Essays on Beliefs and Knowledge

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own. No work by any other author has been used without due acknowledgment. This work has not been submitted for any other degree or professional qualification. Chapter 3 is based on the following publication: Ibrahim Inal, (2016), ”A Metric for Partitions”, Economics Bulletin , Volume 36, Issue 1, pages 588-594.

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To my family, for everything...

To Ferit Öztürk, for teaching me how to think...

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Abstract

The unifying theme of all three chapters of this dissertation is incomplete information games. Each chapter investigates two essential components, namely beliefs and knowledge, of incomplete information games. In particular, the first two chapter studies an alternative equilibrium notion of Sákovics (2001)- mirage equilibrium- and the final chapter introduces a new notion of metric to measure the distance between partitions. All relevant notations and definitions are defined for each chapter so that any of them can be read independently.

In the first chapter, I restudy the Purification theorem of Harsanyi (1973) by relaxing the common knowledge assumption on priors for 2×2 games. I show that the limit of the (Mirage) equilibrium points in perturbed games generically converge to a pure strategy of the original complete information. This result, unlike the original one in which the limit is a mixed equilibrium point, is reminiscent of risk dominance criterion of Carlsson and van Damme (1993). I also study the conditions for different hierarchy levels that yields risk dominant outcome for coordination games. That is, I give conditions (first order stochastic dominance and monotone likelihood ratio order) that yield the risk dominant outcome of a coordination game as the limit of perturbed game á la Harsanyi (1973).

In the second chapter, I attempt to provide a generalization of mirage equilibrium for dynamic games in the context of Cournot duopoly in which costs are private information. The task of extending the definition of mirage equilib-

rium is a nontrivial issue since it is not clear on which level of finite hierarchies of beliefs the update takes place. I take a short-cut to tackle this problem and instead of working on beliefs (probability distributions) directly, I work on the support of them. Broadly speaking, players update their beliefs by eliminating the support of ”types” that do not explain the opponents’ behavior. I show that the limit of this update process converges to a Nash equilibrium of the corresponding complete information game. I also show that the rate of convergence is linear.

In the third chapter, I define a new metric to measure the distance between the partitions of a given finite set. I compare the proposed metric with the ones in the literature through examples.

Contents

Chapter 1

Purification without Common Knowledge of Priors

1.1 Introduction

A mixed strategy Nash equilibrium, unlike a pure one, has been controversial for at least two reasons. Firstly, in a mixed strategy equilibrium, a player's own payoffs do not have any impact on that player's equilibrium playing probabilities. However, this insensitivity has been challenged by many experiments (see, e.g., Ochs (1995), Goeree and Holt (2001)). Secondly, equilibrium points in a mixed strategy fail to satisfy a very basic stability notion. That is, any player in a mixed strategy can deviate from his equilibrium strategy without any cost. Indeed, any pure strategy to which a positive probability is assigned by the equilibrium mixed strategy or any arbitrary probability mixture of such strategies can be used even if the other players do not change their equilibrium mixing probabilities. Therefore, it is necessary to provide a compelling rationale, or at least a justification, for the play of mixed strategies in equilibrium.

The first justification, also known as the classical view, dates back to von Neumann and Morgenstern (1944). They argue that mixed strategies are appealing for players because they prevent their strategies being discovered. Since using a pure strategy can be discovered by an opponent easily, players would like to randomize to protect themselves. Although this explanation is sufficient and persuasive for a certain class of games[1], in a very large class of games players may want to reveal their strategies to coordinate on an equilibrium. In the stag hunt game, for example, each player will be happy to reveal his strategy in order to coordinate on the Pareto dominant outcome. For those games which require coordination or that involve mutual gains, players do not want to conceal their strategies. Therefore, a satisfactory account is needed to explain mixed strategy equilibrium playing in a broader context.

The other approach to justify mixed playing was, also known as the Bayesian view, proposed by Harsanyi (1973). Harsanyi argues that even if players have complete knowledge of their own payoffs, their knowledge about the payoffs of the other players is incomplete. Thus, payoffs in the complete information games capture the situation approximately, however, in reality, players might have some private inclination to play a certain action. The behavior of such players can be considered as random from the perspective of an outsider, whether they are a player or an observer. Thus, mixed strategy stems from fluctuations in a player's utility. Formally, these small fluctuations in utility transform a complete information game into a Bayesian game. In this framework, Harsanyi shows that generically any equilibrium (pure or mixed) can be "purified" as the limit of a pure Bayesian equilibrium of a close-by game.

More recently, Reny and Robson (2004) provide a unification result by consolidating both classical and Bayesian views on this matter. Their explanation, however, does not provide a "real" explanation for the use of mixed strategies. They consider a complete information game and a corresponding incomplete information game in which each player's type is the probability he assigns to

[1]The primary focus of von Neumann and Morgenstern (1944) is zero-sum games in which players have pure conflict.

the event that his mixed strategy in the complete information game is discovered by the other players. This explanation, however, assumes mixed strategy playing in the first place. Since Harsanyi's argument provides the real explanation for large class games, we will take his "fluctuations in payoff" argument as the explanation of mixed strategy playing.[2].

Although Harsanyi's argument is compelling and resolves the instability problem, empirical evidence about mixed strategy is still controversial. Walker and Wooders (2001) use professional tennis players and find that play of these players follow quite closely to the predictions of the theory. Similarly, Chiappori et al. (2002) use penalty kicks in professional soccer games and obtain a similar result. Behavior in the lab (See Walker and Wooders (2008) for details.), however, is inconsistent with the theory. In general, studies which are based on laboratory experiments have generally disagreed with the studies based on field data. More specifically, whilst the latter confirms the theory, the former contradicts it. We explain this dichotomy in the light of our main result, Proposition 1.2. We use Harsanyi's argument with a modification. In particular, we relate mixed playing with the absence or presence of common knowledge of priors. That is, Harsanyi assumes that the fluctuations (or uncertainties about other player's payoff) in players' payoff functions are a random vector and their distribution is known to all players i.e., common knowledge of priors.[3] The salient fact behind the uncertainty assumption is quite intuitive. Since players could not know the exact payoffs of their opponents, their knowledge about the payoffs of the other players could be inexact and this inexactness is represented by this uncertainty. With this line of thinking, however, it is difficult to agree with the claim that the players have somewhat inexact infor-

[2]This explanation is the most common and accepted one among economists. See Osborne and Rubinstein (1994). Note, however, that there might be a mixed strategy equilibria different from the Harsanyi outcome.

[3]Note that there is a tendency in the literature to take priors as common to all players. This assumption, also known as the Harsanyi doctrine, was not imposed in Harsanyi (1973).

mation about the other players' payoff function and that each player knows exactly the form (extent) of this inexactness. This seems to bypass subjective judgments (beliefs or probabilities) which are perhaps the core of the issue. In principle, even if players share the same beliefs about an event this information may not be known by all players. Technically, this information need not be common knowledge among the players. This forces us to question the notion of common knowledge of prior assumptions (CKPA) in this context.

One solution to this problem is to incorporate more types, proposed in Harsanyi (1967, 1968a,b) and constructed in Mertens and Zamir (1985), and to recover CKPA in the context of *universal type space* in which any incomplete information about a strategic situation can be embedded. Even though universal type space is an intriguing mathematical object, it has a highly complex structure due to its constructive nature. To illustrate this point, consider two players with a basic uncertainty about the state of nature with two elements. Suppose for simplicity, it is only player 1 who is uncertain about whether the state is the first or the second. This uncertainty can be described by $(q, 1-q)$ where player 1 ascribes probability q to the state of nature being the first one. Then player 2's belief about q is a probability distribution. Furthermore, player 1's beliefs about 2's beliefs about q, a member of a second-order belief hierarchy, is a distribution over distributions, which is an element of an infinite-dimensional vector space. As it can already be seen, the mathematical structure of the universal type space is far more complicated since it includes an infinite hierarchy. Therefore, in practice, there should be enough common knowledge in order to carry out a tractable analysis.

We adopt an alternative and perhaps a more practical way to handle the situation without CKPA in the purification context. To that end, we employ the Mirage Equilibrium (ME) proposed by Sákovics (2001) as a generalization of

Bayesian Nash Equilibrium (BNE).[4] Experimental studies (see Crawford et al. (2013), for a survey of the literature) reveal that subjects seem to have a finite depth of reasoning in a strategic environment. ME captures this empirically plausible fact by postulating finite-order belief hierarchies. Moreover, ME captures a truly subjective "small" world of a player as in Savage (1972) by allowing a world without CKPA.

1.2 Related Literature

The notion of common knowledge and priors have been extensively investigated in the literature. Unlike us, Rubinstein (1989) and Monderer and Samet (1989) are primarily concerned with the common knowledge concept in general. Our primary concern will be CKPA in the context of the purification theorem since it is hard to justify the existence of commonly known prior distributions or an "objective" probability distribution (Morris, 1995) especially for a one-shot interaction. The CKPA assumption had also been investigated on a more conceptual level for incomplete information games (see, e.g., Gul (1998), Lipman (2003)).

Radner and Rosenthal (1982) study the existence part of the purification theorem with different assumptions. They show that independence of fluctuations across players is important and that the purification theorem fails when there is a correlated information structure. Aumann et al. (1983) show that if the conditional distribution of fluctuations of a player over the fluctuations of other players is atomless then we can escape Radner and Rosenthal (1982) conclusion. In particular, if the independence assumption is not satisfied the purification theorem holds approximately. This paper can be considered as a continuation of these studies. The primary concern in this study, however,

[4] We limit ourselves to a need-based presentation of Mirage equilibrium. For a more thorough and a formal treatment, we refer the reader to the original source.

is not the independence assumption, but common knowledge of priors. In a sense, this study relaxes another assumption of the purification theorem and complements these two studies.

This paper can also be related to Carlsson and van Damme (1993). The global games approach of Carlsson and van Damme (1993) uses the utility fluctuation argument of Harsanyi (1973). However, unlike the purification theorem this approach allows for correlation of signals. As a result, the prediction of this approach is not a mixed strategy, but a pure one. More particularly, the global games approach refines the risk dominant pure equilibrium for 2×2 games. This paper reconciles these two different approaches. See section 1.5 for more on this.

1.3 Notation and Definitions

Let Γ be a 2×2 non-cooperative game. Denote the k^{th} pure strategy of player i as s_i^k and the set of all pure strategies for him as S_i. We shall denote the space of player i's mixed strategies by Σ_i, where any mixed strategy σ_i assigns the probability $\sigma_i(s_i^k)$ to the strategy s_i^k. As usual $supp(f)$ stands for the support of a function f and $conv(A)$ represents the convex hull of a set A. If the players use the pure strategy profile $s = (s_i^k, s_{-i}^m)$ then player i will obtain the payoff

$$U_i(s) = v_i^{k,m}.$$

A given strategy σ_i of player i is a best response to the other player's strategy choice σ_{-i} if

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}), \qquad \forall \sigma_i' \in \Sigma_i.$$

A given strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is a Nash equilibrium if every component

σ_i of σ is a best response of player i to the corresponding strategy of the other players.

Following Harsanyi, we shall define $\sigma = (\sigma_i, \sigma_{-i})$ as strong equilibrium if for all player i,

$$U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i}), \qquad \forall \sigma_i' \neq \sigma_i.$$

If σ is a strong[5] equilibrium point, given the other player's strategy, each player i has a unique best response in equilibrium. Therefore, a strong equilibrium must be a pure strategy equilibrium.

An equilibrium point $\sigma = (\sigma_i, \sigma_{-i})$ is *quasi-strong*, if there is no player i such that

$$U_i(\tilde{\sigma}_i, \sigma_{-i}) \geq U_i(\sigma', \sigma_{-i}), \qquad \forall \sigma_i' \in \Sigma_i,\ \tilde{\sigma}_i \notin conv(supp(\sigma_i)).$$

That is, σ is a quasi-strong equilibrium if all the best responses for player i to the strategy σ_{-i} is a member of the convex hull of the support of σ_i. Since the purification theorem does not hold for games which contain equilibria that are not quasi-strong, we assume Γ does not contain any such equilibrium.

In a perturbed game, $\Gamma^*(\varepsilon)$[6], each player i has a payoff shock that is private information. Thus, the payoff of player i when he chooses his k^{th} strategy and the other players choose their m^{th} strategy can be written as

$$U_i(s) = v_i^{k,m} + \varphi_i^k, \tag{1.1}$$

where φ_i^k is the shock (or the fluctuation) of k^{th} pure strategy of player i. The main idea in Harsanyi's theorem is to observe players' behavior when the effect of fluctuations φ_i^k vanishes. Although this analysis can be carried out in different ways[7] we proceed by decomposing φ_i^k into two parts so that $\varphi_i^k = \varepsilon\theta_i^k$

[5]This equilibrium should not be confused with the strong equilibrium of Aumann (1959)

[6]In order to show the role of ε in the game, we use $\Gamma^*(\varepsilon)$ notation. So, $\Gamma := \Gamma^*(0)$.

[7]See Gibbons (1992) for an example.

where $\varepsilon > 0$, and consider the limit $\varepsilon \to 0$ as in Harsanyi (1973). Hence,

$$U_i(s) = v_i^{k,m} + \varepsilon \theta_i^k. \tag{1.2}$$

We shall assume that random vectors θ_i and θ_j, $j \neq i$, are distributed independently. In order to define the equilibrium strategies we use δ_i where,

$$\delta_i := \theta_i^1 - \theta_i^2.$$

Furthermore, suppose that δ_i is distributed with the continuous density function f_i on the real line. The corresponding cumulative distribution function F_i is assumed to be strictly increasing and is continuous. Similarly, we shall define

$$v_i^m := v_i^{1,m} - v_i^{2,m}$$

which represents player i's gain by choosing his first strategy over the second one when player j chooses his m^{th} strategy.

Lastly, each player's strategy in the Bayesian game $\Gamma^\star(\varepsilon)$ is a function $s_i : \mathbb{R} \times \mathcal{B}_i \to S_i$ where $\mathbb{R}$ denotes real numbers and $\mathcal{B}_i$ represents the belief structure of player i. In equilibrium, we assume each player will follow a threshold strategy of the form that

$$s_i(\delta_i) = \begin{cases} s_i^1, & \text{if } \delta_i \geq z_i(\mathcal{B}_i) \\ s_i^2, & \text{if } \delta_i < z_i(\mathcal{B}_\rangle), \end{cases}$$

where z_i is the threshold level of player i and it depends on the belief structure $\mathcal{B}_i$ of player i. Intuitively, this strategy says that if the benefit of playing first strategy, s_i^1, is sufficiently high, then player i chooses her first strategy, otherwise he chooses his second strategy, s_i^2.

1.4 Analysis

In this section we provide the mixed equilibrium for $\Gamma^\star(0)$; purification theorem and ME for $\Gamma^\star(\varepsilon)$ to make comparisons as easy and explicit as possible.

1.4.1 Mixed Strategy Equilibrium

We first consider the mixed strategy equilibrium of $\Gamma^\star(0)$ in order to motivate Harsanyi's theorem. An easy calculation[8] yields that player i uses his first strategy s_i^1 with the probability $p_i = \frac{v_j^2}{v_j^2 - v_j^1}$.

1.4.2 Purification Theorem

In this section, we reproduce Harsanyi's theorem for our simple set-up. Note that for this subsection, we assume that the distribution functions F_i of each player i is common knowledge. Since players' strategies have already been discussed earlier, we directly present a simple version of the original result.

Proposition 1.1. *As fluctuations disappear, $\varepsilon \to 0$, the limit of the probability distribution induced by (essentially) unique pure Bayesian equilibrium of $\Gamma^*(\varepsilon)$ converges to the mixed equilibrium of $\Gamma^\star(0)$.*

We present a simple version of the proof here in order to facilitate the comparison with the later results.

Proof. Player i chooses strategy s_i^1 if his expected gain is sufficiently high. That is, assuming player j uses threshold z_j player i will play s_i^1 if,

$$(v_i^1 + \varepsilon\delta_i)\Pr(\delta_j \geq z_j) + (v_i^2 + \varepsilon\delta_i)\Pr(\delta_j < z_j) \geq 0. \tag{1.3}$$

[8] See appendix for details.

Rearranging this equation yields the following threshold for player i,

$$\varepsilon z_i = F_j(z_j)(v_i^1 - v_i^2) - v_i^1.$$

Similarly,

$$\varepsilon z_j = F_i(z_i)(v_j^1 - v_j^2) - v_j^1.$$

Since we are interested in the probability of playing a given strategy, say s_i^2, it is not necessary to find the threshold levels of each player explicitly. Observe that $z_i = F_i^{-1}(\frac{\varepsilon z_j + v_j^1}{v_j^1 - v_j^2})$. Thus, the probability of playing s_i^2 is given by,

$$\Pr(\delta_i < z_i) = F_i(z_i) = F_i\left[F_i^{-1}\left(\frac{\varepsilon z_j + v_j^1}{v_j^1 - v_j^2}\right)\right] = \frac{\varepsilon z_j + v_j^1}{v_j^1 - v_j^2}.$$

As ε approaches 0, the probability of playing strategy s_i^2 for player i approaches the mixed strategy equilibrium of $\Gamma(0)$ in which player i plays s_i^2 with probability $v_j^1/(v_j^1 - v_j^2)$.That is,

$$\lim_{\varepsilon \to 0} \Pr(s_i^2) = 1 - p_i = \frac{v_j^1}{v_j^1 - v_j^2},$$

where $\Pr(s_i^2)$ denotes player i's probability of playing his second strategy. □

Although player i has no intention to randomize, the small fluctuations in his utility induce him to use pure strategies with approximately the same probabilities as prescribed by the mixed equilibrium strategy. This result shows that the mixed strategy equilibrium is nothing, but a pure strategy Bayesian equilibrium of a bigger (or close-by) game.

1.4.3 Mirage Equilibrium

We now consider again the perturbed game $\Gamma^*(\varepsilon)$, relaxing the common knowledge of priors by using a finite-level belief system. We assume a simple belief structure. However, the crux of the analysis will not change with a different belief structure so long as the common knowledge assumption is not maintained.

Player i believes that,

i) his opponent's shock difference, δ_j is distributed with a density function f_j^i and a cumulative distribution function(cdf) F_j^i.

ii) player j believes that δ_i is distributed with cdf F_i^{ij}.

iii) player j believes that player i believes that δ_j is distributed with F_j^{iji}.

These beliefs are called parametric beliefs which in general describe the higher-order beliefs of a player about the underlying attribute (payoff) space. Additionally, players have strategic beliefs which capture the strategic uncertainty of players about each other. The main distinction between these beliefs is that while those in the former group are exogenous[9], those in the latter group are endogenous for rational agents. Putting it another way, the parametric beliefs are part of the question/data, while the strategic ones are part of the answer/prediction.

Note that in this context there are different thresholds associated with different beliefs. Hence, we label each threshold with associated distribution so that a threshold like z_i^{ij} - the belief i has about j's belief about i's threshold - is associated with F_i^{ij}.

In a Mirage equilibrium, each player is considered as a Bayesian decision-maker in the sense that his decision is based entirely on his beliefs without any restrictions. Nyarko (2010) and Hellman and Samet (2012) show that the set of states in which the common prior assumption holds is small in the

[9] The real problem is to not identify the source of these beliefs. BNE by using CKPA implicitly assumes that depending on the situation there exists an "objective" probability distribution from which posteriors can be derived, whereas ME allows "subjective" probabilities as in Savage (1972).

measure-theoretic and topological sense, respectively. Mirage equilibrium allows a richer set of states where even inconsistent beliefs are possible. As mentioned before, ME incorporates the idea that players hold finite-order belief systems which are supported in both empirical and theoretical realms[10].

This final point, though a relevant phenomenon, yields the following problem: At each level, each player tries to construct the best response given his belief at that layer. In the last layer of the belief hierarchy, however, since he has no other beliefs available, he cannot construct the best response. In other words, because of finiteness, after k steps, players need a belief for the $k+1^{\text{th}}$ step to construct their best response and to close the system. ME resolves the problem by using the belief of the same player in the previous level. Hence, in the absence of the actual belief, players "*use the closest proxy available*" to substitute the missing belief. More particularly, let z_i be a best reply to z_j^i with the corresponding belief $F_j^i(.)$. Note that z_i can be written as a function of first order strategic and parametric beliefs. Similarly, z_j^i is a best reply to z_i^{ij} with the corresponding belief F_i^{ij}. Finally, z_i^{ij} is a best reply to some strategic belief, say z_j^{iji}, with the corresponding belief F_j^{iji}. This last strategic belief, z_j^{iji}, however, can be picked arbitrarily since there is no higher order parametric belief F_i^{ijij} that can be used to form it. In the absence of such higher order belief, player i would prefer using the rationalized lower level strategic belief z_j^i to picking an arbitrary one. Thus, he substitutes z_j^{iji} with z_j^i and after this substitution, we will be able to solve the system of equations in the backward induction manner to get optimal action as a function of beliefs player i has.

The following result is the main finding of this chapter and it will provide a rigorous presentation of the informal discussion given in the previous paragraph.

Proposition 1.2. *For player i in $\Gamma^*(\varepsilon)$, Mirage equilibrium strategy is characterized*

[10]See Sákovics (2001), for details.

by the solution of the following system,

$$\varepsilon z_i = F_j^i(z_j^i)(v_i^1 - v_i^2) - v_i^1$$
$$\varepsilon z_j^i = F_i^{ij}(z_i^{ij})(v_j^1 - v_j^2) - v_j^1$$
$$\varepsilon z_i^{ij} = F_j^{iji}(z_j^i)(v_i^1 - v_i^2) - v_i^1.$$

The resulting probability of playing his second strategy, $\Pr(s_i^2)$*, is given by,*

$$F_i\left[\left(F_j^i\left[(F_j^{iji})^{-1}\left(\frac{\varepsilon\left(F_i^{ij}\right)^{-1}\left(\frac{\varepsilon z_j^i+v_j^1}{v_j^1-v_j^2}\right)+v_i^1}{v_i^1-v_i^2}\right)\right](v_i^1-v_i^2)-v_i^1\right)\Big/\varepsilon\right]. \tag{1.4}$$

If $F_j^i\left[(F_j^{iji})^{-1}(1-p_j)\right] \neq 1-p_j$ *then either* $\lim_{\varepsilon\to 0}\Pr(s_i^2) = 0$ *or* $\lim_{\varepsilon\to 0}\Pr(s_i^2) = 1$. *That is, as long as* F_j^i *and* F_j^{iji} *do not intersect at the mixed strategy equilibrium point of the unperturbed game* $\Gamma^\star(0)$ *i.e.,* $1-p_j$*, as noise vanishes this probability converges to* 0 *or* 1*, indicating a pure strategy play for player* i *in* $\Gamma^\star(0)$*.*

The proof of the claim follows the pattern of the proof of the proposition 1.1. The role of strategic and parametric beliefs, however, can be seen explicitly. We shall first find strategic beliefs of player i. This will allow us to find probability $\Pr(s_i^2)$ of playing the second strategy for player i. We will evaluate this probability when ε approaches to zero as we did in the proof of 1.1.

Proof. Players' problems will not change structurally except that they solve a similar problem in different layers with (possibly) different beliefs. That is, player i's best response to a strategic belief conditional on his parametric belief in the first layer is given by,

$$\varepsilon z_i = F_j^i(z_j^i)(v_i^1 - v_i^2) - v_i^1. \tag{1.5}$$

which is obtained by simplifying (1.3) according to the belief structure given above.

Since player i's problem depends on the threshold of player j, he needs a belief about it. He considers player j's problem in light of his beliefs. So, player i believes that player j considers the following:

$$\varepsilon z_j^i = F_i^{ij}(z_i^{ij})(v_j^1 - v_j^2) - v_j^1. \tag{1.6}$$

Similarly, this problem requires consideration of another problem that gives information about z_i^{ij}. Indeed, player i needs to consider player j's consideration about player i's problem so that,

$$\varepsilon z_i^{ij} = F_j^{iji}(z_j^{iji})(v_i^1 - v_i^2) - v_i^1.$$

As it can be seen the solution of this problem requires more information. In particular, to write this problem player i needs a belief of z_j^{iji} for which he must have F_i^{ijij}. As explained before, since player i has no such parametric belief to use, he cannot discipline the corresponding strategic belief. In the absence of such strategic belief, he will substitute it with the closest proxy available. That is, player i will use his first layer strategic belief as third layer one. This yields,

$$\varepsilon z_i^{ij} = F_j^{iji}(z_j^i)(v_i^1 - v_i^2) - v_i^1, \tag{1.7}$$

where player i replaces z_j^{iji} with z_j^i.

Now, player i can solve (1.5)-(1.7) to find the optimal threshold, z_i in ME. By rearranging (1.6), (1.7) and (1.5), respectively, we have,

$$z_i^{ij} = (F_i^{ij})^{-1}\left(\frac{\varepsilon z_j^i + v_j^1}{v_j^1 - v_j^2}\right)$$

$$z_j^i = (F_j^{iji})^{-1}\left(\frac{\varepsilon z_i^{ij} + v_i^1}{v_i^1 - v_i^2}\right)$$

$$z_i = \frac{F_j^i(z_j^i)(v_i^1 - v_i^2) - v_i^1}{\varepsilon}$$

Now, by combining these three equations we obtain the probability of using his second strategy for player i, $\Pr(s_i^2)$, as

$$F_i\left[\left(F_j^i\left[(F_j^{iji})^{-1}\left(\underbrace{\frac{\varepsilon\left(F_i^{ij}\right)^{-1}\left(\frac{\varepsilon z_j^i+v_j^1}{v_j^1-v_j^2}\right)+v_i^1}{v_i^1-v_i^2}}_{p}\right)\right](v_i^1-v_i^2)-v_i^1\right)\Big/\varepsilon\right]. \tag{1.8}$$

As ε approaches to zero, z_j^i approaches to $(F^{iji})^{-1}(v_i^1/(v_i^1-v_i^2))$. Thus,

$$\lim_{\varepsilon\to 0} Pr(s_i^2) = \lim_{\varepsilon\to 0} F_i\left[\frac{k(v_i^1-v_i^2)-v_i^1}{\varepsilon}\right]$$

where $k := F_j^i\left[(F_j^{iji})^{-1}(1-p_j)\right]$ and $1-p_j = \frac{v_i^1}{v_i^1-v_i^2}$. Hence, as long as $F_j^i\left[(F_j^{iji})^{-1}(1-p_j)\right] \neq 1-p_j$ this limit converges to 0 or 1. □

Note that in the general version of this problem, player i is going to solve a similar problem with more layers and/or more players. The result holds for these cases as well, but the relationship between higher order beliefs will not be limited with the second-layer, F_j^i, and fourth-layer, F_j^iji. Let us briefly outline the analysis when player i has an n layer. The last layer of the strategic belief will be substituted by a lower level one as shown above. The n^{th} layer and $n-2^{\text{th}}$ layer strategic beliefs will be on the same object e.g., player j's strategy. Extending this argument to the layer $n-4$, $n-6$, …etc. we can conclude that all of them are about the same strategic object. Now, our result will fail if all these beliefs ascribe the same probability to the same event i.e., the event that i's opponent uses his second strategy. So as long as this condition fails our result holds. Clearly, having all those beliefs ascribe the same probability to this event is very unlikely as this event is a zero probability event.[11]

Figure 1.1 makes the mechanics of the result more transparent.[12] The single crossing case represents a class of situations in which a player has noisy higher-

[11]This event is choosing a particular on the interval $[0, 1]$ and it is a zero probability event.
[12]See appendix 1.B for the numerical examples

order beliefs. In particular, his third-order belief is noisier than the first-order one i.e., it has higher variance. What a player does in ME is to first solve the problem by using higher order beliefs to obtain a solution, p in (1.8). Then he puts this solution into the third-order level (red curve) and then puts this inverse back to the first order belief (blue curve) which gives p'. Since p' is higher than p, it becomes arbitrarily big when noise disappears which indicates $F_i(p'-p)/\varepsilon$. This means that the player chooses his second strategy with probability 1. Note that we can conclude what a player does with the same argument, as long as p and horizontal coordinate of E do not coincide. If they do coincide, it is not possible to conclude directly that the player plays a pure strategy for sure.[13]

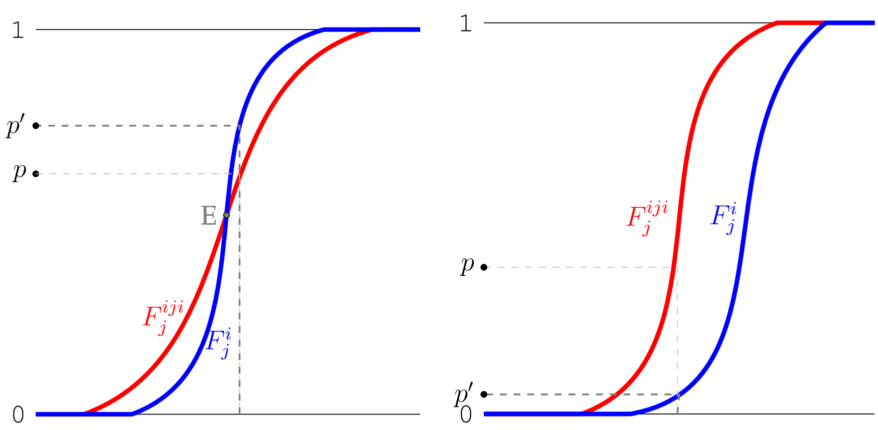
The stochastic dominance case is more trouble-free because there is no intersection point. Therefore, our claim always holds. In this case, again the player uses p initially on his third order level belief to find out the inverse point and uses this point in his first order level belief, which yields p'. This means that the player chooses his second strategy with probability 0, or equivalently he chooses his first strategy with probability 1. Note that, if both players have the same stochastic dominance relationship between the third and the first order beliefs then it is possible for them to coordinate on a particular outcome without any other requirement.

Let us discuss a case where proposition 1.2 fails.[14] If $F_j^i \equiv F_j^{iji}$ so that what player i believes for distribution of j is same as what player i believes j believes i believes about it. In this case (1.8) is reduced to

$$F_i\left[\,(F_i^{ij})^{-1}\Big(\frac{\varepsilon z_j^i + v_j^1}{v_j^1 - v_j^2}\Big)\right]$$

[13] Since this situation yields $F_i(0)$, it is impossible to draw conclusions about any probability without knowing F_i.

[14] When $v_i^1 = v_i^2 = 0$ proposition 1.2 claim also fails, but this means the game has (infinitely) many equilibria that are not quasi-strong.



(a) Single Crossing (b) Stochastic Dominance

Figure 1.1: Player's first order - blue one - and third order - red one - beliefs in Mirage Equilibrium

Again depending on the structure of the cumulative distribution functions, this probability can converge different numbers. However, this case can be considered as non-generic given the richness of the other possibilities. Note also that when $F_i \equiv F_i^{ij}$ we turn back to the purification theorem. Combining with the initial condition, we will restore the purification theorem only if $F_i \equiv F_i^{ij}$ and $F_j^i \equiv F_j^{iji}$. So as long as the parametric beliefs about the same objects are the same, we can purify the limit of Mirage equilibrium points in the Harsanyi sense. To comprehend the strong equality requirements consider, for instance, the first equality of $F_i \equiv F_i^{ij}$. This equality means that player i believes that player j believes the true distribution of shocks of player i. Similarly, $F_j^i \equiv F_j^{iji}$ means that what player i believes, and what player i believes that player j believes that player i believes about the distributions of shocks of player j are the same. Considering the rich possibilities for those beliefs, it would be fair to say that restoring the purification theorem requires strong restrictions on higher order parametric beliefs.

1.5 Coordination Games

The conclusion we reached in the previous section is reminiscent of the risk dominance criterion for coordination games. This class of games is identified with multiple pure strategy Nash equilibria in which players try to coordinate their actions on one of those equilibria. In coordination games with two players, the players try to coordinate their actions on either payoff or risk dominant equilibrium.[15] The refinement approach proposed by Carlsson and van Damme (1993) predicts the risk dominant equilibrium in these games. They also work on a perturbed version of normal form games and carry out a stability analysis. However, while this perturbation allows players to make inferences about their opponent's payoff, in the Harsanyi approach, because of the independence of noises, such inferences are not possible. As a result, these two approaches reach different conclusions. Indeed, while Harsanyi (1973) justifies mixed strategy equilibrium if any exists, Carlsson and van Damme (1993) refines a risk dominant pure equilibrium point. In the light of our result, we pose the following question: Under what conditions does a Mirage equilibrium reconcile these two different approaches?

Consider a generic symmetric[16] two-player coordination game given in figure 1.2. The payoffs in this game are so that $A > B$, $D > C$, $A > D$ and

	H	G
H	$A,\ \ A$	$C,\ \ B$
G	$B,\ \ C$	$D,\ \ D$

Figure 1.2: A generic coordination game

$D - C \geq A - B$.[17] Observe that the first set of specifications about the payoffs (i.e., $A > D > C$ and $A > B$), guarantees three Nash equilibria for this game. We shall say that the strategy pair (H, H) payoff dominates the strategy pair

[15]Mixed equilibrium in these games are Pareto dominated by pure equilibria.
[16]The result holds for asymmetric setup as well
[17]To call this game a coordination game it is enough to have $A > B$, $D > C$

(G, G) if $A > D$. We shall also say that the strategy pair (G, G) risk dominates the strategy pair (H, H) if the product of the deviation losses is highest for (G, G) (Harsanyi and Selten, 1988, p. 216). Thus, (G, G) risk dominates (H, H) if and only if $(D-C)^2 \geq (A-B)^2$. Observe that the specification $D-C \geq A-B$ implies that (G, G) is risk dominates (H, H).

Corollary 1.1. *If F_j^{iji} first-order stochastically dominates F_j^i, $F_j^{iji} \succeq_{FOSD} F_j^i$, then in the limit of ME as the perturbation vanishes player i chooses his second action G.*

Proof. By first-order stochastic dominance, $F_j^i(F_j^{iji}(1-p_j))^{-1} > 1 - p_j$ and the result directly follows from proposition 1.2. □

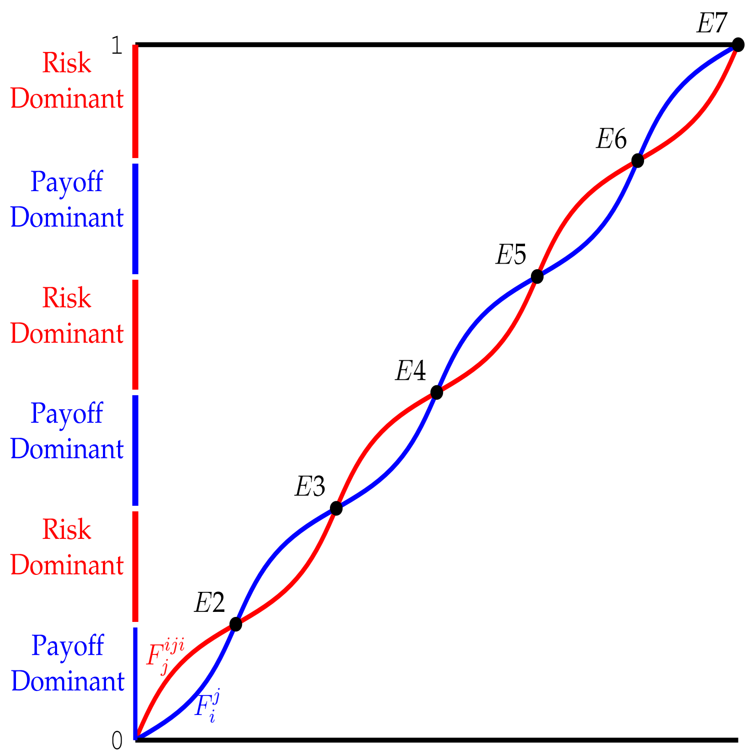


Figure 1.3: ME in Coordination Game

This corollary provides a condition when we can obtain a risk dominant equilibrium if we allow for perturbations á la Harsanyi (1973) without CKPA. Figure 1.3 shows some highly stylized beliefs to summarize the results of the main theorem and the corollary given above. By corollary 1.1, whenever the third-layer parametric belief -red one - stochastically dominates the first-layer belief - blue one -, ME predicts risk dominant action. Points between $E3$-$E4$ and $E5$-$E6$ correspond to action G. Similarly, points between $E2$-$E3$, $E4$-$E5$ and $E6$-$E7$ correspond to action H. Thus, whenever players have the same stochastic dominance in those regions[18], ME predicts as in the figure. For the set $\{E2, E3, E4, E4, E6, E7\}$ of intersection points, we cannot predict the choice of the player without a reference to the actual belief hierarchy. Depending on the distributions, the player may play as in mixed strategy playing or he may randomize with different probabilities.

Our next corollary is based on a statistical property: monotone likelihood ratio (MLR). Roughly, if a probability density function $f(.)$ satisfies MLR with respect to another probability density function $g(.)$ then the higher value of the observation the more (less) likely it is to come from $f(.)$ ($g(.)$). Formally,

Definition 1.1. *The distribution function $f(.)$ of a random variable X (monotone) likelihood ratio dominates the distribution function $g(.)$ of the same random variable if $f(.)/g(.)$ is nondecreasing.*

Corollary 1.2. *If F_j^{iji} is larger than F_j^i, $F_j^i \preceq_{MLR} F_j^{iji}$ for each player i, in the sense of monotone likelihood ratio then as noise disappears players coordinate on a risk dominant equilibrium.*

Proof. It is well known (see Shaked and Shantikumar (2007)) that monotone likelihood ratio implies first-order stochastic dominance, and we have $F_j^i \preceq_{FOSD} F_j^{iji}$. The result, then follows from corollary 1.1. □

Although MLR is a strong property it has various applications in economics (see, for example, Athey (2002)). In the context of proposition 1.2, the property can be attributed to the relative optimism (or, pessimism) in the higher belief system of the players in our set-up. That is, since player i believes that

[18]We do not assume identical distributions across different players. We assume the same dominance relation across layers for both layers, even if distributions are completely different.

player j believes that player i believes player j's payoffs are subject to (relatively) higher shocks, player i's third-order level parametric beliefs are larger, in the sense of monotone likelihood ratio than his first order one. In short, this relative optimism (or pessimism) in the belief system induce the monotone likelihood ratio property.[19]

1.6 Conclusion

The result indicating a pure play in the limit might be useful when interpreting the empirical relevance of mixed strategies. While some experimental evidence is consistent with the play of mixed strategy Nash equilibrium, some others reject it. The first group of papers that accept mixed strategy play is based on the field data where they used data from professional tennis and soccer matches (See, Walker and Wooders (2001), Chiappori et al. (2002) and Palacios-Huerta (2003).). If we accept Harsanyi's explanation for the mixed play, then one would argue that the distributions of the shocks of the players in the field are much closer to common knowledge case than in the lab. Goalkeepers, for instance, may have a pretty good estimate about the mood[20] of the kicker, or conversely, the kicker may know the inclination of the goalkeeper in a probabilistic sense. If we interpret these situations in the light of Harsanyi's explanation of mixed play, we can say that mixed strategy playing would emerge as the shocks i.e., the importance of mood or inclination, becomes smaller. In a lab environment, however, it would be hard to argue that the distribution of these shocks are common knowledge. In general, players do not know each other or they do not know the identity of their opponents.

[19]Although MLR property is strong and we can recover our result with weaker conditions, for instance assuming monotone probability ratio, MPR - see Eeckhoudt and Gollier (1995) -, order also produces the same result, however the intuitive justification for weaker conditions may not be as easy as MLR.

[20]Perhaps the kicker is in the "right mood" and scoring a goal would give a little bit higher payoff than if he were not.

So we can say that the common knowledge of priors assumption is not satisfied in these environments, it is, therefore, expected to reject play of Nash mixed strategy equilibrium in the light of Proposition 1.2. In fact, it may be the case that, one player may get extra "(dis) utility" playing cooperatively in Prisoner's Dilemma game depending on some random variable (mood, time of the day etc.).

We would like to conclude with a warning about the interpretation of our result. We do not claim our result explains the play of mixed strategy. There are many factors[21] that may explain different aspects of mixed play. We claim that if Harsanyi's explanation is relevant for subjects, that is, if there are some private small shocks that cause small changes in payoffs, then in the light of our main result we can say that priors being common knowledge or not may explain the mixed play.

[21] Experience of players, nature of the game etc. See Walker and Wooders (2008) for a good exposition on empirical relevance of mixed play.

Bibliography

Athey, S. (2002). Monotone comparative statics under uncertainty. *Quarterly Journal of Economics*, 117:187–223.

Aumann, R. J. (1959). Acceptable points in general cooperative n-person games. In Kuhn, H. W. and Luce, R. D., editors, *Contributions to the Theory of Games*. Princeton University Press, New Jersey, USA.

Aumann, R. J., Katznelson, Y., Radner, R., Rosenthal, R. W., and Weiss, B. (1983). Approximate purification of mixed strategies. *Mathematics of Operations Research*, 8(3):327–341.

Carlsson, H. and van Damme, E. (1993). Global games and equilibrium selection. *Econometrica*, 61:989–1018.

Chiappori, P. A., Levitt, S., and Groseclose, T. (2002). Testing mixed strategy equilibria when players are heterogeneous: The case of penalty kicks in soccer. *American Economic Review*, 92(4):1138–1151.

Crawford, V. P., Costa-Gomes., M. A., and Iriberri, N. (2013). Structural models of nonequilibrium strategic thinking: Theory, evidence and applications. *Journal of Economic Literature*, 51:5–62.

Eeckhoudt, J. and Gollier, C. (1995). Demand for risky assets and the monotone probability ratio order. *Journal of Risk and Uncertainty*, 11:113–122.

Gibbons, R. (1992). *Game Theory for Applied Economists*. Princeton University Press, Princeton, New Jersey.

Goeree, J. K. and Holt, C. A. (2001). Ten little treasures of game theory and ten intuitive contradictions. *American Economic Review*, 91:1402–1422.

Gul, F. (1998). A comment on Aumann's Bayesian view. *Econometrica*, 66:923–927.

Harsanyi, J. and Selten, R. (1988). *A General Theory of Equilibrium Selection in Games*. MIT Press Classics. Inst. für Math. Wirtschaftsforschung an d. Univ.

Harsanyi, J. C. (1967). Games with incomplete information played by "Bayesian" players, i-iii. part i. the basic model. *Management Science*, 14(3):159–182.

Harsanyi, J. C. (1968a). Games with incomplete information played by "Bayesian" players, i-iii. part ii. Bayesian equilibrium points. *Management Science*, 14(5):320–334.

Harsanyi, J. C. (1968b). Games with incomplete information played by "Bayesian" players, i-iii. part iii. the basic probability distribution of the game. *Management Science*, 14(7):486–502.

Harsanyi, J. C. (1973). Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points. *International Journal of Game Theory*, 2:1–23.

Hellman, Z. and Samet, D. (2012). How common are common priors? *Games and Economic Behavior*, 74(2):517 – 525.

Lipman, B. (2003). Finite order implications of common priors. *Econometrica*, 71:1255–1267.

Mertens, J. F. and Zamir, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory*, 14:1–29.

Monderer, D. and Samet, D. (1989). Approximating common knowledge with common beliefs. *Games and Economic Behavior*, 1:170–190.

Morris, S. (1995). The common prior assumption in economic theory. *Economics and Philosophy*, 11:227–253.

Nyarko, Y. (2010). Most games violate the common priors doctrine. *International Journal of Economic Theory*, 6:189–194.

Ochs, J. (1995). Games with unique, mixed strategy equilibria: An experimental study. *Games and Economic Behavior*, 10:6–38.

Osborne, M. J. and Rubinstein, A. (1994). *A Course in Game Theory*. MIT Press, Cambridge, Massachusetts.

Palacios-Huerta, I. (2003). Professionals play minimax. *The Review of Economic Studies*, 70(2):395–415.

Radner, R. and Rosenthal, R. W. (1982). Private information and pure-strategy equilibria. *Mathematics of Operations Research*, 7(3):401–409.

Reny, P. J. and Robson, A. J. (2004). Reinterpreting mixed strategy equilibria: a unification of the Classical and Bayesian views. *Games and Economic Behavior*, 48:355–384.

Rubinstein, A. (1989). The electronic mail game: The strategic behavior under almost common knowledge. *American Economic Review*, 79:385–391.

Sákovics, J. (2001). Games of incomplete information without common knowledge priors. *Theory and Decision*, 50:347–366.

Savage, L. J. (1972). *The Foundations of Statistics*. Dover, New York,USA, second revised edition edition.

Shaked, M. and Shantikumar, J. G. (2007). *Stochastic Orders*. Springer.

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey.

Walker, M. and Wooders, J. (2001). Minimax play at Wimbledon. *The American Economic Review*, 91(5):1521–1538.

Walker, M. and Wooders, J. (2008). Mixed strategy equilibrium. In Durlauf, S. N. and Blume, L. E., editors, *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, Basingstoke.

Appendix

1.A Mixed Strategy

For each player i, his expected payoff from playing his first strategy is given by,

$$\mathbb{E}(s_i^1) = v_i^{1,1} p_j + v_i^{1,2}(1 - p_j),$$

where p_j denotes player j's probability of playing her first strategy s_j^1. Similarly, player i's expected payoff from playing his second strategy is given by,

$$\mathbb{E}(s_i^2) = v_i^{2,1} p_j + v_i^{2,2}(1 - p_j).$$

Equalizing these expected payoffs yields $p_j = v_i^2/(v_i^2 - v_i^1)$.

1.B Numerical Examples

In this section, we shall give some examples to illustrate how our claim applies to different games. Additionally, we shall demonstrate how our claim fails for games in which players think that their opponents play exactly with the probabilities prescribed in the mixed strategy equilibrium of the unperturbed game. Consider the following games:

For each game Γ_t, we shall define the perturbed game Γ_t^* as in equation 1.2.

	a_2^1	a_2^2
a_1^1	$1,5$	$4,1$
a_1^2	$2,1$	$0,3$

Γ_1

	a_2^1	a_2^2
a_1^1	$5,5$	$0,4$
a_1^2	$4,0$	$2,2$

Γ_2

	a_2^1	a_2^2
a_1^1	$0,0$	$0,-1$
a_1^2	$1,0$	$-1,3$

Γ_3

	a_2^1	a_2^2
a_1^1	$1,-1$	$-1,1$
a_1^2	$-1,1$	$1,-1$

Γ_4

For the sake of simplicity, we choose the following uniform distributions:

$$F_i \sim \mathcal{U}(-\alpha_i, +\alpha_i), \quad F_j^i \sim \mathcal{U}(-\beta_i, +\beta_i), \quad F_i^{ij} \sim \mathcal{U}(-\alpha_i', +\alpha_i'), \quad F_j^{iji} \sim \mathcal{U}(-\beta_i', +\beta_i')$$

For Γ_1 the only equilibrium is the mixed one in which player 1 uses his first strategy with $1/3$ probability and player 2 uses his first strategy with $4/5$ probability. In view of proposition 1.2 player 1 solves

$$\begin{aligned}
\varepsilon z_1 &= -5\frac{z_2^1 + \beta_1}{2\beta_1} + 1 = \frac{-5z_2^1 - 3\beta_1}{2\beta_1} \\
\varepsilon z_2^1 &= 6\frac{z_1^{12} + \alpha_1'}{2\alpha_1'} - 4 = \frac{3z_1^{12} - \alpha_1'}{\alpha_1'} \\
\varepsilon z_1^{12} &= -5\frac{z_2^1 + \beta_1'}{2\beta_1'} + 1 = \frac{-5z_2^1 - 3\beta_1'}{2\beta_1'}
\end{aligned}$$

If we multiply the second equation with ε and combine this with the third equation, we obtain z_2^1 as

$$z_2^1 = \frac{-(9\beta_1' + 2\alpha_1'\beta_1'\varepsilon)}{2\alpha_1'\beta_1'\varepsilon^2 + 15} = \frac{-(9\beta_1' + K\varepsilon)}{K\varepsilon^2 + 15}$$

where $K = 2\alpha_1'\beta_1'$. By using the first equation and z_2^1, we can find player 1's threshold and probability of playing his second strategy. Hence,

$$z_1 = \frac{5(9\beta_1' + K\varepsilon) - 3\beta_1(K\varepsilon^2 + 15)}{2\varepsilon\beta_1(K\varepsilon^2 + 15)}$$

$$\Pr(a_1^2) = \frac{5(9\beta_1' + K\varepsilon) - 3\beta_1(K\varepsilon^2 + 15) + 2\varepsilon\alpha_1\beta_1(K\varepsilon + 15)}{4\varepsilon\alpha_1\beta_1(K\varepsilon^2 + 15)}$$

Similar analysis yields the following equations for player 2:

$$z_2 = \frac{3(5\beta_2' - 3M\varepsilon) - \beta_2(2M\varepsilon^2 + 15)}{\varepsilon\beta_2(2M\varepsilon^2 + 15)}$$

$$\Pr(a_2^2) = \frac{3(5\beta_2' - 3M\varepsilon) - \beta_2(2M\varepsilon^2 + 15) + \varepsilon\alpha_2\beta_2(2M\varepsilon^2 + 15)}{2\varepsilon\alpha_2\beta_2(2M\varepsilon^2 + 15)}$$

where $M = \alpha_2'\beta_2'$. Note that depending on the values of β_i and β_i' playing second strategy for player i, $\Pr(a_i^2)$, converges either to 0 or 1.

For Γ_2, observe that there are two pure Nash equilibria and a mixed one. For this game, players' thresholds and prescribed probabilities in Mirage equilibrium are as following:

$$z_i = \frac{9\beta_i' + 3K\varepsilon + \beta_i(2K\varepsilon^2 - 9)}{2\varepsilon\beta_i(2K\varepsilon^2 - 9)}$$

$$\Pr(a_i^2) = \frac{9\beta_i' + 3K\varepsilon + \beta_i(2K\varepsilon^2 - 9) + 2\varepsilon\alpha_i\beta_i(2K\varepsilon^2 - 9)}{4\varepsilon\alpha_i\beta_i(2K\varepsilon^2 - 9)}$$

As in the previous example this probability converges to 0 or 1 depending on the values of β_i and β_i'. Note that this general pattern related with the upperbound of the support does not hold in general. If we change distributions from uniform to normal for instance, then the parameter that determines the value of convergence becomes the means of the distributions.

The other two games are chosen to show how players play when they believe that the opponent randomizes exactly the same as in the mixed strategy equilibrium. Note that for this event i.e., players' beliefs coinciding with mixed strategy equilibrium playing, our claim about choosing pure strategy might not hold. In addition to crystallizing this point, these examples enable us to compare deviation for each player from his own mixed strategy equilibrium play.

For Γ_3, player 1's problem in Mirage equilibrium is defined by the following system of equations:

$$\varepsilon z_1 = \frac{-z_2^1}{\beta_1}$$

$$\varepsilon z_2^1 = \frac{2z_1^{12} + \alpha_1'}{\alpha_1'}$$

$$\varepsilon z_1^{12} = \frac{-z_2^1}{\beta_1'}$$

This yields $z_2^1 = \dfrac{\varepsilon\alpha_1'\beta_1'}{\alpha_1'\beta_1'\varepsilon^2 + 2}$ that implies as noise disappears player 1 thinks his opponent uses his second strategy with $1/2$ probability which is the same as mixed strategy play of player 2 in the normal form game. Given z_2^1,

$$z_1 = \frac{-\alpha_1'\beta_1'}{\beta_1(\alpha_1'\beta_1'\varepsilon^2 + 2)}$$

$$\Pr(a_1^2) = \frac{-\alpha_1'\beta_1' + \alpha_1\beta_1(\alpha_1'\beta_1'\varepsilon^2 + 2)}{2\alpha_1\beta_1(\alpha_1'\beta_1'\varepsilon^2 + 2)}$$

As ε goes to zero this probability converges to $\dfrac{2\alpha_1\beta_1 - \alpha_1'\beta_1'}{4\alpha_1\beta_1}$ which may or may not be the mixed strategy play in the normal form game. Surprisingly, player 2's probability converges 0 or 1 as in the previous examples. The reason is that, as it could be realized, player 2, unlike player 1, does not think his opponent plays with the same probability as in the mixed strategy equilibrium.

The last example is also in the same spirit with a little twist. Player i's Mirage strategy can be described as the solution of the following system of equations:

$$\varepsilon z_i = 4\frac{z_j^i + \beta_i}{2\beta_i} - 2 = \frac{4z_j^i}{2\beta_i}$$

$$\varepsilon z_j^i = 4\frac{z_i^{ij} + \alpha_i'}{2\alpha_i'} - 2 = \frac{4z_i^{ij}}{2\alpha_i}$$

$$\varepsilon z_i^{ij} = \frac{z_j^i + \beta_i'}{2\beta_i'} - 2 = \frac{4z_j^i}{2\beta_i'}$$

The solution of this system yields $z_i = 0$. Thus, for player i, the Mirage equilibrium predicts that he will use his second strategy with $1/2$ probability as in mixed strategy playing. Note that these results do not hold, if we change the parameters of the uniform distribution or the distribution itself.

Chapter 2

Dynamic Mirage Equilibrium: An Example

2.1 Introduction

Consider a standard Cournot duopoly model in which each firm knows its own cost, but is unsure about the cost of the other firm. How should firms decide their production levels? The difficulty in answering this question stems from sequential expectations (Harsanyi, 1967). That is, firm one's production level depends on what it expects about firm two's production, which in turn depends on firm one's expectation about the cost of firm two. This expectation of firm one about the cost of firm two is called 'first-order expectation'.[1] Since the same logic is true for firm two, then firm one's expectations about the first-order expectation of firm two, called 'second-order expectation', affects this decision. Continuing this process yields an infinite sequence of expectations about the unknown parameters for each player, which may also be called parametric beliefs. Harsanyi (1967) considers this approach very complicated and cumbersome and offers a general framework to transform this situation into

[1] In general, first-order and higher-order beliefs are defined over the state of nature. In this context the state of nature is identified with the vector cost parameters.

the following one: Consider the same duopoly, but assume that the probability distribution P of the cost parameters is commonly known. Then the issue of such sequences of higher and higher-order reciprocals do not arise. Moreover, this transformation (also known as Harsanyi program) does provide an answer to the question: "How should firms decide their production levels?"

Another possible modification of the initial situation is the following: Consider the same duopoly and that each firm is endowed with a *finite* number of parametric beliefs[2] and that these beliefs are part of the private information so the beliefs are not common knowledge. Arguably, many real life situations correspond to this modification. Indeed, it is hard to justify a commonly known probability distribution for real life examples. The complication of original situations arises from the subjective nature of expectations. Indeed, if the underlying uncertainty can not be modeled with an objective distribution, then applying the Harsanyi program may be inappropriate. Indeed, the idea of Bayes equivalence (Harsanyi, 1967, pp. 174-175) critically depends on the existence of an objective probability distribution. In the absence of such an object, we cannot transform the original incomplete information game to an imperfect information one. Hence, the question is: How should firms decide their production in this situation? Or, in general, how do we describe the actions (or the strategies) of the players in an incomplete information game in which priors are also part of the private information?

Sákovics (2001) proposed a solution concept called 'mirage equilibrium' for such games of incomplete information[3]. This solution concept is defined for static incomplete information games and the immediate question is: can this equilibrium be applied to dynamic games? Since many real life problems

[2]Experimental evidence suggests that individuals have a finite depth of reasoning. See Crawford et al. (2013) for a survey of this literature.

[3]Battigalli (2003) calls them "genuine incomplete information". We agree with this classification since incomplete games of incomplete information with a prior are a very small subset of the incomplete information games. See Nyarko (2010) and Hellman and Samet (2012) for more on this issue.

are inherently dynamic, being able to apply this concept beyond the important but limited class of static games is of great interest.

Sákovics (2001) also emphasized the importance of expanding Mirage equilibrium to dynamic games. However, it did not provide a feasible way. This chapter addresses this issue. It does not prove a general result nor does it provide a complete answer, but it shows a way to apply Mirage equilibrium in the context of a dynamic environment. In particular, we apply Mirage equilibrium to the repeated Cournot duopoly model in which uncertainty is about the cost parameter(s).

We show that if players are myopic in the sense that they just consider the instantaneous payoffs without considering the future implications of their actions, then players can learn the true cost parameter of their opponents under Mirage equilibrium. The learning occurs by iterated elimination of unjustifiable types for the observed action. That is, players do not consider the types that do not explain the action of the opponents. Thus, the game eventually becomes a complete information game and players play according to the Nash equilibrium of the underlying complete information game. Because of the framework we are using and of the dynamic nature of the problem, this result can be assessed from different aspects. One important aspect of our result is that learning is driven by unexpected events. So each period a player updates his beliefs if the realized outcome is different from the expected outcome. Unlike, (separating) Bayesian equilibrium, however, the learning does not occur at the end of the first period. That is, the learning described in here is slow but eventually complete.

The learning literature, in general, has two contradicting views about learning to play Nash equilibrium. Papers such as Kalai and Lehrer (1993), Jordan (1995) and Nyarko (1998) argue that the actual play converges to Nash equilibrium in a repeated environment. On the other hand, papers such as Nachbar

(1997) and Foster and Young (2001) argue about the limitations of the assumptions of these convergence results and insist on the difficulty, or even the impossibility, of predicting the behavior of rational agents. Our findings in some sense have some common ground with the papers in the first group, however, since the notion of common knowledge in Mirage equilibrium, unlike the papers mentioned above, is in its weakest form, it is hard to compare our findings with theirs.

The literature in which incomplete information games are investigated is quite substantial yet again no paper in this group is directly comparable with Mirage equilibrium. The substantial part of this literature (see for example Battigalli and Siniscalchi (1999), Battigalli and Siniscalchi (2003) and Dekel et al. (2007)) investigates epistemic issues, formalizes rationality notion and beliefs, and investigates their implications regarding the play in a game. The main ingredients of this literature are the universal type space constructions of Mertens and Zamir (1985) and the rationalizability notion of Pearce (1984) and Bernheim (1984). Battigalli and Bonanno (1999) provide an almost complete summary of this literature. The closest paper to ours in this literature is Nyarko (1997) which studies an incomplete information game with a finite set of attributes where each agent is endowed with an infinite belief hierarchy. He shows that under some conditions[4] players will learn the true attribute of their opponents or "fundamentals of the economy" (as the way he describes it), hence the actual play of the game converges to the Nash equilibrium of the complete information game. We discuss the general logic of convergence results in the last part where we also touch upon differences between Nyarko (1997) and this paper.

The rest of the paper is organized as follows: in section 2.2 we reintroduce mirage equilibrium, then in sections 2.3 and 2.4 we discuss Cournot duopoly

[4]These conditions are; (i) contraction of best responses and (ii) mutual absolute continuity.

model with one-sided asymmetric information; we then move onto two-sided asymmetric case in section 2.5, and conclude with section 2.6 where we discuss some important aspect of the learning process described in this paper.

2.2 Mirage Equilibrium

In this section, we give the Mirage Equilibrium (ME) definition for Bayesian games without common knowledge of priors. The two most relevant properties of ME are as follows:

(a) Unlike Bayes Nash Equilibrium (BNE) it enables us to handle incomplete information games without prior beliefs over the attribute space being common knowledge (CK).

(b) Contrary to the universal type space construction due to Mertens and Zamir (1985), it assumes finite belief hierarchy.

On the one hand we have BNE putting strong and restrictive assumptions on the beliefs of each player where priors are CK, and on the other hand, there is the (complex) universal type space construction in which players have infinite belief hierarchies about the underlying uncertainty, whether on strategies or on fundamental uncertainty. ME, however, discards the restriction on beliefs without being too intricate or complex. In fact, ME can be considered as a belief equivalent version of the level-k theory. Loosely speaking, agents choose best responses according to their cognitive hierarchy in level-k thinking. In ME, however, people choose best responses according to their belief hierarchy which is a collection of finite parametric beliefs about some attribute space. In this equilibrium concept, parametric beliefs - beliefs about attributes,- and strategic beliefs -beliefs about strategies,- are evaluated separately. Therefore, we define the game structure and belief structures separately. For a detailed discussion of belief structures in ME and other equilibrium concepts for incomplete information games, see Sákovics (2001) and references therein.

Consider the quadruple $\Gamma \equiv \langle N, (K_i)_{i\in N}, (S_i)_{i\in N}, (V_i)_{i\in N}\rangle$ where:

- $N = \{1, 2\}$ is a finite set of players;
- $K = \prod_{i\in N} K_i$, where K_i is the attribute (type) space of player i;
- $S = \prod_{i\in N} S_i$, where S_i is a set of feasible mixed actions for player i; and
- $V_i : K \times S \to \mathbb{R}$ is the utility function for player i.

Adding a belief R_i over the attribute space (priors) for each player completes the description of Harsanyi's construction of a Bayesian game, which is assumed to be common knowledge. The general tendency in the literature is to assume a common prior (CP). In this case, beliefs are said to be consistent. The main argument behind CP is not to have "agreeing to disagree" type of arguments (see Binmore (1987) and Binmore (1988) for further on this issue.).

Define player i's parametric beliefs as follows:

(For a set A, we denote the set of probability distributions on A by $\Delta(A)$.)

- $R_i^1 \in \Delta(K)$ is player i's first-order belief. It is a probability distribution over the attribute space.
- $R_i^2 \in \Delta(\Delta(K))$ is player i's second-order belief. It is a probability distribution over the first order beliefs of player j and so on.

Note that in general the set of second order beliefs is represented by $\Delta(K \times \Delta(K))$ which allows correlation for different players' beliefs. We restrict ourselves to the cases where beliefs are independent across layers i.e., correlation is excluded. We were not able to extend our analysis to this general case.

The following remark will be helpful to represent higher order beliefs in a simple way and to represent strategies as a function of the attributes.

Remark 1. To any belief R_i^2, corresponds a belief $r_i \in \Delta(K)$ which almost always assigns the same probabilities as R_i^2 to the same events. To make this claim more rigorous, let $\mathcal{A}$ be σ-field on K, and define $\Delta(K)$ to be the space of all probability measures on $(K, \mathcal{A})$. Now let $\mathcal{F}$ be the σ-field on $\Delta(K)$ generated by the collection of probability maps $P \mapsto P(A)$ for $A \in \mathcal{A}$. Now, for any probability measure R^2 on $(\Delta(K), \mathcal{F})$ we can define $r \in \Delta(K)$ by[5]

$$r(A) = \int_{\Delta(K)} P(A) R^2(\mathrm{d}P).$$

This is the usual integral of a random variable on the probability space $(\Delta(K), \mathcal{F}, R^2)$. It is easy to check that $r(.)$ defines a genuine probability measure on $(K, \mathcal{A})$.

The strategy $\sigma_i(.)$ of player i will be a mapping from his attribute space K_i to his set of mixed actions S_i.

Definition 2.1. *Player i's best response with belief R_i to the strategy profile σ_{-i}, denoted $\sigma_i \in BR(\sigma_{-i}, R_i)$ is given by* [6],

$$\begin{aligned} BR(\sigma_{-i}, R_i) &:= \arg\max_{\sigma_i' \in S_i} EU_i(\sigma_i', \sigma_{-i}), \\ &:= \arg\max_{\sigma_i'} \sum_{k_{-i} \in K_{-i}} R_i(k_i, k_{-i}) V_i\left((\sigma_i', \sigma_{-i}), k_i, k_{-i}\right). \end{aligned}$$

In BNE, there exists a commonly known probability distribution R^*, which is also known as CP, such that $R_i = R^\star$ for every player i. Therefore, each player uses this belief to calculate his best response in BNE. Harsanyi (1967) uses common knowledge assumption to avoid the sequence of expectations which comes out naturally in any incomplete information game as we discussed before. ME turns back to these expectations (or beliefs) by not imposing

[5]Note that for an arbitrary measure μ, $\int f \mathrm{d}\mu$, $\int f(x)\mu(\mathrm{d}x)$ and $\int f(x)\mathrm{d}\mu(x)$ represent the same thing.

[6]Expected utility given here is the ex interim expected utility. For details see Shoham and Leyton-Brown (2008).

common knowledge assumption for a given incomplete information game. So ME uses the belief hierarchy (beliefs and higher order beliefs) as the primitive data.

Definition 2.2. *An n-layer belief hierarchy $B_i^n = (R_i, R_i^j, \ldots, R_i^{j,\ldots,})$ is the collection of the first-order belief , R_i -a probability distribution over the attribute space-; the second-order belief, R_i^j -a probability distribution over the first order beliefs of player j-, and so on until the $n-th$-order belief, $R_i^{j,\ldots,}$.*

Remark 2. Note that all members of B_i^n are in $\Delta(K)$ following remark 1 given before. It is also possible to model more complex beliefs within this framework. Indeed, in a very general construction à la Mertens and Zamir (1985) we can write B_i^∞ where correlation in higher order beliefs are possible. The so-called universal type space construction of Mertens and Zamir (1985) contains all possible B_i^∞ pertaining to underlying uncertainty and higher order beliefs. Thus, a belief hierarchy in ME can be considered a truncated version of a member of universal type space. One important aspect of the universal type space construction is that it can be constructed over any basic uncertainty space[7] whereas in ME the construction is based on the space of (utility) parameters which is naturally called parametric beliefs.

Definition 2.3 (Sákovics, 2001)**.** *Given a 3-layer belief system for player i, a strategy $\sigma_i(.)$ of player i forms part of a Mirage equilibrium profile if and only if there exists strategies $\sigma_j^i(.)$[8], $\sigma_i^{ij}(.)$ with $j \neq i$ such that*

- $\sigma_i \in BR(\sigma_j^i, R_i)$.
- $\sigma_j^i \in BR(\sigma_i^{ij}, R_i^j)$.
- $\sigma_i^{ij} \in BR(\sigma_j^i, R_i^{ji})$.

[7] Ahn (2007), for instance, constructs hierarchies of ambiguous beliefs where players do not have precise beliefs but instead have set of beliefs.

[8] For strategy σ, superscripts represent who thinks about σ and subscript represents whose variable σ is. For belief R, superscripts represent who thinks, whilst subscript represent whose belief it is.

Note that this definition can easily be extended to any finite-layer belief system, but we take a three layer system so that it shows the crucial aspects of Mirage equilibrium and saves us from tedious algebra. The key difference between ME and BNE is that the latter "closes" the model with a restriction on parametric beliefs, namely CK assumption on priors; but the former closes the model with a restriction on strategic beliefs, namely substitution assumption on the last layer strategies. That is, choosing σ_i^{ij} in definition 2.3 agent i should, in principle, respond to σ_j^{iji} which depends on additional belief, namely R_i^{jij}. In the absence of such belief, player i is free to choose any strategic belief. ME disciplines this arbitrariness by using a lower level belief. More particularly, it uses σ_j^i and this closes the system. The behavior of player i can be interpreted as if the higher orders beliefs R_i^{ji} and R_i^j were common knowledge. Thus, in the higher level player i believes that he and player j plays BNE with different priors. He then uses his belief R_i to best respond to what he believes player j does i.e. σ_j^i.

After this very brief introduction of ME, we are going to investigate it in a dynamic environment. The update process of different parametric beliefs is a nontrivial issue requiring a delicate analysis. Our analysis on this matter is a crude first step which does not offer a general method that can be used for any game where ME is applicable.

For the following sections, we shall consider a linear Cournot duopoly model in which there is an informational asymmetry about the cost function.[9] Suppose Ann and Bob are two duopolist who compete in quantities. We assume that they are myopic in the sense that they just consider the current period's profit. It is common knowledge that the inverse demand function has a linear form of $p = m - n(q_i + q_j)$ and the cost function also has a linear form of $C_i(q) = k_i q_i$. The constant unit cost k_i, however, is particular to each player. We

[9] In another very popular version of the Cournot duopoly model demand is unknown. Other than interpretation, our analysis remains intact in this version.

assume that: $(i)\, m > k_i \quad (ii)\, m + k_i - 2k_j > 0$ and $(iii)\, n > 0$.[10] For simplicity, suppose Bob's unit cost is commonly known to be $k_B = 1$. However, Ann's cost is only known to her. To complete the description of the game we need the beliefs of each player about k_A, and to keep analysis simple we assume an elementary belief structure for each player. To make the analysis as explicit as possible, we start our analysis with a situation where the value of k_A comes from a finite set and then we extend our analysis to an infinite set.

2.3 Cournot with Finite Attribute Set

In this section, we start with the simplest case where $k \in \{1, 2\}$. Bob's prior is that, with probability α, $k = 1$, and $k = 2$ with the complement probability $1-\alpha$. Also, he believes that Ann's belief about α is described by the probability density function, $\gamma(.)$. To complete the description we are going to assume Ann's belief system is represented by β i.e., Ann believes that Bob believes, with probability β, $k = 1$.

Bob's Mirage strategy, b, can be deduced from the solution of the following system[11]:

$$
\begin{aligned}
b &= \arg\max_q q\left(m - n\left(q + \alpha a_1^B + (1-\alpha)a_2^B\right) - 1\right), \\
a_1^B &= \arg\max_q q\left(m - n(q + b^{BA}) - 1\right), \\
a_2^B &= \arg\max_q q\left(m - n(q + b^{BA}) - 2\right), \\
b^{BA} &= \arg\max_q \int q\left(m - n\left(q + y a_1^B + (1-y)a_2^B\right) - 1\right)\gamma(y)\mathrm{d}y.
\end{aligned}
$$

These equations represent how Bob evaluates the maximization problems for himself and Ann. For instance, b^{BA}, is what Bob thinks that Ann thinks that

[10] The conditions (i) and (ii) ensures existence whilst the last condition ensures uniqueness.
[11] a_k denotes the production of Ann if her type is k.

Bob plays, and Bob evaluates this, as expected, by using his belief about Ann's belief about Bob's belief, namely $\gamma(.)$. Other equations can be interpreted similarly. Starting from the last equations we can solve this system in backward induction manner to obtain,

$$b^{BA} = \frac{m - \bar{\gamma}}{3n},$$
$$a_1^B = \frac{2m - 3 + \bar{\gamma}}{6n} \qquad a_2^B = \frac{2m - 6 + \bar{\gamma}}{6n},$$
$$b = \frac{4m - (3\alpha + \bar{\gamma})}{12n}$$

where $\bar{\gamma}$ is the mean of $\gamma(.)$. Similarly, the solution of the following system yields Ann's Mirage strategy[12]:

$$\begin{aligned}
a_k &= \arg\max_q q\left(m - n(q + b^A) - k\right), \\
b^A &= \arg\max_q q\left(m - n\left(q + \beta a_1^{AB} + (1-\beta)a_2^{AB}\right) - 1\right), \\
a_k^{AB} &= \arg\max_q q\left(m - n(q + b^A) - k\right).
\end{aligned}$$

Thus,

$$a_k = a_k^{AB} = \frac{2m - 3k + \beta}{6n}, \quad b^A = \frac{m - \beta}{3n}.$$

Suppose Ann's actual production a_k is different from Bob's (conditional) expectations about it, a_k^B. In this case, Bob tries to update the upper-level parametric belief $\gamma(.)$ which is a key component of his expectation about Ann's production. Without loss of generality suppose k's real value is 1. Then, Bob

[12]Note that one can easily observe the similarity of equations across players. The reason of this similarity is the rationality of players. Even if each player knows the maximization problem of the other player, the differences in beliefs causes different expectations for different players. In a sense, each player knows what is going to be maximized objectively, but their assessment about constraints may differ.

tries to reconcile Ann's action with his belief. Even if he does not know the components of a_1 or a_2, he can do a reverse engineering to update his belief. In particular, when Bob observes the actual action of Ann, a_1, he wants to check whether this action may belong to Ann for $k = 2$. This means that a_2^B should have been equal to a_1, but this means that $\bar{\gamma} \geq 3$ which is impossible. So, he concludes that the only way of observing this action is that $k = 1$ for Ann. Hence, even if Bob does not know what belief Ann holds to produce a_1, he knows no belief can justify certain actions for certain type(s). Therefore, Bob figures out the true type of his opponent. Since Ann can do the same analysis she is able to figure out that Bob would learn her true type. It is easy to verify the same conclusion will be reached when $k = 2$ too. Therefore, players learn the true types in one period just like they learn in BNE.[13]

We make the update mechanism a bit more transparent by looking at the same problem with an extended attribute set. Before making a big jump let us probe the same problem with an attribute set $\{k_1 = 1, k_2 = 3/2, k_3 = 2\}$. Let us represent the first order belief as $(\alpha_1, \alpha_2, \alpha_3)$ where α_i is the probability of having k_i and $\sum_i \alpha_i = 1$. Note that the belief we consider for the previous case can be obtained by setting $\alpha_1 = \alpha$, $\alpha_2 = 0$ and $\alpha_3 = 1 - \alpha$. Suppose the integrated out version[14] of third order belief is $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ with $\sum_i \bar{\gamma}_i = 1$. The Mirage strategy of Bob is obtained by solving the following system:

$$b^{BA} = \frac{m-2}{3n} + \frac{1}{3n} \sum_i \bar{\gamma}_i k_i \tag{2.1}$$

$$a_{k_i}^B = \frac{2m - 3k_i + 2}{6n} - \frac{1}{6n} \sum_i \bar{\gamma}_i k_i \tag{2.2}$$

$$b_B = \frac{m-2}{3n} + \frac{1}{4n} \sum_i \alpha_i k_i + \frac{1}{12n} \sum_i \bar{\gamma}_i k_i \tag{2.3}$$

For Ann, let us represent the first order belief $(\beta_1, \beta_2, \beta_3)$. Thus, the mirage

[13] See Appendix 2.A for details.
[14] See Appendix 2.B for details.

strategy of Ann is obtained by solving the following system:

$$b^A = \frac{m-2}{3n} + \frac{1}{3n}\sum_i \beta_i k_i$$
$$a_{k_i}^{AB} = a_{k_i} = \frac{2m - 3k_i + 2}{6n} - \frac{1}{6n}\sum_i \beta_i k_i$$

From this, we can infer boundaries for Bob's expectation of Ann's production, $a_{k_i}^B$. By using equation (2.2), Bob would reason in the following way[15]:

- If the true value of k_i is 1, then the first component of the equation (2.2) will be $\frac{2m-1}{6n}$. The second part which is a convex combination of different values of k_i achieves its minimum value when $\bar{\gamma}_i k_i = 1$ and maximum value when $\bar{\gamma}_i k_i = 2$. Therefore,

$$a_{k_1}^B \in [\frac{2m-3}{6n}, \frac{2m-2}{6n}]$$

- If the true value of k_i is $3/2$ then the first component of the equation (2.2) will be $\frac{2m-2.5}{6n}$. The second part which is a convex combination of different values of k_i achieves its minimum value when $\bar{\gamma}_i k_i = 1$ and maximum value when $\bar{\gamma}_i k_i = 2$. Therefore,

$$a_{k_2}^B \in [\frac{2m-4.5}{6n}, \frac{2m-3.5}{6n}]$$

- If the true value of k_i is 2 then the first component of the equation (2.2) will be $\frac{2m-4}{6n}$. The second part which is a convex combination of different values of k_i achieves its minimum value when $\bar{\gamma}_i k_i = 1$ and maximum value when $\bar{\gamma}_i k_i = 2$. Therefore,

$$a_{k_3}^B \in [\frac{2m-6}{6n}, \frac{2m-5}{6n}]$$

[15]This reasoning is same as reverse engineering mentioned before. The idea is to exploit rationality of the other agent in order to eliminate some of the cost parameters that are inconsistent with the realized outcome.

It is easy to verify by using the a_{k_i} equation that Ann produces in that region if the true value is a_{k_i}. Thus Bob is able to map each k_i with a certain subset of action space.[16] Since these sets are disjoint, the observed action itself is enough to learn the attribute of a player.

As might be realized, enlarging type space causes actions of different types to intertwine. To make this point more explicit we are going to take it one step further by increasing the type set once again. Suppose now the type set is given as $\{k_1 = 1, k_2 = 5/4, k_3 = 7/4, k_4 = 2\}$. As in the previous case, Bob can associate types with subsets of actions. Indeed, $a^B_{k_1} \in [(2m-3)/6n, (2m-2)/6n]$, $a^B_{k_2} \in [(2m-3.75)/6n, (2m-2.75)/6n]$,$a^B_{k_3} \in [(2m-5.25)/6n, (2m-4.25)/6n]$ and $a^B_{k_4} \in [(2m-6)/6n, (2m-5)/6n]$. Unlike previous cases, he is now unable to distinguish k_3 and k_4 or k_1 and k_2 by observing their actions because, in a certain subset of the attribute space, actions for different types overlap. This can be seen in Figure 2.1. For example, the bold area is common actions for k_3 and k_4 hence, when Bob observes an outcome from this area he cannot infer which value of k_i causes this action. Nevertheless, any action in that region indicates that Ann would not take such action had she been k_1 or k_2. In other words, Bob does not learn which type Ann is, but he does learn which type she is not. This implies the new attribute set is $\{k_3 = 7/4, k_4 = 2\}$.

After observing actions Bob puts zero weight on $k_1 = 1$ and $k_2 = 5/4$, he reasons in the way we described above and the expectations become $a^B_{k_3} \in [(2m-5.25)/6n, (2m-5)/6n]$ and $a^B_{k_4} \in [(2m-6)/6n, (2m-5.75)/6n]$. He proceeds to eliminate types with this updated beliefs and he will learn the true type of Ann as in the previous case. In the next section we extend this elimination logic to the case in which the attribute set is infinite.

[16]This is a correspondence from type space to action space.

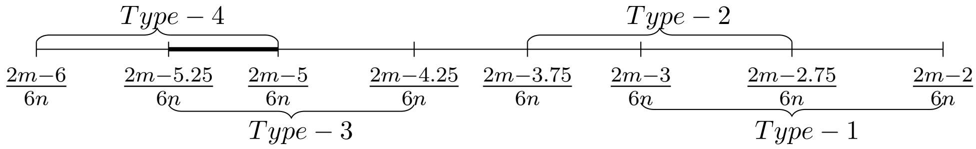


Figure 2.1: Possible actions for all types

2.4 Cournot with Continuum Attribute Set

In this section, we assume a continuum of types. Thus we keep the same set-up with $k \in [1, 2]$.[17] For Bob,

- his prior is given by the distribution function F_1. So, he believes that k is distributed according to the cumulative distribution function (CDF) F_1.
- his belief about Ann's belief about his belief about k is represented by CDF F_2. Note that F_2 is an integrated out version of some other object. So F_2 and r given in the remark above are of the same nature.

Since Ann knows Bob's attribute, we only describe her belief about Bob's belief about Ann's attribute which is given by G which is integrated out version of some other object.[18] In principal, $G \equiv F_2$ is possible, but obviously, the more interesting case is to allow them to be different.

Bob's Mirage strategy can be obtained by solving the following system:

$$\begin{aligned} b_B &= \arg\max_q q\Big(m - n\big(q + \mathbb{E}_1(a_k^B)\big) - 1\Big), \\ a_k^B &= \arg\max_q q(m - n(q + b_B^{BA}) - k), \\ b^{BA} &= \arg\max_q q\Big(m - n\big(q + \mathbb{E}_2(a_k^B)\big) - 1\Big). \end{aligned}$$

where $\mathbb{E}_1(.)$ and $\mathbb{E}_2(.)$ represents expected value operators in which expectations are taken by using $F_1(.)$ and $F_2(.)$, respectively. The solution of this sys-

[17]Note that we dropped subscript i which was used for indexing purposes before.
[18]See Remark 1 and Appendix 2.B.

tem yields:

$$b^{BA} = \frac{m-2}{3n} + \frac{1}{3n}\int_1^2 k\mathrm{d}F_2, \tag{2.4}$$

$$a_k^B = \frac{2m-3k+2}{6n} - \frac{1}{6n}\int_1^2 k\mathrm{d}F_2, \tag{2.5}$$

$$b_B = \frac{m-2}{3n} + \frac{1}{4n}\int_1^2 k\mathrm{d}F_1 + \frac{1}{12n}\int_1^2 k\mathrm{d}F_2. \tag{2.6}$$

Similarly for Ann, the Mirage strategy is obtained by solving the following system:

$$\begin{aligned} a_k &= \arg\max_q q\Big(m - n(q + b_B^A) - k\Big), \\ b^A &= \arg\max_q q\Big(m - n\big(q + \mathbb{E}(a_{k,A}^{AB})\big) - 1\Big), \\ a_k^{AB} &= \arg\max_q q(m - n(q + b_B^A) - k). \end{aligned}$$

The solutions of this system are given by:

$$a_k^{AB} = \frac{2m-3k+2}{6n} - \frac{1}{6n}\int_1^2 kdG, \tag{2.7}$$

$$b^A = \frac{m-2}{3n} + \frac{1}{3n}\int_1^2 kdG, \tag{2.8}$$

$$a_k = \frac{2m-3k+2}{6n} - \frac{1}{6n}\int_1^2 kdG. \tag{2.9}$$

When Bob's expectation about Ann's production , a_k^B, and Ann's real production, a_k, agree with each other there is no need to update the beliefs. The more interesting case is, obviously, when those two values differ. We are going to proceed period by period as in the discrete version.

Initially the players' actions are given by (2.6) and (2.9). From Bob's perspective Ann 's production a_k is a black box since he does not really know the components of a_k.[19] Therefore, he tries to reconcile what he believes and

[19]The asserted arguments hereafter are equally valid for Ann.

what he observes. For notational simplicity we are going to ignore k in a_k and represent action of Ann in period t as a^t. Possible values of k is in interval $I_0 = [l_0, h_0]$ where $l_0 = 1$ and $h_0 = 2$. Given (2.5) and k, Bob's expectation about Ann's action a^B should lie in $[\frac{2m-3k}{6n}, \frac{2m-3k+1}{6n}]$. Suppose the realized value for k is k_0, so

$$a^1 = \frac{2m - 3k_0 + 2}{6n} - \frac{1}{6n} \int_{I_0} k \mathrm{d}G.$$

Once Bob observes a^1, he eliminates types for which a^1 cannot be a best response. In other words, observing a^1 leads Bob to use a new type set which does not contradict a^1. Hence at the end of period 1, the type set is $I_1 = [l_1, h_1] \cap I_0$ where $l_1 = \dfrac{2m - 6na^1}{3}$ and $h_1 = \dfrac{2m + 1 - 6na^1}{3}$.

Note that we take intersection because it is clear that any type lower than 1 or higher than 2 is not possible. If we do not allow such intersection the learning we described here in detail does not work, because some values in the updated interval $[l_1, h_1]$ can justify the action of Ann, even though they are impossible to be materialized. This would essentially be the same as considering a different support than the actual one.[20]

In the second period, Ann's action is:

$$a^2 = \frac{2m - 3k_0 + 2}{6n} - \frac{1}{6n} \int_{I_1} k \mathrm{d}G',$$

where G' is the updated cumulative distribution function. Note that there is no restriction on how update should be for G to obtain G'.

After observing actions the update takes place and Bob's expectation, for a given k, lies in the interval $[(2m - 3k + 2 - h_1)/6n, (2m - 3k + 2 - l_1)/6n]$. Again, once Bob observes a^2 he associates a new type set $I_2 = [l_2, h_2] \cap I_0$ where $l_2 = (2m + 2 - 6na^2 - h_1)/3$ and $h_2 = (2m + 2 - 6na^2 - l_1)/3$, for the given observation.

[20] See the final section for more on this.

In the next period, the player will use this new set, thereby Ann's action is given by

$$a^3 = \frac{2m - 3k_0 + 2}{6n} - \frac{1}{6n}\int_{I_2} k\mathrm{d}G'',$$

where G'' is the updated cumulative distribution function and Bob's expectation, for a given k, lies in the interval $[(2m-3k+2-h_2)/6n, (2m-3k+2-l_2)/6n]$. Therefore, the new set players use is the interval $I_3 = [l_3, h_3] \cap I_0$ where $l_3 = (2m + 2 - 6na^3 - h_2)/3$ and $h_3 = (2m + 2 - 6na^3 - l_2)/3$.

In general the given pattern implies that for each t,

$$a^t = \frac{2m - 3k_0 + 2}{6n} - \frac{1}{6n}\int_{I_{t-1}} k\mathrm{d}G^{t-1} \quad \text{and} \quad I_t = [l_t, h_t] \cap I_0$$

where $l_t = \dfrac{2m + 2 - 6na^t - h_{t-1}}{3}$, $h_t = \dfrac{2m + 2 - 6na^t - l_{t-1}}{3}$ and G^{t-1} is the cumulative distribution function which is obtained from G by updating it $t-1$ times. This process describes how players choose actions and how update process takes place in period t.

We show that the process given above leads Bob to learn the true cost parameter of Ann. The following lemmas pave the way for the main result.

Lemma 2.1. *$I_t \neq \emptyset$ for all t. In particular, $k_0 \in I_t$ for all t.*

Proof. We prove this claim by induction. It is true by definition that $k_0 \in I_0$. Now, suppose $k_0 \in I_{t-1}$. Hence, it is enough to show that $k_0 \in [l_t, h_t]$. By definition of a^t,

$$a^t = \frac{2m - 3k_0 + 2}{6n} - \frac{1}{6n}\int_{I_{t-1}} k\mathrm{d}G^{t-1}.$$

Note that the value of the last integral, which is an expected value, is between $[l_{t-1}, h_{t-1}]$ since l_{t-1} and h_{t-1} are the minimum and maximum values for this expectation, respectively. By definition of l_t and h_t, we can write

$$\begin{aligned} l_t &= \frac{1}{3}\Big(2m + 2 - (2m - 3k_0 + 2 - \int_{I_{t-1}} k \, \mathrm{d}G^{t-1}) - h_{t-1}\Big) \\ &= k_0 + \frac{1}{3}\big(\int_{t-1} k \, \mathrm{d}G^{t-1} - h_{t-1}\big) \\ h_t &= \frac{1}{3}\Big(2m + 2 - (2m - 3k_0 + 2 - \int_{I_{t-1}} k \, \mathrm{d}G^{t-1}) - l_{t-1}\Big) \\ &= k_0 + \frac{1}{3}\big(\int_{I_{t-1}} k \, \mathrm{d}G^{t-1} - l_{t-1}\big) \end{aligned}$$

which implies $k_0 \in I_t$, as desired. □

The next lemma shows that the sequence $\{I_t\}$ of intervals are shrinking. Before the statement and the proof of this lemma, we provide the following definition of the rate of convergence.

Definition 2.4 (Q-Convergence). *Assume* $\lim_{t\to\infty} x_t = x^\star$. *Convergence is said to be with order* Q *if there exists a constant* $c > 0$ *such that*

$$\lim_{t\to\infty} \frac{\mid x_{t+1} - x^\star \mid}{\mid x_t - x^\star \mid^Q} = c.$$

The number Q *is called the order of convergence for the sequence* $\{x_t\}$ *and determines the rate of convergence as follows:*

1. *If* $Q = 1$ *and* $c = 1$ *then convergence is said to be sublinear.*
2. *If* $Q = 1$ *and* $0 < c < 1$ *then convergence is said to be linear.*
3. *If* $Q > 1$ *then convergence is said to be superlinear.*

Lemma 2.2. *The length of intervals* I_t *converges to zero as t increases. Moreover, the convergence is linear.*

Proof. Denote length of intervals with $diam$. Thus

$$diam\, I_t = h_t - l_t = \frac{h_{t-1} - l_{t-1}}{3} = \cdots = \frac{h_0 - l_0}{3^t}$$

Therefore $\lim_{t\to\infty} diam\, I_t = 0$. Also, by the previous definition, it is trivial to see that $Q = 1$ and $c = 1/3$. Hence this convergence is linear. □

Prima facie, this result might be considered as disappointing since linear convergence is relatively slow. However, it is important to note that this convergence speed is roughly the convergence speed of intervals, and that attribute sequence as members of these intervals may well converge faster than the intervals. The linear convergence result can be interpreted as the slowest speed of convergence for the attribute sequence. That is, this result can be considered as the lower bound for the convergence speed of the attribute sequence.

Finally, our main result is given by,

Proposition 2.1. *In the limit of the update mechanism described, the only remaining type set will be the true type* $\{k_0\}$.

Proof. It follows at once from Lemma 2.1 and Lemma 2.2 given above. □

This process might be considered as a statistical test for the true type. Any candidate type passing this test in every period can be evaluated as "possibly true" since it always goes along well with the observed actions. It is rather intuitive to think that this update process ends up with the true parameter because the true type is the only one that will always be considered as "possibly true".

2.5 Two-Sided Asymmetry

So far we have assumed one-sided informational asymmetry in which only one player has uncertainty about the cost parameter of the other party. In this section, we extend our analysis for two-sided asymmetry where each player is uncertain about the cost parameter of the other player.

Reconsider the simplest case with two-sided uncertainty where each player knows his/her cost parameter but does not know about the cost parameter of the opponent. Suppose each player's cost parameter is $k_i \in \{1,2\}$. Note that we consider a symmetric support for the cost parameters because (i) intuitively, this case is more interesting than a case where supports are asymmetric [21] and, (ii) it allows us to avoid introducing additional notations.

Bob's first order belief is still about the cost parameter of Ann and we use α to represent it as before. Bob's second order belief is a distribution $\gamma(.)$ over the first order belief of Ann. Note that we still keep independence assumption. In a general situation, this belief should be a distribution over the Cartesian product of the set of cost parameter of Bob and the set of first order belief of Ann. Lastly, Bob's third order belief is a distribution $\zeta(.)$ over the second order belief of Ann.

Similarly, Ann's first order belief is about the cost parameter of Bob, and we use β to represent it as before. Ann's second order belief is a distribution $\delta(.)$ over the first order belief of Bob. Finally, Ann's third order belief is a distribution $\eta(.)$ over the second order belief of Ann.

As before, Bob's Mirage strategy can be obtained by solving the following system:

$$\begin{aligned}
b_{k_B} &= \arg\max_q q\left(m - n\left(q + \alpha a_1^B + (1-\alpha)a_2^B\right) - k_B\right), \\
a_{k_A}^B &= \arg\max_q q\left(m - n(q + \bar{\gamma} b_1^{BA} + (1-\bar{\gamma})b_2^{BA}) - k_A\right), \\
b_{k_B}^{BA} &= \arg\max_q q\left(m - n\left(q + \bar{\zeta} a_1^B + (1-\bar{\zeta})a_2^B\right) - k_B\right),
\end{aligned}$$

where $\bar{\gamma}$ and $\bar{\zeta}$ are obtained from γ and ζ by integrating them out, respectively.

[21] In asymmetric case, the player who has higher cost may choose not to produce if the difference between the cost parameters is sufficiently high. So to have a positive production for each player we need to have sufficiently close cost parameters.

This gives us:

$$
\begin{aligned}
b_1^{BA} &= \frac{2m-1}{6n} + \frac{1}{6n}\bar{\gamma} - \frac{1}{3n}\bar{\zeta}, & b_2^{BA} &= \frac{m-2}{3n} + \frac{1}{6n}\bar{\gamma} - \frac{1}{3n}\bar{\zeta} \\
a_1^{B} &= \frac{2m-1}{6n} - \frac{1}{3n}\bar{\gamma} + \frac{1}{6n}\bar{\zeta}, & a_2^{B} &= \frac{m-2}{3n} - \frac{1}{3n}\bar{\gamma} + \frac{1}{6n}\bar{\zeta} \\
b_1 &= \frac{2m-1}{6n} + \frac{1}{6n}\bar{\gamma} - \frac{1}{12n}\bar{\zeta} - \frac{1}{4n}\alpha, & b_2 &= \frac{m-2}{3n} + \frac{1}{6n}\bar{\gamma} - \frac{1}{12n}\bar{\zeta} - \frac{1}{4n}\alpha.
\end{aligned}
$$

Similarly, Ann's Mirage strategy can be obtained by solving the following system:

$$
\begin{aligned}
a_k &= \arg\max_q q\left(m - n\left(q + \beta b_1^A + (1-\beta)b_2^A\right) - k_A\right), \\
b_{k_B}^A &= \arg\max_q q\left(m - n\left(q + \bar{\delta} a_1^{AB} + (1-\bar{\delta})a_2^{AB}\right) - k_B\right), \\
a_{k_A}^{AB} &= \arg\max_q q\left(m - n\left(q + \bar{\eta} b_1^A + (1-\bar{\eta})b_2^A\right) - k_A\right),
\end{aligned}
$$

where $\bar{\delta}$ and $\bar{\eta}$ are obtained from δ and η by integrating them out, respectively.

The solutions of this system, then, yields

$$
\begin{aligned}
a_1^{AB} &= \frac{2m-1}{6n} + \frac{1}{6n}\bar{\delta} - \frac{1}{3n}\bar{\eta}, & a_2^{AB} &= \frac{m-2}{3n} + \frac{1}{6n}\bar{\delta} - \frac{1}{3n}\bar{\eta} \\
b_1^{A} &= \frac{2m-1}{6n} - \frac{1}{3n}\bar{\delta} + \frac{1}{6n}\bar{\eta}, & b_2^{A} &= \frac{m-2}{3n} - \frac{1}{3n}\bar{\delta} + \frac{1}{6n}\bar{\eta} \\
a_1 &= \frac{2m-1}{6n} + \frac{1}{6n}\bar{\delta} - \frac{1}{12n}\bar{\eta} - \frac{1}{4n}\beta, & a_2 &= \frac{m-2}{3n} + \frac{1}{6n}\bar{\delta} - \frac{1}{12n}\bar{\eta} - \frac{1}{4n}\beta.
\end{aligned}
$$

As before, we can infer boundaries for each player. For example, Bob's beliefs about Ann's production for each type are,

$$
a_1^B \in [\frac{m-1.5}{3n}, \frac{m}{3n}] \quad \text{and} \quad a_2^B \in [\frac{m-3}{3n}, \frac{m-1.5}{3n}].
$$

Indeed, these intervals are valid for Ann's production, hence, we can say that Bob can distinguish Ann's type by observing her action. Similar consideration confirms that Ann can also distinguish Bob's type by observing her action.

If we enlarge the set of cost parameters as in the one-sided asymmetric info case, we obtain:

Proposition 2.2. *In the limit of the update mechanism, each player learns the true cost parameter of the other side.*

Note that the proof follows the same steps as in the proof of Proposition 2.1. Rather than repeating the proof, we give the logic of the result in a geometric way. Figure 2.1 shows the implication of common knowledge of rationality. A axis represents the production level for Ann (or the action of Ann) and B axis represents the production level for Bob. For player i, the best response curve is represented by BR_i. To distinguish different cost parameters for Ann, we denote her best response curve BR_A^h when her cost parameter is low so that she can be considered as a "high" or an "efficient" type. BR_A^l should be understood in a similar way.

It is a well-known fact[22] that common knowledge of rationality implies that the players will choose only strategies that survive the iterated elimination of strictly dominated strategies. Therefore, Bob and Ann eliminate any action in the region I because they are strictly dominated by the monopoly action and both players know this. Since Bob will not use any action in that region, each player eliminates actions in the region I' since any action in that region would mean that Ann would think that Bob would pick an action from region I which contradicts with the common knowledge of rationality. Similarly, each player eliminates every action of Ann in the region II since every action in that region is strictly dominated by monopoly outputs for each type. This implies the elimination of strategies from the region II'. Note that since strategy spaces are compact and payoff functions are continuous, then the order of deletion does not matter (See, Dufwenberg and Stegeman (2002).). As a result, the limit of iterated elimination of strictly dominated strategies leads us to the rectangle

[22] See, Battigalli and Bonanno (1999) for a thorough discussion.

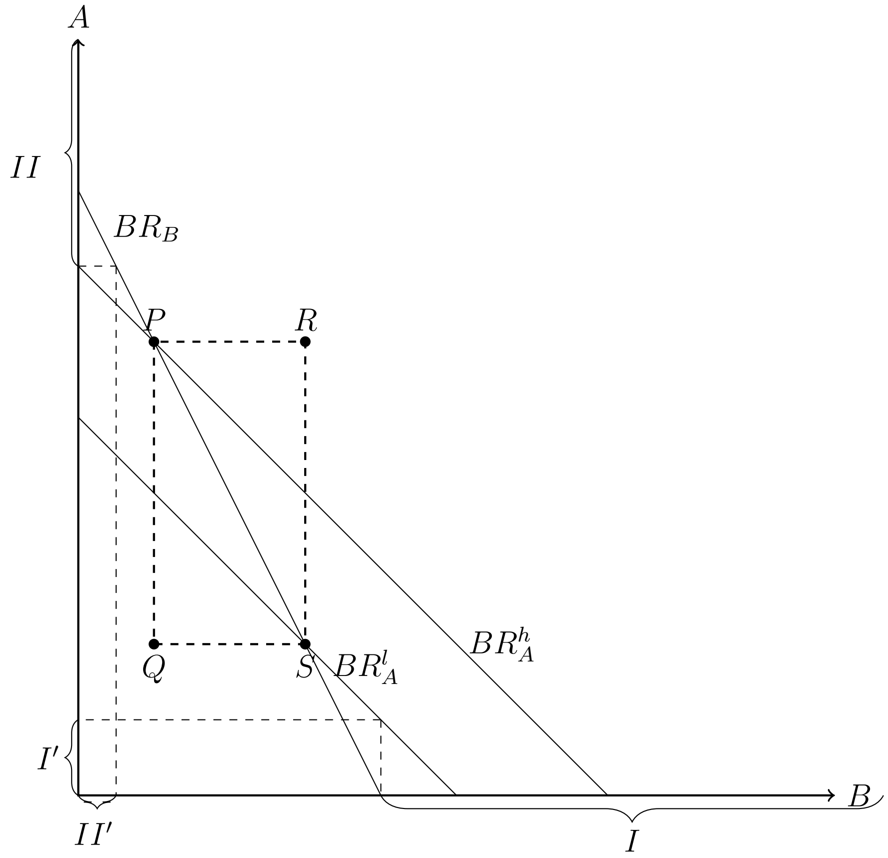


Figure 2.1: Implication of rationality

$PQRS$. Therefore, Ann and Bob in Mirage equilibrium choose their actions in this region since rationality is common knowledge by definition of Mirage equilibrium.[23]

The update mechanism primarily relies on this information. Consider action a for Ann shown in figure 2.2. This action could not be a best response if Ann had a low cost parameter because this would imply that Ann would consider b as the action of Bob which contradicts the (common knowledge) rationality of Bob. Observe that this does not reveal any information about the true cost parameter or the type of Ann, however, it allows Bob to infer that she cannot have a low cost parameter or more precisely this action cannot be justi-

[23]Note that in Mirage equilibrium we take the common knowledge of rationality as granted, however, many epistemic models use much weaker assumptions than this and investigate the equilibrium implications of these weaker assumptions. Therefore, constructions of belief hierarchies in those models contain parametric beliefs and beliefs about rationality. See Dekel and Siniscalchi (2015) for a recent review of such epistemic models.

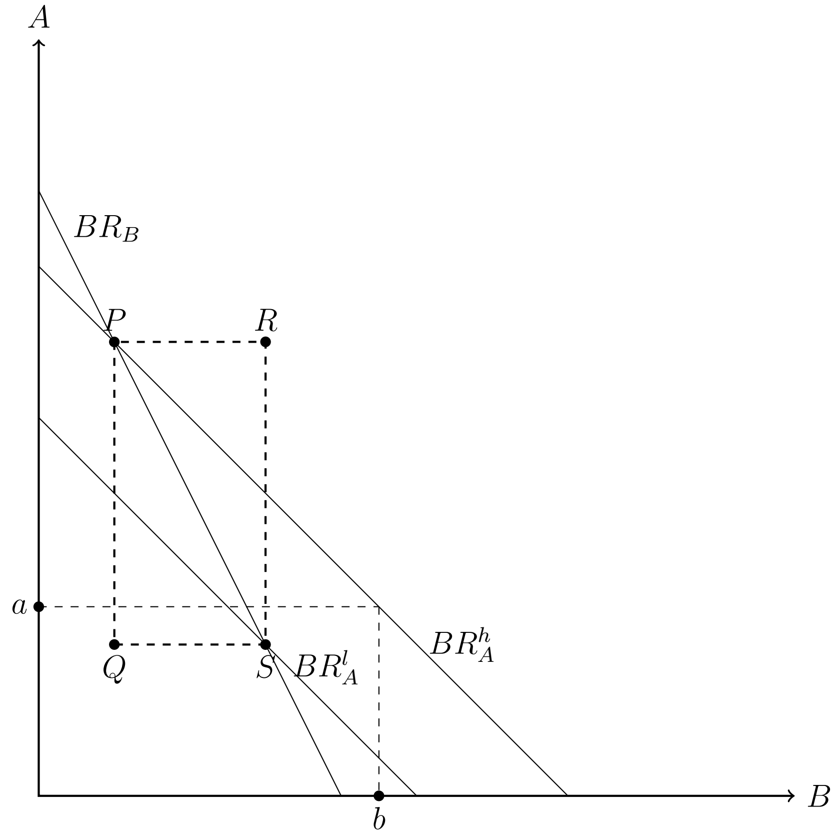


Figure 2.2: Update after observation

fied for best response curve BR_A^h. This implies the elimination of that type or more precisely, elimination of low cost parameter from the set of cost parameters. In general, any best response curve which does not justify the observed action will be eliminated, as well as the cost parameter. By continuing this elimination process players learn the true cost parameter of their opponent.

Figure 2.3 summarizes the same logic for the two-sided asymmetric case. $PQRS$ can be obtained by eliminating strictly dominated strategies as a result of the common knowledge of rationality. Once again, the region we obtain is order independent. Now suppose Ann and Bob pick actions a and b, respectively. If Ann were using best response curve BR_A^h, she would think Bob would pick $\hat{b}$. In other words, "high type" Ann would take action a by conjecturing $\hat{b}$ as Bob's action. However, this conjecture contradicts with common knowledge of rationality, hence, Ann's action a implies Ann cannot be a high type.

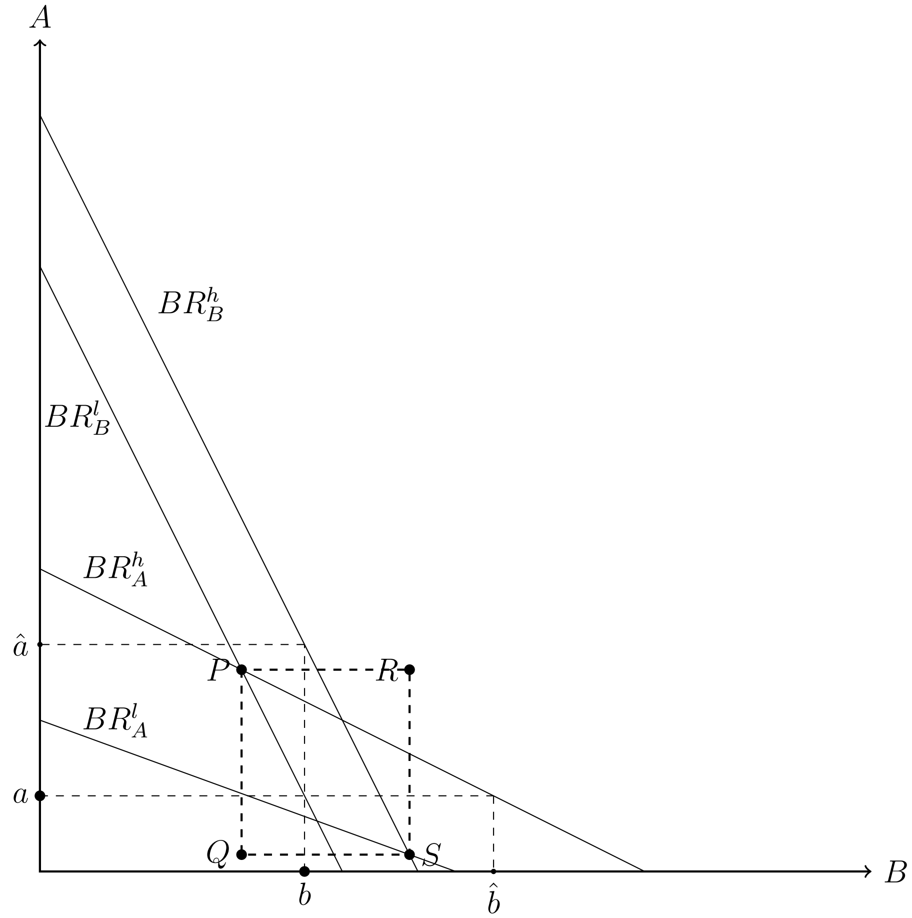


Figure 2.3: Update in two-sided asymmetric case

Similarly, Bob's action b cannot be justified with BR_B^h which requires that Bob's conjecture about Ann's action is $\hat{a}$. The update mechanism is again based on eliminating unjustifiable types for an observed action. We address some important aspects of this learning and put forward some issues regarding implicit assumptions for consideration in the next section.

2.6 Discussion

In general, this result tells us that a player can learn eventually the true type(s) under ME strategy. Moreover, unlike BNE where learning occurs almost immediately, it takes time to learn in ME. Note that this process can always be used as a rule of thumb even for finitely repeated games. For instance, suppose that the players played this game for 10 periods, then at the end of the last period, they could obtain a smaller interval for possible types by using the elimination described before. So, this process can be used as a heuristic in different environments. The other point worth emphasizing is that the update mechanism described here is an implication of rationality. Players update by eliminating types that do not conform to the actions observed. In a sense, the primary concern of agents is the support, not the distribution itself. This means the following: Suppose a player starts with a uniform distribution in the support $[1, 2]$ then after an update, he may obtain a new support while he may still believe the distribution is a uniform one. Assume, however, the true distribution of types is a (truncated) normal distribution for the given support. That is, when an agent is being asked about what is the probability of having any type k, his reply will be calculated upon uniform distribution, whereas the true probability of this event should be calculated by using normal distribution.[24] In this update mechanism, we cannot talk about learning the true distribution - in this case, normal distribution - of types.

Our insight into the learning process is based on eliminating "types" that are not justifying the play of the opponent. Although it is a tedious and non-innovative method, it has one important advantage: It avoids the unpleasant process of defining an update mechanism. That is, if we had to construct the update mechanism with purely probabilistic tools, we would need to be more explicit about the update mechanism because of the zero probability events. In the Cournot example discussed above, players' expectation about the action of the other player is not realized. So the realized outcome is an unexpected event i.e., zero probability event. Therefore, a more sophisticated approach would define the conditional probabilities on measure zero events. Our approach, even if being simple, allows us to avoid this issue so that without putting any

[24]Note that the resulting probability may coincide even though distributions are different

restriction on how this update takes place we can continue our analysis.

The most important aspect of this learning process is that the cost parameter is fixed throughout the game. This allows players to safely eliminate cost parameters that are not justified by observed behavior. If the cost parameter had a "dynamic nature" i.e., changing every period, then our learning process would not work at all simply because although some cost parameters would not justify the past action they may still be realized in future periods. Note that we do not claim no learning occurs. Perhaps, players try to learn the distribution of cost parameters, but the learning we described would fail in this environment.

Another important ingredient of our learning process, although it is not immediate, is the myopic agents. Thus each player cares about the instantaneous payoff without contemplating the implications of their actions or their opponent's actions. For instance, in the one-sided asymmetric information case given above, Ann never considered behaving differently than her true type. In a proper repeated game environment in which players discount future payoffs, Ann might get a higher payoff to behave differently than her type dictates. In this case, players could not eliminate types that would not justify actions. Therefore, it is important to have myopic agents in this result.

Another ingredient of this learning standing in the background, is that the players know the support of the cost parameter of their opponents. To understand the importance of this assumption, consider a situation where Bob contemplates different support from the actual support of the cost parameter of Ann. When he observes the action there are two possibilities: (i) he cannot justify this action for any cost parameter with any belief so he has to consider another support, or (ii) he can justify this action for some cost parameter and for some beliefs. In case (i) Bob has not learned anything and in case (ii) he has learned the wrong thing. Therefore, it is important for Bob to know the true support. One of the weakest forms of this assumption can be absolute continuity (See Nyarko (1998)) which in our context implies that Bob assigns a positive probability to the true cost parameter, not the whole support. We believe anything weaker than this assumption may not lead to the learning described here.

One question that might be relevant in this context is the rate of convergence of this learning process for which we have provided an answer. A more sophisticated approach answering this question would be to invoke a martingale convergence theorem[25] and to use a metric such as Boylan (1971). The problem with this approach is that it requires a probability space which means an "objective" underlying probability measure. Mirage equilibrium, by definition, excludes this. There might be another way to resolve this issue and answer this question in a sensible way, but we were not able to find it.

[25]See Nyarko (1997) for an application of such theorems.

Bibliography

Ahn, D. S. (2007). Hierarchies of ambiguous beliefs. *Journal of Economic Theory*, 136(1):286 – 301.

Battigalli, P. (2003). Rationalizability in infinite, dynamic games with incomplete information. *Research in Economics*, 57(1):1 – 38.

Battigalli, P. and Bonanno, G. (1999). Recent results on belief, knowledge and the epistemic foundations of game theory. *Research in Economics*, 53(2):149–225.

Battigalli, P. and Siniscalchi, M. (1999). Hierarchies of conditional beliefs and interactive epistemology in dynamic games. *Journal of Economic Theory*, 88(1):188 – 230.

Battigalli, P. and Siniscalchi, M. (2003). Rationalization and incomplete information. *Advances in Theoretical Economics*, 3(1).

Bernheim, B. D. (1984). Rationalizable strategic behavior. *Econometrica*, 52(4):1007–1028.

Binmore, K. (1987). Modeling rational players: Part i. *Economics and Philosophy*, 3:179–214.

Binmore, K. (1988). Modeling rational players: Part ii. *Economics and Philosophy*, 4:9–55.

Boylan, E. S. (1971). Equiconvergence of martingales. *The Annals of Mathematical Statistics*, 42(2):552–559.

Crawford, V. P., Costa-Gomes, M. A., and Iriberri, N. (2013). Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and Applications. *Journal of Economic Literature*, 51(1):5–62.

Dekel, E., Fudenberg, D., and Morris, S. (2007). Interim correlated rationalizability. *Theoretical Economics*, 2(1):15–40.

Dekel, E. and Siniscalchi, M. (2015). Epistemic game theory. In Young, H. and Zamir, S., editors, *Handbook of Game Theory, Volume 4*. North Holland, Elsevier, Amsterdam.

Dufwenberg, M. and Stegeman, M. (2002). Existence and uniqueness of maximal reductions under iterated strict dominance. *Econometrica*, pages 2007–2023.

Foster, D. P. and Young, H. P. (2001). On the impossibility of predicting the behavior of rational agents. *Proceedings of the National Academy of Sciences*, 98(22):12848–12853.

Harsanyi, J. C. (1967). Games with incomplete information played by "Bayesian" players, i-iii. part i. the basic model. *Management Science*, 14(3):159–182.

Jordan, J. (1995). Bayesian learning in repeated games. *Games and Economic Behavior*, 9(1):8 – 20.

Kalai, E. and Lehrer, E. (1993). Rational learning leads to Nash equilibrium. *Econometrica*, 61(5):1019–1045.

Mertens, J.-F. and Zamir, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory*, 14(1):1–29.

Nachbar, J. H. (1997). Prediction, Optimization, and Learning in Repeated Games. *Econometrica*, 65(2):275–310.

Nyarko, Y. (1997). Convergence in economic models with Bayesian hierarchies of beliefs. *Journal of Economic Theory*, 74(2):266 – 296.

Nyarko, Y. (1998). Bayesian learning and convergence to Nash equilibria without common priors. *Economic Theory*, 11(3):643–655.

Nyarko, Y. (2010). Most games violate the common priors doctrine. *International Journal of Economic Theory*, 6(1):189–194.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52(4):1029–1050.

Sákovics, J. (2001). Games of incomplete information without common knowledge priors. *Theory and Decision*, 50(4):347–366.

Shoham, Y. and Leyton-Brown, K. (2008). *Multiagent systems: Algorithmic, game-theoretic, and logical foundations*. Cambridge University Press.

Appendix

2.A Bayes Nash Equilibrium (BNE)

In this section, we are going to derive Bayes Nash Equilibrium of the Cournot game described above. Our discussion will be rather loose and we are going to take Bayes Nash Equilibrium as granted without redefining it.

To use BNE we are going to postulate there is a common prior from which every player's beliefs are derived. Let us take Bob's prior α as the common prior and assume that it is common knowledge, so that every higher order belief is putting probability one on α. The following equations define BNE:

$$
\begin{aligned}
b &= \arg\max_q q\left(m - n\left(q + \alpha a_1 + (1-\alpha)a_2\right) - 1\right), \\
a_1 &= \arg\max_q q\left(m - n(q+b) - 1\right), \\
a_2 &= \arg\max_q q\left(m - n(q+b) - 2\right),
\end{aligned}
$$

Note that in BNE, players expectations about the other players' strategies are true in the equilibrium. This is why we do not need any superscript on strategies b, a_1 and a_2. The solution of this system yields,

$$b = \frac{m - \alpha}{3n},$$
$$a_1 = \frac{2m - 3 + \alpha}{6n}, \quad a_2 = \frac{2m - 6 + \alpha}{6n}.$$

So this tells us that Ann chooses a_1 if the value of $k = 1$. Moreover, Bob is expecting this outcome when $k = 1$. The same is true for $k = 2$. Therefore, Bob will learn the true type of Ann at the end of the first period. This conclusion is true even if the set of possible k is uncountable.

2.B Integrating out

In this section, we are going to give details of integrating out expected values when the attribute set is $\{1, 3/2, 2\}$. Since the same operation is trivial for an attribute set with two elements, our choice of the attribute set with three elements makes it easier to understand this operation with larger sets. For notational simplicity, we are going to leave $\arg\max$ operator out of equations, but it is important to keep in mind that final outcome is going to be maximized by choosing variable q.

$$
\begin{aligned}
b^{BA} &= \int_{y_2}\int_{y_1} q\left(m - n(q + y_1 a_1^B + y_2 a_2^B + (1 - y_1 - y_2)a_3^B) - 1\right)\gamma(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2. \\
&= q\int_{y_2}\Big((m-1)\hat{\gamma}(y_2) - n\big(q\hat{\gamma}(y_2) + a_1^B\int_{y_1} y_1\gamma(y_1, y_2)\mathrm{d}y_1 + a_2^B y_2\hat{\gamma}(y_2) \\
&\quad + a_3^B(\hat{\gamma}(y_2) - \int_{y_1} y_1\gamma(y_1, y_2)\mathrm{d}y_1 - y_2\hat{\gamma}(y_2))\big)\Big)\mathrm{d}y_2. \\
&= q\Big((m-1)\int_{y_2}\hat{\gamma}(y_2)\mathrm{d}y_2 - n\big(q\int_{y_2}\hat{\gamma}(y_2)\mathrm{d}y_2 + a_1^B\int_{y_2}\mathbb{E}[Y_1 \mid Y_2 = y_2]\mathrm{d}y_2 \\
&\quad + a_2^B\int_{y_2} y_2\hat{\gamma}(y_2)\mathrm{d}y_2 + a_3^B(\int_{y_2}\hat{\gamma}(y_2)\mathrm{d}y_2 - \int_{y_2}\mathbb{E}[Y_1 \mid Y_2 = y_2]\mathrm{d}y_2 \\
&\quad - \int_{y_2} y_2\hat{\gamma}(y_2)\mathrm{d}y_2))\Big). \\
&= q\Big((m-1) - n\big(q + a_{1,A}^B\mathbb{E}[\mathbb{E}[Y_1 \mid Y_2]] + a_2^B\mathbb{E}[Y_2] + a_3^B(1 - \mathbb{E}[\mathbb{E}[Y_1 \mid Y_2]] - \mathbb{E}[Y_2])\big)\Big). \\
&= q\Big((m-1) - n\big(q + a_1^B\mathbb{E}[Y_1] + a_2^B\mathbb{E}[Y_2] + a_3^B(1 - \mathbb{E}[Y_1] - \mathbb{E}[Y_2])\big)\Big). \\
&= q\Big((m-1) - n\big(q + a_{1,A}^B\bar{\gamma}_1 + a_2^B\bar{\gamma}_2 + a_3^B(1 - \bar{\gamma}_1 - \bar{\gamma}_2)\big)\Big). \\
&= q\Big((m-1) - n\big(q + a_1^B\bar{\gamma}_1 + a_2^B\bar{\gamma}_2 + a_3^B\bar{\gamma}_3\big)\Big)
\end{aligned}
$$

where $\hat{\gamma}(.)$ is the marginal distribution of y_2, $\mathbb{E}[.]$ is the expectation operator, and $\bar{\gamma}_i$ is the expected value of y_i with pdf $\gamma(.,.)$

2.C FOCs

The following is the first order conditions of the mirage system for Bob:

$$
\begin{aligned}
b_B &= \frac{1}{2n}(m - 1 - \mathbb{E}_1(a_k^B)), \\
a_k^B &= \frac{m - k - nb^{BA}}{2n}, \\
b^{BA} &= \frac{1}{2n}(m - 1 - \mathbb{E}_2(a_k^B)).
\end{aligned}
$$

Similarly, we have the following first order conditions for Ann:

$$
\begin{aligned}
a_k &= \frac{m - nb^A - k}{2n}, \\
b^A &= \frac{(m - E(a_k^{AB}) - 1)}{2n}, \\
a_k^{AB} &= \frac{m - b^A}{2n}.
\end{aligned}
$$

Chapter 3

A Metric for Partitions

3.1 Introduction

Modelling knowledge with partitions is very common and almost unique in game theory and information economics. In this construction, each individual is endowed with a partition of the set of possible states which can be interpreted as the knowledge of the players. One interpretation of such models is that a partition, or in particular the cells of a partition, represents the state of the mind (see, for example, Zamir (2008)) for the individual. The aim of this chapter is to offer a metric that measures partitions in the light of this interpretation. The need for this metric stems from the need to measure distances when there is no well-defined probability space for the underlying type space such as Mirage equilibrium. As discussed in earlier chapters, players in Mirage equilibrium uses subjective belief hierarchies and even if there is a true probability space in which these hierarchies can be embedded, players may not be aware of it due to the lack of common knowledge. All the metrics used in the literature require a true probability space since they rely on a unique probability measure to define a distance function. Our construction does not rely on probabilistic tools, but counting or combinatorial tools. Therefore it does not require a true probability measure, but as any combinatorial method, it induces a probability over the set of possible states. According to the proposed metric the distance between two partitions is the weighted average of the non-empty symmetric differences of each cell that contain each element of the set by excluding double counting.

Although it is possible to use this metric as an index for other purposes such as cluster analysis, categorization theory or even data mining given the common usage of partitions in these areas (see Wagner and Wagner (2007) for a thorough coverage), our primary motivation is to measure partitions for game theory or decision theory.

The rest of the paper is organized in the following way: the next section gives the relevant notations and definitions and introduces the proposed metric. After that, we compare some important distance measures defined in the literature with the proposed one through examples.

3.2 Notations and the metric

Let Ω be a finite set with n members. We say that a collection $\mathcal{P} = \{P(\omega)\}_{\omega\in\Omega}$ of sets is a partition of Ω, and call the $P(\omega)$ the atoms of the partition, if

$$P(\omega) \cap P(\omega') = \emptyset \quad \text{for} \quad \omega \neq \omega', \quad \text{and} \quad \bigcup_{\omega} P(\omega) = \Omega.$$

In the interpretation, Ω is the set of all possible states ω that is relevant to the situation at hand. When some state ω_0 is realized a player's knowledge will be represented by some atom that contains ω_0. That is, the player knows that one element of $P(w_0)$ is the true state but he does not know which one. Any member ω in $P(\omega_0)$ is indistinguishable from each other from the viewpoint of the player.

The idea of metric we propose is to give a weight to how far each element is placed in different partitions - we exclude the double counting - and to sum these weights as the distance between partitions. Let $\mathscr{P}$ be the set of all partitions of Ω. Consider two arbitrary members $\mathcal{P}$ and $\mathcal{Q}$ of this set. We can interpret each partition as the prior information of each player. Define the symmetric difference of two arbitrary sets A and B, denoted by $A\Delta B$, as

$$A\Delta B := (A \setminus B) \cup (B \setminus A).$$

Now consider the collection $\mathcal{S}$ of symmetric differences given by,

$$\mathcal{S} := \{P(\omega_i)\Delta Q(\omega_i) \neq \emptyset : P(\omega_i) \in \mathcal{P}, Q(\omega_i) \in \mathcal{Q}, i = 1, \ldots, n\} = \{S(\omega_j)\}_j$$

The collection $\mathcal{S}$ contains symmetric differences of atoms in each partition that contain ω. Note that we may have

$$P(\omega_i)\Delta Q(\omega_i) = P(\omega_j)\Delta Q(\omega_j) \quad \text{for some} \quad i \neq j.$$

To make this point explicit consider the partitions given in Example 3.1 below. It is not difficult to see that

$$P(\omega_2)\Delta Q(\omega_2) = P(\omega_3)\Delta Q(\omega_3) = \{\omega_1\}.$$

Note that $\mathcal{S}$ contains only one of those such repetitive sets and this precludes double counting.

Definition 3.1. *Define* $\rho : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$

$$\rho(\mathcal{P}, \mathcal{Q}) := \sum_{S(\omega)\in\mathcal{S}} \frac{r}{\binom{n}{s}}$$

where $\mid S(\omega) \mid = s$, r *is the total number of sets in* $\mathcal{S}$ *with cardinality* s, *and* r_i *is the cardinality of sets in* $\mathcal{S}$ *with* i *elements.*

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$, this definition takes the collection $\mathcal{S}$ of nonempty symmetric differences that contain each member ω of Ω (note that each member $S(\omega)$ of the collection $\mathcal{S}$ is a set). In this collection $\mathcal{S}$, we count the number of sets with the same cardinality s, given by r. Then we weight this number r with $\binom{n}{s}$ i.e., number of s-subsets. In a sense, we sum the probability of getting an s-subset in the collection $\mathcal{S}$ for different values of s that symmetric differences produce.

To make sense out of the definition consider the following example.

Example 3.1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with two partitions $\mathcal{P} = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$ and $\mathcal{Q} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. Then $\mathcal{S} = \{S(\omega_1), S(\omega_2), S(\omega_3)\} = \{\{\omega_2, \omega_3\}, \{\omega_1\}\}$. To find the distance consider first the set $\{\omega_2, \omega_3\}$ in $\mathcal{S}$ which has the cardinality 2 and it is the only such set. Thus $r = 1$, $s = 2$ and $\binom{3}{2} = 3$. Similarly, for the set $\{\omega_1\}$ we have $r = 1$, $s = 1$ and $\binom{3}{1} = 3$. The distance $\rho(\mathcal{P}, \mathcal{Q})$ is then $2/3$.

Proposition 3.1. *The function ρ is a metric on the set $\mathscr{P}$ of all partitions for a given Ω.*

Proof. (1) $\rho(\mathcal{P}, \mathcal{Q}) = 0$ implies $r = 0$ for each set $S(\omega)$ in $\mathcal{S}$. In other words collection of symmetric differences contain empty sets. That is $P(\omega)\Delta Q(\omega) = \emptyset$ which implies $P(\omega) = Q(\omega)$. Hence every ω is assigned in the same atoms in both $\mathcal{P}$ and $\mathcal{Q}$. Thus, $\mathcal{P} = \mathcal{Q}$.

Conversely, it is immediate to conclude that $\mathcal{P} = \mathcal{Q}$ implies $\rho(\mathcal{P}, \mathcal{Q}) = 0$.

(2) Since symmetric differences in sets by definition satisfy symmetry, we have $\rho(\mathcal{P}, \mathcal{Q}) = \rho(\mathcal{Q}, \mathcal{P})$.

(3) To finish the proof we need to show for arbitrary $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$

$$\rho(\mathcal{P}, \mathcal{Q}) \leq \rho(\mathcal{P}, \mathcal{R}) + \rho(\mathcal{R}, \mathcal{Q})$$

We prove the claim by induction on $| \Omega |= n$. The claim is trivial when $n = 1$.

Suppose now the claim is true for some n i.e., $| \Omega |= n$. Consider now the set $\Omega \cup \{\omega_0\}$ so that the cardinality is $n+1$. Consider arbitrary partitions $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ of $\Omega \cup \{\omega_0\}$. Note that by excluding ω_0 from the atom $P(\omega_0)$ of $\mathcal{P}$ that contains it, we obtain a partition $\mathcal{P}'$ of Ω. Similarly we obtain partitions $\mathcal{Q}'$, $\mathcal{R}'$ from $\mathcal{Q}$ and $\mathcal{R}$ by excluding ω_0, respectively. In other words, partitions $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ can be obtained, by including ω_0 into the appropriate atom, from the partitions $\mathcal{P}'$,$\mathcal{Q}'$ and $\mathcal{R}'$ of Ω, respectively.

There are two possibilities for adding a new element to a given partition. It can either be added as a singleton - the atom contains only the new element - or it can be included into one of the existing atoms. With the abuse of notation, let us denote the former case $\mathcal{X} \oplus x$ and the latter case $\mathcal{X} \uplus x$ for an arbitrary partition $\mathcal{X}$ and an arbitrary new element x.

(3.1) Suppose we obtain each partition by adding ω_0 separately. That is

$$\mathcal{P} := \mathcal{P}' \oplus \omega_0, \quad \mathcal{Q} := \mathcal{Q}' \oplus \omega_0, \quad \mathcal{R} := \mathcal{R}' \oplus \omega_0$$

So by induction hypothesis we have

$$\rho(\mathcal{P}', \mathcal{Q}') \leq \rho(\mathcal{P}', \mathcal{R}') + \rho(\mathcal{R}', \mathcal{Q}').$$

Then since $\{S(\omega_0)\} = \emptyset$ in every collection of symmetric differences, and the other $\{S(\omega)\}$ would not change, the above equation implies that

$$\rho(\mathcal{P}, \mathcal{Q}) \leq \rho(\mathcal{P}, \mathcal{R}) + \rho(\mathcal{R}, \mathcal{Q}).$$

(3.2) Consider now ω_0 is included one of the existing atoms of the given partitions, say $\mathcal{P}'$. That is

$$\mathcal{P} := \mathcal{P}' \uplus \omega_0, \quad \mathcal{Q} := \mathcal{Q}' \oplus \omega_0, \quad \mathcal{R} := \mathcal{R}' \oplus \omega_0$$

Now suppose $P'(\omega_i)$ is the atom that contains ω_0 in $\mathcal{P}$. So the only difference between $\mathcal{P}'$ and $\mathcal{P}$ is that the atom $P'(\omega_i)$ includes ω_0 in addition to other elements. Then symmetric difference $P'(\omega_i)\Delta Q'(\omega_i) = S(\omega_i)$ will include ω_0 which causes change in the left-hand side of the triangle inequality. Also, $S(\omega_0)$ may cause a change. However there will be identical changes in left hand side of the triangle inequality through a change in $P'(\omega_i)\Delta R'(\omega_i)\tilde{S}(\omega_i)$. This is also true for $\tilde{S}(\omega_0)$. Finally, the last component will not change because $\mathcal{Q}$ and $\mathcal{R}$ contain ω_0 as an atom so symmetric differences will not change and $\hat{S}(\omega_0) = \emptyset$. As in the previous case the conclusion follows.

Note that with a similar argument we can show similar cases where ω_0 is added two partitions as a singleton and added into the remaining one as part of the existing atoms. Formally, by the symmetry of arguments, this case implies the same conclusion for the following cases.

$$\mathcal{P} := \mathcal{P}' \oplus \omega_0, \quad \mathcal{Q} := \mathcal{Q}' \uplus \omega_0, \quad \mathcal{R} := \mathcal{R}' \oplus \omega_0,$$
$$\mathcal{P} := \mathcal{P}' \oplus \omega_0, \quad \mathcal{Q} := \mathcal{Q}' \oplus \omega_0, \quad \mathcal{R} := \mathcal{R}' \uplus \omega_0.$$

(3.3) Consider now another case where ω_0 is included only in one partition, say $\mathcal{P}$, as a separate atom. That is

$$\mathcal{P} := \mathcal{P}' \oplus \omega_0, \quad \mathcal{Q} := \mathcal{Q}' \uplus \omega_0, \quad \mathcal{R} := \mathcal{R}' \uplus \omega_0.$$

Now suppose $Q'(\omega_i)$ and $R'(\omega_j)$ are the atoms including ω_0 in $\mathcal{Q}$ and $\mathcal{R}$, respectively. This will cause a change in $P'(\omega_i)\Delta Q'(\omega_i) = S(\omega_i)$

and symmetric differences of every component in these atoms. So left-hand side of the triangle inequality will change, however, symmetrical changes will happen on the right-hand side because of symmetrical differences of atoms of $\mathcal{P}$ and $\mathcal{R}$. If we include symmetric differences of atoms of $\mathcal{Q}$ and $\mathcal{R}$ the result follows.

(3.4) Note that the last case in which ω_0 added to existing atoms of $\mathcal{P}'$, $\mathcal{Q}'$ and $\mathcal{R}'$ follows the same logic with the initial case where ω_0 added as a separate atom to each of the partitions.

The list covers all possible cases and the claim follows by induction. □

The logic of the above proof is that adding a new element to create new partitions would create similar effects on each side of the inequality. The only change then is to have one more element in the denominator of the formula, but this means only rescaling without effecting the direction of the inequality.

3.3 Comparison with other metrics

The distance measures for partitions proposed in the literature are mostly distance indices which can be categorized into three groups[1]: indices constructed with a combinatoric approach, indices constructed with informational approach and metrics that rely on tools from probability (or, measure theoretic metrics). Roughly speaking, the indices in the former group have been constructed by counting the number of pairs that agree in different partitions such as Mirkin and Chernyi (1970) and William (1971) or by counting the number of pairs that disagree in different partitions such as Arabie and Boorman (1973). Note that indices constructed by counting the agreed pairs such as the Rand index are not a metric since the distance between two identical partitions is 1 according

[1]Note that this classification is not common nor uniform.

to them. Counting disagreed pairs, however, solves this problem and establishes a metric.

The construction of the indices in the second group is based on Shannon (1948) entropy such as De Mántaras (1991) and Simovici and Jaroszewicz (2003). The logic behind these indices is to make a random variable by using the partition structure to employ entropy which is introduced for a random variable distribution. The desired random variable is obtained by taking the ratio of cardinality of each atom to the cardinality of the original set. That is given a finite set Ω with cardinality $\mid \Omega \mid= n$, consider the partition $\mathcal{P} = \{P_i\}_{i=1}^k$ where P_i is an atom of $\mathcal{P}$. Assuming that all elements of Ω have the same probability of being picked, and choosing an element ω of Ω at random, the probability that this element is in partition $P_i \in \mathcal{P}$ is $p_i = \frac{|P_i|}{n}$. Then the entropy associated with partition $\mathcal{P}$ is

$$H(\mathcal{P}) = -\sum_{i=1}^{k} p_i \log_2 p_i.$$

Moreover, given another partition $\mathcal{Q} = \{Q_j\}_{j=1}^l$ of Ω, where Q_j is an atom of $\mathcal{Q}$, the mutual information between $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$I(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{k} \sum_{j=1}^{l} p_{i,j} \log_2 \frac{p_{i,j}}{p_i p_j}$$

where $p_{i,j} = \frac{|P_i \cap Q_j|}{n}$. The entropy based metrics mentioned before use this notion of mutual information to define metrics (see Wagner and Wagner (2007) for a thorough discussion of entropy based metrics).

The metrics in the last group are more common in economic theory. One of the earliest forms of such (semi) metric is due to Boylan (1971). The (semi) metric is defined on sub-sigma-algebras of a given measure space and it allows to measure the differences between information structures. Allen (1983), Stinchcombe (1990) and Monderer and Samet (1996) use this metric to measure informational differences and to topologize abstract space of information.

Recently, Mohlin (2015) proposed another metric for the same purpose. This metric weights symmetric differences of each cell with their intersection.

We now compare our metric with the ones in the last group. The main reason for this is that the metrics in the last group have wide usage in economic theory or are proposed for economic theory in mind. The metrics in the other groups are designed for cluster analysis or data mining purposes and measuring distances with these metrics generally produce counter-intuitive results in the context of game theory or decision theory. Also, metric proposed by Mohlin (2015) have a very close relation with the metrics in the first two groups. So assessing one of them would give enough idea about the implications of the other.

Before proceeding let us first give the definitions of these two metrics.

Definition 3.2 (Boylan (1971)). *Let $(\Omega, \mathcal{B}, \mu)$ be a finite measure space. The function ρ_b given by*

$$\rho_b(\mathcal{P}, \mathcal{Q}) := \sup_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} \mu(P \Delta Q) + \sup_{Q \in \mathcal{Q}} \inf_{P \in \mathcal{P}} \mu(P \Delta Q)$$

defined on sub-sigma-algebras of $\mathcal{B}$ is a semi-metric.

Definition 3.3 (Mohlin (2015)). *Let $(\Omega, \mathcal{B}, \mu)$ be a finite measure space. The function ρ_m given by*

$$\rho_m(\mathcal{P}, \mathcal{Q}) := \sum_{P(\omega) \in \mathcal{P}} \sum_{Q(\omega) \in \mathcal{Q}} \mu\Big(P(\omega) \cap \mu(Q(\omega))\Big) \mu\Big(P(\omega) \Delta Q(\omega)\Big)$$

defined on $\mathscr{P}$ is a metric.

In the following example consider μ as the counting measure i.e., the cardinality of each set at hand.

Example 3.2. Consider again $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with the partitions $\mathcal{P} = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$, $\mathcal{Q} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ and $\mathcal{R} = \{\{\omega_1, \omega_2, \omega_3\}\}$. Intuitively, moving from $\mathcal{P}$ to $\mathcal{Q}$ and to $\mathcal{R}$ should have the same distance because of the symmetry of the situation. That is, moving from $\mathcal{P}$ to $\mathcal{Q}$ means distinguishing one more thing and moving from $\mathcal{P}$ to $\mathcal{Q}$ means mixing up one more thing so that the distance from moving $\mathcal{P}$ to $\mathcal{Q}$ and to $\mathcal{R}$ should be the same. The following table summarizes the distance for each of the metrics.

Metric	$\mathcal{P},\mathcal{Q}$	$\mathcal{P},\mathcal{R}$	$\mathcal{R},\mathcal{Q}$
ρ	$2/3$	$2/3$	1
ρ_b	2	3	4
ρ_m	2	4	2

Table 3.1: Distances according to different metrics

Observe that the equality $\rho_m(\mathcal{P},\mathcal{Q}) = \rho_m(\mathcal{R},\mathcal{Q})$ is highly counterintuitive.

Note that the results above can be checked for different partitions and different finite sets as well. The reason that we propose our metric is to measure informational differences as intuitively as possible. Also, our measure offers a unique way of measurement. The other two metrics are sensitive to the measure μ. It is also difficult to make sense of these metrics if we allow two different measures μ and μ' for different individuals. So in that sense, it is difficult to use these metrics with heterogeneous or multiple priors.

To make some of the arguments in the previous paragraph consider the following situation. A planner wants to get certain action A from a group of agents and to achieve that, he can allocate knowledge with some cost function $c(.)$. Without being technical, we assume that cost technology should satisfy $c'(.) > 0$ so that providing (more) knowledge should be (more) costly. For simplicity, suppose that agents will do A so long as all the states are distinguishable i.e., each atom is a singleton. Formally, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be the states of the world and player i will do action A if and only if his knowledge P_i is given by $P_i = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$. Furthermore, suppose that initially agents have no knowledge except the states of the world so that their knowledge is $S_i = \{\{\omega_1, \omega_2, \omega_3\}\}$ for $i = 1, 2$. Assume that the initial beliefs are such that $\mu(\Omega) = 1$ and $\mu(A) = 0$ for any $A \subset \Omega$.[2]

[2]Any non-additive probability measure will do the job.

If we want to measure this cost with the metrics discussed above, that is the cost of information is equal to the distance between two partitions S_i and P_i, we can see that both Boylan metric and the metric proposed by Mohlin (2015) yield 0. This means that there is no cost (!) of providing information for the principal. In this setting our metric measures this distance 1 which at least captures the trade-off between incentive and cost.

3.4 Conclusion

This paper proposed a metric for partitions. Although it can be useful in data mining and clustering analysis, our hope is that it can be applied primarily to game theoretic and decision theoretic situations. One possible set-up where this metric is useful would be information design problems in which a principal sets up an information structure to obtain a certain outcome. If the information design is a costly task, then the metric proposed in this paper can be used to measure the cost of alternative designs.

Bibliography

Allen, B. (1983). "Neighboring information and distributions of agents' characteristics under uncertainty". *Journal of Mathematical Economics*, 12(1):63 – 101.

Arabie, P. and Boorman, S. A. (1973). Multidimensional scaling of measures of distance between partitions. *Journal of Mathematical Psychology*, 10(2):148 – 203.

Boylan, E. S. (1971). "Equiconvergence of martingales". *The Annals of Mathematical Statistics*, 42(2):552–559.

De Mántaras, R. (1991). "A distance-based attribute selection measure for decision tree induction". *Machine Learning*, 6(1):81–92.

Mirkin, B. G. and Chernyi, L. B. (1970). "Measurement of the distance between distinct partitions of a finite set of objects". *Automation and Remote Control*, 31(5):786–792.

Mohlin, E. (2015). "A metric for measurable partitions". *Journal of Mathematical Psychology*, 67:39 – 44.

Monderer, D. and Samet, D. (1996). "Proximity of information in games with incomplete information ". *Mathematics of Operations Research*, 21(3):707–725.

Shannon, C. E. (1948). "A mathematical theory of communication". *Bell System Technical Journal*, 27(3):379–423.

Simovici, D. A. and Jaroszewicz, S. (2003). "Generalized conditional entropy and decision trees". In *Journees francophones d'Extraction et de Gestion de Connaissances (EGC 2003)*, pages 369–380, Lyon: France.

Stinchcombe, M. B. (1990). "Bayesian information topologies". *Journal of Mathematical Economics*, 19(3):233 – 253.

Wagner, S. and Wagner, D. (2007). *Comparing clusterings: an overview*. Universität Karlsruhe, Fakultät für Informatik Karlsruhe.

William, M. R. (1971). "Objective criteria for the evaluation of clustering methods". *Journal of the American Statistical Association*, 66(336):846–850.

Zamir, S. (2008). "Bayesian games: games with incomplete information". Discussion Paper Series dp486, The Federmann Center for the Study of Rationality, the Hebrew University, Jerusalem.