Essays in Game Theory and Bankruptcy

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Ercan Aslan*)

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Abstract

In Chapter 1 I study the iterative strategy elimination mechanisms for normal form games. The literature is mostly clustered around the order of elimination. The conventional elimination also requires more strict knowledge assumptions if the elimination is iterative. I define an elimination process which requires weaker rationality. I establish some preliminary results suggesting that my mechanism is order independent whenever iterative elimination of weakly dominated strategies (IEWDS) is so. I also specify conditions under which the "undercutting problem" occurs. Comparison of other elimination mechanisms in the literature (Iterated Weak Strategy Elimination, Iterated Strict Strategy Elimination, Generalized Strategy Eliminability Criterion, RBEU, Dekel-Fudenberg Procedure, Asheim-Dufwenberg Procedure) and mine is also studied to some extent. In Chapter 2 I study the axiomatic characterization of a well-known bankruptcy rule: Proportional Division (PROP). The rule allocates shares proportional to agents' claims and hence, is intuitive according to many authors. I give supporting evidence to this opinion by first defining a new type of consistency requirement, i.e. *union*$-$consistency and showing that PROP is the only rule that satisfies anonymity, continuity and *union*$-$consistency. Note that anonymity and continuity are very general requirements and satisfied by almost all the rules that have been studied in this literature. Thus, I prove that we can choose a unique rule among them by only requiring *union*$-$consistency. Then, I define a bankruptcy operator and give some intuition on it. A bankruptcy operator is a mapping from

the set of bankruptcy operators to itself. I prove that any rule will converge to PROP under this operator as the claims increase. I show nice characteristics of the operator some of which are related to PROP. I also give a definition for continuity of an operator. In Chapter 3 investigate risk-averse investors' behaviour towards a risky firm. In order to find Pareto Optimal allocations regarding a joint venture, I employ a 2-stage game, first stage of which involves a social-planner committing to an ex-post bankruptcy rule. A bankruptcy rule is a set of suggestions for solving each possible bankruptcy problem. A bankruptcy problem occurs when there is not enough endowment to allocate to the agents each of whom has a claim on it. I devise the game-theoretic approach posed in Kıbrıs and Kıbrıs (2013) and extend it further. In fact, that paper considers a comparison among 4 renowned bankruptcy rules whereas mine do not restrict attention to any particular rule but rather aim to find a Pareto Optimal(PO) one. I start with 2 agent case in order to give some insight to the reader and then, generalise the results to an arbitrary number of investors. I find socially desirable (PO) allocations and show that the same can be achieved through financial markets by the help of some well-known results.

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Chapter 1

Iterative Elimination with Recall

1.1 Motivation and Literature Review

As is well known, the most widely studied and probably the most important issue in game theory is that of making predictions about outcomes of games or at least making predictions about payoffs that might be obtained, to the extent that the predictions about the outcomes allow. Broadly speaking, one can consider two main approaches used in the literature regarding this endeavor. The first approach, which is also widely used in other disciplines such as evolutionary biology, computer science and political science, involves the well-known concept of "equilibrium", which prescribes strategy profiles that might emerge as outcomes. The second approach are iterative methods in which unanticipated strategies are removed from consideration. Such methods focus on which strategies cannot be played rather than which can be played. In both approaches, the state of knowledge the players are in and their ability to use reason and deduce from others' reasonings, play an important role. There is, however, a significant difference between the two approaches in terms of how they use internally consistent belief systems. Nash equilibrium assumes certain restrictions on agents' expectations and argues that agents will expect others to play equilibrium strategies in order to justify the outcomes it suggests. By contrast, iterative methods are concerned with making predictions using rationality alone. Bernheim (1984) says that

> ”...the notion of an equilibrium has little intrinsic appeal within a strategic context. When an agent reaches a decision in ignorance of the strategies adopted by other players, rationality consists of making a choice which is justifiable by an internally consistent system of beliefs, rather than one which is optimal, post hoc. This point of view is not original; indeed, most serious justifications of the Nash hypothesis embrace such an approach, arguing that agents will expect the game to yield a Nash outcome, and consequently will choose their equilibrium strategies. Nevertheless, when we think in terms of maximizing utility subject to expectations rather than realizations, it becomes clear that the Nash hypothesis, far from being a consequence of rationality, arises from certain restrictions on agents' expectations which may or may not be plausible, depending upon the game being played. We are then quite naturally led to ask: are there any restrictions of individuals' expectations (and hence choices) which are required by rationality alone, rather than by (subjective) plausibility?”

The most common assumption is that the players use a common criterion when throwing strategies out, and that this criterion is common knowledge. Strict dominance and weak dominance are at the core of such possible criteria. Although the term ”dominance solvability” is coined by Moulin (1979), the tradition of using such criteria dates back to Luce and Raiffa (1957), and is also used as early as Farquharson (1969) to study voting schemes.

Using such criteria for eliminating strategies and assuming common knowledge may lead to further eliminations as each player will also take into account which strategies her opponents will eliminate, and as a consequence which strategies of her own will become ”eligible” for elimination. However, the players may iterate different sequences of reasoning and draw conclusions that do not agree with each other. Nevertheless, it is commonly known that iterated elimination of strictly dominated strategies *(IESDS)* results in a unique set of strategies in a finite normal form game. Moreover, Moulin (1984) shows that the same applies to Cournot duopoly, *i.e.,* only the Nash equilibrium remains in a Cournot game after sequential elimination of different levels of output. In addition, Dufwenberg and Stegeman (2002) prove that *IESDS* may be an order-dependent procedure when strategy spaces larger than finite sets are considered, and that it may generate spurious Nash equilibria. They also establish that if the strategy spaces are compact in Hausdorff spaces and the payoff functions are upper semi-continuous, *IEWDS* succeeds in yielding a prediction, whereas it is not the case in most of the larger classes of games. They prove an order-independence result for *IESDS* under such a class of games. Additionally, they establish that when the players have well-defined best-response correspondences, *IESDS* preserves Nash equilibria. Gilboa et al. (1990) establish sufficient conditions for order-independence for various types of eliminations and show that *IESDS* satisfy them in finite normal-form games.

On the other hand, the conditions in Gilboa et al. (1990) are not satisfied by iterative elimination of weakly dominated strategies *(IEWDS)*. This is why things are not as straightforward when it comes to *IEWDS* as the order-independence problem persists even if we restrict attention to finite normal-form games. Besides, there has been a long discussion on whether the knowledge of not playing weakly dominated strategies automatically leads to *IEWDS*. Samuelson (1992) argues that the answer to this question is "no". He proves that common knowledge does not guarantee order-independence, nor does it guarantee a solution to the players. Hillas and Samet (2014b) write:

> "Despite the awareness of the problem, no suggestion has been made how to fix the process of iterated elimination of weakly dominated strategies in order to capture common knowledge of weak dominance rationality, due to the lack of formalization of weak dominance rationality".

They show that when it is common knowledge that the players do not play weakly dominated strategies, they must play profiles that survive flaws of the weakly dominated strategies process, which is described by Stalnaker (1994). Hillas and Samet (2014a) establish weak/strong non-probabilistic correlated equilibrium which suggests typically a collection of profiles as it can be perceived in the fashion of correlated equilibrium defined in Aumann (1974).Aumann (1987)employs common knowledge of Bayesian rationality and assumes that beliefs are derived from a common prior. Unlike his paper, Hillas and Samet (2014a) use weak/strong rationality. A player is weakly rational if she does not play strictly dominated strategies, and strongly rational if she does not play weakly dominated strategies.

Besides defining the concept of dominance solvability, Moulin (1979) also uses it to show that some important classes of voting games are dominance solvable. A game is said to be dominance solvable if all outcomes obtained by applying *IEWDS* yield the same payoff profile. Mariotti (2000) defines "maximum games" and shows that they are dominance solvable. He also establishes that an important subclass of such games is dominance solvable on the unique Pareto dominant outcome. Ewerhart (2002) proves that any 2-person strictly competitive game with n outcomes is solvable in $(n-1)$ stages of *IEWDS*. Kukushkin (2012) studies dominance solvability and best-response dominance solvability in finite games. Börgers and Janssen (1995) establish a condition which is necessary and sufficient for a Cournot game to be dominance solvable.

The reason why common knowledge of players' rationality does not directly justify *IEWDS,* is that the principle of rationality is not fully applied. In other words, the strategies that were weakly dominated in some stage of elimination, may become weakly undominated later on. There are numerous solutions offered in order to deal with this issue. One such solution is the reasoning based expected utility procedure *(RBEU)*, suggested by Cubitt and Sugden (2011). *RBEU* comes to a halt, producing a trinary partition of strategies, and this provides a partial answer to the question of order-independence and full employment of common rationality. Unlike many other procedures, *RBEU* generates a trinary partition as there remains a category of strategies about which the mechanism does not make any definite assertions. Nevertheless, it provides a reasoning procedure which removes mutually inconsistent conclusions that may be held by different players.

An alternative approach is chosen by Börgers (1994) by replacing common knowledge of rationality with "approximate common knowledge" which is discussed by Monderer and Samet (1989) and Stinchcombe (1988). Using this approach, he justifies the procedure introduced by Dekel and Fudenberg (1990).The Dekel and Fudenberg procedure utilises maximal elimination of weakly dominated strategies at the first stage, and then continues with *IESDS*.Cheng and Wellman (2007) study a modified version of *IEWDS.* They weaken the weak dominance condition and allow a more aggressive pruning of strategies, and also show some important implications of this technique regarding the equilibria that survive iterated elimination. As with many other papers in the literature, they consider elimination by mixed strategies but impose a condition which does not permit a strategy s_i to be eliminated by a mixture that also includes s_i. They define $\delta-$dominance such that a mixed strategy σ_i can eliminate a pure strategy s_i although it yields payoffs lower by δ against some opponent profiles. In some sense, σ_i can be said to approximately weakly dominate s_i. They also establish that the equilibria of a game obtained by iteratively eliminating $\delta-$dominated strategies will be approximate equilibria of the original game. Unfortunately, their procedure is order-dependent, and the approximate equilibria that survive depend on the order of elimination as well.

The closest procedure to ours in the literature is Asheim and Dufwenberg (2003).They define fully permissible strategy sets using an algorithm that eliminates subsets of the entire strategy set. In the procedure they establish, players treat sets of strategies as choice sets. Their work relies on the assumption that players hold a common belief that each player prefers s_i to t_i if and only if s_i weakly dominates t_i on the set of the opponent's strategies, or on the union of choice sets that are deemed possible for the opponent. This assumption constructs a link between the strategies that survive up to some stage of elimination and the entire set. Notice that in a given stage, a strategy which is not weakly dominated on the set of the opponent's strategies that also survive until that stage, may be eliminated due to being weakly dominated when the entire set is considered. In the present paper, we show that such an *ad hoc* assumption is not necessary in an important class of games. That is, once this condition on choice sets is removed, the Asheim and Dufwenberg procedure produces the same results as ours. However, while they prove their findings for the 2-player case only, we allow for an arbitrary number of players. In other words, we show that EWR is well defined in TDI games and produces the same admissible set of strategies as IEWDS.

It is noteworthy that although many of the papers above consider elimination by mixed strategies, Marx and Swinkels (1997), which is the results-wise closest paper to ours, also considers elimination by pure strategies alone. The following example demonstrates that allowing elimination by mixtures may have an effect on which strategies should be eliminated.

Example 1.

	U	M	D
L	$., 10$	$., 0$	$., 0$
R	$., 0$	$., 4$	$., 10$

In the example above, although M is not weakly dominated by either U or D, it is so by the mixture $(U, 0.5; D, 0.5)$ where both U and D are played with probability 0.5. We also know that if a pure strategy s_i yields a higher payoff than a mixed strategy σ_i against some mixed opponent profile σ_{-i}, then σ_{-i} assigns positive probability to a pure strategy s_j against which s_i generates a higher payoff than σ_i. That is why weak domination against mixtures does not increase the chance of a strategy being recalled, an operation we define in the following section. However, as the effect of deletion of an opponent's strategy on one's own strategies is uncertain, we are far from making any statements on whether allowing mixtures will make our procedure stronger or not.

1.2 Model

Let $I = \{1, 2, ..., n\}$ be the set of players. A typical strategy of player i is denoted by s_i. Let S_i be the finite set of strategies that are available to player i. A strategy profile is a vector which is an ordered collection of strategies denoted by $s = (s_1, s_2, ..., s_n)$ whose ith component shows s_i of i. $S = \times_{i \in I} S_i$ is the set of strategy profiles. $\pi_i : S \to \mathbb{R}$ is the payoff function of player i. The structure of the game is common knowledge and each player is assumed to be rational. Furthermore, we are interested in games where rationality of agents is also common knowledge(so that iterative reasoning applies). $\Pi = (\pi_1, \pi_2, ..., \pi_n)$ is the payoff function. A finite normal-form game is an ordered tuple $\Gamma = (I, S, \Pi)$.

By s_{-i}, π_{-i} we will denote the strategy profile and payoff functions of the opponents' of player i, respectively. $S_{-i} = \times_{j \neq i} S_j$ and $\Pi_{-i} = (\pi_1, \pi_2, ...\pi_{i-1}, \pi_{i+1}, ..., \pi_n)$ will denote the set of strategy profiles and payoff functions of i's opponents, respectively. Let $\Phi = \bigcup_{i \in I} S_i$ and $W \subseteq \Phi$. W is said to be a *restriction* of Φ if it includes at least one strategy of each player, *i.e.,* $W \cap S_i \neq \emptyset$ for each $i \in I$. The strategies in a given restriction W of Φ that belong to player i are denoted by $W_i = W \cap S_i$. The set of strategy profiles that can be constructed by using strategies in a restriction W are given by $S^w = \times_{i \in I} W_i$, with a typical element represented by $s^w \in S^w$. We will denote by $s_i^w \in W_i$ player i's strategy in s^w.

We need to make the distinction between W and s, as W might include more than one strategy of any agent and it does not pair up a player's strategies with the strategies of her opponents.

Definition 1. *i) For any* $t_i, s_i \in S_i$, t_i *strictly dominates* s_i *on* W_{-i}, *if we have* $\pi_i(t_i, s_{-i}^w) > \pi_i(s_i, s_{-i}^w)$ *for all* $s_{-i}^w \in S_{-i}^w$..

ii) For any $t_i, s_i \in S_i$, t_i *weakly dominates* s_i *on* W_{-i}, *if we have* $\pi_i(t_i, s_{-i}^w) \geq \pi_i(s_i, s_{-i}^w)$ *for all* $s_{-i}^w \in S_{-i}^w$ *and* $\pi_i(t_i, z_{-i}) > \pi_i(s_i, z_{-i})$ *for some* $z_{-i} \in S_i^w$.

The exercise of eliminating weakly dominated strategies is performed at the thought level and does not involve any commitment. It is merely a process of reasoning. Therefore, the players are not bound by the remaining strategies but rather benefit from not playing them as a consequence of their rationality. Once the players eliminate weakly dominated strategies, they face a similar situation: A new game possibly comprising strategies which became weakly dominated after the first stage of elimination. It is natural to expect the other players to follow a similar approach. Thus, the same reasoning will apply iteratively until no weakly dominated strategy remains. As there is more than one path of elimination, in order for the players to reach a final set of surviving strategies, some coherency among the reasoning of the players is required. One may argue that in order to further proceed with the iterative elimination, a player needs to be certain about which strategies are deleted in the previous stages by his opponents so that he can adequately choose strategies to delete at that stage. Although there might be more than one path of elimination and the player cannot know which one to follow, she can work out the resulting *"reduced game"* of each path and still act if all paths lead to the survival of the same strategies. If she knows that it is the unique reduced game regardless of the elimination path followed and that other players work it out too, and others also know that each player works out the same outcome and so on, the deletion occurs just as it does when there is a single path. Hence, what matters is the reduced game. With regard to this point, Gretlein (1983) proves order independence of elimination paths for games where players have strict preferences over the outcomes. In such games, given her opponents' strategy vector, a player can be indifferent between two strategies only if both result in the same outcome. Rochet (1980) identifies a class of games which also satisfy order independence, namely, any finite normal form game with a payoff matrix that satisfies the following condition:

$$\pi_i(s) = \pi_i(t) \implies \pi_j(s) = \pi_j(t) \text{ for all } i, j \in I \text{ and for all } s, t \in S.$$

Marx and Swinkels (1997)) prove order independence for a wider class of games, games that satisfy transference of decision maker indifference (TDI).

Definition 2. *A normal form game* $\Gamma = (I, S, \Pi)$ *satisfies TDI if we have for all* $i, \in I$, $r_i, t_i \in S_i$ *and* $s_{-i} \in S_{-i}$;

$$\pi_i(r_i, s_{-i}) = \pi_i(t_i, s_{-i}) \implies \pi_j(r_i, s_{-i}) = \pi_j(t_i, s_{-i})$$

Note that Marx and Swinkels (1997) extend their results to mixed strategies as well.TDI is very similar to non-bossiness condition in social choice. Basically, it states that, given a strategy profile, no player should be able to change some other player's payoff without changing his own payoff by playing a different strategy. Marx and Swinkels (1997) provide many examples that satisfy TDI including patent races, oligopoly with an endogenous number of firms, first price auctions, public good provision games etc. We take the following definition from Marx and Swinkels (1997).

Definition 3. *Let* V *be a restriction of* Φ *and let* W *be a restriction of* V. *Then,* W *is a reduction of* V *by weak dominance if* $W = V \backslash X^1, X^2, ..., X^m$ *where* $\forall k$, $X^k \subset \Phi$ *and* $\forall x_i \in X^k$, $\exists z_i \in V \backslash X_i^1, ..., X_i^k$ *such that* z_i *weakly dominates* x_i *on* $V \backslash X_{-i}^1, ..., X_{-i}^k$. W *is a full reduction of* V *by weak dominance if* W *is a reduction of* V *by weak dominance and no strategies in* W_i *are weakly dominated on* W_{-i} *for all* $i \in I$.

The above definition is saying that a set is a reduction of its superset only if the difference consists of strategies that were weakly dominated on the superset. In other words, in order to obtain a reduction of a set, either some of the weakly dominated strategies should be removed or the set should remain the same. As opposed to EWR new strategies cannot be added. Nor can strategies that are not weakly dominated be eliminated. Note that according to the definition above, a reduction of Φ is also a reduction of itself. Below is the definition from Hillas and Samet (2014b) which is equivalent.

Definition 4. *A process of iterated elimination of weakly dominated strategies consists of sequences of strategy profile sets* $(S^0, S^1, ..., S^m)$*, where* $S^0 = S$*, and for* $k \geq 1$, $S^k = \times_i S_i^k$ *where* S_i^k *is obtained from* S_i^{k-1} *by eliminating some strategies in the latter set which are weakly dominated relative to* S_{-i}^{k-1}*. In the sets* S_i^m *there are no weakly dominated strategies relative to* S_{-i}^m*.*

Next we define iterated elimination with recall (hereafter EWR).

Definition 5. *Let* Ψ *be a restriction of* Φ*. A process of EWR is a sequence of restrictions* $(\Psi^0, \Psi^1, ..., \Psi^m)$ *such that* $\Psi^0 = \Psi$ *and for* $k \in \{1, ..., m\}$ $\forall\ z_i \in \Psi_i^{k-1} \backslash \Psi_i^k$, $\exists\ x_i \in \Psi_i^k$ *where* x_i *weakly dominates* z_i *on* Ψ_{-i}^{k-1}, $\forall z_i \in \Psi_i^k / \Psi_i^{k-1}$, $\nexists$ $x_i \in \Psi_i^k \cap \Psi_{-i}^{k-1}$ *where* x_i *weakly dominates* z_i *on* $\Psi_{-i}^k \cap \Psi_{-i}^{k-1}$ *and* $\forall\ z_i \in \Psi \backslash \Psi^m$, $\exists\ x_i \in \Psi_i^m$ *such that* x_i *weakly dominates* z_i *on* Ψ_{-i}^m*. In the sets* Ψ_i^m *there are no weakly dominated strategies relative to* Ψ_{-i}^m*.*

Unlike $IEWDS$, the size of the set does not necessarily shrink at each step. At each EWR stage, first some of the weakly dominated strategies are eliminated. Then, before proceeding to the next stage, some of the previously removed strategies are recalled back if they are not weakly dominated with respect to the new set. Notice that a strategy which became weakly undominated after some strategies are deleted, does not have to be recalled right away. On the other hand, a strategy which is recalled has to be weakly undominated with respect to the new set obtained after the deletion. This process goes on until there is no weakly dominated strategy left to eliminate and no strategy to recall. Regarding the extent to which rationality and common knowledge of rationality assumptions are employed, we can consider EWR as a criticism of iterated elimination of weakly dominated strategies (hereafter IEWDS). Since a player's opponents' strategies are also going through the process of elimination in IEWDS, some of his strategies which were eliminated at previous stages might become undominated. As applying rationality does not exclude such strategies from being played, his opponents need to consider them valid while further iterating the elimination. Also when deleting strategies at following stages, the player knows that his opponents consider such strategies of his admissible and then, the same kind of iterated logic follows. Consider the following example:

Example 2.

		Player 2				
		a_1	a_2	a_3	a_4	a_5
Player 1	b_1	$5,0$	$5,1$	$3,6$	$4,1,$	$2,6$
	b_2	$3,1$	$2,2$	$3,7$	$3,0$	$2,0$

The game has 3 Nash Equilibria (NE) and these are (a_3, b_1), (a_3, b_2), (a_5, b_1). A possible path of IEWDS is eliminating all weakly dominated strategies of a player at once. Notice that b_1 and a_3 are weakly dominating strategies for player 1 and player 2, respectively. Therefore, the remaining strategies would be a_3 and b_1. Another path of elimination might be (b_2, a_1, a_2, a_4) with the remaining strategies $\{a_3, a_5, b_1\}$. The latter path retains the NE (a_3, b_1) but also (a_5, b_1). On the other hand, all paths of EWR eliminate (a_1, a_2, a_4, a_5) and the strategies $\{a_3, b_1, b_2\}$ survive. For instance, we have

$$\begin{aligned}
\Psi^0 &= \{\{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4, a_5\}\} \\
\Psi^1 &= \{\{b_1\} \cup \{a_3\}\} \\
\Psi^2 &= \{\{b_1\} \cup \{a_3, a_5\}\} \\
\Psi^3 &= \{\{b_1, b_2\} \cup \{a_3, a_5\}\} \\
\Psi^4 &= \{\{b_1, b_2\} \cup \{a_3\}\}
\end{aligned}$$

or

$$\begin{aligned}\Psi^0 &= \{\{b_1, b_2\} \cup \{a_1, a_2, a_3, a_4, a_5\}\} \\ \Psi^1 &= \{\{b_1, b_2\} \cup \{a_3\}\}\end{aligned}$$

which is more straightforward.

The intuition is that each player has a weakly dominating strategy and against player 1's weakly dominating b_1, player 2 wishes to play either a_3 or a_5 against player 2's weakly dominating a_3, player 1 is indifferent between playing b_1 or b_2. However, player 1 is still as well-off by playing either of the 2 strategies given that player 2 chooses a_5(She can play a_5 if she anticipates that player 1 will play b_1). Yet, recalling b_2 will make a_5 weakly dominated again and player 2 will adhere to a_3.

1.2.1 The Undercutting Problem

EWR gets stuck in an infinite cycle when applied to some games. *i.e.* players cannot certainly predict which strategies are admissible, which strategies their opponents think are admissible, which strategies their opponents think they think are admissible and so on ad infinitum. In such games, no matter which successive stages of reasoning are iterated by the players, it is impossible to reach a conclusion about which strategies are permitted as a result of perfectly rational calculation and common knowledge of perfectly rational calculation. Unfortunately, this argument would still be valid even if the players could correlate their reasonings or commit to the same steps of iterated elimination and recall. The epistemic foundations of problems which may arise due to a tension between "common knowledge that the players don't play weakly dominated strategies" and admissibility were laid by Samuelson (1992) without asserting a new procedure of elimination. Cubitt and Sugden (2011) also addresses the same problem and suggests *RBEU.* Asheim and Dufwenberg (2003) resolves this issue by imposing that no strategy which is weakly dominated in the entire set of strategies can be permitted in the choice sets that survive iterated steps of elimination. Our paper shows that without such an additional requirement, *EWR* is sufficient to yield the desired result. More formally, we say that an *EWR* process Ψ ends up in an infinite cycle if there does not exist a sequence of restrictions $(\Psi^0, \Psi^1, ..., \Psi^n)$ where a strategy is weakly dominated if and only if it does not belong to Ψ^n. Consider the example below:

Example 3.

	a_1	a_2
b_1	$1, 1$	$1, 0$
b_2	$1, -1$	$-1, 1$

There is a unique *IEWDS* path which is ;

$$\begin{aligned} \Delta^0 &= \{\{b_1, b_2\} \cup \{a_1, a_2\}\} \\ \Delta^1 &= \{\{b_1\} \cup \{a_1, a_2\}\} \\ \Delta^2 &= \{\{b_1\} \cup \{a_1\}\} \end{aligned}$$

There is also a unique *EWR* path which is an infinite sequence of restrictions;

$$\text{For } k = 0, 1, 2, ... \left\{ \begin{array}{l} \Psi^{4k} = \{\{b_1, b_2\} \cup \{a_1, a_2\}\} \\ \Psi^{4k+1} = \{\{b_1\} \cup \{a_1, a_2\}\} \\ \Psi^{4k+2} = \{\{b_1\} \cup \{a_1\}\} \\ \Psi^{4k+3} = \{\{b_1, b_2\} \cup \{a_1\}\} \end{array} \right\}$$

A presumably fruitful way of attempting to characterize the existence of *EWR* and its uniqueness is to tackle these two problems by considering sets of strategies and relations among those sets rather than dealing with the elimination and recall processes themselves. This is the same approach utilised in Asheim and

Dufwenberg (2003) with some minor changes. It is also along the lines of admissible sets approach and related common knowledge of rationality assumptions which are widely discussed in Samuelson (1992), Brandenburger and Friedenberg (2010), Brandenburger et al. (2008) and Börgers (1994). On that account, the following discussion of "fixed restrictions' and some conjectures we formulise might be helpful and provide some insight on what actually changes in the structure of surviving strategies and the deleted ones when we switch from *IEWDS* to *EWR*.

We will postulate the following conjecture:

Claim 1. *Let* $\Theta = \bigcup_{W \subseteq \Phi \text{ and } \forall i \in I \; W \cap S_i \neq \emptyset} W$ i.e. Θ *is the set of all restrictions. Let* $F = \Theta \twoheadrightarrow 2^{\Theta}$ *be a mapping such that* $F(W) = \bigcup_{1 \leq j \leq n} F_j(W)$ *where* $F_i(W) = \{s_i \in S_i : \; s_i \textit{ is not weakly dominated by some } t_i \in S_i \textit{ on } W_{-i}\}$. *Then,* EWR *is order-independent if and only if for any* W, $W` \in \Theta$ *such that* $F(W) = W$ *and* $F(W`) = W`$*, we have either* $W \subseteq W`$ *or* $W` \subseteq W$.

If $F(W) = W$, then we call W a fixed restriction of Φ. Notice that F maps each restriction W of Φ to a restriction of Φ. The condition required by the claim above is ruling out cases where we have a fixed restriction which contains all other fixed restrictions but also there exists two fixed restrictions each of which contains at least one strategy that is not included in the other. *i.e.* whenever we have $W, W` \subseteq W``$ and $W/W` \neq \emptyset \wedge W`/W \neq \emptyset$ where $F(W) = W$, $F(W`) = W`$and $F(W``) = W``$, *EWR* is order-dependent. In other words, it requires a sequence of restrictions $(W^1, W^2, ..., W^k)$ where $W^1 \subseteq W^2 \subseteq ... \subseteq W^k$. In a 2-person game, however, there might be two strategies constituting a Nash-Equilibrium by yielding extremely large payoffs against each other. Say $s_1 \equiv \max_{s \in S_1} \pi_1(s, s_2)$ and $s_2 \equiv \max_{s \in s_2} \pi_2(s_1, s)$ where s_1 and s_2 are player 1's and player 2's strategies, respectively. *i.e.* there is no other strategy of player 1 which gives a payoff higher than or equal to what s_1 gives against s_2, and vice versa. In this case, there is no *EWR* stage where one of these two strategies can be eliminated. Also, we have $F(\{s_1, s_2\}) = \{s_1, s_2\}$. Still, these two strategies might generate lower payoffs than other strategies that are available. Assume we have $F(W) = W$ where $W = S/\{s_1, s_2\}$. Then, no strategy is eliminated and the only EWR reduction is the set of all strategies.

Instead, we can possibly revise the condition in four ways: First, we can allow for fixed restrictions whose intersection is empty, yet with an exception of a fixed restriction which is superset of other fixed restrictions. Second, we can allow for the first revision above but without such a superset. Third, we can allow for fixed restrictions W and W‘ where $W/W‘ \neq \emptyset \wedge W‘/W \neq \emptyset$ and $W \cap W‘ \neq \emptyset$. Fourth, we can also require a superset as in the first revision and use the third revision along with this requirement. As we also want to account for redundant strategies, arguably the best way to construct the conjecture is the fourth. Hence, we will incorporate the following condition, instead:

"For a given set of fixed restrictions $W = \{W^1, W^2, ..., W^k\}$, $\exists W^h \in W$ with $W^h = \bigcup\limits_{1 \leq j \leq k} W^j$ if and only if EWR is order-independent".

In addition, an alternative and convenient approach may be taken into consideration by defining a fixed restriction in a different way:

Fixed Restriction. For each $i \in I$,let

$$F_i(W) = \left\{ \begin{array}{c} s_j \in S_j :\ j \neq i \text{ and } s_j \text{ is a component of some profile } s_{-i} \text{ in a} \\ \text{restriction on which the set of weakly undominated strategies} \\ \text{that belong to player } i \text{ is } W_i \end{array} \right\}$$

Then, W is a fixed restriction of F if for each $i \in I$ we have $W_{-i} \in F_i(W)$.

Note that with this definition of a fixed restriction, we are actually considering fixed points of n different correspondences.

One might easily recognise that according to the former definition of a fixed restriction, any strategy of player i which is not weakly dominated on W_{-i} is included in $F_i(W)$ whereas we have only opponents' strategies in the set $F_i(W)$ with respect to the latter definition. In fact, there might exist more than one set of opponents' profiles on which the set of weakly undominated strategies that belong to player i is the same. Therefore, each $F_i(W)$ involves a correspondence which links player i's strategy subsets to subsets of opponents' strategies. Notice that although the two definitions are equivalent, they may require different technicalities. With the first sort of fixed restriction used in such assertions as ours, we are more likely to encounter fixed point theorems in the conventional sense. On the other hand, the second definition promises a rather intricate but possibly dynamic and cyclical structure which requires different tools to describe and to deal with.

Conjecture 1. *Undercutting problem occurs if and only if* $\nexists$ *a restriction* W *of* Φ *such that* $F_i(W_{-i}) = W_i$ *for all* $i \in I$*.*

Conjecture 2. *For any* Γ, $\exists$ *a path of IEWDS which gives* $EWR(\Gamma)$ *if* $\exists$ *a non-empty restriction* W *of* Φ *such that* $F_i(W_{-i}) = W_i$ *for all* $i \in I$*.*

Theorem 1. *Let* Γ *be a finite normal form game which satisfies* TDI. *Then,* EWR(Γ) *is well-defined. Furthermore, there exists an* EWR *path that gives the same reduction as* IEWDS.

Proof. Let $\Delta = (\Delta^0, \Delta^1, ..., \Delta^m)$ be a process of *IEWDS* with maximal elimination. *i.e.* Each player removes all weakly dominated strategies at once at a given stage. The *EWR* path Ψ we are going to define involves no recall for the first m stages. *i.e.* $\Psi^h = \Delta^h$ for $0 \leq h \leq m$. Consider the strategies in Δ^{m-1}/Δ^m. Let $s_j \in \Delta^{m-1}/\Delta^m$ be such that it can be recalled at stage $(m+1)$. As s_j is weakly dominated by some other strategy on Δ^{m-1} and $\Delta^m \subset \Delta^{m-1}$, $\exists t_j \in \Delta^m$ such that $\nexists s_{-j} \in \Delta^m_{-j}$ with $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$. Then, $\forall s_{-j} \in \Delta^m_{-j}$, we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$. Define $\Psi^{m+1} = \Delta^m \cup \{s_j\}$. Namely, we recall only a single strategy, if there is any, and it belongs to the set of strategies eliminated at the final stage of *IEWDS.* There is no weakly dominated strategy in Ψ^{m+1} and we recall another strategy $s_i \in \Delta^{m-1}/\Delta^m$, if there is any where $i \in I$. *i.e.* i is not necessarily a different player, we can have $i = j$. Since s_i is also eliminated at stage m, the argument we used for s_j applies to s$_i$ as well. *i.e.* $\exists t_i \in \Delta^m$ such that either $\forall s_{-i} \in \Delta^m_{-i} \cup \{s_j\}$, we have $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ or $\forall s_{-i} \in \Delta^m_{-i}$, $\pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ and for some $s_{-i} \in \Delta^m_{-i}$, $\pi_i(t_i, s_{-i}) > \pi_i(s_i, s_{-i})$ and for some $s_{-ij} \in \Delta^m_{-ij}$ we have $\pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij})$(One of these two conditions should apply to any strategy $t_j \in \Delta^m$ with which s_j generates equal payoffs on Δ^m_{-j}. Hence, it should apply to the strategy which eliminated s_j at stage m as s_j is recalled back to a restriction with same payoff structure as the *IEWDS* reduction) Suppose the latter holds. Then, since $s_{-j}, s_{-ij} \in \Delta^m \implies s_{-j}, s_{-ij} \in \Delta^{m-1}$, s_i wouldn't be weakly dominated by t_i on Δ^{m-1}_{-i}. Thus, $\forall s_{-i} \in \Delta^m_{-i} \cup \{s_j\}$, $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$. As a consequence, $\forall\ s_{-ij} \in \Delta^m_{-ij}$, by *TDI* we have

$$\pi_j(t_j, t_i, s_{-ij}) = \pi_j(s_j, t_i, s_{-ij}) \implies \pi_I(t_j, t_i, s_{-ij}) = \pi_I(s_j, t_i, s_{-ij}) \tag{1.1}$$

$$\pi_i(s_j, t_i, s_{-ij}) = \pi_i(s_j, s_i, s_{-ij}) \implies \pi_I(s_j, t_i, s_{-ij}) = \pi_I(s_j, s_i, s_{-ij}) \tag{1.2}$$

$$\pi_i(t_j, t_i, s_{-ij}) = \pi_i(t_j, s_i, s_{-ij}) \implies \pi_I(t_j, t_i, s_{-ij}) = \pi_I(t_j, s_i, s_{-ij}) \tag{1.3}$$

Hence, by *(*1.1), (2) and (3),

$$\pi_I(t_j, t_i, s_{-ij}) = \pi_I(s_j, t_i, s_{-ij}) = \pi_I(t_j, s_i, s_{-ij}) = \pi_I(s_j, s_i, s_{-ij}).$$

i.e. the payoff structure of Δ^m is preserved by $\Delta^m \cup \{s_i, s_j\}$ as any payoff profile that can be constructed by the latter set can also be constructed by the former. In other words, s_i and s_j are redundant to Δ^m. Furthermore, there is no weakly dominated strategy in the set $\Delta^m \cup \{s_i, s_j\}$. One can easily show that the same argument applies if we continue to recall strategies one by one from the set Δ^{m-1}/Δ^m. If there isn't any strategy to recall in the set Δ^{m-1}/Δ^m to begin with, then we apply the same procedure to $\Delta^{m-2}/\Delta^{m-1}$. If there is no strategy to recall in $\Delta^{m-2}/\Delta^{m-1}$, then we recall a single strategy from $\Delta^{m-3}/\Delta^{m-2}$ and so on.

For $1 < k < m$, assume there exists a strategy $s_j \in \Delta^{m-p}/\Delta^{m-p+1}$ for some $p > k$ which is not weakly dominated on Ψ^k. If for each $t_j \in \Psi^k$, $\exists s_{-j} \in \Psi^k_{-j}$ such that $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$, then since $\Psi^k \subset \Delta^{m-p}$, $\nexists t_j \in \Delta^{m-p}$ such that t_j weakly dominates s_j on Δ^{m-p} and $s_j \in \Delta^{m-p+1}$. Contradiction. Thus, $\exists t_j \in \Psi^k$ such that $\forall s_{-j} \in \Psi^k_{-j}$ we have $\pi_j(t_j, s_{-j}) \geq \pi_j(s_j, s_{-j})$ and as s_j is not weakly dominated on Ψ^k, $\forall s_{-j} \in \Psi^k_{-j}$, $\pi_j(t_j, s_{-j}) = \pi_j(s_j, s_{-j})$. For each step k of Ψ with $k > 1$ and after each strategy s_j we recall at that stage, we are going to recall all strategies that are eliminated after s_j in Δ process and became weakly undominated once s_j is recalled. Let $s_i \in S_i$ be a such strategy. Then, $\exists t_i \in \Psi^k$ such that t_i weakly dominates s_i on Ψ^k_{-i} but $\exists s_{-i} \in \Psi^{k+1}_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. Then, s_j is a component of s_{-i} as $\Psi^{k+1}/\Psi^k = \{s_j\}$. *i.e.* $\pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij})$ for some $s_{-ij} \in \Psi^k_{-ij}$. Since $\pi_j(t_i, s_j, s_{-ij}) = \pi_j(t_i, t_j, s_{-ij})$, by *TDI* we have $\pi_i(t_i, s_j, s_{-ij}) = \pi_i(t_i, t_j, s_{-ij})$. Again by *TDI*, if $\pi_j(s_i, s_j, s_{-ij}) = \pi_j(s_i, t_j, s_{-ij})$, then $\pi_i(s_i, s_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij})$ Thus, $\pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij}) \implies \pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij})$. As t_j and s_{-ij} are not weakly dominated at the stage where s_i is weakly dominated by t_i, we have a contradiction. Since s_i is an arbitrarily chosen among the strategies eliminated after s_j, no strategy eliminated after s_j will be recalled after s_j. Finally we run induction on strategies which are eliminated at a given stage and then, on stages until we get to the first stage of Ψ. Hence, proof is complete. $\square$

Next, we evoke the issue of order-independence and show that a result similar to the one which holds for IEWDS is also true for EWR. In the following theorem, we abuse the notation for convenience and write that a profile is an element of a set of strategies whenever it is constructed solely by strategies from that set. A similar notation is used for opponent strategies etc. All the superscripts are for stages and all the subscripts are for players.

For the first two parts of the following theorem, we employ the same approach.

We take two sets one of which is a subset of the other, $A \subset B$ where A is the set obtained by *IEWDS(EWR)* and B is the set obtained by applying a correspondence (that will be defined in the proof) on the reduction set obtained by *IEWDS* and on the reduction set obtained by *EWR* for the first and the second parts' respectively. We consider an arbitrarily chosen $s_i \in B/A$ and show that s_i cannot be eliminated by a strategy from A, B/A or S/B when *IEWDS* is applied for the first case and *EWR* for the second.

Theorem 2. *For a finite normal form game* Γ *that satisfies* TDI*, we have* $EWR(\Gamma) = IEWDS(\Gamma)$ *where* $IEWDS(\Gamma)$ *is the unique reduction obtained by applying* TDI. *i.e. the reduction obtained by* EWR *and the reduction obtained by* $IEWDS$ *are equivalent up to redundant strategies. Therefore, for such games* EWR *is also order-independent.*

Proof. Let $\Delta = (\Delta^0, \Delta^1, ..., \Delta^m)$ be a sequence of $IEWDS$ restrictions where $\Delta^0 = \Phi$ and Δ^m is the $IEWDS$ reduction of Φ. Let $\Psi = (\Psi^0, \Psi^1, ..., \Psi^n)$ be a sequence of EWR restrictions with $\Psi^0 = \Phi$ and Ψ^n being an EWR reduction of Φ. We want to show that for any $i \in I$ and $s_i \in S_i$, $s_i \in EWR(\Gamma) \iff s_i \in IEWDS(\Gamma)$. In order to do so, we are going to show that neither Ψ^n nor Δ^m is a proper subset of the other and it is not the case that both of them include a strategy which is not an element of the other. *i.e.* $\neg\left[\Psi^n \subset \Delta^m\right]$ and $\neg\left[\Delta^m \subset \Psi^n\right]$ and $\neg\left[\Psi^n/\Delta^m \neq \emptyset \text{ and } \Delta^m/\Psi^n \neq \emptyset\right]$ .Assume $\Delta^m \subset \Psi^n$ with $s_i \in \Psi^n/\Delta^m$. Then, there exists Δ^k and Δ^{k-1} such that $s_i \in \Delta^{k-1}/\Delta^k$ and $t_i \in S_i$ such that t_i weakly dominates s_i on Δ^{k-1}. We also know that if a strategy x_i weakly dominates y_i on a superset of Δ^m, then $\nexists$ any opponent profile $s_{-i} \in \Delta^m_{-i}$ such that $\pi_i(y_i, s_{-i}) > \pi_i(x_i, s_{-i})$ (otherwise, since strategies that construct s_{-i} are also elements of the superset of Δ^m, x_i wouldn't weakly dominate y_i). Since $\Delta^m \subset \Delta^{k-1}$ and t_i weakly dominates s_i on Δ^{k-1}, $\nexists$ any opponent profile $s_{-i} \in \Delta^m_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. If t_i is eliminated in some interim stage r such that $k - 1 < r < m$, then there exists $p_i \in S_i$ such that p_i weakly dominates

t_i on Δ^r_{-i} and $\nexists$ any $s_{-i} \in \Delta^m_{-i}$ such that $\pi_i(t_i, s_{-i}) > \pi_i(p_i, s_{-i})$. Hence, $\nexists$ $s_{-i} \in \Delta^m_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(p_i, s_{-i})$. If p_i is also eliminated at stage s with $r < s < m$, then there exists v_i such that v_i weakly dominates p_i on Δ^s_{-i} and for all $s_{-i} \in \Delta^m_{-i}$ we have $\pi_i(v_i, s_{-i}) \geq \pi_i(p_i, s_{-i})$ and so on. Since we have a finite number of stages and strategies, there exists $z_i \in \Delta^m$ for which $\nexists$ $s_{-i} \in \Delta^m_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(z_i, s_{-i})$. On the other hand, $z_i, s_i \in \Psi^n.i.e.$ z_i does not weakly dominate s_i on Ψ^n_{-i}. If $\forall s_i \in \Psi^n/\Delta^m$ and such $z_i \in \Delta^m$, then $\forall s_{-i} \in \Psi^n_{-i}$ we have $\pi_i(s_i, s_{-i}) = \pi_i(z_i, s_{-i})$ and the claim is true. Assume not. Then, there exists $s_{-i} \in \Psi^n_{-i}$ with at least one strategy $s_j \in \Psi^n/\Delta^m$ such that $\pi_i(s_i, s_{-i}) > \pi_i(z_i, s_{-i})$. By the same token, $s_j \in \Psi^n/\Delta^m \implies \exists t_j \in \Delta^m$ such that $\forall s_{-j} \in \Delta^m_{-j}$, we have $\pi_j(t_j, s_{-j}) \geq \pi_j(s_j, s_{-j})$. Then, either

$$\nexists s_{-j} \in \Psi^n \setminus \Delta^m \quad s.t. \quad \pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j}) \; and$$
$$\exists s_{-j} \in \Delta^m_{-j} \quad s.t. \quad \pi(t_j, s_{-j}) > \pi(s_j, s_{-j}) \tag{1.4}$$

or

$$\forall s_{-j} \in \Psi^n, \pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j}). \tag{1.5}$$

If (1.4) holds, then $s_j \notin \Psi^n$ as it would be weakly dominated by t_j on Ψ^n_{-j}. A contradiction. If (1.5) holds, by TDI $\pi_i(s_j, s_{-j}) > \pi_i(t_j, s_{-j})$, *i.e.*, for all $s_{-ij} \in \Psi^n_{-ij}$,

$$\pi_j(s_i, s_j, s_{-ij}) = \pi_j(s_i, t_j, s_{-ij}) \implies \pi_i(s_i, s_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij}) \tag{1.6}$$

and

$$\pi_j(t_i, s_j, s_{-ij}) = \pi_j(t_i, t_j, s_{-ij}) \implies \pi_i(t_i, s_j, s_{-ij}) = \pi_i(t_i, t_j, s_{-ij}). \tag{1.7}$$

Since $\pi_i(s_i, s_j, s_{-ij}) > \pi_i(t_i, s_j, s_{-ij})$ for some $s_{-ij} \in \Psi^n_{-ij}$, by (1.6) and (1.7) we have

$$\pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij}) \tag{1.8}$$

for such $s_{-ij} \in \Psi^n_{-ij}$.

Therefore, if for each component $s_k \in \Psi^n/\Delta^m$ of s_{-j} (2) holds, then following from (1.8), we have $\pi_i(s_i, t_{-i}) > \pi_i(t_i, t_{-i})$ where $t_{-i} \in \Delta^m_{-i}$. A contradiction. Hence, there exists an $s_j \in \Psi^n/\Delta^m$, an s_{-j} with at least one component from Ψ^n/Δ^m and a $t_j \in \Delta^m$ such that t_j weakly dominates s_j on Δ^m_{-j} and $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$ with s_j being a component of s_{-i} where $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. Therefore, for each $s_i \in \Psi^n/\Delta^m$ for which $\nexists t_i \in \Delta^m$ such that for all $s_{-i} \in \Psi^n_{-i}$ $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ there exists a such s_j. Denote the set of such s_j by $\alpha(s_i, t_i)$. Let $\alpha(s_i) = \bigcup\limits_{t_i \in \Delta^m_i} \alpha(s_i, t_i)$. Let $\alpha(\alpha(s_i)) = \alpha^2$ be the union of set of strategies obtained by applying α to each strategy in $\alpha(s_i)$. Consider the sequence $(\alpha(s_i), \alpha^2(s_i), ...)$. Since we have a finite number of strategies, we have $\alpha^k(s_i) \subseteq \bigcup\limits_{1 \le j \le k-1} \alpha^j(s_i)$ for some finite k. Set $\alpha^k(s_i) = \alpha$. Consider the first strategy eliminated from α by $IEWDS$. For each $t_i \in \Delta^m$ and $s_i \in \alpha$, there exists either

(a) $t_{-i} \in \Delta^m_{-i}$ such that $\pi_i(s_i, t_{-i}) > \pi_i(t_i, t_{-i})$ or

(b) $s_{-i} \in \Delta^m_{-i} \bigcup \alpha$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$

Therefore, the first strategy cannot be eliminated by a strategy from Δ^m.

the first strategy is eliminated by a strategy $g_i \in S_i/\Psi^n_i$, then $\pi_i(g_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ $\forall s_{-i} \in \Delta^m \bigcup \alpha$. Note that we can construct an opponent profile $s_{-i} \in \Delta^m \bigcup \alpha$ since each player has at least one strategy in Δ^m(because it is the reduction obtained by $IEWDS$). We also know that here exists a strategy $e_i \in \Psi^n_i$ such that e_i weakly dominates g_i on Ψ_{-i} and for each $s_i \in \Psi^n_i$ there exists a profile $s_{-i} \in \Psi^n_{-i}$ such that $\pi_i(s_i, s_{-i}) > \pi_i(g_i, s_{-i})$ since otherwise g_i would be recalled back to Ψ^n. By (a) and (b), there exists $s_{-i} \in \Delta^m \bigcup \alpha$ for each $s_i \in \alpha$ and each

$t_i \in \Delta^m$ such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. Therefore, for each $t_i \in \Delta^m$ there exists $s_{-i} \in \Delta^m \bigcup \alpha$ such that $\pi_i(g_i, s_{-i}) > \pi_i(t_i, s_{-i})$. Moreover, for each $y_i \in \Psi^n$ such that $\pi_i(y_i, s_{-i}) = \pi_i(t_i, s_{-i})$ where $t_i \in \Delta^m$ and $s_{-i} \in \Psi^n_{-i}$ (*i.e.* $y_i \in \Psi^n / [\Delta^m \bigcup \alpha]$), we have $\pi_i(g_i, s_{-i}) > \pi_i(y_i, s_{-i})$ for some $s_{-i} \in \Delta^m \bigcup \alpha$. Hence, a strategy which weakly dominates g_i on Ψ^n_{-i} cannot be an element of $\Psi^n / [\Delta^m \bigcup \alpha]$ or Δ^m and has to be an element of α. Since $\pi_i(g_i, s_{-i}) \geq \pi_i(s_i, s_{-i})\ \forall s_{-i} \in \Delta^m \bigcup \alpha$ and $\pi_i(s_i, s_{-i}) \geq \pi_i(g_i, s_{-i})\ \forall s_{-i} \in \Psi^n_{-i}$, we have $\pi_i(g_i, s_{-i}) = \pi_i(s_i, s_{-i})\ \forall s_{-i} \in \Delta^m \bigcup \alpha$. By TDI, $\pi_i(g_i, s_{-i}) = \pi_i(s_i, s_{-i}) \implies \pi_I(g_i, s_{-i}) = \pi_I(s_i, s_{-i})$. *i.e.* For each $s_j \in \alpha$ and $t_i \in \Delta^m$ and $s_{-ij} \in \Delta^m \bigcup \alpha$, we have $\pi_j(s_i, s_j, s_{-ij}) > \pi_j(s_i, t_j, s_{-ij}) \implies \pi_j(g_i, s_j, s_{-ij}) > \pi_j(g_i, t_j, s_{-ij})$. Therefore, $\alpha(s_i) = \alpha(g_i)$ and $\alpha' = \alpha(s_1) \bigcup \alpha(s_2) \bigcup ... \bigcup \alpha(s_{i-1}) \bigcup ... \bigcup \alpha(g_i) \bigcup \alpha(s_{i+1}) \bigcup ... \bigcup \alpha(s_I)$ where I also represents the number of players in the set I. (g_i replaces s_i in $\Delta^m \bigcup \alpha$ and the payoff structures are preserved, so are the weak dominance relations).

Then we consider the first strategy eliminated from $\Delta^m \bigcup \alpha'$ by $IEWDS$. By the same arguments we used for the first strategy eliminated from $\Delta^m \bigcup \alpha$, it cannot be eliminated by a strategy from Δ^m. If it's eliminated by a strategy from S/Ψ^n, then we have a new collection α'' of sets of strategies such that the first strategy eliminated from $\Delta^m \bigcup \alpha''$ by $IEWDS$ cannot be eliminated by a strategy from Δ^m and so on. Since we have a finite number of strategies, the first strategy eliminated from one of those collections must be eliminated by a strategy which is also an element of the same collection. Without loss of generality, say $s_i \in \alpha$ is eliminated by $h_i \in \alpha$. Then, for each $s_{-i} \in \Delta^m \bigcup \alpha$, $\pi_i(h_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$. If there exists $s_{-i} \in \Delta^m \bigcup \alpha$ such that $\pi_i(h_i, s_{-i}) > \pi_i(s_i, s_{-i})$, then there exists $s_{-i} \in \Psi^n$ with at least one component from $\Psi^n / [\Delta^m \bigcup \alpha]$ such that $\pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i})$. For each such component $s_j \in \Psi^n / [\Delta^m \bigcup \alpha]$, since $\pi_j(s_i, s_j, s_{-ij}) = \pi_j(s_i, t_j, s_{-ij})$ for some $t_j \in \Delta^m$, by TDI we have $\pi_i(s_i, s_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij})$ and $\pi_i(h_i, s_j, s_{-ij}) = \pi_i(h_i, t_j, s_{-ij})$ for all $s_{-ij} \in \Psi^n_{-ij}$. Therefore, $\pi_i(h_i, s_j, s_{-ij}) > \pi_i(s_i, s_j, s_{-ij}) \implies \pi_i(h_i, t_j, s_{-ij}) = \pi_i(s_i, t_j, s_{-ij})$. By replacing each such component $s_j \in \Psi^n / [\Delta^m \bigcup \alpha]$ of s_{-i} where $\pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i})$ with $t_j \in \Delta^m$ such that $\pi_j(t_j, s_{-j}) = \pi_j(s_j, s_{-j})$ $\forall s_{-j} \in \Psi^n_{-j}$, we construct $s_{-i} \in \Delta^m \bigcup \alpha$ such that $\pi_i(s_i, s_{-i}) > \pi_i(h_i, s_{-i})$. A contradiction. Hence, $\pi_i(s_i, s_{-i}) = \pi_i(h_i, s_{-i})$ for all $s_{-i} \in \Delta^m \bigcup \alpha$.By TDI, $\pi_i(h_i, s_{-i}) = \pi_i(s_i, s_{-i}) \implies \pi_I(h_i, s_{-i}) = \pi_I(s_i, s_{-i})$. $i.e.$ For each $s_j \in \alpha$ and $t_i \in \Delta^m$ and $s_{-ij} \in \Delta^m \bigcup \alpha$, we have $\pi_j(s_i, s_j, s_{-ij}) > \pi_j(s_i, t_j, s_{-ij}) \implies \pi_j(h_i, s_j, s_{-ij}) > \pi_j(h_i, t_j, s_{-ij})$. Therefore, $\alpha(s_i) = \alpha(h_i)$ and g $\alpha' = \alpha(s_1) \bigcup \alpha(s_2) \bigcup ... \bigcup \alpha(s_{i-1}) \bigcup ... \bigcup \alpha(h_i) \bigcup \alpha(s_{i+1}) \bigcup ... \bigcup \alpha(s_I)$ where I also represents the number of players in the set I. (h_i replaces s_i in $\Delta^m \bigcup \alpha$ and the payoff structures are preserved, so are the weak dominance relations).

Then, we consider the first strategy eliminated from $\Delta^m \bigcup \alpha'$ by $IEWDS$. By the same arguments we used for the first strategy eliminated from $\Delta^m \bigcup \alpha$, it cannot be eliminated by a strategy from Δ^m. If it's eliminated by a strategy from S/Ψ^n, then we have a new collection α'' of sets of strategies such that the first strategy eliminated from $\Delta^m \bigcup \alpha''$ by $IEWDS$ cannot be eliminated by a strategy from Δ^m and so on. Since we have a finite number of strategies, the first strategy eliminated from one of those collections must be eliminated by a strategy which is also an element of the same collection and so on.

On the contrary, assume $\Psi^n \subset \Delta^m$. For some $k \in \mathbb{N}$, we have $s_j \in \Psi^{h-1}/\Psi^h \implies s_j \notin \Delta^m/\Psi^n$ where $1 \leq h \leq k$. Let $s_i \in \Delta^m/\Psi^n$with $s_i \in \Psi^k/\Psi^{k+1}$ such that for $1 \leq h \leq k$ we have $s_j \in \Delta^m/\Psi^n \implies s_j \notin \Psi^{h-1}/\Psi^h$. Namely, s_i is the first strategy eliminated from Δ^m/Ψ^n by EWR. Assume that for k such that $s_j \in \Psi^k \implies s_j \in \Psi^{k+1}$ where $s_j \in \Delta^m/\Psi^n$, we have $s_t \in \Psi^k \implies s_t \in \Psi^{k+1}$ for all $t \in I$. That is, prior to the elimination of the first strategy from Δ^m/Ψ^n, no strategy is eliminated from Ψ^n, either. Consequently, before the stage where s_i is weakly dominated, all the other strategies in Δ^m have survived the previous stages, too. s_i is weakly dominated for the first time at some stage by $t_i \in \Delta^m \implies \left[s_j \in \Delta^m \implies s_j \in \Psi^k\right]$. Since $s_i \in \Delta^m/\Psi^n$, for each $t_i \in \Psi^n_i$ such that t_i weakly dominates s_i on Ψ^n, $\exists s_{-i} \in \Delta^m_{-i}$ with at least one strategy from Δ^m/Ψ^n such that $\pi_i(s_i, s_{-i}) > \pi_i(t_i, s_{-i})$. As the components of s_{-i} are also elements of Ψ^k, s_i cannot be eliminated by $t_i \in \Psi_i^n$

Conversely, assume that $\exists t_j \in \Psi^n$ and $k \in \mathbb{N}$ such that $t_j \notin \Psi^k$ and $s_i \in \Psi^k$. Indeed, let t_j be the first strategy eliminated from Ψ^n (hence recalled at a later stage) before any strategy is eliminated from Δ^m/Ψ^n. (Here, j and i aren't necessarily distinct players as they are throughout the rest of the proof). Since $t_j \in \Psi^n$, we have either

$$\text{For each } t_j \in \Psi_j^n \text{ and } s_j \in S, \exists t_{-j} \in \Psi_{-j}^n \quad s.t. \quad \pi_j(t_j, t_{-j}) > \pi_j(s_j, t_{-j}) \tag{1.9}$$

or

$$\begin{aligned} \exists e_j \in \;& \Psi_j^n \text{ such that } \pi_j(t_j, t_{-j}) = \pi_j(e_j, t_{-j}) \forall t_{-j} \in \Psi_{-j}^n \text{ and } \exists s_{-j} \text{ with} \\ & \text{at least one component from } S/\Psi^n \text{ such that } \pi_j(e_j, s_{-j}) > \pi_j(t_j, s_{-j}) \\ & \text{and there does not exist any } s_{-j} \in \Psi_{-j}^n \text{ such that } \pi_j(t_j, s_{-j}) > \pi_j(e_j, s_{-j}) \\ & \text{where } t_j \text{ is weakly dominated by } e_j \text{ at stage } k. \end{aligned} \tag{1.10}$$

If (1.9) holds, then t_j cannot be the first strategy eliminated from Ψ^n. Assume (1.10) holds. Then, since $e_j, k_j \in \Psi^n \subset \Delta^m$, for all $s_{-j} \in \Delta_{-j}^m$, we have $\pi_j(t_j, s_{-j}) = \pi_j(e_j, s_{-j})$. By *TDI*, $\pi_I(t_j, s_{-j}) = \pi_I(e_j, s_{-j})$ for all $s_{-j} \in \Delta_{-j}^m$. In particular, $\pi_i(s_i, t_j, s_{-ij}) = \pi_i(s_i, e_j, s_{-ij})$ and $\pi_i(t_i, t_j, s_{-ij}) = \pi_i(t_i, e_j, s_{-ij})$ where $s_{-ij} \in \Delta_{-ij}^m$ and s_i is the first strategy eliminated from Δ^m/Ψ^n with t_i being the strategy that weakly dominates it. Since $\pi_i(s_i, t_j, s_{-ij}) > \pi_i(t_i, t_j, s_{-ij}) \implies \pi_i(s_i, e_j, s_{-ij}) > \pi_i(t_i, e_j, s_{-ij})$, the elimination of t_j does not make s_i weakly dominated by t_i.

Assume that before the first strategy from Δ^m/Ψ^n is eliminated and after the first strategy from Ψ^n is eliminated, some strategy $g_k \in \Psi_k^n$ is also eliminated.

(Notice that all the statements are true regardless of whether only one strategy is eliminated from Ψ^n at stage k where $1 \leq h \leq k-1$, $s_j \in \Psi_j^n \implies s_j \in \Psi_j^h$ or many.) Since for each $s_{-j} \in \Delta_{-j}^m$ $\exists e_j$ such that $\pi_j(t_j, s_{-j}) = \pi_j(e_j, s_{-j})$, for each $s_k \in \Delta_k^m$, and each $s_{-kj} \in \Delta_{-kj}^m$ by *TDI* we have*:*

$$\pi_j(g_k, t_j, s_{-kj}) = \pi_j(g_k, e_j, s_{-kj}) \implies \pi_k(g_k, t_j, s_{-kj}) = \pi_k(g_k, e_j, s_{-kj}) \quad (1.11)$$

$$\pi_j(s_k, t_j, s_{-kj}) = \pi_j(s_k, e_j, s_{-kj}) \implies \pi_k(s_k, t_j, s_{-kj}) = \pi_k(s_k, e_j, s_{-kj}) \quad (1.12)$$

From (1.11) and (1.12) we have

$$\pi_k(g_k, t_j, s_{-kj}) > \pi_k(s_k, t_j, s_{-kj}) \implies \pi_k(g_k, e_j, s_{-kj}) > \pi_k(s_k, e_j, s_{-kj}).$$

Hence, $g_k \in \Psi_k^n$cannot be weakly dominated by some strategy $s_k \in \Delta_k^m$. For each $s_k \in \Delta^m/\Psi^n$, $\exists s_{-k} \in \Psi_{-k}^n$ such that $\pi_k(g_k, s_{-k}) > \pi_k(s_k, s_{-k})$(Otherwise s_k would be recalled). One component of $s_{-k} \in \Psi_{-k}^n$ may be eliminated in one of the earlier stages but (8) and (9) apply again. Therefore, g_k can be eliminated only by a strategy also from Ψ_k^n. Thus, the first strategy eliminated from Δ^m/Ψ^n cannot be eliminated by a strategy from Ψ^n.

Assume that the first strategy s_i eliminated from Δ^m/Ψ^n is eliminated by a strategy t_i which is also from Δ^m/Ψ^n. Since $s_i, t_i \in \Delta^m$, we have either

(c)$\forall s_{-i} \in \Delta_{-i}^m$, $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ or

(d) $\exists s_{-i}, t_{-i} \in \Delta_{-i}^m$ such that $\pi_i(s_i, s_{-i}) = \pi_i(t_i, s_{-i})$ and

$\pi_i(t_i, t_{-i}) > \pi_i(s_i, t_{-i})$

If *(d)* holds, since s_i is the first strategy to be eliminated from Δ^m/Ψ^n and all such $s_{-i}, t_{-i} \in \Delta_{-i}^m$ profiles have survived, then s_i is not weakly dominated at this stage. Assume *(c)* holds. Since $s_i \in \Delta^m/\Psi^n$, $\exists k_i$ such that k_i weakly dominates s_i on Ψ^n and $\exists s_{-i} \in \Delta_{-i}^m$ with at least one component from Δ^m/Ψ^n such that $\pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i})$ for such s_i and k_i. Then, we define correspondence β which is similar to α we had defined earlier: $\beta(s_i, k_i) = \{s_j : j \neq i \text{ and } s_j \text{ is a component of some } s_{-i} \text{ such that } \pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i})\}$ Since $\pi_i(s_i, s_{-i}) > \pi_i(k_i, s_{-i}) \implies \pi_i(t_i, s_{-i}) > \pi_i(k_i, s_{-i})$ for all $s_{-i} \in \Delta^m_{-i}$, $\beta(s_i, k_i) = \beta(t_i, k_i)$ By the same reasoning we employed for the case with α, we conclude that the first strategy eliminated from Δ^m/Ψ^n is not eliminated by a strategy from Δ^m/Ψ^n. Therefore, we have $\neg\left[\Psi^n \subset \Delta^m\right]$.

(Notice that the first strategy eliminated from Δ^m/Ψ^n may be recalled at a later stage. Our result still holds, though)

Finally, assume we have both $\Delta^m/\Psi^n \neq \emptyset$ and $\Psi^n/\Delta^m \neq \emptyset$. Then, there are two possible cases, either $\Delta^m \cap \Psi^n \neq \emptyset$ or $\Delta^m \cap \Psi^n = \emptyset$. Assume $\Delta^m \cap \Psi^n \neq \emptyset$. $\exists$ a $k \in \mathbb{N}$ and a restriction Δ^k from the sequence $(\Delta^0, \Delta^1, ..., \Delta^m)$ such that $s_i \in \Psi^n/\Delta^m \implies s_i \in \Delta^h$ for $1 \leq h \leq k$ and $\exists\ s_i \in \Psi^n/\Delta^m$ such that $s_i \in \Delta^{k-1}/\Delta^k$. *i.e.* s_i is the first strategy eliminated from Ψ^n/Δ^m in a given *IEWDS* process. Therefore, $\exists\ t_i \in \Delta^k$ such that for all $s_{-i} \in \Delta^{k-1}$, $\pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$. Since no strategy is yet eliminated from Ψ^n/Δ^m in stage $(k-1)$ and all the strategies in Δ^m(and hence $\Delta^m \cap \Psi^n$) survive *IEWDS,* we have $\Psi^n \subset \Delta^{k-1}$. Then, for each $s_{-i} \in \Psi^n_{-i}$, $\pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ and $t_i \in \Psi^n$. Since s_i is also an element of Ψ^n, we have $\pi_i(t_i, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in \Psi^n_{-i}$. Let G^k be the set of strategies which are elements of Ψ^n/Δ^m and which are eliminated at stage k. Let the strategies $G^k = \{s_i, g_j, ..., x_p\}$ be eliminated by $\{t_i, h_j, ..., y_p\}$, respectively. Then, $\{t_i, h_j, ..., y_p\} \subset \Psi^n$. By *TDI,* $\forall s_{-ijkp} \in \Psi^n_{-ijkp}$, $\forall c_k, d_k \in \Psi^n_k$,

$$\begin{aligned} \pi_i(s_i, g_j, x_p, c_k, s_{-ijkp}) &= \pi_i(t_i, g_j, x_p, c_k, s_{-ijkp}) \implies \\ \pi_k(s_i, g_j, x_p, c_k, s_{-ijkp}) &= \pi_k(t_i, g_j, x_p, c_k, s_{-ijkp}) \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \pi_i(s_i, g_j, x_p, d_k, s_{-ijkp}) &= \pi_i(t_i, g_j, x_p, d_k, s_{-ijkp}) \implies \\ \pi_k(s_i, g_j, x_p, d_k, s_{-ijkp}) &= \pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}). \end{aligned} \tag{1.14}$$

By *(1.13)* and *(1.14)*,

$$\begin{aligned} \pi_k(s_i, g_j, x_p, c_k, s_{-ijkp}) &\geq \pi_k(s_i, g_j, x_p, d_k, s_{-ijkp}) \implies \\ \pi_k(t_i, g_j, x_p, c_k, s_{-ijkp}) &\geq \pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}), \end{aligned} \tag{1.15}$$

without loss of generality. By the same token,

$$\begin{aligned} \pi_j(t_i, g_j, x_p, c_k, s_{-ijkp}) &= \pi_j(t_i, h_j, x_p, c_k, s_{-ijkp}) \implies \\ \pi_i(t_i, g_j, x_p, c_k, s_{-ijkp}) &= \pi_i(t_i, h_j, x_p, c_k, s_{-ijkp}) \implies \\ \pi_k(t_i, g_j, x_p, c_k, s_{-ijkp}) &= \pi_k(t_i, h_j, x_p, c_k, s_{-ijkp}), \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \pi_j(t_i, g_j, x_p, d_k, s_{-ijkp}) &= \pi_j(t_i, h_j, x_p, d_k, s_{-ijkp}) \implies \\ \pi_i(t_i, g_j, x_p, d_k, s_{-ijkp}) &= \pi_i(t_i, h_j, x_p, d_k, s_{-ijkp}) \implies \\ \pi_k(t_i, g_j, x_p, d_k, s_{-ijkp}) &= \pi_k(t_i, h_j, x_p, d_k, s_{-ijkp}). \end{aligned} \tag{1.17}$$

By *(1.15), (1.16), (1.17),* we have

$$\pi_k(t_i, h_j, x_p, c_k, s_{-ijkp}) \geq \pi_k(t_i, h_j, x_p, d_k, s_{-ijkp}),$$

and iterating further

$$\pi_k(t_i, h_j, y_p, c_k, s_{-ijkp}) \geq \pi_k(t_i, h_j, y_p, d_k, s_{-ijkp}),$$

and so on. Notice that for any combination of strategies from G^k, we have a corresponding combination of strategies from $\Delta^m \cap \Psi^n$ which preserves the weak dominance relation between c_k and d_k. Hence, for any $s_{-k} \in \Psi^n_{-k}$, $\exists\ t_{-k} \in \Psi^n/G^k$ such that for all $c_k, d_k \in \Psi^n$, we have $\pi_k(c_k, s_{-k}) \geq \pi_k(d_k, s_{-k}) \implies \pi_k(c_k, t_{-k}) \geq$

$\pi_k(d_k, t_{-k})$.

Consider the strategies eliminated at stage $(k+1)$ in Ψ^n/Δ^m, if any. For each $s_i \in G^{k+1}$, $\exists t_i \in \Delta^{k+1}$ such that t_i weakly dominates s_i on Δ^k. If $t_i \notin \Psi^n$, then $\exists h_i \in \Psi^n$ such that h_i weakly dominates t_i on Ψ^n_{-i}. Since $s_{-i} \in \Psi^n/G^k \implies s_{-i} \in \Delta^k$, we have $s_{-i} \in \Psi^n/G^k \implies \pi_i(t_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$. Then, as $s_{-i} \in \Psi^n/G^k$, we have $\pi_i(h_i, s_{-i}) \geq \pi_i(t_i, s_{-i})$ and therefore, $\pi_i(h_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ Since for each $s_{-i} \in \Psi^n/G^k$ $\exists$ $t_{-i} \in \Psi^n_{-i}$ such that $\pi_i(s_i, s_{-i}) = \pi_i(s_i, t_{-i})$ and $\pi_i(h_i, s_{-i}) = \pi_i(h_i, t_{-i})$ as shown above, we conclude that $\pi_i(h_i, t_{-i}) \geq \pi_i(s_i, t_{-i})$ for all $t_{-i} \in \Psi^n_{-i}$. Since both $s_i$and h_i are elements of Ψ^n_i, then $\pi_i(h_i, t_{-i}) = \pi_i(s_i, t_{-i})$ $\forall t_{-i} \in \Psi^n_{-i}$. Thus, by iterating the argument in the equations above and running induction on $G^k, G^{k+1}, ..., G^m$, for each $s_i \in G^k \cup G^{k+1} \cup ... \cup G^m$, $\exists$ $h_i \in \Delta^m \cap \Psi^n$ such that $\pi_i(h_i, s_{-i}) = \pi_i(s_i, s_{-i})$ $\forall s_{-i} \in \Psi^n_{-i}$. Hence, Ψ^n and $\Delta^m \cap \Psi^n$ are equivalent up to redundant strategies. Consequently, it is enough to show that Δ^m is also strategically equivalent to $\Delta^m \cap \Psi^n$(up to redundant strategies)

Consider the first strategy s_j eliminated from Δ^m/Ψ^n in the *EWR* process Ψ. *i.e.* $\exists k \in \mathbb{N}$ and a restriction Ψ^k from the sequence $(\Psi^0, \Psi^1, ..., \Psi^n)$ such that $s_j \in \Delta^m/\Psi^n \implies s_j \in \Psi^h$ for $1 \leq h < k-1$ and $\exists s_j \in \Delta^m/\Psi^n$ with $s_j \in \Psi^{k-1}/\Psi^k$. Suppose that $\exists t_j \in \Delta^m \cap \Psi^n$ such that t_j weakly dominates s_j on Ψ^{k-1}. As t_j, $s_j \in \Delta^m$, we have either

$$\forall s_{-j} \in \Delta^m_{-j}, \pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j}) \tag{e}$$

or

$$\exists s_{-j}, t_{-j} \in \Delta^m_{-j} \quad s.t. \quad \pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j}) \text{ and } \pi_j(t_j, t_{-j}) > \pi_j(s_j, t_{-j}). \tag{f}$$

If *(f)* holds, then s_j cannot be weakly dominated by t_j as $\Delta^m \subseteq \Delta^{k-1}$. Assume *(e)* holds. Then, s_j is redundant. Then, it follows from the same idea in *(1.15),* *(1.16), (1.17)* that either all strategies in Δ^m/Ψ^n are redundant or the first strategy which is not redundant and eliminated cannot be eliminated by some $t_j \in \Delta^m \cap \Psi^n$.

If the first strategy s_j eliminated from Δ^m/Ψ^n is eliminated by a strategy $t_j \in \Delta^m/\Psi^n$, since $\Delta^m \subseteq \Delta^{k-1}$ and s_j, $t_j \in \Delta^m$, then $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Delta^m_{-j}$. Then, it follows from the same idea in *(1.15), (1.16), (1.17)* that s_j is not weakly dominated by some $t_j \in \Delta^m/\Psi^n$.

Assume s_j is eliminated by some $t_j \notin \Delta^m \cup \Psi^n$, then. Since $t_j \notin \Delta^m$, $\exists g_j \in \Delta^m$ such that $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j})$ for all $g_{-j} \in \Delta^m_{-j}$. If t_j weakly dominates s_j on Δ^m, then $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j}) \geq \pi_j(s_j, g_{-j})$ for all $g_{-j} \in \Delta^m_{-j}$ and $\pi_j(g_j, g_{-j}) \geq \pi_j(t_j, g_{-j}) > \pi_j(s_j, g_{-j})$ for some $g_{-j} \in \Delta^m_{-j}$. *i.e.* g_j also weakly dominates s_j on Δ^m but as $s_j, g_j \in \Delta^m$, we have a contradiction. Thus, $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Delta^m_{-j}$. If $\nexists g_j \in \Delta^m$ such that $\pi_j(s_j, s_{-j}) = \pi_j(g_j, s_{-j})$ for all $s_{-j} \in \Delta^m_{-j}$, then, for each $g_j \in \Delta^m_j$ $\exists s_{-j} \in \Delta^m_{-j}$ such that $\pi_j(s_j, s_{-j}) > \pi_j(g_j, s_{-j})$ As $\forall s_{-j} \in \Delta^m_{-j}$ we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$, then for each $g_j \in \Delta^m_j$ $\exists s_{-j} \in \Delta^m_{-j}$ such that $\pi_j(t_j, s_{-j}) > \pi_j(g_j, s_{-j})$. Hence, $t_j \in \Delta^m_j$. Contradiction. Thus, s_j is not eliminated by some $t_j \notin \Delta^m \cup \Psi^n$ and we cannot have $\Delta^m/\Psi^n \neq \emptyset$ and $\Psi^n/\Delta^m \neq \emptyset$ and $\Delta^m \cap \Psi^n \neq \emptyset$.

On the contrary, suppose $\Delta^m \cap \Psi^n = \emptyset$. Consider the first strategy s_j eliminated from Ψ^n in a given *IEWDS* process. Let $s_j \in \Delta^{k-1}/\Delta^k$. Since $\Psi^n \subseteq \Delta^{k-1}$, for each $t_j \notin \Psi^n$ $\exists s_{-j} \in \Psi^n_{-j}$ such that $\pi_j(s_j, s_{-j}) > \pi_j(t_j, s_{-j})$. Therefore, no $t_j \notin \Psi^n$ can weakly dominate s_j on Δ^{k-1}. Suppose s_j is weakly dominated by some $t_j \in \Psi^n$ on Δ^{k-1}. As $s_j, t_j \in \Psi^n$ and $\Psi^n \subseteq \Delta^{k-1}$, we have $\pi_j(s_j, s_{-j}) = \pi_j(t_j, s_{-j})$ for all $s_{-j} \in \Psi^n_{-j}$. It follows from the same idea in *(1.15), (1.16), (1.17)* that elimination of s_j does not change the payoff structure. Thus, the proof is complete. $\square$

While *EWR* and *IEWDS* suggest payoff equivalent sets of solutions in *TDI* games, one may wonder if one concept prevails over the other when we turn attention to games that don't satisfy *TDI.* We have already seen that *IEWDS* has an advantage over *EWR* when it comes to making predictions about games with 'undercutting problem'. In order to demonstrate that it is not always the case, let's compare *EWR* and *IEWDS* by using our first example in the light of *TDI.* There are two different reductions one can obtain by devising *IEWDS* which are $\{a_3, b_1\}$ and $\{a_3, a_5, b_1\}$. Notice that these reductions are not payoff equivalent as we have $\pi(b_1, a_5) = (6, 2) \neq (6, 3) = \pi(b_1, a_3)$. This is due to the fact that the game does not satisfy *TDI.* We have $\pi_2(b_1, a_3) = \pi_2(b_1, a_5) = 6$, although $\pi_1(b_1, a_3) = 3$ and $\pi_1(b_1, a_5) = 2$. *i.e.* When player 2 changes her strategy from $\{a_3\}$ to $\{a_5\}$ while player 1 is playing $\{b_1\}$, she does not change her own payoff but her opponent's. Nevertheless, we have a unique *EWR* reduction which includes the strategy profile (b_2, a_3) and the associated payoff profile $(3, 7)$, a payoff profile that cannot be obtained by strategy profiles that survive at least one of the two *IEWDS* paths. This counterexample proves two things: First, *EWR* and *IEWDS* are not equivalent solution concepts if we don't restrict attention to *TDI* games but consider the entire set of finite normal-form games. Second, there are games in which *EWR* is order-independent but *IEWDS* is not. An interesting question which might arouse the reader's curiosity as *EWR* and *IEWDS* predict different Nash-Equilibria (NE) in the example mentioned above: Is there any logical relation between Nash-Equilibria deemed possible to arise by *EWR* and *IEWDS*? Under which conditions can one expect a NE to become more likely to be played if the strategies involved survive both concepts of iterative elimination?

Chapter 2

A Characterization of The Proportional Rule

2.1 Motivation and Literature Review

A bankruptcy arises when there is a scarce resource and conflicting claims over it. Since the available resource is in sufficient to honour all the claims, many different suggestons on how to divide may arise. The common aspect of these suggestios is that no claimant gets more than his claim and nobody gets a negative share. A very common example of a bankruptcy problem is the process of liquidation of an insolvent firm among its creditors.

The study of bankruptcy problems has a historical tradition and dates back to the Babylonian Talmud. The Talmud involves two examples regarding bankruptcy situations one of which is about two men conflicting over how to share of a garment. The second one involves a man who leaves mutually inconsistent bequests to his three wifes. Nevertheless, it provides only numerical examples and no generalization. The most popular generalization of those numbers is probably suggested by Aumann and Maschler (1985). Although the problem is very intuitive and old, the formal study of it started as late as O'Neill (1982).Traditionally, there has been two main branches in the literature, axiomatic bankruptcy which aims to characterize bankruptcy rules by some normative but highly regarded properties and game theoretical approach which aims to design games solutions of which coincide with bankruptcy rules. In the present paper, we take the first approach and define *union*−consistency in order to characterize the proportional rule. Consistency and its variations have been widely used for characterizing rules. Young (1987) uses consistency along with equal treatment of equals and continuity to characterize parametric rules. Kaminski (2006) generalizes Young's result. Thomson (2007) develops a technique which determines whether a rule which is defined for 2-agents can be generalized to an arbitrary number of agents Dagan et al. (1997) shows that there may be particular solutions without a consistent extension. Moulin (2000) uses consistency in order to obtain a joint characterization of the proportional rule, the constrained equal awards rule and the constrained equal losses rule. Chun (1999) proves that a consistent rule also satisfies converse consistency Herrero and Villar (2001) considers the same class of rules as in Moulin (2000) and obtain separate characterization results for the constrained equal awards rule, the constrained equal losses rule and the Talmud rule. Since our *union*−Consistency is a group property, it is noexaggeration to say that it is loosely related to Chambers and Thomson (2002) which study group order preservation. They also use a similar approach to ours in their proof.That is why they require claims continuity as well. They characterise PROP. as our paper do but they use group order preservation, claims continuity and consistency. Since claims continuity and consistency are satisfied by a large class of rules, the key assumption in their result is group order preservation which is defined as "given two groups of claimants, suppose that the sum of the claims of the members of the first group is greater than or equal to the sum of the claims of the members of the second group. Then, similar inequalities should hold for the sums of the awards to the members of the two groups, and for the sums of the losses incurred by the members of two groups", as it appears in their paper. However, the assertion of union consistency does not stick to the original claims problem, as the union of groups chosen for comparison does not have to be equal to that of the original problem‘s, but rather decides what each group should get by evaluating the conflicting claims of the groups in a new problem created by considering the sum of the claims of the members of each group as an individual claim. In addition, union consistency requires an equality between the awards received by the groups whereas group order preservation requires a "greater than or equal to". Dagan and Volij (1997) also prove results on extensions of bilateral rules using consistency and average consistency. They analyze how to extend a given bilateral principle to a unique consistent rule and relate it to a family of binary relations.

It is worth mentioning that Thomson (2003) provides a detailed survey on bankruptcy rules.

2.1.1 *union*−Consistency

Definition 6. (*Claims Problem*) *A claims problem is an ordered pair* $(E; d) \in \mathbb{R}_+ \times \mathbb{R}^n_{++}$ *where* $d = (d_1, ..., d_n)$ *and* $\sum_{i=1}^n d_i > E$.

We denote by $N = \{1, 2, ..., n\}$ a set of claimants each of whom has some claim on an endowment E. The class of claims problems involving n agents and the class of all claims problems are denoted by D^n and D, respectively. *i.e* $D = \cup_{N \in \eta} D^N$ where η is the set of all non-empty subsets of $\mathbb{N}$ where it denotes natural numbers.

Definition 7. *An n-tuple vector* $x = (x_1, x_2, ..., x_n)$ *is said to be a solution to or an allocation for the claims problem* $(E; d)$ *if*

i) $0 \leq x_i \leq d_i$ *for all* $i \in N$,

ii) $\sum_{i \in N} x_i = E$.

Here, x_i is interpreted as claimant i's share from E. Let's denote the family of allocations for a given claims problem $(E; d)$ by $A(E; d)$.

Definition 8. (*Bankruptcy Rule*) *A bankruptcy rule* $\phi : D \to \cup_{N \in \eta} R_+^N$ *is a function that maps each each claims problem to an allocation.*

Definition 9. (*Proportional Rule*) *For each* $(E; d) \in D$*,* $PROP(E; d) \equiv (E.\frac{d_i}{\sum_{j \in N} d_j})_{i=1}^n$

Definition 10. (*Anonymity*) *A rule* ϕ *satisfies anonymity if for each* $(E; d) \in D$*, each* $\pi \in \Pi^N$ *and each* $i \in N$*,* $\phi_{\pi(i)}((E; d_{_{\pi(i)}}) = \phi_i(E; d)$ *where* Π^N *denotes the class of bijections from* N *into itself.*

Definition 11. (*Claims Continuity*) *A rule* ϕ *satisfies claims continuity if for each sequence* $\{(E^v; d^v)\}_{v=1}^{\infty}$ *of elements of* D^n *and each* $(E; d) \in D^n$*, if*$(E^v; d^v) \to (E; d)$ *and for each* $v \in \mathbb{N}$ $E^v = E$*, then* $\phi(E^v; d^v) \to \phi(E^v; d^v)$*.*

Definition 12. (*Union−consistency*) *A rule* ϕ *is said to be union−consistent iff for each* $(E; d)$ *and for any* 2 *non-empty, disjoint subsets* S_1 *and* S_2 *of* N *we have;*

$$\sum_{i \in S_1} \phi_i(E; d) = \phi_{S_1}(\sum_{j \in S_1 \cup S_2} \phi_j(E; d); (\sum_{i \in S_1} d_i, \sum_{j \in S_2} d_j))$$

In a sense, union consistency requires fairness among all subgroups. The intuition is that some group of claimants may appeal to court claiming that some other group is favoured against them. From a different perspective, addition of new claimants and extra endowment for the new claimants should not favour one group over the other.

Proposition 1. *PROP is the only rule that satisfies both anonymity, union-consistency and continuity.*

Proof. $(Sufficiency)$ $PROP$ is trivially anonymous. For *union*−consistency, consider an arbitrary $(E; d) \in D$. For any S_1, S_2 with $S_1 \cap S_2 = \emptyset$, $S_1, S_2 \neq \emptyset$ and

$S_1, S_2 \subset N$, we have

$$\sum_{i\in S_1} PROP_i(E;d) = \sum_{i\in S_1\cup S_2} E.\frac{d_i}{\sum_{j\in N} d_j} PROP_1\left(E.\sum_{i\in S_1\cup S_2}\frac{d_i}{\sum_{j\in N} d_j};(\sum_{i\in S_1} d_i, \sum_{j\in S_2} d_j)\right)$$
$$= E.\sum_{i\in S_1\cup S_2}\frac{d_i}{\sum_{j\in N} d_j}.\frac{\sum_{i\in S_1} d_i}{\sum_{j\in S_1\cup S_2} d_j} = \frac{E.\sum_{i\in S_1\cup S_2} d_i}{\sum_{j\in N} d_j}.$$

(*Necessity*) On the contrary, let R satisfy continuity and *union*−consistency. Assume that $\sum_{i\in N} d_i$ is rational and there exists a rational d_j for some $j \in N$. For an arbitrary $(E;d)$, consider the following 3 problems;

i) $(E;(d_1, d_2, ..., d_n))$,

ii) $(E;(d_1, \sum_{j=2}^{N} d_j))$,

iii) $(E;(d_1^{'}, d_2^{'}, ..., d_k^{'}, ...d_{k+M}^{'}))$

where $d_i^{'} = d_j^{'}$ for all $i, j \in \{1, 2, ..., k+M\}$ and $\sum_{j=1}^{k} d_j^{'} = d_1$, $\sum_{j=k+1}^{k+M} d_j^{'} = \sum_{j=2}^{N} d_j$.

Notice that *iii*) is well defined due to our assumption.(By anonymity, we can assume d_1 is rational. Set $d_1 = \frac{a}{b}$ and $\sum_{j=2}^{N} d_j = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}_+$. Then, $\sum_{i\in N} d_i = \frac{a.d+b.c}{b.d}$

Consider $(d_1^{'}, d_2^{'}, ..., d_k^{'}, ..., d_{k+m}^{'})$ where $d_i^{'} = d_j^{'} = \frac{1}{b.d}$ for all $i, j \in \{1, 2, ..., k+M\}$ and $\sum_{j=1}^{k} d_j^{'} = \frac{a}{b}$, $\sum_{j=k+1}^{k+M} d_j^{'} = \sum_{j=2}^{N} \frac{c}{d}$ i.e. $\frac{k}{b.d} = \frac{a}{b} \Rightarrow k = \frac{a/b}{1/b.d} = a.d \in \mathbb{Z}_+$ and $M = \frac{c/d}{1/b.d} = c.b \in \mathbb{Z}_+$. i.e. we have a positive integer number of claimants. However, if either d_i is irrational for all $i \in N$ or $\sum_{i\in N} d_i$ is irrational, then this is not necessarily the case.)

Since by anonymity, $R_i(E;(d_1^{'}, d_2^{'}, ..., d_{k+M}^{'})) = R_j(E;(d_1^{'}, d_2^{'}, ..., d_{k+M}^{'})) = E.\frac{1}{k+M} = E.\frac{1}{a.d+b.c}$, we have by *union*−consistency,

$$R_1(E;(d_1, d_2, ..., d_n) = R_1(E;(d_1, \sum_{j=2}^{N} d_j)) = \sum_{j=1}^{k} R_j(E;(d_1^{'}, d_2^{'}, ..., d_{k+M}^{'}))$$
$$= \sum_{j=1}^{k} \frac{E}{a.d+b.c} = \frac{a.d}{a.d+b.c}.E = E.\frac{d_1}{\sum_{j\in N} d_j} = PROP_1(E;(d_1, d_2, ..., d_N))$$

as desired.

Let either $\sum_{i\in N} d_i$ or each d_i be irrational. Either case, we have at least one irrational d_i. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $\left(d_j^v\right)_{j\in N}$ such that d_j^v is rational and $d_j^v \to d_j$ where d_j is irrational. We also have $d \to d^* \Leftrightarrow d_i \to d_i^*$ for each $i \in N$. Let $(E; d^v)$ be sequence of claims problems with $d_i^v \in \mathbb{Q}$ for $i = 1, 2, ..., N$ and $v = 1, 2, ...$ and $(E; d^v) \to (E; d)$. Since R is *union*-consistent and continuous, we have $R(E; d^v) = PROP(E; d^v)$ and $\lim_{d^v\to d} PROP(E; d^v) = \lim_{d^v\to d} R(E; d^v) = R(E; d) = PROP(E; d)$ as $PROP$ is continuous.

$\square$

Chapter 3

Risk Averse Investors Behavior towards a Risky Firm

We extend the investment game which was first suggested in Kıbrıs and Kıbrıs (2013). In the model, there is an arbitrary number of agents each of which is endowed with a Constant-Absolute Risk Aversion(CARA) utility function. The agents are presented the opportunity to invest in a risky project whose outcome will either be success or be failure based on a Bernoulli Distribution function. In case of a failure, total value of the investment is allocated among the agents according to a bankruptcy rule. Kıbrıs and Kıbrıs (2013) compare 4 most common rules in the literature whereas we relax the assumption that no agent can receive more than his investment, in case of a failure. By doing so, we are allowing the agents to receive amounts which are not possible under standard bankruptcy rules. Hence, we are looking for optimality in a larger set of possible payoffs. In real world situations, there are different types of agents some of which can receive gains on investment even in the case of a failure. For instance, as opposed to shareholders, some creditors may receive more than what they put in. In that sense, our model can be considered as a better approximation to reality. We also drop the assumption that each agent will receive a constant interest rate in case of a success. Our aim is to account for all possible bankruptcy rules, and even other allocations which do not adhere to properties that are imposed by bankruptcy rules, and to determine the optimal allocation a social planner would choose without violating the individual rationality constraints. We first show that only the total amount of investment matters, *i.e.* as long as the level of total investment is the same, how much each investor contributes has no influence on welfare and proceed to show that the optimal allocation can be obtained via a free-market mechanism in which agents can trade payments in different states of the world. We also show that no matter which rule is announced by the social planner, the optimal amount of total investment is equal to the amount which also emerges from the competitive game that utilises the proportional rule.

Following the main idea of Kıbrıs and Kıbrıs (2013) we investigate the Pareto Optimal allocations regarding a joint venture(risky firm).

3.1 The Setting of Kıbrıs and Kıbrıs (2013)

Denote the set of investors/agents by $N = \{1, \dots, n\}$. Each agent i's preferences are represented by a constant-absolute risk aversion (CARA) utility function $u_i : \Re_+ \to \Re$ where $u_i(x) = -e^{-a_i x}$. Here, a_i is agent i's risk-aversion constant and their analysis relies on the assumption that $a_i > 0 \quad \forall i \in N$, that is, the agents are risk averse. Also, $a_1 \leq ... \leq a_n$ without loss of generality.

There is a risky firm whose ex-ante value is determined by the total value of investments made by the agents.*i.e.,* $\sum\limits_i s_i$ where s_i is the investment of agent i. Furthermore, agents choose their investments $s_i \in \Re_+$ simultaneously. They borrow from an outside market and the interest rate is normalized to 0. There are 2 states of the world: with probability $p \in (0, 1)$ the firm succeeds and the total value of the firm becomes $(1 + r) \sum_i s_i$ and with $(1 - p)$ probability it fails and the total value shrinks to $\beta \sum\limits_i s_i$. It is assumed that $r > 0$ and $0 < \beta < 1$. After the realization of the state of the world, the total value is to be distributed among the agents.

The division method used at this point might capture some desirable properties. For example:

1. Given the method, this competitive game might lead to a Pareto Optimal allocation in either the utilitarian sense or the egalitarian sense.
2. It might maximize the volume of investments.

Kıbrıs and Kıbrıs (2013) do not check for Pareto Optimality but rather compare a limited subset of possible division methods evaluated according to utilitarian and egalitarian welfare. While employing the same notions in order to measure welfare, we consider the set of all possible allocations of the ex-post total value and, therefore, characterize the Pareto optimal ones.

Moreover, the welfare levels in Kıbrıs and Kıbrıs (2013) concern only 2 agents with equal Pareto weights (the latter is valid for utilitarian welfare).

Their definitions as they appear in their paper are as follows:

Definition 13 (Utilitarian Social Welfare)**.** *The utilitarian social welfare at the Nash Equilibrium (NE, henceforth) of the 2 agent game induced by an allocation rule F is given by;*

$$UT^F(p, r, \beta, a_1, a_2) = U_1^F(\varepsilon(G^F)) + U_2^F(\varepsilon(G^F))$$

where the game G^F *is defined by the parameters* (p, r, β, a_1, a_2), $U_i^F(\cdot)$ *is agent i's utility and* $\varepsilon(G^F)$ *is an equilibrium of the game.*

Definition 14 (Egalitarian Social Welfare)**.** *The egalitarian social welfare at the N.E. of the 2 agent-game induced by an allocation rule F is given by*

$$EG^F(p, r, \beta, a_1, a_2) = \min\left\{U_1^F(\varepsilon(G^F), U_2^F(\varepsilon(G^F))\right\}$$

where again the game G^F*is defined by the parameters* (p, r, β, a_1, a_2) *and* $\varepsilon(G^F)$ *is* *an equilibrium; it is the minimum utility an agent gets at an equilibrium induced by F.*

3.1.1 Common Allocation (Bankruptcy Rules)

Bankruptcy occurs in the state of the world where the value of output is β of the initial investment, but Kıbrıs and Kıbrıs (2013) assume that in the other state each agent gets the full return $(1+r)$ times their investment.

Definition 15 (Proportional Rule)**.** *For each $i \in N$, $pro_i(s) = \beta s_i$ where s_i is the investment choice of agent $i \in N$ and s is the investment vector.i.e. each agent gets a proportion β of his claim where we equate "claim" with the value of the initial investment.*

Definition 16 (Constrained Equal Losses)**.** *For each $i \in N$, $CEL_i(s) = \max\{s_i - \varphi, 0\}$ where $\varphi \in \Re_+$ satisfies $\sum_{i \in N} \max\{s_i - \varphi, 0\} = \beta \sum_{i \in N} s_i$.*

Note that in order to find equilibrium under CEA and CEL, Kıbrıs and Kıbrıs first prove that not all combinations of parameters lead to a NE under CEL/CEA. They then show that for the 2 agents case if there exists a NE it must be identical to those of EL/EA respectively.where EL/EA are allocations such that agents lose/receive the same amount.

Definition 17. *For each $i \in N$, $CEA_i(s) = \min\{s_i, \varphi\}$ where $\varphi \in \Re_+$ satisfies $\sum\limits_{i \in N} \min\{\varphi, s_i\} = \beta \sum\limits_{i \in N} s_i$.*

Proposition 2 (Kıbrıs and Kıbrıs (2013))**.** *If $\ln\left(\frac{rp}{(1-p)(1-\beta)}\right) \leq 0$, then $Game^{PROP}$ has a unique dominant strategy equilibrium $(0, ..., 0)$. Otherwise, the game has a unique dominant strategy equilibrium s^* in which each agent i chooses a positive investment level s_i^* given by*

$$s_i^* = \frac{1}{a_i(r+1-\beta)} \ln\left(\frac{rp}{(1-p)(1-\beta)}\right)$$

3.2 Our Setting

We relax the assupmtion that no agent can receive more than his investment, in case of a failure. We also drop the assumption that each agent will get an interest rate of r, in case of success. Our aim is to determine the optimal allocation social planner (S.P. hereafter) chooses without violating the individual rationality constraints.

Therefore, we first present the social planner problem for the 2 agent case without imposing the constraints $F_i^s \geq 0$ and $F_i^f \geq 0$ where F_i^s and F_i^f represent player i's shares in the case of success and failure respectively. We shall see that they are not binding for a fairly large set of parameters.

We solve the problem in 2 steps for simplicity: First, we assume that agents' investment levels (x_1, x_2) are given and find the allocations in the case of success and failure as a function of them. Then, we see that what matters to the agents is not the individual investments but rather the total investment. Then, we solve the total utility maximization problem with respect to the total investment.

3.2.1 Social Planner's Problem for 2 Agents

$$\max \lambda_1 \left[pu_1(F_1^s - x_1) + (1-p)u_1(F_1^f - x_1) \right] + \lambda_2 \left[pu_2(F_2^s - x_2) + (1-p)u_2(F_2^f - x_2) \right]$$

such that

$$F_1^s + F_2^s = (1+r)(x_1 + x_2) \quad (1)$$

and

$$F_1^f + F_2^f = \beta(x_1 + x_2). \quad (2)$$

Alternatively, we can plug (1) and (2) in and solve the unconstrained problem and if $0 \leq F_1^s \leq (1+r)(x_1 + x_2)$ and $0 \leq F_1^f \leq \beta(x_1 + x_2)$ then the constraints

$F_i^s \geq 0$ and $F_i^f \geq 0$ for $i = 1, 2$ are not binding Then, the Lagrangian is;

$$\begin{aligned} L &= \lambda_1 \left[-p.e^{-a_1 \left(F_1^s - x_1 \right)} + (1-p)(-e^{-a_1 \left(F_1^f - x_1 \right)}) \right] + \\ & \quad \lambda_2 \left[-p.e^{-a_2((1+r)(x_1+x_2) - F_1^s - x_2)} + (1-p)(-)e^{-a_2(\beta(x_1+x_2) - F_1^f - x_2)} \right]. \end{aligned}$$

Then solving for the 1st order conditions we find:

$$F_1^s = \frac{\ln(\frac{a_1\lambda_1}{a_2\lambda_2}) + a_2(1+r)(x_1+x_2) - a_2x_2 + a_1x_1}{(a_1+a_2)},$$

$$F_2^s = (1+r)(x_1+x_2) - \frac{\ln(\frac{a_1\lambda_1}{a_2\lambda_2}) + a_2(1+r)(x_1+x_2) - a_2x_2 + a_1x_1}{a_1+a_2},$$

$$F_1^f = \frac{\ln(\frac{a_1\lambda_1}{a_2\lambda_2}) + a_2\beta(x_1+x_2) - a_2x_2 + a_1x_1}{a_1+a_2},$$

and

$$F_2^f = \beta(x_1+x_2) - F_1^f = \beta(x_1+x_2) - \frac{\ln(\frac{a_1\lambda_1}{a_2\lambda_2}) + a_2\beta(x_1+x_2) - a_2x_2 + a_1x_1}{a_1+a_2}.$$

This is the solution to the planner's problem given x_1 and x_2. Now we consider the prior choice of investments. Set $\Psi = (x_1 + x_2)$.

We then have

$$\begin{aligned} u_1(F_1^s - x_1) &= -e^{-a_1(\ln(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{1}{a_1+a_2}})}.e^{\frac{-a_1a_2r\Psi}{a_1+a_2}} \\ &= -e^{\frac{-a_1a_2r\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{-a_1}{a_1+a_2}} \end{aligned}$$

$$u_2(F_2^s - x_2) = -e^{\frac{-a_1a_2r\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{a_2}{a_1+a_2}}$$

$$u_1(F_1^f - x_1) = -e^{\frac{-a_1a_2(\beta-1)\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{-a_1}{a_1+a_2}}$$

$$u_2(F_2^f - x_2) = (-1).e^{\frac{-a_1a_2(\beta-1)\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{a_2}{a_1+a_2}}.$$

It is worth mentioning that x_1 and x_2 enter only through Ψ. The intuition that the production function is additive in their investments, and that the preferences are quasi-linear –*i.e.* the fact that return/loss less investment, all goes inside the utility function. The S.P. chooses Ψ to maximize

$$\lambda_1\left[-pe^{\frac{-a_1a_2r\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{-a_1}{a_1+a_2}}.e^{-a_1a_2r\Psi/a_1+a_2}-(1-p)e^{\frac{-a_1a_2(\beta-1)\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{-a_1}{a_1+a_2}}\right]+$$
$$\lambda_2\left[-pe^{\frac{-a_1a_2r\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{a_2}{a_1+a_2}}-(1-p)e^{\frac{-a_1a_2(\beta-1)\Psi}{a_1+a_2}}.(\frac{a_1\lambda_1}{a_2\lambda_2})^{\frac{a_2}{a_1+a_2}}\right]. \qquad (3)$$

We then get:

$$\arg\max(3)=\Psi^*=\begin{cases}\ln(\dfrac{rp}{(1-p)(1-\beta)})\cdot\dfrac{a_1+a_2}{a_1+a_2(1-\beta+r)} & \text{for } rp\geq(1-p)(1-\beta)\\ 0 & \text{otherwise}\end{cases}$$

Ψ^* is independent of λ_i as the quasi-linear structure implies that surplus is maximised independently of λ_i but the Pareto weights determine the split of surplus.

Note that $\Psi^*$is the total investment level in the two agent $Game^{\Pr op}$. Intuitively, if the condition $rp\geq(1-p)(1-\beta)$ holds, on average the project yields a nonnegative return, and when this is strict, a small positive investment will yield a positive first-order gain, while the increase in variance will be second-order, so investment must be positive.

Remark 1. *One should notice that the P.O. allocation might yield utility levels different from those of the competitive game under PROP. With 2 agents. For them to be equal,*

$$\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right)^{a_2/a_1+a_2}=\left(\frac{a_1\lambda_1}{a_2\lambda_2}\right)^{-a/a_1+a_2}=1$$

must hold. Since the S.P. is concerned with sum of utilities, he exploits the fact *that more risk-averse agents are more productive in the case of success and vice versa*

In order to have a feasible allocation, no agent should receive a negative payment in any state of the world. The allocation is feasible if it satisfies the following:

1. $0 \leq F_1^s \leq (1+r)\Psi$,

2. $F_1^s + F_2^s = (1+r)\Psi$,

3. $0 \leq F_1^f \leq \beta\Psi$

4. $F_1^f + F_2^f = \beta\Psi$

One can check that the four conditions above hold for a very large portion of the parameter space.

Proposition 3. *If there are N agents playing the investment game and the S.P.chooses investment levels as well as the allocations in both states of the world by respecting the individual rationality constraints, then the total investment $\Psi^* = 0$ if $pr < (1-p)(1-\beta)$. Otherwise,*

$$\Psi^* = \frac{\ln(\frac{pr}{(1-p)(1-\beta)}) . \sum_j {}^1\!/_{a_j}}{1-\beta+r}$$

i.e. the level of investment under $Game^{PROP}$ is preserved. However, depending on the S.P.'s preferences, $PROP$ is not th only P.O allocation. Moreover, individual levels of investment do not matter.

3.2.2 Social Planner's Problem for N Agents

$$\max_{F_i^s, F_i^f} p \sum_i \lambda_i u_i(F_i^s - x_i) + (1-p) \sum_i \lambda_i u_i(F_i^f - x_i)$$

where $\sum_i F_i^s = (1+r)\sum_i x_i$, $\sum_i F_i^f = \beta \sum_i x_i$ and $F_i^j \geq 0$, $j = s, f$. For all $i \in N$, the last set of constraints, however, might not be binding. Therefore, we ignore them at this point and check whether the condition is satisfied after the maximization.

Solving for the first order conditions, we find:

$$F_i^s = x_i - \frac{\ln\left({}^{\alpha_1}\!/_{p\lambda_i a_i}\right)}{a_i}$$

and

$$F_i^f = x_i - \frac{\ln\left({}^{\alpha_2}\!/(1-p)\lambda_i a_i\right)}{a_i}$$

for all $i \in N$ where

$$\alpha_1 = \sqrt[\frac{1}{\sum_i \frac{1}{a_i}}]{e^{-r.\sum_i x_i.} \prod_i (p\lambda_i a_i)^{\frac{1}{a_i}}}$$

and

$$\alpha_2 = \sqrt[\frac{1}{\sum_i {}^1\!/a_i}]{e^{(1-\beta).\sum_i x_i.} \prod_i ((1-p)\lambda_i a_i)^{{}^1\!/a_i}}.$$

As in the two agent case, the utilities depend on $\Psi = \sum_i x_i$ and not on individual investments levels.

Hence, the S.P. will choose the social utility maximising level of Ψ, i.e.,

$$\arg\max \sum_i \quad \lambda_i[-p.e^{\left(\frac{-r\Psi}{\sum_i \frac{1}{a_i}} + \sum_k [\ln(p\lambda_k a_{ki})]^{\frac{1}{a_k}} \sum_j \frac{1}{a_j} - \ln(p\lambda_i a_i)\right)}$$
$$-(1-p).e^{\left(\frac{(1-\beta)\Psi}{\sum_j \frac{1}{a_j}} + \sum_k \left[\ln((1-p)\lambda_{ki} a_{ki}^{\frac{1}{a_k \sum_j \frac{1}{a_j}}}\right]\right) - \ln(p\lambda_i a_i)}$$

which yields the optimal level of total investment:

$$\Psi^* = \frac{\ln(\frac{pr}{(1-p)(1-\beta)}).\sum_j {}^1\!/a_j}{1-\beta+r}$$

which is equal to the total value of investments in the proportional game.

Note: Notice that if $pr < (1-p)(1-\beta)$ then Ψ^* is not determined. This is because the investment will have a negative expected value in this case and it is optimal not to invest anything at all.

Since the agents are borrowing from outside, S.P. only decides how much each agent will lose or gain. The payment scheme depends on the individual investment so that the earnings don't depend on individual investments but on the sum of investments. (Moreover, there are no externalities in $Game^{PROP}$ so the S.P. is effectively maximizing each agent's utility separately, that's why our result holds and Kıbrıs and Kıbrıs (2013) find a dominant strategy equilibrium).

Proposition 4. *Among the class of bankruptcy rules, PROP is P.O, but it is not unique.*

Another important issue is whether PROP. is a P.O. rule and whether it is unique. In order to see if that‘s the case, we are going to consider all possible allocation schemes in the case of a failure. *i.e.* we will not be interested in how the agents share the surplus in the case of success as in that scenario there is going to be a higher income to share than the amount invested in the venture. Translating this to the language of bankruptcy, the endowment to be distributed will exceed the sum of the claims, hence it will be possible to jointly honour all the claims. Nevertheless, we need to make an assumption on how the success profits will be divided since agents' utilities also depend on this variable. One may expect to see different levels of losses for each agent depending on what the S.P. assigns to each of them in the case of success. Since we have assumed that the production technology is quasi-linear, we may start by assuming that each agent will receive a profit equal to "r" portion of her investment. In other words, we are going to impose proportional allocation of gains. If we do so, however, we can intuitively anticipate proportional division in the case of failure as the ability of S.P. to allocate will be limited to the failure event. *i.e.* even if one of the agents promise a higher marginal utility than the other if the S.P. transfers an infinitesimal unit from the latter to the former when the venture is successful and vice versa in the failure case, since transfers in the case of success won‘t be allowed, an allocaton different than PROP. might lead to a marginal utility discrepancy. Thus, it appears to be reasonable to predict that the S.P. will want to make marginal utilities of agents match with each oter and choose proportional allocation as the utilities are CARA.

One other thing is that the feasibility restrictions have to be tightened up. Instead of requiring that no agent receives more than what's available in the case of success, we confine each agent's maximum earning to a proportion of her investment. The constraints defined for the failure case, however, will remain the same. Therefore, we revisit our baseline model with 2 agents. The S.P.'s problem is :

$$\max_{F_1^f, F_2^f} \lambda_1[pu_1(rx_1) + (1-p)u_1(F_1^f - x_1)] + \lambda_2[pu_2(rx_2) + (1-p)u_2(F_2^f - x_2)]$$

such that

$$F_1^f + F_2^f = \beta(x_1 + x_2) \quad (1)$$

and

$$0 \leq F_1^f \leq \beta(x_1 + x_2). \quad (2)$$

Plugging $F_2^f = \beta(x_1 + x_2) - F_1^f$ in the equation, we write the Lagrangian:

$$\begin{aligned} L \;=\; & \lambda_1[-p.e^{-a_1 r x_1} + (1-p)(-e^{-a_1\left(F_1^f - x_1\right)}] + \\ & \lambda_2[-p.e^{-a_2 r x_2} + (1-p)(-)e^{-a_2(\beta(x_1+x_2)-F_1^f - x_2)}] + \\ & \mu_1 F_1^f + \mu_2(\beta(x_1 + x_2) - F_1^f) \end{aligned}$$

which gives optimality and complementarity conditions;

$$\begin{aligned} a_1\lambda_1(1-p)(-e^{-a_1\left(F_1^f-x_1\right)} - a_2\lambda_2(1-p)e^{-a_2(\beta(x_1+x_2)-F_1^f-x_2)} + \mu_1 - \mu_2 &= 0 \quad (1)\\ \mu_1 F_1^f &= 0 \quad (2)\\ \mu_2(\beta(x_1+x_2)-F_1^f) &= 0 \quad (3)\\ -F_1^f &\leq 0 \quad (4)\\ F_1^f-\beta(x_1+x_2) &\leq 0 \quad (5)\\ \mu_1,\mu_2 &\leq 0 \quad (6) \end{aligned}$$

Then; from (2) and (3), one of the following must hold;

$$\begin{aligned} \mu_1 &= \mu_2 = 0 \quad (1)\\ \beta(x_1+x_2)-F_1^f &= \mu_1 = 0 \quad (2)\\ \beta(x_1+x_2)-F_1^f &= F_1^f = 0 \quad (3)\\ F_1^f &= \mu_2 = 0 \quad (4) \end{aligned}$$

(3) can be satisfied only if the S.P. decides to make no investment at all. Since it may occur only depending on the parameters of the production technology and once we have such parameters, there is not going to be any investment regardless of the risk aversion coefficients or the allocation rule used.Therefore, we rule out (3). If (2) or (4) holds, then we have a corner solution and the S.P.allocates all the remaining endowment to the first agent in case (2), and to the second agent in case (4). Since we have fixed the returns in the success case and for each agent the return cannot be greater than her investment, as there is the risk losing all the investment, it yields negative expected returns for one of the agents and violates individual rationality. With such a rule, regardless of whether the S.P. chooses the investment levels or the agents do, the optimal(or the chosen) investment

would be 0 for one of the agents. (Actually, it would be 0 for the more risk averse investor) Hence, we have $\mu_1 = \mu_2 = 0$. Plugging $\mu_1 = \mu_2 = 0$ in (1) and solving for F_1^f we find:

$$F_1^f = \frac{\ln(\frac{a_1\lambda_1}{a_2\lambda_2}) + a_2\beta(x_1 + x_2) - a_2x_2 + a_1x_1}{a_1 + a_2}$$

which is the same equation we found without fixing the success returns. The reason is that the S.P. treats the two different events separately. Above equation also shows that one can find λ_1, λ_2 such that PROP. is a P.O. rule but it is not unique.

Note that the same result also applies to the case with an arbitrary number of agents.

3.2.3 Difficulties in Applying the P.O. Allocation

In order to apply the P.O. outcome in this setting, the S.P. has to have complete knowledge about the agents' preferences over the lotteries generated by this risky venture. In a realistic marketplace where there are several potential investors, it is too optimistic to say that it is a realistic assumption. Therefore, in order to apply these levels of investments and allocations, the S.P. has to come up with a mechanism which will reveal the preferences correctly.

A solution to this problem might be a free market mechanism where agents can trade their potential gains and potential losses. Since at a given allocation, it is possible to see different marginal utilities, even each agent is receiving exactly the same payments in each state of the world, agents with different risk attitudes might benefit from such trade.

3.2.4 Trading the Different Payments in Different States of the World

Once the investment levels and the corresponding allocation scheme is determined, all agents know how much they are going to be paid in two different outcomes. Regarding the type of utility functions they have, one can view them as traders in a marketplace with two goods as endowment: the payment in the case of "success" and the payment in the case of "failure". Once these two are allowed to be exchanged, we will end up with a pure exchange economy after the allocation is determined. Hence, we will be able to apply two renowned theorems in order to justife this method.

Definition 18. ***(Pure Exchange Economy)*** *A pure exchange economy ε with consumption space X and set of agents N, is*

$\varepsilon = \{e_i, \succ_i\}_{i \in N}$, where $e_i \in X$ and $\succ \in X \times X$ denote the agent i's initial endownment and preferences, respectively.

Theorem 3. ***(First Theorem of Welfare Economics)*** *For any exchange economy ε, where agents preferences are given by continuous, complete, preorders satisfying local non-satiation CE(E) $\subseteq$PO(E).*

Definition 19. *(Local Non-Satiation) A preference relation $\succ \in X \times X$ is locally non-satiated if for all $x \in X$ and all $\varepsilon > 0$ there exists $y \in X$ with $\|y - x\| < \varepsilon$ and $y \succ x$.*

Since the agents in our setting have monotonic prefrences and local non-satiation is implied by monotonicity, one of the requirements to be able to apply the theorem is automatically satisfied.

Theorem 4. *(Social Planner's Problem and Pareto Weights) If every agent i's preferences are represented by a continuous, strictly increasing and concave utility function $u_i : X \to \Re(0 \in X)$, and $e_i >> 0$ then an allocation $x^* \in X^n$ is P.O. if* *and only if there exists* $\theta^* \in \Delta(N)$ *and* x^* *solves the social planner's problem for* ε *at* θ^* *given by* $\max_{x \in F(\varepsilon)} \sum_{i \in N} \theta_i^* u_i(x_i)$.

It is easy to verify that $u_i(.)$ is continuous and strictly increasing in both arguments.Thus, it is enough to check for concavity.

Theorem 5. *(Checking for Concavity) A twice continuously differentiable function f is concave if the Hessian matrix H(x) is negative semi-definite at all points x.*

As a result, we need to check for the eigenvalues of the matrix $H(s, f) =$

$$\begin{matrix} {}^{\partial^2 u_i}/_{\partial s^2} & {}^{\partial^2 u_i}/_{\partial f \partial s} \\ {}^{\partial^2 u_i}/_{\partial s \partial f} & {}^{\partial^2 u_i}/_{\partial f^2} \end{matrix}$$

Where the endownments in the case of "success" and "failure" are denoted by s and f, respectively.

Since the eigenvalues are all negative and the utility functions are concave, then we conclude that the preferences satisfy the desired properties and, thus, the theorems hold. As a result, the price mechanism is efficient and will correct any error the S.P. makes when allocating the liquidated firm.

3.3 Conclusion

The model presented in Kıbrıs and Kıbrıs (2013) does not account for some of the situations which occur in real life. Namely, they do not allow for any investor to receive more than her claim. Moreover, they conduct their analysis for only the 4 most common rules in the literature. However, these are the most favoured rules in the literature. All of them are studied and suggested by various authors on both axiomatic and game theoretic grounds. In order to adopt a more realistic approach to risky investments, we assume that investors' possible gains are not determined by individual contributions. Yet, we assume the same sort of utility function which is also used in the analysis of Kıbrıs and Kıbrıs (2013). This leads to the result that the socially optimal level of welfare is not dependent on individual levels of investment, although the investors have different attitudes towards risk. We get this result by exploiting the fact that the production function is additive in investors' contributions and that the preferences are quasi-linear.

On a modified version of the model, we impose that each agent‘s profit is a fixed fraction of her investment in the case of success. On the other hand, we leave the constraints for the failure case unchanged and solve the S.P.‘s problem. By doing so, not only do we show that PROP. is P.O. among the class of bankruptcy rules but we also establish that it is a P.O. way of distributing the ex-post welfare even if the S.P. is not restricted to give each agent at most as much as she invests. This may explain why PROP. is also used in real ventures. However, it is noteworthy that PROP. is not the only P.O. way of allocation. In fact, we have an abundance of P.O. allocations. Moreover, since different risk aversion coefficients require different Pareto weights, given the Pareto weights of the S.P., a different allocation needs to be chosen since no allocation is P.O. for all levels of risk-aversion.

In that case, S.P. encounters the problem of having incomplete information about agents‘ preferences. In order to cope with the problem, we suggest a market mechanism where agents can trade their potential gains and potential losses. freely. By making use of some well-known results, we conclude that the mechanism ends up with a P.O. allocation.

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