability property of equilibrium. Section 2.6 studies the comparative statics of symmetric equilibrium strategies. Section 2.7 provides a summary of all main results and intuitions of this game. Section 2.8 concludes this chapter.

2.2 The Game

Consider a 2-player entry game. Each player has two choices, activity (or equivalently *in*), or inactivity (or equivalently *out*). They make their decisions after observing their respective private payoff shocks. Then they implement their actions, simultaneously which can be observed by each other. The active firm engages in production, and if both firms are active, a Cournot competition will occur. At the end of the period, each firm collects their own payoffs. The private payoff shocks are assumed to be subject to a bivariate normal distribution $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma, \varsigma^*, \rho)$. Therefore, the correlation coefficient ρ is a natural measure of the dependence between the two players' private payoff shocks. We call it the information correlation coefficient. Hereafter, we use '*' to denote variables of the opponent. Besides, we assume $\varsigma = \varsigma^*$ to ensure the game is symmetric. Since a Cournot competition will happen if both firms are active, monopoly profit (denoted by M) must exceed duopoly profit (denoted by D). The strategic form of this game is depicted as follows:

		Firm i^*	
		inactive (0)	active (1)
Firm i	inactive (0)	0 0	$M+\varepsilon^*$ 0
	active (1)	0 $M+\varepsilon$	$D+\varepsilon^*$ $D+\varepsilon$

It is natural to think that the strategies of the firm may involve a cutoff value: that is, it enters if the privately observed value of ε is above the cutoff value $\bar{\varepsilon}$ and does not enter otherwise. Suppose that a firm believes that the opponent plays such a strategy with a cutoff x^*. Then, the firm forms a belief that the opponent plays out given its own shock ε. The belief is exhibited by the following function:

$$\sigma(x^*, \varepsilon) = \int_{-\infty}^{x^*} f(\varepsilon^*|\varepsilon) d\varepsilon^*$$

where $f(\varepsilon^*|\varepsilon)$ is the conditional density of ε^* given ε. It is easy to show that $\sigma_{x^*}(x^*, \varepsilon) > 0$, $\sigma_\varepsilon(x^*, \varepsilon) < 0$ if $\rho > 0$, and $\sigma_\varepsilon(x^*, \varepsilon) > 0$ if $\rho < 0$. $\sigma_\varepsilon(x^*, \varepsilon) = 0$ at $\rho = 0$. $\sigma_{x^*}(x^*, \varepsilon)$ is the first-order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to x^*, and $\sigma_\varepsilon(x^*, \varepsilon)$ is the first-order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to ε.

However, it is found that, given M, D, ς^2 and ς^{*2}, the cutoff strategy concept cannot be applied for all values of correlation coefficient ρ from -1 to 1. Let us first look at the player's expected payoff of entry, which is given by

$$\begin{aligned} \mathbb{E}\Pi(x^*, \varepsilon) &= \sigma(x^*, \varepsilon)(M+\varepsilon) + (1-\sigma(x^*, \varepsilon))(D+\varepsilon) \\ &= \sigma(x^*, \varepsilon)M + (1-\sigma(x^*, \varepsilon))D + \varepsilon \end{aligned} \tag{2.1}$$

Equation (2.1) indicates that a player's expected payoff is composed of two parts, namely, the payoff induced by strategic uncertainty, $\sigma(x^*, \varepsilon)M + (1-\sigma(x^*, \varepsilon))D$, and the realised payoff shock, ε. If $\rho \leq 0$, given ρ, M, D, ς^2 and ς^{*2}, both parts are non-decreasing with respect to ε. Intuitively, if both firms' private payoff shocks are negatively correlated, a high payoff shock ε for one firm would on average imply a low payoff shock ε^* for the opponent, which provides an incentive that encourages the player to be active. Therefore, the expected payoff should be non-decreasing with respect to ε for $\rho \leq 0$. Thus, for negatively correlated private information situation, a cutoff strategy can always be applied.

However, if ρ is positive, then given all parameter values, the payoff induced by strategic uncertainty $\sigma(x^*,\varepsilon)M + (1-\sigma(x^*,\varepsilon))D$ is decreasing with respect to ε. Thus, whether the expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$ is monotonically increasing with respect to ε depends on the trade-off between the payoff induced by strategic uncertainty and by the realized payoff shock. For positive ρs, if one firm draws a high payoff shock, it can be expected that the opponent also draws a high payoff shock, and hence, it is highly probable that both players will prefer being active, which provides strategic disincentives for entry in a strategic substitutes context. It is also known that ε itself is a part of the payoff and it incentivizes entering. Therefore, whether the firm will choose to be active essentially depends on the trade-off between the two contrasting effects.

If the correlation between players' private information is loose, i.e. ρ is slightly positive, it can be deduced that the positive incentive generated by a high value of ε dominates its negative impact, and hence, in total, its expected payoff should increase with respect to ε. However, if the correlation coefficient between the players' private information is tight, i.e. ρ is close to 1, then it can be reasonably expected that the strategic disincentive induced by the realization of a high value of ε will be strong, and hence, a high payoff shock does not necessarily bring a high expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$. In fact, it is found that there exists a unique boundary $\tilde{\rho} > 0$ in the strategic substitutes discrete game such that if $\rho \leq \tilde{\rho}$, given the opponent's expected cutoff strategy $x^* \in \mathbb{R}$, the expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε, and if $\rho > \tilde{\rho}$, the expected payoff is no longer monotonic; it is also certain that for some x^*, equation $\mathbb{E}\Pi(x^*,\varepsilon) = 0$ has multiple (three) solutions (best responses) of ε and there is one solution below which a payoff shock can make a player choose entry, which contradicts the definition of the cutoff strategy (see Appendix). Therefore, given M, D, ς^2 and ς^{*2}, a player can legitimately use the cutoff strategy to play the game if and only if $\rho \in (-1,\tilde{\rho}]$ in the game, and $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$.[5] Thus, for each player, there exists a boundary of ρ below which a cutoff strategy can be used to play the game. Due to the assumption $\varsigma = \varsigma^*$, the boundary for both players are same, i.e. $\tilde{\rho} = \tilde{\rho}^*$, and therefore, this boundary defines the legitimate range of ρ in which a cutoff strategy can be used

[5]For the opponent, $\tilde{\rho}^* = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}}$.

to solve the game. This result is formally given by the following proposition:

Proposition 1 (Restricions for Implementing Cutoff Strategy in Strategic Substitutes Game): Suppose $M > D$ and $\varsigma^* = \varsigma$. The game can be solved by a cutoff strategy if and only if $\rho \in (-1, \tilde{\rho}]$, where $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2 + (M-D)^2}}$.

Proof: See Appendix. ■

$\pi = 3.14...$ is the ratio of a circle's circumference to its diameter. Given $\rho \in (0, \tilde{\rho}]$ and an $x^* \in \mathbb{R}$, if $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε, it indicates that

$$\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*, \varepsilon)(M - D) + 1 \geq 0$$

for all $x^* \in \mathbb{R}$, and hence,

$$1 \geq -\sigma_\varepsilon(x^*, \varepsilon)(M - D)$$

Because $\sigma_\varepsilon(x^*, \varepsilon) = -\rho f(x^*|\varepsilon)$ (see Appendix A), the above inequality can be written as

$$1 \geq \rho f(x^*|\varepsilon)(M - D)$$

and hence

$$f(x^*|\varepsilon) \leq \frac{1}{\rho(M - D)} \tag{2.2}$$

As ς increases, the variance of the distribution $f(.|\varepsilon)$, which equals $\varsigma^2(1-\rho^2)$, increases and so the density function flattens.[6] Particularly, the maximum value of $f(.|\varepsilon)$, which equals $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ and is taken at the mean $x^* = \rho\varepsilon$, decreases. Hence, (2.2) is easier to be satisfied and it is more certain that at the given value of ρ, $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff increase with respect to ε should be broadened as ς increases, and accordingly, $\tilde{\rho}$ increases.

If $M - D$ decreases, the right-hand side (RHS) of (2.2) increases. Hence, (2.2) is easier to be satisfied and it is more certain that at the given value of ρ, $\mathbb{E}\Pi(x^*, \varepsilon)$

[6] The density function $f(.|\varepsilon)$ refers to $f(\varepsilon^*|\varepsilon)$. For the explicit expression, please refer to Appendix A.

increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff of entry increases with respect to ε should be broadened as $M - D$ decreases, and accordingly, $\tilde{\rho}$ increases.

2.3 The Best Response Function

Given the opponent's cutoff strategy $x^* \in \mathbb{R}$, a firm's best response $g(x^*)$ is determined by $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. That is,

$$\sigma(x^*, g(x^*))(M - D) + D + g(x^*) = 0$$

It is found that $g(x^*) \in [-M, -D]$ because as long as $M > D$, the maximum of $\sigma(x^*, \varepsilon)(M - D) + D$ equals M when $\sigma(x^*, \varepsilon) = 1$ and the minimum of $\sigma(x^*, \varepsilon)(M - D) + D$ equals D when $\sigma(x^*, \varepsilon) = 0$. We define $\Phi(.)$ as the cummulative density function of the standard normal distribution and $\phi(.)$ as the probability density function of the standard normal distribution. Given the joint normal distribution, we obtain the best response function in its reverse form:

$$x^* = \rho \frac{\varsigma^*}{\varsigma} g(x^*) + \varsigma^* \sqrt{1 - \rho^2} \Phi^{-1} \left(\frac{D + g(x^*)}{D - M}\right) \tag{2.3}$$

We can then obtain the derivative of $g(x^*)$ with respect to x^*

$$g'(x^*) = -\frac{\sigma_{x^*}(x^*, g(x^*))(M - D)}{\sigma_{\varepsilon}(x^*, g(x^*))(M - D) + 1} \tag{2.4}$$

In this game, the best response functions are divided into two types: contraction and non-contraction.[7] If the best response functions are contraction functions, according to Zimper (2004), it indicates that the game is dominance solvable, and hence, there exists a unique equilibrium. If the best response functions are non-contraction functions, then the game may contain multiple equilibria. Figure 1 exhibits a numerical example of contraction and non-contraction best response functions . For the prop-

[7]For the description and properties of contraction and non-contraction best response functions, please refer to Appendix H.

erties of the best response functions, they are summarized in the following proposition.

Proposition 2 (Properties of Best Response Functions): Given that $\varsigma = \varsigma^*$ and $M > D$, there exists a $\hat{\rho} = \frac{2\pi\varsigma^{*2} - (M-D)^2}{2\pi\varsigma^{*2} + (M-D)^2}$, which differentiates contraction and non-contraction best response function:

1) for $\rho \in (-1, \hat{\rho})$, $-1 < g'(x^*) < 0$ globally;

2) for $\rho \in [\hat{\rho}, \tilde{\rho}]$,

I. if $g(x^*) \in [-M, -(M-D)\Phi(\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $-1 < g'(x^*) < 0$;

II. if $g(x^*) = -(M-D)\Phi(\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = -1$.

III. if $g(x^*) \in (-(M-D)\Phi(\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -(M-D)\Phi(-\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) < -1$;

IV. if $g(x^*) = -(M-D)\Phi(-\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = -1$.

V. if $g(x^*) \in (-(M-D)\Phi(-\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -D]$, $-1 < g'(x^*) < 0$;

Proof: see Appendix. ■

Due to the negative externalities of payoff specification, the game exhibits strategic substitutes; therefore it is not surprising that the best response function in this game is decreasing. For the opponent, $\hat{\rho}^* = \frac{2\pi\varsigma^2 - (M-D)^2}{2\pi\varsigma^2 + (M-D)^2}$. If both players' best response functions are contractions, the game is dominance solvable, and hence, a unique equilibrium exists. We consider a symmetric game, $\hat{\rho} = \hat{\rho}^*$; therefore, a sufficient condition for the game to have a unique equilibrium is that $\rho \in (-1, \hat{\rho}]$. This sufficient condition can be generalized to asymmetric payoff settings, where each player has different M and D. Therefore, the generalized sufficient condition to ensure a unique equilibrium is $\rho \in (-1, \min\{\hat{\rho}, \hat{\rho}^*\}]$. If we further assume that $\hat{\rho}$ and $\hat{\rho}^*$ are positive, then the condition $\rho \in (0, \min\{\hat{\rho}, \hat{\rho}^*\}]$ is identical to Xu's (2014) sufficient condition to ensure a unique equilibrium by using a normal distribution to model the prior.

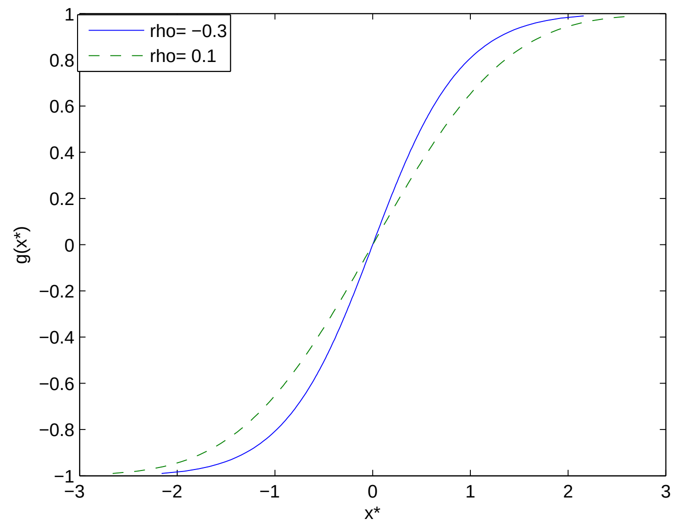


Figure 1: *A numerical example of a contraction best response function and a non-contraction best response function.* $M = 1$, $D = -1$, ς *(and* ς^**)=1 and* $\rho = 0.1$ *(for the solid curve) and 0.3 (for the dashed curve). In this case,* $\hat{\rho} = 0.2220$*.The solid curve represents a contraction best response function, and the dashed curve represents a non-contraction best response function.*

If the best response functions are contraction functions and therefore the game is dominance solvable, then it follows from the implicit function theorem that for all $x^* \in \mathbb{R}$, a player's expected payoff responds more to their own strategy than to the opponent's strategy, i.e.

$$\frac{\partial g(x^*)}{\partial x^*} = -\frac{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}}{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}}|_{\varepsilon=g(x^*)} > -1$$

and hence

$$\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}|_{\varepsilon=g(x^*)} > \frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}|_{\varepsilon=g(x^*)} \tag{2.5}$$

Equation (2.5) can be explicitly written as

$$\sigma_\varepsilon(x^*, g(x^*))(M-D)+1 > \sigma_{x^*}(x^*, g(x^*))(M-D)$$

Furthermore, the above inequality can be written as

$$-\rho f(x^*|g(x^*))(M-D)+1 > f(x^*|g(x^*))(M-D)$$

and we obtain

$$1 > (1+\rho)(M-D)f(x^*|g(x^*))$$

Finally, we find that (2.5) exactly implies the following restriction on the conditional density function:

$$f(x^*|g(x^*)) < \frac{1}{(1+\rho)(M-D)} \tag{2.6}$$

From (2.6), it can be seen that as $\rho \to -1$, the RHS of (2.6) increases to $+\infty$ and hence (2.6) is certainly satisfied. In this situation, it is certain that the best response functions are contraction functions and hence the game is dominance solvable.

Given $\rho \in (-1, \hat{\rho}]$, as ς increases, the variance of $f(.|g(x^*))$, which is equal to $\varsigma^2(1-\rho^2)$, increases, and hence, the density function flattens.[8] Particularly, the maximum value of $f(.|g(x^*))$, which equals $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ and is taken at the mean $\rho g(x^*)$ of the distribution $f(.|g(x^*))$, decreases. Hence, (2.6) is easier to be satisfied and it is more certain that at the given value of ρ, $g'(x^*) > -1$ for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that ensures $g(x^*)$ is a contraction function should be broadened as ς increases, and accordingly $\hat{\rho}$ increases.

If $M-D$ decreases, the RHS of the inequality increases. Hence, (2.6) is easier to be satisfied and it is more certain that at the given value of ρ, $g'(x^*) > -1$ for all $x^* \in \mathbb{R}$. Therefore, if $M-D$ decreases, the range of ρ that ensures $g(x^*)$ is a contraction function should also be broadened, and accordingly $\hat{\rho}$ increases.

2.4 $f(\varepsilon^*|\varepsilon)$ as a Device to Reflect the Opponent's Private Information

In the case of jointly normal variables, if player i gets a draw ε, the mean of the distribution of i^*'s payoff shocks ε^* changes to $\rho\varepsilon$ and the variance is reduced to $\varsigma^2(1-\rho^2)$ (we

[8]The density function $f(.|g(x^*))$ still refers to the function $f(\varepsilon^*|\varepsilon)$, where $\varepsilon = g(x^*)$. For its explicit expression, please refer to Appendix A.

only consider symmetric cases where $\varsigma = \varsigma^*$). The density function $f(\varepsilon^*|\varepsilon)$ remains normal. Thus, $f(\varepsilon^*|\varepsilon)$ shifts (left or right depending on the sign of ρ) and becomes more precise to reflect i^*'s private information ε^*. An extreme situation is that if $\rho = 1$ or -1, $\varsigma^2(1-\rho^2)$ reduces to 0 and $\varepsilon^* = \varepsilon$ or $-\varepsilon$, respectively. Therefore, in the case of $\rho = 1$ or -1, we say that a player can perfectly predict the opponent's private information according to the player's own private information. Obviously, in this context, the density function $f(\varepsilon^*|\varepsilon)$ is a more useful device to predict the opponent's private information ε^* than the belief $\sigma(x^*, \varepsilon)$. The latter measures the probability that the opponent chooses being inactive given ε and a possible entry threshold of the opponent x^*.

For $\rho \in (-1,1)$, the variance $\varsigma^2(1-\rho^2) > 0$. Therefore, given ε, player i cannot predict ε^* as precisely as in the case of $\rho \to \pm 1$. The variance $\varsigma^2(1-\rho^2)$ is composed by ς, which by definition measures the uncertainty of a player's private information, and ρ, which measures the uncertainty between players' private information. In Chapter 3, for a strategic complements game, we obtain a specific result of how ς divides high and low uncertainty of players' private information, which is given by the intuition underlying Corollary 1 in Chapter 3. For specific results of how ρ divides high and low uncertainty between players' private information, please refer the intuition underlying Theorem 1 in Chapter 2 for strategic substitutes games and the intuition underlying Theorem 1 in Chapter 3 for strategic complements games.

Therefore, as ς increases or ρ tends to 0, $\varsigma^2(1-\rho^2)$ will increase. If $\varsigma^2(1-\rho^2)$ is low, i's density function $f(\varepsilon^*|\varepsilon)$ can approximately reflect i^*'s private information ε^* given ε. It might be because the uncertainty of players' private information is low or the uncertainty between players' private information is low. This situation is close to the case where $\rho \to 1$ in the strategic substitutes game or $\rho \to -1$ in the strategic complements game. Therefore, we obtain three equilibria when the uncertainty of or between ε and ε^* is low.

If $\varsigma^2(1-\rho^2)$ is high, $f(\varepsilon^*|\varepsilon)$ is imprecise to reflect the other player's private information. It is either because the uncertainty of players' private information is high, or the uncertainty between players' private information is high. In this situation, players cannot have enough information to assist them to mismatch their action strategies in strategic substitutes games or match their action strategies in strategic complements games in which $D > 0 > M$; hence, they only have an unclear expectation of the opponent's propensity of action choice.[9] Thus, only one equilibrium exists to capture this situation, and it is symmetric.

2.5 Comparative Statics of the Number of Equilibria with respect to the Correlation Coefficient

The equilibria of the game are intersection points of the best response functions. Because the game is symmetrically specified, the two players' best response functions are symmetrically located around the 45° line. Therefore, with the decreasing property of $g(x^*)$, it is reasonable to expect that in the strategic substitutes game, a symmetric equilibrium always exists, which is the intersection point between 45° line and either player's best response function.

We are interested in the stability property of equilibrium. The stability concept adopted in this chapter is Lyapunov stability. An equilibrium is stable (unstable) if and only if it is a stable (unstable) fixed point of the game.

In this symmetric game, an cutoff strategy equilibrium $(x,\, x^*)$ should simultaneously satisfy the following two equations:

$$x = \rho x^* + \varsigma\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+x^*}{D-M})$$

and

$$x^* = \rho x + \varsigma\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+x}{D-M})$$

where $x = g(x^*)$ and $x^* = g^*(x)$. $g(.)$ and $g^*(.)$ are player i's and player i^*'s best response functions, respectively. Therefore, the corresponding Jacobian matrix is

[9] In the strategic complements games, if $D > M > 0$ or $0 > D > M$, players are still able to match their action strategies based on their ex ante expectations of the opponent's behaviour. For details, please refer to Chapter 3.

$$J = \begin{pmatrix} g^{*'}(x) & 0 \\ 0 & g'(x^*) \end{pmatrix}$$

Hence, the eigenvalues of the Jacobian matrix are $g^{*'}(x)$ and $g'(x^*)$, respectively. Thus, if we know the first-order derivative of best response functions at an equilibrium, we can judge the stability of this equilibrium.

According to Zimper (2004), if each player's best response function is a contraction function, i.e. $\rho \in (-1, \hat{\rho}]$ in our context, then the game is dominance solvable. Hence, there exists an equilibrium that is unique, symmetric and stable.

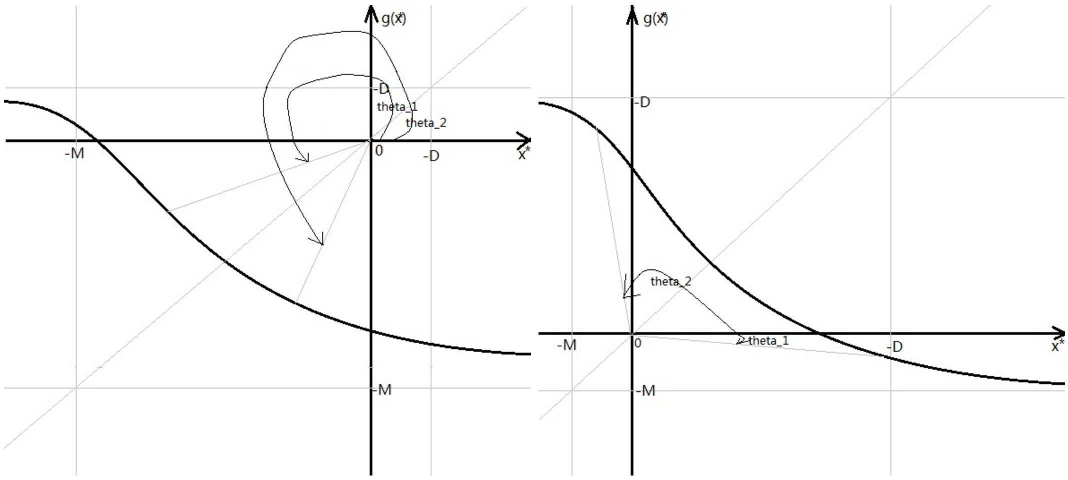
If $g(x^*)$ is no longer a contraction function, it may contain a unique equilibrium for all $\rho \in (\hat{\rho}, \tilde{\rho}]$, or alternatively, for some value of $\rho \in (\hat{\rho}, \tilde{\rho}]$, there exists a unique equilibrium, but for other values of $\rho \in (\hat{\rho}, \tilde{\rho}]$, there are three equilibria. In the following, we derive the comparative statics of the number of equilibria with respect to ρ and the stability of equilibrium. We will provide a complete description of these results in Theorem 1.

Recall the best response function $g(x^*)$ (equation (2.3)). We can express equation (2.3) in polar coordinates. Define $x^* = r\cos\theta$ and $g(x^*) = r\sin\theta$, where $\theta \in [-\frac{\pi}{4}, \frac{7}{4}\pi)$ and $r \geq 0$. Recall that $\varsigma = \varsigma^*$. By substituting $r\cos\theta$ and $r\sin\theta$ in equation (2.3), we find that an equilibrium (θ, r) is a solution of the following equation:

$$\frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma\sqrt{1-\rho^2}} r) = \frac{D}{D-M}$$

We define $p(\theta, r) = \frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma\sqrt{1-\rho^2}} r)$. In this symmetric game, asymmetric equilibria always appear in pairs because we can always find an equilibrium's corresponding equilibrium by switching players' identities. Therefore, a pair of asymmetric equilibria symmetrically locate around the 45° line. The radius of a pair of asymmetric equilibria is denoted by $r = r_a$. In polar coordinates, a pair of asymmetric equilibria have the same radius $r = r_a > 0$. A pair of asymmetric equilibria is denoted by (θ_1, r_a) and (θ_2, r_a), where $\theta_1 < \theta_2$. They satisfy either $\theta_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta_1$ or $\theta_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta_1$.

The symmetric equilibrium is denoted by (s, s). Figure 2 exhibits the two possible



Case I $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$ **Case II** $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$

Figure 2: *The two possible locations of best response functions. The set of radian value is dichotomized according to the sign of the symmetric equilibrium* (s,s)*. In Case I,* $s < 0$ *and hence* $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$*. In Case II,* $s \geq 0$ *and hence* $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$*.*

locations of best response functions. They are classified according to the sign of the symmetric equilibrium, whether $s < 0$ or $s \geq 0$.

The sign of the symmetric equilibrium dichotomizes the set of radian θ into two subsets (cases). In Case I, $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$. In Case II, $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$. Hence, the radians of a pair of asymmetric equilibria accordingly belong to one of the two subsets. Specifically, in Case I, $\frac{\theta_1+\theta_2}{2} = \frac{5}{4}\pi$, and in Case II, $\frac{\theta_1+\theta_2}{2} = \frac{\pi}{4}$.

We define $q(\theta,r) = p(\frac{\pi}{2} - \theta, r) = \frac{r\cos\theta}{M-D} + \Phi(\frac{\sin\theta - \rho\cos\theta}{\varsigma\sqrt{1-\rho^2}} r)$. Irrespective of whether Case I or Case II, an equilibrium (θ, r) is also a solution of the following equation group:

$$\begin{cases} p(\theta,r) = \frac{D}{D-M} \\ q(\theta,r) = \frac{D}{D-M} \end{cases} \tag{2.7}$$

The radius of the symmetric equilibrium is denoted by $r = r_s$. It should be emphasized that 1) given r, the shapes of $p(\theta,r)$ and $q(\theta,r)$ with respect to θ are determined by M, D, ς and ρ; 2) $p(\theta,r)$ and $q(\theta,r)$ are symmetrically located around $\theta = \frac{5}{4}\pi$ or $\theta = \frac{\pi}{4}$; and 3) the symmetric equilibrium is always on the 45° line, i.e. given $r = r_s$, $\theta = \frac{\pi}{4}$ or $\frac{5}{4}\pi$.

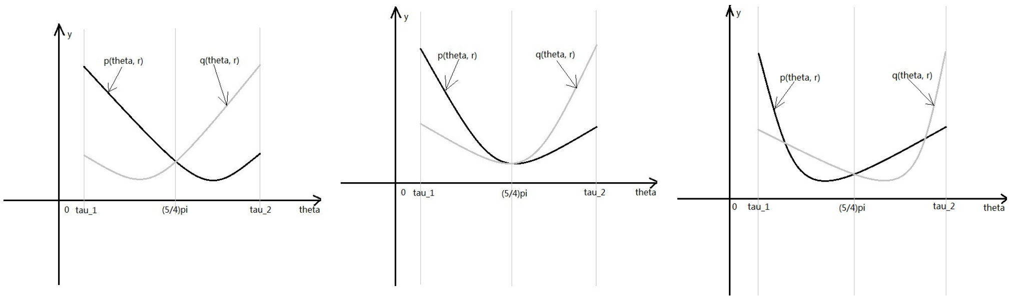
For the relationship between r_s and r_a, it is found that

Lemma 1: In the symmetric strategic substitutes entry game where $M > D$ and $\varsigma = \varsigma^*$, $r_s < r_a$ for all $\theta \in [-\frac{\pi}{4}, \frac{7}{4}\pi)$.

Proof: Consider a pair of asymmetric equilibria (θ_1, r_a) and (θ_2, r_a), where the radians θ_1 and $\theta_2 \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$ and $\theta_1 < \theta_2$. Geometrically, they are the intersection points between $x^{*2} + g(x^*)^2 = r^2$ and $g(x^*)$ because $g(x^*)$ decreases. Therefore, for any function $x^{*2} + g(x^*)^2 = r^2$, where $r > r_a$, its left intersection point with $g(x^*)$ will be on the left side of (θ_1, r_a), and its right intersection point with $g(x^*)$ will be on the right side of (θ_2, r_a). Therefore, these new intersection points are located away from the 45° line, and hence, $r_s \geq r_a$ is impossible. Recall that a symmetric equilibrium is always located on the 45° line. The same analysis applies to the case where the radians of a pair of asymmetric equilibrium belong to $[\frac{3}{4}\pi, \frac{7}{4}\pi)$ and we get the same result. Therefore, in this game, $r_s < r_a$ for all $\theta \in [-\frac{\pi}{4}, \frac{7}{4}\pi)$. *Q.E.D.*

In the following, we explain the comparative statics of the number of equilibria with respect to the correlation coefficient by focusing on Case I, i.e. $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$. For radians $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$, $p(\tau_1, r) > p(\tau_2, r)$ for $r > 0$. There exists a $\tau \in (\pi, \frac{3}{2}\pi)$, $p'_\theta(\theta, r) < 0$ for all $\theta \in (\frac{3}{4}\pi, \tau]$ and $p'_\theta(\theta, r) > 0$ for all $\theta \in (\tau, \frac{7}{4}\pi)$. It could be that $\tau \gtreqless \frac{5}{4}\pi$. According to the comparison relationship between τ and $\frac{5}{4}\pi$, $p(\theta, r)$ and $q(\theta, r)$ have either 1 or 3 intersection points. All possible situations are described in Figure 3. Given $r > 0$, the shape of $p(\theta, r)$ and $q(\theta, r)$ and intersections between $p(\theta, r)$ and $q(\theta, r)$ are determined by M, D and the prior distribution (see Figure 3).

It is found that for all $\theta \in [\pi, \frac{5}{4}\pi)$, $\frac{\partial q(\theta,r)}{\partial r} < \frac{\partial p(\theta,r)}{\partial r} < 0$ and for all $\theta \in (\frac{5}{4}\pi, \frac{3}{2}\pi]$, $\frac{\partial p(\theta,r)}{\partial r} < \frac{\partial q(\theta,r)}{\partial r} < 0$. For all $\theta \in [\frac{3}{4}\pi, \pi)$, $\frac{\partial p(\theta,r)}{\partial r} > \frac{\partial q(\theta,r)}{\partial r}$, $0 > \frac{\partial p(\pi,r)}{\partial r} > \frac{\partial q(\pi,r)}{\partial r}$ and $\frac{\partial p(\frac{3}{4}\pi,r)}{\partial r} > 0 > \frac{\partial q(\frac{3}{4}\pi,r)}{\partial r}$. For all $\theta \in [\frac{3}{2}\pi, \frac{7}{4}\pi)$, $\frac{\partial q(\theta,r)}{\partial r} > \frac{\partial p(\theta,r)}{\partial r}$, $0 > \frac{\partial q(\frac{3}{2}\pi,r)}{\partial r} > \frac{\partial p(\frac{3}{2}\pi,r)}{\partial r}$ and $\frac{\partial q(\frac{7}{4}\pi,r)}{\partial r} > 0 > \frac{\partial p(\frac{7}{4}\pi,r)}{\partial r}$. Besides, for all $\theta \in (\frac{3}{4}\pi, \pi)$, $\frac{\partial^2 p(\theta,r)}{\partial r \partial \theta} < \frac{\partial^2 q(\theta,r)}{\partial r \partial \theta} < 0$, and for all $\theta \in (\frac{3}{2}\pi, \frac{7}{4}\pi)$, $\frac{\partial^2 q(\theta,r)}{\partial r \partial \theta} < \frac{\partial^2 p(\theta,r)}{\partial r \partial \theta} < 0$. Therefore, given $\theta \in [\frac{3}{4}\pi, \frac{5}{4}\pi)$, as r increases, $q(\theta, r)$ always decreases and it decreases relatively faster than $p(\theta, r)$. In addition, given $\theta \in [\frac{5}{4}\pi, \frac{7}{4}\pi)$, as r increases, $p(\theta, r)$ always decreases and it decreases relatively faster than $q(\theta, r)$.



$\tau > \frac{5}{4}\pi$ $\tau = \frac{5}{4}\pi$ $\tau < \frac{5}{4}\pi$

Figure 3: *Given $r > 0$, $p(\tau_1, r) > p(\tau_2, r)$, where $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$. According to the comparison between $\tau \in (\pi, \frac{3}{2}\pi)$ and $\frac{5}{4}\pi$, we obtain three situations describing the relationship between $p(\theta, r)$ and $q(\theta, r)$. The relationship is determined by M, D and the prior distribution. Because a Bayesian Nash equilibrium always exists, we can always find a suitable radius r to locate at least one intersection point on the line $y = \frac{D}{D-M}$. This intersection point is hence an equilibrium according to equation group (2.7).*

Because a symmetric equilibrium always exists, given M, D and the prior, we can always find an $r = r_s$ to make $(\frac{5}{4}\pi, r_s)$ satisfy equation group (2.5). As r increases away from r_s, for cases of $\tau \geq \frac{5}{4}\pi$, $|p(\theta, r) - q(\theta, r)|$ will increase given $\theta \in [\frac{3}{4}\pi, \frac{5}{4}\pi)$ and $\theta \in (\frac{5}{4}\pi, \frac{7}{4}\pi)$. Therefore, for $\tau \geq \frac{5}{4}\pi$, new intersection points will not appear between $p(\theta, r)$ and $q(\theta, r)$ for $r > r_s$. Hence, in these cases, except the symmetric equilibrium, it is impossible to obtain additional equilibrium. Therefore, for cases of $\tau \geq \frac{5}{4}\pi$, there is a unique equilibrium $(\frac{5}{4}\pi, r_s)$.

For the case of $\tau < \frac{5}{4}\pi$, there are three intersection points. As r increases away from r_s, because given $\theta \in [\frac{3}{4}\pi, \frac{5}{4}\pi)$, $q(\theta, r)$ decreases relatively faster than $p(\theta, r)$, the left intersection point decreases vertically and moves towards $\frac{5}{4}\pi$ horizontally. Figure 4 illustrates how the left (and the right) intersection point moves towards the middle intersection point.

In Figure 4, given $r > 0$, at $\theta = c$, $p(c, r) = q(c, r)$. For all $\theta \in (\frac{3}{4}\pi, c)$, $p(\theta, r) > q(\theta, r)$ and for all $\theta \in (c, \frac{5}{4}\pi)$, $q(\theta, r) > p(\theta, r)$. Given θ, as r increases to r', at $\theta = c$, $p(\theta, r') > q(\theta, r')$ because $q(\theta, r)$ decreases relatively faster than $p(\theta, r)$ given θ. For all $\theta \in (\frac{3}{4}\pi, c)$, $p(\theta, r') > q(\theta, r')$. Therefore, for the new left intersection point $(c', q(c', r'))$ where $q(c', r') = p(c', r')$, $c < c' < \frac{5}{4}\pi$. Because for all $\theta \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$, $\frac{\partial q(\theta, r)}{\partial r} < 0$, $q(c', r') < q(c', r)$. Therefore, as r increases, the left intersection point

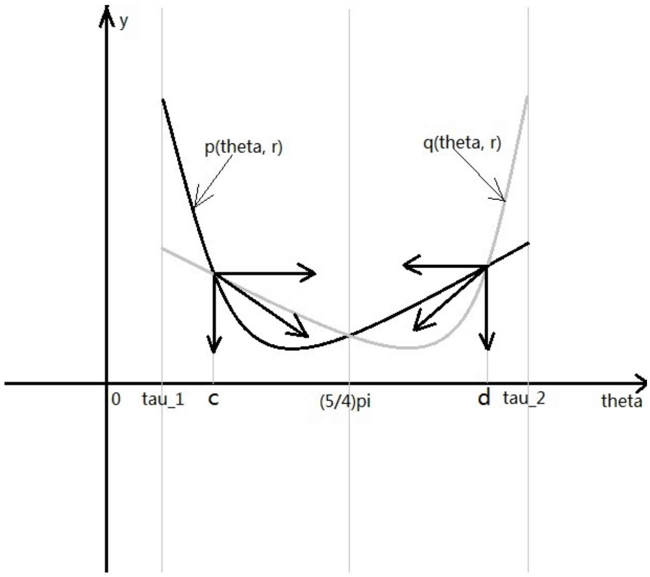


Figure 4: *An illustration of how intersection points (c,r) and (d,r) move towards the middle intersection point as r increases for $\tau < \frac{5}{4}\pi$. $(\tau_1,\tau_2) = (\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$, and $p(\tau_1,r) > p(\tau_2,r)$.*

(c,r) moves towards $\frac{5}{4}\pi$ horizontally and downward vertically. Thus, the aggregate movement as r increases is towards the middle intersection point. Symmetrically, the right intersection point (d,r), where $d \in (\frac{5}{4}\pi, \frac{7}{4}\pi)$, moves towards $\frac{5}{4}\pi$ horizontally and downward vertically. The aggregate movement is therefore still toward the middle intersection point as r increases.

From Lemma 1, it is known that $r_a > r_s$. For the case of $\tau < \frac{5}{4}\pi$, at $r = r_s$, the two outer intersection points are denoted by (θ_1^s, r_s) and (θ_2^s, r_s), where $\theta_1^s < \frac{5}{4}\pi < \theta_2^s$. $p(\theta_1^s, r_s) = p(\theta_2^s, r_s) > \frac{D}{D-M}$. In addition, it is known that as r increases, the middle intersection point moves downward because $\frac{\partial p(\frac{5}{4}\pi,r)}{\partial r} = \frac{\partial q(\frac{5}{4}\pi,r)}{\partial r} < 0$, and the two outer intersection points moves towards the middle intersection point. Thus, as r increases away from r_s, finally the two outer intersection points will be below $y = \frac{D}{D-M}$, i.e. for new intersection points (θ_1', r') and (θ_2', r'), $p(\theta_1', r') = p(\theta_2', r') < \frac{D}{D-M}$, where $r' > r_s$ and $\theta_1' < \frac{5}{4}\pi < \theta_2'$. Therefore, there exists a unique value $r_a > 0$ such that at $r = r_a$, the two outer intersection points locate on $y = \frac{D}{D-M}$, i.e. $p(\theta, r_a) = q(\theta, r_a) = \frac{D}{D-M}$, where $\theta \in \{\theta_1, \theta_2\}$. Hence, in this situation, given M, D and the prior distribution, we get asymmetric equilibria (θ_1, r_a) and (θ_2, r_a), and symmetric equilibrium $(\frac{5}{4}\pi, r_s)$. Therefore, for the case of $\tau < \frac{5}{4}\pi$, there are three equilibria.

According to the above analysis, for $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$, if $\tau \geq \frac{5}{4}\pi$, there is a unique equilibrium, and if $\tau < \frac{5}{4}\pi$, there are three equilibria. Following the same analysis approach, we find that for $\theta \in (-\frac{\pi}{4}, \frac{3}{4}\pi)$, if τ, which now is the global maximum of $p(\theta, r)$, is greater than or equal to $\frac{\pi}{4}$, there is a unique equilibrium. If $\tau < \frac{\pi}{4}$, there are three equilibria (see Appendix). It is found that for $\tau > \frac{5}{4}\pi$ or $\tau > \frac{\pi}{4}$, the symmetric equilibrium is unstable. For $\tau = \frac{5}{4}\pi$ or $\tau = \frac{\pi}{4}$, the stability of the symmetric equilibrium is not determined. For $\tau < \frac{5}{4}\pi$ or $\tau < \frac{\pi}{4}$, the symmetric equilibrium is stable, because the inequalities $\tau \gtreqless \frac{5}{4}\pi$ or $\tau \gtreqless \frac{\pi}{4}$ can be equivalently transformed into the inequalities $g'(s) \lesseqgtr -1$. Therefore, we can differentiate the unique-equilibrium and three-equilibria situations according to the stability of the symmetric equilibrium.

It has been known that for $\rho \in (-1, \hat{\rho}]$, the game is dominance solvable, and hence, the symmetric equilibrium is stable. Therefore, given other parameters, as ρ increases from -1 to $\tilde{\rho}$, the comparative statics of the stability of the symmetric equilibrium performs in the following sequence: stable $\rightarrow$ not determined ($\rightarrow$ unstable). The game can have a unique equilibrium for all $\rho \in (-1, \tilde{\rho}]$ and, hence, in this situation, the symmetric equilibrium cannot be unstable. Therefore, we place a parenthesis at the unstable part of the sequence.

It is found that there exists a unique $\rho = \bar{\rho}$, where the stability of (s,s) is not determined. $\bar{\rho}$ could be greater than $\tilde{\rho}$, or smaller than or equal to $\tilde{\rho}$. If and only if $\bar{\rho} \leq \tilde{\rho}$, the solution (s,s) of the equation group $g(x^*)$ and $g^*(x)$ at $\rho = \bar{\rho}$ can be regarded as an equilibrium. If $\bar{\rho} < \tilde{\rho}$, then for all $\rho \in (\bar{\rho}, \tilde{\rho}]$, the game has three equilibria. Therefore, if the game can exhibit multiple equilibria, i.e. $\bar{\rho} < \tilde{\rho}$, as ρ increases from -1 to $\tilde{\rho}$, the number of equilibria changes from one to three. Otherwise, the game has a unique equilibrium for all $\rho \in (-1, \tilde{\rho}]$.

The analytical expression of $\bar{\rho}$ depends on the sign of $M + D$. Specifically, if $M + D > 0$, then $\bar{\rho}$ is a unique solution of the following equation:

$$\Phi(-\sqrt{\ln\frac{(M-D)^2(1+\bar{\rho})}{2\pi\varsigma^2(1-\bar{\rho})}}) = \frac{D - \sqrt{\frac{\varsigma^2(1+\bar{\rho})}{(1-\bar{\rho})}\ln\frac{(M-D)^2(1+\bar{\rho})}{2\pi\varsigma^2(1-\bar{\rho})}}}{D-M}$$

where $\bar{\rho} > \hat{\rho}$. If $M + D < 0$, then $\bar{\rho}$ is the unique solution of the following equation:

$$\Phi(\sqrt{\ln\frac{(M-D)^2(1+\bar{\rho})}{2\pi\varsigma^2(1-\bar{\rho})}}) = \frac{D+\sqrt{\frac{\varsigma^2(1+\bar{\rho})}{(1-\bar{\rho})}\ln\frac{(M-D)^2(1+\bar{\rho})}{2\pi\varsigma^2(1-\bar{\rho})}}}{D-M}$$

where $\bar{\rho} > \hat{\rho}$. If $M+D=0$, then $\bar{\rho} = \hat{\rho}$.

By summarizing the analysis of both cases of $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi)$ and $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$, we obtain the comparative statics of the number of equilibria with respect to ρ and the stability of equilibrium. It is given by the following theorem.

Theorem 1 (Comparative Statics of the Number of Equilibria with respect to ρ and Stability of Equilibrium in the Strategic Substitutes Game): For a static 2×2 entry game, suppose $M > D$ and $\varsigma = \varsigma^*$. If $\bar{\rho} \geq \tilde{\rho}$, then for all $\rho \in (-1, \tilde{\rho}]$, the game has a unique equilibrium. Conversely, if for all $\rho \in (-1, \tilde{\rho}]$, the game has a unique equilibrium, then $\bar{\rho} \geq \tilde{\rho}$. The equilibrium is stable, except the situation that for $M+D \neq 0$, at $\rho = \bar{\rho} = \tilde{\rho}$, its stability is not determined.

If $\bar{\rho} < \tilde{\rho}$, then for all $\rho \in (-1, \bar{\rho})$, there exists a unique equilibrium. This equilibrium is symmetric and stable. If and only if $M+D=0$, it is $(0,0)$.

At $\rho = \bar{\rho}$, there exists a unique symmetric equilibrium. If $M+D=0$, it is $(0,0)$ and is stable. If $M+D \neq 0$, it is not $(0,0)$ and its stability is not determined.

For all $\rho \in (\bar{\rho}, \tilde{\rho}]$, there exist three equilibria. The symmetric equilibrium is unstable. If and only if $M+D=0$, it is $(0,0)$. The stability of asymmetric equilibrium depends on particular parameter specification.

Proof: see Appendix. ■

The equilibria can be described as solutions of a system of both players' best response functions $g(x^*)$ and $g^*(x)$. As $\rho \to 1$, the limit of $g(x^*)$ is given by [10]

[10] The game at $\rho \to 1$ does not coincide with the game at $\rho = 1$. The game at $\rho \to 1$ is exhibited in the following part in the main context of this chapter. For games at $\rho = 1$, $\varepsilon = \varepsilon^*$ and both players are affected by a common payoff shock which is not known ex ante. However, if it is obtained, the two players play a complete information game. For $\varepsilon \in (-M, -D)$, there are three equilibria: $(1,0)$, $(0,1)$ and a mixed strategy $(-\frac{D+\varepsilon}{M-D}, -\frac{D+\varepsilon}{M-D})$, which is the probability of choosing action 0. Given the cutoff strategy equilibria of games at $\rho \to 1$, for $\varepsilon \in (-M, -D)$, the cutoff strategy equilibrium $(-M, -D)$ implies the action strategy $(1,0)$. Cutoff strategy equilibrium $(-D, -M)$ implies the ac-

$$g(x^*) = \begin{cases} -D & x^* < -\frac{M+D}{2} \\ -\frac{M+D}{2} & x^* = -\frac{M+D}{2} \\ -M & x^* > -\frac{M+D}{2} \end{cases}$$

Because the game is symmetric, $g(x^*)$ and $g^*(x)$ are symmetrically located around the 45° line. The solutions of the equation system of $g(x^*)$ and $g^*(x)$ are the intersection points of the two functions. It is found that there are three solutions at $\rho \to 1$. They are $(-M,-D)$, $(-D,-M)$ and $(-\frac{M+D}{2},-\frac{M+D}{2})$. Figure 5 shows how these results arise.

Now we intuitively analyze the formation of the best response function. The average known payoff of entry is $\frac{M+D}{2}$. Therefore, if a firm wants to choose entry, from an ex ante perspective, at least it should obtain a payoff shock $\varepsilon = -\frac{M+D}{2}$.[11] As $\rho \to 1$, the player is almost sure that the opponent gets the same payoff shock as theirs. Therefore, if player i gets a payoff shock $\varepsilon = -\frac{M+D}{2}$ such that they are indifferent to being active or inactive, then as $\rho \to 1$, player i can almost surely expect the opponent i^* to get the same shock $\varepsilon^* = -\frac{M+D}{2}$. Because i and i^* are identically specified, at $\varepsilon^* = -\frac{M+D}{2}$, i^* is also indifferent to being active or inactive. Therefore, if the opponent's strategy x^* is given by $-\frac{M+D}{2}$, a player's best response will be $-\frac{M+D}{2}$ as well.

If the opponent's strategy $x^* < -\frac{M+D}{2}$, it implies that even if a payoff shock ε^*, where $\varepsilon^* > x^*$, is smaller than $-\frac{M+D}{2}$, which is the average payoff shock that makes i^* indifferent to being active or inactive, i^* will still be expected to choose entry.[12] Therefore, from an ex ante perspective, player i^* becomes more likely to choose entry if $x^* < -\frac{M+D}{2}$. Given such expectation, player i is more likely to get duopoly profit D if the player chooses entry, and hence, from an ex ante perspective, if i's payoff shock

tion strategy $(0,1)$, and the cutoff strategy equilibrium $(-\frac{M+D}{2},-\frac{M+D}{2})$ implies the action strategy $(\int_{-M}^{-\frac{M+D}{2}} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon, \int_{-M}^{-\frac{M+D}{2}} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon)$, which is the unconditional choice probability of choosing action 0 and is independent from the ex post realization of ε. This choice probability is not equal to the mixed strategy equilibrium of the game at $\rho = 1$. Therefore, the incomplete information game at $\rho \to 1$ does not coincide with the game at $\rho = 1$. $g(x^*)$ is continuous with respect to $\rho \in (-1,1)$. Therefore, in this section, the natural benchmark to compare the number of equilibria for games with $\rho \in (-1,\tilde{\rho}]$ is the game at $\rho \to 1$. $\phi(.)$ is the density function of the standard normal distribution.

[11] Ex ante in this chapter means before the payoff shocks are drawn.

[12] We call $-\frac{M+D}{2}$ the average payoff shock that makes i^* indifferent to their action choices because it corresponds to the average known payoff of entry $\frac{M+D}{2}$

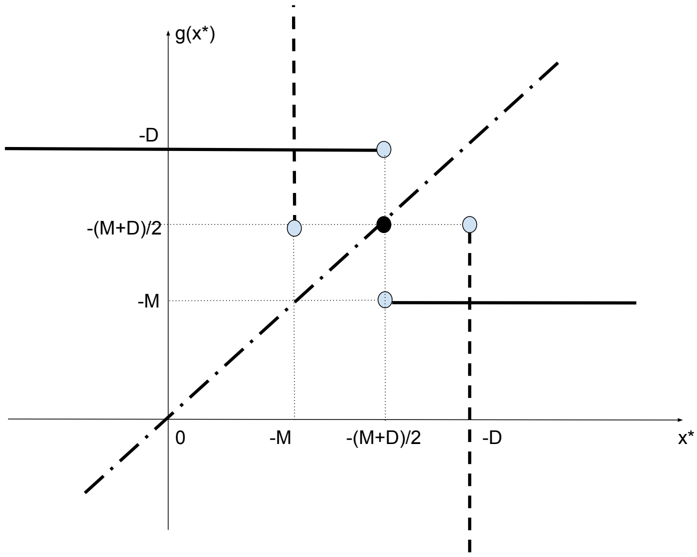


Figure 5: *The solid curve (the two horizontal lines and the point in the middle) represents $g(x^*)$ at $\rho \to 1$ and the dashed curve (the two vertical dashed lines and the point in the middle. The point coincides with the one for $g(x^*)$ at $\rho \to 1$) represents $g^*(x)$ at $\rho \to 1$. The dashed-dot line represent the 45° line. Because the game is symmetric, $g(x^*)$ and $g^*(x)$ are symmetrically located around the 45° line. By drawing a graph of $g(x^*)$ and $g^*(x)$, there are always three intersection points, which are solutions of the equation system of $g(x^*)$ and $g^*(x)$.*

$\varepsilon \geq -D$, player i will choose entry. Therefore, if i^*'s strategy $x^* < -\frac{M+D}{2}$, then i's best response entry threshold $g(x^*) = -D$.

If the opponent's strategy $x^* > -\frac{M+D}{2}$, it implies that even if a payoff shock ε^*, where $\varepsilon^* < x^*$, is greater than $-\frac{M+D}{2}$, which is the average payoff shock that makes i^* indifferent to being active or inactive, i^* will still be expected to choose being inactive. Therefore, from an ex ante perspective, i^* becomes more likely to choose being inactive if $x^* > -\frac{M+D}{2}$. Given such expectation, i is more likely to get monopoly profit M if the player chooses entry, and hence, from an ex ante perspective, if i's payoff shock $\varepsilon \geq -M$, player i will choose entry. Therefore, if i^*'s strategy $x^* > -\frac{M+D}{2}$, then i's best response entry threshold $g(x^*) = -M$.

The intuition of the cutoff strategy equilibrium for the game at $\rho \to 1$ is as follows. If a player expects the opponent to choose entry, then the player will get payoff D if they also choose entry. Thus, the player will adopt a cutoff strategy $-D$ at which they are indifferent to being active or inactive. In this game, $M > D$, so $-M < -D$. Therefore, as the best response, the opponent will adopt a strategy $-M$ at which the opponent is indifferent to the two action choices.

Otherwise, if a player expects the opponent to choose being inactive, the player will get payoff M if they choose to enter. Thus, the player will adopt a cutoff strategy $-M$ at which they are indifferent to being active or inactive. In this game, $-M < -D$, and therefore, as the best response, the opponent will adopt a strategy $-D$ at which the opponent is indifferent to the two action choices. Therefore, we have the equilibria $(-M, -D)$ and $(-D, -M)$.

If a player expects that the opponent is ex ante indifferent to the two action choices, then i thinks that i^* must adopt an entry threshold $-\frac{M+D}{2}$ such that the total average payoff of entry $(\frac{M+D}{2} + \varepsilon^*)$ equals 0. As the best response, player i adopts a cutoff strategy $-\frac{M+D}{2}$. Symmetrically, the opponent will think in the same way and adopt the same strategy. Therefore, we have the equilibrium $(-\frac{M+D}{2}, -\frac{M+D}{2})$.

For games with $\rho \leq \tilde{\rho}$, if given other parameters, at $\rho = \tilde{\rho}$, the uncertainty between each other's private information is low such that $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, then like the situation of $\rho \to 1$, where player i can perfectly predict ε^* given ε, asymmetric equilibria arise. The reason is as follows. In a strategic substitutes game, players always tend to mismatch their action strategies. At $\rho = \tilde{\rho} > 0$, if $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, then players can obtain enough information to assist them to mismatch their action strategies and similar to the $\rho \to 1$ situation, players can show explicit preferences to each action. However, due to the uncertainty between players' private information, the preference to each action is not deterministic. Therefore, in this situation, we can obtain two equilibria. In each equilibrium, one player is more probable to choose entry, and the other player is more probable to choose being inactive. If we translate the representation of these action strategy equilibria into the representation of the cutoff strategy equilibrium, the translated cutoff strategy equilibria should be close to $(-M, -D)$ and $(-D, -M)$, respectively.

In the case of $\rho \to 1$, where players can perfectly predict the other player's private information, it is possible that players are ex ante expected to be indifferent to being active or inactive and this situation is captured by the symmetric equilibrium. At $\rho = \tilde{\rho}$, the uncertainty between players' private information can make a player more uncertain about the other player's propensity of action choice, hence making the indifference situation more reasonable to exist. Symmetrically, the opponent will think and behave in the same way. Thus, this situation is again captured by a symmetric equilibrium. Therefore, at $\rho = \tilde{\rho}$, if $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, the game has three equilibria.

However, if other parameters are given, at $\rho = \tilde{\rho}$, the uncertainty between each other's private information is high and $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε, it can be expected that only a symmetric equilibrium exists. Because $f(\varepsilon^*|\varepsilon)$ is imprecise to reflect ε^* given ε and hence each player does not have enough information to assist them to mismatch each other's action strategies; therefore, an asymmetric equilibrium cannot exist. Still, because the uncertainty between players' private information is high such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε, each player has an unclear expectation of the opponent's propensity of action choice.[13] Conditional on this unclear expectation, a player accordingly chooses a strategy as the best response. Symmetrically, the opponent will think in the same way and adopt the same strategy. Therefore, only the symmetric equilibrium can exist in this situation. The intuition of the games with $M > D$ at $\rho = \tilde{\rho}$ as discussed above applies to games for all $\rho \in (\bar{\rho}, \tilde{\rho}]$ with $M > D$.

Similarly, irrespective of whether there are 3 equilibria or a unique equilibrium at $\rho \in (\bar{\rho}, \tilde{\rho}]$, for ρ taking values that are far away from 1 and -1, or specifically for all $\rho \in (\hat{\rho}, \bar{\rho}]$, the correlation between ε and ε^* is much lower and hence the uncertainty between players' private information reasonably becomes much higher. Therefore, it can be expected that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε. In this situation, following the same intuition in the last paragraph, only a symmetric equilibrium can exist. In addition, as we have discussed in Section 2.3, for $\rho \in (-1, \hat{\rho}]$, it is certain that the best response functions are contraction functions and hence the game exhibits a unique equilibrium. In this situation, as indicated by inequality (2.5), player i's expected payoff of entry is more sensitive to his own strategy than to his opponent's strategy. It means player i is more self-focused and the opponent's private information is less important in player i's decision making, no matter whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect i^*'s private information given ε. Therefore, in this situation, the

[13] An unclear expectation of the opponent's propensity of action choice implies that before the payoff shocks are drawn, player i is not sure whether i^* is more likely to choose being active or being inactive, or more likely to be indifferent to the two action choices.

game is close to an individual decision problem, and hence there exists a unique equilibrium.

Therefore, all of above discussions explain why for all $\rho \in (-1, \tilde{\rho}]$, the game can contain a unique equilibrium, or if there are multiple equilibria for some $\rho \in (-1, \tilde{\rho}]$, as ρ decreases from $\tilde{\rho}$ to -1, the number of equilibria will change from 3 to 1. All of these results are due to the uncertainty between players' private information. Hence, $\bar{\rho}$ is a threshold such that for $\rho > \bar{\rho}$, the uncertainty between players' private information is low, and hence $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε and both players can obtain enough information to mismatch their action strategies. For $\hat{\rho} < \rho \leq \bar{\rho}$, the uncertainty between each other's private information is high such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε, and both players have an unclear expectation of the opponent's propensity of action choice.

Finally, we provide two numerical examples to conclude this section. They exhibit how the number of equilibrium changes with respect to different parameter specifications (see Figures 6 and 7).

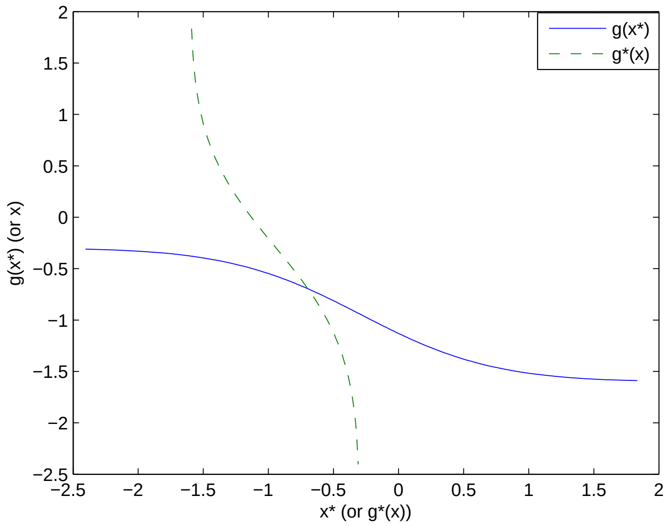


Figure 6: *An example of unique equilibrium. The solid curve represents a player's best response function, and the dashed curve represents the opponent's best response function. In this case,* $M = 1.6$, $D = 0.3$, $\varsigma = \varsigma^* = 1$ *and* $\rho = 0.3$.

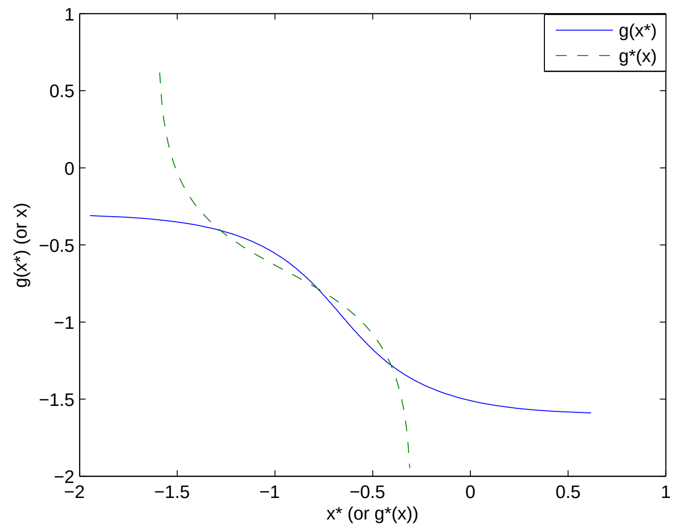


Figure 7: *An example of multiple equilibria. The solid curve represents a player's best response function, and the dashed curve represents the opponent's best response function. In this case,* $M = 1.6$*,* $D = 0.3$*,* $\varsigma = \varsigma^* = 1$ *and* $\rho = 0.7$*.*

2.6 Comparative Statics of Symmetric Equilibrium Strategies

In this game, a symmetric equilibrium always exists and is unique. Hence, for technical convenience, it is natural to adopt the symmetric equilibrium for comparative statics analysis. Assume that players only play the symmetric equilibrium no matter how parameters change. The comparative statics of exogenous parameters on the symmetric equilibrium strategies is given by the following proposition.

Proposition 3: Assume $M > D$ and $\varsigma = \varsigma^*$. Denote a symmetric equilibrium of the game by (s,s) where $-M < s < -D$. It is found that $\frac{\partial s}{\partial M} < 0$ and $\frac{\partial s}{\partial D} < 0$. If $s \leq$ (or $>$) 0, $\frac{\partial s}{\partial \rho} \leq$ (or $>$) 0 and $\frac{\partial s}{\partial \varsigma^2} + \frac{\partial s}{\partial \varsigma^{*2}} \leq$ (or $>$) 0, where the equalities are taken when $s = 0$.

Proof : see Appendix. ■

We are interested in the intuition underlying the results in Proposition 3. We begin from analyzing how a player's best response changes as exogenous parameters change given the opponent's strategy. First, to emphasize the dependence on some parameter ζ, we write $h(x^*, x; \zeta) := \mathbb{E}\Pi(x^*, x)$, where x is the best response towards x^* and

$h(x^*, x; \zeta) = 0$. Therefore, according to implicit function theorem, we obtain,

$$x'(\zeta; x^*) = -\frac{h'_\zeta(x^*, x; \zeta)}{h'_x(x^*, x; \zeta)}$$

where $x'(\zeta; x^*) := \frac{\partial x(x^*)}{\partial \zeta}$. For the jointly changing ς and ς^* case, $x'(\varsigma, \varsigma^*; x^*) = \frac{\partial x(x^*)}{\partial \varsigma} + \frac{\partial x(x^*)}{\partial \varsigma^*}$, and correspondingly, $h'_{\zeta=\{\varsigma,\varsigma^*\}}(x^*, x; \varsigma, \varsigma^*) = h'_\varsigma(x^*, x; \varsigma, \varsigma^*) + h'_{\varsigma^*}(x^*, x; \varsigma, \varsigma^*)$. For $\rho \leq \tilde{\rho}$, $\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \sigma'_\varepsilon(x^*, \varepsilon)(M - D) + D + 1 > 0$, hence $h'_x(x^*, x; \theta) > 0$. Therefore, the sign of $x'(\zeta; x^*)$ is opposite to the sign of $h'_\zeta(x^*, x; \zeta)$.

For parameters M and D, $h'_M(x^*, x; M) = \sigma(x^*, x) > 0$ and $h'_D(x^*, x; D) = 1 - \sigma(x^*, x) > 0$. Thus, $x'(M; x^*) < 0$ and $x'(D; x^*) < 0$. Hence, given x^*, an increase in the profit of entry will make a firm deviate to a lower threshold. Therefore, if both firms are playing symmetric equilibrium (s, s), they will ultimately deviate to a lower equilibrium strategy $s' < s$ after an increase in the profit of entry. Hence, by increasing M or D, players become more likely to choose entry in the new equilibrium, i.e. $\int_{s'}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma}) d\varepsilon > \int_{s}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma}) d\varepsilon$, where $\frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})$ is the density function of the type distribution $\varepsilon \sim N(0, \varsigma^2)$.

We write $\sigma(x^*, x; \rho) = \Phi(\frac{\frac{x^*}{\varsigma^*} - \rho\frac{x}{\varsigma}}{\sqrt{1-\rho^2}})$. Because $\varsigma = \varsigma^*$,

$$\sigma'_\rho(x^*, x; \rho) = -\frac{\rho(\frac{x}{\rho} - x^*)}{\varsigma(1-\rho^2)^{\frac{3}{2}}}\phi(\frac{x^* - \rho x}{\varsigma\sqrt{1-\rho^2}})$$

and $h'_\rho(x^*, x; \rho) = \sigma'_\rho(x^*, x; \rho)(M - D)$. A marginal increase of ρ is to lower the mean ρx if $x < 0$ and raise it if $x > 0$, and thus increasing the variance if $\rho < 0$ and decreasing the variance if $\rho > 0$.

Suppose $x < 0$. If $x^* > \frac{x}{\rho}$ for $\rho > 0$, or $x^* < \frac{x}{\rho}$ for $\rho < 0$, the dominating effect of increasing ρ is the effect from the conditional mean ρx of the opponent's type distribution (See Figures 8 and 9). Intuitively, as the conditional mean decreases, it means that, from one firm's perspective, the opponent's payoff shock on average will become lower. Hence, it is more likely that the opponent chooses inactivity. Therefore, the belief $\sigma(x^*, x; \rho)$ increases, and correspondingly $x'(\rho; x^*) < 0$.

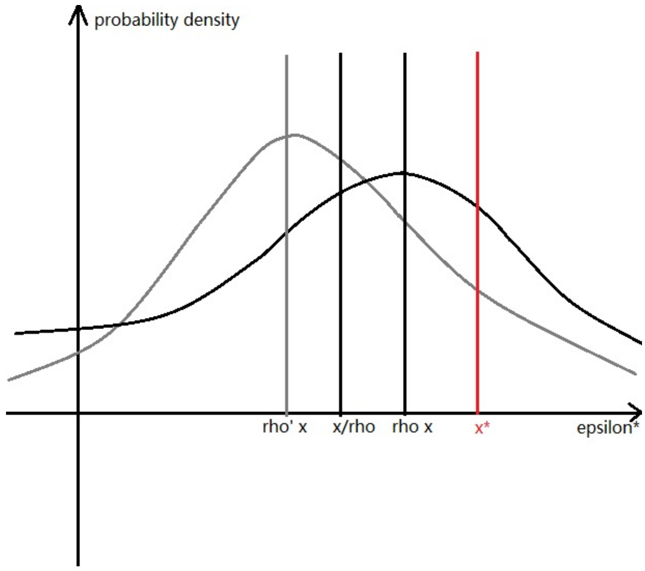


Figure 8: *Suppose $x < 0$ and $\rho > 0$. If $x^* > \frac{x}{\rho}$, the dominant effect by increasing ρ on the belief $\sigma(x^*, x)$ is from the mean ρx. Supposing ρ is increased to ρ', the mean of the conditional distribution ρx decreases and the variance $\varsigma^2(1-\rho^2)$ decreases as well. Without loss of generality, this figure shows that as long as x^* is fixed, the new belief of choosing inactivity by the opponent (the area under the grey curve and on the left-hand side (LHS) of $\varepsilon^* = x^*$) must be higher than the original belief of choosing inactivity by the opponent (the area under the black curve and on the LHS of $\varepsilon^* = x^*$). Both curves represent the conditional density functions of the opponent's type.*

For symmetric equilibrium (s, s), where $s < 0$, it must satisfy $s > \frac{s}{\rho}$ for $\rho > 0$ or $s < \frac{s}{\rho}$ for $\rho < 0$. Hence, given the opponent's strategy, an increase in ρ will make a firm deviate to a lower strategy. Consequently, it is reasonable to expect that the new symmetric equilibrium threshold s' must be lower than s. Therefore, in the new equilibrium, players will become more likely to choose entry, i.e. $\int_{s'}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon > \int_{s}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon$.

Suppose $x > 0$. If $x^* < \frac{x}{\rho}$ for $\rho > 0$ or $x^* > \frac{x}{\rho}$ for $\rho < 0$, the dominating effect of increasing ρ is still from the conditional mean ρx of the opponent's type distribution (see Figures 10 and 11). Intuitively, since the conditional mean increases, from one firm's perspective, the opponent's average payoff shock will increase, and hence, it is more likely that the opponent chooses entry. Therefore, the belief $\sigma(x^*, x; \rho)$ decreases, and consequently $x'(\rho; x^*) > 0$.

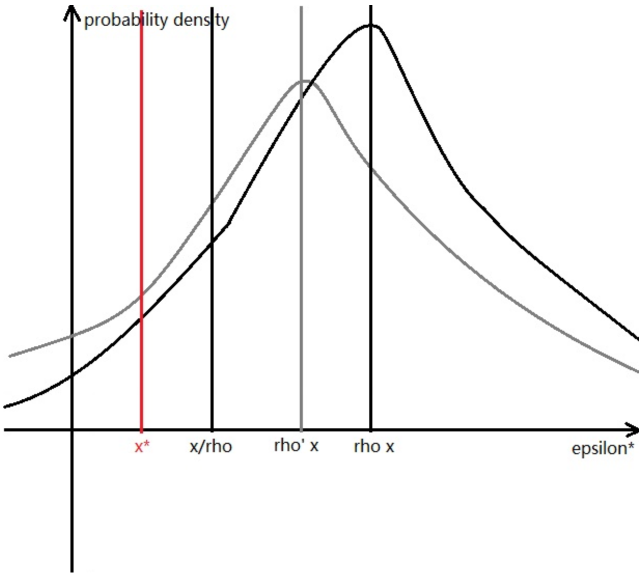


Figure 9: *Suppose $x < 0$ and $\rho < 0$. If $x^* < \frac{x}{\rho}$, the dominant effect by increasing ρ on the belief $\sigma(x^*, x)$ is from the mean ρx. Supposing ρ is increased to ρ', the mean of the conditional distribution of the opponent's type ρx decreases and the variance $\varsigma^2(1-\rho^2)$ increases. Without loss of generality, this figure shows that as long as x^* is fixed, the new belief of choosing inactivity by the opponent (the area under the grey curve and on the LHS of $\varepsilon^* = x^*$) must be higher than the original belief of choosing inactivity by the opponent (the area under the black curve and on the LHS of $\varepsilon^* = x^*$). Both curves represent the conditional density function of the opponent's type.*

For symmetric equilibrium (s,s), where $s > 0$, it must satisfy $s < \frac{s}{\rho}$ for $\rho > 0$ or $s > \frac{s}{\rho}$ for $\rho < 0$. Hence, given the opponent's strategy, an increase in ρ will make the opponent deviate to a higher strategy. Consequently, the new equilibrium threshold s' must be higher than s. Therefore, players will become more likely to choose inactivity in the new equilibrium, i.e. $\int_{-\infty}^{s'} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon > \int_{-\infty}^{s} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon$.

Finally, for symmetric equilibrium (s,s), if $s = 0$, then the conditional mean ρs equals 0. Irrespective of how ρ changes, the belief always keeps $\frac{1}{2}$, and therefore, in this case, the symmetric equilibrium is always $(0,0)$.

Hence, in conclusion, if a firm's original equilibrium strategy $s < 0$, then by increasing ρ, the new equilibrium strategy $s' < s < 0$, while if a firm's original equilibrium strategy $s > 0$, then by increasing ρ, the new equilibrium strategy $s' > s > 0$.

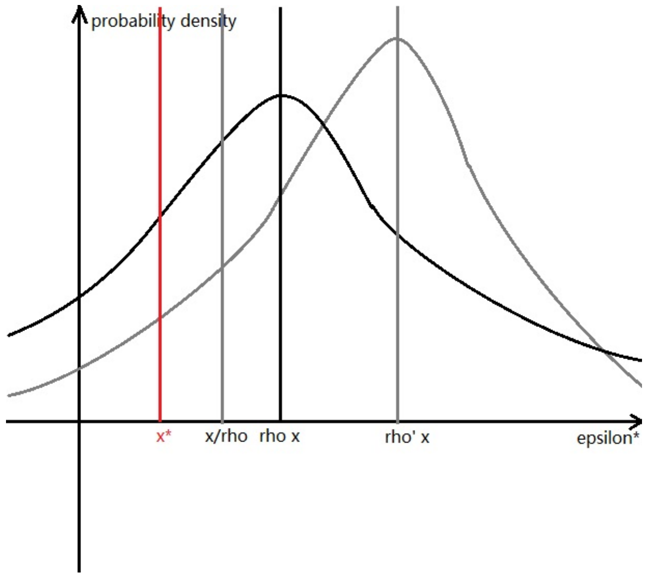


Figure 10: *Suppose $x > 0$ and $\rho > 0$. If $x^* < \frac{x}{\rho}$, the dominant effect by increasing ρ on the belief $\sigma(x^*, x)$ is from the mean ρx. Supposing ρ is increased to ρ', the mean of the conditional distribution of opponent's type ρx increases and the variance $\varsigma^2(1-\rho^2)$ decreases. Without loss of generality, this figure shows that as long as x^* is fixed, the new belief of choosing entry by the opponent (the area under the grey curve and on the RHS of $\varepsilon^* = x^*$) must be higher than the original belief of choosing entry by the opponent (the area under the black curve and on the RHS of $\varepsilon^* = x^*$). Both curves represent the conditional density functions of the opponent's type.*

We now turn to the effect of jointly changing ς and ς^*. Because $\varsigma = \varsigma^*$,

$$\sigma'_{\varsigma}(x^*, x; \varsigma, \varsigma^*) + \sigma'_{\varsigma^*}(x^*, x; \varsigma, \varsigma^*) = -\phi\left(\frac{x^* - \rho x}{\varsigma\sqrt{1-\rho^2}}\right)\frac{x^* - \rho x}{\varsigma^2\sqrt{1-\rho^2}}$$

and $h'_{\varsigma}(x^*, x; \varsigma, \varsigma^*) + h'_{\varsigma^*}(x^*, x; \varsigma, \varsigma^*) = [\sigma'_{\varsigma}(x^*, x; \varsigma, \varsigma^*) + \sigma'_{\varsigma^*}(x^*, x; \varsigma, \varsigma^*)](M - D)$. Apparently, jointly changing ς and ς^* only affects the variance $\varsigma^2(1-\rho^2)$ of the conditional density function of the opponent's type. A marginal joint increase of ς and ς^* is to increase the variance $\varsigma^2(1-\rho^2)$, or vice versa, irrespective of whether $\rho \gtreqless 0$ or $x \gtreqless 0$ (see Figure 12). Intuitively, suppose $x^* < \rho x$. Since the direct consequence of jointly increasing variances is to make the opponent's conditional type distribution assign higher likelihood on low and high payoff shocks, given a low opponent's strategy $x^* < \rho x$, the belief that the opponent chooses inactivity increases, and hence $x'(\varsigma^*, \varsigma; x^*) < 0$. However, if $x^* > \rho x$, since high payoff shocks have been assigned

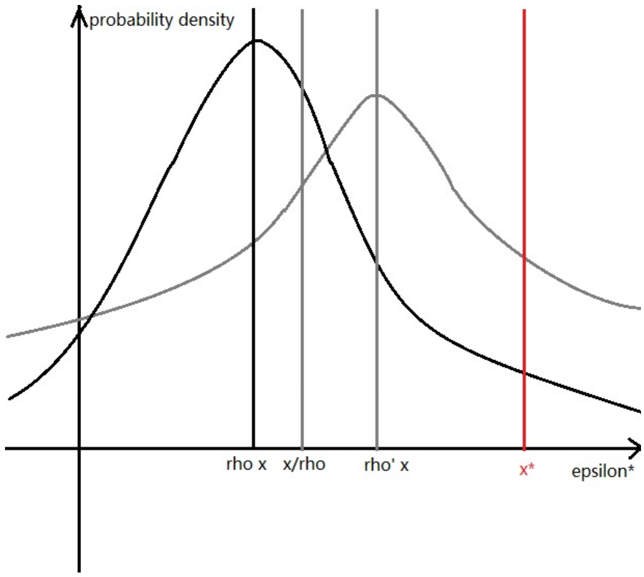


Figure 11: *Suppose $x > 0$ and $\rho < 0$. If $x^* > \frac{x}{\rho}$, the dominant effect by increasing ρ on the belief $\sigma(x^*, x)$ is from the mean ρx. Supposing ρ is increased to ρ', the mean of the conditional distribution of the opponent's type ρx increases and the variance $\varsigma^2(1-\rho^2)$ increases as well. Without loss of generality, this figure shows that as long as x^* is fixed, the new belief of choosing entry by the opponent (the area under the grey curve and on the RHS of $\varepsilon^* = x^*$) must be higher than the original belief of choosing entry by the opponent (the area under the black curve and on the RHS of $\varepsilon^* = x^*$). Both curves represent the conditional density functions of the opponent's type.*

with higher likelihood as well after ς and ς^* are jointly increased, the belief that the opponent chooses entry must increase, and thus $x'(\varsigma^*, \varsigma; x^*) > 0$.

For symmetric equilibrium (s,s), if $s < 0$, then irrespective of whether $\rho \gtrless 0$, we always have $s < \rho s$. Therefore, given the opponent's strategy, a joint increase in ς and ς^* will make a firm deviate to a lower strategy. Consequently, the new equilibrium threshold s' must be lower than s. Therefore, players will become more likely to choose entry in the new equilibrium, i.e. $\int_{s'}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon > \int_{s}^{+\infty} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon$.

While for symmetric equilibrium (s,s), if $s > 0$, then irrespective of whether $\rho \gtrless 0$, we always have $s > \rho s$. Therefore, given the opponent's strategy, a joint increase in ς and ς^* will make a firm deviate to a higher strategy. Consequently, the new symmetric equilibrium threshold s' must be higher than s. Therefore, players will become more likely to choose inactivity in the equilibrium, i.e. $\int_{-\infty}^{s'} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon > \int_{-\infty}^{s} \frac{1}{\varsigma}\phi(\frac{\varepsilon}{\varsigma})d\varepsilon$.

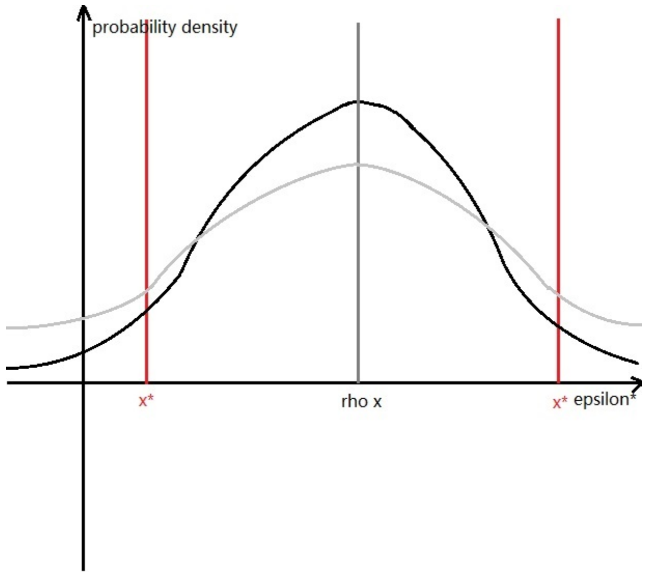


Figure 12: *Given that $\varsigma = \varsigma^*$. If the standard deviations ς and ς^* are jointly increased, the variance of the conditional distribution of the opponent's type will increase. The grey curve represents the new conditional density function after ς and ς^* are jointly increased, while the black curve represents the original conditional density function. Therefore, by jointly increasing ς and ς^*, if $x^* < \rho x$, the belief that the opponent chooses inactivity increases, while if $x^* > \rho x$, the belief that the opponent chooses entry increases.*

Finally, for symmetric equilibrium $(s,s) = (0,0)$, no matter how ς and ς^* change, the belief always maintains $\frac{1}{2}$, and hence, the equilibrium strategy does not change.

2.7 Summary

In this section, we present an organized summary of all main results and intuitions of the game. The game can be summarized by three parameters: $\tilde{\rho}$, $\hat{\rho}$ and $\bar{\rho}$. The relationship between these are as follows: $\hat{\rho} \leq \bar{\rho}$, $\hat{\rho} < \tilde{\rho}$, and $\bar{\rho}$ could be smaller than, equal to or greater than $\tilde{\rho}$.

In Section 2.2, we derive $\tilde{\rho}$. Suppose the game is symmetric. Then, if and only if $\rho \leq \tilde{\rho}$, the game can be solved by cutoff strategies. The intuition is that if $\rho > \tilde{\rho}$, the expected payoff $\mathbb{E}\Pi(x^*, \varepsilon)$ is no longer monotonic with respect to ε, and for some $x^* \in \mathbb{R}$,

$\mathbb{E}\Pi(x^*,\varepsilon) = 0$ has three solutions for ε, and at one of the solutions, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} < 0$; this contradicts the definition of the cutoff strategy used to solve the game. In fact, by assuming $\varsigma = \varsigma^*$, this result can be extended to asymmetric payoff settings, where each player has different M and D. In this situation, the game can be solved by cutoff strategies if and only if $\rho \leq \min\{\tilde{\rho}, \tilde{\rho}^*\}$, where $\tilde{\rho}^* = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M'-D')^2}}$ and M' and D' are player i^*'s known payoffs.

In Section 2.3, we derive $\hat{\rho}$. For $\rho \leq \hat{\rho}$, player i's best response function is a contraction function. In this symmetric game, $\rho \leq \hat{\rho}$ is also the sufficient condition to ensure that the game is dominance solvable. This condition can be generalized to asymmetric payoff settings as described above. In this situation, the sufficient condition is generalized to $\rho \leq \min\{\hat{\rho}, \hat{\rho}^*\}$, where $\hat{\rho}^* = \frac{2\pi\varsigma^2-(M'-D')^2}{2\pi\varsigma^2+(M'-D')^2}$.

For $\bar{\rho}$, if $\bar{\rho} < \tilde{\rho}$, then $\bar{\rho}$ is the threshold to differentiate low and high uncertainty between players' private information. For $\rho > \bar{\rho}$, the uncertainty between players' private information is low, which means the density function $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, and hence, the players can gather enough information to assist them to mismatch their action strategies. For $\hat{\rho} < \rho \leq \bar{\rho}$, the uncertainty between players' private information is high such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε, and hence, both players can only have an unclear expectation of the opponent's propensity of action choice. If $\bar{\rho} \geq \tilde{\rho}$, then for all $\rho \in (\hat{\rho}, \tilde{\rho}]$, the uncertainty between players' private information is high, and therefore, there exists only a symmetric equilibrium, reflecting the fact that they only have an unclear expectation of the opponent's propensity of action choice. For $\rho \in (-1, \hat{\rho}]$, the best response functions are contraction function. In this situation, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem, and hence the game exhibits a unique equilibrium.

For the intuition of the comparative statics of symmetric equilibrium strategies, if we increase M or D, the expected payoff of entry increases and it is not surprising that in the new symmetric equilibrium, both players will adopt a lower entry threshold.

If we increase ρ, then the dominant effect on players' strategies come from the mean of the distribution of opponent's payoff shock given a player's own private payoff shock. The mean is given by ρs, where s is the entry threshold in the symmetric equilibrium. If $s < 0$, increasing ρ will decrease ρs, and hence, a player can expect that the opponent will get a lower payoff shock on average, and hence, the player will adopt a lower cutoff strategy as the best response. This best response adjustment process continues and, consequently, in the new symmetric equilibrium (s', s'), $s' < s$. In contrast, if $s > 0$, then increasing ρ will increase ρs, and hence, a player can expect that the opponent will get a higher payoff shock on average, which discourages the player from choosing entry. As this best response dynamics continues, in the consequent new equilibrium (s', s'), $s' > s$.

If we jointly change ς and ς^*, the impact on players' strategies come from the variance of the distribution of the opponent's payoff shock given a player's own private payoff shock. If we jointly increase ς and ς^*, the tails of the opponent's payoff shocks distribution given a player's own payoff shock will increase; hence, the likelihood of very good payoff shocks or very bad payoff shocks of the opponent increases accordingly. Therefore, if $s < 0$, then the belief will be mainly influenced by the increasing likelihood of very bad payoff shocks of the opponent. In this situation, a player can expect that the opponent becomes less likely to choose entry, which encourages the player to adopt a lower entry threshold. Therefore, as this best response dynamics continues, in the consequent new symmetric equilibrium (s', s'), $s' < s$.

In contrast, if $s > 0$, the belief will be mainly influenced by the increasing likelihood of very good payoff shocks of the opponent. In this situation, a player can expect that the opponent to become more likely to choose entry, which motivates the player to adopt a higher entry threshold. Therefore, as this best response dynamics continues and, consequently, in the new symmetric equilibrium (s', s'), $s' > s$.

2.8 Conclusion

In this chapter, we study how private information correlation affects rational agents' strategic behaviour by investigating a static 2-player entry game based on Pesendorfer and Schmidt-Dengler's (2008) dynamic entry game in their numerical analysis. This game is symmetric, in which players are identically specified. The private information is modelled by a joint normal distribution and the correlation coefficient is a natural measure of the degree of information correlation. This chapter shows that, after introducing information correlation, there exists a restriction on the value of correlation coefficient, allowing the use of a cutoff strategy to solve the game. Information correlation can be used to select a unique equilibrium. In this strategic substitutes game, for certain parameter specification, if the correlation coefficient is less than or equal to a threshold value, a unique equilibrium (symmetric equilibrium) exists, while if the correlation coefficient is above the threshold value, three equilibria will arise: one symmetric equilibrium and two asymmetric equilibria. Alternatively, for the other parameter specifications, the game exhibits a unique equilibrium for any feasible value of the correlation coefficient. To understand how parameter changes affect players' equilibrium behaviour, a comparative statics analysis on the symmetric equilibrium is conducted. It is found that increasing the monopoly profit or the duopoly profit encourages players to enter the market, while increasing the information correlation or jointly increasing the variances of players' type distribution will make players more likely to choose entry if the current equilibrium strategies are negative, and less likely to choose entry if the current equilibrium strategies are positive.

This chapter is also a technical preparation to analytically solve a 2-player dynamic entry game with information correlation. The game studied in this chapter can be viewed as its static version by specifying a discount factor equal to zero. In the static game, we have proven that there are at most three equilibria. In Pesendorfer and Schmidt-Dengler's (2008) numerical experiment of a 2-player dynamic entry game with independent private payoff shocks, they find that their game contains at least five equilibira. To determine the number of equilibria is crucial for identifying and estimating a game. Characterizing the equilibrium set of the static game is one of the major contributions of this chapter, which prepares the research of the dynamic game. Moreover, to analytically understand how information correlation affects players' behaviour in a dynamic setting, this chapter can throw more light on information correlation's role in strategic interactions.

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Chapter 3

Information Correlation in a Strategic Complements Game and an Extension of Purification Rationale

In this chapter, we study a 2×2 *strategic complements Bayesian entry game with correlated private information. The distribution of private information is modelled by a joint normal distribution. We examine the comparative statics of the model, indicating how the number of equilibria varies with the correlation coefficient and variances of the prior distribution. We show that the purification rationale proposed by Harsanyi (1973) can be extended to games with dependent perturbation errors that follow a normal distribution if the correlation coefficient is positive for the strategic complements games or negative for the strategic substitutes games.*

3.1 Introduction

This chapter develops a simple model of firm entry with correlated private information in a 2-player static strategic complements game. This game is symmetric. In the game, after observing their respective private payoff shocks, two firms simultaneously decide whether to enter a market. The private payoff shocks are statistically correlated, and the correlation coefficient of players' joint type distribution measures the degree of information correlation. That is, there are common and idiosyncratic components of each payoff shock, and each firm only observes its own aggregate shock without knowing its component. An example of this situation is two firms that produce complementary inputs entering a local market. Each firm expects its private payoff shocks of entry to be correlated with the other firm's, because the shocks depend on certain common factors of the market.

The game is solved by a cutoff strategy, which is defined as if a player's private payoff shock is above a threshold value, they choose entry, or vice versa. By solving the game, we find a critical value of the correlation coefficient. For correlation coefficients below this critical value, a cutoff strategy cannot be used to solve the game. This result is determined by the normality of the joint prior distribution and the definition of the cutoff strategy. The intuition is that if the correlation coefficient is smaller than this critical value, the expected payoff function is no longer monotonic with respect to the player's own strategies, given any strategy of the opponent. For some strategies of the opponent, there are multiple (three) best responses. One of the three best responses will make a player choose entry if the payoff shock is below the best response cutoff value, which contradicts the definition of cutoff strategies.

Under some parameter specifications, the game exhibits a unique equilibrium. Under other parameter specifications, there may exist two or three equilibria and the number of equilibria changes in the following order as the correlation coefficient increases from the lowest feasible value to 1: $3 \to 2 \to 1$ or $3 \to 1$. The intuition is that the uncertainty between players' private payoff shocks is measured by the correlation coefficient, and the uncertainty between players' private information determines whether a player's conditional density of the opponent's payoff shocks given the player's own payoff shock can approximately or imprecisely reflect the opponent's private information.[1] [2] If the uncertainty between players' private payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private information. If the density function can approximately reflect the other players' private information, then players can obtain enough information to help them match their action strategies, hence leading to multiple equilibria. Otherwise, players cannot obtain enough information to help them match their action strategies

[1] As in Chapter 2, the uncertainty between players' payoff shocks (random variables) indicate that for two random variables ε and ε^*, the relation $\varepsilon^* = a\varepsilon + b + \eta$ holds, where a and b are two real numbers and η is a random variable that is used to reflect the uncertainty between players' private information. Still, we can consider to use the correlation coefficient between the two random variables to measure the uncertainty between them.

[2] To understand how the density function can reflect the opponent's private information given a player's own private information, please refer to Section 2.4 of Chapter 2

and hence the game exhibits a unique equilibrium. In the strategic complements game, high (low) value of correlation coefficient usually represents high (low) uncertainty between players' private information. However, it is also possible that for certain parameter specifications, the game has a unique equilibrium for all feasible values of the correlation coefficient. This is because due to the concerned payoff specification, the two players' ex ante expectations of the opponent's behaviour are unique, irrespective of what payoff shocks will be drawn. Ex ante in this chapter means the expectation is formed before the payoff shocks are drawn, and hence, the expectation is taken for all possible values of payoff shocks. Therefore, we call it ex ante expectation. The expectations are that both players are more likely to choose being inactive, active or not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice. Accordingly, the game exhibits a unique equilibrium to echo the respective expectations.[3]

The equilibrium strategies are represented by the cutoff strategies, which take all real numbers. The comparative statics of the correlation coefficient or variances of prior distribution on players' equilibrium strategies depend on the sign of the equilibrium strategy and the stability of the equilibrium.[4] For a stable equilibrium, increasing the payoff of entry will make a player more likely choose entry. If the given equilibrium cutoff strategies are negative, increasing the information correlation or jointly increasing the variances of the joint prior distribution will make players less likely choose entry. If the given equilibrium cutoff strategies are positive, increasing information correlation or jointly increasing the variances of the joint prior distribution will make players more likely choose entry. If the given equilibrium cutoff strategies equal zero, changing the information correlation or variances of the joint prior distribution does not have any impact on the equilibrium strategies. For unstable equilibrium, increasing the payoff of entry will make a player less likely choose entry, which contradicts

[3]Specifically, if the expectation is that both players are more likely to choose being inactive or that both players are more likely to choose being active, the expectation is dominant in a player's decision making and the uncertainty between players' private information takes a minor role in his decision making. However, if the expectation is that players are not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice, only when the uncertainty between players' private information is high, the expectation is dominant in a player's decision making. These intuitions are established when the best response functions are not contraction function. If the best response functions are contraction function, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem, and hence the game exhibits a unique equilibrium.

[4]The stability concept adopted in this chapter is Lyapunov stability.

our common sense. Because we use a cutoff strategy to solve the game, if the payoff of entry increases, then given the opponent's strategy, a player will more likely choose entry. Because the game exhibits positive externalities in payoffs, the opponent will also be more likely to choose entry as the best response to the player's change of strategies more favouring entry. Given this best response dynamics, no strategy will converge to an equilibrium in which increasing the payoff of entry makes a player less likely to choose entry. This situation satisfies the Lyapunovian instability of an equilibrium and hence such an equilibrium is unstable.

In this symmetric game, the variances of players' prior distribution are assumed to be identical. There is an equivalence relationship between how the number of equilibria varies with the variances and with the correlation coefficient. We find that under certain parameter specifications, the game exhibits a unique equilibrium. Under other parameter specifications, the number of equilibria changes in the following order as variances increase from the lowest feasible value to $+\infty$: $3 \rightarrow 2 \rightarrow 1$ or $3 \rightarrow 1$. The intuition is that the uncertainty of a player's private payoff shock is determined by the variance of the player's prior distribution, and the uncertainty of both players' private payoff shocks determines whether the conditional density of the opponent's payoff shocks given the player's own payoff shock can approximately or imprecisely reflect the opponent's private information.[5] If the uncertainty of both players' private payoff shocks is low (high), then the conditional density function can approximately (imprecisely) reflect the opponent's private information. Still, multiple equilibria arise when the density function can approximately reflect the opponent's private information given the player's own private information. Otherwise, the game exhibits a unique equilibrium when the density function is imprecise to reflect the opponent's private information given the player's own private information. The low (high) value of the variance usually represents low (high) uncertainty of a player's private information in the strategic complements game. However, it is also possible that for certain parameter specifications, the game has a unique equilibrium for all feasible values of variances. Similar to the corresponding case for the uncertainty between players' private information, this is also because due to the concerned payoff specification, the two players' ex ante expectations of the opponent's behaviour are unique, irrespective of what payoff shocks will be drawn, and the expectations are that both players are more likely to

[5] In this chapter, no matter how the variances change, they are always assumed to be identical. Therefore, we consider the uncertainty of both players' private information (payoff shocks).

choose being inactive, active or not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice.[6]

The comparative statics of the number of equilibria with respect to variances is also the necessary and sufficient condition to differentiate unique equilibrium and multiple equilibria. Morris and Shin (2005) study an identically specified game and provide a sufficient condition for unique equilibrium. They focus on how introducing strategic uncertainty can reduce the number of equilibria of a complete information game. The complete information game is symmetric and strategic complements. They also use the cutoff strategy defined in this chapter to solve the game. They argue that when the strategic uncertainty (belief) is sufficiently invariant with respect to all possible strategies, there is a unique equilibrium. Based on this insight, they obtain a sufficient condition to ensure the game exhibits a unique equilibrium. We find that their sufficient condition is essentially the necessary and sufficient to ensure that the best response functions are contraction functions. If both players' best response functions are contractions, then the game is dominance solvable and hence there exists a unique equilibrium. Therefore, we nest Morris and Shin's (2005) result.

The incomplete information entry game can be viewed as a perturbed game of a complete information entry game. According to Harsanyi (1973)'s purification rationale, if the perturbation errors on each player's payoff are independent, a Bayesian Nash equilibrium exists that will converge to the mixed strategy equilibrium as perturbation errors tend to zero. In our game, we specify that the variances of the perturbation-error distribution converge to zero, as the process that uncertainty of perturbed games vanishes. We find that, for the strategic complements complete information games if the perturbation errors are negatively correlated, or for the strategic substitutes complete information games if the perturbation errors are positively correlated, there does not exist a Bayesian game that can be solved by the cutoff strategy as perturbation errors tend to zero. Hence, Harsanyi's purification rationale cannot be applied to this situation. The intuition is that by assuming the variances of both players' type distributions are identical, for negative information correlation in the strategic complements

[6] Specifically, if the expectation is that both players are more likely to choose being inactive or that both players are more likely to choose being active, the expectation is dominant in a player's decision making and the uncertainty of the player's private information takes a minor role in his decision making. However, if the expectation is that players are not sure whether the opponent is more likely to choose being inactive or active, or more likely to be indifferent to either action choice, only when the uncertainty of each player's private information is high, the expectation is dominant in a player's decision making.

game or the positive correlation in the strategic substitutes game, there exists a critical value of variances, below which the expected payoff function is not monotonic with respect to a player's own private payoff shock, and it is possible that given some of the opponent's strategies, the player can have multiple (three) best responses; around one of the best responses, a payoff shock that is below the best response cutoff value can make the player choose entry, which contradicts the definition of the cutoff strategy. Therefore, for negative information correlation in the strategic complements game or positive information correlation in the strategic substitutes game, only if the variances are above the cutoff value, the game can be solved by cutoff strategies.

However, if the information correlation is positive for the strategic complements games or negative for the strategic substitutes games, the purification rationale is still applicable. We find that in these situations, the Bayesian games that are supposed to converge to the complete information game as the perturbation errors degenerate to zero exist, and during the process, the pure-strategy Bayesian Nash equilibrium will converge to the corresponding Nash equilibrium of the underlying complete information game. Therefore, we extend Harsanyi's purification rationale to dependent perturbation-error situations.

The rest of this chapter proceeds as follows. Section 3.2 presents the game. Section 3.3 studies the best response function. Section 3.4 studies the comparative statics of the number of equilibria with respect to the correlation coefficient and the stability of equilibrium. Section 3.5 studies the comparative statics of equilibrium strategies. Section 3.6 presents the comparative statics of the number of equilibria with respect to variances. Section 3.7 explains how purification rationale can be extended to games with dependent perturbation errors. Section 3.8 summarizes all the main results and intuitions of the strategic complements game. Section 3.9 concludes this chapter.

3.2 The Game

Consider a 2-player entry game. Each player has two choices, activity or entry (hereafter, 1), or inactivity (hereafter, 0). Each firm makes its own decision after observing its private payoff shock. Then, both firms implement their decisions, which can be observed by each other. The active firm will enter the market. If both firms are active, a coordination will happen between them and the profit D if the opponent chooses to be active strictly exceeds the profit M if the opponent chooses to be inactive. At the end of the period, both firms collect their respective payoffs. The inactive firm gets payoff zero and the active firm obtains the deterministic payoff (D or M) plus its private payoff shock. It is assumed that the private payoff shocks are subject to a bivariate normal distribution $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma, \varsigma^*, \rho)$. In this chapter, we use '*' to denote variables of the opponent. It is always assumed that $\varsigma = \varsigma^*$ to ensure that the game is symmetric. The strategic form of this game is depicted as follows:

		Firm i^*	
		inactive (0)	active (1)
Firm i	inactive (0)	0 (Firm i), 0 (Firm i^*)	0 (Firm i), $M+\varepsilon^*$ (Firm i^*)
	active (1)	$M+\varepsilon$ (Firm i), 0 (Firm i^*)	$D+\varepsilon$ (Firm i), $D+\varepsilon^*$ (Firm i^*)

Table 1: The incomplete information entry game where $D > M$

Firms adopt cutoff strategies: if payoff shock ε is above a threshold value $\bar{\varepsilon}$, a player chooses to be active, or vice versa. Therefore, the interim belief that the opponent plays out given payoff shock ε is given by $\sigma(x^*, \varepsilon) = \int_{-\infty}^{x^*} f(\varepsilon^*|\varepsilon) d\varepsilon^*$, where $f(\varepsilon^*|\varepsilon)$ is the conditional density of ε^* given ε. $\sigma_{x^*}(x^*, \varepsilon)$ is the first-order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to x^*, and $\sigma_\varepsilon(x^*, \varepsilon)$ is the first order partial derivative of $\sigma(x^*, \varepsilon)$ with respect to ε. It is found that $\sigma_{x^*}(x^*, \varepsilon) > 0$, $\sigma_\varepsilon(x^*, \varepsilon) < 0$ if $\rho > 0$, and $\sigma_\varepsilon(x^*, \varepsilon) > 0$ if $\rho < 0$. $\sigma_\varepsilon(x^*, \varepsilon) = 0$ at $\rho = 0$. So given a player's own payoff shock ε, if the opponent's cutoff strategy becomes higher, then the belief that the opponent chooses being inactive will increase. Given the opponent's strategy, if the correlation coefficient is positive, a high payoff shock of a player indicates that probably the opponent also gets a high payoff shock; thus, the belief that the opponent chooses being inactive decreases. Given the opponent's strategy, if the correlation coefficient is negative, a high payoff shock of a player indicates that probably the opponent gets a negative payoff shock; hence, the belief that the opponent chooses being inactive increases. If the correlation coefficient equals 0, a player's own payoff shock does not have any impact on their belief of the opponent's behaviour. Firm i's expected payoff of entry can be written as

$$\mathbb{E}\Pi(x^*,\varepsilon) = \sigma(x^*,\varepsilon)(M+\varepsilon) + (1-\sigma(x^*,\varepsilon))(D+\varepsilon)$$
$$= \sigma(x^*,\varepsilon)M + (1-\sigma(x^*,\varepsilon))D + \varepsilon \quad (3.1)$$

Equation (3.1) indicates that a player's expected payoff is composed of two parts: the payoff induced by strategic uncertainty, $\sigma(x^*,\varepsilon)M + (1-\sigma(x^*,\varepsilon))D$, and the realised payoff shock, ε. If $\rho \geq 0$, given ρ, M, D, ς^2 and ς^{*2}, both parts are non-decreasing with respect to ε. Intuitively, if both firms' private payoff shocks are positively correlated, a high payoff shock ε for one firm would on average imply a high payoff shock ε^* for the opponent, which provides an incentive that encourages the player to be active in the strategic complements game. Therefore, the expected payoff should be non-decreasing with respect to ε for $\rho \geq 0$. Thus, for a positively correlated private information situation, the cutoff strategy can always be applied.

However, if ρ is negative, then given all parameter values, the payoff induced by strategic uncertainty $\sigma(x^*,\varepsilon)M + (1-\sigma(x^*,\varepsilon))D$ is decreasing with respect to ε. Thus, whether the expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$ is monotonically increasing with respect to ε depends on the trade-off between the payoff induced by strategic uncertainty and by the realized payoff shock. For negative ρs, if one firm draws a high payoff shock, it can be expected that its opponent draws a low payoff shock, and hence, it is highly probable that the opponent chooses being inactive, which provides strategic disincentives for the firm to choose entry in the strategic complements context. It is also known that ε itself is a part of the payoff and it incentivizes entering. Therefore, whether the firm will choose to be active essentially depends on the trade-off between the two contrasting effects.

If the correlation between players' private information is loose, i.e. ρ is slightly negative, it can be deduced that the positive incentive generated by a high value of ε dominates its negative impact, and hence, in total, its expected payoff should increase with respect to ε. However, if the correlation coefficient between the players' private information is tight, i.e. ρ is close to -1, then it can be reasonably expected that the strategic disincentive induced by the realization of a high value of ε will be strong, and hence, a high payoff shock does not necessarily bring a high expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$. In fact, it is found that there exists a unique boundary $\tilde{\rho}$ in the strategic complements discrete game such that if $\rho \geq \tilde{\rho}$, given the expected opponent's cutoff strategy $x^* \in \mathbb{R}$, the expected payoff $\mathbb{E}\Pi(x^*, \varepsilon)$ is increasing with respect to ε, but if $\rho < \tilde{\rho}$, the expected payoff is no longer monotonic; it is then certain that for some x^*, equation $\mathbb{E}\Pi(x^*, \varepsilon) = 0$ has multiple (three) solutions (best responses) of ε and there is one solution below which a payoff shock can make a player choose entry, which contradicts the definition of the cutoff strategy (see Appendix). Therefore, given D, M, ς^2 and ς^{*2}, a player can legitimately use a cutoff strategy to play the game if and only if $\rho \in [\tilde{\rho}, 1)$ in the game and $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$.[7] Thus, for each player, there exists a boundary of ρ and for the value of ρ above the boundary value, a cutoff strategy can be used to solve the game. Due to the assumption $\varsigma = \varsigma^*$, the boundary for both players are the same, i.e. $\tilde{\rho} = \tilde{\rho}^*$, and therefore, this boundary defines the range of ρ for which a cutoff strategy can be used to solve the game. This result is formally given by the following proposition:

Proposition 1 (Restriction of Applying a Cutoff Strategy to Solve the Game) : Suppose $D > M$ and $\varsigma^* = \varsigma$. A cutoff strategy can be applied to solve the game if and only if $\rho \in [\tilde{\rho}, 1)$, where $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$.

Proof: See Appendix. ■

$\pi = 3.14...$ is the ratio of a circle's circumference to its diameter. Given $\rho \in [\tilde{\rho}, 0)$ and an $x^* \in \mathbb{R}$, if $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε, it indicates that

$$\frac{\partial \mathbb{E}\Pi(x^*, \varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*, \varepsilon)(M - D) + 1 \geq 0$$

for all $x^* \in \mathbb{R}$; hence,

$$1 \geq \sigma_\varepsilon(x^*, \varepsilon)(D - M)$$

Because $\sigma_\varepsilon(x^*, \varepsilon) = -\rho f(x^*|\varepsilon)$ (see Appendix A), the above inequality can be written as

$$1 \geq -\rho f(x^*|\varepsilon)(D - M)$$

and hence

[7] For the opponent, $\tilde{\rho}^* = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}}$.

$$f(x^*|\varepsilon) \leq \frac{1}{-\rho(D-M)} \tag{3.2}$$

As ς increases, the variance of the distribution $f(.|\varepsilon)$, which equals $\varsigma^2(1-\rho^2)$, increases, and hence the density function flattens.[8] Particularly, the maximum value of $f(x^*|\varepsilon)$, which equals $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ and is taken at the mean $x^* = \rho\varepsilon$, decreases. Hence, (3.2) is easier to be satisfied and it is more certain that at the given value of ρ, $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff increase with respect to ε should be broadened as ς increases, and accordingly, $\tilde{\rho}$ decreases.

If $D-M$ decreases, the RHS of (3.2) increases. Hence, (3.2) is easier to be satisfied, and it is more certain that at the given value of ρ, $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that makes the expected payoff of entry increase with respect to ε should be broadened as $D-M$ decreases, and accordingly, $\tilde{\rho}$ decreases.

3.3 The Best Response Function

Given the opponent's cutoff strategy $x^* \in \mathbb{R}$, a firm's cutoff best response $g(x^*)$ is determined by $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. That is,

$$\sigma(x^*, g(x^*))(M-D) + D + g(x^*) = 0$$

It is found that $g(x^*) \in [-D, -M]$ because as long as $D > M$, the maximum of $\sigma(x^*,\varepsilon)(M-D)+D$ equals D, where $\sigma(x^*,\varepsilon) = 0$, and the minimum of $\sigma(x^*,\varepsilon)(M-D)+D$ equals M, where $\sigma(x^*,\varepsilon) = 1$. Given the joint normal distribution, we obtain the best response function in its reverse form:

[8] The density function $f(.|\varepsilon)$ refers to $f(\varepsilon^*|\varepsilon)$. For the explicit expression, please refer to Appendix A.

$$x^* = \rho \frac{\varsigma^*}{\varsigma} g(x^*) + \varsigma^* \sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M}) \tag{3.3}$$

where $\Phi(.)$ is the cumulative density function of the standard normal distribution. Then, we can get the derivative of $g(x^*)$ with respect to x^* as follows.

$$g'(x^*) = -\frac{\sigma_{x^*}(x^*, g(x^*))(M-D)}{\sigma_\varepsilon(x^*, g(x^*))(M-D)+1} \tag{3.4}$$

In this game, we can divide the best response functions into two types: contraction and non-contraction.[9] The contraction best response function, according to Zimper (2004), makes the game dominance solvable, and hence, there exists a unique equilibrium. If the best response function is a non-contraction, it may contain multiple equilibria. Figure 1 exhibits a numerical example of a contraction and a non-contraction best response function. The properties of the best response functions are summarized in the following proposition.

Proposition 2 (Properties of Best Response Functions): Given that $\varsigma = \varsigma^*$ and $D > M$, there exists a $\hat{\rho} = \frac{(D-M)^2 - 2\pi\varsigma^{*2}}{(D-M)^2 + 2\pi\varsigma^{*2}}$ which differentiates contraction and non-contraction best response functions:

1) for $\rho \in [\tilde{\rho}, \hat{\rho}]$,

I. if $g(x^*) \in [-D, (D-M)\Phi(-\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $0 < g'(x^*) < 1$;

II. if $g(x^*) = (D-M)\Phi(-\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = 1$;

III. if $g(x^*) \in ((D-M)\Phi(-\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, (D-M)\Phi(\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) > 1$;

IV. if $g(x^*) = (D-M)\Phi(\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = 1$;

V. if $g(x^*) \in ((D-M)\Phi(\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, -M]$, $0 < g'(x^*) < 1$;

[9]For the description and properties of contraction and non-contraction functions, please refer to Appendix H of Chapter 2.

2) for $\rho \in (\hat{\rho}, 1)$, $0 < g'(x^*) < 1$ globally.

Proof: see Appendix. ■

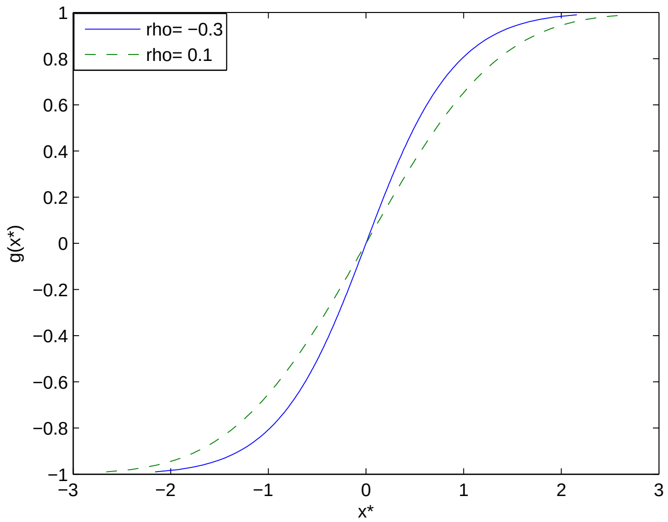


Figure 1: *A numerical example of a contraction best response function and a non-contraction best response function.* $D = 1$, $M = -1$, ς *(and* ς^**)=1 and* $\rho = -0.3$ *(for the solid curve) and 0.1 (for the dashed curve). In this case,* $\hat{\rho} = -0.2220$. *The solid curve represents a non-contraction best response function, and the dashed curve represents a contraction best response function.*

Due to the positive externalities of payoff specification, the game exhibits strategic complements; therefore, it is not surprising that the best response function in this game is increasing. For the opponent, $\hat{\rho}^* = \frac{(D-M)^2 - 2\pi\varsigma^2}{(D-M)^2 + 2\pi\varsigma^2}$. If both players' best response functions are contraction, the game is dominance solvable, and hence a unique equilibrium exists. Because we consider a symmetric game, $\hat{\rho} = \hat{\rho}^*$. Therefore, a sufficient condition to make the game have a unique equilibrium is that given D, M and $\varsigma = \varsigma^*$, $\rho \in [\hat{\rho}, 1)$. This sufficient condition can be generalized to asymmetric payoff settings, where each player has different D and M. Therefore, the generalized sufficient condition to ensure a unique equilibrium is $\rho \in [\max\{\hat{\rho}, \hat{\rho}^*\}, 1)$.

If the best response functions are contraction function and hence the game is dominance solvable, then according to the implicit function theorem, it implies that for all $x^* \in \mathbb{R}$, a player's expected payoff responds more to their own strategy than to the opponent's strategy, i.e.

$$\frac{\partial g(x^*)}{\partial x^*} = -\frac{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}}{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}}|_{\varepsilon=g(x^*)} < 1$$

and hence,

$$\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}|_{\varepsilon=g(x^*)} > -\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}|_{\varepsilon=g(x^*)} \tag{3.5}$$

Equation (3.5) can be explicitly written as

$$\sigma_\varepsilon(x^*, g(x^*))(M-D) + 1 > -\sigma_{x^*}(x^*, g(x^*))(M-D)$$

where $\sigma_\varepsilon(x^*, g(x^*)) = \sigma_\varepsilon(x^*, \varepsilon)|_{\varepsilon=g(x^*)}$ and $\sigma_{x^*}(x^*, g(x^*)) = \sigma_{x^*}(x^*, \varepsilon)|_{\varepsilon=g(x^*)}$. Furthermore, the above inequality can be written as

$$\rho f(x^*|g(x^*))(D-M) + 1 > f(x^*|g(x^*))(D-M)$$

and we obtain

$$1 > (1-\rho)(D-M)f(x^*|g(x^*))$$

Finally, we find that (3.5) exactly implies the following restriction on the conditional density function

$$f(x^*|g(x^*)) < \frac{1}{(1-\rho)(D-M)} \tag{3.6}$$

The inequality (3.6) is held for all $\rho \in [\hat{\rho}, 1)$. From (3.6), it can be seen that as $\rho \to 1$, the RHS of (3.6) increases to $+\infty$, and hence (3.6) is certainly satisfied. In this situation, it is certain that the best response functions are contraction functions and so the game is dominance solvable.

Given $\rho \in [\hat{\rho}, 1)$, as ς increases, the variance of $f(.|g(x^*))$, which equals $\varsigma^2(1-\rho^2)$, increases and hence the density function flattens.[10] Particularly, the maximum value of $f(.|g(x^*))$, which is equal to $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ and is taken at the mean $\rho g(x^*)$ of the distribution $f(.|g(x^*))$, decreases. Hence, (3.6) is easier to be satisfied and it is more

[10] The density function $f(.|g(x^*))$ is exactly the density function $f(\varepsilon^*|\varepsilon)$, where $\varepsilon = g(x^*)$. For its explicit expression, please refer to Appendix A.

certain that at the given value of ρ, $g'(x^*) < 1$ for all $x^* \in \mathbb{R}$. Therefore, the range of ρ that ensures $g(x^*)$ is a contraction function should be broadened as ς increases, and accordingly $\hat{\rho}$ decreases.

If $D-M$ decreases, the RHS of the inequality (3.6) increases. Hence, (3.6) is easier to be satisfied and it is more certain that at the given value of ρ, $g'(x^*) < 1$ for all $x^* \in \mathbb{R}$. Therefore, if $D-M$ decreases, the range of ρ that ensures $g(x^*)$ is a contraction function should also be broadened, and accordingly $\hat{\rho}$ decreases.

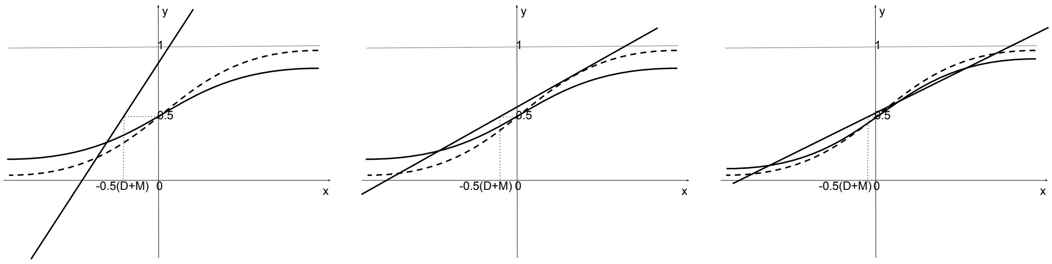
3.4 Comparative Statics of the Number of Equilibria with respect to the Correlation Coefficient

Since the best response function in this game increases and the two players' best response functions are located symmetrically around the 45° line, all equilibria of the strategic complements game must be symmetric and locate at the 45° line. Hereafter, we denote any equilibrium of the strategic complements game by (e,e), which should satisfy $e = g(e)$. Specifically, according to equation (3.3), the following must be satisfied:

$$\Phi\left(\frac{\frac{1}{\varsigma^*} - \frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}e\right) = \frac{D+e}{D-M}$$

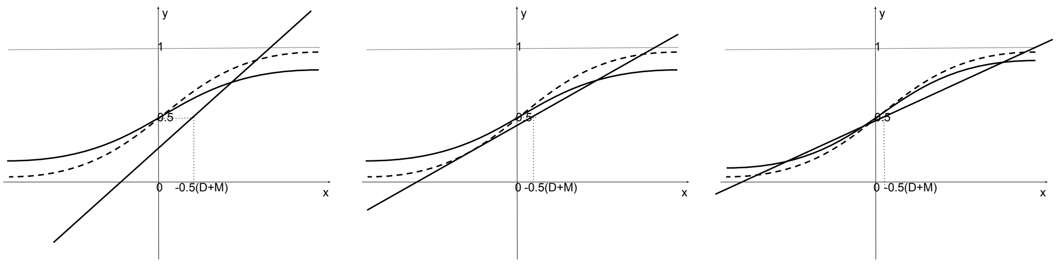
Therefore, equivalently, the equilibria of the strategic complements game are also the intersection points between curves $y = \Phi(\frac{\frac{1}{\varsigma^*} - \frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}x)$ and $y = \frac{D+x}{D-M}$. We define $\alpha(\rho) := \frac{\frac{1}{\varsigma^*} - \frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}$. The line $y = \frac{D+x}{D-M}$ passes through the point $(-\frac{D+M}{2}, \frac{1}{2})$. Hence, we can analyse the equilibria of the game separately in terms of $D+M > 0$, $D+M < 0$ and $D+M = 0$ (see Figures 2–4).

Figures 2 to 4 exhibit all possible cases of the intersections between $y = \Phi(\alpha(\rho)x)$ and $y = \frac{D+x}{D-M}$ for all $\rho \in [\tilde{\rho}, 1)$. The dashed curves describe the limit case $y = \Phi(\alpha(\tilde{\rho})x)$, and the intersection points between $y = \Phi(\alpha(\tilde{\rho})x)$ and $y = \frac{D+x}{D-M}$ will be used to judge whether the game is able to contain multiple equilibria for all $\rho \in [\tilde{\rho}, 1)$. Irrespective of the value of ρ, given D, M and ς (and hence ς^*), $y = \frac{D+x}{D-M}$ always crosses the



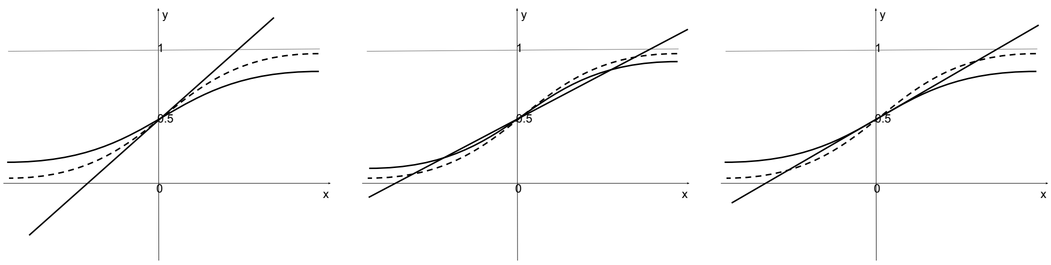
2-1 **2-2** **2-3**

Figure 2: *In the* 2×2 *strategic complements game, for all* $\rho \in [\tilde{\rho}, 1)$*, if* $D+M>0$*, there could be either one, two or three equilibria, which are intersection points between* $y=\Phi(\alpha(\rho)x)$*, where* $\rho \in (\tilde{\rho}, 1)$*, which is represented by the solid curve, and* $y=\frac{D+x}{D-M}$*, which is represented by the solid line. The dashed curve represents the limit case* $y=\Phi(\alpha(\tilde{\rho})x)$*.*



3-1 **3-2** **3-3**

Figure 3: *In the* 2×2 *strategic complements game, for all* $\rho \in [\tilde{\rho}, 1)$*, if* $D+M<0$*, there could be either one, two or three equilibria, which are intersection points between* $y=\Phi(\alpha(\rho)x)$*, where* $\rho \in (\tilde{\rho}, 1)$*, which is represented by the solid curve, and* $y=\frac{D+x}{D-M}$*, which is represented by the solid line. The dashed curve represents the limit case* $y=\Phi(\alpha(\tilde{\rho})x)$*.*



4-1 **4-2** **4-3**

Figure 4: *In the* 2×2 *strategic complements game, for all* $\rho \in [\tilde{\rho}, 1)$*, if* $D+M=0$*, then* $-\frac{D+M}{2}=0$ *and there could be either one or three equilibria, which are intersection points between* $y=\Phi(\alpha(\rho)x)$*, where* $\rho \in (\tilde{\rho}, 1)$*, which is represented by the solid curve, and* $y=\frac{D+x}{D-M}$*, which is represented by the solid line. The dashed curve represents the limit case* $y=\Phi(\alpha(\tilde{\rho})x)$*.*

point $(-\frac{D+M}{2}, \frac{1}{2})$, and $y = \Phi(\alpha(\rho)x)$ always crosses the point $(0, \frac{1}{2})$. The intersection points are the equilibria of the game. We can directly judge the stability of each equilibrium in Figures 2 to 4 by comparing the slopes of $y = \frac{D+x}{D-M}$ and $y = \Phi(\alpha(\rho)x)$ for all $\rho \in [\tilde{\rho}, 1)$. We denote $\phi(.)$ as the probability density function of the standard normal distribution. It is found that at an equilibrium (e,e), $g'(e) < 1$ if and only if $\phi(\alpha(\rho)e)\alpha(\rho) < \frac{1}{D-M}$; $g'(e) > 1$ if and only if $\phi(\alpha(\rho)e)\alpha(\rho) > \frac{1}{D-M}$; $g'(e) = 1$ if and only if $\phi(\alpha(\rho)e)\alpha(\rho) = \frac{1}{D-M}$. Therefore, by comparing the slopes of the two curves at each intersection point, we can learn whether the corresponding eigenvalues at the equilibrium are smaller than one.[11] Hence, we obtain the stability property of all equilibria.

We are interested in the comparative statics of the number of equilibria with respect to ρ. Take Figure 2 as an example. Sub-figures 2-1, 2-2 and 2-3 represent three possible cases of intersections between $y = \Phi(\alpha(\tilde{\rho})x)$ and $y = \frac{D+x}{D-M}$. If at $\rho = \tilde{\rho}$, there exists a unique intersection point between $y = \Phi(\alpha(\tilde{\rho})x)$ and $y = \frac{D+x}{D-M}$, then as ρ increases away from $\tilde{\rho}$, $y = \Phi(\alpha(\rho)x)$ will decrease given $x > 0$ and increase given $x < 0$, and thus, a unique intersection point exists after the change (see sub-figure 2-1). Hence, the games for all $\rho \in [\tilde{\rho}, 1)$ always contain a unique equilibrium (intersection point) given D, M and ς (and hence ς^*). Because in this case, $y = \frac{D+x}{D-M}$ is always steeper than $y = \Phi(\alpha(\rho)x)$, the unique equilibrium is stable. Sub-figure 2-2 represents a boundary situation, which means at $\rho = \tilde{\rho}$, $y = \Phi(\alpha(\tilde{\rho})x)$ and $y = \frac{D+x}{D-M}$ have one intersection point and one tangent point.[12] In this situation, by comparing the slopes of the two curves, the intersection point is stable, while the tangent point's stability cannot be determined. As ρ increases away from $\tilde{\rho}$, again $y = \Phi(\alpha(\rho)x)$ will decrease given $x > 0$ and increase given $x < 0$, and only one intersection point exists. It represents the unique equilibrium for all $\rho \in (\tilde{\rho}, 1)$, and it is stable. Sub-figure 2-3 represents the

[11] In this symmetric strategic complements game, an equilibrium cutoff strategy $(e,\ e)$ should simultaneously satisfy $e = g(e)$ and $e = g^*(e)$. $g(.)$ and $g^*(.)$ are player i's and player i^*'s best response functions, respectively. Therefore, the corresponding Jacobian matrix is

$$J = \begin{pmatrix} g^{*'}(e) & 0 \\ 0 & g'(e) \end{pmatrix}$$

It is straightforward to find that the eigenvalues of the Jacobian matrix are $g^{*'}(e)$ and $g'(e)$, respectively. Thus, if the first-order derivatives of best response functions at an equilibrium are known, the stability of this equilibrium can be judged.

[12] The tangency situation, which represents an equilibrium, can only arise if $D > 0 > M$. If $D > M > 0$ or $0 > D > M$, the game always has a unique equilibrium for $\rho \in [\tilde{\rho}, 1)$ (see Appendix).

most subtle case. At $\rho = \tilde{\rho}$, $y = \Phi(\alpha(\tilde{\rho})x)$ and $y = \frac{D+x}{D-M}$ have three intersection points. By comparing the slopes of the two functions at each intersection point (equilibrium), the middle intersection point (equilibrium) is unstable, while the two outer intersection points (equilibria) are stable. As we increase ρ away from $\tilde{\rho}$, the multiplicity situation continues until $\rho = \bar{\rho}$, where $y = \Phi(\alpha(\bar{\rho})x)$ has one tangent point and one intersection point with $y = \frac{D+x}{D-M}$; hence, the number of equilibria becomes two. As ρ continues to increase away from $\bar{\rho}$, $y = \Phi(\alpha(\rho)x)$ will further decrease given $x > 0$ and further increase given $x < 0$ such that only one intersection point is left, which represents the unique equilibrium and is stable.

We can apply the same approach of deriving how the number of equilibria changes with respect to ρ and the stability of equilibrium to cases of $D+M < 0$ and $D+M = 0$. Figures 3 and 4 describes how many intersections points (equilibria) can exist if $D+M < 0$ or $D+M = 0$. Finally, from previous analysis, it can be determined that $\bar{\rho}$ exists if and only if $D > 0 > M$. If $\bar{\rho}$ exists, $\bar{\rho} \leq \hat{\rho}$, but $\bar{\rho}$ could be smaller than $\tilde{\rho}$, or greater than or equal to $\tilde{\rho}$.

Suppose $D > 0 > M$ and hence $\bar{\rho}$ exists. Its analytical expression depends on the sign of $D+M$. Specifically, if $D+M > 0$, then $\bar{\rho}$ is the unique solution of the following equation:

$$\Phi\Big(\sqrt{\ln\frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}\Big) = \frac{D+\sqrt{\varsigma^2\frac{1+\bar{\rho}}{1-\bar{\rho}}\ln\frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}}{D-M}$$

where $\bar{\rho} < \hat{\rho}$. If $D+M < 0$, then $\bar{\rho}$ is the unique solution of the following equation:

$$\Phi\Big(-\sqrt{\ln\frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}\Big) = \frac{D-\sqrt{\varsigma^2\frac{1+\bar{\rho}}{1-\bar{\rho}}\ln\frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}}{D-M}$$

where $\bar{\rho} < \hat{\rho}$. If $D+M = 0$, then $\bar{\rho} = \hat{\rho}$.

When there are three equilibria, we name the equilibrium located at the middle part of a best response function the middle equilibrium, and the equilibrium located at the outer part of a best response function the outer equilibrium. By summarizing the analysis of all three cases ($D+M \gtreqless 0$), we obtain the comparative statics results of the number of equilibrium with respect to ρ and the stability of equilibrium. It is given by the following theorem.

Theorem 1 (Comparative Statics of the Number of Equilibria with respect to ρ and Stability of Equilibrium in the Strategic Complements Game): For the static 2×2 entry game, suppose $D > M$ and $\varsigma = \varsigma^*$. $\bar{\rho}$ exists if and only if $D > 0 > M$. If $D > 0 > M$ and $\bar{\rho} < \tilde{\rho}$, or if $0 > D > M$ or $D > M > 0$ in which case $\bar{\rho}$ does not exist, then for all $\rho \in [\tilde{\rho}, 1)$, the game has a unique equilibrium. Conversely, if for all $\rho \in [\tilde{\rho}, 1)$, the game has a unique equilibrium, then it is either because $\bar{\rho} < \tilde{\rho}$ if $D > 0 > M$ or because $0 > D > M$ or $D > M > 0$ in which case $\bar{\rho}$ does not exist. The unique equilibrium is stable.

If $\bar{\rho} \geq \tilde{\rho}$, then for all $\rho \in [\tilde{\rho}, \bar{\rho})$, there exist three equilibria. The middle equilibrium is unstable, while the two outer equilibria are stable. In particular, if $D+M=0$, the middle equilibrium is (0, 0).

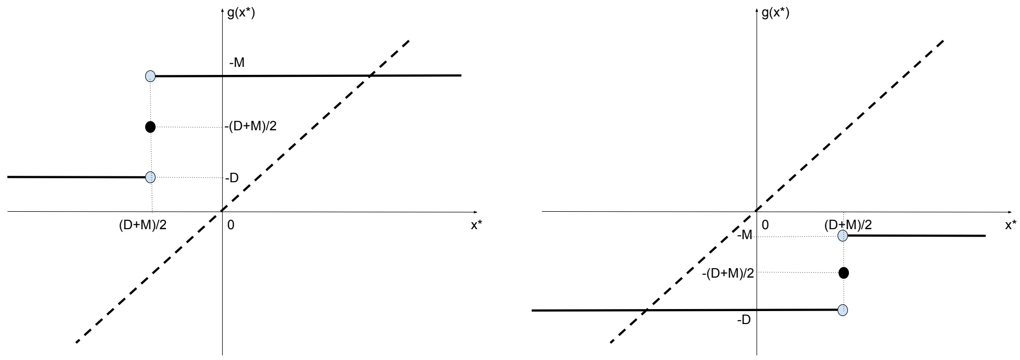
At $\rho = \bar{\rho}$: 1) if $D+M \neq 0$, there are two equilibria. One is stable. Stability of the other is not determined; 2) if $D+M=0$, there exists a unique equilibrium $(0,0)$ and it is stable.

For all $\rho \in (\bar{\rho}, 1)$, there exists a unique equilibrium and it is stable. In particular, if $D+M=0$, the unique equilibrium is (0, 0).

Proof: see Appendix. ■

The equilibria can be described as solutions of the equation system of $g(x^*)$ and $g^*(x)$. As $\rho \to -1$, $g(x^*)$ at the limit is given by [13]

[13] The game at $\rho \to -1$ does not coincide with the game at $\rho = -1$. The game at $\rho \to -1$ is exhibited in the following part in the main context of this chapter. For the game at $\rho = -1$, $\varepsilon = -\varepsilon^*$ and both players are affected by the opposite payoff shocks which are not known ex ante. However, if these shocks are known, the two players play a complete information game. For $\varepsilon \in (-D, -M)$, there are three equilibria: $(0,0)$, $(1,1)$ and a mixed strategy $(\frac{D+\varepsilon}{D-M}, \frac{D+\varepsilon}{D-M})$, which is the probability of choosing action 0. This probability depends on ex post realizations of payoff shocks. In contrast, for the game at $\rho \to -1$, the number of equilibria and the cutoff value of the equilibrium depends on the relation among D, M and 0. Even if we express the equilibria of the game at $\rho \to -1$ in the form of action strategies, none of them depends on the ex post realization of ε and ε^*. Therefore, the incomplete information games at $\rho \to -1$ does not coincide with the games at $\rho = -1$. $g(x^*)$ is continuous with respect to $\rho \in (-1, 1)$. Therefore, in this section, the natural benchmark to compare the number of equilibria for games at $\rho \in [\tilde{\rho}, 1)$ is the game at $\rho \to -1$.



$M < D < 0$ $\qquad$ $0 < M < D$

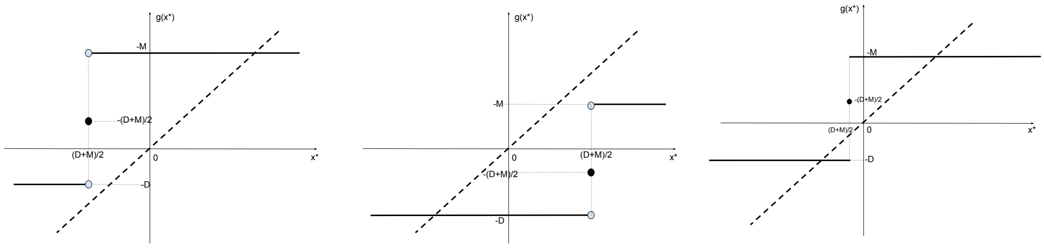
Figure 5: *In each sub-figure, the two horizontal lines and the point between the two lines together represent the piecewise function $g(x^*)$ at $\rho \to -1$. The dashed line represents the $45°$ line. The intersection point between the horizontal line and the dashed line is denoted by (e,e). $g(x^*)$ and $g^*(x)$ are symmetrically located around the $45°$ line, and therefore the solution (e,e) that satisfies $e = g(e)$ is also the intersection point between $g(x^*)$ and the $45°$ line. For $M < D < 0$, the intersection point (solution) is $(-M,-M)$. For $0 < M < D$, the intersection point (solution) is $(-D,-D)$.*

$$g(x^*) = \begin{cases} -D & x^* < \frac{D+M}{2} \\ -\frac{D+M}{2} & x^* = \frac{D+M}{2} \\ -M & x^* > \frac{D+M}{2} \end{cases}$$

Because $g(x^*)$ and $g^*(x)$ are symmetrically located around the $45°$ line as well, all solutions are symmetric and a solution (e,e) of the equation system should be an intersection point between $g(x^*)$ and the $45°$ line. If $M < D < 0$, $g(x^*)$ and $g^*(x)$ have a unique solution $(-M,-M)$. If $0 < M < D$, the equation system has a unique solution $(-D,-D)$. Figure 5 describes how these solutions arise .

If $D > 0 > M$, this situation is complicated. In this situation, if $\frac{D+M}{2} < -D$, then there is a unique solution $(-M,-M)$, and if $\frac{D+M}{2} > -M$, there is a unique solution $(-D,-D)$. Supposing $-D < \frac{D+M}{2} < -M$, if $D+M \neq 0$, there are two solutions $(-M,-M)$ and $(-D,-D)$, while if $D+M = 0$, there are three solutions $(-M,-M)$, $(0,0)$ and $(-D,-D)$. Figure 6 exhibits how these solutions arise.

The intuition of Theorem 1 is as follows. The average known payoff of entry is



$D > 0 > M$ **and** $\frac{D+M}{2} < -D$ | $D > 0 > M$ **and** $\frac{D+M}{2} > -M$ | $D > 0 > M$ **and** $-D < \frac{D+M}{2} < -M$

Figure 6: *In each sub-figure, the two horizontal lines and the point between the two lines together represent the piecewise function $g(x^*)$ at $\rho \to -1$. The dashed line represents the $45°$ line. As in Figure 5, solution(s) (e,e), where $e = g(e)$, are intersection points between $g(x^*)$ and the $45°$ line. Given $D > 0 > M$, if $\frac{D+M}{2} < -D$, there is a unique solution $(-M,-M)$. If $\frac{D+M}{2} > -M$, there is a unique solution $(-D,-D)$. If $-D < \frac{D+M}{2} < -M$ and $D+M \neq 0$, there are two solutions $(-D,-D)$ and $(-M,-M)$, and if $D+M = 0$, there are three solutions $(-D,-D)$, $(-M,-M)$ and $(0,0)$.*

$\frac{D+M}{2}$. Therefore, if a firm wants to choose entry, from an ex ante perspective, it should at least obtain a payoff shock $\varepsilon = -\frac{D+M}{2}$. We call this shock the average payoff shock required for entry. If $-\frac{D+M}{2} > D > M$, the average payoff shock required for entry is even higher than the highest known payoff of being active. Thus, ex ante, i.e. before the payoff shock is drawn, each player will be expected to prefer being inactive to entry. The expectation that the opponent prefers being inactive is formed before the payoff shocks are drawn and there do not exist alternative expectations due to the payoff specification. Hence, the contingent payoff shocks cannot affect the ex ante expectation. Therefore, given this expectation, a player expects that if they choose entry, they will get profit M; therefore, if the payoff shock $\varepsilon \geq -M$, the player will choose entry. In the symmetric strategic complements game, players can behave identically for matching their strategies. Hence, the opponent will think in the same way and adopt cutoff strategy $-M$. This intuition applies to cases of $0 > D > M$ and $D > 0 > M$ with $\frac{D+M}{2} < -D$. Both cases satisfy the requirement $-\frac{D+M}{2} > D > M$. This fact can be judged from concerned sub-figures in Figures 5 and 6.

If $D > M > -\frac{D+M}{2}$, the average payoff shock required for entry is smaller than the lowest known payoff of being active. Thus, ex ante, each player will be expected to prefer being active to being inactive. The expectation that the opponent prefers entry is formed before the payoff shocks are drawn and is determined by such payoff specifications. Hence, the contingent payoff shocks cannot affect the ex ante expectation. Therefore, in this situation, a player expects that if they choose entry, they will get profit D, and thus, if the payoff shock $\varepsilon \geq -D$, the player will choose entry. In the strategic complements game, players behave identically for matching their strategies; hence, the opponent will think in the same way and adopt the strategy $-D$. This intuition applies to the case of $D > M > 0$ and $D > 0 > M$ with $\frac{D+M}{2} > -M$. Both cases satisfy the requirement $D > M > -\frac{D+M}{2}$, which can be judged from concerned subfigures in Figures 5 and 6.

If $D > -\frac{D+M}{2} > M$, the average payoff shock required for entry is between the highest and lowest payoff of being active. This situation happens when $D > 0 > M$, and hence, reasonably in this situation, $-\frac{D+M}{2}$ could be higher or lower than, or equal to 0, which is the payoff of being inactive. Thus, ex ante, each player can either prefer being inactive to being active, or vice versa. Both possibilities are reasonable to happen. If each player prefers being inactive to being active, then the intuition follows the case of $-\frac{D+M}{2} > D > M$ and both players will choose cutoff strategy $(-M, -M)$. If each player prefers being active to being inactive, then the intuition follows the case of $D > M > -\frac{D+M}{2}$ and both players will choose cutoff strategy $(-D, -D)$. Thus, if $D > -\frac{D+M}{2} > M$, two cutoff strategies $(-M, -M)$ and $(-D, -D)$ exist.

In addition, for $D > -\frac{D+M}{2} > M$, if $\frac{D+M}{2} = 0$, then the average known payoff of being active equals that of being inactive, which is 0. Therefore, each player can be indifferent to being active or inactive ex ante. In this situation, conditional on the expectation that the opponent is indifferent to either action choice, if payoff shock $\varepsilon \geq -\frac{D+M}{2} = 0$, the player will choose entry. Symmetrically, the opponent will think in the same way and also choose a cutoff strategy that equals 0. Therefore, if $D > -\frac{D+M}{2} > M$ and $D + M = 0$, another cutoff strategy $(0, 0)$ exists.

At $\rho = \tilde{\rho}$, there exists uncertainty between players' payoff shocks. Players cannot predict each other's private information via the conditional density function $f(\varepsilon^*|\varepsilon)$ as precisely as in the case of $\rho \to -1$.[14] Suppose $0 > D > M$. Then, the payoff of being inactive is higher than the highest payoff of being active. Thus, each player is ex ante

[14]To understand how the density function $f(\varepsilon^*|\varepsilon)$ can reflect the opponent's private information given a player's own private information, please refer to Section 2.4 of Chapter 2.

expected more likely to choose being inactive. In this situation, irrespective of whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect ε^* given ε, the ex ante expectation that the opponent is more likely to choose being inactive is not affected, because this expectation is formed before the payoff shock is drawn and there do not exist alternative expectations due to the payoff specification. In this situation, there still exists a unique equilibrium, and reasonably the equilibrium strategy should be close to $(-M,-M)$ if we translate the action strategy equilibrium into a cutoff strategy equilibrium representation. This intuition can apply to the games for all $\rho \in [\tilde{\rho}, 1)$ given $0 > D > M$.

Suppose $D > M > 0$. Then, the payoff of being inactive is smaller than the lowest payoff of being active. Thus, each player is ex ante expected more likely to choose being active. Although the payoff shocks are negatively correlated at $\rho = \tilde{\rho}$, as in the case of $0 > D > M$, irrespective of whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect ε^* given ε, the players' expectations that both are more likely to choose being active will not be affected, because the expectation is formed before the payoff shock is drawn and there do not exist alternative expectations due to the payoff specification. Therefore, there exists a unique equilibrium; that is, both players are more likely to choose being active. If we translate the action strategy equilibrium into a cutoff strategy equilibrium representation, the equilibrium strategy is expected to be close to $(-D,-D)$. This intuition can apply to the games for all $\rho \in [\tilde{\rho}, 1)$ given $D > M > 0$.

Suppose $D > 0 > M$. In this situation, at $\rho = \tilde{\rho}$, the game could have a unique equilibrium for the following three possibilities:

1) $\frac{D+M}{2} < -D < 0$: the average known payoff of entry is lower than the payoff of being inactive. The intuition in this situation follows the case of $0 > D > M$, in which the payoff specification also satisfies $\frac{D+M}{2} < -D$. The intuition at $\rho = \tilde{\rho}$ can apply to games for all $\rho \in [\tilde{\rho}, 1)$ given $\frac{D+M}{2} < -D$.

2) $\frac{D+M}{2} > -M > 0$: the average known payoff of entry is higher than the payoff of being inactive. The intuition in this situation follows the case $D > M > 0$, in which the payoff specification also satisfies $\frac{D+M}{2} > -M$. The intuition at $\rho = \tilde{\rho}$ can apply to games for all $\rho \in [\tilde{\rho}, 1)$ given $\frac{D+M}{2} > -M$.

3) $-D < \frac{D+M}{2} < -M$: in this situation, the average known payoff of entry could

be higher or lower than, or equal to the payoff of being inactive. If at $\rho = \tilde{\rho}$, the uncertainty between players' private information is very high, then this uncertainty will make players' attempt to match their strategies difficult, because $f(\varepsilon^*|\varepsilon)$ is imprecise to reflect ε^* given ε and players cannot get enough information to match their strategies. In this situation, a player's strategy choice is conditional upon an unclear expectation of the opponent's propensity of action choice.[15] Symmetrically, the opponent will think in the same way and adopt the same strategy. Therefore, the game exhibits a unique equilibrium to capture this unclear situation. Reasonably, it can be expected that in this situation, the value of the equilibrium entry threshold should be between $-D$ and $-M$.

Alternatively, in the situation of $-D < \frac{D+M}{2} < -M$, if at $\rho = \tilde{\rho}$, the uncertainty between players' private information is low such that $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε for player i, then this situation is close to the case of $-D < \frac{D+M}{2} < -M$ with $\rho \to -1$. In this situation, players are either more likely to choose being inactive or to choose being active. In addition, due to the uncertainty existing between players' private information, there is a possibility that each player is unclear about the other player's propensity of action choice. In particular, as $\rho \to -1$, the payoff shocks for both players are opposite, which makes matching strategies difficult; hence, each player's propensity of action choice is blurred and this situation is more likely to happen. Accordingly, a player will choose a strategy conditional on this unclear expectation. Symmetrically, the other player will think in the same way and adopt the same strategy. Hence, the game has another equilibrium to capture this unclear situation. Reasonably, it can be expected that the value of this equilibrium entry threshold is between $-D$ and $-M$. The intuition of the games with $D > 0 > M$ at $\rho = \tilde{\rho}$ as discussed above applies to games for all $\rho \in [\tilde{\rho}, \bar{\rho})$ with $D > 0 > M$.

For ρ taking values that are far from -1 and 1, or specifically for all $\rho \in [\bar{\rho}, \hat{\rho})$, the correlation between ε and ε^* are much lower and hence the uncertainty between players' private information becomes much higher. Therefore, $f(\varepsilon^*|\varepsilon)$ is imprecise to reflect the opponent's private information. In this situation, it is difficult for players to get enough information to match their action strategies. Hence, the possibility that players are unclear about the opponent's propensity of action choice becomes the only reasonable situation to exist. Therefore, the game only has a unique equilibrium, which

[15] An unclear expectation of the opponent's propensity of action choice means that before the payoff shocks are drawn, player i is not sure whether i^* is more likely to choose being active or being inactive, or more likely to be indifferent to either action choice.

captures this unclear situation.

For $\rho \in [\hat{\rho}, 1)$, as we have shown in Section 3.3, it is certain that the best response functions are contraction functions and hence the game has a unique equilibrium. The intuition is that, in this situation, as indicated by inequality (3.5), player i's expected payoff of entry is more sensitive to his own strategy than to his opponent's strategy. It means player i is more self-focused and the opponent's private information is less important in player i's decision making, no matter whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect i^*'s private information given ε. Therefore, in this situation, the game is close to an individual decision problem, and hence there exists a unique equilibrium.

Therefore, $\bar{\rho}$, if greater than $\tilde{\rho}$, is the threshold that differentiates the high uncertainty and low uncertainty between players' private information. The uncertainty between players' private information determines how players can behave (i.e. whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect ε^* given ε, and thus, the players can (cannot) collect enough information to help them match their strategies) and hence how many equilibria could exist.

All of these intuitions mentioned above explain why for all $\rho \in [\tilde{\rho}, 1)$, the strategic complements game can have a unique equilibrium, or if there are multiple equilibria for some values of $\rho \in [\tilde{\rho}, 1)$, as ρ increases from $\tilde{\rho}$ to 1, the number of equilibria will decrease from 3 to 1.

Finally, to conclude this section, we provide a list of numerical examples. They exhibit how the number of equilibrium varies with respect to different parameter specifications (see Figures 7–9).

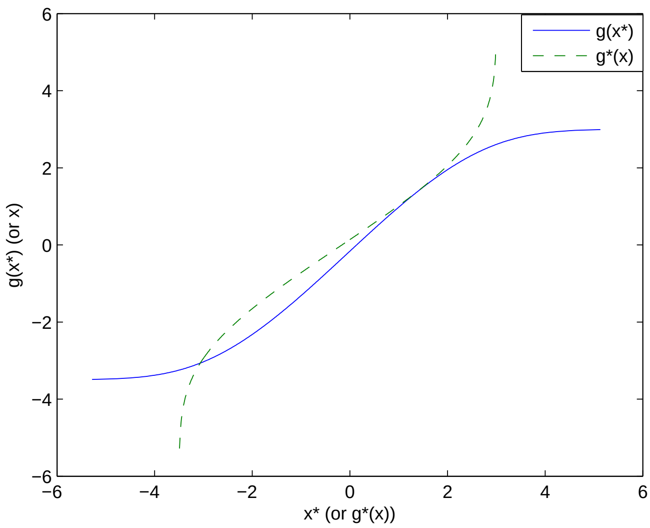


Figure 7: *A numerical example of unique equilibrium in the* 2×2 *strategic complements game. The solid curve represents a player's best response function and the dashed curve represents the opponent's best response function. In this case, D=3.5, M=-3,* $\rho = 0.5$*,* $\varsigma = \varsigma^* = 1.5$.

3.5 Comparative Statics of Players' Equilibrium Strategies

In this section, we present the comparative statics of exogenous parameters on the equilibrium strategy. It is given as

Proposition 3 (Comparative Statics of Players' Equilibrium Strategies): Assume $D > M$ and $\varsigma = \varsigma^*$. We denote an equilibrium of the game by (e, e), where $-D < e < -M$. We obtain that

(3.1) For a stable equilibrium, $\frac{\partial e}{\partial M} < 0$ and $\frac{\partial e}{\partial D} < 0$. If $e \leq$ (or $>$) 0, $\frac{\partial e}{\partial \rho} \geq$ (or $<$) 0 and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}} \geq$ (or $<$) 0, where $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ equal 0 when $e = 0$.

(3.2) For an unstable equilibrium, $\frac{\partial e}{\partial M} > 0$ and $\frac{\partial e}{\partial D} > 0$. If $e \leq$ (or $>$) 0, $\frac{\partial e}{\partial \rho} \leq$ (or $>$) 0 and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}} \leq$ (or $>$) 0, where $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ equal 0 when $e = 0$.

(3.3) For the equilibrium value for which the stability cannot be determined, $\frac{\partial e}{\partial M}$, $\frac{\partial e}{\partial D}$, $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ equals ∞.

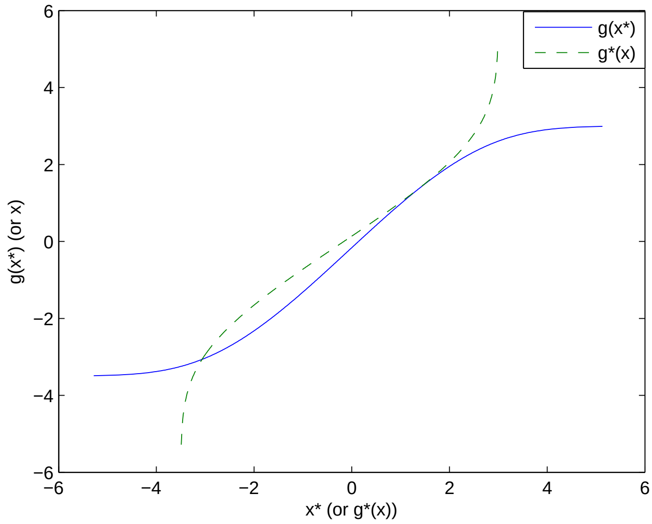


Figure 8: *A numerical example of unique equilibrium in the* 2×2 *strategic complements game. The solid curve represents a player's best response function and the dashed curve represents the opponent's best response function. In this case, D=3.5, M=-3,* $\rho = 0.3$, $\varsigma = \varsigma^* = 1.5$. *This case is close to a boundary case where there exist one intersection point and one tangent point. However, in fact, there is no tangency in this case.*

Proof: see Appendix. ■

It is found that for $\tau \in \{D, M\}$, $sign(\frac{\partial e}{\partial \tau}) = sign(\frac{1}{\phi(\alpha(\rho)e)\alpha(\rho) - \frac{1}{D-M}})$ and for $\tau \in \{\rho, \varsigma$ and $\varsigma^*\}$, $sign(\frac{\partial e}{\partial \tau}) = sign(\frac{1}{\phi(\alpha(\rho)e)\alpha(\rho) - \frac{1}{D-M}}) \times sign(e)$. In Section 3.4, we have shown that at an equilibrium (e, e), $\phi(\alpha(\rho)e)\alpha(\rho) \gtreqless \frac{1}{D-M}$ if and only if $g'(e) \gtreqless 1$. Therefore, in the symmetric strategic complements game, the stability of each equilibrium determines the sign of the comparative statics results.

For a stable equilibrium, increasing the payoff of entry D or M will encourage players to adopt lower cutoff strategies, and hence, they become more likely to choose entry. The signs of $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ are opposite towards the signs of corresponding results in the strategic substitutes game in Chapter 2. This is determined by the strategic complements payoff specification $D > M$. The intuition to explain $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ follows the analysis of $\frac{\partial e}{\partial \rho}$ and $\frac{\partial e}{\partial \varsigma^2} + \frac{\partial e}{\partial \varsigma^{*2}}$ in the strategic substitutes game

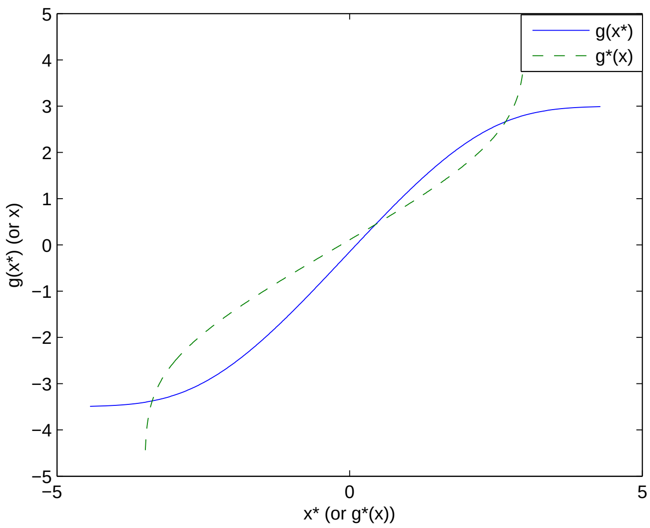


Figure 9: *A numerical example of multiple equilibria in the* 2×2 *strategic complements game. The solid curve represents a player's best response function and the dashed curve represents the opponent's best response function. In this case, D=3.5, M=-3,* $\rho = 0.3$, $\varsigma = \varsigma^* = 1.2$.

presented in Section 2.5 of Chapter 2. Generally, the intuition is that if we change the correlation coefficient, the mean of the conditional distribution of the opponent's payoff shock given the player's own payoff shock changes and the mean has a dominant impact on the player's belief towards the opponent's strategy given the player's own strategy. The change of mean depends on the sign of the equilibrium strategies. If we jointly change the variances of the prior distribution, only the variance of the conditional distribution of the opponent's payoff shock given the player's own shock changes. Increasing the variances will assign higher likelihood on low and high payoff shocks in the conditional distribution of the opponent's payoff shock, and the sign of an equilibrium strategy determines whether this strategy is located in the high or low payoff shock area in the distribution. The different location determines the different impacts of changing variances on a player's belief. The only difference of the analysis for the strategic complements game from that for the strategic substitutes game is that because $D > M$, the effect of increasing ρ or ς^2 and ς^{*2} on the expected payoff of entry is opposite to the effect in the strategic substitutes game, which leads to the opposite comparative statics results in the strategic complements game.

It should be noted that for an unstable equilibrium, increasing D and M will increase e, i.e. increasing the payoff of entry will make a player less likely choose entry. In this game, only the middle equilibrium, when there are three equilibria, is unstable. Therefore, in the middle equilibrium, players' behaviour contradicts with our common sense. The intuition is that because we use a cutoff strategy to solve the game, if the payoff of entry increases, then given the opponent's strategy, a player will become more likely to choose entry. Because the game exhibits positive externalities in payoffs, the opponent will also become more likely to choose entry as the best response to the player's change of strategies more favouring entry. Given this best response dynamics, no strategy will converge to an equilibrium in which increasing the payoff of entry makes a player less likely choose entry. This situation satisfies the Lyapunovian instability of an equilibrium, and hence is unstable.

3.6 Comparative Statics of the Number of Equilibria with respect to Variances

In this section, we study the comparative statics to determine how the number of equilibria changes by simultaneously changing ς^2 and ς^{*2}. Because $\varsigma^2 = \varsigma^{*2}$, in the following, we specify ς^2 and ς^{*2} as the same variable. From Proposition 1, it is known that if and only if $\rho \geq -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$, a cutoff strategy can be used to solve the game. Equivalently, it implies a restriction on the variance:

$$\varsigma^2 \geq \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)} \text{ for } \rho < 0$$

The inequality indicates that given $D > M$ and $\rho < 0$, there exists a lower bound of ς^2, which is denoted by $\tilde{\varsigma}^2$; hence, $\tilde{\varsigma}^2 = \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)}$ and $\tilde{\varsigma} = \sqrt{\tilde{\varsigma}^2}$. Given variances below this lower bound in the case of $D > M$ and $\rho < 0$, the game cannot be solved using a cutoff strategy. The intuition for this result is similar to the intuition of Proposition 1. Let us recall that

$$\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*,\varepsilon)(M-D)+1 = \frac{\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{x^*-\rho\varepsilon}{\varsigma\sqrt{1-\rho^2}})^2)+1$$

From the above expression, it can be seen that if $D > M$ and $\rho \geq 0$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} > 0$ for all $\varsigma \in (0,+\infty)$. For $\rho < 0$, $\tilde{\varsigma}$ exists and it makes $\min \frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \frac{\rho(D-M)}{\tilde{\varsigma}\sqrt{2\pi(1-\rho^2)}} + 1 = 0$.

Therefore, for $\rho < 0$, if $\varsigma \geq \tilde{\xi}$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0\ \forall x^* \in \mathbb{R}$. For $\rho < 0$, if $\varsigma < \tilde{\xi}$, $\mathbb{E}\Pi(x^*,\varepsilon)$ is no longer monotonic with respect to ε, and for some $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*,\varepsilon) = 0$ has multiple (three) solutions of ε, which are the best response threshold values. One of the three solutions has the following property: a payoff shock that is below the threshold value can make a player choose entry, which contradicts the definition of the cutoff strategy. This situation parallels the property of expected payoff function with $\rho < \tilde{\rho}$ given ς and $D > M$ (see Appendix B). Therefore, by assuming $\varsigma = \varsigma^*$, given $D > M$ and ρ, a player can legitimately use a cutoff strategy to play the game if and only if $\varsigma \in [\tilde{\xi}, +\infty)$ for $\rho < 0$ or $\varsigma \in (0, +\infty)$ for $\rho \geq 0$.

Proposition 4: Assuming $\varsigma = \varsigma^*$, given $D > M$ and $\rho \in (-1,1)$, a player can use a cutoff strategy to solve the game if and only if $\varsigma \in [\tilde{\xi}, +\infty)$ for $\rho < 0$ or $\varsigma \in (0, +\infty)$ for $\rho \geq 0$.

Proposition 4 can be generalized to asymmetric payoff settings. Suppose the opponent has different payoffs D' and M' with respect to D and M, respectively. Therefore, for $\rho < 0$, $\tilde{\xi}^{*2} = \frac{\rho^2(D'-M')^2}{2\pi(1-\rho^2)}$. In this situation, Proposition 4 can be generalized such that the games with asymmetric payoff specifications can be solved by a cutoff strategy if and only if $\varsigma \in [\max\{\tilde{\xi}, \tilde{\xi}^*\}, +\infty)$ for $\rho < 0$ or $\varsigma \in (0, +\infty)$ for $\rho \geq 0$.

Assume $\varsigma = \varsigma^*$. For $\rho \in (-1, 0)$, given a $\varsigma \in [\tilde{\xi}, +\infty)$ and an $x^* \in \mathbb{R}$, if $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε, it indicates that

$$\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*,\varepsilon)(M-D) + 1 \geq 0$$

for all $x^* \in \mathbb{R}$. Again, this inequality can be equivalently transformed into the following expression:

$$f(x^*|\varepsilon) \leq \frac{1}{-\rho(D-M)}$$

and it is inequality (3.2).

As the negative ρ increases, the variance of $f(.|\varepsilon)$, which equals $\varsigma^2(1-\rho^2)$, increases and hence the density function flattens. The mean of $f(.|\varepsilon)$, which equals $\rho\varepsilon$, also changes. The maximum value of $f(.|\varepsilon)$, which equals $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ and is taken at the mean $x^* = \rho\varepsilon$, decreases. Meanwhile, the RHS of (3.2) increases as ρ increases. Therefore, as ρ increases, the LHS of (3.2) decreases and the RHS of (3.2) increases; hence, (3.2) is easier to be satisfied and it is more certain that at the given ς, $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, given the $\rho < 0$, the range of ς that makes the expected payoff increase with respect to ε should be broadened as ρ increases, and accordingly, $\tilde{\varsigma}$ decreases.

If $D - M$ decreases, the RHS of (3.2) increases. Hence, (3.2) is easier to be satisfied and it is more certain that at the given $\rho < 0$, $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. Therefore, given $\rho < 0$, the range of ς that makes the expected payoff increase with respect to ε should be broadened as $D - M$ decreases, and accordingly, $\tilde{\varsigma}$ decreases.

Let us recall inequality (3.6), which is the necessary and sufficient condition to ensure $g(x^*)$ is a contraction function for all $x^* \in \mathbb{R}$ and is hence the sufficient condition to ensure that the symmetric game is dominance solvable. This inequality can be explicitly written as

$$\sigma_\varepsilon(x^*, g(x^*))(M - D) + 1 > -\sigma_{x^*}(x^*, g(x^*))(M - D)$$

By rearranging the LHS and RHS, we obtain

$$1 > (\sigma_{x^*}(x^*, g(x^*)) + \sigma_\varepsilon(x^*, g(x^*)))(D - M) \tag{3.7}$$

where $\sigma_{x^*}(x^*, g(x^*)) + \sigma_\varepsilon(x^*, g(x^*)) = \frac{1}{\sqrt{2\pi}\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}\exp(-\frac{1}{2}(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})^2)$.

From inequality (3.7), it can be seen that if the maximum value of $(\sigma_{x^*}(x^*, g(x^*)) + \sigma_\varepsilon(x^*, g(x^*)))(D - M)$, which is $\frac{D-M}{\sqrt{2\pi}\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}$, is smaller than 1, inequality (3.7) and inequality (3.6) always hold and $g(x^*)$ is a contraction function for all $x^* \in \mathbb{R}$. Hence, the game is dominance solvable.

Accordingly, we denote $\hat{\varsigma}$ as the value that makes $\frac{D-M}{\sqrt{2\pi}\hat{\varsigma}}\sqrt{\frac{1-\rho}{1+\rho}} = 1$. That is, $\hat{\varsigma}^2 = \frac{(D-M)^2(1-\rho)}{2\pi(1+\rho)}$. Therefore, for $\varsigma^2 \geq \hat{\varsigma}^2$, $1 \geq \frac{D-M}{\sqrt{2\pi}\varsigma}\sqrt{\frac{1-\rho}{1+\rho}} > (\sigma_{x^*}(x^*, \varepsilon) + \sigma_\varepsilon(x^*, \varepsilon))(D - M)$ for all $x^* \in \mathbb{R}$. Hence, for all $\varsigma^2 \geq \hat{\varsigma}^2$, inequality (3.7) and inequality (3.6) always hold and $g(x^*)$ is a contraction function for all $x^* \in \mathbb{R}$. Hence, the game is dominance solvable in this situation.

By contrast, if $\varsigma^2 < \hat{\varsigma}^2$, $\frac{D-M}{\sqrt{2\pi}\varsigma}\sqrt{\frac{1-\rho}{1+\rho}} > 1$. In this situation, inequality (3.7) and equivalently inequality (3.6) cannot hold for all $x^* \in \mathbb{R}$; hence, $g(x^*)$ is not a contraction function for all $x^* \in \mathbb{R}$. Therefore, $\hat{\varsigma}^2$ is the critical value that differentiate the contraction and non-contraction best response functions. Hence, given other parameters, $\varsigma^2 \in [\hat{\varsigma}^2, +\infty)$ is the necessary and sufficient condition to ensure the best response function of this game to be a contraction function.

Proposition 5: Given $\rho \in (-1,1)$, supposing that $\varsigma = \varsigma^*$ and $D > M$, the game is dominance solvable if and only if $\varsigma^2 \in [\hat{\varsigma}^2, +\infty)$, where $\hat{\varsigma}^2 = \frac{(D-M)^2(1-\rho)}{2\pi(1+\rho)}$.

Proposition 5 can be generalized to asymmetric payoff settings. Suppose the other player i^* has the known payoffs D' and M' with respect to D and M, respectively. In this situation, i^*'s best response function $g^*(x)$ is a contraction function if and only if $\varsigma^{*2} \in [\hat{\varsigma}^{*2}, +\infty)$, where $\hat{\varsigma}^{*2} = \frac{(D'-M')^2(1-\rho)}{2\pi(1+\rho)}$. Hence, Proposition 5 can be generalized such that given $\rho \in (-1,1)$, supposing $\varsigma = \varsigma^*$, $D > M$ and $D' > M'$, the game is dominance solvable if and only if $\varsigma^2 \in [\max\{\hat{\varsigma}^2, \hat{\varsigma}^{*2}\}, +\infty)$.

Inequality (3.7) can be written as

$$1 > (1-\rho)(D-M)f(x^*|g(x^*))$$

because $\sigma_{x^*}(x^*, g(x^*)) = f(x^*|g(x^*))$ and $\sigma_\varepsilon(x^*, g(x^*)) = -\rho f(x^*|g(x^*))$. Moreover, as before, the above inequality implies inequality (3.6):

$$f(x^*|g(x^*)) < \frac{1}{(1-\rho)(D-M)}$$

Given $\rho \in (-1,1)$, (3.6) is held for all $\varsigma^2 \in [\hat{\varsigma}^2, +\infty)$.

Suppose $\rho < 0$. Given $\varsigma^2 \in [\hat{\varsigma}^2, +\infty)$, as ρ increases, the variance of $f(.|g(x^*))$, which equals $\varsigma^2(1-\rho^2)$, increases and hence the density function flattens. As we know, in this situation, the maximum value of $f(.|g(x^*))$ decreases, while the RHS of (3.6) increases. Hence, (3.6) is easier to be satisfied and it is more certain that at the given ς, $g'(x^*) < 1$ for all $x^* \in \mathbb{R}$. Therefore, the range of ς that ensures $g(x^*)$ is a contraction function should be broadened as ρ increases, and accordingly $\hat{\varsigma}$ decreases.

Suppose $\rho > 0$. Given $\varsigma^2 \in [\hat{\varsigma}^2, +\infty)$, as ρ increases, the variance of $f(.|g(x^*))$ decreases and hence the maximum value of $f(.|g(x^*))$, which is $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$, increases. However, the change of $\frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}$ by increasing 0.1 unit of ρ is always smaller than the change of $\frac{1}{(1-\rho)(D-M)}$, because $0 < \frac{\partial \frac{1}{\sqrt{2\pi(1-\rho^2)}\varsigma}}{\partial \rho} < \frac{\partial \frac{1}{(1-\rho)(D-M)}}{\partial \rho}$ given that all parameters satisfy (3.6). Therefore, in this situation, the net effect by increasing ρ is that the RHS of (3.6) relatively increases. Hence, again (3.6) is easier to be satisfied and the range of ς which ensures that $g(x^*)$ is a contraction function should be broadened as ρ increases, and accordingly $\hat{\varsigma}$ decreases.

If $D-M$ decreases, the RHS of (3.6) increases, and hence (3.6) is easier to be satisfied. In this situation, given $\rho \in (-1,1)$, the range of ς which ensures that $g(x^*)$ is a contraction function should be broadened, and accordingly $\hat{\varsigma}$ decreases.

Following the same logic for deriving $\bar{\rho}$ in Theorem 1, correspondingly we obtain $\bar{\varsigma}^2$ as a critical value to differentiate the unique equilibrium and multiple equilibria. Similar to $\bar{\rho}$, $\bar{\varsigma}^2$ exists if and only if $D > 0 > M$. It is found that $\bar{\varsigma}^2 \leq \hat{\varsigma}^2$. If $\rho < 0$ such that $\tilde{\varsigma}^2$ exists, $\bar{\varsigma}^2$ could be smaller than, or greater than or equal to $\tilde{\varsigma}^2$. If $\bar{\varsigma}^2$ exists, its analytical expression depends on the sign of $D+M$. Specifically, if $D+M > 0$, then $\bar{\varsigma}^2$ is the unique solution of the following equation:

$$\Phi(\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}) = \frac{D+\sqrt{\bar{\varsigma}^2\frac{1+\rho}{1-\rho}\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}}{D-M}$$

where $\bar{\varsigma}^2 < \hat{\varsigma}^2$. If $D+M < 0$, then $\bar{\varsigma}^2$ is the unique solution of the following equation:

$$\Phi(-\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}) = \frac{D-\sqrt{\bar{\varsigma}^2\frac{1+\rho}{1-\rho}\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}}{D-M}$$

where $\bar{\varsigma}^2 < \hat{\varsigma}^2$. If $D+M = 0$, $\bar{\varsigma}^2 = \hat{\varsigma}^2$. The comparative statics of the number of equilibrium with respect to ς^2 is thus given by the following corollary.

Corollary 1 (Comparative Statics of the Number of Equilibria with respect to

ς^2 and ς^{*2} in the Strategic Complements Game): For a static 2×2 entry game, suppose $D > M$ and $\varsigma = \varsigma^*$. For $D > M$, $\tilde{\varsigma}^2$ exists if and only if $\rho < 0$. $\bar{\varsigma}^2$ exists if and only if $D > 0 > M$. If $D > 0 > M$ and $\bar{\varsigma}^2 < \tilde{\varsigma}^2$, or if $0 > D > M$ or $D > M > 0$ in which $\bar{\varsigma}^2$ does not exist, then for all $\varsigma^2 \in [\tilde{\varsigma}^2, +\infty)$ for $\rho < 0$ or $\varsigma^2 \in [0, +\infty)$ for $\rho \geq 0$, the game has a unique equilibrium. Conversely, if for all $\varsigma^2 \in [\tilde{\varsigma}^2, +\infty)$ for $\rho < 0$ or $\varsigma^2 \in [0, +\infty)$ for $\rho \geq 0$, the game has a unique equilibrium, then it is either because $D > 0 > M$ and $\bar{\varsigma}^2 < \tilde{\varsigma}^2$ or because $0 > D > M$ or $D > M > 0$ in which $\bar{\varsigma}^2$ does not exist. The unique equilibrium is stable.

Given that $D > 0 > M$ for all $\varsigma^2 \in [\tilde{\varsigma}^2, \bar{\varsigma}^2)$ if $\rho < 0$ and $\bar{\varsigma}^2 > \tilde{\varsigma}^2$ or for all $\varsigma^2 \in (0, \bar{\varsigma}^2)$ if $\rho \geq 0$, there exist three equilibria. The middle equilibrium is unstable, while the two outer equilibria are stable. Particularly, if $D + M = 0$, the middle equilibrium is always (0, 0).

At $\varsigma^2 = \bar{\varsigma}^2$, where $\bar{\varsigma}^2 \geq \tilde{\varsigma}^2$, if $\rho < 0$, then 1) if $D + M \neq 0$, there are two equilibria, of which one is stable and the other's stability is not determined; and 2) if $D + M = 0$, there exists a unique equilibrium $(0, 0)$, which is stable.

$\forall \varsigma^2 \in (\bar{\varsigma}^2, +\infty)$, where $\bar{\varsigma}^2 \geq \tilde{\varsigma}^2$, if $\rho < 0$, there exists a unique equilibrium which is stable. Particularly, if $D + M = 0$, the unique equilibrium is always (0, 0).

Proof: see Appendix. ■

Games with $\varsigma = \varsigma^* = 0$ are complete information games. If $0 > D > M$, the game has a unique action strategy equilibrium $(0, 0)$, which implies a cutoff strategy equilibrium $(-M, -M)$. If $D > M > 0$, the game has a unique action strategy equilibrium $(1, 1)$, which implies a cutoff strategy equilibrium $(-D, -D)$. If $D > 0 > M$, there are three action strategy equilibria: $(0, 0)$, $(1, 1)$ and a mixed strategy $(\frac{D}{D-M}, \frac{D}{D-M})$, which imply cutoff strategy equilibria $(-M, -M)$, $(-D, -D)$ and $(0, 0)$, respectively. In the next section, we explain how these cutoff strategy equilibria are translated into action strategy representations, and we prove that the games with $\varsigma = \varsigma^* = 0$ coincide with the games with ς and $\varsigma^* \to 0$.

Given other parameters, for games with small variances, it is possible that $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, because the uncertainty of each player's private information is low. Therefore, in games where $D > 0 > M$, for variances close to zero, if $\rho \geq 0$, $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, and hence players are able to get enough information to match their strategies. This situation is close to games with $\varsigma = \varsigma^* = 0$, in which players can perfectly predict the opponent's private information. The intuition of the equilibria of games with $D > 0 > M$ and $\varsigma = \varsigma^* = 0$ will be explained in next section. Hence, for $D > 0 > M$, the game can exhibit three equilibria, which are close to $(-M, -M)$, $(-D, -D)$ and $(0, 0)$, respectively.

In contrast, for games with small variances, it is also possible that $f(\varepsilon^*|\varepsilon)$ imprecisely reflect ε^* given ε, because the uncertainty of each player's private information is high. In this situation, for games with $D > 0 > M$, each player is unclear about the opponent's propensity of action choice, and accordingly, they will choose a strategy conditional on this expectation. Therefore, in this situation, the game has only a unique equilibrium that captures this unclear situation.

When variances of payoff shocks increase such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε, the game with $D > 0 > M$ always has a unique equilibrium. The intuition exactly follows the corresponding intuitions for small variances of private payoff shocks but $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^*, conditional on ε given $D > 0 > M$ and $\rho \in (-1, 1)$. This is discussed in the last paragraph. Therefore, for $D > 0 > M$, where $\bar{\varsigma}$ exists, if $\rho < 0$ and $\bar{\varsigma} > \tilde{\varsigma}$ or $\rho \geq 0$, then $\bar{\varsigma}$ is exactly the threshold such that for $\varsigma < \bar{\varsigma}$, the uncertainty of both players' private information is low and hence $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε, and for $\varsigma > \bar{\varsigma}$, the uncertainty of both players' private information is high and hence $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε. At $\varsigma = \bar{\varsigma}$, even if $f(\varepsilon^*|\varepsilon)$ can still approximately reflect ε^* given ε, if the uncertainty of both players' private information increases a little bit, $f(\varepsilon^*|\varepsilon)$ will imprecisely reflect ε^* given ε.

For games with $0 > D > M$ or $D > M > 0$, supposing variances are small, for all feasible values of variances given $\rho \in (-1, 1)$, irrespective of whether $f(\varepsilon^*|\varepsilon)$ can approximately or imprecisely reflect ε^* given ε, there always exists a unique equilibrium. It is because in these situations, players are either expected to be more likely to choose being inactive $(0 > D > M)$ or more likely to choose being active $(D > M > 0)$. These expectations are formed before the payoff shocks are drawn and there do not exist alternative expectations given each payoff specification. Hence the contingent payoff shocks cannot affect these expectations. Therefore, the game exhibits a unique equilibrium based on these expectations and if it is $0 > D > M$, the cutoff strategy equilibrium should be close to $(-M, -M)$, and if it is $D > M > 0$, the cutoff strategy equilibrium should be close to $(-D, -D)$.

Morris and Shin (2005) study the same incomplete information game by assuming $\varsigma^2 = \varsigma^{*2}$, and give a sufficient condition to ensure the game has a unique equilibrium. Their sufficient condition is also expressed by a critical value of variance: if ς^2 is above this critical value, the game exhibits a unique equilibrium. Morris and Shin (2005) focus on how introducing strategic uncertainty can reduce the number of equilibrium of a complete information game. The complete information game is symmetric and strategic complements. They also use the cutoff strategy defined in this chapter to solve the game. They argue that when the strategic uncertainty (belief) is sufficiently invariant with respect to all possible strategies, a unique equilibrium exists. Based on this insight, they get the sufficient condition to ensure that the game exhibits a unique equilibrium.

Specifically, take our game as an example to explain Morris and Shin's rationale. We denote an equilibrium by (e, e). The equilibrium should satisfy the following equation:

$$\mathbb{E}\Pi(e,e) = \sigma(e,e)(M-D) + D + e = 0$$

If $\sigma(e,e)$ is sufficiently invariant with respect to $e \in \mathbb{R}$, then $\mathbb{E}\Pi(e,e)$ is close to be a linear function with respect to $e \in \mathbb{R}$. In this situation, $\mathbb{E}\Pi(e,e) = 0$ has a unique solution of e. Hence, a sufficient condition to make $\sigma(e,e)$ sufficiently invariant with respect to $e \in \mathbb{R}$ is that $\frac{\partial \mathbb{E}\Pi(e,e)}{\partial e} \geq 0$ for all $e \in \mathbb{R}$. Thus, the linear part of $\mathbb{E}\Pi(e,e)$, which is e, dominates the non-linear part of $\mathbb{E}\Pi(e,e)$, which is $\sigma(e,e)(M-D)$. In this way, $\sigma(e,e)$ is sufficiently invariant with respect to e in the sense of Morris and Shin, and due to the monotonicity of $\mathbb{E}\Pi(e,e)$ with respect to e, $\mathbb{E}\Pi(e,e) = 0$ has a unique solution of e.

$\frac{\partial \mathbb{E}\Pi(e,e)}{\partial e} = \phi(\sqrt{\frac{1-\rho}{1+\rho}}\frac{e}{\varsigma})\sqrt{\frac{1-\rho}{1+\rho}}\frac{M-D}{\varsigma} + 1 \geq 0$ for all $e \in \mathbb{R}$. Because $D > M$, equivalently to make $\frac{\partial \mathbb{E}\Pi(e,e)}{\partial e} \geq 0$ for all $e \in \mathbb{R}$, it requires that $\min \frac{\partial \mathbb{E}\Pi(e,e)}{\partial e} = \sqrt{\frac{1-\rho}{1+\rho}}\frac{M-D}{\sqrt{2\pi}\varsigma} + 1 \geq 0$. Therefore,

$$\varsigma \geq \frac{D-M}{\sqrt{2\pi}}\sqrt{\frac{1-\rho}{1+\rho}}$$

The RHS of the above inequality is only $\hat{\varsigma}$ in our study. This inequality indicates that as long as $\varsigma \geq \frac{D-M}{\sqrt{2\pi}}\sqrt{\frac{1-\rho}{1+\rho}}$, $\frac{\partial \mathbb{E}\Pi(e,e)}{\partial e} \geq 0$ for all $e \in \mathbb{R}$, and hence, a unique equilibrium exists. $\hat{\varsigma}^2$ differentiates contraction and non-contraction best response functions. As we mentioned in Section 3.3, if both players' best response functions are contractions, then the game is dominance solvable and hence there exists a unique equilibrium. Corollary 1 provides a complete range of parameter specifications ensuring a unique equilibrium, and the complete range is broader than the range of variances that ensures that the best response functions are contraction function. Therefore, Corollary 1 nests Morris and Shin's (2005) sufficient condition of uniqueness.

3.7 An Extension of Purification Rationale

Now consider the following complete information entry game:

		Firm i^*	
		inactive (0)	active (1)
Firm i	inactive (0)	0, 0	0, M
	active (1)	M, 0	D, D

Table 2: The complete information entry game where $D > M$

Assume $D > 0 > M$. The game has three equilibria, $(0,0)$, $(1,1)$ and $(\frac{M}{M-D}, \frac{M}{M-D})$, where $\frac{M}{M-D}$ is the probability to choose being active. The game shown in Table 1 is the perturbed game of this complete information game. In the following, for simplicity, we call the game shown in Table 2 as the complete information entry game, and the game shown in Table 1 as the perturbed entry game.

Harsanyi (1973) proposed a purification rationale for the play of mixed strategy equilibria. According to Harsanyi (1973), suppose that a player has some small private propensity to choose being active or being inactive, and this propensity is independent of the payoff specification. However, this information is not known to the other player at all. Then, the behaviour of such player will look as if they are randomizing between their actions to the other player. Because of the private payoff perturbation, the opponent will not in fact be indifferent to their actions, but will almost always choose a strict best response. Harsanyi's purification theorem showed that all equilibria of almost all complete information games are the limit of pure strategy equilibria of perturbed games where players have independent small private payoff shocks.

Note that, in Harsanyi's purification theorem, he specifies that the uncertainty of perturbed games vanishes in scale. That is, a constant η times the perturbation error ε, and let $\eta \to 0$. But in our game, we use an alternative approach to model the process that the uncertainty of perturbed games vanishes. That is, to let the variances of the perturbation-error distribution converge to zero. Here we make a clarification. For Harsanyi's (1973) purification rationale, it literally describes the idea that every Nash equilibrium of a complete information game can always be approached by a pure strategy Bayesian Nash equilibrium of a perturbed game. For Harsanyi's (1973) purification theorem, it further requires that the uncertainty of perturbed games vanishes in scale.

Following Morris' (2008) approach to decomposing Harsanyi's purification theorem, we can correspondingly decompose Harsanyi's purification rationale into two parts. The 'purification' part, where all equilibria of the perturbed game are essentially pure, and the 'approachability' part, where every equilibrium of a complete information game is the limit of equilibria of such perturbed games. For the first part, both Harsanyi's purification rationale and Harsanyi's purification theorem use the assumption of sufficiently diffuse independent payoff shocks. For our 2×2 games, the purification rationale indicates that provided that $\rho = 0$, all pure-strategy Bayesian Nash equilibria of the perturbed game obtained by using cutoff strategies (see Table 1) will finally converge to a Nash equilibrium of the complete information game (see Table 2). According to our Corollary 1, given that $D > 0 > M$ and $\rho = 0$, for $\varsigma^2 \in (0, \tilde{\xi}^2)$, the Bayesian games that can be solved by cutoff strategies exist and they have three equilibria. As we will exhibit in the following, these equilibria will finally converge to $(0,0)$, $(1,1)$ and $(\frac{M}{M-D}, \frac{M}{M-D})$, which are action strategy equilibria of the complete information game expressed as in Table 2 as ς and $\varsigma^* \to 0$. Therefore, the purification rationale is still applicable if the uncertainty of perturbed games vanishes as the variances of the perturbation-error distribution converge to zero.

However, what will be the situation if we relax the purification rationale by assuming the perturbation errors are dependent? Will Harsanyi (1973)'s purification rationale be still held for dependent payoff shocks?

Carlsson and van Damme (CvD, Appendix B, 1993) compare their global game model with Harsanyi's model. CvD's game is identical to our game shown in Table 1. Both are symmetric and strategic complements. The only difference is that in their game the ε of our game is additively decomposed into a common shock and an idiosyncratic shock χ, i.e. $\varepsilon = \theta + \chi$. θ and χ are independent and both follow a normal distribution. We denote μ_θ and μ_χ as the mean of θ and χ, respectively, and ς_θ^2 and ς_χ^2 as the variances of θ and χ. Therefore, $\varepsilon \sim N(\mu_\theta + \mu_\chi, \varsigma_\theta^2 + \varsigma_\chi^2)$, where $\mu_\theta + \mu_\chi = 0$ and $\varsigma_\theta^2 + \varsigma_\chi^2 = \varsigma^2$. Thus, ε and ε^* are correlated due to the common payoff shock, i.e. $\rho = \frac{\varsigma_\theta^2}{\varsigma_\theta^2 + \varsigma_\chi^2}$. In contrast, in our games, ε and ε^* can be dependent or correlated in any way, and due to the normal distribution specification, correlation coefficient ρ can reflect the dependence relation between ε and ε^*, rather than a simple correlation relation between the two shocks.

By specifying $\varsigma_\theta^2 \neq 0$ and $\varsigma_\chi^2 \to 0$, their model is the global game, and a unique equilibrium will be selected. The latter result can be accounted by our Theorem 1, which shows that as $\rho \to 1$, the game can only have a unique equilibrium, because during the process, the best response functions become contraction functions. However, CvD's work cannot show whether Harsanyi's (1973) purification rationale can be extended to perturbed games with correlated perturbation errors. It is because CvD's model requires that $\varsigma_\theta^2 + \varsigma_\chi^2 \to 0$, but due to the additive error structure $\varepsilon = \theta + \chi$, as $\varsigma_\theta^2 + \varsigma_\chi^2 \to 0$, $\rho = \frac{\varsigma_\theta^2}{\varsigma_\theta^2 + \varsigma_\chi^2}$ changes as well and $\rho \to 1$. Therefore, CvD's framework cannot isolate ρ's impact on the game as the perturbation errors ε and ε^* degenerate to a constant 0.

In last section of this work, we see that by assuming $D > M$ and $\rho \geq 0$, the games for $\varsigma^2 \in (0, +\infty)$ can be solved by cutoff strategies. The game closest to the complete information entry game is the Bayesian game, where ς and $\varsigma^* \to 0$. If $\varsigma = \varsigma^*$, the best response function in its reverse form is given by

$$x^* = \rho g(x^*) + \varsigma\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$$

where $g(x^*) \in [-D, -M]$ and $x^* \in \mathbb{R}$. Therefore, as ς and $\varsigma^* \to 0$,

$$g(x^*) = \frac{1}{\rho}x^*$$

where $x^* \in [-\rho D, -\rho M]$ if $D > M$ and $\rho > 0$. Let us recall the following definition of equation $g(x^*)$:

$$\mathbb{E}\Pi(x^*, g(x^*)) = \sigma(x^*, g(x^*))(M-D) + D + g(x^*) = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})(M-D) + D + g(x^*) = 0$$

As $\varsigma \to 0$, if $x^* > -\rho M$, $g(x^*) = -M$, and if $x^* < -\rho D$, $g(x^*) = -D$ (see Appendix). Therefore, the best response function of the Bayesian games with ς and ς^* $\to 0$ for all $x^* \in \mathbb{R}$ and $\rho > 0$ is given by

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho}x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M \end{cases}$$

The intuition of the piecewise expression of $g(x^*)$ as ς and $\varsigma^* \to 0$ is as follows. Supposing $D > M$, if the opponent i^* is expected to adopt a very high (low) cutoff strategy, it implies that player i expects that i^* is more likely to choose being inactive (active). In a strategic complements context, players always tend to match their action strategies, and hence as a best response, i will adopt the highest (lowest) cutoff strategy that can be achieved to indicate that the player also prefers being inactive (active). This highest (lowest) strategy is $-M$ ($-D$).

Assuming $\varsigma = \varsigma^*$, as ς and $\varsigma^* \to 0$, the likelihood of the mean of the distribution of the opponent's payoff shock given a player's own payoff shock increases, while the likelihood of the payoff shocks at both sides of the distribution around the mean decreases, because the variance of the conditional payoff shock distribution, $\varsigma^2(1-\rho^2)$, degenerates. Suppose the payoff shock that makes player i indifferent to entry or being inactive equals $g(x^*)$, where reasonably $g(x^*) \in [-D, -M]$ for $D > M$

or $g(x^*) \in [-M, -D]$ for $M > D$, then the mean of the opponent's payoff shock distribution is $\rho g(x^*)$, which happens with a very high likelihood as ς and $\varsigma^* \to 0$.

In symmetric games, no matter whether the game exhibits strategic complements or strategic substitutes, if a player is expected to be indifferent to being active or being inactive, the opponent will also adopt a strategy such that the opponent is also indifferent to entry or being inactive as a best response. Thus, the opponent i^* will choose a strategy x^* indicating indifference to their own action choices.

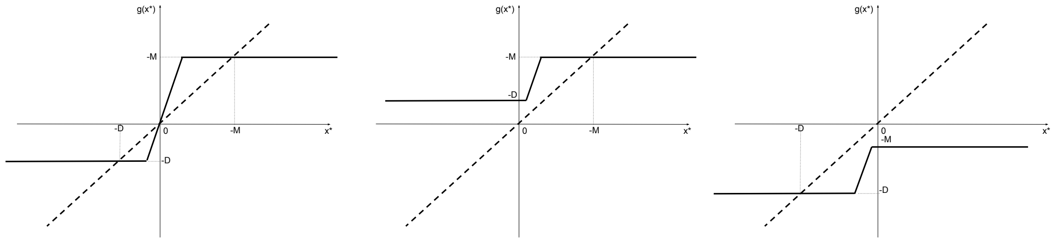
Therefore, based on the analysis from the previous two paragraphs, given $g(x^*)$ between $-M$ and $-D$, i expects that the payoff shock that is most likely to happen for i^* is $\rho g(x^*)$. Because at $g(x^*)$, i is indifferent to either action choice, as a best response, at $\rho g(x^*)$, i^* will also be indifferent to either action choice. Therefore, i^*'s strategy x^* should be equal to $\rho g(x^*)$ when ς and $\varsigma^* \to 0$ if $g(x^*) \in [-D, -M]$ for $D > M$ or $g(x^*) \in [-M, -D]$ for $M > D$. Obviously, this intuition applies to both the strategic complements and strategic substitutes cases.

Because the game is symmetric, for the strategic complements game, the equilibria can be described by the intersection points between $g(x^*)$ and the 45° line. Specifically, if $D > 0 > M$, there are three equilibria (intersection points): $(-M, -M)$, $(-D, -D)$ and $(0, 0)$ (see Figure 10-1).[16] As ς and $\varsigma^* \to 0$, the payoff shocks ε and ε^* converge to 0. Therefore, given cutoff strategy equilibrium $(-M, -M)$, since $-M > 0$, both players always choose action 0 in this equilibrium. Given cutoff strategy equilibrium $(-D, -D)$, since $-D < 0$, both players always choose action 1 in this equilibrium. Given strategy $(0, 0)$, the equilibrium belief $\sigma(0, 0)$ equals $\frac{D}{D-M}$ given any value of $\rho \in (0, 1)$. Thus, in this situation, $\sigma(0, 0)$ is independent of ρ and it is always equal

[16] The intuition of the cutoff strategy equilibrium is that given $D > 0 > M$, a player can expect that the opponent either chooses being active or inactive. If a player expects the opponent to choose entry, the player will get payoff D if they also choose entry. Thus, the player will adopt a cutoff strategy $-D$. As the best response, the opponent will adopt a strategy $-D$.

In contrast, if a player expects the opponent to choose being inactive, then the player will get payoff M if they choose to enter. Thus, the player will adopt a cutoff strategy $-M$. As the best response, the opponent will adopt a strategy $-M$.

If a player expects the opponent is indifferent to being active or being inactive, it indicates that irrespective of what value ε^* is, the expected payoff of entry for opponent i^* is equal to 0. Therefore, player i^*'s cutoff strategy is equal to 0. Hence, given $D > 0 > M$, player i will adopt a strategy 0 as a best response. Therefore, another cutoff strategy equilibrium ς and $\varsigma^* \to 0$ is $(0, 0)$.



10-1: $D > 0 > M$ **10-2:** $0 > D > M$ **10-3:** $D > M > 0$

Figure 10: *The solid curve represents $g(x^*)$ as ς and $\varsigma^* \to 0$. The dashed line represents the $45°$ line. The intersection points between $g(x^*)$ and the $45°$ line are the equilibria of the game with ς and $\varsigma^* \to 0$. If $D > 0 > M$, there are three equilibria, $(-M,-M)$, $(-D,-D)$ and $(0,0)$. If $0 > D > M$, there is a unique equilibrium $(-M,-M)$. If $D > M > 0$, there is a unique equilibrium $(-D,-D)$.*

to the unconditional probability of player i^* choosing action 0. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of the game expressed in the form of action strategies are given by $(0,0)$, $(1,1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $D > 0 > M$. Similarly, if $0 > D > M$ or $D > M > 0$, the equilibrium cutoff strategies are $(-M,-M)$ or $(-D,-D)$ respectively, which imply the action strategies $(0,0)$ or $(1,1)$ (see Figures 10-2 and 10-3).[17] These equilibria are exactly equal to the corresponding equilibria of the games with $\varsigma = \varsigma^* = 0$ and $0 > D > M$ or with $\varsigma = \varsigma^* = 0$ and $D > M > 0$. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of the perturbed games finally converge to the equilibria of the underlying complete information game.

Therefore, if $D > M$ and perturbation errors ε and ε^* follow a joint normal distribution, all equilibria of the complete information entry games are the limit of pure-strategy Bayesian Nash equilibria of perturbed games where players have non-negatively dependent perturbation errors.

[17]The intuitions of these cutoff strategy equilibria are as follows. Suppose $0 > D > M$. As ς and $\varsigma^* \to 0$, it is very likely that each player will choose being inactive. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -M$ since $M + \varepsilon > 0$ and M is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, the cutoff strategy equilibrium $(-M,-M)$ exists in this situation.

Similarly, suppose $D > M > 0$. As ς and $\varsigma^* \to 0$, it is very likely that each player will choose being active. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -D$ since $D + \varepsilon > 0$ and D is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, in this situation, we have the cutoff strategy equilibrium $(-D,-D)$.

However, if $D > M$ and $\rho < 0$, then $\tilde{\varsigma}^2$ arises. In the previous section, we have shown that if and only if $\varsigma^2 \in [\tilde{\varsigma}^2, +\infty)$, the Bayesian games can be solved by cutoff strategies. If $\varsigma^2 \in (0, \tilde{\varsigma}^2)$, the Bayesian games that can be solved by cutoff strategies do not exist due to the violation of the definition of the cutoff strategy concept, as we have exhibited in Section 3.6. Therefore, the sequence of such perturbed Bayesian games that are supposed to converge to the complete information game does not exist. Hence, the 'approachability' part of the purification rationale cannot be satisfied, and so the purification rationale cannot be applied in this situation. Therefore, in the strategic complements games ($D > M$), if and only if $\rho \geq 0$, Harsanyi's purification rationale is still applicable.

Extending purification rationale in the strategic substitutes game where $M > D$ is similar to extending it in the strategic complements game discussed above. In Chapter 2, it has been proven that if and only if $\rho \leq \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$, a cutoff strategy can be used to solve the game (see Wang, 2016). Equivalently, it also implies a restriction on the variance to ensure that the game can be solved by a cutoff strategy:

$$\varsigma^2 \geq \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)} \quad \text{for } \rho > 0 \tag{3.8}$$

This inequality indicates that given $M > D$ and $\rho > 0$, there exists a lower bound of ς^2, which is denoted by $\tilde{\varsigma}^2$ and $\tilde{\varsigma}^2 = \frac{\rho^2(D-M)^2}{2\pi(1-\rho^2)}$. For variances below this lower bound, the game cannot be solved by a cutoff strategy. For $\rho \leq 0$, a cutoff strategy is still applicable for all $\varsigma^2 \in (0, +\infty)$ because for $\rho \leq 0$, the relationship $\rho \leq 0 \leq \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}} = \tilde{\rho}$ always holds for all $\varsigma^2 \in (0, +\infty)$.

The intuition of the existence of $\tilde{\varsigma}^2$ for $M > D$ and $\rho > 0$ is similar to the intuition for $D > M$ and $\rho < 0$. Recall that if $M > D$, for $\rho \leq 0$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} > 0$ for all $\varsigma \in (0, +\infty)$. For $\rho > 0$, there exists $\tilde{\varsigma}$ such that $\min \frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \frac{\rho(D-M)}{\tilde{\varsigma}\sqrt{2\pi(1-\rho^2)}} + 1 = 0\ \forall x^* \in \mathbb{R}$. Therefore, if $\rho > 0$, for $\varsigma \geq \tilde{\varsigma}$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0\ \forall x^* \in \mathbb{R}$. For $\varsigma < \tilde{\varsigma}$ if $\rho > 0$, this situation parallels that of $\rho > \tilde{\rho}$ given $\varsigma = \varsigma^*$ in the strategic substitutes game. In this situation, $\mathbb{E}\Pi(x^*, \varepsilon)$ is no longer monotonic with respect to ε, and for some $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*, \varepsilon) = 0$ has multiple (three) solutions of ε. One of the solutions gets the following property: a payoff shock that is below this threshold can make a player choose entry, which contradicts the definition of the cutoff strategy (see Appendix B in Chapter 2). Therefore, by assuming $\varsigma = \varsigma^*$, given $M > D$ and ρ, a player can use a cutoff strategy to play the game if and only if $\varsigma \in [\tilde{\varsigma}, +\infty)$ for $\rho > 0$ or $\varsigma \in (0, +\infty)$ for $\rho \leq 0$.

If $M > D$ and $\rho \leq 0$, games that can be solved by cutoff strategies exist for all $\varsigma^2 \in (0, +\infty)$. Since the equilibria of a game are solutions of the equation system composed of both players' best response functions, a small perturbation of the equation system will result in a nearby equilibrium. The most closets game is the game with ς and $\varsigma^* \to 0$. Again, the best response function is given by the following piecewise function:

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho}x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M \end{cases}$$

where $x^* \in \mathbb{R}$ and $\rho < 0$.

Although the expression of the best response function is the same as the one for $D > M$ and $\rho > 0$, the intuitions are not exactly the same. For the intuition of $g(x^*) \in [-M, -D]$, we have explained it in the previous part of this section when we analysed the case of $D > M$ and $\rho > 0$. Given that $M > D$, if the opponent i^* is expected to adopt a very high (low) strategy, it means player i expects that i^* is most likely to choose being inactive (active). In a strategic substitutes context, players always tend to mismatch their action strategies, and hence as the best response, i will adopt the lowest (highest) strategy that can be achieved to indicate the player's preference of being active (inactive). This lowest (highest) strategy is $-M$ ($-D$).

Because the game is symmetric, $g(x^*)$ and $g^*(x)$ are symmetrically located around the 45° line. The equilibria are the intersection points between $g(x^*)$ and $g^*(x)$. Specifically, for $M > 0 > D$, if $M > \rho D$ and $D > \rho M$, there are three equilibria (intersection points): $(-\frac{D}{\rho}, -D)$, $(-D, -\frac{D}{\rho})$ and $(0,0)$ (see Figure 11).[18]

[18]The intuition of the cutoff strategy equilibrium is that given $M > 0 > D$, a player can expect that the opponent either chooses being active or inactive. If a player expects the opponent to choose entry, the player will get payoff D if they also choose entry. Thus, the player will adopt a cutoff strategy $-D$. As the best response, the opponent will adopt a strategy $-\frac{D}{\rho}$.

In contrast, if a player expects the opponent to choose being inactive, then the player will get payoff

As ς and $\varsigma^* \to 0$, the payoff shocks ε and ε^* are always equal to 0. Therefore, given cutoff strategy equilibrium $(-\frac{D}{\rho}, -D)$, since $-\frac{D}{\rho} < 0$ and $-D > 0$, in this equilibrium, player i always chooses action 1 and player i^* always chooses action 0. Hence, the action strategy representation of this equilibrium is $(1,0)$. In the same way, the cutoff strategy equilibrium $(-D, -\frac{D}{\rho})$ indicates the action strategy $(0,1)$. Given cutoff strategy equilibrium $(0,0)$, the equilibrium belief $\sigma(0,0)$ is always equal to $\frac{D}{D-M}$ given any value of $\rho \in (-1,0)$. Thus, $\sigma(0,0)$ is always equal to the unconditional probability of player i^* choosing action 0. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of the game expressed in the form of action strategies are given by $(1,0)$, $(0,1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $M > 0 > D$. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of the perturbed games finally converge to the equilibria of the underlying complete information games.

For $M > 0 > D$ and $\rho < 0$, if $M < \rho D$ and $D < \rho M$, there are three cutoff strategy equilibria (intersection points): $(-\frac{M}{\rho}, -M)$, $(-M, -\frac{M}{\rho})$ and $(0,0)$ (see Figure 12).[19] [20] As ς and $\varsigma^* \to 0$, the payoff shocks ε and ε^* are always equal to 0. Therefore, given

M if they choose to enter. Thus, at least when $\varepsilon \geq -M$, the player will consider entry. However, $M > \rho D$ and hence $-M < -\rho D$, where $-\rho D$ is the entry threshold that opponent i^* expects player i to most likely adopt conditional on that i^* expects i will choose entry. Thus, if i gets a payoff shock ε such that $-M < \varepsilon < -\rho D$, the opponent expects that i will not choose entry but in fact i indeed chooses entry. Hence, a contradiction arises and i cannot adopt $-M$. Therefore, based on the opponent's belief that i will choose entry and accordingly i^* will adopt a strategy $-D$, i's best response will be $-\frac{D}{\rho}$.

If a player expects the opponent is indifferent to being active or being inactive, it indicates that irrespective of what value ε^* is, the expected payoff of entry for i^* is equal to 0. Therefore, player i^*'s cutoff strategy is equal to 0. Hence, given $M > 0 > D$, as a best response, player i will adopt a strategy 0. Therefore, another cutoff strategy equilibrium as ς and $\varsigma^* \to 0$ is $(0,0)$.

[19]For $M > 0 > D$ and $\rho < 0$, the following parameter specifications cannot be held: $M > \rho D$ and $D < \rho M$ or $M < \rho D$ and $D > \rho M$. It is because if $\rho = -1$, in either parameter specification, one inequality indicates $M + D > 0$, while the other one indicates $M + D < 0$. Obviously, the two inequalities cannot be held simultaneously.

[20]The intuitions of these cutoff strategy equilibria are similar to the previous case where $M > \rho D$ and $D > \rho M$. Given $M > 0 > D$, a player can expect that the opponent either chooses being active or inactive. If player i expects the opponent to choose being inactive, then the player will get payoff M if they choose entry. Thus, player i will adopt a cutoff strategy $-M$. As the best response, the opponent i^* will adopt a strategy $-\frac{M}{\rho}$.

Otherwise, if player i expects the opponent i^* to choose being active, the player will get payoff D if they choose to enter. Thus, at least $\varepsilon \geq -D$, i will consider entry. However, $D < \rho M$ and hence $-D > -\rho M$, where $-\rho M$ is the entry threshold that i^* expects i is most likely to adopt conditional on that i^* expects i will choose being inactive. Thus, if i gets a payoff shock ε such that $-D > \varepsilon > -\rho M$, the opponent will expect that i will choose being active but in fact i chooses being inactive. Hence, a contradiction arises and i cannot adopt $-D$. Therefore, based on the opponent's belief that i will choose being inactive and accordingly i^* will adopt a strategy $-M$, i's best response will be $-\frac{M}{\rho}$.

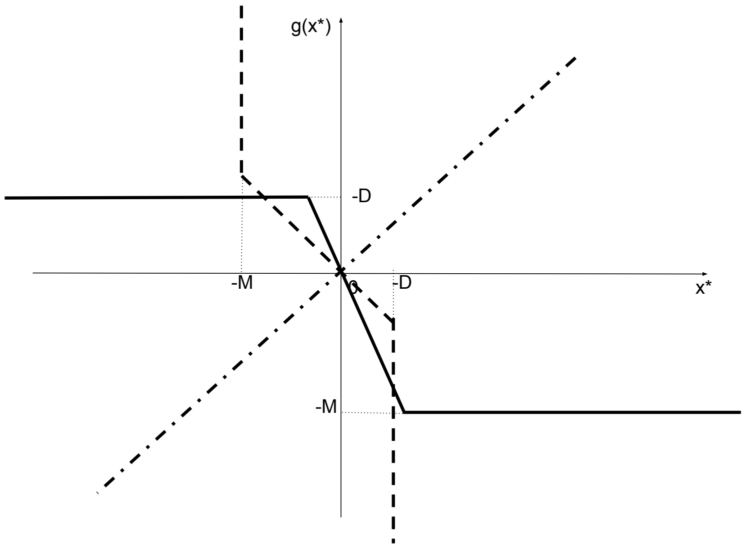


Figure 11: *The solid curve represents $g(x^*)$ as ς and $\varsigma^* \to 0$. The dashed curve represents $g^*(x)$ as ς and $\varsigma^* \to 0$. The dashed-dot line represents the 45° line. The intersection points between $g(x^*)$ and $g^*(x)$ are the equilibria of the game with ς and $\varsigma^* \to 0$. For $M > 0 > D$ and $\rho < 0$, if $M > \rho D$ and $D > \rho M$, there are three equilibria $(-\frac{D}{\rho}, -D)$, $(-D, -\frac{D}{\rho})$ and $(0,0)$.*

cutoff strategy equilibrium $(-\frac{M}{\rho}, -M)$, since $-\frac{M}{\rho} > 0$ and $-M < 0$, in this equilibrium player i always chooses action 0 and player i^* always chooses action 1. Hence, the action strategy representation of this equilibrium is $(0,1)$. In the same way, the cutoff strategy equilibrium $(-M, -\frac{M}{\rho})$ indicates the action strategy $(1,0)$. Given cutoff strategy equilibrium $(0,0)$, the equilibrium belief $\sigma(0,0)$ is equal to $\frac{D}{D-M}$ given any value of $\rho \in (-1,0)$. Hence, it equals the unconditional probability of i^* choosing action 0. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of this game are given by $(1,0)$, $(0,1)$ and $(\frac{D}{D-M}, \frac{D}{D-M})$. These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $M > 0 > D$.

It should be noted that in the case of $M > 0 > D$, irrespective of whether $M > \rho D$ and $D > \rho M$, or $M < \rho D$ and $D < \rho M$, given M, D, and ς and $\varsigma^* \to 0$, as ρ changes, the best response function changes and the cutoff strategy equilibria, except $(0,0)$, change as well. However, when we translate these cutoff strategies with respect to different values of ρ into action strategies, they indicate the same action strategies. For exam-

The intuition of cutoff strategy $(0,0)$ is the same as that in the previous case where $M > \rho D$ and $D > \rho M$.

ple, if $M > \rho D$ and $D > \rho M$, where $\rho < 0$, a cutoff strategy equilibrium is $(-\frac{D}{\rho}, -D)$. For different values of ρ, $(-\frac{D}{\rho}, -D)$ differs, but it always indicates the action strategy equilibrium $(1,0)$.

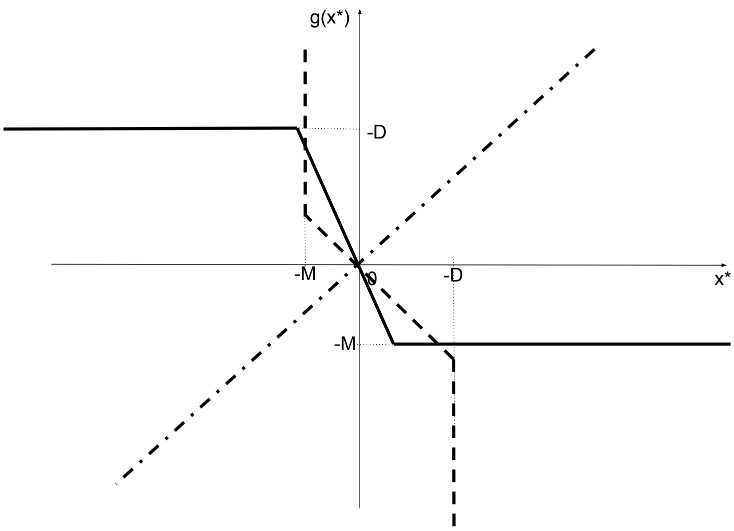
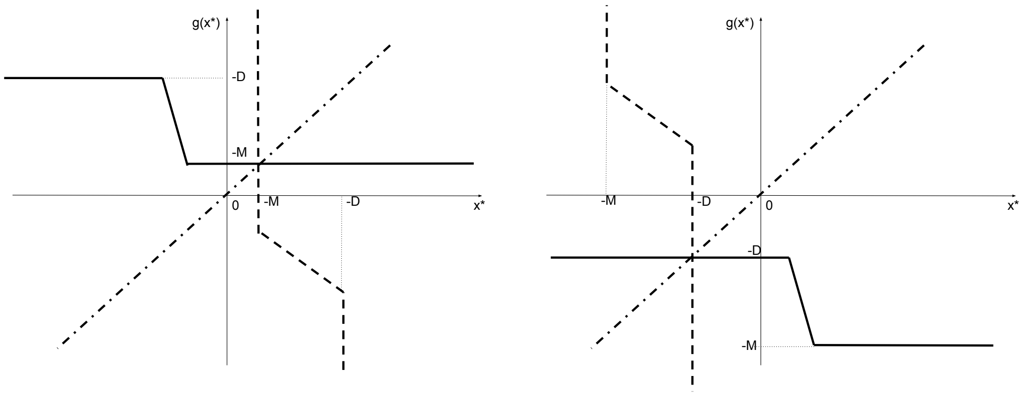


Figure 12: *The solid curve represents $g(x^*)$ as ς and $\varsigma^* \to 0$. The dashed curve represents $g^*(x)$ as ς and $\varsigma^* \to 0$. The dashed-dot line represents the $45°$ line. The intersection points between $g(x^*)$ and $g^*(x)$ are the equilibria of the game with ς and $\varsigma^* \to 0$. For $M > 0 > D$ and $\rho < 0$, if $M < \rho D$ and $D < \rho M$, there are three equilibria $(-\frac{M}{\rho}, -M)$, $(-M, -\frac{M}{\rho})$ and $(0,0)$.*

Similarly, if $0 > M > D$ or $M > D > 0$, the equilibrium cutoff strategies are $(-M, -M)$ and $(-D, -D)$, respectively, which imply the action strategy equilibria $(0,0)$ and $(1,1)$ (see Figures 13-1 and 13-2).[21] These equilibria are exactly equal to the equilibria of the games with $\varsigma = \varsigma^* = 0$ and $0 > M > D$ or with $\varsigma = \varsigma^* = 0$ and $M > D > 0$. Therefore, as ς and $\varsigma^* \to 0$, the equilibria of the perturbed games finally converge to the equilibria

[21] The intuitions of these cutoff strategy equilibria are as follows. Suppose $0 > M > D$. As ς and $\varsigma^* \to 0$, it is very likely that each player will choose being inactive. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -M$ since $M + \varepsilon > 0$ and M is the payoff the player can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, the cutoff strategy equilibrium $(-M, -M)$ exists in this situation.

Similarly, suppose $M > D > 0$. As ς and $\varsigma^* \to 0$, it is very likely that each player will choose being active. Conditional on this expectation, a player choosing entry must get a payoff shock $\varepsilon > -D$ since $D + \varepsilon > 0$ and D is the payoff they can obtain by choosing entry given this expectation. As the best response, the opponent will adopt the same cutoff strategy. Hence, in this situation, we have the cutoff strategy equilibrium $(-D, -D)$.



13-1: $0 > M > D$ **13-2:** $M > D > 0$

Figure 13: *The solid curve represents* $g(x^*)$ *as* ς *and* $\varsigma^* \to 0$. *The dashed curve represents* $g^*(x)$ *as* ς *and* $\varsigma^* \to 0$. *The dashed-dot line represents the* 45° *line. The intersection points between* $g(x^*)$ *and* $g^*(x)$ *are the equilibria of the game with* ς *and* $\varsigma^* \to 0$. *For* $\rho < 0$, *if* $0 > M > D$, *the equilibrium is* $(-M, -M)$, *and if* $M > D > 0$, *the equilibrium is* $(-D, -D)$.

of the underlying complete information games.

Therefore, Harsanyi's purification rationale can also be extended to perturbed games with non-positively dependent perturbation errors in a strategic substitutes context.

However, if $M > D$ and $\rho > 0$, the Bayesian games that can be solved by cutoff strategies do not exist for $\varsigma^2 \in (0, \tilde{\varsigma}^2)$. Therefore, the sequence of perturbed games that are supposed to converge to the complete information game does not exist. Since the 'approachability' requirement cannot be satisfied, Harsayni's purification rationale cannot be applied in this situation.

In conclusion, irrespective of whether the perturbation errors are positively dependent in strategic complements games or negatively dependent in strategic substitutes games, as the perturbation errors degenerate to zero, the Bayesian games that are supposed to converge to the underlying complete information game exist. Supposing the perturbed games exist as variances of the prior distribution tend to 0, given the same primitives except the correlation coefficient, the best response function differs with different values of the correlation coefficient because the slope changes. Except the case of $M > 0 > D$, the value of cutoff strategy equilibria does not depend on the correlation coefficient. For the case of $M > 0 > D$, except the cutoff strategy equilibrium $(0, 0)$, all cutoff strategy equilibria differs with different values of correlation coefficient. However, in any situation, given different values of correlation coefficient, if we translate these cutoff strategy equilibria into action strategy equilibria, they represent the same action strategy equilibria given the same payoffs M and D. These action strategy equilibria are equal to the corresponding Nash equilibria of the complete information game.

Finally, we formally describe the extension of Harsanyi's purification rationale to the normally distributed dependent perturbation-error situations in the following corollary:

Corollary 2: (An Extension of Purification Rationale): In a 2×2 symmetric entry game, described in Table 2, all equilibria are the limit of the pure-strategy Bayesian Nash equilibria of a sequence of perturbed games described in Table 1 as $(\varsigma,\varsigma^*)\to 0$, if and only if $D > M$ and $\rho \geq 0$ or $M > D$ and $\rho \leq 0$. $(\varepsilon,\varepsilon^*)$ follows a joint normal distribution $N(0,0,\varsigma^2,\varsigma^{*2},\rho)$ and the perturbed games are solved by using cutoff strategies, as defined in Section 3.2.

3.8 Summary

In this section, we give an organized summary of all main results and intuitions of the strategic complements game. The game can be described in two ways by six parameters: given $\varsigma=\varsigma^*\in(0,+\infty)$, $\tilde{\rho}$, $\hat{\rho}$ and $\bar{\rho}$, or given $\rho\in(-1,1)$, $\tilde{\varsigma}$, $\hat{\varsigma}$ and $\bar{\varsigma}$. $\tilde{\varsigma}$ exists if and only if $\rho<0$. $\bar{\rho}$ and $\bar{\varsigma}$ can exist if and only if $D>0>M$. The relationships between these parameters are as follows: $\hat{\rho}\geq\bar{\rho}$, $\hat{\rho}>\tilde{\rho}$ and $\bar{\rho}$ could be smaller than, equal to or greater than $\tilde{\rho}$; $\hat{\varsigma}\geq\bar{\varsigma}$, $\hat{\varsigma}>\tilde{\varsigma}$ and $\bar{\varsigma}$ could be smaller than, equal to or greater than $\tilde{\varsigma}$.

In Section 3.2, we derive $\tilde{\rho}$. Supposing the game is symmetric, if and only if $\rho\geq\tilde{\rho}$, the game can be solved by cutoff strategies. The intuition is that if $\rho<\tilde{\rho}$, the expected payoff $\mathbb{E}\Pi(x^*,\varepsilon)$ is no longer monotonic with respect to ε and for some $x^*\in\mathbb{R}$, $\mathbb{E}\Pi(x^*,\varepsilon)=0$ has three solutions of ε and at one of the solutions, $\frac{\partial\mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon}<0$, which contradicts the definition of the cutoff strategy that is used to solve the game. In fact, by assuming $\varsigma=\varsigma^*$, this result can be extended to asymmetric payoff settings, where each player has different D and M. In this situation, the game can be solved by a cutoff strategy if and only if $\rho\geq\max\{\tilde{\rho},\tilde{\rho}^*\}$, where $\tilde{\rho}^*=-\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D'-M')^2}}$ and D' and M'

are player i^*'s known payoffs.

In Section 3.3, we derive $\hat{\rho}$. For $\rho \geq \hat{\rho}$, player i's best response function is a contraction function. In this symmetric game, $\rho \geq \hat{\rho}$ is also the sufficient condition to ensure that the game is dominance solvable. This condition can be generalized to asymmetric payoff settings as described above. In this situation, the sufficient condition is generalized to $\rho \geq \max\{\hat{\rho}, \hat{\rho}^*\}$, where $\hat{\rho}^* = \frac{(D'-M')^2 - 2\pi\varsigma^2}{(D'-M')^2 + 2\pi\varsigma^2}$.

In the strategic complements game, $\bar{\rho}$ exists if and only if $D > 0 > M$. If $\bar{\rho} \geq \tilde{\rho}$, then $\bar{\rho}$ is the threshold to differentiate low and high uncertainty between players' private information. For $\rho < \bar{\rho}$, the uncertainty between players' private information is low, which means $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε; hence, players can gather enough information to assist them to match their action strategies. For $\hat{\rho} > \rho > \bar{\rho}$, the uncertainty between players' private information is high such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflects ε^* given ε; hence, each player has an unclear expectation of the other player's propensity of action choice. At $\rho = \bar{\rho}$, the uncertainty between players' private information is at the margin such that if the uncertainty between players' private information increases slightly, $f(\varepsilon^*|\varepsilon)$ will imprecisely reflect ε^* given ε. If $\bar{\rho}$ does not exist, then for all $\rho \in [\tilde{\rho}, \hat{\rho})$, the game has a unique equilibrium because the payoff shocks do not have any impact on players' ex ante expectations of the opponent's behaviour. The ex ante expectations are that either both players are more likely to choose being inactive $(0 > D > M)$ or more likely to choose being active $(D > M > 0)$. Accordingly, based on these expectations, the game exhibits a unique equilibrium $(-M, -M)$ with respect to the specification $0 > D > M$ or $(-D, -D)$ with respect to the specification $D > M > 0$. For $\rho \in [\hat{\rho}, 1)$, the best response functions are contraction function. In this situation, each player is more focused on the knowledge of himself and the opponent's information becomes less important in a player's decision making. This situation is close to that of an individual decision problem and hence the game exhibits a unique equilibrium.

In Section 3.6, we first derive $\tilde{\varsigma}$. It exists if and only if $\rho < 0$. Then, if and only if $\varsigma \geq \tilde{\varsigma}$ for $\rho < 0$ or $\varsigma > 0$ for $\rho \geq 0$, the game can be solved by a cutoff strategy. The intuition is that if $\rho \geq 0$, for all $\varsigma > 0$, $\mathbb{E}\Pi(x^*, \varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$. If $\rho < 0$, for $\varsigma < \tilde{\varsigma}$, $\mathbb{E}\Pi(x^*, \varepsilon)$ is no longer monotonic with respect to ε. In this situation, for some $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*, \varepsilon) = 0$ has three solutions of ε and at one of the solutions, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} < 0$. This result can also be extended to asymmetric payoff settings. For details, please refer to Section 3.6.

Next, we derive $\hat{\varsigma}$. For $\varsigma \geq \hat{\varsigma}$, player i's best response function is a contraction function. In this symmetric game, $\varsigma \geq \hat{\varsigma}$ is also the sufficient condition to ensure that the game is dominance solvable. This condition can be generalized to asymmetric payoff settings as well (see Section 3.6).

In the strategic complements game, $\bar{\varsigma}$ exists if and only if $D > 0 > M$. By assuming $\varsigma = \varsigma^*$, if $\bar{\varsigma} \geq \tilde{\varsigma}$, then $\bar{\varsigma}$ is the threshold to differentiate low and high uncertainties of a player's private information. For $\varsigma < \bar{\varsigma}$, the uncertainty of players' private information is low, which indicates that $f(\varepsilon^*|\varepsilon)$ can approximately reflect ε^* given ε; hence, players can gather enough information to assist them to match their action strategies. For $\varsigma > \bar{\varsigma}$, the uncertainty of players' private information is high such that $f(\varepsilon^*|\varepsilon)$ imprecisely reflect ε^* given ε; hence, each player has an unclear expectation of the opponent's propensity of action choice. At $\varsigma = \bar{\varsigma}$, the uncertainty of players' private information is at the margin such that if the uncertainty of players' private information increases slightly, $f(\varepsilon^*|\varepsilon)$ will imprecisely reflect ε^* given ε. If $\bar{\varsigma} < \tilde{\varsigma}$, then for $\varsigma \in [\tilde{\varsigma}, +\infty)$, the game has a unique equilibrium. It is because in this situation $f(\varepsilon^*|\varepsilon)$ is imprecise to reflect ε^* given ε.

The final and important result obtained is that based on this game, we extend Harsanyi's (1973) purification rationale to a dependent-perturbation error setting for both strategic complements and strategic substitutes games. In our game, the uncertainty of perturbed games vanishes as the variances of perturbation-error distribution degenerate to zero. By assuming that the perturbed games are solved by cutoff strategies and the perturbation errors follow the joint normal distribution as given in this paper, the purification rationale can be extended to perturbed games with positively dependent perturbation errors if the complete information game exhibits strategic complements or negatively dependent perturbation errors if the complete information game exhibits strategic substitutes. If we assume that the perturbation errors are negatively dependent if the complete information game exhibits strategic complements or positively dependent if the complete information game exhibits strategic substitutes, then the 'approachability' part of the purification rationale cannot be satisfied, and hence, we cannot extend the purification rationale to such situations.

3.9 Conclusion

In this chapter, we study a 2×2 strategic-complements entry game in which players' private information are correlated. The game is symmetrically specified. Given other parameters, there exists a critical value of correlation coefficient below which a cutoff strategy cannot be used to solve the game. We explore the comparative statics of the number of equilibria with respect to the correlation coefficient. As the correlation coefficient increases from the lowest feasible value, $\tilde{\rho}$, to 1, the sequence of the number of equilibria will be $3 \to 2 \to 1$ if $D+M \neq 0$ and $3 \to 1$ if $D+M=0$. Alternatively, under certain parameter specification, the game exhibits a unique equilibrium for all feasible values of the correlation coefficient. The comparative statics of equilibrium strategies with respect to the correlation coefficient and variances of the joint prior distribution depend on the sign of the equilibrium and the equilibrium's stability. For unstable equilibrium, increasing the payoff of entry makes a player less likely to choose entry, which contradicts our common sense.

We obtain the comparative statics of the number of equilibria with respect to variances of the joint prior distribution. It is a necessary and sufficient condition to differentiate unique equilibrium and multiple equilibria. This necessary and sufficient condition nests Morris and Shin's (2005) sufficient condition to ensure a unique equilibrium of the same game. Finally, if the correlation coefficient is negative for the strategic complements games or positive for the strategic substitutes games, there exists a critical value of variance. For variances below this critical value, a cutoff strategy cannot be used to solve the game. With specifying the process that the uncertainty of perturbed games vanishes by letting the variances of the perturbation-error distribution degenerate to zero, this result implies that Harsanyi's (1973) purification rationale cannot be applied for a game with dependent perturbation errors that follow a joint normal distribution with negative correlation coefficient for the strategic complements games or with positive correlation coefficient for the strategic substitutes games.

However, if the correlation coefficient is positive for the strategic complements games or negative for the strategic substitutes games, the purification rationale is still applicable. The Bayesian games that are supposed to converge to the underlying complete information game as perturbation errors degenerate to zero exist, and the pure-strategy Bayesian Nash equilibria of the perturbed games will converge to the corresponding Nash equilibrium of the complete information game during this process.

For future research, we can study the comparative statics of the number of equilibria with respect to variances in a strategic substitutes setting, i.e. $M > D$. In addition, the characterization of equilibria set in the strategic complements game is helpful for further econometric studies such as identification of such type of games. Besides, we will study whether and how the purification with dependent randomization rationale can be applied to more general games.

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Chapter 4

Bayesian Games with Rationally Inattentive Players

We study how scarcity of attention affects strategic choice behaviour in a 2-player incomplete information entry game. Scarcity of attention is a common psychological character among population (Kahnemann, 1973) and it is modelled by the rational inattention approach introduced by Sims (1998). In this game, players acquire information about their private payoff shocks at a cost, which follows a high-low binary distribution. We find that high information cost can generate multiple equilibria and the number of equilibria differs with respect to different ranges of information cost. The number of equilibria could be 1, 5 or 3. Increasing the information cost could encourage or discourage a player to choose entry in some equilibria. This depends on whether the prior probability of high payoff shocks is greater than a given threshold value. We also exhibit a necessary and sufficient condition of parameter specification such that with the same set of parameters satisfying this condition, both the rational inattention Bayesian game and a Bayesian quantal response equilibrium game where the observation errors are additive and follow a Type I extreme value distribution can have a common equilibrium.

4.1 Introduction

Usually, in a study of decision problems or game theoretical problems, a scenario where the economic agents can perform ideally in every aspect is assumed. At least, in terms of dealing with available information, we consider that the agents can pay full attention to any observation without a cost. However, according to a series of psychological studies, human being's ability to pay attention to their observations is found to be actually limited (Kahnemann, 1973). This inevitable feature differentiates real scenario from an ideally perfect economic agent scenario.

Economists have studied the influence of attention scarcity on individual decision problems (Woodford, 2008, 2009, Matějka and McKay, 2015). In this chapter, we study how attention scarcity affects economic agent's strategic choice behaviour in an incomplete information environment in a strategic substitutes context.

Attention scarcity is usually modelled by rational inattention that was first introduced by Sims (1998). In this chapter also, we adopt rational inattention to model attention scarcity for players in a game. By far, the literature most relevant to this chapter is Yang (2014). In an independent work, Yang studies a 2×2 symmetric strategic complements game. Players' payoffs are affected by a common payoff shock θ, which is a continuous random variable. However, each player cannot perfectly observe θ due to scarcity of attention, which is modelled by rational inattention. Players make decisions after observing θ. Yang's game is exhibited in Table 1.

Firm i \ Firm i^*	0	1
0	0 (Firm i), 0 (Firm i^*)	0 (Firm i), $\theta - r$ (Firm i^*)
1	$\theta - r$ (Firm i), 0 (Firm i^*)	θ (Firm i), θ (Firm i^*)

Table 1: *The coordination game in Yang (2014).* θ *is the fundamental state distributed according to a prior distribution* P *with support* $\theta \in \mathbb{R}$. *It is assumed that* P *is absolutely continuous with respect to Lebesgue measure over* $\mathbb{R}$. *Parameter* $r > 0$ *is the cost of miscoordination, which measures the degree of strategic complementarity.*

The game studied in this chapter is a 2×2 symmetric strategic substitutes game. Each player's payoff is affected by a private payoff shock subject to a binary distribution. The players cannot perfectly observe their own payoff shock due to attention scarcity. The game is shown in Table 2 in Section 4.3.

In this chapter, we study how information cost affects players' strategic choice behaviour. By far, no literature has studied this problem yet, including Yang (2014). Yang focuses on players' information acquisition behaviour. By making appropriate assumptions on payoff specifications, he excludes the possible existence of non-information acquisition equilibrium. In our model, we allow the existence of any type of equilibrium, and it is found that in certain asymmetric equilibria, one player acquires information to make choices, while the opponent does not acquire information to make choices.

In terms of the results, Yang (2014) finds that when the information cost is smaller than a threshold value, there are infinitely multiple equilibria. However, it is not clear whether there exists a unique equilibrium when the information cost is greater than or equal to the threshold value. In our model, if there exist multiple equilibria, they arise when the information cost is greater than or equal to a threshold value. If the information cost is smaller than the threshold value, there exists a unique equilibrium. In addition, under proper parameter specification, our game can always exhibit a unique equilibrium for any value of information cost.

Finally, because of our particular focus on information cost's impact, in this chapter, we also study comparative statics of information cost on players' equilibrium behaviour. There is no comparative statics work of information cost in Yang's paper.

This chapter is in line with the literature of entry games. Entry games have been widely studied in industrial organization literature, and it is the most typical form for modelling strategic substitutes behaviour. However, no literature exists that study how psychological factors affect firms' competition. There is a void related to this topic in the industrial organization literature, which this chapter is initially motivated to fill.

In the remaining parts of introduction, we first introduce the evolution of studies on rational inattention choice problems, and present a binary choice example to explain the rationale of modelling a rational inattention discrete choice problem. This binary choice model is a particular case of Matějka and McKay's (2015) general model.

4.1.1 Psychological Motivation of Random Choice

Many experiments suggest that individual choice is not deterministic (e.g. Loomes and Sugden, 1995). The experimental choice behaviour explored in recent literature matches with a much older literature in the branch of experimental psychology known as 'psychophysics', which showed that subjects cannot dependably make the same judgment about the relative strength of two similar but not identical stimuli when facing the same choice on repeated occasions. These experimental data are often explained by models that assume a random factor in the subject's perception of a constant stimulus; however, the randomness is clearly a feature of the subject's nervous system rather than of preferences (Woodford, 2008).

Therefore, in terms of these studies, random choice behaviour results precisely from the decision maker's difficulty in discriminating among different choice situations, a human cognitive limitation extensively documented by the psychophysicists. How does the difficulty in differentiating different choice situations arise? In the psychology literature, this cognitive limit can be accounted for by the scarcity of attention (e.g., Kahneman, 1973). In most economics literature, economic agents have full access to all available information and have no difficulty in paying full attention to all information available. The first attempt to incorporate attention scarcity into an economic model is Sims (1998). Sims' hypothesis of 'rational inattention' is a widely applied approach to model the limited attention and it motivates a very specific theory of the randomized choice (conditional on states) (see Sims, 1998, 2003, 2006; Woodford, 2008, 2009; Matějka and McKay, 2015).

Woodford (2008, 2009) and Matějka and McKay (2015) independently develop the randomized choice theory in the rational inattention framework. Their theories explain how scarcity of attention leads to decision maker's difficulty in discriminating among different choice situations that ultimately results in random choice behaviour. It bridges a fundamental psychological activity—scarcity of attention—and a human cognitive limitation—difficulty in clearly differentiating different choices—via an economic approach.

4.1.2 The Principle of Rational Inattention

To explain the rationale of rational inattention, we now specify a rational inattention binary choice model. This model adopts Sims' (1998, 2003, 2006) hypothesis of 'rational inattention': firms have precisely that information that is most valuable to them, given the decision problem that they face, subject to a constraint on the overall quantity of information that they access. In Woodford (2008, 2009) and Matějka and McKay (2015), rather than specifying a quantity constraint, it is assumed that there is a cost $\lambda > 0$ per unit of information obtained each period by the decision maker and that the total quantity of information obtained is optimal given this cost. This chapter, which studies rationally inattentive players' strategic choice behaviour in a Bayesian game, still follows Woodford (2008, 2009) and Matějka and McKay (2015)'s specification in which λ rather than the overall quantity of information they access, i.e. I, is given. This is because under this specification, the decision problem is a free-constraint utility-maximization problem. Hence, decision makers have complete freedom to allocate their attention, and could certainly allocate more attention to the information that is most relevant to their choice. It makes more sense to suppose that there is a given cost of additional attention, determined by the opportunity cost of reducing the attention paid to other matters, rather than a fixed bound on the attention that can be paid to the discrete choice problems.[1]

4.1.2.1 The Information Cost

Following the rational inattention literature, we shall suppose that any information about the current choice state can be available to the decision maker (DM, hereafter), as long as the quantity of information obtained by the firm without a thorough investigation is within a certain finite limit, representing the scarcity of attention, or information-processing capacity, that is used for this purpose. The quantity of information obtained by the DM is defined as in the information theory of Claude Shannon (1946). In this theory, the quantity of information contained in a given signal is measured by the reduction in the entropy of the DM's posterior over the state space, relative to the prior distribution. Let us suppose that the agents are interested simply

[1]If there exists a constraint k on the overall information an agent can access such that $I \leq k$, then λ becomes the shadow price which varies with respect to k, and thus, the choice probability also varies with respect to k. Therefore, how much information an agent can process to make a choice is ultimately determined by the constraint in this situation.

in information about the current value of the unknown state $\varepsilon \in \{u, d\}$, and the firm's prior is given by the distribution $p = Pr(u)$ and $1 - p = Pr(d)$, where $p \in (0, 1)$. Let $r^s = Pr(u|s)$ and $1 - r^s = Pr(d|s)$ be the firm's posterior, conditional upon observing a particular signal s. The entropy functions associated with a given binary distribution (a measure of the degree of uncertainty) are given by

$$H(p) = -p \ln p - (1-p) \ln(1-p)$$

and

$$H(r^s) = -r^s \ln r^s - (1-r^s) \ln(1-r^s)$$

and as a consequence, the entropy reduction when signal s is received is given by

$$I(s) = [-p \ln p - (1-p) \ln(1-p)] - [-r^s \ln r^s - (1-r^s) \ln(1-r^s)]$$

The average information revealed is therefore

$$I \equiv \mathbb{E}_s[I(s)] = H(p) - \mathbb{E}_s[I(r^s)]$$

where the expected value is taken over the set of signals that were possible ex ante, using the prior probabilities of that each of these signals would be observed. (The prior over s is the one implied by the DM's prior over $\varepsilon \in \{u, d\}$, together with the known statistical relation between ε and the signal s that will be obtained).

According to Matějka and McKay (2015), in a rational inattention binary choice problem, under an optimal information structure $q_s^\varepsilon = Pr(s|\varepsilon)$, the signal s will only contain two possible values, say $s \in \{0, 1\}$ (we will elaborate the notations and the result in Sections 4.1.2.2 and 4.1.2.3). Denote $q_s = pq_s^u + (1-p)q_s^d$. Hence, $q_0 + q_1 = 1$. The entropy functions associated with the information structure are given by

$$H(q_s) = -q_0 \ln q_0 - q_1 \ln q_1$$

and

$$H(q_s^\varepsilon) = -q_0^\varepsilon \ln q_0^\varepsilon - q_1^\varepsilon \ln q_1^\varepsilon$$

Therefore, the amount of information conveyed by the information structure q_s^ε is

$$I(q_s) = H(q_s) - \mathbb{E}_\varepsilon[H(q_s^\varepsilon)]$$
$$= -q_0 \ln q_0 - q_1 \ln q_1 + p[q_0^u \ln q_0^u + q_1^u \ln q_1^u] + (1-p)[q_0^d \ln q_0^d + q_1^d \ln q_1^d]$$

Matějka and McKay (2015) prove that according to the symmetry property of mutual information, $I(q_s) = I$. In the strategic choice problem studied in this chapter, we mainly use $I(q_s)$ to express the mutual information for analytical purpose.

4.1.2.2 Formulation of the Decision Problem

Now, suppose the DM is a firm. The firm faces the following choice problem of whether to enter a market (see Figure 1):

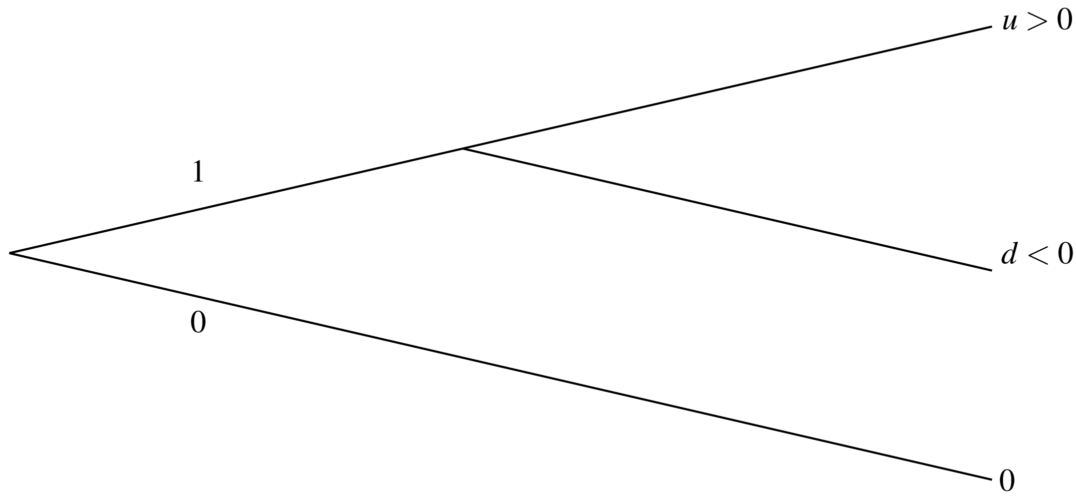


Figure 1: *A firm's entry decision problem*

If the firm chooses entry (1), its payoff will be either $u > 0$ or $d < 0$, and if it chooses inactive (0), its payoff will be 0. The state is drawn before the DM observes it and makes a choice. The problem is that the DM cannot perfectly observe the state because of scarcity of attention; therefore, it has to arrange to acquire a signal s at a cost by paying $\lambda I(s)$ in order to obtain the information that is most relevant to this choice problem. Then, this knowledge about $\varepsilon \in \{u, d\}$ is updated via the posterior r^s and $1 - r^s$. Given the posterior, the DM chooses the action with the highest expected payoff. Let $V(r^s) = \max\{\bar{\mu}(s), 0\}$, where $\bar{\mu}(s) = r^s u + (1-r^s)d$, denote the maximal payoff. The theory of rational inattention reveals the mechanism behind the above (psychological) decision process in a utility maximization framework. It aggregates all possibilities of such a process and posits that both the design of this signal (the set of possible values of s, and the probability that each will be observed conditional upon any given state ε, i.e. $q_s^\varepsilon = Pr(s|\varepsilon)$, also referred to as information structure), and the decision about whether to choose action 1 or 0 conditional upon the signal observed, will be obtained, in the sense of maximizing

$$\mathbb{E}[V(r^s)] - \lambda I$$

Therefore, this solution concept (the information strategy and the ultimate choice behaviour as the essence of this psychological process) can account for when a payoff shock is drawn, what aspects of information about the payoff shock a rationally inattentive agent can and should pay attention to and what choice should be made contingent on this information. The expectation operator sums over possible states $\varepsilon \in \{u, d\}$, possible signals s and possible action choice decisions under the firm's prior probability distribution, which is that payoff shock u happens with probability p and payoff shock d happens with probability $1-p$. $\lambda > 0$ is the cost per unit of information of being more informed when making the action choice decision. This design problem is solved from an ex ante perspective: players must decide how to allocate their attention, which determines what kind of signal players will observe under various circumstances, before learning anything about the current state.

4.1.2.3 Main Results of Rational Inattention Binary Choice Problem

Here, we summarize Matějka and McKay (2015)'s main results of the decision problem. They are expressed in the binary choice context as shown in Figure 1.

The first result of this binary choice problem is that under an optimal information structure q_s^ε, the signal s will take only two possible values, and can be interpreted as a '0-1' signal as to whether under the current state, the firm should enter the market. Since the only use of the signal is to decide whether to enter the market, more elaborate signals (e.g. a third signal) will convey redundant information. In addition, since more informative signal would have a greater cost without improving the quality of the decision, it would be inefficient. Correspondingly, an optimal action decision will necessarily be a deterministic function of the signal (i.e. entry is always chosen if and only if the signal is 1). Therefore, the information structure q_s^ε is identical to the conditional probability of action given the state (see Woodford, 2008, 2009; Matějka and McKay, 2015).

Second, about the choice behaviour, we find that given other primitives, there exists a boundary value of λ denoted by $\bar{\lambda}_{DP}$, which can be equal to $+\infty$. For λ smaller than or equal to $\bar{\lambda}_{DP}$, i.e. the cost of acquiring information is not high for the DM, and its choice will be made by acquiring information. For λ greater than $\bar{\lambda}_{DP}$, since acquiring information is too costly for the DM, it will solely rely on its prior knowledge to make a choice rather than by acquiring information. Therefore, $\bar{\lambda}_{DP}$ is a boundary point differentiating two distinct choice behaviours, and a DM's evaluation of $\bar{\lambda}_{DP}$ depends on its preference. Proposition 1 gives us an accurate description of rationally inattentive agents' choice behaviour.

Define $f_{DP} := w(0,\lambda)$, $F_{DP}(\lambda) := S(\lambda,0)$, $G_{DP}(\lambda) := T(\lambda,0)$ and $\mu_{DP} := e(0)$, where the expressions of $w(x,\lambda)$, $S(\lambda,x)$, $T(\lambda,x)$ and $e(x)$ are given in Appendix A of this chapter. The analytical expression of $\bar{\lambda}_{DP}$ is given by

$$\bar{\lambda}_{DP} = \begin{cases} F_{DP}^{-1}(1) & \text{if } \mu_{DP} < 0 \\ +\infty & \text{if } \mu_{DP} = 0 \\ G_{DP}^{-1}(1) & \text{if } \mu_{DP} > 0 \end{cases}$$

where $\mu_{DP} = pu + (1-p)d$. In addition, define the following function:

$$1\{P\} = \begin{cases} 1 & \text{if P is true} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, Proposition 1 is stated as follows.

Proposition 1 (A Binary-State Binary Choice Model, Matějka and McKay (2015)): Given that $u > 0 > d$ and $\lambda > 0$, the unconditional choice probability is given by

$$\forall\lambda \in (0, \bar{\lambda}_{DP}],\ q = f_{DP} = \frac{p}{1-\exp(\frac{d}{\lambda})} + \frac{1-p}{1-\exp(\frac{u}{\lambda})};$$

$$\forall\lambda \in (\bar{\lambda}_{DP}, +\infty),\ q = 0 \times 1\{\mu_{DP} < 0\} + 1 \times 1\{\mu_{DP} > 0\}.$$

Alternatively, the choice probability can be written in a simplified form as

$$q = 0 \times 1\{f_{DP} < 0\} + f_{DP} \times 1\{0 \leq f_{DP} \leq 1\} + 1 \times 1\{f_{DP} > 1\}$$

Proof: see Appendix. ■

In Proposition 1, the exponential functions in f_{DP} come from the entropy function. At $\bar{\lambda}_{DP}$, the average information revealed by signal I equals 0. Therefore, the agent's action is deterministic and is the interior solution at $\lambda = \bar{\lambda}_{DP}$. If $\lambda > \bar{\lambda}_{DP}$, generally, f_{DP} will be smaller than 0 or greater than 1. Hence, the agent has to choose a boundary solution of q to maximize their utility. The boundary solution is equal to the interior solution at $\lambda = \bar{\lambda}_{DP}$. It is $q = 1$ if $\mu_{DP} > 0$ and $q = 0$ if $\mu_{DP} < 0$.

4.1.3 The Relationship of Rational Inattention Discrete Choice Model and Statistical Decision Problems

Rational inattention discrete choice problems and statistical decision problems (SDPs) are different problems, although the components of these problems (e.g. utility functions, signal, state, prior) are the same. The differences are reflected by two points: 1) an SDP is given the joint distribution of signal and state and the prior distribution of state to derive the optimal decision rule, while a rational inattention discrete choice problem is given the optimal decision rule and the prior distribution of state to obtain the optimal information structure (the joint distribution of signal and state); 2) in rational inattention binary choice problems, the optimal strategy should contain two signals indicating two different action choices, otherwise more elaborate signals are redundant and hence undesirable. The same rationale applies to rational inattention multiple choice problems. In SDPs, the dimension of signal space is not necessarily equal to the dimension of action space. For binary action problems, it allows more than two signals. However, at the optimum, agents can always divide the signal set into two subsets with respect to the two different action choices.

Therefore, rational inattention discrete choice problems and SDPs can be regarded as the same question but viewed from two different perspectives. A rational inattention discrete choice problem is the 'reinterpretation' of an SDP.

4.2 Main Results of the Rational Inattention Bayesian Game

In this chapter, we are interested in agents' strategic choice behaviour if scarcity of attention exists. We examine this subject in a 2×2 incomplete-information entry game and focus on the pure-strategy Nash equilibrium. In industrial organization literature, entry games are typically of strategic substitutes, so is our game. By extending the rational inattention approach into a 2×2 strategic-substitutes incomplete information game, we establish a model that allows us to study how information cost affects an agent's strategic choice behaviour. Here, we present a summary of the main results of this chapter.

First, assuming both players' information costs are identical, we find that there exists a critical value of information cost.[2] If the given information costs of both players are below this value, the game is a Bayesian game, in which the payoff of being active is the production profit plus the private payoff shock. If the given information costs of both players are above or equal to this value, the game becomes a complete information game, in which the payoff of being active is the production profit plus the mean of the distribution of payoff shocks, and players' best responses are made without acquiring information given any strategy of the opponent.

Next, by studying symmetric games, we find that scarcity of attention can generate multiple equilibria, and that different values of information costs lead to different numbers of equilibria.[3] A general rule is that jointly increasing both players' information costs first increases and then decreases the number of equilibria. Specifically, in the

[2] In this chapter, for simplicity, we refer to information cost to indicate the unit information cost.

[3] Symmetric game is defined such that the parameter specification of both players are identical. Particularly, the information costs of both players are identical.

symmetric rational inattention Bayesian games, we find that given other primitives, by jointly raising the information costs from 0 to $+\infty$, the number of equilibria appears in the following sequence: $1 \to 3 \to 5 \to 3$ if multiple equilibria can arise. Alternatively, there always exists a unique equilibrium for any value of the information cost. Besides, we find that in any multiplicity situation, there always exists one pair of asymmetric equilibria in which at least one player plays without acquiring information and relies on their prior knowledge. These results about the game's equilibria are mainly caused by the concavity–convexity property of the part of the second iteration of the best response functions in which both players play the game by acquiring information. Furthermore, the concavity–convexity property is ultimately induced by the structure of entropy functions.[4] Thus, because information processing capacity is modelled by the reduction in the entropy of players' posterior over the state space relative to the prior distribution, there are at most five ways to play the game. For any result, it is either that both players make choices by acquiring information or that one player makes choices by acquiring information and the other player makes choices without acquiring information and only relying on prior knowledge.

For comparative statics of equilibrium strategies, we find that in the symmetric equilibrium and outer asymmetric equilibrium, any improvement in players' expected payoff of entry can increase the probability of entry.[5] If we jointly increase both players' information costs, its impact depends on the relative magnitude between the prior probability of high payoff shock and a threshold value. If the prior probability of high payoff shock is higher (or lower) than the threshold value, increasing the information cost will increase (or decrease) the probability of entry. If the prior probability of high payoff shock equals the threshold value, increasing the information cost does not have any impact on the probability of entry. There is no conclusive result about comparative statics of inner asymmetric equilibrium without particular parameter specification. Finally, in any equilibrium, if we change only one player's information cost, its impacts on both players' equilibrium strategies are not clear without particular parameter specification, but it is found that its impact on one player's strategy is always opposite to its impact on the opponent's strategy.

[4]The concavity–convexity property means that as the value of a player's strategy increases, the part of the second iteration of the best response functions in which both players acquire information exhibits first concavity and then convexity. For details, please refer to Section 4.7.

[5]In this game, there are three types of equilibrium: symmetric equilibrium, outer asymmetric equilibrium and inner asymmetric equilibrium. They are named according to their location at the best response functions.

We also study how information cost affects a player's expected payoff. A player's information cost does not have any impact on the player's expected payoff, but the opponent's information cost can affect the player's expected payoff through the player's belief towards the opponent's behaviour. Except particular parameter specification, there is no conclusive result about at what value of opponent's information cost, the player's expected payoff reaches its highest value.

Finally, we study a game in which the players observe their private payoff shocks with an additive noise that follows Type I extreme value distribution. The solution concept is therefore (Bayesian) Quantal Response Equilibria (QRE). The similar-looking strategic choice models motivate us to further consider under what conditions the two games can be identical. It is found that there exists a specific set of parameter specification under which both games have a common equilibrium $(\frac{1}{2}, \frac{1}{2})$. Except this situation, the two games will not coincide.

The rest of this chapter is organized as follows. Section 4.3 describes the model. Section 4.4 analyses three particular cases of the game with some special value of information cost. Section 4.5 analyses the general case. Section 4.6 studies the impact of information cost on players' best responses. Section 4.7 presents the equilibria set of the game. Section 4.8 studies the impact of information cost on players' equilibrium strategies. Section 4.9 studies how information cost affects a player's expected payoff of entry. Section 4.10 compares a Bayesian quantal response equilibrium game and the rational inattention Bayesian game to determine under what conditions the two types of games can coincide or be identical. Section 4.11 concludes this chapter.

4.3 The Model

We study rationally inattentive players' strategic choice behaviour in the following Bayesian game. Two firms decide whether to enter a market. If a firm enters, it can either get a monopoly profit or a duopoly profit, plus an exogenous payoff shock drawn by nature at the beginning of the time. Each player's payoff shock is independent from each other, and it is the private information for each player. Each firm can only know its own payoff shock. However, although a firm has full access to its own payoff shocks, due to scarcity of attention, it is only able to possess partial information about some aspects of the payoff shock, and this information is obtained by acquiring a signal at a cost. What kind of signal a player (firm) will observe is unconsciously designed in their mind and the player only acquires information about their own shock. A player cannot acquire any information about the opponent's private payoff shock because it is the opponent's private information and it is independent from the player's own payoff shock. Both players make decisions according to observed private signals.

We define our game as follows. The firms' strategic entry behaviour is characterized by the 2×2 Bayesian game with payoffs shown in Table 2.

		Firm i^*	
		0	1
Firm i	0	0 0	$M+\varepsilon^*$ 0
	1	0 $M+\varepsilon$	$D+\varepsilon^*$ $D+\varepsilon$

Table 2: *The Strategic Entry Game*

We use $*$ to denote all variables of the opponent. In this chapter, except in some particular situations, for simplicity we do not describe i^*'s specification separately and its specifications are correspondingly symmetric with i's. Here, $\varepsilon \in \{u, d\}$ is i's private payoff shock, and $\varepsilon^* \in \{u, d\}$ is opponent i^*'s payoff shock. Assume $u > d$. The shocks ε and ε^* have the same distribution, namely $p = Pr(u)$ and $1 - p = Pr(d)$, and we assume that ε and ε^* are independent and $p \in (0, 1)$. The payoff shocks are private information for each player. Nature draws ε and ε^*, respectively, for the players at the beginning of the time, and since they cannot perfectly observe ε or ε^*, they have to acquire a signal at cost, which can reveal some aspects of ε or ε^*. To acquire at a cost what type of signals is designed by each player, the efficient signal should undoubtedly be the most relevant to the player's choice decision. We assume that each player's signal is conditionally independent of the opponent's signal given the player's own payoff shock. The true value of the payoff shock can only be known after they make their choices. The action set of player i is $A = \{0, 1\}$, where 1 stands for 'entry' (being active) and 0 stands for 'staying outside' (being inactive). If both firms enter, they will engage in a Cournot competition and each firm gets a payoff D that is strictly lower than the monopoly profit M.

The game exhibits strategic substitutes because when the probability that the opponent chooses entry increases, the marginal expected payoff gain of being active over inactive decreases.[6] Therefore, the opponent's more aggressive behaviour imposes a negative externality on a player's marginal payoff.

Now let us turn to discuss what kind of signal they will acquire. As stated above, before selecting an action, both players can privately acquire information about their own payoff shocks at a cost. The root of the inability to grasp the full knowledge about ε or ε^* is still scarcity of attention, and the information acquisition process in this game is thus modelled in the rational inattention framework. The signal the players intend to acquire is characterized by the set of realizations of player i's signal, $S \in \mathbb{R}$, and the information structure $q_s^\varepsilon = Pr(s|\varepsilon)$, which is the probability measure of that signal conditional on state ε. In addition, we denote $\sigma(s)$ the probability of choosing action 1 upon observing $s \in S$, i.e. $\sigma : S \to [0,1]$. Then, player i's strategy can be characterized by a triplet $(S, q_s^\varepsilon, \sigma)$.

According to the results of the rational inattention binary choice problem described in the last section, rationally inattentive agent's optimal information structure just contains two signal realizations because essentially, acquiring more elaborate signal not only is costly but also provides no extra benefit to the player, since the player must always take either action 1 or action 0. Hence, without loss of generality, i's strategy can be represented by the following function.

$$q^\varepsilon = Pr(a = 1|\varepsilon)$$

That is, when i's private payoff shock is ε, player i receives signal 1 (signal 0) with probability q^ε (1-q^ε) and then takes action 1 (action 0) as instructed.

Given i^*'s strategy q^{ε^*}, player i's expected payoff of playing q^ε is

[6]The expected payoff could be the one in which a player makes decisions by acquiring information of contingent payoff shocks or the one in which a player makes decisions based only on prior knowledge. We will present the formulation of expected payoffs in detail in the following parts of this chapter.

$$U(q^{\varepsilon},q^{\varepsilon^*}) = pq^u[(1-q^*)(M+u)+q^*(D+u)]+(1-p)q^d[(1-q^*)(M+d)+q^*(D+d)] \tag{4.1}$$

where $q^* = pq^{u^*} + (1-p)q^{d^*}$. Equation (4.1) is directly derived from Table 2. As a standard setup in rational inattention literature, the information cost associated with a strategy q^{ε} is given by $\lambda I(q^{\varepsilon})$, where $I(q^{\varepsilon})$ is the amount of information conveyed by q^{ε}, and $\lambda > 0$ is a scaling parameter that controls the difficulty of acquiring information. Specifically,

$$I(q^{\varepsilon}) = H(q) - \mathbb{E}_{\varepsilon}[H(q^{\varepsilon})]$$
$$= -(1-q)\ln(1-q) - q\ln q + p[(1-q^u)\ln(1-q^u) + q^u \ln q^u] + (1-p)[(1-q^d)\ln(1-q^d) + q^d \ln q^d] \tag{4.2}$$

where $q = pq^u + (1-p)q^d$. According to the symmetry property of mutual information, $I(q^{\varepsilon}) = I = \mathbb{E}_s[I(s)]$, which has been discussed in the rational inattention decision problem in Section 4.1.2.1. Hence, $I(q^{\varepsilon})$ reflects the average information revealed by the designed signals, and $\lambda I(q^{\varepsilon})$ is the average cost of acquiring information.

Taking information cost into account, i's and i^*'s overall expected payoff in terms of q^{ε} and q^{ε^*} are

$$V(q^{\varepsilon},q^{\varepsilon^*}) = U(q^{\varepsilon},q^{\varepsilon^*}) - \lambda I(q^{\varepsilon}) \tag{4.3}$$

and

$$V(q^{\varepsilon^*},q^{\varepsilon}) = U(q^{\varepsilon^*},q^{\varepsilon}) - \lambda^* I(q^{\varepsilon^*}) \tag{4.4}$$

For simplicity, in the rest of this chapter, we abstract from the story of market entry and deal with the problem simply as a 2-player game with preferences (4.3) and (4.4) and strategy profile $(q^{\varepsilon},q^{\varepsilon^*})$. Since q^{ε} and q^{ε^*} are probabilities, we can further restrict the players' strategies to $q^{\varepsilon} \in [0,1]$ and $q^{\varepsilon^*} \in [0,1]$. We write $G(M,D,\lambda,\lambda^*)$ for the game with monopoly profit M, duopoly profit D, i's information cost λ and i^*'s information cost λ^*. Furthermore, to make this incomplete information always interesting, we assume the following:

Assumption 1: Given that $M > D$, the random payoff shocks u and d satisfy $D+u > 0 > M+d$.

Under Assumption 1, random payoff shocks dominates deterministic payoffs in players' decision making. If $\lambda = \lambda^* = 0$, the ex ante choice probability of each action totally depends on the prior distribution, and hence, there exists a unique equilibrium. Therefore, Assumption 1 makes the underlying game a good benchmark to compare games with scarcity of attention.[7]

4.4 Three Particular Cases

Before we analyse the general game, let us begin from three particular cases: (1) $\lambda = \lambda^* = 0$, (2) $\lambda = \lambda^* = +\infty$ and (3) $\lambda = 0$ and $\lambda^* = +\infty$. These particular cases provide useful benchmarks for further analysis. Remember that we only consider pure-strategy equilibrium.

In Case 1, signals are free, and hence players can possess full information about private payoff shocks, and this game then comes back to a typical incomplete information game. According to Assumption 1, in such games, there exists a unique Bayesian Nash equilibrium, $(q,q^*) = (p,p)$. Under Assumption 1, given the payoff shock u, a player will certainly choose action 1 and hence $q^u = 1$. Given the payoff shock d, a player will certainly choose action 0 and hence $q^d = 0$. Therefore, the unconditional probability of choosing action 1 is $q = pq^u + (1-p)q^d$ and in this situation, the unconditional choice probability is a sufficient statistic to describe the equilibrium.

In Case 2, when $\lambda = \lambda^* = +\infty$, any signal is too costly to acquire. Then, a Bayesian player will make a choice by simply comparing the ex ante expected payoffs for each action, which are the expected payoff before payoff shocks are drawn by nature and are simply formulated based on prior knowledge as well as the opponent's strategy. Case 2 can be further analysed in three specific situations:

[7] Underlying games refer to games with $\lambda = \lambda^* = 0$.

1) If $D+pu+(1-p)d>0$, then there exists a unique equilibrium $(q,q^*)=(1,1)$, since under this condition, for all $q^*\in[0,1]$, $(1-q^*)M+q^*D+pu+(1-p)d>0$;

2) If $M+pu+(1-p)d<0$, then there exists a unique equilibrium $(q,q^*)=(0,0)$, since under this condition, for all $q^*\in[0,1]$, $(1-q^*)M+q^*D+pu+(1-p)d<0$;

3) If $M+pu+(1-p)d>0$ and $D+pu+(1-p)d<0$, then generically there exist three equilibria $(q,q^*)=(1,0)$, $(0,1)$, and $(\frac{M+pu+(1-p)d}{M-D},\frac{M+pu+(1-p)d}{M-D})$, where the last equilibrium is essentially a mixed strategy and strictly between 0 and 1. Under these two conditions, $(1-q^*)M+q^*D+pu+(1-p)d$ is not necessarily always greater than or lower than 0 for all $q^*\in[0,1]$. Therefore, in this situation, there exist two (pure strategy) Nash equilibria. Since in this situation players just rely on prior information and do not consider payoff shocks, this game is equivalent to a complete information game with following payoff specifications (see Table 3).

Firm i \ Firm i^*	0	1
0	0, 0	0, $M+pu+(1-p)d$
1	$M+pu+(1-p)d$, 0	$D+pu+(1-p)d$, $D+pu+(1-p)d$

Table 3: *The Strategic Entry Game when* $\lambda=\lambda^*=+\infty$

The three generical equilibria in 3) are obtained in this way and in fact, 1) and 2) can also be analysed in this complete-information-game framework. The equilibria in 1) and 2) are dominant strategies.

Finally in Case 3, $\lambda=0$ and $\lambda^*=+\infty$. In this situation, player i possesses full information about their private payoff shock and knows nothing about player i^*'s private payoff shock, while player i^*, as in Case 2, only relies on prior information and knows nothing about player i's contingent payoff shock either. Then, for player i, the equilibrium strategy is p for the probability of choosing action 1, and hence, $1-p$ for the probability of choosing action 0. Correspondingly, player i^*'s equilibrium strategy is $a^*=1\{(1-p)(M+d)+p(D+u)>0\}$. Therefore, in this situation, there still exists a unique equilibrium. Player i's strategic choice, where $\lambda = 0$, depends on contingent payoff shocks, and player i^*'s decision, where $\lambda^* = +\infty$, depends only on prior information. Note that 1) in all three particular cases, there is no information acquisition, either because players have perfect observation or because signals are too costly to acquire, and 2) the choice behaviour where a player does not acquire information does not just belong to λ (or λ^*) $\to +\infty$. We will show that given the opponent's strategy $q^* \in [0,1]$, a player will not make a choice by acquiring information when λ is greater than a certain value, and in this situation, the choice behaviour is just the one we have presented for $\lambda \to +\infty$.

4.5 General Case

Now, we deal with the general game. A Nash equilibrium of game $G(M,D,\lambda,\lambda^*)$ is a strategy profile $(q^\varepsilon, q^{\varepsilon^*})$ that solves the following problem:

G-1:

$$\max_{q^\varepsilon} U(q^\varepsilon, q^{\varepsilon^*}) - \lambda I(q^\varepsilon)$$
$$s.t. \quad 0 \le q^\varepsilon \le 1$$

$$\max_{q^{\varepsilon^*}} U(q^{\varepsilon^*}, q^\varepsilon) - \lambda^* I(q^{\varepsilon^*})$$
$$s.t. \quad 0 \le q^{\varepsilon^*} \le 1$$

where ε (or ε^*) $\in \{u,d\}$.

By solving G-1, we obtain two equations that contain q^ε and q^{ε^*}. They are

$$\begin{cases} q^\varepsilon = \dfrac{q\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]}{q\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]+(1-q)} \\ q^{\varepsilon^*} = \dfrac{q^*\exp[\frac{(1-q)M+qD+\varepsilon^*}{\lambda^*}]}{q^*\exp[\frac{(1-q)M+qD+\varepsilon^*}{\lambda^*}]+(1-q^*)} \end{cases} \tag{4.5}$$

where ε (or ε^*) $\in \{u,d\}$, and $q = pq^u + (1-p)q^d$ and $q^* = pq^{u^*} + (1-p)q^{d^*}$. According to equation group (4.5), and the relation between unconditional probability and corresponding conditional probabilities, we can see that the strategy profile $(q^{\varepsilon}, q^{\varepsilon^*})$ can be equivalently represented by the unconditional probabilities (q, q^*), since one $(q^{\varepsilon}, q^{\varepsilon^*})$, where ε (or ε^*) $\in \{u, d\}$, corresponds to a specific (q, q^*), and vice versa.

To obtain (q, q^*), rather than using the relation $q = pq^u + (1-p)q^d$ or $q^* = pq^{u^*} + (1-p)q^{d^*}$, we substitute the solutions in (4.5) back to the objective functions in G-1, and reformulate the game as[8]:

G-2:

$$\max_q \lambda\{p\ln[q\exp(\frac{(1-q^*)M + q^*D + u}{\lambda}) + (1-q)] + (1-p)\ln[q\exp(\frac{(1-q^*)M + q^*D + d}{\lambda}) + (1-q)]\}$$
$$s.t. \quad 0 \le q \le 1$$

$$\max_{q^*} \lambda^*\{p\ln[q^*\exp(\frac{(1-q)M + qD + u}{\lambda^*}) + (1-q^*)] + (1-p)\ln[q^*\exp(\frac{(1-q)M + qD + d}{\lambda^*}) + (1-q^*)]\}$$
$$s.t. \quad 0 \le q^* \le 1$$

where q and q^* are strategic choices. We define $f(q^*, \lambda) := w((1-q^*)M + q^*D, \lambda)$. The function $w(x, \lambda)$ is the same one in the decision problem. If interior solutions exist, we obtain the following best response functions:

$$q = f(q^*, \lambda) = \frac{p}{1 - \exp(\frac{(1-q^*)M + q^*D + d}{\lambda})} + \frac{1-p}{1 - \exp(\frac{(1-q^*)M + q^*D + u}{\lambda})} \qquad (4.6)$$

for all $q^* \in [0,1]$, and $q^* = f(q, \lambda^*)$ for all $q \in [0,1]$. We can prove that, given all primitives, the interior solution q from G-2 and corresponding q^{ε} from G-1 satisfy $q = pq^u + (1-p)q^d$, so satisfy $q^* = pq^{u^*} + (1-p)q^{d^*}$. Then, naturally, the question is under what conditions, q and q^* are interior solutions of G-2. To answer this question, we find that, given a $q^* \in [0,1]$, to ensure that the best response q is an interior solution, i.e. $q = f(q^*, \lambda) \in [0,1]$, it is equivalent to letting all parameters satisfy the following conditions. Defining $F(\lambda, q^*) := S(\lambda, (1-q^*)M + q^*D)$ and

[8] If we directly solve $q = pq^u + (1-p)q^d$ to obtain q given any $q^* \in [0,1]$, there are three solutions: 0, 1 and the equation (4.6). However, 0 and 1 cannot maximize player i's utility if we substitute them back to the objective function in G-2.

$G(\lambda, q^*) := T(\lambda, (1-q^*)M + q^*D)$, where $S(\lambda, x)$ and $T(\lambda, x)$ are the same in the decision problem, we have

$$\begin{cases} F(\lambda, q^*) = p\exp[\frac{(1-q^*)M+q^*D+u}{\lambda}] + (1-p)\exp[\frac{(1-q^*)M+q^*D+d}{\lambda}] \geq 1 \\ G(\lambda, q^*) = p\exp[-\frac{(1-q^*)M+q^*D+u}{\lambda}] + (1-p)\exp[-\frac{(1-q^*)M+q^*D+d}{\lambda}] \geq 1 \end{cases} \tag{4.7}$$

The first equation in (4.7) is derived from $f(q^*, \lambda) \geq 0$ and the second equation from $f(q^*, \lambda) \leq 1$.

We are interested in given M, D, and the prior distribution, the range of λ that ensures the existence of interior solutions of G-2. In this chapter, for a bivariate function $y = f(x, v)$, its inverse function with respect to x is expressed as $x = f^{-1}(y; v)$. We define $\mu(q^*) := e((1-q^*)M + q^*D) = (1-q^*)M + q^*D + pu + (1-p)d$. It is found that there indeed exists a $\lambda = \bar{\lambda}_{q^*}$ such that for all $\lambda \leq \bar{\lambda}_{q^*}$, the interior solutions of G-2 exist. It is given by

$$\bar{\lambda}_{q^*} = \begin{cases} F^{-1}(1; q^*) & \text{if } \mu(q^*) < 0 \\ +\infty & \text{if } \mu(q^*) = 0 \\ G^{-1}(1; q^*) & \text{if } \mu(q^*) > 0 \end{cases}$$

If $\lambda > \bar{\lambda}_{q^*}$, $f(q^*, \lambda)$ will be either below 0 or above 1. In this situation, the player's behaviour changes. He will make a choice by comparing the ex ante expected payoff of each action, disregarding any contingent information. This alternative behaviour is grounded on the following reasons. First, this behaviour, reflected by its mathematical representation, matches with the interior solution where the information acquired at the equilibrium is zero. Second, mathematically, it coincides with the boundary solutions of G-2. Hence, the continuity of the best response function $q(q^*)$ with respect to λ is ensured. The description of the best response functions is formally given by Proposition 2.

Proposition 2 (Best Response Functions): Given $q^* \in [0, 1]$, a player's best response $q(q^*)$ is given by

$\forall\lambda \in (0, \bar{\lambda}_{q^*}]$, $q(q^*) = f(q^*, \lambda) = \frac{p}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})} + \frac{1-p}{1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}$;

$\forall(\bar{\lambda}_{q^*}, +\infty)$, $q(q^*) = 0 \times 1\{\mu(q^*) < 0\} + 1 \times 1\{\mu(q^*) > 0\}$.

Alternatively, the best response function can be written in a simplified form as follows:

$$q(q^*) = 0 \times 1\{f(q^*, \lambda) < 0\} + f(q^*, \lambda) \times 1\{0 \leq f(q^*, \lambda) \leq 1\} + 1 \times 1\{f(q^*, \lambda) > 1\} \tag{4.8}$$

Proof: see Appendix. ■

The exponential form of $f(q^*, \lambda)$ comes from the entropy function. At $\lambda = \bar{\lambda}_{q^*}$, the average information revealed I equals 0. In this situation, the interior solution $q(q^*)$ is deterministic. For $\lambda > \bar{\lambda}_{q^*}$, the boundary solutions of G-2 are the same as the solution at $\lambda = \bar{\lambda}_{q^*}$. It is $q(q^*) = 1$ if $\mu(q^*) > 0$ and $q(q^*) = 0$ if $\mu(q^*) < 0$.

From Proposition 2, we can learn that given a $\lambda > 0$, if $q(q^*) = f(q^*, \lambda)$, the best response function is a decreasing curve with respect to q^*. If $q(q^*) = 0 \times 1\{\mu(q^*) < 0\} + 1 \times 1\{\mu(q^*) > 0\}$, the best response function is a horizontal line with respect to q^*. In addition, given Assumption 1, it is found that $\lim_{\lambda\to 0^+} q = \lim_{\lambda\to 0^+} f(q^*, \lambda) = p$, which implies that a player's strategy as $\lambda \to 0^+$ coincides with the strategy at $\lambda = 0$.

Since this game exhibits strategic substitutes, it is not surprising that the best response function is non-increasing. The best response function can reflect two distinct strategic choice behaviour: one by acquiring information, represented by the curvature part of the best response function, and another by comparing ex ante expected payoff of each action, represented by the horizontal parts of the best response function. Only prior knowledge matters for the latter approach. Therefore, the best response of a player in this game reflects not only the player's rational choice of an action but also their choice of behaviour, i.e. the decision is made whether by acquiring information or by comparing ex ante expected payoff of each action.

To conclude this section, we exhibit some numerical examples of the best response function. In these examples, we maintain the values of M, D, u, d, p constant and only change the value of λ from 0.01 to 10. From Figure 2, we see that when λ is large enough, some parts of the best response function become horizontal, which indicates that given the relevant opponent's strategy, a player's best response is made without paying heed to any contingent information and just relying on the prior knowledge (see Figure 2).

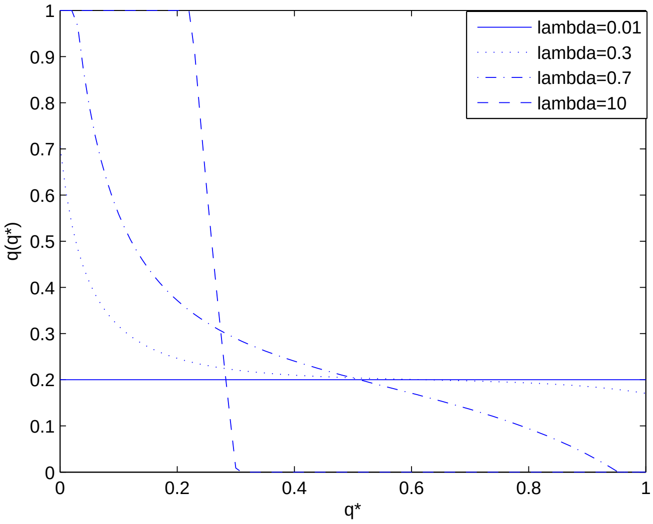


Figure 2: *A series of numerical examples of the best response function. In these examples, M=5, D=3, u=-2, d=-5.1, p=0.2, and* λ *increases from 0.01 to 10. When* $\lambda = 0.01$*, the best response function is close to a horizontal line and approximately equals* $p = 0.2$*. When* $\lambda = 0.3$*, given any* $q^* \in [0, 1]$*, the player always makes the best response by acquiring information, which is reflected by the curvature of the entire best response function. When* $\lambda = 0.7$ *and* 10*, the horizontal parts emerge on the best response function, which indicates that given relevant opponent's strategies, a player's best response is made by comparing ex ante expected payoff of each action.*

4.6 The Impact of Information Cost on a Rational Inattentive Player's Best Response

For the comparative statics analysis in this section, we focus on the case that $q \in (0, 1)$ given $q^* \in [0, 1]$. That means a player's best response is made by acquiring information. In fact, it is not interesting to investigate the boundary cases ($q = 0$ or $q = 1$). If $q = 1$, there is no scope of increasing it further. If $q = 0$ and $\lambda \leq \bar{\lambda}_{q^*}$, entry may be so unattractive to start with that any type of improvement in this action (increasing M, D, u, d or p) will not lead player i to choose it.

By increasing M, D, u and d, the expected payoff of entry increases, and hence $q(q^*)$ increases given $q^* \in [0,1]$. If p increases, the high payoff shock u will happen more often, and it encourages the player to choose entry. Therefore, $q(q^*)$ increases as p increases given q^*.

We define $\bar{p}(q^*) := r((1-q^*)M + q^*D)$, whose expression is given in Appendix A. λ's impact on a player's own best response and the opponent's best response are given by Proposition 3

Proposition 3: Suppose Assumption 1 is satisfied. Given $q^* \in [0,1]$, if and only if $p \geq$ (or $<$) $\bar{p}(q^*)$, $\frac{\partial q(q^*)}{\partial \lambda} \geq$ (or $<$) 0, where the equality is taken when $p = \bar{p}(q^*)$. Besides, $sign(\frac{\partial q(q^*)}{\partial \lambda}) = -sign(\frac{\partial q^*(q)}{\partial \lambda})$.

Proof: see Appendix. ■

To intuitively understand Proposition 3, we have to resort to the analytical expression of $\frac{\partial q(q^*)}{\partial \lambda}$. It is found that

$$\frac{\partial q(q^*)}{\partial \lambda} = -\frac{p\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2}\frac{(1-q^*)M+q^*D+d}{\lambda^2}$$
$$-\frac{(1-p)\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\frac{(1-q^*)M+q^*D+u}{\lambda^2}$$

The parameter λ converts bits of information to utils. Therefore, $\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}$ is the expected payoff of entry measured by bit. By simple calculation, it can be found that the condition $p > \bar{p}(q^*)$ indicates that the first term in the above equation dominates the second term. In this situation, the impact induced by $\frac{(1-q^*)M+q^*D+d}{\lambda}$ overwhelms the impact induced by $\frac{(1-q^*)M+q^*D+u}{\lambda}$. As λ increases, the worst expected payoff of entry, $\frac{(1-q^*)M+q^*D+d}{\lambda}$, increases, and hence it will encourage the player to choose entry. Therefore, in this situation, raising λ increases $q(q^*)$.

To interpret the second result in Proposition 3, we can use the chain rule, according to which the impact of λ on the opponent's best response, $q^*(q)$, can be decomposed as

$$\frac{\partial q^*(q)}{\partial \lambda} = \frac{\partial q^*(q)}{\partial q} \times \frac{\partial q(q^*)}{\partial \lambda}$$

Therefore, the impact of λ on $q^*(q)$ can be decomposed into two separate effects: $\frac{\partial q^*(q)}{\partial q}$, the impact of a player's strategy on the opponent's best response, and $\frac{\partial q(q^*)}{\partial \lambda}$, the impact of the player's information cost on their own best response. The impact of λ on $q^*(q)$ is transmitted through this mechanism. Because $\frac{\partial q^*(q)}{\partial q} < 0$, the impact of λ on the opponent's best response is always opposite to λ's impact on the player's own best response.

4.7 The Equilibrium

Proposition 2 implies that the best response function is continuous with respect to the opponent's strategy. Therefore, according to Brouwer's fixed point theorem, we obtain the following proposition.

Proposition 4 (Existence): Given Assumption 1, a pure-strategy (Bayesian) Nash equilibrium always exists.

Proof: see Appendix. ■

We use parenthesis to denote the word 'Bayesian' because there exists a critical value $\lambda = \lambda_c$, and for $\lambda > \lambda_c$, the game turns into a complete information game. We will show this in detail later in Proposition 6.

4.7.1 Dominance Solvability

The next question is under what conditions the game is dominance solvable. If the game is dominance solvable, the game will exhibit a unique equilibrium. A sufficient condition to ensure that the game is dominance solvable is that both players' best response functions are contraction. Therefore, we have the following proposition.

Proposition 5: Given other parameters, $f(q^*,\lambda)$ is a contraction if and only if $\lambda \in (0,\tilde{\lambda}]$, where $\tilde{\lambda} = \min\{f'^{-1}_{q^*}(-1;q^*=0), f'^{-1}_{q^*}(-1;q^*=1)\}$. Therefore, if $\lambda \in (0,\tilde{\lambda}]$ and $\lambda^* \in (0,\tilde{\lambda}^*]$, the game is dominance solvable.

Proof: see Appendix. ■

$f(q^*,\lambda)$ is expressed by equation (4.6). $f'_{q^*}(q^*,\lambda)$ is its first-order derivative. It is found that at $q^* = 0$ or 1, there exists a $\lambda = \tilde{\lambda}$ such that $f'_{q^*}(q^*,\lambda) = -1$. Because $f'_{q^*}(q^*,\lambda)$ is invertible at $\lambda = \tilde{\lambda}$, according to the expression rule defined in Section 4.5, $\tilde{\lambda} = f'^{-1}_{q^*}(-1;q^*)$.

Now, we look at the intuition of Proposition 5. According to our proof, it is found that the lowest value of $f'_{q^*}(q^*,\lambda)$ happens at either $q^* = 0$ or $q^* = 1$ for any parameter specification. It is caused by the exponential functions in the best response function and Assumption 1. By Assumption 1, $f'_{q^*}(q^*,\lambda)$ either increases, or decreases, or first increases and then decreases, as q^* increases from 0 to 1. Therefore, the lowest value of $f'_{q^*}(q^*,\lambda)$ happens at either $q^* = 0$ or $q^* = 1$.

If $\min\{f'_{q^*}(q^*=1,\lambda), f'_{q^*}(q^*=0,\lambda)\} \geq -1$, then for all $q^* \in (0,1)$, $f'_{q^*}(q^*,\lambda) > -1$ and hence $f(q^*,\lambda)$ is a contraction; similarly, $q(q^*)$ is a contraction. If both players' best response functions satisfy this situation, the game is dominance solvable.

Given a $q^* \in [0,1]$, it is found that as λ increases from 0, $\frac{\partial f'_{q^*}(q^*,\lambda)}{\partial \lambda} < 0$. Because the lowest value of $f'_{q^*}(q^*,\lambda)$ happens at $q^* = 0$ or $q^* = 1$, for all $\lambda \in (0,\tilde{\lambda}]$, where $\tilde{\lambda} = \min\{f'^{-1}_{q^*}(-1;q^*=0), f'^{-1}_{q^*}(-1;q^*=1)\}$, $-1 \leq f'_{q^*}(q^*,\lambda) < 0$, which implies that $-1 \leq \frac{dq(q^*)}{dq^*} \leq 0$. Therefore, we get Proposition 5.

4.7.2 From a Bayesian Game to a Complete Information Game

From Section 4.5, it is found that when the information cost is too high, a player's best response will be made by comparing ex ante expected payoff of each action. Nevertheless, the game may still be a Bayesian game since the player's best response towards some other strategies is still made by acquiring information. However, the game can turn into a complete information game where λ is higher than some critical value. We obtain the following proposition.

Proposition 6: In a symmetric game where $\lambda = \lambda^*$, there exists a $\lambda_c = \max_{q^* \in [0,1]} \bar{\lambda}_{q^*}$ such that $\forall \lambda \in [0, \lambda_c)$, the game is a Bayesian game as shown in Table 2, and $\forall \lambda \in [\lambda_c, +\infty)$, the game is a complete information game as shown in Table 3.

From Proposition 2, it is known that given $q^* \in [0,1]$, $\forall \lambda \in (\bar{\lambda}_{q^*}, +\infty)$, the best response is made by comparing the ex ante expected payoff of each action. If all $\bar{\lambda}_{q^*}$ given $q^* \in [0,1]$ is smaller than the given λ, the player's reaction will always be made by comparing the ex ante expected payoffs. In this situation, the game becomes a complete information game, as expressed by Table 3. Therefore, if $\lambda > \max_{q^* \in [0,1]} \bar{\lambda}_{q^*}$, the game is a complete information game. Hence, Proposition 6 is obtained.

Further, we determine the analytical expression of λ_c. They are given by the following corollary.

Corollary 1: If $\mu(q^* = 0) > 0$ and $\mu(q^* = 1) < 0$, $\lambda_c = +\infty$; if $\mu(q^* = 1) > 0$, $\lambda_c = G^{-1}(1; q^* = 1) < +\infty$; if $\mu(q^* = 0) < 0$, $\lambda_c = F^{-1}(1; q^* = 0)$.

Proof: see Appendix. ■

$\mu(q^*)$ is as defined in Section 4.5. $F(\lambda, q^*)$ and $G(\lambda, q^*)$ are expressed by equation (4.7).

4.7.3 Equilibria of the Game

Here we give a complete characterization of the equilibrium set of the game for all parameter specifications. It is given by Proposition 7.

Proposition 7: For the equilibrium set of the game where $\lambda = \lambda^*$, there are two possibilities:

(1) Under some parameter specifications, there exist λ_1 and λ_2 such that $0 < \lambda_1 <$

$\lambda_2 < \lambda_c$. $\forall\lambda \in [0,\lambda_1)$, there is a unique equilibrium. At $\lambda = \lambda_1$, there are three equilibria. $\forall\lambda \in (\lambda_1,\lambda_2)$, there are five equilibria. $\forall\lambda \in [\lambda_2,\lambda_c)$, there are three equilibria.

$\forall\lambda \in (0,\lambda_2)$, the symmetric equilibrium is stable.[9] At $\lambda = \lambda_2$, the stability of the symmetric equilibrium cannot be determined. $\forall\lambda \in (\lambda_2,\lambda_c)$, the symmetric equilibrium is unstable.

(2) Otherwise, under some parameter specification satisfying $\mu(q^* = 0) < 0$ or $\mu(q^* = 1) > 0$, there is a unique equilibrium $\forall\lambda \in [0,\lambda_c)$, which is stable.

Proof: see Appendix. ■

In the following, we explain Proposition 7. The equilibria are solutions of an equation group comprising the 2 players' best response functions. By putting the opponent's best response function into i's best response function, we get the following function with respect to q, which is the second iteration of the best response functions, $g(q^*(q))$:

$$g(q^*(q)) = 0\times 1\{A(q^*(q))<0\} + A(q^*(q))\times 1\{0\le A(q^*(q))\le 1\} + 1\times 1\{A(q^*(q))>1\} \tag{4.9}$$

where $A(q^*(q)) \equiv \frac{p}{1-\exp(\frac{(1-q^*(q))M+q^*(q)D+d}{\lambda})} + \frac{1-p}{1-\exp(\frac{(1-q^*(q))M+q^*(q)D+u}{\lambda})}$ and $q^*(q)$ is given by equation (4.8). Thus, any equilibrium must be a solution of the following equation:

$$q = g(q^*(q)) \tag{4.10}$$

Geometrically, equation (4.10) shows that the equilibria are intersection points between 45° line and $g(q^*(q))$. $g(q^*(q))$ and $A(q^*(q))$ are continuous and non-decreasing with respect to q for all $\lambda \in (0,+\infty)$. Because the game is symmetric and best response function $q^*(q)$ is non-increasing, there always exists a unique symmetric equilibrium. In addition, if asymmetric equilibria exist, one asymmetric equilibrium always has a corresponding equilibrium obtained by switching players' identities. Therefore, in such games, asymmetric equilibria always appear in pairs and hence the total number

[9] The stability concept adopted in this chapter is Lyapunov stability.

of equilibrium is odd.

We define set $h := \{q|0 \leq f(q,\lambda^*) \leq 1 \text{ and } 0 \leq A(q^*(q)) \leq 1\}$. $A(q^*(q))$ of $q \in h$ represent the part of $A(q^*(q))$ that is between 0 and 1 and no horizontal parts. Its economic sense is that both players' best response functions are made by acquiring information. The horizontal parts of $g(q^*(q))$ indicates that there is at least one player not playing the game by acquiring information. It is proven that when multiple equilibria arise, as q increases, $g(q^*(q))$ where $q \in h$ first exhibits concavity until $q = \tau$, and then exhibits convexity afterwards, where $\tau \in (0,1)$. Therefore, we call this property as concavity–convexity property of $A(q^*(q))$.

In addition, we denote the symmetric equilibrium of the Bayesian game by (s,s). It can be found that $s \in h$. This is because the Bayesian game exhibits strategic substitutes, and the best response function is non-increasing and not constant. Therefore, in the Bayesian game, there is no equilibrium like $(0,0)$ or $(1,1)$. Hence, in the symmetric equilibrium, both players always make decisions by acquiring information.

We define set $k := \{q|0 \leq f(q,\lambda^*) \leq 1\}$. The last component we need in order to explain Proposition 7 is that $\frac{\partial A'_q(q^*(q))}{\partial \lambda} > 0$, where $q \in k$. $A(q^*(q))$ of $q \in k$ could be greater than 1 or smaller than 0. In the following, we first explain the result (1) of Proposition 7.

4.7.3.1 Multiple Equilibria with Stable Symmetric Equilibrium

When λ is small, the game is dominance solvable and the symmetric equilibrium (s,s) is stable. Thus, at $q = s$, $g'_q(q^*(q)) < 1$. Therefore, if we consider the multiplicity situation, we should first consider if the symmetric equilibrium is stable, how many equilibria exist. Because of the concavity–convexity property and that asymmetric equilibria appear in pairs, in this situation, $A(q^*(q))$ of $q \in h$ should have three intersection points with the 45° line. Otherwise, there is a unique intersection point (see Figure 3).

To complete the solid curve in Figure 3 as the geometric representation of $g(q^*(q))$ with respect to $q \in [0,1]$, we should draw two horizontal lines at the two sides of $A(q^*(q))$ of $q \in h$. According to the symmetry property of the game, there are three

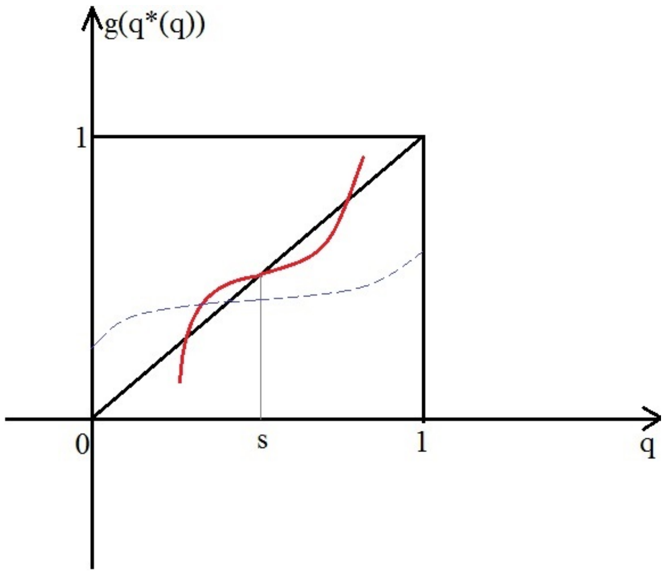


Figure 3: *The solid curve represents $A(q^*(q))$ of $q \in h$ in the multiplicity situation. If the game exhibits multiple equilibria, $A(q^*(q))$ of $q \in h$ has three intersection points with the 45° line. Otherwise, there is a unique intersection point indicated by the dashed curve, which is also a $A(q^*(q))$ of $q \in h$ function.*

possible situations of a complete $g(q^*(q))$ curve. They are determined by whether $g(q^*(0)) = 0$ or not, or $g(q^*(q)) = 1$ or not. Irrespective of the possible situation, $g(q^*(q))$ will have five intersection points with the 45° line (see Figure 4).

Situations such that shown in Figure 5 cannot happen. Figure 5 is characterized by $g(q^*(0)) > 0$ and $g(q^*(1)) < 1$. In Figure 5, (a, b) and (b, a), where a, b $\in (0,1)$, form a pair of asymmetric equilibria in which both players make their equilibrium strategies by acquiring information. However, the intersection points a and b are made by the horizontal part of $g(q^*(q))$ and the 45° line. It implies that there is at least one player not acquiring information, and hence that player's strategy (a or b) is either 0 or 1. However, both a and b are between 0 and 1. Hence, a contradiction arises, and (a, b) and (b, a) in Figure 5 cannot be equilibria of the symmetric rational inattention Bayesian game. Thus, if there are five equilibria, only the three situations in Figure 4 are the correct situations.

At the boundary situation ($\lambda = \lambda_1$), where the game transits from a unique-equilibrium situation to five-equilibria situation, according to the concavity–convexity property and $\frac{\partial A'_q(q^*(q))}{\partial \lambda} > 0 \ \forall q \in k$, there are three equilibria which are intersections between

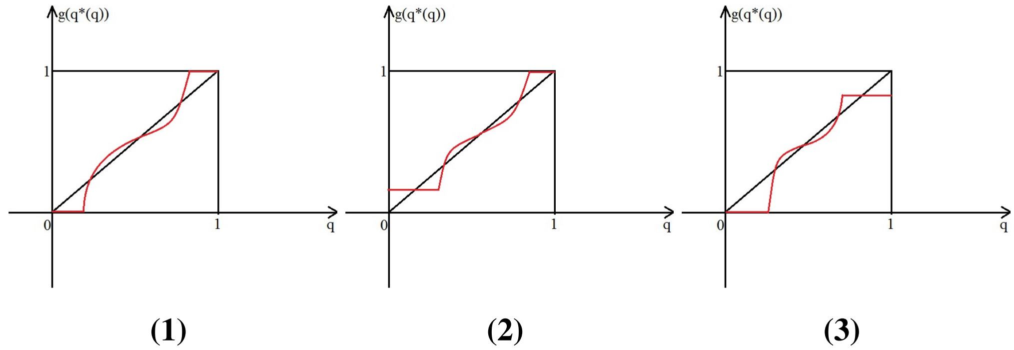


Figure 4: *Three possible situations of $g(q^*(q))$ when there are multiple equilibria and the symmetric equilibrium is stable. There are five intersection points, representing the five equilibria of the symmetric rational inattention Bayesian game.*

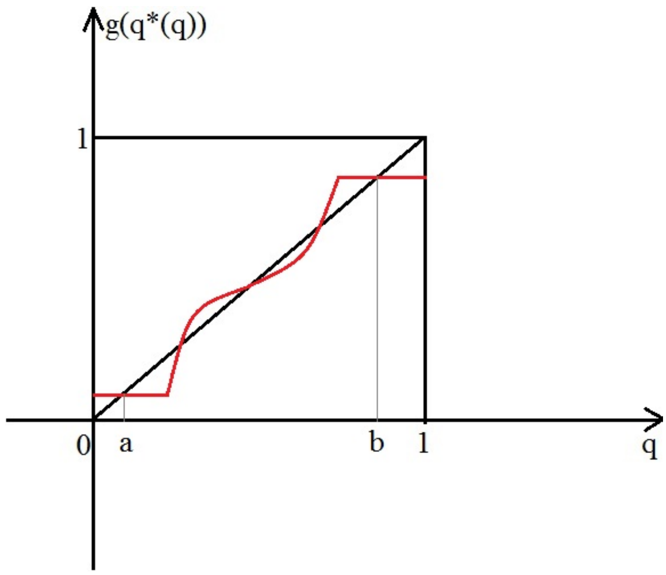
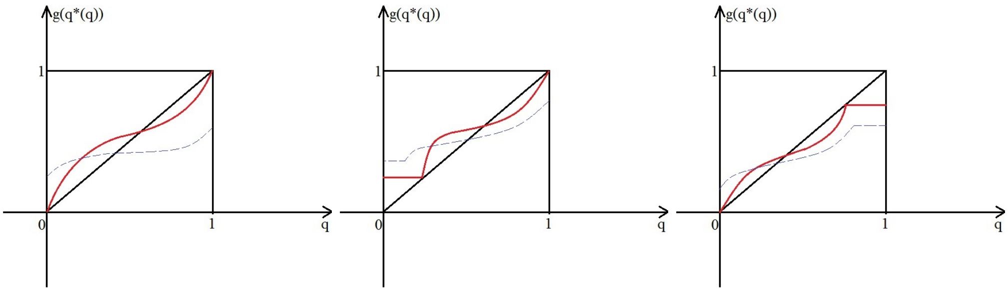


Figure 5: *The situation that cannot happen if there are five equilibria.*

$A(q^*(q))$ of $q \in h$ and the $45°$ line. They are described by the three figures in Figure 6.

From Figure 6, we can see that if we slightly increase λ to $\lambda_1 + \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number, since $\frac{\partial A_q'(q^*(q))}{\partial \lambda} > 0 \ \forall q \in k$, the shape of $g(q^*(q))$ in each sub-figure will come back to the shape in the corresponding sub-figure with the same order number in Figure 4. If we slightly decrease λ to $\lambda_1 - \varepsilon$, still according to $\frac{\partial A_q'(q^*(q))}{\partial \lambda} > 0 \ \forall q \in k$, the slope of $g(q^*(q))$ in each sub-figure of Figure 6 will become the one represented by the dashed curve, which is flatter than the solid curve; hence, there is a unique equilibrium for $\lambda = \lambda_1 - \varepsilon$.



(1) **(2)** **(3)**

Figure 6: *Three possible situations of $g(q^*(q))$ at $\lambda = \lambda_1$. They are represented by the solid curves. Dashed curves represent the $g(q^*(q))$ function at $\lambda = \lambda_1 - \varepsilon$. From these figures, it can be seen that when λ increases from $\lambda_1 - \varepsilon$ to λ_1, the number of equilibria changes from one to three.*

Therefore, as long as the symmetric equilibrium is stable, there are at most five equilibria in this game.

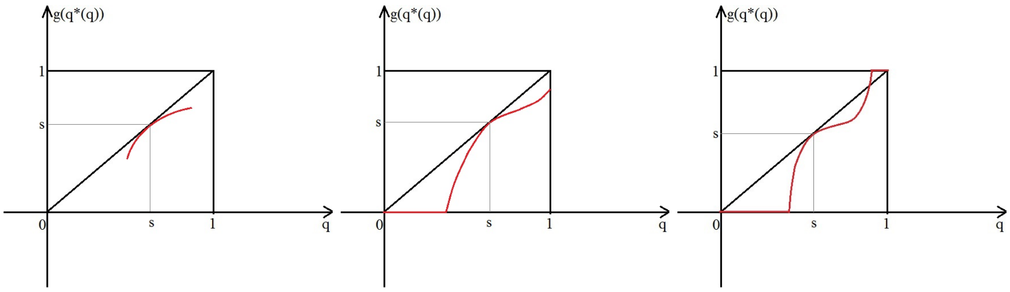
4.7.3.2 Multiple Equilibria with Unstable Symmetric Equilibrium

As λ increases, according to $\frac{\partial f'_{q^*}(q^*,\lambda)}{\partial \lambda} < 0 \ \forall q^* \in [0,1]$, symmetric equilibrium will finally become unstable. At the boundary situation ($\lambda = \lambda_2$), where the symmetric equilibrium will transit from being stable to unstable as λ increases, the stability of the symmetric equilibrium cannot be determined, i.e. $q'(q^*) = -1$ at $q^* = s \in (0,1)$. We can use the undetermined stability of the symmetric equilibrium to characterize λ_2. It can be proven by contradiction that at $\lambda = \lambda_2$, the lowest value of $A'_q(q^*(q))$ of $q \in h$ equals 1 and it occurs at $q = s$.

Suppose that $\forall q \in (s-\varepsilon, s)$, $A'_q(q^*(q)) > 1$, and $\forall q \in (s, s+\varepsilon)$, $A'_q(q^*(q)) < 1$. Then (s,s) is the tangent point of $A(q^*(q))$ with the 45° line, and it should be tangent with the 45° line from below (see Figure 7).

From Figure 7, we can see that if the symmetric equilibrium is tangent with the 45° line, the number of equilibria will be even. Therefore, a contradiction arises.

In addition, situations indicated by Figure 8 also cannot happen. In Figure 8, the stability of the symmetric equilibrium is not determined and there are three equilibria. However, if λ increases to $\lambda_2 + \varepsilon$, because $\frac{\partial A'_q(q^*(q))}{\partial \lambda} > 0$, where $q \in k$, there will be five



(1) **(2)** **(3)**

Figure 7: *The situation that $A'_q(q^*(q)) = 1$ at $q = s$ (1) and its two possible realizations ((2) and (3)). In this situation, $A'_q(q^*(q)) = 1$ at $q = s$ is not the lowest value of $A'(q^*(q))$ of $q \in h$. Because the intersection points and the tangent point are of even number in total, this situation cannot happen.*

intersection points and the middle intersection point s' (symmetric equilibrium) will become stable. This contradicts the prerequisite condition that the symmetric equilibrium should not be stable for $\lambda > \lambda_2$ (see Figure 8).

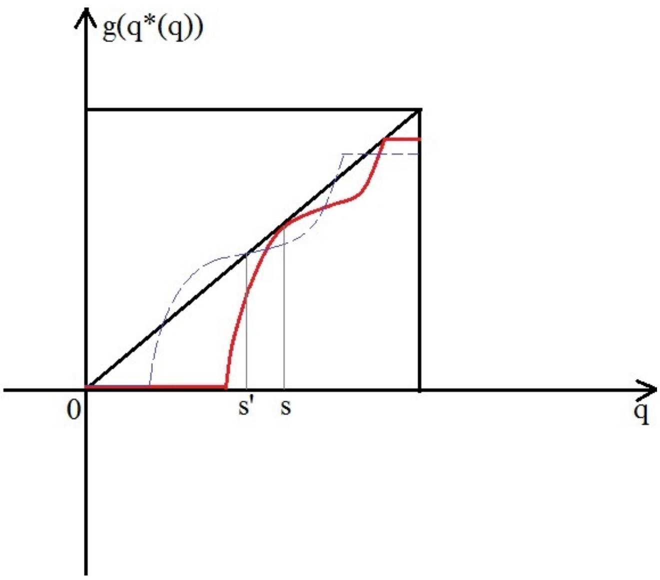


Figure 8: *The situation that $A'_q(q^*(q)) = 1$ at $q = s$. Because we are studying the boundary situation that leads to unstable symmetric equilibrium for $\lambda > \lambda_2$, this situation cannot happen, because here the new symmetric equilibrium s' by increasing λ with $\varepsilon > 0$ is stable.*

Therefore, at $\lambda = \lambda_2$, $A'_q(q^*(q))$ of $q \in h$ reaches its lowest value 1 at $q = s$. In this situation, there are three equilibria. Furthermore, we draw an arbitrary curve whose

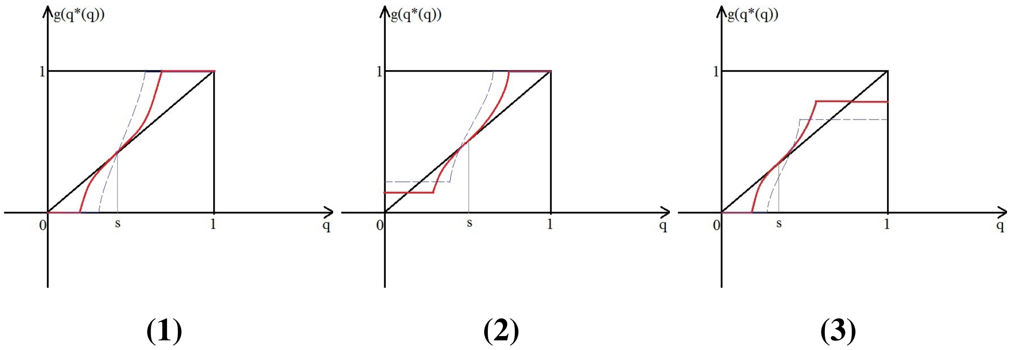


Figure 9: *The three possible situations of $g(q^*(q))$ at $\lambda = \lambda_2$. They are represented by the solid curves. Dashed curves represent $g(q^*(q))$ at $\lambda = \lambda_2 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Therefore, as λ increases from λ_2 to $\lambda_2 + \varepsilon$, the number of equilibria will be maintained as three, and the stability of the symmetric equilibrium will change from undetermined to unstable.*

slope is greater than 1 to represent $A(q^*(q))$ of $q \in h$; there is a unique intersection point with the 45° line. The other two intersection points with the 45° line are realized by the horizontal lines which complete $g(q^*(q))$ with the $A(q^*(q))$ of $q \in h$. Figure 9 provides a complete description of the intersection points between $g(q^*(q))$ and 45° line at $\lambda = \lambda_2$ (see Figure 9).

Because $\frac{\partial A'_q(q^*(q))}{\partial \lambda} > 0$, where $q \in k$, for $\lambda = \lambda_2 + \varepsilon$, $A'_q(q^*(q)) > 1$ for all $q \in k$; hence, there are three equilibria. This situation will be maintained for all $\lambda \in (\lambda_2, \lambda_c)$. Therefore, for all $\lambda \in [\lambda_2, \lambda_c)$, the game has three equilibria.

Therefore, if players with scarcity of attention play the Bayesian game of Table 2 and the scarcity of attention is modelled by the reduction in the entropy of players' posterior over the state space relative to the prior distribution, there are at most five ways to play the game. For any result, either both players make choices by acquiring information or one player makes choices by acquiring information and the other player makes choices without acquiring information and relying only on prior knowledge.

To conclude Sections 4.7.3.1 and 4.7.3.2, we present some numerical examples to visually exhibit how different number of equilibria arise. They are presented in Figure 10. The three figures in the first row represent the situation in which $\mu(q^* = 0) > 0$ and $\mu(q^* = 1) < 0$. The three figures in the second row represent the situation in which $\mu(q^* = 0) < 0$. The three figures in the third row represent the situation in which

$\mu(q^* = 1) > 0$. We can see that in any situation, the number of equilibria in each row of numerical examples follows the 1-5-3 sequence as λ increases (see Figure 10).

4.7.3.3 The Explanation of Result (2) in Proposition 7

Logically, under any parameter specification, as λ increases from 0, the game may exhibit a unique equilibrium, or not, for all $\lambda \in (0, +\infty)$. Result (1) in Proposition 7 belongs to the 'or not' part.

For some parameter specifications satisfying $\mu(q^* = 0) < 0$ or $\mu(q^* = 1) > 0$, the game contains a unique equilibrium for all $\lambda \in (0, +\infty)$. We can find numerical examples to support this fact:

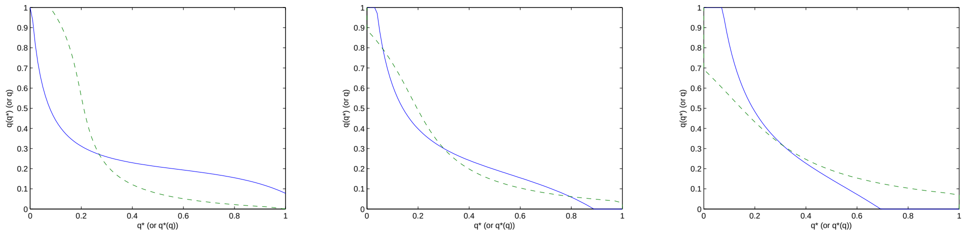
1. $\mu(q^* = 0) < 0$: $p = 0.2, M = 5, D = 3, u = -2$ and $d = -10$;

2. $\mu(q^* = 1) > 0$: $p = 0.8, M = 2, D = 1, u = 1$ and $d = -2.01$.

Therefore, if $\mu(q^* = 0) < 0$ or $\mu(q^* = 1) > 0$, it is possible that there is only one way for both players with scarcity of attention to play the Bayesian game shown in Table 2. In this situation, both players will acquire information to make decisions.

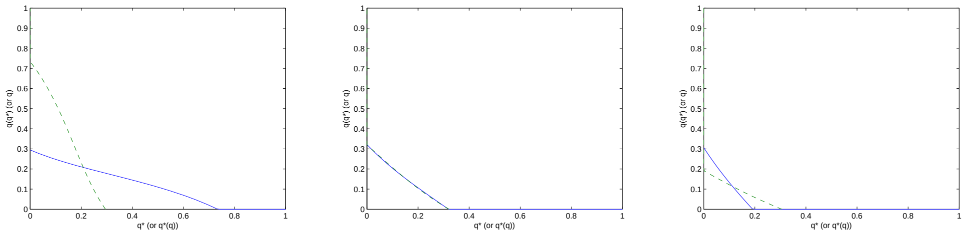
However, if parameters satisfying $\mu(q^* = 0) > 0$ and $\mu(q^* = 1) < 0$, multiple equilibria will surely happen when λ is large enough. For all $q^* \in [\frac{M+pu+(1-p)d}{M-D}, 1]$, $\mu(q^*) < 0$, and $\frac{\partial \bar{\lambda}_{q^*}}{\partial q^*} < 0$. Therefore, as λ increases, $q^* = 1$ is the first point at which the player chooses action 0 deterministically. Then, along the direction from $q^* = 1$ to $q^* = \frac{M+pu+(1-p)d}{M-D}$, as λ increases, the player's best responses gradually turn into 0. At the same time, $\forall q^* \in [0, \frac{M+pu+(1-p)d}{M-D})$, $\mu(q^*) > 0$ and $\frac{\partial \bar{\lambda}_{q^*}}{\partial q^*} > 0$. Therefore, as λ increases, $q^* = 0$ is the first point at which the player chooses action 1 deterministically, and then along the direction from $q^* = 0$ to $q^* = \frac{M+pu+(1-p)d}{M-D}$, the player's best responses gradually turn into 1.

Hence, in the case of $\mu(q^* = 0) > 0$ and $\mu(q^* = 1) < 0$, as λ increases from 0, a player's best response $q(q^*)$ will rise to 1 $\forall q^* \in [0, \frac{M+pu+(1-p)d}{M-D})$, and at the same time, the player's best response will fall to 0 $\forall q^* \in [\frac{M+pu+(1-p)d}{M-D}, 1]$. Therefore, when



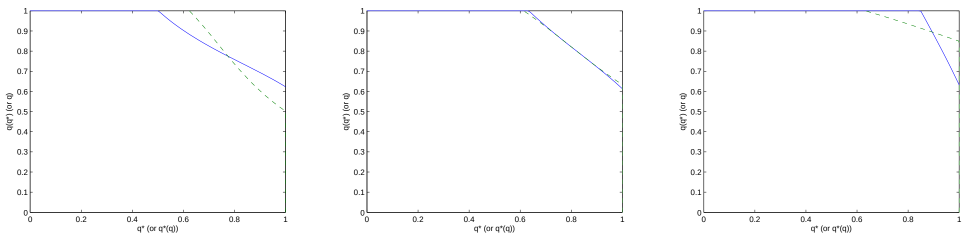
$\lambda = 0.5$ **(1 equilibrium)** $\lambda = 0.8$ **(5 equilibria)** $\lambda = 1.2$ **(3 equilibria)**

Numerical examples corresponding to 1) of Theorem 1: M=5, D=3, u=-2, d=-5.1 and $p = 0.2$.



$\lambda = 1$ **(1 equilibrium)** $\lambda = 2.1$ **(5 equilibria)** $\lambda = 3$ **(3 equilibria)**

Numerical examples corresponding to 2) of Theorem 1: M=5, D=3, u=-2, d=-5.9 and $p = 0.2$.



$\lambda = 1$ **(1 equilibrium)** $\lambda = 1.4$ **(5 equilibria)** $\lambda = 3$ **(3 equilibria)**

Numerical examples corresponding to 3) of Theorem 1: M=5, D=3, u=-2, d=-5.1 and $p = 0.7$.

Figure 10: *Numerical examples of this game. The solid curves represent a player's best response function $q(q^*)$, and the dashed curves represent the opponent's best response function $q^*(q)$. The horizontal axis indicates q^* (or $q^*(q)$) and the vertical axis indicates $q(q^*)$ (or q).*

$q(q^*) = 0$ at $q^* = 1$ and $q(q^*) = 1$ at $q^* = 0$, equilibria (1, 0) and (0, 1) arise. Thus, in this situation, asymmetric equilibria are inevitable.

4.8 Impact of Information Cost on Equilibrium Strategy

In the symmetric strategic substitutes game, the symmetric equilibrium always exists and is unique. When multiple equilibria arise, there are two types of asymmetric equilibria. In one type, there is at least one player choosing action 0 or 1 by comparing ex ante expected payoff of each action. This type of asymmetric equilibria is usually located at the outer part of a player's best response function, hence named outer asymmetric equilibria. In the other type, both players make their best response by acquiring information. This type of asymmetric equilibria is usually located at the inner part of a player's best response function, hence named inner asymmetric equilibria.

For outer asymmetric equilibria, according to the payoff specification, there are three specific results:

1) If $\mu(q^* = 0) > 0$ and $\mu(q^* = 1) < 0$, they are (1, 0) and (0, 1);

2) If $\mu(q^* = 0) < 0$, they are $(t, 0)$ and $(0, t)$, where $t \in (0, 1)$;

3) If $\mu(q^* = 1) > 0$, they are $(j, 1)$ and $(1, j)$, where $j \in (0, 1)$.

In an equilibrium, if a player always chooses 0 or 1 deterministically, the corresponding comparative statics results with respect to any parameter are always equal to 0. Only the equilibrium strategy that is made by acquiring information, i.e. $q \in (0, 1)$, will be further analysed.

For inner asymmetric equilibria, there are no conclusive comparative statics results. It depends on particular parameter specifications.

$\frac{\partial q}{\partial \tau}$ represents the comparative statics of equilibrium strategy q with respect to parameter τ. $\frac{\partial q(q^*)}{\partial \tau}$ represents the comparative statics of best response $q(q^*)$ given q^* with respect to parameter τ. For the symmetric equilibrium and outer asymmetric equilibria in which equilibrium strategy $q \in (0,1)$, the comparative statics results are given by the following two propositions:

Proposition 8: For a symmetric equilibrium strategy or an outer asymmetric equilibrium strategy $q \in (0,1)$, $\frac{\partial q}{\partial \tau} \geq 0$, where $\tau \in \{M, D, u, d, p\}$. The equality is taken for the outer asymmetric equilibria in which $\tau = D$ when $\mu(q^* = 0) < 0$ or $\tau = M$ when $\mu(q^* = 1) > 0$.

Proof: see Appendix. ■

The intuition of Proposition 8 is that increasing $\tau \in \{M, D, u, d, p\}$ can increase the expected payoff of entry; therefore, a player is more willing to choose entry. If $q^* = 0$ and $\mu(q^* = 0) = M + pu + (1-p)d < 0$, duopoly cannot happen, and hence, $\frac{\partial q}{\partial D} = 0$. If $q^* = 1$ and $\mu(q^* = 1) = D + pu + (1-p)d > 0$, from an ex ante perspective, entry is more profitable than being inactive, and hence $\frac{\partial q}{\partial M} = 0$.

Proposition 9: Given that $\lambda = \lambda^*$, for a symmetric equilibrium strategy or an outer asymmetric equilibrium strategy $q \in (0,1)$, if and only if $p \geq$ (or $<$) $\bar{p}(q^*)$, then $\frac{\partial q}{\partial \lambda} \geq$ (or $<$) 0, where the equality is taken when $p = \bar{p}(q^*)$. Here, $q^* = s$ for symmetric equilibrium (s,s), $q^* = 0$ for an outer asymmetric equilibrium in which $\mu(q^* = 0) < 0$, and $q^* = 1$ for an outer asymmetric equilibrium in which $\mu(q^* = 1) > 0$.

Proof: see Appendix. ■

For Proposition 9, it is found that at symmetric equilibrium (s,s), $\frac{\partial q}{\partial \lambda} = \frac{\frac{\partial q(s)}{\partial \lambda}}{C}$, where $C > 1$, and at outer asymmetric equilibrium (q, q^*), where $q \in (0,1)$ and $q^* = 0$ or 1, $\frac{\partial q}{\partial \lambda} = \frac{\partial q(q^*)}{\partial \lambda}$. Therefore, the sign of $\frac{\partial q}{\partial \lambda}$ essentially depends on the sign of $\frac{\partial q(q^*)}{\partial \lambda}$ at equilibrium (s,s) or (q,q^*). According to Proposition 3, it can be deduced that the sign of $\frac{\partial q}{\partial \lambda}$ also depends on the trade-off between the impact of $\frac{(1-q^*)M+q^*D+d}{\lambda}$ and the impact of $\frac{(1-q^*)M+q^*D+u}{\lambda}$ within $\frac{\partial q(q^*)}{\partial \lambda}$.

Finally, for any type of equilibrium, what is the effect of the situation where only one player's information cost changes given the other parameters? The answer is that the impact of changing one player's information cost on both players' equilibrium strategy cannot be determined without particular parameter specification in any equilibrium. We define

$$H = -\frac{M-D}{\lambda^*}[\frac{p}{[1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})]^2}\exp(\frac{(1-q)M+qD+d}{\lambda^*})$$
$$+\frac{1-p}{[1-\exp(\frac{(1-q)M+qD+u}{\lambda^*})]^2}\exp(\frac{(1-q)M+qD+u}{\lambda^*})]$$

and

$$K = -\frac{M-D}{\lambda}[\frac{p}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2}\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})$$
$$+\frac{1-p}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})].$$

The signs of H and K could be positive or negative. It is found that

$$\frac{\partial q^*}{\partial \lambda} = \frac{H}{1-HK}\frac{\partial q(q^*)}{\partial \lambda}$$

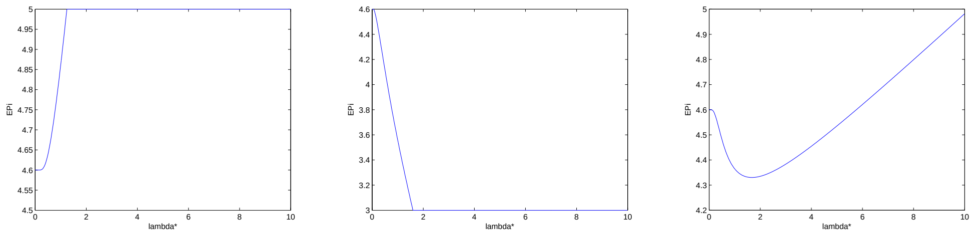
and

$$\frac{\partial q}{\partial \lambda} = \frac{1}{1-HK}\frac{\partial q(q^*)}{\partial \lambda}$$

The signs of $1-HK$ and $\frac{\partial q(q^*)}{\partial \lambda}$ cannot be determined without particular parameter specification. However, $sign(\frac{\partial q^*}{\partial \lambda}) = -sign(\frac{\partial q}{\partial \lambda})$ because $H < 0$. Therefore, the impact of varying only one player's information cost on each player's equilibrium strategy, in any equilibrium, cannot be determined, but its impact on one player's equilibrium strategy is always opposite to its impact on its opponent's equilibrium strategy.

4.9 Impact of Information Cost on Players' Expected Payoffs of Entry

In this chapter, we have two types of expected payoff of entry: the ex ante expected payoff of entry, i.e. $(1-q^*)M+q^*D+pu+(1-p)d$, and the typical expected payoff of entry, i.e. $(1-q^*)M+q^*D+\varepsilon$,where $\varepsilon \in \{u,d\}$. For both types, λ or λ^* can only



p=0.2, M=5, D=3, u=-2, d=-5.9, q=0.6 **p=0.2, M=5, D=3, u=-2, d=-5.1, q=0.1** **p=0.2, M=5, D=3, u=-2, d=-5.1, q=0.3**

Figure 11: *Three numerical examples of* $\mathbb{E}\Pi(\lambda^*)$

affect $(1-q^*)M + q^*D$. We define $\mathbb{E}\Pi = (1-q^*)M + q^*D = M - (M-D)q^*$. Therefore, λ^*, rather than λ, can affect $\mathbb{E}\Pi$ through player i's belief. Therefore,

$$\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*} = -(M-D)\frac{\partial q^*(q)}{\partial \lambda^*}$$

and hence,

$$sign(\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*}) = -sign(\frac{\partial q^*(q)}{\partial \lambda^*})$$

Because if and only if $p \gtreqless \bar{p}(q)$ given $q \in [0,1]$, $\frac{\partial q^*(q)}{\partial \lambda^*} \gtreqless 0$. Therefore, if and only if $p \lesseqgtr \bar{p}(q)$, $\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*} \gtreqless 0$. It is found that the sign of $\frac{\partial \bar{p}(q)}{\partial \lambda^*}$ could be positive or negative. Therefore, without particular parameter specification, it is hard to know the critical λ^* that makes $\mathbb{E}\Pi$ reach its highest value. For example, suppose $\frac{\partial \bar{p}(q)}{\partial \lambda^*} > 0$. It can be found that in this situation, the maximum value of $\mathbb{E}\Pi$ is reached at either $\lambda^* = 0$ or $\lambda^* = \bar{\lambda}_q^*$ (see Appendix). We also exhibit three numerical examples here to show the flexibility of the shape of $\mathbb{E}\Pi$ with respect to λ^* (see Figure 11).

4.10 (Bayesian) Quantal Response Equilibrium and Rational Inattention Bayesian Nash Equilibrium

Consider the following entry game:

		Firm i^*: 0	Firm i^*: 1
Firm i	0	0, 0	0, $M+\varepsilon^*+\eta^*$
	1	$M+\varepsilon+\eta$, 0	$D+\varepsilon+\eta$, $D+\varepsilon^*+\eta^*$

The players make their decisions after observing their respective private payoff shocks ε and ε^*. However, the observation is always affected by an additive error η (or η^*). Therefore, what they actually observe is $\varepsilon+\eta$ and $\varepsilon^*+\eta^*$, assuming that η is independent from η^*. Assumption 1 in Section 4.3 is still held in this game. Therefore, if player i observes $u+\eta$ or $d+\eta$, in this game, the player's choice is probabilistic because $D+u>0>M+d$. The solution concept of this game is therefore (Bayesian) Quantal Response Equilibrium (QRE) (McKelvey and Palfrey, 1995).

It is assumed that η follows Type I extreme value distribution, i.e. $F(\eta)=e^{-e^{-\alpha\eta-\beta}}$. Suppose that players in this game adopt a cutoff strategy. The conditional choice probability $q^{\varepsilon}_{QRE}(q^*)=Pr(a=1|\varepsilon)$ is given by

$$q^{\varepsilon}_{QRE}(q^*)=\frac{\exp\{\alpha[(1-q^*)M+q^*D+\varepsilon]\}}{\exp\{\alpha[(1-q^*)M+q^*D+\varepsilon]\}+1} \tag{4.11}$$

Let us recall the conditional choice probability in the rational inattention Bayesian game. Given $q^*\in[0,1]$ and $\lambda<\bar{\lambda}_{q^*}$ such that $q(q^*)\in(0,1)$, we have

$$\begin{aligned} q^{\varepsilon}_{RI}(q^*) &= \frac{q\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]}{q\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]+(1-q)} \\ &= \frac{\frac{q}{1-q}\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]}{\frac{q}{1-q}\exp[\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda}]+1} \end{aligned} \tag{4.12}$$

Since (4.11) and (4.12) look similar, it is natural to ask under what parameter specifications (4.11) and (4.12) are identical. If they are identical, there will be a clear economic and psychological justification of why the disturbances η should be extreme value distributed in the Bayesian QRE game.

It is found that if and only if

$$\alpha = \frac{1}{\lambda} + \frac{1}{(1-q^*)M + q^*D + \varepsilon} \ln \frac{q}{1-q} \tag{4.13}$$

where $\lambda < \bar{\lambda}_{q^*}$, (4.11) and (4.12) are identical. However, because α and λ are constant, (4.13) is held if and only if $\frac{1}{(1-q^*)M+q^*D+\varepsilon} \ln \frac{q}{1-q}$ is a constant with respect to ε given q^*. Therefore, the only solution to make it as a constant is $q = \frac{1}{2}$. When $q = \frac{1}{2}$, $\alpha = \frac{1}{\lambda}$ given that $\lambda < \bar{\lambda}_{q^*}$.

The best response function of the Bayesian QRE game is given by

$$q_{QRE} = \frac{p}{1+\exp[-\alpha((1-q^*)M + q^*D + u)]} + \frac{1-p}{1+\exp[-\alpha((1-q^*)M + q^*D + d)]} \tag{4.14}$$

Given $q^* \in [0,1]$, $q_{QRE} \in (0,1)$. As $\alpha \to +\infty$, the Bayesian QRE game converges to the benchmark Bayesian game, which is also the limit of the rational inattention Bayesian game as $\lambda \to 0$. In this situation, $q_{QRE} = p$. As $\alpha \to 0$, in the Bayesian QRE game, actions consist of all observational errors such that all actions become indifferent ex ante. Hence, $q_{QRE} = \frac{1}{2}$ in this situation. It does not coincide with the rational inattention Bayesian game of $\lambda \to +\infty$, in which players choose a deterministic action (0 or 1) by comparing ex ante expected payoffs. According to equation (4.14) and equation (4.8), it is further found that $(\frac{1}{2}, q^*)$ is an equilibrium for both types of games under the same parameter specification and $\alpha = \frac{1}{\lambda}$ if and only if $q^* = \frac{1}{2}$. The reason is that because both games are symmetric, if $(\frac{1}{2}, q^*)$ is an equilibrium, $(q^*, \frac{1}{2})$ should be an equilibrium as well. According to the analysis in the previous paragraph, in $(q^*, \frac{1}{2})$, unless $q^* = \frac{1}{2}$, $(q^*, \frac{1}{2})$ will not simultaneously satisfy (4.8) and (4.14).

Therefore, in conclusion, if there is a set of parameters satisfying $\alpha = \frac{1}{\lambda}$ and producing equilibrium $(\frac{1}{2}, \frac{1}{2})$ in the rational inattention Bayesian game, then the same set of parameters will produce the same equilibrium $(\frac{1}{2}, \frac{1}{2})$ in the Bayesian QRE game, and vice versa. With the help of equation (4.14), we can express this result in an analytical form. We define a vector of parameters by $W := (M, D, p, u, d, \alpha, \lambda)$. If and only if $W \in \{W | \frac{1}{2} = \frac{p}{1+\exp[-\alpha(\frac{1}{2}M+\frac{1}{2}D+u)]} + \frac{1-p}{1+\exp[-\alpha(\frac{1}{2}M+\frac{1}{2}D+d)]}, \alpha = \frac{1}{\lambda} \text{ and } \lambda < \bar{\lambda}_{\frac{1}{2}}\}$, both the rational inattention Bayesian game and the Bayesian QRE game can have a common equilibrium $(\frac{1}{2}, \frac{1}{2})$. This is the only coincidence situation of the two types of games.

4.11 Conclusion

In this chapter, we have studied how scarcity of attention affects economic agents' strategic choice behaviour in an incomplete information environment. We use the rational inattention approach to model scarcity of attention. Given the opponent's strategy, as the information cost changes from 0 to $+\infty$, a player's behaviour of making best responses will switch from by acquiring information to by comparing ex ante expected payoff of each action. The latter behaviour solely relies on the player's prior knowledge. This behaviour transition is the behavioural manifestation of the mathematical property that the continuity of the best response function with respect to the opponent's strategy is always ensured no matter how λ changes. Hence, the best response function indeed contains two distinct choice behaviours.

It is particularly interesting to determine the impact of attention scarcity on forming equilibria and affecting players' strategic behaviour. By studying symmetric games, we find that scarcity of attention can bring multiple equilibria and it is the high information cost that generates multiple equilibria. The number of equilibria differs with respect to different ranges of information cost. In any multiplicity situation in symmetric games, there always exists one pair of asymmetric equilibria in which at least one player plays the game without acquiring information.

In a symmetric equilibrium or an outer asymmetric equilibrium (q, q^*), the effect of attention scarcity on a player's information-acquisition choice behaviour depends on whether p is greater than $\bar{p}(q^*)$. We also find that in any equilibrium the impact of a player's information cost on their strategy is always opposite to its impact on the opponent's strategy.

Finally, we have compared the rational inattention Bayesian entry game with a Bayesian QRE entry game. It is found that there exists a set of parameters satisfying

$\alpha = \frac{1}{\lambda}$ and $\lambda < \bar{\lambda}_{\frac{1}{2}}$. Specified by this set of parameters, both Bayesian QRE game and rational inattention Bayesian game have a common equilibrium $(\frac{1}{2}, \frac{1}{2})$. Except this situation, the two games cannot be coincided.

For future research, we will study the situation that players pay full attention to their own information but are inattentive to their opponents' information. We are interested in how players make their strategic choices in such a paradigm. In future study, we will investigate this problem to see how players' behaviour deviates from the behaviour in the game in which scarcity of attention exists for players to their own information.

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Appendix A

Appendix of Chapter 2 Preliminaries and Glossaries of Notations

The standard Gaussian density function is denoted by $\phi(.)$, and the standard Gaussian cumulative density function is denoted by $\Phi(.)$. Given a Gaussian distribution $x \sim N(\mu, \varsigma^2)$, the density function is written as

$$f(x) = \frac{1}{\sqrt{2\pi}\varsigma} \exp(-\frac{(x-\mu)^2}{2\varsigma^2}) = \frac{1}{\varsigma}\phi(\frac{x-\mu}{\varsigma})$$

The joint Gaussian distribution is denoted by $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma^2, \varsigma^{*2}, \rho)$. The density function of the bivariate Gaussian distribution is

$$f(\varepsilon, \varepsilon^*) = \frac{1}{2\pi\varsigma\varsigma^*\sqrt{1-\rho^2}} \exp(-\frac{1}{2(1-\rho^2)}(\frac{\varepsilon^2}{\varsigma^2} + \frac{\varepsilon^{*2}}{\varsigma^{*2}} - \frac{2\rho\varepsilon\varepsilon^*}{\varsigma\varsigma^*}))$$

The conditional density function is

$$f(\varepsilon^*|\varepsilon) = \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\frac{\varepsilon^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})$$

and the conditional cumulative density function is

$$
\begin{aligned}
F(\bar{\varepsilon}^*|\varepsilon) = \int_{-\infty}^{\bar{\varepsilon}^*} f(\varepsilon^*|\varepsilon)d\varepsilon^* &= \int_{-\infty}^{\bar{\varepsilon}^*} \frac{1}{\varsigma^*\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{\frac{\varepsilon^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)d\varepsilon^* \\
&= \int_{-\infty}^{\frac{\frac{\bar{\varepsilon}^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2)du \\
&= \Phi(\frac{\frac{\bar{\varepsilon}^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})
\end{aligned}
$$

We denote a player's belief function by $\sigma(x^*,\varepsilon) = F(x^*|\varepsilon)$, where ε is a player's own private information, and x^* is the expected opponent's cutoff strategy. We get the following results for $\sigma(x^*,\varepsilon)$:

$$
\begin{aligned}
\sigma(x^*,\varepsilon) &= F(x^*|\varepsilon) = \Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}) \\
\sigma_{x^*}(x^*,\varepsilon) &= \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}) \\
\sigma_{\varepsilon}(x^*,\varepsilon) &= -\frac{\rho}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})
\end{aligned}
$$

By assuming $\varsigma = \varsigma^*$, these expressions can be simplified into the following equations, respectively:

$$
\begin{aligned}
\sigma(x^*,\varepsilon) &= \Phi(\frac{x^*-\rho\varepsilon}{\varsigma\sqrt{1-\rho^2}}) \\
\sigma_{x^*}(x^*,\varepsilon) &= \frac{1}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{x^*-\rho\varepsilon}{\varsigma\sqrt{1-\rho^2}}) \\
\sigma_{\varepsilon}(x^*,\varepsilon) &= -\frac{\rho}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{x^*-\rho\varepsilon}{\varsigma\sqrt{1-\rho^2}})
\end{aligned}
$$

The expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ is expressed as follows:

$$
\begin{aligned}
\mathbb{E}\Pi(x^*,\varepsilon) &= \sigma(x^*,\varepsilon)(M+\varepsilon) + (1-\sigma(x^*,\varepsilon))(D+\varepsilon) \\
&= \sigma(x^*,\varepsilon)(M-D) + D + \varepsilon
\end{aligned}
$$

$$= \Phi(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})(M-D) + D + \varepsilon$$

$$= \Phi(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1-\rho^2}})(M-D) + D + \varepsilon$$

The best response function is denoted by $g(x^*)$. In the proof, we often regard $g(x^*)$ as an independent variable and take derivatives of relevant functions with respect to $g(x^*)$ or find an optimum value of relevant functions with respect to $g(x^*)$. For simplicity, we denote $g^{-1'}(x^*) \equiv \frac{dx^*}{dg(x^*)} = \frac{1}{\frac{dg(x^*)}{dx^*}}$, and $\min_{g(x^*)}(\max_{g(x^*)})\rho'(x^*) = \min(\max)\rho'(x^*)$, which is the derivative of a function with x^* as dependent variable and $g(x^*)$ as independent variable, and $\min_{g(x^*)}(\max_{g(x^*)})\rho''(x^*) = \min(\max)\rho''(x^*)$.

Appendix B

Appendix of Chapter 2

Proof of Proposition 1

Lemma B1: There exists a $\tilde{\rho} \in (-1,1)$, if $M > D$, for all $\rho \in (-1,\tilde{\rho}]$ and for all $x^* \in \mathbb{R}$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$, where the equality is obtained at $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x^*}{\rho}$, with $\rho = \tilde{\rho}$.

Proof: For all $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*,\varepsilon) = \sigma(x^*,\varepsilon)(M-D)+D+\varepsilon = \Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+\varepsilon$. Therefore, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*,\varepsilon)(M-D)+1 = -\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)+1$. Hence, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ is equivalent to $-\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)+1 \geq 0$. Therefore, the inequality $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) \geq \frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ is the necessary and sufficient condition for $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$. Apparently, that $\rho(M-D) \leq 0$ is sufficient to make the necessary and sufficient condition hold. Therefore, that $M > D$ and $\rho \leq 0$ is sufficient to guarantee $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) > \frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$, and thus $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} > 0$.

Suppose $\rho(M-D) > 0$. Then, the necessary and sufficient condition $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) \geq \frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ can be equivalently transformed into $(\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. Therefore, under the condition $\rho(M-D) > 0$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ is always held if and only if for all $x^* \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, that $(\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ is always held. Hence, as long as all parameters satisfy $2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}} \leq 0$, the necessary and sufficient condition is always held, and thus $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ given that $\rho(M-D) > 0$. Since $\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}} = 0$ as long as $\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}} = 1$. Given M, D, ς and ς^*, and denoting the solution by $\tilde{\rho}$, then we have $\tilde{\rho}^2 = \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}$. Furthermore, as long as $\rho^2 < \tilde{\rho}^2$,

$2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}} < 0$. Therefore, if $M > D$, and $0 < \rho \leq \tilde{\rho}$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} \geq 0$ is always held, and $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x}{\tilde{\rho}}$. Therefore, combined with the results for the $\rho(M-D) \leq 0$ situation, it can be concluded that if $M > D$, for all $\rho \in (-1,\tilde{\rho}]$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} \geq 0$ is always held, and $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x}{\tilde{\rho}}$, where $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$.

Finally, the game is symmetric and hence $\varsigma = \varsigma^*$. Therefore, $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}} = \tilde{\rho}^*$ for $M > D$. Hence, both players have an identical range to ensure that their respective expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ always increases with respect to $\varepsilon \in \mathbb{R}$. *Q.E.D.*

Proof of Proposition 1: The proof of Proposition 1 is based on the proof of Lemma B1. We denote the set of ρ that makes $\mathbb{E}\Pi(x^*,\varepsilon)$ always increase with respect to ε given x^* by $\Gamma \equiv \{\rho|\rho \leq \tilde{\rho} \text{ if } M > D\}$. From Lemma B1, it has been known that given M and D, the necessary and sufficient condition for $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} \geq 0$ is $\rho \in \Gamma$. Therefore, it is certain that as long as ρ does not belong to Γ, $\mathbb{E}\Pi(x^*,\varepsilon)$ is not monotonic with respect to ε given any $x^* \in \mathbb{R}$. Equivalently, it means that for some ε, $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 < 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. Without loss of generality, Figure B1 geometrically presents a general description of the relationship between $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ and $z(\varepsilon) = 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ given x^*, M, D, ρ, ς and ς^* for all $\rho \notin \Gamma$.

According to the quadratic structure of $y(\varepsilon)$, as long as $\rho \notin \Gamma$, there should be two solutions to solve the equation $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 = 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. They are $\varepsilon_1 = \frac{\varsigma}{\rho\varsigma^*}x - \frac{x^*}{\rho}\sqrt{2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}}$ and $\varepsilon_2 = \frac{\varsigma}{\rho\varsigma^*}x + \frac{x^*}{\rho}\sqrt{2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}}$. Therefore, for $\varepsilon \leq \varepsilon_1$ or $\varepsilon \geq \varepsilon_2$, $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$, then $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} \geq 0$, where the equality is taken when $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$. For $\varepsilon_1 < \varepsilon < \varepsilon_2$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon} < 0$. Based on these results, without loss of generality, Figure B2 geometrically presents a general description of function $\mathbb{E}\Pi(x^*,\varepsilon)$ with respect to ε given any value of $x^* \in \mathbb{R}$, for all $\rho \notin \Gamma$.

Because for all $x^* \in \mathbb{R}$, given all primitives, expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ is always located between the line $M+\varepsilon$ and $D+\varepsilon$, and if $M > D$, increasing x^* will lift $\mathbb{E}\Pi(x^*,\varepsilon)$ upward, it is possible that for some value of x^*, there are two or three solu-

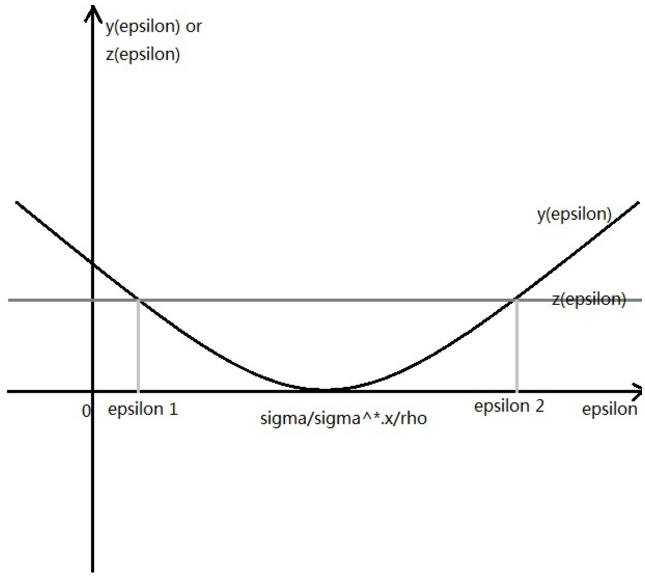


Figure B1: *A geometric description of the relationship between function $y(\varepsilon)$ and $z(\varepsilon)$ as long as $\rho \notin \Gamma$, where $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ and $z(\varepsilon) = 2(1-\rho^2)\ln\frac{\rho(M-D)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. There must be two intersection point which make $f(\varepsilon) = g(\varepsilon)$, and in this figure, they are denoted by ε_1 and ε_2, respectively. The function $y(\varepsilon)$ reaches its global minimum 0 at $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x^*}{\rho}$.*

tions of ε satisfying $\mathbb{E}\Pi(x^*,\varepsilon) = 0$. In Appendix D, we will prove that it is certain that for all $\rho \notin \Gamma$, by using a cutoff strategy, the game always contains a unique symmetric solution $g(s) = s$, such that given s, $\mathbb{E}\Pi(s,\varepsilon) = 0$ has three solutions, and the solution $\varepsilon = s$ is located at the middle where $\mathbb{E}\Pi(s,\varepsilon)$ decreases with respect to ε (see Figure B2). Apparently, the solution (s,s) self-contradicts the definition of the cutoff strategy under which it is derived. Hence, we cannot solve the game using the cutoff strategy concept for all $\rho \notin \Gamma$. Therefore, the set Γ not only indicates that $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$ but also characterizes the set of cutoff strategy Bayesian Nash equilibria of the symmetric strategic substitutes games. Therefore, Proposition 1 is obtained. *Q.E.D.*

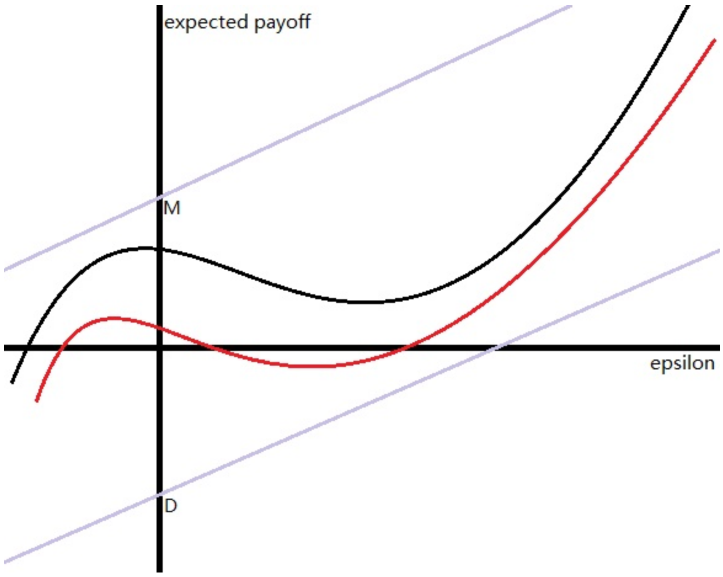


Figure B2: *A general description of expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ with respect to ε given any value of x, for all $\rho \notin \Gamma$. The position of $\mathbb{E}\Pi(x^*,\varepsilon)$ depends on x^*, and $\mathbb{E}\Pi(x^*,\varepsilon)$ is always located within $[D+\varepsilon, M+\varepsilon]$ for all $x^* \in \mathbb{R}$. If $M > D$, increasing x^* will lift $\mathbb{E}\Pi(x^*,\varepsilon)$ upward. In Appendix E, it is proven that as long as a cutoff strategy is used to solve the game, for all $\rho \notin \Gamma$, there always exists a solution (s,s) satisfying $g(s) = s$ such that given s, $\mathbb{E}\Pi(s,\varepsilon)$ behaves non-monotonically and has three intersections with the x-axis; this is indicated by the red curve.*

Appendix C

Appendix of Chapter 2

Derivation of the (Inverse) Best Response Function

The best response function, $g(x^*)$, is defined to satisfy $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. Therefore, we obtain $\sigma(x^*, g(x^*))(M-D)+D+g(x^*) = 0$, and further $\Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+g(x^*) = 0$. This equation can be equivalently transformed into $\frac{D+g(x^*)}{D-M} = \Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})$. Since the cumulative density function of normal distribution is invertible, we obtain $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}}$. Finally, we obtain the inverse best response function $x^* = \rho\frac{\varsigma^*}{\varsigma}g(x^*)+\varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$.

Still, for the definition equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, or $\Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+g(x^*) = 0$, we differentiate this equation with respect to x^* on both sides, and obtain $\mathbb{E}\Pi'_{x^*}(x^*, g(x^*)) + \mathbb{E}\Pi'_{\varepsilon}(x^*, g(x^*))g'(x^*) = 0$. Therefore, $g'(x^*) = -\frac{\mathbb{E}\Pi'_{x^*}(x^*,g(x^*))}{\mathbb{E}\Pi'_{\varepsilon}(x^*,g(x^*))} = -\frac{\frac{\partial\mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}}{\frac{\partial\mathbb{E}\Pi(x^*,\varepsilon)}{\partial\varepsilon}}|_{\varepsilon=g(x^*)} = -\frac{\sigma_{x^*}(x^*,g(x^*))(M-D)}{\sigma_{\varepsilon}(x^*,g(x^*))(M-D)+1} = \frac{1}{\frac{\varsigma^*}{\varsigma}\rho-\frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})^2)}{M-D}}$. $\sigma_{x^*}(x^*, g(x^*)) > 0$, and it is known that as long as $\rho \in \Gamma$, $\frac{\partial\mathbb{E}\Pi(x^*,g(x^*))}{\partial\varepsilon} \geq 0$; hence, if $M > D$, $g'(x^*) < 0$. Therefore, as long as the concept of cutoff strategy Bayesian Nash equilibria is applied to solve the game, i.e. $\rho \in \Gamma$, $g(x^*)$ globally decreases for a strategic substitutes game.

Appendix D

Appendix of Chapter 2

Proof of Proposition 2

Lemma D1: For all $x^* \in \mathbb{R}$, assume $\varsigma = \varsigma^*$ and $M > D$, there exists two functions $\rho'(x^*)$ and $\rho''(x^*)$. Given an $x^* \in \mathbb{R}$ and $\rho \in (-1,1)$, if $\rho \in (-1, \rho''(x^*))$, then $-1 < g'(x^*) < 0$; if $\rho \in (\rho''(x^*), \rho'(x^*))$, $g'(x^*) < -1$; at $\rho = \rho''(x^*)$, $g'(x^*) = -1$; at $\rho = \rho'(x^*)$, $g'(x^*) = \infty$.

Proof: We have $g^{-1'}(x^*) \equiv \frac{dx^*}{dg(x^*)} = \frac{1}{\frac{dg(x^*)}{dx^*}} = \rho + \frac{\varsigma^* \sqrt{2\pi(1-\rho^2)}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$; therefore, $\frac{dg^{-1'}(x^*)}{d\rho} = 1 - \frac{\varsigma^* \rho}{\sqrt{1-\rho^2}} \frac{\sqrt{2\pi}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Hence, given an $x^* \in \mathbb{R}$, if $\rho > 0$ and $M > D$, the function $g^{-1'}(x^*)$ must increase with respect to ρ. Besides, if $\rho = 0$, $g^{-1'}(x^*) < 0$, and if $\rho = 1$, $g^{-1'}(x^*) = 1$. Because $g^{-1'}(x^*)$ is a continuous function with respect to ρ, for all $\rho \in [0,1]$, $g^{-1'}(x^*)$ increases from a negative value to 1 as ρ increases from 0 to 1. Therefore, there must exist a $\rho \in [0,1]$, whose value depends on x^*, and it makes $g^{-1'}(x^*) = 0$. We denote this ρ by $\rho'(x^*)$. Since $g^{-1'}(x^*) < 0$ is equivalent to $g'(x^*) < 0$ and $g^{-1'}(x^*) > 0$ is equivalent to $g'(x^*) > 0$, we can conclude that given an $x^* \in \mathbb{R}$, for all $\rho \in [0, \rho'(x^*))$, $g'(x^*) < 0$, and for all $\rho \in (\rho'(x^*), 1]$, $g'(x^*) > 0$.

We define $A \equiv \frac{\varsigma^* \sqrt{2\pi}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. For $M > D$, $A < 0$. Hence, the equation $g^{-1'}(x^*) = 0$ can be equivalently expressed by $\rho + A\sqrt{1-\rho^2} = 0$. The solution $\rho'(x^*)$ that solves $\rho + A\sqrt{1-\rho^2} = 0$ equals $-\frac{A}{\sqrt{1+A^2}} > 0$. Because $\varsigma = \varsigma^*$, both players' $\rho'(x^*)$ function should be identical.

Because $g'(x^*) = g^{-1'}(x^*) = -1$, $\rho + \frac{\varsigma^* \sqrt{2\pi(1-\rho^2)}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = -1$, and hence, $\rho+1=-\frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)>0$ for $M>D$. Given M, D, ς and ς^*, the ρ that satisfies the equation $\rho+1=-\frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$ must depend on x^*. Thus, we denote the ρ that makes $g'(x^*)=g^{-1'}(x^*)=-1$ as $\rho''(x^*)$. For this incomplete information game, $\rho''(x^*)$ should not be equal to ±1. $g^{-1'}(x^*)=-1$ can be equivalently expressed by $\rho+1=-\sqrt{1-\rho^2}A$. Solving this equation, we get two solutions: $\rho''(x^*)=-1$ and $\rho''(x^*)=\frac{A^2-1}{A^2+1}$; the first solution is excluded on the basis of the previous argument. Therefore, $\rho''(x^*)=\frac{A^2-1}{A^2+1}$.

Because given an $x^*\in\mathbb{R}$, $\rho''(x^*)$ is unique, for all $\rho\in(-1,\rho''(x^*))$ or $\rho\in(\rho''(x^*),\rho'(x^*))$, $g'(x^*)$ is either greater or smaller than -1. To judge in which interval $g'(x^*)$ is smaller or greater than -1, let us recall the derivative of $g^{-1'}(x^*)$ with respect to $\rho\in(-1,1)$:

$$\frac{dg^{-1'}(x^*)}{d\rho}=1-\frac{\varsigma^*\rho}{\sqrt{1-\rho^2}}\frac{\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$$

Because at $\rho=\rho''(x^*)$, $\frac{dg^{-1'}(x^*)}{d\rho}=\frac{1}{1-\rho''(x^*)}>0$, for all $\rho\in(\rho''(x^*)-\varepsilon,\rho''(x^*)+\varepsilon)$, the function $g^{-1'}(x^*)$ increases with respect to ρ. Because at $\rho=\rho''(x^*)$, $g^{-1'}(x^*)=-1$, for all $\rho\in(\rho''(x^*)-\varepsilon,\rho''(x^*))$, $g^{-1'}(x^*)<-1$, and for all $\rho\in(\rho''(x^*),\rho''(x^*)+\varepsilon)$, $g^{-1'}(x^*)>-1$. In addition, because $\rho''(x^*)$ is unique, this result can be extended to the whole interval $\rho\in(-1,1)$. Thus, for all $\rho\in(-1,\rho''(x^*))$, $g^{-1'}(x^*)<-1$, and for all $\rho\in(\rho''(x^*),1)$, $g^{-1'}(x^*)>-1$.

The relationship between $\rho'(x^*)$ and $\rho''(x^*)$: Recall that $\rho'(x^*)>0$. If $\rho''(x^*)\leq0$, then it is certain that $\rho'(x^*)>\rho''(x^*)$. If $\rho''(x^*)$ is positive, $g^{-1'}(x^*)$ increases with respect to ρ when $\rho>0$ and at $\rho=\rho'(x^*)$, $g^{-1'}(x^*)=0$, and at $\rho=\rho''(x^*)$, $g^{-1'}(x^*)=-1$. Therefore, $\rho'(x^*)>\rho''(x^*)$ if $\rho''(x^*)>0$. In conclusion, if $M>D$, $\rho'(x^*)$ is always strictly greater than $\rho''(x^*)$.

Because $\varsigma=\varsigma^*$, both players' $\rho'(x^*)$ and $\rho''(x^*)$ function are identical. Therefore, in conclusion, for $M>D$, the function $g(x^*)$ whose inverse form is $x^*=\rho\frac{\varsigma^*}{\varsigma}g(x^*)+\varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$ has the following property: given an $x^*\in\mathbb{R}$ and $\rho\in(-1,1)$, if $\rho<\rho''(x^*)$, $g'(x^*)>-1$; if $\rho''(x^*)<\rho<\rho'(x^*)$, $g'(x^*)<-1$; if $\rho=\rho''(x^*)$, $g'(x^*)=-1$; if $\rho=\rho'(x^*)$, $g'(x^*)=\infty$; and if $\rho>\rho'(x^*)$, $g'(x^*)>0$. *Q.E.D.*

Lemma D2: For $M>D$, for all $\rho\in[0,\rho'(x^*)]$, $g'(x^*)$ decreases from a negative value to $-\infty$, and for all $\rho \in (\rho'(x^*), 1)$, $g'(x^*)$ decreases from $+\infty$ to 1.

Proof: From the proof of Lemma D1, it has been known that given an $x^* \in \mathbb{R}$, $g^{-1'}(x^*)$ increases with respect to ρ for $\rho \in [0,1)$ and it is continuous with respect to ρ. Therefore, $g'(x^*)$ must perform decreasing property in the interval $\rho \in [0,1)$. At $\rho = 0$, $g^{-1'}(x^*) < 0$ and hence $g'(x^*) < 0$; at $\rho = \rho'(x^*)$, $g^{-1'}(x^*) = 0$ and hence $g'(x^*) = \infty$; and at $\rho = 1$, $g^{-1'}(x^*) = 1$ and hence $g'(x^*) = 1$. Therefore, for function $g'(x^*)$, there is a discontinuity point at $\rho = \rho'(x^*)$. For $\rho \in [0, \rho'(x^*))$, $g'(x^*)$ should decrease from a negative value to $-\infty$, and for $\rho \in (\rho'(x^*), 1)$, $g'(x^*)$ should decrease from $+\infty$ to 1. *Q.E.D.*

Lemma D3: For $M > D$, given an $x^* \in \mathbb{R}$, $g'(x^*)$ is concave for $\rho \in (-1,0)$. It reaches its maximum value at $\rho = -\frac{1}{\sqrt{1+A^2}}$.

Proof: Given an $x^* \in \mathbb{R}$ and $M > D$, for all $\rho \in (-1,0)$, because $\frac{d^2 g'(x^*)}{d\rho^2} = 2(g^{-1'}(x^*))^{-3}(\frac{dg^{-1'}(x^*)}{d\rho})^2 - (g^{-1'}(x^*))^{-2}\frac{d^2 g^{-1'}(x^*)}{d\rho^2}$, $\frac{d^2 g^{-1'}(x^*)}{d\rho^2} = -A\frac{\sqrt{1-\rho^2}+\frac{\rho^2}{\sqrt{1-\rho^2}}}{1-\rho^2} > 0$, and for $\rho < 0$, $g^{-1'}(x^*) < 0$, therefore $\frac{d^2 g^{-1'}(x^*)}{d\rho^2} < 0$ for $\rho \in (-1,0)$. Hence, $g'(x^*)$ is concave for all $\rho \in (-1,0)$. Further, by calculating the first-order condition, it is found that $g^{-1'}(x^*) = \rho + \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$ reaches its maximum value at $\rho = -\frac{1}{\sqrt{1+A^2}} < 0$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. We denote this ρ by $\rho'''(x)$. Thus, given an $x^* \in \mathbb{R}$, the function $g'(x^*)$ reaches its maximum value at $\rho'''(x) = -\frac{1}{\sqrt{1+A^2}}$ for all $\rho \in (-1,0)$, and the maximum value of $g'(x^*)$ is just $-\frac{1}{\sqrt{1+A^2}}$. *Q.E.D.*

For $M > D$, according to Lemmas D1, D2 and D3, the shape of $g'(x^*)$ with respect to ρ given an $x^* \in \mathbb{R}$ can be generally represented as shown in Figure D1.

Lemma D4: Given an $x^* \in \mathbb{R}$ and assuming $M > D$, for $g(x^*) \in (-\frac{M+D}{2}, -D]$, $\frac{d\rho'(x^*)}{dg(x^*)} > 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} > 0$; for $g(x^*) \in [-M, -\frac{M+D}{2})$, $\frac{d\rho'(x^*)}{dg(x^*)} < 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} < 0$.

Proof: Let us recall that $\rho'(x^*) = -\frac{A}{\sqrt{1+A^2}}$ and $\rho''(x^*) = \frac{A^2-1}{A^2+1}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Because $M > D, A < 0$. By the chain rule, $\frac{d\rho'(x^*)}{dg(x^*)} = \frac{d\rho'(x^*)}{dA}\frac{dA}{dg(x^*)}$, and $\frac{d\rho''(x^*)}{dg(x^*)} = \frac{d\rho''(x^*)}{dA}\frac{dA}{dg(x^*)}$. Because $\rho'(x^*) = -\frac{A}{\sqrt{1+A^2}} = \frac{1}{\sqrt{\frac{1}{A^2}+1}}$ and $\rho''(x^*) = \frac{A^2+1-2}{A^2+1} = 1 - \frac{2}{A^2+1}$, as A increases, $\rho'(x^*)$ decreases and $\rho''(x^*)$ decreases. Hence, $\frac{d\rho'(x^*)}{dA} < 0$

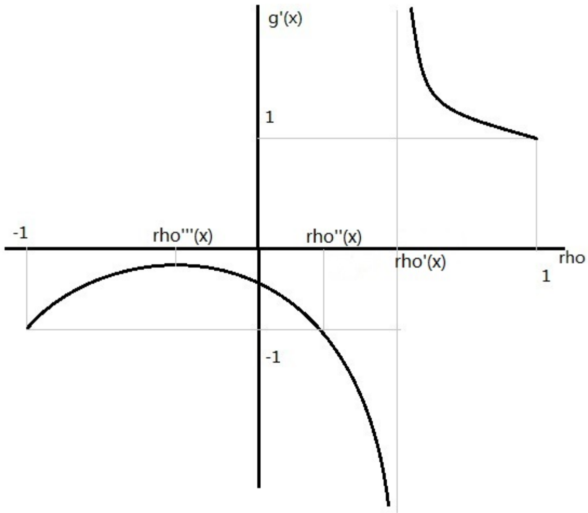


Figure D1: *A general geometric description of function $g'(x^*)$ with respect to ρ for $M > D$ given an $x^* \in \mathbb{R}$.*

and $\frac{d\rho''(x^*)}{dA} < 0$. If $g(x^*) < -\frac{M+D}{2}$, $\frac{dA}{dg(x^*)} > 0$ and if $g(x^*) > -\frac{M+D}{2}$, $\frac{dA}{dg(x^*)} < 0$; therefore, if $g(x^*) \in (-\frac{M+D}{2}, -D]$, $\frac{d\rho'(x^*)}{dg(x^*)} > 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} > 0$, and if $g(x^*) \in [-M, -\frac{M+D}{2})$, $\frac{d\rho'(x^*)}{dg(x^*)} < 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} < 0$. Hence, at $g(x^*) = -\frac{M+D}{2}$, both $\rho'(x^*)$ and $\rho''(x^*)$ reach their global minimum value with respect to $g(x^*)$. The minimum values of $\rho'(x^*)$ and $\rho''(x^*)$ with respect to $g(x^*)$ are $\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}}$ and $\frac{2\pi\varsigma^{*2}-(M-D)^2}{2\pi\varsigma^{*2}+(M-D)^2}$, respectively. *Q.E.D.*

Based on Lemmas D1 and D4, Figure D2 generally depicts functions $\rho'(x^*)$ and $\rho''(x^*)$ with respect to $g(x^*)$. According to Lemmas D1, D2 and D3, given an $x^* \in \mathbb{R}$ and hence $g(x^*)$, as ρ increases from -1 to 1, for $\rho \in (-1, \rho''(x^*))$, $-1 < g'(x^*) < 0$; for $\rho \in (\rho''(x^*), \rho'(x^*))$, $g'(x^*) < -1$; for $\rho \in (\rho'(x^*), +\infty)$, $g'(x^*) > 0$. At $\rho = \rho''(x^*)$, $g'(x^*) = -1$ and at $\rho = \rho'(x^*)$, $g'(x^*) = \infty$. According to these properties, the general shape of $g'(x^*)$ can be illustrated by Figure D2. We choose an arbitrary value of $g(x^*)$ between $-M$ and $-D$, and at this chosen $g(x^*)$, we draw a vertical line from -1 to 1 (the red line in Figure D2). The curves $\rho''(x^*)$ and $\rho'(x^*)$ dissect this line into three parts, on which $g'(x^*) > -1$, $g'(x^*) < -1$ and $g'(x^*) > 0$ from bottom to top. Because $g(x^*)$ is arbitrarily chosen, this result applies for all $g(x^*) \in [-M, -D]$. Therefore, it can be concluded that given M, D, ς and ς^*, for all $g(x^*) \in [-M, -D]$, if $\rho \in (-1, \rho''(x^*))$, $g'(x^*) > -1$ and correspondingly it is the area below the curve $\rho''(x^*)$ in Figure D2; for all $g(x^*) \in [-M, -D]$, if $\rho \in (\rho''(x^*), \rho'(x^*))$, $g'(x^*) < -1$ and correspondingly it is the area between the curve $\rho''(x^*)$ and the curve $\rho'(x^*)$; finally, for all $g(x^*) \in [-M, -D]$, if $\rho \in (\rho'(x^*), 1)$, $g'(x^*) > 0$ and correspondingly it is the area above the curve $\rho'(x^*)$.

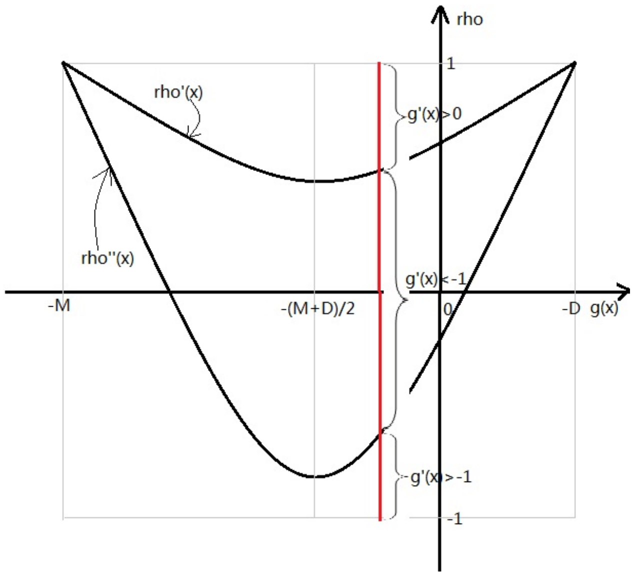


Figure D2: *A general geometric description of functions $\rho'(x^*)$ and $\rho''(x^*)$ with respect to $g(x^*)$ for $M > D$.*

Lemma D5: For $M > D$, given a $\rho \in [\min \rho''(x^*), 1)$, there are one or two values of $g(x^*)$ that make $g'(x^*) = -1$. They are $g(x^*) = -(M-D)\Phi(\pm\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$.

Proof: Given M, D, ς and ς^*, if there are $g(x^*)$s whose derivative $g'(x^*) = -1$, then the corresponding $\rho \in [\min \rho''(x^*), 1)$ and $g(x^*)$ should satisfy $\rho = \frac{A^2-1}{A^2+1}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Therefore, $A^2 = \frac{1+\rho}{1-\rho}$, i.e. $\frac{2\pi\varsigma^{*2}}{(M-D)^2} \exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{1+\rho}{1-\rho}$. Therefore, $\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}} \geq 1$, where the latter equality is held if and only if $\rho = \min \rho''(x^*) = \frac{2\pi\varsigma^{*2}-(M-D)^2}{2\pi\varsigma^{*2}+(M-D)^2}$. Therefore, $[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2 = \ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}$, and $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \pm\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}$. Hence, we get two solutions: $g(x^*)_1 = -(M-D)\Phi(\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$ and $g(x^*)_2 = -(M-D)\Phi(-\sqrt{\ln \frac{1+\rho}{1-\rho} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$. Note that $g(x^*)_1 \leq g(x^*)_2$, where the equality is obtained as long as $\rho = \min \rho''(x^*)$. *Q.E.D.*

Remark: Given all primitives, Lemma D5 must be held in the subinterval $\rho \in [\min \rho''(x^*), \min \rho'(x^*)]$ as well.

Lemma D6: Given a $\rho \in (\min \rho'(x^*), 1)$ if $M > D$, there are two values of $g(x^*)$

that makes $g'(x^*) = \infty$. They are $g(x^*) = -(M-D)\Phi(\pm\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$. If $\rho = \min\rho'(x^*)$ for $M > D$, at $g(x^*) = -\frac{M+D}{2}$, $g'(x^*) = \infty$.

Proof: Given M, D, ς and ς^*, if there are $g(x^*)$s whose derivative $g'(x^*) = \infty$, which means $\frac{1}{g'(x^*)} = 0$, then the corresponding $\rho \in (\min\rho'(x^*), 1)$ for $M > D$, and the $g(x^*)$ should simultaneously satisfy $\rho = \frac{-A}{\sqrt{1+A^2}}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Therefore, $A^2 = \frac{\rho^2}{1-\rho^2}$, i.e. $\frac{2\pi\varsigma^{*2}}{(M-D)^2}\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{\rho^2}{1-\rho^2}$. Hence, $\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}} \geq 1$, where the latter equality is held if and only if $\rho = \min\rho'(x^*) = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}}$ for $M > D$. Thus, $[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2 = \ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}$ and $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \pm\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}$, and we get two solutions: $g(x^*)_1 = -(M-D)\Phi(\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$ and $g(x^*)_2 = -(M-D)\Phi(-\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$. Note that for $M > D$, $g(x^*)_1 \leq g(x^*)_2$, where the equality is obtained as long as $\rho = \min\rho'(x^*)$. If the equality is held, $g(x^*) = g(x^*)_1 = g(x^*)_2 = -\frac{M+D}{2}$. *Q.E.D.*

Proof of Proposition 2: According to Figure D2 and results from Lemmas D5 and D6, we obtain the following conclusion about the shape of $g(x^*)$ given M, D, ς and ς^* for all $\rho \in (-1, 1)$:

1) for $\rho \in (-1, \min\rho''(x^*))$, $-1 < g'(x^*) < 0$ globally;

2) for $\rho \in [\min\rho''(x^*), \min\rho'(x^*)]$,

I. if $g(x^*) \in [-M, -(M-D)\Phi(\sqrt{\ln\frac{1+\rho}{1-\rho}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $-1 < g'(x^*) < 0$;

II. if $g(x^*) \in (-(M-D)\Phi(\sqrt{\ln\frac{1+\rho}{1-\rho}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -(M-D)\Phi(-\sqrt{\ln\frac{1+\rho}{1-\rho}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) < -1$;

III. if $g(x^*) \in (-(M-D)\Phi(-\sqrt{\ln\frac{1+\rho}{1-\rho}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -D]$, $-1 < g'(x^*) < 0$;

IV. if $g(x^*) = -(M-D)\Phi(\pm\sqrt{\ln\frac{1+\rho}{1-\rho}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = -1$.

3) for $\rho \in (\min\rho'(x^*), 1)$,

I. if $g(x^*) \in [-M, -(M-D)\Phi(\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) < 0$;

II. if $g(x^*) \in (-(M-D)\Phi(\sqrt{\ln \frac{\rho^2}{1-\rho^2} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -(M-D)\Phi(-\sqrt{\ln \frac{\rho^2}{1-\rho^2} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) > 0$;

III. if $g(x^*) \in (-(M-D)\Phi(-\sqrt{\ln \frac{\rho^2}{1-\rho^2} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D, -D]$, $g'(x^*) < 0$;

IV. if $g(x^*) = -(M-D)\Phi(\pm\sqrt{\ln \frac{\rho^2}{1-\rho^2} \frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = \infty$.

It is straightforward to find that the description of the shape of $g(x^*)$ is still held even if the payoff specification for both players is asymmetric, because from Lemmas D1 to D6, we only focus on studying the properties of a single best response function.

In the following, we will prove that the shape of $g(x^*)$ in 3), the non-monotonic $g(x^*)$, contradicts the definition of the cutoff strategy concept; hence, for the symmetric strategic substitutes game, using the cutoff strategy concept to solve the game is valid if and only if $\rho \in (-1, \tilde{\rho}]$.

Let us recall expected payoff function $\mathbb{E}\Pi(x,\varepsilon) = \sigma(x^*,\varepsilon)(M-D) + D + \varepsilon$ and the belief $\sigma(x^*,\varepsilon) = \int_{-\infty}^{x^*} f(\varepsilon^*|\varepsilon)d\varepsilon^*$. These formulations are exeactly following the definition of the cutoff strategy concept in the paper. The definition of the cutoff strategy just states that a player's action choice should monotonically non-decrease with respect to the player type. It does not explicitly state that at the equilibrium, given one player's cutoff strategy x^*, another player's best response $g(x^*)$ should be unique. However, if given an opponent's strategy, there are more than one best responses, this situation naturally fails the definition of the cutoff strategy. For example, given an opponent's strategy ε^*, there are two cutoff strategies (best responses) towards ε^*, ε_1 and ε_2, where $\varepsilon_1 < \varepsilon_2$, such that $\mathbb{E}\Pi(\varepsilon^*,\varepsilon) = 0$ and $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$. Then, for a payoff shock $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, according to the definition of the cutoff strategy, because $\varepsilon > \varepsilon_1$, the player should choose being active, but because $\varepsilon < \varepsilon_2$, the player chooses being inactive, leading to a contradiction. Therefore, the definition of the cutoff strategy has implicitly dictated that if a cutoff strategy is adopted, given one player's cutoff strategy, the other player's best response should be unique. This conclusion is held irrespective of the specification of payoffs.

As long as $\rho \notin \Gamma$, in the proof of Proposition 1 (Appendix B), we have proven that given $x^* \in \mathbb{R}$, as ε increases from $-\infty$ to $+\infty$, $\mathbb{E}\Pi(x^*,\varepsilon)$ first increases, then decreases, and finally increases. In addition, we showed that in the strategic substitutes game, for all $\rho > \tilde{\rho}$, as $g(x^*)$ decreases from $-D$ to $-M$, function $g(x^*)$ first decreases, then increases, and finally decreases. In fact, the change of monotonicity of $\mathbb{E}\Pi(x^*,\varepsilon)$ and $g(x^*)$ is synchronized, because $g'(x^*) = -\frac{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}}{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}}|_{\varepsilon=g(x^*)}$. Since $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*} > 0$ for $M > D$, for any point $(x^*, g(x^*))$ from the function $g(x^*)$, if $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}|_{\varepsilon=g(x^*)} \gtrless 0$, $g'(x^*) \lessgtr 0$ for $M > D$, and vice versa.

Solving the symmetric game for all $\rho > \tilde{\rho}$, there still exists a symmetric solution that satisfies $x^* = g(x^*)$. However, at this symmetric solution, $g'(x^*) > 0$, because for a solution that satisfies $x^* = g(x^*)$, its derivative $g'(x^*)$ can be expressed as

$$g'(x^*) = \frac{1}{\rho - \frac{\varsigma^* \sqrt{2\pi(1-\rho^2)} \exp(\frac{1}{2}\frac{1-\rho}{1+\rho}\frac{g^2(x^*)}{\varsigma^2})}{M-D}}$$

In addition, because $\rho^2 > \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}$, we get $\rho^2 > \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2} > \frac{2\pi\varsigma^2(1-\rho^2)}{(M-D)^2}$, and because $g'(x^*)|_{g(x^*)=0} = \frac{1}{\rho - \frac{\varsigma\sqrt{2\pi(1-\rho^2)}}{M-D}}$, for $M > D$ and $\rho > \tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$, we get $g'(x^*)|_{g(x^*)=0} > 0$. For $M > D$, if we regard $g'(x^*)$ as a function with respect to variable $g^2(x^*)$, $\frac{\partial g'(x^*)}{\partial g^2(x^*)} > 0$, then for any symmetric solution (s,s), $g'(x^*)|_{g(x^*)=s} \geq g'(x^*)|_{g(x^*)=0} > 0$. Therefore, for the symmetric strategic substitutes game, as long as $\rho > \tilde{\rho}$, at the symmetric solution (s,s), the derivative $g'(s) > 0$, and correspondingly, $\frac{\partial \mathbb{E}\Pi(s,\varepsilon)}{\partial \varepsilon}|_{\varepsilon=s} < 0$. So as long as $\rho > \tilde{\rho}$, given a symmetric solution (s,s) such that the equation $\mathbb{E}\Pi(s,\varepsilon) = 0$ has a solution $\varepsilon = s$, according to the proof of Proposition 1 in Appendix B, without loss of generality, function $\mathbb{E}\Pi(s,\varepsilon)$ is simply the red curve in Figure B2, and $\varepsilon = s$ is the middle intersection point where the expected payoff function decreases. Therefore, if we still use a cutoff strategy to solve the game for $\rho > \tilde{\rho}$, we will always get a symmetric solution (s,s), for which function $\mathbb{E}\Pi(s,\varepsilon) = 0$ has three values of ε, including $\varepsilon = s$, that make the equation hold; more importantly, the symmetric solution (s,s) itself contradicts the cutoff strategy definition under which it is derived. Therefore, we cannot use the cutoff strategy concept to solve the symmetric strategic substitutes game for $\rho > \tilde{\rho}$ given $M > D$ and $\varsigma = \varsigma^*$. This supplements the existing proof of Proposition 1 in Appendix B, and the proof of Proposition 1 is now complete.

As long as the strategic substitutes game is symmetric, the conclusions 1) and 2) constitute Proposition 2. However, if the payoff specification is asymmetric, these results are still held for describing a single player's $g(x^*)$ function as these results are derived by studying only a single $g(x^*)$ function. Hence, Proposition 2 is held in asymmetric payoff settings also. *Q.E.D.*

Appendix E

Appendix of Chapter 2 Uniqueness/Multiplicity and Stability of Equilibrium

Lemma E1 (Correspondence between the Position of Symmetric Equilibrium and the Sign of $M+D$): We denote the symmetric equilibrium of the strategic substitutes game by (s,s). There is a one-to-one correspondence relationship between the position of the symmetric equilibrium and the sign of $M+D$, which is $M+D \gtreqless 0$ if and only if $0 \gtreqless s \gtreqless -\frac{M+D}{2}$.

Proof: Let us recall the (inverse) best response function $g(x^*)$ that is expressed as $x^* = \rho g(x^*) + \varsigma^* \sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$. We denote symmetric equilibria by (s,s). It should satisfy $g(s) = s$. Hence, we get $s = \rho g(s) + \varsigma^* \sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(s)}{D-M})$. After a series of transformation of the previous equation, we get the following results which necessarily and sufficiently contain the symmetric equilibrium: $\Phi(\sqrt{\frac{1-\rho}{1+\rho}}\frac{s}{\varsigma^*}) = \frac{D}{D-M} + \frac{s}{D-M}$. Apparently, the symmetric equilibrium (s,s) can be equivalently regarded as the solution from the following equation group:

$$y = \Phi(\sqrt{\frac{1-\rho}{1+\rho}}\frac{s}{\varsigma^*})$$

$$y = \frac{D}{D-M} + \frac{s}{D-M} \tag{E.1}$$

Note that the function $y = \frac{D}{D-M} + \frac{s}{D-M}$ always crosses the point $(s,y) = (-\frac{M+D}{2}, \frac{1}{2})$.

Hence, the relative position between $y = \frac{D}{D-M} + \frac{s}{D-M}$ and $y = \Phi(\sqrt{\frac{1-\rho}{1+\rho}}\frac{s}{\varsigma^*})$ depends on the sign of $M+D$, and therefore, the position of the symmetric equilibrium should have some relationship with the sign of $M+D$ as well. Figures E1, E2 and E3 illurstrate all possibilities of the positions of symmetric equilibrium (s,s) in terms of $M+D>0$, $M+D=0$ and $M+D<0$, respectively (see Figures E1, E2 and E3).

According to the analysis from Figures E1–E3, it can be concluded that 1) in the symmetric strategic substitutes game, there always exists a unique symmetric equilibrium; and 2) the position of symmetric equilibrium (s,s) and the sign of $M+D$ have the following correspondence relationship: $M+D \gtreqless 0 \Longleftrightarrow 0 \gtreqless s \gtreqless -\frac{M+D}{2}$. *Q.E.D.*

Remark: The proof and result of Lemma E1 also apply to the case where $\rho > \tilde{\rho}$ for $M > D$. Hence, we can extend the results in Lemma E1 to the region where it is illegitimate to use the cutoff strategy concept to solve the game and correspondingly the 'symmetric equilibrium' should be called 'symmetric solution' instead.

Lemma E2: Assume $M > D$ and $\varsigma = \varsigma^*$. Define function $g(x^*;C)$, which is expressed by its inverse form $x^* = \rho g(x^*;C) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(C + \frac{g(x^*;C)}{D-M})$, where C is an arbitrary constant. If $\rho \leq \tilde{\rho}$, where $\tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(M-D)^2}}$, $g(x^*;C)$ decreases with respect to $x^* \in \mathbb{R}$. For the equation system

$$x^* = \rho g(x^*;C) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(C + \frac{g(x^*;C)}{D-M})$$
$$g(x^*;C) = \rho x^* + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(C + \frac{x^*}{D-M}) \tag{E.2}$$

there always exists a unique solution which satisfies $g(x^*;C) = x^*$. We call it the symmetric solution. If there exists other solutions, for which $g(x^*;C) \neq x^*$, they must appear in even number. We call these solutions the asymmetric solution.

Proof: Define function $F(x^*;\varepsilon) = \Phi(\frac{x^*-\rho\varepsilon}{\varsigma^*\sqrt{1-\rho^2}})(M-D) + C\times(D-M) + \varepsilon$. $g(x^*;C)$ is the function that satsifies $F(x^*;g(x^*;C)) = 0$. Analogous to the proof of Proposition 1, it is easy to show that as long as $\rho \leq \tilde{\rho}$, $F(x^*,\varepsilon)$ globally increases with respect to $\varepsilon \in \mathbb{R}$ and $g(x^*;C)$ globally decreases with respect to $x^* \in \mathbb{R}$. Here, we do not go into details about these results, and interested readers could refer the proof of Proposition 1 to verify these results.

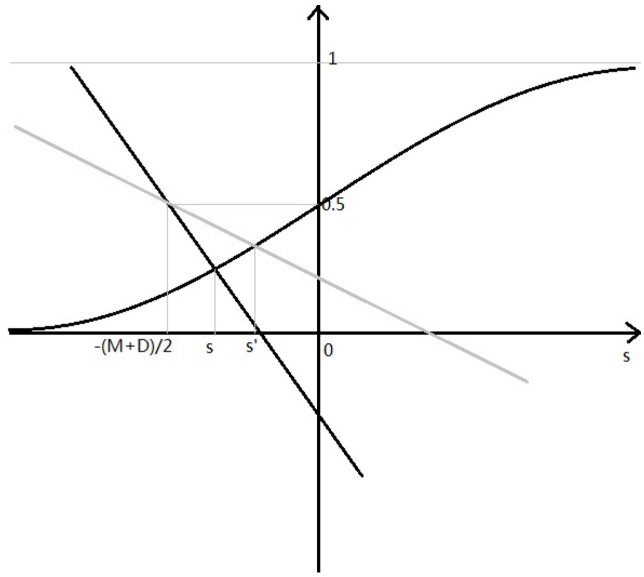


Figure E1: *Given $M > D$, $\varsigma = \varsigma^*$, ρ and $M + D > 0$, there always exists a unique intersection point between $y = \Phi(\sqrt{\frac{1-\rho}{1+\rho}} \frac{s}{\varsigma^*})$ and $y = \frac{D}{D-M} + \frac{s}{D-M}$, which indicates that the symmetric equilibrium always exists and it is unique under the given conditions, and symmetric equilibrium (s,s) must satisfy $-\frac{M+D}{2} < s < 0$.*

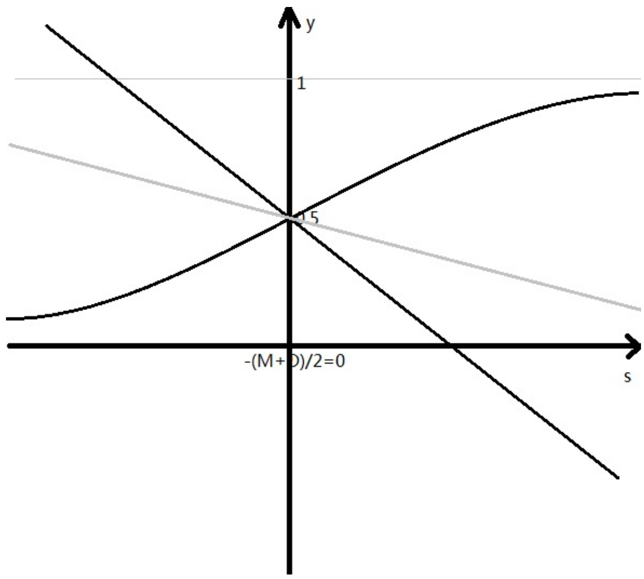


Figure E2: *Given $M > D$, $\varsigma = \varsigma^*$, ρ and $M + D = 0$, there always exists a unique intersection point $(s,y) = (0, \frac{1}{2})$ between $y = \Phi(\sqrt{\frac{1-\rho}{1+\rho}} \frac{s}{\varsigma^*})$ and $y = \frac{D}{D-M} + \frac{s}{D-M}$, which indicates that the symmetric equilibrium always exists and it is unique under the given conditions. Moreover, symmetric equilibrium (s,s) must satisfy $-\frac{M+D}{2} = s = 0$.*

The two equations in equation group E2 can be regarded as the two players' best response functions. They are symmetrically located around the 45° line. By referring

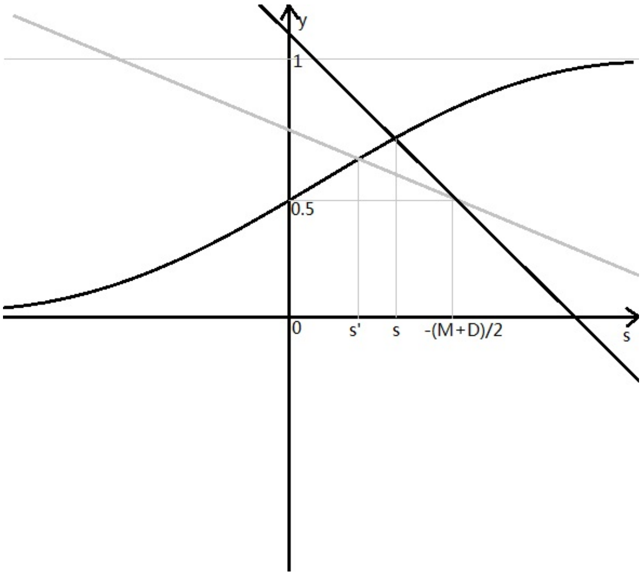


Figure E3: *Given $M > D$, $\varsigma = \varsigma^*$, ρ and $M + D < 0$, there always exists a unique intersection point between $y = \Phi(\sqrt{\frac{1-\rho}{1+\rho}}\frac{s}{\varsigma^*})$ and $y = \frac{D}{D-M} + \frac{s}{D-M}$, which indicates that the symmetric equilibrium always exists and it is unique under the given conditions, and symmetric equilibrium (s,s) must satisfy $0 < s < -\frac{M+D}{2}$.*

to the proof of Lemma E1, it is known that the symmetric solution $g(x^*;C) = x^*$ always exists and it is unique for all $\rho \leq \tilde{\rho}$.

If there exist solutions other than the symmetric solution, it is certain that for these solutions, $g(x^*;C) \neq x^*$. If we get one asymmetric solution, then we can get another one correponding to it by simply switching players' identities. Therefore, if there exist asymmetric solutions, they must appear in pairs, and hence the number of asymmetric solutions must be in even number.

If $C = \frac{D}{D-M}$, then $g(x^*;C)$ becomes the best response function $g(x^*)$ of the game we are analysing. Therefore, it can also be concluded that if there exist asymmetric equilibria, they must appear in even number. *Q.E.D.*

Lemma E3: In the symmetric strategic substitutes game, if there exist asymmetric equilibria, their number is two. Given all primitives, the necessary and sufficient condition that the symmetric equilibrium is not unstable is that the game only contains a unique equilibrium, which is the symmetric equilibrium. Given all primitives, the necessary and sufficient condition that the symmetric equilibrium is unstable is that the game exhibits multiple (three) equilibria.

Proof: Let us recall the inverse best response function $x^* = \rho g(x^*) + \varsigma^* \sqrt{1-\rho^2}\Phi^{-1}(\frac{D}{D-M} + \frac{g(x^*)}{D-M})$, which can be equivalently represented by $\frac{D+g(x^*)}{D-M} = \Phi(\frac{x^*-\rho g(x^*)}{\varsigma^*\sqrt{1-\rho^2}})$. Then, we transform $g(x^*)$ into polar coordinate representation. We define $x^* = r\cos\theta$ and $g(x^*) = r\sin\theta$, where $r \geq 0$ is the radius and θ is the radian. Hence, the best response function can be expressed as

$$\Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{D + r\sin\theta}{D-M}$$

Arranging the terms on RHS and LHS, we obtain the following result:

$$\frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{D}{D-M}$$

We define $p(\theta,r) = \frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)$, and so $p'_\theta(\theta,r) = \frac{r\cos\theta}{M-D} - \phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$. Therefore, the best response function can be equivalently represented by $p(\theta,r) = \frac{D}{D-M}$. Because a player's and the opponent's best response functions should be symmetric around the line $g(x^*) = x^*$ (i.e. the 45° line) in Cartesian coordinates, denoting $q(\theta,r) = \frac{r\cos\theta}{M-D} + \Phi(\frac{\sin\theta-\rho\cos\theta}{\varsigma\sqrt{1-\rho^2}}r)$, the opponent's best response function can be represented by $q(\theta,r) = \frac{D}{D-M}$. Hence, the equilibria of the game (θ,r) are simultaneously determined by functions $p(\theta,r) = \frac{D}{D-M}$ and $q(\theta,r) = \frac{D}{D-M}$.

For the function $g(x^*,C)$, if there exists a point $(x^*, g(x^*,C))$ that reaches $(0,0)$ with zero distance, i.e. $r = 0$, it must be unique according to the monotonicity of function $g(x^*;C)$ for all $\rho \leq \tilde{\rho}$. Therefore, according to Lemma E2, because asymmetric solutions must appear in pairs, their radius does not equal zero. Therefore, in this proof, we only need to consider the $r > 0$ situation.

In terms of all possible values of the radian of asymmetric equilibria (θ,r), the best response function $g(x^*)$ can be separated into the following two cases:

In Case I, the radian of the symmetric equilibrium is $\frac{5}{4}\pi$, and in Case II, it is $\frac{\pi}{4}$. The range of radians for possible asymmetric equilibrium is $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$ in Case I and $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$ in Case II. In addition, it is necessary that a pair of asymmetric equilibria

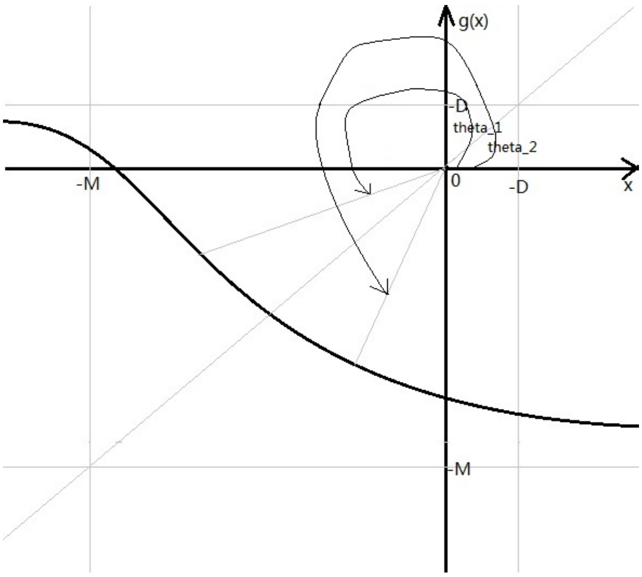


Figure E4: *Case I: The value of radian* θ *ranges in* $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$.

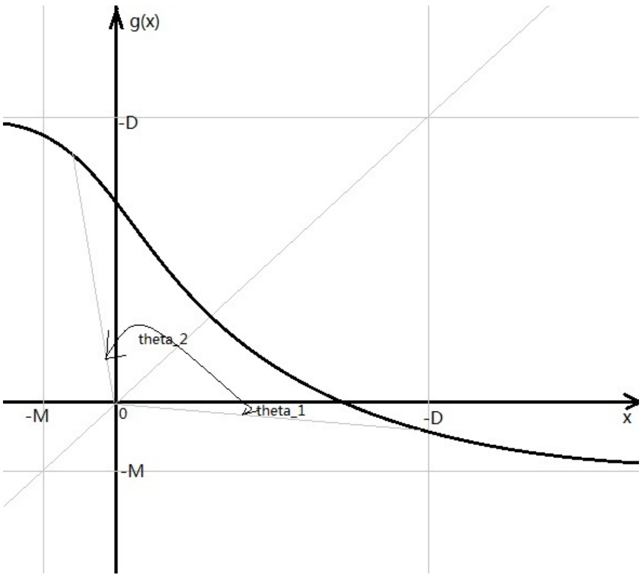


Figure E5: *Case II: The value of radian* θ *ranges in* $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi)$.

(θ_1, r) and (θ_2, r) must satisfy $\theta_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta_1$ in Case I, and $\theta_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta_1$ in Case II.

Let us first study Case I. Since the asymmetric equilibria must be symmetrically located around the 45° line (in Cartesian coordinates), we first focus on $\theta \in [\pi, \frac{3}{2}\pi]$, and then study the entire interval $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$.

Given radius $r > 0$, we have $p'_\theta(\pi, r) = r[\frac{-1}{M-D} + \frac{1}{\sqrt{2\pi}\varsigma^*}[\frac{\rho}{\sqrt{1-\rho^2}}\exp(-\frac{1}{2}\frac{r^2}{\varsigma^{*2}(1-\rho^2)})]]$. Because we consider $\rho \le \tilde{\rho}$, which is equivalent to $\frac{\rho}{\sqrt{1-\rho^2}} \le \frac{\sqrt{2\pi}\varsigma^*}{M-D}$, for all $\rho \le \tilde{\rho}$, $\frac{1}{\sqrt{2\pi}\varsigma^*}\frac{\rho}{\sqrt{1-\rho^2}} \le \frac{1}{M-D}$, and we further have

$$\frac{1}{\sqrt{2\pi}\varsigma^*}\frac{\rho}{\sqrt{1-\rho^2}}\exp(-\frac{1}{2}\frac{r^2}{\varsigma^{*2}(1-\rho^2)}) \le \frac{1}{M-D}\exp(-\frac{1}{2}\frac{r^2}{\varsigma^{*2}(1-\rho^2)}) < \frac{1}{M-D}$$

which is equivalent to

$$\frac{-1}{M-D} + \frac{1}{\sqrt{2\pi}\varsigma^*}[\frac{\rho}{\sqrt{1-\rho^2}}\exp(-\frac{1}{2}\frac{r^2}{\varsigma^{*2}(1-\rho^2)})] < 0$$

Therefore, $p'_\theta(\pi, r) < 0$, for all $\rho \in (-1, \tilde{\rho}]$. Besides, given $r > 0$, $p'_\theta(\frac{3}{2}\pi, r) = \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)r > 0$.

Next, we prove that given an arbitrary value of $r > 0$, for $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$, $p(\theta, r)$ first decreases with respect to θ until $\bar{\theta} \in (\pi, \frac{3}{2}\pi)$, and then increases. For the first step, we restrict our focus within $\theta \in [\pi, \frac{3}{2}\pi]$. We define $l(\theta, r) = -\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}$. Then, $l'_\theta(\theta, r) = \phi'(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)[\frac{1}{r} + r(\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}})^2]$.

Suppose $\rho \in (0, 1)$. Hence, $\arctan\frac{1}{\rho} \in (\frac{5}{4}\pi, \frac{3}{2}\pi)$. If $\cos\theta - \rho\sin\theta > 0$, i.e. $\theta > \pi + \arctan\frac{1}{\rho}$, then $\phi'(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) < 0$ and hence $l'_\theta(\theta, r) < 0$ for all θ in this range. If $\cos\theta - \rho\sin\theta < 0$, i.e. $\theta < \pi + \arctan\frac{1}{\rho}$, then $\phi'(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) > 0$, and hence $l'_\theta(\theta, r) > 0$ for all θ in this range. Only when $\theta = \pi + \arctan\frac{1}{\rho}$, $l(\theta, r)$ reaches its global maximum value for all $\theta \in [\pi, \frac{3}{2}\pi]$. For $\theta = \pi + \arctan\frac{1}{\rho}$, $\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{1}{\sqrt{2\pi}}$, $\sin\theta = -\frac{1}{\sqrt{1+\rho^2}}$, and $\cos\theta = -\frac{\rho}{\sqrt{1+\rho^2}}$. Hence, for $\theta = \pi + \arctan\frac{1}{\rho}$, $\sin\theta + \rho\cos\theta = -\sqrt{1+\rho^2}$ and $l(\pi + \arctan\frac{1}{\rho}, r) = \frac{1}{\sqrt{2\pi}\varsigma^*}\sqrt{\frac{1+\rho^2}{1-\rho^2}}$. In addition, $l(\pi, r) = \frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r)$ and $l(\frac{3}{2}\pi, r) = \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)$. Because $\rho > 0$, $l(\frac{3}{2}\pi, r) > l(\pi, r) > 0$. Without loss of generality, function $l(\theta, r)$ is geometrically represented by Figure E6.

We can equivalently express $p'_\theta(\theta, r) = \frac{r\cos\theta}{M-D} - \phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r \gtreqless 0$ by $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} \gtreqless -\frac{\cos\theta}{M-D}$; In addition, as $p'_\theta(\pi, r) < 0$ and $p'_\theta(\frac{3}{2}\pi, r) > 0$, which implies that if $\theta = \pi$, $-\phi(\frac{\cos\pi - \rho\sin\pi}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\pi + \rho\cos\pi}{\varsigma^*\sqrt{1-\rho^2}} < -\frac{\cos\pi}{M-D}$, and if $\theta = \frac{3}{2}\pi$, $-\phi(\frac{\cos\frac{3}{2}\pi - \rho\sin\frac{3}{2}\pi}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\frac{3}{2}\pi + \rho\cos\frac{3}{2}\pi}{\varsigma^*\sqrt{1-\rho^2}} > -\frac{\cos\frac{3}{2}\pi}{M-D}$. Therefore, combining the above analysis,

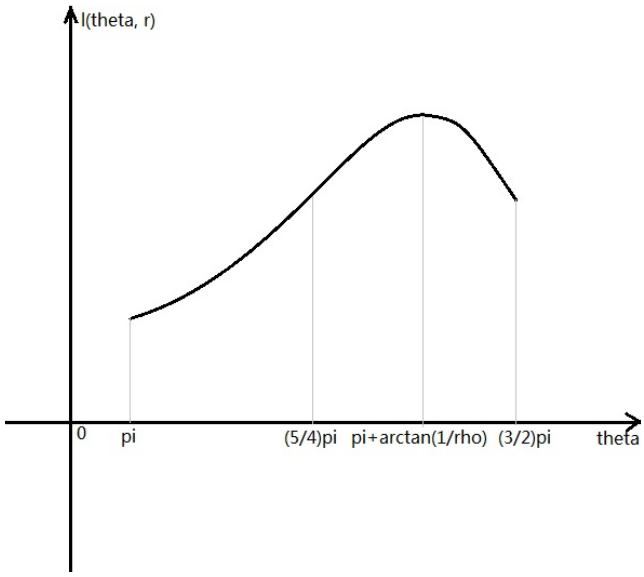


Figure E6: *A general geometric representation of function $l(\theta, r)$ with respect to θ given $r > 0$ in $\theta \in [\pi, \frac{3}{2}\pi]$. The monotonicity, the position of the optimum (i.e. the maximum value of $l(\theta, r)$ is obtained at $\theta = \pi + \arctan\frac{1}{\rho} \in (\frac{5}{4}\pi, \frac{3}{2}\pi)$ for all $\rho \in (0, 1)$), and the relationship $l(\pi, r) < l(\frac{3}{2}\pi, r)$ are always maintained for any parameter specification of $l(\theta, r)$.*

there exist two possibilities about the relative positions between curves $-\frac{\cos\theta}{M-D}$ and $l(\theta, r)$ for all $\theta \in [\pi, \frac{3}{2}\pi]$, and these two possibilities indicate two types of shapes of function $p(\theta, r)$. They are, respectively, described by Figures E7 and E8.

Which shape of $p(\theta, r)$ is correct, the one in Case 1 or in Case 2?

In Case 1, if we draw an arbitrary horizontal line $p(\theta, r) = C$, we can get at most two intersection points between the line and function $p(\theta, r)$, which equivalently mean that there are at most two points on function $g(x^*; C)$ that reach the origin $(0, 0)$ (in Cartesian coordinates) with distance r. However, in Case 2, it is possible that we get three or four intersection points on $p(\theta, r)$ by drawing a horizontal line $p(\theta, r) = C$, which means that there may exist three or four points on $g(x^*; C)$ that reaches the origin $(0, 0)$ with distance r. In fact, it is impossible that the shape of $p(\theta, r)$ is of the one in Case 2, because as long as $\rho \leq \tilde{\rho}$, $g(x^*; C)$ globally decreases with respect to x^*. Hence, the number of intersections between $g(x^*; C)$ and any circle with center $(0, 0)$ and radius $r > 0$ is always two (see Figure E9). Therefore, the shape of Case 2 is incorrect and the shape of Case 1 is correct. Hence, $p(\theta, r)$ has only one interior

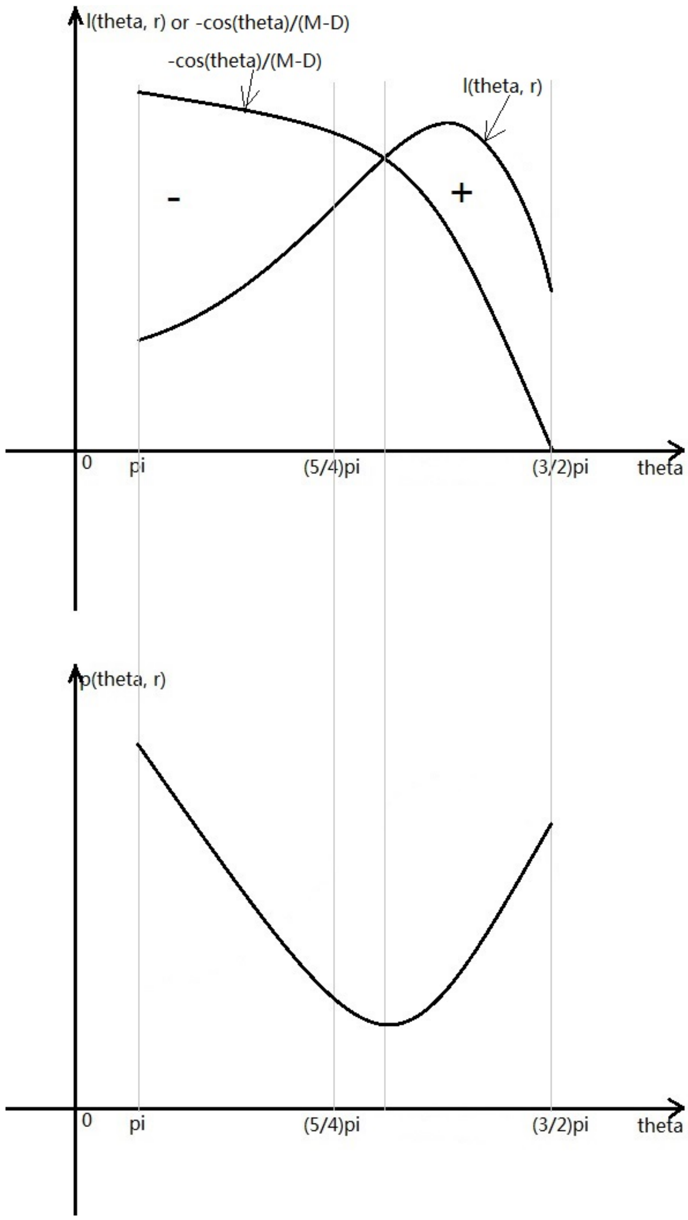


Figure E7: *Case 1: One possibility of the relative position between function* $l(\theta, r)$ *and* $-\frac{\cos\theta}{M-D}$ *for* $\rho > 0$ *and* $\theta \in [\pi, \frac{3}{2}\pi]$*, and the corresponding shape of function* $p(\theta, r)$*, where there just exists one interior optimum. The signs '-' and '+' represent the monotonicity of function* $p(\theta, r)$*, because for* $r > 0$*,* $p'_\theta(\theta, r) = -\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r - (-\frac{r\cos\theta}{M-D})$*. Note that by far, we do not know the relationship between* $p(\pi, r)$ *and* $p(\frac{3}{2}\pi, r)$ *yet. It could be* $p(\pi, r) > p(\frac{3}{2}\pi, r)$ *or* $p(\pi, r) < p(\frac{3}{2}\pi, r)$*.*

optimum (exactly a minimum value) for all $\theta \in (\pi, \frac{3}{2}\pi)$. We denote the θ where $p(\theta, r)$ reaches its (interior) minimum value by $\bar{\theta}$. Hence, given $r > 0$, $p(\theta, r)$ decreases from

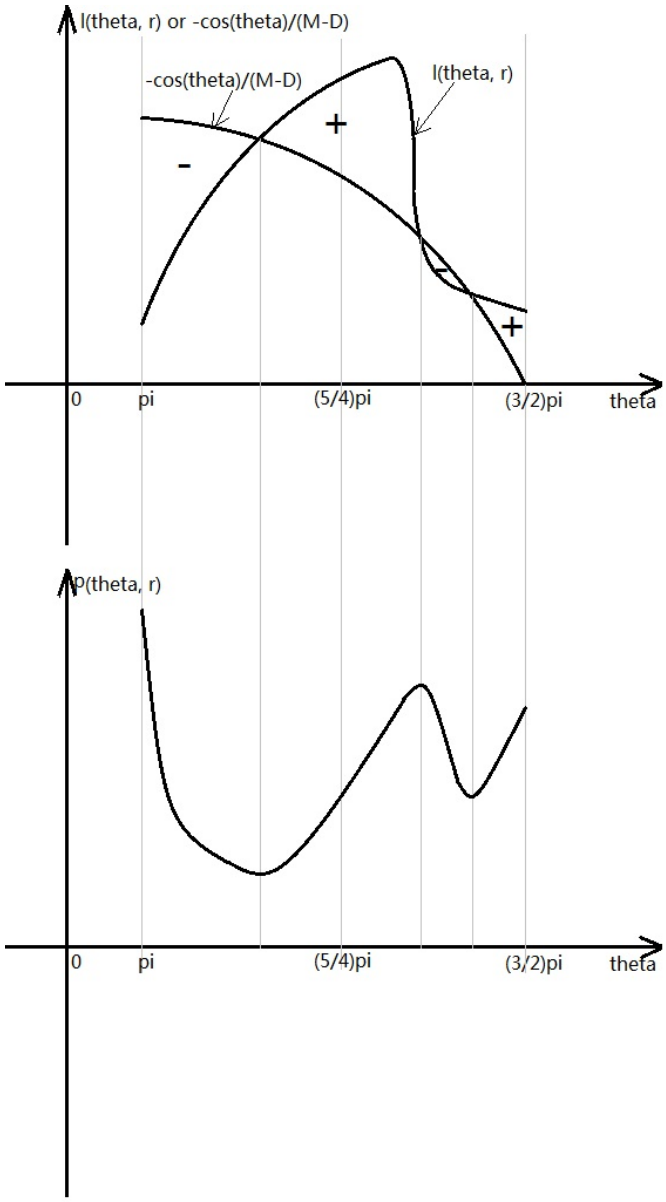


Figure E8: *Case 2: Another possibility of the relative position between function $l(\theta,r)$ and $-\frac{\cos\theta}{M-D}$ for $\rho > 0$ and $\theta \in [\pi, \frac{3}{2}\pi]$, and the corresponding shape of function $p(\theta,r)$, where there exist three interior optima. The signs '-' and '+' represent the monotonicity of function $p(\theta,r)$, because for $r > 0$, $p'_\theta(\theta,r) = -\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r - (-\frac{r\cos\theta}{M-D})$. Note that by far, we do not know the relationship between $p(\pi,r)$ and $p(\frac{3}{2}\pi,r)$ yet. It could be $p(\pi,r) > p(\frac{3}{2}\pi,r)$ or $p(\pi,r) < p(\frac{3}{2}\pi,r)$.*

π to $\bar{\theta}$, and then increases from $\bar{\theta}$ to $\frac{3}{2}\pi$. $\bar{\theta}$ could be either greater than or equal to $\frac{5}{4}\pi$, or smaller than $\frac{5}{4}\pi$.

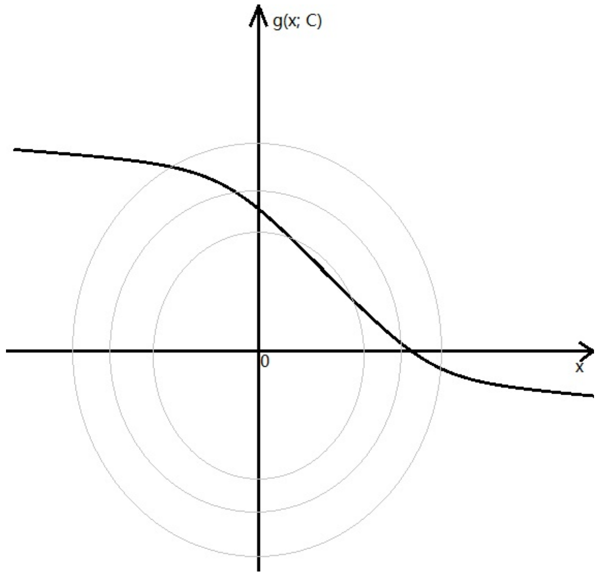


Figure E9: *The number of intersections between $g(x^*;C)$ and any circle with center $(0,0)$ and radius $r>0$ is always two. Therefore, it is impossible that equation $p(\theta,r)=C$ has three or four solutions for all $C\in\mathbb{R}$. This result is held for all $\theta\in[-\frac{\pi}{4},\frac{7}{4}\pi]$.*

What is the shape of $p(\theta,r)$ if $\rho\leq 0$? In the following we prove that the previous conclusion that $p(\theta,r)$ performs a 'U' shape for $\rho>0$ in $\theta\in[\pi,\frac{3}{2}\pi]$ is still established for $\rho\leq 0$ in $\theta\in[\pi,\frac{3}{2}\pi]$. For all $\theta\in[\pi,\frac{3}{2}\pi]$, if $\rho\leq 0$, then $\cos\theta-\rho\sin\theta\leq 0$. Therefore, $\phi'(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\geq 0$, and hence $l'_\theta(\theta,r)\geq 0$, where all of these equalities are held when $\rho=0$ and $\theta=\frac{3}{2}\pi$. It is known that $l(\pi,r)=\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r)$ and $l(\frac{3}{2}\pi,r)=\frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)$. Since $\rho\leq 0$, $l(\pi,r)\leq 0<l(\frac{3}{2}\pi,r)$. Without loss of generality, Figure E10 geometrically describes the relationship between $l(\theta,r)$ and $-\frac{\cos\theta}{M-D}$ for $\rho\leq 0$. It can be observed that, as long as $\rho\leq 0$, for all $\theta\in[\pi,\frac{3}{2}\pi]$, $l(\theta,r)$ and $-\frac{\cos\theta}{M-D}$ always have a unique intersection point in $\theta\in(\pi,\frac{3}{2}\pi)$. It is denoted by $\bar{\theta}$. Therefore, if $\rho\leq 0$, for all $\theta\in[\pi,\bar{\theta}]$, $p(\theta,r)$ decreases, and for all $\theta\in(\bar{\theta},\frac{3}{2}\pi]$, $p(\theta,r)$ increases (see Figure E10).

What is the shape of $p(\theta,r)$ in $\theta\in[\frac{3}{4}\pi,\pi]$ and $\theta\in(\frac{3}{2}\pi,\frac{7}{4}\pi]$?

It is already known that for $r>0$, $p'_\theta(\theta,r)=\frac{r\cos\theta}{M-D}-\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$. For all $\theta\in(\frac{3}{2}\pi,\frac{7}{4}\pi]$, $0<\cos\theta<\frac{\sqrt{2}}{2}$, and $-1<\sin\theta<-\frac{\sqrt{2}}{2}$. Whether ρ is positive or not,

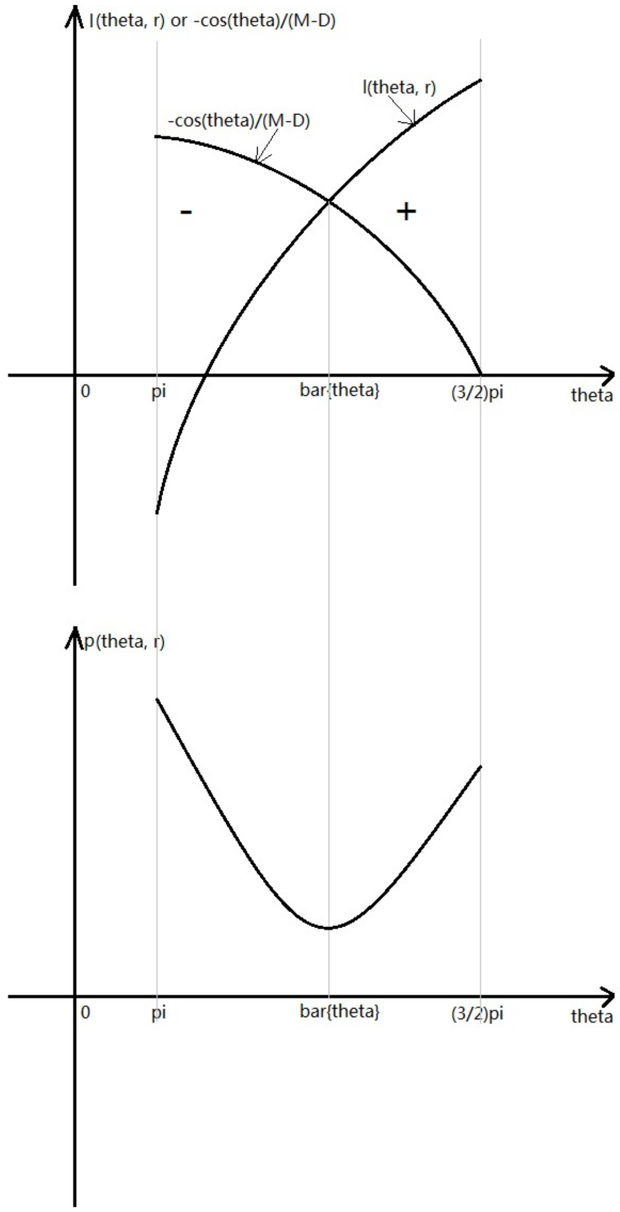


Figure E10: *The relative position between function* $l(\theta,r)$ *and* $-\frac{\cos\theta}{M-D}$*, and the corresponding shape of function* $p(\theta,r)$ *for* $\rho \leq 0$ *and* $\theta \in [\pi, \frac{3}{2}\pi]$*. The signs '-' and '+' represent the monotonicity of function* $p(\theta,r)$*, because for* $r > 0$*,* $p'_\theta(\theta,r) = -\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r - (-\frac{r\cos\theta}{M-D})$*. Note that by far, we do not know the relationship between* $p(\pi,r)$ *and* $p(\frac{3}{2}\pi,r)$ *yet. It could be* $p(\pi,r) > p(\frac{3}{2}\pi,r)$ *or* $p(\pi,r) < p(\frac{3}{2}\pi,r)$*.*

$-\sin\theta - \rho\cos\theta > 0$. Therefore, for $\theta \in (\frac{3}{2}\pi, \frac{7}{4}\pi]$, $p'_\theta(\theta,r) > 0$.

For all $\theta \in [\frac{3}{4}\pi, \pi]$, $0 < \sin\theta < \frac{\sqrt{2}}{2}$ and $-1 < \cos\theta < -\frac{\sqrt{2}}{2}$. If $\rho \leq 0$, then $-\sin\theta -$

$\rho\cos\theta < 0$, which indicates $p'_\theta(\theta,r) < 0$. If $\rho > 0$, then for all $\theta \in [\frac{3}{4}\pi, \pi - \arctan\rho)$, $-\sin\theta - \rho\cos\theta < 0$, and for all $\theta \in [\pi - \arctan\rho, \pi]$, $-\sin\theta - \rho\cos\theta \geq 0$. Therefore, for all $\theta \in [\frac{3}{4}\pi, \pi - \arctan\rho)$, $p'_\theta(\theta,r) < 0$. For all $\theta \in [\pi - \arctan\rho, \pi]$, if $\frac{r\cos\theta}{M-D}$ dominates $-\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$, then $p'_\theta(\theta,r) < 0$; however, if $-\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$ dominates $\frac{r\cos\theta}{M-D}$, then $p'_\theta(\theta,r) > 0$. Based on these analysis, we get two possible shapes of $p(\theta,r)$ given $r > 0$ as shown in Figures E11 and E12, respectively.

Again, for shapes like that of Case 2 (see Figure E12), where there are three optima, it indicates that there could be more than two points on $g(x^*;C)$ that reaches $(0,0)$ with distance r, and as what we have proven, it is impossible. Therefore, the 'U' shape is the only correct shape of function $p(\theta,r)$ in $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$. Hence, for all $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$, $p(\theta,r)$ decreases as θ increases from $\frac{3}{4}\pi$ to $\bar{\theta}$, and then increases from $\bar{\theta}$ to $\frac{7}{4}\pi$, where $\bar{\theta} \in (\pi, \frac{3}{2}\pi)$.

Can there be more than one pair of asymmetric equilibria (θ_1, r) and (θ_2, r) if they exist? To answer this question, first let us recall that a pair of asymmetric equilibria, if they exist, should be the solutions of the following equation group:

$$
\begin{aligned}
p(\theta,r) &= \frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{D}{D-M} \\
q(\theta,r) &= \frac{r\cos\theta}{M-D} + \Phi(\frac{\sin\theta-\rho\cos\theta}{\varsigma\sqrt{1-\rho^2}}r) = \frac{D}{D-M}
\end{aligned}
\tag{E.3}
$$

In addition, it is necessary that the radians of the pair of asymmetric equilibria θ_1 and θ_2 satisfy that $\theta_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta_1$. Then, let us check whether given a pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, there exists another pair of asymmetric equilibria $(\theta'_1, \tilde{r}')$ and $(\theta'_2, \tilde{r}')$ that satisfies equation group E.3. If r is successively changed away from $\tilde{r}$, the new solutions that satisfy $p(\theta,r) = \frac{D}{D-M}$ (or $q(\theta,r) = \frac{D}{D-M}$) cannot make $\theta'_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta'_1$, then it is certain that there exists only one pair of asymmetric equilibria, if they exist.

We first analyse the case when $\theta \in [\pi, \frac{3}{2}\pi]$. Since the functions $p(\theta,r)$ and $q(\theta,r)$ are symmetric around $\theta = \frac{5}{4}\pi$, it indicates that the part of $p(\theta,r)$ for $\theta \in [\frac{5}{4}\pi, \frac{3}{2}\pi]$ is the mirror image to the part of $q(\theta,r)$ for $\theta \in [\pi, \frac{5}{4}\pi)$, i.e. $p(\theta,r) = q(\frac{5}{2}\pi - \theta, r)$. Then, if we restrict θ within the range $\theta \in [\pi, \frac{5}{4}\pi)$, we can analyze the change of $p(\theta,r)$ for all

$\theta \in [\frac{5}{4}\pi, \frac{3}{2}\pi]$ via the corresponding change of $q(\theta, r)$ for all $\theta \in [\pi, \frac{5}{4}\pi)$.

Given r, for all $\theta \in [\pi, \frac{5}{4}\pi]$, we get the following results:

$$\frac{\partial p(\theta,r)}{\partial r} = \frac{\sin\theta}{M-D} + \phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}} < 0$$

$$\frac{\partial q(\theta,r)}{\partial r} = \frac{\cos\theta}{M-D} + \phi(\frac{\sin\theta - \rho\cos\theta}{\varsigma\sqrt{1-\rho^2}}r)\frac{\sin\theta - \rho\cos\theta}{\varsigma\sqrt{1-\rho^2}} < 0$$

Given an arbitrary value of $r > 0$, functions $\frac{\partial p(\theta,r)}{\partial r}$ and $\frac{\partial q(\theta,r)}{\partial r}$ with respect to θ intersect at $\theta = \frac{5}{4}\pi$, because $\frac{\partial^2 p(\theta,r)}{\partial r\partial\theta} = \frac{\cos\theta}{M-D} < 0$ and $\frac{\partial^2 q(\theta,r)}{\partial r\partial\theta} = \frac{-\sin\theta}{M-D} > 0$. Therefore, if and only if $\theta = \frac{5}{4}\pi$, $\frac{\partial p(\theta,r)}{\partial r} = \frac{\partial q(\theta,r)}{\partial r}$, otherwise, the two functions do not intersect. Therefore, for $\theta \in [\pi, \frac{5}{4}\pi)$, $\frac{\partial q(\theta,r)}{\partial r} < \frac{\partial p(\theta,r)}{\partial r} < 0$, for all $r > 0$; symmetrically if $\theta \in (\frac{5}{4}\pi, \frac{3}{2}\pi]$, $\frac{\partial p(\theta,r)}{\partial r} < \frac{\partial q(\theta,r)}{\partial r} < 0$, for all $r > 0$.

Suppose at $r = \tilde{r}$, we get asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, where $\theta_1 < \theta_2$. Then, $\theta_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta_1$. Hence, for all $\theta \in [\pi, \frac{5}{4}\pi)$, given any value of θ, if r successively increases, $q(\theta, r)$ decreases more than $p(\theta, r)$, which equivalently means that $p(\frac{5}{2}\pi - \theta, r)$ decreases more than $p(\theta, r)$ for all $\theta \in [\pi, \frac{5}{4}\pi)$ given the same amount of change of r. Therefore, as r is increased away from $\tilde{r}$ successively, for any new intersection points (θ_1', r) and (θ_2', r) between $p(\theta, r)$ and $\frac{D}{D-M}$, $\theta_2' - \frac{5}{4}\pi > \frac{5}{4}\pi - \theta_1'$; conversely, if we decrease r away from $\tilde{r}$ successively, $p(\theta, r)$ increases less than $p(\frac{5}{2}\pi - \theta, r)$ for all $\theta \in [\pi, \frac{5}{4}\pi]$; hence, for any new pair of intersection points (θ_1'', r) and (θ_2'', r) between $p(\theta, r)$ and $\frac{D}{D-M}$, $\theta_2'' - \frac{5}{4}\pi < \frac{5}{4}\pi - \theta_1''$ (see Figure E13). Therefore, given primitives M, D, ς, ς^* and ρ, if we have found a pair of asymmetric equilibria (θ_1, r) and (θ_2, r), we cannot find another pair of asymmetric equilibria for all $\theta \in [\pi, \frac{3}{2}\pi]$.

How about case when given $\tilde{r}$, one asymmetric equilibrium $\theta_1 \in [\frac{3}{4}\pi, \pi)$, and another asymmetric equilibrium $\theta_2 \in [\frac{3}{2}\pi, \frac{7}{4}\pi]$? Again, if the present asymmetric equilibria are $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, can there exist another pair of asymmetric equilibria by changing the radius r given the same primitives M, D, ς, ς^* and ρ? In the following, we prove that in this case, we still obtain only one pair of asymmetric equilibria if they exist.

As before, it is known that $p(\theta, r) = q(\frac{5}{2}\pi - \theta, r)$. Hence, we can still study the properties and the movement of $p(\theta,r)$ in $\theta \in [\frac{3}{2}\pi, \frac{7}{4}\pi]$ through its mirror image $q(\theta,r)$ in $\theta \in [\frac{3}{4}\pi, \pi]$. Hence, in the following, we restrict our attention to $\theta \in [\frac{3}{4}\pi, \pi]$.

We obtain $\frac{\partial p(\frac{3}{4}\pi,r)}{\partial r} = \frac{\sqrt{2}}{2}\frac{1}{M-D} - \phi(\frac{\sqrt{2}}{2}\frac{1+\rho}{\varsigma\sqrt{1-\rho^2}}r)\frac{1+\rho}{\varsigma\sqrt{1-\rho^2}}\frac{\sqrt{2}}{2}$, $\frac{\partial q(\frac{3}{4}\pi,r)}{\partial r} = -\frac{\sqrt{2}}{2}\frac{1}{M-D} + \phi(\frac{\sqrt{2}}{2}\frac{1+\rho}{\varsigma\sqrt{1-\rho^2}}r)\frac{1+\rho}{\varsigma\sqrt{1-\rho^2}}\frac{\sqrt{2}}{2}$, $\frac{\partial p(\pi,r)}{\partial r} = -\frac{1}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{1}{\varsigma\sqrt{1-\rho^2}}r)$, and $\frac{\partial q(\pi,r)}{\partial r} = -\frac{1}{M-D} + \frac{\rho}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{\rho}{\varsigma\sqrt{1-\rho^2}}r)$. In addition, $-\frac{1}{M-D} < \frac{\partial^2 p(\theta,r)}{\partial r\partial\theta} < -\frac{\sqrt{2}}{2}\frac{1}{M-D}$ and $-\frac{\sqrt{2}}{2}\frac{1}{M-D} < \frac{\partial^2 q(\theta,r)}{\partial r\partial\theta} < 0$. According to above results, it can be found that given $r > 0$, $0 > \frac{\partial p(\pi,r)}{\partial r} > \frac{\partial q(\pi,r)}{\partial r}$ and $\frac{\partial p(\frac{3}{4}\pi,r)}{\partial r} = -\frac{\partial q(\frac{3}{4}\pi,r)}{\partial r}$. Therefore, based on these results, we can conclude that 1) given $r > 0$, function $\frac{\partial p(\theta,r)}{\partial r}$ decreases more rapidly than $\frac{\partial q(\theta,r)}{\partial r}$ as θ increases from $\frac{3}{4}\pi$ to π; 2) $\frac{\partial p(\frac{3}{4}\pi,r)}{\partial r} > 0 > \frac{\partial q(\frac{3}{4}\pi,r)}{\partial r}$; and 3) for all $\theta \in [\frac{3}{4}\pi, \pi]$, $\frac{\partial p(\theta,r)}{\partial r} > \frac{\partial q(\theta,r)}{\partial r}$. Figure E14 gives a general description of functions $\frac{\partial p(\theta,r)}{\partial r}$ and $\frac{\partial q(\theta,r)}{\partial r}$ for all $\theta \in [\frac{3}{4}\pi, \pi]$ given $r > 0$.

Given primitives M, D, ς, ς^* and ρ, suppose we already get one pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, where $\theta_1 \in [\frac{3}{4}\pi, \pi)$ and $\theta_2 \in [\frac{3}{2}\pi, \frac{7}{4}\pi]$. According to the above analysis, if r is increased away from $\tilde{r}$ successively, the new intersection point θ_2' is obtained as θ_2 increases, while another intersection point θ_1' is obtained from θ_1 but θ_1 may increase or decrease and hence $\theta_1' \gtrless \theta_1$. No matter how θ_1 changes, θ_2 always moves faster than θ_1 and along a fixed direction. Therefore, as r increases, the new intersection points θ_1' and θ_2' would never be balanced again, i.e. $\theta_2' - \frac{5}{4}\pi \neq \frac{5}{4}\pi - \theta_1'$ (see Figure E15). The same conclusion can still be obtained by decreasing r away from $\tilde{r}$. Hence, if there exists a pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, where $\theta_1 \in [\frac{3}{4}\pi, \pi)$ and $\theta_2 \in [\frac{3}{2}\pi, \frac{7}{4}\pi]$, there would never exist any other asymmetric equilibria for all $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$.

In conclusion, for all $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$, given all primitives ($M$, D, ς, ς^* and ρ), if asymmetric equilibria exist, the number of asymmetric equilibria is two.

Next, it is natural to ask, under what conditions asymmetric equilibria exist. The asymmetric equilibria are solutions of the following equation group:

$$p(\theta, r) = \frac{D}{D-M}$$

$$q(\theta, r) = \frac{D}{D-M}$$

Then, geometrically, asymmetric equilibria should be the intersections of the following three curves: $y = p(\theta, r)$, $y = q(\theta, r)$ and $y = \frac{D}{D-M}$. According to the possible relationships between $p(\tau_1, r)$ and $p(\tau_2, r)$, where (τ_1, τ_2) is $(\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$, we discuss how asymmetric equilibria arise in the following two distinctive conjectures.

Conjecture 1: $p(\tau_1, r) < p(\tau_2, r)$, where $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$.

In this case, if the symmetric equilibrium is stable, then there must exist asymmetric equilibria. This is 1) because as r increases, $l(\frac{5}{4}\pi, r) = -\phi(\frac{\cos\frac{5}{4}\pi - \rho\sin\frac{5}{4}\pi}{\varsigma^*\sqrt{1-\rho^2}} r)\frac{\sin\frac{5}{4}\pi + \rho\cos\frac{5}{4}\pi}{\varsigma^*\sqrt{1-\rho^2}} > 0$ decreases to 0. If at $r = \hat{r}$, $l(\frac{5}{4}\pi, \hat{r}) < -\frac{\cos\frac{5}{4}\pi}{M-D}$, which implies that $p'_\theta(\frac{5}{4}\pi, r) < 0$, then as r increases away from $\hat{r}$, the relationship $l(\frac{5}{4}\pi, r) < -\frac{\cos\frac{5}{4}\pi}{M-D}$ is held for all $r > \hat{r}$, and therefore, $p'_\theta(\frac{5}{4}\pi, r) < 0$ for all $r > \hat{r}$; and 2) because $p'_\theta(\frac{5}{4}\pi, r) < 0$, the asymmetric equilibrium candidates always exist. The asymmetric equilibrium candidates are the two extreme intersection points between $p(\theta, r)$ and $q(\theta, r)$, and correspondingly the middle intersection point is the symmetric equilibrium candidate. Since $\frac{\partial p(\theta,r)}{\partial r} < 0$ for all $\theta \in [\frac{5}{4}\pi, \tau_2]$ and $\frac{\partial q(\theta,r)}{\partial r} < 0$ for all $\theta \in [\tau_1, \frac{5}{4}\pi]$, the entire function of $p(\theta, r)$ in $\theta \in [\frac{5}{4}\pi, \tau_2]$ and $q(\theta, r)$ in $\theta \in [\tau_1, \frac{5}{4}\pi]$ decreases as r increases. Therefore, suppose the symmetric equilibrium is obtained at $r = \bar{r}$, i.e. it is $(\frac{5}{4}\pi, \bar{r})$, and it satisfies $p'_\theta(\frac{5}{4}\pi, \bar{r}) < 0$. Then, as r increases away from $\bar{r}$, the asymmetric equilibrium candidates fall from positions above the line $y = \frac{D}{D-M}$ to the positions below the line $y = \frac{D}{D-M}$. Therefore, there must exist a moment such that the asymmetric equilibrium candidates pass the line $y = \frac{D}{D-M}$. At that moment, which is recorded by the relevant value of r, the asymmetric equilibrium candidates formally become the asymmetric equilibria that satisfy equation group E.3 (see Figure E16); 3) We denote the symmetric equilibrium (candidate) by (s, s), and equivalently it is denoted by $(\frac{5}{4}\pi, r)$ in polar coordinate representation. According to the following equivalence relationship at the symmetric equilibrium or symmetric equilibrium candidate, for all $r > 0$,

$$g'(s) \gtreqless -1 \iff \frac{1}{(1+\rho)(M-D)} - \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\cos\frac{5}{4}\pi - \rho\sin\frac{5}{4}\pi}{\varsigma^*\sqrt{1-\rho^2}} r) \gtreqless 0 \iff p'_\theta(\frac{5}{4}\pi, r) \lesseqgtr 0$$

Thus, $p'_\theta(\frac{5}{4}\pi, \bar{r}) < 0$ means that the symmetric equilibrium is stable. In conclusion, as long as $p(\tau_1, r) < p(\tau_2, r)$, if the symmetric equilibrium is stable, then there must exist asymmetric equilibria.

However, this conjecture is in fact not correct. Because a fact we have already established is that as long as the best response function is a contraction, there always exists a unique equilibrium. It is symmetric and must be stable. Therefore, for the contraction case, the symmetric equilibrium is stable and no other equilibria exist. Hence, the conjecture 1 that $p(\tau_1, r) < p(\tau_2, r)$ is incorrect.

Conjecture 2: $p(\tau_1, r) > p(\tau_2, r)$, where $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi)$ or $(\frac{3}{4}\pi, \frac{7}{4}\pi)$.

Since conjecture 1 is incorrect, conjecture 2 must be the correct property of $p(\theta, r)$. In this case, if there is no asymmetric equilibrium candidate, $p'_\theta(\frac{5}{4}\pi, r) \leq 0$ must hold, and if there exist asymmetric equilibria candidates, $p'_\theta(\frac{5}{4}\pi, r) > 0$ must hold (see Figures E17 and E18, respectively).

Remember that as r increases, $l(\frac{5}{4}\pi, r)$ decreases. Therefore, supposing at $r = \hat{r}$, $l(\frac{5}{4}\pi, \hat{r}) > -\frac{\cos\frac{5}{4}\pi}{M-D}$, i.e. $p'_\theta(\frac{5}{4}\pi, \hat{r}) > 0$, then as r increases away from $\hat{r}$, $p'_\theta(\frac{5}{4}\pi, \hat{r})$ will decrease gradually from $p'_\theta(\frac{5}{4}\pi, \hat{r}) > 0$ to $p'_\theta(\frac{5}{4}\pi, \hat{r}) = 0$, and finally to $p'_\theta(\frac{5}{4}\pi, \hat{r}) < 0$. If there is no asymmetric equilibrium, it is because at $r = 0$, $l(\frac{5}{4}\pi, 0) \leq -\frac{\cos\frac{5}{4}\pi}{M-D}$, and hence no asymmetric equilibrium candidate exists. Therefore, in this case, as r increases, no asymmetric equilibrium candidates will appear certainly. Alternatively, it is because as r increases from 0, at the beginning, there were asymmetric equilibrium candidates, but before the symmetric equilibrium candidates $(\frac{5}{4}\pi, r)$ passes the line $y = \frac{D}{D-M}$, $p'_\theta(\frac{5}{4}\pi, r)$ changes from a positive value to a non-positive value, and thereafter, asymmetric equilibrium candidates vanish; consequently, only a symmetric equilibrium candidate exists, and hence the game will have only one equilibrium, which is symmetric. For both possibilities, the corresponding symmetric equilibrium $(\frac{5}{4}\pi, \bar{r})$ must satisfy $p'_\theta(\frac{5}{4}\pi, \bar{r}) \leq 0$. Let us recall that for the symmetric equilibrium (candidate) (s, s), $g'(s) \gtreqqless -1 \iff p'_\theta(\frac{5}{4}\pi, r) \lesseqqgtr 0$. Therefore, it can be concluded that if there is no asymmetric equilibrium, the relevant symmetric equilibrium must be not unstable, which means it is either stable or stability is not determined.

Conversely, if the symmetric equilibrium is not unstable, there are no asymmetric equilibria, because 1) as long as the symmetric equilibrium is not unstable, i.e. $g'(s) \geq -1$, $p'_\theta(\frac{5}{4}\pi, \bar{r}) \leq 0$, and therefore, no asymmetric equilibrium candidate exists when $r = \bar{r}$; and 2) since the symmetric equilibrium is not unstable, i.e. $p'_\theta(\frac{5}{4}\pi, \bar{r}) \leq 0$, if r increases away from $\bar{r}$, based on the previous analysis, $p'_\theta(\frac{5}{4}\pi, r) < 0$ must be held; hence, asymmetric equilibrium candidates would never appear. Therefore, an asymmetric equilibrium is impossible to exist and only a symmetric equilibrium survives if $p'_\theta(\frac{5}{4}\pi, \bar{r}) \leq 0$, i.e. the symmetric equilibrium is not unstable.

In conclusion, given all primitives and for all $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$, the symmetric strategic substitutes game contains only a unique equilibrium that must be symmetric if and only if the symmetric equilibrium is not unstable, and equivalently, asymmetric equilibria exist if and only if the relevant symmetric equilibrium is unstable.

Then, we study **Case II**, where $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$. Since the asymmetric equilibria are symmetrically located around the 45° line (in Cartesian coordinates), we first study the game for all $\theta \in [0, \frac{\pi}{2}]$, and then extend our analysis to the entire range of $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$.

Let us recall the following equations: $p(\theta, r) = \frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)$, and $p'_\theta(\theta, r) = \frac{r\cos\theta}{M-D} - \phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$. Given any $r > 0$, $p'_\theta(0, r) = \frac{r}{M-D} - \phi(\frac{r}{\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r$. As $\rho \leq \tilde{\rho}$, $\frac{1}{M-D} - \frac{1}{\sqrt{2\pi}\varsigma^*}\frac{\rho}{\sqrt{1-\rho^2}}\exp(-\frac{1}{2}\frac{r^2}{\varsigma^{*2}(1-\rho^2)}) > 0$. Therefore, for all $\rho \in (-1, \tilde{\rho}]$, $p'_\theta(0, r) > 0$. Given any $r > 0$, $p'_\theta(\frac{\pi}{2}, r) = \frac{r\cos\frac{\pi}{2}}{M-D} - \phi(\frac{\cos\frac{\pi}{2} - \rho\sin\frac{\pi}{2}}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\frac{\pi}{2}+\rho\cos\frac{\pi}{2}}{\varsigma^*\sqrt{1-\rho^2}}r = -\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r < 0$. In addition, $p(0, r) = \Phi(\frac{r}{\varsigma^*\sqrt{1-\rho^2}})$ and $p(\frac{\pi}{2}, r) = \frac{r}{M-D} + \Phi(\frac{-\rho}{\varsigma^*\sqrt{1-\rho^2}}r)$. Finally, recall that $l(\theta, r) = -\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}$, and $l'_\theta(\theta, r) = \phi'(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)[\frac{1}{r} + r(\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}})^2]$. We then prove that given any $r > 0$, for all $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$, $p(\theta, r)$ first increases until $\bar{\theta} \in (0, \frac{\pi}{2})$, and then decreases. In the first step, we restrict our focus within $\theta \in (0, \frac{\pi}{2})$.

Suppose $\rho \in (0, 1)$. Hence, $\arctan\frac{1}{\rho} \in (\frac{\pi}{4}, \frac{\pi}{2})$. If $\cos\theta - \rho\sin\theta < 0$, i.e. $\theta > \arctan\frac{1}{\rho}$, then $\phi'(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) > 0$ and hence $l'_\theta(\theta, r) > 0$ in $\theta \in (\arctan\frac{1}{\rho}, \frac{\pi}{2})$. If $\cos\theta - \rho\sin\theta > 0$, i.e. $\theta < \arctan\frac{1}{\rho}$, then $\phi'(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) < 0$, and hence $l'_\theta(\theta, r) < 0$. Therefore, only at $\theta = \arctan\frac{1}{\rho} \in (\frac{\pi}{4}, \frac{\pi}{2})$, $l(\theta, r)$ reaches its global minimum value for all $\theta \in [0, \frac{\pi}{2}]$. If $\theta = \arctan\frac{1}{\rho}$, $\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{1}{\sqrt{2\pi}}$, $\sin\theta = \frac{1}{\sqrt{1+\rho^2}}$, and $\cos\theta = \frac{\rho}{\sqrt{1+\rho^2}}$. Hence, $\sin\theta + \rho\cos\theta = \sqrt{1+\rho^2}$ at $\theta = \arctan\frac{1}{\rho}$. Therefore, $l(\arctan\frac{1}{\rho}, r) = -\frac{1}{\sqrt{2\pi}\varsigma^*}\sqrt{\frac{1+\rho^2}{1-\rho^2}}$, $l(0, r) = -\frac{\rho}{\sqrt{2\pi}\varsigma^*}\phi(\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r)$ and $l(\frac{\pi}{2}, r) = -\frac{1}{\sqrt{2\pi}\varsigma^*}\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)$. As $\rho > 0$, $l(\frac{\pi}{2}, r) < l(0, r) < 0$. Based on these properties, Figure E19 generally describes the function $l(\theta, r)$ given any $r > 0$.

Because $p'_\theta(\theta,r) = \frac{r\cos\theta}{M-D} - \phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r \gtreqless 0$ is equivalent to $-\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)$ $\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} \gtreqless -\frac{\cos\theta}{M-D}$ given $r > 0$, $p'_\theta(0,r) > 0$ and $p'_\theta(\frac{\pi}{2},r) < 0$, which indicate that for $\theta = 0$, $-\phi(\frac{\cos 0-\rho\sin 0}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin 0+\rho\cos 0}{\varsigma^*\sqrt{1-\rho^2}} > -\frac{\cos 0}{M-D}$, and for $\theta = \frac{\pi}{2}$, $-\phi(\frac{\cos\frac{\pi}{2}-\rho\sin\frac{\pi}{2}}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\frac{\pi}{2}+\rho\cos\frac{\pi}{2}}{\varsigma^*\sqrt{1-\rho^2}} < -\frac{\cos\frac{\pi}{2}}{M-D}$. Given these properties, we find that for $\rho > 0$, there are two possibilities about the relative positions between functions $y = -\frac{\cos\theta}{M-D}$ and $y = l(\theta,r)$ for all $\theta \in [0,\frac{\pi}{2}]$, which give two possible shapes of $p(\theta,r)$. The two possible cases are described in Figures E20 and E21, respectively.

Then, which shape of $p(\theta,r)$ is correct, Case 1 or Case 2?

As we have proven, if $\rho \le \tilde{\rho}$, $g(x^*;C)$ globally decreases with respect to $x^* \in \mathbb{R}$ and there are at most two points on $g(x^*;C)$ that reaches $(0,0)$ with $r > 0$, which equivalently means that $p(\theta,r) = C$ has two solutions at most. This result is applicable for the case where $\theta \in [-\frac{\pi}{4},\frac{3}{4}\pi)$ as well. Hence, the correct shape of $p(\theta,r)$ should be Case 1. In Case 2, it is possible that $p(\theta,r) = C$ contains three or four solutions because $y = p(\theta,r)$ and $y = C$ could have three or four intersection points, which contradicts the fact that $p(\theta,r) = C$ at most has two solutions.

What is the shape of $p(\theta,r)$ if $\rho \le 0$? If $\rho \le 0$, then $\cos\theta - \rho\sin\theta \ge 0$. Therefore, $\phi'(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) \le 0$, and so $l'_\theta(\theta,r) \le 0$. In addition, $l(0,r) = -\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r) \ge 0$ and $l(\frac{\pi}{2},r) = -\frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r) < 0$. Hence, if $\rho \le 0$, $y = l(\theta,r)$ and $y = -\frac{\cos\theta}{M-D}$ will have only a unique intersection point. Based on these analysis, the top graph in Figure E22 generally describes how $y = l(\theta,r)$ and $y = -\frac{\cos\theta}{M-D}$ behave and intersect with each other (see Figure E22). Furthermore, because $p'_\theta(\theta,r) = -\phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r - (-\frac{r\cos\theta}{M-D})$, from the top graph, we can also obtain the results about the monotonicity of $p(\theta,r)$ if $\rho \le 0$, which is described by the bottom graph in Figure E22, where only a unique interior optimum exists (see Figure E22). Therefore, in conclusion, for all $\rho \in (-1,\tilde{\rho}]$, $p(\theta,r)$ increases from 0 to $\bar{\theta}$, and then decreases from $\bar{\theta}$ to $\frac{\pi}{2}$.

What is the shape of $p(\theta,r)$ in $\theta \in [-\frac{\pi}{4},0]$ and $\theta \in [\frac{\pi}{2},\frac{3}{4}\pi]$? It is already known that $p'_\theta(\theta,r) = \frac{r\cos\theta}{M-D} - \phi(\frac{\cos\theta-\rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta+\rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$. For $\theta \in [\frac{\pi}{2},\frac{3}{4}\pi]$, $-\frac{\sqrt{2}}{2} \le \cos\theta \le 0$ and $\frac{\sqrt{2}}{2} \le \sin\theta \le 1$. Moreover, irrespective of whether ρ is positive, $\sin\theta + \rho\cos\theta > 0$. Therefore, for $\theta \in [\frac{\pi}{2},\frac{3}{4}\pi]$, $p'_\theta(\theta,r) < 0$.

For $\theta \in [-\frac{\pi}{4}, 0]$, $-\frac{\sqrt{2}}{2} \leq \sin\theta \leq 0$ and $\frac{\sqrt{2}}{2} \leq \cos\theta \leq 1$. If $\rho \leq 0$, then $\sin\theta + \rho\cos\theta < 0$, which indicates that $p'_\theta(\theta, r) > 0$. If $\rho > 0$, then for all $\theta \in [-\frac{\pi}{4}, -\arctan\rho)$, $\sin\theta + \rho\cos\theta < 0$, and for all $\theta \in (-\arctan\rho, 0]$, $\sin\theta + \rho\cos\theta > 0$, and at $\theta = -\arctan\rho$, $\sin\theta + \rho\cos\theta = 0$. Therefore, for all $\theta \in [-\frac{\pi}{4}, -\arctan\rho)$, $p'_\theta(\theta, r) > 0$. For all $\theta \in [-\arctan\rho, 0]$, if $\frac{r\cos\theta}{M-D}$ dominates $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$, then $p'_\theta(\theta, r) > 0$ in $\theta \in [-\arctan\rho, 0]$; if $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$ dominates $\frac{r\cos\theta}{M-D}$, then $p'_\theta(\theta, r) < 0$. Again, according to these analyses, we get two possible shapes of $p(\theta, r)$ given $r > 0$, which are shown in Figures E23 and E24.

Again, the shape in Case 2 is not established because there could be three or four intersection points between $y = p(\theta, r)$ and $y = C$, where C is a constant, and thus, equation $p(\theta, r) = C$ could contain three or four solutions, which indicates that correspondingly in Cartesian coordinates, there could be three or four points on $g(x^*; C)$ that reaches $(0, 0)$ with distance r. This is impossible as long as $\rho \leq \tilde{\rho}$, as we have proven. Therefore, in conclusion, for all $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$, $p(\theta, r)$ increases with respect to θ from $-\frac{\pi}{4}$ to $\bar{\theta}$, and then decreases with respect to θ from $\bar{\theta}$ to $\frac{3}{4}\pi$, where $\bar{\theta} \in (0, \frac{\pi}{2})$.

Is there more than one pair of asymmetric equilibria (θ_1, r) and (θ_2, r) if they exist? First, let us recall that a pair of asymmetric equilibria should be solutions of the following equation group:

$$p(\theta, r) = \frac{r\sin\theta}{M-D} + \Phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{D}{D-M}$$

$$q(\theta, r) = \frac{r\cos\theta}{M-D} + \Phi(\frac{\sin\theta - \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r) = \frac{D}{D-M}$$

It is necessary that the radians θ_1 and θ_2 satisfy $\theta_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta_1$. Then, the above question becomes that given a pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, whether there exists another pair of asymmetric equilibria $(\theta'_1, \tilde{r}')$ and $(\theta'_2, \tilde{r}')$ that satisfy the same equation group. If r is successively changed away from $\tilde{r}$, and the new solutions that satisfy $p(\theta, r) = \frac{D}{D-M}$ (or $q(\theta, r) = \frac{D}{D-M}$) cannot make $\theta'_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta'_1$, then there certainly exists only one pair of asymmetric equilibria, and hence, the number of asymmetric equilibria is two if asymmetric equilibria exist.

We first analyze the case of $\theta \in [0, \frac{\pi}{2}]$. Since the functions $p(\theta, r)$ and $q(\theta, r)$ are symmetric around $\frac{\pi}{4}$, the part of $p(\theta,r)$ in $\theta \in [\frac{\pi}{4},\frac{\pi}{2}]$ is the mirror image to the part of $q(\theta,r)$ in $\theta \in [0,\frac{\pi}{4})$, i.e. $p(\theta,r) = q(\frac{\pi}{2}-\theta,r)$. Then, methodologically, we can analyze the change of $p(\theta,r)$ for all $\theta \in [\frac{\pi}{4},\frac{\pi}{2}]$ via the corresponding change of $q(\theta,r)$ for all $\theta \in [0,\frac{\pi}{4})$.

For all $\theta \in [0,\frac{\pi}{2}]$, we have the following results:

$$\frac{\partial p(\theta,r)}{\partial r} = \frac{\sin\theta}{M-D} + \phi\left(\frac{\cos\theta - \rho\sin\theta}{\varsigma^* \sqrt{1-\rho^2}} r\right)\frac{\cos\theta - \rho\sin\theta}{\varsigma^* \sqrt{1-\rho^2}} > 0$$

$$\frac{\partial q(\theta,r)}{\partial r} = \frac{\cos\theta}{M-D} + \phi\left(\frac{\sin\theta - \rho\cos\theta}{\varsigma^* \sqrt{1-\rho^2}} r\right)\frac{\sin\theta - \rho\cos\theta}{\varsigma^* \sqrt{1-\rho^2}} > 0$$

For all $r > 0$, $\frac{\partial p(\theta,r)}{\partial r}$ and $\frac{\partial q(\theta,r)}{\partial r}$ intersect at $\theta = \frac{\pi}{4}$, because $\frac{\partial^2 p(\theta,r)}{\partial r \partial\theta} = \frac{\cos\theta}{M-D} > 0$ and $\frac{\partial^2 q(\theta,r)}{\partial r \partial\theta} = -\frac{\sin\theta}{M-D} < 0$. Therefore, if and only if $\theta = \frac{\pi}{4}$, $\frac{\partial p(\theta,r)}{\partial r} = \frac{\partial q(\theta,r)}{\partial r}$, otherwise the two functions do not intersect. Therefore, for all $r > 0$ and for all $\theta \in [0,\frac{\pi}{4})$, $\frac{\partial q(\theta,r)}{\partial r} > \frac{\partial p(\theta,r)}{\partial r} > 0$, and symmetrically, for all $\theta \in (\frac{\pi}{4},\frac{\pi}{2}]$, $\frac{\partial p(\theta,r)}{\partial r} > \frac{\partial q(\theta,r)}{\partial r} > 0$.

Suppose at $r = \tilde{r}$, we get symmetric equilibria $(\theta_1,\tilde{r})$ and $(\theta_2,\tilde{r})$, where $\theta_1 < \theta_2$. Then, the two radians must satisfy $\theta_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta_1$. Hence, for each $\theta \in [0,\frac{\pi}{4})$, if r increases (or decreases), $q(\theta,r)$ increases (or decreases) more than $p(\theta,r)$. If we increase (or decrease) r away from $\tilde{r}$ successively, since $p(\theta,r)$ increases (or decreases) less in $\theta \in [0,\frac{\pi}{4}]$ than in $\theta \in (\frac{\pi}{4},\frac{\pi}{2}]$, for any new pair of intersection points (θ_1',r) and (θ_2',r) (or (θ_1'',r) and (θ_2'',r) if r is decreased) between $y = p(\theta,r)$ and $y = \frac{D}{D-M}$, $\frac{\pi}{4} - \theta_1' > \theta_2' - \frac{\pi}{4}$ (or $\frac{\pi}{4} - \theta_1'' < \theta_2'' - \frac{\pi}{4}$ if r is decreased). Therefore, if we find a pair of asymmetric equilibria, say (θ_1,r) and (θ_2,r), then given the same primitives M, D, ς, ς^* and ρ, we cannot find another pair of asymmetric equilibria for all $\theta \in [0,\frac{\pi}{2}]$ (see Figure E25).

Let us now look at the case when given all primitives and a pair of asymmetric equilibria $(\theta_1,\tilde{r})$ and $(\theta_2,\tilde{r})$, where one asymmetric equilibrium's radian $\theta_1 \in [-\frac{\pi}{4},0]$ and another asymmetric equilibrium's radian $\theta_2 \in [\frac{\pi}{2},\frac{3}{4}\pi]$. Can we find another pair of asymmetric equilibria by changing r away from $\tilde{r}$? In the following, we prove that in this case, we still get only one pair of asymmetric equilibria if they exist.

As before, it is known that by symmetry, $p(\theta,r) = q(\frac{\pi}{2}-\theta,r)$. Hence, we can study function $p(\theta,r)$ in $\theta \in [\frac{\pi}{2}, \frac{3}{4}\pi]$ through its mirror image $q(\theta,r)$ in $\theta \in [-\frac{\pi}{4}, 0)$. In the following, we restrict our attention to $\theta \in [-\frac{\pi}{4}, 0]$. First, by calculation, we get the following results: $\frac{\partial p(-\frac{\pi}{4},r)}{\partial r} = -\frac{\sqrt{2}}{2}\frac{1}{M-D} + \phi(\frac{\sqrt{2}}{2}\frac{1+\rho}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{1+\rho}{\varsigma^*\sqrt{1-\rho^2}}\frac{\sqrt{2}}{2}$, $\frac{\partial q(-\frac{\pi}{4},r)}{\partial r} = \frac{\sqrt{2}}{2}\frac{1}{M-D} - \phi(\frac{\sqrt{2}}{2}\frac{1+\rho}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{1+\rho}{\varsigma^*\sqrt{1-\rho^2}}\frac{\sqrt{2}}{2}$, $\frac{\partial p(0,r)}{\partial r} = \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{1}{\varsigma^*\sqrt{1-\rho^2}}r)$, $\frac{\partial q(0,r)}{\partial r} = \frac{1}{M-D} - \phi(\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\rho}{\varsigma^*\sqrt{1-\rho^2}}$, $\frac{\sqrt{2}}{2}\frac{1}{M-D} < \frac{\partial^2 p(\theta,r)}{\partial\theta\partial r} < \frac{1}{M-D}$, and $0 < \frac{\partial^2 q(\theta,r)}{\partial\theta\partial r} < \frac{\sqrt{2}}{2}\frac{1}{M-D}$. Hence, by continuity, given $r > 0$, $\frac{\partial q(0,r)}{\partial r} > \frac{\partial p(0,r)}{\partial r} > 0$ and $\frac{\partial p(-\frac{\pi}{4},r)}{\partial r} = -\frac{\partial q(-\frac{\pi}{4},r)}{\partial r}$. Therefore, given $r > 0$, function $\frac{\partial p(\theta,r)}{\partial r}$ increases more rapidly than $\frac{\partial q(\theta,r)}{\partial r}$ as θ increases from $-\frac{\pi}{4}$ to 0. Hence, we can obtain that $\frac{\partial q(-\frac{\pi}{4},r)}{\partial r} > 0 > \frac{\partial p(-\frac{\pi}{4},r)}{\partial r}$, and for all $\theta \in [-\frac{\pi}{4}, 0)$, $\frac{\partial q(\theta,r)}{\partial r} > \frac{\partial p(\theta,r)}{\partial r}$ (see Figure E26).

Given primitives M, D, ς, ς^* and ρ, suppose we already get one pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, where $\theta_1 \in [-\frac{\pi}{4}, 0)$ and $\theta_2 \in [\frac{\pi}{2}, \frac{3}{4}\pi]$. According to Figure E27, if r is increased away from $\tilde{r}$ successively, θ_2 increases and the new intersection point $\theta_2' > \theta_2$, while θ_1 could increase or decrease and the new intersection point θ_1' may be greater or smaller than θ_1. No matter how θ_1 changes, θ_2 always moves faster than θ_1 and along a fixed direction. Therefore, as r increases, the new intersection points θ_1' and θ_2' would never be balanced again, i.e. $\theta_2' - \frac{\pi}{4} \neq \frac{\pi}{4} - \theta_1'$ (see Figure E27). The same conclusion can still be obtained by decreasing r away from $\tilde{r}$. Hence, if there exists a pair of asymmetric equilibria $(\theta_1, \tilde{r})$ and $(\theta_2, \tilde{r})$, where $\theta_1 \in [-\frac{\pi}{4}, 0)$ and $\theta_2 \in [\frac{\pi}{2}, \frac{3}{4}\pi]$, there would never exist any other asymmetric equilibria.

Therefore, in conclusion, for all $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$, given all primitives M, D, ς, ς^* and ρ, if asymmetric equilibria exist, the number of asymmetric equilibria is two.

Next, under what conditions asymmetric equilibria exist? Recall that the asymmetric equilibria are solutions of the following equation group:

$$p(\theta,r) = \frac{D}{D-M}$$

$$q(\theta,r) = \frac{D}{D-M} \tag{E.4}$$

Hence, these solutions of the equation group E.4 can be interpreted as the intersections of the three curves $y = p(\theta,r)$, $y = q(\theta,r)$ and $y = \frac{D}{D-M}$, simultaneously. According to the relationship between $p(\tau_1,r)$ and $p(\tau_2,r)$, where (τ_1,τ_2) is $(0, \frac{\pi}{2})$ or $(-\frac{\pi}{4}, \frac{3}{4}\pi)$, we will discuss how asymmetric equilibria arise in the following two distinct conjectures:

Conjecure 1: $p(\tau_1, r) > p(\tau_2, r)$, where $(\tau_1, \tau_2) = (0, \frac{\pi}{2})$ or $(-\frac{\pi}{4}, \frac{3}{4}\pi)$.

In this case, if the symmetric equilibrium is stable, there must exist asymmetric equilibria, because 1) as r increases, $l(\frac{\pi}{4}, r) = -\phi(\frac{\cos\frac{\pi}{4} - \rho\sin\frac{\pi}{4}}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\frac{\pi}{4} + \rho\cos\frac{\pi}{4}}{\varsigma^*\sqrt{1-\rho^2}} < 0$ increases to 0. If at $r = \hat{r}$, $l(\frac{\pi}{4}, \hat{r}) > -\frac{\cos\frac{\pi}{4}}{M-D}$, which implies that $p'_\theta(\frac{\pi}{4}, \hat{r}) > 0$, then as r increases away from $\hat{r}$, it keeps the relationship $l(\frac{\pi}{4}, r) > -\frac{\cos\frac{\pi}{4}}{M-D}$ for all $r > \hat{r}$, and therefore, $p'_\theta(\frac{\pi}{4}, r) > 0$ for all $r > \hat{r}$; 2) Because $p'_\theta(\frac{\pi}{4}, r) > 0$, then the asymmetric equilibrium candidates always exist. Still, the asymmetric equilibrium candidates are the two extreme intersection points between $p(\theta, r)$ and $q(\theta, r)$, and correspondingly the middle intersection point is the symmetric equilibrium candidate. Since $\frac{\partial p(\theta,r)}{\partial r} > 0$ for all $\theta \in [\frac{\pi}{4}, \tau_2]$, and $\frac{\partial q(\theta,r)}{\partial r} > 0$ for all $\theta \in [\tau_1, \frac{\pi}{4}]$, the entire function of $p(\theta, r)$ in $\theta \in [\frac{\pi}{4}, \tau_2]$ and $q(\theta, r)$ in $\theta \in [\tau_1, \frac{\pi}{4}]$ increases as r increases. Therefore, supposing the symmetric equilibrium is obained at $r = \bar{r}$, i.e. it is $(\frac{\pi}{4}, \bar{r})$, and it satisfies $p'_\theta(\frac{\pi}{4}, \bar{r}) > 0$, as r increases away from $\bar{r}$, the asymmetric equilibrium candidates rise from positions below the line $y = \frac{D}{D-M}$ to the positions above the line $y = \frac{D}{D-M}$. Therefore, there must exist a moment such that the asymmetric equilibrium candidates pass the line $y = \frac{D}{D-M}$. At that moment which is recorded by the relevant value of r, the asymmetric equilibrium candidates formally become the asymmetric equilibria (see Figure E28); 3) According to the following equivalence relationship at the symmetric equilibrium or symmetric equilibrium candidate,

$$g'(x^*) \gtreqless -1 \iff \frac{1}{(1+\rho)(M-D)} - \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\cos\frac{\pi}{4} - \rho\sin\frac{\pi}{4}}{\varsigma^*\sqrt{1-\rho^2}}r) \gtreqless 0 \iff p'_\theta(\frac{\pi}{4}, r) \gtreqless 0$$

$p'_\theta(\frac{\pi}{4}, \bar{r}) > 0$ indicates that the symmetric equilibrium is stable. Therefore, in conclusion, as long as $p(\tau_1, r) > p(\tau_2, r)$, if the symmetric equilibrium is stable, then there must exist asymmetric equilibria.

However, this conjecture is not true because of the contradiction with the facts we have obtained. If the best response function is a contraction, there always exists a unique equilibrium, which is symmetric and stable. Hence, the conjecture that $p(\tau_1, r) > p(\tau_2, r)$, where $(\tau_1, \tau_2) = (0, \frac{\pi}{2})$ or $(-\frac{\pi}{4}, \frac{3}{4}\pi)$ is incorrect.

Conjecure 2: $p(\tau_1,r) < p(\tau_2,r)$, where $(\tau_1,\tau_2) = (0,\frac{\pi}{2})$ or $(-\frac{\pi}{4},\frac{3}{4}\pi)$.

Since conjecture 1 is incorrect, conjecture 2 must be the correct one to reflect the true property of $p(\theta,r)$. Given that $p(\tau_1,r) < p(\tau_2,r)$, if there is no asymmetric equilibrium candidate, $p'_\theta(\frac{\pi}{4},r) \geq 0$ must hold, and if there are asymmetric equilibrium candidates, then for the symmetric equilibrium candidate $(\frac{\pi}{4},r)$, $p'_\theta(\frac{\pi}{4},r) < 0$ must hold (see Figures E29 and E30, respectively).

Remember that as r increases, $l(\frac{\pi}{4},r)$ increases. Therefore, if at $r=\hat{r}$, $l(\frac{\pi}{4},\hat{r}) < -\frac{\cos\frac{\pi}{4}}{M-D}$, and hence $p'_\theta(\frac{\pi}{4},\hat{r}) < 0$, then as r increases away from $\hat{r}$, $p'_\theta(\frac{\pi}{4},r)$ will increase gradually from $p'_\theta(\frac{\pi}{4},r) < 0$ to $p'_\theta(\frac{\pi}{4},r) = 0$, and then to $p'_\theta(\frac{\pi}{4},r) > 0$. If there is no asymmetric equilibrium, it is either because for all $r > 0$, even for $r \to 0^+$, $\lim_{r\to 0^+} l(\frac{\pi}{4},r) > -\frac{\cos\frac{\pi}{4}}{M-D}$, and thus no asymmetric equilibrium candidates exist for all $r > 0$, or because as r increases, at the beginning, there were asymmetric equilibrium candidates, but before the symmetric equilibrium candidate $(\frac{\pi}{4},r)$ passes the line $y = \frac{D}{D-M}$, $p'_\theta(\frac{\pi}{4},r)$ changes from a negative value to a non-negative value, and thereafter, asymmetric equilibrium candidates vanish; consequently, the game has only one equilibrium, which is symmetric. For both possiblities, the corresponding symmetric equilibrium $(\frac{\pi}{4},\bar{r})$ must satisfy $p'_\theta(\frac{\pi}{4},\bar{r}) \geq 0$. Let us recall that for the symmetric equilibrium candidate (s,s), $g'(s) \gtreqless -1 \iff p'_\theta(\frac{\pi}{4},r) \gtreqless 0$. Therefore, it can be concluded that given all primitives, in the symmetric strategic-substitutes game, if there is no asymmetric equilibrium, the relevant symmetric equilibrium must be not unstable.

Conversely, if the symmetric equilibrium, which is denoted by $(\frac{\pi}{4},\bar{r})$, is not unstable, there are no asymmetric equilibria, because 1) as long as symmetric equilibrium (s,s) is not unstable, i.e. $g'(s) \geq -1$, $p'_\theta(\frac{\pi}{4},\bar{r}) \geq 0$, and therefore, there do not exist asymmetric equilibrium candidates at $r=\bar{r}$; 2) since the symmetric equilibrium is not unstable, i.e. $p'_\theta(\frac{\pi}{4},\bar{r}) \geq 0$, if r increases away from $\bar{r}$, $p'_\theta(\frac{\pi}{4},r) \geq 0$ must be held for all $r > \bar{r}$ because $p'_\theta(\frac{\pi}{4},r)$ increases as r increases; therefore, the asymmetric equilibrium candidates would never appear for all $r \in [\bar{r},+\infty)$. Hence, it is impossible for the asymmetric equilibria to exist, and only the symmetric equilibrium survives if the symmetric equilibrium is not unstable.

In conclusion, given all primitives and for all $\theta \in [-\frac{\pi}{4},\frac{3}{4}\pi]$, the symmetric strategic substitutes game contains only a unique equilibrium if and only if the symmetric equilibrium is not unstable, and equivalently, asymmetric equilibria exist in the symmetric strategic substitutes game if and only if the relevant symmetric equilibrium is unstable.

Therefore, for the entire symmetric strategic substitutes game (i.e., $\theta \in [-\frac{\pi}{4}, \frac{7}{4}\pi]$), given all primitives (M, D, ς, ς^* and ρ), the necessary and sufficient condition for unique equilibrium is that the symmetric equilibrium, which always exists, is not unstable. Equivalently, the necessary and sufficient condition for the existence of asymmetric equilibria is that the relevant symmetric equilibrium is unstable. Finally, if there exist asymmetric equilibria, the number is two. *Q.E.D.*

Lemma E4: Given that $M > D$ and $\varsigma = \varsigma^*$, the function $f(\rho) = \frac{\sqrt{2\pi}}{M-D}\frac{\sqrt{1-\rho^2}}{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2})$, where s satisfies $\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}s) = \frac{D+s}{D-M}$, strictly decreases with respect to ρ.

Proof: Differentiating $(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2$ with respect to ρ, we obtain that $\frac{d(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2}{d\rho} = \frac{d\frac{(\varsigma-\rho\varsigma^*)^2}{\varsigma^{*2}\varsigma^2(1-\rho^2)}}{d\rho} = \frac{-2\varsigma^*(\varsigma-\rho\varsigma^*)(1-\rho^2)+2\rho(\varsigma-\varsigma^*\rho)^2}{\varsigma^{*2}\varsigma^2(1-\rho^2)^2}$. By differentiating both sides of $\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}s) = \frac{D+s}{D-M}$ with respect to ρ, we get $\frac{ds}{d\rho} = \frac{1}{\phi(\frac{\varsigma-\rho\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}}s)\frac{\varsigma-\rho\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}}+\frac{1}{M-D}}\phi(\frac{\varsigma-\rho\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}}s)\frac{\varsigma^*-\rho\varsigma}{\varsigma\varsigma^*\sqrt{1-\rho^2}}\frac{s}{1-\rho^2}$.

Therefore, given $\varsigma = \varsigma^*$, we get $\frac{d(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2}}{d\rho} = \frac{d(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2}{d\rho}\frac{s^2}{2} + (\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 s\frac{ds}{d\rho}$

$= \frac{-\frac{\varsigma^2(1-\rho)^2}{M-D}}{\phi(\frac{1-\rho}{\varsigma\sqrt{1-\rho^2}}s)\frac{1-\rho}{\varsigma\sqrt{1-\rho^2}}+\frac{1}{M-D}}\frac{s^2}{\varsigma^{*2}\varsigma^2(1-\rho^2)^2}$. Because $M > D$, $\frac{d(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2}}{d\rho} < 0$. Because $\frac{d\frac{\sqrt{1-\rho^2}}{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}}{d\rho} < 0$, as long as $M > D$ and $\varsigma = \varsigma^*$, $f'(\rho) = \frac{\sqrt{2\pi}}{M-D}[\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2})\frac{d\frac{\sqrt{1-\rho^2}}{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}}{d\rho} + \frac{\sqrt{1-\rho^2}}{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2})\frac{d(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2}}{d\rho}] < 0$. In conclusion, given that $M > D$ and $\varsigma = \varsigma^*$, $f(\rho)$ decreases with respect to ρ. *Q.E.D.*

Lemma E5: In the symmetric strategic substitutes game $(M > D)$ and $\varsigma = \varsigma^*$, there always exists a unique boundary $\bar{\rho} \geq \hat{\rho} = \frac{2\pi\varsigma^*-(M-D)^2}{2\pi\varsigma^*+(M-D)^2}$. Given $M > D$ and $\varsigma = \varsigma^*$, the game always contains a unique equilibrium for all $\rho \in (-1, \tilde{\rho}]$ if and only if $\bar{\rho} \geq \tilde{\rho}$. It is symmetric and not unstable. Given $M > D$ and $\varsigma = \varsigma^*$, the game can contain asymmetric equilibria for some value of $\rho \in (-1, \tilde{\rho}]$ if and only if $\bar{\rho} < \tilde{\rho}$. In this case, only the symmetric equilibrium exists if and only if $\rho \in (-1, \bar{\rho}]$. The symmetric equilibrium is not unstable. Multiplicity arises if and only if $\rho \in (\bar{\rho}, \tilde{\rho}]$. There is one symmetric equilibrium, which is unstable, and two asymmetric equilibria.

Given $M > D$ and $\varsigma = \varsigma^*$, the necessary and sufficient condition of $\bar{\rho} < \tilde{\rho}$ is the following equation group:

$$\Phi(\frac{M-D}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\frac{s}{\varsigma}) = \frac{D+s}{D-M}$$

$$s^2 < \frac{2\varsigma^2(\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2})^2}{(M-D)^2}\ln(\sqrt{1+\frac{(M-D)^2}{2\pi\varsigma^2}}+1) \quad \text{(E.5)}$$

where s is an unkonwn variable that is uniquely contained in the first equation of equation group E.5.

Proof: Without loss of generality, for all $\rho \in (-1, \hat{\rho}]$, $g(x^*)$ is a contraction. Therefore, a unique equilibrium always exists. It is symmetric and stable.

For all $\rho \in (\hat{\rho}, \tilde{\rho}]$, $g(x^*)$ is not a contraction. In this case, there could be either one equilibrium or multiple equilibria given all primitives. According to Lemma E3, the criterion to judge whether there exist multiple equilibria is the stability of the symmetric equilibrium. If and only if the symmetric equilibrium is not unstable, i.e. at this equilibrium $g'(x^*) \geq -1$, there exists a unique equilibrium, which is only the symmetric equilibrium. If and only if the symmetric equilibrium is unstable, i.e. at this equilibrium $g'(x^*) < -1$, then asymmetric equilibria exist.

We denote the symmetric equilibrium by (s, s). It should satisfy $g(s) = s$, and at the symmetric equilibrium, each player's best response function's first-order derivative should satisfy the following:

$$g'(s) = \frac{1}{\rho - \frac{\sqrt{2\pi}\varsigma^*\sqrt{1-\rho^2}\exp(\frac{1}{2}(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 s^2)}{M-D}}$$

For all $\rho \in [\hat{\rho}, \tilde{\rho}]$, according to Lemma E3, there exists only a unique (symmetric) equilibrium if and only if the symmetric equilibrium is not unstable, i.e. the first-order derivative of each player's best response function $g'(s) \geq -1$, i.e.

$$0 < \frac{1}{\frac{\sqrt{2\pi}\varsigma^* \sqrt{1-\rho^2}\exp(\frac{1}{2}(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 s^2)}{M-D} - \rho} \leq 1$$

This inequality can be equivalently transformed into $\sqrt{2\pi}\exp(\frac{1}{2}(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 s^2) \geq \frac{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}(M-D)$. In addition, let us recall that the symmetric equilibrium should satisfy $g(s) = s$, i.e. $\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}s) = \frac{D+s}{D-M}$. Therefore, given $M > D$, $\varsigma = \varsigma^*$ and $\rho \in [\hat{\rho}, \tilde{\rho}]$, according to Lemma E3, if there exists only a unique equilibrium, it is necessary and sufficient that symmetric equilibrium (s,s) should simultaneously satisfy the following three conditions:

$$\sqrt{2\pi}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 \frac{s^2}{2}) \geq \frac{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}(M-D) \tag{E.6}$$

$$\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}s) = \frac{D+s}{D-M} \tag{E.7}$$

$$\rho \geq \frac{2\pi\varsigma^* - (M-D)^2}{2\pi\varsigma^* + (M-D)^2} \tag{E.8}$$

In contrast, if the inequality E.6 is replaced by

$$\sqrt{2\pi}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2 \frac{s^2}{2}) < \frac{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}(M-D) \tag{E.9}$$

E.9 with equation E.7 and inequality E.8, the three conditions together indicate that given $M > D$, $\varsigma = \varsigma^*$ and $\rho \in [\hat{\rho}, \tilde{\rho}]$, the symmetric equilibrium is unstable; hence, according to Lemma E3, asymmetric equilibira exist as well.

Furthermore, inequality E.6 can be equivalently written as

$$\frac{\sqrt{2\pi}}{M-D}\frac{\sqrt{1-\rho^2}}{\frac{1}{\varsigma^*}+\frac{\rho}{\varsigma}}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}})^2\frac{s^2}{2}) \geq 1$$

We denote the LHS of above inequality by $f(\rho)$. Then, $f(\rho) \geq 1$ indicates that $g'(s) \geq -1$, while $f(\rho) < 1$ indicates that $g'(s) < -1$. According to Lemma E4, if $M > D$ and $\varsigma = \varsigma^*$, $f(\rho)$ strictly decreases with respect to ρ. Because $\hat{\rho} = \frac{2\pi\varsigma^* - (M-D)^2}{2\pi\varsigma^* + (M-D)^2}$, $\frac{(M-D)^2(1+\hat{\rho})^2}{2\pi\varsigma^{*2}(1-\hat{\rho}^2)} = \frac{(M-D)^2(1+\hat{\rho})}{2\pi\varsigma^*(1-\hat{\rho})} = 1$, and $f(\hat{\rho}) = \frac{\sqrt{2\pi}}{M-D} \frac{\sqrt{1-\hat{\rho}^2}}{\frac{1}{\varsigma^*} + \frac{\hat{\rho}}{\varsigma}} \exp((\frac{\frac{1}{\varsigma^*} - \frac{\hat{\rho}}{\varsigma}}{\sqrt{1-\hat{\rho}^2}})^2 \frac{s^2}{2})$

$= \frac{\sqrt{2\pi}\varsigma}{M-D} \sqrt{\frac{1-\hat{\rho}}{1+\hat{\rho}}} \exp((\frac{\frac{1}{\varsigma^*} - \frac{\hat{\rho}}{\varsigma}}{\sqrt{1-\hat{\rho}^2}})^2 \frac{s^2}{2}) \geq 1$, where s satisfies equation E.7. Because $f(1) = 0$, as ρ increases from $\hat{\rho}$ to 1, $f(\rho)$ decreases from a value greater than or equal to 1 to 0. Therefore, there must exist a unique $\rho = \bar{\rho} \in [\hat{\rho}, 1)$ such that at $\rho = \bar{\rho}$, $f(\bar{\rho}) = 1$.

Suppose $\bar{\rho} < \tilde{\rho} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2} + (M-D)^2}}$. Because $f'(\rho) < 0$, for all $\rho \in [\hat{\rho}, \bar{\rho}]$, $f(\rho) > 1 \iff g'(s) \geq -1$, while for $\rho \in (\bar{\rho}, \tilde{\rho}]$, $f'(\rho) < 1 \iff g'(s) < -1$. Therefore, in this case, according to Lemma E3, given $M > D$ and $\varsigma = \varsigma^*$, the game only has a unique equilibrium if and only if $\rho \in [\hat{\rho}, \bar{\rho}]$, while multiple equilibria exist if and only if $\rho \in (\bar{\rho}, \tilde{\rho}]$.

Suppose $\bar{\rho} \geq \tilde{\rho} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2} + (M-D)^2}}$. Because $f'(\rho) < 0$, for all $\rho \in [\hat{\rho}, \tilde{\rho}]$, the first-order derivative of each player's best response function at symmetric equilibrium (s, s) is $g'(s) \geq -1$, where the equality is obtained as long as $\rho = \tilde{\rho} = \hat{\rho}$. Therefore, in this case, based on Lemma E3, it can be concluded that given $M > D$ and $\varsigma = \varsigma^*$, there always exists a unique equilibrium for all $\rho \in [\hat{\rho}, \tilde{\rho}]$.

From the above analysis, it can be observed that given $M > D$ and $\varsigma = \varsigma^*$, the relationship between $\bar{\rho}$ and $\tilde{\rho}$ equivalently reflects whether asymmetric equilibria can exist. Suppose that given $M > D$ and $\varsigma = \varsigma^*$, the game can contain asymmetric equilibria for some value of ρ, then at least at $\rho = \tilde{\rho}$, the symmetric equilibrium should be unstable, while if at $\rho = \tilde{\rho}$, the symmetric equilibrium is unstable, then the game certainly has asymmetric equilibria. Therefore, we get the following necessary and sufficient condition to guarantee that the game can contain asymmetric equilibria for some value of ρ given $M > D$ and $\varsigma = \varsigma^*$: $\bar{\rho} < \tilde{\rho} \iff$ at $\rho = \tilde{\rho}$, the symmetric equilibrium is unstable $\iff$ given $M > D$ and $\varsigma = \varsigma^*$, asymmetric equilibria must exist in the game for some value of ρ. Next, we turn to find the algebraic representation of the necessary and sufficient condition. At $\rho = \tilde{\rho} = \sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2} + (M-D)^2}}$, the symmetric equilibrium is unstable if and only if the following conditions are simultaneously satisfied:

$$\Phi(\frac{M-D}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\frac{s}{\varsigma})=\frac{D+s}{D-M}$$

$$f(\tilde{\rho})=\frac{\sqrt{2\pi}\varsigma}{M-D}\frac{\sqrt{2\pi\varsigma^2+(M-D)^2}-\sqrt{2\pi\varsigma^2}}{M-D}\exp(\frac{1}{2}\frac{(\sqrt{2\pi\varsigma^2+(M-D)^2}-\sqrt{2\pi\varsigma^2})^2}{(M-D)^2}\frac{s^2}{\varsigma^2})$$

$$=\frac{\sqrt{2\pi\varsigma^2}}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\exp(\frac{1}{2}\frac{(M-D)^2}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\frac{s^2}{\varsigma^2})<1$$

Equivalently, the above conditions can be transformed into the following form:

$$\Phi(\frac{M-D}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\frac{s}{\varsigma})=\frac{D+s}{D-M}$$

$$s^2<\frac{2\varsigma^2(\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2})^2}{(M-D)^2}\ln(\sqrt{1+\frac{(M-D)^2}{2\pi\varsigma^2}}+1)$$

Combining the previous analysis, it can be concluded that the above two conditions are the necessary and sufficient conditions to guarantee $\bar{\rho}<\tilde{\rho}$, and hence, given $M>D$ and $\varsigma=\varsigma^*$, the game can contain asymmetric equilibria for some value of ρ, and which is exactly $\rho\in(\bar{\rho},\tilde{\rho}]$.

Next, we derive the analytical expression of $\bar{\rho}$. At $\rho=\bar{\rho}$, there exists a unique equilibrium/solution; this equilibrium's/solution's stability is not determined. According to the previous analysis, for $\bar{\rho}\in[\hat{\rho},\tilde{\rho}]$, the following conditions should be satisfied simultaneously:

$$\sqrt{2\pi}\exp((\frac{\frac{1}{\varsigma^*}-\frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}})^2\frac{s^2}{2})=\frac{\frac{1}{\varsigma^*}+\frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}}(M-D) \quad \text{(E.10)}$$

$$\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}}s)=\frac{D+s}{D-M} \quad \text{(E.11)}$$

$$\bar{\rho}\geq\frac{2\pi\varsigma^*-(M-D)^2}{2\pi\varsigma^*+(M-D)^2} \quad \text{(E.12)}$$

First, consider the case where $M+D=0$. According to the Lemma E1, it is known that as long as $M+D=0$, symmetric equilibrium $(s,s)=(0,0)$. According to equation

E.10, given $\varsigma = \varsigma^*$ and $(s,s) = (0,0)$, equation E.10 indicates that $\bar{\rho} = \frac{2\pi\varsigma^* - (M-D)^2}{2\pi\varsigma^* + (M-D)^2} = \hat{\rho}$. Hence, if $M+D=0$, $\bar{\rho} = \hat{\rho}$.

Inequality E.12 can be equivalently transformed into $\frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)} \geq 1$, where the equality is held as long as $\bar{\rho} = \frac{2\pi\varsigma^* - (M-D)^2}{2\pi\varsigma^* + (M-D)^2}$. From equation E.10, it is known that $s^2 = \frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}$. If $\bar{\rho} = \hat{\rho}$, then $\frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)} = 1$, and hence $s = 0$. According to Lemma E1, $M+D=0$, which is therefore necessary and sufficient for $\bar{\rho} = \hat{\rho}$.

Now consider the case where $M+D \neq 0$. Then $\bar{\rho} > \frac{2\pi\varsigma^* - (M-D)^2}{2\pi\varsigma^* + (M-D)^2}$, and hence $\frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)} > 1$. Still, $s^2 = \frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}$. According to Lemma E1, it is known that $M+D \gtreqless 0 \iff 0 \gtreqless s \gtreqless -\frac{M+D}{2}$. Therefore, if $M+D>0$, $s = -\sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}$. If $M+D<0$, $s = \sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}$.

According to Lemma E4, function $f(\rho)$ strictly decreases with respect to ρ. It implies that $\bar{\rho}$ must be unique if and only if it simultaneously satisfies E.10, E.11 and E.12, which constitute $\bar{\rho}$'s definition. If there exists multiple $\bar{\rho}$s, it indicates that equation groups of E.10, E.11 and E.12 represent more than one equilibrium/solution of which stability is not determined, which contradicts the fact that there is only one such kind of equilibrium/solution in this symmetric strategic substitutes game. Therefore, given the expressions of s and equation E.11, for $M+D>0$, $\bar{\rho}$ is the unique solution of the equation $\Phi(-\sqrt{\ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}) = \frac{D - \sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}}{D-M}$, while for $M+D<0$, $\bar{\rho}$ is the unique solution of the equation $\Phi(\sqrt{\ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}) = \frac{D + \sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(M-D)^2(\varsigma+\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}}{D-M}$. For $M+D \neq 0$, $\bar{\rho} \gtrless \tilde{\rho}$.

Finally, according to the previous proofs, we can use the stability of the symmetric equilibrium at $\rho = \tilde{\rho}$ to judge whether the game can contain asymmetric equilibria, or equivalently $\bar{\rho} > \tilde{\rho}$ or $\bar{\rho} < \tilde{\rho}$. Here, we give the algebraic representation of the stability of the symmetric equilibrium at $\rho = \tilde{\rho}$. Symmetric equilibrium (s,s) at $\rho = \tilde{\rho}$ is given by

$$\Phi(\frac{M-D}{\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2}}\frac{s}{\varsigma}) = \frac{D+s}{D-M}$$

The solution of this equation is unique. If it is unstable, s satisfies

$$s^2 < \frac{2\varsigma^2(\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2})^2}{(M-D)^2}\ln(\sqrt{1+\frac{(M-D)^2}{2\pi\varsigma^2}}+1)$$

which indicates that $\bar{\rho} < \tilde{\rho}$, and vice versa.

If it is not unstable, s satisfies

$$s^2 \geq \frac{2\varsigma^2(\sqrt{2\pi\varsigma^2+(M-D)^2}+\sqrt{2\pi\varsigma^2})^2}{(M-D)^2}\ln(\sqrt{1+\frac{(M-D)^2}{2\pi\varsigma^2}}+1)$$

which indicates that $\bar{\rho} \geq \tilde{\rho}$, and vice versa.*Q.E.D.*

Proof of Theorem 1: The proof of Theorem 1 completely comprises all proofs in Appendix E, which are the proofs of Lemmas E1 to E5. All of these proofs and their conclusions constitute Theorem 1. *Q.E.D.*

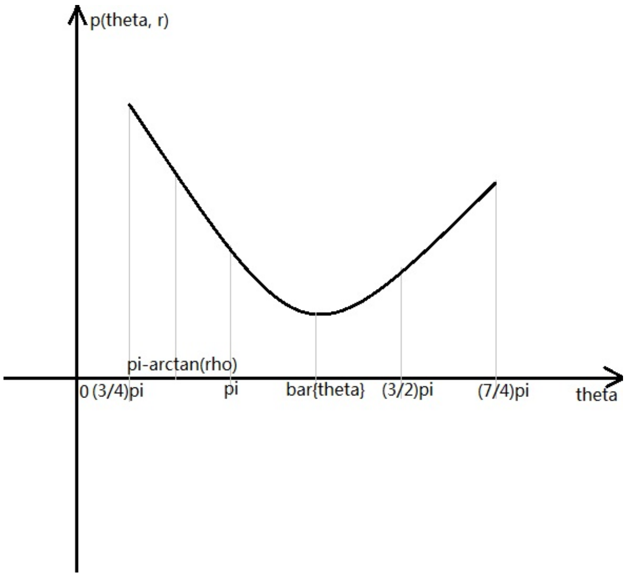


Figure E11: *Case 1: For all* $\theta \in (\frac{3}{2}\pi, \frac{7}{4}\pi]$, $p'_\theta(\theta, r) > 0$ *for all* $\rho \in (-1, \tilde{\rho}]$. *For all* $\theta \in [\frac{3}{4}\pi, \pi]$, *if* $\rho \leq 0$, $p'_\theta(\theta, r) < 0$*; therefore,* $p(\theta, r)$ *gets the 'U' shape in* $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$ *if* $\rho \leq 0$. *If* $\rho > 0$, *for all* $\theta \in [\frac{3}{4}\pi, \pi - \arctan\rho)$, $p'_\theta(\theta, r) < 0$. *For all* $\theta \in [\pi - \arctan\rho, \pi]$, *if* $\frac{r\cos\theta}{M-D}$ *dominates* $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$, *then* $p'_\theta(\theta, r) < 0$, *and therefore in this case,* $p(\theta, r)$ *gets the 'U' shape in* $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$ *if* $\rho > 0$. *Note that by far, we do not know the relationship between* $p(\frac{3}{4}\pi, r)$ *and* $p(\frac{7}{4}\pi, r)$ *yet. It could be* $p(\frac{3}{4}\pi, r) > p(\frac{7}{4}\pi, r)$ *or* $p(\frac{3}{4}\pi, r) < p(\frac{7}{4}\pi, r)$.

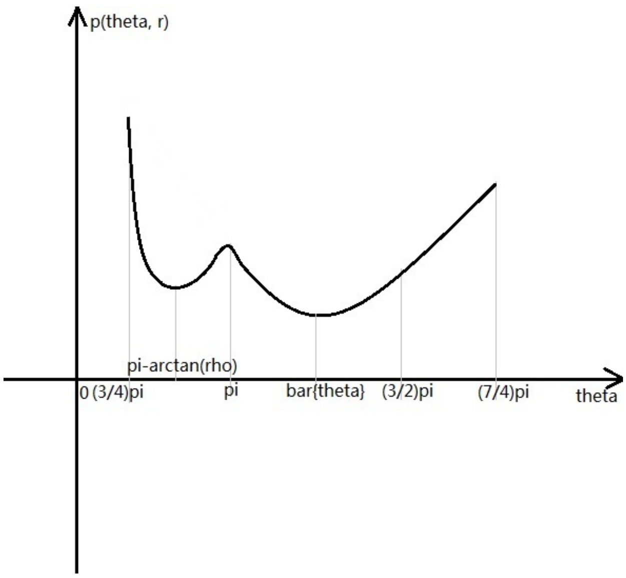


Figure E12: *Case 2: For all* $\theta \in (\frac{3}{2}\pi, \frac{7}{4}\pi]$, $p'_\theta(\theta, r) > 0$ *for all* $\rho \in (-1, 1)$. *If* $\rho > 0$, *for all* $\theta \in [\frac{3}{4}\pi, \pi - \arctan\rho)$, $p'_\theta(\theta, r) < 0$. *For all* $\theta \in [\pi - \arctan\rho, \pi]$, *if* $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r$ *dominates* $\frac{r\cos\theta}{M-D}$, *then* $p'_\theta(\theta, r) > 0$*; therefore, in this case,* $p(\theta, r)$ *gets the shape where there are three optima in* $\theta \in [\frac{3}{4}\pi, \frac{7}{4}\pi]$. *Note that by far, we do not know the relationship between* $p(\frac{3}{4}\pi, r)$ *and* $p(\frac{7}{4}\pi, r)$ *yet. It could be* $p(\frac{3}{4}\pi, r) > p(\frac{7}{4}\pi, r)$ *or* $p(\frac{3}{4}\pi, r) < p(\frac{7}{4}\pi, r)$.

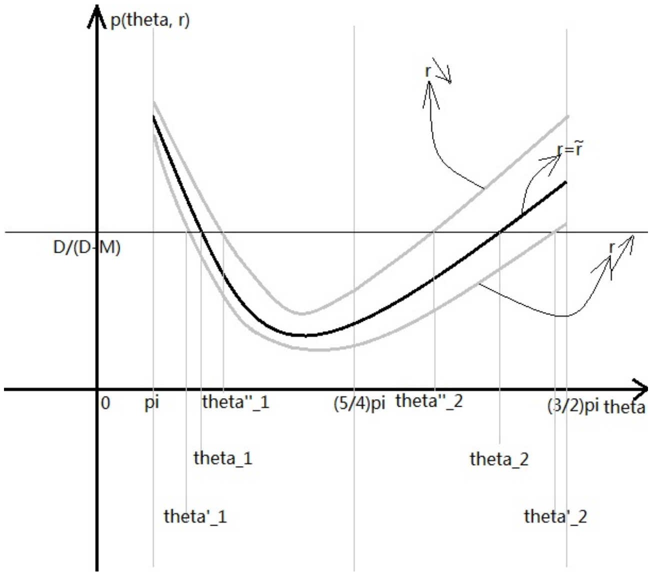


Figure E13: *Since $p(\theta, r)$ always have a larger movement in $\theta \in [\frac{5}{4}\pi, \frac{3}{2}\pi]$ than in $\theta \in [\pi, \frac{5}{4}\pi)$, if r is changed away from $\tilde{r}$, under which $\theta_2 - \frac{5}{4}\pi = \frac{5}{4}\pi - \theta_1$, where (θ_1, θ_2) are intersection points between $p(\theta, r)$ and $\frac{D}{D-M}$ given the value of $r > 0$, the position of the new intersection points will not be balanced, i.e. the new intersection points $(\theta_1, \theta_2) = (\theta_1', \theta_2')$ or (θ_1'', θ_2'') will make $\theta_2 - \frac{5}{4}\pi \neq \frac{5}{4}\pi - \theta_1$. This conclusion is held for either $p(\pi, r) > p(\frac{3}{2}\pi, r)$ or $p(\pi, r) < p(\frac{3}{2}\pi, r)$*

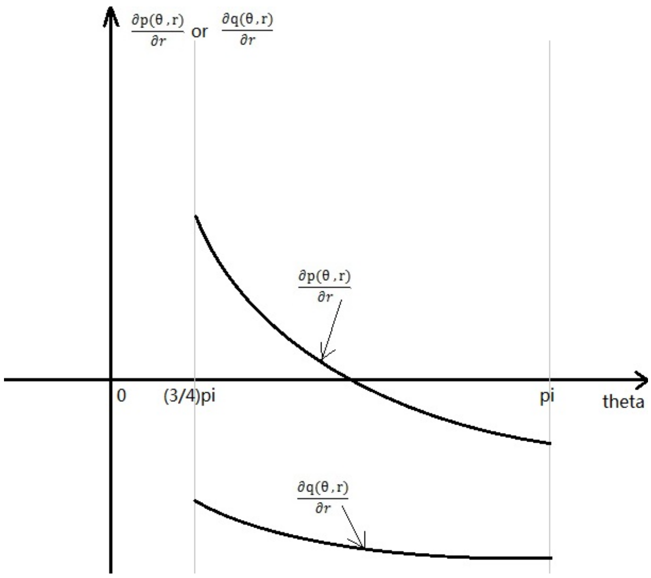


Figure E14: *A geometric description of $\frac{\partial p(\theta,r)}{\partial r}$ and $\frac{\partial q(\theta,r)}{\partial r}$ for all $\theta \in [\frac{3}{4}\pi, \pi]$ given $r > 0$*

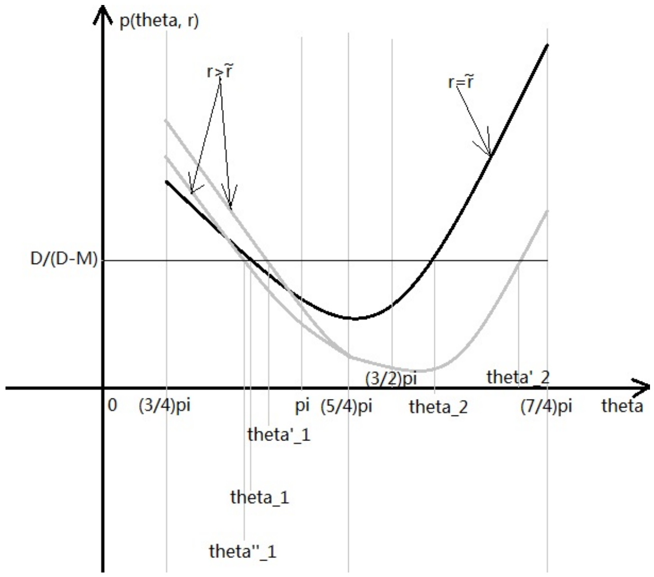


Figure E15: *Given a pair of asymmetric equilibria* $(\theta_1, \tilde{r})$ *and* $(\theta_2, \tilde{r})$*, where* $\theta_1 \in [\frac{3}{4}\pi, \pi)$ *and* $\theta_2 \in [\frac{3}{2}\pi, \frac{7}{4}\pi]$*. If r is increased away from* $\tilde{r}$ *successively, the new intersection point* θ'_2 *is greater than* θ_2*, while another intersection point* θ'_1 *or* θ''_1 *could be greater or smaller than* θ_1*. No matter how* θ_1 *changes,* θ_2 *always moves faster than* θ_1 *and along a fixed direction. Therefore, as r increases, the new intersection points* θ'_1 *(or* θ''_1*) and* θ'_2 *would never be balanced again, i.e.* $\theta'_2 - \frac{5}{4}\pi \neq \frac{5}{4}\pi - \theta'_1$*. This conclusion is held for either* $p(\frac{3}{4}\pi, r) > p(\frac{7}{4}\pi, r)$ *or* $p(\frac{3}{4}\pi, r) < p(\frac{7}{4}\pi, r)$*.*

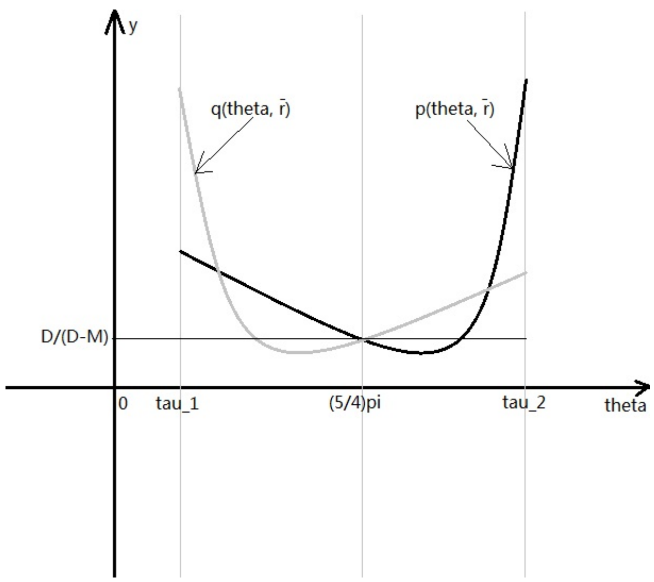


Figure E16: *If* $p(\tau_1, r) < p(\tau_2, r)$*, where* $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi) or (\frac{3}{4}\pi, \frac{7}{4}\pi)$*, the asymmetric equilibrium candidates always exist. They are the two extreme intersection points. Suppose there is a symmetric equilibrium* $(\frac{5}{4}\pi, \bar{r})$*. If r is increased away from* $\bar{r}$*, there must exist a moment that the asymmetric equilibrium candidates fall onto* $y = \frac{D}{D-M}$ *so that they formally become the asymmetric equilibria. From previous analysis and results, it is known that if asymmetric equilibria exist, there will be only two. Hence, the moment, or essentially the coresponding radius r, that asymmetric equilibrium candidates pass the line* $y = \frac{D}{D-M}$ *is unique.*

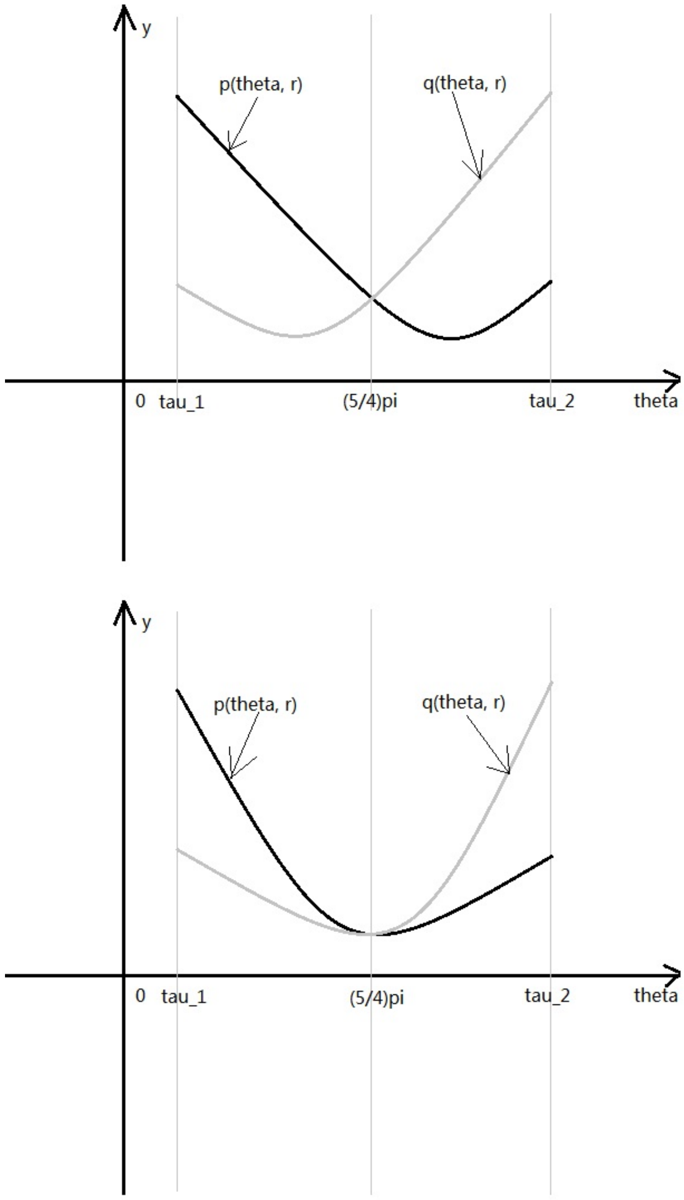


Figure E17: *Because* $p(\tau_1, r) > p(\tau_2, r)$*, where* $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi) or (\frac{3}{4}\pi, \frac{7}{4}\pi)$*, if* $p'_\theta(\frac{5}{4}\pi, r) \leq 0$*, there exists only one symmetric equilibrium candidate and no asymmetric equilibrium candidate. The top figure describes the* $p'_\theta(\frac{5}{4}\pi, r) < 0$ *case, while the bottom figure describes the* $p'_\theta(\frac{5}{4}\pi, r) = 0$ *case.*

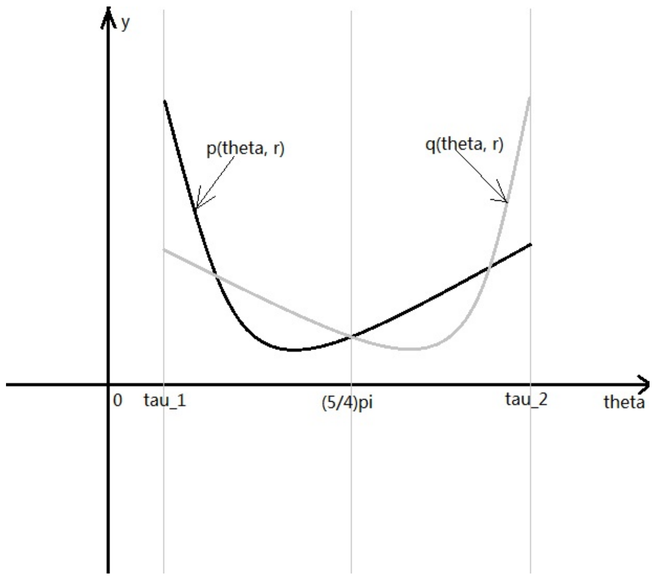


Figure E18: *Because $p(\tau_1, r) > p(\tau_2, r)$, where $(\tau_1, \tau_2) = (\pi, \frac{3}{2}\pi) or (\frac{3}{4}\pi, \frac{7}{4}\pi)$, if $p'_\theta(\frac{5}{4}\pi, r) > 0$, both the symmetric equilibrium candidate (the middle intersection point) and asymmetric equilibrium candidates (the two extreme intersection points) exist.*

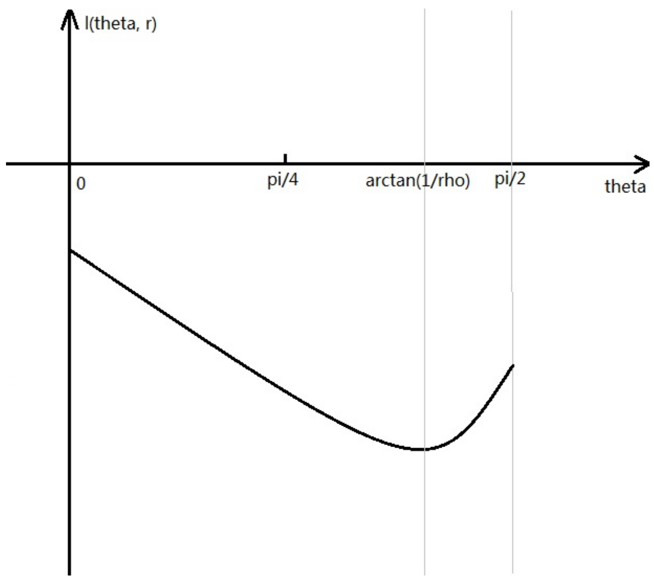


Figure E19: *A general geometric representation of function $l(\theta, r)$ with respect to θ given $r > 0$ in $\theta \in [0, \frac{\pi}{2}]$. The monotonicity, the position of the optimum (i.e., the minimum value of $l(\theta, r)$ is obtained at $\theta = \arctan \frac{1}{\rho} \in (\frac{\pi}{4}, \frac{\pi}{2})$ for all $\rho \in (0, 1)$), and the relationship $l(0, r) > l(\frac{\pi}{2}, r)$ are always maintained for any parameter specification of $l(\theta, r)$.*

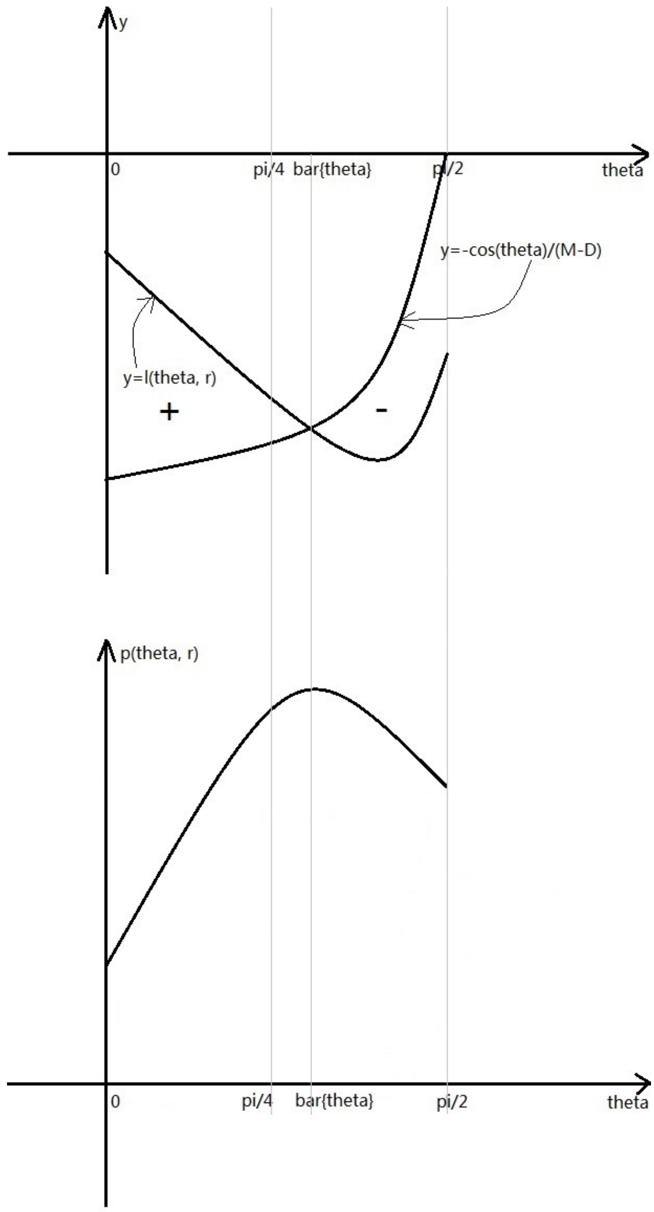


Figure E20: *Case 1: One possibility of the relative position between function $l(\theta, r)$ and $-\frac{\cos\theta}{M-D}$, and the corresponding shape of function $p(\theta, r)$, where there exists only one interior optimum for $\rho > 0$ and $\theta \in [0, \frac{\pi}{2}]$. The signs '+' and '-' represent the monotonicity of function $p(\theta, r)$, because for $r > 0$, $p'_\theta(\theta, r) = -\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}}r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}}r - (-\frac{r\cos\theta}{M-D})$. Note that by far, we do not know the relationship between $p(0, r)$ and $p(\frac{\pi}{2}, r)$ yet. It could be $p(0, r) > p(\frac{\pi}{2}, r)$ or $p(0, r) < p(\frac{\pi}{2}, r)$.*

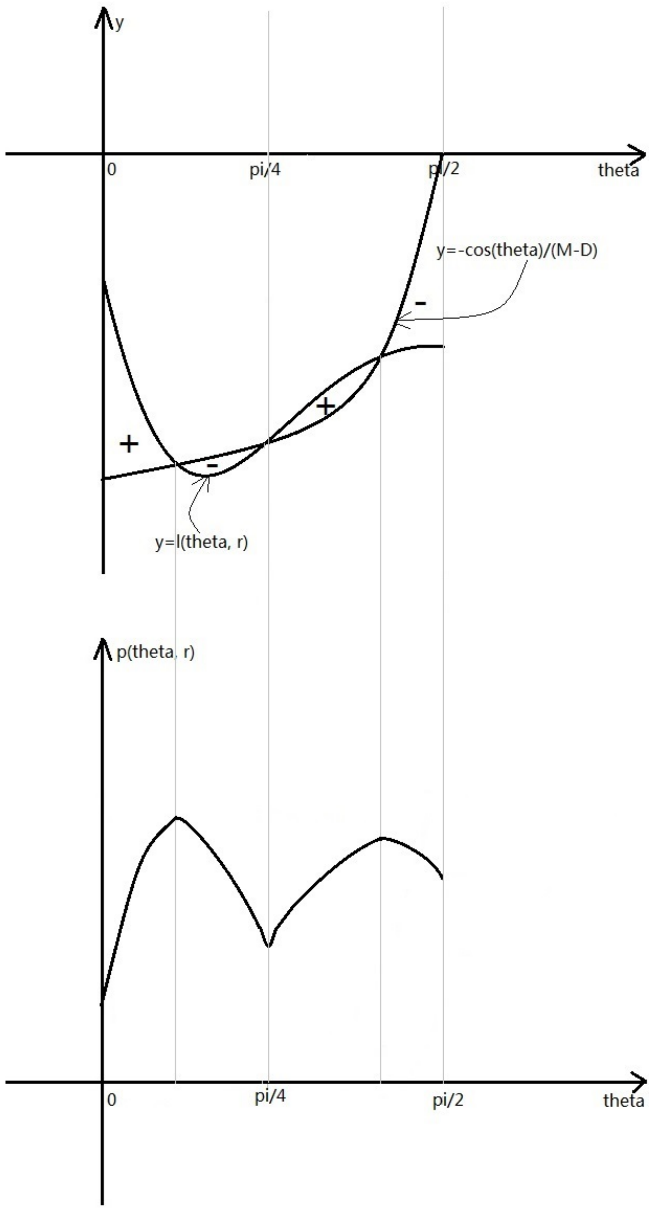


Figure E21: *Case 2: Another possibility of the relative position between function* $l(\theta, r)$ *and* $-\frac{\cos\theta}{M-D}$, *and the corresponding shape of function* $p(\theta, r)$, *where there exist three interior optima for* $\rho > 0$ *and* $\theta \in [0, \frac{\pi}{2}]$. *The signs '+' and '-' represent the monotonicity of function* $p(\theta, r)$, *because for* $r > 0$, $p'_\theta(\theta, r) = -\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}} r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} r - (-\frac{r\cos\theta}{M-D})$. *Note that by far, we do not know the relationship between* $p(0, r)$ *and* $p(\frac{\pi}{2}, r)$ *yet. It could be* $p(0, r) > p(\frac{\pi}{2}, r)$ *or* $p(0, r) < p(\frac{\pi}{2}, r)$.

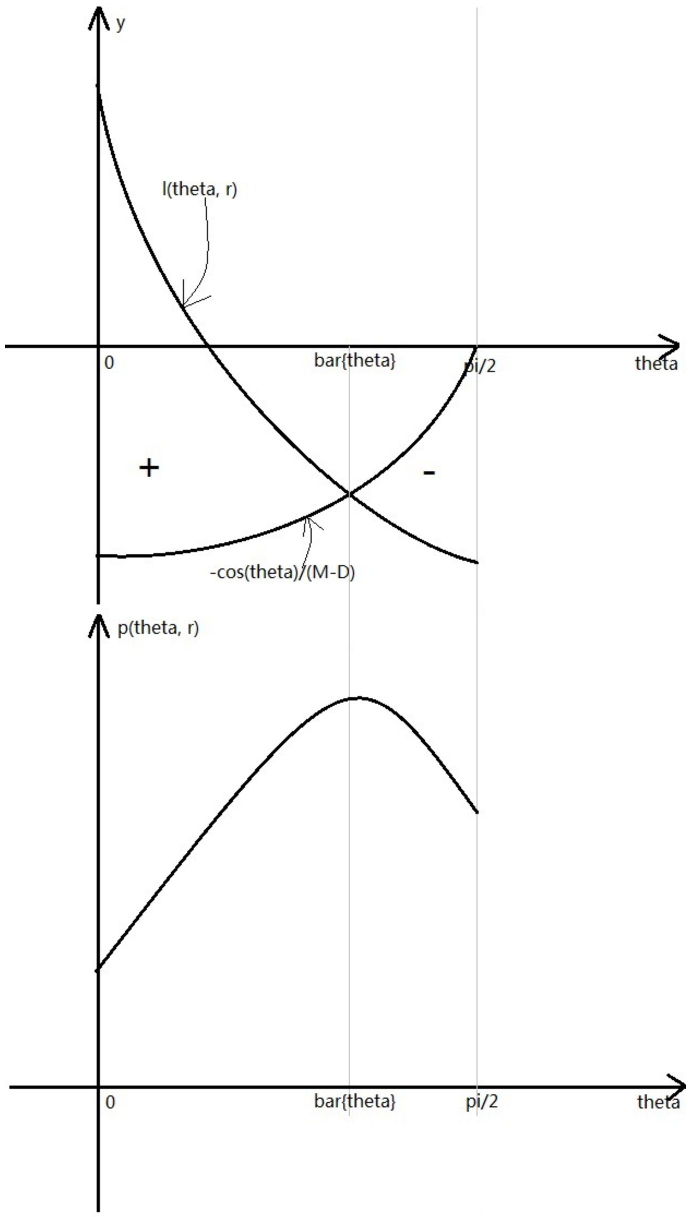


Figure E22: *The relative position between function $l(\theta, r)$ and $-\frac{\cos\theta}{M-D}$, and the corresponding shape of function $p(\theta, r)$ for $\rho \leq 0$ in $\theta \in [0, \frac{\pi}{2}]$. The signs '+' and '-' represent the monotonicity of function $p(\theta, r)$, because for $r > 0$, $p'_\theta(\theta, r) = -\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}} r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} r - (-\frac{r\cos\theta}{M-D})$. Note that by far, we do not know the relationship between $p(0, r)$ and $p(\frac{\pi}{2}, r)$ yet. It could be $p(0, r) > p(\frac{\pi}{2}, r)$ or $p(0, r) < p(\frac{\pi}{2}, r)$.*

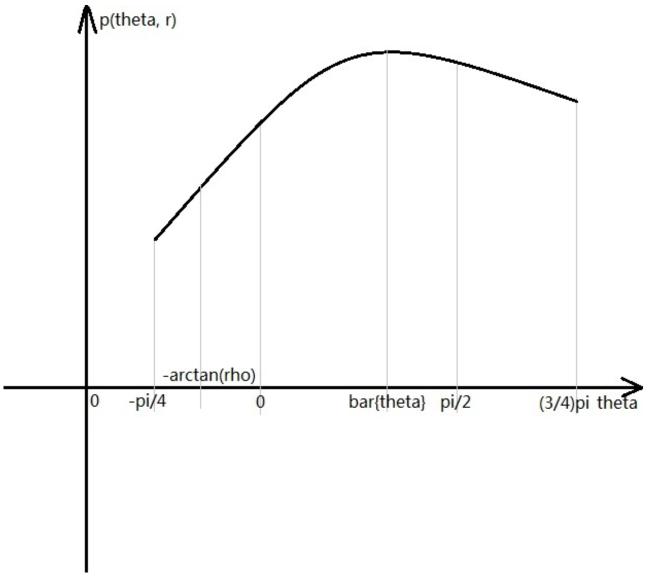


Figure E23: *Case 1: For all* $\theta \in [\frac{\pi}{2}, \frac{3}{4}\pi]$*, irrespective of whether* ρ *is positive,* $p'_\theta(\theta, r) < 0$*. For all* $\theta \in [-\frac{\pi}{4}, 0]$*, if* $\rho \leq 0$*,* $p'_\theta(\theta, r) > 0$*; therefore,* $p(\theta, r)$ *gets the 'cap' shape in* $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$ *if* $\rho \leq 0$*. If* $\rho > 0$*, for all* $\theta \in [-\frac{\pi}{4}, -\arctan\rho)$*,* $p'_\theta(\theta, r) > 0$*. For all* $\theta \in [-\arctan\rho, 0]$*, if* $\frac{r\cos\theta}{M-D}$ *dominates* $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}} r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} r$*, then* $p'_\theta(\theta, r) > 0$*; therefore, in this case also,* $p(\theta, r)$ *gets the 'cap' shape in* $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$ *if* $\rho > 0$*. Note that by far, we do not know the relationship between* $p(-\frac{\pi}{4}, r)$ *and* $p(\frac{3}{4}\pi, r)$ *yet. It could be* $p(-\frac{\pi}{4}, r) > p(\frac{3}{4}\pi, r)$ *or* $p(-\frac{\pi}{4}, r) < p(\frac{3}{4}\pi, r)$*.*

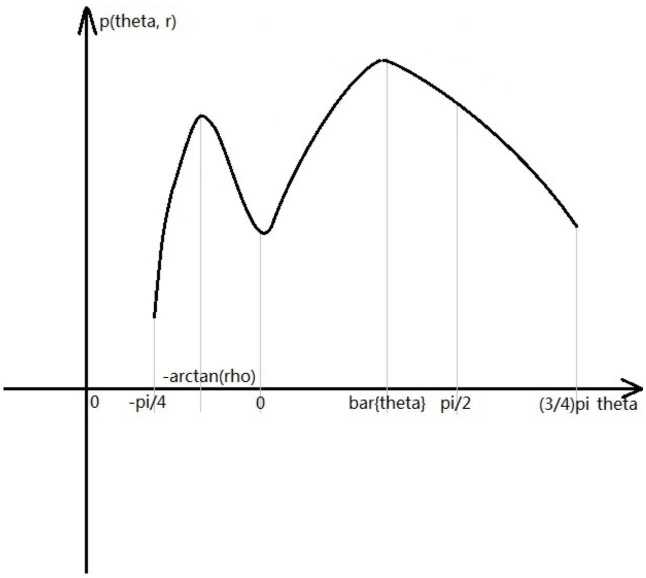


Figure E24: *Case 2: For all* $\theta \in [\frac{\pi}{2}, \frac{3}{4}\pi]$*,* $p'_\theta(\theta, r) < 0$ *for all* $\rho \in (-1, \tilde{\rho}]$*. If* $\rho > 0$*, for all* $\theta \in [-\frac{\pi}{4}, -\arctan\rho)$*,* $p'_\theta(\theta, r) > 0$*. For all* $\theta \in [-\arctan\rho, 0]$*, if* $-\phi(\frac{\cos\theta - \rho\sin\theta}{\varsigma^*\sqrt{1-\rho^2}} r)\frac{\sin\theta + \rho\cos\theta}{\varsigma^*\sqrt{1-\rho^2}} r$ *dominates* $\frac{r\cos\theta}{M-D}$*, then* $p'_\theta(\theta, r) < 0$*; therefore, in this case,* $p(\theta, r)$ *gets the shape such that there are three interior optima in* $\theta \in [-\frac{\pi}{4}, \frac{3}{4}\pi]$*.*

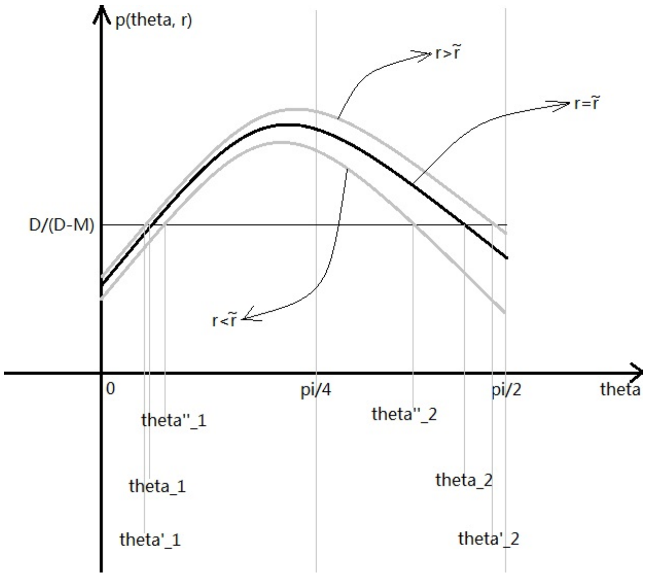


Figure E25: *Since $p(\theta,r)$ always have a larger movement in $\theta \in [\frac{\pi}{4},\frac{\pi}{2}]$ than in $\theta \in [0,\frac{\pi}{4})$, if r is changed away from $\tilde{r}$, under which $\theta_2 - \frac{\pi}{4} = \frac{\pi}{4} - \theta_1$, where (θ_1,θ_2) are intersection points between $p(\theta,r)$ and $\frac{D}{D-M}$ given $r > 0$, the position of the new intersection points will not be balanced, i.e. the new intersection points $(\theta_1,\theta_2) = (\theta_1',\theta_2')$ or (θ_1'',θ_2'') will make $\theta_2 - \frac{\pi}{4} \neq \frac{\pi}{4} - \theta_1$. This conclusion is held for either $p(0,r) > p(\frac{\pi}{2},r)$ or $p(0,r) < p(\frac{\pi}{2},r)$.*

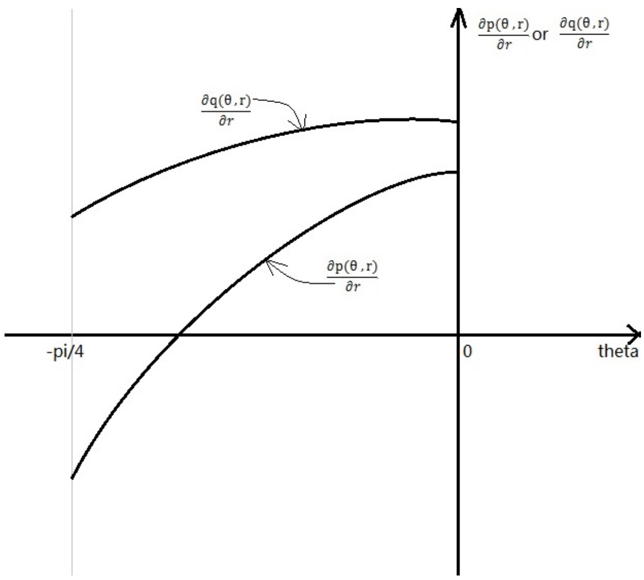


Figure E26: *A geometric illurstration of function $\frac{\partial p(\theta,r)}{\partial r}$ and $\frac{\partial q(\theta,r)}{\partial r}$ for all $\theta \in [-\frac{\pi}{4},0)$ given $r > 0$.*

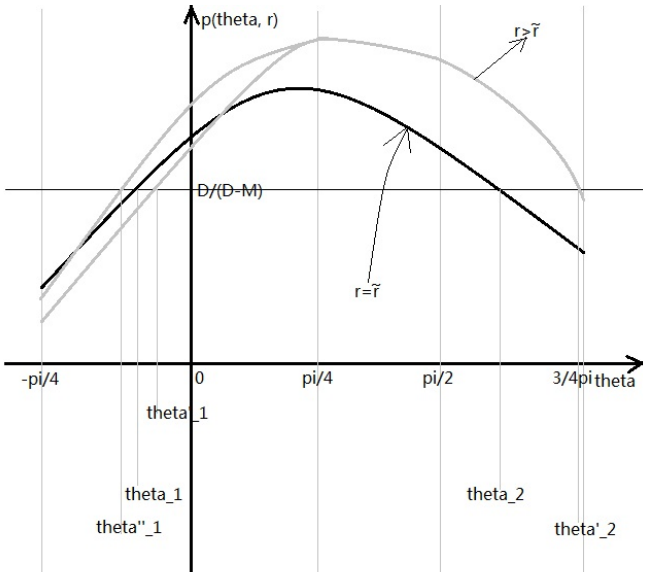


Figure E27: *Given a pair of asymmetric equilibria* $(\theta_1, \tilde{r})$ *and* $(\theta_2, \tilde{r})$*, where* $\theta_1 \in [-\frac{\pi}{4}, 0)$ *and* $\theta_2 \in [\frac{\pi}{2}, \frac{3}{4}\pi]$*. If r is increased away from* $\tilde{r}$ *successively, the new intersection point* θ_2' *is greater than* θ_2*, while another intersection point* θ_1' *or* θ_1'' *could be greater or smaller than* θ_1*. No matter how* θ_1 *changes,* θ_2 *always moves faster than* θ_1 *and along a fixed direction. Therefore, as r increases, the new intersection points* θ_1' *(or* θ_1''*) and* θ_2' *would never be balanced again, i.e.* $\theta_2' - \frac{\pi}{4} \neq \frac{\pi}{4} - \theta_1'$*. This conclusion is held either for* $p(-\frac{\pi}{4}, r) > p(\frac{3}{4}\pi, r)$ *or* $p(-\frac{\pi}{4}, r) < p(\frac{3}{4}\pi, r)$*.*

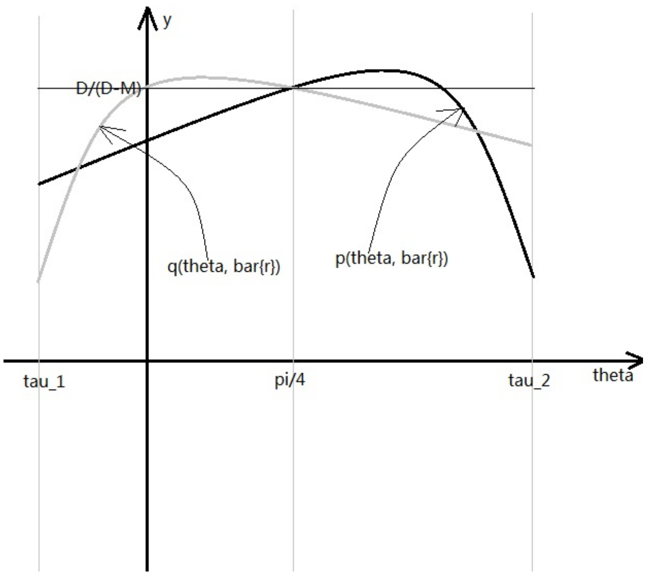


Figure E28: *If* $p(\tau_1, r) > p(\tau_2, r)$*, the asymmetric equilibrium candidates always exist. They are the two extreme intersection points. Suppose there is a symmetric equilibrium* $(\frac{\pi}{4}, \bar{r})$*. If r is increased away from* $\bar{r}$*, there must exist a moment that the asymmetric equilibrium candidates rise and pass the line* $y = \frac{D}{D-M}$ *so that they formally become the asymmetric equilibria. From previous analysis and results, it is known that if there are asymmetric equilibria, there are only two. Hence, the moment, or essentially the coresponding radius r, that asymmetric equilibrium candidates pass the line* $y = \frac{D}{D-M}$ *is unique.*

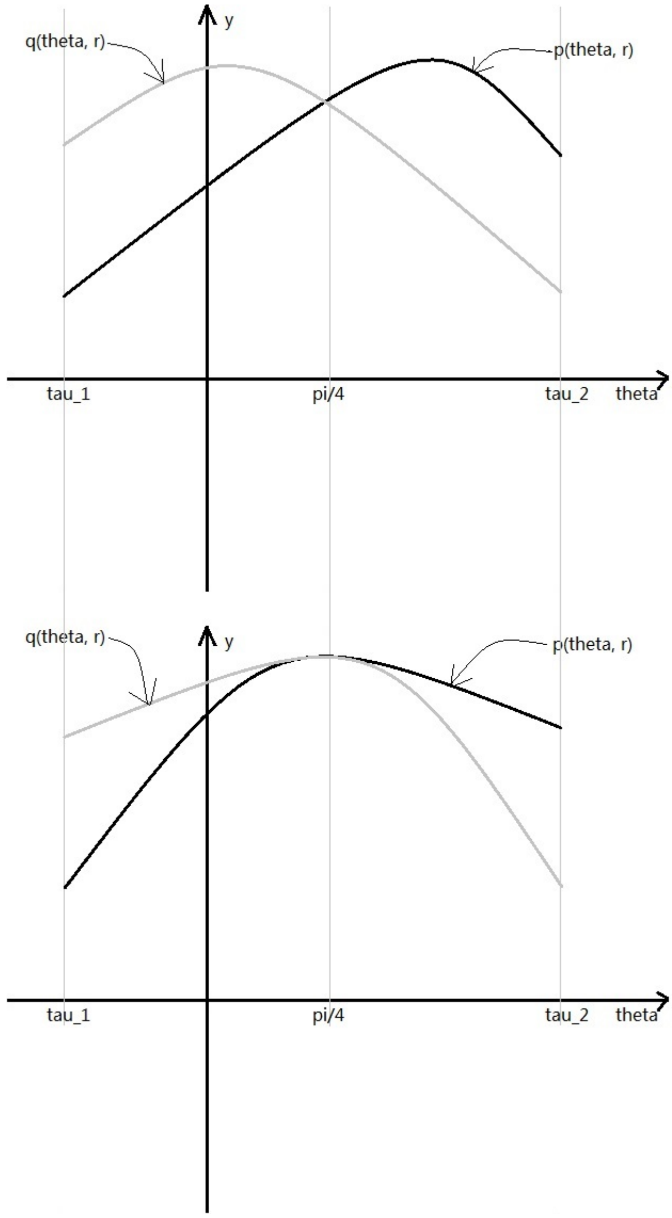


Figure E29: *Because $p(\tau_1, r) < p(\tau_2, r)$, where $(\tau_1, \tau_2) = (0, \frac{\pi}{2})$ or $(-\frac{\pi}{4}, \frac{3}{4}\pi)$, if $p'(\frac{\pi}{4}, r) \geq 0$, there exists only one symmetric equilibrium candidate and no asymmetric equilibrium candidate. The top figure describes the $p'_\theta(\frac{\pi}{4}, r) > 0$ case, while the bottom figure describes the $p'_\theta(\frac{\pi}{4}, r) = 0$ case.*

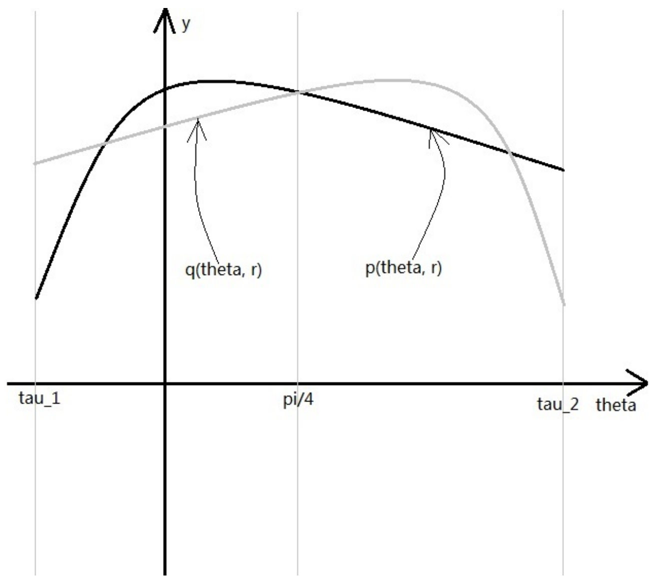


Figure E30: *Because* $p(\tau_1, r) < p(\tau_2, r)$, *where* $(\tau_1, \tau_2) = (0, \frac{\pi}{2})$ *or* $(-\frac{\pi}{4}, \frac{3}{4}\pi)$, *if* $p'_\theta(\frac{\pi}{4}, r) < 0$, *both the symmetric equilibrium candidate (the middle intersection point) and asymmetric equilibrium candidates (the two extreme intersection points) exist.*

Appendix F

Appendix of Chapter 2 Derivation of the Best Response Function at $\rho \to 1$

Let us recall the definition of the cutoff best response $g(x^*)$: $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})(M - D) + D + g(x^*) = 0$. If $x^* = g(x^*)$, $\lim_{\rho \to 1} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \frac{1}{2}$. Therefore, $\frac{1}{2}(M - D) + D + g(x^*) = 0$, and hence $g(x^*) = -\frac{M+D}{2}$. Thus, at $x^* = -\frac{M+D}{2}$, $g(x^*) = -\frac{M+D}{2}$.

If $x^* > g(x^*)$, $\lim_{\rho \to 1} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(+\infty) = 1$, $g(x^*) = -M$. In the equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, if and only if $x^* = -\frac{M+D}{2}$, $g(x^*) = x^*$; therefore, if $x^* > -\frac{M+D}{2}$, $g(x^*) = -M$.

If $x^* < g(x^*)$, $\lim_{\rho \to 1} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(-\infty) = 0$, $g(x^*) = -D$. In the equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, if and only if $x^* = -\frac{M+D}{2}$, $g(x^*) = x^*$; therefore, if $x^* < -\frac{M+D}{2}$, $g(x^*) = -D$.

Appendix G

Appendix of Chapter 2 Comparative Statics of the Symmetric Equilibrium

In this section, we first derive the comparative statics of the best response function at the symmetric equilibrium; next, we derive the comparative statics of the symmetric equilibrium. Both types of comparative statics have been qualitatively discussed in the intuition of comparative statics analysis.

G.1 Proof of Comparative Statics of the Best Response Function

Let us recall that given an opponent's strategy $x^* \in \mathbb{R}$, the best response function is

$$x^* = \rho\frac{\varsigma^*}{\varsigma}g(x^*) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$$

and it can be equivalently represented by

$$\Phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}}) = \frac{D+g(x^*)}{D-M} \tag{G.1}$$

1) $\frac{\partial g(x^*)}{\partial M}$**:** Differentiating both sides of equation (G.1) with respect to M and rearranging terms on both sides, we obtain

$$\frac{\partial g(x^*)}{\partial M} = -\frac{\frac{D+g(x^*)}{(D-M)^2}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

The condition $\rho \leq \tilde{\rho} = \sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$ can be equivalently re-expressed into

$$\frac{1}{D-M} + \frac{1}{\sqrt{2\pi}}\frac{\rho}{\varsigma\sqrt{1-\rho^2}} \leq 0$$

Therefore, when $\rho \leq 0$, $\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}} < 0$, and when $\rho > 0$, then $\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}} \leq \frac{1}{D-M} + \frac{1}{\sqrt{2\pi}}\frac{\rho}{\varsigma\sqrt{1-\rho^2}} \leq 0$. Therefore, for all $\rho \in (-1, \tilde{\rho}]$, $\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}} \leq 0$. Hence, without loss of generality, $\frac{1}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}} < 0$. Because $g(x^*) \in [-M, -D]$, where $-M$ and $-D$ are reached at the asymptote, without loss of generality, $\frac{\partial g(x^*)}{\partial M} < 0$.

2) $\frac{\partial g(x^*)}{\partial D}$: Differentiating both sides of equation (G.1) with respect to D and rearranging the terms on both sides, we obtain

$$\frac{\partial g(x^*)}{\partial D} = \frac{\frac{M+g(x^*)}{(D-M)^2}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

Because $g(x^*) \in [-M, -D]$, where $-M$ and $-D$ are reached at the asymptote, without loss of generality, $\frac{\partial g(x^*)}{\partial D} < 0$.

3) $\frac{\partial g(x^*)}{\partial \rho}$: Differentiating both sides of equation (G.1) with respect to ρ and rearranging the terms on both sides, we obtain

$$\frac{\partial g(x^*)}{\partial \rho} = \frac{\phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho\frac{x^*}{\varsigma^*} - \frac{g(x^*)}{\varsigma}}{(1-\rho^2)^{\frac{3}{2}}}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

At symmetric equilibrium (s, s), where $g(s) = s$ and $s \in [-M, -D]$ in which $-M$ and $-D$ are reached at the asymptote, if $s \gtreqless 0$, $\frac{\partial g(x^*)}{\partial \rho} \gtreqless 0$.

4) $\frac{\partial g(x^*)}{\partial \varsigma} + \frac{\partial g(x^*)}{\partial \varsigma^*}$**:** First, differentiating both sides of equation (G.1) with respect to ς and rearranging the terms on both sides, we obtain

$$\frac{\partial g(x^*)}{\partial \varsigma} = \frac{\phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\sqrt{1-\rho^2}}\frac{g(x^*)}{\varsigma^2}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

Second, differentiating both sides of equation (G.1) with respect to ς^* and rearranging terms on both sides, we obtain

$$\frac{\partial g(x^*)}{\partial \varsigma^*} = -\frac{\phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{1}{\sqrt{1-\rho^2}}\frac{x^*}{\varsigma^{*2}}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

Hence, we have

$$\frac{\partial g(x^*)}{\partial \varsigma} + \frac{\partial g(x^*)}{\partial \varsigma^*} = \frac{\phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho g(x^*) - x^*}{\varsigma^2\sqrt{1-\rho^2}}}{\frac{1}{D-M} + \phi(\frac{\varsigma x^* - \rho\varsigma^* g(x^*)}{\varsigma\varsigma^*\sqrt{1-\rho^2}})\frac{\rho}{\varsigma\sqrt{1-\rho^2}}}$$

At symmetric equilibrium (s,s), where $g(s) = s$ and $s \in [-M, -D]$ in which $-M$ and $-D$ are reached at the asymptote, if $s \gtreqless 0$, $\frac{\partial g(x^*)}{\partial \varsigma} + \frac{\partial g(x^*)}{\partial \varsigma^*} \gtreqless 0$.

G.2 Proof of Comparative Statics of the Symmetric Equilibrium (Proof of Proposition 3)

Symmetric equilibrium (s,s) should always satisfy $g(s) = s$. Therefore,

$$s = \rho\frac{\varsigma^*}{\varsigma}g(s) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(s)}{D-M})$$

This can be equivalently represented as

$$\Phi(\frac{\varsigma - \rho\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}}s) = \frac{D+s}{D-M} \tag{G.2}$$

Then, differentiating both sides of equation (G.2) with respect to M, D, ρ, ς and ς^*, and rearranging the terms on both sides of relevant equations, we obtain the following results on comparative statics of the symmetric equilibrium:

1)

$$\frac{\partial s}{\partial M}=\frac{1}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{D+s}{(M-D)^2}$$

Because $s\in[-M,-D]$ and $-M$ and $-D$ are reached at the asymptote, without loss of generality, $\frac{\partial s}{\partial M}<0$.

2)

$$\frac{\partial s}{\partial D}=\frac{1}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{-M-s}{(M-D)^2}$$

Because $s\in[-M,-D]$ and $-M$ and $-D$ are reached at the asymptote, without loss of generality, $\frac{\partial s}{\partial D}<0$.

3)

$$\frac{\partial s}{\partial \rho}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{\varsigma^*-\rho\varsigma}{\varsigma\varsigma^*(1-\rho^2)^{\frac{3}{2}}}s$$

Therefore, at the symmetric equilibrium, if $s\gtreqless 0$, then $\frac{\partial s}{\partial \rho}\gtreqless 0$.

4)

$$\frac{\partial s}{\partial \varsigma^*}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{s}{\varsigma^{*2}\sqrt{1-\rho^2}}$$

$$\frac{\partial s}{\partial \varsigma}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{-\rho s}{\varsigma^{2}\sqrt{1-\rho^2}}$$

Because $\varsigma=\varsigma^*$, we have

$$\frac{\partial s}{\partial \varsigma}+\frac{\partial s}{\partial \varsigma^*}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}s)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})+\frac{1}{M-D}}\frac{1-\rho}{\varsigma^{*2}\sqrt{1-\rho^2}}s$$

Therefore, at the symmetric equilibrium, if $s\gtreqless 0$, then $\frac{\partial s}{\partial \varsigma}+\frac{\partial s}{\partial \varsigma^*}\gtreqless 0$.

Appendix H

Appendix of Chapter 2 Contraction and Non-contraction Best Response Functions

Contraction is a very useful property of the best response functions. According to Zimper (2004), if the best response functions of a lattice game are contraction, the game is dominance solvable, and therefore, a unique equilibrium exists. In this thesis, the best response functions are real-valued and one-dimensional. In this specific context, we prove that, if and only if the abosulte value of the first-order derivative of a best response function is smaller than one, given any strategy of the opponent, the best response function is a contraction. This result applies to all chapters in this thesis.

Proposition H1: A first-order differentiable best response function $x = g(x^*)$, where $x \in \mathbb{R}$ and $x^* \in \mathbb{R}$, is a contraction if and only if for all $x^* \in \mathbb{R}$, $|g'(x^*)| < 1$.

Proof: In an arbitrary interval $[a,b] \subseteq \mathbb{R}$, $g(x^*)$ as defined is first-order differentiable. Therefore, according to the Lagrange intermdeidate value theorem, there exists a $\eta \in (a,b)$ such that

$$g(b) - g(a) = g'(\eta)(b-a)$$

Therefore, we have $|g(b) - g(a)| = |g'(\eta)|(b-a) \leq \max_{x^* \in (a,b)} |g'(x^*)|.|b-a|$.

If $x = g(x^*)$ is a contraction, according to the formal definition of contraction (see de la Fuente, 2000), we have $\max_{x^* \in (a,b)} |g'(x^*)| < 1$. Because a and b ($b > a$) are arbitrarily valued, for all $x^* \in \mathbb{R}$, $|g'(x^*)| < 1$.

If for all $x^* \in \mathbb{R}$ $|g'(x^*)| < 1$, then at any interval $(a,b) \subseteq \mathbb{R}$, $\max_{x^* \in (a,b)} |g'(x^*)| < 1$, and therefore, $x = g(x^*)$ is a contraction for all $x^* \in \mathbb{R}$. *Q.E.D.*

In this thesis, we will meet two types of best response functions: contraction and non-contraction. Given a best response function $x = g(x^*)$, where $x^* \in \mathbb{R}$, the condition of contraction is that $|g'(x^*)| < 1$ for all $x^* \in \mathbb{R}$. In some situations, $|g'(x^*)| = 1$ at some isolated points on the real line indicating the value of x^* and at the remaining real numbers, $|g'(x^*)| < 1$. For these situations, $x = g(x^*)$ is still a contraction. The proof of Proposition 1 adapts to such situations. The situation of the non-contraction best response functions is that at some interval(s) of $x^* \in \mathbb{R}$, $|g'(x^*)| > 1$.

Reference: de la Fuente, A. (2000), *Mathematical Methods and Models for Economists*, Cambridge University Press.

Appendix A

Appendix of Chapter 3 Preliminaries and Glossaries of Notations

The standard Gaussian density function is denoted by $\phi(.)$, and the standard Gaussian cumulative density function is denoted by $\Phi(.)$. Given a Gaussian distribution $x \sim N(\mu, \varsigma^2)$, the density function is written as

$$f(x) = \frac{1}{\sqrt{2\pi}\varsigma}\exp(-\frac{(x-\mu)^2}{2\varsigma^2}) = \frac{1}{\varsigma}\phi(\frac{x-\mu}{\varsigma})$$

The joint Gaussian distribution is denoted by $(\varepsilon, \varepsilon^*) \sim N(0, 0, \varsigma^2, \varsigma^{*2}, \rho)$. The density function of the bivariate Gaussian distribution is

$$f(\varepsilon, \varepsilon^*) = \frac{1}{2\pi\varsigma\varsigma^*\sqrt{1-\rho^2}}\exp(-\frac{1}{2(1-\rho^2)}(\frac{\varepsilon^2}{\varsigma^2} + \frac{\varepsilon^{*2}}{\varsigma^{*2}} - \frac{2\rho\varepsilon\varepsilon^*}{\varsigma\varsigma^*}))$$

The conditional density function is

$$f(\varepsilon^*|\varepsilon) = \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\frac{\varepsilon^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})$$

and the conditional cumulative density function is

$$
\begin{aligned}
F(\bar{\varepsilon}^*|\varepsilon) = \int_{-\infty}^{\bar{\varepsilon}^*} f(\varepsilon^*|\varepsilon)d\varepsilon^* &= \int_{-\infty}^{\bar{\varepsilon}^*} \frac{1}{\varsigma^*\sqrt{2\pi(1-\rho^2)}} \exp(-\frac{1}{2}(\frac{\frac{\varepsilon^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)d\varepsilon^* \\
&= \int_{-\infty}^{\frac{\frac{\bar{\varepsilon}^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}u^2)du \\
&= \Phi(\frac{\frac{\bar{\varepsilon}^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})
\end{aligned}
$$

We denote a player's belief function by $\sigma(x^*,\varepsilon) = F(x^*|\varepsilon)$, where ε is a player's own private information, and x^* is the opponent's expected cutoff strategy. We obtain the following results of $\sigma(x^*,\varepsilon)$:

$$
\begin{aligned}
\sigma(x^*,\varepsilon) &= F(x^*|\varepsilon) = \Phi(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}) \\
\sigma_{x^*}(x^*,\varepsilon) &= \frac{1}{\varsigma^*\sqrt{1-\rho^2}}\phi(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}}) \\
\sigma_{\varepsilon}(x^*,\varepsilon) &= -\frac{\rho}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})
\end{aligned}
$$

By assuming $\varsigma = \varsigma^*$, these expressions can be simplified into the following equations, respectively:

$$
\begin{aligned}
\sigma(x^*,\varepsilon) &= \Phi(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1-\rho^2}}) \\
\sigma_{x^*}(x^*,\varepsilon) &= \frac{1}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1-\rho^2}}) \\
\sigma_{\varepsilon}(x^*,\varepsilon) &= -\frac{\rho}{\varsigma\sqrt{1-\rho^2}}\phi(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1-\rho^2}})
\end{aligned}
$$

The expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ is expressed as

$$
\begin{aligned}
\mathbb{E}\Pi(x^*,\varepsilon) &= \sigma(x^*,\varepsilon)(M+\varepsilon) + (1-\sigma(x^*,\varepsilon))(D+\varepsilon) \\
&= \sigma(x^*,\varepsilon)(M-D) + D + \varepsilon
\end{aligned}
$$

$$= \Phi(\frac{\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})(M-D) + D + \varepsilon$$

$$= \Phi(\frac{x^* - \rho\varepsilon}{\varsigma\sqrt{1-\rho^2}})(M-D) + D + \varepsilon$$

The best response function is denoted by $g(x^*)$. In the proof, $g(x^*)$ is often regarded as an independent variable and the derivatives of relevant functions with respect to $g(x^*)$ are taken or the optimum value of relevant functions with respect to $g(x^*)$ is found. For simplicity, we denote $g^{-1'}(x^*) \equiv \frac{dx^*}{dg(x^*)} = \frac{1}{\frac{dg(x^*)}{dx^*}}$, and $\min_{g(x^*)}(\max_{g(x^*)})\rho'(x^*) = \min(\max)\rho'(x^*)$, which is the derivative of a function with x^* as a dependent variable and $g(x^*)$ as an independent variable, and $\min_{g(x^*)}(\max_{g(x^*)})\rho''(x^*) = \min(\max)\rho''(x^*)$.

Appendix B

Appendix of Chapter 3

Proof of Proposition 1

Lemma B1: There exists a $\tilde{\rho} \in (-1,1)$, if $D > M$, $\forall \rho \in [\tilde{\rho}, 1)$ and $\forall x^* \in \mathbb{R}$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$, where the equality is obtained at $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x^*}{\rho}$ with $\rho = \tilde{\rho}$.

Proof: For all $x^* \in \mathbb{R}$, $\mathbb{E}\Pi(x^*,\varepsilon) = \sigma(x^*,\varepsilon)(M-D)+D+\varepsilon = \Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+\varepsilon$. Therefore, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = \sigma_\varepsilon(x^*,\varepsilon)(M-D)+1 = \frac{\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)+1$. Hence, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ is equivalent to $\frac{\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}\exp(-\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2)+1 \geq 0$. Therefore, the inequality $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) \geq \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ is the necessary and sufficient condition of $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$. Apparently, that $\rho(D-M) \geq 0$ is sufficient to make the necessary and sufficient condition hold. Therefore, that $D > M$ and $\rho \geq 0$ is sufficient to guarantee $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) > \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$, and thus $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} > 0$.

Suppose $\rho(D-M) < 0$. Then, the necessary and sufficient condition $\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma}}{\sqrt{1-\rho^2}})^2) \geq \frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ can be equivalently transformed into $(\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. Therefore, under the condition $\rho(D-M) < 0$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ always holds if and only if for all $x^* \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, $(\frac{x^*}{\varsigma^*}-\frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ always holds. Hence, as long as all parameters satisfy $2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}} \leq 0$, the necessary and sufficient condition always holds, and thus $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ given $\rho(D-M) < 0$. Since $\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}} = 0$ as long as $\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}} = 1$, given D, M, ς and ς^*, and denoting the solution by $\tilde{\rho}$, we have $\tilde{\rho}^2 = \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}$. Furthermore, as long as $\rho^2 < \tilde{\rho}^2$,

$2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}} < 0$. Therefore, if $D > M$ and $\tilde{\rho} \leq \rho < 0$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ always holds, and $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x}{\tilde{\rho}}$. Therefore, combined with the results for $\rho(D-M) \geq 0$, it can be concluded that if $D > M$, $\forall \rho \in [\tilde{\rho}, 1)$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ always holds, and $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} = 0$ if and only if $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x}{\tilde{\rho}}$, where $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$.

Finally, the game is symmetric and hence $\varsigma = \varsigma^*$. Therefore, $\tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}} = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}} = \tilde{\rho^*}$ for $D > M$. Hence, both players have an identical range to ensure that their respective expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ always increases with respect to $\varepsilon \in \mathbb{R}$. *Q.E.D.*

Proof of Proposition 1: The proof of Proposition 1 is based on the proof of Lemma B1. We denote the set of ρ that makes $\mathbb{E}\Pi(x^*,\varepsilon)$ always increase with respect to ε given x^* by $\Gamma \equiv \{\rho |\ \rho \geq \tilde{\rho}$ if $D > M\ \}$. From Lemma B1, it has been known that given D and M, the necessary and sufficient condition for $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$ is $\rho \in \Gamma$. Therefore, it is certain that as long as ρ does not belong to Γ, $\mathbb{E}\Pi(x^*,\varepsilon)$ is not monotonic with respect to ε given any $x^* \in \mathbb{R}$. Equivalently, it means that for some ε, $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 < 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ for $\rho \notin \Gamma$. Without loss of generality, Figure B1 geometrically gives a general description of the relation between $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ and $z(\varepsilon) = 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$ given x, M, D, ρ, ς and ς^* for all $\rho \notin \Gamma$ (see Figure B1).

According to the quadratic structure of $y(\varepsilon)$, as long as $\rho \notin \Gamma$, there should be two solutions to solve the equation $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 = 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$. They are $\varepsilon_1 = \frac{\varsigma}{\rho\varsigma^*}x - \frac{x^*}{\rho}\sqrt{2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}}$ and $\varepsilon_2 = \frac{\varsigma}{\rho\varsigma^*}x + \frac{x^*}{\rho}\sqrt{2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}}$. Therefore, for $\varepsilon \leq \varepsilon_1$ or $\varepsilon \geq \varepsilon_2$, $(\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2 \geq 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$, then $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} \geq 0$, where the equality is taken when $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$. For $\varepsilon_1 < \varepsilon < \varepsilon_2$, $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon} < 0$. Based on these results, without loss of generality, Figure B2 geometrically gives a general description of function $\mathbb{E}\Pi(x^*,\varepsilon)$ with respect to ε given any value of x, for all $\rho \notin \Gamma$ (see Figure B2).

Because for all $x^* \in \mathbb{R}$, given all primitives, the expected payoff function $\mathbb{E}\Pi(x^*,\varepsilon)$ is always located between the line $D+\varepsilon$ and $M+\varepsilon$, and if $D > M$, increasing x^* will bring $\mathbb{E}\Pi(x^*,\varepsilon)$ downward, it is possible that for some value of x^*, there are two or three solutions of ε satisfying $\mathbb{E}\Pi(x^*,\varepsilon) = 0$. In Appendix D, we will prove that it is

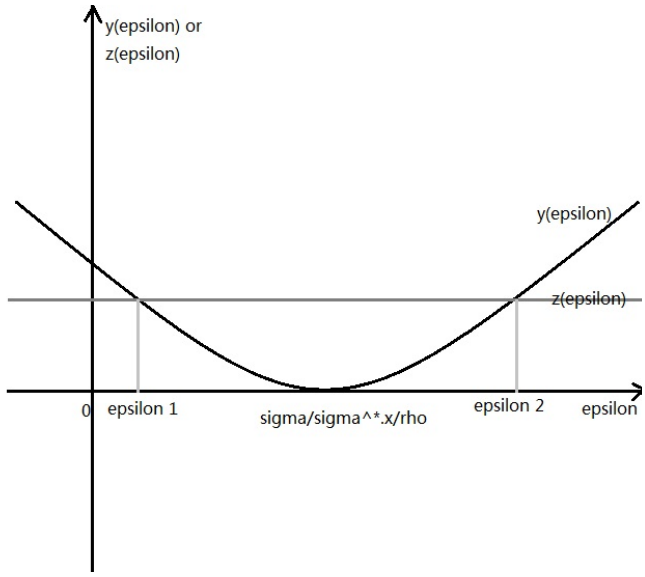


Figure B1: *A geometric description of the relation between function* $y(\varepsilon)$ *and* $z(\varepsilon)$ *as long as* $\rho \notin \Gamma$*, where* $y(\varepsilon) = (\frac{x^*}{\varsigma^*} - \frac{\rho\varepsilon}{\varsigma})^2$ *and* $z(\varepsilon) = 2(1-\rho^2)\ln\frac{-\rho(D-M)}{\varsigma\sqrt{2\pi(1-\rho^2)}}$*. There must be two intersection points which make* $f(\varepsilon) = g(\varepsilon)$*, and in this figure, they are denoted by* ε_1 *and* ε_2*. The function* $y(\varepsilon)$ *reaches its global minimum 0 at* $\varepsilon = \frac{\varsigma}{\varsigma^*}\frac{x^*}{\rho}$.

certain that for all $\rho \notin \Gamma$ by using cutoff strategy, the game always contains a unique symmetric solution $g(e) = e$, such that given e, $\mathbb{E}\Pi(e,\varepsilon) = 0$ has three solutions, and the solution $\varepsilon = e$ is located in the middle where $\mathbb{E}\Pi(e,\varepsilon)$ decreases with respect to ε (see Figure B2). Apparently, the solution (e,e) self-contradicts the definition of the cutoff strategy under which it is derived. Hence, it is impossible to solve the game using the cutoff strategy concept for all $\rho \notin \Gamma$. Therefore, the set Γ not only indicate that $\mathbb{E}\Pi(x^*,\varepsilon)$ increases with respect to ε for all $x^* \in \mathbb{R}$ but also characterizes the set of cutoff strategy Bayesian Nash equilibria of the symmetric strategic complements games. Therefore, Proposition 1 is obtained. *Q.E.D.*

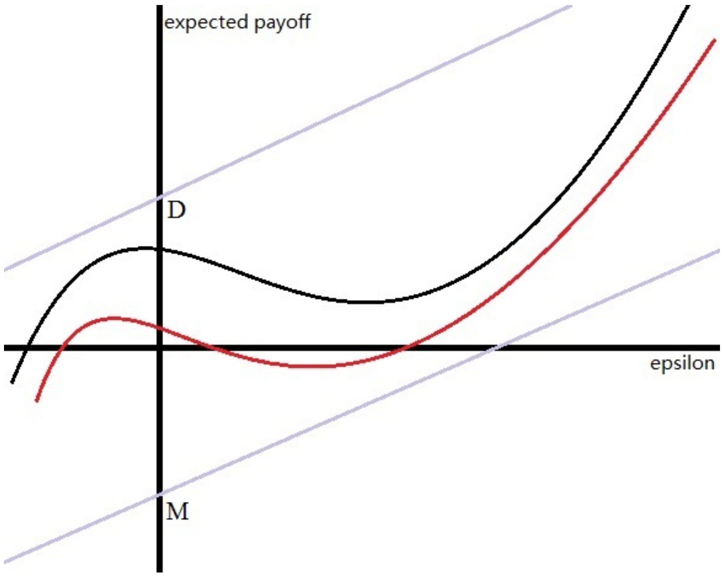


Figure B2: *A general description of expected payoff function* $\mathbb{E}\Pi(x^*, \varepsilon)$ *with respect to* ε *given any value of* x^**, for all* $\rho \notin \Gamma$*. The position of* $\mathbb{E}\Pi(x^*, \varepsilon)$ *depends on* x^**, and* $\mathbb{E}\Pi(x^*, \varepsilon)$ *is always located within* $[M+\varepsilon, D+\varepsilon]$ *for all* $x \in \mathbb{R}$*. If* $D > M$*, increasing* x^* *will bring* $\mathbb{E}\Pi(x^*, \varepsilon)$ *downward. In Appendix D, it is proven that as long as the cutoff strategy concept is used to solve the game, for all* $\rho \notin \Gamma$*, there always exists a solution* (s,s) *satisfying* $g(s) = s$*, such that given* s*,* $\mathbb{E}\Pi(s, \varepsilon)$ *behaves non-monotonically and has three intersections with the x-axis, which is demonstrated by the red curve.*

Appendix C

Appendix of Chapter 3

Derivation of the (Inverse) Best Response Function

The best response function, $g(x^*)$, is defined to satisfy $\mathbb{E}\Pi(x^*, g(x^*)) = 0$. Therefore, it is obtained that $\sigma(x^*, g(x^*))(M-D)+D+g(x^*) = 0$, and further $\Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+g(x^*) = 0$. This equation can be equivalently transformed into $\frac{D+g(x^*)}{D-M} = \Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})$. Since the cumulative density function of normal distribution is invertible, we obtain $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}}$. Finally, we obtain the inverse best response function $x^* = \rho\frac{\varsigma^*}{\varsigma}g(x^*) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$.

Still, for the definition equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, or $\Phi(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})(M-D)+D+g(x^*) = 0$, we differentiate this equation with respect to x^* on both sides, and obtain $\mathbb{E}\Pi'_{x^*}(x^*, g(x^*)) + \mathbb{E}\Pi'_{\varepsilon}(x^*, g(x^*))g'(x^*) = 0$. Therefore, $g'(x^*) = -\frac{\mathbb{E}\Pi'_{x^*}(x^*,g(x^*))}{\mathbb{E}\Pi'_{\varepsilon}(x^*,g(x^*))} = -\frac{\frac{\partial\mathbb{E}\Pi(x,\varepsilon)}{\partial x^*}}{\frac{\partial\mathbb{E}\Pi(x,\varepsilon)}{\partial\varepsilon}}|_{\varepsilon=g(x^*)} = -\frac{\sigma_{x^*}(x^*,g(x^*))(M-D)}{\sigma_{\varepsilon}(x^*,g(x^*))(M-D)+1} = \frac{1}{\frac{\varsigma^*}{\varsigma}\rho - \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}\exp(\frac{1}{2}(\frac{\frac{x^*}{\varsigma^*}-\frac{\rho g(x^*)}{\varsigma}}{\sqrt{1-\rho^2}})^2)}{M-D}}$. $\sigma_{x^*}(x^*, g(x^*)) > 0$, and it is known that as long as $\rho \in \Gamma$, $\frac{\partial\mathbb{E}\Pi(x^*,g(x^*))}{\partial\varepsilon} \geq 0\ \forall x^* \in \mathbb{R}$; hence, if $D > M$, $g'(x^*) > 0$. Therefore, as long as the concept of cutoff strategy Bayesian Nash equilibria is applied to solve the game, i.e. $\rho \in \Gamma$, $g(x^*)$ globally increases for a strategic-complements game.

Appendix D

Appendix of Chapter 3

Proof of Proposition 2

Lemma D1: Assume $\varsigma = \varsigma^*$ and $D > M$. There exist two functions $\rho'(x^*)$ and $\rho''(x^*)$. Given any $x^* \in \mathbb{R}$ and for all $\rho \in (-1,1)$, if $\rho \in (\rho'(x^*), \rho''(x^*))$, $g'(x^*) > 1$; if $\rho \in (\rho''(x^*), 1)$, $0 < g'(x^*) < 1$; at $\rho = \rho''(x^*)$, $g'(x^*) = 1$; at $\rho = \rho'(x^*)$, $g'(x^*) = \infty$.

Proof: We have $g^{-1'}(x^*) \equiv \frac{dx^*}{dg(x^*)} = \frac{1}{\frac{dg(x^*)}{dx^*}} = \rho + \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$; therefore, $\frac{dg^{-1'}(x^*)}{d\rho} = 1 - \frac{\varsigma^*\rho}{\sqrt{1-\rho^2}}\frac{\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Hence, given an $x^* \in \mathbb{R}$, if $\rho < 0$ and $D > M$, then the function $g^{-1'}(x^*)$ must increase with respect to ρ. In addition, if $\rho = -1$, $g^{-1'}(x^*) = -1$, and if $\rho = 0$, $g^{-1'}(x^*) > 0$. Because $g^{-1'}(x^*)$ is a continuous function with respect to ρ, for all $\rho \in (-1, 0]$, $g^{-1'}(x^*)$ increases from -1 to a positive value as ρ increases from -1 to 0. Therefore, there must exist a $\rho \in (-1, 0]$, whose value depends on x^*, and it makes $g^{-1'}(x^*) = 0$. We denote this ρ by $\rho'(x^*)$. Because $g^{-1'}(x^*) < 0$ is equivalent to $g'(x^*) < 0$, and $g^{-1'}(x^*) > 0$ is equivalent to $g'(x^*) > 0$, we can conclude that given an $x^* \in \mathbb{R}$, for all $\rho \in (-1, \rho'(x^*))$, $g'(x^*) < 0$, and for all $\rho \in (\rho'(x^*), 0)$, $g'(x^*) > 0$.

We define $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. For $D > M$, $A > 0$. Hence, the equation $g^{-1'}(x^*) = 0$ can be equivalently expressed by $\rho + A\sqrt{1-\rho^2} = 0$. The solution $\rho'(x^*)$ that solves $\rho + A\sqrt{1-\rho^2} = 0$ equals $-\frac{A}{\sqrt{1+A^2}} < 0$. Because $\varsigma = \varsigma^*$, both players' $\rho'(x^*)$ function should be identical.

Because $g'(x^*) = g^{-1'}(x^*) = 1$, $\rho + \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = 1$, and hence $1-\rho = \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) > 0$ for $D > M$. Given D, M, ς and ς^*, the ρ that satisfies the equation $1-\rho = \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) > 0$ must depend on x^*. Thus, we denote the ρ that makes $g'(x^*) = g^{-1'}(x^*) = 1$ by $\rho''(x^*)$. For this incomplete information game, $\rho''(x^*)$ should not be equal to ± 1. $g^{-1'}(x^*) = 1$ can be equivalently represented by $1-\rho = A\sqrt{1-\rho^2}$. Solving this equation, we get two solutions: $\rho''(x^*) = 1$ and $\rho''(x^*) = \frac{1-A^2}{1+A^2}$. The first solution is excluded according to the previous argument. Therefore, $\rho''(x^*) = \frac{1-A^2}{1+A^2}$.

Because given an $x^* \in \mathbb{R}$, $\rho''(x^*)$ is unique, for all $\rho \in (\rho'(x^*), \rho''(x^*))$ or $\rho \in (\rho''(x^*), 1)$, $g'(x^*)$ is either greater or smaller than 1. To judge in which interval $g'(x^*)$ is smaller or greater than 1, let us recall the derivative of $g^{-1'}(x^*)$ with respect to $\rho \in (-1, 1)$:

$$\frac{dg^{-1'}(x^*)}{d\rho} = 1 - \frac{\varsigma^*\rho}{\sqrt{1-\rho^2}}\frac{\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = 1 - \frac{\rho}{\sqrt{1-\rho^2}}A$$

Because at $\rho = \rho''(x^*)$, $\frac{dg^{-1'}(x^*)}{d\rho} = \frac{1}{1+\rho''(x^*)} > 0$, for all $\rho \in (\rho''(x^*) - \varepsilon, \rho''(x^*) + \varepsilon)$, the function $g^{-1'}(x^*)$ increases with respect to ρ. Because at $\rho = \rho''(x^*)$, $g^{-1'}(x^*) = 1$, $\forall \rho \in (\rho''(x^*) - \varepsilon, \rho''(x^*))$, $g^{-1'}(x^*) < 1$, and $\forall \rho \in (\rho''(x^*), \rho''(x^*) + \varepsilon)$, $g^{-1'}(x^*) > 1$. Moreover, because $\rho''(x^*)$ is unique, this result can be extended to the whole interval $\rho \in (-1, 1)$. Thus, $\forall \rho \in (-1, \rho''(x^*))$, $g^{-1'}(x^*) < 1$, and $\forall \rho \in (\rho''(x^*), 1)$, $g^{-1'}(x^*) > 1$.

The relationship between $\rho'(x^*)$ and $\rho''(x^*)$: Let us recall that $\rho'(x^*) < 0$. If $\rho''(x^*) > 0$, then certainly $\rho'(x^*) < \rho''(x^*)$. Now, consider the case that $\rho''(x^*)$ is negative. Recall that $g^{-1'}(x^*)$ increases with respect to ρ if ρ is negative. At $\rho = \rho'(x^*)$, $g^{-1'}(x^*) = 0$ and hence $g'(x^*) = \infty$, and at $\rho = \rho''(x^*)$, $g^{-1'}(x^*) = 1$ and hence $g'(x^*) = 1$. Therefore, $\rho'(x^*) < \rho''(x^*)$ if $\rho''(x^*) < 0$. In conclusion, if $D > M$, $\rho''(x^*)$ is always strictly greater than $\rho'(x^*)$.

Because $\varsigma = \varsigma^*$, both players' $\rho'(x^*)$ and $\rho''(x^*)$ functions are identical. Therefore, in conclusion, for $D > M$, the function $g(x^*)$ whose inverse form is $x^* = \rho\frac{\varsigma^*}{\varsigma}g(x^*) + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+g(x^*)}{D-M})$ has the following property: given $x^* \in \mathbb{R}$ and $\rho \in (-1, 1)$, if $\rho'(x^*) < \rho < \rho''(x^*)$, $g'(x^*) > 1$; if $\rho \in (\rho''(x^*), 1)$, $0 < g'(x^*) < 1$; if $\rho = \rho''(x^*)$, $g'(x^*) = 1$; if $\rho = \rho'(x^*)$, $g'(x^*) = \infty$; and if $\rho < \rho'(x^*)$, $g'(x^*) < 0$. *Q.E.D.*

Lemma D2: For $D > M$, for all $\rho \in [\rho'(x^*), 0]$, $g'(x^*)$ decreases from $+\infty$ to a positive value, and for all $\rho \in (-1, \rho'(x^*)]$, $g'(x^*)$ decreases from -1 to $-\infty$.

Proof: From the proof of Lemma D1, it has been known that given an $x^* \in \mathbb{R}$, $g^{-1'}(x^*)$ increases with respect to ρ for $\rho \in (-1, 0]$ and it is continuous. Therefore, $g'(x^*)$ must have a decreasing property in the interval $\rho \in (-1, 0]$. We have at $\rho = 0$, $g^{-1'}(x^*) > 0$, and hence $g'(x^*) > 0$; at $\rho = \rho'(x^*)$, $g^{-1'}(x^*) = 0$ and hence $g'(x^*) = \infty$; and at $\rho = -1$, $g^{-1'}(x^*) = 1$, and hence $g'(x^*) = 1$. Therefore, for function $g'(x^*)$, there is a discontinuity point at $\rho = \rho'(x^*)$. For $\rho \in (-1, \rho'(x^*)]$, $g'(x^*)$ should decrease from -1 to $-\infty$, and for $\rho \in (\rho'(x^*), 0]$, $g'(x^*)$ should decrease from $+\infty$ to some positive value. *Q.E.D.*

Lemma D3: For $D > M$, given an $x^* \in \mathbb{R}$, $g'(x^*)$ is convex for $\rho \in [0, 1)$. It reaches its minimum value at $\rho = \frac{1}{\sqrt{1+A^2}}$.

Proof: Given an $x^* \in \mathbb{R}$ and $D > M$, for all $\rho \in (0, 1)$, because $\frac{d^2 g'(x^*)}{d\rho^2} = 2(g^{-1'}(x^*))^{-3}(\frac{dg^{-1'}(x^*)}{d\rho})^2$ $-(g^{-1'}(x^*))^{-2}\frac{d^2 g^{-1'}(x^*)}{d\rho^2}$, $\frac{d^2 g^{-1'}(x^*)}{d\rho^2} = -A\frac{\sqrt{1-\rho^2}+\frac{\rho^2}{\sqrt{1-\rho^2}}}{1-\rho^2} < 0$, and for $\rho > 0$, $g^{-1'}(x^*) > 0$, we have $\frac{d^2 g^{-1'}(x^*)}{d\rho^2} > 0$ for $\rho \in (0, 1)$. Hence, $g'(x^*)$ is convex for all $\rho \in (0, 1)$. Furthermore, by calculating the first-order derivative, it is found that $g^{-1'}(x^*) = \rho + \frac{\varsigma^*\sqrt{2\pi(1-\rho^2)}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$ reaches its minimum value at $\rho = \frac{1}{\sqrt{1+A^2}} > 0$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. We denote this ρ by $\rho'''(x)$. Therefore, given an $x^* \in \mathbb{R}$, the function $g'(x^*)$ reaches its minimum value at $\rho'''(x) = \frac{1}{\sqrt{1+A^2}}$ $\forall \rho \in (0, 1)$, and the minimum value of $g'(x^*)$ is just $\frac{1}{\sqrt{1+A^2}}$. *Q.E.D.*

For $D > M$, according to Lemmas D1, D2 and D3, the shape of $g'(x^*)$ with respect to ρ given an $x^* \in \mathbb{R}$ can be generally represented by Figure D1.

Lemma D4: Given an $x^* \in \mathbb{R}$ and assuming $D > M$, for $g(x^*) \in (-\frac{M+D}{2}, -M]$, $\frac{d\rho'(x^*)}{dg(x^*)} < 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} < 0$; for $g(x^*) \in [-D, -\frac{M+D}{2})$, $\frac{d\rho'(x^*)}{dg(x^*)} > 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} > 0$.

Proof: Let us recall that $\rho'(x^*) = -\frac{A}{\sqrt{1+A^2}}$ and $\rho''(x^*) = \frac{1-A^2}{1+A^2}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}$ $\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Because $D > M, A > 0$. By the chain rule, $\frac{d\rho'(x^*)}{dg(x^*)} = \frac{d\rho'(x^*)}{dA}\frac{dA}{dg(x^*)}$,

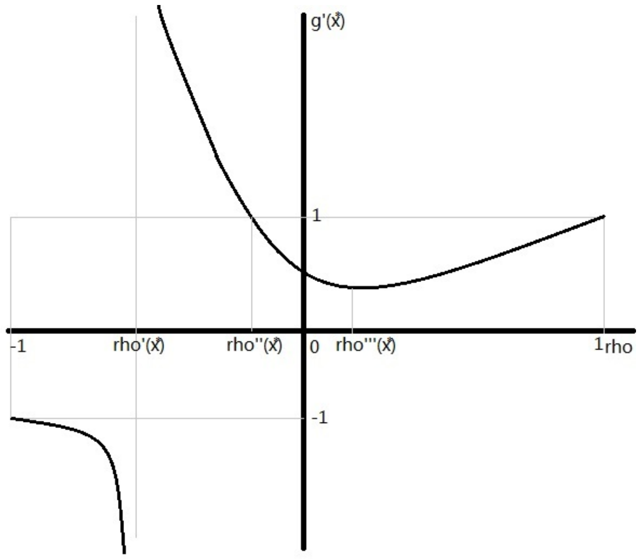


Figure D1: *A general geometric description of function $g'(x^*)$ with respect to ρ for $D > M$ given an $x^* \in \mathbb{R}$.*

and $\frac{d\rho''(x^*)}{dg(x^*)} = \frac{d\rho''(x^*)}{dA}\frac{dA}{dg(x^*)}$. Because $\rho'(x^*) = -\frac{A}{\sqrt{1+A^2}} = -\frac{1}{\sqrt{\frac{1}{A^2}+1}}$ and $\rho''(x^*) = \frac{2-1-A^2}{1+A^2} = \frac{2}{1+A^2} - 1$, as A increases, $\rho'(x^*)$ decreases and $\rho''(x^*)$ decreases. Hence, $\frac{d\rho'(x^*)}{dA} < 0$, and $\frac{d\rho''(x^*)}{dA} < 0$. If $g(x^*) < -\frac{D+M}{2}$, $\frac{dA}{dg(x^*)} < 0$ and if $g(x^*) > -\frac{D+M}{2}$, $\frac{dA}{dg(x^*)} > 0$; therefore, in conclusion, if $g(x^*) \in [-D, -\frac{D+M}{2})$, $\frac{d\rho'(x^*)}{dg(x^*)} > 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} > 0$, and if $g(x^*) \in [-\frac{D+M}{2}, -M]$, $\frac{d\rho'(x^*)}{dg(x^*)} < 0$ and $\frac{d\rho''(x^*)}{dg(x^*)} < 0$. Hence, at $g(x^*) = -\frac{D+M}{2}$, both $\rho'(x^*)$ and $\rho''(x^*)$ reach their global maximum value with respect to $g(x^*)$. The maximum values of $\rho'(x^*)$ and $\rho''(x^*)$ with respect to $g(x^*)$ are $-\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}}$ and $\frac{(D-M)^2-2\pi\varsigma^{*2}}{(D-M)^2+2\pi\varsigma^{*2}}$, respectively. *Q.E.D.*

Based on Lemmas D1 and D4, Figure D2 generally depicts functions $\rho'(x^*)$ and $\rho''(x^*)$ with respect to $g(x^*)$. According to Lemmas D1, D2 and D3, given an $x^* \in \mathbb{R}$ and hence $g(x^*)$, as ρ increases from -1 to 1, for $\rho \in (-1, \rho'(x^*))$, $g'(x^*) < -1$; for $\rho \in (\rho'(x^*), \rho''(x^*))$, $g'(x^*) > 1$; for $\rho \in (\rho''(x^*), 1)$, $0 < g'(x^*) < 1$. At $\rho = \rho''(x^*)$, $g'(x^*) = 1$ and at $\rho = \rho'(x^*)$, $g'(x^*) = \infty$. This change of $g'(x^*)$ can be illustrated by Figure D2. We choose an arbitrary value of $g(x^*)$ between $-D$ and $-M$, and at this chosen $g(x^*)$, we draw a vertical line from -1 to 1 (the red line in Figure D2). The curves $\rho'(x^*)$ and $\rho''(x^*)$ dissect this line into three parts, $g'(x^*) < 0$, $g'(x^*) > 1$ and $g'(x^*) < 1$ from bottom to top. Because $g(x^*)$ is arbitrarily chosen, this result applies for all $g(x^*) \in [-D, -M]$. Therefore, it can be concluded that given D, M, ς

and ς^*, for all $g(x^*) \in [-D,-M]$, if $\rho \in (\rho''(x^*),1)$, $0 < g'(x^*) < 1$ and correspondingly it is the area above the curve $\rho''(x^*)$ in Figure D2; for all $g(x^*) \in [-D,-M]$, if $\rho \in (\rho'(x^*),\rho''(x^*))$, $g'(x^*) > 1$ and correspondingly it is the area between the curves $\rho'(x^*)$ and $\rho''(x^*)$; finally, for all $g(x^*) \in [-D,-M]$, if $\rho < \rho'(x^*)$, $g'(x^*) < 0$ and correspondingly it is the area below the curve $\rho'(x^*)$.

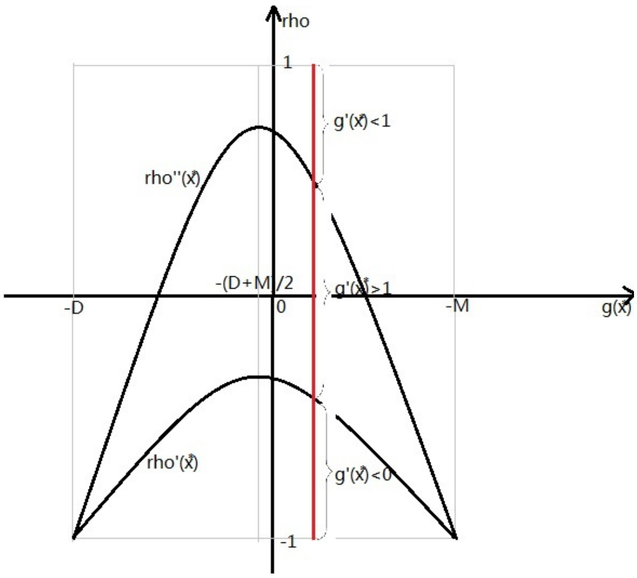


Figure D2: *A general geometric description of functions* $\rho'(x^*)$ *and* $\rho''(x^*)$ *with respect to* $g(x^*)$ *for* $D > M$.

Lemma D5: For $D > M$, given an $\rho \in (-1, \max \rho''(x^*)]$, there are one or two values of $g(x^*)$ which makes $g'(x^*) = 1$. They are $g(x^*) = (D-M)\Phi(\pm\sqrt{\ln \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$.

Proof: Given D, M, ς and ς^*, if there are $g(x^*)$s whose derivative $g'(x^*) = 1$, then the corresponding $\rho \in (-1, \max \rho''(x^*)]$ and $g(x^*)$ should satisfy $\rho = \frac{1-A^2}{1+A^2}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M} \exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Therefore, $A^2 = \frac{1-\rho}{1+\rho}$, i.e. $\frac{2\pi\varsigma^{*2}}{(D-M)^2} \exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{1-\rho}{1+\rho}$. Therefore, $\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}} \geq 1$, where the latter equality is held if and only if $\rho = \max \rho''(x^*) = \frac{(D-M)^2-2\pi\varsigma^{*2}}{(D-M)^2+2\pi\varsigma^{*2}}$. Therefore, $[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2 = \ln \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}}$, and $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \pm\sqrt{\ln \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}}}$. Hence, we get two solutions: $g(x^*)_1 = (D-M)\Phi(-\sqrt{\ln \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$ and $g(x^*)_2 = (D-M)\Phi(\sqrt{\ln \frac{1-\rho}{1+\rho} \frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$. Note that $g(x^*)_1 \leq g(x^*)_2$, where the equality is obtained as long as $\rho = \max \rho''(x^*)$. *Q.E.D.*

Lemma D6: Given a $\rho \in (-1, \max\rho'(x^*))$, if $D > M$, there are two values of $g(x^*)$ which makes $g'(x^*) = \infty$. They are $g(x^*) = -(M-D)\Phi(\pm\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(M-D)^2}{2\pi\varsigma^{*2}}}) - D$. If $\rho = \max\rho'(x^*)$ for $D > M$, at $g(x^*) = -\frac{D+M}{2}$, $g'(x^*) = \infty$.

Proof: Given D, M, ς and ς^*, if there are $g(x^*)$s whose derivative $g'(x^*) = \infty$, which means $\frac{1}{g'(x^*)} = 0$, then the corresponding $\rho \in (-1, \max\rho'(x^*))$ for $D > M$, and the $g(x^*)$ should simultaneously satisfy $\rho = \frac{-A}{\sqrt{1+A^2}}$, where $A \equiv \frac{\varsigma^*\sqrt{2\pi}}{D-M}\exp(\frac{1}{2}[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2)$. Therefore, $A^2 = \frac{\rho^2}{1-\rho^2}$, i.e. $\frac{2\pi\varsigma^{*2}}{(D-M)^2}\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{\rho^2}{1-\rho^2}$. Therefore, $\exp([\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2) = \frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}} \geq 1$, where the latter equality is held if and only if $\rho = \max\rho'(x^*) = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}}$ for $D > M$. Therefore, $[\Phi^{-1}(\frac{D+g(x^*)}{D-M})]^2 = \ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}$ and $\Phi^{-1}(\frac{D+g(x^*)}{D-M}) = \pm\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}$. Hence, we get two solutions: $g(x^*)_1 = (D-M)\Phi(\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$ and $g(x^*)_2 = (D-M)\Phi(-\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$. Note that for $D > M$, $g(x^*)_1 \geq g(x^*)_2$, where the equality is obtained as long as $\rho = \max\rho'(x^*)$. If the equality is held, $g(x^*) = g(x^*)_1 = g(x^*)_2 = -\frac{M+D}{2}$. *Q.E.D.*

Proof of Proposition 2: According to Figure D2 and results from Lemmas D5 and D6, we get the following result for the shape of $g(x^*)$ given D, M, ς and ς^* for all $\rho \in (-1, 1)$:

1) for $\rho \in (-1, \max\rho'(x^*))$,

I. if $g(x^*) \in [-D, (D-M)\Phi(-\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) > 0$;

II. if $g(x^*) \in ((D-M)\Phi(-\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, (D-M)\Phi(\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) < 0$;

III. if $g(x^*) \in ((D-M)\Phi(\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, -M]$, $g'(x^*) > 0$;

IV. if $g(x^*) = (D-M)\Phi(\pm\sqrt{\ln\frac{\rho^2}{1-\rho^2}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = \infty$;

2) for $\rho \in [\max\rho'(x^*), \max\rho''(x^*)]$,

I. if $g(x^*) \in [-D, (D-M)\Phi(-\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $0 < g'(x^*) < 1$;

II. if $g(x^*) \in ((D-M)\Phi(-\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, (D-M)\Phi(\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D)$, $g'(x^*) > 1$;

III. if $g(x^*) \in ((D-M)\Phi(\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D, -M]$, $0 < g'(x^*) < 1$;

IV. if $g(x^*) = (D-M)\Phi(\pm\sqrt{\ln\frac{1-\rho}{1+\rho}\frac{(D-M)^2}{2\pi\varsigma^{*2}}}) - D$, $g'(x^*) = 1$.

3) for $\rho \in (\max\rho''(x^*), 1)$, $0 < g'(x^*) < 1$ globally.

It is straightforward to find that the description of the shape of $g(x^*)$ is still held even if the payoff specification for both players is asymmetric, because in Lemmas D1–D5 and Lemma D6, we only focus on studying a single $g(x^*)$ function.

In the following, we prove that the shape of $g(x^*)$ in 1), the non-monotonic $g(x^*)$, contradicts the definition of the cutoff strategy concept; therefore, for the symmetric strategic complements game, using the cutoff strategy concept to solve the game is valid if and only if $\rho \in [\tilde{\rho}, 1)$. In addition, the definition of the cutoff strategy implicitly dictates that, given the opponent's cutoff strategy, a player's best response towards it should be unique, and it is held irrespective of the specification of payoffs.

As long as $\rho \notin \Gamma$, in the proof of Proposition 1 (Appendix B), we have proven that given an $x^* \in \mathbb{R}$, as ε increases from $-\infty$ to $+\infty$, $\mathbb{E}\Pi(x^*, \varepsilon)$ first increases, then decreases, and finally increases. We just saw that in the strategic complements game, for all $\rho < \tilde{\rho}$, as $g(x^*)$ decreases from $-M$ to $-D$, function $g(x^*)$ first increases, then decreases, and finally increases with respect to the corresponding x^*. In fact, the changes of $\mathbb{E}\Pi(x^*, \varepsilon)$ monotonicity (with respect to ε) and $g(x^*)$ (with respect to x^*) are synchronized, because $g'(x^*) = -\frac{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*}}{\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}}|_{\varepsilon = g(x^*)}$, and since $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial x^*} < 0$ for $D > M$, for any point $(x^*, g(x^*))$ from the function $g(x^*)$, if $\frac{\partial \mathbb{E}\Pi(x^*,\varepsilon)}{\partial \varepsilon}|_{\varepsilon = g(x^*)} \gtrless 0$, $g'(x^*) \gtrless 0$ for $D > M$, and vice versa.

For the symmetric strategic complements game, all solutions are symmetric, i.e. they satisfy $x^* = g(x^*)$. However, we find that if $\rho \in (-1, \tilde{\rho})$, at any (symmetric) solution, $g(x^*)$ must have a negative derivative, i.e. $g'(x^*) < 0$. It is because for solutions that satisfy $x^* = g(x^*)$, its derivative $g'(x^*)$ can be expressed as

$$g'(x^*) = \frac{1}{\rho - \frac{\varsigma^* \sqrt{2\pi(1-\rho^2)}\exp(\frac{1}{2}\frac{1-\rho}{1+\rho}\frac{g^2(x^*)}{\varsigma^2})}{M-D}}$$

Furthermore, because at this situation $\rho^2 > \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}$, we get $\rho^2 > \frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2} > \frac{2\pi\varsigma^2(1-\rho^2)}{(M-D)^2}$, and because $g'(x^*)|_{g(x^*)=0} = \frac{1}{\rho - \frac{\varsigma\sqrt{2\pi(1-\rho^2)}}{M-D}}$, for $D > M$ and $\rho < \tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(M-D)^2}}$, $g'(x^*)|_{g(x^*)=0} < 0$. For $D > M$, if we regard $g'(x^*)$ as a function with respect to variable $g^2(x^*)$, $\frac{\partial g'(x^*)}{\partial g^2(x^*)} < 0$,. Then, for any symmetric solution (e,e), we must have $g'(x^*)|_{g(x^*)=e} \le g'(x^*)|_{g(x^*)=0} < 0$. Therefore, for the symmetric strategic complements game, as long as $\rho < \tilde{\rho}$, at any (symmetric) solution (e,e), the derivative $g'(e) < 0$, and correspondingly, $\frac{\partial \mathbb{E}\Pi(e,\varepsilon)}{\partial \varepsilon}|_{\varepsilon=e} < 0$. Thus, as long as $\rho < \tilde{\rho}$, given any symmetric solution (e,e), the equation $\mathbb{E}\Pi(e,\varepsilon) = 0$ must always have a solution $\varepsilon = e$, and at this point, $\frac{\partial \mathbb{E}\Pi(e,\varepsilon)}{\partial \varepsilon} < 0$. Therefore, according to the proof of Proposition 1 in Appendix B, without loss of generality, the function $\mathbb{E}\Pi(e,\varepsilon)$ with respect to ε is the red curve shown in Figure B2, and the $\varepsilon = e$ is the middle intersection point around which expected the payoff function decreases. Therefore, if we still use a cutoff strategy to solve the game for $\rho < \tilde{\rho}$, all (symmetric) solutions contradict the cutoff strategy definition under which these solutions are derived. Therefore, we cannot use the cutoff strategy concept to solve the symmetric strategic complements game for $\rho < \tilde{\rho}$ given $D > M$ and $\varsigma = \varsigma^*$. This supplements the existing proof of Proposition 1 in Appendix B, and the proof of Proposition 1 is now complete.

It is necessary and sufficient that for $D > M$, as long as $\rho \ge \max \rho'(x^*) = -\sqrt{\frac{2\pi\varsigma^{*2}}{2\pi\varsigma^{*2}+(D-M)^2}}$, $g(x^*)$ globally increases. Because $\varsigma^* = \varsigma$, $\max \rho'(x^*) = \tilde{\rho} = -\sqrt{\frac{2\pi\varsigma^2}{2\pi\varsigma^2+(D-M)^2}}$; therefore, if $g(x^*)$ globally increases, $\rho \ge \tilde{\rho}$ must hold, i.e. the cutoff strategy concept is legitimately used to solve the game, and vice versa. Therefore, combining the previous result in the proof of Proposition 1 in Appendix B, we find that a necessary and sufficient condition for using the cutoff strategy concept to solve the symmetric strategic complements game, or equivalently $\rho \ge \tilde{\rho}$, is that the best response function $g(x^*)$ globally increases.

As long as the strategic complements game is symmetric, the conclusions 2) and 3) obtained at the beginning of the proof form the content of Proposition 2. However, if the payoff specification is asymmetric, these results are still held for describing a single player's $g(x^*)$ function as these results are derived by studying a single $g(x^*)$ function. Hence, Proposition 2 is still held for asymmetric payoff settings. *Q.E.D.*

Appendix E

Appendix of Chapter 3

Proof of Theorem 1

Proof of Theorem 1: The main content of Section 3.4 is essentially about the proof of Theorem 1. Hence, in the Appendix, we supplement Section 3.4 with proofs around $\bar{\rho}$. The proof below, together with Section 3.4, completes the proof of Theorem 1. In the following, we assume that $\bar{\rho}$ exists in the mathematical sense, i.e. $M < 0$ if $D+M > 0$ or $D > 0$ if $D+M < 0$, and therefore, $y = \Phi(\frac{1}{\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}x)$ can be tangent with $y = \frac{D+x}{D-M}$ at $\rho = \bar{\rho}$. If $\bar{\rho}$ cannot exist in the mathematical sense, then the game always contains a unique equilibrium, because it is impossible to make the tangent point arise, let alone the multiple intersection-point situation, as ρ increases from $\tilde{\rho}$ to 1 (recall that as ρ increases, $y = \Phi(\frac{1}{\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}x)$ "stretches"). Therefore, if $\bar{\rho}$ cannot exist in mathematical sense, $\forall\rho \in [\tilde{\rho}, 1)$, the game always contains a unique equilibrium.

From Section 3.4, it is known that at the boundary case $\rho = \bar{\rho}$, there must exist a tangent point (e, e) between $y = \Phi(\frac{\frac{1}{\varsigma^*} - \frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}}x)$ and $y = \frac{D+x}{D-M}$. Hence, $\bar{\rho}$ and (e, e) must simultaneously satisfy the following equation group comprised by equations (E.1) and (E.2):

$$\phi(\frac{\frac{1}{\varsigma^*} - \frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}}e)\frac{\frac{1}{\varsigma^*} - \frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}} = \frac{1}{D-M} \tag{E.1}$$

$$\Phi(\frac{\frac{1}{\varsigma^*} - \frac{\bar{\rho}}{\varsigma}}{\sqrt{1-\bar{\rho}^2}}e) = \frac{D+e}{D-M} \tag{E.2}$$

Reasonably, $\bar{\rho} \leq \hat{\rho} = \frac{(D-M)^2-2\pi\varsigma^{*2}}{(D-M)^2+2\pi\varsigma^{*2}}$. Because $\varsigma = \varsigma^*$, the previous inequality equivalently indicates that $\frac{D-M}{\sqrt{2\pi}} \frac{\varsigma-\bar{\rho}\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}} \geq 1$, where the equality is obtained if and only if $\bar{\rho} = \hat{\rho}$. Given that $\bar{\rho} \leq \hat{\rho}$, by solving equation E.1, we obtain

$$e^2 = \frac{\varsigma^2\varsigma^{*2}(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(D-M)^2}{2\pi} \frac{(\varsigma-\bar{\rho}\varsigma^*)^2}{\varsigma\varsigma^*(1-\rho^2)}$$

From Figures 2–4, it can be concluded that for the sign of the tangent point (e,e), it has the following one-to-one correspondence relationship with respect to the sign of $D+M$: $D+M \gtreqless 0 \iff e \gtreqless 0$. From this equivalence relationship, it is certain that as long as $D+M=0$, $e=0$ and $\bar{\rho} = \hat{\rho}$. Otherwise, if $D+M \neq 0$, then $\bar{\rho} \neq \hat{\rho}$, and vice versa. Therefore, if $D+M > 0$, by substituting the expression of e into E.2, we obtain the following equation, where $\bar{\rho}$ is the unknown variable:

$$\Phi(\sqrt{\ln \frac{(D-M)^2(\varsigma-\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}) = \frac{D+\sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(D-M)^2(\varsigma-\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}}{D-M}$$

Because $\varsigma = \varsigma^*$, the above equation can be equivalently written as

$$\Phi(\sqrt{\ln \frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}) = \frac{D+\sqrt{\varsigma^2 \frac{1+\bar{\rho}}{1-\bar{\rho}} \ln \frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}}{D-M} \tag{E.3}$$

If $D+M < 0$, by substituting the analytical expression of e into E.2, we get the following equation, where $\bar{\rho}$ is the unknown variable:

$$\Phi(-\sqrt{\ln \frac{(D-M)^2(\varsigma-\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}) = \frac{D-\sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}{(\varsigma-\bar{\rho}\varsigma^*)^2} \ln \frac{(D-M)^2(\varsigma-\varsigma^*\bar{\rho})^2}{2\pi\varsigma^{*2}\varsigma^2(1-\bar{\rho}^2)}}}{D-M}$$

Because $\varsigma = \varsigma^*$, the above equation can be equivalently written as

$$\Phi(-\sqrt{\ln \frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}) = \frac{D-\sqrt{\varsigma^2 \frac{1+\bar{\rho}}{1-\bar{\rho}} \ln \frac{(D-M)^2(1-\bar{\rho})}{2\pi\varsigma^2(1+\bar{\rho})}}}{D-M} \tag{E.4}$$

and if $D+M=0$, then $\bar{\rho} = \hat{\rho}$.

Next, we will prove that equations E.3 and E.4 have a unique solution of $\bar{\rho}$ by contradiction. Suppose there is a second solution $\bar{\rho}'$ that is the solution of either equation E.3 or equation E.4. Given this $\bar{\rho}'$, we can obtain another solution from the equation group of E.1 and E.2 (i.e. the tangent point between the functions $y = \Phi(\frac{1}{\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}x)$ and $y = \frac{D+x}{D-M}$). It is impossible. From Figures 2–4, it can be seen that in any situation, given D, M, ς (and ς^*), the functions $y = \Phi(\frac{1}{\varsigma}\sqrt{\frac{1-\rho}{1+\rho}}x)$ and $y = \frac{D+x}{D-M}$ can be tangent with each other at most only once, and hence, the tangent point is unique if it exists. Therefore, a contradiction arises, and thus equations E.3 and E.4 always contain a unique solution of $\bar{\rho}$ if $\bar{\rho}$ exists. Finally, according to Figures 2–4, it is known that, for the strategic complements game, if there exist multiple equilibria (three or two) for $\rho \in [\tilde{\rho}, 1)$, the boundary correlation coefficient $\bar{\rho}$ must be greater than or equal to $\tilde{\rho}$, and vice versa. Therefore, equations E.3 and E.4 are the correct implicit expressions about $\bar{\rho}$ for the symmetric strategic complements games for $D+M \neq 0$ and $\bar{\rho}$ is the unique solution of equations E.3 and E.4. *Q.E.D.*

Appendix F

Appendix of Chapter 3

Derivation of the Best Response Function at $\rho \to -1$

Let us recall the definition equation of the cutoff best response $g(x^*)$: $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})(M-D)+D+g(x^*) = 0$. If $x^* = -g(x^*)$, $\lim_{\rho\to-1}\Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \frac{1}{2}$. Therefore, $\frac{1}{2}(M-D)+D+g(x^*) = 0$, and hence $g(x^*) = -\frac{D+M}{2}$. Thus, at $x^* = \frac{D+M}{2}$, $g(x^*) = -\frac{D+M}{2}$.

If $x^* > g(x^*)$, $\lim_{\rho\to-1}\Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(+\infty) = 1$, $g(x^*) = -M$. In the equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, if and only if $x^* = \frac{D+M}{2}$, $g(x^*) = -x^*$; therefore, if $x^* > \frac{D+M}{2}$, $g(x^*) = -M$.

If $x^* < g(x^*)$, $\lim_{\rho\to-1}\Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(-\infty) = 0$, $g(x^*) = -D$. In the equation $\mathbb{E}\Pi(x^*, g(x^*)) = 0$, if and only if $x^* = \frac{D+M}{2}$, $g(x^*) = -x^*$; therefore, if $x^* < \frac{D+M}{2}$, $g(x^*) = -D$.

Appendix G

Appendix of Chapter 3 Comparative Statics of Equilibria (Proof of Proposition 3)

Proof of Proposition 3: In this symmetric strategic complements game, all equilibria are symmetric. Particularly, recall that $\varsigma = \varsigma^*$ always holds. We denote one by (e,e). Hence, $g(e) = e$. Therefore, they should satisfy

$$e = \rho\frac{\varsigma^*}{\varsigma}e + \varsigma^*\sqrt{1-\rho^2}\Phi^{-1}(\frac{D+e}{D-M})$$

and it can be equivalently represented as

$$\Phi(\frac{\varsigma - \rho\varsigma^*}{\varsigma\varsigma^*\sqrt{1-\rho^2}}e) = \frac{D+e}{D-M} \tag{G.1}$$

Remember that in the strategic-complements game, $D > M$. Therefore, $-D < e < -M$.

For $\rho > \bar{\rho}$, there exists a unique equilibrium. According to the slope relationship expressed in Figures 2–4, such equilibrium should satisfy

$$\phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}s)(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}) < \frac{1}{D-M}$$

As stated in the main context of this paper, this inequality also indicates that the equilibrium is stable.

Similarly, for $\rho < \bar{\rho}$, there are three equilibria. For the two outer ones, they satisfy

$$\phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}s)(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}) < \frac{1}{D-M}$$

They are stable. For the middle equilibrium, they satisfy

$$\phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}s)(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}) > \frac{1}{D-M}$$

It is unstable.

For $\rho = \bar{\rho}$, for the equilibrium expressed as the intersection point in Figures 2 and 3, it satisfies

$$\phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}s)(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}) < \frac{1}{D-M}$$

and therefore, it is stable. For the equilibrium expressed as the tangent point in Figures 2–4, it satisfies

$$\phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}s)(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma^*}}{\sqrt{1-\rho^2}}) = \frac{1}{D-M}$$

and its stability cannot be determined.

Then, differentiating both sides of equation (G.1) with respect to D, M, ρ, ς and ς^* (the relation $\varsigma = \varsigma^*$ is always maintained), and rearranging the terms on both sides of relevant equations, we obtain the following results on comparative statics of symmetric equilibrium:

1)

$$\frac{\partial e}{\partial M}=\frac{1}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{D+e}{(M-D)^2}$$

Because $e\in[-D,-M]$ and $-D$ and $-M$ are reached at the asymptote, without loss of generality, $\frac{\partial e}{\partial M}<0$ for stable equilibria and $\frac{\partial e}{\partial M}>0$ for unstable equilibria.

2)

$$\frac{\partial e}{\partial D}=\frac{1}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{-M-e}{(M-D)^2}$$

Because $e\in[-D,-M]$ and $-D$ and $-M$ are reached at the asymptote, without loss of generality, $\frac{\partial e}{\partial D}<0$ for stable equilibria and $\frac{\partial e}{\partial D}>0$ for unstable equilibria.

3)

$$\frac{\partial e}{\partial \rho}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{\varsigma^*-\rho\varsigma}{\varsigma\varsigma^*(1-\rho^2)^{\frac{3}{2}}}e$$

Therefore, at the symmetric equilibrium, if $e\gtreqless 0$, then $\frac{\partial e}{\partial\rho}\lesseqgtr 0$ for stable equilibria and $\frac{\partial e}{\partial\rho}\gtreqless 0$ for unstable equilibria.

4)

$$\frac{\partial e}{\partial \varsigma^*}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{e}{\varsigma^{*2}\sqrt{1-\rho^2}}$$

$$\frac{\partial e}{\partial \varsigma}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{-\rho e}{\varsigma^{2}\sqrt{1-\rho^2}}$$

Because $\varsigma=\varsigma^*$, we obtain

$$\frac{\partial e}{\partial \varsigma}+\frac{\partial e}{\partial \varsigma^*}=\frac{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)}{\phi(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}}e)(\frac{1-\rho\frac{\varsigma^*}{\varsigma}}{\varsigma^*\sqrt{1-\rho^2}})-\frac{1}{D-M}}\frac{1-\rho}{\varsigma^{*2}\sqrt{1-\rho^2}}e$$

Therefore, at the symmetric equilibrium, if $e \gtreqless 0$, then $\frac{\partial e}{\partial \varsigma} + \frac{\partial e}{\partial \varsigma^*} \lesseqgtr 0$ for stable equilibria and $\frac{\partial e}{\partial \varsigma} + \frac{\partial e}{\partial \varsigma^*} \gtreqless 0$ for unstable equilibria .

For the equilibrium whose stability cannot be determined, any comparative statics result equals ∞. *Q.E.D.*

Appendix H

Appendix of Chapter 3

Proof of Corollary 1

Proof of Corollary 1: Supposing $D+M>0$, we define the following function:

$$F(\rho;\varsigma^2)=\Phi(\sqrt{\ln\frac{(D-M)^2(\varsigma-\varsigma^*\rho)^2}{2\pi\varsigma^{*2}\varsigma^2(1-\rho^2)}})-\frac{1}{D-M}\sqrt{\frac{\varsigma^{*2}\varsigma^2(1-\rho^2)}{(\varsigma-\rho\varsigma^*)^2}\ln\frac{(D-M)^2(\varsigma-\varsigma^*\rho)^2}{2\pi\varsigma^{*2}\varsigma^2(1-\rho^2)}}-\frac{D}{D-M}$$

$$=\Phi(\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}})-\frac{1}{D-M}\sqrt{\varsigma^2\frac{1+\rho}{1-\rho}\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}}-\frac{D}{D-M}$$

The latter equality is obtained by the relation $\varsigma=\varsigma^*$. Differentiating $F(\rho;\varsigma^2)$ with respect to ρ and ς^2, respectively, we obtain

$$\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}=A\times(-\frac{1}{2\varsigma^2})-B\times\frac{1+\rho}{1-\rho}$$

and

$$\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}=A\times(-\frac{1}{1-\rho^2})-B\times\frac{2\varsigma^2}{(1-\rho)^2}$$

where $A=\phi(\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}})\frac{1}{\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}}}>0$ and $B=\frac{1}{2(D-M)}\frac{1}{\sqrt{\varsigma^2\frac{1+\rho}{1-\rho}\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}}}$ $[\ln\frac{(D-M)^2(1-\rho)}{2\pi\varsigma^2(1+\rho)}-1]$. Therefore, $\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}>0\iff A\times(-\frac{1}{2\varsigma^2})-B\times\frac{1+\rho}{1-\rho}>0$ and $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}>0\iff A\times(-\frac{1}{1-\rho^2})-B\times\frac{2\varsigma^2}{(1-\rho)^2}>0\iff A\times(-\frac{1}{1+\rho})-B\times\frac{2\varsigma^2}{1-\rho}>0\iff$

$A\times(-1)-B\times 2\varsigma^2\frac{1+\rho}{1-\rho}>0 \iff A\times(-\frac{1}{2\varsigma^2})-B\times\frac{1+\rho}{1-\rho}>0 \iff \frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}>0$. Thus, mathematically the inequalities $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}\gtreqless 0$ is equivalent to $\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}\gtreqless 0$, respectively.

We define $\bar{\xi}^2$ such that $F(\rho;\bar{\xi}^2)=0$. According to the proof of Theorem 1, it can be known that $\bar{\xi}^2$ exists and it is unique if $D>0>M$ ($\bar{\xi}^2$ makes the curve $y=\Phi(\frac{\frac{1}{\varsigma^*}-\frac{\rho}{\varsigma}}{\sqrt{1-\rho^2}}x)=\Phi(\frac{1-\rho}{\varsigma\sqrt{1-\rho^2}}x)$ and $y=\frac{D+x}{D-M}$ tangent with each other, and $y=\Phi(\frac{1-\rho}{\varsigma\sqrt{1-\rho^2}}x)$ is monotonic with respect to ς^2). Therefore, the reasons of the existence of $\bar{\xi}^2$ and $\bar{\rho}$ are the same. In the same way, if $\bar{\rho}$ is the unique solution of equation E.3, then $\bar{\xi}^2$ is the unique solution of the following function, derived from $F(\rho;\bar{\xi}^2)=0$, and it can be used as the (implicit) analytical expression of $\bar{\xi}^2$ for the $D+M>0$ situation:

$$\Phi(\sqrt{\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\xi}^2(1+\rho)}})=\frac{D+\sqrt{\bar{\xi}^2\frac{1+\rho}{1-\rho}\ln\frac{(D-M)^2(1-\rho)}{2\pi\bar{\xi}^2(1+\rho)}}}{D-M} \tag{H.1}$$

According to the expression of H.1, it can be found that equation H.1 corresponds to equation E.3. In addition, to ensure that $F(\rho;\varsigma^2)$ is valid, the $\bar{\xi}^2$ should satisfy

$$\frac{(D-M)^2(1-\rho)}{2\pi\bar{\xi}^2(1+\rho)}>1 \iff \bar{\xi}^2<\frac{(D-M)^2(1-\rho)}{2\pi(1+\rho)}=\hat{\xi}^2$$

for $D+M\neq 0$. If and only if $D+M=0$, then

$$\frac{(D-M)^2(1-\rho)}{2\pi\bar{\xi}^2(1+\rho)}=1 \iff \bar{\xi}^2=\hat{\xi}^2$$

By putting $\tilde{\xi}^2$ into H.1 to replace $\bar{\xi}^2$, it is found that $\bar{\xi}^2$ could be greater or smaller than or equal to $\tilde{\xi}^2$ if $\rho<0$ and hence $\tilde{\xi}^2$ exists. Because $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}$ and equivalently $\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}$ could be greater or smaller than 0. Therefore, in the following, we discuss the two situations separately. It is found that both situations lead to the same result.

1) For $\rho>\bar{\rho}$, if $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}>0$, then $F(\rho;\varsigma^2)>F(\bar{\rho};\varsigma^2)$, i.e. $F(\rho;\varsigma^2)>0$. Because $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}>0$ is equivalent to $\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}>0$, $F(\rho;\varsigma^2)>F(\rho;\bar{\xi}^2)$. Hence, for such ς^2, $\varsigma^2>\bar{\xi}^2$ can be obtained.

2) For $\rho>\bar{\rho}$, if $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}<0$, then $F(\rho;\varsigma^2)<F(\bar{\rho};\varsigma^2)$, i.e. $F(\rho;\varsigma^2)<0$. Because $\frac{\partial F(\rho;\varsigma^2)}{\partial\rho}<0$ is equivalent to $\frac{\partial F(\rho;\varsigma^2)}{\partial\varsigma^2}<0$, $F(\rho;\varsigma^2)<F(\rho;\bar{\xi}^2)$. Hence, for such ς^2,

$\varsigma^2 > \bar{\varsigma}^2$ can be obtained.

Therefore, for the strategic complements game with $\varsigma^2 > \max\{\bar{\varsigma}^2, \tilde{\varsigma}^2\}$ if $\rho < 0$ or with $\varsigma^2 > \bar{\varsigma}^2$ if $\rho \geq 0$, the situation is equivalent to the game with $\rho > \max\{\bar{\rho}, \tilde{\rho}\}$. Hence, the game contains a unique equilibrium for $\varsigma^2 > \max\{\bar{\varsigma}^2, \tilde{\varsigma}^2\}$ if $\rho < 0$ or $\varsigma^2 > \bar{\varsigma}^2$ if $\rho \geq 0$.

Moreover, from the proof, supposing $\bar{\varsigma}^2 > \tilde{\varsigma}^2$, we can find that for games with $\varsigma^2 \in [\tilde{\varsigma}^2, \bar{\varsigma}^2)$ for $\rho < 0$ or with $\varsigma^2 \in (0, \tilde{\varsigma}^2)$ for $\rho \geq 0$, this situation mathematically corresponds to the game with $\rho \in [\tilde{\rho}, \bar{\rho})$ given ς^2. Hence, the game contains three equilibria. In addition, at $\varsigma^2 = \bar{\varsigma}^2$, supposing $\bar{\varsigma}^2 \geq \tilde{\varsigma}^2$ if $\rho < 0$, because $D+M > 0$, the game contains two equilibria.

Applying the same approach, we can derive $\bar{\varsigma}^2$'s (implicit) analytical expression for $D+M < 0$. Still, the condition $D > 0 > M$ ensures the existence of $\bar{\varsigma}^2$ for $D+M < 0$, which is the unique solution of the following equation:

$$\Phi(-\sqrt{\ln \frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}) = \frac{D - \sqrt{\bar{\varsigma}^2 \frac{1+\rho}{1-\rho} \ln \frac{(D-M)^2(1-\rho)}{2\pi\bar{\varsigma}^2(1+\rho)}}}{D-M}$$

and for $D+M = 0$, $\bar{\varsigma}^2 = \hat{\varsigma}^2$. Still, in these two situations ($D+M \leq 0$), for $\varsigma^2 \in (\max\{\bar{\varsigma}^2, \tilde{\varsigma}^2\}, +\infty)$ if $\rho < 0$ or $\varsigma^2 \in (\bar{\varsigma}^2, +\infty)$ if $\rho \geq 0$, there exists a unique equilibrium, and for $\varsigma^2 \in (\tilde{\varsigma}^2, \bar{\varsigma}^2)$ if $\bar{\varsigma}^2 > \tilde{\varsigma}^2$ and $\rho < 0$ or $\varsigma^2 \in (0, \bar{\varsigma}^2)$ if $\rho \geq 0$, there exist three equilibria. At $\varsigma^2 = \bar{\varsigma}^2$, where $\bar{\varsigma}^2 \geq \tilde{\varsigma}^2$ if $\rho < 0$, there are two equilibria for $D+M < 0$. One equilibrium is stable, and the other equilibrium's stability cannot be determined. For $D+M = 0$, there exists a unique equilibrium, which is stable. *Q.E.D.*

Appendix I

Appendix of Chapter 3

Derivation of the Best Response Function for ς and $\varsigma^* \to 0$

Assume $\varsigma = \varsigma^*$. Suppose $D > M$ and $\rho > 0$. As shown in Section 3.7, as ς and $\varsigma^* \to 0$,

$$g(x^*) = \frac{1}{\rho}x^*$$

where $x^* \in [-\rho D, -\rho M]$. Let us recall the definition equation of the cutoff best response $g(x^*)$: $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})(M-D) + D + g(x^*) = 0$. If $x^* > \rho g(x^*)$, $\lim_{\varsigma\to 0}\Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(+\infty) = 1$ and hence $g(x^*) = -M$. Therefore, if $x^* > -\rho M$, $g(x^*) = -M$.

If $x^* < \rho g(x^*)$, $\lim_{\varsigma\to 0}\Phi(\frac{x^*-\rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(-\infty) = 0$ and hence $g(x^*) = -D$. Therefore, if $x^* < -\rho D$, $g(x^*) = -D$.

Therefore, if $D > M$ and $\rho > 0$,

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho}x^* & -\rho D \le x^* \le -\rho M \\ -M & x^* > -\rho M \end{cases}$$

Suppose $M > D$ and $\rho < 0$. As shown in Section 3.7, as ς and $\varsigma^* \to 0$,

$$g(x^*) = \frac{1}{\rho} x^*$$

where $x^* \in [-\rho D, -\rho M]$. According to the equation $\mathbb{E}\Pi(x^*, g(x^*)) = \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}})(M - D) + D + g(x^*) = 0$, if $x^* > \rho g(x^*)$, $\lim_{\varsigma \to 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(+\infty) = 1$ and hence $g(x^*) = -M$. Hence, if $x^* > -\rho M$, $g(x^*) = -M$.

If $x^* < \rho g(x^*)$, $\lim_{\varsigma \to 0} \Phi(\frac{x^* - \rho g(x^*)}{\varsigma\sqrt{1-\rho^2}}) = \Phi(-\infty) = 0$ and hence $g(x^*) = -D$. Therefore, if $x^* < -\rho D$, $g(x^*) = -D$.

Therefore, if $M > D$ and $\rho < 0$,

$$g(x^*) = \begin{cases} -D & x^* < -\rho D \\ \frac{1}{\rho} x^* & -\rho D \leq x^* \leq -\rho M \\ -M & x^* > -\rho M. \end{cases}$$

Appendix A

Appendix of Chapter 4

Notations

Here, we summarize all notations and expressions of formulas that have been used in this chapter.

1.
$$w(x,\lambda) = \frac{p}{1-\exp(\frac{x+d}{\lambda})} + \frac{1-p}{1-\exp(\frac{x+u}{\lambda})}$$

2.
$$S(\lambda,x) = p\exp(\frac{x+u}{\lambda}) + (1-p)\exp(\frac{x+d}{\lambda})$$

3.
$$T(\lambda,x) = p\exp(-\frac{x+u}{\lambda}) + (1-p)\exp(-\frac{x+d}{\lambda})$$

4.
$$e(x) = p(x+u) + (1-p)(x+d)$$

5.
$$r(x) = \frac{1}{1-\frac{x+d}{x+u}(\frac{\exp(\frac{x+u}{\lambda})-1}{1-\exp(\frac{x+d}{\lambda})})^2\exp(\frac{d-u}{\lambda})}$$

6.
$$1\{P\} = \begin{cases} 1 & \text{if P is true} \\ 0 & \text{otherwise} \end{cases}$$

7. For a bivariate function $y = f(x,v)$, its inverse function with respect to x is expressed by $x = f^{-1}(y;v)$.

Definitions in the binary decision problem:

1.
$$f_{DP} := w(0, \lambda)$$

2.
$$F_{DP}(\lambda) := S(\lambda, 0)$$

3.
$$G_{DP}(\lambda) := T(\lambda, 0)$$

4.
$$\mu_{DP} := e(0)$$

Definitions in the rational inattention Bayesian game:

1.
$$f(q^*, \lambda) := w((1-q^*)M + q^*D, \lambda)$$

If there exists a $\tilde{\lambda}$ given q^* such that $f'_{q^*}(q^*, \tilde{\lambda}) = -1$, then $\tilde{\lambda} = f'^{-1}_{q^*}(-1; q^*)$.

2.
$$F(\lambda, q^*) := S(\lambda, (1-q^*)M + q^*D)$$

If there exists a $\bar{\lambda}_{q^*}$ given q^* such that $F(\bar{\lambda}_{q^*}, q^*) = 1$, then $\bar{\lambda}_{q^*} = F^{-1}(1; q^*)$.

3.
$$G(\lambda, q^*) := T(\lambda, (1-q^*)M + q^*D)$$

If there exists a $\bar{\lambda}_{q^*}$ given q^* such that $G(\bar{\lambda}_{q^*}, q^*) = 1$, then $\bar{\lambda}_{q^*} = G^{-1}(1; q^*)$.

4.
$$\mu(q^*) := e((1-q^*)M + q^*D)$$

5.
$$\bar{p}(q^*) := r((1-q^*)M + q^*D)$$

Appendix B

Appendix of Chapter 4 Proofs of Proposition 1 and Proposition 2

Proof of Proposition 1: Proposition 1 is a particular case of Proposition 2 if $M = D = 0$. Therefore, the proof of Proposition 1 is essentially the proof of Proposition 2. *Q.E.D.*

Proof of Proposition 2: Define $q^{\varepsilon} = \Pr(a = 1|\varepsilon)$, where $\varepsilon \in \{u, d\}$. Player i's utility therefore can be written as

$$U(q^{\varepsilon}, q^{\varepsilon^*}) = pq^u[(1-q^*)(M+u)+q^*(D+u)]+(1-p)q^d[(1-q^*)(M+d)+q^*(D+d)] \tag{B.1}$$

Moreover, the information processing capacity can be written as

$$I(q^{\varepsilon}) = H(q) - \mathbb{E}_{\varepsilon}[H(q^{\varepsilon})]$$

$$= -(1-q)\ln(1-q) - q\ln q + p[(1-q^u)\ln(1-q^u)+q^u\ln q^u]+(1-p)[(1-q^d)\ln(1-q^d)+q^d\ln q^d] \tag{B.2}$$

where $H(.)$ is the entropy measure of relevant probability distribution and $q = pq^u + (1-p)q^d$. Therefore, player i's preference can be expressed as

$$V(q^{\varepsilon}, q^{\varepsilon^*}) = U(q^{\varepsilon}, q^{\varepsilon^*}) - \lambda I(q^{\varepsilon}) \tag{B.3}$$

Next, we solve player i's utility maximization problem to obtain the best response function. Given the opponent's strategy $q^* \in [0,1]$, player i's utility maximization problem can be written as

G-1:

$$\max_{q^u,q^d} pq^u[(1-q^*)(M+u)+q^*(D+u)]+(1-p)q^d[(1-q^*)(M+d)+q^*(D+d)]$$

$$-\lambda\{-(1-q)\ln(1-q)-q\ln q+p[(1-q^u)\ln(1-q^u)+q^u\ln q^u]+(1-p)[(1-q^d)\ln(1-q^d)+q^d\ln q^d]\}$$

s.t. $0 \leq q^u \leq 1, 0 \leq q^d \leq 1.$

Solving G-1, we obtain the conditional choice probabilities q^ε, where $\varepsilon \in \{u,d\}$, which is given by

$$q^\varepsilon = \frac{q\exp(\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda})}{q\exp(\frac{(1-q^*)M+q^*D+\varepsilon}{\lambda})+(1-q)} \tag{B.4}$$

where $\varepsilon \in \{u,d\}$.

Substituting equation (B.4) back to i's objective function in G-1, we obtain the following transformation of the game:

G-2:

$$\max_{q} \lambda\{p\ln[q\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})+(1-q)]+(1-p)\ln[q\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})+(1-q)]\}$$

s.t. $0 \leq q \leq 1.$

If the interior solution in G-2 exists, i.e. the function q with respect to q^* obtained from G-2 is between 0 and 1 ($q \in [0,1]$), then the solution, which is also player i's best response function when i acquires information to make decisions, is given by

$$q = \frac{p}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})} + \frac{1-p}{1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})} \tag{B.5}$$

Given Assumption 1, we find that the second-order condition of G-2: $[\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})-1][\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})-1]<0\ \forall q^*\in[0,1]$. Therefore, given the opponent's strategy q^*, equation (B.5) is the unique maximizer of player i's utility.

Conditions ensuring the existence of (B.5) : Given $q^*\in[0,1]$, for (B.5), that $0\leq q\leq 1$ can be expressed by

$$0\leq \frac{p}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}+\frac{1-p}{1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}\leq 1$$

which can be equivalently expressed into the following inequality system:

$$\begin{cases} p\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})+(1-p)\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})\geq 1 \\ p\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})+(1-p)\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})\geq 1 \end{cases} \tag{B.6}$$

The inequality group (B.6) is the necessary and sufficient condition to ensure that i's information-acquisition best response (B.5) exists given the opponent's strategy. We are interested in the value of λ that ensures (B.5) exists given the other primitives. In the following, we will derive the set of λ that ensures (B.5) exists from (B.6).

Define $F(\lambda)=p\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})+(1-p)\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})$ and $G(\lambda)=p\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})+(1-p)\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})$. It can be obtained that $\lim_{\lambda\to 0^+}F(\lambda)=\lim_{\lambda\to 0^+}G(\lambda)=+\infty$ and $\lim_{\lambda\to+\infty}F(\lambda)=\lim_{\lambda\to+\infty}G(\lambda)=1$. The derivatives of $F(\lambda)$ and $G(\lambda)$ are given by

$$\begin{aligned} F'(\lambda)=&-\frac{(1-q^*)M+q^*D+u}{\lambda^2}\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})p \\ &-\frac{(1-q^*)M+q^*D+d}{\lambda^2}\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p) \end{aligned}$$

$$\begin{aligned} G'(\lambda)=&\frac{(1-q^*)M+q^*D+u}{\lambda^2}\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})p \\ &+\frac{(1-q^*)M+q^*D+d}{\lambda^2}\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p) \end{aligned}$$

In the following, we will analyse the range of λ that ensures the existence of the interior solution (B.5) in three different cases: $(1-q^*)M+q^*D+pu+(1-p)d \lesseqgtr 0$.

Case 1 ($(1-q^*)M+q^*D+pu+(1-p)d<0$) : $F'(\lambda)=0$ indicates that $[(1-q^*)M+q^*D+u]\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})p+[(1-q^*)M+q^*D+d]\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)=0$. Define $LHS(\lambda)=[(1-q^*)M+q^*D+u]\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})p+[(1-q^*)M+q^*D+d]\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)$. Calculating the derivative of $LHS(\lambda)$, we obtain

$$LHS'(\lambda)=-\frac{[(1-q^*)M+q^*D+u]^2}{\lambda^2}\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})p$$
$$-\frac{[(1-q^*)M+q^*D+d]^2}{\lambda^2}\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)<0$$

Therefore, $LHS(\lambda)$ strictly decreases with respect to λ.

Because $\lim_{\lambda\to 0^+}LHS(\lambda)=+\infty$ and $\lim_{\lambda\to+\infty}LHS(\lambda)=(1-q^*)M+q^*D+pu+(1-p)d$, equation $F'(\lambda)=0$ or $LHS'(\lambda)=0$ has a solution if and only if $(1-q^*)M+q^*D+pu+(1-p)d\leq 0$, and the solution must be unique.

Here, we only consider the strict inequality case. Thus, if

$$(1-q^*)M+q^*D+pu+(1-p)d<0 \tag{B.7}$$

then there exists a unique $\hat{\lambda}$ such that $F'(\hat{\lambda})=0$. In addition, for $\lambda<\hat{\lambda}$, $LHS(\lambda)>0$, which equivalently indicates that $F'(\lambda)<0$; for $\lambda>\hat{\lambda}$, $LHS(\lambda)<0$, which equivalently indicates that $F'(\lambda)>0$. Therefore, if (B.7) is held, $F(\lambda)$ decreases with respect to λ for $\lambda<\hat{\lambda}$, and $F(\lambda)$ increases with respect to λ for $\lambda>\hat{\lambda}$.

According to (B.7) and the convexity property of function $x\exp(x)$, we get

$$-\lambda G'(\lambda)=(-\frac{(1-q^*)M+q^*D+u}{\lambda})\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})p$$
$$+(-\frac{(1-q^*)M+q^*D+d}{\lambda})\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)$$
$$\geq[-\frac{(1-q^*)M+q^*D+u}{\lambda}p-\frac{(1-q^*)M+q^*D+d}{\lambda}(1-p)]$$
$$\exp[-\frac{(1-q^*)M+q^*D+u}{\lambda}p-\frac{(1-q^*)M+q^*D+d}{\lambda}(1-p)]>0$$

Therefore, $G'(\lambda)<0\ \forall\lambda\in(0,+\infty)$. Thus, when (B.7) is held, there exists a unique $\lambda=\hat{\lambda}$ such that $F'(\hat{\lambda})=0$. For $\lambda<\hat{\lambda}$, $F(\lambda)$ decreases, and for $\lambda>\hat{\lambda}$, $F(\lambda)$ increases. According to the above analysis, it is known that $F(\hat{\lambda})<1$, and $\forall\lambda\in(0,+\infty)$, $G(\lambda)$

decreases.

Therefore, in this case, there exists a unique $\bar{\lambda} < +\infty$ such that $F(\bar{\lambda}) = 1$. $\forall \lambda < \bar{\lambda}$, $F(\lambda) > 1$, and $\forall \lambda \leq \bar{\lambda}$, $G(\lambda) > 1$. Hence, in conclusion, given a $q^* \in [0,1]$, as long as (B.7) is held, $\forall \lambda \in (0, \bar{\lambda}]$, (B.6) is held, and so the information-acquisition best response (B.5) exists (see Figure B.1).

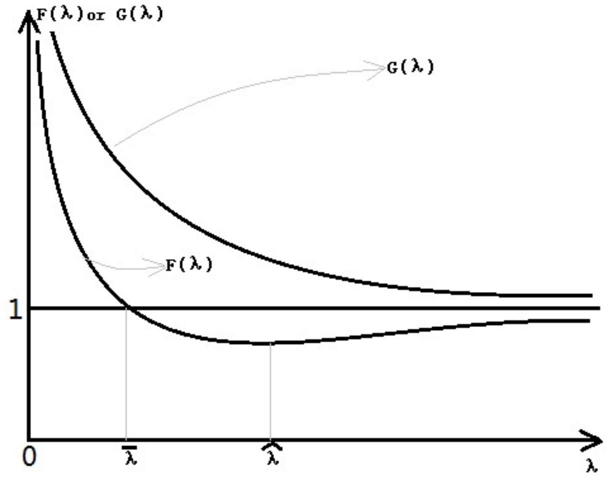


Figure B.1: *Given a* $q^* \in [0,1]$, *if* $(1-q^*)M + q^*D + pu + (1-p)d < 0$, *there exists a* $\bar{\lambda}$ *such that* $F(\bar{\lambda}) = 1$. *In this case,* $\forall \lambda \in (0, \bar{\lambda}]$, *player i's information-acquisition best response exists.*

In this case, for $\lambda > \bar{\lambda}$, the interior solution does not exist. Instead, there is a boundary solution. Given a $q^* \in [0,1]$, since $(1-q^*)M + q^*D + pu + (1-p)d < 0$, $q = 0$ in this situation.[1] The behavioural implication is that for $\lambda > \bar{\lambda}$, player i's best response towards q^* is obtained by comparing the ex ante (i.e. before nature draws payoff shocks for each player) expected payoff of being active and inactive. Note that at $\lambda = \bar{\lambda}$, the interior solution $q = 0$ as well, and in this situation, the amount of information acquired at the optimum is zero. Therefore, the coincidence of the solution at $\lambda = \bar{\lambda}$ with the solution at $\lambda > \bar{\lambda}$ supports that the prior-reliance choice behaviour is the limit of the information-acquisition choice behaviour as λ increases, given a $q^* \in [0,1]$. In the following, we will frequently use the term 'prior-reliance choice behaviour' to represent the choice behaviour by comparing the ex ante expected payoff of each action.

[1] If the constraint $0 \leq q \leq 1$ is ignored, the interior solution q obtained from solving G-2 is negative.

Case 2 ($(1-q^*)M+q^*D+pu+(1-p)d>0$) : $G'(\lambda)=0$ indicates that $[(1-q^*)M+q^*D+u]\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})p+[(1-q^*)M+q^*D+d]\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)=0$. We define $LHS(\lambda)=[(1-q^*)M+q^*D+u]\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})p+[(1-q^*)M+q^*D+d]\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)$. Calculating the derivative of $LHS(\lambda)$, we obtain that

$$LHS'(\lambda)=\frac{[(1-q^*)M+q^*D+u]^2}{\lambda^2}\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda})p$$
$$+\frac{[(1-q^*)M+q^*D+d]^2}{\lambda^2}\exp(-\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)>0$$

Therefore, $LHS(\lambda)$ strictly increases with respect to λ.

Because $\lim_{\lambda\to 0^+}LHS(\lambda)=-\infty$ and $\lim_{\lambda\to+\infty}LHS(\lambda)=(1-q^*)M+q^*D+pu+(1-p)d$, equation $F'(\lambda)=0$ or $LHS'(\lambda)=0$ has a solution if and only if $(1-q^*)M+q^*D+pu+(1-p)d\geq 0$, and the solution must be unique.

Here, we only consider the strict inequality case. Thus, if

$$(1-q^*)M+q^*D+pu+(1-p)d>0 \tag{B.8}$$

then there exists a unique $\hat{\lambda}$ such that $G'(\hat{\lambda})=0$. In addition, for $\lambda<\hat{\lambda}$, $G'(\lambda)<0$ and for $\lambda>\hat{\lambda}$, $G'(\lambda)>0$. Therefore, if (B.8) is held, $G(\lambda)$ decreases with respect to λ for $\lambda<\hat{\lambda}$, and $G(\lambda)$ increases with respect to λ for $\lambda>\hat{\lambda}$.

According to (B.8) and the convexity property of function $x\exp(x)$, we get

$$-\lambda F'(\lambda)=\frac{(1-q^*)M+q^*D+u}{\lambda}\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})p$$
$$+\frac{(1-q^*)M+q^*D+d}{\lambda}\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})(1-p)$$
$$\geq[\frac{(1-q^*)M+q^*D+u}{\lambda}p+\frac{(1-q^*)M+q^*D+d}{\lambda}(1-p)]$$
$$\exp[\frac{(1-q^*)M+q^*D+u}{\lambda}p+\frac{(1-q^*)M+q^*D+d}{\lambda}(1-p)]>0$$

Therefore, $F'(\lambda)<0\ \forall\lambda\in(0,+\infty)$. Thus, when (B.8) is held, there exists a unique $\lambda=\hat{\lambda}$ such that $G'(\hat{\lambda})=0$. For $\lambda<\hat{\lambda}$, $G(\lambda)$ decreases, and for $\lambda>\hat{\lambda}$, $G(\lambda)$ increases. According to the above analysis, it is known that $G(\hat{\lambda})<1$, and $\forall\lambda\in(0,+\infty)$, $F(\lambda)$

decreases.

Therefore, in this case, there exists a unique $\bar{\lambda} < +\infty$ such that $G(\bar{\lambda}) = 1$. $\forall \lambda < \bar{\lambda}$, $G(\lambda) > 1$, and $\forall \lambda \leq \bar{\lambda}$, $F(\lambda) > 1$. Hence, in conclusion, given a $q^* \in [0,1]$, as long as (B.8) is held, $\forall \lambda \in (0, \bar{\lambda}]$, (B.6) is held, and so the information-acquisition best response (B.5) exists (see Figure B.2).

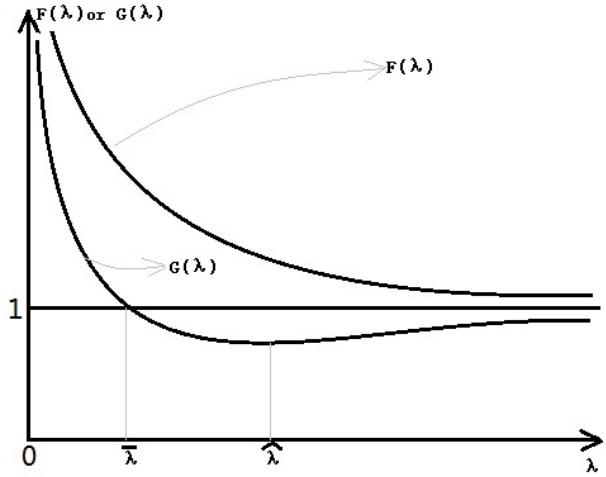


Figure B.2: *Given a* $q^* \in [0,1]$, *if* $(1-q^*)M + q^*D + pu + (1-p)d > 0$, *there exists a* $\bar{\lambda}$ *such that* $G(\bar{\lambda}) = 1$. *In this case,* $\forall \lambda \in (0, \bar{\lambda}]$, *player i's information-acquisition best response exists.*

In this case, for $\lambda > \bar{\lambda}$, the interior solution does not exist. Instead, there is a boundary solution. Given a $q^* \in [0,1]$, since $(1-q^*)M + q^*D + pu + (1-p)d > 0$, $q = 1$ in this situation.[2] Its behavioural implication and its consistence with the information-acquisition choice behaviour are the same as in the analysis of Case 1.

Case 3 $((1-q^*)M + q^*D + pu + (1-p)d = 0)$ **:** If $(1-q^*)M + q^*D + pu + (1-p)d = 0$, then $\lim_{\lambda \to 0^+} F'(\lambda) = -\infty$ and $\lim_{\lambda \to +\infty} F'(\lambda) = 0$; $\lim_{\lambda \to 0^+} G'(\lambda) = -\infty$ and $\lim_{\lambda \to +\infty} G'(\lambda) = 0$. Furthermore, the condition that $(1-q^*)M + q^*D + pu + (1-p)d = 0$ can be equivalently expressed by

$$(1-q^*)M + q^*D + d = -\frac{p}{1-p}[(1-q^*)M + q^*D + u]$$

[2] If the constraint $0 \leq q \leq 1$ is ignored, the interior solution q obtained from solving G-2 is greater than 1.

Therefore, in this case,

$$F(\lambda) = p\exp(\frac{(1-q^*)M+q^*D+u}{\lambda}) + (1-p)\exp(-\frac{p}{1-p}[(1-q^*)M+q^*D+u])$$

$$G(\lambda) = p\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda}) + (1-p)\exp(\frac{p}{1-p}[(1-q^*)M+q^*D+u])$$

Calculating derivatives of $F(\lambda)$ and $G(\lambda)$, we obtain

$$F'(\lambda) = \frac{(1-q^*)M+q^*D+u}{\lambda^2}p\{\exp(-\frac{p}{1-p}\frac{(1-q^*)M+q^*D+u}{\lambda}) - \exp(\frac{(1-q^*)M+q^*D+u}{\lambda})\} < 0$$

$$G'(\lambda) = \frac{(1-q^*)M+q^*D+u}{\lambda^2}p\{\exp(-\frac{(1-q^*)M+q^*D+u}{\lambda}) - \exp(\frac{p}{1-p}\frac{(1-q^*)M+q^*D+u}{\lambda})\} < 0$$

Therefore, according to above results, $\forall\lambda < +\infty$, both $F(\lambda)$ and $G(\lambda)$ decrease from $+\infty$ to 1 as λ increases from 0 to $+\infty$, and hence in this case, only at $+\infty$ can $F(\lambda)$ and $G(\lambda)$ be equal to 1. Therefore, in this case, all $\lambda \in (0,+\infty)$ are applicable to ensure the existence of the interior solution (B.5) (see Figure B.3).

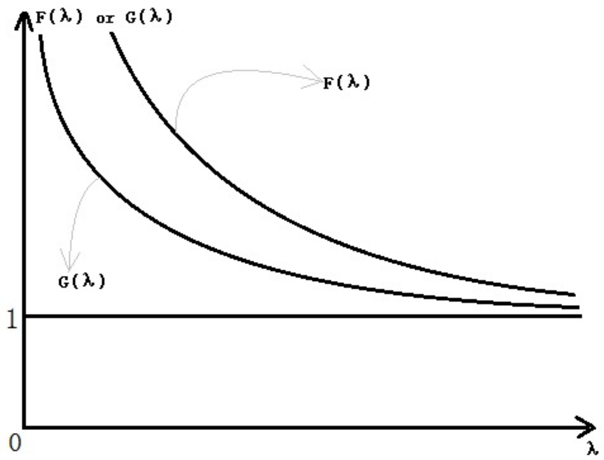


Figure B.3: *Given a* $q^* \in [0,1]$, *if* $(1-q^*)M + q^*D + pu + (1-p)d = 0$, *it is at* $\lambda = +\infty$, $F(\lambda) = G(\lambda) = 1$, *and hence in this case,* $\bar{\lambda} = +\infty$. *Therefore,* $\forall\lambda \in (0,+\infty)$, *the interior solution (B.5) always exists. According to Assumption 1,* $(1-q^*)M+q^*D+u > 0$. *Therefore* $F(\lambda) > G(\lambda)$.

Summary of proof results for all three cases: In conclusion, under Assumption 1, given i^*'s strategy $q^* \in [0,1]$, i's information-acquisition best response (B.5) exists if and only if:

1) if $(1-q^*)M + q^*D + pu + (1-p)d < 0$, there exists a unique $\bar{\lambda} < +\infty$ such that $F(\bar{\lambda}) = 1$, and $\forall\lambda \in (0, \bar{\lambda}]$, i's information-acquisition best response, given by (B.5), exists and is unique.

$\forall\lambda \in (\bar{\lambda}, +\infty)$, $q = 0$ given any $q^* \in [0,1]$.

2) if $(1-q^*)M + q^*D + pu + (1-p)d > 0$, there exists a unique $\bar{\lambda} < +\infty$ such that $G(\bar{\lambda}) = 1$, and $\forall\lambda \in (0, \bar{\lambda}]$, i's information-acquisition best response, given by (B.5), exists and is unique.

$\forall\lambda \in (\bar{\lambda}, +\infty)$, $q = 1$ given any $q^* \in [0,1]$.

3) if $(1-q^*)M + q^*D + pu + (1-p)d = 0$, $\forall\lambda \in (0, +\infty)$, i's information-acquisition best response (B.5) always exists and it is unique. In this case, $\bar{\lambda} = +\infty$. *Q.E.D.*

Appendix C

Appendix of Chapter 4 Proofs of Proposition 4, 5, 7 and Corollary 1

Proof of Proposition 4: The choice probabilities q and q^* are contained in the unit interval. The function $(q(q^*), q^*(q))$, which is represented by the equation system

$$\begin{cases} q = 1 \times 1\{f(q^*,\lambda) > 1\} + f(q^*,\lambda) \times 1\{0 \leq f(q^*,\lambda) \leq 1\} + 0 \times 1\{f(q^*,\lambda) < 0\} \\ q^* = 1 \times 1\{f(q,\lambda^*) > 1\} + f(q,\lambda^*) \times 1\{0 \leq f(q,\lambda^*) \leq 1\} + 0 \times 1\{f(q,\lambda^*) < 0\} \end{cases}$$

where $f(q^*,\lambda)$ is equation (6), is continuous with respect to (q^*,q). Therefore, according to Brouwer's fixed point theorem, there exists a fixed point of the function $(q(q^*), q^*(q))$. The fixed point corresponds to an equilibrium. ■

Proof of Proposition 5: In this proof, we omit the "*" notation for denoting opponent's strategy for simplicity. It will not affect the understanding of this proof because the opponent's strategy only plays the role as an independent variable.

Let us define equation $f(q)$ as

$$f(q) = \frac{p}{1-\exp(\frac{M+d-(M-D)q}{\lambda})} + \frac{1-p}{1-\exp(\frac{M+u-(M-D)q}{\lambda})}$$

First, we derive its first-order and second-order derivatives with respect to q, as follows:

$$f'(q) = -\frac{p\exp(\frac{d}{\lambda})\frac{M-D}{\lambda}\exp(\frac{M-(M-D)q}{\lambda})}{[1-\exp(\frac{d}{\lambda})\exp(\frac{M-(M-D)q}{\lambda})]^2} - \frac{(1-p)\exp(\frac{u}{\lambda})\frac{M-D}{\lambda}\exp(\frac{M-(M-D)q}{\lambda})}{[1-\exp(\frac{u}{\lambda})\exp(\frac{M-(M-D)q}{\lambda})]^2}$$

and

$$f''(q) = p\frac{(M-D)^2}{\lambda^2}\frac{\exp(\frac{M-(M-D)q+d}{\lambda})\{1-[\exp(\frac{M-(M-D)q+d}{\lambda})]^2\}}{\{1-\exp(\frac{M-(M-D)q+d}{\lambda})\}^4}$$
$$+(1-p)\frac{(M-D)^2}{\lambda^2}\frac{\exp(\frac{M-(M-D)q+u}{\lambda})\{1-[\exp(\frac{M-(M-D)q+u}{\lambda})]^2\}}{\{1-\exp(\frac{M-(M-D)q+u}{\lambda})\}^4}$$

Disregarding the constraint $0 \le q \le 1$, $f(q)$ generically has two discontinuity points: $q = \frac{M+u}{M-D}$ and $q = \frac{M+d}{M-D}$. According to Assumption 1, $M+d<0$ and $D+u>0$; therefore, $\frac{M+u}{M-D} > 1$ and $\frac{M+d}{M-D} < 0$. Hence, under Assumption 1, $f(q)$ is continuous with respect to $q \in [0,1]$ (see Figure C.1).

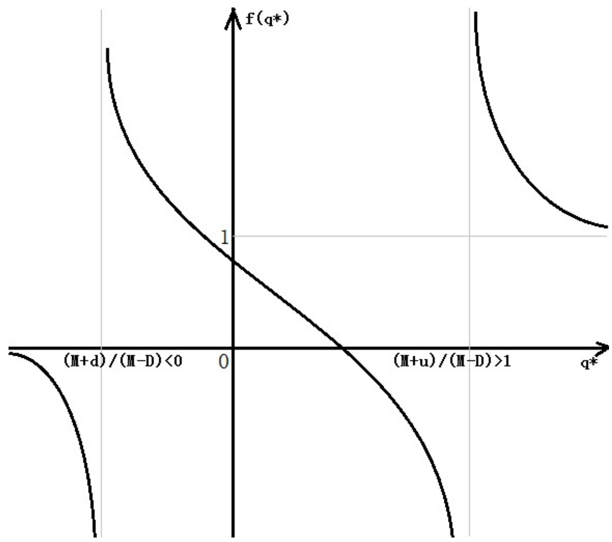


Figure C.1: *A geometric representation of function $f(q^*)$. Note that $f(0)$ and $f(1)$ could be greater than or equal to 1, or less than 1, depending on specific parameter specification.*

Derivation of $\frac{\partial f'(q)}{\partial\lambda}$: Define $\phi(x) = \frac{1+x}{(1-x)^3}$, $c(d) = M-(M-D)q+d$ and $c(u) = M-(M-D)q+u$, and

$$\Phi(\lambda) = -\frac{p\exp(\frac{M-(M-D)q+d}{\lambda})}{[1-\exp(\frac{M-(M-D)q+d}{\lambda})]^2} - \frac{(1-p)\exp(\frac{M-(M-D)q+u}{\lambda})}{[1-\exp(\frac{M-(M-D)q+u}{\lambda})]^2} < 0 \quad \forall\lambda \in (0,+\infty)$$

It is found that $\lim_{\lambda\to 0}\Phi(\lambda) = 0$. The derivative of $\Phi(\lambda)$ is

$$\Phi'(\lambda) = p\frac{M-(M-D)q+d}{\lambda^2}\phi(\exp(\frac{M-(M-D)q+d}{\lambda}))\exp(\frac{M-(M-D)q+d}{\lambda})$$
$$+(1-p)\frac{M-(M-D)q+u}{\lambda^2}\phi(\exp(\frac{M-(M-D)q+u}{\lambda}))\exp(\frac{M-(M-D)q+u}{\lambda})$$

Therefore,

$$\frac{\partial f'(q)}{\partial\lambda} = \frac{d\Phi(\lambda)\frac{M-D}{\lambda}}{d\lambda} = \frac{M-D}{\lambda}[\Phi'(\lambda) - \frac{\Phi(\lambda)}{\lambda}]$$
$$= \frac{M-D}{\lambda}\{p[\frac{c(d)}{\lambda^2}\exp(\frac{c(d)}{\lambda})\phi(\exp(\frac{c(d)}{\lambda})) + \frac{\exp(\frac{c(d)}{\lambda})}{\lambda[1-\exp(\frac{c(d)}{\lambda})]^2}]$$
$$+(1-p)[\frac{c(u)}{\lambda^2}\exp(\frac{c(u)}{\lambda})\phi(\exp(\frac{c(u)}{\lambda})) + \frac{\exp(\frac{c(u)}{\lambda})}{\lambda[1-\exp(\frac{c(u)}{\lambda})]^2}]\}$$

The range of λ that ensures $|f'(q)| < 1$: $|f'(q)| < 1$ can be explicitly written as

$$\{\frac{p\exp(\frac{(1-q)M+qD+d}{\lambda})}{[1-\exp(\frac{(1-q)M+qD+d}{\lambda}]^2} + \frac{(1-p)\exp(\frac{(1-q)M+qD+u}{\lambda})}{[1-\exp(\frac{(1-q)M+qD+u}{\lambda}]^2}\}\frac{M-D}{\lambda} < 1$$

which can be equivalently represented by

$$-\Phi(\lambda)\frac{M-D}{\lambda} < 1$$

Because for $x \in (0,1)$, $\phi(x) > 1$, and for $x \in (1,+\infty)$, $\phi(x) < 0$,

$$-\Phi'(\lambda) = -p\frac{M-(M-D)q+d}{\lambda^2}\phi(\exp(\frac{M-(M-D)q+d}{\lambda}))\exp(\frac{M-(M-D)q+d}{\lambda})$$
$$-(1-p)\frac{M-(M-D)q+u}{\lambda^2}\phi(\exp(\frac{M-(M-D)q+u}{\lambda}))\exp(\frac{M-(M-D)q+u}{\lambda}) > 0$$

We have $\lim_{\lambda\to 0} -\Phi'(\lambda) = 0$, and from the previous analysis, it has been known that $-\Phi(\lambda) > 0$, $-\Phi'(\lambda) > 0$, $\lim_{\lambda\to 0} -\Phi(\lambda) = 0$ and $\lim_{\lambda\to 0} -\Phi'(\lambda) = 0$. Therefore, there should exist an $\varepsilon > 0$ such that given $q^* \in [0,1]$, $\forall\lambda \in (0,\varepsilon)$, $-\Phi'(\lambda) < \frac{1}{M-D}$, and hence $-\Phi(\lambda) < \frac{\lambda}{M-D}$ $\forall\lambda \in (0,\varepsilon)$ and the ε should satisfy $-\Phi(\varepsilon) = \frac{\varepsilon}{M-D}$ (see Figure C.2).

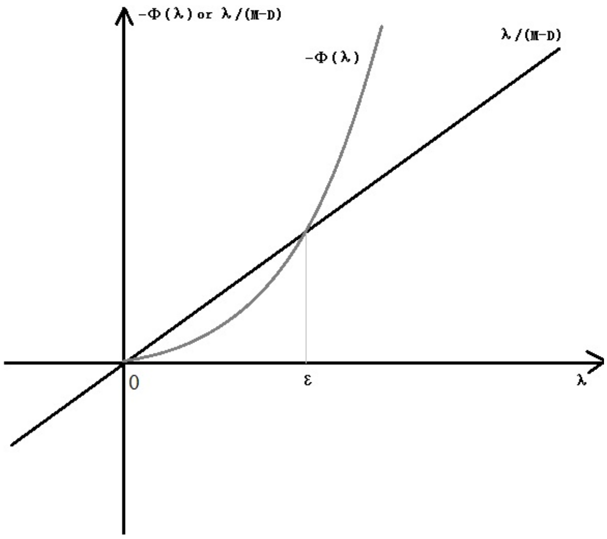


Figure C.2: *A geometric illustration of the proof. For $\lambda \in (0,\varepsilon)$, $-\Phi(\lambda) < \frac{\lambda}{M-D}$, and hence $f'(q) > -1$. From the proof by far, at least it can be known that the λ that makes $f'(q) > -1$ must begin from 0.*

Therefore, we obtain a preliminary result that given the opponent's strategy $q^* \in [0,1]$, there exists an ε such that $\forall \lambda \in (0,\varepsilon)$, $|f'(q)| < 1$, or equivalently, $f'(q) > -1$.

The shape of $f'(q)$: Let us recall $f'(q)$ and $f''(q)$:

$$f'(q) = -\frac{p\exp(\frac{d}{\lambda})\frac{M-D}{\lambda}\exp(\frac{M-(M-D)q}{\lambda})}{[1-\exp(\frac{d}{\lambda})\exp(\frac{M-(M-D)q}{\lambda})]^2} - \frac{(1-p)\exp(\frac{u}{\lambda})\frac{M-D}{\lambda}\exp(\frac{M-(M-D)q}{\lambda})}{[1-\exp(\frac{u}{\lambda})\exp(\frac{M-(M-D)q}{\lambda})]^2}$$

and

$$f''(q) = p\frac{(M-D)^2}{\lambda^2}\frac{\exp(\frac{M-(M-D)q+d}{\lambda})\{1-[\exp(\frac{M-(M-D)q+d}{\lambda})]^2\}}{\{1-\exp(\frac{M-(M-D)q+d}{\lambda})\}^4}$$
$$+(1-p)\frac{(M-D)^2}{\lambda^2}\frac{\exp(\frac{M-(M-D)q+u}{\lambda})\{1-[\exp(\frac{M-(M-D)q+u}{\lambda})]^2\}}{\{1-\exp(\frac{M-(M-D)q+u}{\lambda})\}^4}$$

According to above equations, $f''(q) < 0$ indicates that

$$\frac{p}{1-p}\exp(\frac{d-u}{\lambda}) < [1+\frac{\exp(\frac{u}{\lambda})-\exp(\frac{d}{\lambda})}{\frac{1}{A}+\exp(\frac{d}{\lambda})}][\frac{\exp(\frac{u}{\lambda})-\exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda})-\frac{1}{A}}-1]^3$$

where $A = \exp(\frac{M-(M-D)q}{\lambda}) > 0$ and $A \in [\exp(\frac{D}{\lambda}), \exp(\frac{M}{\lambda})]$ and hence $\frac{1}{A} \in [\exp(-\frac{M}{\lambda}), \exp(-\frac{D}{\lambda})]$.

Define

$$T(A) = [\frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{\exp(\frac{d}{\lambda}) + \frac{1}{A}} + 1][\frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda}) - \frac{1}{A}} - 1]^3$$

We can obtain $T(\exp(\frac{M}{\lambda})) > 0$ and $T(\exp(\frac{D}{\lambda})) > 0$. The derivative of $T(A)$ is given by

$$T'(A) = \frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{A^2}[\frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda}) - \frac{1}{A}} - 1]^2\{\frac{1}{[\exp(\frac{d}{\lambda}) + \frac{1}{A}]^2}\frac{\frac{1}{A} - \exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda}) - \frac{1}{A}}$$

$$- \frac{3}{[\exp(\frac{u}{\lambda}) - \frac{1}{A}]^2}\frac{\exp(\frac{u}{\lambda}) + \frac{1}{A}}{\exp(\frac{d}{\lambda}) + \frac{1}{A}}\}$$

We denote

$$L(s) = \frac{1}{[\exp(\frac{d}{\lambda}) + s]^2}\frac{s - \exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda}) - s} - \frac{3}{[\exp(\frac{u}{\lambda}) - s]^2}\frac{\exp(\frac{u}{\lambda}) + s}{\exp(\frac{d}{\lambda}) + s}$$

where $s \in [\exp(-\frac{M}{\lambda}), \exp(-\frac{D}{\lambda})]$. According to Assumption 1, it can be known that $\exp(\frac{u}{\lambda}) > \exp(-\frac{D}{\lambda}) > s$ and $s > \exp(-\frac{M}{\lambda}) > \exp(\frac{d}{\lambda})$ and

$$[\exp(\frac{u}{\lambda}) - s][s - \exp(\frac{d}{\lambda})] < 3[\exp(\frac{u}{\lambda}) + s][\exp(\frac{d}{\lambda}) + s]$$

which equivalently indicates that $\forall s \in [\exp(-\frac{M}{\lambda}), \exp(-\frac{D}{\lambda})]$, $L(s) < 0$.

Note that $L(s) < 0$ equivalently indicates that $T'(A) < 0$, and we find that the following:

Case 1: if $T(\exp(\frac{D}{\lambda})) < \frac{p}{1-p}\exp(\frac{d-u}{\lambda})$, i.e. $p > \frac{T(\exp(\frac{D}{\lambda}))}{\exp(\frac{d-u}{\lambda}) + T(\exp(\frac{D}{\lambda}))}$, then $f''(q) > 0$. Therefore, in Case 1, $f'(q)$ increases with respect to $q \in [0, 1]$ (see Figure C.3).

Case 2: if $T(\exp(\frac{D}{\lambda})) > \frac{p}{1-p}\exp(\frac{d-u}{\lambda})$ and $T(\exp(\frac{M}{\lambda})) < \frac{p}{1-p}\exp(\frac{d-u}{\lambda})$, i.e. $\frac{T(\exp(\frac{M}{\lambda}))}{\exp(\frac{d-u}{\lambda}) + T(\exp(\frac{M}{\lambda}))} < p < \frac{T(\exp(\frac{D}{\lambda}))}{\exp(\frac{d-u}{\lambda}) + T(\exp(\frac{D}{\lambda}))}$, then there exists an $\bar{A} \in [\exp(\frac{D}{\lambda}), \exp(\frac{M}{\lambda})]$ such that

$$\frac{p}{1-p}\exp(\frac{d-u}{\lambda}) = [\frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{\exp(\frac{d}{\lambda}) + \frac{1}{\bar{A}}} + 1][\frac{\exp(\frac{u}{\lambda}) - \exp(\frac{d}{\lambda})}{\exp(\frac{u}{\lambda}) - \frac{1}{\bar{A}}} - 1]^3$$

Recall the definition of $A \equiv \exp(\frac{M-(M-D)q}{\lambda})$. Therefore, $\bar{A}$ corresponds to a unique value of q and we denote it by $\bar{q}$. According to the above analysis, if $A < \bar{A}$, i.e. $q > \bar{q}$,

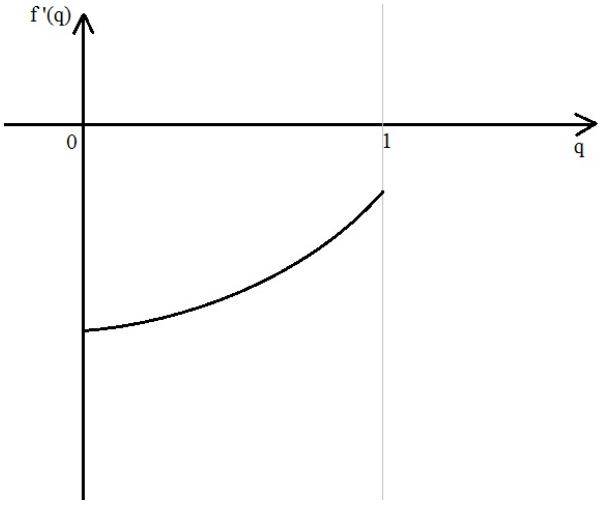


Figure C.3: *A geometric illustration of the shape of $f'(q)$ in Case 1. This figure can only express the monotonicity of $f'(q)$, but that is enough for the remaining proofs.*

then $f''(q) < 0$, and if $A > \bar{A}$, i.e. $q < \bar{q}$, then $f''(q) > 0$. Therefore, for $q \in [0, \bar{q}]$, $f'(q)$ increases, and for $q \in [\bar{q}, 1]$, $f'(q)$ decreases (see Figure C.4).

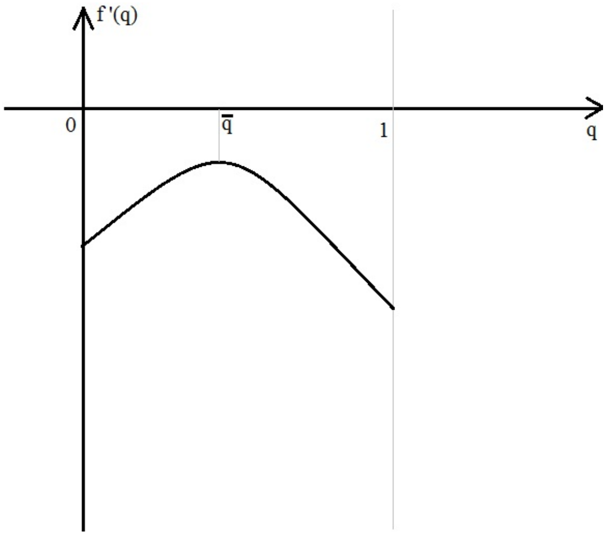


Figure C.4: *A geometric illustration of the shape of $f'(q)$ in Case 2. Similar to Figure C.3, this figure can only express the monotonicity of $f'(q)$, but that is enough for the remaining proofs.*

Case 3: if $T(\exp(\frac{M}{\lambda})) > \frac{p}{1-p}\exp(\frac{d-u}{\lambda})$, i.e. $p < \frac{T(\exp(\frac{M}{\lambda}))}{\exp(\frac{d-u}{\lambda})+T(\exp(\frac{M}{\lambda}))}$, then $f''(q) < 0$ for all $q \in [0,1]$. Therefore, in Case 3, $f'(q)$ decreases with respect to $q \in [0,1]$ (see

Figure C.5).

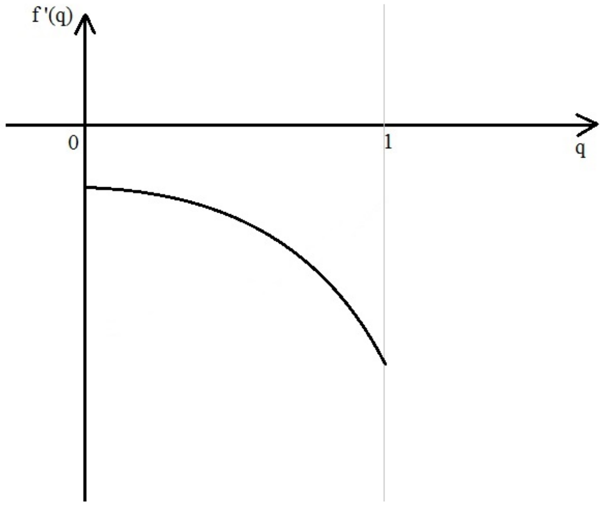


Figure C.5: *A geometric illustration of the shape of $f'(q)$ in Case 3. Similar to Figures C.3 and C.4, this figure can only express the monotonicity of $f'(q)$, but that is enough for the remaining proofs.*

The sign of $\frac{\partial f'(q)}{\partial \lambda}$: Recall the formula of $\frac{\partial f'(q)}{\partial \lambda}$:

$$\frac{\partial f'(q)}{\partial \lambda} = \frac{d\Phi(\lambda)\frac{M-D}{\lambda}}{d\lambda} = \frac{M-D}{\lambda}[\Phi'(\lambda) - \frac{\Phi(\lambda)}{\lambda}]$$

$$= \frac{M-D}{\lambda}\{p[\frac{c(d)}{\lambda^2}\exp(\frac{c(d)}{\lambda})\phi(\exp(\frac{c(d)}{\lambda})) + \frac{\exp(\frac{c(d)}{\lambda})}{\lambda[1-\exp(\frac{c(d)}{\lambda})]^2}]$$

$$+(1-p)[\frac{c(u)}{\lambda^2}\exp(\frac{c(u)}{\lambda}\phi(\exp(\frac{c(u)}{\lambda})) + \frac{\exp(\frac{c(u)}{\lambda})}{\lambda[1-\exp(\frac{c(u)}{\lambda})]^2}]\}$$

Specifically, for $\Phi'(\lambda) - \frac{\Phi(\lambda)}{\lambda}$ the above equation can be rewritten it into the following expression:

$$\Phi'(\lambda) - \frac{\Phi(\lambda)}{\lambda} = \frac{p}{\lambda}\frac{\exp(\frac{c(d)}{\lambda})}{[1-\exp(\frac{c(d)}{\lambda})]^2}[\frac{c(d)}{\lambda}\frac{1+\exp(\frac{c(d)}{\lambda})}{1-\exp(\frac{c(d)}{\lambda})} + 1]$$

$$+\frac{1-p}{\lambda}\frac{\exp(\frac{c(u)}{\lambda})}{[1-\exp(\frac{c(u)}{\lambda})]^2}[\frac{c(u)}{\lambda}\frac{1+\exp(\frac{c(u)}{\lambda})}{1-\exp(\frac{c(u)}{\lambda})} + 1]$$

We define $(A) = \frac{c(d)}{\lambda}\frac{1+\exp(\frac{c(d)}{\lambda})}{1-\exp(\frac{c(d)}{\lambda})}+1$ and $(B) = \frac{c(u)}{\lambda}\frac{1+\exp(\frac{c(u)}{\lambda})}{1-\exp(\frac{c(u)}{\lambda})}+1$. In the following, we first prove that $(A) < 0\ \forall\lambda \in (0,+\infty)$, and then prove that $(B) < 0\ \forall\lambda \in (0,+\infty)$.

1) Define $t = \frac{c(d)}{\lambda}$. For all $\lambda \in (0,+\infty)$, $t \in (-\infty,0)$. Define equation $A(t) = (t-1)\exp(t)+1+t$. Hence, $A'(t) = t\exp(t)+1$.

Define equation $\xi(t) = t\exp(t)$. Hence, $\xi'(t) = (1+t)\exp(t)$. According to the first-order derivative, it can be known that the minimum value of $\xi(t)$ is taken at $t = -1$. It is $\xi(-1) = -\exp(-1) > -1$.

For all $t \in (-\infty,0)$, $t\exp(t) > -1$ and hence $A'(t) > 0$ for all $t \in (-\infty,0)$. Therefore, the maximum value of $A(t)$ is taken at $t = 0$, which is $A(0) = 0$. Hence, $A(\frac{c(d)}{\lambda}) < 0$ for all $\lambda \in (0,+\infty)$.

By re-arranging $A(\frac{c(d)}{\lambda}) < 0$, we can obtain that $(A) = \frac{c(d)}{\lambda}\frac{1+\exp(\frac{c(d)}{\lambda})}{1-\exp(\frac{c(d)}{\lambda})}+1 < 0$ for all $\lambda \in (0,+\infty)$.

2) Define $t = \frac{c(u)}{\lambda}$. For all $\lambda \in (0,+\infty)$, $t \in (0,+\infty)$. Define equation $B(t) = (t-1)\exp(t)+1+t$. Hence, $B'(t) = t\exp(t)+1$.

For all $t \in (0,+\infty)$, $B'(t) > 0$; therefore, the minimum value of $B(t)$ is taken at $t = 0$, which is $B(0) = 0$. Hence, for all $\lambda \in (0,+\infty)$, $B(\frac{c(u)}{\lambda}) > 0$.

By re-arranging $B(\frac{c(u)}{\lambda}) > 0$, we can obtain that $(B) = \frac{c(u)}{\lambda}\frac{1+\exp(\frac{c(u)}{\lambda})}{1-\exp(\frac{c(u)}{\lambda})}+1 < 0$ for all $\lambda \in (0,+\infty)$.

Therefore, in conclusion, $\forall\lambda \in (0,+\infty)$, $(A) < 0$ and $(B) < 0$, which implies that $\Phi'(\lambda) - \frac{\Phi(\lambda)}{\lambda} < 0$, and hence $\frac{\partial f(q^*)}{\partial\lambda} < 0$.

From the previous proof of the range of λ that ensures $|f'(q)| < 1$, we have obtained a preliminary result that given the opponent's strategy $q \in [0,1]$, where $\lambda \in (0,\varepsilon)$, $f'(q) > -1$. Moreover, from the proof of the shape of $f'(q)$, we found that $f'(q)$, in terms of monotonicity, has three kinds of shapes. Based on the analysis by far, it can be learnt that irrespective of the shape of $f'(q)$, given a $q \in [0,1]$, when λ increases from 0, $f'(q)$ will decrease from a value that is greater than -1.

Therefore, according to the proof of shapes of $f'(q)$, combining the above analysis, we obtain the necessary and sufficient condition to ensure $f(q)$ is a contraction function. As a first step, we determine the necessary and sufficient condition to make $f(q)$ a contraction if as λ increases, $f'(q)$ always exhibits a single type of shape (remember there are three possibilities). This condition is still expressed in three cases, consistent with the three possible shapes of $f'(q)$. They are given in terms of the value of λ that ensures $f(q)$ is a contraction:

Case 1: $0 < \lambda \leq \tilde{\lambda}$, where $\tilde{\lambda}$ is the unique solution of $f'(0) = -1$;

Case 2: $0 < \lambda \leq \tilde{\lambda}$, where $\tilde{\lambda}$ is the minimum value between the $\tilde{\lambda}$ which is the unique solution of $f'(0) = -1$ and the $\tilde{\lambda}$ which is the unique solution of $f'(1) = -1$;

Case 3: $0 < \lambda \leq \tilde{\lambda}$, where $\tilde{\lambda}$ is the unique solution $f'(1) = -1$.

As λ increases from 0, we could meet either Case 1, Case 2 or Case 3. The shape of $f'(q)$ may not be confined within a single type for every value of λ. However, no matter what situation it is, the necessary and sufficient condition to ensure $f(q)$ as a contraction is always that, given other primitives,

$$\lambda \in (0, \tilde{\lambda}]$$

where $\tilde{\lambda}$ is the minimum value between the $\tilde{\lambda}$ which is the unique solution of $f'(0) = -1$ ($\tilde{\lambda} = f'^{-1}_{q^*}(-1; q^* = 0)$ in this chapter) and the $\tilde{\lambda}$ which is the unique solution of $f'(1) = -1$ ($\tilde{\lambda} = f'^{-1}_{q^*}(-1; q^* = 1)$ in the chapter) . In one word, it is the value of $f'(q)$ at $q = 0$ or 1 that decides whether $f(q)$ is a contraction.

If $\lambda \in (0, \tilde{\lambda}]$ and $\lambda^* \in (0, \tilde{\lambda}^*]$, where λ may not be equal to λ^*, then the game is dominance solvable. In this situation, both players' best response functions are contraction and the best response could be $q = f(q^*) \times 1\{0 \leq f(q^*) \leq 1\}$, i.e. the information-acquisition best response, or $q = 0$ or 1 that is obtained by comparing the ex ante expected payoff given the opponent's strategy. *Q.E.D.*

Proof of Corollary 1: (1) For $M + pu + (1-p)d > 0$ and $D + pu + (1-p)d <$

0, if the game turns into a complete information game, then at $q^* = \frac{M+pu+(1-p)d}{M-D}$, $f'_{q^*}(q^*, \lambda) = +\infty$.

If $(1-q^*)M + q^*D + pu + (1-p)d = 0$, $\bar{\lambda}_{q^*} = +\infty$. Therefore, at $q^* = \frac{M+pu+(1-p)d}{M-D}$, $\bar{\lambda}_{q^*} = +\infty$. For the other value of q^*, the corresponding $\bar{\lambda}_{q^*} < +\infty$. Therefore, in this situation, $\lambda_c = +\infty$.

Hence, $q^* = \frac{M+pu+(1-p)d}{M-D}$ is the last point at which $f'_{q^*}(q^*, \lambda)$ reaches $+\infty$.

(2) If $(1-q^*)M + q^*D + pu + (1-p)d > 0$, $\bar{\lambda}_{q^*}$ is determined by $G(\bar{\lambda}_{q^*}, q^*) = 1$. It is known that $\frac{\partial G(\lambda,q^*)}{\partial q^*} > 0$ and $\frac{\partial G(\lambda,q^*)}{\partial \lambda}|_{\lambda=\bar{\lambda}_{q^*}} < 0$. Therefore, according to the implicit function theorem, it is obtained that

$$\frac{\partial \bar{\lambda}_{q^*}}{\partial q^*} = -\frac{\frac{\partial G(\lambda,q^*)}{\partial q^*}}{\frac{\partial G(\lambda,q^*)}{\partial \lambda}|_{\lambda=\bar{\lambda}_{q^*}}} > 0$$

If for all $q^* \in [0,1]$, $(1-q^*)M + q^*D + pu + (1-p)d > 0$, thus $D + pu + (1-p)d > 0$. In this situation, λ_c is the solution of the equation $G(\lambda_c, q^* = 1) = 1$, i.e.

$$p\exp(-\frac{D+u}{\lambda_c}) + (1-p)\exp(-\frac{D+d}{\lambda_c}) = 1$$

At $\lambda = \lambda_c$, $G(\lambda, q^* = 1) = 1$ is invertible. Therefore, $\lambda_c = G^{-1}(1; q^* = 1)$.

(3) If $(1-q^*)M + q^*D + pu + (1-p)d < 0$, $\bar{\lambda}_{q^*}$ is determined by $F(\bar{\lambda}_{q^*}, q^*) = 1$. It is known that $\frac{\partial F(\lambda,q^*)}{\partial q^*} < 0$ and $\frac{\partial F(\lambda,q^*)}{\partial \lambda}|_{\lambda=\bar{\lambda}_{q^*}} < 0$. Therefore, according to the implicit function theorem, it is obtained that

$$\frac{\partial \bar{\lambda}_{q^*}}{\partial q^*} = -\frac{\frac{\partial F(\lambda,q^*)}{\partial q^*}}{\frac{\partial F(\lambda,q^*)}{\partial \lambda}|_{\lambda=\bar{\lambda}_{q^*}}} < 0$$

If for all $q^* \in [0,1]$, $(1-q^*)M + q^*D + pu + (1-p)d < 0$, then $M + pu + (1-p)d < 0$. In this situation, λ_c is the solution of the equation $F(\lambda_c, q^* = 0) = 1$, i.e.

$$p\exp(\frac{M+u}{\lambda_c})+(1-p)\exp(\frac{M+d}{\lambda_c})=1$$

At $\lambda=\lambda_c$, $F(\lambda,q^*=0)=1$ is invertible. Therefore, $\lambda_c=F^{-1}(1;q^*=0)$. *Q.E.D.*

Proof of Proposition 7: In this proof, we still omit the notation "*" for simplicity. It will not affect our understanding of the proof because this notation only appears at the opponent's strategy q^*, and q^* only plays the role as an independent variable of a function. Recalling the equation for $f(q)$ defined in the proof of Proposition 5,

$$f(q)=\frac{p}{1-\exp(\frac{M+d}{\lambda}-\frac{M-D}{\lambda}q)}+\frac{1-p}{1-\exp(\frac{M+u}{\lambda}-\frac{M-D}{\lambda}q)}$$

and hence the best response function, equation (8) in the paper, can be written as

$$q^*(q)=0\times 1\{f(q)<0\}+f(q)\times 1\{0\le f(q)\le 1\}+1\times 1\{f(q)>1\}$$

In the following, we denote

$$A(q^*(q))=\frac{p}{1-\exp(\frac{M+d}{\lambda}-\frac{M-D}{\lambda}q^*(q))}+\frac{1-p}{1-\exp(\frac{M+u}{\lambda}-\frac{M-D}{\lambda}q^*(q))}$$

and thus the 2nd iteration of best response functions is given by

$$g(q^*(q))=0\times 1\{A(q^*(q))<0\}+A(q^*(q))\times 1\{0\le A(q^*(q))\le 1\}+1\times 1\{A(q^*(q))>1\}$$

The equilibria of this game should be the solutions of the following equation:

$$q=g(q^*(q))$$

where $q\in[0,1]$ and in particular, we denote the symmetric equilibrium by (s,s). It should satisfy

$$s=q^*(s)=g(q^*(s))$$

where $s\in[0,1]$.

The first- and second-order derivatives of $g(q^*(q))$ are given by

$$\frac{dg(q^*(q))}{dq} = \frac{dA(q^*(q))}{dq^*(q)} \times \frac{df(q)}{dq} \times 1\{0 \leq f(q) \leq 1\} \times 1\{0 \leq A(q^*(q)) \leq 1\} \geq 0$$

and

$$\frac{d^2g(q^*(q))}{dq^2} = 1\{0 \leq f(q) \leq 1\} \times 1\{0 \leq A(q^*(q)) \leq 1\} \times \{\frac{d^2A(q^*(q))}{dq^*(q)^2} \times (\frac{df(q)}{dq})^2 + \frac{dA(q^*(q))}{dq^*(q)} \times \frac{d^2f(q)}{dq^2}\}$$

For symmetric equilibrium $s \in (0,1)$, we have the following relationship:

$$\frac{df(q)}{dq}|_{q=s} = \frac{dq^*(q)}{dq}|_{q=s} = \frac{dA(q^*(q))}{dq^*(q)}|_{q=s} = \frac{dg(q^*(q))}{dq^*(q)}|_{q=s}$$

Rearranging the RHS of equation $\frac{dg(q^*(q))}{dq}$, we obtain

$$\frac{dg(q^*(q))}{dq} = \frac{dA(q^*(q))}{dq^*(q)} \times 1\{0 \leq A(q^*(q)) \leq 1\} \times \frac{df(q)}{dq} \times 1\{0 \leq f(q) \leq 1\}$$

Define set $h := \{q|0 \leq f(q) \leq 1 \text{ and } 0 \leq A(q^*(q)) \leq 1\}$. The function $A(q^*(q))$ where $q \in h$, or equivalently $g(q^*(q))$ where $q \in h$, represents the situation in which both players play the game by acquiring information.

The shape of $g(q^*(q))$**:** There should be four types of best response functions according to whether a player makes the best response at $q^* = 0$ or 1 by acquiring information. Then, based on these four types of best response functions, we can obtain four possible shapes of $g(q^*(q))$ in the symmetric games. As we will present below, these four possible types only reflect monotonicity and information acquisition behaviour, but these properties are enough for the proof. Irrespective of the changes in λ, a particular shape of $q^*(q)$, and hence $g(q^*(q))$, will always belong to one of the four possibilities.

Scenario 1: In this scenario, there are some q in $q^*(q)$ such that $f(q) < 0$, and some q in $q^*(q)$ such that $f(q) > 1$. According to the decreasing property of $f(q)$, $f(q) < 0$ and $f(q) > 1$ must happen at the two extreme parts of $f(q)$. We denote the interval of q that makes $f(q) \in (0,1)$ by $\alpha = (a,b) \subset (0,1)$. This scenario is depicted by Figure C.6-1.

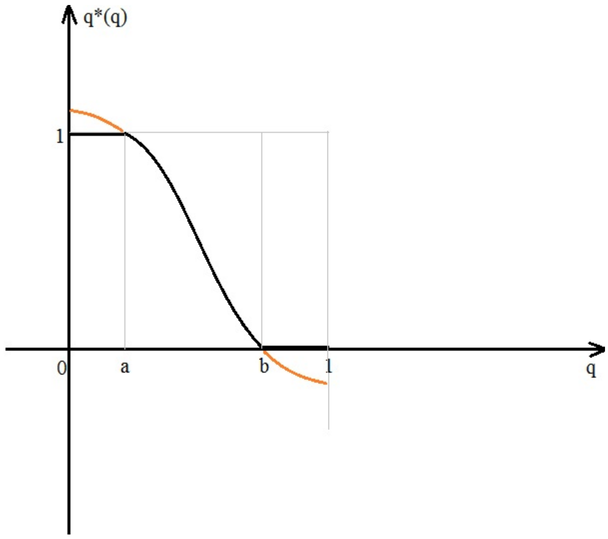


Figure C.6-1: *The first possible type of $q^*(q)$. The red curve represents the generic $f(q)$ function.*

For $q \in (a,b)$, the player makes the best response by acquiring information and hence $q^*(q) = f(q) \in (0,1)$. According to the construction of the 2nd iteration algorithm, in order to obtain $A(q^*(q))$ where $q \in h$, rotate $q^*(q)$ $\forall q \in (a,b)$ around $q = s$ by 180° and stretch as well as compress relevant parts to make the rotated parts fit the interval $[q^{*-1}(b), q^{*-1}(a)]$. According to the continuity and increasing property of $A(q^*(q))$, it can be expected that for $q \in (0, q^{*-1}(b))$, $A(q^*(q)) < 0$ and for $q \in (q^{*-1}(a), 1)$, $A(q^*(q)) > 1$. Therefore, for $g(q^*(q))$, $\forall q \in (0, q^{*-1}(b))$, $A(q^*(q)) = 0$, and $\forall q \in (q^{*-1}(a), 1)$, $g(q^*(q)) = 1$. Figure C.6-2 exhibits the shape of $g(q^*(q))$ geometrically.

For $q \in (q^{*-1}(b), q^{*-1}(a))$, $g(q^*(q))$ that is obtained by rotating $q^*(q)$ for $q \in (a,b)$ should inherit the following property: that $\frac{df'(q)}{d\lambda}$ decreases as λ increases, given any $q \in [0,1]$, and correspondingly, $\frac{dA(q^*(q))}{dq}$ and (hence $\frac{dg(q^*(q))}{dq}$) increases, and $\frac{dA(q^*(q))}{dq}$ (and hence $\frac{dg(q^*(q))}{dq}$) begins going above 1 at either $q = 0$ or 1. The detailed derivation and analysis of $\frac{dg(q^*(q))}{dq}$ will be discussed later.

Scenario 2: In this scenario, there are some q of $q^*(q)$ such that $f(q) < 0$, and for the remaining q, $q^*(q) = f(q) \in (0,1)$, i.e. the player's best response is made by acquiring information. According to the decreasing property of $q^*(q)$, we denote this interval supporting information acquisition by $\beta = [0,a) \in [0,1]$. This scenario is de-

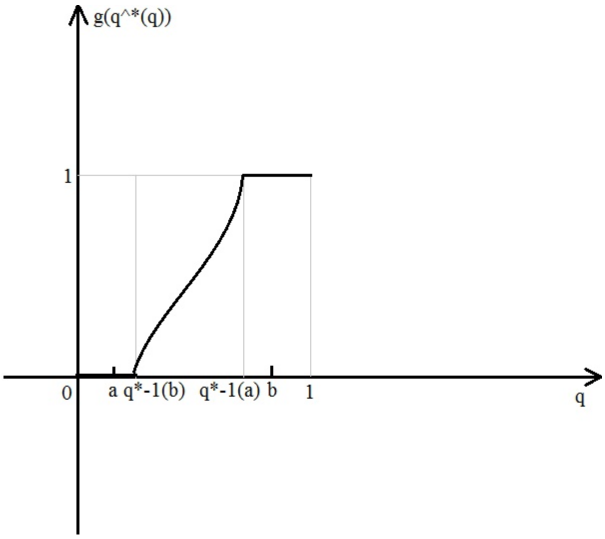


Figure C.6-2: *The first possible type of $g(q^*(q))$. It reflects monotonicity and information acquisition behaviour. $\forall q \in (0, q^{*-1}(b))$ and $\forall q \in (q^{*-1}(a), 1)$, there is at least one player playing the game without acquiring information, indicated by the horizontal line.*

picted by Figure C.7-1.

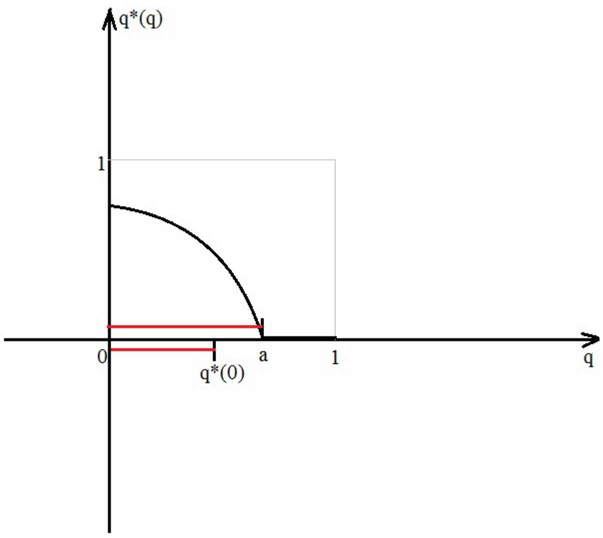
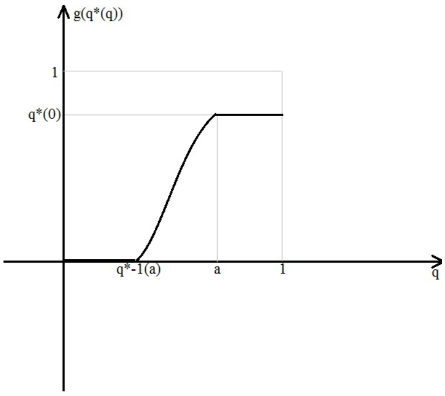
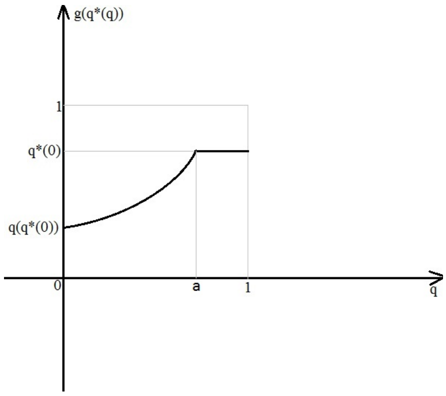


Figure C.7-1: *The second possible type of $q^*(q)$. The red lines indicate two possible parts of $q^*(q)$ that are used to obtain the shape of $g(q^*(q))$. Which part will be used depends on the relative magnitude of a and $q^*(0)$.*

In this scenario, for $q \in [0, a)$, the player makes the best response by acquiring



(a) $q^{*-1}(a) > 0$



(b) $q^{*-1}(a) < 0$

Figure C.7-2: *The second possible type of $g(q^*(q))$. It has two possible shapes which depends on the relative magnitude of $q^{*-1}(a)$ and 0. These figures reflects monotonicity of $g(q^*(q))$ and players' information acquisition behaviour.*

information. According to the construction of the 2nd iteration algorithm, in order to obtain $A(q^*(q))$ where $q \in h$, rotate $q^*(q)$ $\forall q \in [0, \min\{a, q^*(0)\}]$ around $q = s$ by 180°, and compress as well as stretch relevant parts to fit the interval $[\max\{q^{*-1}(a), 0\}, a]$. Therefore, we obtain the $A(q^*(q))$ where $q \in h$. For $q \in [0, \max\{q^{*-1}(a), 0\})$, $g(q^*(q)) = 0$ if $q^{*-1}(a) > 0$ and $g(q^*(q)) = q(q^*(0))$ if $q^{*-1}(a) < 0$. For $q \in [a, 1]$, $g(q^*(q)) = q^*(0)$. This result implies that in Scenario 2, for $q \in [0, \max\{q^{*-1}(a), 0\})$ or $q \in [a, 1]$, there is at least one player not acquiring information when playing the game. Therefore, we obtain the shape of function $g(q^*(q))$ which reflects monotonicity and information acquisition behaviour. It is given by Figure C.7-2.

As in Scenario 1, in Scenario 2, the property of $\frac{dA(q^*(q))}{dq}$ (and hence $\frac{dg(q^*(q))}{dq}$) inherits the property of $\frac{df(q)}{dq}$ (and hence $\frac{dq^*(q)}{dq}$) since the former is in principle a rotated image of the latter. In particular, by inheriting the property of $\frac{df'(q)}{d\lambda} < 0$, correspondingly, as λ increases, $\frac{dA(q^*(q))}{dq}$ increases $\forall q \in [0, 1]$ and it is at $q = 0$ or 1 that $\frac{dA(q^*(q))}{dq}$ begins increasing above 1. The detailed derivation and analysis of $\frac{dg(q^*(q))}{dq}$ will be discussed later.

Scenario 3: In this scenario, there are some q of $q^*(q)$ such that $f(q) > 1$, and for the remaining q, $q^*(q) = f(q) \in (0, 1)$, i.e. the player's best response is made by acquiring information. We denote the interval about information-acquisition behaviour by $r = [b, 1] \subset [0, 1]$. This scenario is depicted by Figure C.8-1.

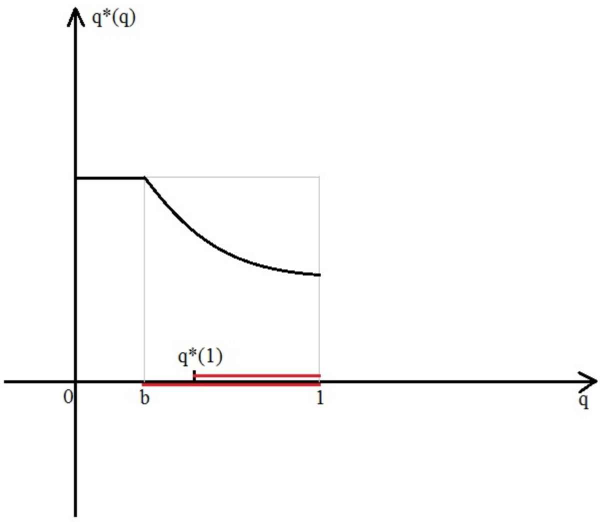
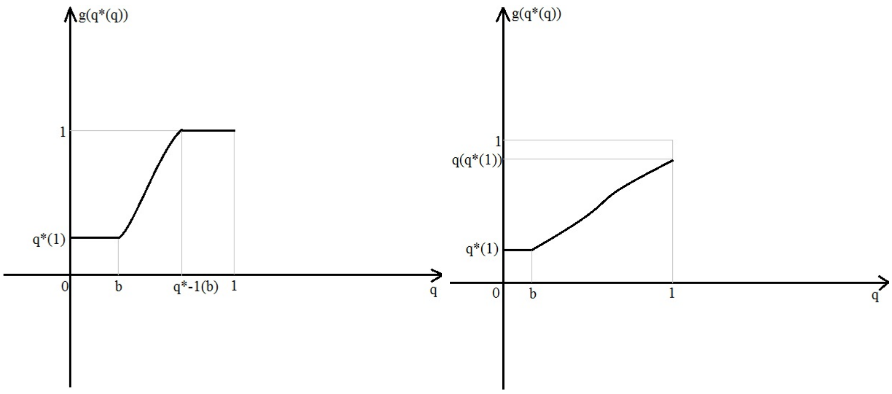


Figure C.8-1: *The third possible type of $q^*(q)$. The red lines denote two possible parts that are used to obtain the shape of $g(q^*(q))$. Which part will be used depends on the relative magnitude of b and $q^*(1)$.*

For $q \in [b,1]$, the player makes the best response by acquiring information. In order to obtain $A(q^*(q))$ where $q \in h$, rotate $q^*(q)$ for all $q \in [\max\{b, q^*(1)\}, 1]$ around $q = s$ by 180°, and compress as well as stretch relevant parts to fit $[b, \min\{1, q^{*-1}(b)\}]$. Therefore, we obtain the shape of $A(q^*(q))$ where $q \in h$, which is the part of $g(q^*(q))$ in which both players play the game by acquiring information. For all $q \in [0,b]$, $g(q^*(q)) = q^*(1)$ and for all $q \in (q^{*-1}(b), 1)$, $g(q^*(q)) = 1$ if $q^*(1) < b$. This result implies that in Scenario 3, for $q \in [0,b]$ or $q \in [\min\{q^{*-1}(b), 1\}, 1]$, there is at least one player not acquiring information when playing the game. Therefore, we obtain the shape (geometric expression) of $g(q^*(q))$ which reflects monotonicity and information acquisition behaviour. They are given by Figure C.8-2.

Scenario 4: In this scenario, for all $q \in [0,1]$, $q^*(q) = f(q) \in [0,1]$, i.e. in the symmetric game, both players always play the game by acquiring information. This scenario is depicted by Figure C.9-1.

In order to obtain $A(q^*(q))$ where $q \in h$, rotate the part of $q^*(q)$ corresponding to $[q^*(1), q^*(0)]$ around $q = s$ by 180°, and compress as well as stretch relevant parts in order to fit the interval $[0,1]$. Without loss of generality, we obtain the shape of function $A(q^*(q))$ where $q \in h$ and hence $g(q^*(q))$. It is given by Figure C.9-2.



(a) $q^{*-1}(b) < 1$ (b) $q^{*-1}(b) > 1$

Figure C.8-2: *The third possible type of $g(q^*(q))$. It has two possible shapes which depends on the relative magnitude between $q^{*-1}(b)$ and 1. These figures reflect monotonicity of $g(q^*(q))$ and players' information acquisition behaviour*

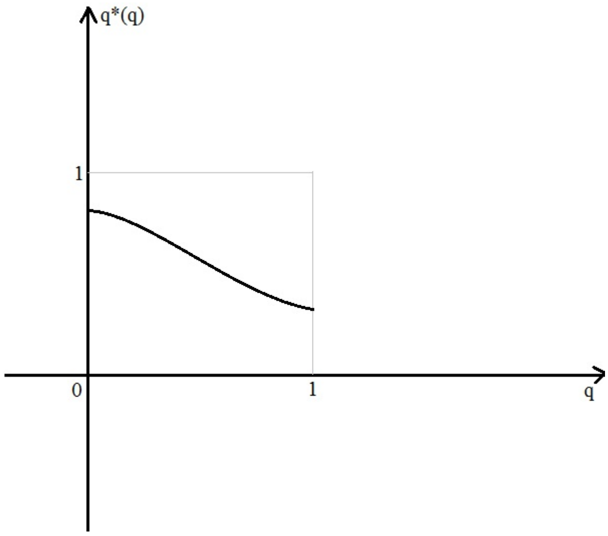


Figure C.9-1: *The fourth possible type of $q^*(q)$*

The property of $\frac{dA(q^*(q))}{dq}$ (and hence $\frac{dg(q^*(q))}{dq}$) inherits the property of $\frac{df(q)}{dq}$ (and hence $\frac{dq^*(q)}{dq}$) since the former is in principle a rotated image of the latter. In particular, it inherits the following property: that as λ increases, $\frac{dA(q^*(q))}{dq}$ increases for all $q \in [0,1]$ and it is at $q = 0$ or $q = 1$ that $\frac{dA(q^*(q))}{dq}$ first begins increasing above 1. The detailed derivation and analysis of $\frac{dg(q^*(q))}{dq}$ will be discussed later.

$\frac{dg(q^*(q))}{dq}$ and equilibria of the game: As we have derived, for $\lambda \in (0, \tilde{\lambda})$, the best response functions should be contraction. Therefore, the 2nd iteration of the best re-

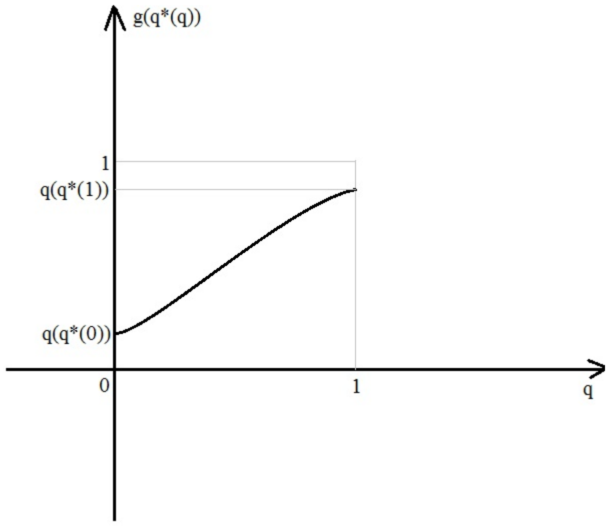


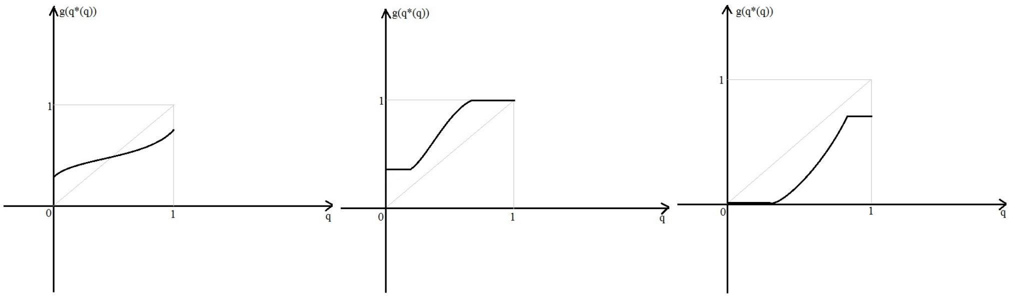
Figure C.9-2: *The fourth possible type of $g(q^*(q))$. As in previous cases, this figure can only reflect monotonicity of $g(q^*(q))$ and players' information acquisition behaviour.*

sponse functions should be contraction as well (recall the equation of $\frac{dg(q^*(q))}{dq}$). The contraction 2nd iteration can be categorized into the following three situations in terms of whether players acquire information at the symmetric equilibrium when the best response functions are contraction functions (see Figure C.10).[1]

However, (2) and (3) are impossible to happen. When $\lambda = 0$, the equilibrium is (p, p), while p is the prior probability and it is assumed that $p \in (0,1)$. For $\lambda \in (0, \varepsilon)$, where ε is a small enough positive number, the information cost is small; hence, the new solution (p', p') should be close to (p, p) and $p' \in (0,1)$. Hence, for the 2nd iteration algorithm we start with, it must be in the form of sub-figure (1) of Figure C.10, where the intersection point between $g(q^*(q))$ and 45° line is between 0 and 1. Moreover, the intersection point is realized by the $A(q^*(q))$ function, not horizontal lines, which implies that both players acquire information at the symmetric equilibrium.

Sub-figure (1) in Figure C.10 represents a situation where for all $q \in [0,1]$, both players play the game by acquiring information. According to previous analysis, in re-

[1]These figures can generally represent monotonity and information acquisition behaviour of players in such games, but as in most previous figures, by far these figures do not necessarily reflect other more precise properties such as curvature of $q^*(q)$ and $g(q^*(q))$. However, we do not need these properties yet. The monotonicity and information-acquisition behaviour shown by these figures have been enough for us in the current proof.



(1) **(2)** **(3)**

Figure C.10: *The three possible cases of the 2nd iteration algorithm $g(q^*(q))$ for $\lambda \in (0, \tilde{\lambda})$ such that the best response function $q^*(q)$ and hence $g(q^*(q))$ are contraction. In (1), in the symmetric equilibrium, both players acquire information. In (2), both players do not acquire information when making decisions and the equilibrium is (1, 1). In (3), both players do not acquire information when making decisions as well and the equilibrium is (0, 0). In (1), (2) and (3), the figures reflect the monotonicity, information-acquisition behaviour of players (at the symmetric equilibrium), and the contraction property of $g(q^*(q))$.*

ality, we have additional two possibilities describing the shape of contraction $g(q^*(q))$. First, for qs close to 0, there is at least one player playing the game without acquiring information; this part is represented by a horizontal line in sub-figure (1)-1 of Figure C.11. Second, for qs close to 1, there is at least one player playing the game without acquiring information; this part is represented by a horizontal line in sub-figure (1)-2 of Figure C.11.

Now, we consider the situation that multiple equilibria begin to arise. As we may start from situation (1) (in Figure C.10), (1)-1 or (1)-2 (in Figure C.11), and each situation has three possibilities in terms of the three possible cases of $f'(q)$, we have to consider 9 possible first-time occurrences of multiple equilibria as λ increases to a level λ_1.[2] These 9 possible situations are described in Figure C.12.

It can be easily proven that in this game, when multiple equilibria arise, it is impossible that at the symmetric equilibrium, both players do not acquire information (a fact indicated by no horizontal line intersection at the symmetric equilibrium). Interested readers can prove this result by themselves.

[2] λ_1 is defined to describe the situation of the first-time occurrences of multiple equilibria.

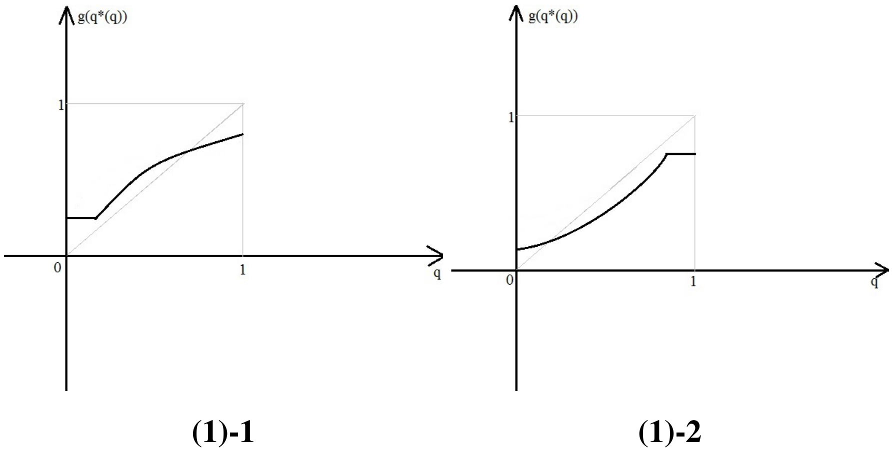
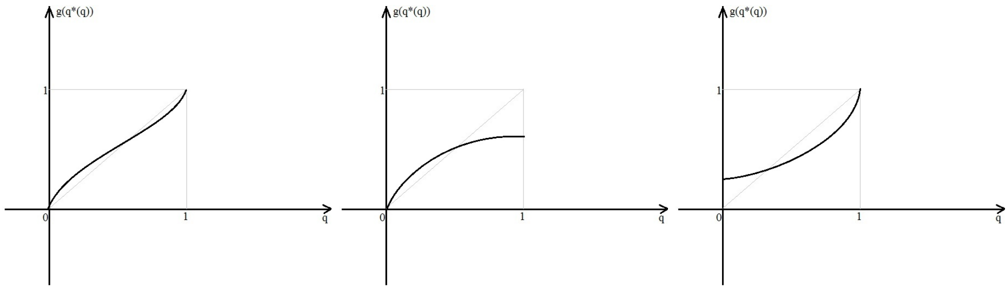


Figure C.11: *Two possible shapes of contraction $g(q^*(q))$. (1)-1 is the case that for qs close to 0, at least one player does not acquire information. (1)-2 is the case that for qs close to 1, at least one player does not acquire information.*

In all nine possibilities, the shapes that match with Cases 1 and 3 of $f'(q)$ are impossible to happen, since there are only two intersection points. The game exhibits strategic substitutes and the best response function is decreasing and continuous with respect to the opponent's strategies, and therefore, if multiple equilibria can arise, the number of equilibria must be odd. Therefore, in all nine possible situations, only cases corresponding to Case 2 can happen when multiple equilibria first appear as λ increases. Therefore, we only get three cases, corresponding to (1), (1)-1 and (1)-2, to describe the first-time appearance of multiple equilibria and the corresponding λ is the λ_1 as indicated from above. In a particular situation, irrespective of whether the multiplicity evolves from (1), (1)-1 or (1)-2, there are always three equilibria, and in the two asymmetric equilibria which are the outer intersection points between $g(q^*(q))$ and the 45° line, there is always one player not acquiring information.

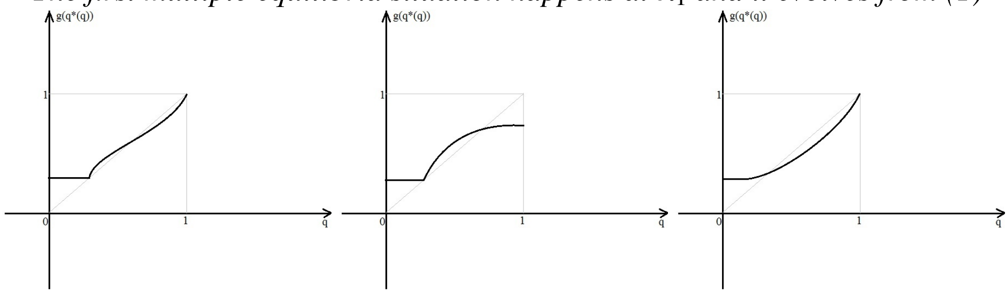
Another result we can derive from the above analysis is that for $\lambda \in (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon)$, where ε is an arbitrarily small positive number, the shape of $q^*(q)$ is always in line with Case 2. It is because of the continuity of $q^*(q)$ and hence $g(q^*(q))$ with respect to λ for all $q \in [0,1]$. In Case 2, the $g(q^*(q))$ where $q \in h$ first exhibits concavity and then exhibits convexity as q increases. We call this property the concavity–convexity property of $g(q^*(q))$.

Therefore, as λ increase away from λ_1, for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$, at each $q \in [0,1]$, $\frac{dg(q^*(q))}{dq}$ increases, and therefore, we get five equilibria in either situation evolving



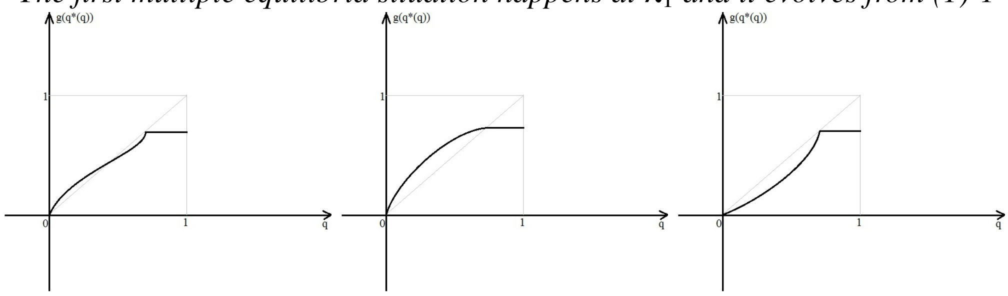
Case 2 **Case 1** **Case 3**

The first multiple equilibria situation happens at λ_1 and it evolves from (1)



Case 2 **Case 1** **Case 3**

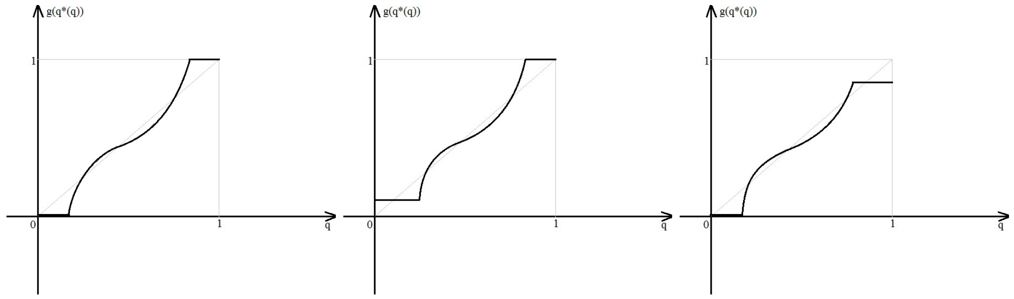
The first multiple equilibria situation happens at λ_1 and it evolves from (1)-1



Case 2 **Case 1** **Case 3**

The first multiple equilibria situation happens at λ_1 and it evolves from (1)-2

Figure C.12: *Nine possible situations describing the first multiple equilibria situations if multiple equilibria can happen in this symmetric game.*



(1) **(1)-1** **(1)-2**

Figure C.13: *The three possible cases of 5 equilibria corresponds to (1), (1)-1 and (1)-2. Based on the previous analysis, it is known that in this situation, the derivative of best response function $q^*(q)$ must follow the shape of Case 2.*

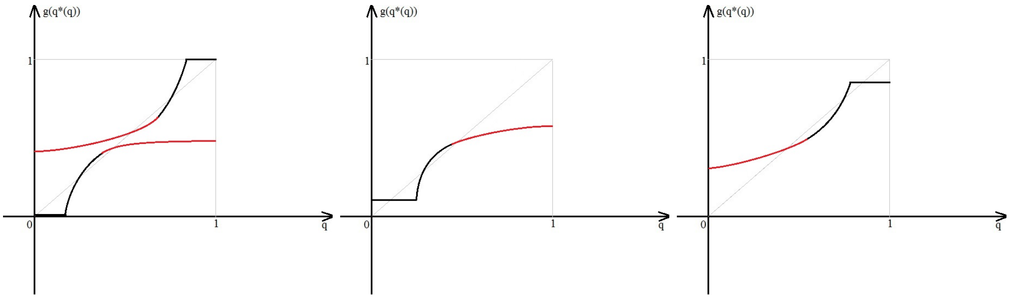
from (1), (1)-1 or (1)-2.[3] Figure C.13 shows a geometric description of these three 5-equilibria situations.

As λ increases, for all $q \in [0,1]$, $\frac{dg(q^*(q))}{dq}$ continues increasing, and geometrically, it indicates that the part of $g(q^*(q))$ in which both players acquire information will grow steeper. This part is essentially $A(q^*(q))$ where $q \in h$, not the horizontal lines. Besides, it can be obtained that as λ increases away from λ_1, before the symmetric equilibrium becomes unstable, $q^*(q)$'s shape is always in line with Case 2, and Cases 1 and 3 are impossible to happen. From the red curves in the following figures (see Figure C.14), it can be found that if Case 1 or Case 3 happens, there are three equilibria and the symmetric equilibrium becomes unstable. More importantly, in one asymmetric equilibrium, both players acquire information (the intersection point of the red curve with the 45° line), and in the other asymmetric equilibrium, at least one player does not acquire information. This contradicts with the symmetry property. Therefore, as λ increases away from λ_1, 5 equilibria will always be maintained and the symmetric equilibrium is stable. Until the symmetric equilibrium becomes unstable as λ increases, we can consider whether the shape of $q^*(q)$ may deviate from Case 2 to Case 1 or Case 3.

As λ increases, the next boundary situation that we need to consider is that at $\lambda = \lambda_2$, the stability of the symmetric equilibrium is not determined, i.e. $\frac{dq^*(q)}{dq}|_{q=s} = -1$; hence, $\frac{dg(q^*(q))}{dq}|_{q=s} = 1$.[4] According to the continuity of $\frac{dq^*(q)}{dq}$ with respect to λ, it can

[3] In the following, for simplicity, we will indicate any situation originating from situation (1) (in Figure C.10), (1)-1 or (1)-2 (in Figure C.11) by (1), (1)-1 or (1)-2. It does not affect the understanding of the proof.

[4] λ_2 is defined to describe this boundary situation.



(1) **(1)-1** **(1)-2**

Figure C.14: *The three possible cases of $g(q^*(q))$ before the symmetric equilibrium becomes unstable. The shapes of $g(q^*(q))$ function are consistent with Case 1 and Case 3. Each curve is combined with a red part and a black part.*

be known that at λ_2, the shape of Case 2 is still maintained.

In addition, by proof of contradiction, it can be shown that at $\lambda = \lambda_2$, $\frac{dg(q^*(q))}{dq}$ reaches its lowest value at $q = s$, which is $\frac{dg(q^*(q))}{dq} = 1$. For other values of q except s, $\frac{dg(q^*(q))}{dq} > 1$ based on its shape indicated by Case 2. Here is the proof. Suppose at $q = s$, the $\frac{dg(q^*(q))}{dq}$ has not reached its lowest value, which without loss of generality can be depicted by Figure C.15.

Around s, for $q \in (s-\varepsilon, s)$, $\frac{dg(q^*(q))}{dq} > 1$ and for $q \in (s, s+\varepsilon)$, $\frac{dg(q^*(q))}{dq} < 1$. Therefore, $(s,\ g(q^*(s)))$ turns to be a tangent point of $g(q^*(q))$ with 45° line (see Figure C.16).

According to the above analysis and Figure C.16, apparently this situation contradicts with the symmetry property of the strategic substitutes game. Therefore, at $\lambda = \lambda_2$, the value of $\frac{dq^*(q)}{dq}$ reaches its lowest value, which equals 1. Therefore, at $\lambda = \lambda_2$, for the value of $q \in (s-\varepsilon, s+\varepsilon)$, $\frac{dq^*(q)}{dq}$ should exhibit in the following shape (see Figure C.17).

Hence, at $\lambda = \lambda_2$, the situations following (1), (1)-1 and (1)-2 turn into the following situations, respectively (see Figure C.18). We can see that at $\lambda = \lambda_2$, $g(q^*(q))$ where $q \in h$ still exhibits the concavity–convexity property. Together with the situation of $\lambda = \lambda_1$, it can be concluded that at any boundary where the number of equilibria will change, $g(q^*(q))$ where $q \in h$ always exhibit the concavity–convexity property.

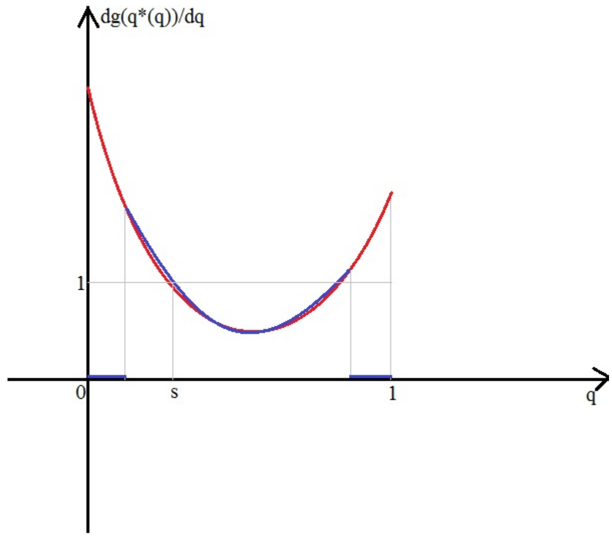


Figure C.15: *The red curve corresponds to the generic $A(q^*(q))$ function. As it is known, the $A(q^*(q))$ function is part of the $g(q^*(q))$ function, to which the piecewise function indicated in blue corresponds. The overlapped parts between the red curve and blue curve correspond to the function $A(q^*(q))$ where $q \in h$ and the symmetric equilibrium s belongs to the set h.*

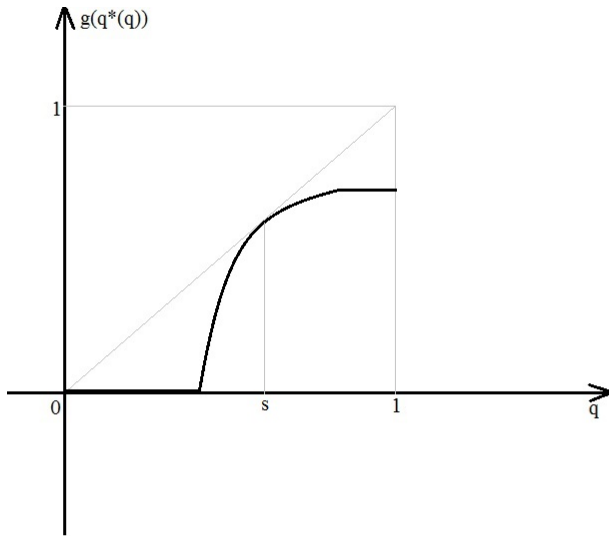


Figure C.16: *Following Figure C.15, in this situation, (s, $g(q^*(s))$) turns to be a tangent point of $g(q^*(q))$ with $45°$ line, and according to value of $\frac{dg(q^*(q))}{dq}$ around s, $g(q^*(q))$ must be located below the $45°$ line. In this symmetric strategic substitutes game, the number of equilibria must be odd, thus a contradiction arises.*

Therefore, at $\lambda = \lambda_2$, at symmetric equilibrium $(s,\, s)$, $\frac{dq^*(q)}{dq} = -1$, and there are

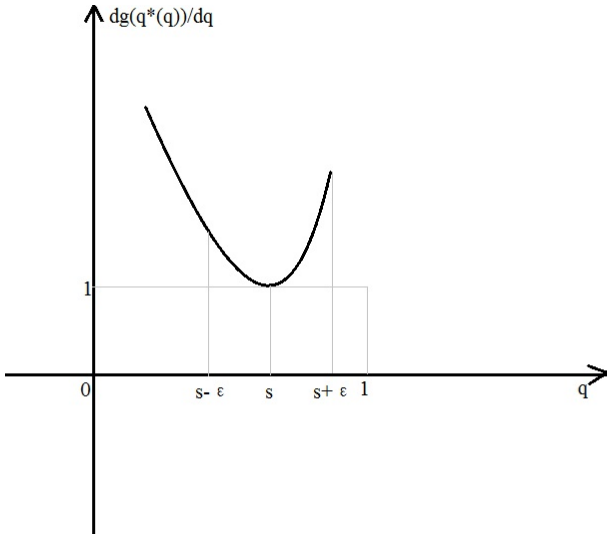
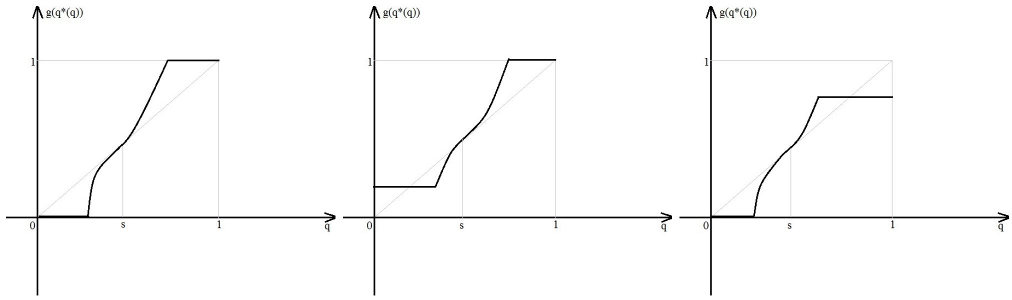


Figure C.17: *The correct shape of* $\frac{dq^*(q)}{dq}$ *at* $\lambda = \lambda_2$



(1) **(1)-1** **(1)-2**

Figure C.18: *At* $\lambda = \lambda_2$, $\frac{dq^*(q)}{dq}|_{q=s} = 1$, *and the shape of* $g(q^*(q))$ *has three possibilities that evolve from (1), (1)-1 and (1)-2, respectively.*

three equilibria.

As λ increases away from λ_2, except the part that at least one player plays the game without acquiring information, which exhibits a horizontal line and happens at the two sides of $g(q^*(q))$, the derivative of $g(q^*(q))$ in which both players play the game by acquiring information is greater than 1 since $\frac{d\frac{dA(q^*(q))}{dq}}{d\lambda} > 0 \; \forall q \in [0,1]$. According to continuity of $g(q^*(q))$ with respect to λ, at the new symmetric equilibrium, both players still play the game by acquiring information. By increasing λ away from λ_2, the shapes of $g(q^*(q))$ evolving from (1), (1)-1 and (1)-2 not only match with Case 2, but the shapes of $g(q^*(q))$ may match with Case 1 or Case 3 as well. Therefore, for $\lambda > \lambda_2$, Cases 1 and 3 again become the candidate shapes of $g(q^*(q))$, and again, we have nine possible shapes of $g(q^*(q))$, which are given by Figure C.19.

It is found that, irrespective of the shape of $g(q^*(q))$, there are always three equilibria, and in either (1), (1)-1, or (1)-2, irrespective of the case (Case 1, Case 2 or Case 3), the concerned properties of equilibria (i.e. the stability of the symmetic equilibrium, players' behaviour at the symmetic equilibrium and the types of asymmetric equilibria) are the same. In all nine possible situations, the symmetric equilibrium is unstable. In the situation of (1), according to symmetry of players' behaviour in a pair of corresponding asymmetric equilibria, the asymmetric equilibria in this situation is (1, 0) and (0, 1). In the situation of (1)-1, again according to symmetry, the asymmetric equilibria are $(1,q)$ and $(q,1)$, where $q \in (0,1)$. And in the situation of (1)-2, the asymmetric equilibria are $(0,q)$ and $(q,0)$, where $q \in (0,1)$. These (outer) asymmetric equilibria have already been there since asymmetric equilibria arise from $\lambda = \lambda_1$.

Before we proceed, let us study some profound issues underlying (1), (1)-1 and (1)-2. We begin from (1)-1. According to the symmetry property of players' behaviour in asymmetric equilibria, as long as asymmetric equilibria arise, as it has been shown, the $(1,q)$ and $(q,1)$ asymmetric equilibria where $q \in (0,1)$ always exist. This means that given an opponent's strategy $q^* \in (0,1)$, player i always chooses action 1 by comparing ex ante expected payoff of each action. Therefore, player i in (1)-1 always has

$$(1-q^*)M + q^*D + pu + (1-p)d > 0 \quad \forall q^* \in (0,1)$$

and equivalently it implies that the payoff specification always satisfies

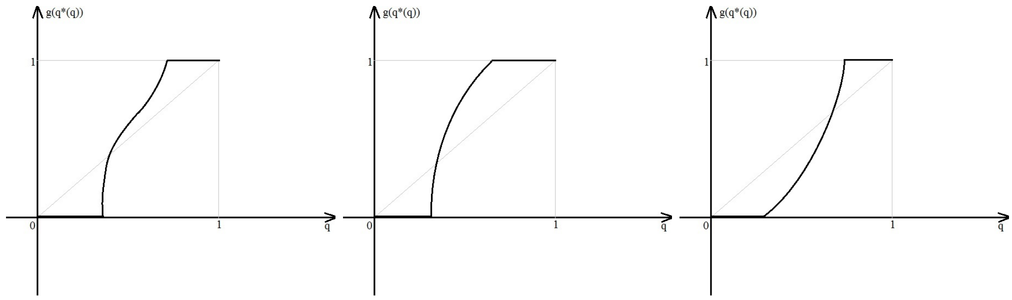
$$D + pu + (1-p)d > 0$$

For (1)-2, as long as asymmetric equilibria arise, the $(0,q)$ and $(q,0)$ equilibria where $q \in (0,1)$ always exist, which means that given opponent's strategy $q^* \in (0,1)$, player i always chooses action 0 when not acquiring information. Hence, in this situation, for player i, we get

$$(1-q^*)M + q^*D + pu + (1-p)d < 0 \quad \forall q^* \in (0,1)$$

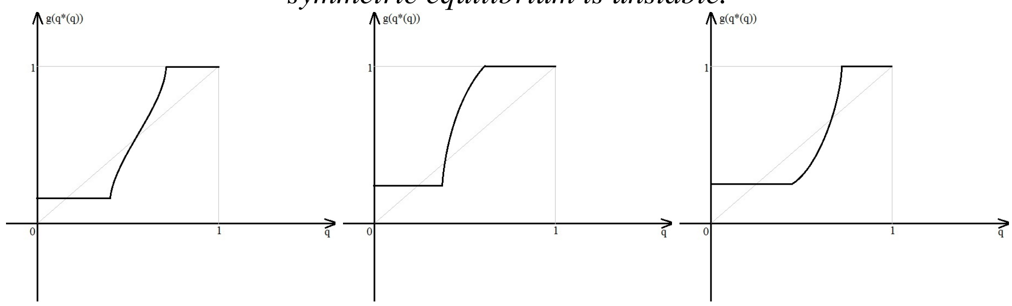
and equivalently it implies that the payoff specification always satisfies

$$M + pu + (1-p)d < 0$$



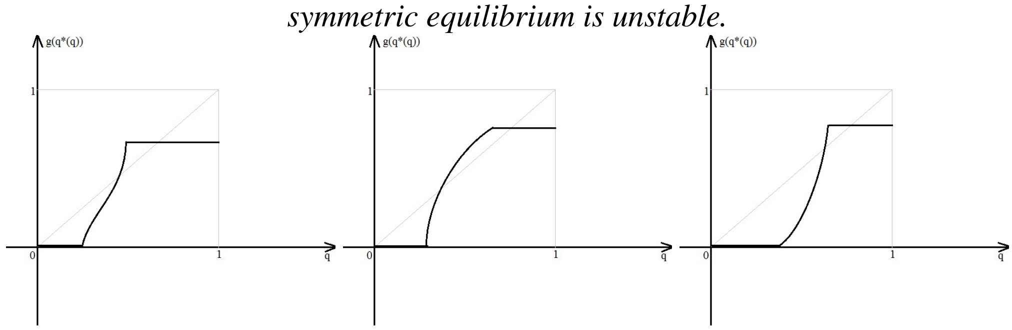
Case 2 **Case 1** **Case 3**

The three-equilibria situation happens for $\lambda > \lambda_2$ and it evolves from (1). The symmetric equilibrium is unstable.



Case 2 **Case 1** **Case 3**

The three-equilibria situation happens for $\lambda > \lambda_2$ and it evolves from (1)-1. The symmetric equilibrium is unstable.



Case 2 **Case 1** **Case 3**

The three-equilibria situation happens for $\lambda > \lambda_2$ and it evolves from (1)-2. The symmetric equilibrium is unstable.

Figure C.19: *Nine possible shapes of $g(q^*(q))$ for $\lambda > \lambda_2$. There are three equilibria in all nine situations, and the symmetric equilibrium is unstable.*

For (1), as long as asymmetric equilibria arise, the $(1,0)$ and $(0,1)$ asymmetric equilibria always exist. It implies that when the player makes the decision without acquiring information, $(1-q^*)M+q^*D+pu+(1-p)d$ is not necessarily always greater than or smaller than 0 for all $q^* \in (0,1)$. Hence, given assumption 1, situation (1) implies that the payoff specification satisfies

$$M+pu+(1-p)d>0$$

and

$$D+pu+(1-p)d<0$$

Now, we go back to the three-equilibria situation that we have just analysed. For (1), as λ increases ($\lambda>\lambda_2$), the $g(q^*(q))$ where $q\in h$ will grow steeper since its derivative with respect to q increases, and at the same time, the three equilibria always exist. Therefore, the limit of this process is reached until $\frac{dg(q^*(q))}{dq}=\frac{dA(q^*(q))\times 1\{0\leq A(q^*(q))\leq 1\}}{dq}=\infty$ at $q=s$. This limit indicates that if the opponent deviates from the symmetric equilibrium s, a player will either choose action 0 or action 1. This is the reaction style for a player in a complete information game. Under the payoff specification $M+pu+(1-p)d>0$ and $D+pu+(1-p)d<0$, the symmetric equilibrium at the limit is just the mixed strategy in the corresponding complete information game which is shown in Table 3, Section 4.4. Therefore, it can be said that when λ is so large that at symmetric equilibrium $q=s$, $\frac{dg(q^*(q))}{dq}=+\infty$, no player will try to make the best response by acquiring information given any strategy of the opponent $q^*\in[0,1]$ and the game turns into the complete information game described in Table 3, Section 4.4. Suppose at $\lambda=\lambda_c$, $\frac{dg(q^*(q))}{dq}$ equals $+\infty$ at $q=s$. Then, for $\lambda>\lambda_c$, the shape of the best response function is fixed because $\frac{dg(q^*(q))}{dq}\geq 0$ always holds; hence, from $\lambda\geq\lambda_c$, the game is always the complete information game (see Figure C.20).

For (1)-1, as λ increases, the $g(q^*(q))$ where $q\in h$ will grow steeper since in this part $\frac{d\frac{dg(q^*(q))}{dq}}{d\lambda}>0$, and at the same time, the three equilibria always exist. It has been known that when λ is greater than λ_c so that players make the best response without acquiring information given any strategy of the opponent, the game becomes the complete information game described in Table 3 and the equilibrium is $(1,1)$. From the three-equilibria situation to the complete information game limit, $\frac{dg(q^*(q))}{dq}$ continues to increase and the three-equilibira situation will always be maintained, unless the ver-

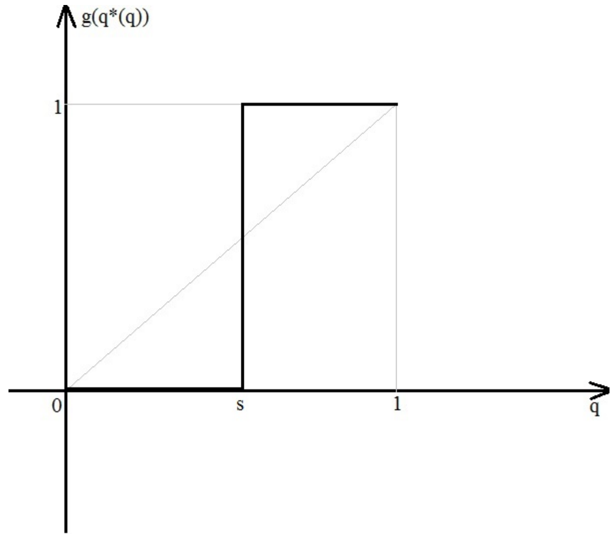


Figure C.20: *The 2nd iteration algorithm $g(q^*(q))$ for $\lambda \geq \lambda_c$ given that $M + pu + (1-p)d > 0$ and $D + pu + (1-p)d < 0$, i.e. situation (1). This algorithm calculates the equilibria of the complete information game described in Table 3.*

tical distance between the two horizontal lines at the two sides of $g(q^*(q))$ decreases at the same time, the complete information limit that its equilibrium is unique and it is (1, 1) would never be reached. Therefore, as λ increases from λ_2, $\frac{dg(q^*(q))}{dq}$ where $q \in h$ continues to increase, and at the same time, the vertical distance between the horizontal lines at the two sides of $g(q^*(q))$ decreases, until $\lambda = \lambda_c$, $g(q^*(q))$ becomes a horizontal line, and the game becomes a complete information game. Because in (1)-1, $D + pu + (1-p)d > 0$, for $\lambda \geq \lambda_c$, $g(q^*(q)) = 1$, and the unique intersection point of $g(q^*(q))$ and the 45° line is just the equilibrium of the complete information game (see Figure C.21).

For (1)-2, the analysis is the same as in (1)-1 and the results parallel with (1)-1. (1)-2 is characterized by $M + pu + (1-p)d < 0$. As λ increases until λ_c in which both players play the game without acquiring information given any strategy of the opponent and hence playing a complete information game, then $g(q^*(q)) = 0$ and so the symmetric equilibrium becomes $(0,0)$. For $\lambda > \lambda_c$, still because $\frac{d\frac{dg(q^*(q))}{dq}}{d\lambda} > 0$, it is known that $g(q^*(q))$ always equals 0 (see Figure C.22).

Therefore, in conclusion, as λ increases from 0, multiple equilibria will arise, and then we get the following results:

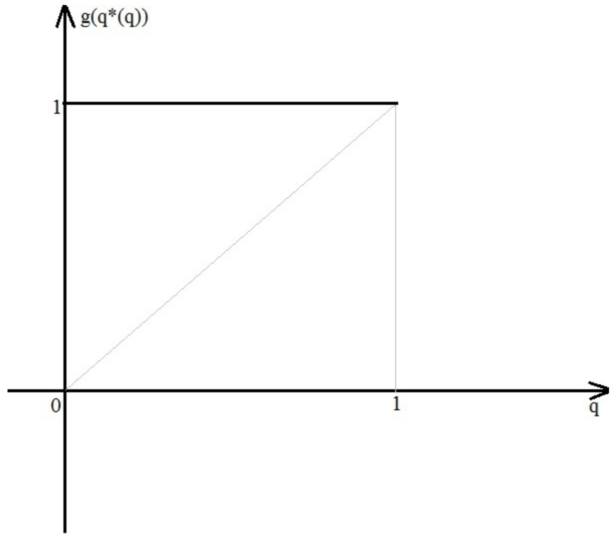


Figure C.21: *The 2nd iteration algorithm $g(q^*(q))$ for $\lambda \geq \lambda_c$ given that $D + pu + (1-p)d > 0$, i.e. situation (1)-1. It is $g(q^*(q)) = 1$. This algorithm calculates the equilibria of the complete information game described in Table 3.*

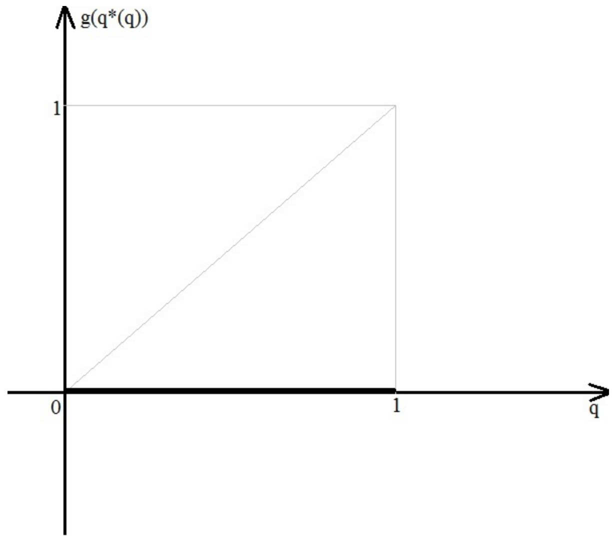


Figure C.22: *The 2nd iteration algorithm $g(q^*(q))$ for $\lambda \geq \lambda_c$ given that $M + pu + (1-p)d < 0$, i.e. situation (1)-2. It is $g(q^*(q)) = 0$. This algorithm calculates the equilibria of the complete information game given in Table 3.*

1) As λ increases from 0, if $M + pu + (1-p)d > 0$ and $D + pu + (1-p)d < 0$, for $\lambda < \lambda_c$, it is always possible that a player makes the best response by acquiring information for some strategies of the opponent. Then, the number of equilibria will change in the following sequence: $1 \to 3 \to 5 \to 3$. During this process, at symmetric equilib-

rium, both players play the game by acquiring information. Finally, when $\lambda \geq \lambda_c$, no player will make the best response by acquiring information given any strategy of the opponent and the game becomes a complete information game shown in Table 3. The limit complete information game has 3 equilibria: $(1,0)$, $(0,1)$ and the mixed strategy $\{(\frac{M+pu+(1-p)d}{M-D}, -\frac{D+pu+(1-p)d}{M-D}), (\frac{M+pu+(1-p)d}{M-D}, -\frac{D+pu+(1-p)d}{M-D})\}$. The mixed strategy is the limit of the symmetric equilibrium as λ increases from 0.

2) As λ increases from 0, if $D+pu+(1-p)d > 0$, for $\lambda < \lambda_c$, it is always possible that a player will make the best response by acquiring information. Then, the number of equilibria will change in the following sequence: $1 \to 3 \to 5 \to 3$. During this process, at the symmetric equilibrium, both players play the game by acquiring information. Finally, when $\lambda \geq \lambda_c$, no player will make the best response by acquiring information given any strategy of the opponent and the game becomes a complete information game shown in Table 3. The limit complete information game has a unique equilibrium $(1,1)$.

3) As λ increases from 0, if $M+pu+(1-p)d < 0$, for $\lambda < \lambda_c$, it is always possible that a player will make the best response by acquiring information for some strategies of the opponent. Then, the number of equilibria will change in the following sequence: $1 \to 3 \to 5 \to 3$. During this process, at the symmetric equilibrium, both players play the game by acquiring information. Finally, when $\lambda \geq \lambda_c$, no player will make the best response by acquiring information given any strategy of the opponent and the game becomes a complete information game shown in Table 3. The limit complete information game has a unique equilibrium $(0,0)$.

Logically, there exists another possibility as λ increases from 0— during the process, there always exists a unique equilibrium. From the previous analysis, we can see that if at symmetric equilibrium $q=s$, $\frac{dg(q^*(q))}{dq} \geq 1$, i.e. the symmetric equilibrium is unstable, then irrespective of the shape of $f(q)$ in Case 1, Case 2 or Case 3, multiple equilibria will definitely arise (which is the situation that $\lambda \geq \lambda_2$ in the previous multiple-equilibria situation). Therefore, a necessary requirement for the uniqueness situation happening is that as λ increases from 0 to $+\infty$, the symmetric equilibrium is always stable. For this situation, when players make the best response by acquiring information at the symmetric equilibrium, $\frac{dq^*(q)}{dq}|_{q=s} > -1$, and hence, $\frac{dg(q^*(q))}{dq}|_{q=s} < 1$. Finally, when players no longer acquire information at the symmetric equilibrium, $\frac{dq^*(q)}{dq}|_{q=s} = 0$, and hence, $\frac{dg(q^*(q))}{dq}|_{q=s} = 0$. In this uniqueness situation, before $g(q^*(q))$ becomes a horizontal line at the symmetric equilibrium, both players play the game by acquiring information at equilibrium and equilibrium strategy $s \in (0,1)$.

When $g(q^*(q))$ becomes a horizontal line at $q = s$, it is impossible that the horizontal line around $q = s$ is located between 0 and 1 because if there exists a unique equilibrium in which no player plays the game by acquiring information, then this equilibrium must be either $(0,0)$ or $(1,1)$ obtained by comparing ex ante expected payoff of each action. Therefore, this fact indicates that this situation can only happen with payoff specifications of $M + pu + (1-p)d < 0$ or $D + pu + (1-p)d > 0$ (see Figure C.23).

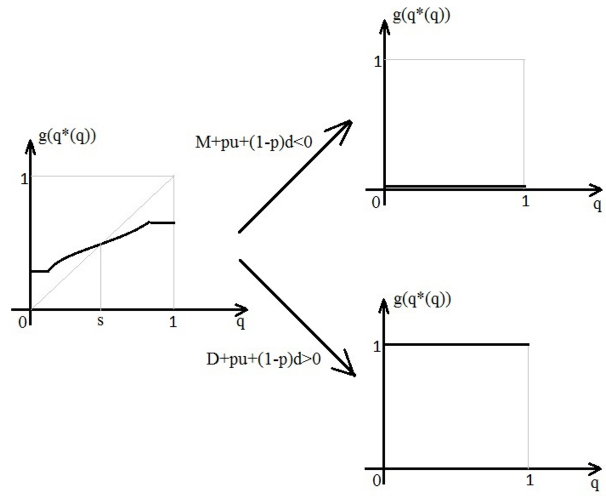


Figure C.23: *An illustration of the unqiueness situation of the game as λ increases from 0 to $+\infty$. Without loss of generality, the process can be described by the LHS sub-figure and RHS sub-figure(s). This uniqueness situation can only happen for $M + pu + (1-p)d < 0$ or $D + pu + (1-p)d > 0$*

Therefore, if $M + pu + (1-p)d < 0$ or $D + pu + (1-p)d > 0$, as λ increases from 0 to $+\infty$, the game is able to always perform a unique equilibrium under appropriate parameter specification. The equilibrium is always stable.

Therefore, all the above proof constitutes the proof of Proposition 7, and Proposition 7 is the summarization of all analytical results in this proof. *Q.E.D.*

Appendix D

Appendix of Chapter 4
Proof of Comparative Statics Analysis (Proposition 3, 8 and 9)

Proof: At an equilibrium (q, q^*) where q and $q^* \in (0,1)$, we denote the comparative statics of parameter τ on player i's equilibrium strategy by $\frac{\partial q}{\partial \tau}$, and the comparative statics of parameter τ on i's best response $q(q^*)$ given q^* by $\frac{\partial q(q^*)}{\partial \tau}$, where $\tau \in \{M, D, u, d, p, \lambda\}$. It is found that players' equilibrium strategies are contained in the following equation system:

$$\begin{cases} \frac{\partial q}{\partial \tau} + A\frac{\partial q^*}{\partial \tau} = \frac{\partial q(q^*)}{\partial \tau} \\ B\frac{\partial q}{\partial \tau} + \frac{\partial q^*}{\partial \tau} = \frac{\partial q^*(q)}{\partial \tau} \end{cases}$$

where

$$A = \frac{M-D}{\lambda}[\frac{p}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2}\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})$$
$$+\frac{1-p}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]$$

and

$$B = \frac{M-D}{\lambda}[\frac{p}{[1-\exp(\frac{(1-q)M+qD+d}{\lambda})]^2}\exp(\frac{(1-q)M+qD+d}{\lambda})$$
$$+\frac{1-p}{[1-\exp(\frac{(1-q)M+qD+u}{\lambda})]^2}\exp(\frac{(1-q)M+qD+u}{\lambda})].$$

We solve this equation system as follows:

$$\begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial \tau} \\ \frac{\partial q^*}{\partial \tau} \end{pmatrix} = \begin{pmatrix} \frac{\partial q(q^*)}{\partial \tau} \\ \frac{\partial q^*(q)}{\partial \tau} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \frac{\partial q}{\partial \tau} \\ \frac{\partial q^*}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial q(q^*)}{\partial \tau} \\ \frac{\partial q^*(q)}{\partial \tau} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \frac{\partial q}{\partial \tau} \\ \frac{\partial q^*}{\partial \tau} \end{pmatrix} = \frac{1}{1-AB} \begin{pmatrix} 1 & -A \\ -B & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial q(q^*)}{\partial \tau} \\ \frac{\partial q^*(q)}{\partial \tau} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} \frac{\partial q}{\partial \tau} = \frac{1}{1-AB}\frac{\partial q(q^*)}{\partial \tau} - \frac{A}{1-AB}\frac{\partial q^*(q)}{\partial \tau} \\ \frac{\partial q^*}{\partial \tau} = \frac{1}{1-AB}\frac{\partial q^*(q)}{\partial \tau} - \frac{B}{1-AB}\frac{\partial q(q^*)}{\partial \tau} \end{cases} \tag{D.1}$$

The comparative statics of best response $\frac{\partial q(q^*)}{\partial \tau}$, where $\tau \in \{M, D, u, d, p, \lambda\}$, are obtained by

$$\frac{\partial q(q^*)}{\partial M} = \{\frac{p\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2} + \frac{(1-p)\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\}\frac{1-q^*}{\lambda} > 0$$

$$\frac{\partial q(q^*)}{\partial D} = \{\frac{p\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2} + \frac{(1-p)\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\}\frac{q^*}{\lambda} > 0$$

$$\frac{\partial q(q^*)}{\partial u} = \frac{\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\frac{1-p}{\lambda} > 0$$

$$\frac{\partial q(q^*)}{\partial d} = \frac{\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2}\frac{p}{\lambda} > 0$$

$$\frac{\partial q(q^*)}{\partial p} = \frac{1}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})} - \frac{1}{1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})} > 0$$

$$\frac{\partial q(q^*)}{\partial\lambda} = -\frac{p\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2}\frac{(1-q^*)M+q^*D+d}{\lambda^2}$$
$$-\frac{(1-p)\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\frac{(1-q^*)M+q^*D+u}{\lambda^2}$$

and therefore

$$\frac{\partial q(q^*)}{\partial\lambda} \gtreqless 0 \iff p \gtreqless \bar{p}, \text{ where } \bar{p} = \frac{1}{1-\frac{(1-q^*)M+q^*D+d}{(1-q^*)M+q^*D+u}[\frac{\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})-1}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}]^2\exp(\frac{d-u}{\lambda})}.$$

Next, we consider the comparative statics of parameter τ, where $\tau \in \{M, D, u, d, p, \lambda\}$, on players' equilibrium strategies:

1. Symmetric Equilibrium: For symmetric equilibrium $q = q^* = s \in (0,1)$, according to equation group (D.1), we get

$$\frac{\partial q}{\partial\tau}|_{q=s} = \frac{\frac{\partial q(s)}{\partial\tau}}{C}$$

where $C = 1 + \frac{M-D}{\lambda}\{\frac{p\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})]^2} + \frac{(1-p)\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}{[1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})]^2}\} > 1$. Therefore, at the symmetric equilibrium, we get

$$sign(\frac{\partial q}{\partial\tau}|_{q=s}) = sign(\frac{\partial q(s)}{\partial\tau})$$

and

$$|\frac{\partial q}{\partial\tau}|_{q=s}| < |\frac{\partial q(s)}{\partial\tau}|$$

2. Outer Asymmetric Equilibrium: At an extreme asymmetric equilibrium (q, q^*), where $q \in (0,1)$ and $q^* = 0$ or 1, because $\frac{\partial q^*}{\partial\tau} = 0$, in such asymmetric equilibrium, according to equation system (D.1), $\frac{\partial q}{\partial\tau} = \frac{\partial q(q^*)}{\partial\tau}$.

3. Inner Asymmetric Equilibrium: At a middle asymmetric equilibrium (q, q^*), where $q \in (0,1)$ and $q^* \in (0,1)$, the comparative statics result in such equilibrium are expressed simply by equation system (D.1). In such asymmetric equilibria, $\frac{\partial q}{\partial\tau}$ is linear with respect to $\frac{\partial q(q^*)}{\partial\tau}$ and $\frac{\partial q^*(q)}{\partial\tau}$. The coefficients $\frac{1}{1-AB}$ and $-\frac{A}{1-AB}$ (or $-\frac{B}{1-AB}$ for $\frac{\partial q^*}{\partial\tau}$) depend on not only the value of parameters but also both players' equilibrium strategy q^* and q. Hence, without specific parameter specification, the signs of $\frac{\partial q}{\partial\tau}$ and $\frac{\partial q^*}{\partial\tau}$ of the middle asymmetric equilibria cannot be determined. *Q.E.D.*

D.1 Proof of the Sensitivity Analysis that Only One Player's Information Cost Changes

Proof: Recall the best response function $q = \frac{p}{1-\exp(\frac{(1-q^*)M+q^*D+d}{\lambda})} + \frac{1-p}{1-\exp(\frac{(1-q^*)M+q^*D+u}{\lambda})}$ for $q \in (0,1)$. Now, we calculate the impact of varying one player's information acquisition cost, namely λ, on each player's equilibrium strategy q and q^*:

$$\frac{\partial q^*}{\partial \lambda} = H \frac{\partial q}{\partial \lambda} \tag{D.2}$$

$$\frac{\partial q}{\partial \lambda} = K \frac{\partial q^*}{\partial \lambda} + \frac{\partial q(q^*)}{\partial \lambda} \tag{D.3}$$

Remember that $\frac{\partial q(q^*)}{\partial \lambda}$ is λ's impact on player i's best response. Solving the equation group comprised by equations D.2 and D.3, we obtain

$$\frac{\partial q^*}{\partial \lambda} = \frac{H}{1-HK} \frac{\partial q(q^*)}{\partial \lambda}$$

and

$$\frac{\partial q}{\partial \lambda} = \frac{1}{1-HK} \frac{\partial q(q^*)}{\partial \lambda}$$

Q.E.D.

Appendix E

Appendix of Chapter 4

Proof for Section 4.9

In this section, we calculate $\frac{\partial \bar{p}}{\partial \lambda^*}$ and explain why its sign is uncertain.

Recall $\bar{p} = \frac{1}{1-\frac{(1-q)M+qD+d}{(1-q)M+qD+u}[\frac{\exp(\frac{(1-q)M+qD+u}{\lambda^*})-1}{1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})}]^2\exp(\frac{d-u}{\lambda^*})}$. Calculating its derivative with respect to λ^*, we obtain

$$\frac{\partial \bar{p}}{\partial \lambda^*} = -\frac{1}{[1-\frac{(1-q)M+qD+d}{(1-q)M+qD+u}[\frac{\exp(\frac{(1-q)M+qD+u}{\lambda^*})-1}{1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})}]^2\exp(\frac{d-u}{\lambda^*})]^2}$$

$$\times -\frac{(1-q)M+qD+d}{(1-q)M+qD+u} \times \frac{1}{\lambda^{*2}}\exp(\frac{d-u}{\lambda^*})\frac{\exp(\frac{(1-q)M+qD+u}{\lambda^*})-1}{[1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})]^3}$$

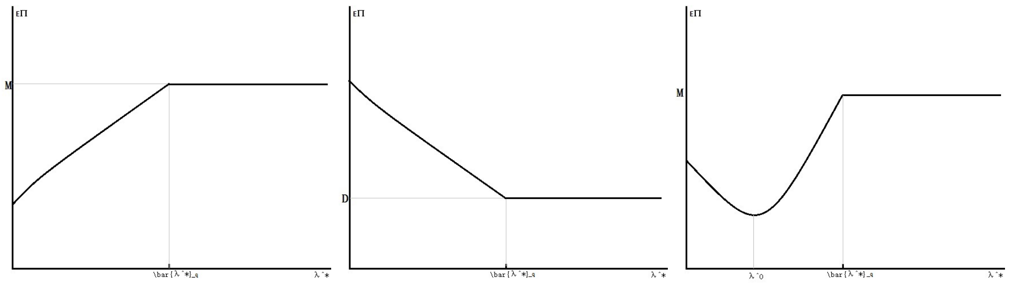
$$\times -\{[1+\exp(\frac{(1-q)M+qD+u}{\lambda^*})][1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})]$$

$$+2\times\frac{(1-q)M+qD+d}{u-d}\times\{\exp(\frac{(1-q)M+qD+u}{\lambda^*})-\exp(\frac{(1-q)M+qD+d}{\lambda^*})\}\}$$

Therefore, under Assumption 1, we find that

$$sign(\frac{\partial \bar{p}}{\partial \lambda^*}) =$$

$$sign(\{[1+\exp(\frac{(1-q)M+qD+u}{\lambda^*})][1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})])$$



$\bar{p^*} \geq p$ $\quad$ $\bar{p^*} < p$ **(Case 1)** $\quad$ $\bar{p^*} < p$ **(Case 2)**

Figure E.1: *Assuming $\frac{\partial \bar{p^*}}{\partial \lambda^*} > 0$, three cases of $\mathbb{E}\Pi(\lambda^*)$ are possible. In the third figure from left, at $\lambda^* = \lambda^0$, $\bar{p^*} = p$.*

$$+2 \times \frac{(1-q)M + qD + d}{u-d} \times \{\exp(\frac{(1-q)M + qD + u}{\lambda^*}) - \exp(\frac{(1-q)M + qD + d}{\lambda^*})\})$$

It can be concluded that the sign of the latter formula cannot be determined without specific parameter specification, because $[1+\exp(\frac{(1-q)M+qD+u}{\lambda^*})][1-\exp(\frac{(1-q)M+qD+d}{\lambda^*})]) > 0$ and $\frac{(1-q)M+qD+d}{u-d} \times \{\exp(\frac{(1-q)M+qD+u}{\lambda^*}) - \exp(\frac{(1-q)M+qD+d}{\lambda^*})\} < 0$. Therefore, $sign(\frac{\partial \bar{p}}{\partial \lambda^*})$ cannot be determined.

In the following, we assume that $\frac{\partial \bar{p^*}}{\partial \lambda^*} > 0$, and study how λ^* affects a player's expected payoff.

Given that $\frac{\partial \bar{p^*}}{\partial \lambda^*} > 0$, suppose when λ^* is close to 0, $\bar{p^*} \geq p$, where p has been given. Then, as λ^* increases, $\bar{p^*}$ becomes higher, and therefore $\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*} > 0$ as λ^* increases from 0 to $+\infty$. In this case, we always have $\frac{\partial q^*(q)}{\partial \lambda^*} < 0$ and when λ^* reaches $\bar{\lambda}^*_q$, as defined in Proposition 2, $q^*(q) = 0$ and thus $\mathbb{E}\Pi = M$, which is the highest of value of $\mathbb{E}\Pi$ (see Figure E.1).

Next, suppose when λ^* is close to 0, $\bar{p^*} < p$. As λ^* increases, two possible scenarios happen:

Case 1: As λ^* increases, $\bar{p^*} < p$ is always maintained. Therefore, $\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*} < 0$. Until $\lambda^* = \bar{\lambda}^*_q$, which is defined in Proposition 1, $q^*(q) = 1$ and $\mathbb{E}\Pi = D$, which is its lowest value.

Case 2: As λ^* increases, at $\lambda^* = \lambda^0 < \bar{\lambda}^*_q$, $\bar{p^*} = p$. For $\lambda^* \lesseqgtr \lambda^0$, $\frac{\partial \mathbb{E}\Pi}{\partial \lambda^*} \lesseqgtr 0$. Therefore, in Case 2, the highest value of $\mathbb{E}\Pi$ is reached at $\lambda^* = \bar{\lambda}^*_q$, and the highest value is M. At $\lambda^* = \lambda^0$, $\mathbb{E}\Pi$ reaches its lowest value.

Therefore, if $\frac{\partial \bar{p^*}}{\partial \lambda^*} > 0$, the highest value of $\mathbb{E}\Pi$ is reached at either $\lambda^* = 0$ or $\lambda^* = \bar{\lambda}_q^*$. The $\bar{\lambda}_q^*$ is defined in Proposition 2 (see Figure E.1).