THE UNIVERSITY of EDINBURGH

Essays on Educational Investment, Income Inequality and Income Mobility

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Doctor of Philosophy in Economics

The University of Edinburgh

2013

Abstract

The first two pieces of work in this thesis look into the strategic decision of inter-generational educational investment and its implication to income inequality, skill distribution and income mobility. The contribution of my work is to incorporate matching frictions into the marriage market and analyze returns from strategic educational investments. The mechanism in the marriage market adopted follows the spirit of the competitive search model which interprets the 'mismatch' phenomenon as the result of coordination frictions in the matching process. The competitiveness and frictions in the family formation process create decreasing returns to high educational investment. The more parental households who choose high educational investment, the less is the return to high educational investment compared to the lower alternative. The fact that rich parental households suffer less from costly high educational investment puts the poor households at a disadvantage and the poor are more likely to be crowded out of the group that have incentives to choose high investment. The model predicts that given a certain parameter region, children of poor parents are more likely to become skilled if the fraction of rich parental households is not too large. In a multi-generational dynamic setting, it further implies the existence of a stationary household income distribution and income mobility rates. An increase in returns to education alone generates a larger stationary fraction of rich households and a larger upward income mobility rate. An increase in the cost of the high educational investment alone generates a smaller stationary fraction of rich households and a smaller upward income mobility rate.

The third piece of work looks into the strategic interaction between passenger carriers over product quality and the location choice in a duopoly scheduled flight market. The model predicts that the two carriers prefer to be specialized in different flight quality (non-stop vs. one-stop) and adopt the same schedule when a higher quality difference makes the consumers less sensitive to the flight frequency. It contributes to literatures on the application of two-dimensional product differentiation in air-travel market analysis.

Acknowledgements

First and foremost I would like to express my deep and sincere gratitude to my supervisors Prof. José V. Rodríguez Mora and Prof. Maia Güell for their great supervision and thoughtful guidance during the past four years. During my PhD studies, they have led me to inspiring research areas, given me constructive advice in modeling and encouraged me to go through the difficulties I met. Their supervision also deepened my understanding of the topics and broadened my horizons.

I am very grateful to Prof. Ed Hopkins for his valuable suggestions, guidance and encouraging comments. I have learned a lot from PhD reading group, from how to value the research work, to how to write a proper paper. I was lucky to have the chance of discussing papers related to my topic in the reading group, which is really helpful in disentangling my puzzles.

I also would like to express my sincere thanks to Dr. Simon Clark and Dr. Kohei Kawamura for their very helpful and important advice.

A special thank you goes to my beloved husband Junwei Fang. Without his support and understanding, I would not be able to go this far. Many Thanks to my parents who have been very supportive and helped me go through the hardest period.

Thanks to my colleagues, Jonathan Spiteri, Rebecca Piggott, Nicholas Myers, Euan Limmack and Petal Hackett, for their helpful comments on my thesis. Thanks to School of Economics for giving me an excellent opportunity to study economics.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own. No other person's work has been used without due acknowledgement. This thesis has not been submitted for any other degree or professional qualification.

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Chapter 1

Introduction and Overview

This thesis provides an insight into the role played by intergenerational educational investment and family formation in the intergenerational transmission of economic status. The next two chapters construct a theoretical framework that investigates the competition in educational investment among parental households and its implications for the interaction between education cost, income inequality and income mobility. The final chapter explores product differentiation of industrial economics. By investigating a two-dimensional product differentiation model in the air-travel market, it sheds some light on the strategic interaction between a hub carrier and a low cost carrier over flight schedule arrangement and flight quality choice.

1.1 Pre-match Investment Competition, Coordination Frictions and Household Income Distribution Over Generations

Economists have done extensive work on income inequality. The research focus has been extended from short-run income inequality to long-run income mobility in recent decades. This tendency partly stems from the notion that opportunity equality is more important than distribution equality at any given time point. Benabou and Ok (2001) argued that individuals living in a society characterized by a great deal of intergenerational income mobility are more tolerant of existing income inequality than those living in a society with very little intergenerational mobility. More recently, research on income dynamics has focused not only on gauging the distributional impact of income changes, but also on the nature and origin of changes in economic well-being (Fields et al, 2007).

Becker and Tomes (1979, 1986) and Becker (1981) predict that intergenerational mobility is affected by the propensity to invest in children, by the degree of inheritability of endowments, and by capital market constraints that limit the ability of families to make self-financing investments. Following this, it is widely agreed that mobility depends essentially on investment (Gall, Legros, Newman, 2009). Among various forms of investment, education is viewed as a major channel of upward intergenerational income mobility in the modern society. Besides financial returns from labor income generated by educational investment, other returns from schooling has been explored in recent research. One of these returns stems from the marriage market, where individuals with better educational attainment are more attractive and are more likely to be matched with a well educated partner (Chiappori, Iyigunm and Weiss, 2009).

Becker (1973, 1974) and the extensions (Becker, 1981; Lam, 1988) first incorporate the notion of assortative mating on spouse's traits into intergenerational mobility analysis. They consider the marriage market in a frictionless environment and the utility gained from a match surplus is transferable[1].

The debate on whether marriage market should be modeled in a transferable framework or a non-transferable framework is still undergoing. Generally speaking, the transferable model is practical in the analysis of marital transfers such as dowry and bride-price which widely exist in traditional societies[2].

Meanwhile, the non-transferable marriage model is used in the studies on savings behavior by Cole and Mailath (1992), sorting in the marriage market with frictions by Smith (2006) and so on. The model in this thesis follows the non-transferable framework in the marriage market, where both parties in a marriage enjoy an equal share of household production.

Though matching in reality presents a strong tendency towards assortative matching, the 'mismatch'[3] phenomenon is also widely observed. This model incorporates coordination frictions into the matching process to capture this phenomenon and the friction becomes one of the major sources of the variation in the intergenerational income distribution. The coordination frictions are embed-

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[1]The concepts: transferable and non-transferable, are commonly used in matching literatures, e.g. see the definition in the work by Smith (1992). Generally speaking, if the intra-match allocation of welfare is determined exogenously or with a fixed sharing rule, matching is with non-transferable utility so that a person cannot "compensate" a potential partner for marrying him or her despite some negative traits; if the intra-match allocation of welfare is negotiable, matching is with transferable utility.

[2]See related work by Rao (1993), Botticini and Siow (2002) etc.

[3]'mismatch' in this thesis refers to a match consisting of a skilled individual and an unskilled individual.

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ded in a directed search (matching) model, which was initially used to model the labor market. It is also called competitive search model following the seminal work by Moen (1997) and Shimer (1996). A nascent version of this model can be traced to the urn-ball matching process analyzed in the work of Montgomery (1991) and Peters (1984, 1991).

Here is an illustration of this matching mechanism. Suppose there are m boys who can make proposals to f girls. Their proposals arrive one by one into girls' mailboxes. If each boy proposes to each girl with an equal probability, any girl gets at least one proposal with probability $1 - (1 - \frac{1}{f})^m$. Taking the limit of this expression as m and f go to infinity with $q = \frac{m}{f}$ held constant, a fraction $1 - e^{-q}$ of girls get at least one proposal in the large marriage market. Hence, even $m = f$, there are some girls not receiving any proposal while some girls receive multiple proposals. In this model, the urn-ball matching process is applied to the first round matching with heterogeneous boys and girls who have different educational attainment. Suppose there are m_1 skilled boys (type 1) and m_2 unskilled boys (type 2). On the other side, there are f_1 skilled girls (type 1) and f_2 unskilled girls (type 2). The number of type 1 boys who choose to propose to a type 1 girl is m_{11}, and the number of type 1 boys who choose to propose to a type 2 girl is m_{12}, where $m_{11} + m_{12} = m_1$. Similarly, the number of type 2 boys who choose to propose to a type 1 girl is m_{21}, and the number of type 2 boys who choose to propose to a type 2 girl is m_{22}, where $m_{21} + m_{22} = m_2$. Girls choose type 1 boys over type 2 boys when receiving proposals from both. Hence a type 1 girl matches with a type 1 boy with probability $1 - e^{\frac{m_{11}}{f_1}}$, and with a type 2 boy with probability $1 - e^{\frac{m_{12}}{f_2}}$. A type 2 girl matches with a type 1 boy with probability $e^{\frac{m_{11}}{f_1}}(1 - e^{\frac{m_{21}}{f_2}})$, and with a type 2 boy with probability $e^{\frac{m_{12}}{f_2}}(1 - e^{\frac{m_{22}}{f_2}})$. Obviously, type 1 individuals have an advantage in the marriage market both in terms of matching quality and matching probability.

Foreseeing the advantages of being a type 1 individual in the matching process, altruistic parents have the incentive to make high educational investment to produce type 1 offspring at the cost of their own consumption. Strategic parental investment in this model endogenizes the type distribution of the offspring's generation. This consideration follows a relatively young literature on strategic pre-match investment. It is called a "matching tournament" with one-sided educational investment in the labor market following the work by Hopkins (2007), or "pre-marital investment" with double-sided investment in the marriage market following the work by Peters and Siow (2002). It has been argued by Bhaskar and Hopkins (2010) that the double-sided pre-marital investment model, where

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investment decisions are made simultaneously with complete information and an expectation of an exogenous assortative matching outcome in the frictionless marriage market, can lead to multiple equilibria due to the strategic interaction between parental households with boys and these with girls. In this model, we assume that each household has one boy and one girl, and invests the same on both of their children regardless of their gender. As a result, it can be considered as an equivalence to the one-sided pre-match investment model. The strategic interaction occurs between the rich and the poor, but there is no strategic interaction based on the offspring's gender. Moreover, the action space for parents is discrete and limited to only two exogenous educational investment levels where costs are also exogenous, which reflects that education sectors are considered as a kind of quasi-market, for the prices and the standard of education are commonly influenced by public authorities (Le Grand and Bartlett, 1993). Strategic concerns in pre-match investment have also been used to account for competitive savings motives which affect the economic growth rate (Cole and Mailath 1992; Wei and Zhang, 2011), where the matching environment is frictionless and the matching outcome is assortative.

The contribution of this model is to examine the strategic thinking of pre-match educational investment with a frictional family formation process and its impact on the variation in income distribution and income mobility over generations. It suggests that the friction in the family formation process is one of the possible factors which can break the "circulation of the elites" (Pareto, 1971). The model also addresses the following research question: does intergenerational educational investment increase the advantage of the rich, which contributes to social stratification, or does it foster income mobility? The results of this model imply that an increase in the cost of high educational investment alone would decrease the equilibrium fraction of rich households and the upward income mobility; while an increase in the return to high educational investment alone would increase the equilibrium fraction of rich households and the upward income mobility. This result is consistent with previous studies on income inequality and income mobility[4].

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[4]See the work by Solon (2004), Corak (2006), D'Addio (2007), Andrews and Leigh (2009), Checchi, Ichino and Rustichini(1999), Hassler, Rodríguez Mora and Zeira(2007).

1.2 Product Line Competition with Vertical and Horizontal Differentiation

Another research interest of this thesis lies in two-dimensional product differentiation in the air-travel market. One-dimensional product differentiation models are typically divided into two categories: horizontal product differentiation (Hotelling, 1929), where consumers have different preferences over differentiated products that are priced at the marginal cost, and vertical product differentiation (Gabszewicz and Thisse, 1979; Shaked and Sutton, 1982), where consumers have the same preference over differentiated products that are priced at the marginal cost. Two-dimensional product differentiation models are addressed by Neven and Thisse (1990), Economides (1989, 1993), Tabuchi (1994), and Degryse(1996). Scheduled flights[5] embody multiple characteristics. Two characteristics are most important to consumers: departure times and flight duration. Hence, a two-dimensional product differentiation model better fits the competition format among passenger carriers than a one-dimensional framework.

My third piece of work attempts to analyze the following research question in the framework of horizontal and vertical product differentiation: how does a low quality service (one-stop flights) provider interact with a high quality service (non-stop flights) provider with regard to product line differentiation of the location and the quality dimension? The results shed some light on the competition format between a hub carrier, who may provide one-stop flight service or non-stop flight service, and a low-cost carrier, who always provides non-stop flight service according to its business model. With regard to this research question, Cento (2008) investigates the price competition between these two types of carriers who adopt complicated price schemes in various market structure, e.g. monopoly, symmetric duopoly, asymmetric duopoly, asymmetric oligopoly, etc. Dunn (2007) empirically examines the factors affecting the entry of non-stop services into 'rim' markets[6]. The results provide evidence on the cannibalization effect[7] between a carrier's own one-stop services and non-stop services, which deters the hub carrier's entry into non-stop services in the 'rim' markets. In contrast to these

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[5]The concept of a scheduled flight is in contrast to that of a charter flight. A scheduled flight, also known as public commercial flights leave at regular intervals with tickets being purchased up to the day of departure, while a charter flight is a private flight scheduled to meet the needs of specific passengers.

[6]The routes consist of two endpoint cities which are typically connected by one-stop flights operated by hub carriers.

[7]The hub carrier's non-stop flights exert a negative effect on the hub carrier's profits with respect to its one-stop flights, which is considered as a form of 'cannibalization'.

work, the model presented in Chapter 4 focuses on the vertical and horizontal product differentiation of flights. It does not consider any price discrimination, neither does it consider the possibility of the hub carrier simultaneously providing one-stop and non-stop flights. In other words, no competition between two types of services within one carrier is studied. Moreover, the model assumes a simple price scheme under which each carrier adopts a uniform price across its own product line. For simplification purposes, this model analyzes a duopoly market structure only, and analysis under other market structure could be an extension.

The flight schedule adopted by a carrier is considered to be the location of its product line (a series of flights), which is modeled by a series of points along a *salop circle*[8]. That is how the carriers differentiate their product line horizontally. This model follows Klemperer (1992), where product line's location is partially endogenized as in the form of only two polar strategies: head-to-head or interlaced. This simplification is due to the complexity of endogenizing location choices in product differentiation models. In addition, the model also endogenizes the quality choice in the form of two possibilities, one-stop or non-stop, which generates different flight durations for any given city-pair market. Flights possess two characteristics, so do the consumers. Consumers differ in their desired departure time and their valuation of the flight duration.

In general, the model set-up closely follows that of Degryse (1996) on banking industry, but differs in four key aspects. First, each of the duopolists provides a product line rather than a single product, so that the scale of product line matters in this analysis, which is reasonable since the frequency of flights (related to the scale of product line) matters in air-travel market. Second, duopolists in this model are allowed to choose their locations in two polar strategies: head-to-head or interlaced, while the locations are fixed and exogenous in Degryse's banking model. This flexibility of product location fits better with the scheduled flight market since choosing a schedule involves strategic thinking in carriers' competition. Third, the competition between the duopolists is asymmetric in the sense that one of them (the low-cost carrier) always provides high quality service (non-stop flights), so it only needs to make decisions on the location. The other player (the hub carrier), however, needs to make decisions on both the location and the quality of its services. Finally, the players move sequentially rather than simultaneously. This follows the conjecture that the hub carrier takes more time in making strategic decisions on product line competition since it needs to decide

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[8]Salop circle model is a variation of Hotelling location model. Originally, it examines consumer preference with regard to geographic location.

both the flight quality and the flight schedule while the low-cost carrier only needs to make decisions on the schedule.

1.3   Overview of the Thesis

This thesis is organized as follows. Chapter 2 constructs a theoretical framework of intergenerational educational investment and the variation in income distribution across two generations. It characterizes the equilibrium consisting of the parental households' investment strategies and the corresponding matching outcomes of their children. It also re-examines the competition of educational investment, which affects the variation in the households' income distribution. It implies that sorting in the family formation process should not worry policymakers from the intergenerational income mobility perspective if the costs of education investment allows enough incentives to make the high education investment.

Chapter 3 is an extension of Chapter 2. It presents a dynamic model, attempting to capture the features of intergenerational educational investment, the variation in income distribution and the mobility over many generations. In both the static and dynamic framework, given that all the households are poor in the first generation, there is an increase in the fraction of rich households from the initial generation to the offspring's generation and this fraction becomes stationary in dynamic settings. The increase is larger in the dynamic setting than that in the static setting, because when investing in education benefits many generations, the poor are better motivated to choose high educational investment. As a result, there is a larger equilibrium fraction of rich households in the dynamic model.

Chapter 4 studies a duopoly air-travel market between a city-pair. This chapter analyzes how a hub-spoke network carrier interact with a point-to-point network carrier with regard to product line arrangement and product quality choice. Conditions are generated under which the hub carrier competes with the low-cost carrier by providing non-stop flights along an interlaced schedule, or by providing one-stop flights at the same schedule adopted by the low-cost carrier. It also indicates that a larger quality difference makes the consumers less sensitive to flight frequency. Hence the two carriers prefer to be specialized in different flight qualities and adopt the same schedule.

Chapter 2

Pre-match Investment Competition and Coordination Frictions in the Marriage Market

Abstract

This chapter considers a competitive pre-match educational investment model. Educational investment before a match determines an individual's type, which is the sole quality valued by potential mates in the family formation process. The model incorporates a directed matching mechanism in the marriage market. Under the assumption of no gender bias in parents' investment behavior, a unique symmetric equilibrium can exist both in the matching stage and the investment stage. Matching can be mixed in equilibrium, as can parents' investment strategies. Hence, offspring from the same type of parental households may end up with different educational attainment and the same type of individuals may end up with different types of partners. Given that the disparity in educational investment costs and the inequality in feasible household income are constant, a smaller fraction of rich parental households leads to a higher propensity of poor parental households to choose the high educational investment.

Key Words: Pre-match investment, directed matching, income inequality

2.1 Introduction

Cross-country empirical studies in recent decades have observed that income inequality can be largely ascribed to the degree of skill dispersion of the labor force (Nickell, 2004). Educational investment that aims to increase human capital is viewed as a major channel of producing skilled individuals in modern society. Besides the understanding of financial returns from labor market generated by educational investments, economists are now paying more attention to other returns generated by schooling which may indirectly affect an individual's life-time economic status. One of these returns stems from the marriage market, where individuals with better educational attainment are more attractive and more likely to be matched with a better educated partner (Chiappori, Iyigunm and Weiss, 2009). This return is worth studying since it is a non-trivial factor in the educational investment decision making process and the family formation process, both of which play an important role in the evolution of income distribution and income mobility through generations.

In light of the theories of family economics, for which Becker (1973, 1974) and the extensions (Becker, 1981; Lam, 1988) provided an important foundation, the notion of assortative mating on spouse's traits has been incorporated into intergenerational income mobility analysis. Studies in other social science fields also provide evidence that marriage has represented one of the primary institutions by which social-economic mobility and social stratification took place in many countries and for centuries (Goody, 1983). However, matching is not perfectly assortative in reality. The literature on search and matching theories provides explanations on the 'mismatch' phenomenon[1], ascribing it to the randomness in the matching process, which is noted as matching frictions. Generally, two types of frictions have been paid special attention to. In the random matching framework, the matching process involves time-consuming search, which generates search frictions in the marriage market(Burdett and Melvyn, 1997; Shimer and Smith 2000). In the directed search (matching) framework, coordination frictions are captured in the matching process. A directed search (matching) mechanism has been applied in labor economics to wage dispersion studies where workers use mixed strategies when applying for jobs, and as a result, identical workers may get matched with different types of firms (Shi, 2001, 2002; Shimer, 2005). Compared to the popularity of arranged marriage in traditional societies, the

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[1]'mismatch' in this thesis refers to a match consisting of a skilled individual and an unskilled individual.

marriage market is generally decentralized in the modern society. However, there are indeed some platforms where people are able to partly control their matching process in order to get a better match. For example, thousands of dating web sites and agencies use various matching algorithms to create potential matches according to clients' characteristics. With this observation, a directed search model is suitable for modeling marriage market too, as it not only captures the frictions, but also the competitiveness in the matching process. In the following context, a directed matching mechanism is applied to establishing the first round matching in the marriage market, and a simple random matching mechanism is adopted in the second round to clear the marriage market. This set-up of the matching process is to some extent reasonable in reality, as people tend to be picky in selecting a mate when they are young, but "easy going" when they become older. Apart from an endogenous number of firms with the free entry assumption in the models of labor economics, we assume a fixed number of marriage market participants and a balanced sex ratio in each generation. Furthermore, any effect on the household's income imposed by the supply-demand conditions or strategic interaction between workers and firms modeled in the studies of labor market has been suppressed in this model. We simply assume an exogenous household production technology and a fixed sharing rule which will be specified in the following section.

Another important stream of literature related to this model is on strategic investment before match, referred to as a "matching tournament" with one-sided educational investment in the labor market following the work by Hopkins (2007), or "pre-marital investment" with double-sided investment in the marriage market following the work by Peters and Siow (2002). The educational investment made by parents on behalf of the children before they enter the marriage market generates two kinds of returns to altruistic parents. First, it enables children to be more productive in the labor market so as to earn a better income. Second, it makes the children more attractive in the marriage market with a better chance to form a high-income household. It has been argued by Bhaskar and Hopkins (2010) that the double-sided pre-marital investment model, where investment decision makers move simultaneously with complete information and an expectation of an exogenous assortative matching outcome in the frictionless marriage market, can lead to multiple equilibria due to the strategic interaction between parental households with boys and these with girls. In this model, under the assumption that each household makes an equal educational investment on behalf of their children, one boy and one girl, it can be considered equivalent to

the one-sided pre-match educational investment model. The strategic interaction exists between the rich and the poor, but there is no strategic interaction based on offspring's gender. Moreover, the action space for parental households is discrete and limited to two exogenous educational investment levels where costs are exogenously determined too. It is reasonable to have these simplifications. First, education sectors are considered as a kind of quasi-market, for the prices and the standard of education are commonly influenced by public authorities (Le Grand and Bartlett, 1993). Second, "costs of investment are often rounded to the smallest monetary denomination available (e.g. to the nearest penny)" in reality[2].

In summary, this chapter attempts to capture the strategic thinking of parents in making pre-match educational investment and its role played in the transmission of household income distribution. The propensity to invest in children is no longer an exogenous parameter as modeled by Becker (1979). Instead, it reflects the competition among parental households with considerations of the consequences of their investment strategies in the competitive marriage market, where their children are forming new households. It also tries to model the marriage market in modern society in a natural manner instead of a centralized frictionless world that is better suited to a model of pre-industrial society. By employing a directed matching mechanism, it is possible to focus on a mixed strategy equilibrium, which provides intrinsic rationale of different match outcomes for identical people other than any stochastic elements in the matching process. The induced investment strategies conducted by heterogeneous parental households can also be mixed under certain parameter regions. Accordingly, a mixed investment strategy in equilibrium becomes the main source of the variation in income distribution in this model. This is another departure from the previous literatures in the sense that the family line becomes scattered either due to the uncertainty in the investment strategies or in the matching outcomes or in both.

This chapter is organized as follows. Section 2 introduces a general model. Section 3 generates the equilibrium in the marriage market by incorporating a directed matching mechanism given any exogenous skill distribution of the participants. Section 4 traces back to the investment stage where parental households endogenize the skill distribution of their offspring by making educational investments given any exogenous household income distribution of their own ge-

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[2]See the similar assumption in the paper by Boudreau, J. W., "Sequential Pre-Marital Investment Games: Implications for Unemployment" (2008). Economics Working Papers. Paper 200845.

neration. Section 5 provides comparative statics, investigating the effects of the feasible household income inequality and education cost structure on equilibrium outcomes. Section 6 gives concluding remarks by generalizing the main findings on the variation of income distribution.

2.2   The Model

We consider an economy composed of a balanced population, where the size of the population is unchanged over time and the sex ratio equals one for every generation. The older generation consists of N households, each of which consists of two parents and two children, one boy and one girl. Parental households as the decision makers of intergenerational educational investment, are divided into two groups, one with a high household income, recognized as the rich, and one with a low household income, recognized as the poor. For simplicity, the educational investment options are limited to two discrete levels, denoted by the index $\chi \in \{1, 2\}$, both of which are available and affordable to all parental households. The cost of education, denoted by c_1 and c_2, where $c_1 > c_2$, are exogenously determined by other sectors in the economy, e.g. government's regulation. Parents decide whether to spend a higher amount on their children's education so as to develop them into skilled persons (type 1) with a better income prospect in future, or opt for a basic education, in which case their children (type 2) will be in the low income category. Apparently, parents can increase their consumption at the expense of their children's welfare, but they are discouraged from doing so since they are altruistic in a manner that their children's utilities are added into their own utilities. Assuming there is no capital accumulation over generations, parental households' maximizing behavior is effectively to choose an optimal educational investment level when they are confronted with the trade-off between their own consumption and their children's.

To make the matching model tractable, the following assumption has been made to ensure the symmetric educational attainment distribution between boys and girls in the marriage market:

Assumption 1: Every parental household chooses an equal educational investment level for each of the children, without gender bias.

This assumption holds when certain conditions are met in the model. Suppose the prospect of one's income level only depends on one's type (skilled if having

a high educational attainment or unskilled if otherwise), in other words, it does not vary across gender. Additionally the costs of educational investment do not vary according to gender. Under these conditions, it is reasonable to assume that altruistic parents would like to treat boys and girls equally in terms of educational investment to maximize the sum of utilities. On the contrary, if the returns to and/or the costs of educational investments differ by gender, this assumption will not hold[3].

This investment game consists of two stages: the investment stage where parental households make strategic investment decisions, and the matching stage where the young participants get matched in the marriage market. This model can be solved by backward induction and we start from the second stage.

2.3 The Matching Stage

The population of participants in the marriage market consists of an equal measure of boys and girls, denoted by N, where $N \to \infty$. According to assumption 1, educational attainment is symmetrically distributed across gender in the marriage market. A fraction v of all the children are skilled, as their parents have made the high educational investment for them. Use the subscripts $i, j \in \{1, 2\}$ to indicate boys' and girls' type respectively. Their type i or j, as a function of their educational attainment $\chi \in \{1, 2\}$ is described as: $i(\chi) = \chi$, $j(\chi) = \chi$, which indicates that the high educational investment produces a skilled individual (type 1) while the low educational investment produces an unskilled (type 2) individual. To keep the population size unchanged, we assume that everyone strictly prefers getting matched than being single, so that everyone will form a household at the end of the matching stage. Their preferences over potential mates are simplified to depend only on the potential household income following the match.

The household production technology adopted in this model is characterized as follows. If a couple consists of two type 1 individuals, the household will have a high income and is categorized as a rich household, enjoying a utility u_r, where $u(.)$ is the general utility function. $u(.)$ is concave and strictly increasing in its sole argument, the household income. If a couple consists of two type 2 individuals, the household will gain a low income, enjoying u_p and is categorized as the poor household. If a couple consists of different types of individuals, the household is equally likely to gain a high income or a low income (becoming rich

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[3]See the similar assumption made in the paper by Bhaskar and Hopkins(2010).

or poor) so that they would enjoy an expected utility $(u_r + u_p)/2$. Considering the population as a whole, it is equivalent to say, half of these the 'mismatched' couples are rich and half of them are poor.

The matching process adopted here takes place over two rounds. To capture the competitive feature in the marriage market, the first round matching follows the configuration of a directed matching model. To clear the marriage market, the second round matching is random for those left unmatched after the first round.

2.3.1 Directed Matching in the First Round

The first round matching mechanism is described as follows. First, the girls simultaneously post their types and selection criteria of boys. Then the boys decide which type of girls to propose to. Every boy proposes to one girl in this single round. If a girl receives multiple proposals from both types of boys, type 1 boys will be selected over type 2 boys. If a girl receives multiple proposals from the same type of boys, each boy has an equal probability of being selected. If she receives only one proposal, she will marry this proposer without further consideration. If no proposal arrives, she has to enter the second matching round. Matching frictions are captured by the failure of coordination among boys due to the large number of participants. Consequently a fraction of the participants will be left unmatched at the end of the first round and enter the second round where matching is random.

Before diving into the analysis of the first round matching, it would be helpful to get some intuition of the concerns of marriage market participants by considering the two rounds matching as a whole. In the first round, boys and girls are picky in the sense that they care about their potential partner's type, preferring a match that would generate a higher household income. On the other hand, they are also afraid of being embarrassed by the matching failure and entering the second round. To capture this, we let boys discount second round payoffs from getting matched to zero. From a boy's point of view in the first round, it is better to marry a type 1 girl, but he needs to consider the competition from other boys. Obviously, type 1 boys have an advantage over type 2 boys in the sense that they only have to be aware of the competition within their type, while type 2 boys have to take into consideration the competition both within their type and that from type 1 rivals. Specifically, after choosing the type of girls, an urn-ball matching process occurs where boys propose to a girl of the chosen type

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randomly by adopting mixed proposing strategies, which means each girl of this type is equally likely to receive a proposal. If there are too many boys choosing to propose to the type j girls, the chance of being selected by type j girls becomes low. To adopt an optimal strategy of choosing the girls' type, boys are confronted with a trade-off between the matching quality (a better mate) and the matching probability (possibly getting rejected in the first round and being left to the second round). While in the second round, matching is random. 'Who marries whom' only depends on the chance of meeting, which is simply determined by the distribution of the opposite gender.

The fraction of type 1 boys, as well as that of type 1 girls, is denoted by v, which is determined by parental households' joint investment strategies in the investment stage. Hence, v is considered as an exogenous variable in the matching stage. Additionally, v is also the fraction of parental households who choose high educational investment. Obviously, when $v = 1$, all the marriage market participants are skilled and all the households formed through the matching process would be rich under the assumed household production technology; when $v = 0$, all of the participants are unskilled and all the households formed up would be poor. We consider the general case where $v \in (0, 1)$.

We focus on the symmetric equilibrium where boys of the same type choose an identical strategy to decide which type of girls to propose to. Type 1 boys' strategy, either a pure or a mixed one, is denoted by a probability, $s_1 \in [0, 1]$, which measures type 1 boys' likelihood of proposing to type 1 girls. $1 - s_1$ denotes type 1 boys' likelihood of proposing to type 2 girls. In the same way, $s_2 \in [0, 1]$ measures type 2 boys' likelihood of proposing to type 1 girls, while $1 - s_2$ denotes their likelihood of proposing to type 2 girls. Take the population as a whole, s_i is also the fraction of type i boys who actually would make proposals to type 1 girls. $1 - s_i$ is the fraction for those who would make proposals to type 2 girls. Since any mixed strategy would not be adopted if there is a dominant pure strategy available, conditions are established that partition the parameter space using the criteria of whether it supports a mixed strategy or not, which will be discussed in the next subsection.

To calculate matching probabilities, it is necessary to cite an important concept used in search models of labor economics, the queue length, which is defined as the expected number of workers applying to a firm. Here, the queue length of

type i boys proposing to type j girls is defined by q_{ij} as follows:

$$q_{11} \;=\; s_1 vN/vN = s_1 \tag{2.1}$$

$$q_{12} \;=\; (1 - s_1)vN/(1 - v)N = (1 - s_1)v/(1 - v) \tag{2.2}$$

$$q_{21} \;=\; s_2(1 - v)N/vN = s_2(1 - v)/v \tag{2.3}$$

$$q_{22} \;=\; (1 - s_2)(1 - v)N/(1 - v)N = 1 - s_2 \tag{2.4}$$

Illustrated below is how one of the above expressions has been deduced. Others can be deduced in a similar way. Let's take q_{11} for an example. $s_1 vN$ is the number of type 1 boys who propose to type 1 girls, and vN is the total number of type 1 girls. With the queue length defined as above, the matching probabilities denoted by P_{ij} (with which a type i boy proposing to a type j girl will succeed in his endeavor) are defined as follows:

$$P_{11} \;=\; (1 - e^{-q_{11}})/q_{11} = \frac{(1 - e^{-s_1})}{s_1} \tag{2.5}$$

$$P_{12} \;=\; (1 - e^{-q_{12}})/q_{12} = \frac{(1 - v)(1 - e^{-\frac{(1-s_1)v}{(1-v)}})}{(1 - s_1)v} \tag{2.6}$$

$$P_{21} \;=\; e^{-q_{11}}(1 - e^{-q_{21}})/q_{21} = \frac{e^{-s_1}v(1 - e^{-\frac{s_2(1-v)}{v}})}{s_2(1 - v)} \tag{2.7}$$

$$P_{22} \;=\; e^{-q_{12}}(1 - e^{-q_{22}})/q_{22} = \frac{e^{-\frac{(1-s_1)v}{1-v}}(1 - e^{-(1-s_2)})}{1 - s_2} \tag{2.8}$$

Based on the matching probabilities defined above[4], it is now obvious to see type 1 boys' advantage over type 2 boys in the marriage market. Type 1 boys have higher matching probabilities regardless of which type of girls they choose to make proposals. However, every boy has to take the risk of matching failure in this process, with its probability given by $1 - P_{ij}$. Given boys' proposing strategies deployed, on the other side of the marriage market, a type j girl who has received proposals, will be matched with a type i boy with probability denoted by $\tilde{P}_{ji}$,

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[4]This is an urn-ball matching process when the number of boys and girls goes to infinity with the queue length fixed. See the explanation in the paper by Shi (2002), or in a general survey of the search-theoretic labor market literature by Rogerson, Shimer and Wright (2005).

which is defined as follows:

$$\tilde{P}_{11} \; = \; 1 - e^{-q_{11}} = 1 - e^{-s_1} \tag{2.9}$$

$$\tilde{P}_{12} \; = \; e^{-q_{11}}(1 - e^{-q_{21}}) = e^{-s_1} \cdot \left(1 - e^{-\frac{s_2 \cdot (1-v)}{v}}\right) \tag{2.10}$$

$$\tilde{P}_{21} \; = \; 1 - e^{-q_{12}} = 1 - e^{-\frac{(1-s_1) \cdot v}{(1-v)}} \tag{2.11}$$

$$\tilde{P}_{22} \; = \; e^{-q_{12}}(1 - e^{-q_{22}}) = e^{-\frac{(1-s_1) \cdot v}{1-v}} \cdot \left(1 - e^{-(1-s_2)}\right) \tag{2.12}$$

$1 - \tilde{P}_{ji}$ indicates the probability of girls' matching failure. Since coordination frictions exist in the matching process, it is possible that some type 1 girls would not receive proposals from type 1 boys even if all the type 1 boys had proposed to type 1 girls. These frictions grant type 2 boys with the opportunity to be accepted by a type 1 girl with positive probability if they propose to her.

2.3.2 Random Matching in the Second Round

Due to coordination frictions in the first round matching, there will be a fraction of marriage market participants left unmatched and they will enter the second round. Random matching is adopted in the second round to clear the marriage market. In the second round, marriage market participants are not picky anymore and the matching probability depends only on the skill distribution of the opposite gender.

In the random matching configuration, the probability of a girl getting matched with a type i boy denoted by $\tilde{P}^{se_i}$, is identical for both types of girls, which is the ratio of type i boys left in the second round to all the boys left in the second round. Similarly, both types of boys will be matched with a type j girl with the same probability in the second round, denoted by P^{se_j}, which is the ratio of type j girls to all the girls in the second round. These probabilities are defined as follows:

$$P^{se_1} = \frac{vN(1 - \tilde{P}_{11} - \tilde{P}_{12})}{vN(1 - \tilde{P}_{11} - \tilde{P}_{12}) + (1-v)N(1 - \tilde{P}_{21} - \tilde{P}_{22})} \tag{2.13}$$

$$\tilde{P}^{se_1} = \frac{vN(1 - s_1^* P_{11} - (1 - s_1^*)P_{12})}{vN[1 - s_1^* P_{11} - (1 - s_1^*)P_{12}] + (1-v)N[1 - s_2^* P_{21} - (1 - s_2^*)P_{22}]} \tag{2.14}$$

where $vN(1 - \tilde{P}_{11} - \tilde{P}_{12})$ denotes the number of type 1 girls left unmatched after the first round matching, $(1-v)N(1 - \tilde{P}_{21} - \tilde{P}_{22})$ denotes the number of type 2 girls left unmatched after the first round matching, and $vN(1 - \tilde{P}_{11} - \tilde{P}_{12}) + (1-$

$v)N(1 - \tilde{P}_{21} - \tilde{P}_{22})$ denotes the total number of girls left unmatched after the first round matching. In the same manner, $vN[1 - s_1^*P_{11} - (1 - s_1^*)P_{12}]$ denotes the number of type 1 boys left unmatched after the first round matching, $(1-v)N[1 - s_2^*P_{21} - (1-s_2^*)P_{22}]$ denotes the number of type 2 boys left unmatched after the first round matching, and $vN[1 - s_1^*P_{11} - (1 - s_1^*)P_{12}] + (1-v)N[1 - s_2^*P_{21} - (1-s_2^*)P_{22}]$ denotes the total number of boys in the second round matching. It is obvious that $1 - P^{se_1} = P^{se_2}$, and $1 - \tilde{P}^{se_1} = \tilde{P}^{se_2}$.

Notice that under the random matching mechanism, a boy's matching probability with either type of girls does not depend on his own type. Unlike in the first round with the directed matching mechanism, type 1 boys lose their advantage in the second round, which is the same situation for type 1 girls. Throughout the matching process, we see that those who fail in matching with a partner in the first round will get matched in the second round, and may even form a match better than expected in the first round. To avoid any speculation behavior over the two rounds of matching, we assume that nobody would like to enter the second round matching at all by setting the discount factor of the payoffs from second round matching to be zero.

2.3.3 Equilibrium in the Marriage Market

An equilibrium in the marriage market consists of the boys' proposing strategy, s_i^*, the probability at which a type i boy or a type j girl forms a rich household, P_{ir}^* and P_{jr}^*, and the fraction of rich households of the young generation, w_r^*, such that given a distribution of types in the marriage market, v, and the feasible household utilities pair, (u_r, u_p) of the young generation, each type i boy chooses a strategy s_i^*, from which nobody within the type i group finds it is in his interest to deviate, and the strategy pair (s_1^*, s_2^*) *per se* determines P_{ir}^*, P_{jr}^* as well as w_r^*.

The above definition of the equilibrium in the matching stage requires that the boys' strategies be optimal. It implies that after observing the girls' type distribution, each boy maximizes his expected payoffs by choosing a particular proposing strategy simultaneously before any match is realized. The equilibrium is symmetric in the sense that boys of the same type are identical and choose the same proposing strategy s_i, and the equilibrium strategy profile denoted by (s_1^*, s_2^*) suggests that no one would have any incentive to deviate from his chosen strategy. Since every boy has only one chance to make a proposal, he will not be able to change his mind after his match outcome is realized (matching with

either a type 1 girl, or a type 2 girl or getting rejected). After the boys propose, the girls will commit to their posting selection rule[5].

This equilibrium strategy profile is obtained under complete information. In particular, the boys are able to observe the type distribution of the marriage market participants and the selection rule posted by girls, and are aware of each other's strategy as well as the coordination frictions in the matching process. Furthermore, they all anticipate the feasible household utilities u_r and u_p obtained from respective feasible household income (high or low) and the household production technology.

Recall that $s_i \in [0, 1]$ denotes a type i boy's likelihood of proposing to a type 1 girl. $s_i = 1$ describes the case where every type i boy chooses a pure strategy of proposing to a type 1 girl. $s_i = 0$ describes another pure strategy of a type i boy: proposing to a type 2 girl for sure. $s_i \in (0, 1)$ indicates that every type i boy chooses a mixed strategy of proposing to a type 1 girl with probability s_i. In an equilibrium, boys of the same type must either choose a mixed strategy $s_i \in (0, 1)$, indifferent between proposing to a type 1 or a type 2 girl, when no pure strategy can be supported as the dominant strategy, or strictly prefer proposing to one type than the other when the corresponding pure strategy can be supported as the dominant one. There are nine possible strategy profiles (s_1, s_2) since s_i can take the value $1, 0$, or a value in the interval $(1, 0)$, a *priori*. However, under Assumption 1, which ensures a symmetrical type distribution of boys and girls in the marriage market, only four of them are plausible in an equilibrium. The above analysis will be elaborated on by Lemma 2.1 and Lemma 2.2.

Denote the ratio of u_p to u_r by α, where u_p is the consumption utility gained from a low household income and u_r is the consumption utility gained from a high household income. α is an exogenous variable and measures income disparity in this model.

Lemma 2.1 *When α is sufficiently low, type 1 boys choose a pure strategy in equilibrium, $s_1^* = 1$, given any value of v and s_2; otherwise, they adopt a mixed strategy $s_1^* \in (0, 1)$. $s_1^* \neq 0$ holds under any parameter region except when $\alpha = 1$. In summary,*

(i)$\alpha \in (0, 1 - 2e^{-1})$, $s_1^* = 1$ holds;

$(ii)\alpha \in (1 - 2e^{-1}, 1)$, $s_1^* \in (0, 1)$ *holds* ;

$(iii)s_1^* = 0$ *holds if and only if* $\alpha = 1$;

(iv) *when* $\alpha \in (1 - 2e^{-1}, 1)$, $s_1^* \in (0, 1)$ *holds, and it can be shown that* $\frac{\partial s_1^*}{\partial \alpha} < 0$, $\frac{\partial s_1^*}{\partial v} > 0$.

Proof. Let $u_{m_{ij}}^e$ denote the expected payoff of a type i boy who proposes to a type j girl. Suppose type 1 boys choose a pure strategy to propose to type 1 girls indicated by $s_1 = 1$, which means all the type 1 boys would propose to type 1 girls. Then their expected payoff would be $u_{m_{11}}^e = (1 - e^{-1})u_r$. If any of the type 1 boys deviates from this pure strategy, switching to proposing to type 2 girls, he would be matched at probability 1, and gain a household utility $(u_r + u_p)/2$. Thus, as long as $u_{m_{11}}^e = (1 - e^{-1})u_r \geq (u_r + u_p)/2$ which can be reduced to $\alpha \leq 1 - 2e^{-1}$, $s_1 = 1$ can be supported as the type 1 boys' equilibrium strategy. Consequently, when $u_{m_{11}}^e = (1 - e^{-1})u_r \leq (u_r + u_p)/2$, or equivalently, $\alpha \geq 1 - 2 \cdot e^{-1}$, type 1 boys have an incentive to deviate from the pure strategy and choose a mixed strategy $s_1 \in (0, 1)$ or the other pure strategy $s_1 = 0$ instead.

Similarly, suppose they all choose $s_1 = 0$, which means all the type 1 boys would propose to a type 2 girl. Then their expected payoff would be $u_{m_{12}}^e = \frac{1-v}{2v}(1 - e^{-\frac{v}{1-v}})(u_r + u_p)$. If any of them deviates from this pure strategy and proposes to a type 1 girl, he would get matched with a type 1 girl at probability 1, and gain a utility u_r. Hence, as long as $u_{m_{12}}^e = \frac{1-v}{2v}(1 - e^{-\frac{v}{1-v}})(u_r + u_p) \geq u_r$, or equivalently $\alpha \geq \frac{2v}{(1-v)(1-e^{-\frac{v}{1-v}})} - 1$, $s_1 = 0$ can be supported as the type 1 boys' equilibrium strategy. Notice that $\lim_{v \to 0} \frac{2v}{(1-v)(1-e^{-\frac{v}{1-v}})} - 1 = 1$ and $\lim_{v \to 1} \frac{2v}{(1-v)(1-e^{-\frac{v}{1-v}})} - 1 = \infty$, therefore only when $\alpha = 1$, this inequality can hold as the sufficient condition to support $s_1^* = 0$.

Based on the analysis above, when $\alpha \in (1 - 2 \cdot e^{-1}, 1)$, $s_1^* \in (0, 1)$, type 1 boys are indifferent between proposing to a type 1 girl and a type 2 girl, where $u_{m_{11}}^e = u_{m_{12}}^e$,

$$P_{11}u_r = P_{12}\frac{(u_r + u_p)}{2}$$

$$\frac{(1 - e^{-s_1})}{s_1}u_r = \frac{(1 - v)(1 - e^{-\frac{(1-s_1)v}{(1-v)}})}{(1 - s_1)v}\frac{(u_r + u_p)}{2} \tag{2.15}$$

s_1^* is the solution to equation(2.15). It is unique over the parameter region:

$\alpha \in (1 - 2e^{-1}, 1)$, $v \in (0, 1)$. It also can be shown that $\frac{\partial s_1^*}{\partial \alpha} < 0$, $\frac{\partial s_1^*}{\partial v} > 0$. This proof is complete. ∎

This lemma states some important properties of the type 1 boys' strategy in an equilibrium of the marriage market. First, when the type 1 boys choose a strategy, either a pure or a mixed one, they do not need to take type 2 boys' response into consideration. The value of s_1 solely depends on the parameters α and v. This is because the girls' selection rule favors type 1 applicants over type 2 applicants.

Second, the way s_1 varies in α shows that the inequality of household utility at the end of the matching stage plays a major role in type 1 boys' decision process. When they foresee huge inequality in future, the benefit from being matched with a type 1 girl outweighs the disutility arising from the risk of matching failure. Every type 1 boy will choose a pure strategy of proposing to a type 1 girl for sure. Conversely, when the inequality is not that sharp, type 1 boys will maximize their expected payoffs by choosing a mixed strategy to find a balance between matching quality and matching probability. The strategy s_1^* adopted by type 1 boys can also be interpreted as the equilibrium fraction of type 1 boys who propose to type 1 girls when considering the population of type 1 boys as a whole.

In addition, since the risk of being left unmatched has an upper bound under this assumed symmetrical type distribution, unless there is foreseeable perfect equality ($\alpha = 1$), none of the type 1 boys would give up proposing to a type 1 girl at all.

Furthermore, in the mixed strategy case, the likelihood of proposing to a type 1 girl is strictly increasing in the fraction of type 1 participants in the marriage market, which indicates that for type 1 boys, more type 1 girls imply a better chance to be matched with them even though the number of type 1 boys increases to the same extent.

Compared with type 1 boys, type 2 boys are in an inferior position under the girls' selection rule. Their optimal proposing strategy as the best response to type 1 boys' strategy also depends on α and v. The extra dependence on s_1^* makes type 2 boys' decision making process different from that of type 1 boys. For example, suppose in the case where type 1 boys choose a pure strategy of proposing to type 1 girls, $s_1^* = 1$, if type 2 boys choose a pure strategy of proposing to type 1 girls too, the expected payoff for them is, $u_{m21}^e = P_{21}(u_r + u_p)/2$. Substitute the expression of P_{21} and $s_1 = 1$ into it, we get:

$$u_{m21}^e = \frac{1}{2}ve^{-1}(1 - e^{-\frac{1-v}{v}})(u_r + u_p) \tag{2.16}$$

If any boy of type 2 deviates from this pure strategy, switching to type 2 girls, his matching probability would be 1, which means he would get a payoff u_p for sure. Hence, the requirement for $s_2 = 1$ be type 2 boys' equilibrium strategy is that:

$$\frac{v}{2(1-v)}e^{-1}(1 - e^{-\frac{1-v}{v}})(u_r + u_p) \geq u_p$$

The above inequality means that every type 2 boy finds that staying with a pure strategy of proposing to a type 1 girl is at least as good as switching to type 2 girls, thus it is not in his interest to deviate from this pure strategy. Define the critical value v_a as the solution to the equation below:

$$\frac{v}{2(1-v)}e^{-1}(1 - e^{-\frac{1-v}{v}})(u_r + u_p) = u_p$$

$$\frac{v}{(1-v)}e^{-1}(1 - e^{-\frac{1-v}{v}}) = \frac{2\alpha}{1+\alpha} \tag{2.17}$$

Given the value of α, if $v \in (0, v_a)$, the inequality discussed above does not hold, which means that type 2 boys have an incentive to deviate from the pure strategy $s_2 = 1$. Conversely, if $v \in [v_a, 1)$, the inequality holds and the pure strategy $s_2 = 1$ is supported in the equilibrium. The following lemma provides a complete characterization of type 2 boys' strategy in the equilibrium.

Lemma 2.2 : *Type 2 boys' strategy in equilibrium is characterized by the following separate cases:*

(i)In the case where type 1 boys choose a pure strategy of proposing to type 1 girls in the equilibrium ($s_1^* = 1$), type 2 boys choose a pure strategy in equilibrium, $s_2^* = 1$, iff α is as low as $\alpha \in (0, (2e-1)^{-1})$, and v is as high as $v \in [v_a, 1)$. Type 2 boys choose a mixed strategy, $s_2^* \in (0, 1)$, if $\alpha \in [(2e-1)^{-1}, 1 - 2e^{-1}]$ and $v \in (0, 1)$, where s_2^* is a unique solution to the following equation:

$$\frac{1+\alpha}{2}e^{-1}\frac{v}{s_2(1-v)}(1 - e^{-\frac{s_2(1-v)}{v}}) = \alpha\frac{1 - e^{s_2-1}}{1 - s_2} \tag{2.18}$$

Furthermore, it can be shown that $\frac{\partial s_2^*}{\partial \alpha} < 0$, $\frac{\partial s_2^*}{\partial v} > 0$;

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(ii)In the case where type 1 boys choose a mixed strategy in equilibrium ($s_1^* \in (0,1)$), type 2 boys choose a mixed strategy in equilibrium, $s_2^* \in (0,1)$, when α and v take any value within $(0,1)$. s_2^* is a unique solution to the following equation:

$$\frac{1+\alpha}{2}e^{-s_1}\frac{v}{s_2(1-v)}(1-e^{-\frac{s_2(1-v)}{v}}) = \alpha e^{-\frac{(1-s_1)v}{1-v}}\frac{1-e^{s_2-1}}{1-s_2} \qquad (2.19)$$

where s_1^* is the solution to equation (2.15).

Moreover, it can be shown that $\frac{\partial s_2^*}{\partial \alpha} < 0$, $\frac{\partial s_2^*}{\partial v} > 0$, and $\frac{\partial s_2^*}{\partial s_1^*} < 0$;

(iii)$s_1^* = s_2^* = 0$ holds if and only if $\alpha = 1$.

Proof. See Appendix A.1. ∎

Lemma 2.1 and Lemma 2.2 indicate that an equilibrium in the marriage market is one of the following four types:

- $s_1^* = s_2^* = 1$, both type 1 and type 2 boys choose a pure strategy of proposing to type 1 girls only, as they can earn a higher expected payoff than if proposing to type 2 girls with any positive probability;

- $s_1^* = 1, s_2^* \in (0,1)$, type 1 boys stick to the pure strategy of proposing to type 1 girls, while type 2 boys choose a mixed strategy by proposing to type 1 and type 2 girls with some positive probability respectively, being indifferent between proposing to the two types of girls;

- $0 < s_2^* < s_1^* < 1$, type 1 and type 2 boys choose a mixed strategy respectively, being indifferent between proposing to type 1 and type 2 girls.

- $s_1^* = s_2^* = 0$, both type 1 and type 2 boys choose a pure strategy of proposing to type 2 girls only. This type of equilibrium would occur if and only if $\alpha = 1$.

The four types of equilibria shown above never occur under the same parameter region, given Assumption 1 is satisfied.

Intuitively, we can see that the case where $s_2 > s_1$ would never happen in an equilibrium. This is because type 1 boys are favored by type 1 girls under the selection rule. In addition, type 1 girls are regarded as the better potential mates by all the boys. Thus type 2 boys always have a worse chance of getting matched with type 1 girls, which consequently lowers their likelihood of proposing to type 1 girls in an equilibrium.

Furthermore, there are some other possible types of strategy profiles not occurring in the equilibrium under any parameter region. One of them is $s_1 = 1, s_2 = 0$, where all the type 1 boys propose to a type 1 girl and all the type 2 boys propose to a type 2 girl. By contrast, $0 < s_2^* < s_1^* < 1$ is one of the possible types that occurs in the equilibrium. That is because the distribution is symmetrical across gender under Assumption 1 and the household production technology is neutral (which will be defined formally and elaborated on later). Consequently, due to the ratio of type 1 boys to type 1 girls (which is equal to 1), no girl can be guaranteed of receiving at least one proposal from a type 1 boy under the directed matching mechanism even in the scenario where every type 1 boy proposes to a type 1 girl for sure. Hence, there is always some chance of getting matched with type 1 girls for a type 2 boy if he proposes to her due to the coordination frictions among type 1 boys. Given the neutral household production technology, type 1 girls are attractive enough to type 2 boys as long as there is a positive chance of getting matched with them. Similarly, in the case where type 1 boys choose a mixed strategy of proposing to each type of girl with some positive probability, type 2 boys would not give up proposing to type 1 girls either as they might be even luckier in getting matched with a type 1 girl who is "ignored" by type 1 boys due to both coordination frictions and an insufficient number of proposals from type 1 boys.

In an equilibrium of the marriage market, the strategy profile (s_1^*, s_2^*) pins down the value of first round matching probabilities $P_{ij}^*, \tilde{P}_{ji}^*$, as well as the second round matching probabilities $P^{se_i*}, \tilde{P}^{se_j*}$. Though the outcome of the second round matching does not affect participants' decisions in the first round, it does affect the probabilities of forming a rich or a poor household at the end of the matching stage for each type of participant, which are calculated as follows:

$$P_{1r}^* = P_{11}^* s_1^* + \frac{1}{2}P_{12}^*(1 - s_1^*) + P^{*se_1}[(1 - P_{11}^*)s_1^* + (1 - P_{12}^*)(1 - s_1^*)]$$
$$+ \frac{1}{2}(1 - P^{*se_1})[(1 - P_{11}^*)s_1^* + (1 - P_{12}^*)(1 - s_1^*)] \quad (2.20)$$

P_{1r}^* is defined as the overall probability of a type 1 boy forming a rich household at the end of the matching stage. $P_{11}^* s_1^*$ indicates the probability of a type 1 boy getting matched with a type 1 girl at the end of the first round matching; $\frac{1}{2}P_{12}^*(1 - s_1^*)$ indicates the probability of a type 1 boy getting matched with a type 2 girl and forming a rich household at the end of the first round matching;

$P^{*se_1}[(1 - P_{11}^*)s_1^* + (1 - P_{12}^*)(1 - s_1^*)]$ represents the probability of a type 1 boy who has been left unmatched after the first round, getting matched with a type 1 girl in the second round, and forming a rich household at the end of the second round of matching; $\frac{1}{2}(1 - P^{*se_1})[(1 - P_{11}^*)s_1^* + (1 - P_{12}^*)(1 - s_1^*)]$ represents the probability of a type 1 boy who has been left unmatched after the first round, but getting matched with a type 2 girl in the second round, and forming a rich household at the end of the second round matching.

$$P_{2r}^* = \frac{1}{2}P_{21}^*s_2^* + \frac{1}{2}P^{*se_1}[(1 - P_{21}^*)s_2^* + (1 - P_{22}^*)(1 - s_2^*)] \quad (2.21)$$

P_{2r}^* is defined as the overall probability of a type 2 boy forming a rich household at the end of the matching stage. $\frac{1}{2}P_{21}^*s_2^*$ indicates the probability of a type 2 boy getting matched with a type 1 girl and forming a rich household at the end of the first round matching; $\frac{1}{2}P^{*se_1}[(1 - P_{21}^*) \cdot s_2^* + (1 - P_{22}^*)(1 - s_2^*)]$ represents the probability of a type 2 boy who has been left unmatched after the first round, getting matched with a type 1 girl in the second round, and forming a rich household at the end of the matching stage.

$$\tilde{P}_{1r}^* = \tilde{P}_{11}^* + \frac{1}{2}\tilde{P}_{12}^* + \tilde{P}^{*se_1} \cdot (1 - \tilde{P}_{11}^* - \tilde{P}_{12}^*) + \frac{1}{2} \cdot (1 - \tilde{P}^{*se_1})(1 - \tilde{P}_{12}^* - \tilde{P}_{12}^*)$$
$$(2.22)$$

$\tilde{P}_{1r}^*$ is defined as the overall probability of a type 1 girl forming a rich household at the end of the matching stage. $\tilde{P}_{11}^*$ indicates the probability of a type 1 girl getting matched with a type 1 boy, and forming a rich household at the end of the first round matching; $\frac{1}{2}\tilde{P}_{12}^*$ indicates the probability of a type 1 girl getting matched with a type 2 boy in the first round and forming a rich household at the end of the first round matching; $\tilde{P}^{*se_1} \cdot (1 - \tilde{P}_{11}^* - \tilde{P}_{12}^*)$ represents the probability of a type 1 girl who has been left unmatched after the first round, getting matched with a type 1 boy and forming a rich household at the end of the second round matching; $\frac{1}{2} \cdot (1 - \tilde{P}^{*se_1})(1 - \tilde{P}_{12}^* - \tilde{P}_{12}^*)$ represents the probability of a type 1 girl who has been left unmatched after the first round matching, getting matched with a type 2 boy in the second round and forming a rich household at the end of the matching stage.

$$\tilde{P}_{2r}^* = \frac{1}{2} \cdot \tilde{P}_{21}^* + \frac{1}{2} \cdot \tilde{P}^{*se_1} \cdot (1 - \tilde{P}_{21}^* - \tilde{P}_{22}^*) \quad (2.23)$$

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$\tilde{P}^*_{2r}$ is defined as the overall probability of a type 2 girl forming a rich household at the end of the matching stage. $\frac{1}{2} \cdot \tilde{P}^*_{21}$ indicates the probability of a type 2 girl getting matched with a type 1 boy and forming a rich household at the end of the second round matching; $\frac{1}{2} \cdot \tilde{P}^{*se_1} \cdot (1 - \tilde{P}^*_{21} - \tilde{P}^*_{22})$ represents the probability of a type 2 girl who has been left unmatched after the first round matching, getting matched with a type 1 boy in the second round matching and forming a rich household at the end of the matching stage.

These expressions from (2.20) to (2.23) are based on the assumption that the marriage market is cleared at the end of the matching stage and the household production technology is neutral. The matching process allows for four routes for young people to form a household: a type 1 boy and a type 1 girl forming a rich household with probability 1, a type 2 boy and a type 1 girl, or alternatively a type 2 boy and a type 1 girl, forming a rich household with probability $\frac{1}{2}$, a poor household with probability $\frac{1}{2}$; a type 2 boy and a type 2 girl forming a poor household with probability 1. We derive some properties on the household formation process according to the set-up of the model:

Property 1 Under Assumption 1 and the household production technology, the probability of forming a rich household is the same to the same type individuals regardless of their genders, which is indicated by $P^*_{1r} = \tilde{P}^*_{1r}$, and $P^*_{2r} = \tilde{P}^*_{2r}$. Furthermore, the equilibrium income distribution, characterized by the fraction of rich households at the end of the matching stage, equals the fraction of skilled individuals in the marriage market, which is indicated by $w^*_r = v$.

Property 1, together with Property 2 (which will be presented in the next subsection), is illustrated by Figure A.1 to Figure A.4 in Appendix A3. Besides no gender difference with regard to the probability of becoming rich, Property 1 also suggests that the household production technology make the matching process, so called, neutral, in the sense that it does not change the overall contribution of educational investment to the economy as a whole. In other words, the outcome of 'who marries whom' does not increase or decrease the total match surplus compared with that under an alternative assignment rule. Thus, the productivity of education sector, which determines the aggregate human capital capacity of the economy, would not been affected by the matching mechanism. However, the matching outcomes will be taken into consideration by each parental household in the investment stage, which would affect their educational investment incentives from an individual household's perspective and thus the value of v. In that sense,

the matching process impacts on the economy through the endogenized v as the measure of the fraction of skilled individuals in the economy.

2.4   The Investment Stage

As stated at the beginning of this chapter, parental households are divided into two types, with a high household income recognized as the rich and a low household income recognized as the poor. In both parental households, parents are altruistic towards their children in the sense that their own utility contain their children's (which is illustrated by (2.24) and (2.25)). Compared to their children, parents are assumed to be more sophisticated about marriage in the sense that they are able to foresee the full picture of marriage market outcome from the first round to the second round matching (they are fully rational). Their children are partially rational in that they discount second round matching heavily due to psychological disutility from matching failure in the first round.

Based on this forward looking behavior, parents sense that the high educational investment would make their children more attractive and give them advantage in the marriage market. However, assortative matching can not be guaranteed due to coordination frictions and the 'speculative behavior' of the less attractive competitors with the low education attainment (type 2) in the first round matching. In addition, the randomness in the second round weakens parental households' control of their children's matching outcome. On the other hand, putting their children at an inferior position in the marriage market by making the low educational investment does not mean their children will definitely end up with a poor household. Their less attractive children still have a chance of matching with skilled individuals, and form a rich household with some positive probability. Though the chance is worse than if making the high educational investment, the less costly educational investment would increase parental households' own consumption as compensation.

2.4.1   Investment Strategy Profile

We still focus on the symmetric equilibrium at the investment stage. Parental households of the same type choose an identical investment strategy, either a pure or a mixed one. We denote the endogenous parental households' investment

strategy profile by two (1×2) row vectors:

$$K_r = [\ k_r, \quad (1 - k_r)\ ], K_p = [\ k_p, \quad (1 - k_p)\ ]$$

where k_r and k_p are the probabilities at which the high educational investment at cost c_1 is chosen by rich and poor households respectively. Hence, $1 - k_r$ and $1 - k_p$ are the probabilities at which the low educational investment at the cost c_2 is chosen by rich households and poor households respectively. Investment strategy profile is represented by K_r or K_p, which is a unit vector if this is a pure strategy.

Secondly, we define the exogenous households' utilities from consumption by two (2×1) column vectors:

$$U_r^c = \begin{bmatrix} u_r^{c_1} \\ u_r^{c_2} \end{bmatrix}, U_p^c = \begin{bmatrix} u_p^{c_1} \\ u_p^{c_2} \end{bmatrix}$$

where $u_r^{c_1}$ denotes the consumption utility of rich households gained from their household income deducting the cost of the high educational investment c_1, and $u_r^{c_2}$ is the consumption utility of rich households gained from their household income deducting the cost of the low educational investment c_2; similarly, $u_p^{c_1}$ and $u_p^{c_2}$ are the consumption utilities gained by the poor households out of their net income (total household income deducting educational investment cost) respectively.

Lastly, we define the endogenous expected offspring household utility by a (2×1) column vector:

$$U^o = \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \begin{bmatrix} P_{1r} u_r + (1 - P_{1r}) u_p \\ P_{2r} u_r + (1 - P_{2r}) u_p \end{bmatrix}$$

where $u_1^e = P_{1r} u_r + (1 - P_{1r}) u_p$ is the expected utility of a type 1 individual in the matching stage, and correspondingly, $u_2^e = P_{2r} u_r + (1 - P_{2r}) u_p$ is the expected utility of a type 2 individual in the matching stage. Based on Property 1 in the previous subsection, boys' probability of forming a rich household equals that of girls. Hence, here we only calculate the boys' probability of forming a rich

household.

Based on the above definitions, the parental households' utilities can be represented by U_r and U_p for the rich and the poor household respectively, which are defined as follows:

$$U_r = K_r U_r^c + \beta \cdot K_r U^o \tag{2.24}$$

$$U_p = K_p U_p^c + \beta \cdot K_p U^o \tag{2.25}$$

From the expressions shown above, we can see that parents' utilities consist of two parts: their own consumption utilities and their children's utilities. β is the discount factor associated with the children's utilities. $K_r U_r^c$ and $K_p U_p^c$ denote the expected consumption utilities of parental households.

K_r and K_p affect the parents' expected utility through two channels, directly through children's utilities indicated by $K_r U^o$ and $K_p U^o$, and indirectly through children's skill distribution indicated by v. Let w_r^{in} be the exogenous parental households' income distribution parameter, which denotes the fraction of rich parental households in the parental generation. The underlying rationale of the indirect effect on parents' expected utility by their investment strategies is that the strategic variables, k_r of K_r and k_p of K_p, together with w_r^{in} determine the value of v. The offspring's pre-match skill distribution indicator, v, is crucial to the equilibrium proposing strategies s_1^* and s_2^*. Consequently v has an effect on the probability of forming a rich household at the end of the matching stage, P_{1r}^* and P_{2r}^*.

By fully anticipating the matching mechanism in the marriage market and household production technology, parents are confronted with the trade-off between a higher consumption utility of their own and a better future for their children. Investment choices available in the investment stage involve strategic thinking by parental households. Their joint investment decisions determine the fraction of skilled individuals in the next generation. Hence, in the investment stage, parental households endogenize v by making the optimal educational investment strategies. Moreover, the following property elaborates on the effects of v on the household formation:

Property 2: For any given value of $\alpha \in (0,1)$ anticipated by the marriage market participants, it can be shown that $P_{1r}^* > P_{2r}^*$, $\partial P_{1r}^*/\partial v > 0$, $\partial P_{2r}^*/\partial v > 0$, and $\partial (P_{1r}^* - P_{2r}^*)/\partial v < 0$.

This property says that both type 1 and type 2 individuals' probabilities of forming a rich household at the end of the matching stage strictly increase in the fraction of skilled individuals v in the marriage market, but the difference between the two probabilities strictly decreases in v. This property is very important as it reveals the monotonicity of the marginal benefits of being a type 1 (skilled) individual as a function of v, which is the key basis for the uniqueness of the equilibrium in the investment stage.

Both k_r and k_p can be equal to 1, 0, or any number within the interval $(0, 1)$. The following lemma characterizes the investment strategy profile which would be adopted by parental households in an equilibrium under different parameter regions.

Lemma 2.3 *There are six types of investment strategy profiles in the equilibrium at the investment stage:*

(i) when $u_r^{c_2} - u_r^{c_1} \geq \beta(u_r - u_p)$, both the rich households and the poor households would choose a dominant pure strategy, $k_r = k_p = 0$, making the low educational investment on behalf of their children;

(ii) when $\frac{1}{2}\beta(u_r - u_p) \leq u_r^{c_2} - u_r^{c_1} < \beta(u_r - u_p) \leq u_p^{c_2} - u_p^{c_1}$, the rich households' strategy may be a mixed one, $k_r \in (0, 1)$, or a pure one, $k_r = 0$ or $k_r = 1$, which depends on the fraction of rich parental households w_r^{in}. The poor households always choose a dominant pure strategy, $k_p = 0$;

(iii) when $u_r^{c_2} - u_r^{c_1} \leq \frac{1}{2}\beta(u_r - u_p) < u_p^{c_2} - u_p^{c_1} \leq \beta(u_r - u_p)$, the poor households' strategy may be a mixed one, $k_p \in (0, 1)$, or a pure one, $k_p = 0$, $k_p = 1$, which depends on w_r^{in}. The rich households always choose a dominant pure strategy, $k_r = 1$;

(iv) when $u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} \leq \frac{1}{2}\beta(u_r - u_p)$, both types of households would choose a dominant pure strategy, $k_r = k_p = 1$;

(v) when $u_r^{c_2} - u_r^{c_1} \leq \frac{1}{2}\beta(u_r - u_p)$, and $u_p^{c_2} - u_p^{c_1} \geq \beta(u_r - u_p)$, the rich households always choose a dominant pure strategy $k_r = 1$ and the poor households always choose a dominant pure strategy $k_p = 0$;

(vi)when $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$, both the rich households and the poor households may choose a mixed strategy, $k_r \in (0,1), k_p \in (0,1)$, or a pure one, which depends on w_r^{in}.

Proof. see Appendix A.2. ∎

Intuitively, this lemma says that when the consumption utility loss (that is caused by a higher cost of the high educational investment) is too high to be compensated by the corresponding utility gain of their offspring, all the households would choose low educational investment. Conversely, if the utility loss is lower than the corresponding utility gain of their offspring, all the households would choose high educational investment. However, the more parental households that make the high educational investment, the larger the fraction of skilled individuals in the matching stage, and the smaller the gap of the expected utilities between being skilled and unskilled, which in return would weaken parental households' incentives to choose high educational investment.

Notice that the rich households can always suffer less than the poor households from costly high educational investment. $u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1}$ represents this feature. The underlying rationale is the concavity of the utility function. Consequently, the rich households are able to tolerate larger v than the poor households before stopping producing type 1 offspring and switching to producing type 2 offspring. Specifically, we define v_p and v_r as the critical values of the fraction of skilled offspring for the rich and the poor parental households respectively, at which their consumption utility loss equals the amount of utility increase from producing type 1 offspring other than type 2 offspring. Based on the analysis shown above, we get $v_p < v_r$.

Definition 1 The critical value of v, the fraction of skilled persons in the next generation, for the poor households, is defined as v_p where v_p satisfies the following equation:

$$
\begin{aligned}
u_p^{c_2} - u_p^{c_1} &= P_{1r}^*|_{v=v_p}u_r + (1 - P_{1r}^*|_{v=v_p})u_p - (P_{2r}^*|_{v=v_p}u_r + (1 - P_{2r}^*|_{v=v_p})u_p) \\
&= (P_{1r}^*|_{v=v_p} - P_{2r}^*|_{v=v_p})(u_r - u_p)
\end{aligned}
$$

$$(2.26)$$

Similarly, the critical value for the rich households is defined as v_r, where v_r

satisfies the following equation:

$$u_r^{c_2} - u_r^{c_1} = P_{1r}^*|_{v=v_r}u_r + (1 - P_{1r}^*|_{v=v_r})u_p - (P_{2r}^*|_{v=v_r}u_r + (1 - P_{2r}^*|_{v=v_r})u_p)$$

$$= (P_{1r}^*|_{v=v_r} - P_{2r}^*|_{v=v_r})(u_r - u_p)$$

(2.27)

The six types of investment strategy profiles are illustrated by Figure A.1 in Appendix. In Figure A.1, v_p and v_r are the X coordinates of the intersection points of the curve (representing the premium of high educational investment) and the horizontal lines (representing the consumption utility loss by costly high educational investment for the rich and the poor parental households respectively).

2.4.2 Equilibrium Outcome

Based on the characteristics of the investment strategy profiles stated in Lemma 2.3 and the two properties of the household formation, we are able to focus on the most interesting case, type (vi) investment strategy profile. By considering an additional exogenous variable, w_r^{in} (the fraction of rich parental households), we have the following proposition:

Proposition 2.1 *A symmetric mixed-strategy equilibrium at the investment stage consisting of parental household investment strategy profile (K_r^*, K_p^*), expected household utility (U_r^*, U_p^*) and corresponding v^* exists and is unique, given the parameter region where $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$ and the initial income distribution parameter, $w_r^{in} \in (0, 1)$. Moreover,*

(i) when $0 < w_r^{in} \leq v_p$, the equilibrium investment strategies are, $k_r^* = 1, k_p^* = \frac{v_p - w_r^{in}}{1 - w_r^{in}}$, and the equilibrium fraction of skilled offspring is, $v^* = v_p$;

(ii) when $v_p \leq w_r^{in} \leq v_r$, the equilibrium investment strategies are, $k_r^* = 1, k_p^* = 0$, and the equilibrium fraction of skilled offspring is, $v^* = w_r^{in}$;

(iii) when $v_r \leq w_r^{in} < 1$, the equilibrium investment strategies are, $k_r^* = \frac{v_r}{w_r^{in}}, k_p^* = 0$, and the equilibrium fraction of skilled offspring is, $v^* = v_r$.

Proof. In the parameter region $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$, both the rich and the poor parental households may adopt a mixed or a pure

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strategy as stated in Lemma 2.3. Since the marginal benefit of producing skilled offspring is continuous and monotonically decreasing in v, the uniqueness and existence of the equilibrium is guaranteed. Whether the parental households choose a mixed or a pure investment strategy depends on the initial income distribution parameter, w_r^{in}. Consequently, the final skill distribution, v^*, is also determined by w_r^{in}. A detailed explanation is made in the following analysis. ∎

Figure 2.1 illustrates Proposition 2.1 as follows. Let R denote the premium
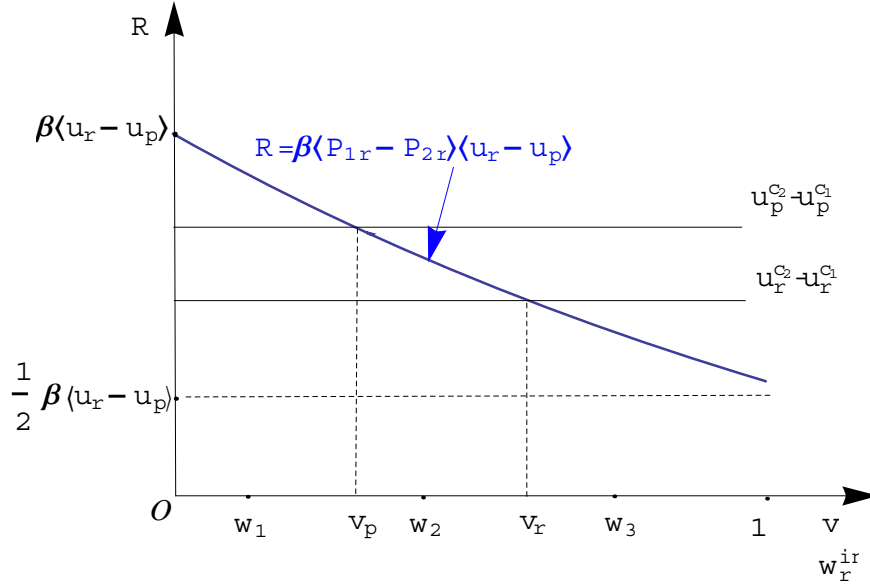
Figure 2.1: The premium of high educational investment and the loss of consumption utility by parental households against the fraction of the skilled offspring

of high educational investment. In Figure 2.1, the downward sloping curve represents the high educational investment premium, $R = (P_{1r}^* - P_{2r}^*)(u_r - u_p)$ as a function of v. Two parallel horizontal lines represent the consumption utility loss from costly high educational investment for the poor and the rich parental household respectively. The two lines intersect with the high educational investment premium curve at two points, $(v_p, u_p^{c2} - u_p^{c1})$ and $(v_r, u_r^{c2} - u_r^{c1})$ respectively.

The economic rationale of parental households can be inferred from this figure. We can see that both the rich households and the poor households prefer the high educational investment before v reaches v_p, as the premium of high educational investment exceeds the consumption utility loss from the associated higher cost $(c_1 > c_2)$. In the region where v lies between v_p and v_r, the poor prefer the

low investment while the rich prefer the high investment because the premium of high educational investment is high enough to compensate the consumption utility loss of the rich, but is not high enough for the poor. As v surpasses v_r, both the rich and the poor prefer low investment, because the premium of the high investment is lower than consumption utility loss of both the rich and the poor parental households.

Besides the comparison between the gains and losses of consumption utility, the fraction of rich parental households, w_r^{in}, also affects the equilibrium outcomes.

Consider the case where w_r^{in} takes an arbitrary value w_1, and $0 < w_1 < v_p$ [6] as shown in Figure 2.1. When all the rich parental households choose the high educational investment, the corresponding v equals w_1, which is still smaller than v_p. Hence, the poor parental households choose a mixed strategy $k_p \in (0, 1)$, which leads to a result where a fraction k_p of the poor households choose high investment, and a fraction $1 - k_p$ of the poor households choose low investment. The optimal value of k_p would make the poor just be indifferent between the two investment choices. It implies that the number of the poor parental household making the high investment just fills the gap between $w_1 N$ and $v_p N$ so that the total number of skilled individuals produced is as large as $v_p N$. Therefore, we obtain the equilibrium investment strategies $k_p^* = \frac{v_p - w_r^{in}}{1 - w_r^{in}}$, $k_r^* = 1$ and the equilibrium fraction of skilled offspring, $v^* = v_p$.

Consider another case where w_r^{in} takes an arbitrary value w_2, and $v_p < w_2 < v_r$ as shown in Figure 2.1. It is obvious to see that all the rich households choose high investment, and produce vN type 1 offspring, where $v = w_2$. On the other hand, all the poor choose low investment as the number of the skilled offspring produced by the rich households is too large to induce any extra poor households to make the high educational investment. Therefore, we obtain the equilibrium investment strategies $k_p^* = 0$, $k_r^* = 1$ and the equilibrium fraction of skilled offspring, $v^* = w_r^{in}$.

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[6] In addition, when $w_r^{in} = v_p$ the poor parental households are just indifferent between the two investment choices and they would invest low as all the rich parental households would make the high investment, $k_p^* = 0$, $k_r^* = 1$. This result can also be obtained by substituting $w_r^{in} = v_p$ into the expression of k_p of case (i) in Proposition 2.1 where $k_p^* = \frac{v_p - w_r^{in}}{1 - w_r^{in}}$. On the other hand, when $w_r^{in} = v_r$, the rich parental households are just indifferent between the two investment choices if all of them make the high investment and they would do so, $k_p^* = 0$, $k_r^* = 1$. This result can also be obtained by substituting $w_r^{in} = v_r$ into the expression of k_r in case (iii) where $k_r^* = \frac{v_r}{w_r^{in}}$. Hence, the equality sign can be taken in all the three cases.

Consider the last case where w_r^{in} takes an arbitrary value w_3, and $v_r < w_3 < 1$. All the poor households would choose the low investment as the number of the rich households is so large. The rich households would choose a mixed strategy $k_r \in (0, 1)$, which leads to a situation where a fraction k_r of the rich households choose high investment, and a fraction $1 - k_r$ of the rich households choose low investment. The optimal value of k_r would make the poor indifferent between the two investment choices. It implies that the number of rich parental households making the high investment is just equal to v_r. Therefore, we obtain the equilibrium investment strategies $k_r^* = \frac{v_r}{w_r^{in}}$, $k_p^* = 0$ and the equilibrium fraction of skilled offspring, $v^* = v_r$.

The relationship between the initial wealth distribution of parental households and the equilibrium skill distribution of offspring is represented by Figure 2.2.
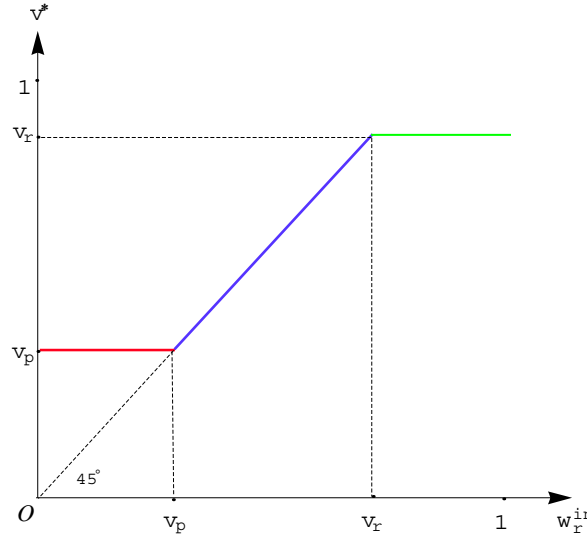
Figure 2.2: The equilibrium fraction of skilled offspring against the fraction of rich parental households

This figure summarizes the analysis shown above. The horizontal axis represents the initial income distribution parameter, w_r^{in}, and the vertical axis represents the fraction of skilled offspring in the equilibrium, v^*. The first line segment in red located in the left lower corner represents the case where $w_r^{in} \in (0, v_p]$, and the corresponding equilibrium skill distribution indicator v^* equals the critical value of the poor parental household v_p; the middle line segment in blue overlapping the 45° line, represents the case where $w_r^{in} \in [v_p, v_r]$, and the corresponding equilibrium skill distribution indicator v^* equals the initial income distribution

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parameter w_r^{in}; the upper right line segment in green represents the case where $w_r^{in} \in [v_r, 1]$, and v^* equals the critical value of the rich parental household v_r.

The corresponding relationship between the initial income distribution parameter, w_r^{in} and the equilibrium investment strategy of the parental households, $k*_r$ and k_p^*, can be illustrated by Figure 2.3.
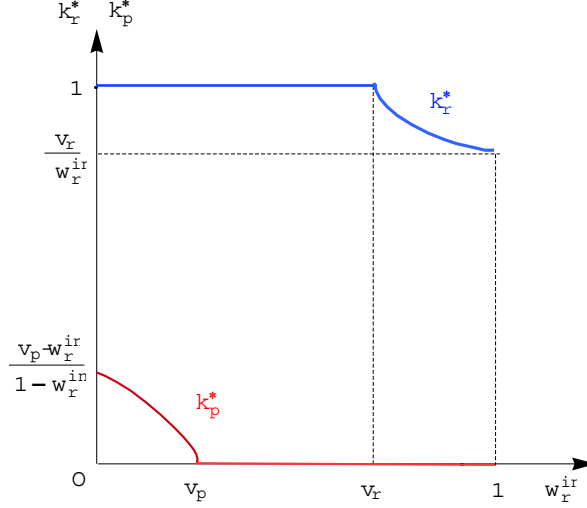
Figure 2.3: Equilibrium investment strategies against the fraction of rich parental households

In this figure, the horizontal axis represents both the initial household income distribution parameter w_r^{in} and offspring skill distribution indicator v (v also equals the fraction of parental households who choose high educational investment). The vertical axis represents the equilibrium parental households' investment strategies, k_r^* and k_p^*, which are the probabilities at which parental households choose high educational investment. The upper curve in blue represents the rich parental households' equilibrium investment strategy, k_r^*. It equals 1 when an arbitrary w_r^{in} is smaller than the rich parental households' critical value v_r, which indicates that the rich would choose a pure investment strategy of making the high educational investment. It starts to decrease in w_r^{in} after w_r^{in} exceeds v_r and equals $\frac{v_r}{w_r^{in}}$, which indicates the rich would choose a mixed strategy, so a fraction $\frac{v_r}{w_r^{in}}$ of the rich parental households would choose the high educational investment. The lower curve in red represents poor parental households' investment strategy k_p^*. It decreases in w_r^{in} and equals $\frac{v_p - w_r^{in}}{1 - w_r^{in}}$ before w_r^{in} reaches v_p, which indicates that a fraction $\frac{v_p - w_r^{in}}{1 - w_r^{in}}$ of the poor parental households would

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choose the high educational investment. It equals zero after w_r^{in} exceeds v_p, indicating that the poor would choose a pure strategy of making the low educational investment.

As stated in Property 1, the fraction of offspring households that are rich, w_r, equals the fraction of offspring who are skilled, v. Hence, we have $w_r^* = v^*$ in equilibrium. It is interesting to compare the income distribution of parental household with that of offspring's households. Considering the parental households income distribution has an effect on that of the offspring as stated in Proposition 2.1, the comparison between the two can be generalized by the following corollary:

Corollary 2.1 *When $w_r^{in} < v_p$, an increase in the number of rich households in the offspring's generation occurs; when $w_r^{in} > v_r$, a decrease in the number of rich households in the offspring's generation occurs; when w_r^{in} lies between v_p and v_r, the number of rich households does not change in the offspring's generation.*

The possible increase in the fraction of rich households of the children's generation as stated in Corollary 2.1 exists because when the fraction of rich parental households is very small, not only do all the rich make the high educational investment, but some of the poor households also do so. On the other hand, the possible decrease in the fraction of rich households exists because when w_r^{in} is very large, some of the rich households and all the poor households choose low investment. Finally, it is possible that the household income distribution becomes stationary across the two generations, with the rich choosing high investment and the poor choosing low investment.

Proposition 2.1 only considers one of the six investment profiles according to separate parameter regions listed in Lemma 2.3. The other five types of investment profiles lead to similar but different equilibrium outcomes, which is summarized in the following proposition:

Proposition 2.2 *The equilibrium outcomes are presented in the following five cases that correspond to type (i), type (ii), type (iii), type (iv) and type (v) investment profiles under separate parameter regions as stated in Lemma 2.3:*

(i) In the parameter region where $u_r^{c_2} - u_r^{c_1} \geq \beta(u_r - u_p)$, $k_r^* = k_p^* = 0$, $v^* = w_r^* = 0$.

(ii) In the parameter region where $\frac{1}{2}\beta(u_r - u_p) \leq u_r^{c_2} - u_r^{c_1} < \beta(u_r - u_p) \leq u_p^{c_2} - u_p^{c_1}$, $k_p^* = 0$, $k_r^* = 1$ and $v^* = w_r^* = v_p$ if $0 < w_r^{in} \leq v_r$; $k_p^* = 0$, $k_r^* = \frac{v_r}{w_r^{in}}$ and

$v^* = w_r^* = v_r$ *if* $v_r \le w_r^{in} < 1$.

(iii) In the parameter region where $u_r^{c2} - u_r^{c1} \le \frac{1}{2}\beta(u_r - u_p) < u_p^{c2} - u_p^{c1} \le \beta(u_r - u_p)$, $k_r^* = 1$, $k_p^* = 0$ *and* $v^* = w_r^* = w_r^{in}$ *if* $v_p \le w_r^{in}$; $k_r^* = 1$, $k_p^* = \frac{v_p - w_r^{in}}{1 - w_r^{in}}$ *and* $v^* = w_r^* = v_p$ *if* $0 < w_r^{in} \le v_p$.

(iv) In the parameter region where $u_r^{c2} - u_r^{c1} < u_p^{c2} - u_p^{c1} \le \frac{1}{2}\beta(u_r - u_p)$, $k_r^* = k_p^* = 1$, $v^* = w_r^* = 1$.

(v) when $u_r^{c2} - u_r^{c1} \le \frac{1}{2}\beta(u_r - u_p)$, *and* $u_p^{c2} - u_p^{c1} \ge \beta(u_r - u_p)$, $k_r^* = 1$, $k_p^* = 0$ *and* $v^* = w_r^* = w_r^{in}$, *given any value of* w_r^{in}.

Proof. Refer to the proof of Lemma 2.3 and Proposition 2.1. ∎

Figure A.1 illustrates all types of investment profiles and provides some intuition in interpreting this proposition. Case (i) in Proposition 2.2 that corresponds to the type (i) investment profile indicates that the premium from the high educational investment is so low that no one would choose high investment, and all the offspring's households would be poor. In case(ii), the premium is sufficient low that all the poor parental households choose low investment, while it is possibly high enough for the rich to choose high investment and the investment strategy adopted by rich parental households determines the income distribution. In case (iii), the premium is sufficient high that all the rich parental households choose high investment, while it is possibly not high enough for the poor to choose high investment and it is the poor parental households' investment strategy that determines the income distribution. Case (iv) refers to the situation where the premium is so high that every parental household would choose high investment, and all the offspring's households would be rich. Case (v) represents the scenario where education cost structure perfectly separates the skill prospect of the offspring from the rich parental household and that of the poor parental household, and the fraction of rich households would not change across the two generations.

2.5   Comparative Statics

Based on the analysis shown above, this section analyzes the effects of an exogenous change in parameters of feasible household income inequality and education

costs on the equilibrium outcomes. First, changes in the ratio of rich households' utility to poor households' utility, α, are analyzed in the following context.

The overall effect of α on equilibrium outcomes is through returns to educational investment, which can be decomposed into two parts: a direct effect on the difference in the offspring's household utilities stemming from the inequality in feasible household income $u_r - u_p = u_p(\frac{1}{\alpha} - 1)$ and an indirect effect on the difference in the skilled versus unskilled offspring's chance of forming a rich household: $P_{1r} - P_{2r}$.

Proposition 2.3 *Both $u_r - u_p$ and $\partial(P_{1r} - P_{2r})/\partial v$ are decreasing in α.*

This proposition can be verified intuitively. By the definition of α, the difference in utility of a rich household and that of a poor household (both consume up all the household income) shrinks. It also implies a less attractive household income prospect of getting matched with a type 1 (skilled) individual. By anticipating this change, type 1 boys and type 2 boys put more weight on matching probability in the first round matching when they are confronted with the trade-off between matching probability and matching quality. This result can also be inferred from Lemma 2.1 and Lemma 2.2, which state that s_1^* generally decreases in α, and s_2^* decreases in α too, but at a lower speed[7]. Hence, the chance of forming a rich household for a skilled individual, P_{1r}^*, decreases as α increases, as does the chance of forming a rich household for an unskilled individual, P_{2r}^*. However, P_{2r}^* decreases in α at a lower speed than P_{1r}^*, which leads to the result that the gap between the two probabilities is shrinking.

Based on Proposition 2.3, we can further infer the effect of α on the changes of the premium of high educational investment by the following Corollary:

Corollary 2.2 *An increase in α decreases the premium of high educational investment $\beta(P_{1r} - P_{2r})(u_r - u_p)$ under any value of the skilled fraction of offspring, v, and this increase also decreases the speed at which the premium decreases in v.*

Figure A.6 in Appendix illustrates the premium of high educational investment under different values of α. As it is shown in the figure, the premium curve shifts downwards as α increases and it also becomes flatter at a higher value of α.

---

[7]Recall that in Lemma 2.1, $s_1^* = 1$ when α is low enough, and otherwise $s_1^* \in (0, 1)$ and is decreasing in α. In Lemma 2.2, s_2^* is also decreasing in α as well as s_1^* when $s_2^* \in (0, 1)$. It can be inferred that s_2^* decreases in α at a lower speed than s_1^* due to its interdependency on s_1^*.

Based on the results given above, we then look at the possible changes of equilibrium outcomes according to the changes in α across parameter regions, under which different types of investment profiles are considered (refer to Figure A.1 in the Appendix to get the intuition). It can be summarized as follows:

Remark 2.1 *An increase in α discourages parental households from choosing the high educational investment as it decreases the critical values, v_p^*, or v_r^*, or both the two values, if the original investment profile belongs to type (ii), (iii), or (vi). In type (i) investment profile, only an increase in α would affect equilibrium outcomes. In type (iv) and type (v) investment profiles, only a sufficiently large increase in α would decrease the value of v_p^*, or v_r^*, or both.*

Moreover, a stationary income distribution is more likely to happen due to an increase in α as the particular interval (v_p, v_r), within which lies w_r^{in} that would lead to a stationary fraction of rich households in the next generation, would be enlarged due to a larger distance between v_r and v_p when the investment profile belongs to type (ii),(iii) or (vi). If this increase is sufficient, this effect also applies to type (iv) and type (v) investment profiles.

Next, let's look at the effect of the costs in education on equilibrium outcomes. We are more interested in the difference in costs between the high and the low educational investment, hence, for simplicity, we hold the cost of the low educational investment as a constant, e.g. $c_2 = 0$. By doing this, any change in the cost difference would be represented by a value of the high educational investment cost alone. The education cost variation leads to a change in the relative suffering from the utility loss of consumption by the parental households.

An increase in the cost of the high educational investment, c_1, would cause an increase of utility loss of consumption from the high educational investment by the two types of parental households, $u_p^{c_2} - u_p^{c_1}$ for the rich, $u_r^{c_2} - u_r^{c_1}$ for the poor. It also increases the distance between the two utility losses, $(u_p^{c_2} - u_p^{c_1}) - (u_r^{c_2} - u_r^{c_1})$. Take the type (vi) investment profile as an example, this effect can be represented by a leftwards shifting of v_p and v_r where v_p would shift more than v_r, and an increasing distance between the two points will be observed. As the cost of high educational investment increases, an exogenous w_r^{in}, which would lead to an increase in v^* in the next generation under the original education cost structure (the case where $w_r^{in} < v_p$), is very likely to result in an unchanged v^* instead. This is because w_r^{in} is possibly between the new v_p and the new v_r. On the other hand, if this exogenous w_r^{in} leads to an unchanged v^* under the original education

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cost structure (the case where $v_p < w_r^{in} < v_r$), it is possible to cause a decrease in v^* under the new education cost structure instead.

Generally speaking, a larger difference between the cost of the high educational investment and that of the low one leads to a larger difference in the suffering from consumption utility loss of the rich and the poor parental households. The increase in inequality of education costs may lead to a larger difference between the chance of forming a rich household of a skilled individual and that of an unskilled individual, for it may reduce the fraction of skilled offspring by discouraging the parental households, especially those with a low income, from making the high educational investment. When the difference in suffering from consumption utility loss of the rich and of the poor is large enough, education system just perfectly separates the educational investment behavior of the rich and that of the poor (which is illustrated by the type (v) investment profile), even though it seems that both types of households can 'afford to pay' these costs. In that case, stationary income distribution occurs no matter what the income distribution in the parental generation is.

2.6   Concluding Remarks

According to the static model described above, we characterize the equilibrium consisting of the parental households' investment strategies and the corresponding matching outcomes of their children, which also generate the fraction of rich households in the offspring's generation. By focusing on the most interesting case, under the parameter region of type (vi) investment profile, we are able to re-examine the competitive feature of the family formation process through educational investment and the marriage market, which affects the household income distribution of the younger generation. The results can be summarized as follows:

- An increase in the number of rich households in the young generation occurs in the case where the initial fraction of the rich households is relatively small and the inequality of the feasible household income is large enough, so that the poor find the competition from the rich is not too intense while the reward of high educational investment is attractive, which encourages the poor to make the high educational investment.

- A decrease in the number of rich households in the young generation occurs in the case where the initial fraction of the rich households is relatively large

and the inequality of feasible household income is not large enough, so that the competition within the rich as well as between the rich and the poor is too intense while the reward of high educational investment turns out to be relatively less attractive, which discourages both the poor and the rich from making the high educational investment.

- A stationary fraction of the rich households across the two generations occurs in the case where the initial fraction of rich households is just in-between the 'relatively' small and the 'relatively' large, together with inequality of the feasible household income prospects, which generates a level of competition between the rich and the poor that ensures all of the rich choose high investment, crowding out the poor, who all choose low investment.

In each case discussed above, the probability that the offspring from a rich household forming a poor household (the downward income mobility rate) and the probability that the offspring from a poor household forming a rich household (considered as the upward income mobility rate) are both positive, even when the income distribution across the two generations is stationary as mentioned in the third case shown above.

By introducing matching frictions, this model implies that assortative mating itself alone does not contribute to intergenerational income stratification. On the contrary, take type (iv) investment profile as an example, all the parental households would choose the high educational investment if there is no matching friction and as a result pure positive assortative matching in their children's generation is guaranteed. In that case, the premium of high educational investment does not change with the fraction of parental households who actually choose the high educational investment (the fraction of skilled individuals in the children's generation)[8]. Without coordination frictions in the matching process, the rewards to the high educational investment from the marriage market and the labor market are certain according to the settings of the model. This certainty encourages educational investment because there is no chance for an unskilled individual to form a rich household. By contrast, due to the coordination frictions in the matching process, an unskilled individual may end up with a rich household while a skilled individual may end up with a poor household and this uncertainty discourages parental households' incentive to make the high educational investment.

---

[8]Graphically, the origional downwards sloping premium curve in Figure 2.1 becomes a horizontal line which is higher than the other two horizontal lines representing the loss of consumption by the poor and the rich respectively.

However, this implication is not consistent with the general opinion on the role played by assortative mating in the intergenerational income mobility. Typically, people think assortative mating facilitates income stratification. This viewpoint may arise based on the observation that assortative mating (assured by arranged marriage) and social stratification co-exist in the traditional society where social status, occupations and wealth are inherited. It is widely recorded about traditional society across countries that income earned by different social class is rigidly stratified and there is a little chance to realize intergenerational income mobility by making intergenerational investment. Therefore, as long as marriage is arranged to guarantee pure positive assortative mating, family income is determined by householders' family background and intergenerational income mobility is very low. However, when we look at the modern society where income is determined by individual's productivity that largely depend on his/her human capital accumulation, the rewards from labor market and marriage market can be internalized in the decision-making process of educational investment made by parental households. From the altruistic parents' point of view, educational investment provides a channel to realize high income for their children. In this model, if the educational cost structure allows enough rewards and all the parental households have the access to the high educational investment, assortative mating would enhance the incentive to make the high educational investment and boost upward income mobility.

Based on the analysis above, though it is natural to associate assortative mating and intergenerational income stratification, there is no inevitable cause-effect relationship between the two social phenomenon. Specifically, it is the education system and labor market institution that are essential to the intergenerational income mobility in the modern economy. In the next chapter, this static model is extended to a dynamic one to reveal the transmission of household income distribution over many generations. We will elaborate on the discussion of the properties of income mobility as it is more appropriate in that context.

Appendix A

A.1 Proof of Lemma 2.2

To prove Lemma 2.2, this proof proceeds in two parts that discuss two cases respectively: the one when $s_1^* = 1$, and the other when $s_1^* \in (0,1)$.

a. In the case where $s_1^* = 1$, let $f(v) = \frac{v}{(1-v)} e^{-1}(1 - e^{-\frac{1-v}{v}})$, $g(\alpha) = \frac{2\alpha}{1+\alpha}$. It can be shown that $\frac{\partial f(v)}{\partial v} > 0$, $\frac{\partial g(\alpha)}{\partial \alpha} > 0$, and $\lim_{v \to 1} f(v) = e^{-1}$. Hence, if $g(\alpha) \geq e^{-1}$, (2.17) is held. The solution to $g(\alpha) = e^{-1}$ is: $\alpha = (2e - 1)^{-1}$. Thus, as long as $\alpha \in ((2e-1)^{-1}, 1 - 2e^{-1})$, (2.17) does not hold regardless of the value of v. On the other hand, if $g(\alpha) < e^{-1}$, the comparison between $f(v)$ and $g(\alpha)$ depends on the relationship between v and α. $v_a(\alpha)$ is defined as the solution to the equation which is derived from (2.17) by taking the equality sign:

$$\frac{v}{(1-v)} e^{-1}(1 - e^{-\frac{1-v}{v}}) = \frac{2\alpha}{1 + \alpha}$$

Hence, when $\alpha \in (0, (2e-1)^{-1})$, as long as $v \in [v_a, 1)$, (2.17) holds; otherwise, given α lies in the same interval, if $v \in (0, v_a)$, (2.17) does not hold.

Finally, consider the situation when all the type 2 boys choose to propose to a type 2 girl, in other words, $s_2 = 0$, then the expected payoff for the type 2 boys would be: $u_{m22}^e = P_{22} u_p = (1 - s_2)^{-1}(1 - e^{s_2 - 1})u_p$. If any boy of type 2 deviates from this pure strategy, switching to a type 1 girl, his expected payoff would be the product of the probability that a type 1 girl does not get any proposal from type 1 boys from any of the type 1 boys, and the expected household utility generated by a type 1 girl and a type 2 boy: $e^{-1}\frac{u_r + u_p}{2}$. To support $s_2 = 0$ as the equilibrium solution, we need:

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$$e^{-1}\frac{u_r + u_p}{2} \leq u_{m22}^e$$
$$e^{-1}\frac{\alpha + 1}{2} \leq (1 - e^{-1})\alpha \qquad (A.1)$$

The inequality (A.1)is reduced to $\alpha \geq \frac{e^{-1}}{3e^{-1}-1}$, which contradicts the condition in this case:$0 < \alpha \leq 1 - 2e^{-1}$. Hence, $s_2 = 0$ can not be supported as the type 2 boys' strategy in the equilibrium when all the type 1 boys propose to type 1 girls.

b. In the case where $s_1^* \in (0, 1)$, if all the type 2 boys choose to propose to type 1 girls, $s_2 = 1$, then the expected payoff for the type 2 boys would be: $u_{m21}^e = P_{21}(u_r + u_p)/2 = \frac{v}{1-v}e^{-s_1}(1 - e^{-\frac{1-v}{v}})\frac{u_r+u_p}{2}$. If any boy of type 2 deviates from this pure strategy, switching to type 2 girls, his expected payoff would be the probability that a type 2 girl does not get any proposal from type 1 boys, times the expected household utility generated by a type 2 boy and a type 2 girl: $e^{-\frac{(1-s_1)v}{1-v}}u_p$. To support $s_2 = 1$ as type 2 boys strategy in the equilibrium, we need:

$$e^{-\frac{(1-s_1)v}{1-v}}u_p \leq \frac{v}{1-v}e^{-s_1}(1 - e^{-\frac{1-v}{v}})\frac{u_r + u_p}{2} \qquad (A.2)$$

where the relationship between v and s_1 is determined by (2.15). Specifically, according to (2.15), the RHS of inequality (A.2) $e^{-\frac{(1-s_1)v}{1-v}} = 1 - \frac{2s_1(1-e^{-s_1})}{1+\alpha}\frac{v(1-s_1)}{1-v}$. Substitute it back into (A.2), it can be shown that (A.2) does not hold in this case. Hence, $s_2^* \neq 1$.

In the same way, if all type 2 boys choose to propose to type 2 girls, $s_2 = 0$, then the expected payoff for the type 2 boys would be: $u_{m22}^e = P_{22}u_p = \frac{e^{-\frac{v(1-s_1)}{1-v}}}{1-s_2}(1-e^{s_2-1})u_p$. If any boy of type 2 deviates from this pure strategy, switching to type 1 girls, his expected payoff would be the product of the probability that a type 1 girl does not get any proposal from type 1 boys, and the expected household utility generated by a type 2 boy and a type 1 girl: $e^{-s_1}(u_r + u_p)/2$. To support $s_2 = 0$ as type 2 boys strategy in the equilibrium, we need:

$$e^{-s_1}\frac{(u_r + u_p)}{2} \leq \frac{e^{-\frac{v(1-s_1)}{1-v}}}{1 - s_2}(1 - e^{s_2-1})u_p \qquad (A.3)$$

where the relationship between v and s_1 is determined by (2.15). It can be shown that (A.1) does not hold in this case. Hence, $s_2^* \neq 0$.

Finally, if the type 2 boys choose a mixed strategy $s_2 \in (0, 1)$, being indifferent between type 1 and type 2 girls, which indicates:

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$$\frac{ve^{-s_1}}{s_2(1-v)}(1 - e^{-\frac{-s_2(1-v)}{v}})\frac{u_r + u_p}{2} = \frac{e^{-\frac{v(1-s_1)}{1-v}}}{1 - s_2}(1 - e^{s_2-1})u_p \qquad (A.4)$$

again, the relationship between v and s_1 is determined by (2.15). It can be shown that there is a unique solution satisfying the equation (A.4). Hence, in the case $s_1^* \in (0,1)$, type 2 boys adopt a mixed strategy $s_2^* \in (0,1)$ in equilibrium.

A.2 Proof of Lemma 2.3

To prove Lemma 2.3, this proof proceeds in 3 parts: Part a, Part b and Part c. Part a shows the marginal benefit of making the high educational investment is monotonic decreasing in the fraction of skilled offspring v; part b generates the conditions under which parental households have the incentive to deviate from the pure strategy $k_r = k_p = 0$, which indicates all the parental households uniformly investing low and consequently $v = 0$; part c generates the conditions under which parental households have the incentive to deviate from the pure strategy $k_r = k_p = 1$, which indicates all the parental households uniformly investing high and consequently $v = 1$.

a. Given any foreseeable joint educational investment strategy outcome, represented by the fraction of skilled persons in the offspring's generation: $v \in \{0,1\}$, the marginal benefit for parental households of switching from investing high to investing low, or in other words, the marginal value of being a type 1 person in the matching stage is:

$$
\begin{aligned}
R_1 &= \beta\{P_{1r}(v + \Delta v)u_r + [1 - P_{1r}(v + \Delta v)]u_p - \{[P_{2r}(v)u_r] + [1 - P_{2r}(v)]u_p\}\} \\
&= \beta\{[P_{1r}(v + \Delta v)u_r - P_{2r}(v)]u_r + [P_{2r}(v) - P_{1r}(v + \Delta v)]u_p\} \\
&= \beta\{[P_{1r}(v + \Delta v) - P_{2r}(v)](u_r - u_p)\}
\end{aligned}
$$

$$(A.5)$$

where P_{1r} is shown as a function of v. R_1 indicates when an incremental fraction of type 1 offspring are produced, how much more utility they would generate as the contribution to their parents' utility than if they are type 2 person instead.

Taking Δv to zero, let $R = \lim_{\Delta v \to 0} R_1$, we get:

$$
\begin{aligned}
\lim_{\Delta v \to 0} R_1 &= \lim_{\Delta v \to 0} \beta \{ [P_{1r}(v + \Delta v) - P_{2r}(v)](u_r - u_p) \} \\
&= \beta \{ [P_{1r}(v) - P_{2r}(v)](u_r - u_p) \} \\
&= \beta [s_1^{-1}(1 - e^{-s_1}) - \frac{1}{2} \frac{v}{s_2(1-v)} e^{-s_1} (1 - e^{-\frac{s_2(1-v)}{v}})](u_r - u_p)
\end{aligned}
$$
(A.6)

It can be shown that $\frac{\partial R}{\partial v} < 0$.

b. Consider the scenario where all the parental households choose the pure strategy, $k_r = k_p = 0$, all making the low educational investment. If a small fraction of households deviate from this pure strategy to make the high educational investment and sacrifice part of their consumption utility, skilled offspring would be produced who can get matched without coordinate frictions due to the small number, and form rich household at probability close to 1. The benefit of this deviation differs between the rich and the poor parental households, which is represented by d_r and d_p respectively as follows:

$$
\begin{aligned}
d_r &= u_r^{c1} + \beta u_r - u_r^{c2} - \beta u_p \\
&= u_r^{c1} - u_r^{c2} + \beta(u_r - u_p)
\end{aligned}
$$
(A.7)

$$
\begin{aligned}
d_p &= u_p^{c1} + \beta u_r - u_p^{c2} - \beta u_p \\
&= u_p^{c1} - u_p^{c2} + \beta(u_r - u_p)
\end{aligned}
$$
(A.8)

Due to the concavity of utility function $u(.)$, the poor parental household suffer more by the high investment than the rich parental household: $u_r^{c2} - u_r^{c1} < u_p^{c2} - u_p^{c1}$, equivalently, $u_r^{c1} - u_r^{c2} > u_p^{c1} - u_p^{c2}$, and $d_r > d_p$. Hence, so long as $d_p > 0$, both the rich and the poor households have incentives to deviate from this pure strategy; $d_p < 0 < d_r$, the rich households have the incentive to deviate from this pure strategy while the poor household would like to stay; $d_p < d_r < 0$, both types of household would like to stay with this pure strategy.

c. Consider the scenario where all the parental households choose the pure strategy, $k_r = k_p = 1$, all making the high educational investment. If a small fraction of households deviates from this pure strategy to make the low educational investment and enjoy a higher consumption utility, unskilled offspring would be produced, their children's probability of forming a household with a skilled person would approximately equal to 1 as v is very close to 1. The benefit of this

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deviation are then represented by d'_r and d'_p respectively as follows:

$$d'_r = u_r^{c_2} + \beta \frac{(u_r + u_p)}{2} - u_r^{c_1} - \beta u_r$$
$$= u_r^{c_2} - u_r^{c_1} + \beta \frac{1}{2}(u_p - u_r) \tag{A.9}$$

$$d'_p = u_p^{c_2} + \beta \frac{(u_r + u_p)}{2} - u_p^{c_1} - \beta u_r$$
$$= u_p^{c_2} - u_p^{c_1} + \beta \frac{1}{2}(u_p - u_r) \tag{A.10}$$

In the same way, due to the concavity of utility function $u(.)$, $u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1}$, which leads to $d'_r < d'_p$. Hence, so long as $d'_r > 0$, both the rich and the poor households have incentives to deviate from this pure strategy; $d'_r < 0 < d'_p$, the poor households have the incentive to deviate from this pure strategy while the rich household would like to stay with $k_r = 1$; $d'_r < d'_p < 0$, both types of households would like to stay with this pure strategy.

Substitute the expressions of d_r and d_p, d'_r and d'_p back into the analysis at the end of part b and part c, we get the results shown in Lemma 2.3. In addition, the Figure A.1 illustrate the six types of investment profiles as stated in Lemma 2.3.

A.3 Illustration of Properties of the Household Formation

Properties of household formation can be illustrated by Figure A.2 to Figure A.5. The four figures correspond to the four cases discussed in Lemma 2.1 and Lemma 2.2 according to the separate parameter regions. Each of the following figures consists of six sub-figures. From the top left to the top right, each of the three sub-figures represents: the relationship between type 1 boys' strategy in the marriage market equilibrium s_1^* and the fraction of skilled individuals at the beginning of the matching process, v; the relationship between type 2 boys' strategy in the equilibrium s_2^* and v; the relationship between the difference in the probabilities of forming a rich household as a type 1 boy and a type 2 boy $P_{1r}^{m*} - P_{2r}^{m*}$ and v. From the bottom left to the bottom right, each of the three sub-figures represents: the relationship between the difference in the probabilities of forming a rich household for a type 1 girl and a type 2 girl, $P_{1r}^{f*} - P_{2r}^{f*}$ and v; the average of the differences in probabilities, $\frac{1}{2}[(P_{1r}^{m*} - P_{2r}^{m*}) + (P_{1r}^{f*} - P_{2r}^{f*})]$

against v; the relationship between v and w_r^*.

Figure A.2 is a numerical example of the case where $\alpha \in (0, (2e-1)^{-1})$ and $v \in [v_a, 1)$. The subfigures show $s_1^* = s_2^* = 1$; $P_{1r}^{m*} - P_{2r}^{m*} = P_{1r}^{f*} - P_{2r}^{f*}$, both decreasing in v; $v = w_r^*$.

Figure A.3 is a numerical example of the case where $\alpha \in (0, (2e-1)^{-1})$ and $v \in (0, v_a)$. The subfigures show $s_1^* = 1$, s_2^* is increasing in v, and $P_{1r}^{m*} - P_{2r}^{m*} = P_{1r}^{f*} - P_{2r}^{f*}$, both decreasing in v; $v = w_r^*$. Figure A.4 is a numerical example of the case where $\alpha \in [(2e-1)^{-1}, 1 - 2e^{-1}]$, $s_1^* = 1$ and s_2^* lies between the interval $(0, 1)$, increasing in v, and $P_{1r}^{m*} - P_{2r}^{m*} = P_{1r}^{f*} - P_{2r}^{f*}$, both decreasing in v; $v = w_r^*$.

Figure A.5 is a numerical example of the case where $\alpha \in (1 - 2e^{-1}, 1)$, both s_1^* and s_2^* lies between the interval $(0, 1)$, increasing in v, and $P_{1r}^{m*} - P_{2r}^{m*} = P_{1r}^{f*} - P_{2r}^{f*}$, both decreasing in v; $v = w_r^*$.

A.4 Comparative Statics

Figure A.6 gives a numerical example about how the premium of high educational investment decreases as the fraction of skilled offspring increases under different values of α.
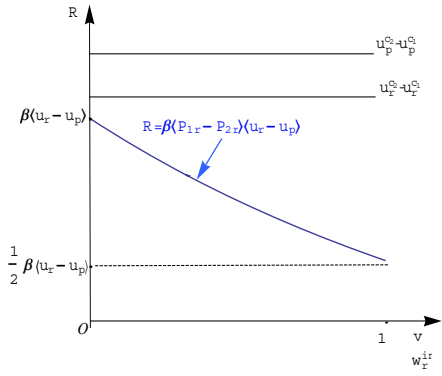
(a) Type (i) investment profile
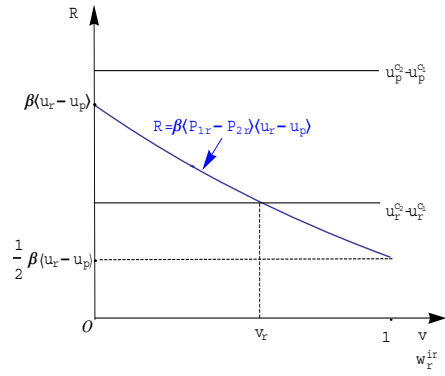
(b) Type (ii) investment profile
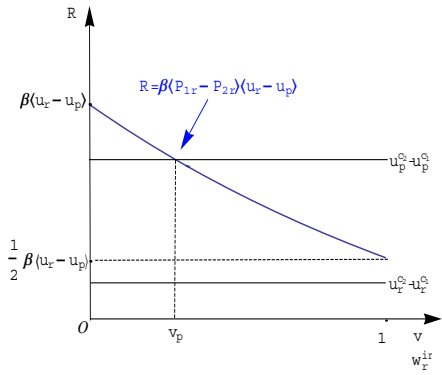
(c) Type (iii) investment profile
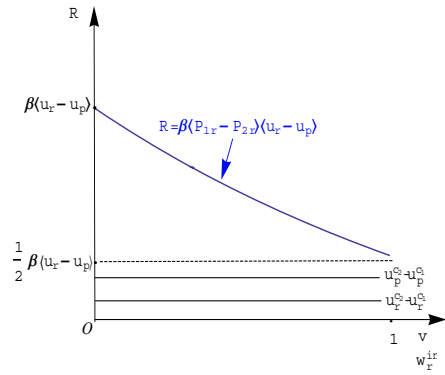
(d) Type (iv) investment profile
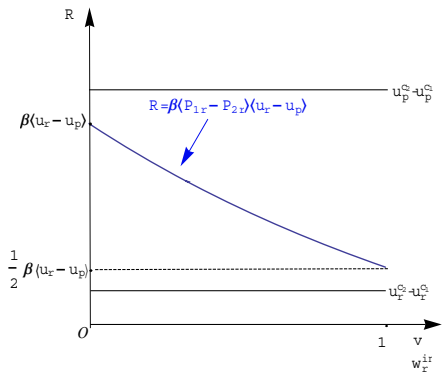
(e) Type (v) investment profile
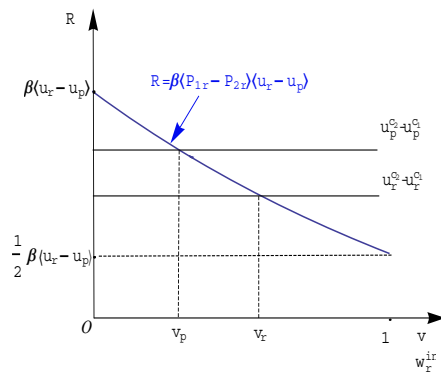
(f) Type (vi) investment profile

Figure A.1: Six types of investment profiles
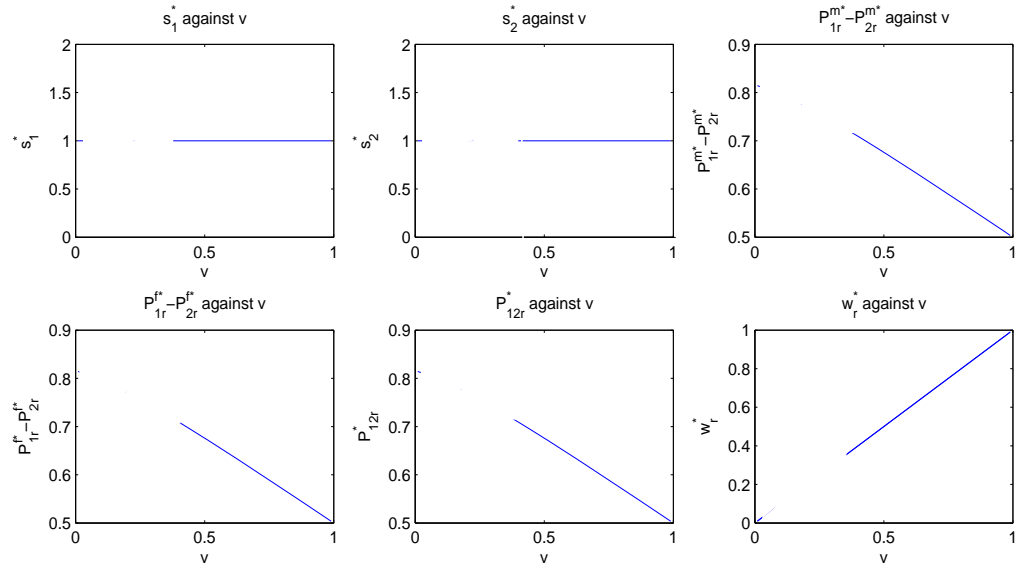
Figure A.2: Illustration of properties of the household formation, given $\alpha = 0.1$ and $v \in (0.38, 1)$
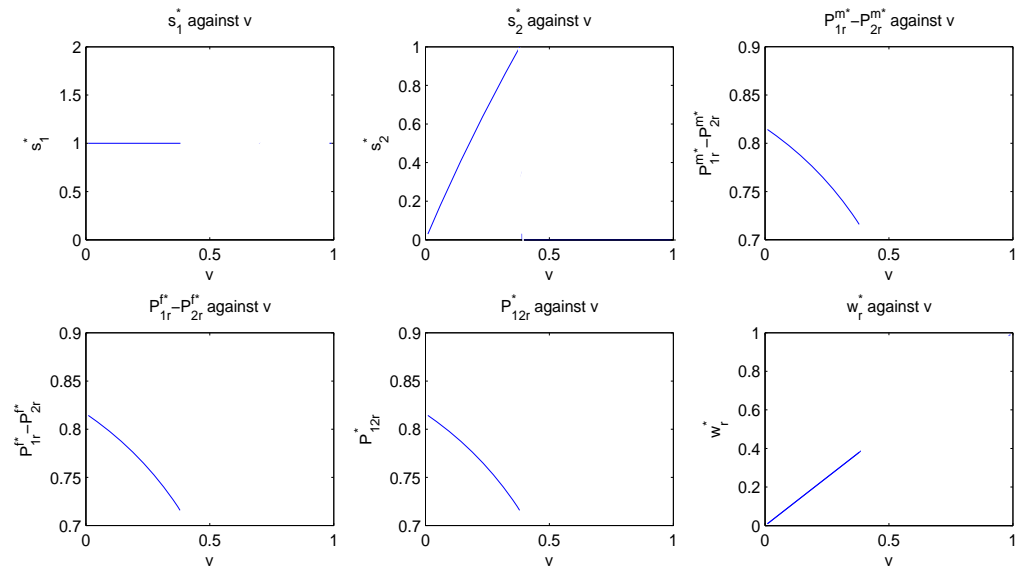
Figure A.3: Illustration of properties of the household formation, given $\alpha = 0.1$ and $v \in (0, 0.38)$
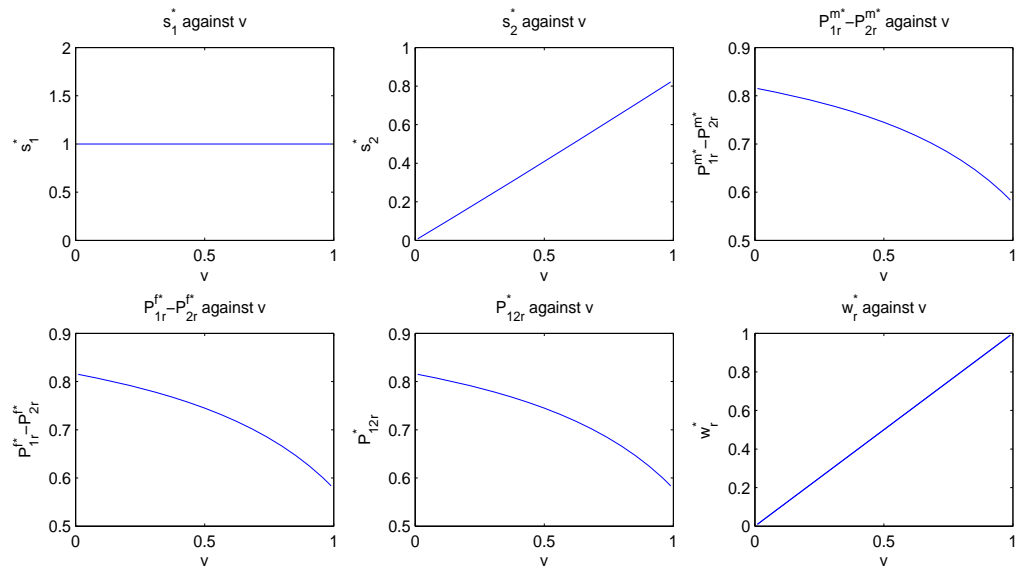
Figure A.4: Illustration of properties of the household formation, given $\alpha = 0.25$
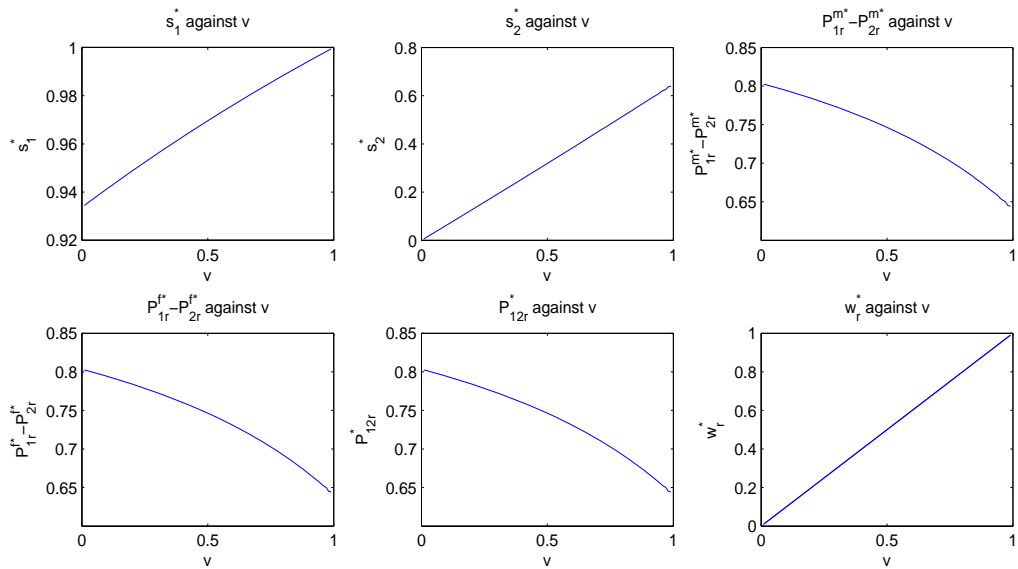
Figure A.5: Illustration of properties of the household formation, given $\alpha = 0.3$
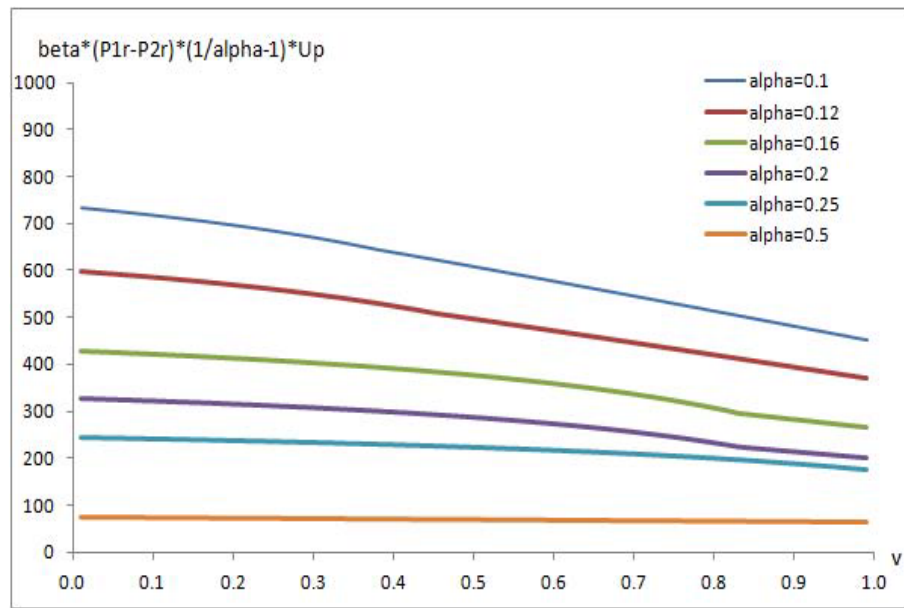
Figure A.6: The premium of high educational investment against the fraction of skilled offspring under different values of α, given $\beta u_p = 100$

Chapter 3

Intergenerational Educational Investment and Household Income Distribution over Generations

Abstract

This chapter extends the static model in the previous chapter into a dynamic one. The household income disparity is held constant and exogenous over time, while the fraction of rich households is endogenously determined over finite generations. The mechanism of the variation in income distribution through the channel of educational investment and household formation follows the spirit of the static model. A stationary fraction of the rich households is generated by backward induction. Compared with the results of the static model, a larger increase in the fraction of rich households is more likely to happen in the dynamic setting than in the static setting given a certain initial income distribution. This is because the poor are better motivated to choose high educational investment when they need to take more generations' welfare into consideration.

Key Words: Income distribution, income mobility, educational investment

3.1 Introduction

Following the theoretical framework established by the static model in the previous chapter, the extended dynamic model presented in this chapter attempts to capture the strategic thinking of parental households about the competition in the offspring family formation process over many generations, and its implications for intergenerational educational investment, the variation in income distribution and income mobility.

Educational investment has been identified as a channel of upward income mobility both by theoretical work[1] and empirical studies[2] on intergenerational income mobility. Compared with developing countries where education is still mainly financed by parental income, many developed countries have mature financial system and government subsidies to support young people from poor families in completing higher education. With these financial aids, the poor are given more chance to have skilled offspring who in return gain a higher lifetime income. However, the potential pre-college education costs of providing a good learning environment are huge and involve many aspects of households' economic activities, such as family location choice, extra curriculum allowance for the children and so on. Rich parental households have advantages in financing better but more expensive educational investment, through which their children, on average, acquire more advanced skills and better platforms for their careers and thus are more likely to get well paid positions in the job market. In short, the financial advantages of rich households in making educational investment decisions are common across countries, only differing in the extent. These advantages become prominent when the costs of education go up, which worries the public in the social mobility aspect. A very key question is, whether intergenerational educational investment has become the channel of enhancing the financial advantages of the rich, leading to a stratification of the social structure, or whether it still functions as a major channel of upward income mobility?

To answer this research question, it is worth understanding the inequality in education sector. Inequality in returns to education influences the incentive to make educational investment, while inequality in access to education affects the ability to make educational investment. Incentives in educational investment,

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[1]See the seminal work by Becker (1962), Becker and Tomes (1979, 1986). More recently see Benabou (2001).

[2]There is a large volume of empirical studies using data set across countries indicate that educational investment contributes to upward income mobility. For a survey, see the work by D'Addio (2007).

inequality in return to education and education availability are well investigated in the literature on the relationship between income inequality and income mobility.[3] Among these studies, Hassler, Rodríguez Mora and Zeira (2007) distinguish two types of inequality which explain the different relationship between income inequality and income mobility under a macro theoretical framework. It reveals that a negative relationship between income inequality and income mobility may arise when the variation of country specific inequality in the education sector (e.g. barriers to becoming skilled, costs of education) dominates. This is because an unequal education system favors the skilled (the high income group) who are able to accumulate advantages over generations. This is very likely to result in a rigid social structure. By contrast, a positive relationship may arise when the variation of country specific inequality in production sector dominates (skill-biased wage inequality). This is because a large skill-biased wage inequality increases people's incentives to make educational investment, which contributes to the upward income mobility.

In line with this insight in the relationship between inequality in education sector and income mobility, this model (both the static one and the following dynamic one) captures the inequality in returns to education by a constant ratio of the household utility obtained from a high income to that from a low income. Meanwhile, to capture the inequality in access to education, this model adopts a differential cost structure. In addition, a concave utility function is employed to reveal the fact that the poor parental household always suffers more than the rich parental household from making the costly educational investment. The implications of this model are consistent with the insights of previous studies mentioned above. An increase in cost of the high educational investment is harmful to the upward income mobility, while an increase in the inequality of feasible household income has an opposite effect.

Besides educational investment, intergenerational income mobility depends on many other factors as explored firstly in Becker (1979), of which the inheritance of physical capital, genetic heterogeneity and other endowments from parents like caste, religion, etc have interested economists in recent decades. In this chapter and the previous one, to make the model tractable, we assume that neither non-human capital investment nor other forms of endowment are available to parents as the channel of intergenerational economic status transmission.

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[3]Recent research reveals both a positive and a negative relationship between income inequality and income mobility by making comparisons among countries. On the negative relationship, see Solon (2004), Corak (2006), D'Addio (2007), Andrews and Leigh (2009), and on the positive relationship see Checchi, Ichino and Rustichini(1999).

In addition to educational investment, family formation process also plays an important role in intergenerational economic status transmission. Positive assortative matching in income in the marriage market, which has been observed in empirical studies, contributes substantially to the rigidity of social structure[4]. The contribution of this model is that it incorporates the family formation process in a frictional matching environment into the analysis of strategic thinking over intergenerational educational investment and its implications for income mobility. Following the static model in the previous chapter, matching is not perfectly assortative here and the 'mismatch' phenomenon is captured by coordination frictions generated under the directed matching mechanism. It acknowledges that the friction in the matching process is one of the sources of intergenerational income mobility.

As mentioned in the previous chapter, by using a neutral household production technology, these frictions do not affect the productivity of the education sector in the sense that 'who marries whom' does not change the fraction of rich households formed through the marriage market. The contribution of education sector to the overall economy is determined only by the fraction of skilled individuals who receive high educational investment. But frictions do affect the incentives of intergenerational educational investment because the frictional matching process creates chance for an unskilled individual of forming a rich household and a skilled individual forming a poor household.

To elaborate on the role played by this neutral household production technology in the income distribution's variation across generations, it is worth comparing with other possible matching production technologies. In the directed search literature on labor market, production technology is often assumed as skill-biased, or in the terminology: supermodular. Among these studies, Shi (2002) explains within-type wage dispersion as the result of directed search mechanism under skill-biased production technology, where matching between heterogeneous firms, who make endogenous investment, and heterogeneous workers with exogenous type distribution, is partially mixed and socially efficient. In the literature fo-

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[4]See the work by Lam and Schoeni (1994), which interprets the greater effect of father-in-law's schooling than that of father's schooling on the wages of male workers in Brazil as an indication of a high degree of assortative mating in the marriage market. Chadwick and Solon (2002) present evidence on daughters' intergenerational mobility in the United States with data from the Panel Study of Income Dynamics (PSID). They find that the assortative mating is an important element in the intergenerational transmission process. In the study using German Socio-Economic Panel and British Household Panel Survey, Ermisch, Francesconi and Siedler (2005) find the assortative mating contributes 40%-50% to the covariance between parents' and own permanent family income. This effect is driven by a strong spousal correlation in human capital, which is larger in Germany than Britain.

cusing on the efficiency of matching in the non-transferable[5]utility environment, where workers make endogenous investment, submodular production technology leads to overinvestment by the high type and underinvestment by the low type (Gall, Legros, Newman, 2009). Intuitively, we conjecture that when the household production technology changes to be supermodular, separating equilibrium (skilled only propose to skilled, unskilled only propose to unskilled) in the marriage market is plausible. It would take more generations to reach a stationary household income distribution, and the fraction of rich households is lower than that in this model. By contrast, if the household production technology is submodular, the stationary fraction of rich households would be higher than that in this model. This is because coordination frictions reduce the inefficiency in overall output, which is caused by segregation under the non-transferable utility environment and submodular household production technology.

The arrangement of this chapter is as follows. Section 2 introduces the framework of this dynamic model based on the static one in the previous chapter. Section 3 generates the equilibrium income distribution and income mobility rates, where the effects of income disparity and education cost structures on the equilibrium outcomes are also investigated. Section 4 provides concluding remarks and discussions of the model's main findings.

3.2   The Model

We consider a dynamic model across many but finite generations, using backward induction to find out the variation pattern of income distribution. As in the static model of the previous chapter, parental households decide whether to make a high investment or a basic one on their children, to maximize their total utilities by finding a balance between their own consumption utility and the welfare of their offspring. To keep simplicity on the forward looking behavior, we have this assumption as follows:

Assumption 1: People are partial rational in the marriage market before forming households, in the sense that they anticipate that the inequality in feasible

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[5]A term commonly used in the matching literature, which is in contrast to transferable utility. The theoretical work on marriage by Becker (1973) is considered as under the transferable utility framework, as is the work by Shimer and Smith (2000). The studies by Roth and Sotomayor (1990) is considered as under the non-transferable utility framework, as is the work by Smith(2006).

household income will be a constant, which is denoted by α, where $\alpha \in (0,1)$[6].

This assumption simplifies the strategies of marriage market participants, which can be forecast by their parents when making educational investment decisions. Compared with offspring's partial rationality, parents can fully anticipate variations in the difference between forming a rich household and forming a poor household in different generations. Specifically, after the young generation grow up, they become aware of the income distribution in their own generation and its impact on the relative value of being rich or poor. Since parents possess the ability of full rationality when making investment decisions on their own children, the parental households in every generation are capable of foreseeing the behavior of all the descendants' behavior in the successive generations. They make consumption-investment decisions with complete information of the income distributions in all the future generations. They believe that all the grown-up descendants would choose an investment strategy that is the same as what they would choose in that state.

Assume that all the households in the first generation are identical, earning low income, recognized as the poor households. There is an educational investment opportunity that could give their children some chance to form a household with a higher income, recognized as the rich households. Given a parental household's income state vector $(Q_r^0, Q_p^0) = (0,1)$ in the first generation, generation 0, the possibility of their descendants being rich or poor in generation t is denoted by a 1×2 row vector, (Q_r^t, Q_p^t), where

$$Q_r^1 = K_r^0 P_{ir}^1, \quad Q_p^1 = K_r^0 P_{ip}^1$$
$$Q_r^2 = (Q_r^1, Q_p^1)(K_r^1 P_{ir}^2, K_p^1 P_{ir}^2)', \quad Q_p^2 = (Q_r^1, Q_p^1)(K_r^1 P_{ip}^2, K_p^1 P_{ip}^2)'$$
$$Q_r^3 = (Q_r^2, Q_p^2)(K_r^2 P_{ir}^3, K_r^2 P_{ir}^3)', \quad Q_p^3 = (Q_r^2, Q_p^2)(K_r^2 P_{ip}^3, K_r^2 P_{ip}^3)'$$
$$\vdots \qquad \vdots$$

where K_r^t and K_p^t are the investment strategy vectors of the rich parental household and the poor parental household of generation t respectively, which are defined in section 2.4.1 of the previous chapter. Denote the probabilities of forming a rich household for the generation t descendants by the column vector, P_{ir}^t

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[6]The concept of partial rational is borrowed from the paper by Burdett and Melvyn (1997) "Marriage and class", where agents believe that the sex ratio in the current marriage market stays constant even though it may not.

and the probabilities of forming a poor household by the column vector, P_{ip}^t:

$$P_{ir}^t = \begin{bmatrix} P_{1r}^t \\ P_{2r}^t \end{bmatrix}, P_{ip}^t = \begin{bmatrix} P_{1p}^t \\ P_{2p}^t \end{bmatrix}$$

Both P_{ir}^t and P_{ip}^t are determined by the exogenous variable α, and the fraction of skilled individuals, v^t, in generation t, which is pinned down by the rich and the poor parental households' joint investment strategies in generation $t-1$ as discussed in the static model. Based on the formulas shown above, it is obvious to see that the descendants in generation t could possibly be rich or poor at the probability Q_r^t and Q_p^t, $t \geq 1$, respectively, which follows the law of motion expressed as:

$$Q_r^t = (Q_r^{t-1}, Q_p^{t-1})(K_r^{t-1} P_{ir}^t, K_p^{t-1} P_{ir}^t)' \tag{3.1}$$

$$Q_p^t = (Q_r^{t-1}, Q_p^{t-1})(K_r^{t-1} P_{ip}^t, K_p^{t-1} P_{ip}^t)' \tag{3.2}$$

where $Q_r^t + Q_p^t = 1$.

(Q_r^t, Q_p^t) represents the state variable vector of the descendants. $K_r^{t-1} = (k_r^{t-1}, 1 - k_r^{t-1})$, as the control variable vector for the rich households in generation $t-1$, denotes the investment strategy adopted by rich parental households in generation $t-1$ $(t \geq 1)$, while $K_p^{t-1} = (k_p^{t-1}, 1 - k_p^{t-1})$ denotes the investment strategy chosen by the poor. The product of $K_r^{t-1} P_{ir}^t$ is the probability of a rich household in generation $t-1$ $(t \geq 1)$, having rich offspring in generation t; $K_r^{t-1} P_{ip}^t$ is the probability of a rich household in generation $t-1$ having poor offspring in generation t; $K_p^{t-1} P_{ir}^t$ is the probability of a poor household in generation $t-1$ having rich offspring in generation t; $K_p^{t-1} P_{ip}^t$ is the probability of a poor parental household in generation $t-1$ having poor offspring in generation t.

The dependence of Q_r^t or Q_p^t on both K_r^{t-1} and K_p^{t-1} stems from the fact that the rich/poor descendants in generation t may come from either the rich or the poor households in generation $t-1$. In particular, the rich parental households and the poor parental households interplay with each other in generation $t-1$, and their investment strategies jointly determine the fraction of skilled people in the next generation and ultimately pin down the chance of forming a rich or a poor household in generation t, P_{ir}^t or P_{ip}^t. Three elements altogether pin down the state vector of their next generation, (Q_r^t, Q_p^t), according to the law of motion described by the equation (3.1) and (3.2): first, the probability of forming a rich household, P_{ir}^t, and a poor household P_{ip}^t, of their next generation; second, their

investment strategy profile, represented by the control variables, (K_r^{t-1}, K_p^{t-1}); third, the state of their own generation, (Q_r^{t-1}, Q_p^{t-1}).

Hence, any change to the control variable vectors, K_r^{t-1} and K_p^{t-1}, will yield different pairs of the state variables (Q_r^t, Q_p^t) and those in the following generations. That is the effect of the current investment strategies on the state vectors in the future generations. In addition, K_r^{t-1} and K_p^{t-1} also have effects on the payoffs of households in current generation directly. In each generation t $(t \geq 1)$, the payoff of the rich household (the poor households, resp.) is the expected consumption utility relying on their own investment strategy K_r^t (K_p^t, resp.), which is equal to $K_r^t(u_r^{c_1}, u_r^{c_2})'$ ($K_p^t(u_p^{c_1}, u_p^{c_2})'$, resp.). The vectors $(u_r^{c_1}, u_r^{c_2})$ and $(u_p^{c_1}, u_p^{c_2})$ are assumed to be constant over time, only depending on the exogenous parameters, household income levels and the costs of educational investments.

Given an initial state $(Q_r^0, Q_p^0) = (0, 1)$, and a sequence $(K_r^{(0,T-1)}, K_p^{(0,T-1)}) = \{(K_r^t, K_p^t); t = 0, 1, 2, ..., T-1\}$ of investment strategies in each generation, the evolution of the descendants' chance of being rich or poor in the next generation is then determined by the law of motion according to equation (3.1) and (3.2). Therefore, (Q_r^0, Q_p^0) induces a sequence of $(Q_r^{(1,T)}, Q_p^{(1,T)})$. Denote $z_{0,T} = \{(K_r^{(0,T-1)}, K_p^{(0,T-1)}) \cup (Q_r^{(1,T)}, Q_p^{(1,T)})\}$ as the feasible sequence. Given a sequence $z_{0,T}$, we can calculate the income distribution $w_r^t, (t \geq 1)$ in each generation, where $w_r^t = Q_r^t$, and $1 - w_r^t = Q_p^t$. The law of motion described in equation (3.1) and (3.2) can be rewritten as:

$$w_r^t = (w_r^{t-1}, 1 - w_r^{t-1})(K_r^{t-1}P_{ir}^t, K_p^{t-1}P_{ir}^t)' \tag{3.3}$$

$$1 - w_r^t = (w_r^{t-1}, 1 - w_r^{t-1})(K_r^{t-1}P_{ip}^t, K_p^{t-1}P_{ip}^t)' \tag{3.4}$$

Recall that we analyzed how the rich interplay with the poor when choosing investment strategies to maximize their payoffs under a given initial income distribution in the static model of two consecutive generations. Similarly, in the dynamic setting, the rich interplay with the poor through the control variable vector K_r^t and K_p^t along the optimal path through generations. (Q_r^t, Q_p^t) could have non-zero elements, which means descendants can be rich or poor at some positive probability in generation t. Because of this possibility, decision-makers in generation $t-1$ know that their offspring in future generations are likely to form a rich household or a poor household. Hence they will have to foresee both K_r^t and K_p^t when choosing an optimal control K_r^{t-1} or K_p^{t-1} by imagining themselves being rich and poor respectively under every possible income distribution of the corresponding generations. Formally, given $w_r^{t-1}, u_r^{c_1}, u_r^{c_2}, u_p^{c_1}, u_p^{c_2}$ and the value of

being rich or poor in generation t, under every possible income distribution w_r^t, the maximum attainable value of the parental households in generation $t-1$ can be achieved by choosing the optimal K_r^{t-1*}, or K_p^{t-1*}:

$$V_r^{t-1} = \max_{K_r^{t-1}}\{K_r^{t-1}[(u_r^{c_1}, u_r^{c_2})' + \beta V_r^t P_{ir}^t + \beta V_p^t P_{ip}^t]\} \tag{3.5}$$

$$V_p^{t-1} = \max_{K_p^{t-1}}\{K_p^{t-1}[(u_p^{c_1}, u_p^{c_2})' + \beta V_r^t P_{ir}^t + \beta V_p^t P_{ip}^t]\} \tag{3.6}$$

$s.t.(3.3), (3.4), w_r^0 = 0, V_r^T = u_r, V_p^T = u_p$

Equations (3.5) and (3.6) are the Bellman's Equations (BE) for the rich parental households and the poor parental households respectively. These equations characterize the optimal choice of the current control variables K_r^t and K_p^t as the solution to a static optimization problem in which the future consequences of current parental households' investment strategies are summarized by incorporating the next generation's value function into the current generation's return function. The solution to this static maximization problem gives the optimal values of the current controls, K_r^{t-1} and K_p^{t-1} as a function g_{t-1} of the current state, w_r^{t-1}, which is defined by (3.9).

In the following analysis, we will focus on the symmetric equilibrium under the same exogenous parameters region as stated in Proposition 2.1, $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$, which corresponds to the type (vi) investment profile as categorized in Lemma 2.3. To compute the optimal control sequence, we use backward induction starting from the last two generations and working backwards. The second last generation's maximization problem becomes:

$$V_r^{T-1}(w_r^{T-1}, T-1) = \max_{K_r^{T-1}}\{K_r^{T-1}[(u_r^{c_1}, u_r^{c_2})' + \beta S_r^T P_{ir}^T + \beta S_p^T P_{ip}^T]\} \tag{3.7}$$

$$V_p^{T-1}(w_r^{T-1}, T-1) = \max_{K_p^{T-1}}\{K_p^{T-1}[(u_p^{c_1}, u_p^{c_2})' + \beta S_r^T P_{ir}^T + \beta S_p^T P_{ip}^T]\} \tag{3.8}$$

$st.(3.3), (3.4), w_r^0 = 0, S_r^T = u_r, S_p^T = u_p$

where S_r^T and S_p^T are the scrap value functions. The solution to the last two generations' problem is exactly the same as in the static model stated by Proposition 2.1. Every possible combination of the parental households' investment decisions in generation $T-1$ jointly determines a corresponding fraction of rich households w_r^T in their next generation. Rational parental households in generation $T-1$

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would take into account the consequences of every possible w_r^T to their payoffs. They also foresee that their children are the last generation who will consume up all the household income so that the value of forming a rich household is equal to u_r and the value of forming a poor household is equal to u_p (both u_r and u_p are constant according to the model set-up).

To compute the value functions in generation $T-2$, we need to know the relationship between the controls K_r^{T-2}, K_p^{T-2} and the associated return in utilities of offspring in generation $T-1$. Let g_{T-1} denote the last policy function yielded by the solution to (3.7) and (3.8), where g_{T-1} satisfies:

$$g_{T-1}(w_r^{T-1}) = (K_r^{T-1*}, K_p^{T-1*}) \tag{3.9}$$

This policy function g_{T-1} maps w_r^{T-1} into the equilibrium investment strategies of the parental households in generation $T-1$. Recall that every possible pair of the control (K_r^{T-1*}, K_p^{T-1*}) generated under different state variable w_r^{T-1}, leads to a unique value of v_T^* as the fraction of skilled individuals in generation T, so we can deduce the relationship between w_r^{T-1} and v_T^* (where $w^{T*}=v^{T*}$ according to Property 1 of the household formation generated in the previous chapter) from the policy function in the same manner as in the static model, which is shown in Figure 2.2.

Moreover, by finding out the optimal investment strategies (K_r^{T-1*}, K_p^{T-1*}) under every possible w_r^{T-1} as the solution to (3.7) and (3.8), we can compute $V_r(w_r^{T-1}, T-1)$ that represents the value of being rich in generation $T-1$ under any given value of w_r^{T-1}, and $V_p(w_r^{T-1}, T-1)$ that represents the value of being poor in generation $T-1$ under any given value of w_r^{T-1}. In particular, each w_r^{T-1} corresponds to a unique pair of $(V_r(w_r^{T-1}, T-1), V_p(w_r^{T-1}, T-1))$. Each pair of controls (K_r^{T-2}, K_p^{T-2}) leads to a unique value of w_r^{T-1}. The relationship is pinned down between the controls K_r^{T-2}, K_p^{T-2} and the associated utilities of offspring in generation $T-1$, V_r^{T-1} and V_p^{T-1}. By knowing this, the value functions of the parental households of generation $T-2$ can be computed.

$$V_r^{T-2}(w_r^{T-2}, T-2) = \max_{K_r^{T-2}}\{K_r^{T-2}[(u_r^{c_1}, u_r^{c_2})' + \beta V_r^{T-1}P_{ir}^{T-1} + \beta V_p^{T-1}P_{ip}^{T-1}]\}$$

$$V_p^{T-2}(w_r^{T-2}, T-2) = \max_{K_p^{T-2}}\{K_p^{T-2}[(u_p^{c_1}, u_p^{c_2})' + \beta V_r^{T-1}P_{ir}^{T-1} + \beta V_p^{T-1}P_{ip}^{T-1}]\}$$

$s.t.(3.3), (3.4), w_r^0 = 0, S_r^T = u_r, S_p^T = u_p$

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Based on the value function in $T-2$ described above, we can derive the corresponding policy function g_{T-2}, $g_{T-2}(w_r^{T-2}) = (K_r^{T-2*}, K_p^{T-2*})$, where g_{T-2} maps w_r^{T-2} into the equilibrium investment strategy pair K_r^{T-2*}, K_p^{T-2*}. The relationship between w_r^{T-2} and v^{T-1*} (where $w^{T-1*}=v^{T-1*}$) can also be deduced from this policy function. Proceeding in this manner, we eventually reach the initial generation and solve equation (3.5) and (3.6) to derive the value functions of the first generation and the first policy function, g_0. The whole sequence of policy functions $\{g_t; t = 0, 1, 2, ...T-1\}$, the corresponding income distribution and the investment strategies can also be traced.

3.3 Equilibrium Income Distribution and Income Mobility

Following the setting up of the static model in the last chapter, we employ a neutral household production technology and the parameter region corresponding to the type (vi) investment profile in the dynamic model setting of this chapter. Therefore Property 1 and Property 2 of the household formation apply here as well. According to Assumption 1, people are partial rational in the marriage market, therefore the probabilities of forming a rich household, denoted by P_{1r}^t and P_{2r}^t, are determined by the fraction of skilled individuals in their own generation, v^t and the exogenous variable α under the matching mechanism described in the static model. $(P_{1r}^t - P_{2r}^t)$ changes in α as illustrated in Proposition 2.3 of the previous chapter.

Recall that we derived the relationship between the parental households' income distribution and their investment strategies by looking into the trade-off between the consumption utility loss and the premium of high educational investment. Instead of foreseeing the difference in utilities of being rich and being poor in the last generation, $u_r - u_p$ (that is a constant since the last generation consume up all their income), the parental households in generation $T-2$ would anticipate that this inequality equals the difference between the value of being rich and that of being poor in the offspring generation $T-1$, $(V_r^{T-1} - V_p^{T-1})$ (that is not a constant, since the households in generation $T-1$ have to make educational investment). As discussed in the static model, the rich and the poor parental households interplay with each other in generation $T-1$, and the opti-

mal investment strategies, denoted by K_r^{T-1*} and K_p^{T-1*}, vary across three segments of w^{T-1} $(v^{T-1})^7$ according to Proposition 2.1. Specifically, $K_r^{T-1*} = (1,0)$ and $K_p^{T-1*} = (\frac{v_p^T - w_r^{T-1}}{1-w_r^{T-1}}, 1 - \frac{v_p^T - w_r^{T-1}}{1-w_r^{T-1}})$ when $w_r^{T-1} \in [0, v_p^T]$; $K_r^{T-1*} = (1,0)$, and $K_r^{T-1*} = (0,1)$ when $w_r^{T-1} \in [v_p^T, v_r^T]$; $K_r^{T-1*} = (\frac{v_r^T}{w_r^{T-1}}, 1 - \frac{v_r^T}{w_r^{T-1}})$, $K_r^{T-2*} = (0,1)$, when $w_r^{T-1} \in [v_r^T, 1])$. Substitute the optimal values of K_r^{T-1} and K_p^{T-1} into (3.7) and (3.8), the value of being rich and being poor in generation $T-1$, $V_r^{T-1}(w_r^{T-1}, T-1)$ and $V_p^{T-1}(w_r^{T-1}, T-1)$ are generated as follows (the arguments w_r^{T-1} and $T-1$ are suppressed):

$$
V_r^{T-1} = \begin{cases}
(1,0)[(u_r^{c1}, u_r^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [0, v_p^T] \\
(1,0)[(u_r^{c1}, u_r^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [v_p^T, v_r^T] \\
(\frac{v_r^T}{w_r^{T-1}}, 1 - \frac{v_r^T}{w_r^{T-1}})[(u_r^{c1}, u_r^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [v_r^T, 1)
\end{cases}
$$
(3.10)

$$
V_p^{T-1} = \begin{cases}
(\frac{v_p^T - w_r^{T-1}}{1-w_r^{T-1}}, 1 - \frac{v_p^T - w_r^{T-1}}{1-w_r^{T-1}})[(u_p^{c1}, u_p^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [0, v_p^T] \\
(0,1)[(u_p^{c1}, u_p^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [v_p^T, v_r^T] \\
(0,1)[(u_p^{c1}, u_p^{c2})' + \beta u_r P_{ir}^T + \beta u_p P_{ip}^T] & w_r^{T-1} \in [v_r^T, 1)
\end{cases}
$$
(3.11)

The high educational investment premium $R^{T-2} = \beta(P_{1r}^{T-1} - P_{2r}^{T-1})(V_r^{T-1} - V_p^{T-1})$ is a function of the fraction of skilled offspring in generation $T-1$, v^{T-1} (w^{T-1}), and it is generally decreasing in v^{T-1} (w^{T-1}). In particular, the premium of high educational investment, with which the generation $T-2$ parental households are confronted, varies across three segments of w^{T-1} as follows:

- when $w^{T-1} \in [0, v_p^T]$, equivalently $v^{T-1} \in [0, v_p^T]$, according to Proposition 2.1, the rich parental household in generation $T-1$ chooses a dominant pure strategy in equilibrium, $k_r^{T-1*} = 1$, while the poor parental household chooses a mixed strategy in equilibrium, $k_p^{T-1*} \in (0,1)$, being indifferent between the high investment and the low investment. Hence, the difference between being the rich household and being the poor household in generation $T-1$, denoted by $(V_r^{T-1} - V_p^{T-1})$, is a constant that equals $u_r^{c1} - u_p^{c1}$. Consequently, the high educational investment premium, with which the generation $T-2$ parental households are confronted, R^{T-2}, is decreasing in v_{T-1} (w_{T-1}), and this decrease is fully contributed by the shrinking gap

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between the probability of forming a rich household of the skilled and that of the unskilled in generation $T-1$, $(P_{1r}^{T-1} - P_{2r}^{T-1})$;

- When $w^{T-1} \in [v_p^T, v_r^T]$, equivalently $v^{T-1} \in [v_p^T, v_r^T]$, the equilibrium parental household's investment strategy profile in generational $T-1$, (k_r^{T-1*}, k_p^{T-1*}) would be, $(1, 0)$, the rich investing high and the poor investing low. Hence, the difference between being rich and being poor in generation $T-1$, $(V_r^{T-1} - V_p^{T-1})$, is no longer a constant. Instead, it contains a fixed part, $u_r^{c_1} - u_p^{c_2}$, which is the difference in consumption utility between the rich and the poor of generation $T-1$, indicating that the rich invest high and the poor invest low in generation $T-1$. It also contains a variable part, $R^{T-1} = \beta[(P_{1r}^{T-1} - P_{2r}^{T-1})(u_r - u_p)]$, the premium of high educational investment for the generation $T-1$ parental households, which is decreasing in v_T. Hence, the gap between the probability of the skilled and that of the unskilled to form a rich household in generation $T-1$, $(P_{1r}^{T-1} - P_{2r}^{T-1})$, has both a direct and an indirect effect (with respect to the variable part of $V_r^{T-1} - V_p^{T-1}$) on the premium of high educational investment for generation $T-2$ parental households, $R^{T-2} = \beta[(P_{1r}^{T-1} - P_{2r}^{T-1})(V_r^{T-1} - V_p^{T-1})]$, which is then decreasing in v_{T-1} (w_{T-1}) at a higher speed than $(P_{1r}^{T-1} - P_{2r}^{T-1})$ decreases in v_{T-1} (w_{T-1});

- When $w^{T-1} \in [v_r^T, 1)$, equivalently $v^{T-1} \in [v_r^T, 1)$, the equilibrium parental household's investment strategy profile in generation $T-1$ would be, $k_r^{T-1*} \in (0, 1)$ and $k_p^{T-1*} = 0$. The rich parental household chooses a mixed strategy, being indifferent between making the high investment and the low investment, while the poor parental household chooses a pure strategy making the low investment. Hence, the difference between being the rich parental household and being the poor parental household of generation $T-1$ $(V_r^{T-1} - V_p^{T-1})$ comes back to be a constant equal to $u_r^{c_2} - u_p^{c_2}$ (by setting $c_2 = 0$ for simplicity, we have $u_r - u_p = u_r^{c_2} - u_p^{c_2}$). The premium R^{T-2} decreases in v_{T-1} (w_{T-1}) at the same rate as $(P_{1r}^{T-1} - P_{2r}^{T-1})$ decreases in v_{T-1} (w_{T-1}).

Figure 3.1 visualizes the premium of high educational investment in generation $T-2$ by the red curve. As described above, this red premium curve consists of three consecutive segments, illustrating the premium for parental households in generation $T-2$, $R^{T-2} = \beta(P_{1r}^{T-1} - P_{2r}^{T-1})(V_r^{T-1} - V_p^{T-1})$, against the fraction of skilled individuals in generation $T-1$, v^{T-1}. The first segment of R^{T-2} in red overlaps the upper blue curve (representing $\beta(P_{1r}^{T-1} - P_{2r}^{T-1})(u_r^{c_1} - u_p^{c_1})$)

since $V_r^{T-1} - V_p^{T-1} = u_r^{c_1} - u_p^{c_1}$ according to the first case discussed above. The second segment of R^{T-2} in red is steeper than the two blue curves since $(V_r^{T-1} - V_p^{T-1}) = u_r^{c_1} - u_p^{c_2} + R^{T-1}$, and as a result, R^{T-2} is decreasing in w_{T-1} (v_{T-1}) at a higher speed than $(P_{1r}^{T-1} - P_{2r}^{T-1})$ decreases in w_{T-1} (v_{T-1}). The third segment of R^{T-2} in red overlaps the lower blue curve representing R^{T-1} $(R^{T-1} = \beta[(P_{1r}^{T-1} - P_{2r}^{T-1})(u_r - u_p)])$, since $V_r^{T-1} - V_p^{T-1} = u_r^{c_2} - u_p^{c_2} = u_r - u_p$.

Point A is the intersection of the red premium curve and the horizontal line that represents the rich households' utility loss of consumption from making the high educational investment, at which $R^{T-2} = u_r^{c_2} - u_r^{c_1}$, and it determines the critical value of the poor parental household of generation $T-2$, v_r^{T-1}, which is the same as that of generation $T-1$. Point B is the intersection of the red premium curve and the horizontal line that represents the poor households' utility loss of consumption from making the high educational investment, at which $R^{T-2} = u_p^{c_2} - u_p^{c_1}$ and it determines the critical value of the poor parental household of generation $T-2$, v_p^{T-1}.

Figure 3.1: The premium of high educational investment and the loss of consumption utility by parental households in generation $T-2$ against the fraction of skilled offspring in generation $T-1$

Proceeding in the manner shown above, the premium curves of the high educational investment can be derived as shown in Figure 3.2 and Figure 3.3. They illustrate the trade-off between the premium derived from the high educational investment and the utility loss of consumption caused by the higher cost of high

R^{t-1}

$\beta\left(u_r^{c_1} - u_p^{c_1}\right)$

R^{T-3} $=\beta\langle P_{1r}^{T-2} - P_{2r}^{T-2}\rangle\langle V_r^{T-2} - V_p^{T-2}\rangle$

$u_p^{c_2} \text{-} u_p^{c_1}$

$\beta\left(u_r^{c_2} - u_p^{c_2}\right)$

B C

$u_r^{c_2} \text{-} u_r^{c_1}$

A

R^{T-1} $=\beta\langle P_{1r}^{T} - P_{2r}^{T}\rangle\langle u_r^{c_2} - u_p^{c_2}\rangle$

O v_p^T $v_p^{T-1} v_p^{T-2} v_r^T$ 1 v^t

Figure 3.2: The premium of high educational investment and the loss of consumption utility by parental households in generation $T - 3$ against the fraction of skilled offspring in generation $T - 2$

educational investment for parental households in generation $T - 3$, as well as in the previous generation $T - 4$ respectively. In Figure 3.2, Point C corresponds to the critical value of the poor parental household of generation $T - 3$, v_p^{T-2}, at which $R^{T-3} = u_p^{c_2} - u_p^{c_1}$, and v_r^{T-2} is still represented by Point A, at which $R^{T-3} = u_r^{c_2} - u_r^{c_1}$. In Figure 3.3, Point D corresponds to the critical value of the poor parental household of generation $T - 4$, v_p^{T-3}, at which $R^{T-4} = u_p^{c_2} - u_p^{c_1}$, and v_r^{T-3} is still represented by Point A, at which $R^{T-4} = u_r^{c_2} - u_r^{c_1}$.

Through the figures shown above, we can see that the dynamics of this trade-off over generations that is revealed by the colored curves in a backward manner. Notice that those colored curves each consist of three consecutive segments, and converge to a fixed one in generation $T - S$ when S is large enough, which is illustrated by Figure 3.4. Point Z corresponds to the critical value of the poor parental household of generation $T - S$, $v_p^{T-S+1} = v_p^1$, at which $R^{T-S} = u_p^{c_2} - u_p^{c_1}$, and v_r^{T-S+1} is still represented by Point A, at which $R^{T-S} = u_r^{c_2} - u_r^{c_1}$.

The features of the premium curves through generations determine the pattern of the corresponding policy functions, as they indicate that the pattern of how the poor households' critical value, v_p^t changes. v_p^t increases at a decreasing rate in a backward manner through generations, which is in contrast to the rich households' critical value, v_r^t which is pinned down in the last generation and remains constant
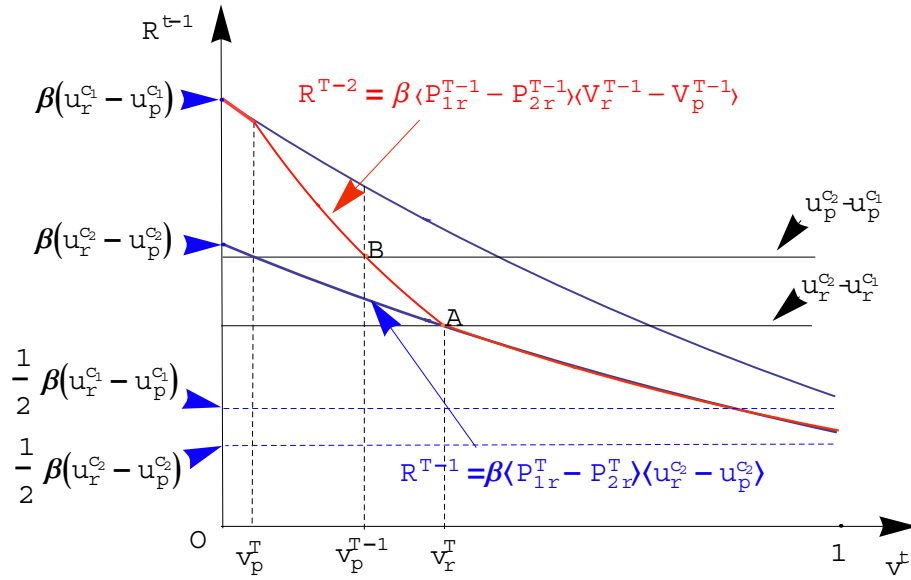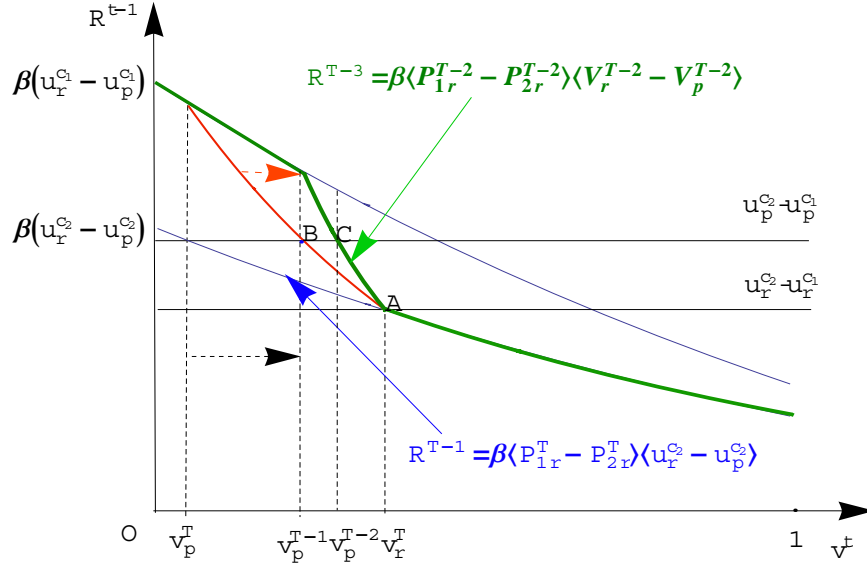
Figure 3.3: The premium of high educational investment and the loss of consumption utility by parental households in generation $T - 4$ against the fraction of skilled offspring in generation $T - 3$

all the way back to the first generation.

The policy function through generations can be illustrated by Figure 3.5. In Figure 3.5, the horizontal axis represents the state variable, w_r^{t-1}, which is the fraction of rich parental households in each generation $t - 1$, and the vertical axis represents the values of control variables, k_r^{t-1*} and k_p^{t-1*}, which are the optimal investment strategies adopted by the rich parental households and the poor parental households respectively. The rich parental households' investment strategy is still represented by the blue curve, as it is in Figure 2.3, while the red curve is for the poor parental households' investment strategy. In this figure, $k_r^{T-1*} = k_r^{0*}$, which shows that the rich parental households' investment strategy under any given income distribution indicator w_r^t in their own generation does not change through time. It equals 1 when the fraction of rich parental households w_r^{t-1} is below the critical value v_r^t, and starts to decline as w_r^{t-1} exceeds v_r^t. The underlying reason for its time-invariant pattern over generations is, the rich households' critical value v_r^t stays the same over generations, as indicated by $v_r^t = v_r^1$, $t = 1, 2, ...T$. On the other hand, the poor households' investment strategies vary across the last several generations.

The colored curves in the left lower corner of the graph in Figure 3.5 repre-
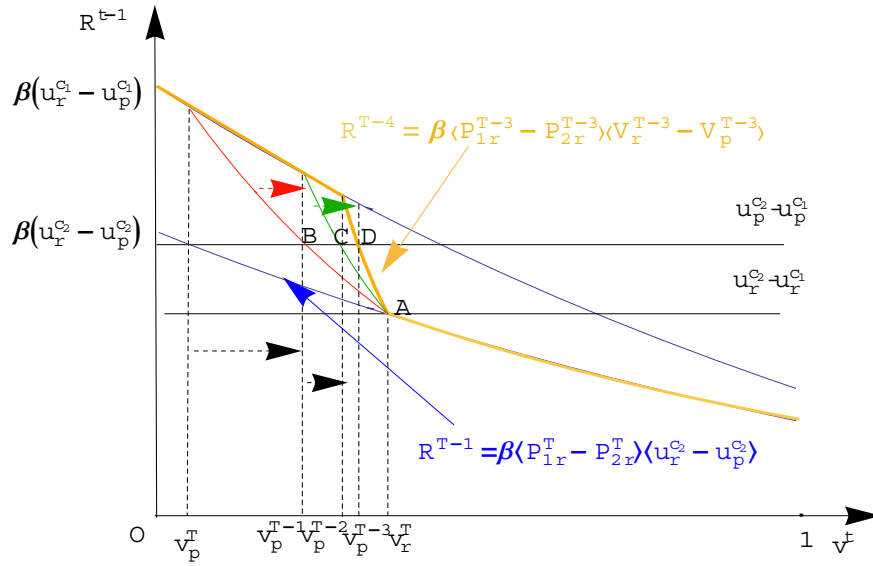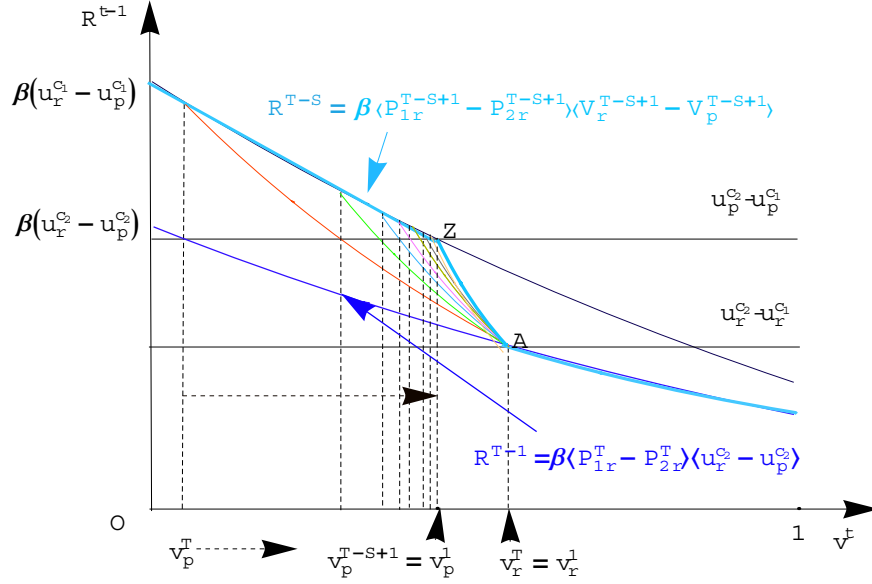
Figure 3.4: The premium of high educational investment and the loss of consumption utility by parental households in generation $T - S$ against the fraction of skilled offspring in generation $T - S + 1$

sents the poor households' investment strategies over the last several generations. The one in red representing k_p^{T-1*} shifts rightwards as the time goes back, and converges to a fixed one in blue representing K_p^{T-S*} at some generation $T - S$ when S is large enough. In other words, as the generation becomes closer to the last one, the curve representing the poor households' investment strategy k_p^{t-1*} shifts leftwards. Taking any one of them as an example, we see that k_p^{t-1*} equals $\frac{v_p^t - w_r^{t-1}}{1 - w_r^{t-1}}$ when w_r^{t-1} is below v_p^t. It declines as w_r^{t-1} increases. k_p^{t-1*} equals 0 when w_r^{t-1} reaches v_p^t and beyond. As t gets closer to T after generation $T - S$, the poor households are less tolerant of the competitiveness from the increased fraction of skilled in the next generation. They tend to choose a pure investment strategy, $k_p^{t-1} = 0$, giving up the high educational investment when $v_p^T < w_r^{t-1} < v_p^1$.

To summarize the analysis shown above, we get the conclusion that, given the parameter region, $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$, the policy function $g(w_r^t)$ mapping the w_r^{t-1} into the optimal value of the current control of the rich parental household, K_r^{t-1*}, does not change over generations, while it does change when mapping the w_r^{t-1} into the optimal value of the current control of the poor parental household, K_p^{t-1*}, if the system gets close enough to the last generation. Specifically, when w_r^{t-1} is as low as in a certain interval, e.g $v_p^T < w_r^{t-1} < v_p^1$, the poor households are less tolerable of the competitiveness in
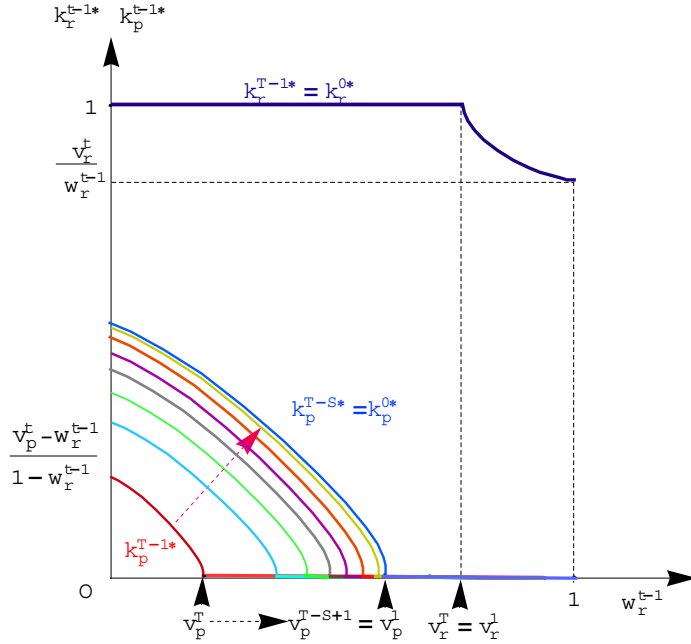
Figure 3.5: Policy function through generations: optimal investment strategies against the fraction of rich households in each generation

making the high educational investment as the generation gets closer to the last one.

Compared to the second last generation, the poor parental households in any previous generation would be more willing to compete with the rich households in making the high educational investment. The intuition could be that, unlike the parental households in the second last generation, those parents in the previous generations consider that their children are going to make educational investment on their grandchildren, and they know that the poor households always suffer more than the rich ones when making educational investment. The poor parental households in generation t where $t < T-1$ are more eager than those in generation $T - 1$ to catch up with the rich in order to relieve their children from being in an inferior position. This eagerness reduces as the system gets closer to the last one, for there are less and less generations of their descendants having to make educational investment. Compared to those in the earlier generations, the poor parental households in the second last generation are the least anxious ones to relieve their children from being poor. This is because the disadvantages of being in a poor household in the last generation are less than those in the previous

generations as people do not need to make any educational investment when they are the last generation.

Based on the pattern of the policy function described above, we can also map the parental households' income distribution indicator w_r^{t-1} into their children's skill distribution indicator v^{t*} in equilibrium as we did in the static model. As v_p^t starts to decrease when t is close enough to T, this relationship changes correspondingly in the manner represented by Figure 3.6.
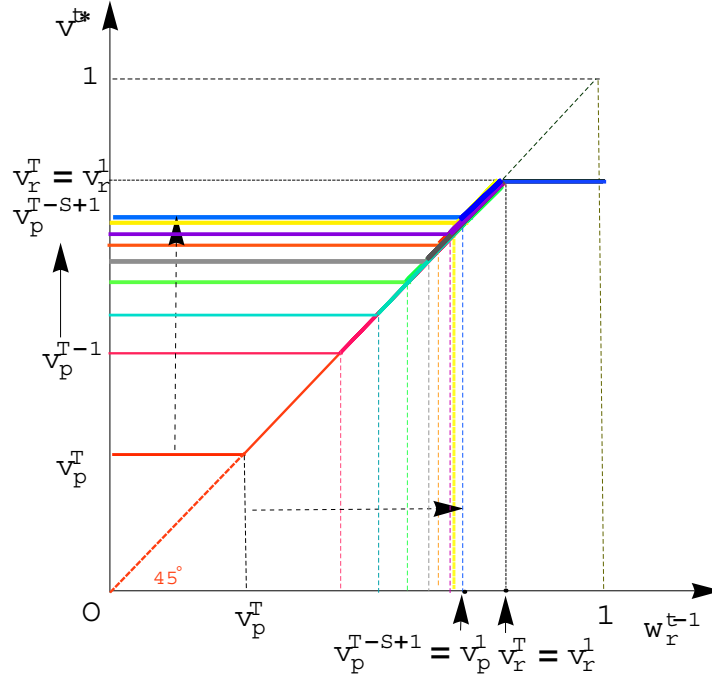
Figure 3.6: The fraction of rich households against the equilibrium fraction of skilled individuals through generations

The horizontal axis in Figure 3.6 represents the fraction of rich households in generation $t-1$, which is the parental generation from the perspective of people in generation t. The vertical axis represents the equilibrium fraction of skilled individuals in generation t. Like the static model, this relationship is illustrated by three segments of the line. When any exogenous value of w_r^{t-1} is below v_p^t, we have $v^{t*} = v_p^t$, which is represented by the horizontal line segment in the left lower corner; when w_r^{t-1} lies between the two critical values v_p^t and v_r^t, we have $v^{t*} = w_r^{t-1}$, which is represented by the 45° line segment; when w_r^{t-1} exceeds the rich parental households' critical value v_r^t, we have $v^{t*} = v_r^t$, which is represented by the horizontal line segment in the right upper corner. Compared with Figure

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2.2 in the previous chapter, the relationship between v^{t*} and w_r^{t-1} varies in the last several generations due to the variation in the critical value, v_p^t, which is generated in this dynamic system. This variation is illustrated by a group of colored curves in the left lower corner in Figure 3.6. Given this relationship between w_r^{t-1} and v^{t*}, we arrive at the equilibrium income distribution and investment strategies for the rich and the poor households in the dynamic system.

Proposition 3.1 *Given the parameter region, $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$ and $w_r^0 = 0$, the income distribution becomes stationary after the first generation along the equilibrium path through generations, where $w_r^{t*} = v_p^1$ and $t > 0$. Meanwhile K_r^{t*} and K_p^{t*} are constant vectors, $K_r^{t*} = (1,0)$, $K_p^{t*} = (0,1)$.*

Proof. See Appendix B.1 ∎

This proposition can be illustrated by Figure 3.7. The right half of the graph replicates Figure 3.6 and it illustrates the relationship between the fraction of rich parental households and the equilibrium fraction of skilled offspring in the next generation. The left half of the graph visualizes Property 1 of the household formation in the previous chapter which states that the fraction of rich households equals the fraction of skilled individuals in every generation, $w_r^t = v^t$. By combining the two subfigures, it is possible to visualize the dynamics of income distribution over generations.

Let's start from the first generation where an initial income distribution is imposed as $w_r^0 = 0$. The relationship between the fraction of rich parental households in first generation and the equilibrium fraction of skilled individuals in the next generation is illustrated by the blue curve that consists of three segments. Along the blue arrow, we can see that when $w_r^0 = 0$, the corresponding v^{1*} is generated by the blue curve, where $v^{1*} = v_p^1$. According to Property 1 of the household formation illustrated by the left half of the graph, the second generation's income distribution w_r^1 is equal to v^1, where $w_r^{1*} = v^{1*} = v_p^1$. Taking this w_r^{1*} back to the right half of the graph, we see that even though the relationship between the fraction of rich parental households in the second generation and the equilibrium fraction of skilled individuals in the next generation may change, as represented by the yellow curve[8], v^{2*} is still equal to v^{1*}, since w_r^{1*} lies between v_p^2 and v_r^t. Hence, we have $w_r^{2*} = w_r^{1*} = v_p^1$. Proceeding in this manner, even

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[8]Notice that only the line segment in the left lower corner can be seen in yellow since the other two segments are overlapped by other lines in blue

though v_p^t moves leftwards from some generation, it has no effects on the equilibrium income distribution in generation t, w_r^t, because $v_p^t \leq v_p^1$. The income distribution becomes stationary after the first generation, $w_r^{t*} = v_p^1$ $(t \geq 1)$ as stated in Proposition 3.1.

Figure 3.7: The fraction of rich households along the equilibrium path through generations

Notice that for any exogenous value of the initial income distribution indicator in the first generation, w_r^0, where $w_r^0 \leq v_p^1$, the stationary fraction of rich households in equilibrium would be, $w_r^{t*} = v_p^1$ $(t \geq 1)$. For any value of w_r^0 where $v_p^1 \leq w_r^0 \leq v_r^1$, the stationary fraction of rich households in equilibrium would be, $w_r^{t*} = w_r^0$. For any value of w_r^0 where $v_r^1 \leq w_r^0 < 1$, the stationary fraction of rich households in equilibrium would be, $w_r^{t*} = v_r^1$.

It is also interesting to check the income mobility rates for the rich households and the poor households respectively. Here the mobility rate of the rich is the probability of their children forming a poor household in the next generation

(downward income mobility rate), and that of the poor is the probability of their children forming a rich household in the next generation (upward income mobility rate). Let P_{rp}^t and P_{pr}^t denote the two concepts here respectively, where:

$$P_{rp}^t = K_r^{t-1} P_{ip}^t$$
$$P_{pr}^t = K_p^{t-1} P_{ir}^t$$

According to Proposition 3.1, w_r^{t*} remains stationary after the first generation, as is v^{t*}. Hence, P_{ir}^{t*} and P_{ip}^{t*}, K_r^{t-1*} and K_p^{t-1*} ($t \geq 1$) are constant vectors. Consequently, P_{rp}^{t*} and P_{pr}^{t*} are constants too. Specifically, we have $K_r^{t-1*} = (1,0)$ and $K_p^{t-1*} = (0,1)$, which leads to $P_{rp}^{t*} = P_{1p}^{t*}$ and $P_{pr}^{t*} = P_{2r}^{t*}$. Proposition 3.2 elaborates on the properties of intergenerational income mobility in this dynamic model.

Proposition 3.2 *Along the equilibrium path, if $w_r^{t*} \geq \frac{1}{2}$, $P_{pr}^{t*} \leq P_{rp}^{t*}$, $t \geq 1$; if $w_r^{t*} \leq \frac{1}{2}$, $P_{pr}^{t*} \geq P_{rp}^{t*}$, $t \geq 1$.*

Proof. See Appendix B.2 ∎

Proposition 3.2 implies that when the income distribution becomes stationary as stated in Proposition 3.1, a smaller fraction of skilled individuals leads to a higher matching probability of getting matched with an unskilled individual. As a result, this dynamic system yields a higher probability of forming a poor households in general. On the other hand, a larger equilibrium fraction of skilled individuals leads to a higher matching probability of getting matched with a skilled individual, and consequently a higher probability of forming a rich household in general.

Based on the results in Proposition 3.2, the equilibrium fraction of rich households w_r^* affects income mobility. Since the income disparity parameter α impacts on w_r^{t*}, it also has an indirect effect on income mobility rates, which is elaborated on as follows.

Remark 3.1 *Given education costs c_1, c_2 and the household utility gained from the low income u_p unchanged, a decrease in the income inequality parameter α leads to larger critical values, v_r^1 and v_p^1, which generates a higher w_r^* in equilibrium. Consequently, P_{rp}^* is increasing in α, while P_{pr}^* is decreasing in α.*

This remark is consistent with the results of comparative statics in the previous chapter. It is illustrated by Figure B.3 in Appendix. By holding the same parameter region, we restrict our focus on type (vi) investment profile as is analyzed in this chapter. This remark says that when the income disparity increases, equivalently as α decreases, an individual from a rich parental household is less likely to form a poor household, and an individual from a poor parental household is more likely to form a rich household. This is mainly due to the fact that a smaller α creates more incentive for households to make the high educational investment, which leads to a larger fraction of rich households as well as a larger fraction of skilled individuals in the equilibrium.

On the other hand, the effect of inequality in education cost has an opposite effect on the equilibrium outcome, which is elaborated upon as follows:

Remark 3.2 *Given α and c_2 unchanged, an increase in the cost of the high educational investment, c_1, would increase the suffering from making the high educational investment for both the rich and the poor. It also increases the difference in this suffering of the rich and that of the poor. According to Proposition 3.1 and 3.2, this increase leads to a smaller equilibrium fraction of rich households, w_r^*. Consequently, P_{rp}^* is increasing in c_1, while P_{pr}^* is decreasing in c_1.*

Remark 3.2 says that when the high educational investment cost increases, all the households suffer more from investing high but the poor's suffering is more severe by this increase than the rich. The equilibrium downward income mobility rate P_{rp}^* increases in c_1, while the equilibrium upward mobility rate P_{pr}^* decreases in c_1. A larger inequality in education costs leads to a smaller fraction of rich households, which is harmful to the total output of the economy and to the upward mobility rate.

In summary, we so far have extended the static model to a dynamic one, where family members span several generations. It follows the parameter region of type (vi) investment profile defined in the static model, where $\frac{1}{2}\beta(u_r - u_p) < u_r^{c_2} - u_r^{c_1} < u_p^{c_2} - u_p^{c_1} < \beta(u_r - u_p)$. It is possible that both the rich and the poor household may choose a mixed investment strategy under a certain exogenous income distribution of their own generation. For simplicity, we set the initial income distribution $w_r^0 = 0$ in generation 0. It will not change the main results if any other value of w_r^0 is imposed. The household income distribution becomes stationary in equilibrium as stated in Proposition 3.1. Therefore, given an identical initial income distribution, e.g. $w_r^0 = 0$, we can compare the equilibrium household income distribution and

income mobility rate in the static model with these in the dynamic one. It is obvious that in both systems, there is an increase in the fraction of rich households from the initial generation to the offspring generations given $w_r^0 = 0$, and this fraction becomes stationary in dynamic setting. The amount of increase is larger in the dynamic setting than that in the static setting, because the poor are better motivated to choose high educational investment when they need to take more generations' welfare into consideration. As a result, a larger fraction of skilled offspring is generated in equilibrium under dynamic system.

3.4   Concluding Remarks

In the model discussed above, this dynamic system generates a stationary household income distribution and household income mobility rates along the equilibrium path. The randomness that stems from frictions in the family formation process and the induced mixed investment strategies adopted by parental households contribute to the variation in income distribution before it arrives at the stationary state. As both the inequality in education costs and the inequality in returns to education affect the equilibrium fraction of rich households, this model implies that an increase in the cost of high educational investment alone would decrease the equilibrium fraction of rich households as well as upward income mobility; while an increase in returns to the high educational investment alone would increase the equilibrium fraction of rich households as well as upward income mobility. This result is consistent with previous studies on inequality and mobility mentioned in the introduction.

Following the spirit of the static model, the directed matching mechanism which is used to model the family formation process generates critical values for the parental households in terms of the fraction of skilled offspring in the next generation. As the premium of high educational investment declines in the fraction of households who choose high educational investment (which is equal to the fraction of skilled offspring in the next generation), it indicates that there is a "given" number of skilled being produced so that the loss in consumption utility can be compensated by the high educational investment premium.

Due to the difference in suffering from consumption utility loss by costly educational investment, the rich are able to crowd out the poor if they can produce this "given" number of skilled offspring all by themselves. Hence, we can not evaluate the education cost structure's effects alone on the future income distribution and income mobility, as it is a relative term compared to the current

income distribution and future income inequality. In the case when the cost of the high educational investment goes up, the poor may still be willing to pay the education costs if the associated labor market returns to this investment is large enough, or from the family formation process aspect, if there are not enough rich households in the current generation so that they are not able to crowd out the poor from making the high educational investment. Otherwise, this congestion effect of family formation competition would make the poor find the chance of their children forming a rich household too small and give up making the high educational investment.

The policy implication of this model is that a low cost for people to access the high education would enhance the incentives to make educational investment. This effect is stronger to the poor than to the rich, which would boost the upward income mobility and the overall output of the economy. On the other hand, the inequality of labor income should not worry policy makers as long as the labor market institution is effecient. A high degree of inequality of income would encourage people to make educational investment if the costs of education do not hurt them too much. Considering economic growth based on human capital accumulation in the context of the dynamic setting, it is essential to preserve a certain fraction of skilled workers in the economy especially when the technology improves rapidly. Public financial support aiming at reducing the high educational costs would boost upward income mobility, which makes the poor more tolerate about the inequality of labor income from sociology viewpoint. In addition, it would also generate larger fraction of skilled workers in the long run, which improves a country's production technology level and economic output.

Appendix B

B.1   Proof of Proposition 3.1

According to the properties of household formation stated in the static model, given any value of the parental household income distribution w_r^{t-1}, the fraction of skilled offspring in the next generation v^{t*} generated along the equilibrium path equals the income distribution of that generation w_r^{t*}. Given $w_r^0 = 0$, all the households are poor and they would choose a mixed investment strategy $k_p^{0*} = v_p^1$, which results in $v^{1*} = v_p^1$ and consequently $w_r^{1*} = v_p^1$. Though the relationship between w_r^{t-1} and v^{t*} along the equilibrium path represented by Figure 7 varies over the last several generations due to the movement of the critical value v_p, w_r^{t-1} lies between the v_r^t and v_p^t as v_p^1 is the maximum of v_p^t. Hence, after first generation, $w_r^{t*} = v^{t*} = v_p^1$, and $K_r^{t*} = (1, 0)$, $K_p^{t*} = (0, 1)$ according to Lemma 2.3.

B.2   Proof of Proposition 3.2

Given $t > 0$, $P_{1r}^{t+1*} \geq P_{2r}^{t+1*}$, and $P_{rp}^{t*} = P_{1p}^{t+1*} = 1 - P_{1r}^{t+1*}$, $P_{pr}^{t*} = P_{2r}^{t+1*}$, which then leads to: $P_{rp}^{t*} \leq 1 - P_{pr}^{t*}$. According to Proposition 3.1, the w_r^{t*} becomes stationary along the equilibrium path after generation 0, which indicates that the number of the rich households that fall to be poor equals the number of the poor households that rise to be rich between any two consecutive generations, represented by: $P_{rp}^{t*} w_r^{t*} = P_{pr}^{t*}(1 - w_r^{t*})$. Hence, we get $\frac{P_{1p}}{P_{2r}} = \frac{1}{w_r} - 1$. Then comes the conclusion: if $w_r^{t*} \geq \frac{1}{2}$, $P_{pr}^{t*} \leq P_{rp}^{t*}$; if $w_r^{t*} \geq \frac{1}{2}$, $P_{pr}^{t*} \geq P_{rp}^{t*}$.

Note that it is possible that $v_p^1 = v_r^1$. In that case, the poor households and the rich households' critical values equal to each other, indicating that the rich households in the first generation will not have a space of actions which can crowd out the poor from making the high educational investment. In other words,

the rich and the poor will choose the same investment strategy, which results in $P_{pr}^1 = P_{rp}^1$ under any value of w_r^0.

B.3   Illustration of Remark 3.1
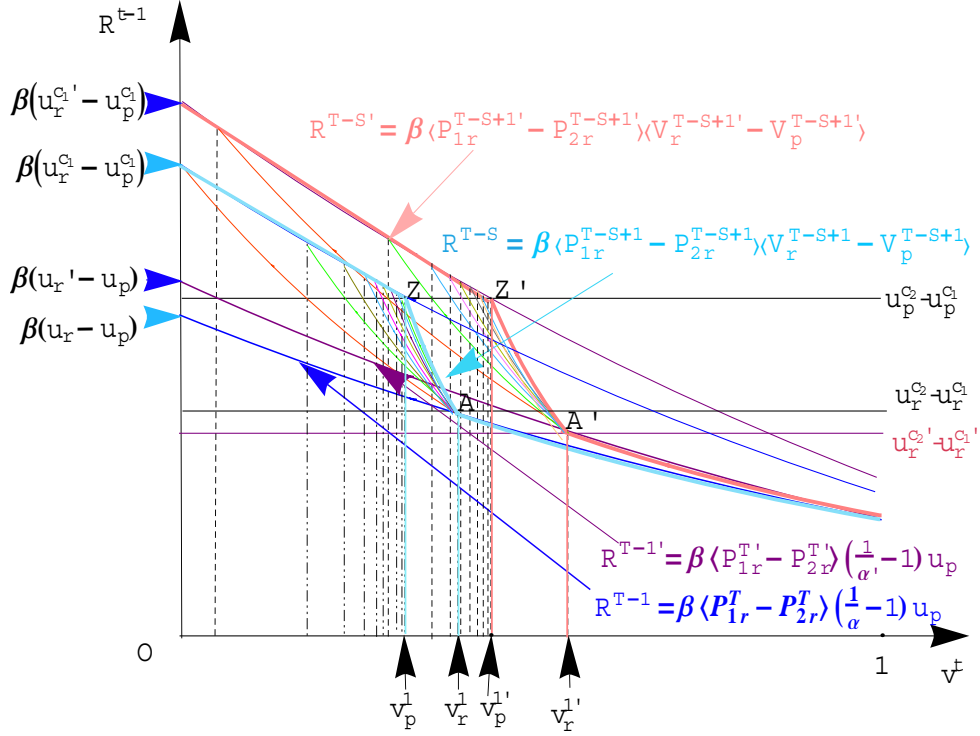
Figure B.1: v_p^1 and v_r^1 increase as α decreases

In Figure B.1, as α decreases, the upper blue curve representing $\beta(P_{1r}^T - P_{2r}^T)(u_r^{c_1} - u_p^{c_1})$ shifts to the upper purple curve representing $\beta(P_{1r}^T - P_{2r}^T)(u_r^{c_1'} - u_p^{c_1})$; the lower blue curve representing $R^{T-1} = \beta(P_{1r}^T - P_{2r}^T)(u_r - u_p) = \beta(P_{1r}^T - P_{2r}^T)(\frac{1}{\alpha} - 1)u_p$ shifts to the lower purple curve representing $R^{T-1'} = \beta(P_{1r}^T - P_{2r}^T)(\frac{1}{\alpha'} - 1)u_p$. The horizontal line representing $u_r^{c_2} - u_r^{c_1}$ in black shifts to the purple horizontal line representing $u_r^{c_2'} - u_r^{c_1'}$. The original premium curve $R^{T-S} = \beta(P_{1r}^{T-S+1} - P_{2r}^{T-S+1})(V_r^{T-S+1} - V_p^{T-S+1})$ in aqua changes to a new one $R^{T-S'} = \beta(P_{1r}^{T-S+1'} - P_{2r}^{T-S+1'})(V_r^{T-S+1'} - V_p^{T-S+1'})$ in pink. Consequently, the intersection points A and Z change to A' and Z'. v_p^1 and v_r^1 increase to $v_p^{1'}$ and $v_r^{1'}$.

B.4   Illustration of Remark 3.2

In Figure B.2, as c_1 increases, the horizontal line representing $u_p^{c_2} - u_p^{c_1}$ in black
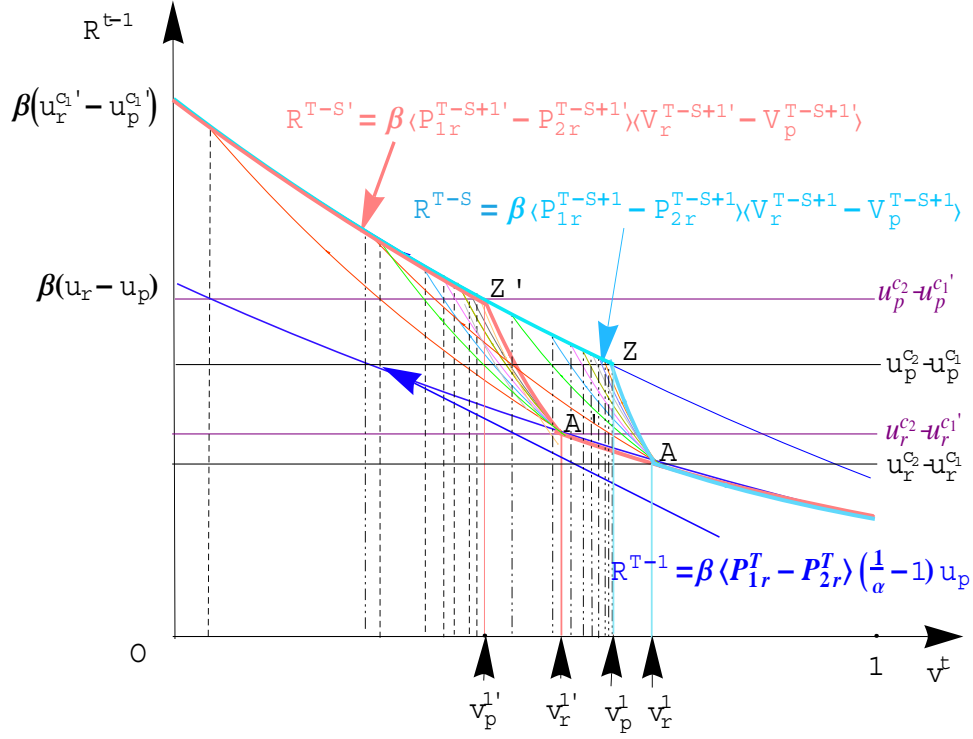
Figure B.2: v_p^1 and v_r^1 decrease as c_1 increases

shifts to the upper purple horizontal line representing $u_p^{c_2} - u_p^{c_1'}$; the horizontal line representing $u_r^{c_2} - u_r^{c_1}$ in black shifts to the lower purple horizontal line representing $u_r^{c_2} - u_r^{c_1'}$. The original premium curve $R^{T-S} = \beta(P_{1r}^{T-S+1} - P_{2r}^{T-S+1})(V_r^{T-S+1} - V_p^{T-S+1})$ in aqua changes to a new one $R^{T-S'} = \beta(P_{1r}^{T-S+1'} - P_{2r}^{T-S+1'})(V_r^{T-S+1'} - V_p^{T-S+1'})$ in pink. Consequently, the intersection points A and Z change to A' and Z'. v_p^1 and v_r^1 decrease to $v_p^{1'}$ and $v_r^{1'}$.

Chapter 4

Product Line Competition with Vertical and Horizontal Differentiation: An Application to the Scheduled Flight Market

Abstract

This chapter studies a duopoly air-travel market between a city-pair. Scheduled flights can be differentiated along the quality dimension: one-stop or non-stop, and the variety dimension: departure times. Consumers are heterogeneous in their valuation over the quality of flights and their preferences over departure times. Conditions are generated under which a hub-and-spoke network carrier provides non-stop flights along an interlaced schedule, or one-stop flights at the same schedule adopted by its competitor, a point-to-point network carrier. It also indicates that higher quality difference makes the consumers less sensitive to the flight frequency. Hence the two carriers prefer to be specialized in different quality of flights and adopt the same schedule.

Key Words: Vertical differentiation, horizontal differentiation, product line competition

4.1 Introduction

The multi-faceted nature of the modern-day air travel market presents various obstacles and challenges for analysis. Chief among these obstacles is the involvement of various parties and stakeholders, including authorities, airlines, aircraft manufactures, airports, and passengers who, combined with exogenous economic and political forces, have contributed towards the current state-of-affairs within the industry. One of the major trends over the last two decades has been the meteoric rise of low-cost carriers in the North American and European air travel markets. In contrast to the traditional carriers' hub-and-spoke network which provides connection flights as well as direct flights, the low-cost carriers have adopted a point-to-point network which provides only direct flights between city-pairs.

Hub-and-spoke network operation emerged and has been widely adopted by traditional carriers after the deregulation of airline industry of US in late 1970s. The major objective of the hub-and-spoke network design is to reduce costs (both marginal cost and fixed cost) by providing connection flights, taking advantage of economies of scale, scope, density. There is a large literature providing empirical evidence in support of cost saving by the hub-and-spoke network (Brueckner, Dyer and Spiller, 1992; Brueckner and Spiller, 1994; Berry, 1990, 1996). Another reason why the hub airport is preferred by the hub carriers is that it can be used as a 'fort', which enables its major users to increase entry barriers and set higher prices for hub-originating passengers (Borenstein, 1989, 1991; Berry, 1990, 1996). These effects suggest that the hub-and-spoke network structure generates significant market power that would make it possible for the traditional carriers (hub-and-spoke network carriers) to sustain high prices and deter potential entry into the market. However, this market power has been threatened by a new business model implemented by Southwest, a US 'no frills' discount airline. This business model takes a new way of costs saving. Typically, it largely reduces costs by increasing seat density, daily aircraft utilization and reducing crew costs, sales costs etc. Further, by using secondary airports and operating in the 'rim' route[1], low-cost carriers largely steer clear from the entry barriers. This business model was studied and copied by many firms in America (e.g. Frontier, AirTran Airways, Jetblue, etc.), Europe(e.g. Ryanair, Easyjet, Vueling, Air Berlin etc.) and some other countries. While the market development of low-cost airlines has reached

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[1]The route consists of two endpoint cities which are typically connected by one-stop flights operated by hub carriers.

its maturity in America, currently believed to be in its consolidation stage in Canada, UK and Europe Mainland, and still in proliferation stage in the rest of the world (European Parliament Study, 2007). Some of the hub carriers have launched 'no frill' division (airline in airline, e.g. Buzz in KLM, Delta Express in Delta Airlines, MetroJet in US Airways, etc.) operating direct flights as a response to the penetration of the low-cost carriers. However, most of these divisions (airline in airline) of the hub carriers turn out to perform badly due to various factors associated with their parental companies, such as limits of fleet expansion, union's opposition and cost reduction challenges (Doganis, 2001).

The competition between the two business models has been elaborately studied from various aspects of the underlying cost-reduction strategies. For example, it is generally agreed that longer routes (about 1,000 miles and above) are more suitable for the hub-spoke network' style of cost reduction due to aircraft configurations, while the shorter ones seem to favor the 'no frill' style. In the past two decades, low-cost carriers have occupied much of the market for intra-European travel that traditional carriers left open by reducing the number of flights, while cross-continent travel markets are dominated by the traditional carriers with their hub-and-spoke network. On the other hand, one cabin class, minimum in-flight service and standerized fleet are the typical features of cost-reduction made by low-cost carriers, which provide less comfort flight to the leisure travellers. However, recent trend in airline industry shows a tendency of convergence of the two business model. For example, European second largest low-cost carrier Easyjet starts to provide two cabin classes from 2012 to draw more business customer; Norwegian shuttle, another low-cost carrier, have launched long-haul routes from the Scandinavian capitals to New York and Bangkok recently; while SAS, the flag-carrier of the Scandinavian countries with three hubs in Oslo, Stockholm and Copenhagen, began to reduce its unit cost from 2008 by learning from low-cost carriers' experience.

As industrial observers point out, all the airlines may look the same in the near future and it may be trivial to clarify the identity of carriers (low-cost or the traditional) when analyzing the competition of airline industry. However, what kind of flights are provided in the market and how much of them have always been the major concerns of air-travellers. The very likely valuation of flights in the future air-travel market may be less about other characteristics of flights but the most important two of them: flight duration and departure times. Following this viewpoint, it is very interesting to investigate the competition between a hub-and-spoke network carrier and a point-to-point network carrier in a city-pair

air-travel market where both of them do not have relative cost advantages against each other (it is feasible for a one-stop flight incurring the same costs as a non-stop one). The directness of non-stop flights generates less flying hours, and it is regarded as the feature of the higher quality compared with the undesirable one-stop flights. Carriers have the options to set different departure times of their flights, which is recognized as horizontal differentiation since consumers have different preferences over departure times.

The research question of this chapter is: how does a hub-and-spoke network carrier interact with a point-to-point network carrier in regard to product line location arrangement and product quality choice? The results have implications on the demand of different sizes of aircrafts. For convenience, the hub-and-spoke network carrier is referred to as the hub carrier and the point-to-point network carrier is referred to as the PTP carrier in the following context. In the model, a hub carrier has the choice to provide one-stop flights or non-stop flights by operating on its hub-and-spoke network, and a PTP carrier can only provide non-stop flights by operating on its point-to-point network. Consumers are assumed to be heterogeneous in their valuation of the flight quality (flight duration) and preferred departure time. In addition, those with high valuation of the quality suffer more from a flight at an undesirable departure time. Any price discrimination, based on the type of cabin class or advance booking days, towards targeted group of consumers is not considered in this model, and the costs of the two service types are assumed to be zero in the model for simplification purposes.

The framework is based on the literature on two-dimensional product differentiation (Degryse, 1996; Economides, 1989; Neven and Thisse, 1990). Specifically, this paper closely follows the model proposed in Degryse (1996), which deals with product differentiation within the banking industry. The model specified below differs from Degryse's model in four key aspects. First, each one of the duopolists provides a product line in this model rather than a single product by each firm, so that the scale of product line matters. It is reasonable since frequency of flights (related to the scale of product line) matters in the air-travel market. Second, duopolists in this model are allowed to choose their product line location in two polar strategies: head-to-head or interlaced, while the locations are fixed and exogenous in Degryse's banking model. This flexibility of product location fits better with scheduled flight market since choosing a schedule involves strategic thinking in carriers' competition. Third, the competition between the duopolists is asymmetric in the sense that one of them (the PTP carrier) always provides high quality services (non-stop flights), so it only needs to make decisions on the

location, while the other player (the hub carrier) needs to make decisions on both the location and the quality of its services. Finally, the players move sequentially in this model rather than simultaneously. The sequence of moves follows the conjecture that the hub carrier takes more time in making strategic decisions on product line competition since it needs to decide both the flight quality and the flight schedule while the PTP carrier only needs to make decisions on the schedule.

With regard to the second point mentioned above, the model by Klemperer (1992) also endogenizes location choices partially as in the form of only two polar strategies: head-to-head and interlaced. This simplification is due to the complexity of endogenizing location choices in product differentiation models. In Klemperer's model, it is argued that shopping cost's existence may cause head-to-head competition less competitive, which sheds some light on the results of this model as well. One result of this model is that by competing head-to-head in terms of the flight schedule, both carriers are able to charge higher prices and earn higher profits when they specialize in different quality of flight (one-stop flights vs. non-stop flights). Moreover, when shopping costs are considered as the second dimension of product characteristic, its conclusion is consistent with a more general prediction of multi-characteristics space competition model by Irmen and Thisse (1998), which states that firms tend to maximize differentiation in the dominant characteristic and to minimize differentiation in other characteristics.

There is a growing literature focused on how the hub carriers responds to PTP carriers' entry. Cento (2008) investigates the price competition between airlines in various market structures, e.g. monopoly, symmetric duopoly, asymmetric duopoly, asymmetric oligopoly, etc. With the assumption that consumers are both horizontally and vertically heterogeneous (which refers to the consumers' physical distance from airports and their valuation of the cabin classes) equilibrium outcome varies across different market structures. Dunn (2007) empirically examines the factors affecting non-stop services in 'rim' markets. The results provide evidence on the cannibalization effect[2] between a carrier's own one-stop services and non-stop services, which deters the hub carrier's entry into non-stop services in the 'rim' markets.

Different from the related models mentioned above, the main concern of the following model is the strategic interaction between the hub carrier and the PTP

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[2]The hub carrier's non-stop flights as the modified products exert a negative effect on the hub carrier's profits with respect to its one-stop flights, which is considered as a form of 'cannibalization'.

carrier on the product line competition and two-dimensional differentiation rather than complicated price schemes designed by carriers. Hence, the model assumes a simple price scheme under which each carrier sets a uniform price across its own product line. For simplification purpose, no competition between two types of services within one carrier is considered in this model. The hub carrier chooses to provide only one type of flights: one-stop or non-stop, while the PTP carrier always provides non-stop flights. Moreover, the analysis is made in a duopoly market only, and studies with the same research interest but under other market structures could be an extension of this model.

The chapter is organized as follows. Section 2 discusses the set-up of the model. Section 3 generates equilibrium prices in the final stage of the game. Section 4 characterizes Subgame Perfect Nash Equilibria. Section 5 provides a discussion of the model's main findings.

4.2 The Model

Consider a duopolistic scheduled flight market consisting of a hub-and-spoke network carrier (the hub carrier for short) H and a point-to-point network carrier (the PTP carrier for short) L both operating n flights per week between a city pair. This model focuses on a short run strategic interplay between the two types of carriers in terms of product line position, product quality choice and pricing. By short run, it means both carriers arrange in advance the flight schedule, the flight type (one-stop vs. non-stop) and the corresponding prices of a short calendar period (e.g. one month). To focus on the product line differentiation, this model only considers a simplified pricing scheme. It rules out any kind of price discrimination[3], which is commonly applied in the air-travel industry (also referred to as 'yield management' or 'revenue management' in practice). In this model, each carrier choose a uniform price across its own product line instead.

The feature of schedules available to the two carriers is characterized as follows. Schedules denote the position/location of the product line in this model. As the characteristic of horizontal differentiation, it is modeled in *Salop circle* style. The whole circle represents a time period of one week. Each carrier's n scheduled flights in the city-pair market are located at equal distances from each other on the circle with the circumference of $C = 1$. The hub carrier's one-stop

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[3]Airlines use combinations of different types of price discrimination regularly, as they sell travel products and services simultaneously to different market segments. There is also a large volume of academic literature on price discrimination of this industry, e.g. Borenstein and Rose (1995), Geradi and Shapiro (2009).

flights can only be operated along a fixed schedule in a short run[4], which is modeled by n fixed points at an equal distance with each other along a circle. Both carriers are able to choose an alternative schedule, which can only be adopted for non-stop flights. This alternative schedule is modeled by another series of n points, each of which is located right in the middle of two fixed points along this circle. In the graph, this alternative schedule is interlaced with the fixed one[5]. Each consumer takes one flight every week. Their preferred departure time x is uniformly distributed along the circle. The cost to any consumer of substituting a flight which departs at x', for a desired flight which departs at x, is a strictly increasing function of the minimum distance around the circle between x' and x, $\tau(|x - x'|)$, where τ is constant and measures the substitution rate between different flights' departure times.

In addition, the feature of flight qualities available to the two carriers is characterized as follows. The hub carrier has the option to operate a series of one-stop scheduled flights as it owns a hub-spoke network, or to start a series of non-stop scheduled flights along an alternative schedule instead. On the other hand, the PTP carrier only provides non-stop scheduled flight services according to its business model. Vertical product differentiation occurs as non-stop flights bear shorter flying hours than one-stop flights. All the consumers prefer shorter flight hours of a single trip, but they are heterogeneous in the way they suffer from this duration. By tradition, this heterogeneity is captured by a parameter $\theta \in [\underline{\theta}, \overline{\theta}]$, which measures different valuation of time cost incurred by a flight. The space of consumers' characteristics (x, θ) is represented by a cylinder $C \times [\underline{\theta}, \overline{\theta}]$ (see Figure 4.1 to 4.4 for illustrations). Consumers are distributed uniformly on the surface of the cylinder with density $\frac{1}{\overline{\theta} - \underline{\theta}}$, and are of mass $M = \overline{\theta} - \underline{\theta}$.

Competition for traffic flow is modeled as a three-stage non-cooperative game with complete information. In the first stage, the PTP carrier decides whether to overlap the hub carrier's one-stop flights' fixed schedule, or not to overlap but to interlace with it. In the second stage, the hub carrier has three options: to operate its one-stop flights along the fixed schedule, to provide non-stop flights instead but still along the same fixed schedule, or to provide non-stop flights

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[4]Though all the carriers change their route network from time to time, especially according to the seasonal demand fluctuation, an overall route network can be considered as static for a short period since carriers sell tickets as early as several months in advance of the departure time. In particular, hub carriers provide one-stop flight services between two secondary cities, which requires connecting flights through a third city. As the overall network is static in short run operation, the one-stop flights schedule can be considered as fixed compared to the non-stop flight schedule

[5]Figure 4.3 and 4.4 illustrate the layout of interlaced schedule.

along an alternative schedule interlaced with the fixed one. In the third stage, both carriers announce their prices simultaneously. The sequence of moves is made following the conjecture that the PTP carrier takes less time to determine its product line position than its opponent, since it only needs to choose the schedule while its opponent needs to choose both the schedule and the quality of flight. The information of schedule is easily uncovered by the airport, so the information can be considered as complete. Moreover, as prices are easier to set than product line positioning, the strategic pricing interplay takes place in the last stage.

Assuming the marginal cost equals zero, the two carriers maximize the profit function, $\pi = p_{ij}q_{ij}$, where $i \in \{H, L\}$, and j indicates whether it provides a non-stop flight, denoted by $j = a$, or a one-stop flight, denoted by $j = b$. Let t_a denote the flying duration of a non-stop flight, and t_b denote the flying duration of a one-stop flight. $z = |x - x'|$ denotes the minimum distance between a consumer's preferred departure time and the closest departure time available. The indirect utility function for consumer k is:

$$v_k = r - p_{ij} - \theta_k t_j - \tau \theta_k z_k \tag{4.1}$$

where r represents consumers' reservation price, which is assumed to be large enough to keep all the consumers active in the market; i denotes the type of carriers: the hub carrier or the PTP carrier, $i \in \{H, L\}$, and j denotes the type of flight quality: non-stop or one-stop, $j \in \{a, b\}$ (a is for non-stop and b is for one-stop). The business model of PTP carrier is to provide non-stop flights only, while that of the hub carrier is to provide either non-stop flights or one-stop flights. Hence, $j = a$ if $i = L$ and $j \in \{a, b\}$ if $i = H$. The difference in business models indicates that there are only three possible types of flights: non-stop by the hub carrier, one-stop by the hub carrier and non-stop by the PTP carrier. p_{ij} is the price of a type j flight provided by the carrier i.

The term $\theta_k t_j$ represents the time cost generated by the flight duration that is exogenously-determined. Let $\delta = t_b - t_a$ denote the time saved by non-stop service in comparison with one-stop service. Assume $\overline{\theta} > 2\underline{\theta}$ to ensure that the market is able to accommodate two different flight quality levels. The last term $\tau \theta z$ represents the overall substitution costs. This interaction term indicates that the two characteristics of consumers, represented by x and θ are not independent of each other. It comprises of substitution costs stemming from the distance between a preferred departure time and the closest departure time available, τz,

and that from the taste (valuation) of time, θ. Intuitively, it is not surprising that those who have a higher valuation on time-saving flight are more sensitive to the accuracy of departure time. In other words, these people with a higher valuation of time-saving flights suffer more 'transportation cost' by an alternative flight other than their desired one. The overall substitution rate $\tau\theta$ that is increasing in θ captures this interaction between the two characteristics of consumers.

4.3   Equilibrium Prices

As mentioned above, the hub carrier has to choose flight quality (non-stop vs. one-stop) and flight location (flight schedule), while the PTP carrier only has to choose flight location according to its business model. The location choices are restricted to two cases: (i) exactly on the potential one-stop schedule; (ii) interlaced with the potential one-stop schedule. By borrowing the concepts in Klemperer (1992), the possible product line location layout is at either of the two polar: head-to-head or interlaced. By head-to-head product lines, both carriers provide n flights per week at the same time slots, e.g. located at $x = m/n$, $m = 1, 2, 3...n$. By interlaced product lines, one of the two carriers provides n flights per week at the locations $x = m/n$, $m = 1, 2, 3...n$ and the other carrier provides n flights at the locations $x = (m - \frac{1}{2})/n$, $m = 1, 2, 3...n$. Given the first two stages' decisions on the product location and quality, four types of subgame occur in the last stage:

- The hub carrier provides non-stop flights to compete with the PTP carrier along the same schedule, namely head-to-head homogeneous product line competition (see Figure 4.1 for illustration), hereafter HHO;

- The hub carrier provides one-stop flights and the PTP carrier provides non-stop flights along the same schedule, namely head-to-head heterogeneous product line competition (see Figure 4.2 for illustration), hereafter HHE;

- The hub carrier chooses a schedule interlaced with that adopted by the PTP carrier, and both carriers provide non-stop flights, namely interlaced homogeneous product line competition (see Figure 4.3 for illustration), hereafter IHO.

- The hub carrier provides one-stop flights along a schedule interlaced with that adopted by the PTP carrier who provides non-stop flights, namely

interlaced heterogeneous product line competition (see Figure 4.4 for illustration), hereafter IHE.
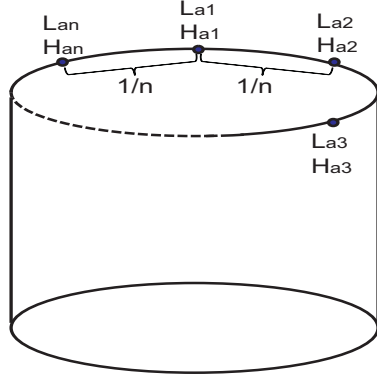

Figure 4.1: Head-to-head homogeneous product line competition

The concepts head-to-head and interlaced refer to the location dimension of the product line (the alignment of schedules), while homogeneous and heterogeneous refer to the quality dimension of the product line (non-stop flights vs. one-stop flights). Note that in the standard one-dimensional product differentiation literature, these concepts are used to imply different scenarios than those used in this model. It is straightforward to see that the equilibrium prices in head-to-head homogeneous product competition would be zero, generating zero profit for both carriers. Other than that case, there are three types of subgame left to be analyzed in the following context.

4.3.1 Head-to-Head Heterogeneous Product Line Competition

With products located at an equal distance with each other, and no location differentiation between the hub carrier and the PTP carrier in head-to-head competition, this subgame is a case of the standard vertical differentiation model with n equal-sized submarkets. The demand of each submarket is $\frac{1}{n}(\overline{\theta} - \underline{\theta})$. The two carriers maximize their own profits by setting an optimal price, which is the same for their own product line respectively. According to (4.1), the marginal consumer who is indifferent between non-stop flight and one-stop flight satisfies $r - p_{Hb} - \hat{\theta}t_b = r - p_{La} - \hat{\theta}t_a$:

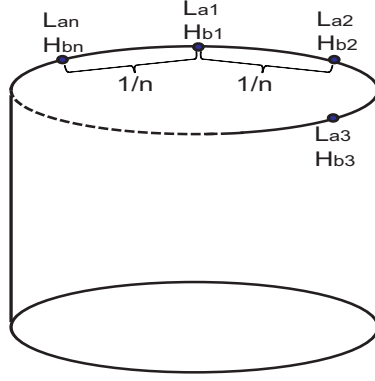$$\hat{\theta} = \frac{1}{\delta}(p_{La} - p_{Hb}) \tag{4.2}$$

Figure 4.2: Head-to-head heterogeneous product line competition

where $\delta = t_b - t_a$, $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$.

Consumers of type $\theta > \hat{\theta}$ take non-stop flights operated by the PTP carrier, and those with $\theta < \hat{\theta}$ take one-stop flights operated by the hub carrier. The demand functions are of the form:

$$
q_{Hb}(p_{Hb}, p_{La}; \delta) = \begin{cases} 0 & p_{Hb} > p_{La} - \underline{\theta}\delta \\ \frac{p_{Hb} - p_{La}}{\delta} - \underline{\theta} & p_{La} - \overline{\theta}\delta \leq p_{Hb} \leq p_{La} - \underline{\theta}\delta \\ \overline{\theta} - \underline{\theta} & p_{Hb} < p_{La} - \overline{\theta}\delta \end{cases} \quad (4.3)
$$

$$
q_{La}(p_{Hb}, p_{La}; \delta) = \begin{cases} \overline{\theta} - \underline{\theta} & p_{La} < p_{Hb} + \underline{\theta}\delta \\ \overline{\theta} - \frac{p_{Hb} - p_{La}}{\delta} & p_{Hb} + \underline{\theta}\delta \leq p_{La} \leq p_{Hb} + \overline{\theta}\delta \\ 0 & p_{Hb} + \overline{\theta}\delta < p_{La} \end{cases} \quad (4.4)
$$

The equilibrium prices as the solutions to the first-order conditions of the two carriers' profit maximization problems are as follows:

$$
p_{Hb}^{hhe*} = \frac{1}{3}(\overline{\theta} - 2\underline{\theta})\delta
$$

$$
p_{La}^{hhe*} = \frac{1}{3}(2\overline{\theta} - \underline{\theta})\delta
$$

The superscript *hhe* represents head-to-head heterogeneous competition. With the assumption $\overline{\theta} > 2\underline{\theta}$, both prices are positive in the equilibrium. The corresponding equilibrium quantities are:

$$
q_{Hb}^{hhe*} = \frac{1}{3}(\overline{\theta} - 2\underline{\theta})
$$

$$
q_{La}^{hhe*} = \frac{1}{3}(2\overline{\theta} - \underline{\theta})
$$

Under the assumption that the marginal cost for both carriers equals zero, the corresponding equilibrium profits are:

$$\pi_{Hb}^{hhe*} = \frac{1}{9}(\overline{\theta} - 2\underline{\theta})^2 \delta$$

$$\pi_{La}^{hhe*} = \frac{1}{9}(2\overline{\theta} - \underline{\theta})^2 \delta$$

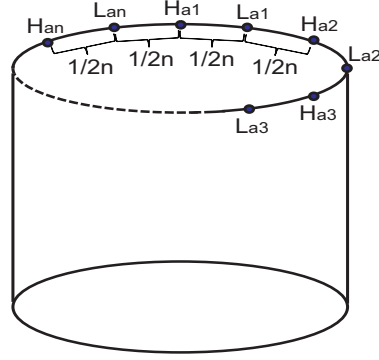## 4.3.2 Interlaced Homogeneous Product Competition

Figure 4.3: Interlaced homogeneous product line competition

This subgame describes the case when both carriers choose the same quality of product, non-stop flights, to compete against each other, while horizontally differentiating their non-stop flights by making an interlaced schedule so as to relax price competition. According to (4.1), the marginal consumer locates at:

$$\hat{x}(\theta) = \frac{p_{La} - p_{Ha}}{2\tau\theta} + \frac{1}{4n} \tag{4.5}$$

Notice that $\frac{\partial \hat{x}}{\partial \theta} \leq 0$ if $p_{La} - p_{Ha} \geq 0$, and $\frac{\partial \hat{x}}{\partial \theta} \geq 0$ if $p_{La} - p_{Ha} \leq 0$. The demand function $q_{Ha}(p_{Ha}, p_{La}; \tau, \theta, n)$ of the hub carrier is specified below::

$$q_{Ha} = \begin{cases} 0 & p_{Ha} \geq p_{La} + \overline{\theta}\tau/2n \\ 2n\int_{\theta_1}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} + \underline{\theta}\tau/2n \leq p_{Ha} \leq p_{La} + \overline{\theta}\tau/2n \\ 2n\int_{\underline{\theta}}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} - \underline{\theta}\tau/2n \leq p_{Ha} \leq p_{La} + \underline{\theta}\tau/2n \\ 2n(\int_{\underline{\theta}}^{\theta_2} \frac{1}{2n}d\theta + \int_{\theta_2}^{\overline{\theta}} \hat{x}(\theta)d\theta) & p_{La} - \overline{\theta}\tau/2n \leq p_{Ha} \leq p_{La} - \underline{\theta}\tau/2n \\ \overline{\theta} - \underline{\theta} & p_{Ha} \leq p_{La} - \overline{\theta}\tau/2n \end{cases} \tag{4.6}$$

where $\theta_1 = \frac{2n}{\tau}(p_{Ha} - p_{La})$, and $\theta_2 = \frac{2n}{\tau}(p_{La} - p_{Ha})$. According to the strategic variable p_{Ha}, the demand function varies across five successive segments represented by (4.6) (see Figure C.1 in Appendix for illustration).

The hub carrier would grab all the market share if it sets a price low enough compared to its opponent's as in the last segment, where $q_{Ha} = \overline{\theta} - \underline{\theta}$ as represented by the last line of (4.6). On the other hand it would lose all the market share to the PTP carrier if its price is set too high as represented by the first line of (4.6). In the second segment of p_{Ha} (the second line of (4.6)), the hub carrier loses the submarket consisting of customers with $\theta \in (\underline{\theta}, \theta_1)$ but stays competitive in the rest of the submarkets. In the fourth segment of p_{Ha} (the fourth line of (4.6)), the hub carrier gains the whole submarket share consisting of customers with $\theta \in (\underline{\theta}, \theta_2)$ and faces competition from the PTP carrier in the rest of the submarkets. In the third segment of p_{Ha} (the third line of (4.6)), the two carriers stay competitive throughout the whole market.

The corresponding demand function of the PTP carrier is:

$$q_{La}(p_{Ha}, p_{La}; \tau, \theta, n) = \overline{\theta} - \underline{\theta} - q_{Ha}(p_{Ha}, p_{La}; \tau, \theta, n)$$

The two carriers maximize their profits, which generates two first-order conditions. A symmetric equilibrium exists and lies in the third segment (see the derivation in Appendix 1). Both carriers set the same price in the equilibrium:

$$p_{Ha}^{iho*} = p_{La}^{iho*} = \frac{\tau(\overline{\theta} - \underline{\theta})}{2n(ln\overline{\theta} - ln\underline{\theta})}$$

Here interlaced homogeneous competition is represented by the superscript iho. The two carriers split the total market demand equally in the equilibrium: $q_{Ha}^{iho*} = q_{La}^{iho*} = \frac{1}{2}(\overline{\theta} - \underline{\theta})$. Both carriers earn the same profit:

$$\pi_{Ha}^{iho*} = \pi_{La}^{iho*} = \frac{\tau(\overline{\theta} - \underline{\theta})^2}{4n(ln\overline{\theta} - ln\underline{\theta})}$$

4.3.3 Interlaced Heterogeneous Product Competition

Interlaced heterogeneous product line competition happens when the two carriers not only vertically differentiate the flights (non-stop vs. one-stop) but also horizontally differentiate the flights by setting interlaced schedules. According to
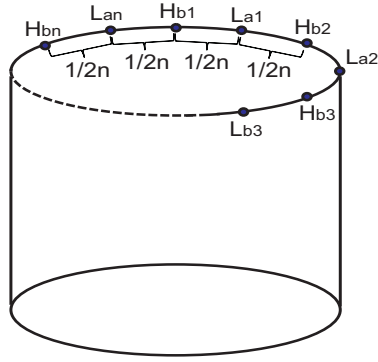
Figure 4.4: Interlaced heterogeneous product line competition

(4.1), the marginal consumer then locates at:

$$\hat{x}(\theta) = \frac{p_{La} - p_{Hb}}{2\tau\theta} + \frac{1}{4n} - \frac{\delta}{2\tau} \tag{4.7}$$

The location of the marginal consumer $\hat{x}$ depends on the price difference between the two carriers, valuation of the time cost θ, quality difference δ between non-stop and one-stop flights and the scale of product line n. To compute the equilibrium, two exclusive cases arise from separate regions of exogenous variables. When $\delta < \frac{\bar{\theta}+\underline{\theta}}{\bar{\theta}-\underline{\theta}}\frac{\tau}{2n}$, there is a unique equilibrium where both carriers grab a positive market share in each submarket partitioned by θ, namely Horizontal Dominance(HD) by tradition in the literature (see Figure 4.5 for illustration).
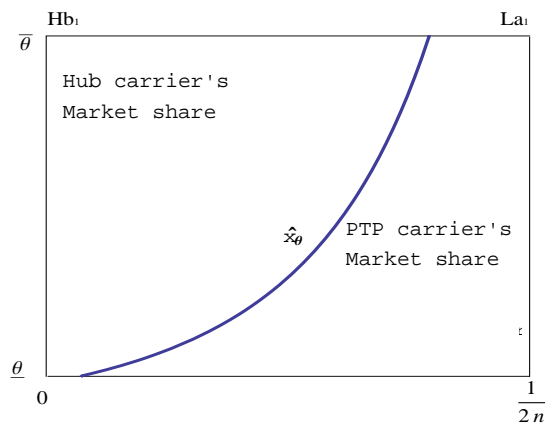

Figure 4.5: HD equilibrium

Specifically, the demand function is of the form when $\delta < \frac{\tau}{2n}$:

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$$q_{Hb} = \begin{cases} 0 & p_{Hb} \geq p_{La} - \overline{\theta}(\delta - \tau/2n) \\ 2n\int_{\theta_1}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} - \underline{\theta}(\delta - \tau/2n) \leq p_{Hb} \leq p_{La} - \overline{\theta}(\delta - \tau/2n) \\ 2n\int_{\underline{\theta}}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} - \underline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta - \tau/2n) \\ 2n(\int_{\underline{\theta}}^{\theta_2} \frac{1}{2n}d\theta + \int_{\theta_2}^{\overline{\theta}} \hat{x}(\theta)d\theta) & p_{La} - \overline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta + \tau/2n) \\ \overline{\theta} - \underline{\theta} & p_{Hb} \leq p_{La} - \overline{\theta}(\delta + \tau/2n) \end{cases}$$

$$(4.8)$$

where $\theta_1 = (p_{La} - p_{Hb})/(\delta - \frac{\tau}{2n})$, and $\theta_2 = (p_{La} - p_{Hb})/(\delta + \frac{\tau}{2n})$. The PTP carrier's demand function is:

$$q_{La}(p_{Hb}, p_{La}; \tau, \theta, n) = \overline{\theta} - \underline{\theta} - q_{Hb}(p_{Hb}, p_{La}; \tau, \theta, n)$$

Note that (4.8) differs from (4.6) in two aspects: the function form of $\hat{x}$ and the segments partitioned by the strategic variable p_{Hb}. In (4.6), indifference location $\hat{x}$ is not determined by the quality difference δ since there is no difference in product quality in the IHO competition. By contrast, consumers need to consider the time cost of flight duration as the carriers provide flights of different flying hours in this IHE competition.

The difference in segment partition from (4.8) to (4.6) indicates that given its opponent's price is fixed, the hub carrier has to set a much lower price to increase its market share when it encounters IHE competition than IHO competition. For example, in order to gain a positive market share, the hub carrier needs to set price $p_{Ha} \leq p_{La} + \overline{\theta}\tau/2n$ if both carriers provide non-stop services and their schedules are interlaced with each other, while the upper bound of p_{Ha}: $p_{La} - \overline{\theta}(\delta - \tau/2n)$ is lower when it provides one-stop flights along an interlaced schedule with its opponent's non-stop flights.

In this subgame, the demand function is slightly different in terms of the segment partition when $\frac{\tau}{2n} \leq \delta < \frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$ (see Appendix C.2). However it does not affect the equilibrium outcome. A unique equilibrium lies in the third segment (see the derivation in Appendix C.2) with the equilibrium prices:

$$p_{Hb}^{hd*} = (\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} - \frac{\delta}{3})ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$$

$$p_{La}^{hd*} = (\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} + \frac{\delta}{3})ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$$

Note that HD and VD are valid only in the IHE competition, the superscript

hd representing Horizontal Dominance indicates the type of this subgame as well. The corresponding quantities in equilibrium are:

$$q_{Hb}^{hd*} = \frac{n}{\tau}(\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} - \frac{\delta}{3})$$

$$q_{La}^{hd*} = \frac{n}{\tau}(\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} + \frac{\delta}{3})$$

The corresponding equilibrium profits of the two carriers in IHE subgame HD case are (see the derivation in Appendix C.2):

$$\pi_{Hb}^{hd*} = \frac{n}{\tau}(\overline{\theta} - \underline{\theta})^2(\frac{\tau}{2n} - \frac{\delta}{3})^2 ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$$

$$\pi_{La}^{hd*} = \frac{n}{\tau}(\overline{\theta} - \underline{\theta})^2(\frac{\tau}{2n} + \frac{\delta}{3})^2 ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$$

Based on the equilibrium outcome shown above, the hub carrier sets a lower price and earns a lower profit than the PTP carrier in HD equilibrium of IHE competition.

Compared with HHE competition, the relationships between equilibrium prices p_{Hb}^{hhe*} and p_{Hb}^{hd*}, demand functions q_{Hb}^{hhe*} and q_{Hb}^{hd*}, as well as profits π_{Hb}^{hhe*} and π_{Hb}^{hd*}, are ambiguous, which depend largely on exogenous variables, in particular, the ratio of the quality difference to the relative substitution rate $\delta : \frac{\tau}{2n}$, and the extent of heterogeneity in consumers' valuation of time, $\phi = \underline{\theta}/\overline{\theta}$. For instance, $\pi_{Hb}^{hhe*} < \pi_{Hb}^{hd*}$ is more likely to happen at a lower ratio of $\delta : \frac{\tau}{2n}$. This indicates that when the non-stop flight saves insignificant time compared with the one-stop flight, the effect of substitution rate between departure time slots becomes relatively stronger, so choosing an interlaced schedule becomes more effective in attracting consumers. The same effect applies to the PTP carrier's profit.

Compared with IHO competition, the hub carrier is worse off in HD equilibrium of IHE competition. It earns a lower profit with the one-stop flights. This is because that though it sets a lower price but gets less demand in HD equilibrium of IHE competition than in IHO competition. On the other hand, the PTP carrier apparently is better off in HD equilibrium of IHE competition than in IHO competition, since it is able to set a higher price and grab a higher demand while its opponent stays with the less appealing one-stop service.

Different from Horizontal Dominance equilibrium, Vertical Dominance equi-

librium (VD) describes the situation where the hub carrier occupies the whole lower end market share where $\theta \in [\underline{\theta}, \theta_2]$, splits a middle market with its opponent where $\theta \in [\theta_1, \theta_2]$ and loses the whole upper end market share where $\theta \in [\theta_1, \overline{\theta}]$ (see Figure 4.6 for illustration). Vertical Dominance equilibrium occurs when the
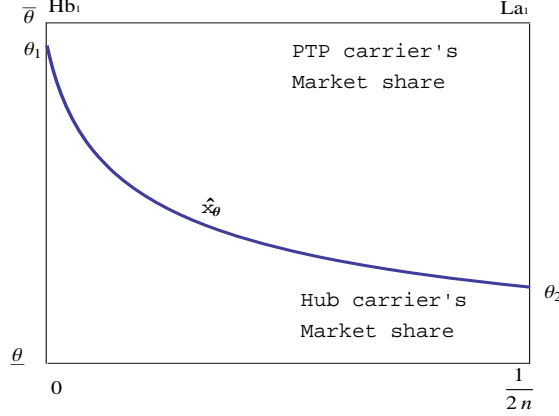

Figure 4.6: VD equilibrium

quality difference is large enough, in particular, $\delta \geq \frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$. The demand function is then of the form:

$$
q_{Hb} = \begin{cases}
0 & p_{Hb} \geq p_{La} - \underline{\theta}(\delta - \tau/2n) \\
2n\int_{\theta_1}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} - \underline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta - \tau/2n) \\
2n(\int_{\underline{\theta}}^{\theta_2} \frac{1}{2n}d\theta + \int_{\theta_2}^{\theta_1} \hat{x}(\theta)d\theta) & p_{La} - \overline{\theta}(\delta - \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta + \tau/2n) \\
2n(\int_{\underline{\theta}}^{\theta_2} \frac{1}{2n}d\theta + \int_{\theta_2}^{\overline{\theta}} \hat{x}(\theta)d\theta) & p_{La} - \overline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \overline{\theta}(\delta - \tau/2n) \\
\overline{\theta} - \underline{\theta} & p_{Hb} \leq p_{La} - \overline{\theta}(\delta + \tau/2n)
\end{cases}
$$

$$(4.9)$$

Again, $\theta_1 = (p_{La} - p_{Hb})/(\delta - \frac{\tau}{2n})$, and $\theta_2 = (p_{La} - p_{Hb})/(\delta + \frac{\tau}{2n})$ and the PTP carrier's demand function is: $q_{La}(p_{Hb}, p_{La}; \tau, \theta, n) = \overline{\theta} - \underline{\theta} - q_{Hb}(p_{Hb}, p_{La}; \tau, \theta, n)$. A unique equilibrium lies in the third segment where $p_{La} - \overline{\theta}(\delta - \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta + \tau/2n)$. The VD equilibrium prices and profits are as follows:

$$
p_{Hb}^{vd*} = \frac{\tau}{3n}(\overline{\theta} - 2\underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}
$$

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$$p_{La}^{vd*} = \frac{\tau}{3n}(2\overline{\theta} - \underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}$$

In the similar way, the superscript vd representing Vertical Dominance indicates interlaced heterogeneous product competition in the context. Note that the equilibrium quantities in VD case of IHE competition equal that in HHE competition, which are:

$$q_{Hb}^{vd*} = \frac{1}{3}(\overline{\theta} - 2\underline{\theta})$$

$$q_{La}^{vd*} = \frac{1}{3}(2\overline{\theta} - \underline{\theta})$$

The corresponding profits of the two carriers in VD equilibrium of IHE competition are:

$$\pi_{Hb}^{vd*} = \frac{\tau}{9n}(\overline{\theta} - 2\underline{\theta})^2 ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}$$

$$\pi_{La}^{vd*} = \frac{\tau}{9n}(2\overline{\theta} - \underline{\theta})^2 ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}$$

In VD equilibrium, the hub carrier sets a lower price but gains a lower market demand than its opponent, which leads to a lower profit as well. Note that it is the difference in the exogenous ratio of $\delta : \frac{\tau}{2n}$ that leads to different equilibrium outcomes: HD or VD equilibrium in the interlaced heterogeneous competition. VD happens when δ reaches a threshold, which is a little larger than $\frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$, depending on the value of $\phi = \underline{\theta}/\overline{\theta}$ (see the derivation in Appendix C.3).

When VD occurs, both carriers gain the same market demand in HHE and IHE competition, indicated by $q_{Hb}^{vd*} = q_{Hb}^{hhe*}$ and $q_{La}^{vd*} = q_{La}^{hhe*}$. But both carriers are able to set a higher price in the equilibrium of HHE competition than in VD equilibrium of IHE competition. The gap between the pair p_{Hb}^{vd*} and p_{Hb}^{hhe*} or π_{Hb}^{vd*} and π_{Hb}^{hhe*} strongly depends on the ratio of the quality difference to the relative substitution rate $\delta : \frac{\tau}{2n}$, and the extent of the heterogeneity in time cost valuation $\phi = \underline{\theta}/\overline{\theta}$. However, in general, $p_{Hb}^{vd*} < p_{Hb}^{hhe*}$, $p_{La}^{vd*} < p_{La}^{hhe*}$, both carriers are better off in head-to-head product line competition when they specialize in different quality of flights (non-stop vs. one-stop).

Proposition 4.1 *Both carriers earn a larger profit with a higher price in equilibrium of subgame HHE than subgame IHE when the quality difference δ is large enough to ensure the existence of VD equilibrium in subgame IHE.*

Proof See Appendix C.4.

The intuition of Proposition 4.1 is as follows. The two carriers' ability to charge a price premium is increased by a larger quality disparity in both HHE competition and IHE competition, because by staying apart further along the quality dimension (flight duration), both carriers have relaxed price competition pressure. This is a typical result in standard vertical differentiation model. However, the reduction in price competition pressure by a higher quality disparity is weakened in IHE competition due to the influence of the second characteristic: schedule (product location). The two carriers need to compete further in price to win the consumers as they provide different schedules, which pushes the price down. On the other hand, without consideration of the quality dimension, interlaced competition also relaxes price competition pressure because by staying apart along the schedule, the two carriers are able to sustain higher prices as well. Hence, this is a matter of which characteristic is dominant in terms of the influence on consumers' choices. In HD equilibrium, the consumers place more importance on their departure time preferences, indicating that they will accept an inferior departure time slot only if significantly compensated by a much cheaper price and/or time cost saving by non-stop flights (which represents their valuation of quality). The two carriers find that by staying apart along the variety dimension (schedule) is the better way to relax price competition. However, as the quality disparity increases, the departure time becomes less important to the consumers, and the two carriers prefer to relax the price competition pressure by increasing their quality disparity to the greatest extent. In VD equilibrium, δ is large enough so that flight quality other than the schedule becomes the dominant characteristic to the consumers, which means consumers give more weight to the duration of flight when making purchasing decisions. The two carriers relax price competition pressure further by minimizing their schedule differences that in return reinforce the disparity along the quality dimension.

In comparison with the IHO competition, the PTP carrier, as in HD equilibrium, is better off in VD equilibrium of IHE competition as well. It is able to set a higher equilibrium price and gain a higher demand in VD equilibrium. On the other side, the hub carrier will not necessarily become worse off in VD equilibrium than in IHO competition. As δ becomes larger in VD equilibrium, the hub carrier is more likely to earn a greater profit by staying with one-stop service in IHE competition as the price competition is relaxed due to the improvement of its competitor's flights quality.

4.4 Product Line Decision Making

Back to the second stage, it is the hub carrier's turn to make a move given its opponent's decision in the first stage. To simplify the game, any move leading to HHO competition is omitted as it leads to zero profit and strictly dominated by other forms of competition. Two types of subgames occur at this stage:

- Given that the PTP carrier chooses to overlap the hub carrier's potential one-stop flight schedule, the hub carrier chooses between operating its one-stop flights, which ends in HHE competition, or starting a series of non-stop flights along a schedule interlaced with that adopted by the PTP carrier, which ends in IHO competition;

- Given that the PTP carrier chooses an alternative schedule interlaced with the hub carrier's potential one-stop flight schedule, the hub carrier chooses between operating its one-stop flights, which ends in IHE competition, or starting a series of non-stop flight along a schedule interlaced with that adopted by the PTP carrier, which ends in IHO competition.

By foreseeing the hub carrier's strategy, the PTP carrier decides whether to overlap its opponent's potential one-stop flights schedule or not in the first stage. In the second stage, the hub carrier compares the profit from IHO competition and that from HHE competition if the PTP carrier chooses overlapping strategy (π_{Ha}^{iho} vs. π_{Hb}^{hhe}). It compares the profit from IHO competition and that from IHE competition if the PTP carrier chooses interlacing strategy (π_{Ha}^{iho} vs. π_{Hb}^{hd}, if HD equilibrium occurs in subgame IHE, or π_{Ha}^{iho} vs. π_{Hb}^{vd} if VD equilibrium occurs). Since IHE competition generates different equilibrium outcomes under different values of exogenous variables, which affects the hub carrier's decision in the second stage and further impacts on the PTP carrier's decision in the first stage, it is necessary to divide the following analysis into two parts distinguished by whether HD or VD prevails in IHE competition. When HD prevails in subgame IHE, the following pure strategy Subgame Perfect Nash Equilibria are presented in Proposition 4.2.

Proposition 4.2 *Given that HD is the unique equilibrium in subgame IHE, the following pure strategy Subgame Perfect Nash Equilibria occur:*

(i)The PTP carrier chooses to overlap the potential one-stop flights schedule, and the hub carrier provides non-stop flights with an interlaced schedule in equilibrium, which is characterized as IHO competition.

(ii)The PTP carrier chooses a schedule interlaced with the potential one-stop flights schedule, and the hub carrier provides non-stop flights with its own potential one-stop flights schedule in equilibrium, which is characterized as IHO competition.

Proof. See Appendix C.5.

Proposition 4.2 implies that when HD equilibrium prevails in subgame IHE, the PTP carrier is indifferent between overlapping and interlacing with its opponent's potential one-stop schedule, because the hub carrier's best response is to provide interlaced non-stop flights anyway and it is always capable of realizing this response in the model. Though there are two pure strategy Subgame Perfect Equilibria, both of them end up with the same form of product line competition, where both carriers provide non-stop flights along an interlaced schedule with each other (IHO competition).

On the other hand, when VD prevails in subgame IHE, the corresponding Subgame Perfect Equilibria are characterized as follows.

Let $\phi = \underline{\theta}/\overline{\theta}$, $\delta = \alpha(1 + \frac{2\phi}{1-\phi})\frac{\tau}{2n}$, where α is an arbitrary natural number greater than 1 measuring how large the quality difference δ is above the threshold $(1+\frac{2\phi}{1-\phi})\frac{\tau}{2n}$ in VD case. Denote $\gamma_1(\alpha, \phi) = ln\frac{\alpha(1+\frac{2\phi}{1-\phi})+1}{\alpha(1+\frac{2\phi}{1-\phi})-1}$, and $\gamma_2(\phi) = ln\frac{1}{\phi}$. Denote α_1 the solution to the equation (4.10), α_2 the solution to the equation (4.11), and α_3 be the solution to the equation (4.12).

$$[\alpha(1 + \frac{2\phi}{1-\phi}) - 1]\gamma_1 = \frac{2}{3}(1 + \phi) \tag{4.10}$$

$$[\alpha(1 + \frac{2\phi}{1-\phi}) + 1]\gamma_1 = \frac{2}{3}(1 + \frac{1}{\phi}) \tag{4.11}$$

$$\frac{9}{2}\frac{(1-\phi)^2}{(1-2\phi)^2} = \gamma_2\alpha(1 + \frac{2\phi}{1-\phi}) \tag{4.12}$$

Proposition 4.3 *Given that VD is the unique equilibrium in subgame IHE, the*

following pure strategy Subgame Perfect Equilibria occur:

(i)If $\alpha_1 \le \alpha < \alpha_3$, where $0.197477 \le \phi < 0.5$ (the lower bound of α would be α_2 where $\phi \le 0.197477$), the PTP carrier is indifferent between overlapping and interlacing the potential one-stop flights schedule, and the hub carrier provides non-stop flights with a schedule interlaced with that adopted by the low cost carrier, which leads to IHO competition.

(ii)If $\alpha > \alpha_3$, the PTP carrier chooses to overlap the potential one-stop flights schedule, and the hub carrier provides one-stop flights with the same schedule, which leads to HHE competition.

Proof. See Appendix C.6.

Case (i) in Proposition 4.3 says that there are two Subgame Perfect Nash Equilibria when α is low enough, both of which lead to subgame IHO, while case(ii) generates the conditions under which HHE competition emerges in SPNE when α exceeds some threshold. Specifically, when the ratio of quality difference δ to the relative substitution rate $\frac{\tau}{2n}$ is low enough, IHO competition occurs in SPNE. The intuition is as follows. Given a constant relative substitution rate, $\frac{\tau}{2n}$, lower δ makes the consumers give more weight to the desirability of the departure time. The hub carrier finds it is optimal to relax the price competition by maximizing the location disparity while minimizing the quality disparity with its opponent. No matter if the PTP carrier chooses overlapping strategy or not, the hub carrier's best response is to interlace with the PTP carrier's schedule and provide the same quality flights. In addition, given a constant quality difference δ, a higher absolute substitution rate τ or a lower measure of the product line scale n has the same effect with regard to increasing consumers' weight on the desirability of the departure time.

By contrast, when the ratio of quality difference δ to the relative substitution rate $\frac{\tau}{2n}$ exceeds some threshold, consumers would put more weight on the quality dimension of flights. The hub carrier would find that it is optimal to stay with its one-stop flights to relax the price competition with its opponent along the quality dimension. Foreseeing the hub carrier's response, the PTP carrier chooses to overlap the hub carriers' one-stop flight schedule so that it would end up in HHE competition which generates a higher profit.

Proposition 4.3 and Proposition 4.2 reveal that IHE product line competition is never going to occur in SPNE when HHE product line competition is possible.

The underlying economic rationale is that both carriers are better off by competing head-to-head than being interlaced in terms of product line positioning, in the case that they specialize in different quality of flights (one-stop vs. non-stop). This conclusion is consistent with the results in Proposition 4.1.

The above results stem from the assumptions that all the consumers stay active in the market and the demand is inelastic (everyone wants exactly one unit of product). Hence if we relax some of them, e.g. $\overline{\theta} > 2\underline{\theta}$, the findings are different.

When the assumption $\overline{\theta} > 2\underline{\theta}$ is relaxed, the hub carrier is not able to set a positive price in HHE competition since the market is no longer capable of accommodating two different flights' quality levels. It does not affect the results in Proposition 4.2 where HD prevails in equilibrium. But in VD case, when α reaches some threshold (α_4, see Appendix C.6 for its definition), the hub carrier chooses interlaced non-stop services against its opponent's overlapping strategy that leads to subgame IHO and interlaced one-stop services against its opponent's interlacing strategy that leads to subgame IHE. By foreseeing the hub carrier's strategy, the PTP carrier chooses to interlace with its opponent's one-stop flight. Hence, as HHE competition is no longer an option when $\overline{\theta} \leq 2\underline{\theta}$, IHE product line competition emerges in equilibrium.

4.5   Discussion

The model discussed above provides some insight into the strategic interaction between a hub-and-spoke network carrier and a point-to-point network carrier over flights' schedule arrangement, flights' quality choice and pricing in a short run operation. Scheduled flights[6] embody multiple characteristics. Two characteristics are most important to consumers: flight duration and departure times. Hence, two-dimensional product differentiation model better fits the competition between passenger carriers than a one-dimensional framework.

Given that there is a sufficiently-large heterogeneity in consumers' valuation over the quality of flights that keeps all the consumers active in the market, the two carriers end up in HHE competition if the quality difference between the

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[6]The concept of a scheduled flight is in contrast to that of a charter flight. A scheduled flight, also known as public commercial flights leave at regular intervals with tickets being purchased up to the day of departure, while a charter flight is a private flight scheduled to meet the needs of specific passengers.

non-stop flight and the one-stop flight is large enough, where the hub carrier provides one-stop flights along the same schedule as is adopted by the PTP carrier who provides non-stop flights. Conversely, if the quality difference is not large enough, the hub carrier would provide non-stop flights along the schedule interlaced with that adopted by the PTP carrier who provides non-stop flights. When the assumption regarding the degree of heterogeneity in consumer valuation of quality is relaxed, the two carriers would end up in IHE competition if the quality difference is large enough, where the hub carrier would provide one-stop flights against the PTP carrier's non-stop flights on an interlaced schedule. Intuitively, it indicates that it is not always the case that the hub carrier would improve its flight quality when confronted with the competition from non-stop flights provided by the PTP carrier. By staying with one-stop flight, the hub carrier may relax price competition with the PTP carrier either along the same schedule or along an interlaced one.

This model also provides insights of the product structure in air-travel market from a new perspective other than the cost concern. Consumers value long-haul flights more than short-haul flights because the former ones, especially those crossing continents, are less likely to be substituted by ground transportation. Accordingly, the reservation price of the long-haul flights is much higher than that of the short-haul ones, which leaves enough room for the hub carriers to design profitable one-stop routes in these long-haul markets. Hence, the co-existence of one-stop flights and non-stop flights is more likely to happen in long-haul air-travel market. Based on airlines' general operation experience, one-stop flights through a hub requires more seats to reduce the unit cost by taking the advantage of economics of scale, while non-stop flights need less seats to increase load factor to boost revenue. Following these facts, results of this model also have implications for the demand of different sizes of aircrafts.

The model has a number of limitations. First, it only considers a short run strategic interaction between two carriers. Flights schedules and prices change frequently in the air-travel markets. A dynamic model may be better at capturing the strategic interactions between carriers. Second, this model does not consider the possible product differentiation and competition *within* a carrier. In other words, there is no mixed product structure within a carrier that provides both one-stop and non-stop services, so the cannibalization effect within a firm is not taken into consideration either. It would be interesting to incorporate this internal cannibalization effect into the analysis of hub carriers' product differentiation strategy in future research. Lastly, this model assumes an equal marginal cost

set to zero for both carriers. Though this assumption may be somewhat justifiable since both of the one-stop flights and non-stop flights have aggressively sought out cost minimizing strategies, a more practical model would incorporate this difference in cost structure.

Appendix C

C.1 Equilibrium in IHO Competition

To prove that in the interlaced homogeneous competition, a unique equilibrium exists in the 3rd segment of the demand function, this proof proceeds with two parts. In part a, it proves that in the 2nd and the 4th segment, no equilibrium exists. In part b, it proves that the equilibrium exists in the 3rd segment.

a. In the 2nd segment, $p_{Ha} \in [p_{La} + \tau\underline{\theta}/2n, p_{La} + \tau\overline{\theta}/2n]$, the first-order conditions for the maximization problem are:

$$\frac{\partial \pi_{Ha}}{\partial p_{Ha}} = (p_{La} - 2p_{Ha})[ln\frac{\overline{\theta}\tau}{2n(p_{La} - p_{Ha})} + 1] + p_{Ha} + \frac{\tau\overline{\theta}}{2n} = 0 \qquad \text{(C.1)}$$

$$\frac{\partial \pi_{La}}{\partial p_{La}} = (p_{Ha} - 2p_{La})[ln\frac{\overline{\theta}\tau}{2n(p_{La} - p_{Ha})} + 1] + p_{La} + \frac{\tau\overline{\theta}}{2n} - \frac{\tau\underline{\theta}}{n} = 0 \qquad \text{(C.2)}$$

If there is a solution to the system, this equation exists: $\frac{(C.1)}{p_{La} - 2p_{Ha}} - \frac{(C.2)}{p_{Ha} - 2p_{La}} = 0$ if $p_{Ha} \neq 2p_{La}$, which is:

$$p_{La}^2 - p_{Ha}^2 + \frac{3\tau\overline{\theta}}{2n}(p_{La} - p_{Ha}) = \frac{\tau\underline{\theta}}{n}(p_{La} - 2p_{Ha}) \qquad \text{(C.3)}$$

Let $p_{Ha} = \alpha(p_{La} + \tau\underline{\theta}/2n) + (1 - \alpha)(p_{La} + \tau\overline{\theta}/2n)$ where $\alpha \in [0, 1]$, and substitute it back into C.3. It has the following solution when $\alpha \neq 0$:

$$p_{La} = \frac{1}{(\overline{\theta} - \underline{\theta})(1 - \alpha)}\frac{1}{4}[\alpha\underline{\theta} + (1 - \alpha)\overline{\theta}](4 - \alpha)(\underline{\theta} - \overline{\theta}) < 0 \qquad \text{(C.4)}$$

If $\alpha = 0$, $p_{La} = -\overline{\theta} < 0$. Furthermore, when $p_{Ha} = 2p_{La}$, $p_{La} = -\frac{\tau}{n}(\frac{1}{2}\overline{\theta} - \underline{\theta}) < 0$ under the assumption $\overline{\theta} > 2\underline{\theta}$. $p_{La} < 0$ yields a contradiction. Thus, no equilibrium exists in the 2nd segment where $p_{Ha} \in [p_{La} + \tau\underline{\theta}/2n, p_{La} + \tau\overline{\theta}/2n]$. In the

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similar way, it can be shown that no equilibrium exists in the 4th segment where $p_{Ha} \in [p_{La} - \tau\overline{\theta}/2n, p_{La} - \tau\underline{\theta}/2n]$.

b. In the 3rd segment, the equilibrium prices are: $p_{Ha}^* = p_{La}^* = \frac{\tau(\overline{\theta}-\underline{\theta})}{2n(ln\overline{\theta}-ln\underline{\theta})}$. This equilibrium exists if and only if:

$$\pi_{Ha}^*(p_{Ha}^*, p_{La}^*; \tau, \theta, n) > \pi_{Ha}(p_{Ha}, p_{La}^*; \tau, \theta, n)$$

$$\pi_{La}^*(p_{Ha}^*, p_{La}^*; \tau, \theta, n) > \pi_{Ha}(p_{Ha}^*, p_{La}; \tau, \theta, n)$$

The profit function is concave when $p_{Ha} \in [p_{La} - \tau\underline{\theta}/2n, p_{La} - \tau\underline{\theta}/2n]$, which covers the whole 3rd and 4th segment. Thus, if profits reach their maximum in the 3rd segment, no maximum lies in the 4th segment. Since $\pi_{Ha} = 0$ in the 1st segment where p_{Ha} is relatively larger compared to the 2nd segment, if the derivative of the profit function with respect to p_{Ha} were positive in the 2nd segment, π_{Ha} would be negative within this segment. Therefore, the maximum of profit should not be lying within the 2nd segment either. The following is to prove $\frac{\partial \pi_{Ha}}{\partial p_{Ha}} > 0$ when $p_{Ha} \in [p_{La} + \tau\underline{\theta}/2n, p_{La} + \tau\overline{\theta}/2n]$. Again, let $p_{Ha} = \alpha(p_{La} + \tau\underline{\theta}/2n) + (1 - \alpha)(p_{La} + \tau\overline{\theta}/2n)$ where $\alpha \in [0, 1]$, then substitute it and $p_{La}^* = \frac{\tau(\overline{\theta}-\underline{\theta})}{2n(ln\overline{\theta}-ln\underline{\theta})}$ back into C.1, which would have the following form:

$$ln\frac{\overline{\theta}\tau}{\alpha\tau\underline{\theta} + (1-\alpha)\tau\overline{\theta}} + \frac{\alpha(\underline{\theta} - \overline{\theta})(ln\overline{\theta} - ln\underline{\theta})}{\overline{\theta} - \underline{\theta} + (2\alpha\underline{\theta} - 2\alpha\overline{\theta} + 2\overline{\theta})(ln\overline{\theta} - ln\underline{\theta})} = 0 \qquad (C.5)$$

Note that when $\alpha = 0$, C.5= 0, and when $\alpha = 1$, C.5> 0. Thus, if $\frac{\partial C.5}{\partial \alpha} > 0$, C.5 > 0 for all $\alpha \in [0, 1]$. That is $\frac{\partial C.5}{\partial \alpha} > 0$ iff:

$$\Delta[\Delta(1 - 2\alpha ln\Phi) + 2\overline{\theta}ln\Phi]^2 - (2\overline{\theta}\Delta ln^2\Phi + \Delta^2 ln\Phi)(\overline{\theta} - \Delta\alpha) > 0 \qquad (C.6)$$

where $\Delta = \overline{\theta} - \underline{\theta}$, $\Phi = \overline{\theta}/\underline{\theta}$.

The minimum of the LHS of C.6 is:

$$7\Delta^2 - 4\overline{\theta}^2 ln^2\Phi + 12\Delta\overline{\theta}ln\Phi \qquad (C.7)$$

C.7> 0 iff $ln\Phi < \frac{7}{2}(1 - \Phi)$ is fulfilled, which requires $\Phi > 1$ and that is true in the context. Thus, C.6 exists, which proves $\frac{\partial \pi_{Ha}}{\partial p_{Ha}} > 0$ when $p_{Ha} \in [p_{La} + \tau\underline{\theta}/2n, p_{La} + \tau\overline{\theta}/2n]$. The proof for the maximum of profit not lying in the 2nd segment is then complete.

In short, part b implies that π_{Ha} has a global maximum in the interior of the 3rd segment. By symmetry, the same conclusion holds for $\pi_{La}^*(p_{Ha}^*, p_{La}^*; \tau, \theta, n) > \pi_{Ha}(p_{Ha}^*, p_{La}; \tau, \theta, n)$.

C.2 Equilibrium in IHE Competition HD Case

The demand function for the hub carrier's one-stop service in interlaced heterogeneous competition is of the form when $\frac{\tau}{2n} \leq \delta < \frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$:

$$
q_{Hb} = \begin{cases}
0 & p_{Hb} \geq p_{La} - \underline{\theta}(\delta - \tau/2n) \\
2n \int_{\underline{\theta}}^{\theta_1} \hat{x}(\theta)d\theta & p_{La} - \overline{\theta}(\delta - \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta - \tau/2n) \\
2n \int_{\underline{\theta}}^{\overline{\theta}} \hat{x}(\theta)d\theta & p_{La} - \underline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \overline{\theta}(\delta - \tau/2n) \\
2n(\int_{\underline{\theta}}^{\theta_2} \frac{1}{2n}d\theta + \int_{\theta_2}^{\overline{\theta}} \hat{x}(\theta)d\theta) & p_{La} - \overline{\theta}(\delta + \tau/2n) \leq p_{Hb} \leq p_{La} - \underline{\theta}(\delta + \tau/2n) \\
\overline{\theta} - \underline{\theta} & p_{Hb} \leq p_{La} - \overline{\theta}(\delta + \tau/2n)
\end{cases}
\tag{C.8}
$$

In HD case of interlaced heterogeneous competition, the profit function is continuous across five successive segments of the demand function. Consider the hub carrier's profit function first. The profit function is concave in the 3rd and 4th segment of demand function. Given any value of the PTP carrier's price $p_{La}^{hd} \in [0, +\infty]$, it can be shown that π_{Hb}^{hd} is quasi-concave in p_{Ha} across these five successive segments, which guarantees that it has a unique maximum with respect to p_{Ha}. Thus, so long as this global maximum occurs in the 3rd segment, the corresponding price is the best response of the hub carrier to its opponent, which requires that $p_{La} - \underline{\theta}(\delta + \tau/2n) \leq p_{Hb}^* \leq p_{La} - \underline{\theta}(\delta - \tau/2n)$ when $\delta < \frac{\tau}{2n}$, or $p_{La} - \underline{\theta}(\delta + \tau/2n) \leq p_{Hb}^* \leq p_{La} - \overline{\theta}(\delta - \tau/2n)$ when $\frac{\tau}{2n} \leq \delta < \frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$. The requirement on the PTP carrier's price p_{La}^* as the best response to p_{Hb}^* in the price game can be deducted in the similar way. To sum up, the prices that lead to the maximum of profits in the 3rd segment are the equilibrium prices if they satisfy:

$$
\underline{\theta}(\delta - \frac{\tau}{2n}) \leq p_{La}^{hd*} - p_{Ha}^{hd*} \leq \underline{\theta}(\delta + \frac{\tau}{2n})
\tag{C.9}
$$

when $\delta < \frac{\tau}{2n}$. or

$$
\overline{\theta}(\delta - \frac{\tau}{2n}) \leq p_{La}^{hd*} - p_{Ha}^{hd*} \leq \underline{\theta}(\delta + \frac{\tau}{2n})
\tag{C.10}
$$

when $\frac{\tau}{2n} \leq \delta < \frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n}$.
Substitute $p_{Hb}^{hd*} = (\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} - \frac{\delta}{3})ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$ and $p_{La}^{hd*} = (\overline{\theta} - \underline{\theta})(\frac{\tau}{2n} + \frac{\delta}{3})ln^{-1}\frac{\overline{\theta}}{\underline{\theta}}$ back into

(C.9) and (C.10). Let $\phi = \underline{\theta}/\overline{\theta}$, $\lambda = \frac{3}{2}\frac{ln\frac{1}{\phi}}{1-\phi}$, where $\phi < \frac{1}{2}$.

The inequality (C.9) is of the form as follows then:

$$\delta \leq \frac{\phi\lambda}{1 - \phi\lambda}\frac{\tau}{2n} \tag{C.11}$$

The LHS and the RHS of (C.10) are of the form respectively as follows:

$$\delta \leq \frac{\lambda}{\lambda - 1}\frac{\tau}{2n} \tag{C.12}$$

$$\delta \leq \frac{\phi\lambda}{1 - \phi\lambda}\frac{\tau}{2n} \tag{C.13}$$

Note the following inequalities exist in specific region of ϕ:

$$\begin{cases} 0 < \frac{\phi\lambda}{1-\phi\lambda} < 1 & 0 < \phi < 0.148999 \\ 1 \leq \frac{\phi\lambda}{1-\phi\lambda} < 1 + \frac{2\phi}{1-\phi} < \frac{\lambda}{\lambda-1} & 0.148999 \leq \phi < 1 + \frac{2\phi}{1-\phi} < 0.197477 \\ 0 < \frac{\lambda}{\lambda-1} \leq \frac{\phi\lambda}{1-\phi\lambda} \leq 1 + \frac{2\phi}{1-\phi} & 0.197477 \leq \phi < 0.466411 \\ \frac{\phi\lambda}{1-\phi\lambda} < 0 & \phi > 0.466411 \end{cases} \tag{C.14}$$

Based on the above, the following regions partitioned by the relationship between δ and $\tau/2n$ are generated, which guarantees the existence of the unique equilibrium lying in the 3rd segment of the demand function:

$$\begin{cases} \delta \leq \frac{\phi\lambda}{1-\phi\lambda}\frac{\tau}{2n} & 0 < \phi < 0.197477 \\ \delta \leq \frac{\lambda}{\lambda-1}\frac{\tau}{2n} & 0.197477 \leq \phi < 0.5 \end{cases} \tag{C.15}$$

Note that when $\phi = 0.197477$, we have $\frac{\phi\lambda}{1-\phi\lambda} = 1 + \frac{2\phi}{1-\phi}$. Hence, as long as $0 < \phi < 0.197477$, HD equilibrium of IHE subgame exists in the region $\delta \leq \frac{\phi\lambda}{1-\phi\lambda}\frac{\tau}{2n}$. But when $\phi \neq 0.197477$, either $\frac{\phi\lambda}{1-\phi\lambda}$ or $\frac{\lambda}{\lambda-1}$ is less than $1 + \frac{2\phi}{1-\phi}$, which indicates that there exists a region where $(\frac{\phi\lambda}{1-\phi\lambda})\frac{\tau}{2n} < \delta < (1 + \frac{2\phi}{1-\phi})\frac{\tau}{2n}$ or $(\frac{\lambda}{\lambda-1})\frac{\tau}{2n} < \delta < (1 + \frac{2\phi}{1-\phi})\frac{\tau}{2n}$, no equilibrium exists in subgame IHE. So we are not able to investigate the overall SPNE in that parameter region.

C.3   Equilibrium in IHE Competition VD Case

In the same way as the HD case, it can be shown that the profit function is continuous and quasi-concave in price, thus there exists a unique equilibrium.

The prices generated by FOCs. are the equilibrium prices iff they belong to the 3rd segment of demand function, which requires the following inequality to be fulfilled:

$$\underline{\theta}(\delta + \tau/2n) \leq p^*_{La} - p^*_{Hb} \leq \overline{\theta}(\delta - \tau/2n) \tag{C.16}$$

This inequality can not be solved, yielding an implicit inequality. However, it is possible to characterize the regions where VD equilibrium exists.

VD case occurs within the region: $\delta \geq (1 + \frac{2\phi}{1-\phi})\frac{\tau}{2n}$, where again, $\phi = \underline{\theta}/\overline{\theta}$. . let $\delta = \alpha(1 + \frac{2\phi}{1-\phi}\frac{\tau}{2n})$, where α is an arbitrary constant and $\alpha \geq 1$. δ takes the lowest value applicable in VD case when $\alpha = 1$. Substitute δ, $p^{vd*}_{Hb} = \frac{\tau}{3n}(\overline{\theta} - 2\underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}$, and $p^{vd*}_{La} = \frac{\tau}{3n}(2\overline{\theta} - \underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}$ back into (C.16), generating:

$$\frac{1}{2}[\alpha(1 + \frac{2\phi}{1-\phi}) - 1]ln\frac{\alpha(1 + \frac{2\phi}{1-\phi}) + 1}{\alpha(1 + \frac{2\phi}{1-\phi}) - 1} \geq \frac{1}{3}(1 + \phi) \tag{C.17}$$

$$\frac{1}{2}[\alpha(1 + \frac{2\phi}{1-\phi}) + 1]ln\frac{\alpha(1 + \frac{2\phi}{1-\phi}) + 1}{\alpha(1 + \frac{2\phi}{1-\phi}) - 1} \leq \frac{1}{3}(1 + \frac{1}{\phi}) \tag{C.18}$$

(C.17) is generated from LHS of (C.16), while (C.18) is generated from RHS of (C.16). It can be shown that RHS of (C.17)is increasing in α and RHS of (C.18) is decreasing in α. Thus, if $(C.17)$ is fulfilled at the lowest value of α, it can be held for all $\alpha \in [1, \infty)$; in the same way, if $(C.18)$ is fulfilled at $\alpha = 1$, it can be held for all $\alpha \in [1, \infty)$. Take $\alpha = 1$, (C.17) and (C.18) become:

$$\frac{\phi}{1-\phi}ln\frac{1}{\phi} \geq \frac{1}{3}(1 + \phi) \tag{C.19}$$

$$\frac{1}{1-\phi}ln\frac{1}{\phi} \leq \frac{1}{3}(1 + \frac{1}{\phi}) \tag{C.20}$$

(C.19) and (C.20) can not be fulfilled at the same time unless $\phi = 0.197477$. In particular, when $0 < \phi < 0.197477$, only (C.18) is fulfilled for all α, so additional requirement on δ and $\frac{\tau}{2n}$ is generated by (C.17); when $0.197477 < \phi < 0.5$, only (C.17) is fulfilled for all α, additional requirement on δ and $\frac{\tau}{2n}$ is then generated by (C.18). In general, the additional requirement generates $\alpha \geq g(\phi) > 1$ when $\phi \neq 0.197477$, which indicates a region where $\frac{\overline{\theta}+\underline{\theta}}{\overline{\theta}-\underline{\theta}}\frac{\tau}{2n} < \delta < g(\phi)\frac{\tau}{2n}$, no equilibrium exists in subgame IHE. So we are not able to investigate the overall SPNE in that region.

C.4 Proof of Proposition 4.1

To prove Proposition 4.1, we only need to compare the following price pairs: p_{Hb}^{vd*} vs.p_{Hb}^{hhe*} and p_{La}^{vd*} vs. p_{La}^{hhe*} because the demand for the hub carrier's one-stop service and the PTP carrier's non-stop service in subgame HHE equals their counterparts in VD equilibrium of subgame IHE respectively. Recall that in VD case, $\delta \geq (1+\frac{2\phi}{1-\phi})\frac{\tau}{2n}$, $\phi = \frac{\overline{\theta}}{\underline{\theta}}$. Let $\delta = \alpha(1+\frac{2\phi}{1-\phi})\frac{\tau}{2n}$, the comparisons are presented by the followings:

$$
\begin{aligned}
\frac{p_{Hb}^{hhe*}}{p_{Hb}^{vd*}} &= \frac{\frac{1}{3}(\overline{\theta} - 2\underline{\theta})\delta}{\frac{\tau}{3n}(\overline{\theta} - 2\underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}} \\
&= \frac{\alpha}{2}(1 + \frac{2\phi}{1 - \phi})ln\frac{\alpha(1 + \frac{2\phi}{1-\phi}) + 1}{\alpha(1 + \frac{2\phi}{1-\phi}) - 1}
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
\frac{p_{La}^{hhe*}}{p_{La}^{vd*}} &= \frac{\frac{1}{3}(2\overline{\theta} - \underline{\theta})\delta}{\frac{\tau}{3n}(2\overline{\theta} - \underline{\theta})ln^{-1}\frac{\delta + \frac{\tau}{2n}}{\delta - \frac{\tau}{2n}}} \\
&= \frac{\alpha}{2}(1 + \frac{2\phi}{1 - \phi})ln\frac{\alpha(1 + \frac{2\phi}{1-\phi}) + 1}{\alpha(1 + \frac{2\phi}{1-\phi}) - 1}
\end{aligned}
\tag{C.22}
$$

Given any pair of $\alpha \in [1, \infty)$ and $\phi \in (0, \frac{1}{2})$, it can be shown that $\frac{p_{Hb}^{hhe*}}{p_{Hb}^{vd*}} = \frac{p_{La}^{hhe*}}{p_{La}^{vd*}} > 1$, and since the demand shares in equilibrium of HHE competition and that in VD equilibrium of IHE competition are the same for both carriers, we have $\frac{\pi_{Hb}^{hhe*}}{\pi_{Hb}^{vd*}} = \frac{\pi_{La}^{hhe*}}{\pi_{La}^{vd*}} > 1$. This completes the proof of Proposition 1.

C.5 Proof of Proposition 4.2

To prove Proposition 4.2, we need to find out whether the hub carrier would choose to provide non-stop flight services or one-stop flight services in each subgame at the 2nd stage. This proof proceeds with two cases according to the separate region of ϕ.

If $0 < \phi < 0.19477$, $\delta \leq \frac{\phi\lambda}{1-\phi\lambda}\frac{\tau}{2n}$, where again, $\lambda = \frac{3}{2}\frac{ln\frac{1}{\phi}}{1-\phi}$. In the subgame following the PTP carrier's decision of overlapping the hub carrier's potential one-stop flight schedule, we need to compare the respective payoff of the hub carrier by choosing providing head-to-head one-stop service or interlaced non-

stop services.

$$\frac{\pi_{Ha}^{iho*}}{\pi_{Ha}^{hhe*}} = \frac{9}{4}\frac{\tau}{n}\frac{(1-\phi)^2}{(1-2\phi)^2}\frac{1}{\delta ln\frac{1}{\phi}}$$

$$\geq \frac{3(1-\phi-\frac{3}{2}\phi ln\frac{1}{\phi})(1-\phi)^2}{\phi ln^2\frac{1}{\phi}(1-2\phi)^2} > 1 \qquad (C.23)$$

Thus, $\pi_{Ha}^{iho*} > \pi_{Ha}^{hhe*}$, which indicates that the hub carrier would choose non-stop services to interlace with the PTP carrier's services. In the second subgame following the PTP carrier's decision of interlacing with its opponent's potential one-stop flights schedule, we need to compare π_{Ha}^{iho*}, and π_{Ha}^{hd*}:

$$\frac{\pi_{Ha}^{iho*}}{\pi_{Ha}^{hd*}} = (1 + \frac{\frac{\delta}{3}}{\frac{\tau}{2n} - \frac{\delta}{3}})^2 \qquad (C.24)$$

Since $\delta \leq \frac{\phi\lambda}{1-\phi\lambda}\frac{\tau}{2n} = \frac{\frac{3}{2}\frac{\tau}{2n}\phi ln\frac{1}{\phi}}{1-\phi-\frac{3}{2}\phi ln\frac{1}{\phi}}$, we get that:

$$\frac{\tau}{2n} - \frac{\delta}{3} \geq \frac{\tau}{2n}(1 - \frac{\frac{1}{2}\phi ln\frac{1}{\phi}}{1-\phi-\frac{3}{2}\phi ln\frac{1}{\phi}}) > 0 \qquad (C.25)$$

Hence, $\frac{\pi_{Ha}^{iho*}}{\pi_{Ha}^{hd*}} > 1$, the hub carrier would rather start a series of non-stop flights interlacing with its opponents' services, which is characterized as IHO competition.

If $0.19477 \leq \phi < 0.5$, $\delta < \frac{\lambda}{\lambda-1}\frac{\tau}{2n}$, we then get the following results:

$$\frac{\pi_{Ha}^{iho*}}{\pi_{Ha}^{hhe*}} = \frac{9}{4}\frac{\tau}{n}\frac{(1-\phi)^2}{(1-2\phi)^2}\frac{1}{\delta ln\frac{1}{\phi}}$$

$$\geq \frac{3(\frac{3}{2}\phi ln\frac{1}{\phi}+\phi-1)(1-\phi)^2}{\phi ln^2\frac{1}{\phi}(1-2\phi)^2} > 1 \qquad (C.26)$$

Since $\delta \leq \frac{\lambda}{\lambda-1}\frac{\tau}{2n} = \frac{\frac{3}{2}\frac{\tau}{2n}\phi ln\frac{1}{\phi}}{\frac{3}{2}\phi ln\frac{1}{\phi}+\phi-1}$, we get that:

$$\frac{\tau}{2n} - \frac{\delta}{3} \geq \frac{\tau}{2n}(1 - \frac{\frac{1}{2}\phi ln\frac{1}{\phi}}{\frac{3}{2}\phi ln\frac{1}{\phi}+\phi-1}) > 0 \qquad (C.27)$$

Hence, $\frac{\pi_{Ha}^{iho*}}{\pi_{Ha}^{hd*}} > 1$. To sum up, no matter if $0 < \phi < 0.19477$ or $0.19477 \leq \phi < 0.5$, the hub carrier chooses to interlace with the PTP carrier by providing non-stop services in each subgame at the 2nd stage, which ends up with the

IHO competition in the final stage of the game. Thus, the PTP carrier would be indifferent between overlapping and interlacing with its opponent's potential one-stop flight schedule as described by Proposition2. This completes the proof.

C.6 Proof of Proposition 4.3

To prove Proposition 4.3 which characterizes the SPNE when VD prevails in subgame IHE, we should first investigate the hub carrier's decision at the 2nd stage. In the subgame following the overlapping strategy of the PTP carrier at the 1st stage, the hub carrier needs to compare its payoffs of providing interlaced non-stop services or head-to-head one-stop services: π_{Ha}^{iho*} and π_{Hb}^{hhe*}:

$$\frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{hhe*}} = \frac{9}{2} \frac{(1-\phi)^2}{(1-2\phi)^2 \gamma_2} \frac{1}{\alpha(1 + \frac{2\phi}{1-\phi})} \tag{C.28}$$

Let α_3 be the solution to $\frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{hhe*}} = 1$. Since $\partial \frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{hhe*}}/\partial\alpha < 0$, any $\alpha > \alpha_3$ leads to $\pi_{Ha}^{iho*} < \pi_{Hb}^{hhe*}$, and *vice versa*.

In the subgame following the interlacing strategy of the PTP carrier, the hub carrier needs to compare its payoffs of providing interlaced non-stop services or interlaced one-stop services: π_{Ha}^{iho*} and π_{Hb}^{vd*}:

$$\frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{vd*}} = \frac{9}{4} \frac{(1-\phi)^2}{(1-2\phi)^2} \frac{\gamma_1}{\gamma_2} \tag{C.29}$$

Let α_4 be the solution to $\frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{vd*}} = 1$. Since $\partial \frac{\pi_{Ha}^{iho*}}{\pi_{Hb}^{vd*}}/\partial\alpha < 0$, any $\alpha > \alpha_4$ leads to $\pi_{Ha}^{iho*} < \pi_{Hb}^{vd*}$, and *vice versa*. It can be verified that $\alpha_3 < \alpha_4$. Furthermore, the restrictions on α to guarantee the existence of VD equilibrium are: $\alpha \geq \alpha_2$ if $\phi \leq 0.197477$ and $\alpha \geq \alpha_1$ if $0.197477 \leq \phi < 0.5$ according to Appendix C.3.

Hence, the hub carrier's strategy at the 2nd stage can be summarized as follows:

(i) $\alpha_2 \leq \alpha < \alpha_3$ when $\phi \leq 0.197477$ (the lower bound of α would be α_1 if $0.197477 \leq \phi < 0.5$), no matter what strategy the PTP carrier adopts the 1st stage, the hub carrier would choose interlaced non-stop services, which leads to subgame IHO in the final stage;

(ii) $\alpha_3 < \alpha < \alpha_4$, the hub carrier chooses head-to-head one-stop services against the PTP carrier's overlapping strategy and interlaced non-stop services

against the PTP carrier's interlacing strategy, which lead to subgame HHE and subgame IHO respectively in the final stage.

(iii) $\alpha_4 < \alpha$, the hub carrier chooses head-to-head one-stop services against the PTP carrier's overlapping strategy and interlaced one-stop services against the PTP carrier's interlacing strategy, which lead to subgame HHE and subgame IHE respectively in the final stage.

When case (i) happens, the PTP carrier is indifferent between overlapping and interlacing strategy in the first stage because the game will end up with IHO competition anyway. When case (ii) happens, the PTP carrier needs to compare its payoff in subgame HHE and subgame IHO: π_{La}^{iho*} and π_{La}^{hhe*}.

$$\frac{\pi_{La}^{iho*}}{\pi_{La}^{hhe*}} = \frac{9}{2} \frac{(1-\phi)^2}{\gamma_1 \gamma_2 (2-\phi)^2} \tag{C.30}$$

Since $\partial \frac{\pi_{La}^{iho*}}{\pi_{La}^{hhe*}} / \partial \alpha < 0$ and the maximum of $\frac{\pi_{La}^{iho*}}{\pi_{La}^{hhe*}}$ is less than 1 when $\alpha = 1$. It can be verified that $\pi_{La}^{iho*} < \pi_{La}^{hhe*}$. Hence the PTP carrier would choose overlapping strategy in the first stage. When case (iii) happens, the PTP carrier needs to compare its payoff in subgame HHE and subgame IHE: π_{La}^{ihe*} and π_{La}^{hhe*}. According to Proposition 1, we get that $\pi_{La}^{hhe*} > \pi_{La}^{ihe*}$. Hence the PTP carrier would choose overlapping strategy in the first stage. This completes the proof of Proposition 3.
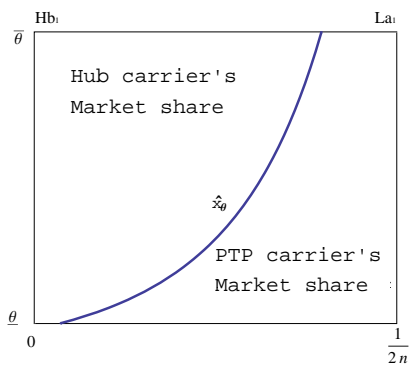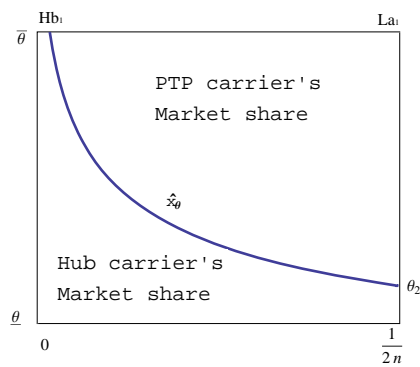
C.7 Illustration of Demand
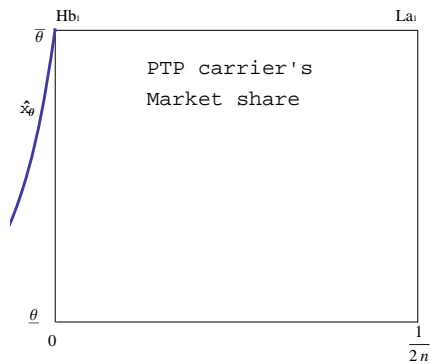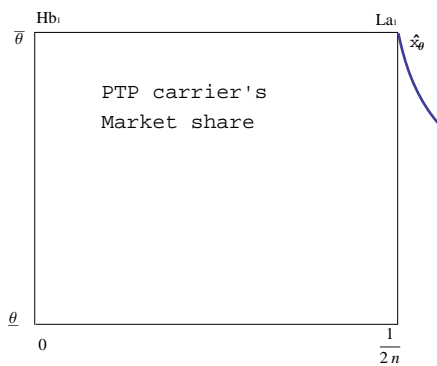
(a) First segment of demand
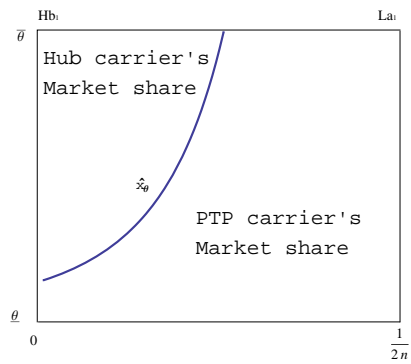
(b) Second segment of demand

(c) Third segment of demand

(d) Fourth segment of demand

(e) Fifth segment of demand

Figure C.1: The demand across five segments

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