An Analysis of
The Chinese College Admission System

Haibo Zhang

PhD in Economics

The University of Edinburgh

2009

Acknowledgements

I would like to express my gratitude to all those who gave me the possibility to complete this thesis.

I am deeply indebted to my supervisors, Dr. Simon Clark and Professor Jonathan Thomas, for their greatly helpful comments and encouragement.

Special thanks to Dr. Ahmed Anwar for his great help.

Many thanks to Professor Ed Hopkins for his encouragement and helpful advice during my PhD study.

I want to thank Professor Andy Snell, SGPE and the School of Economics for providing me the scholarship and teaching opportunities.

All my colleagues from the School of Economics supported me in my research work. I want to thank them for all their help, support, interest and valuable hints.

Last but not least, thank my family, my dear friends, especially Ms. Wei Zhang for their support.

Declaration

I confirm that all this work is my own except where indicated. I have clearly referenced/listed all sources as appropriate, referenced and put in inverted commas all quoted text (from books, web, etc), not made any use of the thesis of any other student(s) either past or present, and not sought or used the help of any external professional agencies for the work.

Abstract

This thesis focuses on the problems of the Chinese University Admission (CUA) system. Within the field of education, the system of university admissions involves all of Chinese society and causes much concern amongst all social classes. University admissions have been researched since the middle of last century as an issue which has economic impact. However, little attention has been paid to the CUA system from the perspective of economics. This thesis explores a number of interesting aspects of the system. As a special case of the priority-based matching mechanism, the CUA system shares most properties of the Boston Mechanism, which is another example of a priority-based matching mechanism. But it also has some unique and interesting characteristics. The first chapter will introduce the main principles of the CUA system in detail and discuss stability, efficiency, strategy-proofness, and other properties under different informational assumptions.

There is a heated debate about whether the CUA system should be abandoned or not. Educational corruption is one of the issues that have been raised. Corruption is a major issue of the CUA system as well as university admission systems in other areas in the world, e.g. India, Russia, etc. We contrast the performance of markets and exams under the assumption that there exists corruption in the admission process. The problem will be analyzed under perfect capital markets and also under borrowing constraints. We use auction theory to obtain equilibria of the market system and the exam system and analyse the effects of corruption on the efficiency of the two systems. We conclude that the exam system is superior to the market system if we only consider the issue of corruption.

In the third chapter, we construct a model to reveal the forces that positively sort students into different quality universities in a free choice system under assumptions of supermodular utility and production functions. Given a distribution of student ability and resources, we analyse the planner's decisions on the number of universities and the design of the "task level" for each university, as well as the allocation of resources between universities. Students gain from completing requirements (tasks) in universities, while having to incur costs of exerting effort. In contrast to previous literature, our model includes qualifications as well as cost in the student's utility function, and educational outputs depend on qualification, ability and resources per capita. Our main focus is on the design of task levels. Our result differs from the literature as regards the optimal number of colleges. A zero fixed cost of establishing new colleges does not necessarily result in perfect tailoring of tasks to students. Furthermore, if the fixed cost is not zero, then the planner has to take fixed costs into account when deciding the number of universities.

Introduction to the Thesis

China is the largest developing country in the world. During the last thirty years[1], demand for higher education in China has been increasing rapidly. As a result, concerns over college admission[2] have become a focal point of discussion in China society. Due to historical and cultural reasons, the college admission in today's China involves the whole society and causes concerns of various social classes (Zhang (2008)). Although public attention on this issue has been drawn widely, most discussions occur in the popular media, and little theoretical research, specially studies from the economic perspective, has been carried out. This thesis tries to analyse the Chinese college admission system theoretically. Empirical analysis may be more convincing, but the collection of data about the higher education in China is difficult. Nevertheless, it will be a direction for further studies in the future. This introduction consists of the following parts: a brief introduction about the Chinese College Admission System; some criticisms of the system; and the aims and contributions of this thesis.

The Chinese College Admission System

The Chinese College Admission System may be one of the most unique college admission systems in the world. It was established in 1952 to meet political and educational needs, as well as to serve as one of the accelerators for the economic transition of the new People's Republic of China. The system was interrupted from 1966 to 1976 during the "Cultural Revolution". In that period, almost all education activities were suspended. It was restored in 1977. Although there have been changes to the admission system until present day, the principle of the system remains the same, which is to admit students into higher education institutions through uniform national examinations.

The establishment of the original admission system in 1952 was based on five arguments, including external and internal factors. (Zheng, 2007)

1. There are political needs and consideration of economic construction for the new country. As a new country reborn from wars, almost every aspect of the society was indeed for rebuilding, and hence all kinds of professionals were in short supply. Past experiences of higher education in Chinese history had

[1] The higher education system in China was interrupted due to domestic political complication during the Cultural Revolution and was restored back to normal in 1977.

[2] The phrase of "college admission" is used to represent the admission system of undergraduate education which is the main focus of this thesis.

suggested that a national examination system could solve the unbalanced recruiting problems between different regions, colleges and disciplines. Also it was thought that an examination system for admission would increase the efficiency of using limited higher education resources.

2. There was the need of educational adjustment. A large scale adjustment of academic disciplines was underway in 1952, involving most universities and colleges. The higher education sector needed a more scheming admission system to ensure all institutions could recruit enough academically eligible freshmen.

3. As China has such a large population, the scale of college admission is enormous. Therefore equality and efficiency are the most important considerations for large-scale exam. Under such condition, a uniform national admission system organised by the government is more efficient than independent admission operated by colleges.

4. In a few centuries before the establishment of new country, China had been in the war and falling into pieces and infirmness. People of the country desired for the unification. This desire had very deep effect on the implementation of centralism in many aspects in China, including the college admission system.

5. The last but not the least, it was influenced by culture.

The admission system consists of two stages. Stage one is a standard exam, called the National College Entrance Examination, and stage two is the recruitment procedure which starts immediately after exam results are released. Although students may sit different papers in different provinces or in the four municipalities directly under the Central Government (i.e. Beijing, Chongqing, Shanghai, and Tianjin) for the same academic subject, the structure and administration of the exam are the same across the country.[3] The exam normally lasts for 3 days. Three subjects are mandatory for all students: Chinese, Mathematics and a foreign language[4]. Apart from the three mandatory subjects, six other subjects are also being examined selectively depending on the course the student

[3]Most information and references about the exam and admission systems are from public sources of the Ministry of Education of the People's Republic of China. http://www.moe.edu.cn/

[4]It is usually English but may also be substituted by Japanese or Russian in some northern provinces.

wishes to study in higher education.[5] Three of these six subjects for the category of science include Physics, Chemistry, Biology and three for arts include History, Geography and Political Education. These examinations are essentially the only criterion for higher education admissions. The details of the exam are arranged by the Ministry of Education,[6] such as the exam date, subjects, qualification of examinees, criterion of check both in politics and health, matriculation principles. There is a National College Entrance Exam committee overseeing the operations of the exams, which include setting exam questions, making reference answers and grading guidance. However, the implementation and administration of the exams are arranged by the local government at provincial level. As a tradition, the exams take place at the same time across the country. Students who are dissatisfied by the results of their first attempt may repeat the last year study of high school and take the exams again in the following year. These exams are traditionally and culturally the most important event for Chinese students as it is the only possible way to get into colleges.

Although the exam result is the only criterion, the admission also depends on students' order of preferences of colleges on their preference lists. The whole admission process takes place in four phases. Phase one is known as "Early Admissions", which deals with applications to degrees in education-related courses, applications to institutions of the armed forces and the police force, as well as applications to institutions in Hong Kong and Macau[7]. Phase two is known as "Key Undergraduate Admissions" which deals with applications to institutions administered by central government departments and institutions, in other words, the top universities in the country. Phase three is called "General Undergraduate Admissions" which deals with applications to institutions located in the capital of each province; these institutions are usually the top ones within the province. The fourth phase deals with the applications to the remaining institutions. Students are allowed to list four to six choices of institutions and courses in each of the admission phase. Currently, sequences of enlisting the choice of institutions and courses are available to students depending on where they take the exams. In the first sequence, students are required to report their preferences before taking the exams; in the second sequence, students can list their preferences after exams but before results come out; and in the third sequence, students are allowed to

[5]There are two main categories in Chinese higher education system: Science/engineering or art/humanities.

[6]There some exceptions, e.g., Shanghai, where local government provides different exam questions.

[7]Due to historical and political reasons, education systems in Hong Kong and Macau are different from mainland China under the "one country, two systems" policy.

report their preferences after exam results are released. For the admission operation in 2005, five provinces adopted the first sequence of enlisting preferences, sixteen provinces adopted the second sequence, and ten provinces adopted the third sequence.

The specific admission rules of each college are not necessary the same, but most colleges adopt a policy called "Preference Clearance".[8] "Preference Clearance" implies that, first, a college will consider offering a place to a student who does not rank it as the first choice only when the college can not fill up the quota by those who rank it as the first choice. Second, the allocation of degree programmes in a college follows the same principle. Although in theory it is possible that a student who does not put a college as the first choice could still be admitted, the reality is that the chance is very small because of the large number of applicants. The fact is that the number of students applying to an institution is far greater than the quotas available. Moreover, as an internal policy, some colleges do not admit those students who do not rank them as the first choice even when they can not fill up their quota by accepting students ranking them as the first choice. For example, Beijing University will only admit a student who does not put it as the first choice if the quota is not filled up by those who do rank it as the first choice, given the student's score is above the average grade of those who have been already admitted. Given this policy and the increasing number of applicants, how to list the preferences concerns most students and families. By making an inappropriate decision in listing preference, an applicant may be rejected by all colleges on his preference list.

Given the large scale of higher education application (e.g. there are 10 million applicants in 2007) and the limited resources available, it is not difficult to imagine how highly competitive it is when it comes to the admission procedure. On the other hand, it is not difficult to understand how huge the influence is to a student's future. Therefore any reform of the system will affect millions of students and their families.

Criticisms of the System

We have talked about the reasons why the higher education system has been running for such a long time given its importance to the Chinese people. One fact is that criticisms never stop appearing. People question the whole system in two aspects: the examination itself and the recruitment policies.

[8] See Chapter 4 of "Guide of Higher Education Entrance Examination Recruitment and Listing Preferences (2009)".

Some critics point out that the design of the exam lacks flexibility. Different types and levels of higher education institutions have differing requirements on their students, but in the current system, it is difficult to pick the suitable students merely according to their performance in the once-in-a-life-time exam. In other words, the system can not include a flexible personal assessment. It is also difficult to tell a student's character by simply looking at exam scores. A student may have special strength or talent in some areas, but he could be weak in exams or in the exam of a particular subject; if this is the case, this student may fail to enter higher education. The current system is also criticized for its emphasis on rote learning because examinations are basically tests of how much knowledge students are able to memorise in their years at school. Critical thinking and personal development are rarely emphasized in Chinese classrooms, especially in high schools where all students have only one objective: getting ready for exams. The once-in-a-life-time exams bring many social and psychological problems to students as most students and their families believe the result of the exam will decide their entire lives.

Recruitment policy is another aspect being widely criticized. "Preference Clearance" could lead to an inefficient outcome[9]. A student is likely to lose the opportunity of being admitted by a college if he does not list it in his preferences or does not rank it as the first choice. Therefore we might see the following outcomes: a student wants to enter a college, which has places available, but he will fail to do so because he did not list that college as one of the preferences; a student is rejected by a college, which is on his preference list because that college was not the first choice, although the student has met the exam result requirement. We will explain this in more details in the next chapter. Another problem of the recruitment policy is regional discrimination. Under current rules, colleges set a fixed admission quota for each province, and the number of places available to students from the province which the institution is located is normally higher than other provinces. As education quality and the number of universities are highly uneven in different geographical locations across China, it is argued that students are being discriminated in the admission process as those who are from the area where the institution is located are given priority of being admitted to that particular institution over students outside the area. For example, students from outside Beijing would need a significantly higher exam score to enter a university or college located in Beijing compared to students from the city of Beijing. Such regional discrimination policy leads to an abnormal phenomenon,

[9]We will define efficiency in the first chapter.

called exam immigrants. Currently, students are not allowed to take the higher education entrance exams in locations other than the place of permanent address where the student's household is registered; however, for the sole purpose of increasing students' chances of entering university, some families choose to relocate their household registrations to a different city or even province. Under the Soviet-style population movement control, changing the location of household registration is not easy. Although the actual population movement is free in China nowadays, it would still take a large effort and may cost a lot in financial terms to change the permanent location where a household is officially registered.

Due to the competition in securing a higher education opportunity, officials who can influence the admission result have the incentive to seek personal gains. Two scandals being reported in the news below expose the tip of the iceberg of malpractice and corruption affecting the higher education admission system.

"The Xi'an Conservatory of Music in Northwest China's Shaanxi Province had asked for 30,000 yuan (US$3,620) from each enrolled student." [10]

"China Central Television (CCTV) reported on Friday three staff members of the Beijing University of Aeronautics and Astronautics (Beihang) had extorted at least 550,000 yuan (US$66,505) from seven students in South China's Guangxi Zhuang Autonomous Region." [11]

In the two cases, students who refused to pay the bribe were also threatened with expulsion from the university. These scandals have indicated a number of loopholes in the university admission system including lack of transparency and investigation, lax management and lack of professionalism among admission personnel. The government has invested much effort in keeping the admission processes transparent and corruption-free, however, the outcome is not satisfactory. Leaking of exam materials, bribery, and other abuses are still being discovered every year. These problems together with other irregular practices have existed for long time in national college admission procedures, and the whole system has largely been suffering from the so-called "hidden corruption"[12]. According to the report, it is an open secret that in most colleges, admission job is a lucrative post, and accepting bribes is a common practice for these personnel. Corruption has rapidly evolved into open extortion of money because the investigation is not strict enough. The problem of corruption in admission has harmed the equality and efficiency of the higher education system; moreover, it has a negative impact on young students' sense of fairness and justice.

[10]The People's Daily, August 17 2004.
[11]The People's Daily, August 19 2004.
[12]The People's Daily, August 19 2004.

One aspect of the debates over abolition of the current system concerns corruption. The opposition argues that the standard exam system has given too much power to the administrative personnel, while the supporting side believes that corruption is more likely to be something that has its root in the culture. At the end of the day, it is not surprising to see some degrees of corruption when the competition for university places is strong and the investigation of the use of admission tutors' power is weak. It has also been argued that corruption would be even more serious if the current exam system for higher education admission was replaced by a free market system. In a market system, students can apply to colleges freely, and applications are not determined only by a one-off exam. We will compare these systems from the perspective of corruption in the second chapter.

In order to avoid the problems induced by the standard exam admission system, many reforms or even alternative mechanisms have been proposed. One of them is "easy admission but strict graduation". Tang (1995), Li (1996) and some other scholars argued that this mechanism is a precious opportunity for higher education to break the fetters of a planned economy and is a way to deepen reforms and to promote the development of education. They believe that the proposed policy is ready to be put in trial cities and regions with certain degrees of economic powers. Liu (1995) argued that "easy admission but strict graduation" will not work in China because of the unfailing and influential belief of "networking through petticoat influence" in Chinese culture and traditions; therefore the implementation of "easy admission" will most likely result in "easy graduation" without students paying much effort in studying. Zhang (1996) suggested that there is neither necessity nor practicality for implementing the "easy admission but strict graduation" in the near future. Generally speaking, there were more opponents than proponents for such a view Zhang (2008). It is interesting that the idea of "easy admission but strict graduation" has been implemented in the French higher education system for a long time. There are two systems in France.[13] The rules adopted in the open system are exactly "easy admission but strict graduation". The literature related to the problem of sorting suggests that a a positive assortative matching, i.e., the higher ability students are accepted by the higher quality institutions and the lower ability students are allocated to the lower quality institutions, will be produced in competitive procedures, e.g., the Chinese exam system. It has not been discussed why there will be a positive sorting result in an "easy admission but strict graduation" mechanism as well as

[13] A more detailed introduction of the two systems can be found in the third chapter.

the competitive systems. The third chapter of this thesis will try to find out the driving force.

Aims and Contributions

Although the research resources on Chinese higher education are very limited, there have been many studies on higher education in other countries over the past several decades. The literature in this area mostly focuses on mechanism and policy designs. Efficiency, fairness and welfare of the admission results are important topics of concern. It is interesting that we can find different mechanisms for higher education admissions in different countries. In some countries, the mechanisms have been reformed many times, but some countries have been using the same mechanism for a long time. The main content of this thesis focuses on the Chinese college admission mechanism, and we try to discuss three issues arising in this system. The first chapter describes the Chinese College Admission (CCA) mechanism, explores its main properties and compares it with other mechanisms currently in use in other countries. In the second chapter, we compare two college admission mechanisms from the perspective of corruption in an auction theory framework. The last chapter explores the driving force of a positive assortative sorting in the college admission problem when there are no entry requirements, or under the policy of "easy admission but strict graduation". The first two issues concern the current admission mechanism in China, while the third chapter discusses an alternative system, which has been used in France. In this section, there are brief literature reviews for each chapter and more detailed reviews can be found in the introduction of each chapter.

College admission modelling is one of the applications of matching theory. An outcome of such modelling is a matching of students to colleges, such that each student is matched to at most one college, and each college is matched to at most its quota of places available. Therefore, a college admission model is an example of many-to-one matching models. The discussion on the college admission problems has existed since the celebrated article "College Admissions and the Stability of Marriage" by Gale and Shapley (1962). In their model, students and colleges have their own preferences over the opposite party. They defined the stability and proposed a famous agent optimal stable mechanism in a college admission problem, in which the outcomes are stable and Pareto efficient.

The CCA mechanism is a special case of a priority matching mechanism. In a priority-based mechanism, the key phase is submitted preferences determining priorities. The school choice problem is one of the most important priority-

based matching problems. The difference between school choice problem and traditional college admission model lies in priority takes the place of preference of college. Priorities of students in schools are determined by students' submitted preferences over schools as well as some other rules. The algorithm behind the CCA is similar to the Boston Mechanism (Abdulkadiroğlu, Pathak, Roth, Sönmez (2005); Abdulkadiroğlu, Sönmez (2003); Chen, Sönmez (2003); Ergin, Sönmez (2006)). The priority of a student at a college in the CCA model is dependent on the student's submitted preferences and score obtained in the exam. Under the CCA mechanism, a student who is not offered a place at his top-ranked college will only be considered for his reported second choice college after those who have top-ranked that college. Therefore a student will lose his priority to be admitted at a college unless he lists it as his first choice. The match priority is lexicographic in the CCA mechanism, which implies that any student-college pair that ranks each other first has the highest match priority. It first considers the student preferences and only then the college priorities.

In the first chapter, we will review the general college Two-Sided Matching Market model and Priority Matching Mechanism, and introduce some basic properties of an outcome of a priority-based matching game. We briefly introduce how the CCA mechanism works and explore properties of this mechanism under two different assumptions about the availability of information. Fairness and efficiency are the two characteristics being mainly focused in every mechanism. Given perfect information, the equilibrium strategy of the preference revelation game induced by the CCA mechanism is not truthful revelation of preferences equilibrium, but it is fair and hence stable based on either true preferences or stated preferences, and Pareto efficient. If we relax the assumption of perfect information as students do not know scores of others, then it will be proved that the allocation is not necessarily fair, and telling the truth is not always an optimal strategy. Efficiency will be discussed separately under the assumption of imperfect information.

Although the CCA mechanism has been used for several decades in China, debates about the system never stopped. Criticisms mainly focus on the fairness and equality of opportunity for all candidates; design of exam questions; the pressure induced by exam on students; and corruption. There are voices to replace the current exam system by a free market system. In the market system, applications are not determined only by a one-off exam, but with the support of other materials including reports on student's performance in high school, references from teachers, and face-to-face interviews between the student and the

college. The argument supporting the current system indicates that corruption has been a very serious problem in the exam system, and it would be worse under a free market system since there are no standard criteria. This would lead to a more severe outcome that corruption results in unfairness and inequality of the student being educated at college. The second chapter of this thesis addresses the problem of corruption in college admissions by contrasting the two systems.

Corruption affects many aspects of higher education. It has a negative impact on the quality of colleges; it increases inequality in the access to higher education, and causes unfairness. Hallak and Poisson (2002) defined corruption in the education sector as "the systematic use of public office for private benefit, whose impact is significant on the availability and quality of educational goods and services, and, as a consequence on access, quality or equity in education". Corruption in education was observed in many places, and previous cases were seen in countries such as Russia, India, France, China, etc. The second chapter compares the market system and the exam system in a college admission environment from the perspective of corruption. We employ the auction model to discuss the efficiency and degree of corruption between the market system and the exam system. The principle of efficiency is to allocate the best resources (places of best college) to those agents (students) who can use them most efficiently. This principle is based on the assumption of complementarity in student's valuation of college. (Becker (1973))

Fernandez (1997, 1998) examined the performance and properties of the markets system and the exam system as alternative allocation devices with perfect capital markets. She shows that both systems would achieve efficient allocation results. However, when borrowing constraints exist, the exam system dominates the market system in terms of matching efficiency. In this thesis, we will contrast the performance of both systems under no borrowing constraints as well as under borrowing constraints. The difference from Fernandez's work is that we use auction theory to obtain the equilibrium and effects of corruption. To our knowledge, no researcher has previously used auction theory to analyse corruption in education with borrowing constraints. We will derive the equilibria of the two systems in a case with perfect capital markets and in a market with borrowing constraints, and establish the allocation results of both systems are efficient in a perfect market, but inefficient under borrowing constraints. We will also prove that the degree of corruption would be higher in the market system than in the exam system under both assumptions.

In France, there are two different systems for higher education admission. One

is through competitive examination, entrance examination or application form, with an interview where appropriate; the other is an open system, where all baccalaureate holders have the right to enter this system without any prior selection procedure. The former system is a simple two-sided college admission problem, which can be analysed by the original two-sided matching model or the priority-based mechanism model. The mechanism with competitive procedures naturally produces a positive sorting if a student's performance is positively correlated to his ability and the mechanism is well designed. Our question is, since students have the right to enter the system without any prior selection in the latter system, whether or not the sorting would still be positive assortative? If the answer is yes, then what is the driving force? This question will be answered in the last chapter of this thesis.

This piece of work is closely related to the literatures on the assortative sorting and allocation of resources in higher education. In the theory of marriage, Becker (1973) defines positive assortative matching as a positive correlation between the values of the traits of husbands and wives. He argues that positive assortative matching is generally optimal in most circumstances. The agent of likes or unlikes is optimal as traits are complements or substitutes, because superior types reinforce each other when traits are complements and offset each other when traits are substitutes. The condition in the theorem is commonly referred to as the (strict) supermodularity condition of the matching output function. Topkis (1998) gives a comprehensive mathematical treatment of supermodularity, and Milgrom and Roberts (1990) and Vives (1990) presents applications in game theory and economics. Arnott and Rowse (1987) find that any type partition in a case of allocating students to various classes within an elementary or secondary school is possible. The partition depends on the strength of peer effects.

Given a distribution of students' abilities and a limited pool of resources, we model the planner's decision to establish colleges, to design a "task level" for each college, and to optimize the allocation of resources. The cost of accomplishing the task is the critical factor of sorting students positively. Given all other factors being equal, students with different abilities will have different decisions due to different costs incurred. Kremer (1993) highlights the role of positive assortative matching in economic development. In his model of one-sided, many-to-many matching market, each firm consists of a fixed number of workers each employed for a production task. Workers have different skills, with a higher-skilled worker less likely to make mistakes in performing his task. Self-matching is obtained in equilibrium where each firm employs workers of identical skills. Kremer uses this

form of positive assortative matching to explain the large wage and productivity differences between developing and developed countries that cannot be accounted for by their differences in levels of physical or human capital. Epple and Romano (1998) study a competition between private and public schools and analyse the effects of vouchers under peer effects assumption. Epple, Romano and Sieg (2006) look through the admission, tuition, and financial aid policies in higher education market. They claim that colleges attempt to attract students with higher abilities by designing appropriate tuition and admission policies. Fernandez and Gali (1997), Fernandez (1998) argue that prices and borrowing constraints play the role of sorting students in the market system, and prices and exam results decide the matching of students to colleges in the exam system. In our model, we include costs into the student's utility function, and ignore the tuition and income effects on a student's utility.

The crucial assumption of our model is supermodularity in utility function and education output function of a particular student. Sallee, Resch and Courant (2008) claim that the assumptions of supermodularity in production function and fixed costs of building up colleges are sufficient to construct an optimal tiered system that sorts students by ability. Our model only assumes the supermodularity and ignores the peer effects as we will reach the same positive assortative sorting as well.

The main contribution of the third chapter, in contrast with the literature, is that we claim that student's consideration of cost is crucial when they decide whether or not and where to be educated. Based on the assumption of supermodularity for utility and production functions, we present a general equilibrium from a perspective of a central planner who has to sort students into colleges with different qualities. It can be applied to either the education market or any other occasions concerning sorting and resource allocation. The result of our model gives the design of task levels and resource allocation. Some numerical examples are presented to discuss the optimal number of colleges given the fixed cost of setting up a college as we are unable to solve the equilibrium in a general case. The conclusion is different from Sallee, Resch and Courant (2008)'s work. It would be shown that the planner only sets up a finite number of colleges even though the fixed cost of establishing a college is zero.

Contents

1 Chapter 1:
The Chinese College Admission Mechanism with Perfect and Imperfect Information about Student Scores **3**
1.1 Introduction . . . 3
1.1.1 Two-Sided Matching Market . . . 4
1.1.2 Priority-Based Allocation Mechanisms . . . 6
1.1.3 Introduction to the CCA Mechanism . . . 9
1.1.4 Comparisons and Contributions . . . 13
1.2 The Model . . . 15
1.3 Properties of the CCA Mechanism . . . 17
1.3.1 Perfect Information . . . 17
1.3.2 Imperfect Information . . . 24
1.4 Conclusion . . . 39
1.5 Appendix . . . 40

2 Chapter 2:
A Comparison between Two College Admission Systems From the Perspective of Corruption **47**
2.1 Introduction and Literature Review . . . 47
2.2 Description of the Economy . . . 52
2.2.1 Student, Authority, and Official . . . 52
2.2.2 Mechanisms in the Two Systems . . . 56
2.3 Model with One college, One Place under Perfect Capital Markets 58
2.3.1 Market System . . . 58
2.3.2 Exam System . . . 59
2.3.3 Market System vs Exam System . . . 62
2.4 Model with Multiple Places under Perfect Capital Markets . . . 63
2.4.1 One College with k Places . . . 63
2.4.2 Two Colleges with the Same Quality . . . 65
2.4.3 k Colleges with Different Qualities . . . 69
2.5 Model with Borrowing Constraints . . . 75
2.5.1 Market System . . . 76
2.5.2 Exam System . . . 79
2.5.3 Market System vs Exam System . . . 80
2.6 Conclusion . . . 83
2.7 Appendix . . . 85

3 Chapter 3:
The Cost of Attending College and Positive Self-Sorting 111
3.1 Introduction and Literature Review 111
3.2 The Model 118
3.2.1 Description of the Economy 118
3.2.2 The Decision Problem of Students 120
3.2.3 The Planner's Decision Problem 123
3.3 Equilibrium 126
3.3.1 Definition of Equilibrium 126
3.3.2 The Optimal Design When $N = 1$ 126
3.3.3 The Number of Colleges 132
3.3.4 Students' Welfare 143
3.4 Conclusion 145
3.5 Appendix 147

1 Chapter 1: The Chinese College Admission Mechanism with Perfect and Imperfect Information about Student Scores

1.1 Introduction

This chapter discusses a matching problem that focuses on a centralized college[1] admission system via standard exams. This model tries to mimic the current college admission system running in China. For simplicity, we call it as the Chinese College Admission (CCA) system. The CCA is operated as a centralized mechanism. The admission office in a province acts as the agent of the central government's Ministry of Education to be responsible for processing applications of admission to colleges in that province. Those students who have successfully completed their senior high school leading to a Certificate of General Education (equivalent to A-level in the UK) are eligible to take a nationwide unified examination in June of each year. The score of this exam is released one month later. Having obtained the score, students have to report their preferences of colleges to the provincial admission office. After the reporting deadline, the admission office then assigns places of each college to students based on their scores and reports about their orders of preferences over colleges. This process is a standard CCA procedure, and each admission office is only responsible for dealing with applications made in the province where the office is located. In the CCA system, students with higher scores have higher priority of being admitted. However, exam score is not the only determinant in the CCA system as the admission office takes account of students' reported preferences. This piece of work analyses the CCA mechanism from the perspective of two-sided matching market.

This introduction is organised as follows: The first two sections are brief literature reviews on two-sided matching market and priority-based matching mechanism; the third section is a detailed description of the CCA system; and the last section is to explain the difference between the CCA system and other related mechanisms as well as our findings and contributions, such as analysis on the CCA mechanism under perfect information and imperfect information.

[1] We simply use college to represent all undergraduate education institutions in China.

1.1.1 Two-Sided Matching Market

The college admission problem was initially studied by Gale and Shapley (1962). They introduce a marriage market model and a college admission model, which are regarded as the beginning of study on matching theories. In the college admission model, a set of places in colleges are allotted to a set of applicants. Each applicant ranks the colleges according to his preference, omitting those colleges to which he would never go under any circumstances. It is assumed that applicants are allowed to manipulate their preferences if they believe it would make them better off. They assume that there would not be any ties, which implies that no colleges are in the same order on a stated preference ranking list. Each college similarly has preference over applicants, excluding those students who will never be admitted under any circumstances.

Formally a college admission problem (Gale and Shapley (1962)) consists of:

1. a set of students $S = \{s_1, ..., s_n\}$,
2. a set of colleges $C = \{c_1, ..., c_m\}$,
3. a capacity vector $q = \{q_1, ..., q_m\}$, where q_j is the capacity of college c_j,
4. a list of student preferences $P^s = \{p_1^s, ..., p_n^s\}$, where p_i^s is the preference relation of student s_i over colleges including the no-college option,
5. a list of college preferences $P^c = \{p_1^c, ..., p_n^c\}$, where p_j^c is the preference relation of college c_j over students.

Each applicant has a strict preference on $C \cup \{c_0\}$, where c_0 denotes the no-college option. Let $cp_i^s c'$ denotes that student s_i strictly prefers college c to c'. Thus, an assignment could be defined given each triple lists of preferences of students, preferences of colleges, and capacities, i.e. (P^s, P^c, q).

Let $\mu(s)$ denote the assignment of student s under matching μ, and $\mu^{-1}(c)$ denote the set of students each of whom is assigned to college c under matching μ. An outcome of the college admissions model is a matching of students to colleges, $\mu : S \to C \cup \{c_0\}$, which is a function from the set of students to the set of colleges such that each student is matched to at most one college, and each college is matched to at most its quota of students, i.e., $|\mu^{-1}(c_j)| \leq q_j$.

Roth and Sotomayor (1990) introduce and provide a comprehensive account of college admission problems and other two-sided matching applications. They define a matching μ as blocked if, for a student-college pair (s, c), either

(1) Student s prefers college c to assignment $\mu(s)$ and college c has empty places under matching μ, or

(2) Student s prefers college c to assignment $\mu(s)$ and college c prefers student s to at least one of the students in $\mu^{-1}(c)$.

A pair that satisfies either (1) or (2) is called a block pair.

Definition 1 *A matching is stable if and only if there are not any block pairs.*

Gale and Shapley (1962) prove that a stable matching always exists by the deferred acceptance algorithm, which also produces an optimal solution for one side of the matching market. For example, in the college admission model, there are at least two stable results, which are selected by the student optimal stable algorithm and the college optimal stable algorithm. We describe the student optimal matching mechanism as follows.

Algorithm 1 *Student Optimal Matching Mechanism (Gale and Shapley(1962)):*

Step 1: Each student applies to his first choice. If the null object is the first choice of a student, then he is allotted the null object. Each college tentatively assigns places to its proposers one at a time according to its preference. Any remaining proposers are rejected.

In general, at

Step k: Each student who was rejected in step $k-1$ applies to his next favourite college. If the null object is the next favourite of an agent, then he is allotted the null object. Each college considers the set consisting of the students it has been holding and its new proposers, and tentatively assigns its places to those who have high order on its preference list. Any remaining students after all the places are assigned are rejected.

The algorithm terminates when each student is either holding a place at any college or has been allotted the null object.

At termination, each student is assigned his final tentative college or the no-college option. In fact, the student optimal stable mechanism was employed to address the pre-registration problem in the American medical labour market in the 1950's (Roth (1984)).

Gale and Shapley (1962) show that "Every applicant is at least as well off under the assignment given by the deferred acceptance procedure as he would be under any other stable assignment." The outcome of the student optimal stable mechanism has the following properties: stability, strategy-proofness, fairness,

resource monotonicity, and population monotonicity. But it violates other properties such as efficiency, consistency, and group strategy-proofness.[2] Dubins and Freedman (1981) and Roth (1982) establish that truthful-preference revelation is always in students' best interest in one-to-one matching such as marriage market. Roth (1985) and Roth and Sotomayor (1990) show it also holds for the many to one matching problem which is the case studied here. Roth (1982) shows that no stable matching mechanism exists that makes it a dominant strategy for all agents on both sides of the market to state their true preferences, which implies that, in the college admission problem, there does not exist any stable matching mechanisms that are strategy-proof for all colleges and students.

1.1.2 Priority-Based Allocation Mechanisms

Two-sided matching and resource allocation based on priorities are a commonly observed problem. A priority-based matching problem was first discussed by Roth (1991). A priority-based matching mechanism was used to match medical school graduates (interns) to supervising consultants in three regions of the UK from the late 1960's, but were eventually abandoned (Roth (1991), Ergin and Sönmez (2003)). The product of the student's ranking of the consultant and the consultant's ranking of the student is used as the basis for the priorities, thus, which are determined by student and consultant's preferences. The highest priority is referred as a (1, 1) match when a consultant and a student ranks each other as the first choice. If the consultant ranked the student first but the student ranked the consultant second, referred as a (1,2) match, they had a second highest priority, as did a consultant who ranked a student second but was ranked first by the student, a (2, 1) match. The two schemes differed in how they broke ties. It could be either in consultant's favour, so that a (1,2) match would have a higher priority than a (2, 1) match, or in the student's favour. Roth (1991) shows these schemes may produce unstable matching. Moreover, he shows any priority matching scheme will sometimes produce unstable matching.

[2]The definitions of these properties are as follows: (see Kesten(2004)

Strategy proofness: No agent ever benefits by misrepresenting his preferences.

Resource monotonicity: All agents are affected in the same direction (in welfare terms) whenever the set of available objects shrinks or expands.

Population monotonicity: All agents are affected in the same direction (in welfare terms) whenever some agents leave without their allotments.

Consistency: The recommendation for any given problem does not change after the departure of some of the agents with their allotments.

Group stategy proofness: No group of agents ever benefit by jointly misrepresenting their preferences.

Formal definitions of fairness and efficiency can be found in the next section.

In a priority-based mechanism, the key phase is submitted preferences determining priorities. In the case described in the last paragraph, priorities are purely determined by the rankings reported by consultants and students. In the following case, the school choice problem, priorities are determined by the preferences over colleges reported by students plus schools' orderings over students, but students' reported rankings are considered first.

The school choice problem is one of the most important priority-based matching problems. The most famous one is the Boston school choice mechanism,[3] which is in use at several U.S. school districts including Boston, Cambridge, Charlotte, Minnesota, Seattle and St. Petersburg-Tampa. The key difference between the school choice model and the two-sided matching model is that in the former schools are indivisible objects which shall be assigned to students based on student preferences and schools priorities whereas in the latter parties in both sides of the market are agents who have preferences over the other side and whose welfare are taken into consideration. While schools priorities are determined by the submitted preferences over schools as well as some other rules[4] imposed by law and do not necessarily represent schools tastes, one can formally treat colleges priorities as schools preferences and hence obtain a two-sided matching market (see Abdulkadiroğlu and Sönmez (2003), Balinski and Sönmez (1999), Ergin (2002), and Ergin and Sönmez (2006)).

In a school choice problem, there is a set of students each of whom will be placed in a school from a set of schools. For each school a strict ordering of students is determined according to the rules before admission process. Each student submits a preference ranking of the schools, which together with fixed priorities determine the choice of a matching. Under the Boston mechanism a student who is not assigned to his top ranked school is considered for his second choice only after the students who have top-ranked that student's second choice. Therefore a student may lose his priority at a school unless he ranks it as his first choice. The match priority likewise is lexicographic under the Boston school mechanism, which implies that any student-school pair that the student ranks the school as the first choice and the student is on the top of the school's ordering has the highest match priority. In fact, it is also lexicographic concerning preferences: It first considers the student preferences and only then the schools' ordering. A

[3]See Balinski and Sönmez (1999); Chen and Sönmez (2003); Abdulkadiroğlu and Sönmez (2003); Abdulkadiroğlu, Pathak, Roth and Sönmez (2005); Ergin and Sönmez (2006); Kesten (2004).

[4]Such as whether a student is handicapped, the proximity of his residence to the school, whether he has a sibling attending the same school etc.

similar mechanism was used in Edinburgh in 1967 and 1968 (Roth (1991)). The mechanism first considers the (1,1) match, i.e., the student ranks the school as the first choice and he has the first priority at that school; (1,2), (1,3), (1,4), and so forth follow the (1,1) match. Only when all student's first choice had been exhausted were other matches, (2,1),...; (3,1),...; etc.

The following algorithm is a brief description of the Boston mechanism.

Algorithm 2 *Boston Mechanism (Ergin and Sönmez (2006)):*

For each school a strict ordering of students is determined, each student submits a preference ranking of the schools, and the key phase is the choice of a matching based on fixed orderings and submitted preferences.

Round 1 : In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign places of the school to these students one at a time following the school's ordering until either there are no places left or there is no student left who has listed it as his first choice.

In general, at

Round k: Consider the remaining students. In Round k only the kth choices of these students are considered. For each school with still available places, consider the students who have listed it as their kth choice and assign the remaining places to these students one at a time following the school's ordering until either there are no places left or there is no student left who has listed it as his kth choice.

The procedure terminates when each student is assigned a place at a school.

Ergin and Sönmez (2006) suggest a transition from the Boston Mechanism to an alternative mechanism, the student-optimal stable mechanism, is likely to result in potentially significant welfare gains. This transition also eliminates the needs of students for gaming strategy because truthful preference revelation is a dominant strategy under the student-optimal stable mechanism.

Balinski and Sönmez (1999) study another priority-based matching problem, called student placement model. Their model precisely mimics the current Turkish college admissions practices, which was named "multi-category serial dictatorship". The Turkish college admission system is centralized and uses a student placement office to assign students to colleges, in fact to the particular faculties (e.g., engineering, medical, dental, business) of colleges. The "category" here refers to a faculty with a particular required skill. The multi-category serial dictatorship allocates places of a college in a particular category to applicants according to their ranking scores in the test of that category. They define fairness in a priority-based allocation mechanism as students with better test scores

being assigned their better choices. They show that the Turkish mechanism is not Pareto efficient and not even a second-best mechanism[5] among fair mechanisms. It is not strategy-proof, and does not necessarily respect improvements in scores. They also show the student optimal stable mechanism is the only second-best mechanism, and the only strategy-proof mechanism for college admission in Turkey.

There are arguments about whether subjective preference or objective ability should be the first consideration. Some[6] argue that among all fair assignments society should prefer efficient ones in which students are assigned according to their comparative advantages (e.g., ability which determines scores in the CCA mechanism), rather than by their personal preferences. Balinski, Sönmez (1999) think students should be given the greatest freedom of choice consistent with their aptitudes, since motivation is more important in ultimate success than mere scores on standardized exams.

1.1.3 Introduction to the CCA Mechanism

The CCA mechanism is a special case of the priority-based allocation mechanism. It is operated and managed by a non-executive agency in the Ministry of Education in the central government. Within the system, colleges are either directly administrated by the Ministry of Education (such institutions are classified as "key colleges of the state", these are usually referred as "the good/better colleges". The power of personnel appointment is held by the Ministry of Education.) or managed by local governments (these colleges are administrated by provincial governments or local governments; such institutions are usually referred as "ordinary colleges", the power of personnel appointment is held by the government at the local level.). Every year this authority offers a standardized exam that all students[7] who are planning to enter colleges have to take.

The admission system consists of two stages. Stage one is a standard exam, called National College Entrance Examination, and stage two is the recruitment procedure which starts immediately after exam results are released. In the exam, three subjects are mandatory for all students: Chinese, Mathematics and a foreign language. Apart from the three mandatory subjects, six other subjects are also being examined selectively depending on the courses students wish to study

[5] The second best mechanism does not induce a Pareto efficient outcome, but the best among all possible outcomes.

[6] See Fernandez (1998), Epple & Romano (2006), Sallee, Resch & Courant (2008).

[7] Some students do not need to take the test if they have been recommended to universities or colleges by their schools. The exceptions are few compared to total applicants.

in colleges. Besides scores obtained in the exam, the admission also depends on students' submitted order of preferences of colleges. The submitted preferences form consists of four parts, in which colleges in four phases are filled. Phase one is known as "Early Admissions", which deals with applications to degrees in education-related courses, applications to institutions of the armed forces and the police force, as well as applications to institutions in Hong Kong and Macau. Phase two is known as "Key Undergraduate Admissions" which deals with applications to institutions administrated by central government departments and institutions, in other words, the top universities in the country. Phase three is called "General Undergraduate Admissions" which deals with applications to institutions located in the capital of each province; these institutions are usually the top ones within the province. The fourth phase is dealing with the applications to the remaining institutions. Students are allowed to list four to six choices of institutions and courses in each of the admission phase.

The CCA system is similar to the Boston mechanism. The principle is that it first considers student preferences and only then colleges' orderings. For each college, a strict ordering of students is determined by exam scores. A college will consider sending offers to students who do not rank it as their first choice only after those who rank it as their first choice. The specific admission rules of each college are not necessarily the same, but most colleges adopt a policy called "Preference Clearance". "Preference Clearance" implies that, first, a college will consider to offer a place to a student who does not rank it as his first choice only when the college can not fill up the quota by those who rank it as their first choice. Second, the allocation of degree programmes in a college follows the same principle. Although in theory it is possible that a student who does not put a college as his first choice may still be admitted, the reality is that the chance is very small because of the large number of applicants. The following example illustrates the principle explicitly.

Suppose n students are applying to two colleges, c_1 and c_2, with quota q_1 and q_2. Firstly, students have to take the standard exam and report their preferences before or after the results of the exam are released. The admission office will consider the applications to $c_j\,(j = 1, 2)$ only from those who rank the college as their first choice on their submitted preference lists. The top q_j students (according to scores) who rank c_j as their first choice will be allocated a place at c_j. If the quota at c_j is not filled by students ranking the college as the first choice, the admission office will then allocate the spare places to those who rank c_j as the second according to their score ranking. Therefore, if there are more

students than colleges, then it is possible that a student fails to get a place in a college even if he has a higher score than at least one student in that college because he does not put that college at the top of his submitted preference.

Next we describe how the CCA mechanism operates. For simplicity, we assume there is only one phase in the admission process. Students take the exam and obtain scores. Then every student submits a preference list by ranking a limited number of colleges. Once scores and submitted preference lists are ready, the CCA mechanism operates in the following algorithm.

Algorithm 3 *The CCA mechanism:*

Round 1: Only the first choices of students are considered. Each college only considers to make offers to students who list that college as their first choice one at a time following their ranking scores, until either the admission quota is filled, or until all students who list the college as their first choice are offered a place.

Round 2: Only the second choices of students are considered. Each college considers to make offers to students who list that college as their second choice one at a time following their ranking scores, until either the admission quota is filled, or until all students who list the college as the second choice are offered a place.

.....

In general, at

Round k: All remaining students who were not offered a place in previous rounds of admission are considered. Each college with places available after previous rounds of admissions will consider and make offers to the remaining students who list that college as their kth choice one at a time following students' ranking scores, until either all remaining places are taken or until all students who list that college as the kth choice are offered a place.

The procedure terminates when each student is assigned a place at a college or there are no more places available in colleges.

The CCA mechanism has some special features that we need to make clear.

Base Score Base score is the minimum required score to enter a particular college. This requirement is not set by colleges but by the governmental education authority's admission office in each province. Since higher education admissions is administrated at the provincial level, each provincial admission office sets the base score after reviewing the overall performance of students in the exam as well as their submitted preferences of colleges. Admission office in each province sets

a base score for every college in that province and that base score is only valid for applicants who took the entry exams in that particular province where the admission office is located. Therefore, it is possible that a college have different base scores across different provinces. A student will surely be offered a place if he ranks a college as his first choice and his score is higher or at least the same as that college's base score. The central admission office publishes a Guide of Admission in every year, in which students can find the base scores of all colleges in different provinces in previous years as well as the recruiting plans of current year. By reading and working on this guidance, each student forms his own expected base scores of different colleges, and then decides his submitted preference of colleges.

Timing The current CCA mechanism may be operated in three possible sequences. The first sequence is that students have to report their preferences over colleges before taking the exam; the second sequence is that students report their preferences over colleges after exam but before results are released; and the third sequence is that students report their preferences over colleges after results are released. In the admission operation in 2005, five provinces adopted sequence one, sixteen provinces adopted sequence two, and ten provinces adopted sequence three.[8]

Obviously, it is the most difficult for students to decide their submitted preferences before taking the exam as they would not be certain about their performance in the exam as well as the average performance of other students. Sequence two makes students' lives easier since they may know how they did in the exams before reporting their preferences of colleges; but they would be uncertain about the results and how others did. Sequence three allows students to report their preferences of colleges after receiving exam scores. It is a huge advantage for students to maximise the possibility of being admitted since information released by the admission office would contain information on the distribution of overall student performance in the exam as well as the average scores and the highest individual scores. The more information students have on the overall exam performance, the less uncertainty there will be. As a general trend, more provinces are using sequence three in admission operations; and therefore, we only consider the system with sequece three.

Regional discrimination There exists regional discrimination in the CCA mechanism. As a general practice, colleges allocate more admission quota to

[8]Data was from the official website of Ministry of Education of the People's Republic of China: http://www.moe.edu.cn/.

the province where they locate than other provinces. Together with different distributions of students' abilities, such discrimination may cause within a single college, it has different base scores from different provinces. This discrimination induces problems in efficiency and revelation of preferences. This issue is caused by some degrees of political considerations and influences and will not be discussed in the current context.

1.1.4 Comparisons and Contributions

The CCA mechanism is a special case of a priority-based matching mechanism. As we mentioned in the last section, a priority-based matching mechanism is different from the Gale and Shapley's college admission model. The CCA mechanism is not an exception. In Gale and Shapley's model, the outcome of matching depends on the preferences of colleges over students and students over colleges. In the CCA mechanism, students are the only agents, and colleges are merely objects to be consumed as public goods. So the main difference between the CCA mechanism and Gale and Shapley's mechanism is that scores take the place of preferences of colleges over students and together with submitted preferences over colleges determine allocations. Therefore, in our model, we do not consider welfare issues or strategic behavior for the college side. Notions such as the stability that are central for college admissions problems do not have any direct meaning in the CCA problems. However, this does not mean that the findings in Gale and Shapley's model are irrelevant in the CCA mechanism. We can formally treat scores as college preferences and hence obtain a two-sided matching market. Consequently concepts/findings in a two-sided matching have their counterparts in our model. Stability is the central notion in the two-sided matching problem and it is still important in a priority-based matching problem. Ergin and Sönmez (2006) define stability in a priority-based matching market as: a matching is not stable if there is a student-college pair (s, c) such that (1) student s prefers college c to his assignment; (2) either school c has some empty places or student s has higher priority than another student who is assigned a place at college c. Balinski and Sönmez (1999) show that a matching is individually rational, non-wasteful and fair[9] for a priority matching problem if and only if it is stable for its associated college admissions problem.

The CCA system is similar to the Boston mechanism. They both belong to the priority-based matching michanism, in which reported preferences of students

[9]We will define individually rational, non-wasteful, and fair allocation in next section.

determine students' priorities. The main difference between the CCA mechanism and the Boston school choice mechanism is colleges' orderings over students in the CCA system are only determined by students' scores, which are obtained from a standard nationwide exam, while schools' orderings in the Boston Mechanism are dependent upon several aspects. As in the Boston Mechanism, students in the CCA mechanism may have incentives to misrepresent their preferences. The critical reason for such manipulation is that students can possibly lose their priorities of being admitted to a college unless they rank that college as the first choice. If those students who report the true preferences are rejected by the reported first choice, then they may have to accept offers from other colleges which are on lower ranks of their list of true preferences, or they may lose their priorities in all colleges and hence end up being excluded from entering higher education in the worst scenario.

Let us take a look at a simple example that shows the CCA mechanism may induce unstable matching according to true preferences. Suppose two student s_1 and s_2 with scores t_1 and t_2 and $t_1 > t_2$. s_1 prefers college c_2 to c_1, but he ranks c_1 as the first choice because he is not sure if he can be allocated a place at c_2. In the end, assume that s_1 is assigned a place at c_1, but it is possible that c_2 has some open places as some other applicants may have the same thought as s_1, or s_2 has been allotted a place at the college c_2 because he ranks c_2 as the first choice. So, it is unstable according to the true preferences. The interesting part is the result is stable based on reported preferences. The example shows that, in the CCA mechanism, a student may have incentive to misrepresent his preference because he needs to keep his priority in the second choice or even less favourite universities when he is not guaranteed to be allocated a place in his favoured college.

Compared to litreatures on the Boston mechanism, the main contribution of the chapter includes three aspects. First, we describe the CCA system from the perspective of economics. Second, we build up a formal model for the system under the assumption of perfect information on exam scores. In Nash equilibrium of the mechanism, a student who realises that there are no vacancies in his favourite college may misrepresent his preference, and rank higher the best of those colleges whose base scores are lower or equal to his score. We will show that the outcome of the CCA mechanism is the same as the one selected by a recursive algorithm and the student optimal stable matching mechanism. The set of Nash equilibrium outcomes is equal to the set of stable matching under the students' true preferences. Given perfect information, the CCA mechanism

is not truthful revelation of preferences, but it is fair, stable based on either true preferences or stated preferences, and Pareto efficient. Third and last, we present a discussion on the CCA mechanism with imperfect information about the exam score. There are two colleges and three students, and a finite number of scores dividing students into two different types. The probability of a student being high type is public knowledge. We discuss the symmetric pure Bayesian Nash equilibria and the symmetric mixed strategy Bayesian Nash equilibria. Whether the model reaches a symmetric pure equilibrium or a symmetric mixed strategy equilibrium depends on the utility of the better college and the probability of being a high type student. Given imperfect information, the mechanism is fair and strategy-proof when the utility is high and the probability is low. In that case, we have a symmetric pure strategy equilibrium that both types of students rank the better college as the first choice. We also discuss the effects of changes of the utility and the probability on equilibria. The conclusion of the simple model is used to explain two interesting observations in reality. Contrast to the well known Boston School system literature, our work discusses the CCA mechanism, which has some different properties from the Boston School system,

The organisation of the rest of the chapter is as follows. In section 2, we describe the model for the CCA mechanism. In section 3, we illustrate the properties of the mechanism under the assumption of perfect information. In section 4, we discuss the main results of the CCA mechanism under the assumption of imperfect information. In section 5, we present some concluding remarks and possible further directions of research.

1.2 The Model

We define the ingredients in the CCA mechanism as follows.

1. a set of students $S = \{s_1, ..., s_n\}$;
2. a set of colleges $C = \{c_1, ..., c_m\}$;
3. a capacity vector $q = \{q_1, ..., q_m\}$, total number of places $Q = \sum_{j=1}^{m} q_j$;
4. a list of students' preferences $P_s = \{p_1, ..., p_n\}$, where p_i is the strict preference relation of student s_i over colleges including the no-college option;
5. a list of students' scores $t = \{t_1, ..., t_n\}$;

6. a list of reported preferences $R_S = \{r_1, ..., r_n\}$;[10]

7. a list of the base scores of colleges $B = \{b_1, ..., b_m\}$;

Each applicant has a strict preference on $C \cup \{c_0\}$, where c_0 denotes the no-college option and $q_0 = |S|$. Let $c_i p_i c_j$ denote that student s_i strictly prefers college c_i to c_j. Given each triple lists of reported preferences of students, student test scores, and capacities of colleges (R_S, t, q), we can define an assignment via the CCA mechanism.

To implement the CCA mechanism, described earlier, in the context of the model, it is assumed that students have to submit a strict preference ranking over all colleges and the no-college option. We will relax this assumption later.

A matching is an allocation of college places to students such that no student occupies more than one place. Formally it is a function $\mu : S \to C \cup \{c_0\}$ such that $|\mu^{-1}(c_j)| \leq q_j$, $\forall c_j \in C$. Student s_i is not assigned any college place if $\mu(s_i) = c_0$. Given a preference relation p_i of student s_i, initially defined over $C \cup \{c_0\}$, it is extended to the set of matching in the following natural way: student s_i prefers the matching μ to the matching μ' if and only if she prefers $\mu(s_i)$ to $\mu'(s_i)$. Student s_i will be assigned a place at college c_j if 1) he ranks c_j as the first choice in his submitted preference and $t_i \geq b_j$ (his score is higher than the base score of college c_j); 2) he ranks c_j as the kth choice, $t_i \geq b_j$ and the admission quota of c_j can not be filled up in last $k-1$ rounds.

Next we define some properties of student placement models summed up by Balinski and Sönmez (1999).

Individual Rationality: A matching μ is individually rational if no student is assigned to a college that is worse than the no-college option. In other words, a student will not be assigned a place at a college that he would never want to attend. Formally a matching μ is individually rational if whenever $\mu(s_i) \neq c_0$, $\mu(s_i) p_i c_0$ for all $s_i \in S$.

Non-Wastefulness: A matching μ is non-wasteful if, whenever a student s_i prefers c_j to his assignment c_i, c_j has all its places filled up. Formally a matching μ is non-wasteful if $c_j p_i \mu(s_i)$ and $|\mu^{-1}(c_j)| = q_j$ $\forall$ all $s_i \in S$ and $c_j \in C$.

Pareto Efficiency: A matching μ Pareto dominates a matching μ' if no student prefers μ' to μ and there is at least one student who prefers μ to μ'. Formally, a matching μ Pareto dominates a matching μ' if $\mu r_i \mu'$ $\forall$ all $s_i \in S$ and $\mu p \mu'$ for at

[10]Here we assume that students can include as many colleges as they want. But as we described in the introduction, in the practice, students can only list a limited number of colleges in their reported preferences.

least one student.[11] A matching μ is Pareto efficient if it is not Pareto dominated by any other matching.

Strategy-Proofness: A matching μ is strategy-proof only if the truthful-preference revelation of preferences is always a dominant strategy for all students.

Fairness: Students with higher test scores should be assigned their more preferred choices. Formally a matching μ is fair if $\mu(s_l) = c_j$, then $c_j p_i \mu(s_i)$ implies $t_l > t_i \ \forall$ all s_i and $s_l \in S$.

A mechanism is individually rational, non-wasteful, Pareto efficient, strategy-proof and fair if it always selects an individually rational, non-wasteful, Pareto efficient, strategy-proof and fair matching. Pareto efficiency implies individual rationality and non-wastefulness, but the reverse is not necessarily true. An allocation is fair if no agent envies any other agent whose allotment he has higher priority for. The mechanism is fair if it always selects fair allocations. Balinski and Sönmez (1999) argue that a matching is individually rational, non-wasteful and fair for a placement problem if and only if it is stable for its associated two-sided matching problem. Although these properties are summed up for a placement problem, the first four can be related to a general matching model and all of them can be applied to a general priority-based matching problem.

The CCA mechanism satisfies individual rationality because no students will be allocated to any colleges which are not on their submitted preferences. As regards other properties, we will discuss them for the CCA mechanism under different assumptions in the next section.

1.3 Properties of the CCA Mechanism

1.3.1 Perfect Information

Perfect information means any student's scores and preferences over colleges are known by all students and this is common knowledge. At this state, we assume there are no ties between scores

Before we establish the equilibrium of the model, let us look at a simple example, which illustrates how the mechanism works under the assumption of perfect information.

Example 1 *Let $S = \{s_1, s_2\}$ be the set of students, $C = \{c_1, c_2\}$ be the set of colleges, and $q = \{1, 1\}$ be the college capacity vector. Scores and preferences of*

[11] Here $\mu r_{s_i} \mu'$ implies that μ is at least as good as μ' for s_i.

students are as follows:

$$t = \{t_1, t_2\}\,,\ \textit{with}\ t_1 > t_2,$$

$$P_s = \{p_1, p_2\}\,, \textit{with}\ p_1 = p_2 = \{c_1 \succ c_2 \succ c_0\}.$$

Student s_1 knows his score is higher than s_1, thus he does not need to manipulate his preference, thus $r_1 = \{c_1 \succ c_2 \succ c_0\}$. Student s_2 has two options: $r_2 = \{c_1 \succ c_2 \succ c_0\}$ or $r_2' = \{c_2 \succ c_1 \succ c_0\}$. We have the reported preferences as follows:

$$R_S = \{r_1, r_2\}\,,$$

or

$$R_S' = \{r_1, r_2'\}\,.$$

If the reported preference set is R_S, then the system works as follows:

In Round 1, only students' first choices are considered. College c_j, $j = 1, 2$, makes offers to students who list it as their first choice by following their ranking scores until either all places are taken or all students who list the college as first-choice are offered a place. Student s_1 is assigned a place at college c_1, student s_2 is rejected by college c_1 and the process goes to next round;

In Round 2, only students' second choices are considered by the college with places available. Thus, college c_2 assign the remaining place to student s_2. The procedure terminates here since each student is assigned a place at a college.

If the reported preference set is R_S', then the system works as follows:

In Round 1, student s_1 is assigned a place at college c_1, s_2 is assigned a place at college c_2 and the admission process is over.

In both cases, the matching induced by the CCA mechanism in this example is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix},$$

i.e., $\mu(s_i) = c_i$, $i = 1, 2$. Therefore, R_S and R_S' are two Nash equilibria for this example.

Since information is perfect, the student with the higher score knows he would have a higher priority in any college that he ranks as the first choice. The student with the lower score knows he would not have a higher priority no matter which college he ranks as the first choice. The student with the lower score will be assigned a place at the college which he prefers but is not preferred by the student with the higher score,[12] otherwise he would have to consider accepting the offer

[12] In this example, it is the case if $p_2 = \{c_2 \succ c_1 \succ c_0\}$.

from his less-preferred college, or he might end up with c_0 if he rejects that less preferred offer.[13] Within the matching of this example, the student with the higher score is assigned a more preferred place and there are no empty places left, thus the outcome is individually rational, fair, non-wasteful, and hence stable. Therefore, if there are only two agents on both sides of the market, given the information is perfect, the matching induced by the CCA mechanism would be stable, and truthful revelation of preferences is a Nash equilibrium strategy for both students.

Next we explain what would happen if there are more students and more colleges. Firstly, we keep $n = 2$, and increase the total quota of colleges, i.e., $Q > 2$. Holding at the number of students, but increasing the number of places will not affect the outcome in Example 1. The only difference is that students have more choices. A student with a higher score obtains a place which he most prefers, and a student with a lower score is assigned a place which he most prefers apart from the one assigned to student with higher score. Therefore, the result must be stable and truthful revelation of preferences is a Nash equilibrium strategy for both students.

Secondly, suppose $m = 2$, $\{q_1 = 1, q_2 = 1\}$, and $n > 2$. Consider the following Example:.

Example 2 *Assume there are three students* $S = \{s_1, s_2, s_3\}$*, and two colleges* $C = \{c_1, c_2\}$*, the quota is* $q = \{q_1 = 1, q_2 = 1\}$*. Students' preferences and scores are as follows:*

$$P_s = \{p_1, p_2, p_3\},$$

with

$$p_1 = p_2 = p_3 = \{c_1 \succ c_2 \succ c_0\};$$

$$t = \{t_1, t_2, t_3\},$$

with

$$t_1 > t_2 > t_3.$$

s_1 will submit his true preference because he has the highest priority in either college. s_2 realised that he has lower priority than s_1 in c_1 and s_1 will rank c_1 as his first choice. Besides, s_2 will lose his priority in c_2 if he reports his true preference and s_3 reports $r_3 = \{c_2 \succ c_1 \succ c_0\}$. Therefore, s_2's best strategy is to list c_2 as his first choice with c_1 as the second choice. s_3 knows he will be assigned a place at college c_2 if he manipulates his preference and s_2 tells the

[13] In this example, it is the case if $p_2 = \{c_1 \succ c_0 \succ c_2\}$.

true preference. Therefore, we can have a Nash equilibrium strategy for reported preferences as follows:[14]

$$\begin{aligned} r_1 &= \{c_1 \succ c_2 \succ c_0\}, \\ r_2 &= r_3 = \{c_2 \succ c_1 \succ c_0\}. \end{aligned}$$

College c_1 assigns its place to student s_1, college c_2 assigns its place to s_2, and student s_3 has to take c_0. Thus the outcome is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_0 \end{pmatrix}.$$

Since there are only three students in this example, the base score of college c_j is just the score of the student who is admitted by c_j. Thus, $B = \{t_1, t_2, t_3\}$. In a general case, the base score of a particular college is the equilibrium score of the least successful placed student in that college if it is filled up. Student s_i can observe the ranking of scores and preferences of other students, hence he is able to work out the base score of each college. s_i will not rank his true first option, say c_1, as the first choice unless $t_i \geq b_1$. In other words, s_i's first choice is his most favourite among those colleges whose base scores are lower than or the same as t_i. The matching in the example is individually rational, fair, and non-wasteful again as students with higher priority are assigned a more preferred place and all places have been filled up. In fact, the result of this example can be generalised as follows.

Proposition 1 *Given perfect information, the CCA mechanism is individually rational, non-wasteful, fair and Pareto efficient; but truthful revelation of preferences may not be a dominant strategy for* ***all*** *students.*

Proof. See Appendix. ■

Given perfect information, we have a Nash equilibrium in the CCA mechanism, which has the property that a student for whom there are no vacancies in his favourite colleges may misrepresent his preference, and rank higher his most favourite among those colleges whose base scores are lower than or the same as his score. Thus, the Nash equilibrium must be characterised as follows: the student with the highest score must get his most preferred option; the student with the second highest score must get his most preferred option from the remaining

[14]The other Nash equilibrium would be $r_1 = r_3 = \{c_1 \succ c_2 \succ c_0\}$, and $r_2 = \{c_2 \succ c_1 \succ c_0\}$. Both equilibria would lead to the same output.

places, ..., etc. The process would terminate in the first round by this Nash equilibrium.

This is not the only Nash equilibrium. For example, the student with the lowest score may have different strategies. Let $\{s_1, s_2, ..., s_3\}$ denote the set of students with $t_1 > t_2..., t_{n-1} > t_n$. If s_n is only able to find available places at his least preferred college, then he may rank any college as his first choice. If he ranks the college with place available as his first choice, which is the first equilibrium, then the admission process ends in the first round, two rounds otherwise. In the end, $\{s_1, ..., s_{n-1}\}$ are allocated a place at their reported first choices and s_n is allocated a place at his least preferred college.

Given perfect information, the CCA mechanism is equivalent to the so-called "recursive algorithm". While the two algorithms are different in general, they always yield the same outcome. Given a set of students $S = \{s_1, ..., s_n\}$ with $n > 2$; a set of colleges $C = \{c_1, ..., c_m\}$; a capacity vector $q = \{q_1, ..., q_m\}$; a list of student preferences $P_s = \{p_1, ..., p_n\}$; a list of student test scores $t = \{t_1, ..., t_n\}$ with $t_1 > t_2..., t_{n-1} > t_n$, the recursive algorithm is as follows:

Algorithm 4 *Recursive Algorithm:*

Round 1: s_1 reports his preference and he will be allocated a place at his reported first choice;

Round 2: s_2 reports his preference and he will be allocated a place at his reported first choice if there is a place available; otherwise he will go to the no-college option and the process goes to the next round;

.....

In general, at

Round k: s_k reports his preference and he will be allocated a place at his first choice if there is place available, otherwise he will go to the no-college option and the process goes to the next round

The procedure terminates when each student is assigned a place or there are no places available.

It is easy to obtain a Nash equilibrium for the recursive algorithm. In the first round, s_1 reports p_1, and he will be allocated a place at his most preferred college; in the second round, s_2 ranks his most preferred college among those remaining colleges as his first choice, and he will be allocated a place at that college; and so on.

In contrast with the equilibria of the CCA mechanism, the process in the recursive algorithm lasts k rounds while the CCA mechanism may only last one

or two rounds as we noted, but they will reach the same outcome. In both mechanisms, students with higher priority are considered before students with lower priority.

Remark 1 *Given perfect information, the CCA mechanism and the recursive algorithm select the same matching.*

We know the outcomes induced by the CCA mechanism given perfect information are fair and stable under students' true preferences. Ergin and Sönmez (2006) show that Nash equilibrium outcome of the Boston School mechanism is equivalent to the stable matching under the true preferences. This claim holds in the CCA mechanism.

It is well known that, in a priority matching game, the Deferred Acceptance Algorithm Pareto dominates any other fair mechanisms (Balinski and Sönmez (1999)), and it is strategy-proof (Dubins and Freedman (1981)). The following algorithm describes how the Deferred Acceptance Algorithm is applied to the CCA circumstance.

Algorithm 5 *Deferred Acceptance Algorithm:*

Round 1: Each student applies to his favourite college. If c_0 is the favourite object of a student, then he is allotted c_0. If the total number of students who apply to college is greater than its quota, say q_j, then q_j students with the highest scores are assigned a place temporarily. The remaining students are rejected.

Round 2: Each remaining student applies to his favourite college. If c_0 is the favourite object of a student, then he is allotted c_0. If the total number of students who apply to a college including those students who were temporarily allocated a place at Round 1 is greater than its quota, say q'_j, then q'_j students with the highest scores are assigned a place temporarily. The remaining students are rejected.

.....

Round k, $k \geq 2$: Each student who is rejected by a college at step $k-1$ applies to his next favourite college. If c_0 is the favourite object of a student, then he is allotted c_0. If the total number of students who apply to a college, say c_k, is greater than its quota, q_k, then q_k students among them with the highest scores are assigned a place temporarily. The remaining students are rejected.

The procedure terminates when each student is assigned a place at a college or there are no places left in colleges.

This mechanism is also referred to the "student optimal stable mechanism". The main differences between the student optimal stable mechanism and the

CCA mechanism under perfect information is: it is always optimal for students to report their true preferences in the former mechanism; students may manipulate their preferences by ranking colleges, which have low order on their true preference lists, as their first choices in order to secure a place at those colleges in the latter mechanism.

Summary 1 *Given perfect information, any Nash equilibrium in the CCA mechanism, the recursive algorithm, and the student optimal stable mechanism has the same matching result, and satisfy individually rationality, non-wastefulness, fairness and Pareto efficiency. The first two mechanisms do not satisfy strategy-proofness but the last one does.*

One of assumptions in the perfect information case is that no ties exist between scores. Now we use an example to illustrate the model may violate fairness if we relax this assumption to allow the existence of ties between scores, and varying the example slightly, that Pareto efficiency can fail if students can only list a limit number of colleges in their reported preferences and also there are score ties.

Suppose all students are risk neutral, and a place will be allocated to a student with $1/k$ probability if there are k students with the same scores applying to this place.

Example 3 *Assume there are three students* $S = \{s_1, s_2, s_3\}$*, and two colleges* $C = \{c_1, c_2\}$*, the quota is* $q = \{q_1 = 1, q_2 = 1\}$*. Students' preferences and scores are as follows:*

$$P_s = \{p_1, p_2, p_3\},$$

with

$$p_1 = p_2 = p_3 = \{c_1 \succ c_2 \succ c_0\};$$

$$t = \{t_1, t_2, t_3\},$$

with

$$t_1 = t_2 > t_3.$$

The utility of c_1*,* $u_1 = 1$*, the utility of* c_2*,* $u_2 \equiv 0.1$*, and the utility of* c_0*,* $u_0 = 0$ *to all students.*

Given that s_1 ranks c_1 as the first choice, if s_2 ranks c_1 as his first choice, then his expected payoff is $0.5u_1 = 0.5$; if s_2 ranks c_2 as his first choice, then his expected payoff is $u_2 = 0.1$. Therefore, s_2's optimal strategy is to rank c_1 as his first choice. s_1 has the same optimal strategy as does s_2. Since all this

information is publicly known by all students, s_3 would rank c_2 as his first choice because he knows he has chance to be admitted by c_2. In the end, the place at c_1 will be allocated to s_1 or s_2 randomly, and the place at c_2 is allocated to s_3. If, say, s_1 gets the place at c_1, then the outcome is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_0 & c_2 \end{pmatrix}.$$

Clearly, the outcome is not fair.

Next we assume that $t_1 = t_2 = t_3$. Students can only put one college in their submitted preferences, and a student will end up with c_0 if he is not allocated any place. Since $1/3u_1 > 0.1$, all students will only put c_1 in their preferences. One of the three students will get the place at c_1 randomly, and others have to select c_0. If, say, s_1 gets the place at c_1, then the outcome is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_0 & c_0 \end{pmatrix}.$$

It is clear that the outcome is not Pareto efficient. Note that if there are no score ties, then the outcome selected by the CCA mechanism would be the same as the unrestricted case even though the number of colleges in reported preferences is restricted.

The two examples imply that the CCA mechanism may violate fairness and Pareto efficiency if we relax assumptions such as no score ties, or an unlimited number of colleges in reported preferences.

1.3.2 Imperfect Information

In the last section, we discussed the CCA mechanism under the assumption that students have all relevant information about each other, including preferences over colleges and scores. In practice, scores and preferences are not public knowledge, and hence students have imperfect information. In this section, we look through the CCA mechanism under the assumption of imperfect information. As a starting point, we simply assume that all the students have identical preferences and this is common knowledge. Therefore, the only private information is their scores, which are determined by a number of factors, such as the student's academic ability, his performance on the exam day, and luck. How scores are realised is not our concern; thus we simply assume that the actual realisation of scores is observed only by the student, but its ex ante probability distribution is commonly known among all students. Now the mechanism is a Bayesian game:

Nature makes the first move, choosing realisations of the scores that determine each student's type; each student only observes the realisation of his own score but the distribution of scores is publicly known; each student decides how to fill in his preferences form based on available information. Thus, here we are looking for a Bayesian Nash equilibrium.

Our concern for the CCA mechanism is on those students whose scores are on the margin of two colleges. The problem of those students is to decide whether to rank the better college or the safe alternative as the first choice. For the same set of colleges (a good one and an ordinary one), marginal students should have similar academic abilities among them. For simplicity, we assume these students have the same academic ability, and then they have the same probabilities of obtaining a high score and a low score given other symmetric factors. Given these assumptions, we simplify the problem as follows.

Assume there are two colleges: c_1 with utility $u > 1$, and c_2 with utility 1;[15] three students: $s_i\,(i = 1, 2, 3)$; and two possible scores: $t_h > \overline{b}_1 > t_l$, where $\overline{b}_1$ denotes the base score of c_1 in the previous year. We call the students with t_h as high type, and the students with t_l as low type. All students know their own scores and the probability that another student is high type, denoted by p, and the probability that another student is low type, denoted by $1 - p$. The place of a college will be randomly allocated to a student if there is more than one applicant to that college with same score. A college will consider sending offers to students who do not rank it as the first choice only if no students ranked it as the first choice.

Since there are two colleges, students have two possible strategies: Either ranking c_1 as the first choice or ranking c_2 as the first choice. Denote the first and second strategy by r_1 and r_2 respectively. Define x as the probability that a high type student plays r_1, and y as the probability that a low type student plays r_1, that is, assuming a symmetric equilibrium in which strategies do not depend on the student's identification. Therefore, the probability that a high type student plays r_2 is $1 - x$, and the probability that a low type student plays r_2 is $1 - y$. Given this structure of mixed strategies, there are four possible pure strategies: $x = 1$, $y = 1$, i.e., both types play r_1; $x = 1$, $y = 0$, i.e., high type plays r_1 and low type plays r_2; $x = 0$, $y = 1$, i.e., high type plays r_2 and low type plays r_1; $x = 0$, $y = 0$, i.e., both types play r_2.

In order to find equilibria of the game, we need to derive the expected utilities for the two types of students. Consider a particular student, say s_1. Nature

[15] Here we simply assume that there is only one place at each college.

decides his type to be high or low with probability vector $(p, 1-p)$. We will work out s_1's expected utilities when he is a high type and when he is a low type respectively.

Claim 1 *(1)* $t_1 = t_h$.

When he is a high type student, s_1*'s expected utilities from playing* r_1 *and* r_2 *are respectively as follows:*

$$EU_{r1}^h = u\,(1-px) + \frac{p^2x^2}{3}\,(1+u) + pxy\,(1-p)\,,$$

$$EU_{r2}^h = \frac{1}{3}p^2(x-1)\,[x+u(x-3y+2)-1] + p(x-1)\,[1+u(y-1)] + 1.$$

(2) $t_1 = t_l$.

When he is a low type student, s_1*'s expected utilities from playing* r_1 *and* r_2 *are respectively as follows:*

$$\begin{aligned} EU_{r1}^l &= \frac{1}{3}(p-1)y\,[p(y-3x)-y] \\ &\quad + \frac{1}{3}u\left[3-3y+y^2+p\left(3y-6x+3xy-2y^2\right)+\frac{1}{3}p^2(3x^2-3xy+y^2)\right], \end{aligned}$$

$$\begin{aligned} EU_{r2}^l &= p^2x^2 - px(p-1)(1+y) \\ &\quad + \frac{1}{3}(p-1)^2(1+y+y^2) + \frac{1}{3}u(p-1)(y-1)\,[1-y+p(2-3x+y)]\,. \end{aligned}$$

Proof. See Appendix. ■

For the symmetric pure strategy Bayesian Nash equilibria, we have the following proposition:

Proposition 2 *There are two possible symmetric pure Bayesian Nash equilibria in our model under the assumption of private information about scores:*

1. $x = 1$, $y = 1$, *i.e., both types playing* r_1, *is a symmetric pure strategy Bayesian Nash equilibrium, if*

$$u \geq \frac{2-p+2p^2}{(p-1)^2};$$

2. $x = 1$, $y = 0$, *i.e., high types playing* r_1 *and low types playing* r_2, *is a symmetric pure strategy Bayesian Nash equilibrium, if*

$$\frac{3-p^2}{3-3p+p^2} \leq u \leq \frac{1+p+p^2}{2(p-1)^2}$$

and $p \geq 0.2787$.

Proof. See Appendix. ■

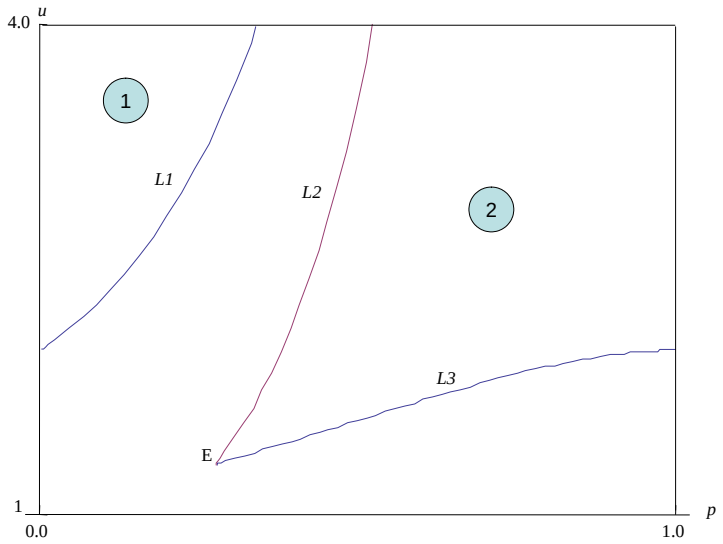

Figure 1: Pure Strategy Equilibria

For $u \in [1, 4]$, Figure 1 shows possible symmetric pure strategy equilibria. All the points in Region 1 meet the requirement for the first equilibrium, which is $x = 1$, $y = 1$. Region 2 includes all of the possibilities that satisfy the condition for the second equilibrium, which is $x = 1$, $y = 0$. Point E denotes the starting point of the second equilibrium with coordinate $(0.279, 1.304)$.[16]

On the curve $L1$, a low type student is indifferent between choosing $r1$ and r_2 if both types of students play r_1. Thus, the curve $L1$ is determined by the solution to the following equation:

$$EU^l_{r1}(x = 1, y = 1) = EU^l_{r2}(x = 1, y = 1).$$

On $L1$, $EU^h_{r1}(x = 1, y = 1) > EU^h_{r2}(x = 1, y = 1)$.

On the curve $L2$, a low type student is indifferent between playing $r1$ and r_2 if all high type students play r_1 and all other low type students play r_2. Therefore, the curve $L2$ is determined by the solution to the following equation:

$$EU^l_{r1}(x = 1, y = 0) = EU^l_{r2}(x = 1, y = 0).$$

[16]The coordinates keep three figures after the decimal point.

On $L2$, once again, $EU^h_{r1}(x=1,y=0) > EU^h_{r2}(x=1,y=0)$.

On the curve $L3$, a high type student is indifferent between playing $r1$ and playing r_2 if all other high type students play r_1 and all low type students play r_2. Curve $L3$ is determined by the solution to the following equation:

$$EU^h_{r1}(x=1,y=0) = EU^h_{r2}(x=1,y=0).$$

On $L2$, $EU^l_{r1}(x=1,y=0) < EU^h_{r2}(x=1,y=0)$.

In other regions, we have mixed strategies. See Figure 2.

Proposition 3 *There are three possible symmetric mixed strategy equilibria as follows:*

1. *In Region 3, high type students play a symmetric mixed strategy with* $(x, 1-x)$ *and low type students play the symmetric pure strategy of choosing* r_2*, i.e.,* $0 < x < 1$ *and* $y = 0$*;*

2. *In Region 4, low type students play a symmetric mixed strategy with* $(y, 1-y)$ *and high type students play the symmetric pure strategy of choosing* r_1*, i.e.,* $0 < y < 1$ *and* $x = 1$*;*

3. *In Region 5, both types play symmetric mixed strategies with* $\{(x, 1-x), (y, 1-y)\}$*, i.e.,* $0 < x < 1$ *and* $0 < y < 1$*.*

Proof. See Appendix. ■

In Region 3, a low type student's expected utility of playing r_2 is higher than playing r_1. But the difference is falling as u decreases until the curve $L4$ is reached, on which the low type student is indifferent between playing r_2 and playing r_1 given that high type students play the equilibrium mixed strategy and all other low type students play r_2. Therefore, the curve $L4$ is determined by the solution to the following simultaneous equations:

$$\begin{aligned} EU^h_{r1}(y &= 0) = EU^h_{r2}(y=0), \\ EU^l_{r1}(y &= 0) = EU^l_{r2}(y=0). \end{aligned}$$

In Region 4, a high type student's expected utility of playing r_1 is higher than playing r_2. But the difference is falling as u decreases until the curve $L5$, on which the high type student is indifferent between playing r_1 and r_2 given low type students play the equilibrium mixed strategy and all other high type

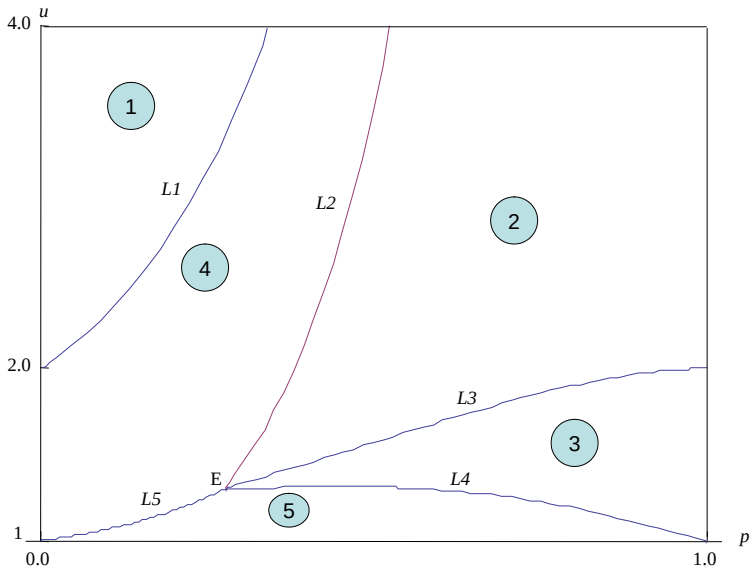

Figure 2: Mixed Strategy Equilibia

students play r_1. Thus the curve $L5$ is determined by the solution to the following simultaneous equations:

$$\begin{aligned} EU^h_{r1}(x &= 1) = EU^h_{r2}(x = 1), \\ EU^l_{r1}(x &= 1) = EU^l_{r2}(x = 1). \end{aligned}$$

Now we set $p = 0.3$, and see how these equilibria vary.[17] See Figure 3. We start from the bottom of Figure 3. In Region 5, the utility of c_1 is close to the utility of c_2, thus both types play mixed strategies such that

$$\begin{aligned} EU^h_{r1}(x^*, y^*) &= EU^h_{r2}(x^*, y^*), \\ EU^l_{r1}(x^*, y^*) &= EU^l_{r2}(x^*, y^*). \end{aligned}$$

Given $p = 0.3$, at each level of u, we can find explicit solutions of (x^*, y^*) from the functions above. The equilibrium condition is that both types are indifferent

[17] The reason we chose 0.3 is that it is across all the five regions.

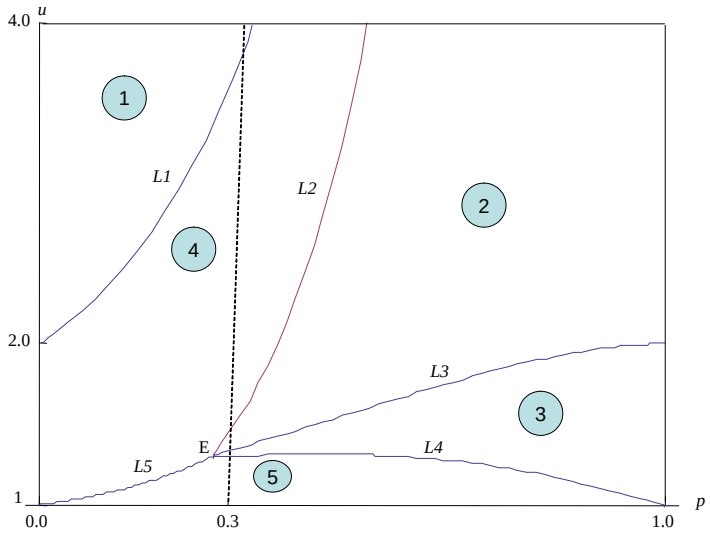

Figure 3: Equilibria ($p = 0.3$)

between playing r_1 and playing r_2:

$$\begin{aligned} D_h^5|_{p=0.3} &: = EU_{r1}^h(x^*, y^*) - EU_{r2}^h(x^*, y^*) \\ &= -2.19 + x(0.63y - 0.72) + u\left[2.28 - x(0.09 + 0.63y) + 0.63y\right] \\ &= 0, \\ D_l^5|_{p=0.3} &: = EU_{r1}^l(x^*, y^*) - EU_{r2}^l(x^*, y^*) \\ &= -0.49 - 0.63x - 0.27x^2 + u(1.88 - 1.17x + 0.27x^2 - 0.49y) - 0.49y \\ &= 0. \end{aligned}$$

We do not give explicit solutions for (x^*, y^*) in terms of u as it is too complicated. Table 1 presents ten equilibrium solutions when u rises from 1.010 to 1.308, and Figure 4 will show us the changes of $x^*$and y^* over u in Region 5.

u	1.01	1.05	1.09	1.13	1.17	1.21	1.25	1.25	1.3	1.304
x	0.516	0.577	0.637	0.695	0.752	0.809	0.865	0.921	0.935	0.946
y	0.482	0.414	0.347	0.282	0.218	0.155	0.092	0.029	0.013	0

Table 1

When $u = 1$, which implies that the two colleges have the same utility for all the students, both high type and low type students will choose either c_1 or c_2 with a fifty percent probability. As u rises, high type students will increase their probability of playing r_1. Thus, the equilibrium x increases. Although increasing u also attracts low type students to play r_1, they realise that high type students

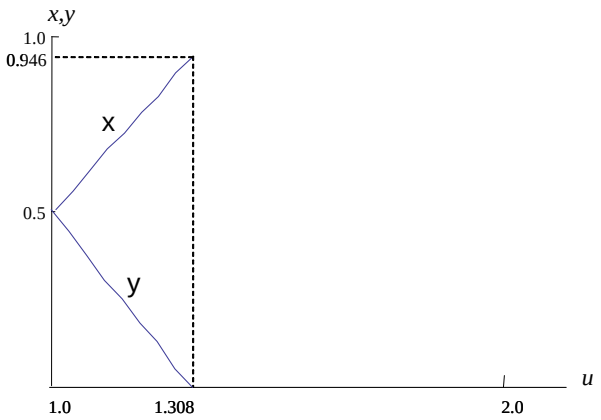

Figure 4: Changes of x and y over u in Region 5

must have raised their probabilities of playing r_1 and hence the probability that they can be admitted by c_1 is very low. Therefore, given a low level u, low type students' probability of playing r_1, i.e., y, is falling as u increases. When u arrives at 1.308, $y = 0$, which means that low type students will play a pure strategy by ranking c_2 as the first choice; and $x = 0.946$, which implies that high type students play r_1 by the probability of 0.946. This is the point on $L4$ when $p = 0.3$.

In Region 3, by solving the following equation:

$$D_h^3|_{p=0.3} := EU_{r1}^h(x, y = 0) - EU_{r2}^h(x, y = 0) = 0,$$

we can obtain the equilibrium mixed strategy for high type students:

$$x = \frac{2.19 - 2.28u}{-0.72 - 0.09u}.$$

Thus, the equilibrium in Region 3 is $\left(x = \frac{2.19-2.28u}{-0.72-0.09u}, y = 0\right)$, and

$$\frac{dx}{du} = \frac{1.839}{(0.72 + 0.09u)^2}.$$

High type students' probability of playing r_1 is increasing in u. High type students will play this symmetric mixed equilibrium strategy until $u = 1.329$, which is on the curve $L3$, and then change to play the symmetric pure strategy of choosing r_1. Although y is always zero in Region 3, $D_l^3|_{p=0.3}$ is decreasing as u rises.

In Region 2, we have the symmetric pure strategy equilibrium, $(x = 1, y = 0)$, i.e., high type students play r_1 and low type students play r_2. In equilibrium, we

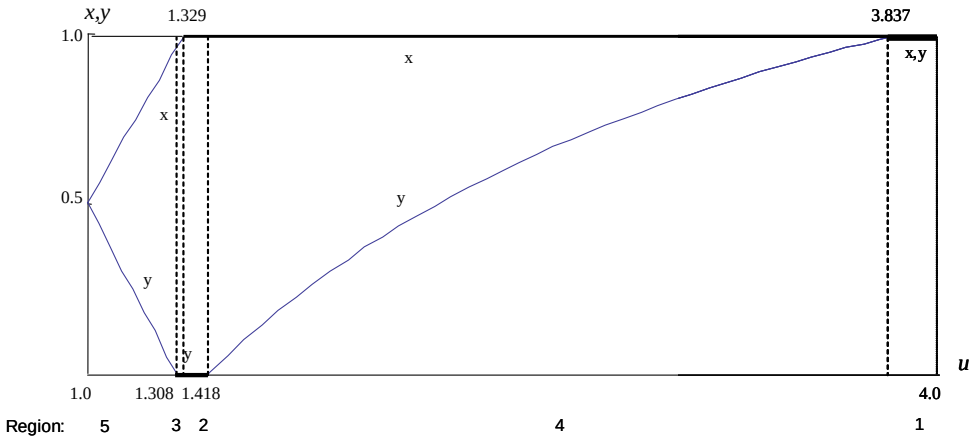

Figure 5: Changes of x, y over u when $p = 0.3$.

have the following equations:

$$\begin{aligned} D_h^2|_{p=0.3} &: = EU_{r1}^h(x=1,y=0) - EU_{r2}^h(x=1,y=0) \\ &= -2.91 + 2.19u, \\ D_l^2|_{p=0.3} &: = EU_{r1}^l(x=1,y=0) - EU_{r2}^l(x=1,y=0) \\ &= -1.39 + 0.98u. \end{aligned}$$

We can see that $\frac{d\left(D_l^2|_{p=0.3}\right)}{du} > 0$. That is because high type students start to use the symmetric pure equilibrium of r_1 once u reaches 1.329, and hence the only factor affecting low type students' decisions is the value of u from that point. However, low type students will not change their pure strategy until $u = 1.418$, which is on the curve $L2$. On $L2$, $D_l^2|_{p=0.3} = 0$ and $D_h^2|_{p=0.3} > 0$.

When $u > 1.418$, low type students will play a symmetric mixed equilibrium with a positive y. In Region 4,

$$y = \frac{1.39 - 0.98u}{-0.49 - 0.49u}.$$

Since $\frac{d(y)}{du} > 0$, low type students' probability of playing r_1 is increasing in u. On $L1$, when $u = 3.837$, as well as in Region 1, both types play the symmetric pure strategy of ranking c_1 as the first choice.

How x and y vary with u in all regions are presented in Figure 5.

From the preceding analysis, it can be concluded that

Remark 2 *(1) High type students are not always choosing a pure strategy by simply ranking c_1 as their first choice. They would play a mixed strategy if the*

utility of c_1 is sufficiently low.

(2) Low type students have two possible pure strategies: When the utility of c_1 is low, they rank c_2 as the first choice; and when the utility of c_1 is sufficiently high, they rank c_1 as the first choice.

Item 1 implies that a high type student has the incentive to rank c_2 as his first choice for securing the place in c_2 by playing a symmetric mixed strategy if c_1 is not sufficiently attractive.

Item 2 tells us that a low type student's decision is strongly influenced by the decisions of high type students. His willingness to choose c_1 as the first choice is decreasing in x when both types play the symmetric mixed strategies. Once high type students start playing pure strategy r_1, the low type student will increase the probability of choosing r_1 as u increases. When u is sufficiently large, he would just play the symmetric pure strategy of choosing r_1 because his gain from c_1 is sufficiently high for him to take the risk.

Not only the utility of the better college would affect the equilibria but also would the probability that a student is high type affect the equilibria. If p is low, then low type students' pure strategy of choosing r_2 may disappear (when $p < 0.279$, which is the x coordinate of point E). Moreover, they only need a lower level of u to start using the symmetric pure strategy of choosing r_1 (shown by $L1$ in Figure 2). It is because low type students realise that a lower p means there is little chance that competitors are high type, thus they have higher opportunity of being admitted to c_1 if they rank c_1 as their first choice. When p is greater than the threshold point ($p > 0.279$), $L4$ is still climbing up until $p = 0.453$, but low type students may play the symmetric pure strategy of r_2 given some appropriate values of u. When $p > 0.453$, the threshold curve $L4$ starts falling in p. Low type students need a higher level of u to go back to the symmetric mixed strategy and to play the symmetric pure strategy of r_1 in p. This is shown by the ascending curve $L2$ and $L1$. It is simpler for high type students. As shown by curve $L5$ and $L3$, they need a higher value of u to trigger the symmetric pure strategy of r_1 as p increases from 0 to 1 because the opportunity that competitors are high type rises.

The effects of p vary in u. If $u \in [2, 4]$, then y will fall from 1 to a value between 0 and 1 and then to 0 as p increases from 0 to 1, while x is always equal to 1. It is shown in Figure 2, from Region 1 to Region 4 and then to Region 2. y is decreasing in p in Region 4. If $u \in [1.324, 2]$, then y falls from a value between 0 and 1 to 0, while x falls from 1 to a value between 0 and 1 as p increase from 0 to 1. It is reflected in Figure 2, from Region 4 to Region 2 and to Region 3.

If $u \in [1, 1.324]$, then y decreases from a value between 0 and 1 to 0 and x falls from 1 to a value between 0 and 1 as p increases. It is the change from Region 4 to Region 5 in Figure 2. These changes of strategies indicate that it is more likely that both types rank c_2 as their first choice when p increases.

The thresholds are graphed by points on those curves, across which at least one type's strategy changes, and they depend on values of p. All curves except for $L4$ go up as p rises. The thresholds on $L4$ increases to the maximum, where $u = 1.324$ and $p = 0.453$, and then falls in p.

We define fairness as the property that students with higher test scores should be assigned their better choices. Therefore, if a high type student is rejected by both colleges and a low type student is allocated a place, then the allocation is not fair.

In all regions except for Region 1, it is possible to have such a scenario that two high type students choose r_1 and a low type student chooses r_2. The resulting allocation will be that one of high type students is admitted by c_1, the low type student is admitted by c_2, and the other high type student is rejected by both colleges. In this outcome, the high type student who are rejected by both colleges envies the low type student, thus the allocation is not fair.

In Region 5, since both type students play mixed strategies, we may see that high type students rank c_2 as the first choice, while at least one of low type students ranks c_1 as the first choice. One of low type students will be allocated the place in c_1 and one of high type students will be allocated the place in c_2. The probability that this scenario happens is very low as it needs all high type students choose r_2 and at least one low type student chooses r_1. In Region 3, suppose there is only one high type student, who may rank c_2 as his first choice with a very low probability, and two low type students who play r_2. If the high type student ranked c_2 as his first choice, then he will be allocated the place at c_2 and both low type students will be rejected by c_2, but one of them would be admitted by c_1. In all these possible outcomes, the high type student who lost his opportunity for higher education envies at least one low type student, and hence fairness is violated.

Fairness always holds in Region 1, where all the students rank c_1 as the first choice, and a high type student will obtain the place in c_2 in the second round over a low type even if he is rejected by c_1 in the first round.

Summary 2 *If scores are private information, the resulting allocation may violate the property of fairness and strategy-proofness.*

The following example shows us an unfair outcome.

Example 4 *Let $S = \{s_1, s_2, s_3\}$ be the set of students, $C = \{c_1, c_2\}$ be the set of colleges, $q = \{1, 1\}$ be the college capacity vector. Scores are as follows:*

$$t_1 = t_h, t_2 = t_h, t_3 = t_l.$$

The utility of c_1 is 1.4 *and the probability of being a high type student is* 0.3.

From Figure 5, we know that this set of u and p will induce a symmetric pure strategy equilibrium $(x = 1, y = 0)$. Therefore, the reported preferences are as follows:

$$\begin{aligned} r_1 &= \{c_1, c_2\}, \\ r_2 &= \{c_1, c_2\}, \\ r_3 &= \{c_2, c_1\}. \end{aligned}$$

Assume that the place in c_1 is allocated to s_1, then the outcome is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_0 & c_2 \end{pmatrix}.$$

s_2's score is higher than s_3's, but he is rejected by both colleges and s_3 is allocated a place in c_2. Therefore, the result is not fair.

Proposition 4 *If u is sufficiently high and p is sufficiently low, then the symmetric pure Bayesian Nash equilibrium is strategy-proof.*

Proof. See Appendix. ■

The conclusions of our simple model can be used to explain some observations in practice.

(1) There are many more applicants being rejected by top colleges than by ordinary colleges.

Top colleges usually have high relative utilities over the next level of colleges. That corresponds to a high level of u in our model. If u is sufficiently high, then low type students have incentives to risk themselves by ranking the top college as their first choice. It is the case in Region 1 or Region 4. Given the same probability that a student is of high type, the rejection rate is higher in the top colleges.

(2) When scores in the current year are universally lower than in previous years,[18] *there will be more applicants applying to better colleges, vice versa.*

[18] The reason may be the exam is easier than before, or the quality of students is higher than before.

This can be explained by the last conclusion we had above. Scores in the current year are universally lower than previous years would correspond to a lower value of p. Each student realises that the probability that any other student has a higher score is low. Therefore, those whose scores are higher will be confident to rank c_1 as their first choice, and those whose scores are lower would take a chance by ranking c_1 as their first choice. On the other hand, we will have the opposite result when scores in the current year are universally higher than previous years.

We have shown that the CCA mechanism is Pareto efficient under the assumption of perfect information. Next we will discuss the efficiency of the CCA mechanism in a more general case where scores are private information.

Recall that a matching μ Pareto dominates a matching μ' if no student prefers μ' to μ and there is at least one student who prefers μ to μ'. A matching μ is Pareto efficient if it is not Pareto dominated by any other matching. Since we assume that all students have identical preferences over colleges, swapping any two students can not improve their welfare simultaneously. Therefore, the only case that an outcome is inefficient is: if at least one student s_i prefers a college c_j than his assignment, $c_j \succ \mu(s_i)$, and there is at least one place available at c_j.

Proposition 5 *Given imperfect information about students' scores and identical preferences, the CCA mechanism is Pareto efficient.*

Proof. See Appendix. ■

In our model, we assume that students have identical preferences, and students are allowed to include as many colleges as they want in their submitted preference forms. The efficiency result will fail if we relax either of the two assumptions.

Remark 3 *Given imperfect information about students' scores, the CCA mechanism might not be Pareto efficient if*

1. *Students have different preferences, or*
2. *Students are not allowed to include an unlimited number of colleges in their reported preferences.*

We use two simple examples to illustrate the two scenarios. The first one assumes that students have different preferences, and the second one assumes students can only include a limited number of colleges on their preference forms.

Example 5 *Let $S = \{s_1, s_2, s_3, s_4\}$ be the set of students, $C = \{c_1, c_2, c_3\}$ be the set of colleges, $q = \{1, 1, 2\}$ be the college capacity vector, and $\{t_1, t_2, t_3, t_4\}$ be*

students' scores. Preferences of students are as follows:

$$p_1 = \{c_1 \succ c_2 \succ c_3 \succ c_0\},$$

$$p_2 = \{c_1 \succ c_2 \succ c_3 \succ c_0\},$$

$$p_3 = \{c_1 \succ c_2 \succ c_3 \succ c_0\},$$

$$p_4 = \{c_2 \succ c_1 \succ c_3 \succ c_0\}.$$

Suppose that it is publicly known that $t_2 = t_3 > t_4$, $t_1 > t_2 = t_3$ with a large probability and $t_1 < t_4$ with a small probability.

Without the uncertainty about t_1, and $t_1 > t_2 = t_3$, s_1 reports his true preference, s_2 and s_3 will rank c_2 as the first choice, and s_4's decision does not matter. If, say, the place at c_2 is randomly allocated to s_2, then the outcome is as follows:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ c_3 & c_2 & c_3 & c_3 \end{pmatrix}.$$

A Nash equilibrium for this example is as follows: Given the uncertainty, s_2 and s_3 will not change their strategy because a 50% chance of being admitted by c_2 is better than a tiny chance of going to c_1. This must be true for a small enough probability that $t_1 < t_4$. However, for s_4, he would rank c_1 as the first choice because he has a small chance of being allocated a place at c_1, but no chance of going to c_2. For s_1, he would put c_1 as the first choice if $t_1 > t_2 = t_3$, and rank any college as his first choice if $t_1 < t_4$.

If in fact, say, $t_1 > t_2 = t_3$, then the outcome is the same as the case with perfect information. If, say, $t_1 < t_4$ and the place at c_2 is randomly allocated to s_2, then the outcome is as follows:

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ c_3 & c_2 & c_3 & c_1 \end{pmatrix}.$$

μ' is Pareto dominated by either of the following matching:

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ c_3 & c_1 & c_3 & c_2 \end{pmatrix},$$
$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ c_3 & c_3 & c_1 & c_2 \end{pmatrix}.$$

Therefore, the CCA mechanism is not Pareto efficient.

Example 6 *Let $S = \{s_1, s_2\}$ be the set of students, $C = \{c_1, c_2\}$ be the set of*

colleges, $q = \{1, 1\}$ *be the college capacity vector, and* $\{t_1, t_2\}$ *be students' scores. Preferences of students are as follows:*

$$p_1 = p_2 = \{c_1 \succ c_2 \succ c_0\}.$$

The utility of c_1 *is* 1*, the utility of* c_2 *is* 0.1*, and the utility of* c_0 *is* 0 *to both students. Suppose that students can only have one choice in their reported preferences and there are no test score ties. Assume* t_2 *is known by both students, but* t_1 *is uncertain, so* s_1 *knows whether he has the highest score but* s_2 *is uncertain. Suppose the probability that* $t_1 > t_2$ *is 0.5, there are no score ties, and hence the probability that* $t_1 < t_2$ *is 0.5 as well.*

With no uncertainty, if it turns out that $t_1 > t_2$, then $r_1 = \{c_1 \succ c_2 \succ c_0\}$, $r_2 = \{c_2 \succ c_1 \succ c_0\}$, and hence the outcome is

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix};$$

if $t_1 < t_2$, then $r_1 = \{c_2 \succ c_1 \succ c_0\}$, $r_2 = \{c_1 \succ c_2 \succ c_0\}$, and hence the outcome is

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix}.$$

Clearly, both outcomes are Pareto efficient. Now consider s_2. If he ranks c_1 as his first choice, then his expected payoff is at worst $0.5u_1 + 0.5 * 0 = 0.5$; if he ranks c_2 as his first choice, then his expected payoff is at most $u_2 = 0.1$. Thus, his optimal strategy is to rank c_1 as his first choice. For s_1, given s_2's strategy, he would rank c_1 as his first choice if $t_1 > t_2$ and c_2 as his first choice if $t_1 < t_2$. If $t_1 < t_2$ in the end, then the outcome would be

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix},$$

which is Pareto efficient. However, if $t_1 > t_2$ in the end, then the outcome would be

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_0 \end{pmatrix}.$$

Clearly, the outcome is not Pareto efficient as s_2 will be better of if he was allocated a place at c_2 without hurting s_1. μ is Pareto dominated by the following matching:

$$\mu = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix}.$$

Therefore, the CCA mechanism is not Pareto efficient even though preferences are identical and there are no ties.

1.4 Conclusion

In this chapter we presented an analysis on the equilibria of the Chinese College Admission (CCA). We explained how the mechanism works, and constructed two formal models under different assumptions about information concerning scores. Firstly, we looked through the simple model under the assumption of perfect information. We established that there is a pure Nash equilibrium in the model. A student who realises that there are not vacancies in his favourite colleges will misrepresent his preference, and ranks his most preferred college among those colleges whose base scores are lower than his score or the same as his score. We also presented a number of examples to show that the allocation results are fair and hence stable and strategy-proofness is sometimes violated. Moreover, we showed that the set of Nash equilibrium outcomes of this game is equal to the set of stable matching under the true preferences, and it is Pareto efficient. In the model under the assumption of imperfect information and identical preferences, we described a particular model where there are two colleges and three students with two possible scores. The model has two pure symmetric Bayesian Nash equilibria and three possible mixed symmetric strategy Bayesian Nash equilibria. Which equilibrium obtains depends on the utility of the best college and the probability of being a high type student. When the utility is high and probability is low, both types of students rank the better college as the first choice, and hence the mechanism satisfies strategy-proofness and produces a fair allocation result. However, in any other cases, the allocation could be inefficient and strategy-proofness might fail. We discussed how the equilibrium varies with the utility of the best college and the probability of being a high type student. The conclusions were used to explain two common observations in practice. In the end, we discussed efficiency of the CCA mechanism given imperfect information. The CCA mechanism is always Pareto efficient if students have same preference and are allowed to include as many colleges as they want in the preference form. The outcome of the CCA mechanism may fail to be Pareto efficient if either of these assumptions does not hold.

1.5 Appendix

Proof of Proposition 1:

Proof. Since Pareto efficiency implies individual rationality and non-wastefulness, if we can prove the result is Pareto efficient, then it must be individual rational and non-wasteful.

Suppose the outcome is not fair. Then there are two students s_i and s_j with $\mu(s_j) p_i \mu(s_i)$ but $t_i > t_j$. Under the CCA system, this matching can be selected only in the following process. In round k with $k < m$, s_j is allocated a place at $\mu(s_j)$ as he submits $\mu(s_j)$ as the kth choice, while s_i must have submitted $\mu(s_j)$ as the $(k+j)$th choice with $k + j \leq m$ and $j \geq 1$. In this case, if s_i ranks $\mu(s_j)$ as the first choice, then s_i would be allocated a place at $\mu(s_j)$ in the first round, and hence be better off. Therefore, the outcome of the CCA mechanism under perfect information is fair.

Suppose the outcome is not Pareto efficient. Then there is a matching μ' which dominates the matching selected by the mechanism. So, $\mu' r \mu \ \forall$ all $s \in S$ and $\mu' p \mu$ for at least one student, say s_i, which means s_i can be made strictly better off. However, if s_i is better off, then he must get a place from another student, say s_j, who is ranked above s_i because when s_i makes his choice, he can only choose the best from the remaining places after students ranked above him. Unless s_j is in turn allocated a place from one ranked above him, he will be worse off. If s_j is allocated a place from one ranked above him, then we can repeat the argument with s_j rather than s_i. Eventually, we will find a student ranked above s_i who is worse off. Hence, this can not be a Pareto improvement. Thus, the outcome of the CCA mechanism is Pareto efficient, and hence individual rational and non-wasteful.

Suppose all students report their true preferences. If the number of students who rank c_i as their first choice is greater than q_i, then those students whose scores are lower than b_i (the lowest of top q_i students who rank c_i as their first choice) would have incentives to manipulate their preferences by ranking their second true choices as their submitted first choices. Therefore, truthful revelation of preferences is not always a dominant strategy for all students. ■

Proof of Claim 1:

Proof. (I)

1) s_1 uses r_1.[19]

[19] We only explain the probabilities in this part.

a. Both s_2 and s_3's scores are t_h. $\Pr(t_2 = t_3 = t_h) = p^2$.

The probability of going to c_1 is as follows:

$\Pr(c_1) = \Pr(s_1$ is admitted by c_1 conditional on s_2 and s_3 using $r_1)$

$\times \Pr(s_2$ and s_3 use $r_1)$

$+ \Pr(s_1$ is admitted by c_1 conditional on either s_2 or s_3 using $r_1)$

$\times [\Pr(s_2$ uses r_1 and s_3 uses $r_2) + \Pr(s_2$ uses r_2 and s_3 uses $r_1)]$

$+ \Pr(s_2$ and s_3 use $r_2)$

$= \frac{1}{3}x^2 + \frac{1}{2}2x(1-x) + (1-x)^2$.

The probability of going to c_2 is as follows:

$\Pr(c_2) = \Pr(s_1$ is rejected by c_1 conditional on s_2 and s_3 using $r_1)$

$\times \Pr(s_1$ is admitted by c_2 conditional on s_2 and s_3 use r_1 and s_1 being rejected by $c_1)$

$\times \Pr(s_2$ and s_3 use $r_1)$

$= \frac{2}{3} \times \frac{1}{2}x^2$.

b. s_2 or s_3's score is t_h, and the other's is t_l. $\Pr(t_2 = t_h, t_3 = t_l) = p(1-p)$ and $\Pr(t_2 = t_l, t_3 = t_h) = p(1-p)$.

$\Pr(c_1) = \Pr(s_1$ is admitted by c_1 conditional the student with t_h using $r_1)$

$\times \Pr($the student with t_h uses $r_1)$

$+ \Pr($the student with t_h uses $r_2)$

$= \frac{1}{2}x + (1-x)$;

$\Pr(c_2) = \Pr(s_1$ is rejected by c_1 conditional the student with t_h using $r_1)$

$\times \Pr($The student with t_h uses $r_1) \times \Pr($The student with t_l uses $r_1)$

$= \frac{1}{2}xy$.

c. Both s_2 and s_3's scores are t_l. $\Pr(t_2 = t_3 = t_l) = (1-p)^2$.

$\Pr(c_1) = 1$; $\Pr(c_2) = 0$.

2) s_1 uses r_2.

a. Both s_2 and s_3's scores are t_h. $\Pr(t_2 = t_3 = t_h) = p^2$.

$\Pr(c_1) = \frac{2}{3} \times \frac{1}{2}(1-x)^2$; $\Pr(c_2) = x^2 + \frac{1}{2} \times 2x(1-x) + \frac{1}{3}(1-x)^2$.

b. s_2 or s_3's score is t_h, and the other's is t_l. $\Pr(t_2 = t_h, t_3 = t_l) = p(1-p)$ and $\Pr(t_2 = t_l, t_3 = t_h) = p(1-p)$.

$\Pr(c_1) = \frac{1}{2}(1-x)(1-y)$; $\Pr(c_2) = x + \frac{1}{2}(1-x)$.

c. Both s_2 and s_3's scores are t_l. $\Pr(t_2 = t_3 = t_l) = (1-p)^2$.

$\Pr(c_1) = 0$; $\Pr(c_2) = 1$.

Therefore, expected utilities are as follows:

$$EU_{r1}^h = \left\{ \begin{array}{c} p^2\left[\frac{1}{3}x^2 + \frac{1}{2} \times 2x(1-x) + (1-x)^2\right] \\ +2p(1-p)\left[\frac{1}{2}x + (1-x)\right] + (1-p)^2 \end{array} \right\} u$$

$$+ \left[p^2\left(\frac{2}{3} \times \frac{1}{2}x^2\right) + 2p(1-p)\left(\frac{1}{2}xy\right) + (1-p)^2 0\right]$$

$$= \left\{p^2\left[1 - x + \frac{1}{3}x^2\right] + 2p(1-p)\left[1 - \frac{1}{2}x\right] + (1-p)^2\right\} u$$

$+\left[\frac{1}{3}p^2x^2 + p\left(1-p\right)xy\right]$

$= u\left(1-px\right) + \frac{p^2x^2}{3}\left(1+u\right) + pxy\left(1-p\right);$

$EU^h_{r2} = \left\{ \begin{array}{c} p^2\left[\frac{2}{3}\times\frac{1}{2}\left(1-x\right)^2\right] \\ +2p\left(1-p\right)\left[\frac{1}{2}\left(1-x\right)\left(1-y\right)\right] + \left(1-p\right)^2 0 \end{array} \right\} u$

$+ \left[\begin{array}{c} p^2\left[x^2 + \frac{1}{2}\times 2x(1-x) + \frac{1}{3}\left(1-x\right)^2\right] \\ +2p\left(1-p\right)\left[x + \frac{1}{2}(1-x)\right] + \left(1-p\right)^2 1 \end{array} \right]$

$= \left\{p^2\left[\frac{1}{3}\left(1-x\right)^2\right] + 2p\left(1-p\right)\left[\frac{1}{2}\left(1-x\right)\left(1-y\right)\right]\right\} u$

$+p^2\left[x + \frac{1}{3}\left(1-x\right)^2\right] + p\left(1-p\right)\left(x+1\right) + \left(1-p\right)^2$

$= \frac{1}{3}p^2(x-1)\left[x + u(x-3y+2) - 1\right] + p(x-1)\left[1 + u(y-1)\right] + 1.$

(II)

1) s_1 uses r_1

a. Both s_2 and s_3's scores are t_h. $\Pr\left(t_2 = t_3 = t_h\right) = p^2$.

$\Pr(c_1) = \left(1-x\right)^2$; $\Pr(c_2) = 0$.

b. s_2 or s_3's score is t_h, and the other's is t_l. $\Pr\left(t_2 = t_h, t_3 = t_l\right) = p\left(1-p\right)$ and $\Pr\left(t_2 = t_l, t_3 = t_h\right) = p\left(1-p\right)$.

$\Pr(c_1) = (1-x)(1-y) + \frac{1}{2}y(1-x)$; $\Pr(c_2) = \frac{1}{2}xy$.

c. Both s_2 and s_3's scores are t_l. $\Pr\left(t_2 = t_3 = t_l\right) = \left(1-p\right)^2$.

$\Pr(c_1) = \frac{1}{3}y^2 + \frac{1}{2}2y(1-y) + \left(1-y\right)^2$; $\Pr(c_2) = \frac{2}{3}\frac{1}{2}y^2$.

2) s_1 uses r_2

a. Both s_2 and s_3's scores are t_h. $\Pr\left(t_2 = t_3 = t_h\right) = p^2$.

$\Pr(c_1) = 0$; $\Pr(c_2) = x^2$.

b. s_2 or s_3's score is t_h, and the other's is t_l. $\Pr\left(t_2 = t_h, t_3 = t_l\right) = p\left(1-p\right)$ and $\Pr\left(t_2 = t_l, t_3 = t_h\right) = p\left(1-p\right)$.

$\Pr(c_1) = \frac{1}{2}(1-x)(1-y)$; $\Pr(c_2) = xy + \frac{1}{2}x(1-y)$.

c. Both s_2 and s_3's scores are t_l. $\Pr\left(t_2 = t_3 = t_l\right) = \left(1-p\right)^2$.

$\Pr(c_1) = \frac{2}{3}\frac{1}{2}\left(1-y\right)^2$; $\Pr(c_2) = y^2 + \frac{1}{2}2y(1-y) + \frac{1}{3}\left(1-y\right)^2$.

Therefore, expected utilities are as follows:

$EU^l_{r1} = \left[2p\left(1-p\right)\left(\frac{1}{2}xy\right) + \left(1-p\right)^2\left(\frac{1}{3}y^2\right)\right]$

$+ \left\{ \begin{array}{c} p^2\left[\left(1-x\right)^2\right] + 2p\left(1-p\right)\left[(1-x)(1-y) + \frac{1}{2}y(1-x)\right] \\ +\left(1-p\right)^2\left[\frac{1}{3}y^2 + y(1-y) + \left(1-y\right)^2\right] \end{array} \right\} u$

$= \left[p\left(1-p\right)xy + \frac{1}{3}\left(1-p\right)^2 y^2\right]$

$+ \left\{ \begin{array}{c} p^2\left[\left(1-x\right)^2\right] + 2p\left(1-p\right)\left[(1-x)(1-y) + \frac{1}{2}y(1-x)\right] \\ +\left(1-p\right)^2\left[1-y+\frac{1}{3}y^2\right] \end{array} \right\} u$

$= \frac{1}{3}(p-1)y\left[p(y-3x) - y\right]$

$+\frac{1}{3}u\left[3 - 3y + y^2 + p\left(3y - 6x + 3xy - 2y^2\right) + 1/3p^2(3x^2 - 3xy + y^2)\right];$

$EU^l_{r2} = \left\{p\left(1-p\right)\left[(1-x)(1-y)\right] + \left(1-p\right)^2\left[\frac{1}{3}\left(1-y\right)^2\right]\right\} u$

$+\left[p^2x^2 + 2p\left(1-p\right)\left[xy + \frac{1}{2}x(1-y)\right] + \left(1-p\right)^2\left[y^2 + \frac{1}{2}2y(1-y) + \frac{1}{3}\left(1-y\right)^2\right]\right]$

$= \left\{p\,(1-p)\,[(1-x)(1-y)] + \frac{1}{3}\,(1-p)^2\,(1-y)^2\right\} u$
$+p^2x^2 + p\,(1-p)\,[x+xy] + (1-p)^2\left[y + \frac{1}{3}\,(1-y)^2\right]$
$= p^2x^2 - px(p-1)(1+y) + \frac{1}{3}(p-1)^2(1+y+y^2)$
$+\frac{1}{3}(p-1)u(y-1)\,[1-y+p(2-3x+y)]$. ■

Proof of Proposition 2:

Proof. In the first case, if $x = 1$, $y = 1$ is an equilibrium, then $EU^h_{r1}(x = 1, y = 1) \geq EU^h_{r2}(x = 1, y = 1)$ and $EU^l_{r1}(x = 1, y = 1) \geq EU^l_{r2}(x = 1, y = 1)$.

$EU^h_{r1}(x = 1, y = 1) - EU^h_{r2}(x = 1, y = 1) \geq 0$,
$p^2(u-2) + 3(u-1) - 3p(u-1) \geq 0$,
$u \geq \frac{3-3p+2p^2}{3-3p+p^2}$.
$EU^l_{r1}(x = 1, y = 1) - EU^l_{r2}(x = 1, y = 1) \geq 0$,
$p^2(u-2) + u - p(2u-1) - 2 \geq 0$,
$u \geq \frac{2-p+2p^2}{(p-1)^2}$.

Thus, the above two inequalities hold when $u \geq \frac{2-p+2p^2}{(p-1)^2}$ as $\frac{2-p+2p^2}{(p-1)^2} > \frac{3-3p+2p^2}{3-3p+p^2}$ given $p \in [0, 1]$.

In the second case, if $x = 1$, $y = 0$ is an equilibrium, then $EU^h_{r1}(x = 1, y = 0) \geq EU^h_{r2}(x = 1, y = 0)$ and $EU^l_{r1}(x = 1, y = 0) \leq EU^l_{r2}(x = 1, y = 0)$.

$EU^h_{r1}(x = 1, y = 0) - EU^h_{r2}(x = 1, y = 0) \geq 0$,
$3(u-1) - 3pu + p^2(1+u) \geq 0$,
$u \geq \frac{3-p^2}{3-3p+p^2}$.
$EU^l_{r1}(x = 1, y = 0) - EU^l_{r2}(x = 1, y = 0) \leq 0$,
$2u + p^2(2u-1) - p(4u+1) - 1 \leq 0$,
$u \leq \frac{1+p+p^2}{2(p-1)^2}$.

Thus, the above two inequalities hold when $\frac{3-p^2}{3-3p+p^2} \leq u \leq \frac{1+p+p^2}{2(p-1)^2}$. This can only be the case when $\frac{3-p^2}{3-3p+p^2} \leq \frac{1+p+p^2}{2(p-1)^2}$ if $p \geq 0.2787$. In Figure 1, $L1$ graphs $u = \frac{2-p+2p^2}{(p-1)^2}$, $L2$ graphs $u = \frac{1+p+p^2}{2(p-1)^2}$, and $L3$ graphs $u = \frac{3-p^2}{3-3p+p^2}$.

$x = 0$, $y = 1$ and $x = 0$, $y = 0$ can not be equilibria as high type students will be better off by playing r_1 in both cases. ■

Proof of Proposition 3:

Proof. In Region 2, $x = 1$ and $y = 0$. We have

$$\begin{aligned} EU^h_{r1}(x &= 1, y = 0) = \frac{1}{3}(p^2 + (3 - 3p + p^2)u), \\ EU^h_{r2}(x &= 1, y = 0) = (1-p)^2 + 2(1-p)p + p^2, \\ EU^l_{r1}(x &= 1, y = 0) = (1-p)^2 u, \\ EU^l_{r2}(x &= 1, y = 0) = \frac{1}{3}\left(1 + p + p^2\right) + \frac{1}{3}u(1-p)^2, \end{aligned}$$

and

$$\begin{aligned} EU^h_{r1}(x &= 1, y = 0) > EU^h_{r2}(x = 1, y = 0), \\ EU^l_{r1}(x &= 1, y = 0) < EU^l_{r2}(x = 1, y = 0). \end{aligned}$$

Define $D^2_h = EU^h_{r1}(x = 1, y = 0) - EU^h_{r2}(x = 1, y = 0)$ and $D^2_l = EU^l_{r1}(x = 1, y = 0) - EU^l_{r2}(x = 1, y = 0)$. In Region 2, $D^2_h > 0$ and $D^2_l < 0$. Given any value of p,

$$\frac{d\left(D^2_h\right)}{du} > 0, \frac{d\left(D^2_l\right)}{du} > 0.$$

Consider decreasing u holding p fixed. D^2_h and D^2_l fall until the curve $L3$, on which $D^2_h = 0$ but $D^2_l < 0$. If u keeps falling, then the high types will no longer play r_1 as $D^2_h < 0$. Define $D^3_h = EU^h_{r1}(x, y = 0) - EU^h_{r2}(x, y = 0)$ and $D^3_l = EU^l_{r1}(x, y = 0) - EU^l_{r2}(x, y = 0)$. $EU^j_{ri}(y = 0)$, where $i = 1, 2$ and $j = h, l$, implies that the expected utility of the j type by playing r_i when the high types play mixed strategy $(x, 1 - x)$ and the low type plays r_2. At the equilibrium, $D^3_h = 0$. By solving $D^3_h = 0$, we have the symmetric mixed equilibrium strategy in terms of u and p, denoted by x^*:

$$x^* = \frac{-3 + 3p - p^2 + 3u - 3pu + 2p^2u}{p(3 - 2p + pu)}.$$

If $D^3_l\left(x = x^*, y = 0\right) < 0$, then $x = x^*, y = 0$ is an equilibrium. It can shown that $D^3_l\left(x = x^*, y = 0\right) < 0$ if u is between $L3$ and $L4$. On $L4$, $D^3_l\left(x = x^*, y = 0\right) = 0$.

If u keeps falling below $L4$, then $D^3_l\left(x = x^*, y = 0\right) > 0$. Define $D^5_h = EU^h_{r1}(x, y) - EU^h_{r2}(x, y)$ and $D^5_l = EU^l_{r1}(x, y) - EU^l_{r2}(x, y)$. In Region 5 ($p \geq p_E$), we will find the symmetric mixed equilibria for both types by solving $D^5_h = 0$ and $D^5_l = 0$.

Define $D^1_h = EU^h_{r1}(x = 1, y = 1) - EU^h_{r2}(x = 1, y = 1)$ and $D^1_l = EU^l_{r1}(x =$

$1, y = 1) - EU^l_{r2}(x = 1, y = 1)$. In Region 1, we have

$$\frac{d\left(D^2_h\right)}{du} > 0, \frac{d\left(D^2_l\right)}{du} > 0.$$

Both D^1_h and D^1_l fall as u decreases until $L1$, on which

$$EU^l_{r1}(x = 1, y = 1) = EU^l_{r2}(x = 1, y = 1)$$

and

$$EU^h_{r1}(x = 1, y = 1) > EU^l_{r2}(x = 1, y = 1).$$

Below $L1$, $D^1_h > 0$ but $D^1_l < 0$, then the low type would play a mixed strategy. Define $D^4_h = EU^h_{r1}(x = 1, y) - EU^h_{r2}(x = 1, y)$ and $D^4_l = EU^l_{r1}(x = 1, y) - EU^l_{r2}(x = 1, y)$. In Region 4, $D^4_l = 0$ at the equilibrium. By solving $D^4_l = 0$, we have

$$y^* = \frac{-1 - p - p^2 + 2u - 4pu + 2p^2u}{(p-1)^2(1+u)}.$$

If $D^4_h\left(x = 1, y = y^*\right) > 0$, then $x = 1, y = y^*$ is an equilibrium. It can shown that $D^4_h\left(x = 1, y = y^*\right) > 0$ if u is between $L1$ and $L2$ or $L1$ and $L5$. On $L2$, $D^4_l\left(x = 1, y = 0\right) = 0$ and $D^4_h\left(x = 1, y = 0\right) > 0$; on $L5$, $D^4_h\left(x = 1, y = y^*\right) = 0$ and $D^4_l\left(x = 1, y = y^*\right) = 0$. Below $L5$, $D^4_h\left(x = 1, y = y^*\right) < 0$, then the high type will play a mixed strategy as well as the low type.

In Region 5 ($p \leq p_E$), once again we will find the symmetric mixed equilibria for both types by solving $D^5_h = 0$ and $D^5_l = 0$. ■

Proof of Proposition 4:

Proof. When u and p fall into Region 1, all the students report the true preference no matter what type they are. The symmetric pure Bayesian Nash equilibrium is that both types rank c_1 as the first choice. Since both types of students' preference is identical: $c_1 \succ c_2$, the equilibrium is strategy-proof. ■

Proof of Proposition 5:

Proof. Assume the identical preference is as follows:

$$c_1 \succ c_2 ... \succ c_m \succ c_0.$$

If any colleges are ranked below c_0, then they can simply be ignored as they won't be selected in equilibrium.

Consider $Q \leq n$, i.e., when the number of applicants is greater than the total quota. Suppose the outcome is inefficient, then there must be at least one place avalaible at a randomly selected college, say c_j, $j = 1, 2, ..., m$. Since $Q \leq n$, there must be at least one student choosing c_0, say s_i. In s_i's reported preferences, c_0 must be above c_j, otherwise he would be allocated a place at c_j. So, if s_i simply switched positions of c_0 and c_j, then he would be allocated a place at c_j, and hence be better off. Therefore, there would not be any unfilled places at the end of admission. The allocation of the mechanism must be Pareto efficient.

Consider when $Q > n$, i.e., when the total quota is greater than the number of students. Assume a sub quota $Q_1 = \sum_{j=1}^{k-1} q_j < n$ and $Q_2 = \sum_{j=1}^{k} q_j \geq n$, where $k \leq m$. So, if we can prove there is not any place available at a randomly selected college c_j, $j < k$, in equilibrium, then the result is Pareto efficient. Since $Q_1 < n$, then $n - Q_1$ students are allocated to c_k. c_{k-1} will not have any places available because all students who go to c_k must have ranked c_{k-1} in front of c_k. For the same reason, there will not be any places available in c_j, $j = 1, 2, ..., k-2$. Thus, the outcome of the CCA mechanism is Pareto efficient. ■

2 Chapter 2: A Comparison between Two College Admission Systems From the Perspective of Corruption

2.1 Introduction and Literature Review

Millions of Chinese high school graduates sit China's national higher education entry exam in June every year. The exam results determine whether they enter colleges, or end up not receiving any higher education at all. The exam and the following admission procedures constitute the Chinese college admission system. The importance of the system is increasing as the need for highly educated employees in the ongoing modernization of China is growing. In order to meet this need, there has been a huge increase in the number of students entering the system and the higher education sector. As a result, the number of colleges increased by 536 from 1985 to 2003, and the number of new students enrolled in the system rose from 61.9 to 382.2 per 10,000 people.[20] The system plays an important role for every single candidate and the whole society. China has the largest population in the world, and so far she remains a developing country despite the marked development she has achieved. In order to be competitive in the job market and have a bright future, taking the exam seems to be the only way for all young Chinese. The number of applicants therefore far exceeds available places, particularly for entry into prestigious colleges or colleges in large cities such as Beijing or Shanghai. The fierce competitiveness has resulted in parents and teachers placing considerable pressure on their children, on their students and on themselves. Such pressure could start from primary school, or even earlier.

Like any other college admission systems, the Chinese college admission system has limitations. The first criticism concerns the fairness and equality of opportunity. There exists regional discrimination in the system. Students who live in the same city as a college can be admitted by the college with lower scores than students from other areas. The second criticism is about the exam itself, e.g., the style of questions. Some educators indicate that the exam focuses too much on subject knowledge and theory rather than the ability to solve problems in practice. As a result, Chinese students are very good at theoretical subjects such as mathematics, but lack the ability of carrying out practical tasks com-

[20] China Statistical Yearbook, 2004

pared to western students. The third criticism indicates that the exam is like a "gamble" with betting on the candidate's future life. Failure in the exam really seems to be the end of the world for students. As a consequence, the childhoods of most Chinese children are full of studying as well as the fear of failure in the exam. *"I spend my waking hours studying and even my spare time is dedicated to after-school curricula. Life is hard and all my friends worry about failing our exams. Sometimes I feel I can't cope but I just don't want to let my parents down." (Davey, Higgins (2005, p.32)* The pressure also falls on the shoulders of schools and teachers, causing the school teaching to serve only for the purpose of preparation for the exam rather than learning ability and study skills of students. The last concern is about corruption. The system has long been hailed as an effective mechanism to ensure equal education opportunities for all students, disregarding their backgrounds. But admission scandals have exposed many flaws as well as laying bare its vulnerability to power abuse and corruption. The widely publicized scandals have also triggered severe doubts about the government's call for higher education institutions to become business-like and profit oriented. It seems that the system has not been able to serve its intended purpose, as it seems to fail providing equal education opportunities for all students.

Due to these criticisms, there has been a heated debate about the college admission system in China. The main argument here is: Should the current exam system be replaced by a free market system? In the market system, students are able to apply to colleges freely, and applications are not determined only by a one-time exam. The proponents of the current system argue that corruption has posed a very serious problem in the current admission system. The competitiveness of gaining a place in a Chinese higher education institution has given some individuals the motive to be corrupt. How does a free market system deal with that?

Corruption in the education sector can be defined as *"the systematic use of public office for private benefit, whose impact is significant on the availability and quality of educational goods and services, and, as a consequence on access, quality or equity in education"* (Education corruption 29, Hallak and Poisson, 2002). Educational corruption has an enormous negative impact on society. It will undermine public trust in the system or even the government, exacerbates the quality of education, negatively affects young professionals and students, and it spreads them distorted values and culture. It is even more distinct in China as the higher education system has undertaken such an important task to the development of the country. Despite the serious consequences of corruption in

education, attention was only drawn to it in recent years, especially from the economic perspective. Although developing a deep and solid understanding of the structure of corruption in education is necessary for building strong theories on its potential causes and impacts on fairness, social welfare and efficiency, we will not introduce it in detailed context. We only look into one aspect of educational corruption.

Our focus is corruption in the admission process, thus the definition of corruption is based on the general definition of the public sector corruption, which is the use of public resources for private gains. In the current exam system, all admission procedures are operated by the admission office of each college. The serious competition for a higher education opportunity has given the admission office considerable power, and motive for private gains. Both formal (World Bank, 2003)[21] and informal reports posit that bribes are increasingly necessary to gain admissions into college programs. The admission office is assumed to be an official who acts as an individual agent in either a market system or an exam system. In order to win a place in a preferred college, a student has to pay illegal bribes to the admission officials. Corruption in the market system is hard to detect as there is not standard criteria to tell a candidate's ability. Corruption in the exam system is conducted in the following ways: selling entry exam papers in advance to high-paying candidates; the evaluations of oral examinations are subjective and difficult to monitor, etc. In some former Soviet states, admission to colleges is for sale: Well-connected applicants or those who bribe or otherwise influence the academic authorities responsible for admissions are the ones who are admitted regardless of their academic qualifications (Hallak, Poisson, 2007). The authority's objective is to minimize the degree of corruption and the cost of investigation. When the cost of investigation is relatively high, the officials of college have the incentive to be corrupt.

In the market system, the official accepts or rejects a student only on consideration from the application materials, which indicate the student's performance at high school, and the student's personal statement, and references from teachers, etc. When the power of investigation is high, the officials will not take bribes, and then students with higher abilities are always admitted prior to students with lower abilities. When the cost of investigation is high and power is low, however, the officials will have incentives to take bribes. Application materials and support documents are the only object that the authority is able to monitor, making the

[21] World Bank. (2003). Governance and service delivery in Kazakhstan—Results of diagnostic surveys.

market system relatively subjective. We therefore assume the object that the authority monitors in the market system are identical across different students.

Examinations have become a universal method of selecting qualified candidates and distributing limited resources. In the exam system, like the current one in China, a centralized test allocates college places to individuals according to the order of exam results. Unfortunately, the exam system can not escape from corruption because of the considerable private benefit motivation. For the same reason, when the power of investigation is high, the officials will not take bribes, then students with higher exam scores are always admitted prior to students with lower scores. When the cost of investigation is high and power is low, however, the officials will have incentives to be corrupt. For example, a place of a college could be allocated to a student with an unqualified score as long as the student bribes a sufficient amount. We do not provide how exactly the cheating works as it is not our concern.

We try to look into the two systems in terms of corruption from the perspective of the authority or the planner of higher education sector. We will discuss three issues. First is to tell whether or not corruption has influence on the efficiency in both systems; if it does, what is the way. Second is to compare the two systems in terms of the degree of corruption. The last is to discuss the effect of borrowing constraints.

This work is related to several branches of the literature. There is a massive literature on matching problems, in which a set of heterogeneous individuals is mapped into a set of heterogeneous objects or individuals (e.g., marriage market) with the payoff from each match depending on some characteristic of both sides of match.[22] In this literature, the outcome is produced by either a free market or a centralized mechanism. A common characteristic is they assume both sides of the matching have preferences over the other side, and the outcome is driven by the preferences and a particular mechanism. Lazear and Rosen (1981), Green and Stokey (1983) and Nalebuff and Stiglitz (1983) examine the performance of tournament-based compensation schemes relative to individualistic reward schemes. They are concerned with the relative efficiency of tournaments in environments with moral hazard and how to extract effort from homogeneous agents. Balinski and Sönmez (1999) study a model of student placement where scores play the role of matching students to colleges. Roth(1991), Abdulkadiroğlu, Sönmez (2003); Abdulkadiroğlu, Pathak, Roth, Sönmez (2005) and Ergin, Sön-

[22]See Gale and Shapley (1962), Becker (1973), Cole, Mailath, and Postlewaite 91992), Kremer (1993), Sttinger (1005), Acemoglu (1995), Shimer and Smith (1996), Burdett and Coles (1996), Epple, Romano and Sieg (2003, 2006), etc.

mez (2006) study the Boston School Choice Mechanism. Fernandez (1997, 1998) examines the performance and properties of markets (prices) system and exam (tournament) system as alternative allocation devices with perfect capital markets. He shows both systems achieve the efficient allocation results. However, when borrowing constraints are present, exams (tournaments) dominate markets (prices) in terms of matching efficiency.

In this context, we use some very basic auction theory to analyse corruption in the admission problem. Most of the theoretical framework is in Krishna (2002). The literature in this area views corruption in auctions either as a manipulation of the quality assessment in complex bids or as bid rigging. The former was introduced in a seminal paper by Laffont and Tirole (1991), who assume that the auctioneer has some leeway in assessing complex multidimensional bids, and is predisposed to favour a particular bidder. That framework was later adopted by Celantani and Ganuza (2002), who employ it to assess the impact of increased competition on the equilibrium corruption and show that corruption may increase if the number of competing bidders is increased. More recently, Burguet and Che (2004) consider a scoring auction, make the assignment of the auctioneer's favourite agent endogenous, and assume that bribery competition occurs at the same time as contract bidding. Their main result is that corruption may entail inefficiency, and that *"...the inefficiency cost of bribery is in the same order of magnitude as the agent's (i.e. auctioneer's) manipulation capacity"* (Burguet and Che, 2004). Lengwiler and Wolfstetter (2005) propose a model of corruption in which the auctioneer orchestrates bid rigging by inviting a bidder to either lower or raise his bid, whichever is more profitable.

The aim of this paper is to contrast the performance of the market system and the exam system under no borrowing constraints and under borrowing constraints. With a market mechanism prices do not discriminate among individuals except with respect to their willingness and ability to pay; accordingly, individuals with the same level of bribe expenditures attend the same college, regardless of their abilities. In the centralized exam system, identical bribe by students with different scores (different abilities) do not lead to the same allocation results. By bribing the same amount, the higher-score student wins the place. Therefore, the higher-ability students are less credit constrained than those with identical wealth but lower ability, thus the former is more likely to be admitted by the college. This paper attempts to show the superiority of the centralized test system under the assumption of corruption. Corruption in our model is simply a bribe to the official during the admission process.

The main contributions of our work are that we use auction theory to obtain equilibria of the market system and the exam system under the assumption of perfect capital markets and borrowing constraints. We discuss the effects of corruption on efficiency in both systems. To our knowledge, the literature has not considered corruption in higher education sector from this angle. The admission process can be taken as an auction game as students are bidding for college places by paying a bribe in the market system, or by paying a bribe and score in the exam system. The corrupt official is the auctioneer, applicants are bidders, and the bribe money is the bid in the market system, while bribe money and score are taken as a binding bid in the exam system. College places are allocated to students who have the highest bids. We show that the allocation results of both systems are efficient in a perfect capital market. Borrowing constraints can prevent the two systems from attaining efficient allocations. We also show the degree of corruption in the market system is always higher than in the exam system. The degree of corruption is measured by the expected revenue received by officials. Thus, expected revenues in each scenario will be provided. In all analysis, we use some well-known results from standard auction theories, particularlly from first prize auction model. However, in the discussion for exam system, we establish an auction model, in which students compete for college places by combining their bribes and scores.

We build up our model incrementally throughout the rest of the context. Section 2 describes the general model. Section 3 illustrates the equilibrium of the model without borrowing constraints, and contrasts the efficiency and the degree of corruption between the two systems. Section 4 analyses equilibrium allocation if there are borrowing constraints, and compares the two systems in terms of efficiency and degree of corruption. Section 5 summarizes.

2.2 Description of the Economy

2.2.1 Student, Authority, and Official

This model describes a college admission problem in two different systems, one is market-based and the other is exam-based. From now on, we just call the market-based system as the market system and the exam-based system as the exam system. There are two sides in the market system: students and colleges. First of all ,we impose several assumptions on the student's side. Assume there are n students, each of whom is characterized by an endowment of "ability", a, and initial wealth w. For simplicity, we assume that a and w are independently

distributed according to continuous, differentiable cumulative distributions, F and H, with probability density functions denoted by f and h and a finite supports given by $[0,1]$ respectively. Thus, each student is characterized by a point in $I^2 \equiv [0,1] \times [0,1]$, and these attributes are un-correlated across students. Assume both ability and wealth are unobservable, but the probability distributions of ability and wealth are commonly known. On the other side, assume there are m colleges, and the quality q is exogenous and belongs to $[0,1]$.

Apart from students, in our model, there is an authority, e.g., the government; officials, who are the intermediaries between students and colleges. The authority's objective is to minimize the degree of corruption, and the process of investigation incurs a cost to the authority. We define the objective function of the authority as

$$\begin{aligned} \max_{\delta} w &= W\left(\left(\Gamma - tc\left(\delta\right)\right), dc\left(\delta\right)\right), \\ s.t \;\; tc\left(\delta\right) &\leq \Gamma, \\ \delta &\geq 0, \end{aligned}$$

where w denotes the authority's utility, δ denotes the power of investigation, $dc\left(\delta\right)$ denotes the degree of corruption,[23] Γ is the budget of authority, and $tc\left(\delta\right)$ is the total cost of investigation. The authority finds a value of δ to maximise his utility.

If we assume he is risk neutral, then an official's objective is to maximise his expected payoff. One important assumption is that officials can not elicit bribes from students, and hence he can only consider accepting or rejecting bribes. Officials will not take a student's bribe if the student failed to be admitted. Therefore, officials pick the bribes which maximise their expected payoffs from all students' intended bribes and reject others. The reason for this assumption is that eliciting bribes from students or taking all bribes may entail an official to expose him to an exceedingly high risk of detection and punishment. Let ϕ denote the expected payoff of officials. If the punishment is assumed to be losing his job, then an official's expected payoff function is as follows.

$$\begin{aligned} \phi &= (Y+b)P + 0\left(1-P\right) = (Y+b)P \qquad \text{when the official is corrupt,} \\ &= Y \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{when the official is not corrupt,} \end{aligned}$$

where Y denotes the official's income without taking bribes, b is a particular bribe,

[23] dc is measured by officials' expected revenue from the corruption. A higher total expected revenue implies a higher degree of corruption.

and P measures the probability that the official is not caught and punished. Having received a bribe, if $bP < Y(1-P)$, then the official would reject the bribe; but if $bP > Y(1-P)$, then the official would take the bribe. Thus, the official will not be corrupt when either the probability of being caught or the legal income is sufficiently high. In practice, an official's income is much lower compared to bribes. Therefore, in the current context, we simply assume the legal income is low enough to be ignored, and hence the official's expected payoff can be rewritten as:

$$\phi = bP.$$

Since $bP \geq 0$, this function implies officials would take any bribes.

We use an exponential function for the probability: $P = x^{\delta}$, where $x \in [0, 1)$ denotes the object that the authority monitors. x differs in the two systems. In the market system, $x = \lambda$, where λ denotes the degree of non-transparency of the admission process. A higher value of λ means higher probability to keep the bribe safely, and it is identical for all applicants in the market system. In the exam system, $x = s$, where s denotes a student's score. Given the same bribe, an official will obtain a higher expected payoff from a student with a higher score than from a student with a lower score for the same reason. If we substitute P in the payoff function by x^{δ}, then the official's objective function can be rewritten as $\phi = bx^{\delta}$. In the extreme cases, such as $\delta = 0$, and hence $P = 1$, $\phi = b$, which implies there is no investigation and the official can safely keep the bribes; as δ goes to infinity, x^{δ} and hence ϕ approach zero, so the official will not take any bribes. Here we assume these objective functions are publicly known by all agents.

A risk neutral student's objective is to maximise his expected payoff from education, which is the difference between his valuation of education and the cost of bribe. Here the tuition fee and other costs are not considered in our model, and hence bribes are the only cost to students. Assume there is an outside option for all students. Let y^c denote the expected aggregate income of a student in the future if the student receives higher education; let y^o denote the expected aggregate income of a student in the future if the student takes the outside option. We can take y^o as the gain for a student from a college with quality being zero. So, a student's valuation for education at a particular college is the difference between y^c and y^o: $v \equiv y^c - y^o$. A student's y^c obtained from a college is assumed to be determined by the student's ability and the quality of the college. Assume y^o only depends on the student's ability. There are two more assumptions for y^c and y^o as follows:

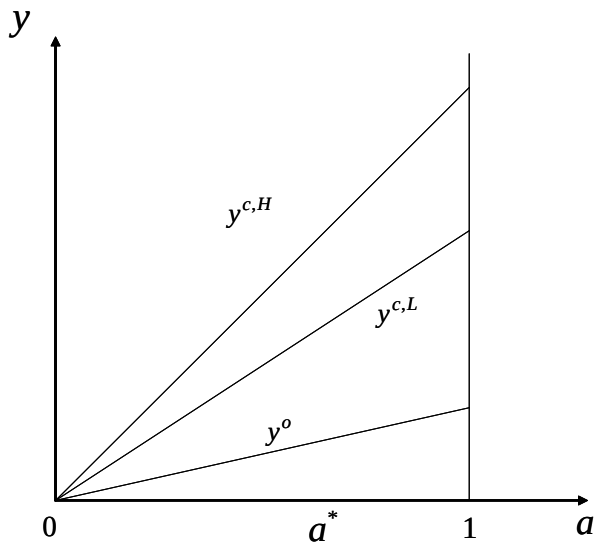

Figure 6: Valuations

1. y^c and y^o are continuous, differentiable and increasing in a, $\frac{\partial y^c}{\partial a} > 0$ and $\frac{\partial y^o}{\partial a} > 0$; y^c is increasing in q, $\frac{\partial y^c}{\partial q} > 0$.

2. Education output function has supermodularity between its arguments. Students with higher ability can gain more from the same improvement in education quality than students with lower ability, $\frac{\partial^2 y^c}{\partial a \partial q} > 0$.

The second assumption immediately implies that $\frac{\partial v}{\partial a} > 0$. Since a particular student values a college with higher quality more, we know that $v_q > 0$. For simplicity, we assume both y^c and y^o are linear functions of ability, hence v is a linear function of ability as well. Figure 6 shows an example for valuation. The distance between $y^{c,H}$ and y^o measures the value of the college with a higher quality, and the distance between $y^{c,L}$ and y^o measures the value of the college with a lower quality. Figure 6 shows that the values of both colleges are increasing in ability. The difference of values between the higher quality college and the lower quality college is increasing in ability, which is in accordance with the second assumption.

We define the valuation function as: $v = aq$.[24] In this function, q is assumed to be exogenous as we do not consider peer effects.[25]

The main difference between the market system and the exam system is that

[24] Assume $y^c = \alpha_c aq$, $y^{nc} = \alpha_{nc} a$, if set $\alpha_{nc} = 1$, $\alpha_c = 1 + \frac{1}{q}$, then $v = aq$.

[25] Epple and Romano (1998) use $a(\theta, b) = \theta^\gamma b^\beta$ as a student's achievement function, where b is student's ability, and θ denotes the mean ability of the student body in the school attended. Since we do not consider peer effects, θ is a constant in our model as the quality of college.

score plays a key role in the latter system. A particular student's score is determined by how much effort the student invests into studying and his performance in the entry exam. How much effort to invest depends on the student's valuation of the higher education; his performance in the exam depends on both his ability and some other disturbance factors. For simplicity, we assume all other factors are symmetric between students, and hence the student's performance in the exam is only determined by his ability. We use a weighted sum function to express score.

$$s = \gamma' \overline{v} + (1 - \gamma')\, a,$$

where $\overline{v}$ denotes the average valuation for higher education, γ' is a constant and $\gamma' < 1$. Given $v = aq$, we have

$$\begin{aligned} s &= \gamma' a \overline{q} + (1 - \gamma')\, a \\ &= (\gamma' \overline{q} + (1 - \gamma'))\, a, \end{aligned}$$

where $\overline{q}$ denotes the average quality of colleges. Since $\overline{q} \leq 1$, then $(\gamma' \overline{q} + (1 - \gamma')) < 1$. Therefore, we can simply use a constant γ to substitute $(\gamma' \overline{q} + (1 - \gamma'))$. Hence the score is determined by the following function:

$$s = \gamma a,$$

where $\gamma < 1$.

2.2.2 Mechanisms in the Two Systems

In the market system, students apply to their favourite colleges and intend to bribe the officials of those colleges. The official of a particular college observes the intended bribes and allocates the places at that college to those students from whom he gets the highest expected payoffs, which are determined by the following function:

$$\phi^m = b^m \lambda^\delta.$$

Since λ is identical for all students, an official allocates the places to students only according to the ranking of their bribes. Officials would take any value of bribes unless δ goes to infinity. So, students' bribes and hence the degree of corruption will be dependent on the power of investigation only if it goes to infinity. If δ goes to infinity, then there will not be any corruption in both systems. Thus, for the purpose of comparison between the two systems, in the current context, we

simply assume δ as a positive value, and hence the game induced by the market system is the same as a standard sealed-bid first price auction.

In the exam system, students attend exams, obtain scores, and then choose which colleges to apply and how much to bribe. The official of a particular college computes his expected payoff from all students applying to the college by substituting students' bribes and scores to his expected payoff function:

$$\phi^e = b^e s^\delta.$$

The places at that college will be allocated to the students from whom this official gets the highest expected payoffs. Note that the superscripts m and e in the functions denote the market and exam systems, respectively.[26]

A student will go to the college at which he obtains the highest payoff. The student's payoff depends on his valuation for a college and cost for bribing the official of that college, so his choice is not necessarily the college with the highest quality because he may need to bribe too much to be admitted into that college. In equilibrium, student's optimal bribe will be determined only by valuations of colleges in the market system but by valuations and scores in the exam system. In the final section of this work, the model will incorporate financial constraints into a student's decision making process. When there exist borrowing constraints, students have to consider his budget, and we will have some different results.

To define an efficient allocation result, we assume complementarity in a student's valuation of college. This may also be called supermodularity. It means that, at any given level of quality of college, higher ability students produce more when given a marginal increase in quality. Complementarity leads to positive assortative matching, which is the underlying mechanism in the marriage market model of Becker (1973).[27] We will also believe that our focus from a social perspective is of value. The distributions of college qualities and student abilities are exogenous, and there are no peer effects or externalities. Therefore, the principle of efficiency is to allocate the best resources (places of the best colleges) to those agents (students) who can use them most efficiently, take them most valuable and are, thus, willing to pay more for the resources. The efficient allocation result will lead to maximised total educational output if we consider a student's valuation as the product of education. We also will consider the competition among officials

[26] Without special explication, from now on, the superscripts m and e in any functions denote the market system and the exam system, respectively. If there is no superscript, then it is for a general case.

[27] The literature viewing positive matching will be introduced in the next chapter.

from different colleges when colleges have identical quality, and the relationship between the degree of corruption and the difference of quality between different colleges.

Next, we start with the model without borrowing constraints.

2.3 Model with One college, One Place under Perfect Capital Markets

In this section, we will construct a model with perfect capital markets. Perfect capital markets mean that students are able to borrow money from external perfect capital markets at a zero rate of interest (for simplicity). Assuming the capital is external and risk free allows us to avoid endogenous interest rate which is not the focus of our work. The objective of this part is not only to find the equilibrium bribes in the market system and the exam system respectively, but also to compare the degree of corruption and efficiency in the market system to the exam system.

We start with the simplest model, where there is only one college with only one place for admission. This simple scenario is not only a starting point, but also an application to such circumstances where there is only one vacancy and many applicants, for instance, the competition for a PhD place at a popular institution. In the next section, we will extend this simple model to several more general cases with more places.

2.3.1 Market System

Recall when market is the allocation mechanism, the official's expected pay is $\phi^m = b^m \lambda^\delta$. Since λ is identical across students, the official gains the highest expected payoff from the highest bribe, therefore, he will make offer to the student who bribes the most. So, the competition of the place can be considered as a standard sealed-bid first price auction.

Next we aim to find a Bayesian Nash equilibrium bribe for a student in this game.

The following notations will be used in the whole chapter:

Given a student i's ability follow the distribution of F, we let Y_k denote the kth highest of other $n-1$ abilities, $F_k(\cdot)$ denote the distribution of Y_k, and $f_k(\cdot)$ be the corresponding density function.

In this case, a student i with ability a will win the place whenever $b(Y_1) < b(a)$. By using the standard approach in the first price auction model (see, for

example, Riley and Samuelson (1981)), we can have a symmetric equilibrium bribe as follows.

Claim 2 *Given perfect capital markets and one college with one place, the symmetric equilibrium bribe in the market system satisfies*

$$b^m(a) = qE[Y_1|Y_1 < a].$$

Proof. See Appendix. ■

The optimal bribe of a student is the expected ability of the expected value of the highest of other $n-1$ abilities multiplied by q, conditional on his ability being greater than Y_1. The equilibrium bribe is increasing in the student's ability. Therefore, i will win the place whenever $Y_1 < a_i$. The equilibrium can be written as

$$\begin{aligned} b^m(a) &= q\left[a - \int_0^a \frac{F_1(y)}{F_1(a)} d_y\right] \\ &= v - q\int_0^a \frac{F_1(y)}{F_1(a)} d_y. \end{aligned}$$

This expression shows that the bribe is less than the student's valuation v. For any given $F(\cdot)$, as the number of students increases, the equilibrium bribe approaches v.

The following example where ability follows uniform distribution gives us a more intuitive impression for the equilibrium strategy.

Example 7 *If abilities are uniformly distributed on* $[0,1]$*, then*

$$b^m(a) = \frac{n-1}{n} aq = \frac{n-1}{n} v.$$

At the equilibrium of this example, each student bribes a fraction of his valuation. Clearly, the result is efficient as the student with the highest ability will bribe the most and be allocated the place.

2.3.2 Exam System

The exam system in this case works in the following procedure. Consider a particular student.

1. The student takes an exam, and obtains his score, which is linearly determined by his ability as we assumed, $s = \gamma a$, $0 < \gamma < 1$.

2. In order to maximise his expected payoff, the student bribes the official of the college which he is applying to.

3. The official observes the student's score and bribe as well as other students' scores and bribes.

4. The official would allocate the place to the student from whom he gets the highest expected payoff.

Recall in the exam system, the official's expected payoff function is

$$\phi^e = b^e s^\delta,$$

where s^δ denotes the probability that the official can keep the money safely, i.e. the probability that corruption is not discovered. The probability is increasing in the exam score as corruption is more difficult to be discovered if the place has been allocated to students with higher scores than other students with lower scores. Therefore, the probability that a student is allocated the place is dependent on his bribe and score. This expected payoff function is public knowledge in students, so they will decide their bribes based on this function.

Recall a follows a continuous, differentiable cumulative distribution, $F(a)$, with density function $f(x)$ and a finite support on $[0,1]$; and also $s = \gamma a$. So, exam score follows a distribution with the same distribution function but on a different interval $[0,\gamma]$.

Now we define some notations for the exam system: Let $F_s(\cdot)$ denote the distribution of s, $f_s(\cdot)$ denote the corresponding density function. Let $Y_{s,k}$ denote the kth highest of other $n-1$ scores and let $F_{s,k}(\cdot)$ denote the distribution of Y_k, and $f_{s,k}(\cdot)$ be the corresponding density function.

In the exam system, a student i with ability a will win the place whenever $b^e(Y_{s,1}) < b^e(a)$. A symmetric equilibrium bribe is as follows.

Claim 3 *Given perfect capital markets and one college with one place, the symmetric equilibrium bribe in the exam system satisfies*

$$b^e(s) = \frac{q}{\gamma}\frac{E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right]}{s^\delta}.$$

Proof. See Appendix. ■

The equilibrium bribe in the exam system immediately implies the expected payoff for the official from a student with score s satisfies

$$\phi^e(s) = \frac{q}{\gamma} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right].$$

So, we have the following proposition.

Proposition 6 *Given perfect capital markets and one college with one place, the allocation result is efficient in the exam system.*

Proof. See Appendix. ■

This claim implies that the place will be allocated to the student with the highest score, and hence a student i with score s will win the place whenever $Y_{s,1} < s$. In the market system, students with higher valuations are willing to pay higher bribes for the place, and bribes are the only measurement; thus, the result in the market system is efficient as the student with the highest ability will be allocated the place. In the exam system, an integration of score and bribe measures the extent that the student is eager for education. Hence, in the exam system, students with higher valuations for education will provide higher expected payoffs for the official. The official allocates the place to the student who has the highest score as well as the highest ability, and hence the outcome is efficient.

Claim 3 also implies that students will take the power of investigation into account in the exam system.

Proposition 7 *Given perfect capital markets and one college with one place, the equilibrium bribe in the exam system is decreasing in the power of investigation.*

Proof. See Appendix. ■

The following example with a uniform distributed ability illustrates the result.

Example 8 *If abilities are uniformly distributed on* $[0,1]$*, then* $F(a) = a$*, and the symmetric equilibrium satisfies*

$$\phi^e(s) = \frac{q}{\gamma} \frac{n-1}{n+\delta} s^{1+\delta},$$

and hence

$$b^e(s) = \frac{q}{\gamma} \frac{n-1}{n+\delta} s,$$

or

$$b^e(v) = \frac{n-1}{n+\delta} v.$$[28]

[28] This result is by the following functions: $v = aq$ and $s = \gamma a$.

The result is efficient and it shows that the equilibrium bribe is decreasing in the power of investigation since $\frac{\partial b^e}{\partial \delta} < 0$. When $\delta = 0$, the optimal bribe is exactly the same as in the market system. When δ goes to the infinity, no students will bribe as they know the investigation is so strict that no officials would risk themselves to take the chance.

2.3.3 Market System vs Exam System

Next we compare the two systems in terms of equilibrium bribes, efficiency and degree of corruption.

Claim 4 *Given perfect capital markets and one college with one place, if $\delta \neq 0$, then $b^e(a) \leq b^m(a)$ and a strict inequality holds for all $a \in (0,1]$.*

Proof. See Appendix. ■

Claim 4 implies that every type's equilibrium bribe is higher in the market system than in the exam system. As regards the efficiency and the degree of corruption, we have the following proposition:

Proposition 8 *Given perfect capital markets and one college with one place, the allocation results in both systems are efficient; if $\delta \neq 0$, then the degree of corruption in the market system is higher than in the exam system.*

Proof. See Appendix. ■

This claim states the degree of corruption is higher in the market system than in the exam system, although they can both select efficient allocation results. Given perfect capital markets, the existence of corruption does not affect the efficiency of the allocations, but the existence of the exam lowers the degree of corruption. The following example gives us a clearer idea about this result.

Example 9 *If abilities are uniformly distributed on $[0,1]$, then the expected revenues in the market system and the exam system are respectively*

$$\begin{aligned} E(\pi^m) &= \frac{(n-1)}{n+1}q, \\ E(\pi^e) &= \frac{n(n-1)}{(n+1)(n+\delta)}q. \end{aligned}$$

Thus,

$$E(\pi^m) - E(\pi^e) = \frac{(n-1)\delta q}{(n+1)(n+\delta)} > 0.$$

The official's expected revenue is higher in the market system than in the exam system, and therefore the degree of corruption is higher in the market system than in the exam system.

2.4 Model with Multiple Places under Perfect Capital Markets

In the last section, we have looked through the simplest case of the model where there is only one place for admission in one college. Now we extend the model to discuss the scenario with more than one place for admission. As we have assumed, a student's valuation for the place at a particular college is determined by the student's ability and the college's quality. As a result, we need to take the quality of colleges into account when there are multiple places. Education quality would be the same for all places in a particular college, and it could either differ or be the same across different colleges. Next, we will go through the following scenarios: One college with k places; two colleges with the same quality; k colleges with different qualities.

2.4.1 One College with k Places

Consider the case where n students are applying to one college with k places. The model has become to a discriminatory "price" model and every student has single-unit demand. In both systems, the official assigns the places to the students from whom the official gets the highest expected payoffs. In the market system, the official's only concern is the bribes, so the places would be allocated to the students whose bribes are above the $(k+1)$th highest bribe; however, in the exam system, the official has to consider exam scores as well as bribes.

Market System Given that the expected payoff function in the market system is $\phi^m = b^m \lambda^\delta$, the official allocates the k places to the students whose bribes are above the $(k+1)$th highest bribe. The quality of college does not matter as all places in one college are assumed to have the same quality. Based on this information, there exists a symmetric equilibrium bribe for all students. We use the standard approach in the auction model (see, for example, Krishna (2002), p195) to obtain the equilibrium bribe.

Claim 5 *Given perfect capital markets and one college with k places, the symmetric equilibrium bribe in the market system satisfies*

$$b^m(a) = qE[Y_k | Y_k < a].$$

Proof. See Appendix. ■

Since all students follow the same strategy, it is clear that the equilibrium bribe is increasing in a. Therefore, the k places will be allocated to the students with the k highest ability types, so the allocation result is efficient.

Exam System The equilibrium bribe in terms of scores is as follows.

Claim 6 *Given perfect capital markets and one college with k places, the symmetric equilibrium bribe in the exam system satisfies*

$$b^e(s) = \frac{q}{\gamma} \frac{E\left[Y_{s,k}^{1+\delta} | Y_{s,k} < s\right]}{s^\delta}.$$

Proof. See Appendix. ■

By using the results in Proposition 6 and 7, we have the following corollary.

Corollary 1 *Given perfect capital markets and one college with k places, the allocation result of the exam system is efficient; the equilibrium bribe is decreasing in the power of investigation.*

Market system vs Exam system Next we compare the two systems in terms of efficiency and degree of corruption. Proposition 8 immediately implies the following corollary.

Corollary 2 *Given perfect capital markets and one college with k places, the allocation results in both systems are efficient; if $\delta \neq 0$, then the degree of corruption in the exam system is lower than in the market system.*

Proof. See Appendix. ■

This claim implies that both systems produce efficient outcomes given perfect capital markets. The existence of scores lowers the degree of corruption in the exam system. Clearly, as the investigation is getting stricter, the degree of corruption in the exam system decreases, while the degree of corruption in the market system keeps the same unless the investigation is strict enough to eliminate all kinds of corruptions.

2.4.2 Two Colleges with the Same Quality

Now we assume the case where there are two colleges, $\{c_1, c_2\}$ (each college has one place, and the admission of each college is managed by one official) and n students, with $n \geq 3$.[29] Here we have a price competition game. This competition game is similar to a Bertrand model.[30] Recall we assume officials can not elicit bribes from students, but they may compete for a student by reducing the bribe he needs to pay. For simplicity, we assume a particular student would bribe the same amount to the two officials because the values of the two colleges are the same. For example, suppose two students $\{i, j\}$ apply to two colleges $\{c_1, c_2\}$. If it turns out i will provide a higher expected payoff to the two officials, say $\phi_i > \phi_j$, then one official has incentives to reduce i's bribe to ϕ_i' and get i to accept his offer. Since both officials will use the same strategy, in the end, ϕ_1' will be slightly higher than ϕ_2. For simplicity, it is assumed that $\phi_1' = \phi_2$ in the equilibrium. In a general game, we assume each official observes students' intended bribes, and then compete for the student from whom the official gets the highest expected payoff.

The game induced by the two systems is described as follows:

1. Students realise their abilities and valuations for both colleges. In the market system, students go to the next step; in the exam system, students take the exam, obtain scores and go to the next step.

2. Students submit their strict intended bribes, $\left\{b_1^{IB}, b_2^{IB}, ..., b_n^{IB}\right\}$, to both officials simultaneously.

3. Each official observes the intended bribes, the expected payoff $\left\{\phi_1^{IB}, \phi_2^{IB}, ..., \phi_n^{IB}\right\}$ by these intended bribes and $\phi_1^{IB} > \phi_2^{IB} > \phi_3^{IB}, ..., \phi_{n-1}^{IB} > \phi_n^{IB}$ and then offer student 1 a required amount, $b_{1,j}$ with $b_{1,j} \leq b_1^{IB}$, $j = 1, 2$.

4. Student 1 selects the offer with a lower b_1, or choose one randomly if $b_{1,1} = b_{1,2}$, and then student 1 bribes the official with $b_{1,j}$ if he chooses college c_j.

[29] We are currently unable to analyse the case with $k > 2$ colleges.

[30] Bertrand competition is a model of competition used in economics, named after Joseph Louis Francis Bertrand (1822-1900). Specifically, it is a model of price competition between duopoly firms with the same marginal cost and producing homogeneous products. The game results in each charging the price that would be charged under perfect competition, known as marginal cost pricing. Competing in price means that firms can easily change the quantity they supply, but once they have chosen a certain price, it is very hard, if not impossible, to change it.

5. Suppose student 1 chooses college c_1, then the official of college c_2 would make an offer to student 2 with $b_2 = b_2^{IB}$; student 2 bribes the official of college c_2 with b_2.

6. The officials receive the actual expected payoffs, $\{\phi_1, \phi_2\}$, and then make admission offers to student 1 and 2.

Referring to the standard Bertrand model, the equilibrium strategy of both officials will be offering student 1 an amount such that $\phi_1 = \phi_2^{IB}$. Since the two officials will have the same strategy, s_1 receives two identical offers, and s_1 will choose one randomly, say c_1, and then the official of c_2 will make an offer to s_2 with $b_2 = b_2^{IB}$.

The officials' decision making process is public knowledge, and hence it will be in the students' consideration. Next, we aim to find students' optimal strategies in each system respectively.

Market System In the market system, the officials' expected payoff function is $b^m \lambda^\delta$, and hence the competition is on the student with the highest intended bribe. In equilibrium, $b_1^m = b_2^{m,IB}$. Given the officials' strategy, the following claim gives students' equilibrium bribes in the market system.

Claim 7 *Given perfect capital markets and two colleges (one place at each college) with the same quality, the symmetric equilibrium bribe in the market system satisfies*

$$b^m(a) = q\left(a - \int_0^a \left(\frac{F(y)}{F(a)}\right)^{n-2} dy\right).$$

Proof. See Appendix. ■

Again, we use the uniformly distributed ability example to illustrate the equilibrium.

Example 10 *If abilities are uniformly distributed on* $[0,1]$*, then* $F(a) = a$*, and the equilibrium satisfies*

$$\begin{aligned} b^m(a) &= q\left(a - \int_0^a \left(\frac{y}{a}\right)^{n-2} dy\right) \\ &= \frac{n-2}{n-1} aq. \end{aligned}$$

It will be worthwhile to compare the results between the following cases: one college with one place; one college with 2 places; 2 colleges with the same quality and each college has one place. Let $b^{m,1}(a)$ denote the equilibrium bribe in the first case; $b^{m,2}(a)$, denote the equilibrium bribe in the second case; and $b^{m,3}(a)$, denote the equilibrium bribe in the third case. We have the following claim.

Proposition 9 *Given perfect capital markets, in the market system,* $b^{m,2}(a) < b^{m,3}(a) < b^{m,1}(a)$*; and the expected revenue for the only official in the case of one college with* 2 *places is the same as the total expected revenues for the two officials in the case of* 2 *colleges with the same quality and each college having one place.*

Proof. See Appendix. ■

The first part of this claim imply that a student shades more of his valuation and hence when there are more objectives, the same student bribes more in the first case than in the last two cases; the competition between officials gets students to be more aggressive because a student realises that if his bribe is the highest, then he will only need to pay the second highest bribe, and hence he also bribes more in the last case than in the second case. The second part of this claim states that the degree of corruption is the same in the second and third cases although students are more aggressive in the third case

.

Exam System In the exam system, the officials' expected payoff function is $b^e s^\delta$, and hence in equilibrium, $b_1^e s_1^\delta = b_2^{e,IB} s_2^\delta$. Given the officials' strategy, the following claim gives students' equilibrium bribes in the market system.

Claim 8 *Given perfect capital markets and two colleges (one place at each college) with the same quality, the symmetric equilibrium bribe in the exam system satisfies*

$$b^e(s) = \frac{q}{\gamma}\left(s - (1+\delta)\int_0^s \left(\frac{y}{s}\right)^\delta \left(\frac{F_s(y)}{F_s(s)}\right)^{n-2} dy\right).$$

Proof. See Appendix. ■

Proposition 9 immediately implies the following corollary. Let $b^{e,1}(s)$ denote the equilibrium bribe in the case of one place; $b^{e,2}(a)$, denote the equilibrium bribe in the case of one college with 2 places; and $b^{e,3}(a)$, denote the equilibrium bribe in the case of 2 colleges with the same quality and each college having one place.

Corollary 3 *Given perfect capital markets, in the exam system,* $b^{e,2}(s) < b^{e,3}(s) < b^{e,1}(s)$*; the expected revenue for the only official in the case of one college with* 2 *places is the same as the total expected revenues for the two officials in the case of* 2 *colleges with the same quality and each college having one place.*

The comparison between the market system and the exam system is complicated in the case with a general distribution function, so we take uniform distribution as an example.

Example 11 *If abilities are uniformly distributed on* $[0,1]$*, then* $F(a) = a$*. Hence* $F_s(s) = \frac{s}{\gamma}$*. So,*

$$b^e(s) = \frac{q}{\gamma}\left(s - (1+\delta)\int_0^s \left(\frac{y}{s}\right)^{\delta}\left(\frac{y}{s}\right)^{n-2} dy\right) = \frac{n-2}{n-1+\delta}\frac{q}{\gamma}s,$$

which can be transformed to

$$b^e(a) = \frac{n-2}{n-1+\delta}aq.$$

Compared to the market system, where the equilibrium in the same example is

$$b^m(a) = \frac{n-2}{n-1}aq.$$

The equilibrium in the exam system has included the power of investigation in the denominator. Therefore, the amount of any student's bribe is lower in the exam system.

As regards efficiency and degree of corruption, we have the following conclusions.

Corollary 4 *Given perfect capital markets and two colleges (one place at each college) with the same quality, the allocation results in both systems are efficient.*

Proposition 10 *Given perfect capital markets and two colleges (one place at each college) with the same quality, the degree of corruption in the exam system is lower than in the market system.*

Proof. See Appendix. ■

2.4.3 k Colleges with Different Qualities

We next turn to a more realistic case where there are k colleges, $\{c_1, c_2, ...c_k\}$, with strictly ranking qualities, $q_1 > q_2 > ... > q_k$, and each college has only one place available for admission; there are $n > k$ students. Student i has different valuations for different colleges, for example student i's valuation of college c_j is $v_{ij} = V(a_i, q_j)$, which differs in colleges. Considering the total social welfare, colleges with better qualities should have higher priority to admit students with higher abilities because of the supermodularity of education production. Therefore, we assume that colleges admit students sequentially, i.e., the college with the highest quality starts the admission procedure first, and then the college with the second highest quality, and so on.

The game is described as follows:

1. Students realise their abilities and valuations for all colleges. In market system, students go to the next step; in exam system, students take the exam, obtain scores and go to the next step.

2. The admission commences from c_1. Students decide how much to bribe the official at c_1. The official at c_1 will allocate the place to the student from whom he gets the highest expected payoff among n students. The remaining students are rejected and enter the next round.

3. All remaining students bribe the official at c_2. The official of c_2 then chooses the student from whom he gets the highest expected payoff among the $n-1$ students. The remaining students are rejected and enter the next round.

   .....

4. In the round k, the last remaining $n-k+1$ students bribe the official at college c_k. The official of c_k then chooses the student from whom he gets the highest expected payoff among the $n-k+1$ students.

The officials' decision making process is public knowledge in students, and hence it will be in students' consideration. Next, we aim to find students' optimal strategies in each system respectively.

Market System In the market system, one official will allocate the place at his college to the student with the highest bribe. We begin with the simplest model, in which there are two colleges c_H and c_L, with quality $q_H > q_L >$

0; $n > 2$ students compete for the two places. Student i's valuations of c_H and c_L depend on his ability and qualities of c_H and c_L, i.e. $v_{i,j} = a_i \times q_j$, $\left(i = 1, ...n; j = H, L; v'_a > 0, v'_q > 0\right)$. In a healthy system, without bribes, c_H will take the student with the highest ability, and c_L takes the student with the second highest ability based on their applications. Under the assumption of corruption, however, students have to decide how much to bribe the official in c_H when they apply to c_H, and how much to bribe the official of c_L if they failed to win the place in c_H. A symmetric equilibrium consists of two bribe functions (b_H^m, b_L^m), denoting the bribe strategies in the first and second admissions, respectively. Assume that the information of the winning student will not be released, and then the equilibrium bribes in both rounds will be depending on the student's values.[31] We are interested in the equilibria that are sequentially rational, which implies that a student chooses sequential strategies to form the equilibria in both rounds. We begin with the second round.

In the second round, the admission has completed in c_H, but the winner's value is not released, thus, b_L is independent of the first round. The game is almost the same as in the one-college, one-place case with $(n-1)$ students. Back to the first round, a student has to take the expected result of the second round into account when he decides the optimal bribe in the first round. Suppose that all students are following the same strategies, b_H in the first round, and b_L in the second round regardless of what happens in the first round. The following claim shows the equilibrium bribes in this game.

Claim 9 *Given perfect capital markets and two colleges with different qualities, the symmetric equilibrium bribes in the market system satisfy*

$$\begin{aligned} b_L^m &= q_L E\left[Y_2 | Y_2 < a < Y_1\right], \\ b_H^m &= \left(q_H - q_L\right) E\left[Y_1 | Y_1 < a\right] + q_L E\left[Y_2 | Y_1 < a\right]. \end{aligned}$$

Proof. See Appendix. ■

The equilibrium bribe in the first round, b_H^m, depends on not only the quality of c_H, q_H, but also the quality of c_L, q_L. b_H^m is increasing in q_H but decreasing in q_L.

Let us take an example where a follows uniform distribution.

Example 12 *If abilities are uniformly distributed on* $[0, 1]$*, then the equilibrium*

[31] In the second round, the only updated information of any one of remaining students is that at least one other student's valuation (ability) is higher than his, but this information will not change his belief about distribution of other students' abilities.

bribes satisfy

$$\begin{aligned} b_H^m(a) &= \frac{n-1}{n} a q_H - \frac{1}{n} a q_L, \\ b_L^m(a) &= \frac{n-2}{n-1} a q_L. \end{aligned}$$

The gap of quality between the two colleges may cause a student's bribe to the college with high quality is less than his bribe to the college with low quality. When

$$\frac{q_H}{q_L} \geq 1 + \frac{n-2}{(n-1)^2},$$

$b_H^m(a) \geq b_L^m(a)$; otherwise $b_H^m(a) < b_L^m(a)$. When $b_H^m(a) < b_L^m(a)$, it indicates a student with ability a who is active in the first round but fails to win the place in c_H, will be more aggressive in the second round. This is due to the deterioration of available supply relative to current demand given a small difference of qualities between the two colleges.

Suppose there are k colleges with different qualities, $q_1 > q_2 > ... > q_k$. We will derive symmetric bribing strategies $(b_1^m, b_2^m, ..., b_k^m)$ by working backward from the last round. So first consider the kth round, the equilibrium bribe is

$$b_k^m(a) = q_k E\left[Y_1^{(n-k)} | Y_1^{(n-k)} < a\right].$$

The equilibrium in the last round does not depend on the bribes in other rounds. Consider the jth round for some $j < k$. Now look at student i with ability a and assume that all other students are following the jth round strategy $b_j^m(a)$, and suppose all students including i will follow the strategies $b_{j+1}^m, b_{j+2}^m, ..., b_k^m$, in the subsequent games.

Claim 10 *Suppose there are k colleges with different qualities $q_1 > q_2, ..., > q_k$ and n students applying sequentially. Given perfect capital markets, a set of symmetric equilibrium strategies in the first, jth, $2 \leq j \leq k-1$, and kth round in the market system is as follows:*

$$b_1^m(a) = \sum_{L=1}^{k-1} (q_L - q_{L+1}) E[Y_L | Y_1 < a] + q_k E[Y_k | Y_1 < a];$$

$$b_j^m(a) = \sum_{L=j}^{k-1} (q_L - q_{L+1}) E[Y_L | Y_j < a < Y_{j-1}] + q_k E[Y_k | Y_j < a < Y_{j-1}];$$

$$b_k^m(a) = q_k E\left[Y_k | Y_k < a < Y_{k-1}\right].$$

Proof. See Appendix. ■

Example 13 *Abilities are uniformly distributed on* $[0,1]$. *In the last round, the equilibrium bribing strategy is*

$$b_k^m(a) = \frac{n-k}{n-k+1} a q_k.$$

Proceeding inductively, it may be verified that the bribing strategy in the j*th round,* $2 \leq j \leq k-1$, *is*

$$b_j^m(a) = \frac{n-k}{n-j+1} a q_k + \sum_{L=j}^{k-1} \frac{n-L}{n-j+1} a\left(q_L - q_{L+1}\right).$$

The equilibrium bribe of a particular student in the jth round is determined by the number of students and colleges, the student's ability, and qualities of all the colleges.

Exam System As usual, we begin with the simplest model with two colleges as in the last part. Now an official has to take scores and bribes into account and allocate the place at his college to the student from whom he gets the highest expected payoff.

We use the same notations as in the market system. A symmetric equilibrium consists of two bribe functions (b_H^e, b_L^e), denoting the bribe strategies in the first and second admissions, respectively. The bribe in the first round depends on the student's values and score. Assume that the information of the winning student in the first round will not be released. Thus, in the second round, the bribe strategy in the second round will only depend on the value of the low quality college and score.

Claim 11 *Given perfect capital markets and two colleges with different qualities, the equilibrium bribes in the exam system satisfy*

$$\begin{aligned} b_L^e(s) &= \frac{q_L}{\gamma s^\delta} E\left[Y_{s,2}^{1+\delta} | Y_{s,2} < s < Y_{s,1}\right], \\ b_H^e(s) &= \frac{q_H - q_L}{\gamma s^\delta} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right] + \frac{q_L}{\gamma s^\delta} E\left[Y_{s,2}^{1+\delta} | Y_{s,1} < s\right]. \end{aligned}$$

Proof. See Appendix. ■

Example 14 *Abilities are uniformly distributed on* $[0,1]$*. If a follows uniform distribution, then s follows uniform distribution as well, but with a different distribution,* $F_s = \frac{s}{\gamma}$*, and hence*

$$\begin{aligned} b_H^e(s) &= \frac{n-1}{n+\delta}\frac{q_H - q_L}{\gamma}s + \frac{(n-1)(n-2)}{(n+\delta)(n+\delta-1)}\frac{q_L}{\gamma}s \\ &= \frac{n-1}{n+\delta}\frac{q_H}{\gamma}s - \frac{(n-1)(1+\delta)}{(n+\delta)(n+\delta-1)}\frac{q_L}{\gamma}s, \\ b_L^e(s) &= \frac{n-2}{n+\delta-1}\frac{q_L}{\gamma}s. \end{aligned}$$

It is easy to compare the two systems with the uniformly distributed ability.

Proposition 11 *Given perfect capital market and two colleges with different qualities, if abilities are following uniform distribution, then the degree of corruption in the market system is higher than in the exam system.*

Proof. See Appendix. ■

Now consider there are k colleges with different quality, $q_1 > q_2 > ... > q_k$. In what follows, $Y_{s,j}^{(n-1)}$ denotes the jth highest of $n-1$ scores, $F_{s,k}$ denotes the distribution of $Y_{s,k}^{(n-1)}$, and $f_{s,k}$ denotes the corresponding density. We will derive symmetric bribing strategies $(b_1^e, b_2^e, ..., b_k^e)$ by working backward from the last round. First consider the kth round, the equilibrium bribe is

$$b_k^e(s) = \frac{q_k}{\gamma s^\delta} E\left[\left(Y_{s,1}^{(n-k)}\right)^{1+\delta} | Y_{s,1}^{(n-k)} < s\right],$$

and the contribution to the official of c_k in equilibrium is

$$\phi_k^e(s) = \frac{q_k}{\gamma} E\left[\left(Y_{s,1}^{(n-k)}\right)^{1+\delta} | Y_{s,1}^{(n-k)} < s\right].$$

Next we use the same method as in the market system to find the equilibrium in every other round. Consider the jth round for some $j < k$. Now look at student i with score s, and assume that all other students are following the jth round strategy $b_j^e(s)$, and suppose all students including i will follow the strategies $b_{j+1}^e, b_{j+2}^e, ..., b_k^e$, in the subsequent games.

Claim 12 *Given perfect capital markets, a set of symmetric equilibrium strategies in the first,* j*th,* $2 \leq j \leq k-1$*, and* k*th rounds in the exam system is as*

follows:

$$b_1^e(s) = \sum_{L=1}^{k-1} \frac{(q_L - q_{L+1})}{\gamma} \frac{E\left[Y_{s,L}^{1+\delta} | Y_{s,1} < s\right]}{s^\delta} + \frac{q_k}{\gamma} \frac{E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right]}{s^\delta},$$

$$b_j^e(s) = \sum_{L=j}^{k-1} \frac{(q_L - q_{L+1})}{\gamma} \frac{E\left[Y_{s,L}^{1+\delta} | Y_{s,j} < s < Y_{s,j-1}\right]}{s^\delta} + \frac{q_k}{\gamma} \frac{E\left[Y_{s,k}^{1+\delta} | Y_{s,j} < s < Y_{s,j-1}\right]}{s^\delta},$$

$$b_k^e(s) = \frac{q_k}{\gamma} \frac{E\left[Y_{s,k}^{1+\delta} | Y_{s,k} < s < Y_{s,k-1}\right]}{s^\delta}.$$

Proof. See Appendix. ■

The following example shows the result of the model when ability follows uniform distribution.

Example 15 *Abilities are uniformly distributed on* $[0,1]$*. In the last round, the equilibrium contribution to the official in* c_k *is*

$$\phi_k^e(s) = \frac{n-k}{n-k+\delta+1} \frac{q_k}{\gamma} s^{1+\delta},$$

and the symmetric equilibrium bribe strategy is

$$b_k^e(a) = \frac{n-k}{n-k+\delta+1} \frac{q_k}{\gamma} s.$$

Proceeding inductively, it will be verified that the bribing strategy in the j*th round is*

$$b_j^e(s) = \frac{n-k}{n-j+\delta+1} \frac{q_k}{\gamma} s + \sum_{L=j}^{k-1} \frac{n-L}{n-j+\delta+1} \frac{(q_L - q_{L+1})\, s}{\gamma},$$

which can be transformed to

$$b_j^e(a) = \frac{n-k}{n-j+\delta+1} a q_k + \sum_{L=j}^{k-1} \frac{n-L}{n-j+\delta+1} a\,(q_L - q_{L+1}).$$

The equilibrium bribe of a particular student in the jth round is determined by the number of students and colleges, the student's score, the investigation power and qualities of all the colleges. Given the same conditions and uniform distribution ability, the degree of corruption in the exam system is lower than in the market system, but both systems will produce efficient outcome.

Corollary 5 *Given perfect capital markets and two colleges with different qualities, if the ability follows uniform distribution, then the allocation results in both*

systems are efficient but the degree of corruption in the market system is higher than in the exam system.

The analysis on the CCA model under corruption given perfect capital markets presents students' equilibrium strategies in different scenarios, and also compare of the two systems in terms of efficiency and degree of corruption. Although the allocation results in both systems are efficient, the exam system dominates the market system in terms of degree of corruption. It implies that the officials in the market system would get more expected revenue than in the exam system.

2.5 Model with Borrowing Constraints

In this section, we look at the model assuming that there are no capital markets at all. The reason for this market failure is that there is not a well constructed capital system for students. Another possible reason is that the inability to penalize recalcitrant borrowers or an unverifiable output level would be sufficient to close down capital markets. (Fernandez, 1998) As a result, those who can not afford the cost of bribing for education have to consider their budgets.

Assume that there are n students, each of whom is characterized by an endowment of "ability", a, and initial wealth w. a and w are independent and distributed on the area of $[0,1]\times[0,1]$ according to the joint distribution $F(a,w)$ with a density function $f(a,w)$. These attributes are independent across students. Assume both ability and wealth are not observed by other students, but each student knows the probability distribution of other students' abilities and wealth. We will refer to the pair (a_i, w_i) as the type of student i. A student's strategy should be a function of his ability and budget in the market system, and a function of his score and budget in the exam system. Firstly, we look at the simplest case of one college with one place and n applicants. For the purpose of comparison between systems, we use a to substitute s in the equilibrium bribe of the exam system.[32]

Che and Gale (1995, 1998) show that, in a first price auction with borrowing constraints, there exists a unique, symmetric equilibrium given sufficient conditions. We employ their results in our model as follows.

Let $G^c(a,w)$ denote the probability that a random chosen student's type, say (a',w'), satisfies $a'<a$ or $w'<w$. We have

$$G^c(a,w)\equiv 1-\int_a^1\int_w^1 f(\widetilde{a},\widetilde{w})\,d\widetilde{w}d\widetilde{a}.$$

[32]In the exam system, score can be transformed to a function of ability by $s=\gamma a$.

In the x $(x = market, exam)$ system, we consider the equilibrium strategies of the form $B^x(a,w) = \min\{\beta^x(a), w\}$, where $\beta^x(a)$ is some continuous, strictly increasing function in a.[33] Consider a student with $(a,1)$. This student effectively never faces financial constraints and his equilibrium bribe would be $B^x(a,1) = \min\{\beta^x(a), 1\} = \beta^x(a)$. A random selected student (a', w') has a lower bribe than type $(a,1)$ if $\min\{\beta^x(a'), w'\} \leq \beta^x(a)$ with probability $F^c(a) \equiv G^c(a, \beta^x(a))$. Therefore, the problem facing a student with $(a,1)$ is the same as if all students do not have borrowing constraints, with abilities drawn from the distribution $F^c(\cdot)$. By using the standard technique, we can easily obtain the equilibria in both systems. However, the existence of $\beta^x(\cdot)$ is not immediate because $F^c(\cdot)$ is determined by $\beta^x(a)$ itself. Che and Gale (1998) show that a technical assumption can ensure the existence of a unique equilibrium bribe function.[34]

Next we analyse the two systems respectively.

2.5.1 Market System

In the market system, the equilibrium bribe is of the following form:

$$B^m(a,w) = \min\{\beta^m(a), w\}.$$

The result of Claim 2 implies $\beta^m(a)$ must satisfy

$$\beta^m(a) = v - q\int_0^a \frac{F_1^c(y)}{F_1^c(a)} d_y,$$

or

$$\beta^m(a) = qE[Y_1^c | Y_1^c < a],$$

where Y_1^c is the highest of $n-1$ draws from the distribution F^c, and $F_1^c(\cdot) \equiv F^c(\cdot)^{n-1}$ is the distribution of Y_1^c.

If a student with (a_0, w_0) and $\beta^m(a_0) < w_0$, then he would follow $\beta^m(a_0)$, otherwise he just bribes his wealth. The strategy entails Leontief isobid curves. Figure 7 depicts the set of types who bribe the same amount as does type (a_0, w_0). We can find a point $(a', 1)$ which satisfies $B^x(a_0, w_0) = \beta^x(a')$. Clearly, there are

[33] Che and Gale (1995) show that any symmetric equilibrium in a first price auction with borrowing constraints must take the form as $B(\beta, w)$, and $\beta^x(\cdot)$ is continuous and strictly increasing.

[34]
$$(n-1)w + \frac{G^c(a,w)}{G_1^c(a,w)}$$
is strictly increasing in w for all $a \in (0,1)$.

two possible groups. The first group consists of those students whose $\beta^m(\cdot)$ is less than their wealth, e.g., (a_0, w_0), and the horizontal line with w_0 as its abscissa; the second includes those students whose $\beta^m(\cdot)$ is greater than their wealth and the vertical line with a' as its vertical coordinates. All students in the first group will bribe their wealth, and the bribes of all students in the second group follow $\beta^m(\cdot)$.

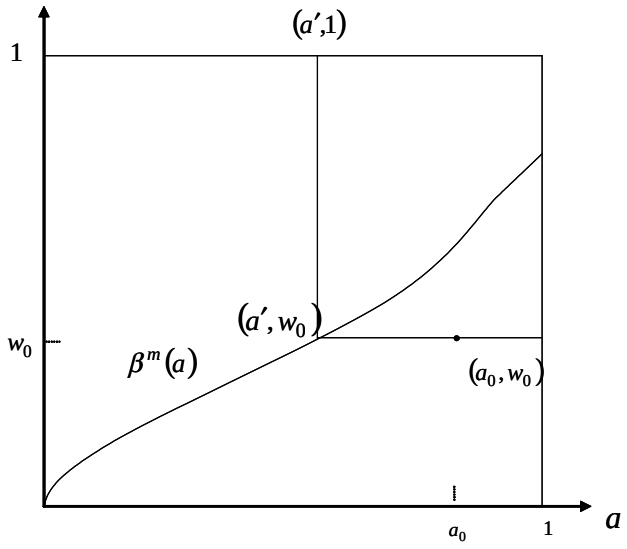

Figure 7: Budget Constraint in Market System

Claim 13 *With borrowing constraints, the official's expected revenue in the market system is*

$$E[\pi^m] = qE[Y_2^c],$$

where Y_2^c is the second-highest of n draws from the distribution $F^c(\cdot)$.

Proof. See Appendix. ■

This claim implies that the expected revenue in the case with borrowing constraints is of the same form as in perfect capital markets model, but with a different distribution for abilities.

This research is from a social planner's perspective, and hence efficiency of outcomes concerns us. If we consider students' valuation as the product of education, then the efficient allocation results imply the maximal total educational output. Therefore, the principle of efficiency in our work is therefore to allocate the places of the best colleges to those students who can use the education opportunities most efficiently. In the market system, the officials decide the admission

only based on the amount of bribe paid by students. Given perfect capital markets, the allocation result is efficient because students have no financing problem and those with higher abilities will bribe more and be accepted. However, under borrowing constraints, students with higher ability may lose their opportunities to receive higher education. Let us look at the following example.

Suppose two students, (a_0, w_0) and (a_1, w_1) with $a_0 > a_1$, are competing for one place. Without borrowing constraints, student (a_0, w_0) wins the place because $b^m(a_0) > b^m(a_1)$ even if $b^m(a_0) > w_0$. With budget constraints, suppose $\beta^m(a_0) > w_0$ and $w_0 < \beta^m(a_1) < w_1$. Since (a_0, w_0) can only bribes w_0, which is less than $\beta^m(a_1)$, he will be rejected by the official although he has a higher ability. This scenario is graphed in Figure 8. Point (a_0, w_0), (a_1, w_1) denote the two types respectively, and $\beta^m(a)$ represents an increasing, symmetric function of a. So, $B^m(a_0, w_0) = \beta^m(a')$ and $B^m(a_1, w_1) = \beta^m(a_1)$. Since $\beta^m(a') < \beta^m(a_1)$, $B^m(a_0, w_0) < B^m(a_1, w_1)$, and hence (a_1, w_1) will be accepted. Generally speaking, any types which fall in the shadow area in Figure 8, e.g. (a_1, w_1), will bribe more and win the place with higher probability than student (a_0, w_0) although they have lower abilities. Note that the shadow area above the curve $\beta^m(a)$ indicates the types who are not subject to the budget, so their bribes follow $\beta^m(a)$. The shadow area below the curve represents the types who are limited by their budgets, so they bribe their budgets, which however are higher than w_0. This example suggests that allocation results with borrowing constraints may be

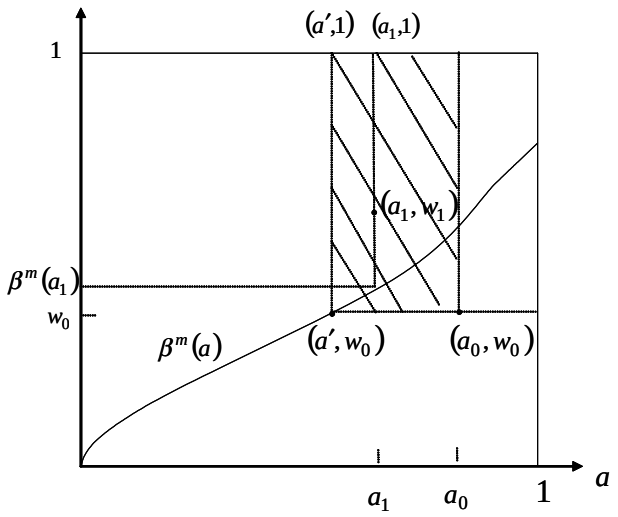

Figure 8: Inefficiency of the Market System

inefficient.

Degree of corruption is another feature concerning us. Recall the expected

received bribe in the perfect capital case satisfies

$$E\left[\pi^{m,p}\right] \equiv qE\left[Y_2^p\right];$$

and the expected received bribe in the case with borrowing constraints satisfies

$$E\left[\pi^{m,c}\right] \equiv qE\left[Y_2^c\right],$$

where Y_2^p and Y_2^c denote the second highest of n draws from the distribution $F(\cdot)$ and $F^c(\cdot)$ respectively. Therefore, if the relationship between $F(\cdot)$ and $F^c(\cdot)$ satisfies $E\left[Y_2^p\right] > E\left[Y_2^c\right]$, then the degree of corruption is greater in the unconstrained case than in the constrained case.

2.5.2 Exam System

In the exam system, the equilibrium strategy is assumed to be of the following form:

$$B^e(a, w) = \min\left\{\beta^e(a), w\right\},$$

for some function $\beta^e(a)$ increasing in a. Note we do not use score in the function as we can substitute s by γa to make the model simpler. For the same reason as in the market system, it must be that $\beta^e(a) < a$.

The result of Claim 3 implies $\beta^e(a)$ must satisfy

$$\beta^e(a) = v - (1+\delta) q \int_0^a \frac{F_1^c(y) y^\delta}{F_1^c(y) s^\delta} dy,$$

or

$$\beta^e(a) = q \frac{E\left[\left(Y_1^c\right)^{1+\delta} | Y_1^c < a\right]}{a^\delta}.$$

Claim 14 *With borrowing constraints, the official's expected revenue in the exam system is*

$$E\left[\pi^{c,e}\right] = \frac{q}{a^\delta} E\left[\left(Y_2^c\right)^{1+\delta}\right].$$

where Y_2^c is the second-highest of n draws from the distribution $F^c(\cdot)$.

Proof. See Appendix. ■

For the same reason, the allocation result in the exam system may be inefficient because a higher ability student could fail to win a place due to the

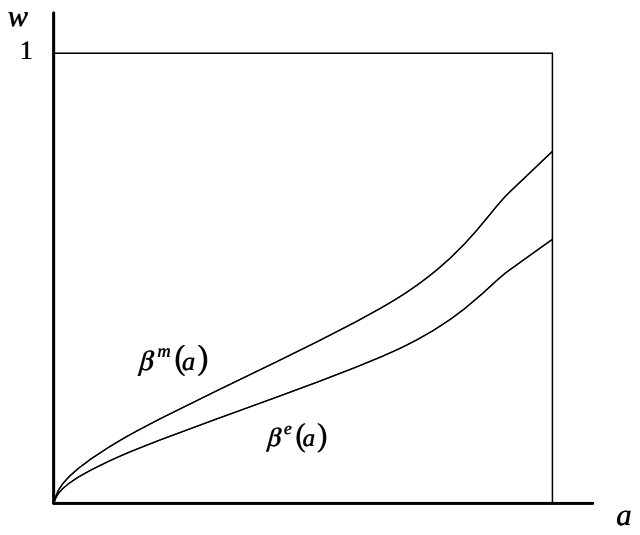

Figure 9: Market System and Exam System I

borrowing constraints. If $F(\cdot)$ and $F^c(\cdot)$ satisfies $E[Y_2^p] > E[Y_2^c]$, then the degree of corruption is greater in the unconstrained case than in the constrained case.

2.5.3 Market System vs Exam System

Claim 3 immediately implies that $\beta^e(a) \leq \beta^m(a)$ and a strict inequality holds for all $a \in (0, 1]$. This result is graphed in Figure 9. The curve $\beta^m(a)$ is above the curve $\beta^e(a)$ for all $a \in (0, 1]$.

We use a simple example to compare the two systems in terms of efficiency. Consider a student S_1 with type (a_0, w_0), who has higher ability but lower wealth than student S_2 with type (a', w') and $a_0 > a'$, $w_0 < w'$. S_1 and S_2 are competing for one place. An efficient outcome is supposed to allocate the place to S_1 since S_1 has a higher ability than S_2. Given Claim 3, we have the following possible outcomes as graphed in Figure 10, Figure 11 and Figure 12.

1. See Figure 10. Consider S_1's endowment point is below β^m and β^e. Since $a' < a_0$, and $w_0 < w'$, S_2's three possible endowments are in area $a_0 S_1 w_0$.

   (a) When $a' \in (a_1, a_0)$, for instance $S_2^{(1)}$, S_2 wins the place in both systems. The allocations are inefficient in both systems as the student with lower ability wins the place.

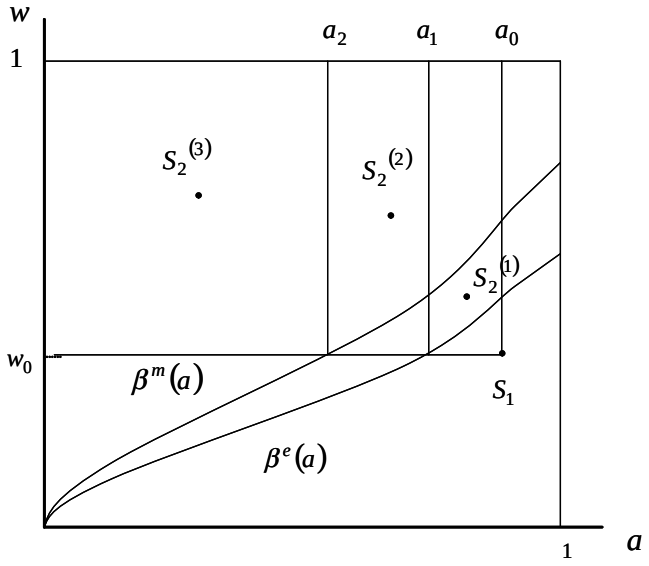

Figure 10: Market System and Exam System II

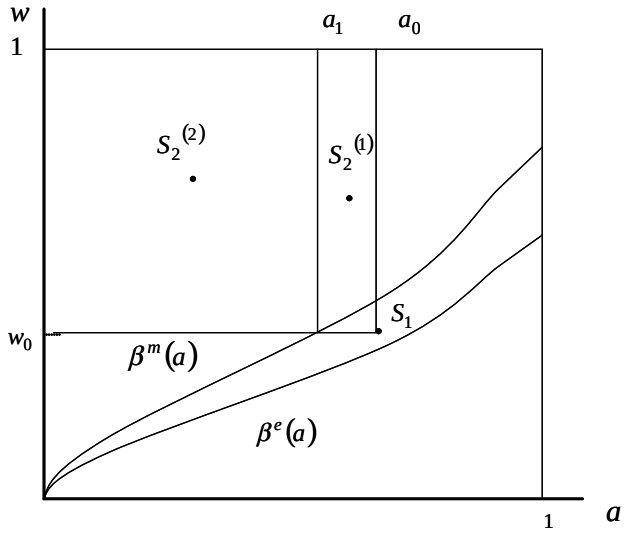

Figure 11: Market System and Exam System III

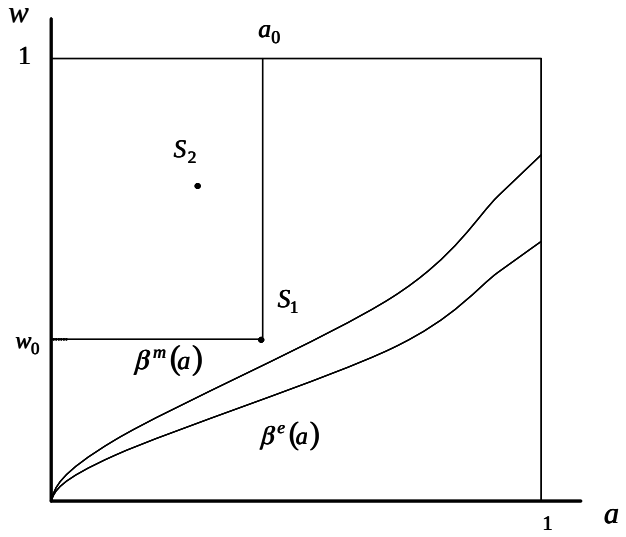

Figure 12: Market System and Exam System IV

(b) When $a' \in (a_2, a_1)$, for instance $S_2^{(2)}$, S_2 wins in the market system, but S_1 wins in the exam system. The allocation is efficient in the exam system, but inefficient in the market system.

(c) When $a' \in (0, a_2)$, for instance $S_2^{(3)}$, S_1 wins in both systems. The allocations are efficient in both systems.

2. See Figure 11. Consider S_1's endowment point is below β^m and above β^e. Since $a' < a_0$, and $w_0 < w'$, S_2's two possible endowments are in area $a_0 S_1 w_0$.

(a) When $a' \in (a_1, a_0)$, for instance $S_2^{(1)}$, S_2 wins the place in the market system, but S_1 wins in the exam system. The allocation is efficient in the exam system , but inefficient in the market system.

(b) When $a' \in (0, a_1)$, for instance $S_2^{(2)}$, S_1 wins in both systems. The allocations are efficient in both systems.

3. See Figure 12. Consider S_1's endowment point is above β^m and β^e. Since $a' < a_0$, and $w_0 < w'$, S_2's only possible endowment is in area $a_0 S_1 w_0$. There is only one possible result. When $a' \in (0, a_1)$, S_1 wins in both systems. The allocations are efficient in both systems.

The outcomes above show that allocation results could be inefficient in both systems because of borrowing constraints. However, for the same set of students

and colleges, the allocation results induced by the exam system are always efficient if the allocation results induced by the market system are efficient, and the allocation results induced by the exam system may be efficient even if the allocation results induced by the market system are inefficient. So, we conclude the exam system dominates the market system in terms of efficiency.

As regards degree of corruption, the following claim implies that, once again, the exam system is less corrupt than the market system with borrowing constraints.

Proposition 12 *Given borrowing constraints, the allocation results in both systems may be inefficient, but the exam system dominates the market system in terms of efficiency; the degree of corruption in the market system is greater than in the exam system.*

Proof. See Appendix. ■

The intuition here is that the existence of exams and scores has lowered the amount of bribe, and hence the effect of borrowing constraints on efficiency. With borrowing constraints, allocation results in both systems may be inefficient, however, exams possess greater allocation efficiency than markets. Different scores imply differing probabilities of being caught, and thus exams give students with higher a ability but a lower budget a higher chance of being admitted. The official's expected revenue is higher in the market system than in the exam system even with borrowing constraints.

2.6 Conclusion

The national college entry examination is used in China to decide whether students are admitted by colleges or not. There are many criticisms of this system. One of them is that many students may lose the opportunity of being educated at colleges because of poor performance in the exam even though they are talented but are not capable of taking exams. Another criticism focuses on corruption in the admission process. These critics suggest abandoning the exam system and to adopt a market system, in which each college decides to accept or reject a student by face-to-face interviews or base on average performance in high school which could only be shown in application materials. In the sense of telling the real ability of students, the market system could perform better than the exam system. However, the conclusion may be different if we include the considerations of corruption. Corruption in this piece of work mainly implies that students have to

bribe admission officials with the result that investment may not be productive. The more important is that the allocation result could be inefficient as a result of corruption. We define efficiency as when students with the highest valuations of higher education have a higher priority of being admitted. Students' valuations are determined by their abilities and qualities of colleges. In this chapter, we find the equilibria of the two systems under different assumptions, and then compare the two systems in terms of efficiency and the degree of corruption.

We firstly assume that students have no borrowing constraints. They can borrow money for bribing and pay that back after graduation. It has been shown that the efficiency of allocation would not be affected in either the market or the exam system. Places are allocated to students according to the ranking of abilities. However, the degree of corruption is higher in the market system than in the exam system because of the higher equilibrium bribe in the market system. This result holds in both the simplest model with one place and the models with multiple places or multiple colleges. The reason is that exams and scores provide criteria for the authority to supervise admission officials, and students include it into their decision making process. As intuition would suggest, the equilibrium bribe is decreasing in the power of investigation.

The allocation may not be efficient when students have borrowing constraints because a higher ability student could lose his priority as his budget would not allow him to bribe as much as he would like. We show that the exam system is better than the market system in terms of efficiency. The exam system may be inefficient, but it dominates the market system because the existence of exams and scores reduces the effect of budget constraints on efficiency. Once again, the expected amount of bribe in the exam system is lower than the market system, and hence the degree of corruption in the market system is greater than that in the exam system.

The conclusion is favours the exam system. However, there are many other aspects that further analysis should take into account, such as the cost of taking the exam, etc. The conclusion may change if we include these factors. Another way that the conclusion might change is if corruption in the market system can be avoided or made less serious by reinforcing the investigation system or using a centralised admission system. All these aspects are possible directions of further study.

2.7 Appendix

Proof of Claim 2:

Proof. Assume there exists such a symmetric equilibrium involves student i bribing $b^m(a)$ where $b^m(a)$ is differentiable and increasing in a, $\frac{\partial b^m}{\partial a} > 0$. Student i is following an optimal strategy to maximise his expected payoff if he realises that everyone else is bribing according to b^m . We have a Bayesian Nash equilibrium since we have assumed that the game is symmetric. Consider what happens if student i with ability a bribes $b^m(z)$ instead of $b^m(a)$, which is the equilibrium strategy.

The expected payoff of student i is

$$\Pi_i = F_1(z) \times (v - b^m(z)).$$

Maximizing the expected payoff with respect to z yields the first order condition,

$$f_1(z) \times (v - b^m(z)) - F_1(z) b^{m\prime}(z) = 0.$$

At a symmetric equilibrium it is optimal to report $z = a$, so we obtain

$$\begin{aligned} f_1(a) \times (v - b^m(a)) - F_1(a) b^{m\prime}(a) &= 0, \\ f_1(a) \times (aq - b^m(a)) - F_1(a) b^{m\prime}(a) &= 0 \end{aligned}$$

as $v = aq$.

$$\frac{\partial}{\partial a}\left[b^m(a) F_1(a)\right] = aq f_1(a).$$

Since $b^m(0) = 0$, student with zero ability will bribe nothing. Integrating and rearranging the equation gives

$$\begin{aligned} b^m(a) &= \frac{q}{F_1(a)} \int_0^a \left[y f_1(y)\right] d_y \\ &= qE\left[Y_1 | Y_1 < a\right]. \end{aligned}$$

■

Proof of Claim 3:

Proof. Assume there exists such a symmetric equilibrium involves student i bribing $b^e(s)$ where $b^e(s)$ is differentiable and also satisfies that $\phi^e(s) = b^e(s) s^\delta$ is differentiable and increasing in s, $\frac{\partial \phi^e}{\partial s} > 0$.We now look at a particular student

i. Suppose student i bribes $b^e(z)$ instead of equilibrium strategies, $b^e(s)$.

Student i's expected payoff is

$$\begin{aligned}\Pi &= \Pr(\text{win})\,(v - b^e(z)) \\ &= \Pr\left(b^e(z)\,s^\delta \geq \phi(Y_{s,1})\right)(v - b^e(z)).\end{aligned}$$

Note we use s^δ as the probability of keeping the bribe safely. This is because the student's score is observed by the official, and hence the probability of being caught does not change even if i pretended to be some other types.

Now we let S denote the inverse function of ϕ, i.e. $s = S\left(b^e(s)\,s^\delta\right)$.

$$\begin{aligned}\Pr\left(b^e(z)\,s^\delta \geq \phi(Y_{s,1})\right) &= \Pr\left(S\left(b^e(z)\,s^\delta\right) \geq Y_{s,1}\right) \\ &= F_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right).\end{aligned}$$

So, student i's expected payoff can be written as

$$\Pi = F_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right)(v - b^e(z)).$$

The first order condition in terms of z is

$$f_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right)S'\left(b^e(z)\,s^\delta\right)b^{e\prime}(z)\,s^\delta(v - b^e(z)) - F_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right)b^{e\prime}(z) = 0,$$

$$f_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right)S'\left(b^e(z)\,s^\delta\right)s^\delta(v - b^e(z)) = F_{s,1}\left(S\left(b^e(z)\,s^\delta\right)\right).$$

In equilibrium it is optimal to bribe $b^e(s)$. Setting $z = s$ in the first order condition results in

$$f_{s,1}\left(S\left(b^e(s)\,s^\delta\right)\right)S'\left(b^e(s)\,s^\delta\right)\left(vs^\delta - b^e(s)\,s^\delta\right) = F_{s,1}\left(S\left(b^e(s)\,s^\delta\right)\right),$$

$$f_{s,1}(s)\frac{\partial s}{\partial \phi}\left(vs^\delta - \phi^e(s)\right) = F_{s,1}(s),$$

$$f_{s,1}(s)\left(vs^\delta - \phi^e(s)\right) = F_{s,1}(s)\,\phi^{e\prime}(s),$$

or

$$F_{s,1}(s)\,\phi^{e\prime}(s) + f_{s,1}(s)\,\phi^e(s) = vs^\delta f_{s,1}(s),$$

or

$$F_{s,1}(s)\,\phi^e(s) = \int_0^s vy^\delta f_{s,1}(y)\,dy,$$

which has a solution

$$\begin{aligned}\phi^e(s) &= \frac{1}{F_{s,1}(s)}\int_0^s f_{s,1}(y)\left(\frac{q}{\gamma}y^{1+\delta}\right)dy \\ &= \frac{q}{\gamma}\left[s^{1+\delta}-(1+\delta)\int_0^s \frac{F_{s,1}(y)}{F_{s,1}(s)}y^{\delta}dy\right].\end{aligned}$$

Since

$$E\left[Y_{s,1}^{1+\delta}|Y_{s,1}<s\right]=\frac{1}{F_{s,1}(s)}\int_0^s f_{s,1}(y)\left(y^{1+\delta}\right)dy,$$

we have

$$\phi^e(s)=\frac{q}{\gamma}E\left[Y_{s,1}^{1+\delta}|Y_{s,1}<s\right].$$

$$\begin{aligned}b^e(s) &= \frac{\phi^e(s)}{s^{\delta}} \\ &= \frac{q}{\gamma}\left[s-(1+\delta)\int_0^s \frac{F_{s,1}(y)\,y^{\delta}}{F_{s,1}(s)\,s^{\delta}}dy\right] \\ &= \frac{q}{\gamma}\frac{E\left[Y_{s,1}^{1+\delta}|Y_{s,1}<s\right]}{s^{\delta}}.\end{aligned}$$

■

Proof of Proposition 6:

Proof. The place will be allocated to the student from whom the official gets the highest expected payoff, i.e., the highest ϕ^e. Given

$$\phi^e(s)=\frac{q}{\gamma}E\left[Y_{s,1}^{1+\delta}|Y_{s,1}<s\right],$$

we have

$$\phi^e(s)=\frac{q}{\gamma}\left[s^{1+\delta}-(1+\delta)\int_0^s \frac{F_{s,1}(y)}{F_{s,1}(s)}y^{\delta}dy\right],$$

$$\frac{\partial\phi^e}{\partial s}=\frac{q}{\gamma}\left[(1+\delta)s^{\delta}-(1+\delta)\left(s^{\delta}-\frac{f_{s,1}(s)\int_0^s F_{s,1}(y)\,y^{\delta}dy}{F_{s,1}(s)^2}\right)\right],$$

or

$$\frac{\partial\phi^e}{\partial s}=\frac{q}{\gamma}\left[(1+\delta)\frac{f_{s,1}(s)\int_0^s F_{s,1}(y)\,y^{\delta}dy}{F_{s,1}(s)^2}\right]>0.$$

Thus, the official gets the highest expected payoff from the student with the

highest score, and hence he will allocate the place to this student. Since the student with the highest score has the highest ability, therefore, the result must be efficient. ■

Proof of Proposition 7:

Proof. Let $\Gamma(s,\delta)$ denote $E\left[Y_{s,1}^{1+\delta}|Y_{s,1}<s\right]$. We have $\Gamma_s>0$ and $\Gamma_\delta<0$ as $Y_{s,1}<s<1$.

Substituting Γ into the equilibrium function induces $b^e=\frac{q}{\gamma}\frac{\Gamma}{s^\delta}$, hence $b_\delta^e=\frac{q}{\gamma}\left(\Gamma_\delta s^{-\delta}+\Gamma s^{-\delta}\ln s\right)$. We have $b_\delta^e<0$ as $\Gamma_\delta<0$, $s^{-\delta}>0$ and $\Gamma s^{-\delta}>0$, $\ln s<0$. Thus, the equilibrium bribe is decreasing as the power of investigation. ■

Proof of Claim 4:

Proof. The equilibrium bribe of type a in the market system with one place is as follows:

$$b^m(a)=\frac{q}{F_1(a)}\int_0^a\left[yf_1(y)\right]d_y.$$

The equilibrium bribe of type a in the exam system with one place is as follows:

$$b^e(a)=\frac{q}{a^\delta F_1(a)}\int_0^a\left[y^{1+\delta}f_1(y)\right]d_y,$$

which is transformed from

$$b^e(s)=\frac{1}{s^\delta F_{s,1}(s)}\int_0^s f_{s,1}(y)\left(\frac{q}{\gamma}y^{1+\delta}\right)dy$$

since $s=\gamma a$.

Since for all $a\in(0,1]$, if $\delta\neq0$, then

$$\int_0^a\left[\frac{qy}{F_1(a)}\left(\frac{y}{a}\right)^\alpha f_1(y)\right]d_y<\int_0^a\left[\frac{qy}{F_1(a)}f_1(y)\right]d_y,$$

and hence

$$\frac{q}{a^\delta F_1(a)}\int_0^a\left[y^{1+\delta}f_1(y)\right]d_y<\frac{q}{F_1(a)}\int_0^a\left[yf_1(y)\right]d_y.$$

Therefore, we have $b^e(a)\leq b^m(a)$ and a strict inequality holds for all $a\in(0,1]$. ■

Proof of Proposition 8:

Proof. In both systems, the place is allocated to the student with the highest ability, i.e. highest valuation. So, the results are efficient in both systems.

To compare the degree of corruption between the two systems, we need to derive the official's expected revenue.

The expected revenue in system x $(x = market, exam)$ is

$$E(\pi^x) = nE(b^{x,ex}),$$

where $E(\pi^x)$ denotes the expected revenue, and $E(b^{x,ex})$ denotes the *ex ante* expected bribe of a particular student with ability a in the market system, marked by m, and the exam system, marked by e, respectively.[35] We know that

$$E(b^{m,ex}) = \int_0^1 E(b^m) f(a) da,$$

and

$$E(b^{e,ex}) = \int_0^1 E(b^e) f(a) da.$$

Clearly, $E(b^{m,ex}) \geq E(b^{e,ex})$ if and only if $E(b^m) \geq E(b^e)$.

Given

$$E(b^x) = \Pr(\text{Win in the } x \text{ system}) \times b^x(a)$$

and for a particular student, the probabilities of being allocated the place in the two systems are the same, therefore the inequality of $E(b^m) \geq E(b^e)$ is equivalent to $b^m(a) \geq b^e(a)$.

Claim 3 immediately implies that $E(b^m) \geq E(b^e)$ and a strict inequality holds for all $a \in (0, 1]$. Therefore, $E(b^{m,ex}) > E(b^{e,ex})$, and hence $E(\pi^m) > E(\pi^e)$. The expected winning bribe in the exam system is lower than in the market system, which implies the degree of corruption is lower in the exam system. ■

Proof of Claim 5:

Proof. Consider what happens if student i bribes $b^m(z)$ instead of $b^m(a)$, which is the strategy in equilibrium.

The expected payoff of student i is therefore

$$\Pi_i = F_k(z) \times (v - b^m(z)).$$

[35] See definition of ex ante expected value in Krishna (2002) p20.

Maximizing the expected payoff with respect to z yields the first order condition,

$$f_k(z) \times (v - b^m(z)) - F_k(z) b^{m\prime}(z) = 0.$$

At a symmetric equilibrium it is optimal to report $z = a$, so we obtain

$$\begin{aligned} f_k(a) \times (v - b^m(a)) - F_k(a) b^{m\prime}(a) &= 0, \\ f_k(a) \times (aq - b^m(a)) - F_k(a) b^{m\prime}(a) &= 0 \end{aligned}$$

as $v = aq$.

$$\frac{\partial}{\partial a}\left[b^m(a) F_k(a)\right] = aq f_k(a).$$

Since $b^m(0) = 0$, student with zero ability will bribe nothing. Integrating and rearranging the equation gives

$$\begin{aligned} b^m(a) &= \frac{q}{F_k(a)} \int_0^a \left[y f_k(y)\right] d_y \\ &= qE\left[Y_k | Y_k < a\right]. \end{aligned}$$

■

Proof of Claim 6:

Proof. Consider what happens if student bribes $b^e(z)$ instead of $b^e(s)$ which is equilibrium strategy. Student i's expected payoff is

$$\begin{aligned} \Pi &= \Pr(\text{win})(v - b^e(z)) \\ &= \Pr\left(b^e(z) s^\delta \geq \phi^e(Y_{s,k})\right)(v - b^e(z)). \end{aligned}$$

Note we use the product of $b^e(z) s^\delta$ as the contribution to the official by i because even if i bribes $b^e(z)$ which is different with his equilibrium strategy $b^e(z)$, the probability of keeping the bribe is still s^δ for the official. Now we let S denote the inverse function of ϕ^e.

$$\begin{aligned} \Pr\left(b^e(z) s^\delta \geq \phi^e(Y_{s,k})\right) &= \Pr\left(S\left(b^e(z) s^\delta\right) \geq Y_{s,k}\right) \\ &= F_{s,k}\left(S\left(b^e(z) s^\delta\right)\right). \end{aligned}$$

So, student i's expected payoff can be written as

$$\Pi = F_{s,k}\left(S\left(b^e(z) s^\delta\right)\right)(v - b^e(z)).$$

The first order condition in terms of z is

$$f_{s,k}\left(S\left(b^e\left(z\right)s^\delta\right)\right)S'\left(b^e\left(z\right)s^\delta\right)b^{e\prime}\left(z\right)s^\delta\left(v-b^e\left(z\right)\right)-F_{s,k}\left(S\left(b^e\left(z\right)s^\delta\right)\right)b^{e\prime}\left(z\right)=0,$$

$$f_{s,k}\left(S\left(b^e\left(z\right)s^\delta\right)\right)S'\left(b^e\left(z\right)s^\delta\right)s^\delta\left(v-b^e\left(z\right)\right)=F_{s,k}\left(S\left(b^e\left(z\right)s^\delta\right)\right).$$

In equilibrium it is optimal to bribe $b^e\left(s\right)$. Setting $z=s$ in the first order condition results in

$$f_{s,k}\left(S\left(b^e\left(s\right)s^\delta\right)\right)S'\left(b^e\left(s\right)s^\delta\right)\left(vs^\delta-b^e\left(s\right)s^\delta\right)=F_{s,k}\left(S\left(b^e\left(s\right)s^\delta\right)\right),$$

$$f_{s,k}\left(s\right)\frac{\partial s}{\partial\phi^e}s^\delta\left(vs^\delta-\phi^e\left(s\right)\right)=F_{s,k}\left(s\right),$$

$$f_{s,k}\left(s\right)\left(vs^\delta-\phi^e\left(s\right)\right)=F_{s,k}\left(s\right)\phi^{e\prime},$$

$$\frac{\partial}{\partial s}\left[\phi^e\left(s\right)F_{s,k}\left(s\right)\right]=\left(\frac{q}{\gamma}s^{1+\delta}\right)f_{s,k}\left(s\right).$$

Combine with $b^e\left(0\right)=0$, which implies that student with zero ability will bribe nothing. Integrating and rearranging the equation gives

$$\begin{aligned}\phi^e\left(s\right) &= \frac{q}{\gamma}\frac{1}{F_{s,k}\left(s\right)}\int_0^s\left[y^{1+\delta}f_{s,k}\left(y\right)\right]d_y\\ &= \frac{q}{\gamma}E\left[Y_{s,k}^{1+\delta}|Y_{s,k}<s\right]\\ &= \frac{q}{\gamma}\left(s^{1+\delta}-\left(1+\delta\right)\int_0^s\frac{F_{s,k}\left(y\right)}{F_{s,k}\left(s\right)}y^\delta dy\right).\end{aligned}$$

Thus, the symmetric equilibrium bribe is

$$\begin{aligned}b^e\left(s\right) &= \frac{\phi^e\left(s\right)}{s^\delta}\\ &= \frac{q}{\gamma}\frac{E\left[Y_{s,k}^{1+\delta}|Y_{s,k}<s\right]}{s^\delta}\\ &= \frac{q}{\gamma}\left(s-\left(1+\delta\right)\int_0^s\frac{F_{s,k}\left(y\right)}{F_{s,k}\left(s\right)}\left(\frac{y}{s}\right)^\delta dy\right).\end{aligned}$$

■

Proof of Claim 7:

Proof. Let I^m denote the symmetric equilibrium expected payment of a particular student. Suppose other students are following the equilibrium strategy b^m. If this student bribes $b^m\left(z\right)$ instead of $b^m\left(a\right)$. Then the expected payoff for this

student is defined as

$$\Pi^m(z,v) \equiv F_2(z)\, v - I^m.$$

The first order condition gives

$$\frac{\partial \Pi^m(z,v)}{\partial z} \equiv f_2(z)\, v - \frac{\partial I^m}{\partial z} = 0.$$

At the equilibrium, it is optimal to bribe $z = a$, so we obtain that for all y,

$$\frac{\partial I^m}{\partial y} = q f_2(y)\, y.$$

Thus,

$$\begin{aligned} I^m &= I^m(0) + q\int_0^a y f_2(y)\, dy \\ &= q\int_0^a y f_2(y)\, dy. \end{aligned}$$

Now we consider the same student at the equilibrium. If he is allocated a place, then his bribe could be either the highest which means $a > Y_1$ or the second highest which implies $Y_2 < a < Y_1$. In the first possible case, student 1 pays the second highest bribe, $b(Y_1)$; in the second case, he pays $b(a)$.

Now we derive the expected payment when his bribe is the highest. The density function of Y_1, conditional on the event that $Y_1 < a$, can be written as

$$f_1(y|Y_1 < a) = \frac{f_1(y)}{F_1(a)} = \frac{(n-1) f(y) F(y)^{n-2}}{F(a)^{n-1}}.$$

Thus, the expected payment when his bribe is the highest can then be written as

$$\begin{aligned} I^{m,1} &\equiv F_1(a)\, E[b(Y_1)\,|Y_1 < a] \\ &= F_1(a)\int_0^a b(y)\, f_1(y|Y_1 < a)\, dy \\ &= F(a)^{n-1}\int_0^a b(y)\frac{(n-1) f(y) F(y)^{n-2}}{F(a)^{n-1}} dy \\ &= (n-1)\int_0^a b(y)\, f(y)\, F(y)^{n-2}\, dy. \end{aligned}$$

The expected payment when he is the second highest can then be written as

$$I^{m,2} \equiv (n-1)(1 - F(a))\, F(a)^{n-2}\, b(a).$$

So the total expected payment satisfies

$$\begin{aligned} I^m &= I^{m,1} + I^{m,2} \\ &= (n-1)\left[\int_0^a b(y) f(y) F(y)^{n-2} dy + (1-F(a)) F(a)^{n-2} b(a)\right]. \end{aligned}$$

Given

$$I^m = q\int_0^a y f_2(y)\, dy,$$

equalising the two equations implies that

$$\begin{aligned} & (n-1)\left[\int_0^a b^m(y) f(y) F(y)^{n-2} dy + (1-F(a)) F(a)^{n-2} b^m(a)\right] \\ = \; & q\int_0^a y f_2(y)\, dy. \end{aligned}$$

Differentiating with respect to a on both sides of the equation gives that

$$\begin{aligned} & (n-1)\left[\begin{array}{c} b^m(a) F(a)^{n-2} f(a) \\ +\left((n-2) F(a)^{n-3} f(a)(1-F(a)) - F(a)^{n-2} f(a)\right) b^m(a) \\ +(1-F(a)) F(a)^{n-2} b^{m\prime}(a) \end{array}\right] \\ = \; & aq f_2(a). \end{aligned}$$

Since

$$F_2(a) = F(a)^{n-1} + (n-1)(1-F(a)) F(a)^{n-2},$$

we have

$$f_2(a) = (n-1)(n-2) F(a)^{n-3} f(a)(1-F(a)).$$

Hence,

$$\begin{aligned} & b^m(a) F(a)^{n-2} f(a) + \left((n-2) F(a)^{n-3} f(a)(1-F(a)) - F(a)^{n-2} f(a)\right) b^m(a) \\ & +(1-F(a)) F(a)^{n-2} b^{m\prime}(a) \\ = \; & ap\left[F(a)^{n-2} f(a) + (n-2) F(a)^{n-3} f(a) - (n-1) F(a)^{n-2} f(a)\right]. \end{aligned}$$

Dividing both sides of the equation by $F(a)^{n-3}$ gives us

$$\begin{aligned} & b^m(a) F(a) f(a) + ((n-2) f(a)(1-F(a)) - F(a) f(a)) b^m(a) \\ & + (1-F(a)) F(a) b^{m\prime}(a) \\ = \; & aq\left[F(a) f(a) + (n-2) f(a) - (n-1) F(a) f(a)\right]. \end{aligned}$$

Rearrange it as

$$(n-2) f(a) aq = F(a) b^{m\prime}(a) + (n-2) f(a) b^m(a).$$

The solution to the function is as follows:[36]

$$\begin{aligned} b^m(a) &= q\frac{(n-2)\int_0^a y f(y) F(y)^{n-3} dy}{F(a)^{n-2}} \\ &= q\frac{\int_0^a y d_{F(y)^{n-2}}}{F(a)^{n-2}} \\ &= q\left(a - \int_0^a \left(\frac{F(y)}{F(a)}\right)^{n-2} dy\right). \end{aligned}$$

■

Proof of Proposition 9:

Proof. Now we compare the current model to the case of one college with one place. Let $b^{m,1}(a)$ denote the equilibrium in case of one place, $b^{m,2}(a)$ denote the equilibrium bribe in case of two colleges, and $b^{m,3}(a)$ denotes case of one college with two places.

$$\begin{aligned} b^{m,1}(a) &= qE[Y_1 | Y_1 < a] = q\left(a - \int_0^a \left(\frac{F(y)}{F(a)}\right)^{n-1} d_y\right), \\ b^{m,2}(a) &= qE[Y_2 | Y_2 < a] = q\left(a - \int_0^a \frac{F_2(y)}{F_2(a)} d_y\right), \\ b^{m,3}(a) &= q\left(a - \int_0^a \left(\frac{F(y)}{F(a)}\right)^{n-2} dy\right). \end{aligned}$$

[36] We used the following formula to solve the function:

$$y' + p(x) y = q(x),$$

$$y = e^{-\int p(x) dx}\left[\int q(x) e^{\int p(x) dx} dx + c\right].$$

Since $qE[Y_2|Y_2 < a] < qE[Y_1|Y_1 < a]$, clearly $b^{m,2}(a) < b^{m,1}(a)$. Since $\frac{F(y)}{F(a)} < 1$, then $\left(\frac{F(y)}{F(a)}\right)^{n-1} < \left(\frac{F(y)}{F(a)}\right)^{n-2}$, hence $b^{m,3}(a) < b^{m,1}(a)$.

Substitute $F_2(a) = F(a)^{n-1} + (n-1)(1-F(a))F(a)^{n-2}$ into $b^{m,2}(a)$ and rearrange it, then we obtain

$$b^{m,2}(a) = q\left(a - \int_0^a \frac{(n-1)F(y)^{n-2} - (n-2)F(y)^{n-1}}{(n-1)F(a)^{n-2} - (n-2)F(a)^{n-1}} d_y\right).$$

Since

$$\frac{(n-1)F(y)^{n-2} - (n-2)F(y)^{n-1}}{(n-1)F(a)^{n-2} - (n-2)F(a)^{n-1}} = \left(\frac{F(y)}{F(a)}\right)^{n-2}\left(\frac{(n-1)-(n-2)F(y)}{(n-1)-(n-2)F(a)}\right),$$

and $\left(\frac{(n-1)-(n-2)F(y)}{(n-1)-(n-2)F(a)}\right) > 1$, we have

$$\frac{(n-1)F(y)^{n-2} - (n-2)F(y)^{n-1}}{(n-1)F(a)^{n-2} - (n-2)F(a)^{n-1}} > \left(\frac{F(y)}{F(a)}\right)^{n-2}.$$

Thus,

$$q\left(a - \int_0^a \left(\frac{F(y)}{F(a)}\right)^{n-2} dy\right) > q\left(a - \int_0^a \frac{(n-1)F(y)^{n-2} - (n-2)F(y)^{n-1}}{(n-1)F(a)^{n-2} - (n-2)F(a)^{n-1}} d_y\right),$$

and hence $b^{m,2}(a) < b^{m,3}(a)$.

Recall that the expected payments of a particular student in the second and third cases are the same, and satisfy

$$E\left(b^{m,2}\right) = E\left(b^{m,3}\right) = q\int_0^a y f_2(y)\, dy,$$

and hence the *ex ante* expected bribe of the student would be the same as

$$E\left(b^{ex}\right) = \int_0^1 E(b) f(a)\, da.$$

Therefore, the expected revenue for the only official in the second case is the same as the expected revenues for the two officials in the third case. ■

Proof of Claim 8:

Proof. Let I^e denote the symmetric equilibrium expected payment of a particular student. Suppose other students are following the equilibrium strategy b^e. If this student bribes $b^e(z)$ instead of $b^e(s)$. Then the expected payoff for this student is defined as

$$\Pi^e(z) \equiv F_{s,2}\left(S\left(b^e(z)\, s^\delta\right)\right) v - I^e,$$

where $F_{s,2}$ is the probability he wins or the probability that $b^e(z)\, s^\delta$ exceeds the second highest competing $\phi^e(Y_{s,2})$.

First order condition implies that

$$\frac{\partial \Pi^e(z)}{\partial z} = f_{s,2}\left(S\left(b^e(z)\, s^\delta\right)\right) S'\left(b^e(z)\, s^\delta\right) b^{e\prime}(z)\, s^\delta v - \frac{\partial I^e}{\partial z} = 0,$$

or

$$f_{s,2}\left(S\left(b^e(z)\, s^\delta\right)\right) S'\left(b^e(z)\, s^\delta\right) s^\delta v = \frac{\partial I^e}{\partial z}\frac{\partial z}{\partial b^e} = \frac{\partial I^e}{\partial b^e}.$$

Now we let φ be the symmetric equilibrium expected payoff by the student. If this student bribes $b^e(z)$ instead of $b^e(s)$, then $\varphi = I^e s^\delta$. So, $\frac{\partial \varphi}{\partial z} = s^\delta \frac{\partial I^e}{\partial z}$.

At an equilibrium it is optimal to report $z = s$, thus,

$$\begin{aligned} f_{s,2}(s)\, S'\left(b^e(s)\, s^\delta\right) s^\delta v &= \frac{\partial I^e}{\partial b^e}, \\ f_{s,2}(s)\, s^\delta v &= \frac{\partial I^e}{\partial b^e}\frac{\partial \phi^e}{\partial s}, \\ &= \frac{\partial I^e}{\partial s}\frac{\partial \phi^e}{\partial b^e} \\ &= s^\delta \frac{\partial I^e}{\partial s}. \end{aligned}$$

Since at the equilibrium, $\frac{\partial \varphi}{\partial s} = s^\delta \frac{\partial I^e}{\partial s}$, we have

$$f_{s,2}(s)\, s^\delta v = \frac{\partial \varphi}{\partial s}$$

$\varphi(0) = 0$, thus

$$\varphi = \frac{q}{\gamma}\int_0^s y^{1+\delta} f_{s,2}(y)\, dy.$$

On the other hand, consider this student, winning implies his score s is either the highest which means s exceeds the highest of the other $n-1$ abilities, $Y_{s,1} < s$ or the second highest which means s is lower than the highest of the other $n-1$ scores, but higher than the second highest of the other $n-1$ scores, $Y_{s,2} < s < Y_{s,1}$. The bribe this student pays is an amount such that his contribution to the official

is the second highest bribe, $\phi(Y_{s,1})$, $b = \phi(Y_{s,1})/s$, when he has the highest score, and his own bribe, $b(s)$, when he has the second highest score.

First, we need to derive the expected payment when his score is the highest. The density of $Y_{s,1}$, conditional on the event that $Y_{s,1} < s$, can be written as

$$f_{s,1}^{(n-1)}(y|Y_{s,1} < s) = \frac{f_{s,1}^{(n-1)}(y)}{F_{s,1}^{(n-1)}(s)} = \frac{(n-1) f_s(y) F_s(y)^{n-2}}{F_s(s)^{n-1}},$$

so the expected payoff to the official from the student with the highest score satisfies

$$\begin{aligned}
\varphi^1 &\equiv F_{s,1}^{(n-1)}(s) E[\phi(Y_{s,1}) | Y_{s,1} < s] \\
&= F_{s,1}^{(n-1)} \int_0^s \phi(y) \frac{(n-1) f_s(y) F_s(y)^{n-2}}{F_{s,1}^{(n-1)}} dy \\
&= (n-1) \int_0^s \phi(y) f_s(y) F_s(y)^{n-2} dy.
\end{aligned}$$

The expected payoff to the official from the student with the second highest score satisfies

$$\varphi^2 = (n-1)(1 - F_s(s)) F_s(s)^{n-2} \phi(s).$$

The total expected payment is

$$\begin{aligned}
\varphi &= \varphi^1 + \varphi^2 \\
&= (n-1) \int_0^s \phi^e(y) f_s(y) F_s(y)^{n-2} dy + (n-1)(1 - F_s(s)) F_s(s)^{n-2} \phi^e(s).
\end{aligned}$$

Hence

$$\begin{aligned}
&(n-1) \int_0^s \phi^e(y) f_s(y) F_s(y)^{n-2} dy + (n-1)(1 - F_s(s)) F_s(s)^{n-2} \phi^e(s) \\
= &\frac{q}{\gamma} \int_0^s y^{1+\delta} f_{s,2}(y) dy.
\end{aligned}$$

Differentiating with respect to s implies that

$$(n-1) \begin{bmatrix} \phi^e(s) F_s(s)^{n-2} f_s(s) + \\ \left((n-2) F_s(s)^{n-3} f_s(s) - (n-1) F_s(s)^{n-2} f_s(s)\right) \phi^e(s) \\ + (1 - F_s(s)) F_s(s)^{n-2} \phi^{e\prime}(s) \end{bmatrix}$$

$$= \frac{q}{\gamma} s^{1+\delta} f_{s,2}(s).$$

Since

$$F_{s,2}(y) = F_s(s)^{n-1} + (n-1)(1-F_s(s))F_s(s)^{n-2},$$

$$f_{s,2}(y) = (n-1)(n-2)F_s(s)^{n-3} f_s(s)(1-F_s(s)).$$

Thus,

$$\begin{aligned}
&(n-1)\left[\begin{array}{c}\phi^e(s)F_s(s)^{n-2}f_s(s)\\ +\left((n-2)F_s(s)^{n-3}f_s(s)(1-F_s(s))-F_s(s)^{n-2}f_s(s)\right)\phi^e(s)\\ +(1-F_s(s))F_s(s)^{n-2}\phi^{e\prime}(s)\end{array}\right]\\
= \;& \frac{q}{\gamma}s^{1+\delta}(n-1)(n-2)F_s(s)^{n-3}f_s(s)(1-F_s(s)).
\end{aligned}$$

Rearrange it,

$$(n-2)f_s(s)\frac{q}{\gamma}s^{1+\delta} = F_s(s)\phi^{e\prime}(s) + (n-2)f_s(s)\phi^e(s).$$

The solution of the function is as follows:

$$\begin{aligned}
\phi^e(s) &= \frac{q}{\gamma}\frac{(n-2)\int_0^s y^{1+\delta}f_s(y)F_s(y)^{n-3}dy}{F_s(s)^{n-2}}\\
&= \frac{q}{\gamma}\frac{\int_0^s y^{1+\delta}d_{F_s(y)^{n-2}}}{F_s(s)^{n-2}}\\
&= \frac{q}{\gamma}\left(s^{1+\delta}-(1+\delta)\int_0^s y^{\delta}\left(\frac{F_s(y)}{F_s(s)}\right)^{n-2}dy\right).
\end{aligned}$$

Thus,

$$b^e(s) = \frac{\phi^e(s)}{s^\delta} = \frac{q}{\gamma}\left(s-(1+\delta)\int_0^s\left(\frac{y}{s}\right)^{\delta}\left(\frac{F_s(y)}{F_s(s)}\right)^{n-2}dy\right).$$

■

Proof of Proposition 10:

Proof. The expected winning bribe in system x $(x = m, e)$ is

$$E(\pi^x) = nE(b^{x,ex}).$$

Since $E(b^{x,ex}) = \int_0^1 E(b^x) f(a)\,da$, then we only need to compare $E(b^x)$.

$$E(b^e(s)) = \frac{\phi^e(s)}{s^\delta} = \frac{q}{\gamma}\int_0^s y\left(\frac{y}{s}\right)^{\delta} g_s(y)\,dy.$$

We substitute s by a since $s = \gamma a$,

$$E\left(b^e\left(a\right)\right) = q\int_0^a y\left(\frac{y}{a}\right)^\delta g\left(y\right)dy.$$

In the market system, the expected revenue is

$$E\left(b^m\left(a\right)\right) = q\int_0^a yg\left(y\right)dy.$$

Therefore,

$$E\left(b^m\left(a\right)\right) - E\left(b^e\left(a\right)\right) = q\int_0^a y\left[1-\left(\frac{y}{a}\right)^\delta\right]g\left(y\right)dy > 0,$$

as $1 > \left(\frac{y}{s}\right)^\delta$. Thus, $E\left(\pi^m\right) > E\left(\pi^e\right)$.

Therefore, the degree of corruption in the exam system is lower than the market system given two same quality colleges. ■

Proof of Claim 9:

Proof. We use the conclusion in the first section to obtain b_L. The symmetric equilibrium bribe in the second round is as follows.

$$\begin{aligned} b_L^m\left(a\right) &= \frac{q_L}{F_1^{(n-2)}\left(a\right)}\int_0^a yf_1^{(n-2)}\left(a\right)dy \\ &= q_L E\left[Y_1^{(n-2)}|Y_1^{(n-2)} < a\right] \\ &= q_L E\left[Y_2|Y_2 < a < Y_1\right], \end{aligned}$$

where $Y_1^{(n-2)}$ is the highest of remaining $n-2$ abilities and $F_1^{(n-2)}\left(a\right)$ denotes the distribution of $Y_1^{(n-2)}$.

Student has to take the expected result of the second round into account when he decides b_H^n, the optimal bribe in the first round. Suppose that all other students are following the same first round strategy b_H^m, and all students will follow b_L^m in the second round regardless of what happens in the first round. Assume student bribes $b_H^m\left(z\right)$, $z \neq a$, instead of $b_H^m\left(a\right)$, which is the equilibrium bribe. The total payoff is

$$\Pi\left(z,a\right) = F_1\left(z\right)\left[v_H - b_H^m\left(z\right)\right] + \left(n-1\right)\left(1-F\left(z\right)\right)F\left(a\right)^{n-2}\left[v_L - b_L^m\left(a\right)\right],$$

where the first term results from the event $Y_1 < z$, i.e., the student wins the place

in the first round with a bribe of $b_H^m(z)$. The second term results from the event $Y_2 < a \leq z \leq Y_1$, i.e., he fails in the first round, but wins in the second.

The first order condition is

$$f_1(z)[v_H - b_H^m(z)] - F_1(z) b_H^{m\prime}(z) - (n-1) f(z) F(a)^{n-2} [v_L - b_L^m(a)] = 0.$$

In equilibrium, $z = a$ and the first order condition can be rearranged as

$$\begin{aligned} b_H^{m\prime}(a) &= \frac{f_1(a)}{F_1(a)}[v_H - b_H^m(a)] - \frac{f_1(a)}{F_1(a)}[v_L - b_L^m(a)] \\ &= \frac{f_1(a)}{F_1(a)}[v_H - v_L] + \frac{f_1(a)}{F_1(a)}[b_L^m(a) - b_H^m(a)], \end{aligned}$$

as $f_1(a) = (n-1) f(a) F(a)^{n-2}$.

Then we have

$$\frac{d}{da}[F_1(a) b_H^m(a)] = f_1(a)[v_H - v_L] + f_1(a) b_L^m(a),$$

together with $b_H^m(0) = 0$, we have

$$\begin{aligned} F_1(a) b_H^m(a) &= \int_0^a f_1(y)[v_H - v_L]\, dy + \int_0^a f_1(y) b_L^m(y)\, dy \\ &= [q_H - q_L] \int_0^a f_1(y)\, y dy + \int_0^a f_1(y) b_L^m(y)\, dy. \end{aligned}$$

Therefore,

$$\begin{aligned} b_H^m(a) &= [q_H - q_L] \int_0^a \frac{f_1(y)\, y}{F_1(a)} dy + \frac{1}{F_1(a)} \int_0^a f_1(y) b_L^m(y)\, dy \\ &= [q_H - q_L] E[Y_1 | Y_1 < a] + E[b_L^m(Y_1) | Y_1 < a] \\ &= [q_H - q_L] E[Y_1 | Y_1 < a] + E[q_L E[Y_2 | Y_2 < Y_1] | Y_1 < a] \\ &= [q_H - q_L] E[Y_1 | Y_1 < a] + q_L E[Y_2 | Y_1 < a] \\ &= (q_H - q_L) E[Y_1 | Y_1 < a] + q_L E[Y_2 | Y_1 < a]. \end{aligned}$$

■

Proof of Claim 10:

Proof. The equilibrium bribe in the last round is regardless of the previous rounds, and hence it can be taken as a independent game. The result can be

found easily by using the previous result,

$$b_k^m(a) = q_k E\left[Y_k | Y_k < a < Y_{k-1}\right].$$

Now consider the equilibrium bribe for student i, $b_j^m(a)$, in the jth round, with $j = 1, ..., k-1$. Consider if he bribes slightly higher, say $b_j^m(a+\varepsilon)$. If $Y_j < a$, he would win but pay more than he would do if he bribe $b_j^m(a)$. His expected payoff increases by

$$F_j(a) \times \left[b_j^m(a+\varepsilon) - b_j^m(a)\right].$$

On the other hand, if $a < Y_j < a+\varepsilon$, he would have failed in the jth round with his equilibrium bribe, whereas bribing higher results his winning. Now there are two sub cases. In the event that $Y_{j+1} < a < Y_j < a+\varepsilon$, he would have failed in the jth round but won in the $j+1$st. In the event that $a < Y_{j+1} < Y_j < a+\varepsilon$, however, he would have failed in both the jth and the $j+1$st rounds, and possibly won in a later round, say the lth for some $l > j+1$. When ε is small, however, the probability that $a < Y_{j+1} < Y_j < a+\varepsilon$ is very small and it is of second order in magnitude. Thus, the contribution to the expected gain from all events in which the student fails in both the jth and the $j+1$st rounds can be safely neglected when ε is small. The overall expected gain from bribing $b_j^m(a+\varepsilon)$ is the probability that $a < Y_j < a+\varepsilon$ times the difference in the equilibrium bribe paid tomorrow and the bribe paid today, which is approximately

$$\left[F_j(a+\varepsilon) - F_j(a)\right] \times \left[\left(v_j - b_j^m(a+\varepsilon)\right) - \left(v_{j+1} - b_{j+1}^m(a)\right)\right].$$

Equating the two equations, dividing by ε, and taking the limit as $\varepsilon \to 0$, we obtain the differential equation

$$b_j^{m\prime}(a) = \frac{f_j(a)}{F_j(a)}\left[(v_j - v_{j+1}) - \left(b_j^m(a) - b_{j+1}^m(a)\right)\right],$$

together with the boundary condition $b_j^m(0) = 0$. Thus, the solution is

$$\begin{aligned}
b_j^m(a) &= \frac{1}{F_j(a)}\left[\int_0^a f_j(y)\left[v_j - v_{j+1}\right]dy + \int_0^a f_j(y)\, b_{j+1}^m(y)\, dy\right] \\
&= (q_j - q_{j+1})\frac{1}{F_j(a)}\int_0^a f_j(y)\, y dy + \frac{1}{F_j(a)}\int_0^a f_j(y)\, b_{j+1}^m(y)\, dy \\
&= (q_j - q_{j+1})\, E\left[Y_1^{(n-j)} | Y_1^{(n-j)} < a\right] + E\left[b_{j+1}^m\left(Y_1^{(n-j)}\right) | Y_1^{(n-j)} < a\right] \\
&= (q_j - q_{j+1})\, E\left[Y_j | Y_j < a\right] + E\left[b_{j+1}^m(Y_j) | Y_j < a < Y_{j-1}\right].
\end{aligned}$$

In order to solve the function, we work backward from the second last round.

$$\begin{aligned} b_{k-1}^{m}(a) &= (q_{k-1}-q_k)\,E\left[Y_{k-1}|Y_{k-1}<a\right]+E\left[b_k^m\left(Y_{k-1}\right)|Y_{k-1}<a<Y_{k-2}\right] \\ &= (q_{k-1}-q_k)\,E\left[Y_{k-1}|Y_{k-1}<a\right]+q_kE\left[E\left[Y_k|Y_k<Y_{k-1}\right]|Y_{k-1}<a<Y_{k-2}\right] \\ &= (q_{k-1}-q_k)\,E\left[Y_{k-1}|Y_{k-1}<a\right]+q_kE\left[Y_k|Y_{k-1}<a<Y_{k-2}\right], \end{aligned}$$

and proceeding inductively in this fashion results in the solution for all $j = 2, ..., k-1$,

$$b_j^m(a)=\sum_{L=j}^{k-1}(q_L-q_{L+1})\,E\left[Y_L|Y_j<a<Y_{j-1}\right]+q_kE\left[Y_k|Y_j<a<Y_{j-1}\right].$$

■

Proof of Claim 11:

Proof. We begin with the second round. In the second round, b_L^e is independent of the result of the first round. We can derive the equilibrium bribe in the second round by using the conclusion in the first section. The symmetric equilibrium bribe and equilibrium expected contribution to the official of c_L are as follows.

$$\begin{aligned} b_L^e(s) &= \frac{q_L}{\gamma}\frac{E\left[Y_{s,2}^{1+\delta}|Y_{s,2}<s<Y_{s,1}\right]}{s^\delta} \\ &= \frac{q_L}{\gamma}\left[s-(1+\delta)\int_0^s\frac{F_{s,2}(y)\,y^\delta}{F_{s,2}(s)\,s^\delta}dy\right], \end{aligned}$$

$$\phi_L^e(s)=\frac{q_L}{\gamma}E\left[Y_{s,2}^{1+\delta}|Y_{s,2}<s<Y_{s,1}\right].$$

Student has to take the expected result of the second round into account when he decides how much to bribe in the first round. Suppose that all other students are following the first round strategy b_H^e, and all students will follow b_L^e in the second round, regardless of what happens in the first round. Assume student bribes $b_H^e(z)$, $z \neq a$, instead of $b_H^e(a)$, which is the equilibrium bribe. The total payoff is

$$\begin{aligned} \Pi(z,s) &= F_{s,1}\left(S\left(b_H^e(z)\,s^\delta\right)\right)\left[v_H-b_H^e(z)\right] \\ &\quad+(n-1)\left(1-F_s\left(S\left(b_H^e(z)\,s^\delta\right)\right)\right)F_s(s)^{n-2}\left[v_L-b_L^e(s)\right], \end{aligned}$$

where the first term results from the event $Y_{s,1} < S\left(b_H^e(z)\,s^\delta\right)$, i.e., the student wins the place in the first round with a bribe of $b_H^e(z)$. The second term results

from the event $S\left(b_H^e(z)\, s^\delta\right) < Y_{s,1}$, but $Y_{s,2} < s$ i.e., he fails in the first round, but wins in the second.

The first order condition is

$$\begin{aligned} 0 &= f_{s,1}\left(S\left(b_H^e(z)\, s^\delta\right)\right) S'\left(b_H^e(z)\, s^\delta\right) b_H^{e\prime}(z)\, s^\delta \left[v_H - b_H^e(z)\right] \\ &\quad -F_{s,1}\left(S\left(b_H^e(z)\, s^\delta\right)\right) b_H^{e\prime}(z) \\ &\quad -(n-1)\, f_s\left(S\left(b_H^e(z)\, s^\delta\right)\right) S'\left(b_H^e(z)\, s^\delta\right) b_H^{e\prime}(z)\, s^\delta F_s(s)^{n-2}\left[v_L - b_L^e(s)\right]. \end{aligned}$$

In equilibrium, $z = s$ and the first order condition can be rearranged as

$$\begin{aligned} f_{s,1}(s)\,\phi_H^e(s) + F_{s,1}(s)\,\phi_H^{e\prime}(s) &= s^\delta f_{s,1}(s)\, v_H - f_{s,1}(s)\left[s^\delta v_L - \phi_L^e(s)\right], \\ \left[F_{s,1}(s)\,\phi_H^e(s)\right]' &= s^\delta f_{s,1}(s)\left(v_H - v_L\right) + f_{s,1}(s)\,\phi_L^e(s). \end{aligned}$$

Thus, the equilibrium ϕ_H is

$$\begin{aligned} \phi_H^e(s) &= \frac{1}{F_{s,1}(s)}\int_0^s y^\delta f_{s,1}(y)\left(v_H - v_L\right) dy + \frac{1}{F_{s,1}(s)}\int_0^s f_{s,1}(y)\,\phi_L^e(y)\, dy \\ &= \frac{q_H - q_L}{\gamma} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right] + E\left[\phi_L^e\left(Y_{s,1}^{1+\delta}\right) | Y_{s,1} < s\right]. \end{aligned}$$

Substituting ϕ_L into the above equation gives

$$\begin{aligned} \phi_H^e(s) &= \frac{q_H - q_L}{\gamma} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right] + \frac{q_L}{\gamma} E\left[E\left[Y_{s,2}^{1+\delta} | Y_{s,2} < Y_{s,1}\right] | Y_{s,1} < s\right] \\ &= \frac{q_H - q_L}{\gamma} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right] + \frac{q_L}{\gamma} E\left[Y_{s,2}^{1+\delta} | Y_{s,1} < s\right]. \end{aligned}$$

Thus, the equilibrium bribe in the first round is

$$\begin{aligned} b_H^e(s) &= \frac{\phi_H^e(s)}{s^\delta} \\ &= \frac{q_H - q_L}{\gamma s^\delta} E\left[Y_{s,1}^{1+\delta} | Y_{s,1} < s\right] + \frac{q_L}{\gamma s^\delta} E\left[Y_{s,2}^{1+\delta} | Y_{s,1} < s\right]. \end{aligned}$$

■

Proof of Proposition 11:

Proof. As we have shown, we only need to compare the expected winning bribes in the two systems to tell which system is better. The total expected payment to

two officials in the market system is

$$\begin{aligned} E(\pi^m) &= \frac{n-1}{n}E(a_1)\,q_H - \frac{1}{n}E(a_1)\,q_L + \frac{n-2}{n-1}E(a_2)\,q_L \\ &= \frac{n-1}{n+1}q_H + \frac{n-3}{n+1}q_L. \end{aligned}$$

The total expected payment to two officials in the exam system is

$$\begin{aligned} E(\pi^e) &= \frac{n-1}{n+\delta}\frac{E(s_1)}{\gamma}q_H - \frac{(n-1)(1+\delta)}{(n+\delta)(n+\delta-1)}\frac{E(s_1)}{\gamma}q_L + \frac{n-2}{n+\delta-1}\frac{E(s_s)}{\gamma}q_L \\ &= \frac{n(n-1)}{(n+\delta)(n+1)}q_H + \frac{(n-1)(n^2-3n-2\delta)}{(n+1)(n+\delta)(n+\delta-1)}q_L. \end{aligned}$$

Since

$$\begin{aligned} \frac{n(n-1)}{(n+\delta)(n+1)} &< \frac{n-1}{n+1}, \\ \frac{(n-1)(n^2-3n-2\delta)}{(n+1)(n+\delta)(n+\delta-1)} &< \frac{n-3}{n+1}, \end{aligned}$$

the degree of corruption in the exam system is lower than the market system in this model with different-quality colleges. ■

Proof of Claim 12:

Proof. The equilibrium bribe in the last round is regardless of the previous rounds, and hence it can be taken as an independent game. The result can be found easily by using the previous result,

$$b_k^e(s) = \frac{q_k}{\gamma}\frac{E\left[Y_{s,k}^{1+\delta} | Y_{s,k} < s < Y_{s,k-1}\right]}{s^\delta}.$$

Suppose the equilibrium contribution to the official of c_j from student i is $\phi_j^e(s)$ in the jth round but consider if he bribes slightly higher, say $b_j^e(s+\varepsilon)$, and contribute $\phi_j^e(s+\varepsilon)$. If $Y_{s,j} < s$, he would win but pay more than he would do if he contributes $\phi_j^e(s)$. His expected payment increases by

$$F_{s,j}(a) \times \frac{\left[\phi_j(s+\varepsilon) - \phi_j(s)\right]}{s^\delta}.$$

On the other hand, if $s < Y_{s,j} < s+\varepsilon$, he would have failed in the jth round with his equilibrium bribe, whereas bribing higher results his winning. Now there are two sub cases. In the event that $Y_{s,j+1} < s < Y_{s,j} < s+\varepsilon$, he would have failed

in the jth round but won in the $j+1$st. In the event that $s < Y_{s,j+1} < Y_{s,j} < s+\varepsilon$, however, he would have failed in both the jth and the $j+1$st rounds, and possibly won in a later round, say the lth for some $l > j+1$. When ε is small, however, the probability that $s < Y_{s,j+1} < Y_{s,j} < s+\varepsilon$ is very small, it is of second order in magnitude. Thus, the contribution to the expected gain from all events in which the student fails in both the jth and the $j+1$st rounds can be safely neglected when ε is small. The overall expected gain from contributing $\phi_j^e(s+\varepsilon)$ is the probability that $s < Y_{s,j} < s+\varepsilon$ times the difference in the equilibrium bribe paid tomorrow and the bribe paid today, which is approximately

$$\left[F_{s,j}(s+\varepsilon) - F_{s,j}(s)\right] \times \left[\left(v_j - \frac{\phi_j^e(s+\varepsilon)}{s^\delta}\right) - \left(v_{j+1} - \frac{\phi_{j+1}^e(s)}{s^\delta}\right)\right].$$

Equating the two equations, dividing by ε, and taking the limit as $\varepsilon \to 0$, we obtain the differential equation

$$\begin{aligned}
\frac{\phi_j^{e\prime}(s)}{s^\delta} &= \frac{f_{s,j}(s)}{F_{s,j}(s)}\left[(v_j - v_{j+1}) - \frac{\left(\phi_j^e(s) - \phi_{j+1}^e(s)\right)}{s^\delta}\right], \\
\phi_j^{e\prime}(s) &= \frac{f_{s,j}(s)}{F_{s,j}(s)}\left[s^\delta(v_j - v_{j+1}) - \left(\phi_j^e(s) - \phi_{j+1}^e(s)\right)\right], \\
\left[\phi_j^e(s) F_{s,j}(s)\right]' &= f_{s,j}(s)\, s^\delta (v_j - v_{j+1}) + f_{s,j}(s)\, \phi_{j+1}^e(s).
\end{aligned}$$

Together with the boundary condition $\phi_j^e(0) = 0$, we have

$$\begin{aligned}
\phi_j^e(s) &= \frac{1}{F_{s,j}(s)}\left[\int_0^s f_{s,j}(y)\, y^\delta [v_j - v_{j+1}]\, dy + \int_0^s f_{s,j}(y)\, \phi_{j+1}^e(y)\, dy\right] \\
&= \frac{(q_j - q_{j+1})}{\gamma}\frac{1}{F_{s,j}(s)}\int_0^s f_{s,j}(y)\, y^{1+\delta} dy + \frac{1}{F_{s,j}(s)}\int_0^s f_{s,j}(y)\, \phi_{j+1}^e(y)\, dy \\
&= \frac{(q_j - q_{j+1})}{\gamma} E\left[\left[Y_{s,1}^{(n-j)}\right]^{1+\delta} | Y_{s,1}^{(n-j)} < s\right] + E\left[\phi_{j+1}^e\left(Y_{s,1}^{(n-j)}\right) | Y_{s,1}^{(n-j)} < a\right] \\
&= \frac{(q_j - q_{j+1})}{\gamma} E\left[Y_{s,j}^{1+\delta} | Y_{s,j} < s\right] + E\left[\phi_{j+1}^e(Y_{s,j}) | Y_{s,j} < a < Y_{s,j-1}\right].
\end{aligned}$$

In order to solve the function, we work backward from the last round.

$$\begin{aligned}
\phi_{k-1}^{e}(s) &= \frac{(q_{k-1}-q_k)}{\gamma}E\left[Y_{s,k-1}^{1+\delta}|Y_{s,k-1}<s\right] \\
&\quad +E\left[\phi_k^e(Y_{s,k-1})\,|Y_{s,k-1}<s<Y_{s,k-2}\right] \\
&= \frac{(q_{k-1}-q_k)}{\gamma}E\left[Y_{s,k-1}^{1+\delta}|Y_{s,k-1}<s\right] \\
&\quad +\frac{q_k}{\gamma}E\left[E\left[Y_{s,k}^{1+\delta}|Y_{s,k}<Y_{s,k-1}\right]|Y_{s,k-1}<s<Y_{s,k-2}\right] \\
&= \frac{(q_{k-1}-q_k)}{\gamma}E\left[Y_{s,k-1}^{1+\delta}|Y_{s,k-1}<s\right] \\
&\quad +\frac{q_k}{\gamma}E\left[Y_{s,k}^{1+\delta}|Y_{s,k-1}<s<Y_{s,k-2}\right].
\end{aligned}$$

Proceeding inductively in this fashion results in the solution for all j,

$$\begin{aligned}
\phi_{j}^{e}(s) &= \frac{(q_{j}-q_{j+1})}{\gamma}E\left[Y_{s,j}^{1+\delta}|Y_{s,j}<s\right] \\
&\quad +E\left[\phi_{j+1}^e(Y_{s,j})\,|Y_{s,j}<s<Y_{s,j-1}\right] \\
&= \frac{(q_{j}-q_{j+1})}{\gamma}E\left[Y_{s,j}^{1+\delta}|Y_{s,j}<s\right] \\
&\quad +\frac{q_k}{\gamma}E\left[E\left[Y_{s,k}^{1+\delta}|Y_{s,j+1}<Y_{s,j}\right]|Y_{s,j}<s<Y_{s,j-1}\right] \\
&= \sum_{L=j}^{k-1}\frac{(q_L-q_{L+1})}{\gamma}E\left[Y_{s,L}^{1+\delta}|Y_{s,j}<s<Y_{s,j-1}\right] \\
&\quad +\frac{q_k}{\gamma}E\left[Y_{s,k}^{1+\delta}|Y_{s,j}<s<Y_{s,j-1}\right].
\end{aligned}$$

Thus, the equilibrium bribe is

$$\begin{aligned}
b_j^e(s) &= \frac{\phi_j^e(s)}{s^\delta} \\
&= \sum_{L=j}^{k-1}\frac{(q_L-q_{L+1})}{\gamma}\frac{E\left[Y_{s,L}^{1+\delta}|Y_{s,j}<s<Y_{s,j-1}\right]}{s^\delta} \\
&\quad +\frac{q_k}{\gamma}\frac{E\left[Y_{s,k}^{1+\delta}|Y_{s,j}<s<Y_{s,j-1}\right]}{s^\delta}.
\end{aligned}$$

■

Proof of Claim 13:

Proof. The expected bribe by a randomly selected student (a_0, w_0) from the group of students whose types are graphed by the Leonitief isobid curve in Figure

7 is the same as the expected bribe by the type $(a', 1)$. It satisfies

$$\begin{aligned} E\left[B^m\left(a', 1\right)\right] &= \Pr\left[\text{Win}\right] \times B^m\left(a', 1\right) \\ &= F^c\left(a'\right)^{n-1} \times \beta^m\left(a'\right) \\ &= q\int_0^{a'} yf^c\left(y\right)^{n-1} dy. \end{aligned}$$

The expected revenue of the official or the expected winning bribe is just the sum of the ex ante (prior to knowing their abilities) expected bribes of the students. The ex ante expected bribe of $(a', 1)$ satisfies

$$\begin{aligned} E\left[B^{m,ex}\left(a', 1\right)\right] &= \int_0^1 E\left[B^m\left(a', 1\right)\right] f^c\left(a'\right) da' \\ &= q\int_0^1 \left(\int_0^{a'} yf^c\left(y\right)^{n-1} dy\right) f^m\left(a'\right) da' \\ &= q\int_0^1 \left(\int_y^1 f^c\left(a'\right) da'\right) yf^c\left(y\right)^{n-1} dy \\ &= q\int_0^1 y\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1} dy. \end{aligned}$$

The expected payment to the official is just n multiplied by $E\left[B^{m,ex}\left(a', 1\right)\right]$, and hence

$$\begin{aligned} E\left[\pi^{c,m}\right] &\equiv n \times E\left[B^{m,ex}\left(a', 1\right)\right] \\ &= nq\int_0^1 y\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1} dy. \end{aligned}$$

Note that the distribution of Y_2^c is

$$F_2^c\left(y\right) \equiv nF^c\left(y\right)^{n-1} - \left(n-1\right) F^c\left(y\right)^n.$$

The associated density function is

$$\begin{aligned} f_2^c\left(y\right) &= n\left(n-1\right)\left(1 - F^c\left(y\right)\right) F^c\left(y\right)^{n-2} f^c\left(y\right) \\ &= n\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1} \\ &= n\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1}. \end{aligned}$$

Thus, the expected revenue can be written as

$$\begin{aligned} E\left[\pi^{c,m}\right] &= q\int_0^1 y f_2^c\left(y\right)dy \\ &= qE\left[Y_2^c\right], \end{aligned}$$

where Y_2^c is the second-highest of n draws from the distribution $F^c\left(\cdot\right)$. ■

Proof of Claim 14:

Proof. For a randomly selected student (a_0, w_0), there must be a type with $w = 1$, say $(d, 1)$, whose equilibrium bribe satisfies

$$B^e\left(a_0, w_0\right) = B^e\left(d, 1\right).$$

The expected bribes by the type $(d, 1)$ is as follows:

$$\begin{aligned} E\left[B^e\left(d,1\right)\right] &= \Pr\left[\text{Win}\right] \times B^e\left(d,1\right) \\ &= F^c\left(d\right)^{n-1} \times \beta^e\left(d\right) \\ &= \frac{q}{a^\delta}\int_0^d y^{1+\delta} f^c\left(y\right)^{n-1} dy. \end{aligned}$$

The ex ante expected bribe of $(d, 1)$ satisfies

$$\begin{aligned} E\left[B^{e,ex}\left(d,1\right)\right] &= \int_0^1 E\left[B^e\left(d,1\right)\right] f^c\left(d\right) dd \\ &= \frac{q}{a^\delta}\int_0^1 \left(\int_0^d y^{1+\delta} f^c\left(y\right)^{n-1} dy\right) f^c\left(d\right) da' \\ &= \frac{q}{a^\delta}\int_0^1 \left(\int_y^1 f^c\left(d\right) dd\right) y^{1+\delta} f^c\left(y\right)^{n-1} dy \\ &= \frac{q}{a^\delta}\int_0^1 y^{1+\delta}\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1} dy. \end{aligned}$$

The expected revenue of the official is just n multiplied by $E\left[B^{e,ex}\left(d,1\right)\right]$, and hence

$$\begin{aligned} E\left[\pi^{c,e}\right] &\equiv n \times E\left[B^{m,ex}\left(d,1\right)\right] \\ &= \frac{nq}{a^\delta}\int_0^1 y^{1+\delta}\left(1 - F^c\left(y\right)\right) f^c\left(y\right)^{n-1} dy. \end{aligned}$$

Since

$$f_2^c(y) = n(1 - F^c(y)) f^c(y)^{n-1}.$$

Thus, the expected revenue can be written as

$$\begin{aligned} E[\pi^{c,e}] &= \frac{q}{a^\delta} \int_0^1 y^{1+\delta} f_2^c(y)\, dy \\ &= \frac{q}{a^\delta} E\left[(Y_2^c)^{1+\delta}\right]. \end{aligned}$$

■

Proof of Proposition 12:

Proof. The first part of this claim is clearly established in Figure 10, 11, 12. They show three possible outcomes: both systems lead to inefficient results; both of them bring efficient allocations; only the exam system has an efficient result. Therefore, although both systems may produce inefficient results, the exam system is likely to be better than the market system while here is no possibility that the market system performs better than the exam system in terms of efficiency.

For degree of corruption, we need to look at the expected revenues for the official. In the x $(x = m, e)$ system, $E[\pi^x] = nE[B^{x,ex}]$. The ex ante expected bribe of a particular student, say (a, w) in the market system under borrowing constraints is

$$\begin{aligned} E[B^{m,ex}] &= \int_0^1 E[B^m(a', 1)] f^m(a')\, da' \\ &= q \int_0^1 \left(\int_0^{a'} y f^{m,1}(y)\, dy \right) f^m(a')\, da', \end{aligned}$$

where a' is a ability such that $B^m(a', 1) = B^m(a, w)$. The ex ante expected bribe of a particular student with type (a, w) in the exam system under borrowing constraints is

$$\begin{aligned} E[B^{e,ex}] &= \int_0^1 E[B^e(d, 1)] f^e(d)\, dd \\ &= q \int_0^1 \left(\frac{\int_0^d y^{1+\delta} f^{e,1}(y)\, dy}{d^\delta} \right) f^e(d)\, dd, \end{aligned}$$

where d is a value such that $B^e(d, 1) = B^e(a, w)$.

Given a same particular student with type (a, w), $d \leq a'$. We can have this

result from different cases in Graph 9, 10 and 11. In the cases of Graph 9 and 10, $d < a'$, while in the case of Graph 11, $d = a'$. Therefore, we have

$$\frac{\int_0^d y^{1+\delta} f^{e,1}(y)\, dy}{d^\delta} < \int_0^d y f^{e,1}(y)\, dy \leq \int_0^{a'} y f^{e,1}(y)\, dy.$$

In addition, $f^e(d) \leq f^e(a')$ if $d \leq a'$, hence

$$E[B^{e,ex}] < q \int_0^1 \left(\int_0^a y f^{e,1}(y)\, dy \right) f^e(a')\, da'.$$

Combing with the inequality $f^e(a') < f^m(a')$ and hence $f^{e,1}(y) < f^{m,1}(y)$, we have $E[B^{e,ex}] < E[B^{m,ex}]$. So, $nE[B^{e,ex}] < nE[B^{m,ex}]$, and hence $E[\pi^e] < E[\pi^m]$. The expected winning bribe is greater in the market system than in the exam system, therefore the degree of corruption in the exam system is lower than in the market system. ■

3 Chapter 3: The Cost of Attending College and Positive Self-Sorting

3.1 Introduction and Literature Review

The principle of "easy admission but strict graduation" was proposed by some scholars in the last decade to replace the current Chinese college admission system. In fact, one of two college admission systems in France follows that proposed principle. This chapter explores the properties of this type admission mechanism in France from the economic perspective.

At the start of the 2006-2007 higher education academic year, there were 2.287 million students in France enrolled into the higher education admission system. Among these students, 1.357 million of them applied to the universities, 113,500 of them applied to the university institutes of technology and 76,000 of them applied to preparatory courses for the top graduate schools.[37] Higher education in France covers all studies after the baccalaureat ('A' level equivalent). There are two systems that coexist side by side:

(1) An open system in the universities: Most students study under this system (1.357 million out 2.287 million in 2006-2007). All baccalaureat holders have the right to enter this system without any prior selection procedure. Universities offer an extremely wide range of studies;

(2) A selective system with a limited number of places: Admission is by competitive examination, entrance examination or application form, with an interview where appropriate. This is the system in use in the grandes ecoles (top graduate schools such as the Ecole Nationale d'Administration – French Senior Civil Service School – Ecole Nationale Superieure – national post-graduate school – and the top engineering and business schools), the institutes universitaires de technologie (IUTs – university institutes of technology) and the institutes universitaires professionnalises (IUPs – university institutes of professional education). These establishments train mainly public-sector and private-sector senior and middle managers.

The second system is more similar to the prevalent college admission mechanism in other countries, e.g., USA, Britain, etc. In the prevalent system, in which admission is operated through competitive procedures, a positive assortative matching may be produced, where the higher ability students are accepted

[37] Source: Ministry of National Education, Advanced Instruction, and Research, France.

by the higher quality institutions and the lower ability students are allocated to the lower quality institutions. The mechanism with competitive procedures naturally produces the positive sorting if a student's performance is positively correlated to his ability, and the mechanism is well designed. However, in the first admission system of the French higher education sector, students have the right to enter universities without any prior selection procedure. Is the sorting still positive assortative? If the answer is yes, what is the driving force?

The purpose of this research is to discover the driving force that sorts students into different quality colleges in a free choice system such as the open system at French universities. It has important implications at both a practical and theoretical level. From the practical perspective, it provides an analysis of an alternative mechanism for the current CCA mechanism. In theory, it looks through the allocation of students and resources from a new angle. At the end of this introduction, we will explain why our approach is different from the literature.

Given a distribution of student ability and a limited pool of resources, we model the planner's decision to establish colleges and set a "task level" for each college, and also the allocation of resources to colleges. Ignoring entry frictions, the main cause that drives a student to select a college may be the cost of accomplishing the task. At a particular college, the task level and students' abilities are the determinants of cost for students to complete the educational qualification. For example, students have two options: One is to choose a good college but with high requirement, and the other is to choose an ordinary college but with low requirement. The cost that a higher ability student completes the requirement of the better college is lower than a lower ability student. If all students obtain the same qualification from the same college, then the higher ability student may select the first choice, while the lower ability student may choose the second college.

Certainly, there are other factors that could influence student's choice, such as tuition, budget constraint, geography consideration and therefore transportation costs, spatial considerations, and preference of subjects, etc. If these factors are assumed to be symmetric, then different ability students will have different decisions only due to variant costs. The cost depends on abilities and task levels. Thus, in our work, abilities are still the key of sorting them to institutions as the central feature of other related works. In Epple, Romano and Sieg (2006), colleges attempt to attract the higher ability students by designing appropriate tuition and admission policies. In Fernandez and Gali (1997), Fernandez (1998), prices and borrowing constraints play the role of sorting students in a market

system, and prices and exam results decide the matching of students to colleges in an exam system. In our model, we include the cost into student's utility function, and drop tuition and income effects on student's utility. We believe this approach is reasonable because the difference of tuition for undergraduate study across colleges in some countries is not large enough to influence student's decision.

Our model may apply to a more general case other than to the higher education sector. In any circumstances where a central planner is maximizing output by dividing a group of people into different task levels and allocating them with limited resources, our model is able to solve for an optimal solution if the planner does not set any entry requirements, this group of people's abilities follow some identical distribution, and the cost of completing a task level differs among people.

For example, a firm is designing a mechanism to stimulate workers' output. The manager is not able to observe any individual worker's ability, but the distribution of workers' abilities. The manager sets up some groups with different levels of rewards or qualifications and allocates allocable resources to each group. These rewards or qualifications require workers to accomplish some specific tasks. Since the manager does not have knowledge of a particular worker's ability, he would have to let workers choose appropriate levels of tasks by themselves. A worker's concern is the qualification and the cost of accomplishing the required task of that qualification. All workers in a task will obtain the same qualification; thus their gains would be the same. The costs, however, will differ as workers vary in their abilities. The manager's purpose is to maximise the total output, which is the aggregate of all individual outputs. Individual output depends on the worker's ability, resources per worker, and the motivation (qualification) obtained upon completing the task. In such a game, an equilibrium can be reached by working backword from the end of the game. At the end of the game, workers select the task that maximises their utilities and produce outputs. At the beginning, the manager sets up the optimal task levels and allocation of resources to maximise the expected total output by taking workers' decisions into account.

Although this model can apply to a wide range of scenarios, we concentrate our focus on the higher education sector. Therefore in our model, the market is composed of the higher education planner, colleges (for simplicity, we use colleges to represent higher education institutions in the rest context), and students. In such an economy, a central planner has limited amount of resources and he needs to decide the number of colleges, a task level for each college and the allocation of

resources; a group of continuum students whose abilities follow some distribution can either choose an outside option, or go to college. The planner does not have the information about the ability of any particular student, but he knows the distribution of students' abilities. The planner's objective is to maximise the total education output which is the aggregate output of individual education. The individual output of a particular student depends on the student's ability, resources per capita, and the task level of that college. In order to maximise total output, the planner determines the number of colleges, designs an identical task level for all students at each individual college, and allocates the allocable resources to all colleges.

Our model is driven by a few simple assumptions: (1) there is complementarity between resources and ability, task and ability, task and resources in the education output function. In other words, the production function satisfies supermodularity; (2) students have a cost function which has converse property to the production function, we call it submodularity: given any level of task, the student with higher ability faces a lower rise in cost when given a marginal increase in the task level; (3) a student's utility depends on the value gained from the completion of the given task and the cost incurred, the value is concave while the cost is strictly converse in task level. These assumptions yield that the planner would set up a tiered structure with sorted task levels, and students select the most suitable college. Therefore the system produces a tiered structure that sorts students by their abilities and results in an optimal output overall.

This work is closely related to the literatures on the assortative sorting and allocation of resources in higher education. Using the ability of individuals to sort or segregate themselves across various dimensions and in different spheres is a topic of concern in many countries, e.g., different races, incomes, and abilities are sorted into different residences, workplaces, schools, and households (Fernandez (2002)). Kremer and Maskin (1996) present some evidence that sorting by skill level at the workplace has increased. They find that the efficient matching in their model depends on the distribution of skill in the matching market, because the trade-off between the asymmetry and the complementarity in the match output function depends on the relative scarcity of highly-skilled workers. As we mentioned, even without an explicit plan, students will sort themselves into different quality colleges. Students are distributed among these colleges largely according to their academic abilities. The requirement of completing the study and obtaining the qualification in each tier, and the public spending per student at each level are strongly and positively associated with students' average ability.

The driving force of the positive sorting was firstly discussed by Becker (1973). In his theory of marriage, Becker (1973) defined positive assortative mating as a positive correlation between the values of the traits of husbands and wives. He argued that the positive assortative mating is generally optimal in most circumstances.

Theorem 1 *Positive assortative mating-mating of likes-is optimal when*

$$\frac{\partial^2 Z\left(A_m, A_f\right)}{\partial A_m \partial A_f} > 0,$$

where Z denotes aggregate output, A_m denotes male's ability and A_f denotes female's ability, because aggregate output is then maximised. Negative assortative mating-mating of unlikes-is optimal when the inequality is reversed. (Becker (1973))

This theorem indicates that, in the marriage market, higher-quality men and women marry each other rather than selecting lower-quality mates when these qualities are complements: a superior woman raises the productivity of a superior man and vice versa. Beck's theorem can be applied to other matching or sorting problems when the concern is the aggregate output produced by two sides of the market, e.g. colleges and students, firms and employees, etc. The agent of likes or unlikes is optimal as traits are complements or substitutes, because superior types reinforce each other when traits are complements and offset each other when traits are substitutes. The condition in the theorem is commonly referred to as the (strict) supermodularity condition of the match output function Z. See Topkis (1998) for a comprehensive mathematical treatment of supermodularity, and Milgrom and Roberts (1990) and Vives (1990) for applications in game theory and economics.

Arnott and Rowse (1987) find that any type partition in a case of allocating students to various classes within an elementary or secondary school is possible. The partition depends on the strength of peer effects, which was initially defined in Coleman (1966). Peer effects have been employed in a large amount of literatures to explain the positive sorting.[38] These literatures find that having better peers tends to improve a student's own academic performance, and many find the effects to be larger for students with low abilities than for those with

[38]See Coleman (1966), Henderson, Mieszkowski, and Sauvageau (1976), Hoxby and Terry (2000), Zimmer and Toma (2000), Checchi and Zollino (2001), Robertson and Symons (2003), McEwan (2003), Epple and Romano (1998), Epple, Romano and Sieg (2003, 2006).

high abilities. Epple and Romano (1998) construct a model of private and public secondary schools in order to analyse the effects of voucher reforms. Epple, Romano and Sieg (2003) and Epple, Romano and Sieg (2006) present a general equilibrium model of the market for higher education. In their model, colleges seek to maximise the quality of the educational experience provided to their students. The quality of colleges is assumed to depend on peer ability and income of the student body, and also the instructional expenditures per student. Since peer effect is an important component of college quality, students will seek out colleges with high ability peers. Meanwhile, colleges will attempt to attract high ability students in order to improve quality. Their model yields the hierarchy of colleges in equilibrium about the distribution of students according to income and ability, and the price policy adopted by colleges. In most cases that consider peer effects, outcomes are influenced by the distribution of income (exogenous endowment). We do not include peer effects into our model as our assumption of supermodularity is enough to obtain an assortative matching outcome.

Kremer (1993) highlights the role of positive assortative matching in economic development. In his model of a one-sided, many-to-many matching market, each firm consists of a fixed number of workers each employed for a production task. Workers have different skills, with a higher-skilled worker less likely to make mistakes in his task performance. Self-matching would be obtained in equilibrium where each firm employs workers of identical skills. Kremer uses this form of positive assortative matching to explain the large wage and productivity differences between developing and developed countries that cannot be accounted for by their differences in the levels of physical or human capital.

Fernandez and Gali (1997) and Fernandez (1998) compare the market system to the exam (tournament) system under the assumption of complementarity between student ability and college quality. They assume a continuum of exogenous quality for schools, and the education output depends on student ability and college quality. Students sort themselves into different tier schools by maximizing education output. They also discuss the effects of borrowing constraints on the efficiency. Sallee, Resch and Courant (2008) have a perspective of central planner similar to ours. They assume complementarity and fixed costs of building up colleges are sufficient to construct an optimal tiered system that sorts students by ability and results in discontinuous spending and educational output per student for essentially identical students at the margin between schools. Although the assumptions from existing literatures are different, the peer effects and complementarity are looking at the same thing. In the end, these different approaches

would have given the same result. Our model only assumes the supermodularity and ignores the peer effects, but we will reach the same positive assortative sorting.

The main contribution of this work, in contrast with existing literature, is that we pay more attention to students' considerations of costs when they decide whether or not and where to be educated. Based on the assumption of supermodularity for utility function and production function, we present a general equilibrium from a planning perspective and explain how students sort themselves into colleges with different quality optimally. This work could apply to both the admission sorting of education market, or any other scenarios concerning sorting and resource allocation. To our knowledge, this piece of work raises some issues that do not appear in the previous literature. Although we reach the same positive assortative matching result as of in most of the literature, all roads lead to Rome, and our approach is different from others. The presence of a submodular cost function for students and other assumptions allows us to construct the model with central planning but free choice for students. We are standing for a central education planner whose output will be maximised, but students are not allocated by the planner, which is in most literature with social planners, e.g., Sallee, Resch and Courant (2008). In our model, the central planner can not allocate students to colleges and students are free to make decisions. Our model assumes that the planner is concerned about the total outputs and a student cares only his utility. The output of a student in a college depends on his ability, resources per student, and the task level of that college. A student's utility is the difference between gains from the college and costs of accomplishing the task, so it depends on the student's ability and the task level. The assumption of supermodularity of the production and utility functions provides the planner with the motivation to set up different task levels. Our model yields a result that the hierarchy of colleges emerges in equilibrium, the optimal design of task levels and resource allocation, and students' choice making such as to have a positive sorting. In the discussion on the optimal number of colleges, we have inconsistent conclusion with the work of Sallee, Resch and Courant (2008). The planner will only set up a finite number of colleges even if the fixed cost is zero. This different result is firstly because we let students select colleges freely by considering students' maximised utilities. We think this makes sense in such systems as "easy admission but strict graduation", where students do not have any requirements to enter higher education institutions. Secondly, we include task level into education production model, which combining with free choices of students determine

a small number of colleges even though the fixed cost is low. By including task level into our model, we think requirements of colleges do have positive effects on higher education outputs.

The work is organised as follows. Section 2 describes the economy and lays out the key elements of the general model. Section 3 defines the equilibria in a one-college case and a two-college case; compares the two systems in several aspects, and considers the model by taking fixed cost into account. Section 4 presents the conclusion.

3.2 The Model

3.2.1 Description of the Economy

The economy in our model consists of two sides:

1. We think of the ministry of education as a social planner, who has a total resource Ω which is free to allocate. The planner decides the number of colleges, N. The fixed cost of setting up a college is denoted by θ. The planner also sets a vector of task levels for each college $\{T_1, ..., T_j..., T_N\}$. After subtracting the fixed cost from total resources, the planner allocates the remainder to colleges, which is denoted by R, $R = \Omega - N\theta$. The proportions of total resources allocated to each college are denoted by vector $\{p_1, ..., p_j..., p_N\}$. The planner has an objective function W, which denotes the total education output.

2. There is a continuum of potential students who differ with respect to their ability levels. Let x denote a student's ability, which follows a continuous distribution. A particular student's educational output is determined by the production function $q(x, r, T)$ (where $r = \frac{pR}{w}$, pR measures the amount of resource available to the college, and w is the measure of students in the particular college). It is thus assumed that a student's output depends on resources per capita in his college. The student's gain from his college is determined by a function $V(T)$, and his cost function is $C(x, T)$. Let $u = U(x, T) = V(T) - C(x, T)$ be the utility function. There is an outside option with constant utility u_0 available to all students.

Note that students have a different output function from the utility function. The difference is that resources per student have been included in the output function. Excluding resources from the utility function is a controversial assumption. Our opinion is that the qualification achieved from colleges is the only

thing matters to students, while the planner is concerned about what students would have actually contributed to colleges. Task level in our model is taken as the requirements to achieve qualifications; thus it is in the student's utility function as a higher task level means a better qualification. The same qualification gives the same gain to different students. Employers in a job market are only concerned about students' qualifications, not how much resources students occupy in colleges. Therefore, resources per student are not in the gain function. In order to benefit from a better qualification, students have to make more effort which incurs higher cost. Given the same task level, costs are different for different types of students; thus we have $C(x, T)$. Another reason of excluding resources from the utility function is that it is not easy for students to realise the resources allocation. Student's ability and resources per student are employed as inputs in the production function can be found in many literatures (Sallee, Resch and Courant (2008); Epple (2003); Arnott and Rowse (1987)). Task level in our model, however, is the third determinant of output. q is capturing not only the output of students but also a short-hand for college output, i.e., it adds the two together. If a college has good students (higher a) and a higher T, then it is likely to be a better college and attracts better academics, etc. So it is plausible that extra resources to that college have a higher marginal product. For example, the college can allow these good academics to have more time for research since the resources could pay for teaching assistants etc.; or if the students are research students, then the resources allow for better equipment and facilities. The research output will be improved. Therefore, higher r, x and T increase the output of the college, and hence the planner's welfare.

This is a two sided market between the planner and students. From the planner's point of view, total outputs are to be maximised; on the other side, students maximise their own utilities. The moves of the planner and students obey the following order:

1. The planner is endowed with total resources Ω and he observes student's ability distribution F as well as the fixed cost of setting up a college θ.

2. The planner decides the number of colleges N, designs task levels $\{T_1, ..., T_N\}$ for each college, and allocates allocable resources R to each college $\{r_1, ..., r_N\}$.

3. A set comprising a continuum of students receives abilities, which are drawn from distribution $F(x)$. Having observed $\{T_1, ..., T_N\}$ and an outside option, students simultaneously select colleges that maximise their utilities.

The planner knows the individual production function, ability distribution, utility function, and the outside option. It is a simple sequential decision making game. We therefore use backward induction to discuss the model. Let us look at the decision problem of students first.

3.2.2 The Decision Problem of Students

In this section, we discuss the student's decision given the set of task levels and the outside option. As we described in the last section, there is a continuum of potential students who differ in their ability levels x in the economy.

Assumption 1: *The distribution of ability* $F(x)$ *is continuous and differentiable with a corresponding density function by* $f(x)$ *over a finite interval* $[\underline{x}\ ,\ \overline{x}]$.

Next we turn to focus on how a student's selection is made. A student's utility has been assumed to equal his gain from the college qualification minus the cost of accomplishing the task. The gain for a particular college j is the same for all students involved in that college, and is expressed as $v_j = V(T_j)$.

Assumption 2: $V(T)$ is *twice differentiable, increasing, and concave in* T*, i.e.,* $V_1(T) > 0$, $V_{11}(T) \leq 0$*; and* $V(0) = 0$.

The cost function of student i who studies at the college is given by $c_i = C(x_i, T)$.

Assumption 3: $C(x,T)$ *is twice differentiable, increasing, and strictly convex in* T*, i.e.,* $C_2(x,T) > 0$ *and* $C_{22}(x,T) > 0$*; it is decreasing in* x*, i.e.,* $C_1(x,T) < 0$*; it satisfies* $\frac{dc}{dT} \to \infty$ *as* $x \to 0$ *and* $C(x,0) = 0$*; it is submodular, i.e.,* $C_{12}(x,T) < 0$.

Submodularity of the cost function implies that given any level of task, a higher ability student's cost rises more slowly than that of a lower ability student given a marginal increase in the task level. In other words, it is relatively easier for the high ability student to accomplish a tougher task. The second part of Assumption 3 indicates the cost rises to the infinity when ability is zero. The last part of Assumption 3 implies the cost is zero when the task level is zero. This assumption rules out the possibility for zero ability students to enter any college with positive task level.

Given the gains and costs functions, the utility function of student i obtained from college j is

$$U(x_i, T_j) = V(T_j) - C(x_i, T_j)$$

Combining Assumption 2 and Assumption 3 implies that the utility function is strictly concave and single-peaked in T, i.e., $U_{22}(x,T) < 0$;[39] $U(x,0) = 0$; supermodularity, i.e., $U_{12}(x,T) > 0$,[40] which means, given the same task level, a higher ability student earns more utility than a lower ability student at the same college.

Supermodularity means that, at the same level of task, the marginal utility of achieving a higher level task is higher for students with higher abilities. Note that resources are not in student's utility function. However, in the next section, we will see that resources per student are one of determinants in the output function.

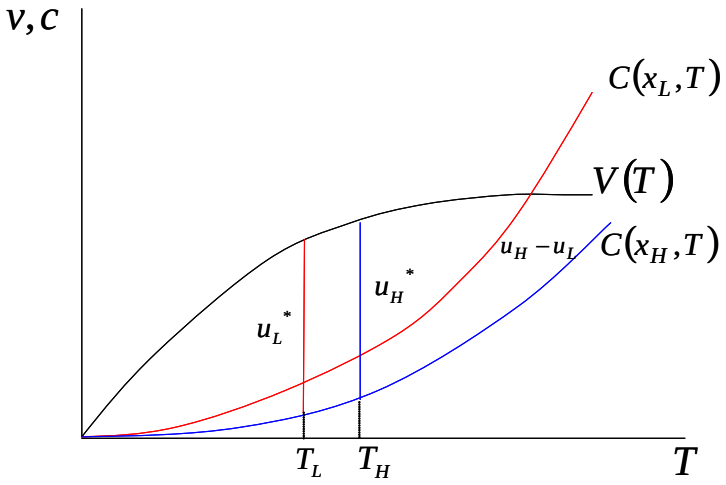

Figure 13: Utility

Figure 13 shows a simple example presenting the utilities of two students at the same college. One has ability of x_H, and the other has x_L. They share the same gains curve since they attend the same college, but the costs curve of the higher ability student is flatter than the lower ability student due to the submodularity of the cost function. The distance between the gains curve and the costs curve measures the utility. These curves immediately imply $U_{22}(x,T) < 0$ as already noted. For instance, the distance between $V(T)$ and $C(x_L,T)$ starts from zero, where both gains and costs are zero, and increases to the maximum, where the task level is marked as T_L, and then falls to zero where costs equals gains again. As we can tell from Figure 13, the difference between two cost curves is increasing in T, which is consistent with our assumption of $U_{12}(x,T) > 0$. Let T_m denote the task level that maximises the utility of the corresponding type.

[39] $U_{22}(x,T) = V_{11} - C_{22}(x_i,T_j) < 0$, thus U is strictly concave, and hence single-peaked.
[40] If c is submodular, $-c$ is supermodular, and hence $u = v - c$ is supermodular.

Proposition 13 *The utility of a student with lower ability is maximised at a lower level of T than a student with higher ability, $\frac{dT_m}{dx} > 0$.*

Proof. See Appendix. ■

In Figure 13, we use T_H to denote the task level maximizing the high ability type's utility. Proposition 13 yields that $T_L < T_H$. Since they share the same gains curve, and the high ability type's cost curve is flatter, the high ability student's utility is greater than the low ability type's for any *positive* task level by Assumption 2 and Assumption 3.

Assumption 4: *Each student has an outside option of* u_0, independent of x.

Given the outside option, student i's objective is to find a college j that satisfies the following condition:

$$u_{i,j} \geq \max[u_0, u_{i,k}] \quad \forall \text{ all } k \neq j.$$

Let T_0 denote no higher education; thus all students face $N+1$ options. The equilibrium will be a many-to-one mapping from the set of students to the set of colleges with different task levels including the outside option $\{T_0, T_1, ...T_N\}$. Our concern is whether or not students will be positively sorted to the colleges which have been ranked by task levels, in other words, whether or not the higher ability types would go to colleges with the higher task levels.

Now we define positive sorting.

Definition 2 *Given a set of ranked colleges including the outside option with task levels $\{T_0, T_1, ...T_N\}$ with $T_0 < T_1 < T_2, ..., T_{N-1} < T_N$ and two types x, y with $x < y$, a mapping result does not have positive sorting if given $T_{j'} < T_j$, type x chooses college j and type y selects college j'.*

If a mapping result has positive sorting, then a student selects college j if and only if $x_{j,l} \leq x \leq x_{j,h}$, where $x_{j,l}$ and $x_{j,h}$ denote the lowest type and the highest types at college j, respectively.[41]

Our assumptions about the utility function and the design of colleges yield the following proposition.

Proposition 14 *Under Assumption 3 and 4, given a set of colleges including the outside option ranked by task levels as $T_0 < T_1 < T_2 < ... < T_N$, the equilibrium assignment has positive sorting.*

[41] Sallee and Resch (2008) define a partition to be monotonic if and only if, for the least and greatest elements $\underline{x}_k$ and $\overline{x}_k$ in each school, a student x is assigned to school k if and only if $\underline{x}_k \leq x \leq \overline{x}_k$.

Proof. See Appendix. ■

Proposition 14 implies that, in equilibrium, the group of students with the highest ability select the college with the highest task level, the group of students with the second highest ability choose to attend the college with the second highest task level, and so on. Students who choose the outside option would be the ones with the ability lower than those with the lowest ability who choose the college with the lowest task level.

Proposition 15 *Any two adjacent colleges in terms of task levels share the same marginal types, i.e., there are no overlaps or gaps of abilities between two adjacent colleges.*

Proof. See Appendix. ■

The highest type of any lower level college or outside option is the same as the lowest type of the adjacent higher level college. Marginal types are indifferent between the two adjacent colleges. They will earn the same utility from the two colleges.

3.2.3 The Planner's Decision Problem

The planner aims to maximise the total output, which is the aggregate of the continuum of students' output. Any individual student's output is assumed to be a function of that student's ability, resources per student and task level in a particular college. The output of student i in college j is given by

$$q_{i,j} = q\left(x_i, r_j, T_j\right),$$

where recall that r_j denotes the resources per student in the college and $r_j = \frac{p_j R}{w_j}$. We assume all arguments are complementary (supermodular). In contrast to the assumption of submodularity in the student's cost function, supermodularity implies positive cross partial derivatives. The formal assumption for the output function is as follows.

Assumption 5: *The output function* $q(\cdot) : R_+^3 \to R$, *is a twice differentiable, increasing, concave, and supermodular function of its arguments, and hence* $q_1(x,r,T) > 0$, $q_2(x,r,T) > 0$, $q_3(x,r,T) > 0$; $q_{11}(x,r,T) < 0$,$q_{22}(x,r,T) < 0$, $q_{33}(x,r,T) < 0$; $q_{12}(x,r,T) > 0$, $q_{13}(x,r,T) > 0$, $q_{23}(x,r,T) > 0$. *The output is zero if one of the arguments is zero, i.e.,* $q(0,r,T) = q(x,0,T) = q(x,r,0) = 0$.

The cross partial derivatives represent the supermodularity. $q_{12}(x,r,T) > 0$ means at any given level of resources per student, the higher ability student

produces more when given a marginal increase in resource; $q_{13}(x, r, T) > 0$ implies that at any given level of task in each college, the higher ability student produces more when given a marginal increase in task level; $q_{23}(x, r, T) > 0$ represents that at any given level of task, a student in a college with higher resources per student produces more when given a marginal increase in task level.

We use utilitarian social welfare function (Sallee, Resch and Courant (2008)) to express the total output. The aim of the planner is to maximise the aggregate output of all colleges. The total output function is expressed as

$$W = \int_{x_{1,l}}^{x_{1,h}} q\left(x, \frac{p_1 R}{w_1}, T_1\right) f(x)\, dx + ... + \int_{x_{N,l}}^{x_{N,h}} q\left(x, \frac{p_N R}{w_N}, T_N\right) f(x)\, dx,$$

where recall that $x_{j,l}$ and $x_{j,h}$ represent the lowest and highest abilities in college j $(\forall j = 1, 2, ..., N)$, w_j denotes the size of college j, i.e., the total measure of students in college j, $w_j = \int_{x_{j,l}}^{x,h} f(x)\, dx$. Given this output function, the planner will simultaneously choose the optimal number of colleges, N, design task level for each college, $\{T_1, ..., T_N\}$, and decide the optimal allocation of total resources to each college. Therefore, the planner's decision problem is

$$\max_{N, \{T_1, ... T_N\}, \{p_1, ... p_N\}} W.$$

The maximization is subject to the following constraints.

1. Budget constraint:

$$\theta N + R \leq \Omega,$$

where Ω is the total resources. The sum of total fixed costs and the total allocable resources is less than or equal to the total resources.

$$\sum_{k=1}^{N} p_k = 1.$$

We assume that the planner will invest all resources into colleges.

2. Feasibility constraint:

$$p_k \geq 0 \quad \forall \quad k.$$

This rules out negative allocations.

3. Identity constraint:

$$0 < T_1 < T_2 < ... < T_N.$$

The identity constraint implies that colleges are identified by the task levels, and we assume they are ranked from the lowest (denoted by 1) to the highest (denoted by N). It is clearly inefficient for the planner to set up a college with a zero task level as output will then be zero. Consider there are two colleges with the same task level, say T_j. Students whose abilities satisfy $U(x, T_j) \geq \max(u_0, U(x, T_{k\neq j}))$ would select one of the two colleges randomly. One would think it would be sub-optimal as a single college would lead to the same output and save on θ. This is not quite a complete argument as the output would vary if r differed between colleges. The planner would allocate more resources to the college having a higher average ability. However, it will not be optimal to set the same task level for two colleges with different average abilities and resources per student. Therefore, the planner will not set the same task level for two or more colleges in equilibrium. Note that Proposition 14 and Proposition 15 imply that, given $0 < T_1 < T_2 < ... < T_N$, $x_{1,h} = x_{2,l}; x_{2,h} = x_{3,l}; ...x_{N-1,h} = x_{N,l}$, hence from now on we use x_j to denote the lowest type in college j, which is also the highest type in college $j-1$ $\forall j = 2, 3, ..., N$. In the college with the highest task level, the highest type should be $\overline{x}$ since type $\overline{x}$ does not have any other options better than joining college N. However, type x_1 is indifferent between college 1 and the outside option. Note that all marginal types are different from each other otherwise an empty college may exist. The bottom type x_1 in the lowest level college, however, may be the same as the lower bound $\underline{x}$, in which case all types enter into colleges. Therefore, the planner's objective function can be rewritten as

$$\max_{N,\{T_1,...T_N\},\{p_1,...p_N\}} W = \int_{x_1}^{x_2} q\left(x, \frac{p_1 R}{w_1}, T_1\right) f(x)\, dx + ... \int_{x_N}^{\overline{x}} q\left(x, \frac{p_N R}{w_N}, T_N\right) f(x)\, dx,$$

s.t.

$$\theta N + R \leq \Omega,$$

$$\sum_{k=1}^{N} p_k = 1,$$

$$p_k \geq 0 \quad \forall \quad k,$$

$$0 < T_1 < T_2 < ... < T_N.$$

In equilibrium, x_1, x_2, ..., x_N are determined by $\{T_1, ..., T_N\}$. Since we have constructed both sides of the market, next we are able to define the equilibrium of the model.

3.3 Equilibrium

3.3.1 Definition of Equilibrium

To define the equilibrium, we need to find the optimal number of colleges, the equilibrium task levels and allocation proportions of resources for the planner, and to consider the choices made by students. In our work, the economy will reach a "free choices" equilibrium, where students select colleges as they will. We assume that the planner anticipates how students allocate themselves to colleges and only needs to choose the optimal number of colleges, to design the task levels and allocate resources to each college. Students select any college or the outside option freely to maximise utilities. The equilibrium the Nash equilibrium of this game and is defined as follows:

Definition 3 *The equilibrium of the economy consists of strategies of students and the planner such that:*

1. *By taking account of students' decisions and the distribution of students' abilities, the planner decides the number of colleges, and the vector of resources allocation and vector of task levels to maximise total output.*

2. *Given N^* colleges, the outside option and task levels, student i selects college j if and only if*

$$u_j \geq \max_k (u_0, u_k),$$

where u_k denotes the utility from any other college.

Resources and ability, task levels are complements in production. This immediately yields the following conclusion.

Proposition 16 *Given any two colleges k_i and k_j with $0 < T_i < T_j$, if $w_i \geq w_j$ in equilibrium, then $0 < r_i < r_j$ and $0 < R_i < R_j$.*

Proof. See Appendix. ■

This proposition implies that if the measure of students at a higher level college is less than a lower level college, then total allocable resources and resources per capita are higher at the higher level college than the lower level college.

3.3.2 The Optimal Design When $N = 1$

The planner's decision consists of two steps: (1) choosing the optimal number of colleges; (2) setting the optimal level of tasks and the optimal allocation of

resources. We isolate the two steps by beginning with the second stage. Fixing the number of colleges will simplify our model so we derive the equilibrium only consisting of $\{T_{1,...}T_N\}$ and $\{p_{1,...}p_N\}$. If the number of colleges is given, the fixed cost is not our concern at this stage, and only the amount of allocable resources and the distribution of students' ability matter.

We start with the simplest model, where there is only one college. In this case, the planner does not have to consider how to allocate resources as he will invest all resources into the only college. Therefore, the planner only needs to figure out the optimal design of task level for this college in order to maximise total output by considering the compatibility condition of students. For simplicity, assume that a student's ability is uniformly distributed on $[0,1]$, hence $f(x) = 1$. This assumption will not substantively affect our model. Under these assumptions, the planner's objective model is

$$\begin{aligned} \max_{T} W(T, x_1) &= \int_{x_1}^{1} q\left(x, \frac{R}{1-x_1}, T\right) dx, \\ \text{s.t.} \quad 0 &< T, \end{aligned}$$

where x_1 denotes the lowest ability in the college. Since the ability follows a uniform distribution on $[0,1]$, the measure of students in the college is $w = 1 - x_1$.

The compatibility condition implies students will join the college if and only if $u \geq u_0$, where u denotes the utility obtained from the college, and u_0 denotes the utility obtained from the outside option. At the equilibrium, the marginal student is indifferent between the outside option and attending college, i.e.,

$$U(x_1, T) = u_0.$$

This equation tells the planner how to find the marginal ability in the college. By solving this equation for x_1, which must be unique, the bottom ability is determined by a function g:

$$x_1 = g(T, u_0).$$

The task level has a direct positive impact on individual output. However, it has a negative effect on the size of the college.

Proposition 17 *At the optimal task level, T^*, if $x_1 = g(T^*, u_0)$, then $U_2(x_1, T^*) \leq 0$ and x_1 is increasing in T^*.*

Proof. See Appendix. ■

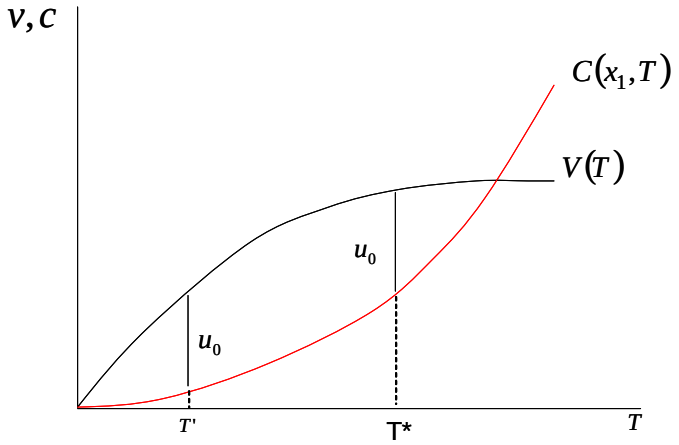

Figure 14: g_T

The intuition behind Proposition 17 is shown in Figure 14. Given the outside option u_0, there must be a marginal type who is indifferent between attending the college and choosing the outside option. $V(T)$ and $C(x_1, T)$ denote the gain and cost of that type of students. Since $U(x_1, T)$ strictly concave and single-peaked in T, we can find two task levels such that $U(x_1, T) = u_0$, which are denoted by T^* and T' with $T' \leq T^*$. T^* and T' will be equal if type x_1 maximises his utility at T^*. The proposition implies that the planner will only choose T^* as the task level. It is because T^* and T' will attract the same set of students whose abilities are higher than x_1, but the higher task level produces more education output. The second part looks at how the marginal type changes in the equilibrium task level. If the planner increases the task level from T^*, then type x_1 will leave the college as cost increases more than gain. Any increase in the task level will cause the marginal type to leave the college.[42]

Type zero will not choose to attend any colleges under the assumption that $\frac{dc}{dT} \to \infty$ as x goes to zero; the planner would not set a task level such that $x_1 = 1$ as it means the top college is empty and unused. Therefore, the optimum will be an interior solution. First-order necessary condition for an interior solution follows from the unconstrained optimization problem. The first order condition of with respect to T will give us the optimal solution for T by substituting out x_1 with function g.

[42] An equivalent way of looking at this is to note that g is initially decreasing but beyond some point increasing, and an optimal government policy must choose a point on the upward sloping part for the reasons we outline.

Claim 15 *The F.O.C. of the one-college case is:*

$$-q\left(x_1, r, T\right) g_1\left(T, u_0\right) + \left[\frac{R}{\left(1-x_1\right)^2}\int_{x_1}^{1} q_2 dx\right] g_1\left(T, u_0\right) + \int_{x_1}^{1} q_3 dx = 0,$$

where q_2 *denotes* $q_2\left(x_1, r, T\right)$, q_3 *denotes* $q_3\left(x_1, r, T\right)$ *and* $r = \frac{R}{1-x_1}$.

Proof. See Appendix.[43] ■

This equation states that the marginal output of a unit increase in the task level must be zero in equilibrium. Given the same level of total resources, production of each student depends not only on the task level, but also on resources per student, which is determined by resources and the measure of students. An increase in the task level will directly increase output because output of each student is increasing in task level; this is represented by the third term. However, output will decrease due to the change of size of the college. The first term measures the loss of output due to losing marginal students. The second term represents the increase in output because those students still in the college produce more due to a smaller size of the college. The overall effect is zero in equilibrium.

Next we use an example to illustrate the model.

Assume that $C\left(x, T\right) = \frac{T^2}{x}, V\left(T\right) = T$, and hence $U\left(x, T\right) = T - \frac{T^2}{x}$. These functions satisfy the assumptions about the cost, gain and hence utility functions. In the one-college case, $U\left(x_1, T\right) = u_0$; thus we have

$$x_1 = \frac{T^2}{T - u_0}.$$

Note that the outside option needs to satisfy $u_0 \leq 0.25$ in order that $x_1 \leq 1$;[44] and when $u_0 = 0$, $x_1 = T$.

For output, assume that

$$q\left(x, r, T\right) = \left(xr\right)^{\alpha} T^{\beta}.$$

The Cobb-Douglas education output function has the properties as we assumed.

[43] We did not derive the second derivative of W with respect to T, but it does not matter too much as the interior solution has to satisfy the first order condition.

[44] $\frac{T^2}{T-u_0} \leq 1$, so $T^2 - T + u_0 \leq 0$. Therefore, $\frac{1-\sqrt{1-4u_0}}{2} \leq T \leq \frac{1+\sqrt{1-4u_0}}{2}$. If $u_0 > 0.25$, then $T^2 - T + u_0 > 0$, and hence $\frac{T^2}{T-u_0} > 1$.

The total output function in the one-college case can be rewritten as

$$\begin{aligned} \max_{T} W(T) &= \int_{x_1}^{1} (xr)^{\alpha} T^{\beta} dx \\ \text{s.t.} \quad 0 &< T < 1 \end{aligned}$$

where $x_1 = \frac{T^2}{T-u_0}$ and $r = \frac{R}{1-x_1}$.

The first part of the first order condition is

$$-T^{\beta}\left(1-\frac{T^2}{T-u_0}\right)^{-\alpha}\left(\frac{T}{T-u_0}\right)^{\alpha}\left(1-\left(\frac{u_0}{T-u_0}\right)^2\right),$$

the second part is

$$\frac{\alpha}{1+\alpha}T^{\beta}\left(1-\frac{T^2}{T-u_0}\right)^{-(1+\alpha)}\left(1-\left(\frac{T^2}{T-u_0}\right)^{1+\alpha}\right)\left(1-\left(\frac{u_0}{T-u_0}\right)^2\right)$$

and the last part is

$$\frac{\beta}{1+\alpha}T^{-1+\beta}\left(1-\frac{T^2}{T-u_0}\right)^{-\alpha}\left(1-\left(\frac{T^2}{T-u_0}\right)^{1+\alpha}\right).$$

As we can see, the amount of total resources does not affect the equilibrium, but the outside option does. The following numerical example provides some properties which can not be analysed in a general case.

Conjecture 1 *If* $U(x,T) = T - \frac{T^2}{x}$ *and* $q(x,r,T) = (xr)^{0.5} T^{0.5}$, *the maximised output falls and the ability of the bottom type rises as* u_0 *goes up, but the optimal task level does not vary monotonically.*

We are not able to prove this claim in a general case, but we believe it to be true. The increase in u_0 implies the college is less attractive; thus some bottom types of students will leave the college if the planner does not change the task level. Given the increase in u_0 at a relatively low level, the planner could lower the task level, which is high, to keep some of the students who are leaving. At this stage, the effect represented by the first term in the first order condition dominates the effect measured by the other two terms. But the planner will only keep part of all leaving types as the output would drop if the task level is set too low. Therefore, x_1 rises at this stage. When u_0 approaches the upper limit, the increase in output by retaining students is less than the decrease due to a lower task level, which means the last two terms in the first order condition

U0	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.22	0.24	0.25
T*	0.581	0.567	0.553	0.539	0.524	0.509	0.495	0.481	0.471	0.465	0.467	0.475	0.490	0.500
X1	0.581	0.588	0.596	0.607	0.618	0.633	0.653	0.678	0.713	0.759	0.817	0.885	0.960	1
r	2.387	2.427	2.475	2.545	2.618	2.725	2.882	3.106	3.484	4.149	5.464	8.696	25.000	N/A
W*	0.437	0.430	0.421	0.412	0.401	0.390	0.376	0.360	0.340	0.314	0.279	0.227	0.138	0

U0: Outside option;
T*: Optimal task level;
r: Resources per student at equilibrium;
X1: Bottom type;
W*: Maximized output.

Table 2

dominates. So, the optimal task level increases. The outside option is like a rival of the college, so an increase in the outside option must reduce the output of the college. Therefore, W falls as u_0 increases. An increasing task level plus u_0 increasing cause x_1 to rise at this stage as well. The following numerical result and the diagram support the claim.

Table 2 presents the optimal task levels, bottom types, and maximal outputs given different values of u_0.[45] The ability of the bottom type rises as u_0 increases from 0 to 0.24. The optimal task level decreases until u_0 reaches 0.18, and then starts increasing. It goes to 0.5 as u_0 approaches 0.25, and x_1 approaches 1.

Figure 15 presents how total output curves vary in the task level given different values of u_0. Given a value of T, the outside option is smaller, more is total output. When the outside option is too large, plausible task levels are few. For example, suppose $u_0 = 0$, the planner has the largest number of plausible task levels, and total output is the greatest given any value of T.

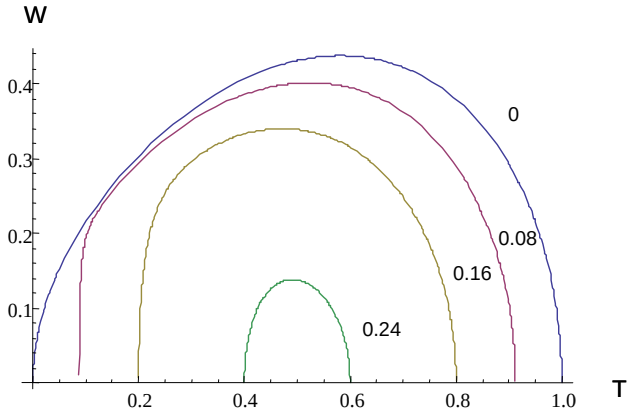

Figure 15: Total Output Given Different u_0

[45] The numerical and graphical results in this section are calculated by Mathematica and Excel.

3.3.3 The Number of Colleges

In the last section, we considered the case where there is only one college. But the planner has to decide the optimal number of colleges before setting the task levels and allocation of resources. The number of college should be determined by the total resources and the fixed cost of setting up a college. Since this is a sequential game, the planner as the first mover has to take students' decisions into account. Apart from the task levels, students' concerns include the outside option. Therefore, the outside option should be in the planner's consideration. In this section, we discuss the number of colleges given the total resources, fixed costs and outside option.

Two-College Case Under the same assumptions as in the one-college case, the planner designs the task level for each college and allocates a proportion of total allocable resources to each college in order to maximise total outputs.

Assume there are two colleges, and denote the one with the lower task level by k_1 and the other one with the higher task level by k_2. Since there are two colleges, the planner has to consider the allocation of resources. By choosing the proportion of resources to k_1, p, the proportion of resources to k_2, $1-p$, and the task levels in each college, T_1 and T_2, the planner can have all students sorted themselves into each college or to stay in the outside option. Note that the results of this case are easily translatable to the case of more colleges.

The objective function in this case (with uniform distributed ability) is

$$\begin{aligned} \max_{T_1,T_2,p} W &= \int_{x_1}^{x_2} q\left(x, \frac{pR}{x_2-x_1}, T_1\right) dx + \int_{x_2}^{1} q\left(x, \frac{(1-p)R}{1-x_2}, T_2\right) dx \\ \text{s.t.} \quad 0 &< T_1 < T_2. \end{aligned}$$

Where x_1 and x_2, the marginal abilities, are functions of T_1 and T_2, which are the factors that the planner is able to control. Student i's decision is according to the rule that he selects k_1 if $U(x_i, T_1) \geq U(x_i, T_2)$ and $U(x_i, T_1) \geq u_0$, or k_2 if $U(x_i, T_2) \geq U(x_i, T_1)$ and $U(x_i, T_2) \geq u_0$, or the outside option if $U(x_i, T_1)$ and $U(x_i, T_2)$ are less than u_0.

In equilibrium, the bottom type in 1 satisfies

$$U(x_1, T_1) = u_0$$

and the top type in 1, which is identical to the bottom type in 2, satisfies

$$U(x_2, T_1) = U(x_2, T_2).$$

The first equation states that the bottom ability type student is indifferent between the outside option and k_1; the second equation implies that the marginal type student is indifferent between k_1 and k_2.

Let g^1 and g^2 denote the solutions to the following two equations:[46]

$$x_1 = g^1(T_1, u_0)$$

$$x_2 = g^2(T_1, T_2).$$

The bottom type of k_1 only depends on T_1 and the outside option. The top type in k_1 which is the same as the bottom type in k_2 is determined by T_1 and T_2. The bottom type in k_1 still follows Proposition 17.

Proposition 18 *x_2 is increasing in both T_1 and T_2, i.e., $g_1^2(T_1, T_2) > 0$ and $g_2^2(T_1, T_2) > 0$.*

Proof. See Appendix. ■

Proposition 18 can be extended to a general case with any number of colleges. Given any number of colleges, we use x_{k+1} to denote the top type in a particular lower level college c_k, which is identical to the bottom type in the adjacent higher level college c_{k+1}, and we define $g^{k+1}(T_k, T_{k+1}) = x_{k+1}$. Then:

Corollary 6 *$g_1^{k+1}(T_k, T_{k+1}) > 0$ and $g_2^{k+1}(T_k, T_{k+1}) > 0$.*

We can use the simple example from the one-college case to illustrate the claim. If $U(x, T) = T - \frac{T^2}{x}$, then $x_1 = \frac{T_1^2}{T_1 - u_0}$ and $x_2 = T_1 + T_2$. More generally, $x_{k+1} = T_k + T_{k+1}$, $\forall k = 1, ..., N-1$. Clearly, $g_1^{k+1}(T_k, T_{k+1}) > 0$ and $g_2^{k+1}(T_k, T_{k+1}) > 0$.

In the two-college model, we can derive the first order condition for an interior solution of the objective function. If we substitute the solutions to x_1 and x_2 into the initial objective function, then we have

$$\widehat{W}(T_1, T_2, p) = W\left(T_1, T_2, p, g^1(T_1, u_0), g^2(T_1, T_2)\right)$$

[46] The solutions are unique by the fact that $U_1(x, T) > 0$ and $u_{12}(x, T) > 0$.

The first order condition of the new objective function is as follows:

$$\begin{aligned}
\widehat{W}_{T_1} &= W_4 g_1^1(T_1, u_0) + W_5 g_1^2(T_1, T_2) + W_1 = 0, \\
\widehat{W}_{T_2} &= W_5 g_2^2(T_1, T_2) + W_2 = 0, \\
\widehat{W}_p &= W_3 = 0,
\end{aligned}$$

where W_i denotes $W_i(T_1, T_2, p, g^1(T_1, u_0), g^2(T_1, T_2))$, $i = 1, 2, 3, 4, 5$.

The first and second equations represent the equilibrium conditions for the design of task levels. The changes of T_1 and T_2 have direct and indirect effects on output. Since the task level is a factor of individual output function, any change of the task level will have a direct impact on total output. The indirect effects consist of the change of measure effect and the congestion effect[47].

The first term of the first equation measures the overall indirect effect of T_1 through x_1, which includes the loss due to the students who left k_1, and the increase in output of those still in the college. The second term measures the overall indirect effect of T_1 through x_2, which is a little more complicated than the effect through x_1. Consider an increase in the task level of k_1: the bottom type in k_2 will therefore go to k_1, i.e., x_2 rises. The indirect effect through x_2 consists of the loss of students in k_2, and the corresponding increase in k_1; and the combined marginal benefit of changing size of k_1 and k_2. The last term measures the direct effect of T_1, which is positive, therefore the first two indirect effects must have an overall negative effect.

The first term of the second equation measures the overall indirect effect of T_2 through x_2, and the second term measures the direct effect of T_2.

Lemma 2 $W_{x_2} < 0$.

Proof. Since $W_2 > 0$ and $g_2^2(T_1, T_2) > 0$, $W_{x_2} < 0$ follows immediately from $\widehat{W}_{T_2} = W_5 g_2^2(T_1, T_2) + W_2 = 0$. ■

This implies that the effect on total output of an increase in x_2 is negative at the equilibrium, which means the loss due to losing students from k_2 is larger than the contribution of rising resources per student due to the falling size.

The third equation states the equilibrium condition for resource allocation. We can rewrite it as

$$\int_{x_1}^{x_2} \frac{1}{w_1} q_2\left(x, \frac{pR}{w_1}, T_1\right) dx = \int_{x_2}^{1} \frac{1}{w_2} q_2\left(x, \frac{(1-p)R}{w_2}, T_2\right) dx.$$

[47] Congestion effect: Changes of the measure of students will influence resources per capita for those who are already in the college, and hence outputs.

The equation states that the marginal output of a unit resource spent on either college must be the same in equilibrium. The individual output depends on the resources per student, so that one unit extra resource has to be divided by the number of students, $\frac{1}{w_i}, i = 1, 2$. By multiplying the derivatives by $\frac{1}{w_i}$, we have the individual marginal increase in output given one unit increase in resource. By integrating, we get the total marginal output.

The next section investigates the relationship between the task level in the one-college and the task levels in the two-college case.

The results of the two-college case are easily transferrable to any value of N. At the equilibrium, the compatibility condition for type x_i to choose college k_j $(1 \leq j \leq N)$ is as follows:[48]

$$u_j \geq \max(u_0, u_i) \quad \forall\, i \neq j.$$

When $2 \leq j \leq N$, denote the bottom marginal type in k_j by x_j. Proposition 15 implies that x_j is indifferent between c_j and c_{j-1} in equilibrium:

$$U(x_{j+1}, T_{j+1}) = U(x_{j+1}, T_j).$$

At college k_1, the bottom type x_1 is indifferent between k_1 and the outside option in equilibrium:

$$U(x_1, T_1) = u_0.$$

We use g^j to denote the function expressing the bottom type in college j. It is easy to conclude that each marginal type is determined by task levels of two adjacent colleges, hence $x_j = g^j(T_j, T_{j-1})$ and $x_1 = g^1(T_1, u_0)$.

The objective function is as follows. (Under the assumption of uniformly distributed ability)

$$\max_{\{T_1,\ldots T_N\},\{p_1,\ldots p_N\}} W = \int_{x_1}^{x_2} q\left(x, \frac{p_1 R}{x_2 - x_1}, T_1\right) dx + \ldots \int_{x_N}^{1} q\left(x, \frac{\left(1 - \sum_{k=1}^{N-1} p_k\right) R}{1 - x_N}, T_N\right) dx$$

$$s.t. \quad \sum_{k=1}^{N} p_k = 1,\ p_k > 0.$$

[48] See Definition 2.

We obtain the optimal task levels and allocation of resources by solving the first order condition for an interior solution of the objective function.

Comparison Between the One-College System and the Two-College System

θ=0: Sallee, Resch and Courant (2008) argue that if there are no fixed costs, the optimal solution features a unique level of funding (a unique school) for each student's ability that is funded at a positive level. In the proof of the proposition in their model, Sallee, Resch and Courant (2008) argue that the planner has incentives to set up more schools to feature different level of resources for different student ability because of the supermolarity of the production function. In our model, we include the task level in the production function, and assume that the planner can not allocate students arbitrarily. Without the first difference, our model would have the same result as theirs. If there is no fixed cost, then the planner sets up an infinite number of colleges, yielding a unique task level (unique college) for each positive ability type. At the optimum, each college has such a task level that maximises the corresponding student's utility. This follows straightforwardly from the supermodularity of student's utility function and the planner's production function. In our model, the same result obtains although the planner can not allocate students arbitrarily. The supermodularity of student's utility function ensures that they will sort themselves into colleges with different task levels.

Now we look at the first difference. The following proposition presents a result of our model by including the task level in the production function. Let W_1^* and W_2^* denote the maximised total outputs in the one-college case and two-college case respectively.

Proposition 19 *Given $\theta = 0$,*

a. if T is allowed to be zero, then the two-college system is at least as good as the one-college system, $W_2^* \geq W_1^*$;

b. when the optimal task level in the one-college system is smaller than the task level that maximises the utility of the highest type, the two-college system always does better than the one-college system, $W_2^* > W_1^*$.

Proof. See Appendix. ■

Part a of this proposition implies that the two-college system will not be worse than the one-college system under the strong assumption that T is allowed

to be zero. Part b states that the two-college system always does better than the one-college system under some circumstances. However, if we withdraw the assumption that T is allowed to be zero, then this proposition does not always hold. Consider the case when $u_0 = 0$ and T^* is equal or greater than the task level that maximises the utility of the type with the highest ability. It is likely that the planner can not find any positive task level such that total output of the two-college system is higher than the one-college system. The following numerical example will illustrate this scenario.

Suppose a student's utility function is $U(x,T) = T - \frac{T^2}{x}$, and his output function is $q(x,r,T) = (xr)^{\alpha} T^{\beta}$.[49]

In the first place, we assume $\theta = 0$. The problem is whether or not the planner has incentives to set up a new college. We have had the optimal task levels and maximal output from the one-college case.[50] Thus, the planner will establish a new college if the maximal total output from the two-college system is higher than output from the one-college system.

If we assume $\alpha = \beta = 0.5$, and $R = 1$, then the total output function in the two-college case is as follows:

$$\begin{aligned} \max_{T_1,T_2,p} W &= \int_{x_1}^{x_2} (xr_1)^{0.5} T_1^{0.5} dx + \int_{x_2}^{1} (xr_2)^{0.5} T_2^{0.5} dx \\ \text{s.t.} \quad 0 &< T_1 < T_2, \end{aligned}$$

where

$$x_1 = \frac{T_1^2}{T_1 - u_0},$$

and x_2 is determined by the following condition:

$$\begin{aligned} T_2 - \frac{T_2^2}{x_2} &= T_1 - \frac{T_1^2}{x_2}, \\ x_2 &= T_1 + T_2. \end{aligned}$$

Since R is assumed to be 1, the allocation of resources is determined by

$$r_1 = \frac{p}{x_2 - x_1}, r_2 = \frac{1-p}{1-x_2}.$$

We are not able to work out a general expression for the equilibrium task level in terms of u_0. But by giving different values to u_0, we can calculate the optimal task levels as shown in Table 3. In this case, the planner will not set up a new

[49] We assumed $\alpha = \beta = 0.5$ in the one-college case.

[50] See Table 2.

U0	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.22	0.24	0.25
T*	0.581	0.567	0.553	0.539	0.524	0.509	0.495	0.481	0.471	0.465	0.467	0.475	0.490	0.500
T1*	0	0.023	0.050	0.083	0.120	0.163	0.209	0.258	0.306	0.353	0.397	0.438	0.479	0.500
T2*	0.581	0.573	0.564	0.554	0.544	0.534	0.525	0.517	0.511	0.506	0.503	0.501	0.500	0.500
P*	0	0.019	0.042	0.073	0.109	0.154	0.204	0.261	0.314	0.368	0.414	0.435	0.459	N/A
X1	0	0.176	0.250	0.300	0.360	0.422	0.491	0.564	0.641	0.720	0.800	0.880	0.960	1
X2	0.581	0.596	0.614	0.637	0.664	0.697	0.734	0.775	0.817	0.856	0.900	0.939	0.980	1
r1	0	0.045	0.115	0.217	0.359	0.560	0.840	1.237	1.784	2.706	4.140	7.373	22.950	N/A
r2	2.387	2.428	2.482	2.554	2.652	2.792	2.992	3.284	3.749	4.389	5.860	9.262	27.050	N/A
W*	0.437	0.433	0.427	0.421	0.412	0.402	0.390	0.373	0.352	0.324	0.285	0.230	0.139	0
ΔW	0	0.003	0.006	0.009	0.011	0.012	0.014	0.013	0.012	0.010	0.006	0.003	0.001	0

U0: Outside option;
T*: Optimal task level of one-college system;
T1*: Optimal lower task level;
T2*: Optimal higher task level;
P*: Optimal allocation of resources;
X1: Bottom type of the college with lower task level;
X2: Bottom type of the college with higher task level;
r1: Resources per student of the college with lower task level at the equilibrium;
r2: Resources per student of the college with higher task level at the equilibrium;
W*: Maximized output under two task levels;
ΔW: Difference of output between the one-college case and two-college case.

Table 3

college when $u_0 = 0$.

Proposition 20 *Given $\theta = 0$, if $q(x, r, T) = (xr)^{0.5} T^{0.5}$ and $U(x, T) = T - \frac{T^2}{x}$, then the two-college system does not do better than the one-college system when $u_0 = 0$; the two-college system does better than the one-college system when $0 < u_0 < 0.25$.*

Proof. See Appendix. ■

The following numerical results support our claim. Table 3 shows us the optimal task levels, allocations of resources, bottom types of the two colleges, maximal outputs and the increase in output from the new system as u_0 varies. Figure 16 and Figure 17 present the contrasts of the two systems.

The high task level falls and the low task level rises as u_0 increases. Note the task level in the one-college case does not change monotonically. The difference between the two new task levels decreases as u_0 rises. When u_0 goes to 0.25, which makes the available types contract to 1, the difference disappears. The bottom types in the new systems have lower abilities than the bottom types in the one-college system. It implies that more students take higher education in the new systems. But the difference decreases as u_0 rises. The increase in total output from the new system rises when u_0 is small and then falls after the peak where $u_0 = 0.12$ as u_0 increases. The result about p supports Proposition 16, which states that the college with the higher task level has higher resources per student.

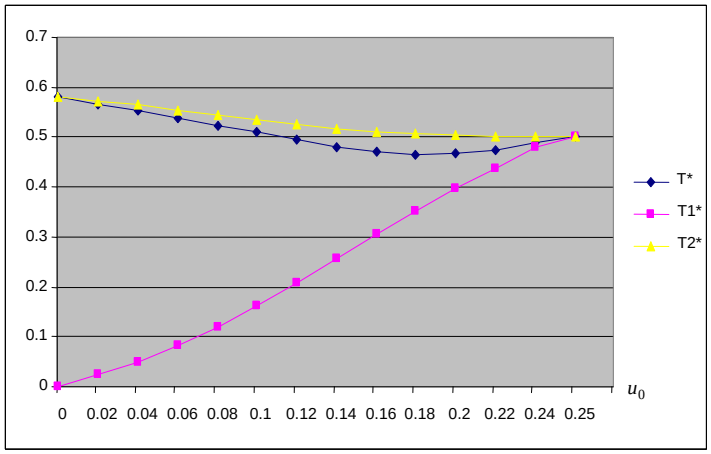

Figure 16: Task levels.

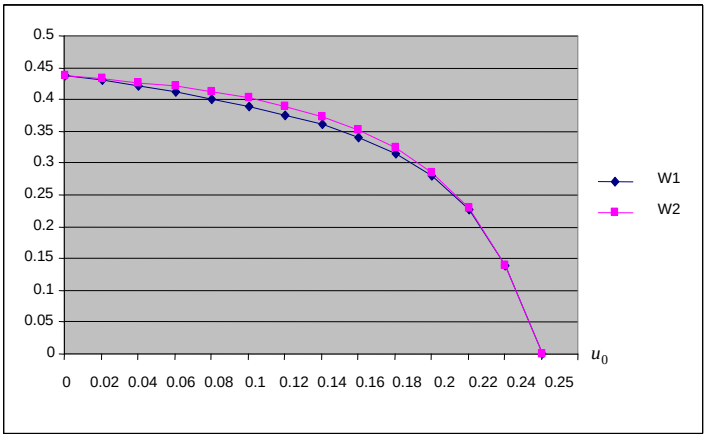

Figure 17: Outputs.

The intuition behind these results is as follows. When u_0 is small, the optimal task level in the one-college case would be high, so that the direct effect of a rising task level (increasing output) is small and the indirect effect (decreasing output) is large, and hence the difference between T_2^* and the original task level is small. Meanwhile, T_1^* needs to be low enough to yield a large positive indirect effect to balance the large direct effect by T_2^*. In the extreme case, where $u_0 = 0$, it is not possible to find a T_1^* to balance the effect of changing the higher task level. As u_0 rises, the optimal task level in the one-college system falls until it reaches the minimum point. At this point, the direct effect of a rising task level increases and the indirect effect falls. Therefore T_2^* falls and the difference between T_2^* and the original task level rises. On the other hand, as u_0 rises, the possible downward room for T_2^* decreases because $T_1^* - u_0 > 0$. This negative effect increases as u_0 rises. Therefore, the increase in total output between the two cases stops rising when the negative effect exceeds the positive effect, then starts falling.[51] The increase in T_1^* has a positive effect on T_2^*, hence the decrease of T_2^* slows down. In the stage when the optimal task level of the one-college case increases, T_2^* still falls but at a decreasing rate because of the effect of T_1^*, and the increase in total output keeps falling. As u_0 approaches 0.25, T_1^* and T_2^* approach 0.5, and the increase in total output goes to zero.

In this example, the output of the one-college system is higher than the total output of any two colleges which are not empty when $u_0 = 0$. This result would change if we transform the production function. Suppose $q(x, r, T) = (xr)^\alpha T^\beta$. Numerical results show that when $u_0 = 0$, the planner will set up the second college only if $\alpha + \beta < 1$, i.e. the production function has decreasing returns to scale.[52] This results also show that the increase in output of the two-college system over the one-college system is rising as α decreases. On the other hand, when $\alpha + \beta$ is too large and u_0 is sufficiently small, we found that the two-college system does not do better than the one-college system. For instance, when $\alpha = \beta = 0.9$ and $u_0 = 0.01$, the total output of the two-college system approaches the maximum only if the low task level goes to zero and the high task level goes to the optimal task level of the corresponding one-college system.

Table 4 and Figure 18 show the trends of optimal task levels and outputs in both cases as α changes. When $\alpha < 0.5$, it is more profitable for the planner to establish a two-college system as α goes to zero. On the other side, this observation supports our analysis on the effects of the outside option in the two-

[51] See Table 3, $u_0 = 0.12$.

[52] See Table 4, where we simply assume $\alpha = \beta$ and hence the production function is $q(x, r, T) = (xr)^\alpha T^\alpha$. $W_2^* > W_1^*$ for $\alpha \neq \beta$ as long as $\alpha + \beta < 1$.

α	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
T*	0.116	0.239	0.360	0.475	0.581	0.680	0.771	0.854	0.930
W1*	0.673	0.542	0.476	0.444	0.437	0.452	0.492	0.564	0.695
W2*	0.747	0.605	0.514	0.454	0.437	0.452	0.492	0.564	0.695
ΔW	0.074	0.063	0.038	0.010	0.000	0.000	0.000	0.000	0.000

α : Power of inputs in the production function;
T*: Optimal task level in the one-college case;
W1*: Maximial output in the one-college case when u0=0;
W2*: Maximial output in the two-college case when u0=0;
ΔW: W2*-W1*

Table 4

college case. The optimal task level in the one-college case falls as α decreases, so that the upward room for a higher task level and the downward room for a lower task level increases, and hence the output of the two-college system is possibly greater than the one-college system.

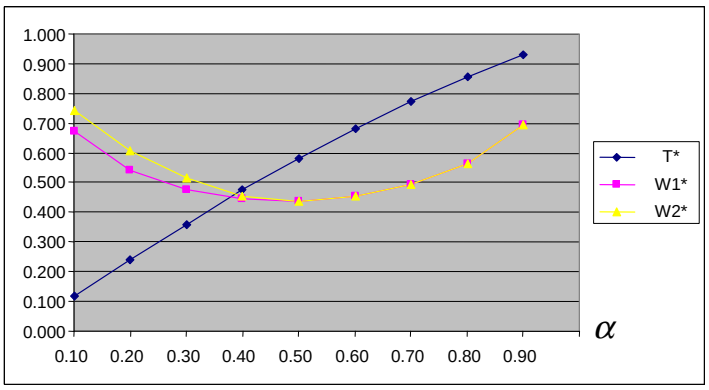

Figure 18: Task levels and outputs.

We have bee unable to establish whether or not the three (or more)-college system would do better than the two-college system, but we believe that it should depend on the elasticity of inputs, i.e. α and β. When the elasticity is very low, it could increase the output to break one college into two or more colleges. On the other side, when the elasticity is high, the planner would rather keep a small number of colleges. Sallee, Resch and Courant (2008) propose that if there are no fixed costs, the optimal solution features a unique level of funding (a unique school) for each student ability that is funded at a positive level. Thus, there must be an infinite number of colleges if students' abilities follow a continuous distribution. The analysis on our model has shown that the number of colleges is not necessarily infinite.

$\theta \neq$**0:** If $\theta \neq 0$, then the resource constraint is $\theta N + R \leq \Omega$. In the optimum, $\theta N + R = \Omega$. Given Ω, the planner's decision is dependent upon the optimal total outputs of the two systems. Since $\theta \neq 0$, the allocable resources are decreasing as N rises. Therefore, the planner would set up the second college only if the new system can produce a higher output than the one-college system although $R_2 < R_1$. When the fixed cost is too high, the decrease of allocable resources may cause the total output of the two-college system to be lower than the output of the one-college system. In that case, the planner will not establish a new college. Let us see the following example.

Suppose $u_0 = 0.12$.[53] If $\theta = 0$ and the total resources are Ω, then the total output in the one-college case is $W_1^0 = 0.376\Omega^{0.5}$, and the total output in the two-college case is $W_2^0 = 0.39\Omega^{0.5}$.[54] Figure 19 shows how the total output varies in Ω when $\theta = 0$. The total output of the two-college system is higher than the one-college system for all positive Ω.

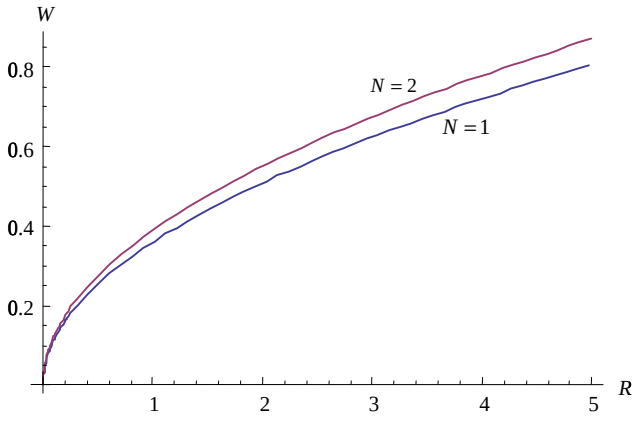

Figure 19: Outputs vs total resources when $\theta = 0$.

If we let $\theta = 0.1$, then the total output in the one-college case is $W_1^{0.1} = 0.376(\Omega - 0.1)^{0.5}$, and the total output in the two-college case is $W_2^{0.1} = 0.39(\Omega - 0.2)^{0.5}$. Figure 20 presents the changes of W over Ω when $\theta = 0.1$. When $R < 1.54$, the output is higher in the one-college case. When $R > 1.54$, the output is higher in the two-college case. This example shows that given a constant fixed cost, the optimal number of colleges is weakly increasing as Ω rises. Figure 21 shows how total output changes as Ω rises in a general case when θ increases from zero. The planner will start setting up the first college when $\Omega = \theta$, and W_1 rises

[53]The result can be applied to all cases with different u_0.

[54]Since equilibrium conditions do not depend on R, the optimal task levels do not change even though R varies. Therefore, we have this output functions simply by substituting the optimal task levels and allocation of resources into the original output function.

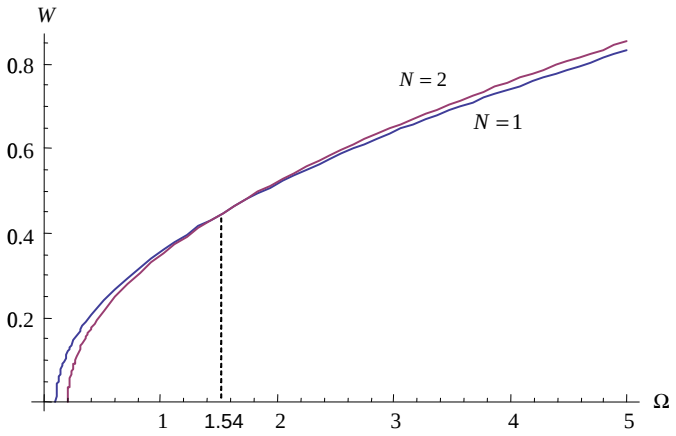

Figure 20: Outputs vs total resources when $\theta = 0.1$.

as Ω increases. The planner decides to set up a new college when $W_2 = W_1$. W_2 increases until the establishment of the third college, and so on.

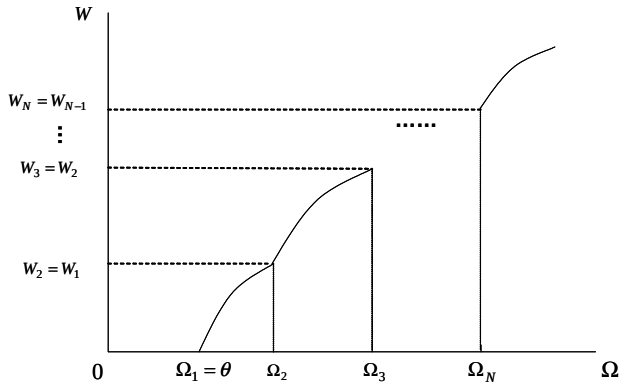

Figure 21: Outputs vs total resources in a general case.

On the other hand, if we assume $\Omega = 1$ and the fixed cost changes, then the total outputs in the one-college and two-college cases are functions of θ. When $u_0 = 0.12$, $W_1^\theta = 0.376(1-\theta)^{0.5}$ and $W_2^\theta = 0.39(1-2\theta)^{0.5}$. Figure 22 shows the changes of the total outputs of the two cases as θ changes. When $\theta < 0.066$, the output is higher in the two-college case; and when $\theta > 0.066$, the output is higher in the one-college case. This example implies that given a constant total resources, the optimal number of colleges is weakly decreasing as θ rises.

3.3.4 Students' Welfare

The last point that concerns us is students' welfare. We do not provide any general conclusions, but a numerical example to illustrate the effects of the two

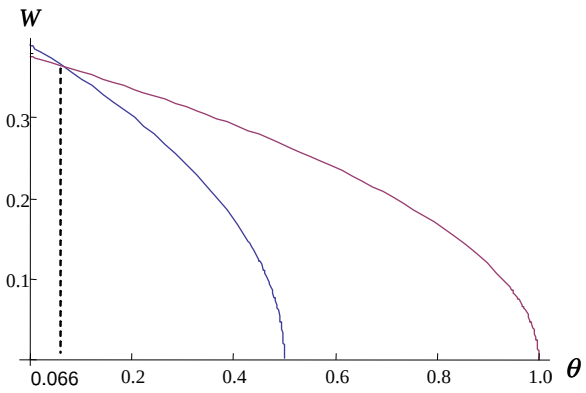

Figure 22: Outputs vs fixed costs.

systems on students' welfare. Consider the case where $u_0 = 0.12$. Our question is whether or not all students are better off if the planner sets up a new college. A particular student with ability x is better off if $u_1 < u_2$, where u_j denotes his utility in the j-college case, $j = 1, 2$. Let k_1 denote the college with the low task level and k_2 denote the college with the high task level. Those students who attend k_1 rather than taking the outside option must be better off and the bottom type of k_1 is indifferent between k_1 and the outside option. For other students in k_1, who attended the college in the one-college case, their utilities from the college in the one-college system are

$$u_1 = U(x, 0.495) = 0.495 - \frac{0.495^2}{x},$$

and their utilities from k_1 in the two-college system is

$$u_2 = U(x, 0.209) = 0.209 - \frac{0.209^2}{x}.$$

When $x < 0.704$, $u_1 < u_2$, and hence all the types between 0.491 and 0.704 are better off; when $x > 0.704$, $u_1 > u_2$, and hence those whose ability is between 0.704 and 0.734 are worse off. On the other hand, if x chooses k_2, then his utility from attending k_2 is

$$u_2 = U(x, 0.525) = 0.525 - \frac{0.525^2}{x}.$$

$u_1 > u_2$ for all $x \in [0.734, 1]$.[55] Therefore, all the types in k_2 is worse off compared to the one-college case. The reason that those students whose abilities are between 0.734 and 1 choose k_2 instead k_1 is the task level of k_1 is too low for them. However, the utilities gained from k_2 is less than the utility from the college in the one-college case because the task level of k_2 is too high.

Remark 4 *A system with more colleges does not necessarily lead to higher utilities for all students.*

3.4 Conclusion

This chapter presented a framework for sorting students and allocating resources to colleges from a planner's point of view. The intuition comes from the French higher education admission system. Without any entry requirements, how do students sort themselves into different colleges with various education qualities? Do they all choose to go to the college with the highest quality? This piece of work tries to build up a model to analyse these questions by introducing the cost of completing a specific task into the model. In the French higher education admission system, different quality colleges have different levels of performance requirements which can be treated as "Tasks" in our model. Students have to take the cost into account when they choose colleges. Although there is no entry requirement, students will sort themselves by maximizing their utilities. There are undoubtedly many other factors that may affect a student's utility, such as the interest rate, tuition fee, distance to the school, living cost, financial constraints, and so on. Indeed, all benefits of attending a particular school could be included into the gains from that school, while all disadvantages could be included in the costs. However, that will make the model asymmetric and too complicated. Therefore, this model abstracted from such factors. These asymmetric factors combining with incomplete information could explain why the central planner has difficulty in controlling the system and why some universities in Paris are overcrowded.

Our work simply assumes a student's gain from college only depends on the task level, which is similar to the degree of a university, and the cost of completing the task is determined by the task level and the student's ability. An important assumption about the cost function is submodularity between task level and ability. A student selects a college or the outside option that maximises his utility.

[55] Note that 0.491 and 0.734 are the lower marginal types at each college respectively. See Table 3.

We defined the concept of positive sorting as students sort themselves into different levels of colleges according to their abilities. The higher ability types go to the higher level colleges. We showed that given a positive ranked colleges by task levels, students will sort themselves positively.

The planner's objective is to maximise total output which is the aggregate of all individual students' outputs. The crucial assumption is supermodularity between arguments in production function. We derived the equilibrium in the simple models of one college and two colleges, and the results can be translated to an N college case. At the optimum, the planner has maximised the total output by designing task levels and allocation of resources and students have maximised utilities as well. We use a specific utility and production function to illuminate our model. We concluded that the optimal number of colleges is not necessarily infinite even if there are no fixed costs. When there are fixed costs, the planner has to balance the benefit of the new college against the reduction of the allocable resources. The optimal number is determined by total resources, fixed costs and the outside option. An increase in the total output does not necessarily mean all students are better off from a system with more colleges.

This model is rather simple and may miss out some other important factors. Nevertheless, it captures a number of key features about such a sorting problem where one side of the market has to incur certain costs in order to accomplish the qualification provided by the other side. It explains why high ability students are more willing to attend a higher quality institution than low ability students. Also, our model offers an explanation for a tiered system, and why higher quality universities are allocated more resources.

3.5 Appendix

Proof of Proposition 13:

Proof. Given the utility function, $U(x,T)$, the first order condition for maximizing utility function with respect to T is

$$U_2(x,T^m)=0.$$

By taking the first derivative with respect to x on both sides of $U_2(x,T^m)=0$, we have

$$U_{21}(x,T^m)+U_{22}(x,T^m)\frac{dT^m}{dx}=0.$$

Thus, $\frac{dT^m}{dx}=-\frac{U_{21}(x,T^m)}{U_{22}(x,T^m)}$. Since $U_{21}(x,T^m)>0$ and $U_{22}(x,T^m)<0$, then $\frac{dT^m}{dx}>0$. ■

Proof of Proposition 14:

Proof. For any college j, which is not empty, consider the highest ability and the lowest ability, denoted by $x_{j,l}$ and $x_{j,h}$. In order to prove the equilibrium sorting is positive, it is sufficient to show that any type in college $j+1$ (if $j+1$ is not empty and $j\leq N-1$), denoted by $X(j+1)$, are higher than $x_{j,h}$ and any type in college $j-1$ (if $j-1$ is not empty and it denotes the outside option if $j=1$), denoted by $X(j-1)$, is lower than $x_{j,l}$.

If positive sorting fails, then $\exists\ x_k\in X(j+1)$, with $x_k<x_{j,h}$ or $\exists\ x_k\in X(j-1)$, with $x_k>x_{j,l}$. In the first case, as type x_k selects $j+1$, $U(x_k,T_j)\leq U(x_k,T_{j+1})$; type $x_{j,h}$ selects j, then $U(x_{j,h},T_j)\geq U(x_{j,h},T_{j+1})$. Therefore we have

$$U(x_{j,h},T_{j+1})-U(x_{j,h},T_j)\leq U(x_k,T_{j+1})-U(x_k,T_j),$$

which contradicts the assumption of supermodular utility given $x_k<x_{j,h}$, hence type x_k will not choose college $j+1$ if $x_k<x_{j,h}$. In the second case, $\exists\ x_k\in j-1>x_{j,l}$, which means for type x_k, $U(x_k,T_j)\leq U(x_k,T_{j-1})$; for type $x_{j,l}$, $U(x_{j,l},T_j)\geq U(x_{j,l},T_{j-1})$, therefore we have

$$U(x_k,T_j)-U(x_k,T_{j-1})\leq U(x_{j,l},T_j)-U(x_{j,l},T_{j-1}),$$

which again contradicts the assumption of supermodular utility given $x_k>x_{j,l}$, hence type x_k will not choose college $j-1$ if $x_k>x_{j,l}$.

Therefore, in equilibrium, the result is positive sorting. ■

Proof of Proposition 15:

Proof. Suppose that the highest type in college $j-1$ is different from the lowest type in the adjacent higher level college, say college t, i.e., $x_{j-1,h} \neq x_{j,l}$. There must exist a type z such that $x_{j-1,h} < z < x_{j,l}$ and does not choose college $j-1$ or college j because we have assumed $x_{j-1,h}$ and $x_{j,l}$ are the highest and lowest marginal type in each college respectively. If type z selects college $j+1$ or even a higher level college, say j^+, then we have $U(z, T_j) \leq U(z, T_{j^+})$; but since $x_{j,l}$ has chosen j, so $U(x_{j,l}, T_j) \geq U(x_{j,l}, T_{j^+})$; thus

$$U(x_{j,l}, T_{j^+}) - U(x_{j,l}, T_j) \leq U(z, T_{j^+}) - U(z, T_j),$$

which contradicts the assumption of supermodular utility given $z < x_{j,l}$. If type z selects $j-2$ ($j \geq 2$) or even a lower level college or the outside option, say $(j-1)^-$, then we have $U(z, T_{j-1}) \leq U\left(z, T_{(j-1)^-}\right)$; for type $x_{j-1,h}$, $U\left(x_{j-1,h}, T_{(j-1)^-}\right) \leq U(x_{j-1,h}, T_{j-1})$; thus

$$U(z, T_{j-1}) - U\left(z, T_{(j-1)^-}\right) \leq U(x_{j-1,h}, T_{j-1}) - U\left(x_{j-1,h}, T_{(j-1)^-}\right),$$

which contradicts the assumption of supermodular utility given $x_{j-1,h} < z$. Therefore, such a type of z that $x_{j-1,h} < z < x_{j,l}$ does not exist, hence

$$x_{j-1,h} = x_{j,l};$$

i.e., there are no overlaps or gaps between any two adjacent colleges. ■

Proof of Proposition 16:

Proof. Without loss of generality, suppose $j < N$. Define the total output of k_j as $h(x_j, x_{j+1}, r_j, T_j) = \int_{x_j}^{x_{j+1}} q(x, r_j, T_j)\, dx$; the total output of k_i as $h(x_i, x_{i+1}, r_i, T_i) = \int_{x_i}^{x_{i+1}} q(x, r_i, T_i)\, dx$. Proposition 14 implies that $x_j, x_{j+1} > x_i, x_{i+1}$ given $T_j > T_i$.

Suppose k_j has a lower resources per student than k_i, i.e., $r_j < r_i$, then $R_j < R_i$ as $w_j \leq w_i$ at the equilibrium. The total output of the two colleges is written as

$$\begin{aligned} W &= h(x_j, x_{j+1}, r_j, T_j) + h(x_i, x_{i+1}, r_i, T_i) \\ &= h\left(x_j, x_{j+1}, \frac{R_j}{w_j}, T_j\right) + h\left(x_i, x_{i+1}, \frac{R_i}{w_i}, T_i\right). \end{aligned}$$

Now consider a positive transfer of resources from k_i to k_j. Since a student's decision does not depend on the allocation of resources, the marginal types would not alter if the planner only changed the allocation of resources and kept the same task levels. Therefore, marginal types at k_i and k_j would not change if there is such a transfer. Therefore, the total output after the swap is

$$W' = h\left(x_j, x_{j+1}, \frac{R_j + \varepsilon}{w_j}, T_j\right) + h\left(x_i, x_{i+1}, \frac{R_i - \varepsilon}{w_i}, T_i\right).$$

By the fact of supermodularity and $\frac{1}{w_j} \geq \frac{1}{w_i}$, we have the following inequality:

$$\frac{\partial W'}{\partial \varepsilon} = \frac{1}{w_j} h_3\left(x_j, x_{j+1}, \frac{R_j + \varepsilon}{w_j}, T_j\right) - \frac{1}{w_i} h_3\left(x_i, x_{i+1}, \frac{R_i - \varepsilon}{w_i}, T_i\right) > 0.$$

So, it is optimal to set $\varepsilon > 0$, which contradicts the optimality of the proposed solution. Therefore, $r_i < r_j$ and $R_i < R_j$. ■

Proof of Proposition 17:

Proof. Consider the marginal type x_1. Recall $V(0) = C(x_1, 0) = 0$, $V_{11}(T) \leq 0$, $C_{22}(x, T) > 0$, and hence $U_{22} < 0$. Then there must be two positive solutions (identical or different) for the following function:

$$V(T) - C(x_1, T) = u_0.$$

Let $T_1 \leq T_2$ denote the two solutions. Therefore, $U_2(x_1, T) \geq 0$ at T_1 and $U_2(x_1, T) \leq 0$ at T_2, where strict inequalities hold if $T_1 < T_2$. The optimal task level should be the identical solution or one of the two different solutions.

Since the marginal type is the same for the two task levels,

$$\int_{x_1}^{1} q\left(x, \frac{R}{1 - x_1}, T_2\right) dx \geq \int_{x_1}^{1} q\left(x, \frac{R}{1 - x_1}, T_1\right) dx,$$

where strict inequalities hold if $T_1 < T_2$. Therefore, the planner will choose T_2. Thus, $T^* = T_2$, and hence $U_2(x_1, T) \leq 0$ at T^*, where strict inequalities hold if $T_1 < T_2$.

At the optimal task level, T^*, taking the first derivative on both sides of the following equation:

$$U(x_1, T) = u_0,$$

with respect to T gives

$$U_1(x_1, T)\frac{\partial x_1}{\partial T} + U_2(x_1, T) = 0.$$

Therefore, at T^*,

$$g_T = \frac{\partial x_1}{\partial T} = -\frac{U_2(x_1, T)}{U_1(x_1, T)}.$$

Since $U_1(x_1, T) > 0$, then $g_T \geq 0$. Therefore, x_1 rises given any increase in T^*.[56] ■

Proof of Claim 15:

Proof. The first derivative of W with respect to T is

$$W_T = W_2(T, x_1)\frac{\partial x_1}{\partial T} + W_1(T, x_1),$$

where

$$W_2(T, x_1) = -q(x_1, r, T) + \frac{R}{(1-x_1)^2}\int_{x_1}^{1} q_2(x, r, T)\, dx.$$

Hence

$$W_T = \left[-q(x_1, r, T) + \frac{R}{(1-x_1)^2}\int_{x_1}^{1} q_2(x, r, T)\, dx\right]\frac{\partial x_1}{\partial T} + \int_{x_1}^{1} q_3(x, r, T)\, dx.$$

F.O.C. sets $W_T = 0$, and as in equilibrium $x_1 = g(T, u_0)$, it can be rearranged as

$$-q(x_1, r, T)\, g_1(T, u_0) + \left[\frac{R}{(1-x_1)^2}\int_{x_1}^{1} q_2 dx\right] g_1(T, u_0) + \int_{x_1}^{1} q_3 dx = 0,$$

where q_2 denotes $q_2(x_1, r, T)$, q_3 denotes $q_3(x_1, r, T)$ and $r = \frac{R}{1-x_1}$. ■

Proof of Proposition 18:

Proof. In the equilibrium, since the top marginal type in the lower level college is identical to the bottom marginal type in the higher level college, the marginal students in college 1 and college 2 must have the same utility, $U(x_2, T_1) = U(x_2, T_2)$. In Figure 23, T_1 and T_2 must be on two sides of the task level that induces the maximal utility for x_2.

[56] If $g_T = 0$, then $V_T - C_2(x_1, T) = 0$ at T^*, which means type x_1's utility is maximised. Therefore, given any changes of T^*, type x_1 will leave the college and the bottom ability will rise.

Now consider a marginal increase from T_1 to T_1': the utility of the top type student in college 1 is higher than the utility of the bottom type student in college 2, i.e., $U(x_2, T_1') > U(x_2, T_2)$, by the single peakedness of student's utility. Thus the bottom marginal student in college 2 will switch to college 1, hence the bottom type in college 2 rises, which implies $g_1^2(T_1, T_2) > 0$.

By the same logic, a marginal increase in T_2 will lead to $U(x_2, T_2') < U(x_2, T_1)$, which implies that the increase squeezes the bottom marginal student in college 2 out to college 1, i.e., $g_2^2(T_1, T_2) > 0$. ■

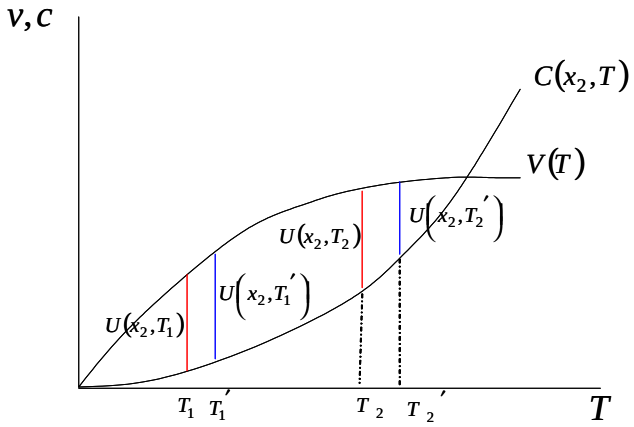

Figure 23: Effects on x_2.

Proof of Proposition 19:

Proof. a. If we found a two-college system that replicates the output of the optimal one-college system, then part a of this proposition would have been proved. The planner can simply set up a new college with zero task level, keep the college in the one-college system the same and allocate all the resources to the original college. The total output is the same as before, $W_2 = W_1^*$. Note that W_2 needs not to be maximised. The planner is able to change the task levels of the two colleges as well as the allocation of resources to maximise the total output. Therefore, $W_2^* \geq W_2 = W_1^*$.

b. Denote the optimal task level in the one-college system by T^* and the task level that maximises the utility of the highest type by T_t^*. If $T^* < T_t^*$, then $U_2(x, T)$ must be positive at T^* for some types whose abilities are higher than x_1, which is the bottom type in the one-college case. Suppose the planner sets up a new college with task level T' and $T^* < T' < T_t^*$. For those types,

$U(x, T^*) < U(x, T')$, and hence they will go to the new college. Denote the lowest type of those going to the new college by x_2. Note that x_1 does not changed because T^* keeps the same.

Resources per student in the one-college case is $r = \frac{R}{1-x_1}$, and the resources per student in the new system is $r_1 = \frac{pR}{x_2-x_1}$, $r_2 = \frac{(1-p)R}{1-x_2}$ for each college respectively. If $p = \frac{x_2-x_1}{1-x_2}$, then $r_1 = r_2 = r$. Since the output of a particular student is increasing in the task level given the same resources per capita, we obtain

$$\int_{x_2}^{1} q(x, r, T^*)\, dx < \int_{x_2}^{1} q(x, r, T')\, dx.$$

Hence

$$\int_{x_1}^{x_2} q(x, r, T^*)\, dx + \int_{x_2}^{1} q(x, r, T^*)\, dx < \int_{x_1}^{x_2} q(x, r, T^*)\, dx + \int_{x_2}^{1} q(x, r, T')\, dx,$$

$$\int_{x_1}^{1} q(x, r, T^*)\, dx < \int_{x_1}^{x_2} q(x, r, T^*)\, dx + \int_{x_2}^{1} q(x, r, T')\, dx.$$

Therefore, $W_2 > W_1^*$, where W_2 denotes total output of the new system. Note that T^* and T' need not to be the optimal task levels, so the $W_2^* \geq W_2 > W_1^*$. ■

Proof of Proposition 20:

Proof. When $u_0 = 0$, the output of the one-college system is 0.437. The total output of the two-college system approaches 0.437, which is the maximum, when T_1 goes to zero and T_2 stays at the optimal task level of the one-college system. Therefore, the two-college system can not do better than the one-college system when $u_0 = 0$.

When $0 < u_0 < 0.25$, there are two possible circumstances.

First, $U_T = 0$ at T^* for the bottom type. T^* must be lower than the task level that maximises the utility of the highest type, otherwise the college in the one-college system would be empty. Proposition 19 has shown that the two-college system produces more total output than the one-college system in this case.

Second, $U_2(x_1, T) \neq 0$ at T^* for the bottom type. There must be two positive solutions for T to the following function:

$$U(x_1, T) = u_0.$$

One of the solutions is T^*, and denote the other one by T'. At the equilibrium $U(x_1, T') = U(x_1, T^*)$. By using Proposition 17 and Figure 14, we know that

$T' < T^*$ and $U_2(x_1, T) < 0$ at T^* and $U_2(x_1, T) > 0$ at T' for type x_1. Suppose the planner sets up a new college with task level T'. Consider those students whose abilities are higher than the bottom type, $x > x_1$. The supermodularity of the utility function, $U_{12}(x, T) > 0$, immediately implies $U(x, T') < U(x, T^*)$. Therefore, these types would stay in the college with T^*. Consider those students whose abilities are lower than the bottom type, $x < x_1$. They will not go to the college with T^* and T' because $U(x, T^*) < U(x_1, T^*) = u_0$ and $U(x, T') < U(x_1, T') = u_0$. In the end, the new college does not recruit any students.

Now suppose the planner changes the task level of the new college from T' to T_1 and $T_1 = T' + \epsilon$, where ϵ is a positive number, and allocate a small amount of resources to that college. Our aim is to show the total output of the new system is increasing as ϵ rises from zero. In other words, the first derivative of the total output in terms of ϵ is positive when ϵ goes to zero.

Let x_1' denote the bottom type of the new college. The bottom type of the original college is $x_2 = T_1 + T^* = T' + T^* + \epsilon$. From the function $T - \frac{T^2}{x_1} = u_0$, we obtain $T' + T^* = x_1$, and hence $x_2 = x_1 + \epsilon$. Therefore, the total output of the new system is

$$\begin{aligned} W_2 &= \int_{x_1'}^{x_1+\epsilon} \left(x \frac{p}{x_1 + \epsilon - x_1'} \right)^{0.5} (x_1 - T^* + \epsilon)^{0.5} \, dx \\ &\quad + \int_{x_1+\epsilon}^{1} \left(x \frac{1-p}{1 - x_1 - \epsilon} \right)^{0.5} (T^*)^{0.5} \, dx \\ &= \frac{1}{1.5} \left(\frac{p}{x_1 + \epsilon - x_1'} (x_1 - T^* + \epsilon) \right)^{0.5} \left((x_1 + \epsilon)^{1.5} - (x')^{1.5} \right) \\ &\quad + \frac{1}{1.5} \left(\frac{1-p}{1 - x_1 - \epsilon} T^* \right)^{0.5} \left(1 - (x_1 + \epsilon)^{1.5} \right). \end{aligned}$$

Together with

$$\begin{aligned} x_1' &= \frac{T_1^2}{T_1 - u_0} \\ &= \frac{(x_1 - T^* + \epsilon)^2}{x_1 - T^* + \epsilon - u_0}, \end{aligned}$$

we obtain the first derivative of W_2 with respect to ϵ

$$\begin{aligned}
\frac{\partial W_2}{\partial \epsilon} &= -1.5\left(T^*\right)^{0.5}\left(\frac{1-p}{1-x_1-\epsilon}\right)^{0.5}\left(x_1+\epsilon\right)^{0.5} \\
&+\frac{0.5\left(T^*\right)^{0.5}(1-p)\left(1-\left(x_1+\epsilon\right)^{1.5}\right)}{(1-p)^{0.5}\left(1-x_1-\epsilon\right)^{1.5}} \\
&+\frac{0.5\left(\left(x_1+\epsilon\right)^{1.5}-\left(x_1'\right)^{1.5}\right)\left(\frac{p}{x_1+\epsilon-x_1'}\right)^{0.5}}{\left(x_1-T^*+\epsilon\right)^{0.5}} \\
&-\frac{0.5p\left(x_1-T^*+\epsilon\right)^{0.5}\left(\left(x_1+\epsilon\right)^{1.5}-\left(x_1'\right)^{1.5}\right)\left(\begin{array}{c}1+\frac{\left(x_1-T^*+\epsilon\right)^2}{\left(x_1-T^*+\epsilon-u_0\right)^2} \\ -\frac{2\left(x_1-T^*+\epsilon\right)}{\left(x_1-T^*+\epsilon-u_0\right)}\end{array}\right)}{(p)^{0.5}\left(x_1+\epsilon-x_1'\right)^{1.5}} \\
&+1.5\left(x_1-T^*+\epsilon\right)^{0.5}\left(\frac{p}{x_1+\epsilon-x_1'}\right)^{0.5} \\
&\left(\left(x_1+\epsilon\right)^{0.5}-\left(x_1'\right)^{0.5}\left(\frac{2\left(x_1-T^*+\epsilon\right)}{\left(x_1-T^*+\epsilon-u_0\right)}-\frac{\left(x_1-T^*+\epsilon\right)^2}{\left(x_1-T^*+\epsilon-u_0\right)^2}\right)\right).
\end{aligned}$$

When ϵ goes to zero, x_1' approaches x_1. Therefore,

$$\begin{aligned}
\lim_{\epsilon\to 0}\left[\frac{\partial W_2}{\partial \epsilon}\right] &= -1.5\left(\frac{x_1T^*(1-p)}{1-x_1}\right)^{0.5}+\frac{0.5\left(T^*\right)^{0.5}(1-p)\left(1-\left(x_1\right)^{1.5}\right)}{(1-p)^{0.5}\left(1-x_1\right)^{1.5}} \\
&+\lim_{\epsilon\to 0}\left[A+B+C\right].
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{0.5\left(\left(x_1+\epsilon\right)^{1.5}-\left(x_1\right)^{1.5}\right)\left(\frac{p}{\epsilon}\right)^{0.5}}{\left(x_1-T^*\right)^{0.5}}, \\
B &= -\frac{0.5p\left(x_1-T^*\right)^{0.5}\left(\left(x_1+\epsilon\right)^{1.5}-\left(x_1\right)^{1.5}\right)\left(1+\frac{\left(x_1-T^*\right)^2}{\left(x_1-T^*-u_0\right)^2}-\frac{2\left(x_1-T^*\right)}{\left(x_1-T^*-u_0\right)}\right)}{(p)^{0.5}(\epsilon)^{1.5}} \\
C &= 5\left(x_1-T^*\right)^{0.5}\left(\frac{p}{\epsilon}\right)^{0.5}\left(x_1\right)^{0.5}\left(\left(\frac{2\left(x_1-T^*\right)}{\left(x_1-T^*-u_0\right)}-\frac{\left(x_1-T^*\right)^2}{\left(x_1-T^*-u_0\right)^2}\right)\right).
\end{aligned}$$

$\lim_{\epsilon\to 0}\left[A+B+C\right]$ is determined by term C. The limit of the last term in the square brackets is infinity and the sign is directed by the sign of the following expression

$$\frac{\left(x_1p\right)^{0.5}u^2\left(x_1-T^*\right)^{0.5}}{\left(x_1-T^*-u_0\right)^2},$$

which is positive.

Therefore, W_2 is increasing as ϵ goes up from zero, and hence W_2 is greater

than W_1^* for some positive ϵ. Once again, W_2 is not necessarily the maximal output. The planner is able to find the optimal T_1^* and T_2^* to maximise the total output. ■

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