RELATIONSHIP BETWEEN LEARNERS' MATHEMATICS-RELATED BELIEF SYSTEMS AND THEIR APPROACHES TO NON-ROUTINE MATHEMATICAL PROBLEM SOLVING: A CASE STUDY OF THREE HIGH SCHOOLS IN TSHWANE NORTH DISTRICT (D3), SOUTH AFRICA

by

MUNYARADZI CHIROVE

submitted in accordance with the requirements for the degree of

DOCTOR OF PHILOSOPHY

in the subject

MATHEMATICS EDUCATION

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROF L D MOGARI

JUNE 2014

Declaration

I declare that the thesis "Relationship between learners' mathematically-related belief systems and their approach to mathematical problem solving: A case study of three High Schools in Tshwane North District (D3), South Africa" is my own work and that all the sources I have used or quoted have been acknowledged by means of complete references.

---------------------------------   -------------------------------------

MR CHIROVE, M.   DATE

Abstract

The purpose of this study was to determine the relationship between High School learners' mathematics-related belief systems and their approaches to mathematics non-routine problem-solving. A mixed methods approach was employed in the study. Survey questionnaires, mathematics problem solving test and interview schedules were the basic instruments used for data collection.

The data was presented in form of tables, diagrams, figures, direct and indirect quotes of participants' responses and descriptions of learners' mathematics related belief systems and their approaches to mathematics problem solving. The basic methods used to analyze the data were thematic analysis (coding, organizing data into descriptive themes, and noting relations between variables), cluster analysis, factor analysis, regression analysis and methodological triangulation.

Learners' mathematics-related beliefs were grouped into three categories, according to Daskalogianni and Simpson (2001a)'s macro-belief systems: utilitarian, systematic and exploratory. A number of learners' problem solving strategies were identified, that include unsystematic guess, check and revise; systematic guess, check and revise; trial-and-error; logical reasoning; non-logical reasoning; systematic listing; looking for a pattern; making a model; considering a simple case; using a formula; numeric approach; piece-wise and holistic approaches. A weak positive linear relationship between learners' mathematics-related belief systems and their approaches to non-routine problem solving was discovered. It was, also, discovered that learners' mathematics-related belief systems could explain their approach to non-routine mathematics problem solving (and vice versa).

Key terms: Non-routine mathematical problem; Mathematical problem solving; Non-routine problem solving; Approach to non-routine mathematical problem solving; Mathematical problem solving strategies; Mathematics problem solving theories; Mathematics-related beliefs; Mathematics-related belief systems; Mathematics-related belief theories; High school learner

Dedication

To my dearest wife, Freddia, and sons, Wayne, Wendy and Walter.

Acknowledgements

During the course of my study at the University of South Africa, extra-ordinary people met my needs. For their good will and understanding, I am most grateful. There are many people to whom I wish to express my sincere appreciation, gratitude and indebtedness for their direct and indirect contribution towards the completion of my entire PhD in mathematics education degree study.

Firstly, I would like to thank God for seeing me through this project. Without him this degree would have been impossible. Secondly, I would like to extend my special thanks and deepest gratitude to Professor Mogari, D., Project Supervisor, whose generosity, support, encouragement, understanding and scholarly advice proved significant to the completion of this study. Thirdly, I wish to offer special thanks to the three High School principals, heads of mathematics departments and students for interest and cooperation. Without their invaluable contributions, this research study could not have been a success.

Fourthly, special thanks go to Ms Makwakwa, E. and Mr Kotoka, J. for the provision of SPSS 16.0 software that was indispensable in conducting this study. Lastly, I would like to express my deepest respect, love and gratitude to Freddia, my dear wife, who unselfishly deprived herself for my three years of education at the University of South Africa. Your unlimited love, services, support, and continuing faith in my abilities to achieve my educational goal have always been very special and important to me. Thanks for riding out the ups and downs with me. You gave me the strength and courage to carry on.

Abbreviations

BQ   :   Beliefs questionnaire

HSRC: Human Sciences Research Council

IPT   :   Information Processing Theory

IS   :   Interview Schedule

KMO :   Kaiser-Meyer-Olkin measure of sampling adequacy

LPS   :   Learning through problem solving

MRBQ: Mathematics related belief questionnaire

NCS   :   National Curriculum Statement

NCTM: National Council of Teachers of Mathematics

OQ   :   Open questionnaire

PBL   :   Problem based learning

PT   :   Problem Test

RQ   :   Retrospective questionnaire

SPSS :   Statistical Package for Social Sciences

TIMSS: Trends in International Mathematics and Science Study

CONTENTS **PAGE**

List of tables

List of figures

CHAPTER 1

1.1 <u>INTRODUCTION</u>

In this chapter, the researcher, firstly, discussed the background to the study. Secondly, he stated the statement of the problem. Thirdly, he stated the research questions. Fourthly, the study was justified. Lastly, the key terms of the study were defined and the structure of the thesis was presented.

1.2 <u>BACKGROUND TO THE STUDY</u>

Studies conducted over the past twenty years on South African learners' problem solving abilities discovered that the learners lack adequate problem-solving strategies and skills, thus perform poorly in mathematics. For example, the Trends in International Mathematics and Science Study (TIMSS) (2003) results reflected that South African mathematics learners, in general, were not adequately involved in problem solving activities and this adversely affected their problem solving strategies and skills. The TIMSS 2003 collected data from grade 8 learners and teachers on how often the learners engage in three problem solving activities: relating classroom mathematics to their daily lives, explaining their answers, and deciding procedures for solving complex problems (Mullis, Martin, Smith, Garden, Gregory, Gonzalez, Chrostowski & O' Connor, 2003). Still in the same study, teachers indicated that less emphasis was placed on deciding problem-solving procedures in classrooms (36%) (see table 1.1). This rating was lower than that of learners (64%), and the learner and teacher international averages of 53% and 45% respectively.

Table 1.1: South African learners and teachers' reports on problem solving related emphasis in classroom activities.

Adapted from Mullis et al. (2003, p. 283, 284).

| | Percentage of learners/ learners whose teachers reported learners doing the activity about half of the lessons or more | | |
	Relate what is being learned in mathematics to learners' daily lives	Explain answers	Decide procedures for solving complex problems
Learner	74 (44)	78 (67)	64 (53)
Teacher	59 (50)	67 (78)	36 (45)

[Note: International averages are in brackets]

Mullis et al. (2003) describe the percentages that reflect emphasis placed on deciding problem-solving procedures in classrooms as a matter of concern in South Africa. When solving non-routine mathematical problems, learners need to exercise independent and critical thinking. Learners should, therefore, be exposed enough to problem solving situations that require creativity and deep thinking. A heavy dependence on the teacher on how to approach problems tends to cause learners to develop negative beliefs related to mathematical problem-solving, e.g., mathematics problems can only be solved by ready-available procedures; and procedures to solve non-routine mathematical problems are decided by the teacher (Schoenfeld, 1983).

In comparison with other five lowest performing countries that participated in TIMSS 1999 and TIMSS 2003, South Africa scored lowest in mathematics than the rest (Human Sciences Research Council [HSRC], 2006). The performance of South Africa in TIMSS 2003 was slightly lower than the 1999's (see figure 1.1). However, the HSRC (2006) concluded that the decrease in average mathematics scores (from 275 in 1999 to 264 in 2003) was not significant.

A TIMSS (2003) study was also done on learners' mathematics performance in relation to other constructs, e.g., attitudes towards learning mathematics (self-confidence in learning mathematics, enjoyment of mathematics, and valuing mathematics) (HSRC, 2006). The results of the study showed no significant

variations in achievement scores between South African learners who indicated high positive attitudes to mathematics and those who did not. The HSRC (2006) concluded that the results obtained might not be reflecting the actual situation on the ground because the learners might be providing some socially desirable responses other than their real attitudes towards mathematics. Perhaps there is a need to gather data on learners' espoused mathematics-related beliefs in conjunction with learners' written work to infer their real beliefs in action and, also, to determine the learners' real mathematics-related beliefs in action that guide their problem solving actions.
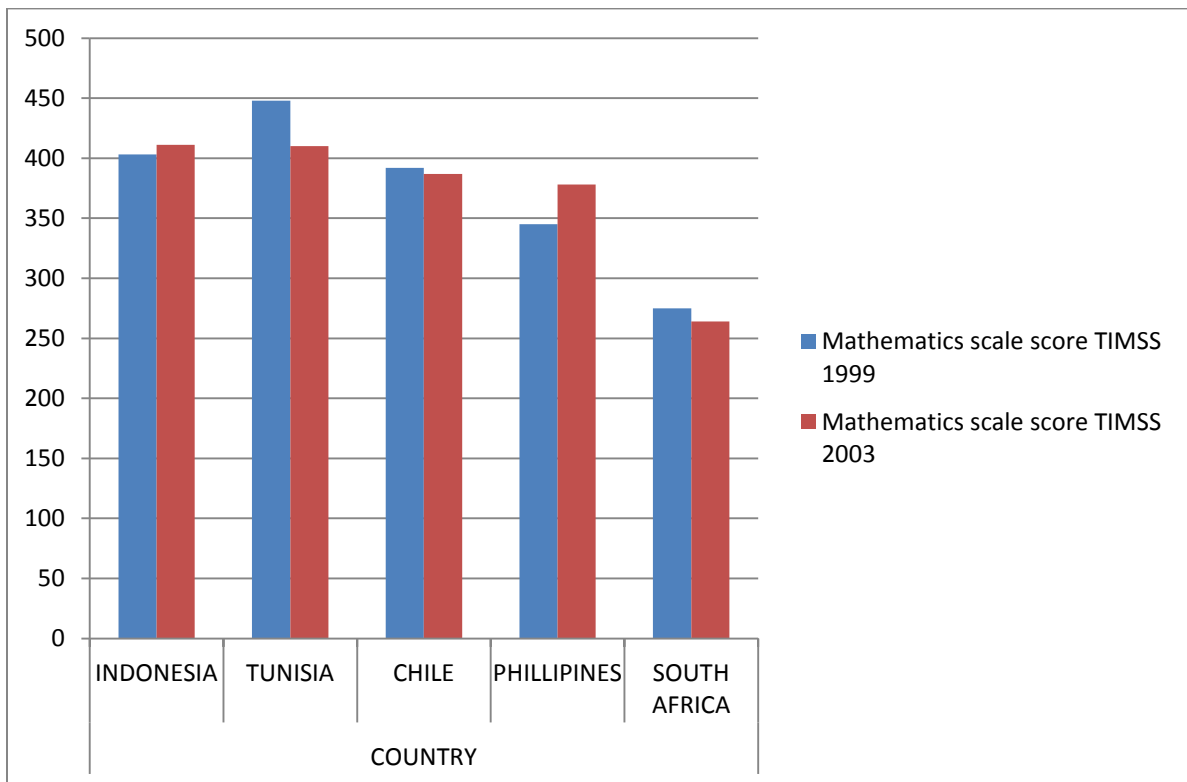
Figure 1.1: Change in mathematics performance from TIMSS 1999 to TIMSS 2003 by country (HSRC, 2006)**.**

A study by Maree, Aldous, Hattingh, Swanepoel and Vander Linde (2006) in Mpumalanga, South Africa, on predictors of learner performance in mathematics reveals that learners' mathematics problem solving skills are inadequate (indicated by their poor performance in mathematics). They discovered that conservative teaching methods, e.g., chalk and talk method, practiced by teachers have a great negative impact on learners' performance. They also discovered that learners are rarely asked to ask questions and link mathematics to everyday life. In South Africa, Bopape (n.d.) posits that teachers prefer drill work to investigative approach to teach mathematics because the latter is blamed for wasting time and delaying completion of the syllabus.

A study by Webb and Webb (2004), in South Africa, on Eastern Cape teachers' beliefs of the nature of mathematics reveals that teachers' teaching approaches are not learner-centered. They do not assume a role of a learning 'facilitator', but assume that of a 'instructor' whereby they emphasise mastery of skills and obtaining correct answers. Maree et al. (2006) cite traits that might develop in learners due to exposure to these undesirable teaching and learning experiences, such as use of complex vocabulary, conformity to rules and reticence. Maree et al. (2006) believe that, due to the teaching and learning approaches that prevail in mathematics classrooms, e.g., drill work, learners often acquire knowledge of basic concepts that is rote and deficient. In the light of this discussion, the researcher believes that it is from their classroom experiences that learners develop unhealthy mathematics beliefs which adversely affect them in mathematical problem solving. Learners cannot transfer or apply their conceptual mathematical knowledge to real world, non-routine mathematical problems, if there is no connectedness in concepts learnt. Ideally, learners should understand and make sense of what they learn in order to succeed in non-routine mathematical problem-solving.

According to Wessels (2012), the main cause of poor performance of South African learners in mathematics in TIMSS studies was lack of mathematical problem solving skills or competences. Wessel (2012) places the blame on our

out-going National Curriculum Statement (NCS) saying, though it proved to be able to improve learners' affective values of mathematics, it failed to improve their quality of performance in mathematics. Bopape (n.d.) locates the root of the problem as from the apartheid era whereby the main approach used in the mathematics teaching and learning by teachers was rote learning. The fact that most of the current crop of teachers in South Africa are a product of the apartheid education system, they are more likely to perpetuate the traditional approaches to mathematics teaching and learning inherited from their past education system.

Kolovou, Van den Heuvel-Panhuizen, Bakker and Elia (2011) posit that the school mathematics do not adequately prepare learners to solve real-life and non-routine mathematics problems. Kolovou et al. (2011) believe that teachers have difficulties in preparing learners for non-routine mathematical problem-solving. Wessels (2012) argue that South African learners' problems in solving non-routine mathematical problems, probably, root from their teachers whose pedagogical and mathematics content knowledge lack conceptual depth, clarity and connectedness (integration).

Lester's (2013) view on this matter is different from that of Wessels (2012). Lester (2013) cites meta-cognitive abilities (ability to monitor and regulate cognitive behaviours) as one of the key ingredients to success in problem solving. The problem mathematics teachers face is of how to enhance learners' meta-cognitive abilities in non-routine mathematics problem solving. This will even be a bigger problem if teachers lack conceptual depth. Lester (2013) helplessly states that, as of to date, among all the research studies conducted aimed at enhancing learners' meta-cognitive abilities none has identified the teaching and learning approaches, nor the capabilities the teachers need to achieve this. In this regard, for mathematics teachers to be helpful in enhancing learners' problem solving skills, Lester (2013) asserts that they should themselves be serious learners of problem solving rather than being experts in problem solving.

Wessels (2012) identified South African learners' challenges in mathematics problem-solving such as, e.g., poor language proficiency, inability to handle and utilize mathematics to solve problems and lack of the ability to approach mathematical problems in a meaningful and constructive way. In an attempt to address learner performance in mathematics problem solving, Wessels (2012) advocates for a problem-centered teaching and learning approach. Similar research findings were obtained by Zanzali and Nam (1997) from Malaysian Secondary School learners, who identified that the learners, in general, lack good mastery of the problem solving strategies. They pointed out that learners had limited exposure to plan and carry out their problem solving strategies. Similarly, Mogari and Lupahla (2013) discovered that Namibian grade 12 learners had weaknesses in understanding non-routine problems (probably, because of poor English command) and devising their own problem solving strategies. As a result, they faced difficulties in solving non-routine problems. The ability to device and carry out a problem solving strategy is one of the skills important to solving non-routine mathematical problems. This study intends to look into learners' approaches to non-routine problem solving.

Though the NCS outcome based education system in South Africa promoted a shift from a traditional approach (that is associated with rote learning, learning without the necessary insight, a lack of creativity, a tendency to be too teacher-oriented and a lack of learner activity) to a problem-centered approach in mathematics teaching and learning, it seems to have failed to yield satisfactory results (based on the current learner unsatisfactory mathematics performance trend) (Wessels, 2012). Brijlall (2008) posits that a positive change in learners' beliefs about problem solving might be achieved if teachers modify their approach to mathematical problem solving in the classroom and make it, e.g., more centered on problem solving.

This background highlights what happens in our South African school mathematics classrooms; the results of the teaching and learning practices in terms of learner performance in mathematics and other behavioural traits; and

the possible causes of learners' poor performance in mathematics problem solving. Learners' set of mathematics-related beliefs is, to a larger scale, influenced by their school experiences.

The present study determined if there is a relationship between high school learners' mathematics-related belief systems and their approaches to non-routine mathematical problems. As the classroom instruction do not prepare learners to solve non-routine mathematical problems effectively (Kolovou et al., 2011), the researcher conjectures that learners' mathematics-related belief systems do guide their mathematics problem solving behaviour.

1.3 **STATEMENT OF THE PROBLEM**

This study will determine if there is a relationship between grades 10, 11, and 12 learners' mathematics-related beliefs and their approaches to non-routine mathematical problem-solving.

1.4 **RESEARCH QUESTIONS**

1.4.1 What are the grades 10, 11, and 12 learners' approaches to non-routine mathematical problem solving?
1.4.2 What are the grades 10, 11, and 12 learners' mathematics-related belief systems?
1.4.3 Is there any relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematical problem-solving?

1.5 **JUSTIFICATION FOR THE STUDY**

The results of this study will raise learners' awareness of their probable belief systems and how they influence their learning and achievement in mathematics. Learners' awareness of how beliefs develop, change over time and affect learning might assist them to develop a healthy and productive relationship with mathematics. Teachers will be made aware of the impact of learners' belief

systems on their mathematics learning and performances. Teachers can be better placed to explain some learning behaviours exhibited by learners in mathematics classrooms. In addition, since beliefs are a product of one's experiences (Pehkonen & Pietila, 2003), from a learner's beliefs concerning the nature of mathematics, one can indirectly infer and evaluate the kind of instruction, for example, the learner had received.

The study will also raise awareness to teachers of how their teaching styles, kinds of tasks they set and assessment methods remarkably influence learners' development of either positive or negative mathematics-related beliefs. Since a strong relationship between learning outcomes and mathematics-related belief systems is believed to exist (Kislenko, Grevholm & Lepik, 2005; Jin, Feng, Liu & Dai, 2010), teachers may effectively assess or evaluate learners' mathematical knowledge in awareness of their existing beliefs. In addition, Pehkonen and Torner (1999) argue that any changes that might be done to the teaching of mathematics in schools should, partly, take into account the learners' beliefs as a possible force that affects change.

Vinner (1999) stresses that awareness of our beliefs enables one to be able to reflect on his/her behaviour and be well positioned to change it. A study by De Corte and Malaty (2010) reveals that teachers' endeavour to improve the mathematics problem solving in the classroom might be made difficult by learners' beliefs. For example, if learners think that mathematics means solving text-book tasks only, they might have difficulties in solving real-life problem situations. Educators' awareness of how learners' beliefs develop will basically be important for the planning and implementation of appropriate and effective intervention programs aimed at changing learners' negative mathematics-related beliefs that adversely affect their learning and performance. It is important that learners develop a set of beliefs which supports and empowers them for further learning, and see the relevance of the skills acquired in mathematics class to situations encountered in the world beyond the classroom.

From their study on "intervention on learners' problem-solving beliefs", Stylianides and Stylianides (2011) assume that the more learners are aware of their existing beliefs, the higher is their chance to problematise and change these beliefs when they engage in a problem-solving situation that challenges their existing beliefs and encourage the formation of alternative, possibly, health beliefs. It is of paramount importance to diagnose learners' naïve beliefs and apply an appropriate remedy to break the cycle of influence on learners' learning behaviour. In this regard, Daskalogianni and Simpson (2001b) argue that learners go to tertiary education together with the mathematics-related beliefs developed at high school. As a result, the high school mathematics beliefs affect their proper adaptation to tertiary mathematics.

Dating back from early nineties, literature on beliefs reveals some theoretical deficiencies that warrant attention of mathematics education researchers. For example, Pehkonen and Torner (1999) note that mathematics education researchers face difficulties in defining the concept of belief. As a result, they provide different definitions of belief which, at times, contradict each other. Schoenfeld (1992) observes some deficiencies in mathematics education research, such as lack of convincing and effective new research methodologies and ways of explaining research findings. In this regard, Schoenfeld (1992) suggests that research studies should move away from "telling good stories" that something happens to providing solid explanations as to how and why it happens. In support to Schoenfeld (1992), De Corte and Op't Eynde (1999) argue that mathematics researchers rarely study intensively learners' mathematics-related belief systems. What is common is studying beliefs in isolation from each other rather than as a complex system, whose components cannot be studied in isolation from each other. Op't Eynde, De Corte and Verschaffel (2006) also identify that researchers lack an agreement on the exact categories of mathematics-related beliefs that affect learners' mathematics learning and problem solving.

To the mathematics education community, at large, this study will contribute to the development of theory on beliefs and problem-solving, and how learners' mathematics-related beliefs are related to mathematics teaching and learning. The study will shed further light on the kinds of mathematics-related belief systems learners hold, and the relationship between the beliefs and learners' approaches to solving non-routine mathematical problems. The study will attempt to bridge the theory-practice gap in mathematics education, for example, by suggesting some teaching and learning methods that promote sense making and development of healthy mathematical beliefs like teaching and learning through problem-solving. As such, an effective mathematics teaching practice should address the learners' belief outcomes.

1.6 **DEFINITION OF KEY TERMS**

To provide clarity when reading this research document, the following definitions of key terms will be used. Definitions of some terms under discussion will be presented in the research document whenever necessary.

1.6.1. **Learners' mathematics-related belief systems**

Mathematics-related belief systems are learners' deep seated perceptions that are informed by the learners' knowledge of and attitudes towards mathematics.

1.6.2. **Problem Solving**

In this study, problem solving refers to a systematic and logical procedure carried out to determine a solution to a problem.

1.6.3. **Non-routine problem solving**

Non-routine problem solving is a process that involves searching for heuristics or inventing algorithms to resolve a problem which cannot be resolved by readily available direct methods, procedures or algorithms (Wilson, Fernandez & Hadaway, 1993).

1.6.4. Learners' approaches to non-routine mathematical problem-solving

Strategies or patterns of action learners employ to solve a problem such as listing all possibilities, applying standard or invented algorithms, logical reasoning, using algebraic expressions or equations, trial-and-error, guessing, logical estimation, using tables, experimenting with simple cases, etc (Elia, Van den Heuvel-Panhuizen & Kolovou, 2009; Mabilangan, Limjap & Belecina, 2011).

1.7. <u>Structure of the thesis</u>

Chapter 1 contains the background to the study, statement of the problem, research questions, justification of the study, and definition of key terms.

Chapter 2 is basically composed of three sections: mathematical problems and problem solving, mathematics-related belief systems and theoretical framework. Under mathematical problems and problem solving, the researcher, firstly, discussed some definitions of a mathematical problem, mathematical exercise and mathematical problem solving suggested by some scholars. Secondly, the researcher discussed some approaches to problem solving. Thirdly, he discussed some factors that affect problem solving. Fourthly, he outlined some difficulties faced by learners in problem solving. Lastly, he discussed some characteristics of good problem solvers.

Under mathematics-related belief systems, the researcher, firstly, discussed some definitions of mathematics-related beliefs, and belief systems suggested by several researchers in mathematics education. Secondly, he listed some typical learners' mathematics-related beliefs. Thirdly, he discussed several categories of mathematics related belief systems suggested by some scholars. Lastly, he discussed the relationship between belief systems and mathematics learning and problem-solving. Under theoretical framework, the researcher, firstly, discussed mathematics problem solving and mathematics-related belief theories that guided, supported and informed this study. Lastly, he discussed the relationship

between belief systems and mathematics learning and problem solving discovered by some scholars (e.g., Callejo & Vila, 2009; Jin et al., 2010).

Chapter 3 presents the research methodology. It is composed of five sections: the research design, the population and learners selected for study, data collecting instruments, data collection procedure and ethical considerations.

Chapter 4 presents how the collected data was presented and analyzed in the study. It describes methods employed in analysing the mathematics-related beliefs questionnaire, non-routine mathematics problem solving test, interview schedule, open-ended questionnaire, retrospective questionnaire and the relationship between belief systems and approaches to problem solving.

Chapter 5 presents the research findings in three sections: learners' mathematics-related belief systems, learners' mathematical problem solving strategies, and relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematics problem solving.

Chapter 6 presents discussion of the research findings. Learners' approach to mathematical problem solving was discussed in conjunction with their predominant mathematics-related belief systems. The relationship between learners' approach to non-routine mathematical problem solving and their belief systems was discussed.

Chapter 7 presents summary of the study, conclusions and recommendations. The researcher's reflections on his intellectual journey were presented.

Chapter 2

Problem solving and beliefs

2.1. <u>Introduction</u>

The researcher, firstly, discussed mathematical problems, mathematical exercises and problem solving. Secondly, he discussed mathematics related belief systems. Lastly, he discussed the theoretical framework of the study.

2.2. <u>Mathematical problems and problem solving</u>

In this section, the researcher discussed some definitions of a mathematical problem, mathematical exercise and mathematical problem solving as suggested by various scholars. Differences between a mathematical problem and a mathematical exercise were also discussed. Some categories of mathematical exercises were discussed, identifying the kinds of mathematical exercises that would be used in this study. Some problem solving approaches, strategies (heuristics) and algorithms were discussed. Factors affecting problem solving, difficulties faced by learners in problem solving and characteristics of good problem solvers are some issues discussed in this section. At last, differences between expert and novice problem solvers were discussed.

2.2.1. <u>What is a mathematical problem?</u>

There has been different definitions of a 'mathematical problem' formulated over the years by different scholars in mathematics education (e.g., Andre, 1986; Xenofontos & Andrews, 2008; Branca, 1980; Blum & Niss, 1989; Carson, 2007; Focant, Gregoire & Desoete, 2006; Hoosain, 2001; Kee, 1999; Zanzali & Nam, 1997). A mathematical task can be referred to as a mathematical problem when a learner who faces it wants to solve it, has no an immediately available procedure to solve it and must, actually, attempt to solve it (Kee, 1999).

A mathematical problem can be defined as a situation in which a learner really wants to look for a solution to it, but does not know how to find it (Andre, 1986). Carson (2007) views a mathematical problem as a situation that is quantitatively expressed or otherwise, that is faced by a learner who wants to resolve it, but has no an immediate direct way to do it. Hoosain (2001) views a mathematical problem as a non-routine problem whose solution process requires more than ready-to-hand procedures or algorithms.

According to Hooisan (2001), a mathematical problem can be regarded as a task or experience which an individual encounters for the very first time of which he/she has no known procedures to handle it. In order to solve a mathematical problem, the individual should devise his/her own approach or method by utilizing the resources at his/her disposal such as the various skills, knowledge and strategies which were previously learned.

An analysis of mathematical problems done by Andre (1986) reveals that mathematical problems have four components: (1) The goal or goals (what you want to do in a situation); (2) The givens (what is available to you to start in a problem situation); (3) The obstacles (The elements or factors that get in the way of a solution); and (4) The methods or operations (The procedures that may be used to solve the problem). For a learner to effectively solve a mathematical problem, he/she should clearly identify these four components at the initial or approach stage of problem solving.

Though there is no a precise consensus on the definition of a mathematical problem from researchers as (Hoosain, 2001) noted, from the interpretations given by the authors such as Andre (1986); Carson (2007); Hoosain (2001) and Kee (1999), this researcher, however, observes that there seems to be a general agreement that a mathematical problem should be a situation that confronts an individual who desires a solution and for which an algorithm which leads to a solution is unavailable or not known by the individual. A mathematical problem, then exists if the situation is new and not recognisable by the potential problem solver, and the problem solver does not possess direct methods or algorithms

that are enough to resolve the problem. This means that a mathematical question will not be considered a 'mathematical problem' if the individual has previously solved the problem or can easily solve the problem by applying algorithms that were previously learnt.

In this study, the researcher used the term 'mathematical problem' to refer to a situation which carries with it either open questions or non-routine questions that challenge somebody intellectually who does not immediately possess direct methods that are enough to resolve the problem (Blum & Niss, 1989). Only mathematical problems were given to learners who participated in this study. In choosing the mathematical problems, the researcher attempted to ensure that the learners did not previously solve the mathematical problems or they did not previously learn the requisite direct methods or procedures to solve them.

2.2.2. <u>What is a mathematical exercise?</u>

The Schoenfeld (1985) defines a mathematical exercise as a routine exercise in which a learner applies some learnt mathematical facts and procedures. The main purpose of a mathematical exercise is to enable the learner to master the relevant mathematical matter. The introduction of several similar worked examples in preparation for learners to attempt a mathematical exercise on their own eliminates or minimises the challenge of the task. In essence, the potential mathematical problems to learners are reduced to mere mathematical exercises.

In this study, learners attempted resolving mathematical questions that were new to them, which demanded originality and creative thinking. Learners devised their own strategic problem solving approaches they considered appropriate to solve the given mathematical problems. No similar worked examples were provided.

2.2.3. <u>Difference between a mathematical problem and a mathematical exercise</u>

The difference between a mathematical problem and a mathematical exercise is that, in a mathematical problem, an algorithm which will lead to a solution is unavailable to the problem solver, whereas, in a mathematical exercise, one

determines the algorithm first and applies it in problem solving (Hooisan, 2001). When learners are given an opportunity to practice how to solve a mathematical problem, then, the mathematical problem becomes a mathematical exercise. Schoenfeld (2007) classifies this act as a 'degradation' of mathematical problems into mathematical exercises. Mathematical problems are expected to pose a real challenge to the learner. Focant et al. (2006), Hooisan (2001), and Wilson et al. (1993) argue that a mathematical problem is relative to the individual(s) involved. A mathematical question or task that is a mathematical problem to one learner might be a mathematical exercise to another because of the absence of a real challenge (blockage) or acceptance of the goal to problem solving. This means that a mathematical situation can only be defined as a mathematical problem relative to specific learners.

In a way, based on Hooisan (2001) and Schoenfeld's (2007) view, one may argue that mathematical problems are non-routine, whereas mathematical exercises are routine (see section 2.2.4). A mathematical exercise is thus a mathematical question which a learner knows how to resolve immediately, whilst a mathematical problem is a mathematical question which requires a learner to apply creative thinking and resourcefulness in search of the appropriate problem solving approach. In another way, one may argue that a mathematical exercise can be either routine (procedural or algorithmic in nature) or non-routine (non-procedural or non-algorithmic in nature) as evident in school mathematics textbooks which present both routine and non-routine questions under the same heading 'exercise' or 'activity'. Based on the definition of a mathematical exercise and the fact of degradation of mathematical problems into mathematical exercises stated by Schoenfeld (1985, 2007) (see sections 2.2.2 & 2.2.3), the classification of both routine and non-routine questions as an exercise done in school mathematics textbooks can be regarded as appropriate if learners were previously taught or learned by themselves how to solve similar questions. Otherwise, it is not appropriate if learners did not previously have an opportunity to learn how to solve questions of similar nature. In the light of this discussion, the researcher classified non-routine mathematical exercises as mathematical

30

problems in this study after checking that the learners participating in the study were not previously taught how to resolve questions of similar nature.

This researcher, however, concedes that there are mathematical exercises that are non-routine simply because their resolutions are not obvious and learners have not practiced resolving them before. As such, a mathematical question qualifies to be a mathematical problem if it is novel and the learner cannot solve it immediately. According to Bunday (2013), a mathematical exercise serves to drill a learner in some technique or procedure and requires little, if any, original thought (because of the provision of worked examples of similar nature), while a mathematical problem requires thought on the part of the learner. When resolving a mathematical problem, the learner has to devise his/her own strategic attacks which might be subject to failure or success.

The distinction between routine/non-routine mathematical exercises and mathematical problems was of paramount importance to this study. If a learner could readily resolve a mathematical question posed by applying the previously practiced procedures, the objectives of the study would not have being met (see section 1.4). This would mean that the questions posed in the study were routine mathematical exercises and not non-routine mathematical exercises (or mathematical problems). Learners' approaches to real and suitable mathematical problems were expected to reflect, somehow, their mathematics-related belief systems as they search for solutions by applying their own original strategies. The solution to the mathematical problem should be an original product of the learner than a reflection of someone else's thought.

2.2.4. **Categories of mathematical exercises**

2.2.4.1. **Routine mathematical exercise**

Routine mathematical exercises are mathematical questions which the learners solving them possess a previously established procedure for finding one. Brunning, Schraw and Ronning (1999) view a routine mathematical exercise as a well-defined problem whose single correct solution can be obtained by applying a well

known guaranteed method or procedure. For example, solving a quadratic equation by using the quadratic formula produces one solution through pre-determined steps.

Posamentier and Schulz (1996), and Kirkley (2003) view routine mathematical exercises as well-structured problems that always use the same step by step solution. Some distinguishable characteristics of these mathematics questions are that they have a solution strategy that can be predicted, have one correct answer, and contain all information needed to solve the problem. Such types of mathematical questions were not suitable for use in this study, as they could not distinguish learners' different problem solving approaches or behaviours. Different problem solving strategies learners used to resolve non-routine mathematical problems were interpreted as a manifestation of different belief systems in this study.

Landa (1983) views routine mathematical exercises as algorithmic problems. An algorithmic mathematical question or task enables the problem solver to resolve it by following a predefined sequence of operations. Landa (1983) posits that, for any given algorithmic mathematical question or task, a learner can clearly state all the mental processes he/she has to undergo in order to successfully solve it. It is clear from this discussion that algorithmic tasks limit learners' independent mathematical reasoning or creativity; and might promote development of, for example, a belief that a mathematical problem has one exact correct answer that can be obtained by applying predetermined procedures.

2.2.4.2 **Non-routine mathematical exercise**

Non-routine mathematical exercises are mathematical questions which the learners attempting to solve them possess neither a known answer nor a previously established (routine) procedure for finding one (Branca, 1980). Non-routine mathematical exercises are referred to as non-routine mathematical problems or simply mathematical problems in this study. It is important to note that mathematical questions which are non-routine to somebody may be routine

to another. There were attempts in the current study to ensure that the mathematical problems were non-routine as far as possible to the participating learners by checking that they have not attempted to solve the problems previously and that they have not been taught standard methods of solving the types of problems involved.

Brunning et al. (1999) classify a non-routine mathematical exercise as an ill-defined mathematical problem that has several acceptable solutions that can be obtained by several unique strategies. There is no a single strategy that is universally agreed upon to resolve a non-routine problem. Each new unique non-routine problem requires different new approach strategies to resolve it. Kirkley (2003) views non-routine mathematical exercises as ill-structured problems that have vague and unclear goals. The problems are characterised by having multiple perspectives, goals and solutions; solution that is not well defined, predictable or agreed upon, and some needed information, often, must be determined for effective problem solving. In a similar way, Polya (1985) classifies non-routine mathematical problems into two categories: (a) Problems to find, the principal parts of which are the unknown, the data, and the condition; and (b) Problems to prove which comprise a hypothesis and a conclusion.

The researcher defines the mathematical problems used in this study as ill-defined or ill-structured because they could not be solved by use of mathematical algorithms or any pre-existing formula. Some necessary information needed to be determined before the problem is successfully solved. Chamberlain and Moon (2008) and Cai (2010) argue that the demand of these types of questions and problem statements elicit or produces various appropriate responses as well as various levels of correctness. Another advantage of these tasks is that they enable learners to be flexible in exercising mathematical reasoning. As a result, multiple correct answers and multiple solutions were expected. Different approaches/strategies that the learners applied in solving these problems enabled the researcher to group learners with similar problem-solving behaviours

or learners whose problem-solving strategies could be influenced by similar predominant belief systems.

Landa (1983) views non-routine mathematical exercises as creative or heuristic problems. Unlike algorithmic mathematical problems, creative or heuristic mathematical problems have no clear predetermined instructions or procedures a learner can rigidly follow in order to resolve them. Rather, heuristics or instructions that serve as guidelines that can be applied to various non-routine mathematical problem-solving situations can be formulated. The heuristics do not guarantee a success in problem solving but serve, merely, as a guideline.

The mathematical problems in this study required learners to apply certain heuristics for successful resolution, e.g., working backwards, finding a pattern, drawing a diagram, and intelligent guessing and testing.

2.2.4.3 **Single-step mathematical exercises**

As the name suggests, these mathematical questions can be solved by carrying out one step. Blum and Niss (1989) identify recognition exercises as an example of one-step mathematical exercises. Recognition exercises ask the problem solver to recognise or recall a specific fact, definition or statement of a problem. For example, which of the following are rational numbers? (a)... (b)... (c)... Single-step mathematical exercises can be either routine or non-routine mathematical questions (see sections 2.2.4.1 & 2.2.4.2).

2.2.4.4 **Multi-step mathematical exercises**

The questions can be successfully solved by carrying out a multiple number of steps. The questions can be algorithmic exercises that can be solved by applying a well known step by step predetermined procedure (e.g., solve $2x^2 - 3x - 5 = 0$) or heuristic or open-search mathematical questions that do not contain a strategy for solving the problem in their statements. For example, "Prove that ..."; "Find all ...". In this study, only multi-step heuristic mathematical problems were utilised. These mathematical problems were intended to give learners room to decide on

their own problem solving approaches and reveal different levels of correctness of the solutions. Learners' line of thinking or problem solving patterns of action could be traced from the solution process presented.

Table 2.1 below gives a summary of the categories of mathematical exercises discussed above.

Table 2.1: Categories of mathematical exercises

	Branca (1980)	Landa (1983)	Posamentier & Schulz(1996)	Brunning et al. (1999)	Hooisan (2001)
Routine	Recognition Algorithmic	Algorithmic	Well-structured	Well- Defined	Single-step text book
Non-routine	Application Open-search Problem- situation	Creative/ Heuristic	Ill- structured	Ill- defined	Multi-step Non-text book

It is important to note that the grouping of categories the researcher have done in Table 2.1 is mainly based on the extent to which the type of problems satisfies the definition of a mathematical problem discussed above. Some categories may not precisely fall on one side. For example, some multi-step problems might be algorithmic problems as well. The type of mathematical problems the researcher used in this study can be described as non-routine mathematical questions which are open-search, heuristic, ill-structured, ill-defined, non-routine or multi-step.

2.2.5. <u>Definition of mathematical problem solving</u>

Branca (1980), Wilson et al. (1993), Xenofontos and Andrews (2008), and Zhu (2007) agree that problem solving has too many facets to be considered when defining it. As such, the authors give different meanings of it because they view it from different angles. Because of different interpretations attached to problem solving, Branca (1980), and Xenofontos and Andrews (2008) propose that

whenever we encounter the term problem solving, we should consider which interpretation is intended.

Problem solving can be viewed as a goal, process and basic skill (Branca, 1980; Meiring, 1980; Wilson et al., 1993; Xenofontos & Andrews, 2008; and Xenofontos, 2010). Wilson et al. (1993) consider problem solving as the heart of mathematics. As such, learning how to solve problems is considered as the primary reason for studying mathematics. In appreciation of the importance of problem solving in mathematics learning, Romberg (1994) includes 'developing into a competent mathematical problem solver' as one of the five general goals for learners.

Grouws and Cebulla (2000), Kitchen, DePree, Pattichis and Brinkerhoff (2007) and Wilson et al. (1993) posit that teachers can use problem solving as an instructional teaching approach to facilitate the acquisition of basic mathematical facts, concepts and problem solving skills by learners. Cai (2010) cites two new roles for teachers to play in the classroom based on teaching through problem solving: (1) Selecting appropriate tasks and (2) organizing the classroom discourse in order to guide learners engaging in problem solving appropriately. De Corte and Malaty (2010) discourage an approach to teaching that is teacher-centered, whereby the teacher is viewed as a sole source of mathematical knowledge and solutions. They advocate for a teaching approach that is learner-centered, whereby the teacher facilitates learning to take place by encouraging and accommodating learners' initiatives, interaction and collaboration in the classroom.

By viewing problem solving as a basic skill learners have to master in learning mathematics, teachers have to put more emphasis in understanding the essentials or the basics of problem solving. This includes the problem content, problem types and solution methods. Sweller, Clark and Kirschner (2010) suggest that learners can acquire problem-solving strategies by constantly practicing application of specific mathematical problem solving strategies to a

relevant set of problems. They stress the importance of worked examples when it comes to teaching domain-specific mathematical problem-solving skills.

Problem solving can be viewed as a process whereby a learner applies some relevant previously learnt mathematical knowledge to a new and unfamiliar situation. In this regard, what are considered important in problem solving are the methods, procedures, strategies, and heuristics that learners use in solving the problems. A study by Elia et al. (2009) on Dutch learners' strategy use and strategy flexibility in non-routine mathematical problem-solving reveals that learners rarely apply heuristic strategies in solving problems. Since learners are incompetent in problem solving, Wilson et al. (1993), Branca (1980), Goliath (2008), Cai (2010), and Gale Encyclopedia of Education (2010) suggest that mathematics should be taught through problem solving.

Krulik and Rudnick (1980) accept in their study Polya (1980)'s definition of problem solving which highlights the following aspects of problem solving: 'one has to look for a way to resolve a situation that he/she has no way at hand to resolve it', 'in the face of a difficulty or obstacle, one has to look for a way to resolve the problem', and 'at the absence of immediate and direct ways of resolving a problem, one has to obtain the desired solution'. From Polya (1980)'s definition, the researcher can infer that the problem solver should find some solution to a mathematical situation that is a problem to him/her or that poses some level of difficulty that acts as an obstacle to achieve some desired goals.

In problem solving, a learner is expected to synthesise and coordinate some previously acquired knowledge and skills and apply them to a novel problem situation (Carson, 2007; Lester, 2013). The problem solving process might demand a learner to 'take stock' of some relevant previous experiences and learnt knowledge, and develop new understanding, representations and patterns of inference in order to successfully solve an unfamiliar problem situation (Lester, 2013). Lester's (2013) definition of problem solving highlights that for a learner to be successful in problem solving he/she must have adequate and relevant previous experiences in learning how to solve problems, a strong mathematical

knowledge base, knowledge of various mathematical models or representations and have the ability to model or represent mathematical situations and construct or draw patterns of inferences.

The mathematical knowledge base is composed of four categories of knowledge or skills which are resources, heuristics, control, and belief systems (Schoenfeld, 1985). In a nutshell, a learner should have proposition and procedural knowledge of mathematics as essential 'resources' in problem solving. He/she should also have knowledge of heuristic strategies (see section 2.2.7), skills of controlling use and application of resources and strategies, and belief systems that determine behaviour or approach to be adopted to a problem (see section 2.3.2) in order to successfully solve a problem.

In this study, the researcher viewed problem solving as a process because of his interest in analyzing learners' approaches to problem solving. Non-routine mathematical problems that are suitable for high school education were given to learners who participated in the study. These learners were expected to demonstrate a commitment, zeal and endurance to problem solving. In this regard, Meiring (1980) posits that problem solving is more than knowing what to do but involves having a commitment to problem solving- a willingness to tackle a problem even when one does not know what to do and to keep on searching for solution strategies until one finds a reasonable solution.

2.2.6. **Problem solving approaches**

There exist several problem solving approaches. Out of these approaches, the ones proposed by Polya (1985), Burton (1984), Kirkley (2003), Cherry (2011) and the Lorain County Community College (2011) appealed to me because of their circular nature.

2.2.6.1. **Polya's (1985) problem solving approach**

According to Polya (1985), problem solving consists of four stages namely understanding the problem, devising a plan, carrying out the plan, and looking

back. A learner is, initially, expected to understand the verbal statement of the problem. The problem solver is expected to analyse the problem situation to identify the unknown, the data, and the condition. When devising a plan, the problem solver should check whether he/she used all the relevant data. When carrying out the plan, the learner should check each step and verify if it is correct. Finally, the learner is expected to look back at the solution, to check if the solution and the line of reasoning employed are plausible and indeed correct. Wilson et al. (1993) devise a framework that illustrates the dynamic and cyclic interpretation of Polya's stages (See Figure 2.1).
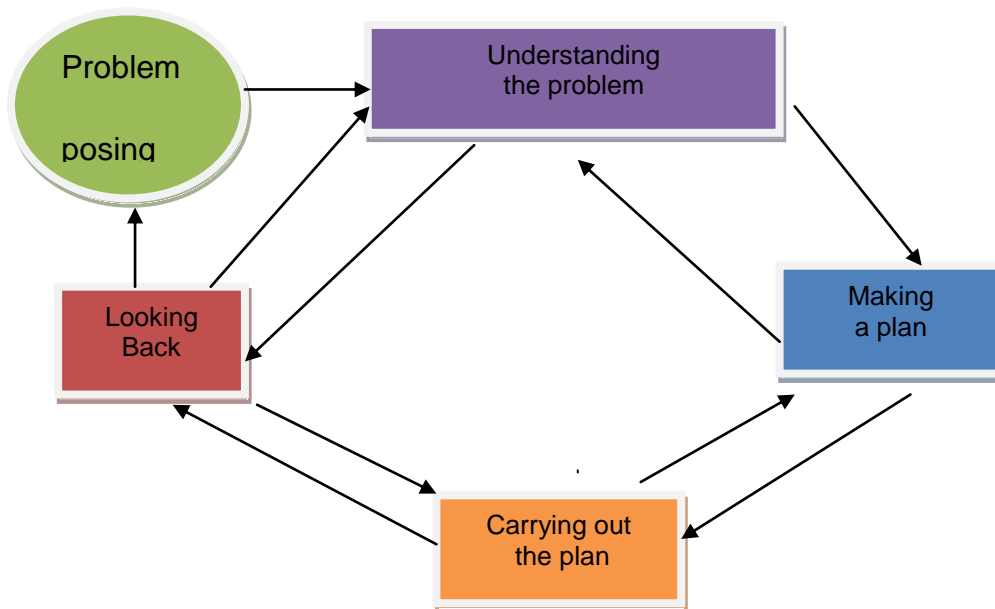
Figure 2.1: Wilson et al.'s (1993) framework of problem-solving that illustrates the dynamic, cyclic interpretation of Polya's problem solving stages.

[Adapted from http://jwilson.coe.uga.edu]

Wilson et al.'s (1993) framework of problem solving shows that genuine problem solving is dynamic and cyclic. The researcher says so, in the sense that a learner may begin by attempting to understand the problem posed. When understood the

problem at hand, the learner may, then, attempt to make a plan of how to resolve the problem. In the process of making the plan, the learner might discover that he/she does not understand the problem fully. As a result, he/she makes further effort to understand the problem. After understanding the problem and revising the plan, the learner may proceed to carrying it out in solving the problem. While carrying out the plan, the learner might see a need to revise the plan and implement it in solving the problem. After carrying out the plan, the learner might, then, look back to the solution, try to understand the problem further or pose another related problem to resolve.

2.2.6.2. **Burton's (1984) problem solving approach**

Burton (1984) proposes four phases of problem solving: entry, attack, review and extension. The phases of problem solving are considered to be cyclic. For instance, a learner initially seeks understanding the problem and devises approaches or strategies to 'enter' into problem solving. When a strategy is devised, the learner might go further to 'attacking' the problem by implementing the strategies devised. While attacking the problem, the learner might see a need to go back to the entry phase, probably, in search of more understanding of the problem or devising more effective strategies. After revising the entry strategies, the learner goes further to re-attacking the problem. When a solution is obtained, the learner reviews it and, even, extends the problem by reposing another problem, and the cycle starts again. Burton (1984) devises a framework of problem solving that depicts the cyclical nature of problem solving (see Figure 2.2).
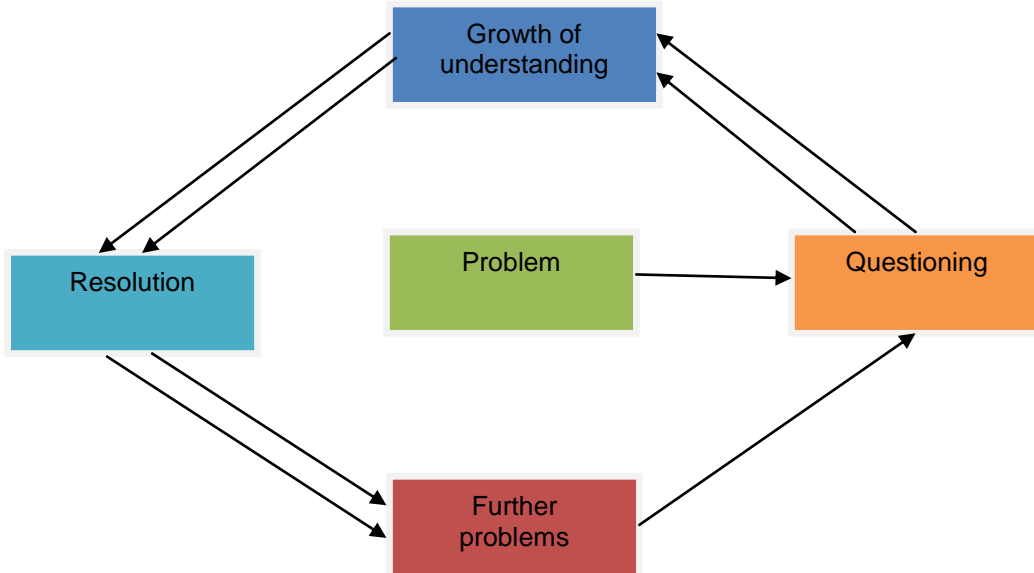
Figure 2.2: Burton's (1984) framework of problem solving.

Adapted from Burton (1984, p. 22)

Burton's (1984) framework depicts that problem solving is spiral. The framework shows that out of an initial problem and its first resolution can come out more new questions, new resolutions, and further problems. The spiral nature of the problem solving process makes it more engaging, interesting and even personal as a learner can choose to focus on any different aspect of the solution in order to ask further questions of his/her interest to resolve. As such, one of the roles of the problem solver is resolving rather than solving.

2.2.6.3. **Kirkley's (2003) problem solving approach**

Kirkley (2003) views the process of problem solving as cyclic. To illustrate the cyclic nature of problem-solving, Kirkley (2003) adopted Gick's (1986) model of problem-solving in his/her study (see Figure 2.3).
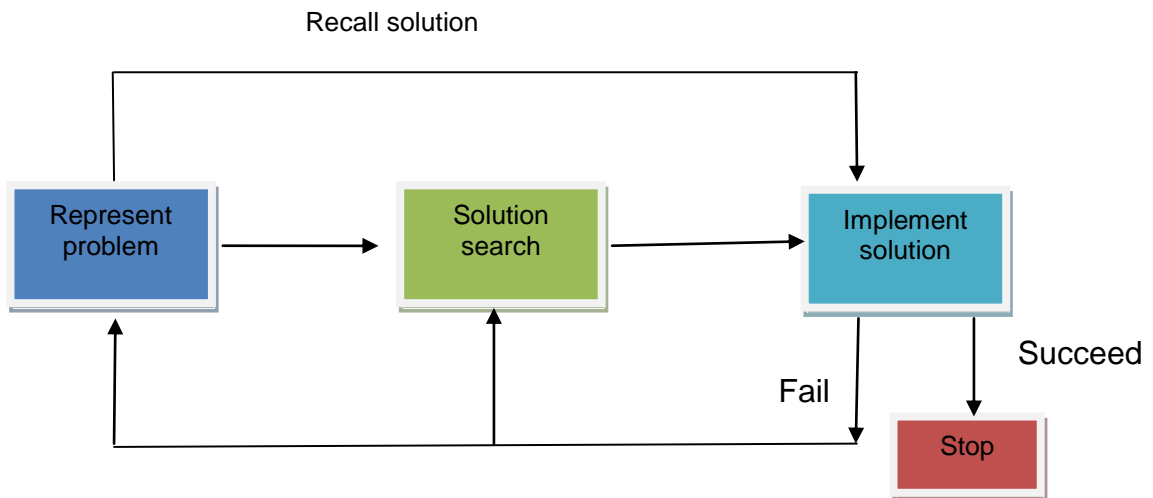
Figure 2.3: Kirkley's (2003) model of the problem solving process.

Adapted from Kirkley (2003, p. 4).

Kirkley's (2003) model of problem solving shows that if a learner has previously solved a similar problem, he/ she can employ a short cut route of just recalling the previous solution strategy and apply it again. If the problem is unfamiliar, the learner begins by representing the problem, searching for a solution and implementing the solution. If the solution implemented yields the desired results, then, the problem solving process stops. If it fails to yield the desired results, the learner might either search for another solution or look for another way to represent the problem, and the cycle begins again until a successful solution implementation is obtained.

2.2.6.4. **Cherry's (2011) problem solving approach**

Cherry (2011) suggests a 7- step cyclic approach to problem solving. The steps range from 'identifying the problem', 'defining the problem', 'forming a strategy', 'organizing information', 'allocating resources', 'monitoring progress', to 'evaluating the results'. Even though this cycle is portrayed sequentially, Cherry highlights that problem solvers do not follow rigidly these series of steps to find a

solution. A learner might skip some steps forwards and backwards a number of times in search of a desired solution. The cycle stops when the satisfactory solution is obtained.

2.2.6.5. **The Lorain County Community College (2011)'s problem solving approach**

The Lorain County Community College (2011) also propose six cyclic steps in problem solving, which are 'problem definition', 'problem analysis', 'establish your goals', 'generate possible solutions', 'analyze the solution', and 'implementation'. The six steps are considered to be cyclic in the sense that a learner begins problem solving by defining and analyzing the problem. Then, he/she establishes his/her goals (What he/she wants to do in a given situation). If the goals set are not clear, the learner may go back to re-defining and analyzing the problem. When clear goals are established, the learner may, then, generate possible solutions, analyse the solutions and implement them in problem solving. If the implemented solution is not effective, the learner may generate more solutions. If an effective solution is found, then the problem is regarded as being solved; otherwise the process has to be started again.

2.2.7. **Problem solving strategies (or heuristics)**

Schoenfeld (1985) defines heuristic as rules of thumb that can be used when solving problems. A similar definition is suggested by Mueller (1997) who defines a heuristic as consisting of either non-elementary operations which are not previously known by a learner or elementary operations that are not executed in a regular or uniform way in any given similar situations. This means that a group of learners might, possibly, present different solutions to the same problem even though they are using the same heuristics. Elia et al. (2009) classify heuristic strategies into cognitive and meta-cognitive strategies.

2.2.7.1. **Cognitive heuristic strategies**

Cognitive heuristic strategies include general strategies, such as working backwards, finding a pattern; using analogies; considering extreme cases;

modeling; systematic guessing and checking; and logical reasoning, just to mention a few (Depaepe, 2009; Dorlan, & Williamson, 1983; Engel, 1998; Ewen, 1996; Logsdon, 2007; Malouff, 2011; Muis, 2004; Russell, 2007a; Russell, 2007b; Stephenson, 2001). However we should note that although a heuristic guides a learner on how to obtain a solution in any given problem situation, it does not guarantee him/her success in obtaining the desired solution.

In this study, learners' cognitive approaches or strategies to problem solving were analysed in connection with their mathematics-related belief systems. The researcher traced the learner's line of thinking in problem solving in an attempt to reveal his/her pattern of thinking and action in resolving non-routine mathematical problems.

Concerning non-routine problems, Sweller et al. (2010) claim that seemingly problem solving experts apply a 'means-ends-analysis' strategy to problem solving. This unusual strategy involves, firstly, identifying the differences between a given problem situation and one's goal, and then looking for methods or problem solving strategies that aim at reducing those differences. The problem with this strategy is that there is no evidence of it being teachable to or learnable by learners. Problem solvers seem to apply it intuitively when faced with problems they have not previously encountered.

2.2.7.2. **Meta-cognitive heuristic strategies**

Meta-cognitive strategies are the strategies a learner applies in self-regulating his/her problem solving process. The self-regulating strategies include decomposing the problem situation, monitoring the correctness of the solution process, evaluating the solution process, and verifying the correctness of the final solution (Elia et al., 2009) (see section 2.2.9). Meta-cognitive strategies ensure a learner's success in problem solving. In problem solving, a learner uses his/her own discretion to choose the appropriate strategy to apply to a problem. When the chosen strategy proves fruitless, the learner may choose another alternative strategy to resolve the problem. This can only be possible if a learner

has an adequate bank of strategies to choose from. The ability to determine whether the chosen strategy is inefficient and to decide to abandon it in favour of another strategy is an important problem solving skill. In this study, the researcher analysed the solutions presented by the learner to check if he/she monitors and verifies or proves the correctness of the solution process. The learner's choice of a strategy and its use was analysed in an attempt to unravel his/her problem solving behaviour.

2.2.8. <u>Problem solving algorithms</u>

Mueller (1997) views an algorithm as consisting of relatively elementary operations that a learner executes in a uniform and similar fashion in solving certain types of problems. Since algorithms guarantee a learner success in problem solving, the process of carrying out an algorithm is not considered as problem solving (see section 2.3.). A learner can be considered as engaging in problem solving if he/she, firstly, creates an algorithm that he/she, then, applies to solve an unfamiliar situation (Wilson et al., 1993). Examples of mathematical algorithms are quadratic formula, Pythagoras theorem, and the sum of n-terms of an arithmetic sequence.

2.2.9. <u>Factors affecting problem solving</u>

Op't Eynde et al. (2006), Pimta, Tayruakham and Nuangchalerm (2009), Schoenfeld (1985), and Zhu (2007) posit that when people engage in mathematics problem solving a number of factors shape their behaviour. Some examples of such factors that affect learners' problem solving behaviour are 'knowledge base', 'heuristic methods', 'meta-knowledge', 'mathematics- related beliefs' and 'self- regulatory skills'.

The knowledge base is composed of all the contents of mathematics as a discipline, for example, mathematics concepts, theorems, symbols, and formulae. As such, the knowledge base forms the foundation on which problem solving is done. The possession or not of the relevant body of knowledge determines how the learner approaches problem solving. If a learner possesses

relevant knowledge, but is not able to apply it in problem solving, it may be necessary to assess why that information was not accessed or used during problem solving in order to understand his/her problem solving behavior (Muis, 2004).

Livingston (1997), Op't Eynde et al. (2006), and Panaoura, Philippou and Christou (2010) classify meta-knowledge into meta-cognitive knowledge and meta-volitional knowledge. Meta-cognitive knowledge is composed of three types of knowledge, which are person, task and strategy variables (Livingston, 1997). Knowledge of person variables consists of both general knowledge of how people learn and individual knowledge of how one learns and processes new material. In this regard, for instance, a learner should be able to identify conditions which are conducive to him/her for learning and avoid learning distractions for effective learning.

The researcher conjectures that a learner's beliefs on how mathematics content is learnt and processed might influence his/her approach to problem solving. For instance, if a learner heavily depends on the teacher to learn and solve mathematical problems, this might have a negative impact on exercising independent thinking and choice of problem-solving approach strategies.

Knowledge of task variables consists of one's knowledge of the level of difficultness of the task at hand and the level of thinking or reasoning to be applied to it. For example, a learner may be able to identify that some problems require more time to analyse and understand them, and plan their solutions before solving them than other problems. Knowledge of strategy variables consists of one's knowledge of cognitive and meta-cognitive strategies, and conditional knowledge of the right strategy to apply at a right time to an appropriate problem situation (Livingston, 1997). An example of a meta-cognitive strategy is self-questioning (Goliath, 2008), whereby a learner asks him/herself questions such as 'What am I asked to solve?', 'Is the problem completely solved?', and 'Is there other possible solutions?'

A learner's beliefs on the amount of time to take to solve a mathematical problem might determine the amount of time he/she allocates to resolve any given mathematical problem (Schoenfeld, 1992). Analysis of responses to mathematical problems given by learners in this study revealed, to some extent, the amount of time a learner had engaged in resolving each problem. Some of the guiding questions were, 'Did a learner allocate enough time to analyse and understand the problem before giving a solution to the problem?' and 'Does the solution process presented by a learner reveal that adequate thought has been given to the problem?' The kinds of problem-solving strategies a learner employed in solving a problem were valuable in this study. The different kinds of strategies a learner used to resolve a problem were noted and analysed in conjunction with his/her mathematics-related beliefs.

Meta-volitional knowledge refers to one's knowledge of what motivates him/ her to do something (Livingston, 1997). For instance, if a learner knows what motivates him/her to do mathematics, he/she can make use of that knowledge to engage in problem solving effectively. A learner's belief about his/her ability to do mathematics might motivate him/her to learn mathematics. Pimta et al. (2009) and Nichols (2008) contend that, for instance, self-efficacy enhances a learner's motivation to engage in problem solving. They argue that one's success in mathematics, for example, is a result of one's acknowledgement of his/her ability, setting a higher target or goal to achieve, and motivation to work hard to meet the goal. A learner's beliefs on his/her ability to learn and solve mathematical problems were of value in this study, as they could influence a learner's motivation to resolve a problem. One of the questions answered was, 'How does the learner behave in face of an obstacle in problem solving?'

According to Op't Eynde et al. (2006), a learner's mathematics-related beliefs consist of his/her subjective conceptions about mathematics education, the self as a mathematics learner and the mathematics classroom as a context of mathematics learning. Schoenfeld (1985) defines beliefs as a learner's view of mathematics that determines how he/she approaches problem solving. The way

the learner perceives mathematics tends to create a psychological context in which he/she learns mathematics or approaches problem-solving. In other words, beliefs shape how a learner engages in problem solving. Schoenfeld (1983) contends that a learner's consciously or unconsciously held beliefs about what is important in mathematics might affect effective retrieval of mathematics knowledge or 'cognitive resources' as he/she engages in problem solving. The effect of such beliefs is evidently seen when learners cannot easily retrieve and apply some information stored in their memory because of beliefs that make them inaccessible. In this regard, Otten (2010) argues that due to belief systems that do not allow connection to be made among learnt concepts, learners may be unable to solve problems, even if they have necessary resources required for problem solving (see section 2.3).

Self-regulatory skills include skills of one's ability to regulate his/her cognitive and volitional processes in problem solving (Lucangeli & Cabrele, 2006; Schoenfeld, 1992). In mathematics problem solving, self regulation involves one's strategic and effective application of meta-cognitive knowledge to obtain the desired results. Panaoura et al. (2010) state some of the self regulatory behaviours in mathematics problem solving, such as, identifying and clarifying problem goals, applying appropriate knowledge to solve the problem and monitoring the solution process for omissions, errors and progress.

For a learner to be successful in problem solving, he/she should have knowledge of when and where to use a problem-solving strategy. Failure to regulate one's problem solving process makes learners, for instance, to keep on pursing a wrong path towards a solution or accepting a solution that makes no sense with respect to the given problem situation. As such, self regulation enables a learner to learn, apply and consciously control cognitive problem solving strategies in mathematics. The use of either an appropriate or inappropriate problem-solving strategy to resolve a given problem was analysed in an attempt to reveal a learner's mathematics-related beliefs that were attached to the choice of appropriate strategies. In this study, knowledge of factors influencing problem

solving enabled the researcher to have a deep understanding of why learners exhibited certain problem solving behaviours.

2.2.10. <u>**Difficulties faced by learners in problem solving**</u>

In light of the stage models of problem solving proposed by mathematics education scholars (see section 2.2.6), learners' incompetencies in problem solving might be due to failure or difficulties in carrying out each stage. The researcher discussed difficulties faced by learners in applying a problem solving approach proposed by Brunning et al. (1999) of 'identifying the problem'; 'selecting an appropriate strategy'; 'implementing the strategy'; and 'evaluating solutions'. Firstly, learners face difficulties in identifying the problem. According to Brunning et al. (1999), some obstacles that hinder effective problem solving are as follows: Most learners do not have the habit of actively searching and isolating problems; learners exhibit deficiencies in comprehending the problem posed; learners cannot define a problem; and learners lack the skills to analyse a problem and to identify the 'givens' and the 'unknowns' in a problem.

Learners lack the skill of analysing and reflecting enough on the problem posed and their solution. The common phenomenon observed from the learners is that they rush into problem solving before they fully understand the problem situation. Brunning et al. (1999) argue that there is an association between time spent by a learner on initial stages of solving the problem and considering all possible solutions, and success in problem solving.

In this study, the solution process presented by a learner revealed his/her understanding of the problem situation, and, possibly, the amount of time taken to resolve a problem. Learners' beliefs on the importance of understanding both a question and its solution process, and reflecting on the solution process in an attempt to devise alternative solution strategies were unraveled. According to Polya (1985), there is no a problem one can exhaust all its possible solutions.

There is, always, room for one to improve his/her understanding of the problem as well as its solution.

Secondly, learners also face difficulties in representing the problem. There are two forms of problem representation a learner can do: (1) thinking about the problem abstractly without writing anything on a paper, and (2) modeling the problem situation in form of, e.g., a diagram, picture, table, or equation (Brunning et al., 1999; Posamentier & Schulz, 1996). Failure by a learner to represent a problem on paper might mean failure to identify and understand the problem, and, consequently, failure to effectively 'attack' the problem.

The advantage of representing a problem on a paper is that it summarises a word problem to a convenient state the mind can comfortably work on. For example, a paragraph of word statements can be reduced to one or a few algebraic equations. As such, a seemingly complex problem will be reduced to a simple problem. The use of a tangible external representation of a problem situation (e.g., table, graph, equation) can put to light all the 'givens' and 'unknowns' of a problem and minimises the amount of information a learner needs to remember to effectively solve the problem (Brunning et al., 1999). Thinking about a problem abstractly without representing it visually has a possible disadvantage of over burdening the mind by having so much information to remember. A learner can omit using some necessary details or conditions in the problem, and other possible solutions to the problem.

A study carried out in United States of America by Hegarty and Kozhevnikov (1999) on the relationship between visual-spatial representations and mathematical problem solving reveals two significant findings: (1) there is a positive relationship between use of schematic representations and success in mathematical problem solving, and (2) there is a negative relationship between use of pictorial representations and success in mathematical problem solving. A visual imagery is defined by Hegarty and Kozhevnikov (1999) as a representation of an object's appearance that we can see or imagine with our senses such as its

50

shape or colour. As such, a pictorial imagery is a construction of a clear visual image.

Other than concentrating on the outward appearance of an object, a learner can focus on the parts that make up an object and represent the spatial relationships between them. Hegarty and Kozhevnikov (1999) referred to this as production of a spatial imagery. Schematic imagery is defined by Hegarty and Kozhevnikov (1999) as a representation of the spatial relationships between objects. In mathematics, schematic imagery might refer to representing relationships between concepts, topics, and transformations of shapes.

The study revealed a negative association between use of concrete pictorial imagery and performance in problem solving because pictorial imagery cause the learners to focus on some irrelevant information on a problem situation other than the important relationships between the problem statements. The study revealed a positive association between schematic representation and performance in problem solving because it made learners to focus on mathematical relationships between or among problem statements and not consider concrete irrelevant details. This discussion highlights that not all visual spatial representations may enhance learners' success in problem solving. Some representations might distract learners from focusing on relevant relationships between objects that are essential in problem solving.

A study carried out in Cyprus by Pantziara, Gagatsis and Pitta-Pantazi (2004) on the use of diagrams in solving non-routine problems reveals that an efficient use of a diagram by a learner did not mean that he/she will resolve the problem correctly; and also, presentation of a correct solution to a problem did not mean that the learner used the diagram provided efficiently in problem solving. In their study, they discovered that learners perceived problems which were accompanied with diagrams as different from those same problems asked without diagrams. The learners failed to perceive the diagrams as an additional aid offered to them to solve the problems. In this regard, the diagrams provided

diverted the learners' focus from effective analysis of the problem at hand to other visible characteristics of the diagrams.

A seemingly contradictory discovery was done on grade 12 Namibian learners by Mogari and Lupahla (2013) who found out that learners performed relatively well on solving non-routine problems accompanied with diagrams than on problems without diagrams. They argued that diagrams might have aided learners in problem solving by simplifying the problem posed and illustrating the abstract concepts concretely. Pantziara et al. (2004) suggest that, for learners to make use of diagrams as an effective tool for problem solving, educators should carefully design instructional activities that promote diagrammatic literacy among learners.

In this study, learners' mathematical models or any form of mathematical representations of the problem situations were noted as an approach or strategy to problem solving. A learner's partial or complete use of all necessary conditions or restrictions in a problem and the ability to connect different parts in a model were examined in light of his/her mathematics-related beliefs.

Thirdly, learners face difficulties in selecting and implementing appropriate problem solving strategies. A study by Goliath (2008) on the cognitive strategy instruction aimed at improving mathematical problem solving of learners with learning disabilities reveals the following weaknesses learners have in problem solving: Learners have difficulties in abandoning ineffective problem solving strategies. They kept on persevering on the wrong path, even though it pointed to a dead end. The learners could not adapt problem solving strategies to other similar problem situations. They, also, faced difficulties in generalizing the strategies to some other similar problem situations and settings. A study carried out in Malaysia on the levels of problem solving abilities in Mathematics by Zanzali and Nam (1997) also reveals that learners lack a good mastery of problem solving strategies. They observed that some learners did not use suitable, effective strategies in problem solving. As a result, some learners failed to solve the mathematics problems within the prescribed time period because of

wasting a lot of time and effort in following wrong directions. Failure to abandon wrong strategies and try other alternatives caused the learners to fail solving problems they had great chances to resolve.

A study by Kaur (1998) on problem-solving strategies used by Singapore learners at Primary and Secondary school level discovered that learners, most of the time, worked out the solution for a problem using only one strategy. They lacked flexibility of trying another strategy if the first one did not work. Some contradictory findings were discovered by Elia et al. (2009) on their study on dutch learners' strategy use and strategy flexibility in non-routine mathematical problem-solving. They found that learners who exhibited inter-task strategy flexibility (switching strategies between problems) performed well in problem solving than learners who applied the same problem solving strategies throughout the problems. They, also, discovered that learners who exhibited intra-task strategy flexibility (switching strategy within a problem) did not perform well in problem solving than those who exhibited inter-task strategy. This last finding seems to oppose what is expected in problem solving on strategy application.

Though Elia et al.'s (2009) research study revealed that intra-task strategy flexibility did not promote successful problem solving among learners; in the current study, the researcher argues that learners should apply at least one approach or strategy in solving each of the non-routine problems. As such, the researcher encouraged the learners to resolve each problem using as many strategies as possible.

Similar findings on the difficulties faced by learners in problem solving were identified from a study by Yeo (2008) in Singapore on secondary learners' difficulties in solving non routine problems. Yeo (2008) identified the following difficulties faced by learners: Learners face difficulties in comprehending the problem situation posed. They lack knowledge of problem solving strategies. They face difficulties in modeling or representing problems mathematically. They

are not able to use correct mathematical facts, concepts and strategies in problem solving.

Lastly, some authors (e.g., De Corte & Malaty, 2010; Ernie, LeDocq, Serros & Tong, 2009; Gale Encyclopedia of Education, 2010; Op`t Eynde et al., 2006; Schoenfeld, 1985, 1992) contend that learners' beliefs related to mathematics can be an obstacle to problem solving. For instance, some learners were observed to give up too easily after only a short period of time because they viewed problem solving as a time-limited activity. To be specific, Schoenfeld (1985, 1992) found that learners solving mathematical word problems tended to give up after five minutes on the assumption that if the solution did not occur during this period it would not occur at all. Consequently, a belief that mathematics problems are a "five-minute problems" acts as an obstacle to effective problem solving.

In summary, Yeo (2008) cites 'traits', 'dispositions' and 'experiential background' as some characteristics of a problem-solver that cause difficulties in problem-solving. Traits include a learner's ability to draw relationships between mathematical concepts. Dispositions include a learner's mathematics-related beliefs. Experiential background includes a learner's past learning and experience in solving different types of problems. In this study, the researcher inferred learners' characteristics that could explain the difficulties they face in problem solving from their written responses or through interviewing.

Though a learner's success in problem solving (i.e., obtaining a correct answer) was not a matter of concern in this study, identifying learners' difficulties in problem solving was useful in explaining their problem solving behaviour in face of difficulties. Some of the questions to answer were: 'How does a learner approach a problem in face of difficulties?' and 'Is there a trend in problem solving behaviour exhibited by a learner in face of common problem-solving difficulties?'

2.2.11. <u>Some characteristics of good problem solvers</u>

Hardin (2002), Foster (2010), and Schoenfeld (1992) shed light on how good problem solvers resolve mathematical problems. Foster (2010) attempts to differentiate the problem solving behaviour of an expert from that of a novice by identifying four behaviours that can be inferred from an expert solution: (1) An expert, initially, makes an intensive qualitative analysis of a problem. (2) He/she, then, creates a mathematical representation of the problem situation based on the initial analysis made. (3) He/she begins problem solving from using general strategies to using specific strategies that meet the goals. (4) He/she makes a plan before engaging in problem solving. The characteristics and problem solving behaviours of good problem solvers can be used to identify learners within a class who might be good mathematics problem solvers.

From the study of college and high school learners solving unfamiliar problems in America, Schoenfeld (1992) discovered some behaviours that novices exhibit in problem solving. The learners read the problems quickly and did not analyse it fully. They quickly chose a problem solving approach to apply to it. They kept on working on the same approach strategy, even though it is not leading to the desired solution for the whole time period allocated for problem solving. As a result, the learners failed to resolve the problem successfully because they could not regulate and control their problem solving process.

Schoenfeld (1992) contrasts the above mentioned novice- approach to problem solving with that of an expert. By observing one of the mathematics faculty members solving a mathematics problem, Schoenfeld observed that an expert spends most of his/her time trying to understand the problem. The expert analyses and explores the problem until he/she has a clear and sure direction of resolving it. Unlike the learners who could not carefully monitor their solution, the expert was seen pursuing his/her approach strategies and abandoning strategies that seem to be fruitless. The expert could solve the problems because of the exhibition of this kind of problem solving behaviour.

2.2.12 <u>Difference between expert and novice problem solvers</u>

Expert and novice problem solvers can be differentiated from each other by use of three attributes: 'conceptual understanding', 'basic, automated skills', and 'domain-specific strategies' (Hardin, 2002). Conceptual understanding of mathematics enables an individual to make sense of new mathematical problem situations by use of his/her previously acquired knowledge of mathematics. The individual can link old and new concepts learnt and build a deep understanding of mathematics. Basic, automated skills are the mathematics problem solving skills that become deeply rooted in an individual because of repeated and sufficient learning of them. The skills become habitual and can be applied unconsciously by the individual without any effort of recalling them. This improves an individual's problem solving efficiency because it enhances his/her speed and accuracy in problem solving. The mathematics problem solving procedures that demand conscious thought from any individual are called 'domain-specific strategies' (Hardin, 2002).

The expert-novice differences that can be stated in terms of the above mentioned three attributes are as follows: As compared to novices, experts possess better conceptual understanding of mathematics. Experts possess and apply more automated skills and domain-specific strategies in problem solving than novices. Experts have declarative conceptual understanding of mathematics and procedural basic skills and strategies of problem solving (Hardin, 2002). Note that, declarative knowledge is one's knowledge of mathematical concepts, facts, theorems, objects, and so on; while procedural knowledge refers to one's knowledge of how to carry out a certain process or procedure.

From the discussion on some approaches to problem solving, factors affecting problem solving, some difficulties faced by learners in problem solving, and some novice-expert problem solving behaviours done above, the following deductions about the characteristics of good problem solvers can be made (Branca, 1980, p. 36):

1. The ability to understand mathematical concepts and terms.
2. The ability to note likenesses, differences and analogies.
3. The ability to identify critical elements and to select correct procedures and data.
4. The ability to note irrelevant detail.
5. The ability to estimate and analyse.
6. The ability to visualize and interpret quantitative or spatial facts and relationships.
7. The ability to generalize on the basis of few examples.
8. The ability to switch methods readily.
9. Higher scores for self-esteem and confidence, with good relationships with other children.
10. Lower scores for test anxiety.

In this study, a learner's problem solving behaviour could be explained in light of the problem solving characteristics or attributes he/she exhibits (or that can be inferred from his/her verbal or written responses). For instance, a learner's response to a mathematical problem might reveal his/her level of understanding mathematical concepts or terms, ability to note relevant and irrelevant details, ability to select correct and appropriate approach strategies, and the ability to switch methods readily; to mention a few. A learner's beliefs, e.g., in his/her ability to solve mathematical problems, number of possible solutions a problem might have, and amount of thinking to devote to a problem, may, somehow, influence his/her approach to problem solving.

2.3. **Mathematics-related belief systems**

In this section, firstly, the researcher discussed some definitions of mathematics-related beliefs, and belief systems suggested by several researchers in mathematics education. Secondly, he listed some typical learners' mathematics-related beliefs. Thirdly, he discussed several categories of mathematics related

belief systems suggested by some scholars. Lastly, he discussed the relationship between belief systems and mathematics learning and problem-solving.

2.3.1. <u>**What are beliefs?**</u>

Benbow (2004), Callejo and Vila (2009), Goldin, Rosken and Torner (2009), Kislenko et al. (2005), Lindenskov and Hetmar (2009), Osterholm (2009), Op`t Eynde, De Corte and Verschaffel (2002), Furinghetti and Pehkonen (2002), and Pehkonen and Pietila (2003) point out that the task of defining the construct of belief is a problem. The literature on beliefs reveals that mathematics education researchers disagree on both the definition of a belief and what are the categories of beliefs (Benbow, 2004). Lindenskov and Hetmar (2009) indicate that researchers disagree on whether beliefs should be regarded as phenomenon or as situated process and action. As such, researchers end up formulating their own definitions of 'belief' which might even be in contradiction with others (Pehkonen & Torner, 1999; Furinghetti & Pehkonen, 2002).

Callejo and Vila (2009), and Furinghetti and Pehkonen (1999) perceive beliefs as a learner's subjective knowledge of a specific mathematical content, him/ herself as a mathematics learner, and mathematics problem solving. Similarly, Pehkonen and Torner (1999) view beliefs as composed of a learner's interpretations of and conclusions about his/her everyday experiences or perceptions. They, also, view beliefs as under continuous evaluation and change as a learner compares his/her beliefs with his/her new mathematical experiences, and with the beliefs of other learners.

Pehkonen and Pietila (2003) understand beliefs as an individual's implicit and subjective knowledge constructed from his/her experience of some subject matter. Muis (2004) defines beliefs as constituting an individual's knowledge about the world. Muis (2004) sees beliefs as critical components that help to create meaning and establish overall goals that define the contexts for learning mathematics.

Schoenfeld (1992) perceives beliefs as made up of an individual's understanding and feelings which determine how he/she engages in mathematics problem solving. Lazim, Abu Osman and Wan Salihin (2004) view beliefs as an individual's personal knowledge that was constructed from some past experience that he/she uses, either consciously or unconsciously, to make sense of new experiences and to guide how to engage in mathematics learning and problem solving. Goldin (2002) defines beliefs as composed of an individual's cognitive or affective configurations to which he/she attaches some truth value. Ledger, Pehkonen and Torner (2002) view beliefs as a variable that is hidden to both mathematics teachers and learners that, unknowingly, affect learners' learning and problem solving.

A close analysis of these definitions of beliefs reveals some difference in the way scholars view beliefs. Some of them include the phrases 'beliefs are', 'beliefs constitute', and so on, in their definitions of beliefs which might be indicating that beliefs are static (Furinghetti & Pehkonen, 2002). Some of the scholars, for example, Schoenfeld (1992), Pehkonen and Torner (1999), and Lazim et al. (2004), perceive beliefs as dynamic; under continuous review in light of new experiences.

In the light of these definitions, the researcher views mathematics-related beliefs as an individual's subjective knowledge of the world, constructed from experience that guide and shape one's behaviour in mathematics problem solving. The researcher conjectures that learners' beliefs depend largely on their social experiences. As such, their beliefs are in continuous change due to new experiences that arise from interaction with mathematics content, teachers, parents, peers and any other influential parties. The beliefs they hold on mathematics determine their approaches to problem solving.

From the findings of several researchers (e.g. Cai, 2010; Ernie et al., 2009; Lucangeli & Cabrele, 2006; Mason, 2003; Schoenfeld, 1992) mathematics learners' beliefs that are related to mathematics include:

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem; usually the rule the teacher has most recently demonstrated to the class.
- Ordinary learners cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.
- Mathematics is a solitary activity done by individuals in isolation.
- Learners who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has little or nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention.
- Mathematics is associated with certainty.
- The difficulty in solving a mathematical problem is determined by the size of the numbers contained in the problem.
- All problems may be solved by the application of one or more arithmetical operations.
- The correct operation is determined by keywords which usually appear in the last question or sentence of the problem (it becomes thus not necessary to read it all).
- Mathematics is basically computation.
- The decision of checking or not the solution is not routinely necessary but it depends on the time which is available.
- Any problem assigned by a teacher always has a solution.
- Some learners just were not born to do mathematics.
- The learner's role is to receive mathematical knowledge by paying attention in class and to demonstrate that it has been received by producing correct answers.
- Mathematics teacher's role is to transmit mathematics knowledge and to verify that learners have received that knowledge.

The list of learners' probable mathematics-related beliefs was of great use in the current study in designing the beliefs questionnaires. The researcher checked if learners hold some of these beliefs and determined how the beliefs learners hold influence their problem solving behaviour.

2.3.2. **What is a belief system?**

Several scholars expressed their views on the definition of belief systems (e.g. Benbow, 2004; Callejo & Vila, 2009; Furinghetti & Pehkonen, 2002; Op`t Eynde et al., 2006; Schoenfeld, 1985). Schoenfeld (1985) views belief systems as a learner's perspective through which he/she views mathematics and how to approach problem solving. Pehkonen and Torner (1999), and Furinghetti and Pehkonen (2002) view an individual's belief system as made up of clusters of his/her conscious or unconscious beliefs. They argue that beliefs do not exist in isolation but in clusters which together form a person's belief system.

Furinghetti and Pehkonen (2002) describe the stages through which belief systems develop within an individual. Firstly, it begins as simple perceptual or authority beliefs. Secondly, new beliefs, expectations, conceptions, opinions and convictions develop from simple perceptual beliefs. Lastly, one develops a general conception of life that guides his/her actions. Benbow (2004) views a belief system as a description of the relationship and organization of one's beliefs. As such, Benbow (2004) likens a belief system to a cognitive structure. Under this perspective, a learner's belief systems can be viewed as undergoing continuous change as he/she evaluates his/her present beliefs against new mathematical experiences. A similar definition of a belief system is given by Callejo and Vila (2009) who, in adoption of Rokeah's (1968) definition in their study, define it as composed of one's psychologically, but not necessarily logically, organized beliefs about mathematics learning and problem solving.

Jin et al. (2010) describe belief systems as made up of a learner's beliefs about the nature of mathematics, how mathematics is taught and learnt, and the self as

a learner of mathematics. Goldin (2002) defines a belief system as a structurally organized set of beliefs that is socially or culturally shared. Op`t Eynde et al. (2006) define learners' mathematics-related belief systems as composed of a learner's consciously or unconsciously held subjective conceptions about mathematics as a subject, about him/herself as a learner of mathematics and about the mathematics classroom context. Op't Eynde et al. (2006) argue that beliefs interact with each other within a system and determine how a learner learns and engages in mathematics problem solving.

The difference this researcher observes on the above definitions is that some researchers define a belief system in terms of its components (Op`t Eynde et al.*, 2006; Schoenfeld, 1985); while some researchers define it as a way one's beliefs are organized (Benbow, 2004; Callejo & Vila, 2009). In this study, the researcher viewed mathematics related belief systems as perceived by both Schoenfeld (1985) and Op`t Eynde et al. (2006). As such, the researcher viewed mathematics-related belief systems as a learner's consciously and unconsciously held subjective conceptions about mathematics education, about him/herself as a learner of mathematics and about the mathematics classroom context that determine his/her approach to mathematics and mathematical tasks.

It is important to note that it is possible for learners to have the same set of beliefs which might not be organized into the same belief systems (Callejo & Vila, 2009). The difference in belief systems might occur as a result of differences in learners' dominant beliefs that influence their problem solving behaviour and the organization or structure of their belief systems. This explains why learners with the same set of beliefs, but different belief systems, might approach mathematical problem solving differently. The researcher discussed below some categories of mathematics-related belief systems.

2.3.2.1. **<u>Categories of mathematics-related belief systems.</u>**

Brunning et al. (1999), Daskalogianni and Simpson (2001a), De Corte and Op`t Eynde (1999), Diego-Mantecon, Andrews and Op't Eynde (2006), Jin et al. (2010), Lazim et al. (2004), Op`t Eynde et al. (2006), and Schoenfeld (1992) are some of the researchers who gave different categories of mathematics-related belief systems. The categories proposed by these researchers reveal a lack of consensus among them on the precise categories of beliefs. The researcher briefly discussed below the categories of belief systems suggested by these researchers.

Op`t Eynde et al. (2006) identify three components of learners' mathematics-related belief systems, which they illustrate in form of a triangle (see Figure 2.4).
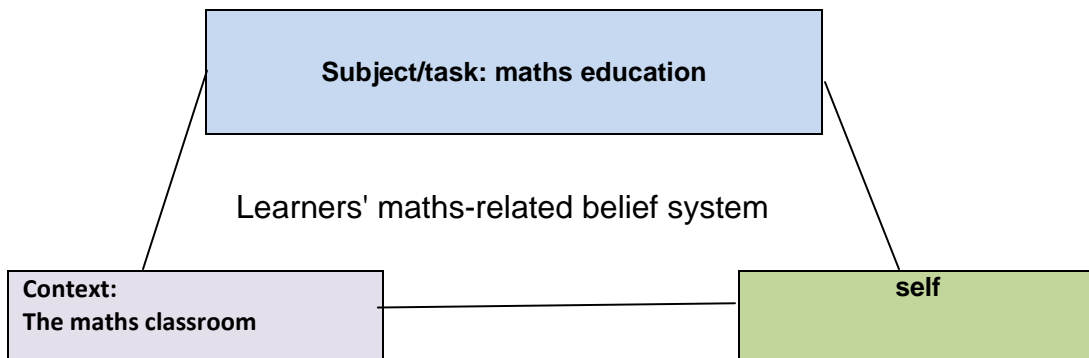
Figure 2.4: Components of mathematics-related belief systems.

Adapted from Desoete & Veenman (2006, p. 88).

Op`t Eynde et al. (2006, p. 88) identify the following categories and sub-categories of learners' mathematics-related belief systems:

1. Beliefs about mathematics education

    1.1. Beliefs about mathematics
    1.2. Beliefs about mathematical learning and problem solving
    1.3. Beliefs about mathematics teaching

2. Beliefs about the self as a mathematician

    2.1. Intrinsic goal orientation beliefs
    2.2. Extrinsic goal orientation beliefs
    2.3. Task-value beliefs
    2.4. Control beliefs.
    2.5. Self-efficacy beliefs

3. Beliefs about the mathematics class context

    3.1. Beliefs about the role and the functioning of their teacher
    3.2. Beliefs about the role and the functioning of the learners in their class, and
    3.3. Beliefs about the socio-mathematical norms in their own class

The triangle of belief systems depicts that learners' beliefs about mathematics education originate from their mathematics classroom, and serve their individual psychological and social needs, desires, and goals. Overall, the composition of a learner's beliefs about mathematics as a subject and how it is taught and learnt, beliefs about the self as a learner of mathematics, and beliefs about the classroom context make up his/her mathematics-related belief system.

The beliefs the learners hold do not only serve their individual psychological and social needs, but, also, control the way they perceive, judge or approach social situations (Op't Eynde & De Corte, 2003). In light of this discussion, in this study, after identifying learners' mathematics-related beliefs, the researcher revealed the psychological factors that determine the development of the beliefs. This enabled him to better explain how the identified beliefs were related to the mathematical problem-solving behaviours exhibited by the learners.

A study of upper sixth-form learners' beliefs about mathematics in Britain by Daskalogianni and Simpson (2001a) reveals that learners hold three dominant

macro-belief systems which can be used to predict their mathematics problem solving behavior. From the discoveries of their study, Daskalogianni and Simpson (2001a) describe learners' beliefs as structured; macro-beliefs make up the central belief system, while the micro-beliefs make up the peripheral belief system.

They identified "systematic", "exploratory" and "utilitarian" as the three key central macro-belief systems. They categorized the peripheral micro-belief systems into four belief systems that pertain to "nature of mathematics", "focus of exercises", "working in mathematics", and "didactical contract" (see table 2.2).

Table 2.2: Dsakalogianni and Simpson (2001a)'s Macro- and micro-belief systems. Adopted from Daskalogianni and Simpson (2001a, p. 15).

		MACRO-BELIEFS		
		Systematic	Exploratory	Utilitarian
MICRO-BELIEFS	Nature of mathematics	Methodical, logical	Problem-solving, linking things	Tool for other subjects, applied in life
	Focus of exercises	Follow series of steps	Understand different ways of thinking	Obtain correct exam answer
	Working in mathematics	Exact answer, similar exercises	Explore things, enjoy challenge	Known algorithms, study techniques
	Didactical contract	Dependence on notes and teacher	Dependence on own abilities	Dependence on teacher

We should note that a learner can hold all the three macro-belief systems, but at varying strengths. A learner's problem solving behavior is greatly influenced by his/her predominant belief system (Daskalogianni & Simpson, 2001a). In the light of this discussion, the researcher used learners' predominant beliefs as a criterion for categorizing them into these belief systems. The researcher briefly discussed below the micro- and macro- belief systems shown in table 2.2.

Learners whose macro belief systems can be classified as 'systematic' believe that mathematics is a static subject that is made up of a rigid body of knowledge.

They are at easy with exercises where they have to apply previously practiced methods or strategies. As such, they also view mathematics as a logical and methodical subject, and view mathematical exercises as tasks that have exact answers and whose solution process involves a series of steps. When working on problems, they constantly refer back to notes and depend much on the teacher.

Learners whose macro belief systems can be classified as 'exploratory' view mathematics as dynamic. They believe that new mathematical truths, concepts and approaches to problem solving are under discovery. They view mathematics as involving problem solving, and having more than one correct answer. They also enjoy the challenge of new exercises and they do look for connections or links between the concepts learnt. They depend much on their own abilities on learning and solving mathematical problems.

Learners, whose macro belief systems can be classified as 'utilitarian', view mathematics as a tool for other subjects and as a subject that can be applied to solving real life problems. They focus much on study techniques and expect to obtain correct answers in exercises or exams. They approach mathematical problem solving by applying well known algorithms and numerical techniques. Utilitarian believers tend to depend much on the teacher in learning mathematics and problem solving.

A number of researchers have shown that learners differ markedly on the way they view mathematics (e.g., Brunning et al., 1999; Hare, 1999; Lazim et al., 2004; and Schoenfeld, 1985). Ernest (1988, 1996) outlines three possible ways in which learners view mathematics. The first view of mathematics is the instrumentalist view in which learners perceive mathematics as an accumulation of facts, concepts, skills, rules and strategies that are used in problem solving. Consequently, they view mathematics as made up of some unrelated concepts, facts and rules that serve to solve problems. The instrumentalist view of mathematics relates much to the utilitarian macro-belief system suggested by Daskalogianni and Simpson (2001a).

The second view of mathematics is the Platonist view which perceives mathematics as a unified body of knowledge that is both certain and static (Ernest, 1988, 1996). According to this perspective, mathematics is not a product of humankind, but it is discovered. This view of mathematics has much in common with the 'systematic macro-belief system'. The problem solving view is the third view of mathematics (Ernest, 1988, 1996) that is related to the exploratory macro-belief system. Learners with this view perceive mathematics as created and invented by 'man'. As such, mathematics is seen as a cultural product that is dynamic (Lazim et al., 2004). Mathematics is also viewed as involving problem solving (see section 2.2.5) rather than solving routine problems or exercises.

Brunning et al. (1999) and Ernest (1991) identify three kinds of belief systems learners have about mathematical knowledge: dualist, multiplist, and relativist. The dualist learners view knowledge as either right or wrong. In addition, dualists tend to view knowledge as being absolute, universally certain, and accessible only to authorities (e.g. teachers). Multiplists view mathematical knowledge as sometimes right and sometimes wrong. Relativists view knowledge as uncertain and relative. They believe that knowledge must be evaluated on a personal basis by using the best available evidence (Brunning et al., 1999). They also believe that the truth of knowledge depends on its context.

Brunning et al. (1999) further identify four dimensions of beliefs about knowledge. (1) Simple knowledge; which refers to the belief that knowledge is discrete and unambiguous. Learners believe that learning is equivalent to accumulating a vast amount of factual knowledge. This belief system relates to the instrumentalist view of mathematics.

(2) Certain knowledge; which refers to the belief that mathematical knowledge is constant. Mathematical facts are perceived as remaining true forever once they are established and believed to be true. This belief system relates to the Platonist view of mathematics. (3) Fixed ability; which refers to the belief that a learner has some inborn abilities to learn mathematics. As such, a learner's ability to learn

mathematics cannot be improved by any use of effort or strategy in teaching and learning. (4) Quick learning; which refers to the belief that if a learner fails to understand mathematical concepts within the shortest possible time, then he/she will never understand them? An example of a belief in quick learning is "If a problem cannot be solved within ten minutes, it will never be solved" (Brunning et al., 1999, p. 164).

According to Schoenfeld (1985), learners' mathematics-related beliefs can be categorized into empiricist or rationalist belief systems. The learners with an empiricist view believe that insight into solving problems comes from accurate drawings or step-by-step procedures (Muis, 2004). Furthermore, Schoenfeld (1985) proposed that empiricists do not typically use mathematical argumentation, proof-like procedures, or logic, but they heavily rely on trial-and-error exploration of the problem situation to identify and test hypotheses. Schoenfeld argues that, since hypotheses are tested one by one, they engage in little metacognitive monitoring and control. In addition, all solutions reached are verified by empirical means.

In contrast, learners with a rationalist perspective believe that mathematics argumentation is important to the discovery of mathematics. Mathematics proof and deductive logic are viewed as important to problem solving and discovery of solutions. When relevant information is not available, rationalists derive the necessary information through the use of argumentation. During problem solving, rationalists constantly monitor progress at both the tactical and strategic levels, and plans are continually assessed and acted upon. Rationalists rely on mathematics argumentation, proof, and logic for the discovery and verification of solutions.

It is important to note the suggestion by Schoenfeld (1985) that empiricists or rationalists do not rely solely on one form. As such, empiricists can apply rational approaches as can rationalists apply empirical approaches to problem solving. What largely influences how they engage in mathematics problem solving is their predominant belief system. Consequently, empiricists predominantly rely on

empirical means to solve problems whereas rationalists predominantly rely on logical means to solve problems.

Ernest (1991, 1996), and Bishop (1996) identify two ways in which learners can view mathematics: absolutist and fallibilist. The absolutists view mathematics as made up of a body of knowledge that is certain, absolute, universally true and static. The fallibilists view mathematics as composed of uncertain knowledge that is developed through conjectures, proofs and refutations. The absolutist view of mathematics relates to both the Platonist and instrumentalist views, while the fallibilist view has much in common with the problem solving view of mathematics.

Jin et al. (2010) classified the Chinese high school learners' beliefs about mathematics, mathematical learning and teaching according to Ernest's (1991) five ideologies of mathematics education which are named industrial trainers, technological pragmatist, old humanist, progressive educator, and public educator (see table 2.3). From the analysis of their findings, they discovered that the Chinese high school learners hold multiple peripheral beliefs, and have belief systems that are in constant change. Their beliefs could easily change with time. For instance, the beliefs identified as peripheral could either disappear or change to core-beliefs after some time.

Table 2.3: Ernest's (1991) five ideologies of mathematics education.

Adopted from Jin et al. (2010, p. 140).

	Mathematics	Mathematics learning	Mathematics teaching
Industrial trainers	Set of Truth and rules	Hard work, effort, practice, rote	Authoritarian-transmission, drill, no 'frills'
Technological pragmatist	Unquestioned body of useful knowledge	Skill acquisition, practical experience	Skill instructor, motivate through work-relevance
Old humanist	Body of structured pure knowledge	Understanding and application	Explain, motivate pass on structure
Progressive educator	Personalized Mathematics	Activity, play, exploration	Facilitate personal exploration, prevent failure
Public Educator	Social constructivism	Questioning, decision making, negotiation	Discussion, conflict, questioning of content & pedagogy

In brief, the learners who are classified as industrial trainers view mathematics as composed of facts, skills and theories that remain true and unchangeable forever (Bishop, 1996). They believe that the aim of mathematics education is to enable learners to become numerate and grasp the "basics". The teaching and learning approach should be drill and rote, with emphasis on hard working. They might approach problem solving by applying memorized facts and rules. It is possible for them to apply wrong procedures or strategies to some problems because of failure to make sense of and relate what they learn in mathematics. As such, industrial trainers' view of mathematics relates to that of absolutists.

The technological pragmatist learners also hold views of mathematics that are related to those of absolutists. They view mathematics as an unquestioned body of useful information. Mathematics should equip learners with everyday life skills. In this respect, mathematics is seen as serving the industrial or economic needs of the country. In the light of this discussion, these learners can also be described as holding a utilitarian view of mathematics. The old humanist students view mathematics as a body of objective, pure knowledge. As such, they hold a platonic view of mathematics. In this regard, they view mathematics education as

the transmission of this pure and well structured mathematical knowledge. In order to effectively pass on this structured mathematics knowledge, the teacher has to assume a role of an explainer or lecturer. The teacher will act as the sole source of mathematical knowledge.

Learners who are classified as technological pragmatists or old humanists might heavily depend on their mathematics teachers for problem solving. When faced with a problem, they strive to recall facts, skills and procedures their teacher taught them or quoted in their text books. They rarely depend on their own logical reasoning when solving a problem. They are bound to fail solving non-routine problems because they cannot think outside the box.

Progressive educator learners also hold an absolutist view of mathematics; that can be best described as progressive absolutism. They view mathematics as personally constructed knowledge that can be learnt through investigation, exploration and cooperative work. The teacher's role is to facilitate learning other than transmitting knowledge. Public educator learners view mathematics as socially constructed; thereby rejecting absolutism. They see mathematics as knowledge that can be questioned and revised. As such, mathematics can be best learnt through discussion and cooperative group work (Ernest, 1991).

Learners who are classified as progressive educators or public educators might approach problem solving through reasoning. They are likely to seek different ways of solving a problem. Since they strive to make sense of and relate what they learn, they are likely to create different representations to a problem situation.

Table 2.4 gives a summary of the categories of mathematics-related belief systems as discussed above.

Table 2.4: Summary of categories of mathematics-related belief systems.

	Mathematics-related belief systems		
Daskalogianni & Simpson (2001a)	Systematic	Utilitarian	Exploratory
Enerst (1988, 1994 , 1996)	Platonist	Instrumentalist	Problem-solving
Enerst (1991) , Ernest (n.d.)	Industrial Trainers, Old humanist	Technological pragmatist, Progressive Educator	Public Educator
Schoenfeld (1985)	Empiricist		Rationalist
Ernest (1991), Bishop (1996)	Absolutist		Fallibilist
Brunning et al. (1999), Ernest (1991)	Dualist, Multiplist		Relativist

Though there is quite a number of belief systems, in this study, the researcher classified learners' beliefs according to Daskalogianni and Simpson (2001a)'s belief systems (systematic, exploratory and utilitarian). Then he discussed the learners' predominant beliefs that determine their mathematical problem-solving behaviour.

2.4. <u>Theoretical framework</u>

The theories that guided, supported and informed this research study were problem solving theories (behavioural, cognitive, and information processing), and mathematics-related belief theories (foundations, coherence and complexity theories). The researcher discussed the theories because "all belief systems are based on theories and ideologies" (Sorochan, 2011, p. 2); whereas beliefs drive learners' behaviour in non-routine mathematics problem solving (Levy, West & Rosenthal, 2012).

2.4.1. **Mathematics problem solving theories**

2.4.1.1. **Behavioural problem solving theory**

Thorndike is one of the historical behaviourists who contributed to the behavioural approach to problem-solving (Andre, 1986; Brunning et al., 1999). Behaviourists emphasise the role of both positive and negative reinforcements in enhancing the development of problem solving skills in learners (Hardin, 2002). For example, a teacher can use both positive and negative comments to encourage good problem solving practices among learners. Trial-and-error is one of the common approaches to problem solving promoted by the behaviourists. Trial-and-error is an approach to mathematical problem solving that involves the use of different methods in search for a solution. The trials of different methods end when a desired solution is found. Though this behaviourists' view is heavily criticised for allowing little room for thought and planning in problem solving, trail-and-error is one of the heuristic strategies learners apply today in solving some class of problems.

In the current study, the researcher checked if learners applied trial-and-error approach to solve some non-routine mathematical problems. An attempt was made to identify beliefs that drove learners to adopt certain mathematics problem solving approaches within given problem situations. Rokeah (1968) claims that a learner's behaviour in problem solving is a complex function of sets of beliefs. The belief systems predispose a learner to approach non-routine mathematical problem-solving in certain ways.

2.4.1.2. **Cognitive problem solving theory**

John Dewey, Kohler, and Wallas are some of the historical cognitive psychologists that contributed to the cognitive approach to problem-solving (Hardin, 2002). Cognitivists identified the mental stages through which problem-solving proceeded. In contrast with Thorndike who emphasises the behavioural

component of problem solving, Dewey views problem solving as composed of both thinking (cognitive) and behavior (Brunning et al., 1999). In problem solving, a learner consciously and deliberately applies thinking. Dewey proposes a problem solving approach that involves the application of a sequence of steps: (1) Presentation of the problem, (2) Defining the problem, (3) Developing hypotheses, (4) Testing the hypotheses and (5) Selecting the best hypothesis. Dewey describes the problem solving steps as occurring in a natural sequence in problem solving.

Kohler, a Gestalt psychologist, developed an approach to problem solving that emphasises the role of a sudden reorganization of mental elements into a structure that provides a solution to a problem (Andre, 1986). This sudden reorganization was called insight experience. From the Gestalt's point of view, it is evident that they consider the bulk of problem solving process as done mentally without recording all the thinking processes one has undergone. The problem solver will then, communicate on paper when a sudden insight to the problem has emerged. Given the great demands of problem solving, the failure to breakdown the problem solving process into stages, as done by Dewey, and evidently showing some form of communication on paper at each stage of problem solving might over-burden the memory, makes problem solving even more difficult.

A learner needs some form of external representation of every solution phase to alleviate the pressure on his/her short term memory (see section 2.4.1.3), and be in a position to review the solution process. In this study, the researcher encouraged the participants to either think aloud or put on paper all the thinking processes they underwent to reach a solution. This could, possibly, assist them to monitor and regulate their problem solving process. This, also, assisted the researcher to keep track of and be able to analyse their approaches to non-routine problem solving.

Wallas analyses problem solving into a series of stages: (1) preparation, (2) incubation, (3) inspiration, and (4) verification (Andre, 1986). In the preparation

stage, the problem solver analyses the problem, and gathers relevant information. In the incubation stage, the problem is considered subconsciously while the problem solver is relaxing or considering something else. During the inspirational stage, the solution to the problem comes to the learner unexpectedly – in a similar fashion to the insight experience of the Gestalt psychologists. The verification stage involves checking the solution and working out the details (Andre, 1986).

What is unique to Wallas' problem solving model is the inclusion of two stages of incubation and inspiration that are not commonly found in other step models proposed by other researchers. These two stages describe the process of problem solving as subconscious unlike other researchers (e.g. Dewey) who describe problem solving as consisting of a rational series of steps (Brunning et al., 1999). Consequently, the analysis of the incubation and inspiration stages is difficult because of their subconscious nature.

2.4.1.3. Information processing problem-solving theory (IPT)

The IPT is a theory that explains how learners perceive, store, and remembers the information they receive everyday (Ganly, 2007). Huitt (2003) illustrates in form of a diagram how information flows from sensory memory, via short-term memory, to long-term memory. Huitt (2003), also, indicates attention, rehearsal, chunking, encoding, and retrieval as processes that enable information to move or transfer from one memory stage to the next (see figure 2.5).
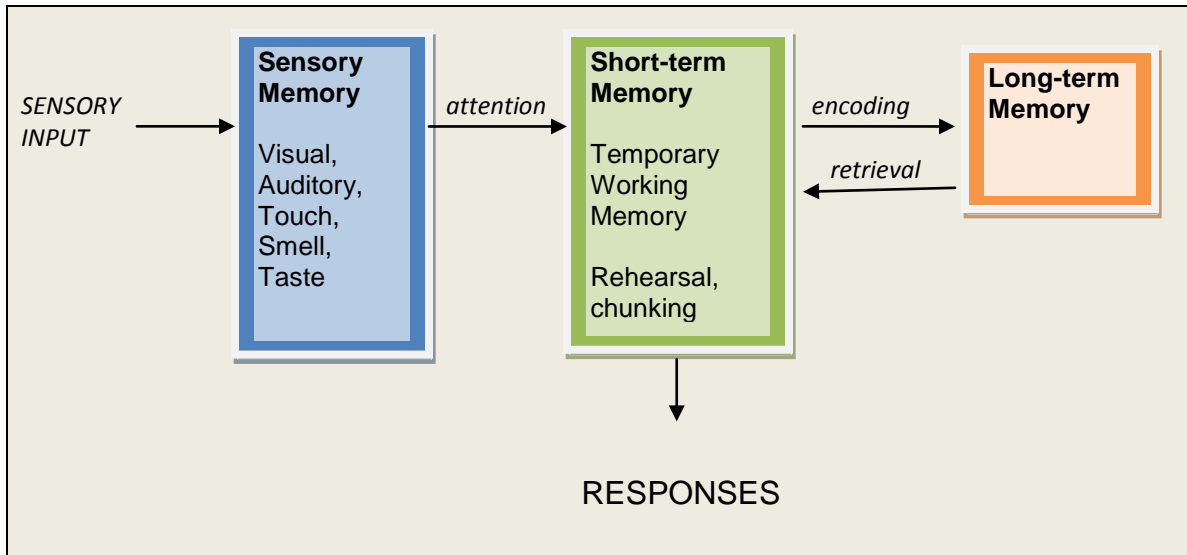
Figure 2.5: How information flows according to information processing theory.

Adapted from Huitt (2003).

The information processing problem-solving theory views a learner's capacity of the working memory, the filing system of information in long-term memory, and retrieval of relevant information from long-term memory as factors that are important in mathematics problem solving (Hardin, 2002). A learner's belief system might affect how he/she processes the mathematics information received. Sorochan (2011) likens a belief system to an operating system of a computer that controls how one sorts and files all the input data. Upon receipt of mathematical information, a learner's personal operating system (mathematics-related belief system) breaks down, reorganises, and links all input mathematical facts and concepts in line with what he/she believes. For example, a learner who believes that problems are best solved by applying textbook procedures may resort to memorizing the procedures without understanding why they work in solving the problems. Because of failure to link concepts learnt and file them appropriately, the learner might fail to retrieve the relevant knowledge learnt in face of a problem.

In the light of the IP theory, the researcher views a learner's mathematics-related beliefs as outputs of the information processed in his/her mind. The information about mathematics learning and problem solving that learners perceive, remember and think about basically comes from their immediate mathematics classroom, school or family environment. The researcher conjectures that learners might be subjected to the same environmental stimuli or mathematical information, but construct or develop different beliefs out of it. Different predominant belief systems that learners hold would, then, be manifested in their problem solving behaviours.

2.4.2. **Mathematics-related belief theories**

2.4.2.1. **Foundations mathematics-related belief theory**

The foundations mathematics-related theory claims that a learner accepts or holds beliefs that he/she has justification for their existence (Gardenfors, 1989). As such, every learner needs to keep a traceable record of the justifications for his/her mathematics-related beliefs. Gardenfors (1989) posits that a belief might be justified by one or several other beliefs or by itself. Gardenfors (1989), also, argues that if a belief is justified by several independent beliefs, then, the belief may remain held by a learner even when some of the justifying beliefs were removed.

A learner's endorsement of belief systems is believed to be affected by several factors such as his/her family members (e.g., parents, siblings), the community members (e.g., school, peers, friends), and the mass media (e.g., television, radio) (Levy et al., 2012). Levy et al. (2012) believe that a learner holds belief systems that serve his/her epistemic, psychological and social needs. An example of a learner's epistemic need is understanding and explaining mathematical concepts, facts, theories and procedures. An example of a psychological need is developing self-esteem. An example of a social need is forming and maintaining good relationships with teachers, classmates or peers,

and family members. As such, within a given environment, a learner tends to receive and hold belief systems that serve his/her individual needs.

In the current study, learners were asked to justify their confessed mathematics-related beliefs. The reasons revealed other beliefs that served as justifications for holding the beliefs in question by the learners. The justifications furnished by the learners revealed the epistemic, as well as psychological, centrality of the beliefs. Thus, the reasons assisted in unraveling the learners' mathematics-related belief systems. In the light of Levy et al.'s (2012) theory on belief systems, learners' differences in choice and application of approach strategies to non-routine mathematics problem solving could be explained in terms of learners' receptiveness to different predominant belief systems that support their individual needs. Thus, each learner's prevalent belief system largely influences his/her behaviour in a given mathematics problem situation. Levy et al.'s (2012) theory on beliefs points to the power of beliefs in driving learners' behaviour in mathematics problem-solving.

2.4.2.2. **Coherence mathematics-related belief theory**

In contrast to the foundations theory, the coherence theory claims that beliefs do not require any justification but are justified just as they are (Gardenfors, 1989). A belief system is expected to be logically consistent or structured. As such, a belief should cohere with the other beliefs within the system. According to this theory, even though a learner holds multiple (even contradictory) beliefs, 'inconsistency' is an irrelevant consideration because all beliefs are considered to be constructed from, and situated in, environmental situations a learner experiences (Callejo & Vila, 2009). All the beliefs of a learner are considered as equally fundamental. The main drawback of the coherence theory is its failure to recognize some beliefs as justifications for other beliefs.

Since one's beliefs are coherent when they are mutually supporting, the coherence theory guided me in interpreting the link among a learner's beliefs.

Hamilton and Mineo (1996) discovered that a learner's belief system is structured hierarchically, ranging from peripheral beliefs to the central beliefs (see figure 2.2). Similarly, Daskalogianni and Simpson (2001a) discovered three coherent categories of central macro-beliefs (systematic, exploratory and utilitarian) and their peripheral micro-beliefs. The central belief systems are believed to change relatively slower than the peripheral belief systems that could easily change in light of new experiences.
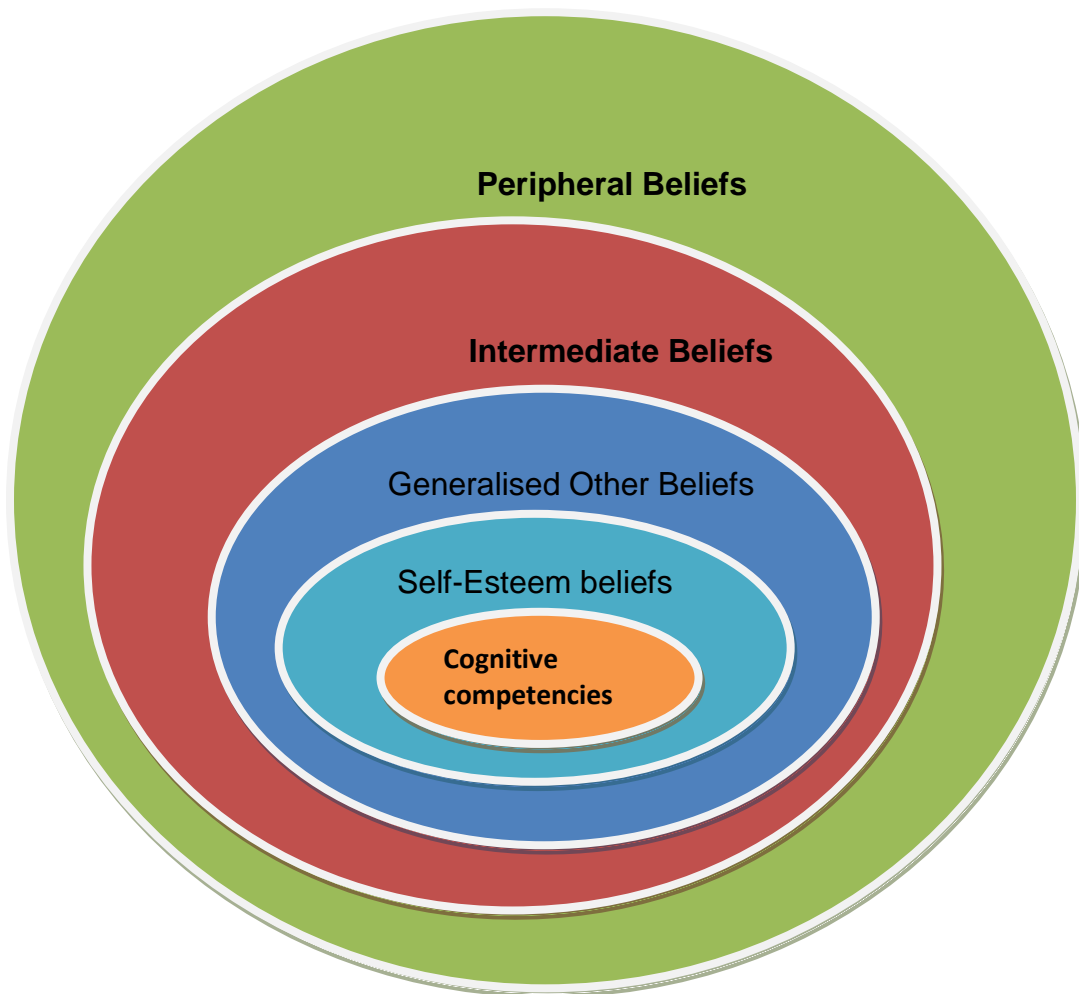
Figure 2.6: Hierarchical structuring of the belief system (Hamilton & Mineo, 1996)**.**

Hamilton and Mineo (1996) view a belief system as composed of three regions: (a) central region (cognitive competencies, self concept beliefs and generalized other beliefs), (b) intermediate region (intermediate beliefs), and (c) peripheral region (peripheral beliefs). The number of connections of beliefs within a belief system is viewed as increasing from the peripheral region to the central region. As such, central beliefs can be regarded as relatively more stable than the peripheral beliefs. Hamilton and Mineo (1996) view beliefs as developing over time in a hierarchical sequence; cognitive competencies develop into self-esteem beliefs, self-esteem beliefs develop into generalized other beliefs, generalized other beliefs develop into intermediate beliefs, while intermediate beliefs finally develop into peripheral beliefs.

2.4.2.3. **Complexity theory of mathematics-related belief systems**

The complexity theory views a system as composed of a number of different components that are related to each other. The interactions of a system's constituent parts result in change of the system. As such, systems are viewed as dynamic. The complexity theory tries to explain non-linear relationships that exist between dynamic systems (Manson, 2001). According to the complexity theory, relatively simple interactions between or among components of a system might result in emergent complex behaviors. A system could be better understood if one traces the relationships among its components. Manson (2001) contends that a system's emergent or synergistic characteristics can be better understood by studying and revealing the relationships that exist among its components.

A learner's belief system can be conceptualized as a complex system, in the sense that it is dynamic. Complexity theory can be used to understand and explain learners' mathematics-related belief systems which seem to be complex. A belief system is composed of individual beliefs that autonomously interact among each other. The interaction of beliefs within a system result in emergence of other beliefs, and ultimate behaviours linked to the beliefs. Learners' behaviours in non-routine problem solving could be better understood when one

studies the relationship between a behavior and a belief system as a whole than relationship between a behaviour and an individual belief within a system.

Beswick (2006) views belief systems as complex systems which have properties or characteristics that transcend those of the individual beliefs that constitute the whole system. In other words, Manson (2001) posits that the influence of a belief system on a learner's problem solving behaviour is greater than the sum of the effects of its constituent individual beliefs. In the light of the complexity theory, a learner's problem solving behaviour could be attributed to the interactions among the beliefs within a system, not to the additive effects of single beliefs. In addition, learners' problem solving behaviours cannot be attributed to the actions of any individual beliefs within a belief system. In the current study, learners' beliefs were studied as a system, not as isolated beliefs within a system.

The internal structure of a belief system is defined by the degree of relationships among its constituent beliefs (Manson, 2001). Individual beliefs that are strongly related are believed to form sub-systems. As a point of note, it is possible for any given single belief to be inter-related to multiple sub-systems. It is also possible for the belief systems or sub-belief systems to be nested with the other belief systems. For example, a learner's beliefs about learning mathematics could be nested with beliefs about oneself as a mathematics learner.

The possible interconnectedness of belief systems gives rise to the problem of unraveling the possible relationship between a specific belief system and a problem solving behaviour of a learner. All the interacting belief systems act as a complex system that has a collective influence to a learner's behaviour in mathematical non-routine problem solving. In the current study, the researcher assumed that a learner's prevalent or predominant belief system has more influence on the learner's problem solving behaviour than the other belief systems. As a result, the study was limited to the relationship between a learner's predominant mathematics-related belief system and his/her approach to non-routine mathematical problem solving. Daskalogianni and Simpson (2001a, 2001b) and Levy et al. (2012) espouse such an assumption.

2.4.3. **Relationship between belief systems and mathematics learning and problem solving**

A number of scholars in mathematics education have shown greater interest in studying learners' mathematics-related beliefs (e.g. Callejo & Vila, 2009; Daskalogianni & Simpson, 2001a, 2001b; Jin et al., 2010; Spangler, 1992). Their studies focused on describing and categorizing learners' mathematics-related beliefs and unraveling the relationship between beliefs and other constructs such as learners' performance, problem-solving behaviour, mathematics learning, motivation and meta-cognition.

From the study of learners' belief systems in relation to approaches to mathematical problem-solving, Goldin et al. (2009), and Callejo and Vila (2009) discovered a complex relationship between them. As a result, they could not identify any causal relationship between learners' belief systems and their problem solving strategies.

Daskalogianni and Simpson (2001a), Jin et al. (2010), Lazim et al. (2004), and Spangler (1992) focused their studies mainly on describing and categorising learners' mathematics-related beliefs. However, Spangler (1992) went further to inferring the relationship between beliefs and learning, and observed a cyclic relationship between them. For instance, learners' classroom learning experiences might influence development of beliefs about mathematics learning, while, on the other hand, learners' beliefs about mathematics might influence how they learn mathematics. Daskalogianni and Simpson (2001a) also extended their study to inferring the relationship between learners' belief systems and the way they respond to mathematical problems. In this regard, they argue that belief systems might be used to predict learners' mathematical problem solving behaviours and approaches in a given problem (see section 2.3.2.1).

De Corte and Op't Eynde (1999), and Schoenfeld (1992) unraveled learners' belief systems that relate to mathematics learning and problem solving. Their

studies revealed that a learner's mathematics-related beliefs determine how he/she approaches a problem. They, also, discovered that the beliefs a learner holds about mathematics determine the choice of problem solving strategies the learner applies. In this regard, Schoenfeld (1992) argues that the problem solver's beliefs about mathematics might determine the 'cognitive resources' that will be available to him/her in problem solving. This argument is in line with Hardin's (2002) argument that beliefs might hinder the retrieval of relevant information from a learner's memory in mathematics problem solving.

Dorman, Adams and Ferguson (2003), Hassi and Laursen (2009), Kislenko et al. (2005), Mason (2003), and Marcou and Philippou (2005) studied beliefs in relation to achievement or performance in mathematics. Generally, their studies reveal that there is a relationship between learners' mathematics beliefs and their achievement in mathematics. For instance, Mason (2003) inferred that the learner's mathematics-related beliefs can act as a predictor of his/her achievement in mathematics. Hassi and Laursen (2009) discovered a positive correlation between learners' beliefs and their gains in mathematics courses. They argue that learners' level of attendance to and learning mathematics might be determined by their beliefs about mathematical knowledge and problem-solving, and about their mathematical problem solving capability.

Muis (2004) discovered that learners who strongly believe in quick-learning may set a maximum time they will engage in a particular task without considering the complexity or difficulty of the task. As a result, learners rush into problem solving without enough problem analysis and understanding. In contrast, learners who strongly believe in gradual learning are more likely to examine the problem and then decide how much time is needed to solve the problem. As a result, a well planned, strategic approach toward problem-solving may be used. Similarly, Schoenfeld (1992) discovered that learners who believe that those who understand the content can solve a mathematical problem posed within a shortest possible time might give up solving the problem after some few minutes of unsuccessful attempts, even though they have a potential to resolve it.

Lucangeli and Cabrele (2006), and Pehkonen and Pietila (2003) studied the relationship between beliefs and knowledge in mathematics education. In their study, Pehkonen and Pietila (2003) describe a learner's beliefs as his/her subjective knowledge that is based on personal experience. They also clarify that beliefs represent some kind of tacit knowledge that is unique to learners because of their different experiences and interpretations of teaching and learning situations. In this regard, Muis (2004) discovered that learners who believe that mathematical problem solving is basically application of facts, rules and formulae tend to rely on memorization as the main method for learning. Similarly, learners who believe that problems presented in mathematical textbooks can only be solved by the methods suggested in those textbooks tend to rely on memorizing the methods presented in the textbook. When solving textbook exercises, they attempt to remember the methods given in the book rather than attempting the problems through reason.

In addition, Muis (2004) discovered that learners who believe that the source of mathematical knowledge is some authority figure, e.g., the teacher, might not attempt to derive the knowledge on their own; they tend to rely on memorizing formulae and procedures and do not engage in attempting to understand the nature of the question. Similarly, learners who believe that knowledge consists of isolated facts, tend to memorise lists of definitions as a strategy for understanding. Failure to consider relationships among facts enables these learners to fail to engage in effective transfer of knowledge learnt to some similar problem situations.

The above discussion illustrates some behavioural characteristics of learners who hold certain mathematics-related belief systems. The discussion provides some examples of behaviours and sets of beliefs that guided the researcher in classifying learners into belief systems. The interpretations of probable learners' behaviours in mathematics learning and problem-solving served in this study as background knowledge of how to interpret learners' behaviours in mathematics problem-solving in relation to their espoused beliefs or beliefs 'in action'. This

detail also served as a yardstick to compare findings in this study with the findings of research studies carried out at other different settings by other researchers; and point out similarities or differences that emerged.

Though a quotable number of studies were done on beliefs in relation to other constructs such as learners' performance, problem-solving behaviour, mathematics learning, motivation, meta-cognition, just to mention a few, there is still a lack of clarity on how learners' beliefs are related to their mathematical problem-solving. Callejo and Vila (2009) point out that research findings regarding the relationship between beliefs and problem-solving behaviours present seemingly contradictory conclusions. For example, Daskalogianni and Simpson (2001a) assume a causal relation; Spangler (1992) views the relationship as cyclic, in the sense that beliefs influence problem-solving behaviours while, on the other hand, problem-solving behaviours influence development of beliefs; and Callejo and Vila (2009), and Goldin et al. (2009) assume no causality, but rather view the relationship between them as complex.

De Corte and Op't Eynde (1999) point to inadequacy of theory on learners' mathematics-related beliefs. They argue that this results from studying specific learners' belief systems in isolation from other belief systems or constructs. An intensive study of learners' mathematics-related beliefs in relation to other constructs, e.g., mathematics problem solving, might yield more theory on how beliefs function.

In agreement with De Corte and Op't Eynde (1999), Jin et al. (2001) posit that within a system interact with each other, and are, therefore, interrelated. In this regard, Di Martino (2004, p. 273) argues that it is possible for a single belief to be linked to different beliefs in different learners. By virtue of belonging to different belief systems in different learners, the same single belief might elicit different problem solving behaviours in different learners. This discussion highlight the fact that the problem solving behaviour a learner exhibits might not be simply linked to a single belief, but to the interaction of beliefs within a system.

Beswick (2011) acknowledges that an individual's beliefs are complexly related to one another; and, as such, it is difficult to unravel their relationship with mathematical behaviour. From Beswick (2011)'s discussion of belief systems, it is possible that beliefs might be interwoven with other beliefs, even with other beliefs that are situated in other different clusters, to such an extent that conflicting beliefs might be held by one person; and the relationship between behaviour and a specific belief system is difficult to identify.

In the light of the above discussion, the researcher argues that the study on learners' mathematics-related beliefs should be done in relation to other mathematical constructs. De Corte and Op't Eynde (1999) believe that a study of learners' mathematics-related belief systems in relation to other construct (e.g., problem solving) might highlight, for instance, how beliefs are related to mathematics problem solving.

In this context, this study attempted to determine if there is an observable relationship between, the 10th, 11th and 12th graders' mathematics-related belief systems and their approaches to non-routine mathematical problem-solving. In summary, a diagrammatic illustration of my theoretical framework is as follows:
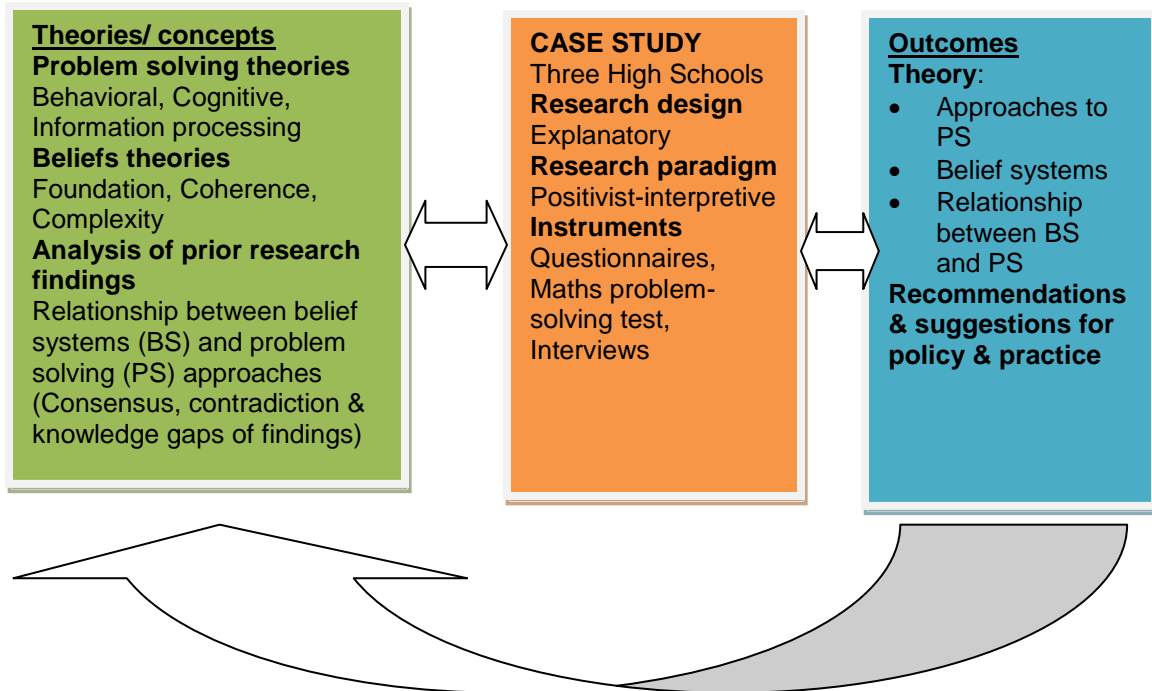
Figure 2.7: Theories/concepts, research case study and research outcomes.

The diagram shows that the theories guided and informed the research study done. The diagram depicts the stages undergone in the study as cyclical (indicated by the arrows). For instance, after designing the research methods, there was need to re-view the related literature again and possibly revise the research methods. After obtaining and analysing results, there was, also, need to compare them with some findings of other researchers who studied some similar problems. The effectiveness of the research methods used in the study was, also, assessed in the light of the research findings discovered.

2.5.   **Summary and conclusion of the chapter**

Categories of mathematical problems were discussed in an attempt to clarify and distinguish non-routine mathematical problems from other kinds of mathematical problems. Some definitions of mathematical problem solving proposed by other scholars were discussed. A working definition of non-routine mathematical problem solving was derived from insights obtained from definitions suggested by

the other scholars, for example Xenofontos and Andrews (2008). Examples of strategies learners employ in non-routine problem solving were discussed. When analysing learners' written responses to mathematics non-routine problems, it might be necessary to check if learners used some of these strategies.

Several categories of mathematics-related belief systems were discussed. Out of these belief systems (see section 2.3.2.1), the researcher decided to categorise learners' beliefs into belief systems according to Daskalogianni and Simpson (2001a)'s macro-belief systems. Daskalogianni and Simpson (2001a)'s macro-belief systems appealed to the researcher because they provided examples of beliefs that fall into each category which guided him to classify learners' beliefs more appropriately into the respective belief systems. Theories on problem solving and belief systems were discussed. The theory shed light on how learners solve problems and how beliefs develop and function within a learner. The chapter concluded by discussing the findings of other scholars on the relationship between approach to non-routine problem solving and mathematics-related belief systems. Their findings were compared with the discoveries in this study.

Chapter 3

Methodology

3.1. <u>Introduction</u>

The researcher, firstly, discussed the research design. Secondly, he described the population under study. Fourthly, the instruments used to collect the data were described. Fifthly, the procedure employed in data collection was described. Lastly, some ethical issues taken into consideration in the study were explained.

3.2. <u>Research Design</u>

This study employed a mixed methods approach to determine and explain the nature of relationship between high school learners' mathematics-related belief systems and their approach to non-routine mathematical problem solving. A mixed methods approach was adopted in consideration of the complexity of unraveling the relationship between the two constructs (Callejo & Vila, 2009). According to Cresswell, Klassen, Plano Clark and Smith (2011), a mixed methods research approach is the use of both quantitative and qualitative methods to collect data that answer the research questions. It has an advantage of combining the strengths of both qualitative and quantitative research methods in conducting a research study. Specifically, an explanatory sequential method design was used (see figure 3.1), and it is a design in which the qualitative data collected will be used to explain, in depth, the quantitative data collected (Cresswell et al., 2011; Ngulube, 2013). In addition, the researcher adopted a positivist-interpretive paradigm in conducting the research study.
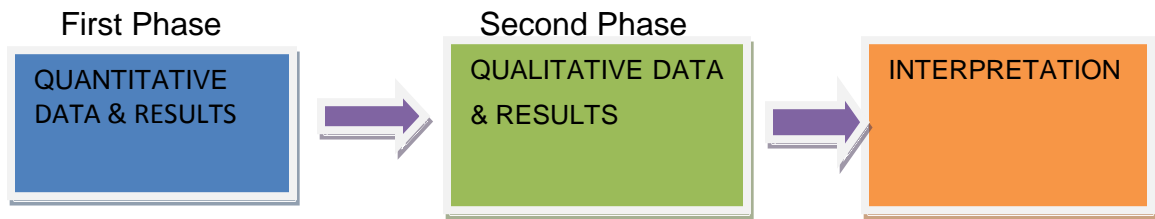
Figure 3.1: Explanatory sequential research design

Adapted from Creswell (2007)

The primary purpose of this explanatory study was to use a beliefs questionnaire (close-ended) and non-routine mathematics problem solving test to determine learners' mathematics-related belief systems and approaches to problem solving, respectively. All the selected learners participated in this first phase of the study of answering close-ended beliefs questionnaires and solving non-routine mathematics problems. The primary data set was used to select learners who participated in the second phase of the study (answering open-ended beliefs questionnaires and interviews). The secondary purpose was to gather qualitative data, by use of interviews and open-ended questionnaires, which explained the learners' beliefs and approaches to problem solving. The qualitative study explored, in depth, high school learners' mathematics-related beliefs and their approaches to non-routine mathematical problem solving at Tshwane North District (D3). The researcher was able to find learners' explanations of their mathematical problem-solving approaches and beliefs which uncovered the possible causes to the development of beliefs, the psychological centrality of the beliefs and the possible mathematics-related belief systems.

In this study, the qualitative data was used to explain in depth the quantitative data previously collected and, thereby, enhancing understanding of the problem under study (Cresswell et al., 2011). This explanatory research study brought more insight into the problem of how beliefs relate to problem-solving approaches that could possibly be used as a starting point for more investigation.

3.3. __Population and case selection__

The population of the study was grades 10, 11 and 12 mathematics learners of Tshwane North District, Gauteng province of South Africa. A convenience sample of 425 grade 10, 11 and 12 learners was selected for participation in the study. The learners were drawn from three schools that were easily accessible and willing to participate in the study.

All the selected learners, firstly, completed the mathematics beliefs questionnaire and then answered the mathematics problem solving test. The mathematics beliefs questionnaire was used to measure learners' mathematics related beliefs. The mathematics problem test was used to measure learners' approaches to mathematical problems. The 425 learners were clustered into two groups based on their beliefs determined through the questionnaire. In line with Field's (2001) suggestion of specifying the number of clusters one expects the SPSS to produce, SPSS was also used to produce two clusters using the Hierarchical (complete linkage) method (see Appendix G). See also Appendix G for a copy of the dendrogram produced by the SPSS to illustrate the clusters identified. Cluster analysis is a way of grouping learners based on the similarities of their responses to several variables (Field, 2001; Simbamoorthi, 1999).

The researcher, then, purposefully selected six learners, three from each cluster, on the basis of their degree of representative of learners belonging to each cluster for interviews. Another factor the researcher considered in selecting participants for interviews was the approaches or strategies the learners applied in solving the mathematics problems. Specifically, the researcher purposefully selected learners who, at least, applied different approaches to solving the mathematics problems. This enabled him to understand the different approaches learners applied to solve the problems in relation to their mathematics related belief systems. The selected learners participated in interviews and answered an open-ended questionnaire and a retrospective questionnaire.

3.4. **Data collection instruments**

The instruments the researcher used for data collection were mathematics beliefs questionnaire (BQ), open-ended questionnaire (OQ), retrospective questionnaire (RQ), mathematics problem solving test (PT) and the interview schedule (IS).

3.4.1. **Mathematics Beliefs Questionnaire (BQ)**

The researcher developed a 63-items mathematics-related beliefs questionnaire which he modeled on different beliefs questionnaires developed by Lazim et al. (2004); Op't Eynde et al. (2006); Physick (2010) and Muis (2004). In order to improve the quality of the initial 63 items questionnaire, the researcher carried out a pilot study with a sample size of 30 composed of grades 10, 11 and 12 learners at another school outside the study. Although the researcher encouraged the learners in the pilot study to complete the questions individually and write marginal comments on them, upon analyzing the beliefs questionnaire, he found that there was no need for improvements. Hence, he used the beliefs questionnaire as it was in the main study.

The researcher conducted principal component factor analysis to assess 'construct validity' of the beliefs questionnaire. Factor analysis was based on how learners answered the questionnaire. As part of the main factor analysis, the researcher created a matrix of inter-correlations between the variables (see appendix H). The matrix of correlations between variables shows how the variables are related to each other. The largest correlation coefficient value is 0.629. Neither perfect correlations of -1 or +1 nor zero between the variables were obtained. Therefore, the researcher did not expect any problems to emerge from the data of multicollinearity and singularity. Multicollinearity or singularity is whereby the variables are strongly or perfectly correlated, respectively (Field, 2005).

In factor analysis, singularity causes difficulties in determining and distinguishing the unique contribution of the strongly correlated variables to a factor (Field, 2005). Consequently, the researcher did not exclude any items from the

questionnaire at the preliminary stage on the basis of multicollinearity or singularity.

The researcher computed the Kaiser-Meyer-Olkin (KMO) measure of sampling adequacy as well as the Bartlett's Test of sphericity to check the appropriateness of conducting factor analysis on the collected data (see table 3.1). The researcher obtained an acceptable KMO value of 0.881, which indicated that factor analysis could be conducted well on the data. Hence, he proceeded with factor analysis confident that it will produce factors that are distinct and reliable (Field, 2005). The Barlett's test has a significance value less than 0.001, which is a highly significant value (see table 3.1). Therefore, it was appropriate to carry out the factor analysis. According to Field (2005), the significance value of the Barlett's test should be less than 0.05 for it to be significant.

Table 3.1: KMO and Bartlett's Test results showing significance values for factor analysis

Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		.881
Bartlett's Test of Sphericity	Approx. Chi-Square	8.231E3
	Df	1953
	Sig.	.000

The researcher used SPSS to draw a scree plot to guide him on the number of factors that can represent the 63 items loaded (see figure 3.2). The scree plot shows that the eigenvalues seem to level off at five factors. The five factors extracted using this data have eigenvalues greater than 1.7 and they account for 35.75% of cumulative variance (see table 3.2).
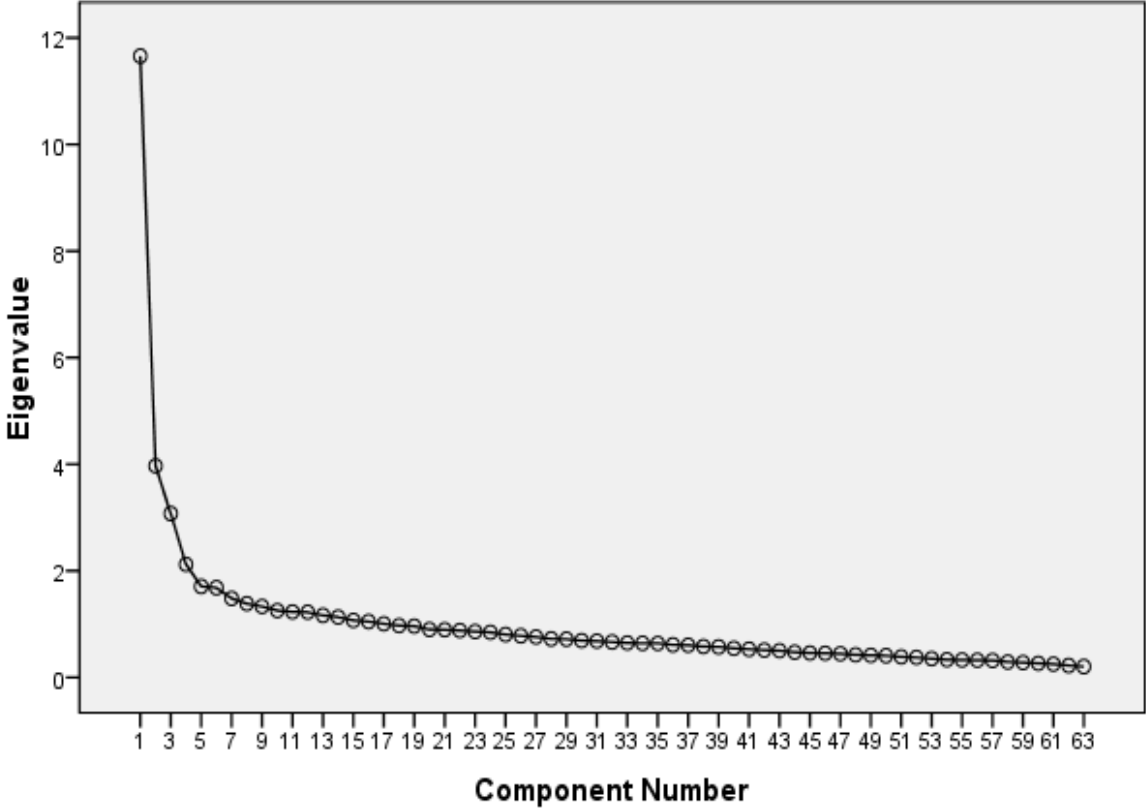
Figure 3.2: Scree Plot showing number of components and eigenvalues of the correlation matrix

Table 3.2: Extract of Factors extracted using Principal Component Analysis method

Total Variance Explained

Component	Initial Eigenvalues			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	11.659	18.507	18.507	7.130	11.318	11.318
2	3.968	6.298	24.805	4.978	7.902	19.220
3	3.074	4.879	29.684	4.974	7.895	27.114
4	2.120	3.366	33.049	3.153	5.004	32.119
5	1.706	2.709	35.758	2.293	3.639	35.758
6	1.682	2.670	38.427			
7	1.481	2.351	40.778			
8	1.384	2.197	42.975			
9	1.330	2.110	45.086			
10	1.253	1.989	47.075			
11	1.230	1.952	49.027			
12	1.223	1.941	50.968			

The researcher analyzed the data using orthogonal rotation (Varimax with Kaiser Normalization) because of his assumption that the factors to be produced are unrelated to or independent of one another. He, also, analyzed the 63 items using principal component analysis (with Varimax with Kaiser Normalization) in order to identify a relative small number of factors that could be used to represent the relationships among this large set of 63 interrelated items. The researcher started with a relatively large number of variables (N = 63) so that after reduction he may end up with a relatively sufficient number of variables to analyse in the study. Brauer (n.d.) suggests that at least five variables should be included for each factor for factor analysis and after factor analysis each factor should have at least three variables that load highly on it. In line with Brauer (n.d.)'s general rule on which variables to interpret, the researcher considered loadings greater than 0.3. For easy interpretation of factor analysis, the researcher used SPSS to

suppress all loadings less than 0.3 and also order the variables according to size of their loadings (see table 3.3).

Table 3.3: Factor loadings of the Mathematics Beliefs Questionnaire

Rotated Component Matrix[a]

	Component				
	1	2	3	4	5
I am very interested in mathematics	.735				
I like doing mathematics	.667	.326			
Mathematics is my favorite subject	.664	.341			
Mathematics has always been my worst subject	.659				
I am sure that I can learn mathematics	.652				
I know I can do well in mathematics	.612				
Mathematics is a mechanical and boring subject	.596				
I am not good in mathematics	.583				
I am not the type to do well in mathematics	.577				
Mathematics is difficult	.557				
I think I could handle more difficult math	.539	.361			
I can understand the course material in mathematics	.515	.324			
I enjoy pondering mathematical exercises	.513	.467			
I feel confident in my ability to solve mathematics problem	.498	.467			
To me mathematics is an important subject	.441				
It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem	.426				
The problems we work on in mathematics class have no relationship to daily life	.417		.329		
Considering the difficulty of this course, the teacher, and my skills, I think I will do well in mathematics	.412				
There is only one way to find the correct solution of a mathematics problem	.347		.308		
I think I will be able to use what I learn in mathematics also in other courses	.328		.315		
It is my own fault if I do not learn the material in this course					
I try to play around with ideas of my own and relate them to what I am learning in this course		.653			
I try to relate ideas in this subject to those in other courses whenever possible.		.626			
When I find a solution, I always look for other ways of solving the problem.		.572			
Whenever I read or hear an assertion or conclusion in this class, I think about possible alternatives		.552			
I try to understand the material in this class by making connections between the readings and the concepts from the lesson		.508	.313		

	C1	C2	C3	C4	C5
I ask myself questions to make sure I understand the material I have been studying in this class		.473			
I try a different approach when my first attempt fails		.466			
When I have the opportunity I choose mathematical assignments that I can learn from even if I am not at all sure of getting a good grade		.430	.327		
I always prepare myself carefully for exam		.419			
I prefer mathematics tasks for which I have to work or think hard in order to find a solution	.339	.401			
I am hard working by nature		.365			
I usually understand a new idea in mathematics quickly		-.356			
Learning mathematics requires a lot of effort			.727		
There are several ways to find the correct solution of a mathematics problem			.621		
For me the most important thing in learning mathematics is to understand			.595		
Mathematics is about solving problems			.587		
I need mathematics in order to study what I would like after I finish high school			.583		
Mathematics is used by a lot of people in their daily life			.522		
If I study in appropriate ways, then I will be able to learn the material in this course			.522		
When I have finished working on the problem, I look back to see whether my answer makes sense.		.305	.503		
Making mistakes is part of learning mathematics			.482		
One learns mathematics through doing exercises			.447		
When I cannot understand the material in this course, I ask another student or my teacher for help		.411	.418		
Group work facilitates the learning of mathematics			.390		
Solving a mathematics problem is demanding and requires thinking, also from smart students			.364		
If I try hard enough, then I will understand the course material of the mathematics class					
One learns mathematics best by memorizing facts and procedures				.631	
Mathematics is numbers and calculations				.631	
Mathematics is a set of rules and techniques				.587	
The problem solving process is linear: you advance directly towards the solution		-.301		.554	
There are always numbers in formulations of mathematics problems				.550	
I am only satisfied when I get a good grade in mathematics				.505	
Those who are good in mathematics can solve any problem in a few minutes				.404	
The teacher must always show me which method to solve a given mathematics problem	.336			.347	
Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone.				.338	
Mathematics is a static and rigid body of knowledge: No new things about mathematics are yet to be discovered.					.456

I rarely find time to review my notes before an exam				.409	.454
It is not important to understand why a mathematical procedure works as long as it gives the correct answer					.452
Mathematics enables a man to better understand the world he lives in					.450
I feel the most important thing in mathematics is getting the correct answer				.348	.438
Mathematics is continuously evolving. New things are still being discovered			.389		.416
Anyone can learn mathematics					

Extraction Method: Principal Component Analysis.
Rotation Method: Varimax with Kaiser Normalization.
a. Rotation converged in 10 iterations.

20 items loaded heavily onto factor 1. Some of the 20 items, together with their loadings, which loaded highly onto factor 1 were: "I am very interested in mathematics (0,735)"; "I like doing mathematics (0,667)"; "Mathematics is my favorite subject (0,664)"; "I am sure that I can learn mathematics (0,652)"; "I know I can do well in mathematics (0,612)"; and "I think I could handle more difficult mathematics (0,539)". Most of the items under factor 1 have a common theme of doing well in mathematics. As a result, the researcher named factor 1, "I can do well in mathematics".

12 items loaded highly onto factor 2. Some of the items, together with their loadings, that loaded onto factor 2 were: "I try to play around with ideas of my own and relate them to what I am learning in this course (0,653)"; "I try to relate ideas in this subject to those in other courses whenever possible (0,626)"; "Whenever I read or hear an assertion or conclusion in this class, I think about possible alternatives (0,552)"; and "I try to understand the material in this class by making connections between the readings and the concepts from the lesson (0,508)". The items have a common theme of understanding mathematics. As such, the researcher named factor 2, "I make sense of what I learn".

13 items loaded heavily onto factor 3. Some of the items, together with their loadings, that loaded onto factor 3 were: "Learning mathematics requires a lot of effort (0,727)"; "There are several ways to find the correct solution of a mathematics problem (0,621)"; and "For me the most important thing in learning mathematics is to understand (0,595)". The items that loaded onto factor 3 have a common theme of relying on one's own ability and working in collaboration with mathematics teachers and other learners. The researcher named factor 3, "Group work facilitates learning of mathematics".

9 items loaded onto factor 4. Some of the items, together with their loadings, that loaded onto factor 4 were: "One learns mathematics best by memorizing facts and procedures (0,631)"; "Mathematics is numbers and calculations (0,631)"; and "Mathematics is a set of rules and techniques (0,587)". The items that loaded highly onto factor 4 will determine if "Mathematics is numbers, rules and techniques". 6 items loaded highly onto factor 5. Some of the items, together with their loadings, that loaded onto factor 5 were: "Mathematics is a static and rigid body of knowledge: No new things about mathematics are yet to be discovered (0,456)"; and "Mathematics is continuously evolving: New things are still being discovered (0,416)". The researcher named factor 5, "Mathematics is continuously evolving" because of some items under this factor that have a common theme of making new discoveries in mathematics (see table 3.3).

Only three items did not load significantly onto any of the factors, that is, their loading was less than 0.3. As a result, the researcher excluded them from the final questionnaire. Ideally, each variable is expected to load highly on only one factor (Brauer, n.d.). Table 3.3 shows that some variables loaded onto two factors. As such, the researcher classified the variable under the factor it loaded to more highly (Field, 2005).

The survey was made up of closed form questions and used five points of Likert Scale: strongly disagree, disagree, uncertain, agree and strongly agree. For a complete questionnaire developed and analyzed in this study see Appendix A. However, it is worthwhile to mention some of the weaknesses of the closed form

questionnaire that the researcher took into consideration when he constructed it. Firstly, the closed form questionnaire provides little room for relating the inquiry to particular individuals and circumstances. Secondly, the closed form questionnaire tends to use standard wording that may make questions and the responses artificial and irrelevant. Lastly, presentation of one's categories of experiences and beliefs in closed form questions might constrain the respondents to fit their experiences and beliefs within those of the researcher. This might be perceived as being manipulative, inconsiderate and impersonal (Keeves, 1998). The researcher tried to minimize these weaknesses by making use of an open-ended questionnaire in the current study that was made up of unrestricted or open- form items.

3.4.1.1. **Reliability**

To measure the reliability of the mathematics beliefs questionnaire, the researcher calculated the Cronbach's alpha on each factor (see table 3.4). Factor 1 (I can do well in mathematics) had a very high reliability coefficient, **α =** 0,900. Similarly, factors 2 (I make sense of what I learn); 3 (Group work facilitates learning mathematics); and 4 (Mathematics is numbers, rules and techniques) had high reliability coefficients of 0,766; 0,816 and 0,696, respectively. Factor 5 (Mathematics is continuously evolving) had a slightly lower alpha of 0,557 as compared to the other four factors. After combining all the five factors, the researcher concluded that the mathematics beliefs questionnaire was reliable.

Table 3.4: Reliability statistics of the five factors that represent the mathematics beliefs questionnaire

Factor	Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
1	0.900	0.904	20
2	0.766	0.771	12
3	0.816	0.827	13
4	0.696	0.701	9
5	0.557	0.560	6

3.4.1.2. **Convergent and Discriminant validity**

According to Shuttleworth (2009), convergent validity tests and ascertains the existence of a relationship between variables that are expected to be related. Discriminant or divergent validity tests ascertain that the variables that should not be related are not related, indeed. To assess both convergent and discriminant validity of the beliefs questionnaire instrument, the researcher computed the correlation coefficients between the factors and represented them in form of a matrix (see table 3.5).

Table 3.5: Extract of matrix of correlations between factors.

		FACTOR 1				FACTOR 2				FACTOR 3			
		Q16	Q1	Q1	Q4	Q3	Q3	Q36	Q3	Q2	Q27	Q30	Q38
FACTOR 1	Q1	1	.62	.63	.43	.33	.25	.227	.25	.17	.172	.119	.069
	Q1	.629	1	.64	.44	.31	.25	.310	.27	.06	.135	.094	.056
	Q1	.636	.64	1	.48	.33	.26	.271	.27	.10	.122	.111	.003
	Q4	.437	.44	.48	1	.25	.24	.189	.17	.15	.129	.095	.056
FACTOR 2	Q3	.333	.31	.33	.25	1	.44	.389	.35	.13	.138	.227	.024
	Q3	.255	.25	.26	.24	.44	1	.340	.37	.16	.158	.208	.049
	Q3	.227	.31	.27	.18	.38	.34	1	.37	.11	.109	.197	.044
	Q3	.254	.27	.27	.17	.35	.37	.376	1	.24	.229	.246	.196
FACTOR 3	Q2	.173	.06	.10	.15	.13	.16	.118	.24	1	.370	.397	.355
	Q2	.172	.13	.12	.12	.13	.15	.109	.22	.37	1	.292	.311
	Q3	.119	.09	.11	.09	.22	.20	.197	.24	.39	.292	1	.404
	Q3	.069	.05	.00	.05	.02	.04	.044	.19	.35	.311	.404	1

Trochim (2006) posits that correlations between items that measure the same 'thing' should be 'high', while correlations between items that measure different 'things' should be 'low'. Since there is no a distinct cut off point of how "high" or "low" the correlations should be, we should always expect convergent correlations to be higher than the discriminant correlations. The extract of correlations shows, for example, that the inter-correlations between items of factor 1 are higher than the correlations between items of factor 1 and other factors. Similarly, correlations between items of factor 2 are higher than the correlations between items of factor 2 and other factors.

Higher correlations between items within the same scale or factor indicate that the measures or items are converging on the same thing; while lower correlations between items of different scales indicate that the set of scales are discriminated

from each other (Trochim, 2006). The highlighted blocks on the extract of matrix of correlations represent the convergent coefficients. Since the convergent coefficients are higher than the discriminant ones, the correlation matrix provides some evidence for both convergent and discriminant validity. By virtue of having evidence for both convergent and discriminant validity, the researcher concluded that the mathematics beliefs questionnaire was valid (Trochim, 2006).

3.4.2. **Open-ended Questionnaire (OQ)**

The OQ was made up of seven questions that the researcher wanted to follow up from the case studies. The respondents were asked to describe or explain in their own words, (1) what is mathematics, (2) what is a mathematics problem, (3) types of questions they encounter in mathematics problems, (4) what is solving a mathematical problem, (5) what they like or dislike about mathematics as a discipline, (6) examples of mathematical problems, exercises and other activities they enjoy doing or solving most, and justify why they enjoy them most (see Appendix C).

The OQ did not provide possible responses the respondents can select from. As such, it expected the respondents to give responses that were expressed in their own words. This enabled the researcher to get additional information about what the learners generally thought of mathematics. Best and Kahn (1993) argue that the OQ enables the respondents to state their own point of view without being guided by alternative responses of the researcher. It, also, gives the respondents room to state the reasons for their responses. However, its major weaknesses are that it demands a lot from a respondent in terms of time, thinking and effort, and, because of that, returns are often less than expected. In addition, because of detailed word responses, the responses to open form items might, sometimes, challenge the researcher in terms of summarizing and interpreting the data.

3.4.3. __Mathematics Problem solving Test (PT)__

The PT was made up of six multi-step, real life problems (see Appendix D). The variables the researcher considered in selection of the problems were: (1) the problem should be comprehensible by the learners, (2) the problem requires no specialist mathematical knowledge, (3) facts and concepts to be applied, (4) strategies or approaches to be applied in solving the problem, and (5) the level of difficulty of the problem- demands of the questions should match the academic level of learners.

Problem 1 (P1) was an arithmetic sequence word problem. It required learners to look for pattern, list possible values or use an algorithm either derived from pattern analysis or from recognition. Problem 2 (P2) could be solved by use of logic (reasoning), or simple proportion, Problem 3 (P3) could be solved by use of an algebraic approach (forming an inequality). It expected learners to be able to manipulate inequalities, and look back to check if the answer makes sense.

Problem 4 (P4) could be solved by either logic (reasoning) or systematic trial and error. Problem 5 (P5) could be solved by an algebraic method (forming an equation) or by systematic trial and error. Problem 6 (P6) had no clear mathematics referents (i.e. had no numbers in its formulation). It could be solved by reasoning and algebraic method. Learners were also expected to be able to manipulate inequalities.

The problems chosen could be approached in different ways by the learners. The different approaches learners applied in solving the problems could be interpreted as a manifestation of their different mathematics beliefs. Learners' beliefs that guided or determined their problem solving behaviour could be inferred from their responses to the problems. As such, the responses to the problems enabled the researcher to compare learners' professed beliefs with their beliefs in-action (beliefs inferred from their responses to the problems).

3.4.3.1. <u>**Validity and Reliability**</u>

The content of the mathematics problem test was validated by two public high school heads of mathematics department and three high school mathematics teachers. To measure the reliability of the test, the researcher computed Spearman-Brown coefficient measure of reliability (see table 3.6). The researcher obtained a coefficient of 0,509 which he regarded as lower than expected (0.70 and higher) (Maree et al., 2006) probably because most learners faced difficulties on solving the non-routine problems (see Table 6.2).

Table 3.6: Reliability statistics of mathematics problem solving test

Spearman-Brown Coefficient	Equal Length	.509
	Unequal Length	.509
Guttman Split-Half Coefficient		.483

a. The items are: Money problem, Cars problem, Calves problem.

b. The items are: Tea and Cakes problem, Rabbits and hutches problem, Fish counting problem.

3.4.4. <u>**Retrospective questionnaire (RQ)**</u>

The RQ gave the respondents an opportunity to reflect on how they solved each problem, explain the approaches they employed, state any obstacles or difficulties met in solving problems, and their perception on whether they had solved the problems correctly. Basically, it tested learners' metacognitive skills of predicting, monitoring and evaluating their problem solving process.

The RQ was made up of both closed and open questions (see Appendix E). Below is an example of a RQ that was used.

Reflect on how you answered Problem 1 and answer the following questions:
1.1. How confident were you that you could solve it correctly? (Prediction)
    Circle your best response.

    1. Absolutely sure that I can do it correctly.
    2. Quite sure that I can do it correctly.
    3. Not sure, I didn't know how correctly I could do it.
    4. Really not sure, I thought that I could not succeed.
    5. Absolutely sure that I could not do it correctly.

1.2. Think over what you have done and try to describe what kind of strategies you used to solve the task.
    (monitoring)
    _____
    _____
    _____
    _____

1.3. State any difficulties or obstacles you met when solving the problem. (monitoring)
    _____
    _____
    _____
    _____

1.4. Now that you have answered the question, try to say if you are sure that you have done it correctly.
    (Evaluation)
    Circle your best response.

    1. You are absolutely sure that you have done it in the right way.
    2. You are quite sure that you have done it in a right way.
    3. You are not sure; you don't know how correctly you have done it.
    4. You are really not sure, and you think you have probably made a mistake.
    5. You know that you have made a mistake.

    [Adapted from Lucangeli and Cabrele, 2006, p. 130]

3.4.5. **Interview Schedule**

The interview was semi-structured and pre-sequenced as indicated on the schedules (see Appendix F). The researcher designed two schedules for the interviews. Schedule 1 was made of questions that were based on learners' responses to the beliefs questionnaire. The researcher chose, at least, one item to represent each sub-scale of beliefs (see Table 3.7). As a result, Schedule 1 was made up of five basic common questions (see Appendix F).

Table 3.7: Belief scales and examples of beliefs in interview schedule 1

Scale 1: I can do well in mathematics.
  1.1. It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.
Scale 2: I make sense of what I learn.
  2.1. I prefer mathematics tasks for which I have to work or think hard in order to find a solution.
Scale 3: Group work facilitates learning of mathematics
  3.1. Making mistakes is part of learning mathematics.
Scale 4: Mathematics is numbers, rules and techniques
  4.1. Even if I have trouble learning the material in this class, I try to do the work on my own without help from anyone.
Scale 5: Mathematics is continuously evolving
  5.1. It is not important to understand why a mathematical procedure works as long as it gives the correct answer.

The researcher requested the respondents to justify their degree of agreement to the statements. That is, they stated reasons why they chose the specific rating for each belief item. In this regard, Di Martino (2004, p. 277) argues, "the reasons provided by the respondents allow the researcher to highlight other beliefs linked to the declared belief and the psychological centrality of the declared beliefs, thus giving information about the belief system containing it".

Schedule 2 was made up of eight open questions that sought learners' personal statements about their belief systems. Examples of questions asked in Schedule 2 were: What is the best way you think you can learn mathematics? Can you state, in brief the strategies you employ in studying or learning mathematics? Why do you think mathematics is or is not important to you? What are your comments on the statement: "Mathematics is a static and rigid body of knowledge. No new things about mathematics are yet to be discovered." (see Appendix F). The researcher asked the respondents to justify their responses whenever necessary in order to gain a deeper understanding of their declared beliefs.

Any other interesting leads that emerged during the interview were also discussed. Therefore the schedules suggested were not rigid, but only served to

give direction and remind the researcher some topics or questions of interest he had to address and to remain focused towards the main objectives of the study.

3.4.5.1. **Validity and reliability**

The researcher views validity as a fit between what he records as data and what actually occurs at the setting under study. As such, the researcher gave a detailed record of what transpired. In addition, the incorporation of multiple sources of data in this study enabled the researcher to use triangulation to interpret converging evidence and ultimately, point to a clear conclusion. For instance, data from semi-structured interviews and questionnaires was cross-validated through convergent validity. Converging as well as diverging responses were noted and interpreted, thereof. In this regard, Anderson (1990) argues that conclusions suggested by different data sources are far stronger than those suggested by one alone. Reliability was determined by the consistency of the responses the researcher got from the interviewees. In order to minimize variability in participants' responses, all the interviewees were subjected to similar conditions: (a) they all answered the same basic questions which were worded and sequenced the same; and (b) they were given room to express themselves freely.

3.5. **Data collection procedure**

The data was collected by the use of beliefs questionnaire (BQ), open-ended questionnaire (OQ) and retrospective questionnaire (RQ), problem test (PT), and interview schedule (IS). The use of multiple methods for data collection had an advantage of giving different types of data which provided validity checks within the study (Moistus, 2001). In this section, the researcher explained how the research instruments were administered to the learners.

3.5.1. <u>**Administration of research instruments**</u>

The OQ was administered in one session. The BQ, PT, IS and RQ were administered or used in two sessions (See table 3.8).

Table 3.8: Administration of research instruments

Session 1	Session 2
BQ 1	BQ 2
P1 , P2, P3	P4, P5, P6
OQ	
IS 1	IS 2
RQ1 , RQ2 , RQ3	RQ4 , RQ 5 , RQ 6

The BQ and PT were administered to all 425 students, while OQ, RQ and IS are the additional instruments that were administered to the six selected learners. The research instruments were administered in following sequence:

Table 3.9: Sequence of administration of instruments

Day	Instrument(s)
1	BQ1
2	BQ2
3	P1, P2, P3
4	P4, P5, P6
5	RQ1, RQ2, RQ3
6	RQ4, RQ5, RQ6
7	OQ
8	IS1, IS2

3.5.1.1. <u>**Questionnaire administration**</u>

The researcher administered the questionnaires personally to the learners. This gave him an opportunity to introduce himself, explain the rationale for the study, and explain questions that learners faced difficulties in understanding them. The purpose of the BQ was two-fold in this study. The first purpose of the BQ was to gather information about the belief systems of learners and determine how they might be 'clustered' into "belief subsets". The second purpose of the BQ was to select six learners who exhibit different belief systems to participate in the second part of the study.

To encourage truthfulness in survey responses, the researcher numbered each survey and instructed the learners not to record their names on the paper. He then kept a separate record that matched names to the numbers. He assured the learners that the survey data would be analyzed anonymously and that only the six learners selected for part two of the study will have their names revealed to him. The researcher also informed them that, after the six learners were chosen for part two of the study, the key that matched names to numbers would be destroyed. Muis (2004) encourages and espouses such practice.

3.5.1.2. **Problem Test (PT) administration**

The researcher sequenced the questions from the less demanding to more demanding problems. He personally invigilated the test. The PT was broken down into two sessions, each of about thirty minutes, due to the limited amount of time that was available to carry out the study, to avoid straining the respondents, and to have the entire test attended to by the respondents. The learners were instructed to show all necessary calculations done and even rough calculations done on the spaces provided when resolving the non-routine mathematical problems.

3.5.1.3. **Interviews**

The researcher conducted two sessions of interviews with each learner. Learners were given an opportunity to browse the interview questions before the interview commences to encourage a greater depth in responses from them. This, also, offered the learners the opportunity to think of and prepare in advance some responses to the general questions to be asked. As a result, the learners responded to the interview questions with confidence. All the interviews were audio-recorded and transcribed verbatim.

3.6. <u>Ethical considerations</u>

All protocols in conducting a research study were observed by the researcher. A letter of request to carry out research at schools was obtained from the institution (see Appendix L). A permission letter to conduct research at the selected and named schools was obtained from the department of education (see Appendix J). The researcher wrote letters to the school management teams or principals seeking permission to conduct research studies at their schools (see Appendix I). Since the research study was to be conducted after school contact time, the researcher wrote letters to parents/guardians requesting them to allow their children to participate in the research study.

Principals and learners, who were willing to participate in the study, were given an informed consent form which spelt out, in a simple understandable language, the purpose of the study, the nature of participation, issues concerning privacy, anonymity and confidentiality, the names and contact details of the researcher and research supervisor and the name of the institution (see Appendix I). In a nutshell, the purpose of the data collecting instruments, the use of the collected data and the implications of the results obtained to policy and practice were briefly explained to the participants. Learners were informed that an audio-recorder was to be used in interviews. They were, also, informed that some of their written responses to the problem solving test were to be scanned and presented in the research document to be produced. They were assured that their responses will be treated confidentially and used for research purposes only. They were not allowed to disclose their identity when participating in the research study. They were to be kept anonymous throughout the study. They were also informed that they were to participate at their own free will, and could withdraw from participating in the research study at any time they felt doing so without any undesirable consequences attached to them.

3.7. <u>Summary and conclusion of the chapter</u>

The study employed a mixed methods approach whereby both quantitative and qualitative methods were used to collect the data. The data were collected from 425

learners using a closed form mathematics beliefs questionnaire and a mathematics non-routine problem solving test. A representative group of six learners was selected from the 425 learners for interviews and answering a retrospective questionnaire and an open-ended beliefs questionnaire. The research instruments were confirmed to be valid and reliable for data collection by use of statistical tests and significant mathematics teachers (e.g., senior mathematics teachers). Ethical issues considered on conducting the research were mentioned. The open-ended beliefs questionnaire, retrospective questionnaire and interviews proved valuable on revealing learners' beliefs that could not be tapped by the closed-form mathematics-related beliefs questionnaire. The non-routine problem solving test was accessible to learners by a number of different problem solving strategies.

Chapter 4

Data analysis

4.1. <u>Introduction</u>

The researcher mentioned how the data was presented and analyzed in the study. He discussed how the research instruments used in the study were analyzed. He, also, discussed how the relationship between learners' belief systems and their approaches to non-routine mathematical problem solving was analyzed.

4.2. <u>Data presentation and analysis</u>

The data was presented in the form of tables (e.g. a table of the summary of learners' belief systems) (see chapter 5), direct quotes from the questionnaires, extracts of learners' written solutions to problems, excerpts of interviews, and descriptive data on learners' beliefs about mathematics and approaches they applied in problem solving. The basic approaches to data analysis used in the current study were coding, factor analysis, thematic analysis (organizing data into descriptive themes, noting relations between variables), cluster analysis (using the Statistical Package for Social Sciences (SPSS)), and methodological triangulation.

4.3. <u>Analysis of mathematics-related beliefs questionnaire (BQ)</u>

To facilitate analysis of BQ, the items were encoded in such a way that positive beliefs (that is, desirable or healthy beliefs about mathematics learning and problem solving) always gave a high value, while negative beliefs (that is, undesirable or unhealthy beliefs about mathematics learning and problem solving) gave a low value. For example, the belief: 'It is not important to understand why a mathematical procedure works as long as it gives the correct answer' was coded from 1 = "strongly agree" to 5 = "strongly disagree", and "For me the most important thing in learning mathematics is to understand" was coded from 5 = "strongly agree" to 1 = "strongly disagree".

To establish learners' beliefs about each scale used, the researcher calculated the mean value of their responses to each scale (see section 5.3). The researcher considered a learner having an average score greater than 3 as holding that belief; and a mean score of 3 as holding a neutral belief. As such, a higher average belief score meant that the belief was strongly held by the learner (Jin et al., 2010; Kislenko et al., 2005). The researcher used Daskalogianni and Simpson (2001a)'s three key macro-belief systems (systematic, exploratory and utilitarian) to classify all the beliefs questionnaire items used in this study. Then, using the average score of each category in conjunction with thematic analysis, the researcher identified learners' predominant beliefs that largely influenced their behaviour in problem-solving (see chapter 5). The cluster analysis was used to classify learners holding similar beliefs into two clusters. Three learners were purposefully selected from each cluster as case studies. Belief profile of each of the selected six learners was drawn. Each learner was then classified into his/her belief system category based on his/her predominant belief system (see section 5.3). For example, learner A26's utilitarian mean belief score was higher than his systematic and exploratory mean belief scores. As a result, he was classified as a utilitarian believer.

4.4. Analysis of non-routine mathematics problem solving test (PT)

To identify the approaches applied by learners in solving problems, the researcher analyzed their written responses to the PT. Examples of approaches he sought from their written work were: random trial and error, systematic trial and error, guessing, seeking patterns, systematic search for all possibilities, listing in some order, use of logic (reasoning), drawing tables or graphs, counting, using simple cases, working backwards, piece-wise, holistic, and non-attempts (blank answer sheet). The strategies applied by learners in solving each of the six problems were noted down. To facilitate analysis of the learners' responses to the PT, the researcher formulated a coding scheme for each problem (Elia et al., 2009; Mabilangan et al., 2011) (see table 4.1). The researcher used the learners' approaches to problem-solving to infer their belief systems that explained their problem solving behaviour.

Table 4.1: Coding scheme for the problem solving strategies

Adapted from Elia et al. (2009) and Mabilangan et al. (2011)

Category	Variable name	Explanation
Systematic listing	SL	Making an organized list which is composed of at least three values. The steps are of the same size and trials 'move' in one direction.
Modeling	MD	Use of algebra (linear equations, simultaneous equations, linear inequalities), drawing diagrams, or sketches
Trial-and-Error	TE	Making at least two trials of which the last value given is the answer. The steps are not of the same size, and the 'movement' of the trials does not necessarily go in one direction.
Guess, Check and Revise [systematic(sys)/ unsystematic(unsys)]	GCR	Sys:- Making a reasonable guess, checking the guess and revising the guess if necessary Unsys: - Making a guess and lack checking or revision to improve the guess. Making one trial only. Giving the answer only.
Use a formula	F	Selecting a formula to use or substituting values into a formula.
Elimination	E	Eliminating incorrect answers or eliminating possible solutions based on the given information in the problem.
Logical reasoning	LG	Using logical reasoning to justify statements or reach a conclusion. Writing logical statements.
No logical reasoning	NLG	Statements lack logic or does not make sense. Unreasonable (absurd). Naïve, impulsive or unthinking. Not answering the question asked. Unable to detect method used.
Look for patterns	LP	Identifying some common characteristics that can be generalized and used to solve the problem.
Consider a simple case	SC	Includes repeating information from the problem formulation or rewording the problem; dividing the problem into simpler problems; using smaller numbers or working backwards.

As suggested by Mabilangan et al. (2011), the researcher classified the problem solving strategies applied by learners into three main categories: (1) Thorough or insightful use of strategies, (2) Partial use of strategies, and (3) Limited strategies (see table 4.2). To facilitate analysis of the approaches or strategies employed by learners to problem solving, a point system was used: 5 points for insightful use of strategies, 3 points for partial use of strategies, and 1 point for limited use of strategies. 2 points were assigned to an approach in between limited and partial

use of strategies. Similarly, 4 points were assigned to an approach in between partial and insightful use of strategies (Mabilangan et al., 2011). A score of 0 was assigned to a blank answer sheet. As such, the researcher used the table that classifies problem solving strategies as a marking rubric in this study.

Table 4.2: Classification of problem solving strategies

Adopted from Mabilangan et al. (2011, p. 28).

Thorough/ Insightful use of strategies	Partial use of strategies	Limited strategies
The strategies show some evidence of insightful thinking to explore the problem. The learner's work is clear and focused. The strategies are appropriate and demonstrate some insightful thinking. The learner gives possible extensions or generalizations to the solution or the problem.	The strategies have some focus, but clarity is limited. The learner applies a strategy which is only partially useful. The learner starts the problem appropriately, but changes to an incorrect focus. The learner recognizes the pattern or relationship, but expands it incorrectly.	The strategies lack a central focus and the details are sketchy or not present. The procedures are not recorded (i.e., only the solution is present). Strategies are random. The learner does not fully explore the problem and look for concepts, patterns or relationships. The learner fails to see alternative solutions that the problem requires.

4.5. **Analysis of interview schedule (IS), open-ended questionnaire (OQ) and retrospective questionnaire (RQ)**

The verbatim transcripts of interviews and responses to the OQ and RQ were used to add more clarity on the data obtained from BQ and inferences made on the Problem Test (PT). The researcher also used them to test his interpretation of learners' belief systems and approaches to problem-solving. The textual data was analyzed using thematic analysis. Thematic analysis is viewed by Braun and Clarke (2006) as a recursive 6-steps approach to data analysis that involves familiarization with the data, generating initial codes, searching for themes,

reviewing themes, defining and naming themes, and producing the report. Specifically, the researcher analyzed the textual data using theoretical thematic analysis. Mathematics-related beliefs revealed from learners' responses were coded in relation to the themes (or belief systems), namely, systematic belief, utilitarian belief and exploratory belief, developed by Daskalogianni and Simpson (2001a). In this respect, Braun and Clarke (2006) argue that it is impossible to free the researcher from his/her theoretical and epistemological orientation. As such, the themes developed or identified were driven by the researcher's theoretical or analytical interest in the area of study.

The extracts of learners' responses were used as evidence of the beliefs held by the learners (see section 5.3). Learners' repeated pattern of responses revealed their predominant set of beliefs. To confirm the findings and assess data quality, methodological triangulation whereby more than one methodology of inquiry is employed was used (Bogdan and Biklen, 1982; Magagula; 1996, Wragg, 1994). Specifically, in this study the researcher used the following data collection methods: questionnaire, problem test and interviews. The researcher checked if the data from these various sources led to the same conclusions. Triangulation involves checking if one's research findings and interpretations are in agreement with those of other researchers. It, also, involves cross checking findings from different sources (e.g., fellow researchers, teachers, learners, learners' written work) to note similarities and differences (Wragg, 1994).

4.6. **Analysis of the relationship between belief systems and approaches to problem solving**

By using each learner's problem solving average score and belief systems mean score, a scatter plot was drawn and Pearson's correlation coefficient was computed to analyse the relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematical problem-solving (see chapter 5, diagram 6 and table 18). Learners' belief systems were closely examined in connection to their approaches to problem solving. The relations between belief systems and approaches to problem solving that

117

emerged were noted. The relationship between belief systems and approaches to problem solving was discussed in detail, supported by the concrete evidence obtained from the study. The researcher showed evidence of learners' beliefs and approaches to problem solving in form of extracts from interviews, questionnaires or written solutions to problems tested (see chapter 5).

4.7. <u>Summary and conclusion of the chapter</u>

The data was presented in a number of different forms, for example, tables, diagrams, direct quotes from questionnaires and interviews and extracts of learners' written work. Examples of data analysis approaches used were coding, factor analysis, cluster analysis, regression analysis, thematic analysis and methodological triangulation. The problem solving strategies and belief systems of learners were analysed to determine the relationship between them. Scatter plots and Pearson's correlation coefficient were used as a measure of the relationship between beliefs and approaches to non-routine problem solving. The thematic analysis approach provided valuable explanations of how the learners' beliefs are related to their problem solving strategies.

CHAPTER 5

Research findings

5.1. <u>Introduction</u>

Firstly, the researcher discussed the strategies applied by learners when resolving non-routine mathematical problems. Extracts of learners' written work were presented as evidence to the researcher's discussion and interpretations. Secondly, the researcher discussed learners' mathematics-related belief systems. Learners' responses to beliefs questionnaires and interviews were quoted as evidence to arguments raised and findings discovered by the researcher. Lastly, the researcher discussed the relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematical problem solving. Scatter plots were drawn to represent the relationship that exists between the two constructs diagrammatically. Pearson's correlation coefficient was calculated as a numerical measure of the strength of the relationship between beliefs and approaches to problem solving.

5.2. <u>Learners' approach/strategies to non-routine mathematical problem solving</u>

The researcher focused on the solution strategies and answers of the six learners he selected since they represented each category of learner clusters as indicated in section 3.3.The six selected learners solved the problems using the following strategies: Systematic Listing (SL); Modeling (MD); Trial-and-error (TE); Use a Formula (F); Systematic Guess, Check and Revise (GCR(sys)); Unsystematic Guess, Check and Revise (GCR(unsys)); Consider a simple case (SC); Logical reasoning (LG); No logical reasoning (NLG); and Look for patterns (LP). (See table 5.1).

Table 5.1: Summary of strategies applied by learners in solving the six non-routine problems.

<table>
<tr><td rowspan="2"></td><td rowspan="2"></td><td colspan="6">Problem</td></tr>
<tr><td>P1</td><td>P2</td><td>P3</td><td>P4</td><td>P5</td><td>P6</td></tr>
<tr><td rowspan="6">Learner</td><td>A26</td><td>GCR(unsys)</td><td>SC; LG</td><td>NLG</td><td>NLG; GCR(unsys)</td><td>NLG</td><td>SC; MD</td></tr>
<tr><td>A31</td><td>SL</td><td>LG</td><td>GCR(unsys)</td><td>NLG; GCR(unsys)</td><td>SC; GCR(unsys)</td><td>MD; SC; LG</td></tr>
<tr><td>B08</td><td>GCR(unsys)</td><td>SC; LG</td><td>GCR(unsys)</td><td>MD; GCR(unsys)</td><td>GCR(unsys)</td><td>MD</td></tr>
<tr><td>B57</td><td>SL;LP; MD</td><td>SC; LG</td><td>MD</td><td>MD; TE</td><td>NLG; MD</td><td>MD; SC;TE</td></tr>
<tr><td>C27</td><td>GCR(unsys)</td><td>LG</td><td>GCR(unsys)</td><td>NLG; GCR(unsys)</td><td>MD; GCR(unsys)</td><td>MD</td></tr>
<tr><td>F43</td><td>GCR(unsys)</td><td>LG</td><td>SC</td><td>MD; LG</td><td>MD</td><td>MD; LG</td></tr>
</table>

5.2.1. <u>**Problem 1 (P1)**</u>

P1 was basically solved using five strategies: GCR(unsys); SL; LP; MD; and F. Learners A26, B08, C27 and F43 solved P1 using the same strategy (GCR(unsys)), but applied in different approaches. They all guessed the number of days the two people will take to have the same amount of money and verified the correctness of their guesses through calculations (see figures 5.1, 5.2, 5.3 & 5.4). As shown on Figure 5.3, learner C27 attempted to use a formula. After stating the formula, he did not show any application of it in solving the problem.

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114)

A26

ANSWER	ROUGH CALCULTIONS
Mpho: R25 wage; 4 days; P=R40,00	$25 \times 4 + 40 = 140$
R25 × 4 = R100	$10 \times 4 + 100 = 140$
R100 + R40 = 140,00	After 4 days
Thabang: R10,00 wage; 4 days; P=R100	Mpho R25 × 4 = R100
R10,00 × 4 = R40,00	R100 + R40
R40,00 + R100 = R140,00	= R140.
After 4 days they will have the same amount of money.	Thabang R10 × 4 = R40
	R40 + R100
	= R140

GCR (unsys?)

5

Figure 5.1: Learner A26's solution to P1

B08

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114)

B08

ANSWER	ROUGH CALCULTIONS
	Thabang = x
	mpho = y
They both have to work for 4 days	Thabang = 10 × 4
	= 40 + 100
	= 140
	mpho = 25 × 4
	= 100 + 40
	= 140

GCR (unsys)

5

Figure 5.2: Learner B08's solution to P1

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114)

C27

ANSWER	ROUGH CALCULTIONS
$\frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[2a + (n-1)d\right]$ $\frac{n}{2}\left[$	
Thabang $10 \times 4 = 40$ $40 + 100 = 140$ Mpho $25 \times 4 = 100$ $100 + 40 = 140$	

F

GCR
(unsya)

④

Figure 5.3: Learner C27's solution to P1

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114) F43

ANSWER	ROUGH CALCULTIONS
If thabang and Mpho works four days they will have same amount of money. Thabang 100 + 40 = R140 mpho 40+ 100 = R140 Thabang = mphoi on the fourth day	Thabang 4 days x R10 = R40 Mpho 4 days X R25 = 100

G(CR (mess))
5

Figure 5.4: Learner F43's solution to P1

When asked to describe the kind of strategies he used to solve P1, learner C27 said, "I have noticed that for Mpho, if I multiply his salary by 4 days, it will give me R100, and, then, adding R40 of his pocket money, I get R140. For Thabang, I multiplied his salary by 4 days, and got R40, and added his pocket money to get R140. So, they had the same amount after 4 days".

Learner A31 used SL strategy to solve P1. She calculated the amounts accumulated by each person from day one up to the day they have the same amount of money (see figure 5.5). Learner B57 used SL, LP and MD strategies to solve P1. She applied SL strategy in a different approach to that of A31 (see figure 5.6). B57 used systematic lists to identify patterns and, thereafter, derive the general rules to calculate the number of days the two people will take to have

the same amount of money. Unfortunately, she failed to monitor her final solution
and expressed the units of the period taken in years other than days.

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary
   jobs at   different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If
   they do not spend their pocket money or their daily wages, after how many days will they have
   the same amount of money?

   (Adopted from Muis, 2004, p. 114)

   A31

ANSWER			ROUGH CALCULTIONS
Thabang	Mpho	days	
$100 + 10 = 110$	$R40 + 25 = 65$	1	
$110 + 10 = 120$	$R65 + 25 = 90$	2	
$\frac{120}{110} + 10 = 130$	$R90 + 25 = 115$	3	
$130 + 10 = 140$	$R115 + 25 = 140$	4	

∴ they will both have the
same amount = R 140 after
4 days

Figure 5.5: Learner A31's solution to P1.

124

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114)

Figure 5.6: Learner B57's solution to P1.

5.2.2. **Problem 2 (P2)**

P2 was solved by the learners using two strategies: SC and LG. Learners A31, C27, and F43 used LG to solve P2 in different approaches. Learner A31, firstly, showed the distance in km travelled by each type of a car at a cost of R2 and, then, calculated the number of gallons of gasoline each type of a car uses per year, which she wrongly considered as the cost of using each type of a car for a period of a year. She did not consider all the necessary information given, e.g., the cost of a gallon. As a result, she failed to solve the problem satisfactorily (see

125

figure 5.7). A31 faced difficulties of understanding the requirements of the problem. This was evident in her statement: "I could not really understand what the question wanted me to answer and I found it quiet difficult to do any of the calculations….".

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

(Adopted from Greenes et al., 1986, p. 12)

A31

ANSWER	ROUGH CALCULTIONS
R 2 = 16 km if u drive 15 000 km/year you are spending $\frac{15\ 000}{16}$ = R 9375/yr R 2 = 26 if u drive 15 000 km/year you are spending $\frac{15\ 000}{26}$ = 576,92/yr that is R360,58 less than R 9375	15 000 km/year $\frac{15\ 000}{16}$ = R9 37,5/year 16 km + 10 km = 26 km $\frac{15\ 000}{26}$ = R576,92/year 937,5 - 576,92 = 360,58

Figure 5.7: Learner A31's solution to P2.

Learners C27 and F43 used LG to solve P2 using piece-wise approaches that were slightly different from each other. In a similar fashion as A31, learner C27, firstly, restated the distances covered by each type of a car at a cost of R2 and,

126

then, calculated the costs of using each type of a car for a year (see figure 5.4). Unlike A31, C27 managed to convert the number of gallons each car uses a year to the cost of using each car per year. However, C27 failed to monitor if his solution process (the steps taken to reach the solution) makes sense. Learner F43 approached P2 in a similar manner as C27 and presented mathematically sensible statements (see figure 5.9).

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

   (Adopted from Greenes et al., 1986, p. 12)

C27

ANSWER	ROUGH CALCULTIONS
Old Car $$\frac{15000}{16} = 937,5 \times 2$$ $$\therefore R\ 1875,00$$ Mew Car $$\frac{15000}{26} = 576,92 \times 2$$ $$\therefore R\ 1153,84$$ Save = Cost (old) − Cost(new) = 1875 − 1153,84 = R 721,16	old car $$1g = 16\ km$$ $$1g = R2.00$$ Hew car $$1g = 10+16$$ $$1g = 26$$

L9
5

Figure 5.8: Learner C27's solution to P2.

127

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

(Adopted from Greenes et al., 1986, p. 12)

F43

ANSWER	ROUGH CALCULTIONS
$\frac{15000}{16} = 937,5$ gallons a year on the old car $937,5 \times R2 = R1875$ a year: old car $\frac{15000}{26} = 576,92$ gallons a year: new car $576,9230769 \times R2 = R1153,85$ a year new car Old car — new car $= R1875 - R1153,85$ $= R721,15$ thats the money will be saved in a year using an a new car.	$2 \cdot 16$ 1875 per year $1153,85$

LG
(5)

Figure 5.9: Learner F43's solution to P2.

Learners A26, B08 and B57 used both SC and LG strategies to solve P2, but in different approaches. They all restated the information given in the problem. After restating the given information, B57 presented an incomplete solution to the problem. The parts presented required to be developed further to deduce the cost of using each car for a year (see figure 5.10). When asked to state any difficulties or obstacles met on solving P2, B57 said "I thought of many methods

to use in order to solve this question, but I was not confident enough to pull through as I was intimidated by the question". Asked further of the strategies she applied to solve the problem, she responded: "I tried to compare the cars and the amount of gasoline they used. Afterwards, I thought of using financial mathematics methods, but, due to time, I was unable to finish". When confronted by the problem, B57 devoted much of her time to attempting to fit the situation to the usual problems solved before and searching for an appropriate procedure to apply, e.g. financial mathematics formulae learnt in class.

The solution process of learner B08 was similar to that of C27 in that the learner managed to calculate the cost of using each car for a year by applying a piece-wise approach and presenting steps that does not make mathematical sense (see figure 5.11). Learner A26's solution process was similar to that of F43 in that the learner approached the problem in a piece-wise manner and presented sensible statements (see figure 5.12).

Figure 5.10: Learner B57's solution to P2.

B08

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs
   R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the
   gallon?

   (Adopted from Greenes et al., 1986, p. 12)   B08

ANSWER	ROUGH CALCULTIONS
$\dfrac{15000}{12} = 1250$ km every month $\dfrac{15000}{365} = 41,1$ km each day	16 km per gallon 15 000 km per year 1 gallon = R2,00 New car = 16+10 = 26 km
$\dfrac{15000}{16} = 937,5$ $= 937,5 \times 2$ $= 1875$ $\dfrac{15000}{26} = 576,9 \times 2$ $= 1153,85$ Difference $= 1875 - 1153,85$ $= 721,15$ The person can save R721,15^c	New car = 16+10 $= 26$ km

SC
LG
(S)

Figure 5.11: Learner B08's solution to P2.

130

A26

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

(Adopted from Greenes et al., 1986, p. 12)

A26

ANSWER	ROUGH CALCULTIONS
Old car: 16km; 15000 pe a year $$\frac{15000}{16}$$ = 937,5 km 937,5 × 2 = R1875 New car: 26 km; 15000 km $$\frac{15000}{26}$$ = 576,92 576,92 × 2 = 1153,84 R1875 − R1153,84 = R721,16 He/she is going to save R721,16	Old car: 16 km : 15000 km a year $$\frac{15000}{16}$$ = 937,5 km 937,5 × 2 = 1875 New car: 26 km : 15000 km a year $$\frac{15000}{26}$$ = 576,92 576,92 × 2 1875−1153,84 = 1153,84 =721,16 old He is going to save R721,16

SC
LG

Figure 5.12: Learner A26's solution to P2.

5.2.3. <u>**Problem 3 (P3)**</u>

P3 was solved using four strategies: NLG, GCR(unsys), MD and SC. Learner A26 used NLG strategy to solve P3. The learner simply added the percentages given in the problem that relate to the same gender and, thereafter, wrote a conclusion that was not derived from the calculations presented (see figure 5.13). The calculation presented reveals that A26 did not use all the necessary given information to solve the problem. When asked to state difficulties met on solving

131

P3, A26 said that he did not understand the problem and, as a result, could not identify the appropriate method to tackle the problem. This was evident in his statement, "I met problems on how I must do it, must I add, divide or multiply, and the statement was a little bit confusing".

Figure 5.13: Learner A26's solution to P3.

Learners A31, B08 and C27 used GCR(unsys) strategy to solve P3 in different ineffective ways. Learner A31 used a partial or piece-wise approach to solve P3. She could not connect the two sets of data that pertain to male and female calves in the solution process. Although the information provided in the problem was sufficient to solve it, she felt that more information was supposed to be provided for a successful problem resolution. This was evident in her response, "I felt that they should've given us more information, and, so, I found it hard to

complete the question the way I needed it to be, the way I felt it could be right". On solving P3, B08 seemed to search for well known methods of solving the problem and was much worried of obtaining the correct answer. This was evident in her statement, "I did not know how to respond or which method to apply in order to obtain the correct answer".

What was common in their problem solving behaviour was that they all, initially, guessed the number of male and female calves born to be 100 each and failed to revise and improve their guess work. They, also, failed to check back if their final answer makes sense because if we are to guarantee that 100 calves born survive the first year, then the number born should not be less than 100. Their final solutions were all values less than 100 (see figures 5.14; 5.15 & 5.16).

3. A game management scientist found that 90% of the male calves born and 95% of the female calves born survive their first year. If 50% of the calves that are born are male, how many calves must be born to guarantee that 100 survive the first year?

(Adopted from Greenes et al., 1986, p. 41)

A31

ANSWER	ROUGH CALCULTIONS

ANSWER

90% of 100 = 90

96% of 100 = 96

50% of 100 = 50

∴ 50 calves must be born

OR

90 + 95 = 185

50% = $\frac{185}{2}$

= 92,5

∴ 92,5 calves must be born

ROUGH CALCULTIONS

90% → male

95% → female

50% → male

(handwritten in red margin: GCR(mss) ?? ①)

Figure 5.14: Learner A31's solution to P3.

Figure 5.15: Learner B08's solution to P3.

Figure 5.16: Learner C27's solution to P3.

Learner B57 used MD strategy to solve P3. The learner introduced the x and y variables to stand for the unknown number of male and female calves, respectively. However, the learner did not use all or correctly the necessary given information to form algebraic statements used to solve the problem (see figure 5.17). Learner F43 used SC strategy to solve P3. He simply restated part of the given information and failed to use any given information to solve the problem (see figure 5.18).

3. A game management scientist found that 90% of the male calves born and 95% of the female calves born survive their first year. If 50% of the calves that are born are male, how many calves must be born to guarantee that 100 survive the first year?

B57 (Adopted from Greenes et al., 1986, p. 41)

ANSWER	ROUGH CALCULTIONS

1st year survival %
male calves = 90%
50% female calves = 95%

$90x + 95y = 1$ ...①
$x = 50$ ...②
sub ② → ①
$90(50) + 95y = 1$
$4500 + 95y = 1$
$95y = 1 - 4500$
$95y = -4499$
$\overline{95} \quad 95$
$y =$

MD
①

Figure 5.17: Learner B57's solution to P3.

136

Figure 5.18: Learner F43's solution to P3.

5.2.4. **Problem 4 (P4)**

The learners solved P4 using the following strategies: MD; GCR(unsys); NLG; and LG. Learners A26, A31, B08 and C27 used GCR(unsys) strategy to solve P4 in different approaches. C27 and A31 guessed that each person had one cup of tea and one piece of cake and failed to check and improve the guess. This initial guess gave them a fractional number of people, who took tea, but they could not evaluate that their solution did not make sense; hence, an improvement was needed (see figures 5.19 & 5.20).

4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and cakes cost R 5.00 each. Everyone had the same number of cakes and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had?

(Adopted from Burton, 1984, p. 80)

C27

ANSWER	ROUGH CALCULTIONS
Cake + tea $= 5 + 3 = 8$	
$\dfrac{133,00}{8} = 16$	
number of people $\times 8 = 133,00$	
$n = \dfrac{133,00}{8}$	
$n = 16,6$	
$\dfrac{\text{Number of people}}{\text{cost of each}}$	
$\dfrac{16}{8}$	
$= 2$ for each	

CiCR
(margi)

①

NLa

Figure 5.19: Learner C27's solution to P4.

138

4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and cakes cost R 5.00 each. Everyone had the same number of cakes and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had?

(Adopted from Burton, 1984, p. 80)

A31

ANSWER	ROUGH CALCULTIONS
GCR(unsys)	R5 + R3 = R8
tea cup = R3 one person	tea & cake
cake = R5 = R8	
NLG	16, 625 × R3 =
	49, 875
R133 ÷ 8 = 16, 625	
	16, 625 × R5
① 16, 625 cakes were boaght	
and 16, 625 cups of tea were	= 83, 125
bought	
	83, 125 + 49, 875
	= R133

Figure 5.20: Learner A31's solution to P4.

Learner A26 used GCR(unsys) strategy in a different approach as learners C27 and A31. A26 divided the cost of a cup and the cost of a piece of cake each into the total bill. This approach seems to be non-logical and absurd, and the conclusion reached made no sense (see figure 5.21). Learners B08 and B57 both used MD strategy to solve P4. They introduced x and y variables to stand for number of tea cups and number of pieces of cakes taken by each person. However, they both proceeded incorrectly in forming algebraic expressions and making some guesses. (See figures 5.22 & 5.23). Both learners, B08 and B57, did not consider the third variable, the total number of people who took tea, in forming their algebraic expressions. They did not, also, check if their intermediate steps make sense, e.g., the values of x and y should be natural numbers.

139

Figure 5.21: Learner A26's solution to P4.

Figure 5.22: Learner B08's solution to P4.

140

4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and cakes cost R 5.00 each. Everyone had the same number of cakes and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had?

(Adopted from Burton, 1984, p. 80)

B57

	ANSWER	ROUGH CALCULTIONS
MD ③ 4E		$x = $ tea $y = $ cakes $x \leq 3,00$ $y \leq 5,00$ $3x + 5y \leq 133.00$ $5 \times 5 = 25$ $15 \times 3 = $ s8 slice of cake $= 5.00$ $2 \times 5 = 10$ $10 \times 5 = 75$ $3x + 5y = 133.00$ $3x + 0 = 133$ $\dfrac{3x = 133}{3 \quad 3}$ $x = 44\tfrac{1}{3}$ $x = 44{,}33$ $\therefore x = 44$

Figure 5.23: Learner B57's solution to P4.

In solving P4, B57 was much worried on the correct method to apply and obtaining a correct answer. This was evident in the statement she said when asked to state the difficulties met on solving the problem: "Finding the right method to use which will lead me to the correct answer". When asked to describe the kind of strategies she used to solve the problem, she responded, "Firstly, I attempted to solve the question using simultaneous equations, but later thought it will not produce an accurate answer. Therefore, I used linear programming method and inequalities". Her response indicates that, other than using the given information to create appropriate mathematical models to solve the problem, she attempted to apply methods and formulae learnt in class. Her search for an

141

accurate answer to the problem led her to doubt the use of the models of her own creation. For example, equations derived from the data given in the problem.

Learner F43 used both MD and LG strategies to solve P4. Though the reasoning applied was logical and produced the correct solution, there were some gaps in the solution process that were not explicitly brought to light (see figure 5.24). The learner, probably, guessed the number of people who took tea and thereby worked backwards to determine the number of cups and number of pieces of cakes taken by each person.

4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and cakes cost R 5.00 each. Everyone had the same number of cups and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had?

(Adopted from Burton, 1984, p. 80)

F43

ANSWER | ROUGH CALCULTIONS

There are 7 people and they had 3 cups of tea each and two pieces of cake, each paid an amount of R19 on the bill.

MD
LG
(5)

Let tea cost = x
cakes cost be y

$x + y = 133$
$3 + 5 = 133$

$\frac{133}{7} = R19$

Figure 5.24: Learner F43's solution to P4.

Learner F43 confessed that he met problems on formulating equations that could be used to solve the problem. When asked to describe the kind of strategies he used to solve P4, F43 said, "I have tried to use simultaneous equations and then

divide the bill with the number of people I got from my equations". However, the learner did not show how he got the total number of people, or how he deduced the number of cups or number of pieces of cakes taken by each person. The cancellations of different numbers evident in the solution process might mean that the learner tried substituting different numbers mentally into the equation until he reached the conclusion that was presented.

5.2.5. **Problem 5 (P5)**

The learners solved P5 using the following strategies: NLG; SC; GCR(unsys) and MD. Learner A26 solved P5 using NLG approach. The pattern of reasoning applied by the learner was not logical. He added together part of the two sets of data provided in the problem and got a sum that did not make any sense (see figure 5.25). Learner A31 used GCR(unsys), in conjunction with SC, to solve P5. She repeated writing the given information and guessed the number of hutches. She did not check if the guessed solution satisfies the conditions stated in the problem. As a result, she failed to improve her solution (see figure 5.26).

Figure 5.25: Learner A26's solution to P5.

5. There are some rabbits and some rabbit hutches. If seven rabbits are put in each rabbit hutch, one rabbit is left over. If nine rabbits are put in each rabbit hutch, one hutch is left empty.

Can you find how many rabbit hutches and how may rabbits there are?

(Adopted from Burton, 1984, p. 64)

A31

ANSWER	ROUGH CALCULTIONS
6 × 1 hutch	7 rabbits / hutch
9 × 6 = 64	1 rabbit is left over
∴ 54 hutches	∴ 6 rabbit fit /hutch
54 × 6 = 324	10 hutchs
∴ 324 rabbits	9 rabbits
Prove: 324 rabbits	
54 hutches	
= 6 rabbit /hutch	

SC (CCR(unsys))
①

Figure 5.26: Learner A31's solution to P5.

Learner B08 used GCR(unsys) strategy to solve P5 (see figure 5.27). The learner guessed the number of rabbits and hutches that were there, but failed to check if the initial guess either makes sense or satisfies the conditions provided. For example, if the rabbits present could be possibly grouped exactly in groups of nines, then the possible number of rabbits should be a multiple of 9.

B08

5. There are some rabbits and some rabbit hutches. If seven rabbits are put in each rabbit hutch, one rabbit is left over. If nine rabbits are put in each rabbit hutch, one hutch is left empty.

Can you find how many rabbit hutches and how many rabbits there are?

(Adopted from Burton, 1984, p. 64)

B08

ANSWER	ROUGH CALCULTIONS
Rabbits and Rabbit hutches are both equal to 8	Rabbits = 7 + 1
	= 8
	hutches = 9 - 1
	= 8

GCR (unsys)
①

Figure 5.27: Learner B08's solution to P5.

B57 solved P5 using MD and NLG strategies. She used variables, x and y, to represent the number of rabbits and hutches that were present. However, she went further on to formulating algebraic expressions that made no sense and, hence, failed to complete solving the problem (see figure 5.28). She did not use all the necessary given information in modeling the situation. For example, she did not take into consideration the rabbit left over when grouping them in sevens or the hutch left empty when grouping the rabbits in nines. As a result, the approach employed to solve the problem was not effective.

Figure 5.28: Learner B57's solution to P5.

Learner C27 solved P5 using MD and GCR(unsys) strategies. He made a single guess of the possible number of groupings in nines and in sevens. Then, by taking into consideration all the conditions in the problem, he was able to identify the number of hutches and rabbits that were there. In addition, he also checked or illustrated diagrammatically that conditions stated in the problem were met (see figure 5.29).

145

5. There are some rabbits and some rabbit hutches. If seven rabbits are put in each rabbit hutch, one rabbit is left over. If nine rabbits are put in each rabbit hutch, one hutch is left empty.

Can you find how many rabbit hutches and how many rabbits there are?

(Adopted from Burton, 1984, p. 64)

C27

ANSWER	ROUGH CALCULTIONS

GCR (unugs)

MD

(S)

$9 \times 4 = 36$

$7 \times 5 = 35 + 1 = 36$

5 rabbit hutches

36 rabbits

1 left

⑦   ⑦ ⑦ ⑦ ⑦

⑨ ⑨ ⑨ ⑨ ◯

empty hutch

Figure 5.29: Learner C27's solution to P5.

F43 solved P5 using MD strategy. He used diagrams to deduce the number of hutches and rabbits that were there. The pattern of reasoning was effective because the solution presented showed that F3 took into consideration all the constraints described in the problem. As a result, he was able to state the correct number of hutches and rabbits present (see figure 5.30).

146

Figure 5.30: Learner F43's solution to P5.

5.2.6. **Problem 6 (P6)**

P6 was solved using SC, MD and LG strategies. All the learners applied MD strategy, but in different approaches. Learners A31, B57 and F43 introduced letters to represent the number of fish caught by each person. Then, they represented the word statements in form of algebraic statements involving inequalities. Learners A31 and B57 introduced numbers to work with instead of letters, since the problem involved no numbers. A31 verified the correctness of the guesses by substituting the numbers for the letters in the algebraic expressions and used reasoning to improve the guess (see figure 5.31). The pattern of action employed by A31 was effective.

.

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.
- Annah and Refilwe together caught as many as Joel and Thabo
- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth?

(Adopted from Callejo& Villa, 2009, p. 115)

A31

ANSWER	ROUGH CALCULTIONS

$A = 2 + R = 7$
 $= 9$
 	
 ⊗ $T > J$ ⊗ $A + R = T + J$
 $5 > 4$ $2 + 7 = 5 + 4$
                      $9 = 9$
 	
 ⊗ $A + T \neq R + J$
    $2 + 5 \neq 7 + 4$
     $7 \neq 11$
 	
 ∴ Refilwe caught 7
 She is the one who caught the most
 	
 Thabo came in second with 5
 Joel was third with 4
 And Annah came in fourth
 with 2. She caught the least
	Say:
	 $T = 5$
	 $J = 4$ } 9
	 $R = 7$
	 $A = 2$ } 9
	
	 $A + R = T + J$
	 $2 + 7 = 5 + 4$
	 $A + T \neq R + J$
	 $2 + 5 \neq 7 + 4$
	    $7 \neq 11$
	
	 $R + J > A + T$
	 $7 + 4 > 2 + 5$
	 $11 \; ? \; 7$
	   true
	
	 $A + R \gtrless J + T$
	 $2 + 7 \gtrless 4 + 5$
	 $9 \quad\quad 9$
	   true

Figure 5.31: Learner A31's solution to P6

When asked to describe the kind of strategies she used to solve P6, A31 responded: "I did as many examples in my head as possible that I thought

148

related to the matter. I wrote what I was given and tried hard to make the teacher understand my point of view as I was working it out". The conclusions reached were justified by the calculations done.

Unlike learner A31, B57 did not show in her working how she used the numbers presented. However, the cancellations of different numbers in the solution presented might imply that the learner improved her solution by trying out different numbers and checking mentally if they satisfied the algebraic inequalities presented (see figure 5.32). Though she did not put on the paper all her reasoning processes involved in solving the problem, the approach undertaken seemed effective because, finally, B57 could arrange the people in order of the number of fish they caught.

Learner F43 could not solve the problem completely. He was able to state the person who caught the most fish by using the algebraic expressions and applying reasoning (though the reasoning stated in the solution was not satisfactory) (see figure 5.33). F43 described the strategies he used to solve P6 as follows: "On this question, I used colored pencils to stand for each person. Then, I tried to substitute different numbers for each person until I arrive at the proper person who got the most fish". His description of the strategies used to solve the problem shows that he used tangible objects to represent the people in question and also applied trial and error method to reach a conclusion. However, the different numbers substituted in the inequalities were not presented in the solution.

ANSWER	ROUGH CALCULTIONS
1. Refilwe caught $\cancel{8}$ 7 fishes 2. Thabo caught $\cancel{8}$ 6 3. Joel caught 4 4. Annah caught 3	$w = Annah$ $x = Refilwe$ $y = Joel$ $z = Thabo$ $\cdot z > y$ $\cdot w + x \cancel{>} y + z$ $\cdot w + z < x + y$ Estimation They all caught 20 fishes $Thabo = 6$ $Joel = 4$ $Annah = 3$ $Refilwe = 7$ $\cdot 6z > 4y$ $\cdot 3w + 7x \leq 4y + 6z$ $\cdot 3w + 6z < 7x + 6y$

Figure 5.32: Learner B57's solution to P6.

Learners A26, B08 and C27 approached P6 in a similar way. They all expressed the statements given in form of mathematical statements that involve names and inequalities. In this case, the name acted as a variable. In addition to modeling the situation, A26 used SC strategy in form of repeating writing the given information. All of them (A26, B08 and C27) presented the people in order of the number of fish they caught, but did not show in the solution process how they deduced the order of the people (see figures 5.34; 5.35 & 5.36).

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.

- Annah and Refilwe together caught as many as Joel and Thabo

- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth?

(Adopted from Callejo& Villa, 2009, p. 115) **F43**

	ANSWER	ROUGH CALCULTIONS
MD ③	Refilwe cause she alway appeared on the pair of people which had more caught More fishes.	T > J A $ R > J T A $ T $ R $ J

Figure 5.33: Learner F43's solution to P6.

151

A26

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.

- Annah and Refilwe together caught as many as Joel and Thabo

- Annah and Thabo together did not catch as many as Refilwe and Joel.

    Who caught the most? Who came in second, third and fourth?

A26 (Adopted from Callejo& Villa, 2009, p. 115)

ANSWER

Refilwe caught most
Thabo came in second
Joel third
Annah fourth

MD
SC

(4)

ROUGH CALCULTIONS

Thabo more than Joel
Annah and Refilwe as
many as Joel and Thabo
Annah and Thabo together
did not catch as many
as Refilwe and Joel
         big
Thabo > Joel Small
    big small   small big
Annah and Refilwe = Joel
and Thabo
Annah and Thabo < Refilwe
and Joel

Figure 5.34: Learner A26's solution to P6.

152

B08

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.

- Annah and Refilwe together caught as many as Joel and Thabo

- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth?

(Adopted from Callejo& Villa, 2009, p. 115)

B08

ANSWER	ROUGH CALCULTIONS
Thabo > Joel	Thabo > Joel
Annah + Refilwe = Thabo + Joel	
Annah + Thabo < Refilwe and Joel	
Refilwe caught the most	
Thabo came the second	
Joel came the third whilst	
Annah came the fourth	
Thabo > Joel	
Annah + Refilwe = Joel and Thabo	
Annah + Thabo < Refilwe and Joel	
Refilwe caught the most	
Thabo came the second	
Annah came the third whilst	
Joel is the fourth one	

Figure 5.35: Learner B08's solution to P6.

153

Figure 5.36: Learner C27's solution to P6.

5.3. <u>Learners' mathematics-related belief systems</u>

As mentioned in section 4.3, the items of the beliefs questionnaire were classified into three sub-categories according to Daskalogianni and Simpson (2001a)'s three key macro-belief systems, namely systematic, exploratory and utilitarian. By calculating the mean score of each of the three belief systems (see table 5.2), the learners' strongly held belief systems were identified as follows: The researcher considered a learner having a mean score greater than 3 as holding a

certain belief, and the greater the score the more strongly held the belief is (Jin et al., 2010).

Table 5.2: Selected learners' mathematics related belief systems mean scores

Learner	Mean belief score		
	systematic	exploratory	utilitarian
A26	3.39	3.65	3.68
A31	4.11	3.87	4.21
B08	3.94	3.48	4.05
B57	4.50	4.65	4.63
C27	3.94	3.87	3.74
F43	3.83	3.87	3.95

The table shows that each learner holds all the three belief systems, since their scores are all more than 3 in all the belief systems. Similar results were obtained by Daskalogianni and Simpson (2001a) on their study of learners' mathematics-related belief systems and their mathematical problem-solving behaviour. They discovered that learners whom they categorized as, e.g. utilitarian believers also held some other belief systems (exploratory and systematic). The learner was categorized as a utilitarian believer, for example, because he/she holds the utilitarian belief system more strongly than the exploratory or systematic belief systems. The researcher considered the belief systems with higher scores as the predominant set of beliefs that could possibly explain the learners' approaches to problem solving. According to Schoenfeld (1985) a learner's mathematics problem solving behaviour is largely influenced by his/her predominant belief system.

The researcher used the table to identify the predominant belief systems of the learners as follows: A26 has a highest mean score of 3.68. As such, he holds utilitarian beliefs more strongly than other belief systems. A31 has a highest score of 4.21. The researcher classified her as a utilitarian believer. B08 has a highest score of 4.05. The researcher classified her as a utilitarian believer. B57

has a highest score of 4.65. The researcher classified her as an exploratory believer. C27 has a highest score of 3.94. The researcher classified him as a systematic believer. Finally, F43 has a highest score of 3.95; that classifies him as holding a utilitarian belief system more strongly than other belief systems.

In brief, four learners, A26, A31, B08 and F43, were classified as utilitarian believers; one learner, B57, was classified as an exploratory believer; and one learner, C27, was classified as a systematic believer. In order to understand the constituents of each belief system, the researcher analyzed learners' responses to beliefs questionnaire in conjunction with their responses to retrospective questionnaire and interviews. The researcher, also, inferred some beliefs from learners' solutions to non-routine problems. Below are the learners' mathematics-related beliefs the researcher discovered in this study.

5.3.1. <u>Learner A26 (Utilitarian believer's) mathematics-related beliefs</u>

A26 describes mathematics as a subject that involves working with numbers, solving problems using different methods and has daily life applications. In his own words he said, "Mathematics is something whereby you work with numbers, solve problems using different methods and is something we use in our daily life". Beliefs that can be inferred from A26's definition of mathematics are: Mathematics is numbers and calculations. There are always numbers in formulations of mathematics problems. Mathematics is a set of rules and techniques. Though A26 faced difficulties in explaining what is solving a mathematics problem, the following pieces of facts could be picked from his statements: "…given numbers,…solve using any method as long as you will get the correct answer". This explanation reveals that A26 holds a belief that solving a problem is looking for the correct answer.

A26 views mathematics as an important subject. He strongly believes that he will be able to use what he learns in mathematics also in other courses and to study what he would like after he finishes high school. The following dialogue reveals this belief:

Interviewer: Can you state a reason why you strongly agree with the statement: "To me maths is an important subject".

A26:   Maths is an important subject because we use it in our daily life, and, also, the career that I want needs mathematics.

He prefers tasks that are easy and do not involve long and tedious calculations. In his own words, he said, "What I dislike about mathematics is that it needs much practice, and other problems need long calculations". Although he strongly believes that he can handle more difficult mathematics, he thinks he can do well in mathematics if he works closely with his subject teacher and other mathematics students. The following dialogue reveals this belief:

Interviewer: Can you state a reason why you rated the following statement 'uncertain': "Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone".

A26:   Sometimes I try to solve problems on my own even if I see that those problems want me to think hard. I work on them until I see that there is no progress on what I am doing, and, then, that is when I look for someone to help me.

A26 believes that he will be able to learn the material in mathematics if he studies in appropriate ways. Moreover, he thinks that learning mathematics requires a lot of effort. He thinks that, on solving some other types of problems, understanding why the mathematical procedure works is not important as long as it gives the correct answer. The following extract from the interviews indicates this belief:

Interviewer: Can you state a reason why you rated the following item 'uncertain': "It is not important to understand why a mathematical procedure works as long as it gives the correct answer".

A26:   I think it is important, and, at the same time, it is not important. According to me, it will depend on what types of problems are we dealing with.

Learner A26 solved P1 by presenting an exact and straight forward solution (see section 5.2.1). From his solution to P1, the researcher can infer that he believes that mathematics has exact answers and the solution process is linear. Learner A26 solved P3 and P5 by simply adding the numbers given in the problems. As a result, he obtained sums that made no sense. His approach to P3 and P5 might

157

reveal that he believes that it is not important to understand the solution one presents (see sections 5.2.3 and 5.2.5). A26 solved P6 by presenting an answer without a process that shows how it was obtained. From this behaviour, the researcher can infer that he believes that what is important is a solution to the problem, not the solution process (see section 5.2.6).

According to Daskalogianni and Simpson (2001a), the following set of beliefs held by learner A26 classifies him as a 'utilitarian' believer: Mathematics involves numbers; mathematics is applied in daily life and other courses; mathematics is a set of rules and techniques; it is not important to understand the solution one presents; Mathematics is a static and rigid body of knowledge; a mathematics problem has an exact answer; what is important is a solution to a problem, not the solution process; prefers easy tasks; prefers short calculations; depends on the subject teacher or other students to learn mathematics effectively; learning mathematics requires a lot of effort; and one has to study in appropriate ways to be able to learn mathematics.

5.3.2. <u>**Learner A31 (Utilitarian believer's) mathematics-related beliefs**</u>

A31's responses to the beliefs questionnaire, open questionnaire and interview indicate the following beliefs: She views mathematics as a set of rules and techniques. She believes that, in most cases, there are numbers in formulations of mathematics problems. She confirms this belief by saying, "…sometimes you get questions where you have to give reasons, but most times there are numbers in formations of mathematics problems". She practically put this belief in action on solving P6 by introducing her own simple numbers to work with because the problem had no numbers in its formulation (see section 5.2.6). Asked to describe what a mathematics problem is, she said, "A mathematics problem is a situation whereby something needs to be done right or placed in a correct form, so it could be easy to understand, and it has to be done in numeric form". She believes that

the aim of solving a mathematical problem is to make the problem understandable by everybody. The following dialogue reveals this belief:

Interviewer: What is solving a mathematical problem?
A31:   Simply finding a solution for a given problem (in numbers and, sometimes, in words) and making it as clear as possible so you and other people can understand it.

A31, also, has an open view of mathematics. She believes that mathematics is continuously evolving. As such, new aspects of mathematics are continuously being discovered. She strongly believes that making mistakes is part of learning mathematics. She justified this belief saying, "Nobody can ever be perfect at everything, you have to make mistakes so that you can learn more". Asked why she disagrees with the statement "It is not important to understand why a mathematical procedure works as long as it gives the correct answer" she responded, "It is very important to understand why a procedure works for it will help you to know how to solve the problem, and, after you know it, you will be able to solve any type of the given problem". She believes that there are several ways to find the correct solution of a mathematics problem. She practically put this belief in action on solving problems 4, 5 and 6 (see sections 5.2.4, 5.2.5 & 5.2.6) by applying more than one strategy to solve each problem.

She believes that the most important thing in learning mathematics is to understand. To confirm this belief she said, "You don't have to memorize facts and procedures, but you just have to practice them so that you know them because all mathematics procedures work in every situation in life". She, also, believes that learning mathematics requires a lot of effort, and views solving a mathematics problem as demanding and requiring much thinking. To justify her disagreement to the statement, "It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem" she said, "… it makes us think out of the box and exercise our brain, it trains us to be good in mathematics". She strongly believes that one learns mathematics through working on exercises, and, whenever a blockage arises, one should seek help from other students or the subject teacher. To confirm this belief she said, "Most

times I do the work on my own to see just how much I understand, try to fix my problem areas and get more practice on them. Yes, sometimes, I do need help here and there, but I prefer doing it on my own first to see how good I am in this subject".

Below are some citations from open questionnaire and interviews that confirm her beliefs about mathematics, mathematics problem solving and herself, as a mathematics learner:

Interviewer: What do you like or dislike about mathematics as a discipline or subject?
A31:   I like mathematics as a subject because it opens many opportunities in careers, and it helps one's mind to think sharply, clearly and wisely. What I dislike is the fact that it has too much algebra, solving for 'x', but other than that, it's a good subject.

Interviewer: Could you tell me why you rated this item 'strongly agree': I am only satisfied when I get a good grade in mathematics.
A31:   I, personally, get satisfied when I get a good grade because getting good grades mean I understand what I am doing and how to solve the given problems.

Learner A31 solved P1 by searching for the answer systematically (see section 5.2.1). This approach to problem solving might mean that she believes that the solution process is important. When solving P2, A31 did not use all the necessary information provided (see section 5.2.2). This might be due to failure to analyse and understand fully the problem. This problem solving behavior might result from beliefs that there is a certain amount of effort and time needed to solve a particular kind of problems and not all conditions given are necessary to solve a problem. Learner A31 solved P3, P4 and P5 by unsystematically guessing the solutions and not improving the guesses (see sections 5.2.3; 5.2.4 & 5.2.5). As a result, non-sensible solutions were presented. This problem solving behaviour might be due to the belief that it is not necessary to make sense of the solution one presents. P5 was solved by introducing her own simple numbers to work with because the problem had no numbers in its formulation (see section 5.2.6). This behaviour might be resulting from a belief that mathematics problems always involve numbers in their formulation. However, some of the inferred beliefs are at odds with the learner's confessed beliefs. For example, the belief that 'there is a certain amount of effort and time needed to

solve a particular kind of problems' seem to oppose the learner's confessed belief 'solving a mathematics problem requires a lot of thinking'.

In brief, learner A31 holds the following set of beliefs about mathematics and mathematics problem-solving which classifies her as a 'utilitarian' believer according to Daskalogianni and Simpson (2001a): Mathematics is a set of rules and techniques; Mathematics is applied in everyday life; Mathematics develops one's mind to think better; Mathematics involves numbers; Mathematics problems are solved by applying well known procedures; All the information required to solve a problem is provided in the problem situation; Mathematics is learnt through practice; Learning mathematics requires a lot of effort; Solving a mathematics problem is looking for a solution that enables everybody to understand the situation well; Solving a mathematics problem is demanding and requires much thinking; One should rely on the subject teacher to learn mathematics; and One is satisfied by obtaining a good grade in mathematics.

5.3.3. <u>**Learner B08 (Utilitarian believer's) mathematics-related beliefs**</u>

B08 defines mathematics as "a subject which includes solving problems using numbers, variables, methods and rational thinking". Her definition of mathematics shows that she views mathematics as all about solving problems. She views mathematics as numbers, algebraic terms and calculations. She believes that a mathematical problem is made up of numbers in its formulation. This belief is revealed in her statement, "Mathematics problem is any situation involving numbers that require one to apply skills and knowledge of mathematics in order to come up with a solution". The process of problem solving is seen as involving application of known methods, formulae and rational thinking. Her definition of mathematical problem solving is "... interpreting the questions, coming up with mathematical strategies on how to solve the problem and applying mathematical formulae to find the relevant answer".

She prefers solving problems with known algorithms more than problems which could be best solved by using reasoning. She justifies her preference by saying, "I enjoy solving algorithmic problems because you just apply the formula, substitute and get the answer". She is not comfortable with non-routine word problems as well as challenging mathematical problems. The following dialogue reveals this feeling:

Interviewer: Given the following two mathematical problems, which one will you like to answer? (Don't try to solve it). Give reasons for your choice.

(a) Find the value of x in: $\frac{1}{4}x + 5 = 25$.

(b) Mpho spends a quarter of her money on chips and R5 on soft drinks. Together, she had spent R25. How much did she have initially?

B08:   Part (a) because it would not take me a very long time to answer it and I am sure I can answer it correctly.

Interviewer: How much comfortable are you in learning new material? How do you feel in face of challenging material?

B08:   I am not too much comfortable because I don't know if I can understand all the concepts of a new chapter.

She believes that new discoveries of mathematics are still being made. She said, "I think new things about mathematics can be discovered and new formulae can be derived". She believes that making mistakes is part of learning mathematics. She justifies this belief in her response: "Making mistakes and being corrected is part of learning mathematics because most of us learn from our mistakes". However, she also holds some conflicting beliefs. For example, on one hand, she strongly believes that mathematics is a solitary subject that should be worked on alone without help from anyone, while, on the other hand, she views group work as important in facilitating the learning of mathematics. In addition, she agrees that when she cannot understand the material in mathematics, she asks other students or her teacher for help. The following dialogue confirms this belief:

Interviewer: Can you state, in brief, the strategies you employ in studying or learning mathematics.

B08:   I attempt to resolve questions that we worked on in class and compare my solutions, and identify where I go wrong. If I can't get anything we did in class well, I consult my teacher or one learner who understands better than me.

She views mathematics as an important subject. She thinks that she will be able to use what she learns in mathematics also in other courses. Unfortunately, she believes she will not need mathematics in order to study what she would like after finishing high school. When asked her main goal in learning mathematics, she responded: "I want to be able to apply it in my daily life where it is required". She does not believe that anyone has the potential to learn mathematics. She justifies her belief, basing the argument from her observation, by saying, "… there are learners or people who cannot even make sense out of a mathematical problem and never mind attempting to solve it". She considers practice as the best way to learn mathematics. This is evident in her statement: "The best way to learn mathematics is through practice, making mistakes and be corrected". She believes that learning mathematics requires a lot of effort. Below are some extracts of her responses to the open-ended questionnaire and interviews that reveal some of her beliefs:

Interviewer: Could you tell me why you rated this item 'strongly disagree': It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.

B08:   It's not a waste of time when the teacher makes us think on our own because it helps us to come up with strategies on how to go about resolving a mathematical problem; it helps us think 'out-of-the-box'.

Interviewer: Could you tell me why you strongly disagreed with the statement: It is not important to understand why a mathematical procedure works as long as it gives the correct answer.

B08:   I think it is very important to know why a mathematical procedure works because it can contribute to us coming out with our own procedures. It is important to understand the root or origin of something before you can apply it.

Learner B08 solved P1, P2 and P6 by presenting only the exact correct answers (see sections 5.2.1; 5.2.2 & 5.2.6). Beliefs that could be inferred from solutions of P1, P2 and P6 are that mathematics problems have exact solutions; what is important is a correct answer, not the process and understanding the problem is important. B08's solution to P5 reveals that she believes that there is a certain amount of effort and time needed to solve a particular kind of problems (not enough thought and time should be given to a problem one cannot easily determine how to solve it) (see section 5.2.5).

In a nutshell, according to Daskalogianni and Simpson (2001a), the following set of beliefs held by learner B08 classifies her as a 'utilitarian' believer: Mathematics is about solving problems; Mathematics is numbers, algebraic terms and calculations; problem solving involves application of known methods, formulae and rational thinking; Can do well in solving algorithmic problems; It takes a short time to solve the type of problems one is good at; Do not enjoy challenging exercises; rely on the teacher or other students to learn mathematics; Mathematics is applied in other courses and in everyday life; and mathematics is best learnt through practice.

5.3.4. <u>**Learner B57 (Exploratory believer's) mathematics-related beliefs**</u>

B57's responses to the open-ended questionnaire, beliefs questionnaire and interviews reveal that she holds the following beliefs: She understands mathematics as a subject that involves numbers, and is about solving problems. She views mathematics as a set of rules and techniques. These beliefs are revealed in her definition: "mathematics is a science of numbers whereby laws, principles and theories are studied and applied in order to acquire solutions to mathematics problems that might benefit us in our daily lives in terms of improvement". Her definition, also, brings out an idea that mathematics is a solution to people's some daily life problems.

She considers mathematics as an important subject that is applicable in our daily lives. She, also, thinks that mathematics is an important requirement for the course she wishes to pursue after high school. To justify why she strongly agree that mathematics is important to her, she said, "Yes it is, as I enjoy learning it, it provides knowledge and other challenges which build and develop one's brain, therefore, making it possible to solve daily problems, and, also, the career I have chosen to pursue requires mathematics". She enjoys the challenge that mathematics sometimes brings and exploring new things. She believes that making mistakes is part of learning mathematics. This is evident in the following dialogue:

Interviewer: What do you like or dislike about mathematics as a discipline or subject?
B57:   I like maths as a subject because it challenges one's mind and provide a sense of adventure that one will be pleased and enjoy exploring the journey as, after a while, the development of the brain occurs as a result of the knowledge gained from the various exercises done. I dislike nothing about it, it's my favorite subject.

Interviewer: Could you tell me why you rated this item 'strongly disagree': It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.
B57:   I strongly disagree with this statement because, as learners, one of the basic things we should have is the need to learn more and not being afraid of a challenge. If the teacher lets us think and try to solve questions on our own, then we will improve and discover great mathematical mysteries.

Interviewer: How much comfortable are you in learning new material? How do you feel in face of challenging material?
B57:   I'm very comfortable in learning new material because I'm confident and know that with my determination, focus and hard work that will be applied to the new material, I will understand at the end. I'm not a quitter, and challenges only build me than breaking me.

B57 believes that learning mathematics requires a lot of effort. She, also, believes that one can understand mathematics through hard working. She thinks that she will be able to learn mathematics when she studies in appropriate ways, of which one of the ways is doing daily exercises. Although B57 is confident in her ability in mathematics, she, also, believes that mathematics can be best learnt by working in collaboration with other students and the subject teacher. These beliefs are evident in the following dialogue:

Interviewer: What is the best way you think you can learn mathematics?
B57:   The best way to learn mathematics is, firstly, to understand it, secondly, practice daily for better results, such as writing a test, solving previous question papers in order to get used to how to solve exam questions, and, lastly, if you are struggling, you have to swallow your pride and ask for help from those whom you think are able to solve the questions that you are unable to solve, e.g. educators, learners or even parents.

B57 believes that understanding the mathematics concepts learnt is very important for one to be a successful mathematics problem solver. She builds understanding of the material in class by making connections between the readings and the concepts from the lessons. She confessed that she, also, tries to relate the ideas in mathematics to those in other courses whenever possible. Some of these beliefs are evident in the following dialogue:

Interviewer: Could you tell me why you rated this item 'strongly disagree': One learns mathematics best by memorizing facts and procedures.
B57:   I strongly disagree because you cannot learn mathematics by, only, memorizing facts and procedures; you have to understand, as mathematics questions are presented in different ways in tests or exams. Therefore, if one does not understand how to solve the questions, then he/she will struggle and fail.

Interviewer: Could you tell me why you rated this item 'strongly disagree': It is not important to understand why a mathematical procedure works as long as it gives the correct answer.
B57:   I strongly disagree as this does not show that one is growing mentally and understanding mathematics laws and principles. You cannot do something without knowing the reason behind it, e.g. why 1 + 1 = 2. You have to know reasons in order to understand problems easily and, also, not to apply the laws wrongly. By understanding and knowing reasons, along with practicing daily, one is able to study and love doing mathematics.

B57 defines solving a mathematical problem as "applying mathematics principles and laws or theories to find a solution, and simplify it to have some form of understanding of the problem itself so that one can be able to use the solution or knowledge gained to solve other problems in our daily lives". From this definition, the researcher can infer that B57 believes that simplifying the problem to a form that is understandable is part of solving the problem. She believes that a solution can be obtained by applying well known mathematical rules or techniques. She,

166

also, believes that a solution to one problem can be applied to solve other problems. She believes that there are many ways to solve one mathematical problem.

She prefers solving, for example, algebraic equations other than word problems that need to be modeled into algebraic equations. She justified her choice saying, "I enjoy doing algebraic equations rather than word problems as I sometimes get confused and unable to understand what the questions require me to do, but, with algebraic equations, I'm able to adapt and understand which mathematics principle or law I can use to solve them within a few seconds". She enjoys solving problems with known algorithms more than problems which can be best solved by using logic or reasoning. She justified her preference saying, "I think those with equations, known formulae, as I can use the knowledge gained in order to create my equations which are simplified and I can apply the formulae to solve the problems. Word questions are a bit confusing, as some may not make sense. Therefore, this might prevent me from solving the problem fully….".

Learner B57 solved P1 by applying three strategies: systematic listing, looking for pattern and modeling (see section 5.2.1). P4 was solved by modeling and try-and-error (see section 5.2.4). The possible beliefs behind this problem solving behaviour might be that a mathematics problem can be solved by more than one strategy; solving a mathematics problem involves studying the relationship among numbers and linking things and the solution process is important. B57 presented short and incomplete solutions to P2 and P5 (see sections 5.2.2 & 5.2.5). This behaviour might have been driven into being by the belief that if one is capable to solve the problem, then, the solution should be obtained within a shortest possible time. B57's solution to P6 reveals a belief that when solving a problem, it is important to consider the 'whole picture' of the problem (see section 5.2.6).

In brief, according to Daskalogianni and Simpson (2001a), the following set of beliefs classifies learner B57 as an 'exploratory' believer: Mathematics involves numbers; mathematics is about solving problems; it is important to consider the

'whole picture' of the problem; mathematics is evolving; mathematics develops one's thinking ability; mathematics is applied in everyday life; mathematics is a prerequisite to study a career; enjoys challenges that mathematics brings; enjoys exploring new things; making mistakes is part of learning mathematics; learning mathematics requires a lot of effort; confident in one's ability in mathematics; mathematics can be best learnt in collaboration with other students and the subject teacher; understanding is important in learning mathematics; understanding is built by making connections between readings and concepts from lessons; and there are many ways to solve a mathematics problem.

5.3.5. **Learner C27 (Systematic believer's) mathematics-related beliefs**

C27 defines mathematics as "the study of numbers, how they relate to each other, numerical patterns and formulae….". This definition reveals that the learner views mathematics as, partly, made up of numbers, relationships among numbers and formulae. Contrary to this view, the learner strongly disagrees that mathematics is numbers and calculations in response to the beliefs questionnaire. He believes that there are no new things about mathematics that are yet to be discovered. He bases his argument on the fact that all mathematics problems asked at school could be solved with well known methods or formulae. In his own words, he said, "…. any problem can be solved without discovering a new method to solve it".

C27 defines a mathematical problem as a "situation that is amenable to being analyzed and possibly solved with the methods of mathematics". This view of a mathematical problem emphasizes application of mathematical methods or formulae in solving a problem. He defines solving a mathematical problem as looking for and obtaining a solution to the problem. He enjoys solving problems with known algorithms more than problems which could be best solved by using reasoning. He justified his choice saying, "Because you have to identify your variables in your formula and solve for the unknown variables". Interest in

formulae or algorithms might mean that the learner prefers problems that could be solved by following some predetermined steps. This leaves less room for creativity and reasoning.

He feels that the most important thing in mathematics is getting the correct answer. In his own words, he said, "I think mathematics is about accuracy. If you use correct procedures, you will get an accurate answer". This view might mean that, in solving a mathematical problem, the learner, ultimately, expects to get an exact answer. He considers understanding of concepts as important in learning mathematics. He, also, considers understanding why a procedure works as more important than knowing how to apply it to obtain correct answers. These beliefs are evident in the following dialogue:

Interviewer: Could you tell me why you rated this item 'disagree': One learns mathematics best by memorizing facts and procedures.
C27:   In mathematics you should not memorize, you should know how to approach problems and apply correct formulae by means of studying.

He relies on other significant people for help whenever he confronts a problem he could not solve. The following dialogue reveals this belief:

Interviewer: Could you tell me why you strongly disagree with the statement: Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone.
C27:   Once you notice that you have trouble learning the material, you should consult someone who is doing well in the material for help.

Interviewer: What is the best way you think you can learn mathematics?
C27:   By making an effective study, meaning, studying in a peaceful and quiet environment, solving problems, allowing your mind to think, and consulting anyone who is able to solve problems you cannot solve.

He views mathematics as an important subject. He needs mathematics in order to study what he would like to do after finishing high school. He thinks learning mathematics can make him an effective problem solver of both school mathematics problems and real life problems. The following dialogue reveals these views:

Interviewer: Could you tell me why you 'agree' with the statement: To me mathematics is an important subject.

C27:   Is important because it increases the rate at which I thinks; and be faster in approaching and solving some problems.

Interviewer: What is your main goal in learning mathematics?

C27:   To be able to solve any problem I come across.

Learner C27 solved P1 by using a formula and unsystematic guessing (see section 5.2.1). Beliefs that could be inferred from the solution are that mathematics problems can be best solved by well known procedures or algorithms and all the information required to solve a problem is provided in the problem situation. C27 solved P2 by presenting a step-by-step straight forward solution (see section 5.2.2).The possible beliefs behind this problem solving approach are that the problem solving process is linear; one advances directly towards the solution and an answer is presented in a series of steps. C27's solutions of P3 and P4 reveal a belief that it is not necessary to understand or make sense of the solution process presented (see sections 5.2.3 & 5.2.4). C27 presented an answer only of P6 (section 5.2.6). This might be due to beliefs that what is important in mathematics is obtaining the correct answer and some mathematics problems are solved by applying rational thinking. C27's solution to P5 reveals a belief that all the necessary conditions given in a problem should be used in search of a solution (see section 5.2.5).

According to Daskalogianni and Simpson (2001a), the following set of beliefs C27 holds classifies him as a 'systematic' believer: Mathematics a study of relationships among numbers; mathematics is about solving problems; mathematics is a static and rigid body of knowledge; a mathematics problem is solved by applying well known methods or formulae; solving a problem requires logical thinking; enjoys solving algorithmic problems; the solution process follows some predetermined steps; getting an accurate correct answer is what is important in mathematics; it is important to understand fully concepts and procedures in mathematics; and one should rely on the subject teacher and other learners to learn mathematics effectively.

170

5.3.6. **Learner F43 (Utilitarian believer's) mathematics-related beliefs**

F43 agrees that mathematics is about solving problems. He defines mathematics as a "branch of knowledge that deals with measurements, numbers and quantities". Although he views mathematics as involving numbers, he disagrees with the statement that 'there are always numbers in formulations of mathematics problems'. He views solving a mathematics problem as doing calculations, using different methods, to get the correct answer. He confesses that he always looks for other ways of solving the problem whenever he finds a solution. He justifies his action saying, "There can be a simpler method that can be done when attempting solving the problem". As such, he believes that there is more than one way to find the correct solution of a mathematics problem.

He thinks that he can learn mathematics best by practicing mathematics problem-solving every day. The strategies he employs in studying mathematics are "I use a home time-table and even free periods at school to study. When solving problems, I use methods, procedures and examples that I have been taught by my teacher". He depends heavily on his mathematics teacher to learn mathematics. He considers his teacher as the main source of mathematics knowledge and consults him/her whenever he encounters an obstacle in his studies. He, also, considers group work as important in facilitating the learning of mathematics. The following dialogue reveals some of these views:

Interviewer: Could you tell me why you rated this item 'uncertain': Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone.
F43:   I am uncertain because when I cannot find help from other people, I will think I am doing right whilst doing wrong.

Interviewer: Why do you 'strongly agree' with the statement: Group work facilitates the learning of mathematics.
F43:   I strongly agree because, when solving mathematics problems with other people, you share new things and even shortest methods one didn't know.

He prefers solving problems with known algorithms than problems which could be best solved using reasoning. He justified his choice saying, "Problems with

171

known algorithms are easy to solve and they don't take a lot of time solving them". This reason reveals that F43 enjoys solving problems that can be completely solved within a short period of time. He is comfortable in learning mathematics material he can easily understand. In the face of challenging material, he said, "I feel like I'm losing my ability to do mathematics, but I strive to solve the challenges".

He considers mathematics as an important subject that has applications in other courses and in everyday life. When asked his main goal in learning mathematics, he said, ".… my career needs mathematics and to be able to solve real life situations easily using mathematics". Although he thinks that new discoveries might be there in future, he believes that, as at present, there are no new discoveries being made in mathematics, simply, because they have been learning at school the same mathematics material for years without noticing anything new being added to their text books. In his words, he said, ".…we are still learning the theorem of Pythagoras, and I never heard of new things being discovered, but I think they will be there in the future".

He believes that one should understand why mathematical procedures work and be able to apply them correctly in problem solving. He thinks that understanding enables one to file the learnt concepts properly in the brain and be able to retrieve and apply them correctly when the need arises. The following dialogue reveals these views:

Interviewer: Could you tell me why you are 'uncertain' that 'one learns mathematics best by memorizing facts and procedures'.

F43:   I am uncertain because if you can memorize facts and procedures without correct application of them, it will not help.

Interviewer: Could you tell me why you 'disagree' with the statement: It is not important to understand why a mathematical procedure works as long as it gives the correct answer.

F43:   I disagree; we need to understand the procedures so that they are captured in our brains.

Learner F43 solved P1 by presenting an exact answer without a process showing how it was obtained (see section 5.2.1). Possible beliefs that might be inferred from this solution are that mathematics problems have exact solutions and the solution process is not important. F43 solved P2 and P5 by presenting all the step by step processes in a meaningful way. He even verified that the conditions of P5 were satisfied by modeling the situation in form of circular diagrams (see sections 5.2.2 & 5.2.5). This approach to problem solving might mean that F43 believes that the solution process is important (It is necessary to write meaningful and correct mathematical statements) and all the conditions given in a problem should be used in searching for a solution. The problem solving behaviour he portrayed on P2 and P5 seem to contradict that shown on P1. A similar finding was made by Callejo and Vila (2009). However, the coherence belief theory claims that all the beliefs of a learner are equally important in understanding his/her problem solving behaviour irrespective of their contradictory nature (see section 2.4.2.2). F43's solutions to P3 and P6 reveal a belief that not enough thought and time should be given to a problem one has no immediate idea of how to solve it (see sections 5.2.3 & 5.2.6).

According to Daskalogianni and Simpson (2001a), F43 can be classified as a 'utilitarian' believer because of the following set of beliefs he holds: Mathematics is about solving problems; mathematics is numbers and calculations; solving a mathematics problem involves using different methods and formulae to obtain a correct answer; mathematics can be best learnt by practicing problem solving everyday; depends on the subject teacher to learn mathematics; enjoys solving algorithmic problems; prefers problems that can be solved within a short period of time; comfortable in learning easy material; mathematics has applications in other courses and in everyday life; and it is important to understand procedures and apply them correctly in problem solving.

5.4   **Relationship between learners' mathematics related belief systems and their approaches to mathematics problem solving**

The researcher drew a scatter diagram by plotting the learners' problem solving approaches mean scores against their predominant mathematics related belief systems mean scores (see Figure 5.37). The pattern of the dots does not depict any special significant statistical relationship (e.g. linear, exponential, quadratic, circular) between the two constructs (belief system and problem solving approach). Pearson's correlation coefficient of 0.242 means that there is a very weak statistical linear relationship between mathematics problem-solving approach and mathematics-related belief systems (see table 5.3). The correlation coefficient of determination, $R^2 = 0.0586$ means that only 5.86% of the change in approach to problem solving is explained by the mathematics-related belief systems and vice versa. Since there is no special significant statistical relationship between the two constructs, a qualitative analysis of the relationship between them could possibly reveal the nature of their relationship. The researcher, also, analysed the relationship between approach to problem solving and each belief system (see sections 5.4.1; 5.4.2 & 5.4.3).

Table 5.3: Pearson's correlation coefficient, Problem solving versus Belief system

		PROBLEM SOLVING AVERAGE SCORE	BELIEF SYSTEM MEAN SCORE
PROBLEM SOLVING AVERAGE SCORE	Pearson Correlation	1	.242[**]
	Sig. (2-tailed)		.000
	N	425	425
BELIEF SYSTEM MEAN SCORE	Pearson Correlation	.242[**]	1
	Sig. (2-tailed)	.000	
	N	425	425

**. Correlation is significant at the 0.01 level (2-tailed).
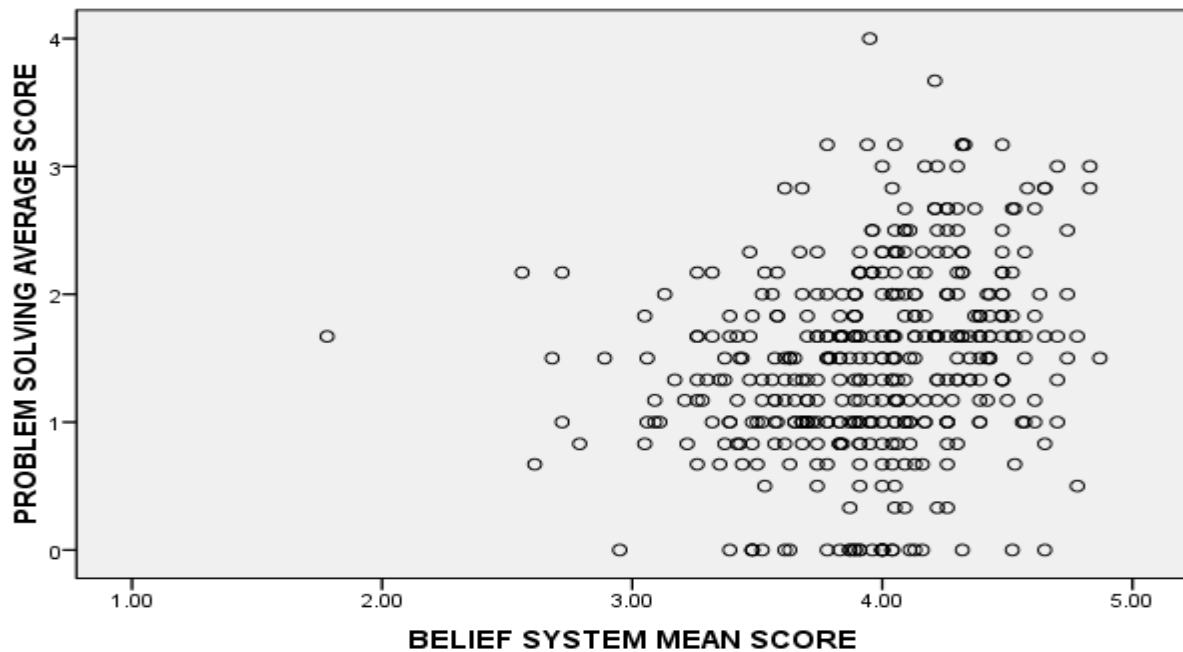
Figure 5.37: Scatter Plot showing relationship between learners' belief systems and approach to problem solving.

5.4.1 **Relationship between Utilitarian belief system and approach to problem solving**

The researcher drew a scatter diagram by plotting the learners' problem solving approaches mean scores against their utilitarian mathematics-related belief system mean scores (see Figure 5.38). The pattern of dots is similar to that of figure 5.37. Pearson's correlation coefficient, r = 0.241 (see table 5.4) means that there is a very weak positive linear relationship between approach to problem solving and utilitarian belief system. The correlation coefficient of determination, R^2 = 0.058 means that only 5.8% of the change in approach to problem solving is explained by the utilitarian belief system and vice versa. In order to explain the nature of the relationship between learners' utilitarian belief systems and their approach to non-routine problem solving a qualitative analysis was done (see section 5.4.1.1.).

Table 5.4: Pearson's correlation coefficient, Problem solving versus Utilitarian belief system

		PROBLEM SOLVING AVERAGE SCORE	UTILITARIAN BELIEF SYSTEM MEAN SCORE
PROBLEM SOLVING AVERAGE SCORE	Pearson Correlation	1	.241**
	Sig. (2-tailed)		.000
	N	425	425
UTILITARIAN BELIEF SYSTEM MEAN SCORE	Pearson Correlation	.241**	1
	Sig. (2-tailed)	.000	
	N	425	425

**. Correlation is significant at the 0.01 level (2-tailed).
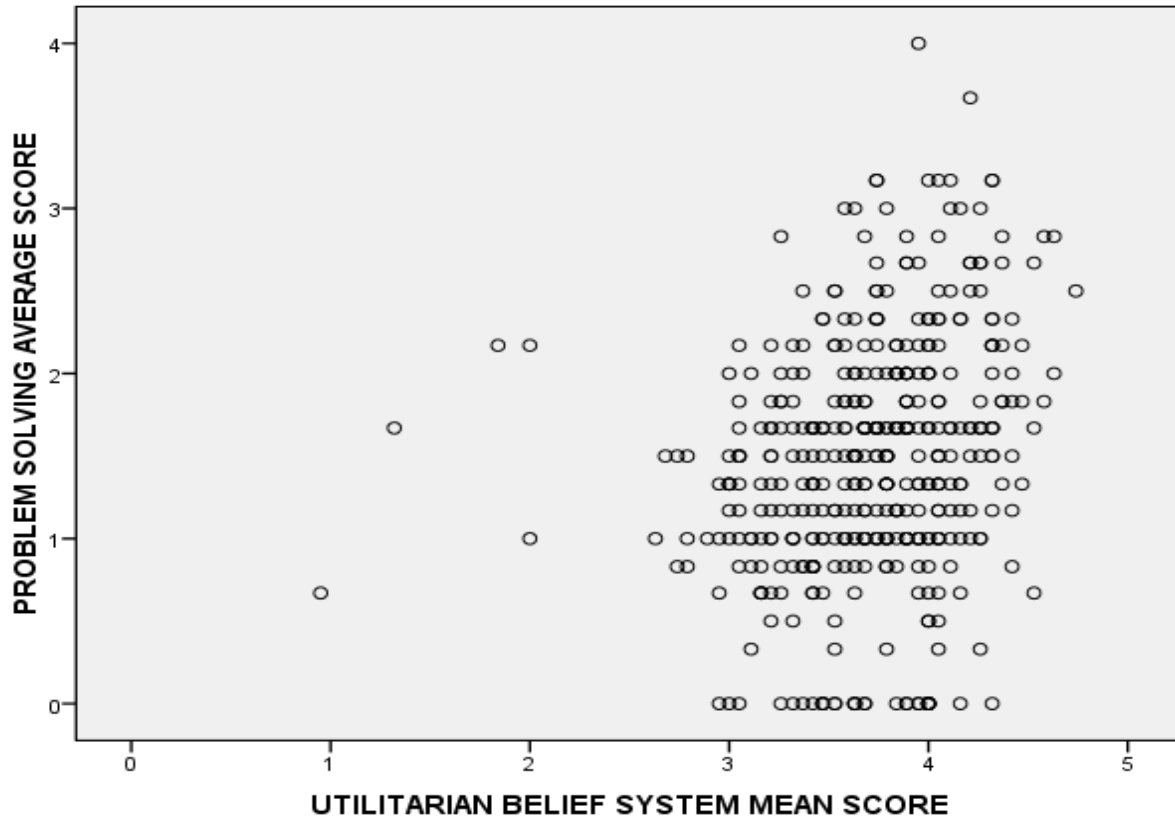
Figure 5.38: Scatter plot showing relationship between approach to problem solving and utilitarian belief system.

5.4.1.1. <u>**Learner A26: Utilitarian believer**</u>

The approach applied by A26 to P1 was guess work. He guessed '4' as the correct number of days the two people will take to have the same amount of money and verified the guess by calculating the total amount they will each have after 4 days. His utilitarian beliefs, e.g. mathematics problems have exact answers, and problem solving process is linear, could have led him to apply an unsystematic guess and check approach to problem solving. The utilitarian set of beliefs might have made him to avoid showing a systematic search and improvement of his guesses until he obtains the solution to the problem. The learner did the calculations somewhere and presented only a straight forward

solution on the answer space provided. The utilitarian beliefs seem to be at play in presenting piece-wise, logical and mathematically correct statements in verifying the solutions.

When approaching P2, A26 started by listing some of the given data and, then, solved the problem by using logical reasoning. His utilitarian set of beliefs might have led him to answer P2 by applying a piece-wise approach; presenting separate logical statements, and not considering a holistic approach to problem solving (not presenting a statement that takes into account the whole picture of the problem). When solving P3, A26 did not consider all the given necessary information, but, rather, he simply added together percentages given in the problem that relate to the same gender. Because of his set of utilitarian beliefs, he could not link the given set of data in the problem, and failed to derive meaning from the problem. Faced with this unusual problem that required one to consider the whole picture and, possibly, use a mathematical model to solve it, A26 could not draw up meaning of the situation and get an effective strategy to solve the problem. As a result, he approached the problem blindly. An extract from the retrospective questionnaire below justifies this analysis:

Interviewer: State any difficulties or obstacles you met on solving the problem.
A26:   I met a lot of problems on how must I do it; must I add, divide or multiply, and the
        statement was a little bit confusing.

When solving a real life problem, P4, A26 applied an unsystematic guess-work approach. Simply dividing the cost of a piece of cake into the total bill and concluding, stating the number of cups taken by each person on the basis of this division, was, rather, a non-logical and absurd approach to problem-solving. His utilitarian belief system left him with no option when no known procedures or methods could be applied, except to give a non-sensible solution. The learner could not monitor his solution process if it made sense. For example, decimal fractions could not apply to this problem when looking for number of people or number of cups of tea taken by each person. After answering P4, A26 was asked to predict if he was confident that he solved it correctly. In response, he circled,

178

"Not sure, I didn't know how correctly I could do it". It seems, when faced with problems he feels is not capable to solve them, he resorts to guessing and giving absurd or non-sensible solutions.

A26 approached P5 in a similar way to P4. The approach was non-logical. When faced with a problem that could not be tackled by any known algorithms, his utilitarian belief system led him to simply add together part of the two data sets provided in the problem and gets a solution that makes no sense. The learner could not check back if the solution satisfies the conditions stipulated in the problem. A26 approached P6 by repeating writing the given information and, then, modeling the statements using inequalities. His utilitarian set of beliefs blocked him from linking the 3 inequalities. As a utilitarian believer, he could not get a starting point because the problem had no mathematical referents (e.g. numbers). With no way at hand to tackle the problem further, A26 resorted to guessing the order of people according to the number of fish they caught.

5.4.1.2. **Learner A31: Utilitarian believer**

When solving P1, A31 applied a systematic listing approach. She calculated the amounts accumulated by each person from day 1 up to the day they have same amount of money. Though this mechanical approach was effective for this particular problem because the days taken to have the same amount of money were few, it could be inconvenient when the number of days taken is extremely large. Her set of utilitarian beliefs could be seen being at play in solving this particular problem. Firstly, she considered this money problem as an everyday life situation that people face; that could be effectively solved by some mathematical means. Finally, she searched for the solution, systematically, day after day, until the solution was obtained.

When faced with a real life problem, P2, learner A31, initially, approached it using logical reasoning. She calculated the number of gallons of gasoline each type of a car uses per year correctly. However, she did not consider the 'whole picture' of the problem. As a result, she could not use all the necessary information

179

provided to solve the problem effectively. Faced with a problem that requires one to consider the whole picture of the problem and rational, logical reasoning to solve it, her utilitarian set of beliefs led her to jump to a conclusion without giving the problem the necessary time and thinking it deserves.

This problem solving behaviour seems to be at odds with some of her confessed beliefs, e.g. mathematics is a demanding subject that requires a lot of effort to learn it, and solving a mathematics problem requires a lot of thinking. On this problem, her utilitarian set of beliefs that were in action (e.g. there is a certain amount of effort and time needed to solve a particular kind of problems), in conjunction with her lack of confidence to solve the problem, might have led her to give herself insufficient time to solve the problem and check if it was fully solved. The following response to the question in the retrospective questionnaire confirms this analysis:

Interviewer: How confident were you that you could solve P2 correctly?
A31:   Not sure, I didn't know how correctly I could do it.

When solving P3, A31 could not relate the pieces of information given and create a model to solve it. Faced with this unusual problem, A31 could not get a straight forward way to resolve it, but, rather, resorted to unsystematic guess work. Because of her set of utilitarian beliefs that were in action (e.g. what is important is obtaining the correct answer, and the process of problem solving is linear), she could not improve her guess and check if the solution obtained makes sense. For example, she gave the number of calves to be born in decimal form. She believed that all the information required to solve the problem should be provided. This part of utilitarian belief system, coupled with her prediction that she could not solve the problem correctly, blocked her from getting a starting point; as she could not determine the information she thought was necessary to resolve the problem. The only way out was to apply an ineffective approach to problem solving (guess work). The following dialogue confirms this analysis:

Interviewer: State any difficulties or obstacles you met on solving the problem.

A31: I felt that they should've given us more information. So, I found it hard to complete the question the way I needed it to be, the way I felt it could be right.

In solving P4 and P5, her utilitarian belief system led her to repeat writing the given information, probably, in search of known mathematical procedures or algorithms that could be applied. When she could not identify any well known procedures to apply to resolve the problems, she resorted to unsystematic guess work. Since, e.g. she believes in obtaining exact answers, she could not improve her initial guesses and evaluate if the solutions make sense.

In solving P6, A31 applied an effective approach of modeling the situation and applying logical reasoning to resolve the problem. However, her utilitarian set of beliefs could be seen in action, as she introduced and played around with numbers in her solution process. Since the problem had no mathematical referents, she could solve the problem effectively by working and experimenting with simple numbers of her choice other than with algebraic terms. She did not present on the answer all the numbers she tried substituting into her models. She only presented the numbers that satisfied the inequalities formulated. Her utilitarian beliefs, e.g. mathematics problems have exact answers, and the problem solving process is linear, could, possibly, be behind this behaviour.

Learner A31 could predict before solving P6 that she was absolutely sure that she could do it correctly. Her confidence that she could solve the problem, coupled with her utilitarian belief system, led her to put as much effort and time as possible in resolving the problem. An outstanding problem solving behaviour of A31 is that when faced with a blockage in solving a problem, she applies an unsystematic guess and check approach, non-logical reasoning, and allocates little effort and time to it.

5.4.1.3. **Learner B08: Utilitarian believer**

In solving P1, P3 and P5, learner B08 applied unsystematic guess and check approach. On solving P1, B08 guessed the number of days when the two people will have the same amount of money and verified the correctness of her guess.

Her utilitarian set of beliefs are evident in her behaviour of presenting an exact answer. Even though the learner was prompted to show all rough calculations on the space provided, she, probably, did rough calculations on a separate piece of paper or mentally, and, then, presented on the answer sheet an exact, straight forward, solution.

B08's approaches to P3 and P5 were ineffective and similar to the approach done on P1. She concluded on the basis of a single guess done. The solutions presented were unreasonable and reflect lack of enough analysis of the requirements of the problems. For example, on solving P3, she used only part of the given necessary data to solve the problem and ignored the rest. She confessed, "I didn't know how to respond or which method to apply in order to obtain the correct answer". In the face of these difficulties, her utilitarian belief system might have led her to present abrupt and absurd guessed solutions that were reached as a result of not considering the 'whole picture' of the problems. Her utilitarian belief system, in conjunction with lack of motivation and confidence to solve challenging problems, could have limited the amount of effort and time she devoted to solving the problems.

In solving P2, B08 restated the given data and, then, applied a piece-wise approach to problem-solving. Though the approach was effective in reaching the desired solution, the separate, logical stages linked in the solution process produced non-logical mathematical statements. Before solving the problem, she predicted that she was not sure on how to solve it correctly. Faced with a real life problem she had no known procedures to apply to, her utilitarian beliefs were evident in considering a simple case (repeating writing the given information), tackling the problem by 'parts' and applying mathematical logical reasoning.

In solving P4, she created a mathematical model (equation with x and y variables). Because of failure to consider all the given information in modeling the situation, the equation formed was not correct. She could not use it to solve the problem. After meeting an obstacle in solving the problem and observing that the type of the problem could not be solved by ready available procedures, she

rushed into applying unsystematic guess and check approach. Unfortunately, she could not check or verify the correctness of her guess.

Similarly, in solving P6, she created correct mathematical models (inequalities), but did not show in writing how she used the inequalities to reach her conclusions. P6 was a problem that had no mathematical referents. Her utilitarian beliefs (e.g. mathematics problem involves numbers) could have left her with no starting point after formulating inequalities from the problem situation. In the face of this problem (failure to link or relate the available inequalities and deduce the possible emerging relationships), she guessed the order of the people according to the number of fish they caught.

5.4.1.4. **Learner F43: Utilitarian believer**

In the face of a real life problem that could not be solved by known algorithms, F43 solved P1 using unsystematic guess work. Only an exact and verified solution was presented. He did not show any trials and improvements made to obtain the solution presented. This kind of behaviour could have been guided by his utilitarian belief system. In solving P2, he applied a piece-wise and logical reasoning approach. His utilitarian belief system could be at play in presenting exact and logical steps in the solution process. The learner predicted that he was absolutely sure that he could solve it correctly. Having a feeling that he could succeed in solving the problem, he devoted adequate time and effort to resolving it.

In solving P3, his prediction on how confident he was that he could solve it correctly was "Really not sure, I thought, probably, that I could not succeed". Faced with a situation he had no confidence of solving it successfully, his set of utilitarian beliefs that were in action (e.g. mathematics problems are solved using known procedures, one can solve mathematics problems he/she is good at within a short period of time) could have guided his problem solving behavior. He only restated part of the given information, and failed to give the problem the adequate time and thinking it deserved for successful resolution.

F43 solved P4 by applying modeling; guess, check and revise; and logical reasoning strategies. His set of utilitarian beliefs could have influenced his approach to the problem as follows: He realized that different mathematical strategies could be applied in combination to resolve a real life problem. He made a guess of the number of people who took tea and used his initial guess to deduce the other unknown variables. The several different numbers guessed and checked in the model were not explicitly shown in the solution process. The cancellations of, and overwriting on, different numbers done by the learner in the solution process imply that several trials and their improvements were done in search of the correct solution.

An exact and correct solution was presented, though there is a question on how he determined the number of cups and pieces of cakes taken by each person. The solution process was silent on this aspect. The learner could have, probably, deduced the numbers mentally and did not commit himself to writing the processes he underwent in resolving the problem. He considered the correct answer as more important than the process that produced the solution.

In solving P5, his set of utilitarian beliefs at action was evident in the mechanical way he approached the problem. He drew diagrams to represent the hutches and 'acted out' the situation to determine the number of hutches and rabbits that were present. Probably, he put the rabbits in the drawn circular hutches in groups of sevens and nines, from one hutch to the next, until the conditions of the problem were met. Finally, an exact correct answer was presented. In face of P6 that had no mathematical referents, he modeled the situation using inequalities. Unfortunately, he could not use them to resolve the problem. He could not get a starting point after creating the three inequalities that involved no numbers.

The researcher observed him playing around with four different coloured pencils which, probably, represented the people in question. When asked to describe the kind of strategies he used to solve P6, he responded, "….I used different coloured pencils to stand for each person. Then, I tried to use numbers in place of pencils to arrive at the proper person who got the most fish". The solution

184

process presented was silent on the different numbers he tried to substitute for each person and how he used the inequalities to arrive at the solution. The correct answer given was supported by a non-logical reason. His utilitarian belief system could have influenced him to exhibit this kind of mathematical problem-solving behaviour.

5.4.2 **Relationship between exploratory belief system and approach to problem solving**

The researcher drew a scatter diagram by plotting the learners' problem solving approaches mean scores against their exploratory mathematics-related belief system mean scores (see Figure 5.39). The pattern of dots does not depict any special significant relationship between the two constructs. The pattern of dots is similar to that of figure 5.37. Pearson's correlation coefficient, r = 0.175 (see table 5.5) means that there is a very weak positive linear relationship between approach to problem solving and exploratory belief system. The correlation coefficient of determination, $R^2 = 0.031$ means that only 3.1% of the change in approach to problem solving is explained by the exploratory belief system and vice versa. Since there is no special significant statistical relationship between the two constructs, a qualitative analysis of the relationship between them was done to explain the nature of their relationship.
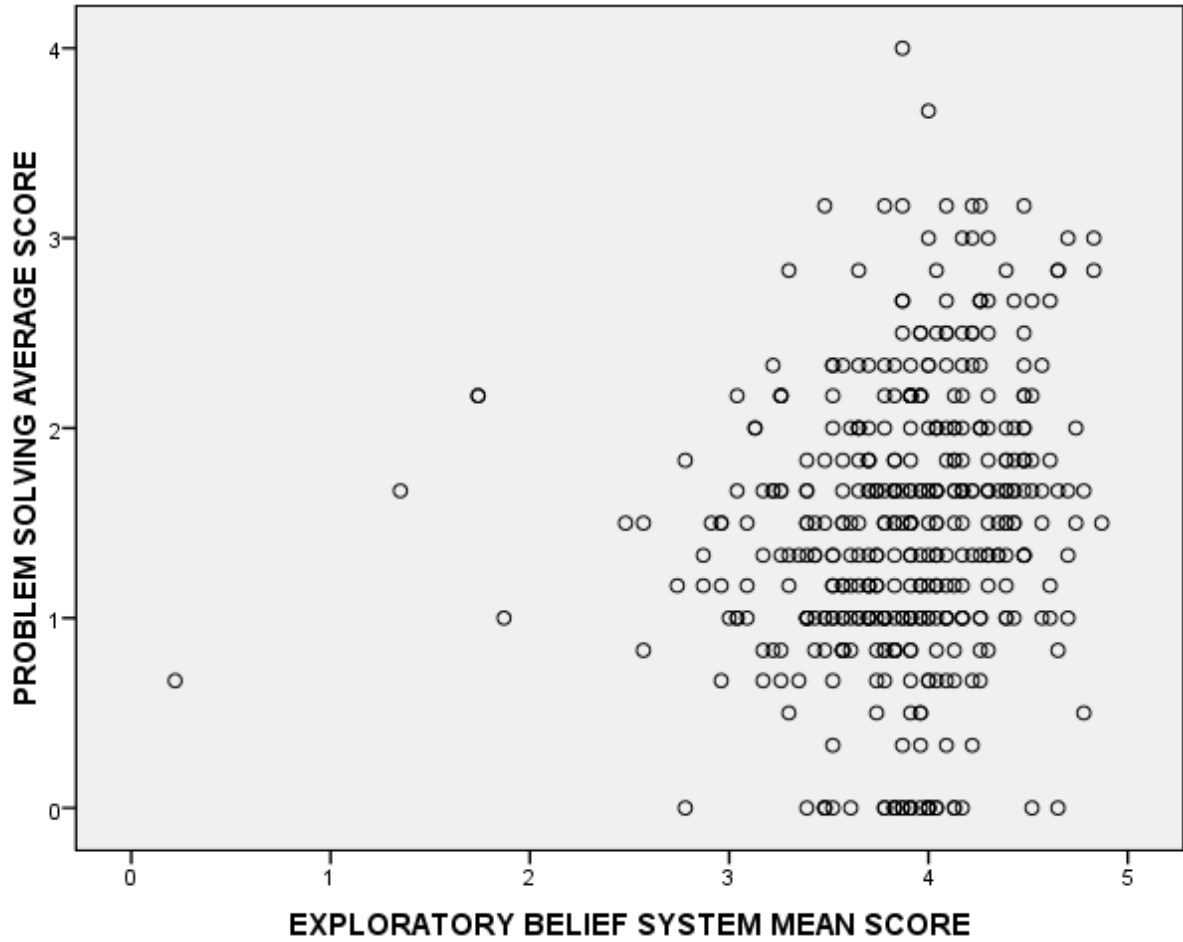
Figure 5.39: Scatter plot showing relationship between approach to problem solving and exploratory belief system.

Table 5.5: Pearson's correlation coefficient, Problem solving versus Exploratory belief system

		PROBLEM SOLVING AVERAGE SCORE	EXPLORATORY BELIEF SYSTEM MEAN SCORE
PROBLEM SOLVING AVERAGE SCORE	Pearson Correlation	1	$.175^{**}$
	Sig. (2-tailed)		.000
	N	425	425
EXPLORATORY BELIEF SYSTEM MEAN SCORE	Pearson Correlation	$.175^{**}$	1
	Sig. (2-tailed)	.000	
	N	425	425

**. Correlation is significant at the 0.01 level (2-tailed).

5.4.2.1. Learner B57: Exploratory believer

Learner B57 approached P1 by applying three methods: systematic listing, looking for a pattern and modeling. Her approach was effective. Her set of exploratory beliefs were at play on being confident in her ability to solve the problem, considering the 'whole picture' of the problem, tackling the problem using different methods, and modeling the relationship among numbers. In her prediction of how confident that she could solve it correctly, she circled, "absolutely sure that I could do it correctly". Having a feeling that she could solve the problem, she devoted effort and time to exploring and solving the problem.

In the face of real life problems, P2, P3 and P5, B57 predicted that she was not sure and did not know how correctly she could solve them. She approached P2 by attempting a simple case and, thereafter, applying logical reasoning. However, because of her feeling that she could not solve the problem correctly, she could not give it enough time, thinking and attention it deserves to be resolved effectively. Her description of the strategies she used to solve the problem reveals that she devoted much time to trying to fit the problem to the

usual ones and searching for well known procedures to apply. As a result, she presented an incomplete response.

In solving P3 and P5, she modeled the situations using equations. Her exploratory belief system was evident, e.g., in the ability or attempt to link data for male and female calves into one equation. When creating the equations for P3, she only used the first condition and ignored the second condition. To produce an effective model, these two conditions were supposed to be combined to create a single mathematical statement. Similarly, when creating equations for P5, she omitted crucial conditions stipulated in the problem (e.g. the rabbit left over when grouping them in sevens or the hutch left empty when grouping the rabbits in nines). She allocated less time to analyzing and understanding the problems, and presented, rather, 'rushed' trivial responses that lacked adequate thought.

B57 approached P4 by modeling the situation (using equations and inequalities) and applying guess work. She tried different numbers that could satisfy the inequality. Her solution had evidence of cancellation, erasing and rewriting in search of a solution. In face of a challenge, her exploratory set of beliefs were at play in attempting different methods of solving a single problem, linking different sets of data into a single mathematical model, and enduring the hardships faced in resolving the problems. Her approach was not effective due to failure to consider the 'whole picture' of the problem. She did not consider the total number of people who took tea in her model. This omission was a 'tumbling block' in resolving the problem.

The following dialogue reveals how her exploratory belief system guided her in decision making and controlling her problem solving process. When one approach fails to yield the desired solution, she was able to try other different approaches.

Interviewer: Think over what you have done and try to describe the kind of strategies you used to solve the task.

B57: Firstly, I attempted to solve the question with simultaneous equations, but realized they will not produce an accurate answer. Therefore, I, then, used inequalities and linear programming.

She solved P6 by modeling and trying a simple case. She created inequalities from the situation and introduced numbers to work with, since the problem had no mathematical referents. Her modeling and numerical approach to solving the problem was effective. Her exploratory belief system could be noticed being at play in her problem solving behavior: taking into account the 'whole picture' of the problem, trying different ways of solving the problem, and introducing and experimenting with different numbers in search of a solution.

In general, learner B57's exploratory belief system influenced her to consider the 'whole picture' of the problem situation and, thereafter, choose a strategy and applied the thinking pattern she believed was appropriate to the given situation.

5.4.3 **Relationship between systematic belief system and approach to problem solving**

The researcher drew a scatter diagram by plotting the learners' problem solving approaches mean scores against their systematic mathematics-related belief system mean scores (see Figure 5.40). The pattern of dots is similar to that of figures 5.38 and 5.39. Pearson's correlation coefficient, r = 0.270 (see table 5.6) means that there is a very weak positive linear relationship between approach to problem solving and systematic belief system. The correlation coefficient of determination, R^2 = 0.073 means that only 7.3% of approach to problem solving average score is explained by the systematic belief system and vice versa. Since there is no special significant statistical relationship between the two constructs, a qualitative analysis of the relationship between them was, also, done to explain the nature of their relationship.
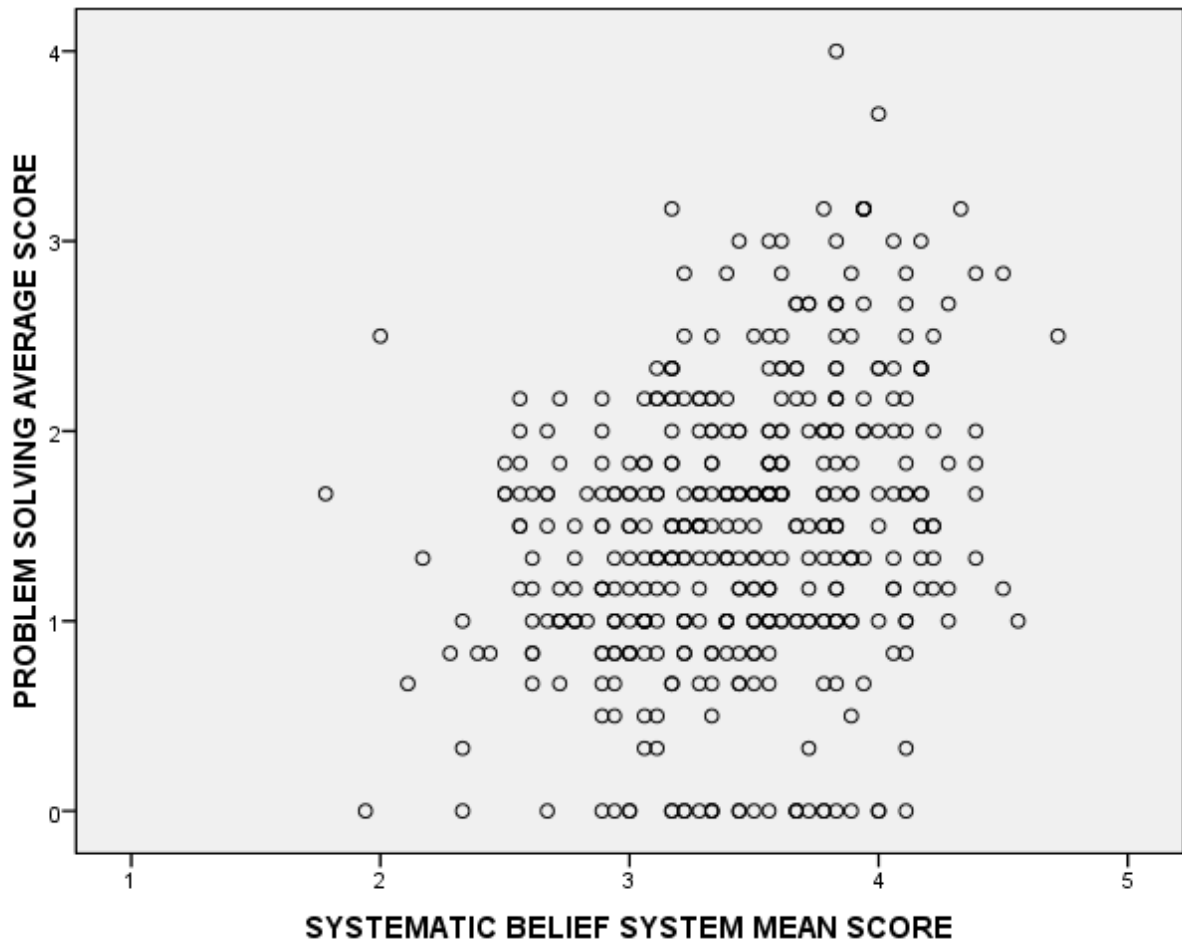
Figure 5.40: Scatter plot showing relationship between systematic belief system and approach to problem solving. N=425.

Table 5.6: Pearson's correlation coefficient, Problem solving versus Systematic belief system

		PROBLEM SOLVING AVERAGE SCORE	SYSTEMATIC BELIEF SYSTEM MEAN SCORE
PROBLEM SOLVING AVERAGE SCORE	Pearson Correlation	1	.270**
	Sig. (2-tailed)		.000
	N	425	425
SYSTEMATIC BELIEF SYSTEM MEAN SCORE	Pearson Correlation	.270**	1
	Sig. (2-tailed)	.000	
	N	425	425

**. Correlation is significant at the 0.01 level (2-tailed).

5.4.3.1 <u>Learner C27: Systematic believer</u>

C27 applied a formula and unsystematic guess work on solving P1. Faced with a real life problem that could not be solved by readily available procedures, his systematic belief system influenced his problem solving behaviour in choice of the method to use and in presentation of the solution. He attempted to use a formula, at first, and failed to do so because it (the formula) required other variables to be determined their numerical values (e.g. a and b) at first in order to solve for the unknown variable, n . Discovering that the formula could not be directly used to solve the problem, he resorted to 'unsystematic guess, check and revise approach. However, he did not show all the trials or guesses done, checked and revised. He presented an exact answer that he checked its correctness by calculating the total amount of money each person would have after the mentioned period.

In solving P2, C27's systematic belief system could have influenced his choice of method and presentation of the solution in the following possible ways: He restated the given data. After observing that it is a problem that could not be resolved by well known methods, he applied logical reasoning. He presented his

solution process in a series of steps. Unfortunately, some of the stages that he linked resulted in mathematically incorrect statements (e.g. equating quantities of different values).

P3 and P4 were unusual real life problems. C27 could not understand and interpret the situations correctly. His prediction on how confident he was that he could solve them correctly was "Really not sure, I thought, probably, that I could not succeed". In the face of a task that did not fit into any similar exercises done before, he applied guess work. He failed to consider all the given constraints in solving the problems. An exact solution presented as a solution to P3 was not reasonable in the light of the conditions stated in the situation. For example, giving an answer as 50 calves to be born when we want a guarantee that 100 calves born survive the first year is an absurd response. The learner could not check back if the problem was solved, but his focus was on obtaining and presenting an answer. Similarly, in solving P4, the learner could not see that the number of people could not assume decimal numbers. Hence, an improved guess was required.

C27's set of systematic beliefs were also evident in his methodical approach to solving P5. He drew circular diagrams to model the situation and derive the number of hutches and rabbits that were present. The approach applied was effective. He presented exact correct answers. In solving P6, he modeled the situation using inequalities. However, the solution process had no evidence of use of the inequalities created. Only a smart, possibly guessed response was presented, thereafter. In response to the retrospective questionnaire that asked him to describe the kind of strategies he used to solve P6, he confessed that he used the statements, and worked mentally, trying different combinations to come up with the order of people he presented on the answer sheet. This method of solving the problem (working mentally and presenting an exact answer) was, probably, influenced by his systematic belief system.

Analysis of the relationship between each belief system (utilitarian, systematic or exploratory) and approaches to non-routine problem solving revealed a weak

positive linear relationship between them. In general, the findings show that mathematics-related belief systems are positively related to approaches to non-routine problem solving.

5.5. <u>**Summary and conclusion of the chapter**</u>

It was discovered that learners hold three mathematics-related belief systems: utilitarian, exploratory and systematic. The belief system each learner held stronger than the other was defined as his/her predominant belief system. The utilitarian, exploratory and systematic predominant mathematics-related belief systems that possibly guided the learners' behaviour in non-routine problem solving are presented below.

Utilitarian believers' predominant mathematics beliefs are:

- Mathematics involves numbers and calculations.
- Mathematics is applied in daily life and in other courses.
- Mathematics is a set of rules and techniques.
- Mathematics is a static and rigid body of knowledge.
- Mathematics is about solving problems.
- Mathematics problems are solved by applying well known procedures.
- The problem solving process is linear.
- It is important to understand and apply procedures correctly.
- A mathematics problem has an exact correct answer.
- Mathematics develops one's thinking ability.
- A solution to a problem enables everybody to understand the problem situation well.
- Mathematics is learnt through practice.
- Learning mathematics requires a lot of effort.
- It takes a short time to solve the type of problems one is good at.
- Prefers easy or algorithmic tasks.
- Prefers tasks that can be solved within a short period of time.
- One should depend on the subject teacher or other learners to learn mathematics effectively.
- Group work facilitates learning of mathematics.

Exploratory believers' predominant mathematics-related beliefs are:

- Mathematics involves numbers and calculations.
- Mathematics is about solving problems.

- Mathematics develops one's thinking ability.
- Mathematics is evolving.
- There are many ways to solve a mathematics problem.
- A solution to a problem enables everybody to understand the problem situation well.
- Making mistakes is part of learning mathematics.
- Learning mathematics requires a lot of effort.
- Understanding is important in learning mathematics.
- Group work facilitates the learning of mathematics.
- Confident in one's ability in mathematics.
- Enjoys learning challenging material.
- Enjoys exploring new things in mathematics.

Systematic believers' predominant mathematics-related beliefs are:

- Mathematics involves numbers and calculations.
- Mathematics is a study of relationships among numbers.
- Mathematics is a static and rigid body of knowledge.
- A mathematics problem is solved by applying well known methods or formulae.
- Solving a problem requires logical thinking.
- It is important to understand concepts and procedures in mathematics.
- Enjoys solving algorithmic problems.
- What is important in mathematics is getting an accurate correct answer.
- One should rely on the subject teacher and other learners to learn mathematics effectively.

Learners solved non-routine mathematics problems using a number of strategies, for example, systematic listing, modeling, trial-and-error, use a formula, look for patterns, consider a simple case, and logical reasoning. Extracts of learners' written work were used as evidence of the observations done or findings discovered. The relationships between beliefs and approaches to problem solving were studied. A weak positive linear relationship was discovered between mathematics-related belief systems and approaches to non-routine mathematical problem solving.

CHAPTER 6

Discussion of the results

6.1. <u>Introduction</u>

Section 5.4 indicates that there are some beliefs that are common between or among the belief systems. For example, the beliefs, "Mathematics involves numbers and calculations", "Mathematics is about solving problems", and "It is important to understand concepts and procedures in mathematics" were common among all the belief systems. This is an indication that belief systems do not exist in isolation from each other, but are interwoven with each other (Jin et al, 2001). On this regard, Di Martino (2004) argues that it is possible for a single belief to be linked to different belief systems in different learners. As a result, the same belief can, possibly, influence learners to approach problem solving differently.

As noted by Di Martino (2004), we may not expect learners to exhibit similar problem solving behaviours simply because they hold some common beliefs between or among them. The beliefs which are held in different belief systems interact with other beliefs in the same belief systems, or other nearby belief systems they are linked to, and elicit different problem solving behaviours in different individuals (Beswick, 2011). As such, a learner's approach to problem solving is largely influenced by his/her predominant belief system as a whole, not by a single belief he/she holds. In this regard, Callejo and Vila (2009)'s research findings highlight the need to take into consideration a learner's belief system other than his/her single specific beliefs in an attempt to interpret his/her mathematics problem solving behaviour. In the light of this discussion, the approaches to problem solving exhibited by learners in this study, discussed below, were, largely, a result of their predominant belief systems.

6.2. Utilitarian, exploratory and systematic learners' approach to mathematical problem solving.

An analysis of the problem solving strategies applied by utilitarian, systematic and exploratory learners (see table 6.1) revealed that 'unsystematic guess, check and revise' (GCR(unsys)) strategy was commonly applied by the utilitarian and systematic believers. The exploratory believer approached most of the non-routine problems by modeling the situations. Non logical statements were commonly presented by utilitarian believers. Approaches or strategies that were common among the utilitarian, systematic and exploratory believers were 'logical reasoning' (LG), 'modeling' (MD), and 'non-logical reasoning' (NLG). Strategies that were common between utilitarian and exploratory believers were 'consider a simple case' (SC), and 'systematic listing' (SL). Approaches or strategies that were unique to exploratory believers were 'looking for a pattern' (LP), and 'try-and-error' (TE). A strategy that was unique to systematic believers was 'use a formula' (F).

Table 6.1: Problem solving strategies applied by utilitarian, systematic and exploratory believers.

STRATEGY	QUESTIONS SOLVED		
	UTILITARIAN	SYSTEMATIC	EXPLORATORY
GCR(unsys)	P1, P3, P4, P5	P1, P3, P4, P5	
SC	P2, P6		P2, P6
LG	P2, P4, P6	P2, P6	P2
NLG	P3, P4, P5	P4	P5
MD	P4, P6	P5	P1, P3, P4, P5, P6
SL	P1	P6	P1
LP			P1
TE			P4, P6
F		P1	

6.2.1. <u>Utilitarian believers' approach to mathematical problem solving.</u>

The learners' predominant utilitarian belief system might have led them to approach mathematical problem solving as follows: The learners approached problem solving by repeating writing the given information. Possible reasons behind this behaviour could be trying to fit the problem to the usual, routine problems solved previously, and looking for algorithmic procedures that could be applied to solve the problem. When faced with a blockage in problem solving, when no known methods or procedures could be applied, and when they felt that they were not capable to solve the problem, the learners, often, resorted to 'unsystematic guess, check and revise' approach to problem solving. The learners, often, gave non-sensible solutions. Because of this kind of approach to problem solving, the learner, often, failed to monitor and control their solution process, and, as a result, could not look back to check if the conditions stated in the problem were met or if the problem was really solved.

Due to failure to make connections between or among mathematical statements, the learners approached the problems by parts, that is, they applied a piece-wise approach to problem-solving. As such, in their effort to make the solution process as clear as possible to everybody, they presented piece-wise, logical and mathematically correct statements in their solution process. The learners, often, did the calculations somewhere on a rough piece of paper and presented only a straight forward solution on the answer space provided. This behaviour could be largely influenced by the beliefs that mathematics problems have exact correct answers, and the problem solving process is linear- you move directly towards the solution.

The learners often made a 'rushed', 'blind', 'absurd' or 'non logical reasoning' approach to problem solving. The learners exhibited this kind of behaviour, probably, due to failure to allocate enough time to problem solving, and failure to link the given set of data in the problem. The learners, often, started correctly, e.g., by applying logical reasoning or creating models on solving some problems, but later on diverted to guess work and non-logical reasoning approach to

197

problem solving. This might be due to limiting the amount of time and effort they have to engage in problem solving. They lacked confidence and trust in their own ability to solve the non-routine problems or in models of their own creation in producing the exact and correct solutions they were looking for. Similar findings were discovered by Kolovou et al. (2011), who discovered that learners, at first, gave a correct solution and later gave an incorrect solution. They described this problem solving process as the 'bouncing effect'.

The learners searched for the solutions empirically. For example, when solving problem 1 (P1), the utilitarian believers calculated the total amount of money each person would have day by day up to the day the two people, in question, will both have the same amount of money. They engaged less in logical or rational reasoning in verifying their solutions. The learners, at times, used tangible objects, and 'acted out' or 'dramatized' the problem situation when resolving some problems. When faced with a mathematical problem that had no clear mathematical referents (e.g. numbers, formulae), the learners applied a numerical approach to problem solving. This could be largely due to the belief that mathematics involves numbers and calculations. Hence, a starting point in problem solving could be obtained if numbers exist in the problem situation.

6.2.2. **Exploratory believer's approach to mathematical problem solving.**

The learner's predominant exploratory belief system might have influenced his/her approach to mathematical problem solving as follows: The learner, often, combined different approach strategies to resolve a single mathematical problem. This behaviour could be attributed, largely, to the beliefs that there are several ways to solve a problem; solving a mathematics problem is demanding and it requires much rational thinking; and the learner enjoys exploring new mathematical problems. The learner could flexibly switch approach strategies when resolving a problem. When one approach fails to produce the desired solution, the learner was able to try other different approaches. This behaviour

increased his/her chances of success in solving the problems. Similarly, Mabilangan et al. (2011) discovered that learners who were proficient in using solution strategies performed relatively well than those who were not. In solving problems the learner felt confident to solve correctly, he/she exhibited persistence and endurance in problem solving. In the face of challenges and hardships, the learner, often, did not give up, but searched for the solution through application of different approach strategies.

In solving problems the learner felt he/she could not solve them correctly, he/she did not allocate enough time to analyze, understand and solve the problems. Often, even though the learner was in the right direction, he/she could not engage him/her-self enough to problem solving and presented incomplete solutions. At times, the learner devoted much time to trying to fit the problem to the routine ones and searching for well known procedures to apply; a behaviour that was at odds with the ways he/she approached most of the problems. There seemed to be a time limit the learner engaged in resolving a single mathematical problem.

In solving problems the learner felt he/she could solve them correctly, he/she, often, made a 'rushed' response to resolving the problems. As a result, the learner made an oversight of some conditions stated in the problem situation and created incorrect mathematical models. This behaviour might be, largely, due to the belief that one takes less time to resolve problems he/she is good at. Similarly, Schoenfeld (1992) and Muis (2004) discovered that learners who strongly believe in quick learning or who believe that those who understand the content can solve mathematical problems posed within a shortest possible time tend to set the maximum time they engage in solving a particular problem. They withdraw from solving the problem when the time set elapses even though they have a potential to resolve it. Similar findings were, also, discovered by Callejo and Vila (2009) who discovered that learners' level of involvement in solving a problem, monitoring and regulating the solution process was determined by their

beliefs about difficultness of the task and beliefs about one's confidence in solving a particular problem.

The learner considered the 'whole picture' of the problem situation and, e.g., applied modeling and a numerical approach to resolve a problem that had no clear mathematical referents. The learner could link different sets of data within the problem to create a mathematical model. He/she introduced and experimented with simple numbers of his/her choice in search of a solution. Though this approach strategy has limitations on solving some kinds of non-routine problems, the learner effectively applied it on solving a problem that had no numbers in its formulation. Similarly, Callejo and Vila (2009) discovered that learners approached problems without numbers in their formulations by making use of numbers as a starting point for solving a mathematical problem.

6.2.3. **Systematic believer's approach to mathematical problem solving.**

The learner's predominant systematic belief system might have influenced his/her approach to mathematical problem-solving as follows: The learner used a formula to solve problems that could be solved effectively by making a systematic list, looking for a pattern or modeling the situation. In face of a problem, the learner, firstly, tried to fit the problem to the usual ones, and then match a well known formula to the problem situation.

The learner, often, restated the given data, possibly, as a means to derive meaning of the given situation or in search of well known methods or procedures to apply. In the face of real life problems that could not be solved by readily available procedures, the learner, at times, applied logical reasoning, and presented the solution process in a series of logical steps. The learner applied 'unsystematic guess, check and revise' approach to problem solving when resolving tasks that did not fit into any similar usual exercises done before and faced with a blockage in problem solving. Under this situation, the learner, often, presented non-sensible solutions due to failure to monitor and control his/her solution process. Similarly, Daskalogianni and Simpson (2001a) discovered that

systematic believers are more confident in working on problems that require application of strategies they had learned and practiced before and they rarely take risks or enjoy exploring a problem.

The learner's approach to some problems was methodical. The learner could draw diagrams to model the situation or create inequalities and use his/her models to derive the solution to the problem. Whenever he/she failed to use the models created, he/she, often, resorted to guess work. The learner, also, did much of his/her calculations mentally or somewhere on a rough piece of paper, and presented on the answer space provided only a smart and exact answer.

6.3. **Approach to non-routine problem solving versus belief systems**

A statistical analysis of the relationship between learners' mathematics-related belief systems and their approach to mathematical non-routine problem solving revealed that there exist a weak positive linear relationship between them. The proportion of change in approach to problem solving that could be explained by belief systems is very low (5.86%). The possible causes of a weak relationship and low correlation coefficient of determination (R^2) might be that the majority of learners (85.6%) performed poorly and scored low points in mathematics non-routine problem solving test (see table 6.2). The mean problem solving score of all the learners (N=425) was 1.62 out of the possible mean score of 5 (see table 6.3). On average, the learners performed poorly in solving non-routine mathematical problems.

Similarly, Maree et al. (2009) discovered that South African learners performed poorly in mathematics problem solving. The TIMSS (2003) discovered that, approximately, 30% of the South African grade 8 learners were able to solve non-routine mathematical problems. Similar results were discovered in other countries as well. For example, Mogari and Lupahla (2013) and Kolovou et al. (2011) discovered that Namibian and Dutch learners, respectively, performed poorly in mathematics non-routine problem solving even though they were classified as high performers in school mathematics.

Table 6.2: Problem solving mean score frequencies

Mean Score (Nearest whole no.)		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	0	36	8.5	8.5	8.5
	1	157	36.9	36.9	45.4
	2	171	40.2	40.2	85.6
	3	53	12.5	12.5	98.1
	4	8	1.9	1.9	100.0
	Total	425	100.0	100.0	

Table 6.3: Problem solving mean score of all learners (N=425).

Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
Problem Solving mean score	425	0	4	1.62	.877
Valid N (listwise)	425				

In this study, most of the learners (99.1%) scored points between three and five, inclusively, on beliefs questionnaire (see table 6.4). This percentage (99.1%) could be extremely high and questionable. On average, the learners' mean belief score was 3.73 (see table 6.5). A mean score of 3.73 in beliefs indicated, in this study, that the learners held the beliefs (see section 5.3) in question. However, when answering the beliefs questionnaire, it is possible for a learner to choose the socially acceptable beliefs and fail to reveal his/her actual beliefs in mathematics problem solving. A similar finding was discovered by HSRC (2006) (see section 1.2) who discovered that South African learners stated some socially acceptable attitudes to mathematics other than their real attitudes.

Table 6.4: Beliefs mean score frequencies

Mean Score (Nearest whole no.)	Frequency	Percent	Valid Percent	Cumulative Percent
Valid 1	1	.2	.2	.2
2	3	.7	.7	.9
3	112	26.4	26.4	27.3
4	303	71.3	71.3	98.6
5	6	1.4	1.4	100.0
Total	425	100.0	100.0	

Table 6.5: Belief system mean score of all the learners (N=425).

Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
Beliefs mean score	425	1	5	3.73	.504
Valid N (listwise)	425				

However, the results indicate that there is a positive relationship between mathematics-related belief systems and approach to problem solving. This might mean that if a learner develops more positive healthy beliefs in mathematics learning and problem solving, his/her approach to non-routine problem solving improves positively as well (and vice versa). Similar finding was discovered by Spangler (1992), Schoenfeld (1992) and Mason (2003) (see section 2.4.3) who discovered that there is a positive relationship between beliefs and mathematics learning and problem solving. Schoenfeld (1992) concluded that beliefs determine the approach a learner applies to a given problem. Spangler discovered that mathematics-related beliefs and learning influence each other. Hence, Spangler (1992) described the relationship between them as cyclic. Mason (2003) concluded that learners' beliefs can be used to predict their achievement in mathematics. A contradictory finding was discovered by Goldin et al. (2009) and Callejo and Vila (2009). They discovered a complex relationship

between beliefs and approach to problem solving. As a result, they could not identify if beliefs influence problem solving behaviour and vice versa.

A qualitative analysis of the relationship between learners' mathematics-related belief systems and their approach to non-routine mathematical problem solving took into consideration learners' beliefs that were derived and inferred from responses on interviews, open-ended questionnaire and non-routine mathematics problem solving test to counteract the weaknesses of the closed-form beliefs questionnaire whereby learners might choose only the socially acceptable beliefs and hide their actual beliefs. This decision was done in line with the 'foundations belief theory' (see section 2.4.2) which claims that a learner justifies why he/she holds a certain belief by use of other beliefs (Gardenfors, 1989). Analysis of learners' problem solving behaviour indicated that some problem solving behaviours were common among learners holding different predominant belief systems. For example, both utilitarian and exploratory learners applied a numerical approach to problem 6.

Systematic and utilitarian believers, often, resorted to unsystematic guess work in the face of a blockage or an unusual task, and engage in little meta-cognitive monitoring and control. There were some behavioural traits that were unique to a specific belief system. For example, exploratory learners applied different approaches or strategies to resolve a single mathematical problem. They were flexible in switching from one approach/strategy to the other as they explore the problem situation. They applied a holistic approach to problem solving. They considered the 'whole picture' of the problem situation, and the approach or the pattern of thinking to be applied was determined by their belief system. Their levels of engagement in problem solving were determined by their rating of the level of difficultness of the problem and their motivation to resolve the problem (possibly derived from the challenge imposed by the problem, the opportunity to learn something and their feeling that they are capable to solve the problem).

Utilitarian believers, often, apply a 'piece-wise' approach to problem solving and present partial logical statements. They, initially, classify the type of a problem

they are dealing with, and, then, approach the problem in the way they believe is appropriate to such type of problems. Systematic believers, often, use a formula or a methodical approach to solve problems, and present on the answer space only what they consider as the correct answer. These results accords with Muis's (2004) findings, who discovered that empiricists (classified as utilitarian or systematic in this study) engaged in little meta-cognitive monitoring and control because of their heavy dependence on step-by-step procedures, accurate drawings and unsystematic trial-and-error exploration of the problem situation to identify the correct answer.

6.4. **Summary and conclusion of the chapter**

The problem solving strategies employed by utilitarian, exploratory and systematic learners on solving non-routine problems were discussed (see section 6.2). The relationship between approach to non-routine problem solving and mathematics-related belief systems was discussed (see section 6.3). The meaning of the positive relationship between belief systems and approach to problem solving was discussed in section 6.3. Similar findings, as well as contradictory findings, of some scholars were compared and contrasted with findings of this study. The discussions done on findings revealed that learners' mathematics-related belief systems could, possibly, explain their approaches to non-routine mathematical problem solving and vice versa.

CHAPTER 7

Summary, conclusions and recommendations

7.1. <u>Introduction</u>

Firstly, the researcher summarised the research study. Secondly, he stated the findings that relate to each research question. Thirdly, he proposed some general recommendations that related to the research findings. Fourthly, the researcher suggested some directions for future research studies. Fifthly, the limitations of the study were spelled out. Lastly, the researcher gave his reflections on the research study done.

7.2. <u>Summary of the study</u>

The main research problem of this study was 'the relationship between learners' mathematics related belief systems and their approaches to mathematical problem solving: A case study of three high schools in Tshwane North District (D3), South Africa'. This topic was chosen in an attempt to provide answers to factors that relate to learners' mathematics problem solving at high schools. Though several factors that affect South African learners' mathematics learning and problem solving were identified (see TIMSS, 2003; Maree et al., 2006; Wessels, 2012), Goldin (2002) points to beliefs as a possible hidden variable that affects learners' mathematics problem solving. This researcher was motivated to contribute some theory on this problem because some scholars, e.g., De Corte and Op't Eynde, 1999, point to inadequacy of theory on how learners' beliefs relate to other mathematical constructs, e.g., problem solving. Though several scholars in mathematics education (e.g., Jin et al., 2010; Callejo & Vila, 2009; Daskalogianni & Simpson, 2001a, 2001b; Spangler, 1992; Schoenfeld, 1985) studied beliefs in relation to other constructs, the relationship between beliefs and approach to problem solving seem to be understudied and remain unclear.

This study was carried out in an attempt to determine if there is a relationship between grades 10, 11, and 12 learners' mathematics-related belief systems and

their approaches to non-routine mathematical problem solving. The study sought to answer the following three research questions: "What are the grades 10, 11, and 12 learners' approaches to non-routine mathematical problem solving?", "What are the grades 10, 11, and 12 learners' mathematics-related belief systems?", and "Is there any relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematical problem solving?"

The study employed a mixed methods approach, whereby both quantitative and qualitative data were collected and used to answer the research questions. As such, the researcher adopted a positivist-interpretive perspective in conducting the study. Six learners were purposefully selected for interviews from the previous group of 425 learners who participated in the first phase of the study (completing beliefs questionnaires and solving non-routine mathematics problems). The selected learners also answered open-ended and retrospective questionnaires. The data collected were presented in the form of tables, direct quotes from the questionnaires, extracts of learners' written solutions to problems, excerpts of interviews, and descriptive data on learners' beliefs about mathematics and approaches to problem solving. The approaches used to analyse the data were factor analysis, cluster analysis, regression analysis, coding, organizing data into descriptive themes, noting relations between variables and methodological triangulation.

The learners solved the non-routine problems using the following strategies: Systematic Listing (SL); Modeling (MD); Trial-and-error (TE); Use a Formula (F); Systematic Guess, Check and Revise (GCR(sys)); Unsystematic Guess, Check and Revise (GCR(unsys)); Consider a simple case (SC); Logical reasoning (LG); No logical reasoning (NLG); and Look for patterns (LP). (See section 5.2). Learners' mathematics-related beliefs were categorized into belief systems according to Daskalogianni and Simpson (2001a)'s three key macro-belief systems (systematic, exploratory and utilitarian). It was discovered that learners held all the three belief systems. As a result, they were classified into these three macro-belief systems by considering their predominant mathematics beliefs (see section 5.3). The

relationship between a learner's predominant belief system and his/her approach to problem solving was studied.

A weak positive linear relationship between approach to non-routine problem solving and mathematics-related belief systems was discovered. It was, also, discovered that learners' predominant mathematics-related belief systems could explain their mathematical non-routine problem solving approaches (and vice versa).

7.3. **Conclusion**

Though learners faced difficulties in solving non-routine mathematical problems, they employed several strategies to resolve the problems. Some of the strategies applied in solving problems were Systematic Listing (SL); Modeling (MD); Trial-and-error (TE); Use a Formula (F); Systematic Guess, Check and Revise (GCR(sys)); Unsystematic Guess, Check and Revise (GCR(unsys)); Consider a simple case (SC); Logical reasoning (LG); No logical reasoning (NLG); and Look for patterns (LP).

Grades 10, 11 and 12 learners hold mathematics related beliefs that could be classified into belief systems, namely, systematic, exploratory and utilitarian. Among the three belief systems, it was discovered that every learner held one of the belief systems stronger than the others. The predominant belief system influenced the behaviour of the learners in solving non-routine mathematical problems.

It was discovered that there is a weak positive linear relationship between high school learners' mathematics-related belief systems and their approaches to non-routine mathematical problem solving. The existence of a positive relationship might mean that a positive change of a learner's mathematics-related beliefs (i.e., development of more healthy beliefs) is likely to result in a positive improvement in use and application of problem solving strategies. A weak positive relationship might, also, mean that a relatively large change in development of positive, healthy mathematics-related belief systems among learners would result in some noticeable improvement in learner performance in mathematics (as measured by effective use

and application of problem solving strategies). Though the correlation coefficient of determination (R^2) was relatively low, in general, the learners' predominant mathematics-related belief systems could explain their approach to mathematics non-routine problem solving (and vice versa). This means that one can possibly predict a learner's likely approach to a certain non-routine problem by taking stock of his/her predominant mathematics-related beliefs (and vice versa).

7.4. <u>General recommendations</u>

In the light of the findings of this study, the researcher recommends the following teaching and learning practices: Mathematics should be taught through problem solving in order to gain knowledge of how to solve non-routine problems and exercise satisfactory meta-cognitive monitoring and control. Teachers should assess and be aware of learners' active belief systems that adversely affect their mathematical problem solving. Spangler (1992) posits that learners manifest mathematics-related beliefs in the classroom as they engage in learning activities such as answering questions, asking questions, approaching new problems and working on problems. Learners should be made aware of their belief systems and the possible effects of their naïve beliefs to mathematical problem solving. Teachers should incorporate learners' belief systems in their teaching and learning process in an attempt to encourage the development of positive, health and enlightened mathematics-related belief systems. Teachers should expose learners to a teaching and learning environment that challenges their existing naïve belief systems and, possibly, lead them to re-examine their beliefs and, ultimately, modify them. In other words, learners should be exposed to mathematical experiences that enrich their mathematics-related belief systems.

7.5. <u>Recommendation for future studies</u>

This study investigated two components of mathematics-related belief systems; 'beliefs about mathematics education' and 'beliefs about the self as a mathematician'. It did not delve much into learners' beliefs about their mathematics classroom context- a third component of students' mathematics-related belief

systems identified by Op't Eynde et al. (2006). As such, the researcher suggests a comprehensive study of the relationship between learners' mathematics-related belief systems and their approaches to non-routine mathematical problem solving to be done that incorporates all the three components of a mathematics-related belief system: beliefs about mathematics education, beliefs about the self as a mathematician, and beliefs about the mathematics classroom context. The researcher suggests a similar study to be carried out by use of a random sample of participants so that the results can be generalized to the South African population, as a whole. He, also, suggests a small scale international comparative research study to be carried out on the relationship between learners' mathematics-related belief systems and their approach to mathematical problem-solving.

7.6. **Limitations of the study**

The study was done at three High Schools in Tshwane North District (D3). As such, the results of the study might only apply to the schools under study. Data collecting instruments were in English only, considered as the official language of instruction at the three schools under study. Based on the findings of this single explanatory case study, the researcher could not draw conclusive and generalisable conclusions. Further research is required on this problem, probably, using more learners in data collection in order to obtain a more robust evidence for the findings.

7.7. **Reflections on my intellectual journey**

7.7.1 **Reflections on research questions**

Research question 1: What are the grades 10, 11, and 12 learners' approaches to mathematical non-routine problem-solving?

Even though learners encountered problems with the resolution of non-routine problems, they employed various solution strategies to solve the problems (see sections 5.2 and 6.2). The researcher, therefore, concludes that the question was answered.

Research question 2: What are the grades 10, 11, and 12 learners' mathematics-related belief systems?

The researcher discovered that grades 10, 11, and 12 learners hold systematic, exploratory and utilitarian belief systems (see section 5.3). Among the three belief systems, it was discovered that every learner holds one of the belief systems stronger than the others (see section 5.3). The researcher can conclude that this research question was answered.

Research question 3: Is there any relationship between learners' mathematics-related belief systems and their approach to mathematical non-routine problem solving?

Learners' approaches or strategies to non-routine problem solving were discussed in the light of their predominant belief systems. It was discovered that the way learners approach non-routine problem solving was related to their belief systems (see sections 5.4 and 6.2). Specifically, a weak positive linear relationship was discovered. One could, possibly, foretell a learner's likely approach to a non-routine problem by using his/her predominant belief system. The researcher can conclude that this research question was answered.

7.7.2. <u>Reflections on literature review and theoretical framework</u>

The literature review clarified key concepts to the study of the relationship between belief systems and approaches to problem solving (see chapter 2). Findings of some scholars on the problem under study were noted. Areas understudied, that remain unclear and with inadequate theory were discussed (see section 2.4.3). Theories that guided, supported and informed the researcher in conducting the research were discussed (see section 2.4). The discussion of literature on non-routine problem solving, mathematics-related belief systems and relationship between them gave the researcher a better understanding of the problem under study and equipped him with different approaches to conducting this study.

7.7.3. **Reflections on the research design, and methods of data collection, presentation and analysis**

The mixed methods design adopted in this study seemed appropriate on unraveling the relationship between belief systems and approach to non-routine problem solving (see section 3.2). Though the conveniently selected group of 425 learners who participated in the first phase of the study was large enough to make results generalisable, the application of non-probability methods to selection of the six learners from the group of 425 learners for further intensive study limited the generalisability of the findings. Different data collection instruments used in the study provided different types of data that enhanced triangulation and validity of the findings (see section 3.4). Presentation of data and findings in form of tables, extracts of learners' written work, excerpts of interviews and direct quotes from questionnaires provided tangible evidence that supported and substantiated the discussions of findings (see chapter 5). The use of several data analysis methods (see chapter 4) proved effective in determining the relationship between belief systems and approach to problem solving.

7.7.4. **Reflections on findings and conclusions**

Learners classified into different belief systems shared some common beliefs and approach strategies to non-routine problem solving (see sections 5.2 and 5.3). This made the task of revealing the relationship between belief systems and approach to problem solving difficult. Learners showed a limited knowledge of problem solving heuristics. This might have had an effect on learners' approach to solving non-routine mathematical problems. This study reveals that learners' belief systems are positively related to their approach to non-routine problem solving. This conclusion accords with the findings of Spangler (1992), Mason (2003) and Daskalogianni and Simpson (2001a), and contradicts the finding of Callejo and Vila (2009). Practical implications were drawn from the findings (see section 7.4).

References

Anderson, G. (1990). *Fundamentals of educational research*. London: The Salmar Press.

Andre, T. (1986). Problem solving and education. In Phye, G. D., & Andre, T. *Cognitive classroom learning: Understanding, thinking and problem solving*. Florida: Academic Press, Inc.

Benbow, R. (2004). Mathematical and theological beliefs: *A cognitive science perspective*. Retrieved April 01 2011 from http://www.acmsonline.org/journal/2004/Benbow95.pdf

Best, J. W., & Kahn, J. V. (1993). *Research in education* (7th Ed.). Boston: Allyn & Bacon.

Beswick, K. (2006). Implications of complexity science for the study of belief systems. In Hewitt, D. (2006). *Proceedings of the British Society for Research into Learning Mathematics 26 (3) November 2006*.

Beswick, K. (2011). Knowledge/ beliefs and their relationship to emotion. In K. Kislenko (Ed.), *Current state of research on mathematical beliefs XVI: Proceedings of the MAVI-16 conference June 26-29, 2010* (pp.43-59). Institute of Mathematics and Natural Sciences, Tallin University, Estonia.

Bishop, S. (1996). *Beliefs shape mathematics*. Spectrum London association of Christian teachers, 28, 131-142

Blum, W., & Niss, M. (1989). Mathematical problem solving, modeling, applications and links to other subjects – state, trends and issues in mathematics instruction. In Blum, W., Niss, M., & Huntley, I. *Modeling, applications and applied problem solving: Teaching mathematics in a real context*. Chichesteer: Ellis Horwood Limited.

Bogdan, R. C., & Bklen, S. K. (1982). *Qualitative research for education: An introduction to theory and methods.* London: Allyn and Bacon, Inc.

Bopape, M. (n.d.). The South African new mathematics curriculum: *People's mathematics for people's power?* Retrieved July 05 2013 from http://www.nottingham.ac.uk/csme/meas/papers

Branca, N. A. (1980). Problem solving as a Goal, Process and Basic Skill. In Krulik, S, & Rey, R. E. *Problem solving in school mathematics, National Council of Teachers of Mathematics, 1980 Yearbook.* Virginia: NCTM.

Brauer, M. (n.d.) Data analysis: *Factor analysis.* Retreived February 12 2014 from lapsco.univ-bpclermont.fr/persos/brauer/pdf/HO Factor analysis.pdf

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology*. Qualitative research in psychology.* 3 (2), pp. 77-101.ISSN: 1478-0887

Brijlall, D. (2008). Collaborative learning in a multilingual class. *Pythagoras*, 68, 52-61.

Brunning, R. H., Schraw, G. J., & Ronning, R. R. (1999). *Cognitive Psychology and instruction* (3rd Ed.). New Jersey: Prentice-Hall Inc.

Bunday, K. (2013). Problems versus exercises. Retrieved June 10 2013 from http://www.epsiloncamp.org/faq/faq_1.php

Burton, L. (1984). *Thinking things through: Problem solving in mathematics.* Oxford: Basil.

Cai, J. (2010). What research tells us about teaching mathematics through problem solving. In F. Lester (Ed.). *Research and issues in teaching mathematics through problem solving.* Reston, VA: National council of teachers of mathematics.

Callejo, M. L., & Vila, A. (2009). Approach to mathematical problem solving and students' belief systems: two case studies. *Educational studies in mathematics, vol 72,* pp.111-126.

Carson, J. (2007). A problem with problem solving: *Teaching thinking without teaching knowledge. The mathematics educator 2007, vol 17, no 2, 7-14.*

Chamberlin, S. A., & Moon, S. M. (2008). *How does the problem based learning approach compare to the Model- Eliciting Activity approach in mathematics?* Retrieved March 30 2011 from http://www.cimt.plymouth.ac.uk/journal/chamberlin.pdf

Chan, P. (2005). What learners learn through problem solving; *CDTLink, Nov 2005 Vol. 9 No. 3.*

Cherry, K. (2011). What is problem-solving? Retrieved June 29 2011 from http://psychology.about.com/od/problemsolving/f/problem-solving-steps.htm

Cohen, L., Maion, L., & Morrison, K. (1996). *Aguide to teaching practice* (4th edn). New York: Routledge.

Cresswell, J. W., & Plano Clark, V.L. (2007). *Designing and conducting mixed methods research.* Thousand Oaks, CA: Sage

Cresswell, J. W., Klassen, A. C., Plano Clark, V. L., & Smith, K. C. (2011). *Best practices for mixed methods research in health sciences.* Washington DC: National Institute of Health.

Darling, J. (1994). *A child-centred education and its critics.* London: Paul Chapman Publishing Ltd.

Daskalogianni, K., & Simpson, A. (2001a) A categorisation of upper soxth-form students` beliefs about mathematics. In Rowland, T. (Ed.). *Proceedings of the British Society for Research into Learning mathematics 21(1) March 2001*.

Daskalogianni, K., & Simpson, A. (2001b). Beliefs Overhang: The transition from school to University. In Winter, J. (Ed.). *(Proceedings of BCME5) Proceedings of the British Society for Research into Learning Mathematics 21(2) July 2001*.

De Corte, E., & Malaty, G. (2010). Working group 1: *Powerful learning environments for mathematics problem solving.* Retrieved March 25 2011 from http://math.unipa.it/˜grim/21silicy_group1_02.pdf

De Corte, E., & Op`t Eynde, P. (1999). Unraveling students' belief systems relating to mathematics learning and problem solving. Retrieved March 25 2011 from http://math.unipa.ir/ ˜grim/siDeCorte.PDF

Depaepe, F. (2009). The culture and practices in sixth-grade mathematics classrooms: *An attempt to unravel relationships between social and individual aspects in problem-solving lessons.* Retrieved March 29 2011 from http://lirias-kuleuven.be/bitstream/.../233766/.../doctoraat+Fien+Depaepe.pdf

Desoete, A., & Veenman, M. (2006). *Metacognition in mathematics education.* New York: Nova Science Publishers, Inc.

Di Martino, P. (2004). From single beliefs to belief systems: A new observational tool . *Proccedings of the 28th conference of the International Group for the Psychology of Mathematics Education, 2004, Vol2* pp271-278.

Diego-Mantecon, J., Andrews, P., & Op`t Eynde, P. (2006). Refining the mathematics-related beliefs questionnaire (MRBQ). Retrieved April 01 2011 from http://ermeweb.free.fr/CERME5/WG2/2_Diego_Mantecon.pdf

Dorlan , D. T., & Williamson, J. (1983). *Teaching problem-solving strategies.* London: Addison Wesley Publishing company.

Dorman, J. P., Adams, J. E., & Ferguson, J. M. (2003). A cross-national investigation of students' perception of mathematics classroom environment and academic efficacy in secondary schools. *International journal of mathematics teaching and learning, 15 April.*

Elia, I., Van den Heuvel-Panhuizen, M., & Kolovou, A. (2009). Exploring strategy use and strategy flexibility in non-routine problem solving by primary school high

achievers in mathematics. *ZDM Mathematics Education (2009) 41*: 605-618. DOI 10.1007/s11858-009-0184-6.

Engel, A. (1998). *Problem-solving strategies.* New York: Springer-verlag.

Ernest, P. (1988). The impact of beliefs on the teaching of mathematics. *Paper presented at the 6th international congress of mathematics education*, Budapest, August 1988.

Ernest, P. (1991). *The philosophy of mathematics education.* London: The Falmer Press.

Ernest, P. (1996). The nature of mathematics and teaching. *Philosophy of mathematics education newsletter 9.*

Ernie, K., LeDocq, R., Serros, S., & Tong, S. (2009). Mathematical reasoning: challenging students' beliefs about mathematics. In Gurung, R. A. R., Chick, N. L., & Haynie, A. *Exploring signature pedagogies: Approaches to teaching disciplinary habits of mind.* Virginia: Stylus Publishing, LLC.

Ewen, I. (1996). Strategies for problem solving. In Posameties, A. S., & Schulz, W. *The art of problem solving: A resource for the mathematics teacher.* California: Corwin Press, Inc.

Field, A. (2001). Postgraduate statistics: *Cluster analysis.* Retrieved July 08 2011 from http://www.statisticshell.comcluster.pdf

Field, A. (2005). *Discovering statistics using SPSS* (2nd edition). London: Sage.

Focant, J., Gregoire, J., & Desoete, A. (2006). Goal setting, planning and control strategies and arithmetical problem solving at Grade 5. In Desoete, A., & Veenman, M. *Metacognition in mathematics education.* New York: Nova Science Publishers INC.

Ford, M. I. (2010). *Teachers' beliefs about mathematical problem solving in an elementary school.* Doi :10.1111/j.1949-85

Foster, T. (2010). Open-ended problem analysis (coding Rubric). Retrieved April 01 2011 from http://groups.physics.umn.edu/physed/Research/CRP/coding.html

Furinghetti, F., & Pehkonen, E. (1999). *A virtual panel evaluating characterizations of beliefs.* doi : 10.1.1.119.4110

Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G.C. Leder, E. Pehkonen, & G. Torner (Eds). *Beliefs: A hidden variable in Mathematics education?* 39-57. Dordrecht: Kluwer Academic Publishers.

Gale Encyclopedia of Education. (2010). Mathematics learning: Complex problem solving. Retrieved March 22 2011 from http://www.answers.com/topic/mathematics-learning-complex-problem-solving#ixzz1HKWzoIrm

Ganly, S. (2007). The information processing theory and its effect on children and learning. Retrieved August 22 2013 from http://voices.yahoo.com/the-information-processing-theory-its-effect-on-633559.html

Gardenfors, P. (1989). The dynamics of belief systems: *Foundations versus coherence theories.* doi :10.1.1.23.5375

Goldin, G. A. (2002). Affect, meta-affect, and mathematical belief structures. In G.C. Leder, E. Pehkonen, & G. Torner (Eds.). *Beliefs: A hidden variables in mathematics education?* 59-72. Dordrecht: Kluwer Academic Publishers.

Goldin, G., Rosken, B., & Torner, G. (2009). Beliefs- No longer a hidden variable in mathematical teaching and learning processes. In MaaB, J., & Schoglmann, W. (Eds.). *Beliefs and attitudes in mathematics education: New research results*, pp.1-18. Rotterdam: Sense Publishers.

Goliath, G. (2008). Self-regulation strategies to improve mathematical problem solving for students with learning disabilities. Retrieved April 01 2011 from http://goliath.ecnext.com/coms2/gi_0199-7693942/self-regulation_strategies-to-improve.html.

Greenes, C., Immerzeel, G., Ockenga, E., Schulman, L., & Spungin, R. (1986). *Secondary (grades 9 -12) problem resource book.* New York: Dale Seymour Publications.

Grouws, D. A., & Cebulla, K. J. (2000). Improving student achievement in mathematics Part1: *Research findings.* Retrieved February 25 2011 from http://www.ericdigests.org/2003-1/math2.html

Hamilton, M. A., & Mineo, P. J. (1996). Central to belief system, Peripheral to belief system. Retrieved August 08 2013 from http://Coms.uconn.edu/directory/faculty/mhamilton/persuassion/chapter%207d-t.pdf

Hardin, L. E. (2002). Educational strategies: *Problem solving concepts and theories.* JVME 30(3), 227-230.

Hare, A. Y. V. M. (1999). Revealing what urban early childhood teachers think about mathematics and how they teach it: *Implications for practice.* Retrieved January 26 2011 from http://www.people.ex.ac.uk/Pernest.hare/thm

Hassi, M. L., & Laursen, S. (2009). Studying undergraduate mathematics: *Exploring students' beliefs, experiences and gains.* Retreived March 22 2011 from http://www.pmena.org/2009/proceedings/.../affect+RR369923replacement.pdf

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spartial representations and mathematical problem solving. *Journal of educational psychology, Vol 91 No. 4,* 684-689. Doi:10.1.1.167.8485

Hooisan, E. (2001). What are mathematical problems? Humanistic mathematics network journal, (24), May 2001.

HSRC. (2006). Grade 9 mathematics and science achievement in TIMSS 2003. Retrieved January 26 2011 from http://www.hsrcpress.ac.za

Huitt, W. (2003). The information processing approach to cognition. *Educational psychology interactive.* Valdosta, GA: Valdosta State University.

Jin, M., Feng, X., Liu, J., & Dai, F. (2010). Comparison study on high school students mathematics belief systems between Han and Chaoxian nationality. *Journal of mathematics education June 2010; Vol. 3, No. 1*, pp. 138-151.

Kaur, B. (1998). Mathematical problem solving in Singapore Schools. *Teaching and learning, 19(1)*, 67-78. Singapore Institute of Education.

Kee, T. C. (1999). Problem solving in mathematics. Retrieved March 30 2011 from http://library.thinkquest.org/254959/learning/problem...

Keeves, J. P. (ed.). (1998). Educational research methodology and measurement: *An international handbook.* Oxford: Pergamon Press.

Kirkley, J. (2003). Principles of teaching problem solving; *Technical paper #4.* doi: 10.1.1.117.8503

Kislenko, k., Grevholm, B., & Lepik, M. (2005). "Mathematics is important but boring": students' beliefs and attitudes towards mathematics. In C. Bergsten, B. Grevholm, H. Masoval, & F. Ronning (Eds.). *Relating practice and research in mathematics education. Proceedings of norma5, fourth Nordic conference on mathematics education, pp. 349-360.* Trondheim: Tapir Akademisk Forlag.

Kitchen, R. S., DePree, J., Pattichis, S. C., & Brinkerhoff, J. (2007). Mathematics education at highly effective schools that serve the poor. *Strategies for change.* London: Lawrence Erlbaum Associates, Inc. Publishers.

Kolovou, A., Van den Heuvel-Panhuizen, M., Bakker, A., & Elia, I. (2011). An ICT environment to assess and support students' mathematical problem-solving performance in non-routine puzzle-like word problems. ICME 11.

Krulik, S., & Rudnick, J. (1980). Problem solving: *A handbook for teachers.* Newton, MA: Allyn & Bacon.

Kyllonen, P. C., & Lee, S. (2005). Assessing problem solving in context. In Wilhelm, O., & Engle, R.W. *Handbook of understanding and measuring intelligence.* London: Sage publications.

Landa, L. (1983). Algo-heuristic theory. Retrieved March 18 2011 from
http://tip.psychology.org/landa.htm

Lazim, M. A., Abu Osman, M. T., & Wan Salihin, W. A. (2004). The statistical evidence
in describing the students' beliefs about mathematics. *International journal for
mathematics teaching and learning, October 12th*.

Ledger, G. C., Pehkonen, E. & Torner, G. (Eds.). (2002). Beliefs: *A hidden variable in
mathematics education?* 39-57. Dordrecht: Kluwer Academic Publishers

Lester, F. K. (2013). Thoughts about research on mathematical problem-solving
instruction. *The mathematics enthusiast*, ISSN 1551-3440, Vol. 10, nos. 1 & 2,
pp. 245-278.

Levy, S. R., West, T., & Rosenthal, L. (2012). The contributing role of prevalent belief
systems to intergroup attitudes and behhaviours. *Online readings in psychology
and culture, 5(1).* doi: org/10,9707/2307.0919.1044

Lindenskov, L., & Hetmar, V. (2009). Proposal: *Comparative research project on
component of analytical competences.* Retrieved March 29 2011 from
http://www.dpu.dk/fileadmin/www.dpu.dk/.../conferences/.../resources_2801.pdf

Livingston. J. A. (1997). Metacognition: *An overview*. Retrieved March 18 2011 from
http://gse.buffalo.edu/fas/shuell/cep564/metacog.htm

Logsdon, A. (2007). How to solve math word problems- learn the SQRQCQ strategy to
solve math problems. Retrieved April 01 2011 from
http://learningdisadisabilities.about.com/od/insttructionalmaterials/ht /sqrqcq.html

Lorain Country Community College (2011). Problem solving strategies. Retrieved June
29 2011 from
http://www.loraincc.edu/Lcccweb/Extras/Printable.aspx?guid={E575C498-F937-
46BD-A552-CDFB8009B028}

Lucangeli, D., & Cabrele, S. (2006). The relationship of metacognitive knowledge, skills
and beliefs in children with and without mathematics learning disabilities. In

Desoete, A., & Veenman, M. *Metacognition in mathematics education.* New York: Nova Science Publishers, Inc.

Lunzer, E. (1986). Cognitive development: Learning and the mechanisms of change. In Phye, G.D., & Andre, T. *Cognitive classroom learning: understanding, thinking, and problem solving.* Florida: Academic Press, Inc.

Mabilangan, R. A., Limjap, A. A., & Belecina, R. R. (2011). Problem solving strategies of high school students on non-routine problems: A case study. *Journal of basic education Vol 5,* 23-46.

Magagula, C. (1996). The issue of paradigms in educational research. Keeping the debate alive. *Zimbabwe journal of educational research, Vol8 No3 Nov 1996* p.233-254

Malouff, J. (2011). Psychology-Over fifty problem solving solving strategies explained. Retrieved April 01 2011 from http://www.une.edu.au/bcss/psychology/john.malouff/problem-solving.php

Manson, S. M. (2001). Simplifying complexity: a review of complexity theory. *Geoforum 32 (2001)* 405-414.

Marcou, A., & Philippou, G. (2005). Motivational beliefs, self-regulated learning and mathematical problem solving. In chick, H. L., & Vincent, J .L. (Eds.). *Proceedings of the 29th conference of the international group for the psychology of mathematics education, vol.3* pp, 297-304. Melborne: PME.

Maree, K., Aldous, C., Hattingh, A., Swanepoel, A, & Van der Linde, M. (2006). Predictors of learner performance in mathematics and science according to a large-scale study in Mpumalanga. *South African Journal of Education. Vol. 26 (2)* 229-252.

Mason, L. (2003). High school students' beliefs about mathematics, mathematical problem solving, and their achievement in mathematics: A cross-sectional study. *Educational journal of psychology, 23(1),* 73-84.

McQuade, F. (2010). Problem solving strategies. Retrieved March 30 2011 from
http://pred.boun.edu.tr/ps/ps4/html

Meiring, S. P. (1980). Problem solving: A basic mathematics goal. *A resource for
problem solving 2.* Columbus: Ohio department of education.

Mogari, D. & Lupahla, N. (2013). Mapping a group of Northern Namibian grade 12
learners' algebraic non-routine problem solving skills. *African iournal of research
in mathematics, science and technology education, 17:1-2*, 94-105.

Moistus, K. (2001). Pupils' interaction with the exhibits according to the learning
behaviour model. Retrieved January 26 2011 from
http://www.sciencecommunication.se/master_thesis_01_moistus.pdf

Moscucci, M. (2007). About mathematical belief systems awareness. *CERME 5*.

Mueller, C. (1997). Landamatics: Learning theory of Lev Landa. Retrieved March 18
2011 from http://tecfa.unige.ch/staf/staf9698/mullerc/3/landa.htm

Muis, K. R. (2004). Epitemic styles and mathematics problem solving: *Examining
relations in the context of self –regulated learning.* Retrieved March 30 2011 from
http://ir.lib.sfu.ca/bitstream/1892/9935/1/b37359599.pdf

Mullis, I. V. S., Martin, M. O., Smith, T. A., Garden, R. A., Gregory, K. D., Gonzalez, E.
J., Chrostowski, S. J., & O'Connor, K. M. (2003). *TIMSS Assessment frameworks
and specifications 2003* (2nd ed.). Chestnut Hill, MN: Boston College.

Murray, H., Olivier, A., & Human, P. (1998). Learning through problem solving. In
Olivier, A., & Newstead, K. (Eds.). *Proceedings of the Twenty-second
International conference for the Psychology of mathematics Education; vol. 1*.
(pp. 169-185). Stellenbosch, South Africa.

Ngulube, P. (2013). Mixed methods research designs: *College of Education, mixed
methods research design workshop.* University of South Africa, Pretoria.

Nichols, B. (2008). Improving student achievement: *50 research-based strategies*. Columbus, Ohio: Linworth Publishing, Inc.

Op`t Eynde, P., De Corte, E., & Verschaffel, L. (2002). Framing students` mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G.C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 13-37). Dordrecht: Kluwer Academic Publishers.

Op't Eynde, P. and De Corte, E. (2003). Students' mathematics-related belief systems: Design and analysis of a questionnaire. ERIC identifier: ED475708

Op't Eynde, P. and Hannula, M. S. (2006). The case study of Frank. Educational studies in mathematics (2006) 63: 123-129 DOI: 10.1007/s10649-006-9030-8

Op't Eynde, P., De Corte, E., & Verschaffel, L. (2006). Beliefs and metacognition: An analysis of Junior High School students' mathematics-related beliefs. In De Corte, A., & Veenman, M. *Metacognition in mathematics education*. New York: Nova Science Publishers,Inc.

Osterholm, M. (2009). Beliefs: A theoretically unnecessary construct? *Working group 1: Proceedings of cèrme 6, Jnuary 8th-February 1st 2009*, pp. 154-163.

Otten S. (2010). Thirty years of problem solving in mathematics education: *Policy and promise*. Retrieved June 29 2011 from http://www.msu.edu/~Ottensam/DSME_comps2010_otten_breadth1.doc

Panaoura, A., Philippou, G., & Christou, C. (2010). Young pupils' metacognitive ability in mathematics. Proceedings of cèrme 3.

Pantziara, M., Gagatsis, A., & Pitta-Pantazi, D. (2004) The use of diagrams in solving non-routine problems. *Proceedings of the 28th conference of the international group for the psychology of mathematics education, 2004; Vol. 3*. pp. 489-496.

Pehkonen, E. (1999). *Beliefs as obstacles for implementing an educational change in problem solving*. Doi:10.1.1.119.4110

Pehkonen, E. and Torner, G. (1999). *Introduction to the abstract book for the Oberwolfach meeting on research into beliefs*. Doi:10.1.1.119.4110

Pehkonen, E., & Pietila, A. (2003). On relationships between beliefs and knowledge in mathematics education. Paper presented at the cerme 3: Third conference of the European society for research in mathematics education, Bellaria, Italy.

Physick, M. D. (2010). Exploring mathematics-related belief systems. Retrieved March 30 2011 from http://www.peterliljedahl.com/wp-content/uploads/Thesis-Mike-Physick.pdf

Pimta, S. Tayruakham, S., & Nuangchalerm, P. (2009). Factors influencing mathematics problem-solving ability of sixth grade students. *Journal of social sciences 5 (4):*381-385, 2009.

Polya, G. (1980). On solving mathematical problems in High School. In krulik, S., & Rey, R. E. (1980). *Problem solving in school mathematics, National Council of Teachers of Mathematics, 1980 Yearbook.* Virginia: NCTM.

Polya, G. (1985). *How to solve it* (2nd ed). Princeton, NJ: Princeton University Press.

Posamentier, A. S., & Schulz, W. (Eds.). (1996). The art of problem solving: *A resource for the mathematics teacher.* California: Corwin Press, Inc.

Reddy, V., Kanjee, A., Diedericks, G., & Winnaar, L. (2006). *Mathematics and science achievement at South African schools in TIMSS 2003.* Cape Town: HSRC Press.

Rokeah, M. (1968). *Beliefs, attitudes and values.* San Francisco: Jassey-Bass.

Romberg, T. A. (1994). Classroom instruction that fosters mathematical thinking and problem solving: Connections between theory and practice. In Schoenfeld, A. H. *Mathematical thinking and problem solving.* New Jersey: Lawrence Erlbaum Associates, Publishers.

Russell, D. (2007a). Problem solving strategies. Retrieved April 01 2011 from http://math.about.com/cs/testprep/∞/ps.html

Russell, D. (2007b) Problem solving in mathematics. Retrieved on April 01 2011 from http://math.about.com/od/1/a/problemsolv.html

Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive science, volume 7 , issue 4.* p. 329-363.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In Grouws, D. A. (Ed.). *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1994). *Mathematical thinking and problem solving.* New Jersey: Lawrence Erlbaum Associates, Publishers.

Schoenfeld, A. H. (Ed.). (2007). *Assessing mathematical proficiency.* Mathematical Science Research Institute: Cambridge University Press. ISBN: 978-0521 87492-2

Sellinger, M. (1994). *Teaching mathematics.* London: Routledge.

Shuttleworth, M. (2009). Convergent and discriminant validity. Retrieved July 23 2012 from http://www.experiment-resources.com/convergent-validity.html

Simbamoorthi, N. (1999). Hierarchical cluster analysis: *Some basics and algorithms.* Retrieved July 08 2011 from http://www.crmportals.com/hierarchical_cluster_analysis.pdf

Sorochan, W. (2011). What is a belief system. Retrieved August 08 2013 from http://www.freegrab.net/beliefsystems.htm

Spangler, D. A. (1992). Assessing students' beliefs about mathematics. *Arithmetic teacher, 40 (2),* 148-152.

Stephenson, D. (2001). Math problem solving techniques: *Eighty problem solving strategies and practice activities for 9 to 12 year olds.* Greenwood, WA: Ready-Ed Publishers. ISBN: 186397 1106

Stice, J. E. (2010). Teaching problem solving. Retrieved June 29 2011 from http://www.csi.unian.it/educa/problemsolving/stice_ps.html

Stylianides, G. J., & Stylianides, A. J. (2010). An intervention on students' problem-solving beliefs. *Working group 8, cerme 7.*

Sweller, J., Clark, R., & Kirschner, P. (2010). Teaching general problem-solving skills is not a substitute for, or a viable addition to, teaching mathematics. *Notices of the AMS, vol. 57, no. 10.* pp.1303-1304.

Trochim, W. M. K. (2006) Convergent and Discriminant validity. Retrieved March 25 2011 from http://www.socialresearchmethods.net/kb/convdisc.php

Vinner, S. (1999). *Beliefs we live by and quite often are even not aware of their possible impact on teaching and learning mathematics*. Doi: 10.1.1.119.4110

Webb, L., & Webb, P. (2004). Eastern Cape teachers' beliefs of the nature of mathematics: implications for the introduction of in-service mathematical literacy programmes for teachers. *Pythagoras, 60*, 13-19.

Wessels, D. C. J. (2012). The possibility of a modeling perspective for school mathematics. Retrieved May 10 2013 from http://hdl.handle.net/10019.1/19737

Willoughby, S. S. (1990). *Mathematics education for a changing world.* Alexandria: Association for supervision and curriculum development

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. Retrieved on March 22 2011 from http://jwilson.coe.uga.edu/emt725/PSsyn/Pssyn.html

Wragg, E. C. (1994). *An introduction to classroom observation*. London: Routledge.

Xenofontos, C. (2010). International comparative research on mathematical problem solving: Suggestions for new research directions. *Proceedings of CERME6, January 28th February 1st 2009,* Lycon, France.

Xenofontos, C., & Andrews, P. (2008). Teachers' beliefs about mathematical problem solving, their problem solving competence and the impact on instruction: A case study of three Cypriot primary teachers. Retrieved March 22 2011 from http://tsg.icme11.org/document/get/465

Yeo, K. K. J. (2008). Secondary 2 students' difficulties in solving non-routine problems Retrieved march 22 2011 from http://www.cimt.plymouth.ac.uk/journal/yeo.pdf

Zanzali, N. A. A., & Nam, L. L. (1997). Evaluating the levels of problem solving abilities in mathematics. Retrieved April 01 2011 from http://math.unipa.it/□grim/jzanzalinam.PDF

Zhu, Z. (2007). Gender differences in mathematical problem solving patterns: A review of literature. *International Educational Journal, 2007, 8(2),* 187-203.

Mathematics Beliefs questionnaire

[Adapted from Op't Eynde et al. (2006); Op't Eynde and Hannula (2006); Physick (2010); Muis (2004); and Lazim et al. (2004)]

(Factor loadings for this research study are in brackets)

Scale 1: I can do well in mathematics

1.1. I am very interested in mathematics (0.735).

1.2. I like doing mathematics (0.667).

1.3. Mathematics is my favourite subject (0.664).

1.4. Mathematics has always been my worst subject (0.659).

1.5. I am sure that I can learn mathematics (0.652).

1.6. I know I can do well in mathematics (0.612).

1.7. Mathematics is a mechanical and boring subject (0.596).

1.8. I am not good in mathematics (0.583).

1.9. I am not a type to do well in mathematics (0.577).

1.10. Mathematics is difficult (0.577).

1.11. I think I could handle more difficult mathematics (0.539).

1.12. I can understand the course material in mathematics (0.515).

1.13. I enjoy pondering mathematical exercises (0.513).

1.14. I feel confident in my ability to solve mathematics problems (0.498).

1.15. To me mathematics is an important subject (0.441).

1.16. It is waste of time when the teacher makes us think on our own about how to solve a new mathematical problem (0.426).

1.17. The problems we work on in math class have no relationship to daily life (0.417).

1.18. Considering the difficulty of this course, the teacher, and my skills, I think I will do well in mathematics (0.412).

1.19. There is only one way to find the correct solution of a mathematics problem (0.347).

1.20. I think I will be able to use what I learn in mathematics also in other courses (0.328).

Scale 2: I make sense of what I learn

2.1 I try to play around with ideas of my own and relate them to what I am learning in this course (0.653).

2.2 I try to relate ideas in this subject to those in other courses whenever possible (0.626).

2.3 When I find a solution, I always look for other ways of solving the problems (0.572).

2.4 Whenever I read or hear an assertion or conclusion in this class, I think about possible alternatives (0.552).

2.5 I try to understand the material in this class by making connections between the readings and the concepts from the lessons (0.508).

2.6 I ask myself questions to make sure I understand the material I have been studying in this class (0.473).

2.7 I try a different approach when my first attempt fails (0.466).

2.8 When I have the opportunity, I choose a mathematical assignment that I can learn from even if I'm not at all sure of getting a good grade (0.430).

2.9 I always prepare myself carefully for exams (0.419).

2.10 I prefer mathematics tasks for which I have to work or think hard in order to find a solution (0.401).

2.11 I am a hard worker by nature (0.365).

2.12 I usually understand a new idea in mathematics quickly (0.356).

Scale 3: Group work facilitates learning of mathematics

3.1. Learning mathematics requires a lot of effort (0.727).

3.2. There are several ways to find the correct solution of a mathematics problem (0.621).

3.3. For me the most important thing in learning mathematics is to understand (0.595).

3.4. Mathematics is about solving problems (0.587).

3.5. I need mathematics in order to study what I would like after I finish high school (0.583).

3.6.   Mathematics is used by a lot people in their daily life (0.522).

3.7.   If I study in appropriate ways, then I will be able to learn the material in this course (0.522).

3.8.   When I have finished working on the problem, I look back to see whether my answer makes sense (0.503).

3.9.   Making mistakes is part of learning mathematics (0.482).

3.10.   One learns mathematics through doing exercises (0.447).

3.11.   When I can't understand the material in this course, I ask another student or my teacher for help (0.418).

3.12.   Group work facilitates the learning of mathematics (0.390).

3.13.   Solving a mathematics problem is demanding and requires thinking, also from smart students (0.364).

Scale 4: Mathematics is numbers, rules and techniques

4.1.   One learns mathematics best by memorizing facts and procedures (0.631).

4.2.   Mathematics is numbers and calculation (0.631).

4.3.   Mathematics is a set of rules and techniques (0.587).

4.4.   The problem –solving process is linear: you advance directly towards the solution (0.554).

4.5.   There are always numbers in formulations of mathematics problems (0.550).

4.6.   I'm only satisfied when I get a good grade in mathematics (0.505).

4.7.   Those who are good in mathematics can solve any problem within a few minutes (0.404).

4.8.   The teacher must always show me which method to use to solve a given mathematics problem (0.347).

4.9.   Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone (0.338).

Scale 5: Mathematics is continuously evolving

5.1.   Mathematics is a static and rigid body of knowledge. No new things about mathematics are yet to be discovered (0.456).

5.2.   I rarely find time to review my notes before an exam (0.454).

5.3. It is not important to understand why a mathematical procedure works as long as it gives the correct answer (0.452).

5.4. Mathematics enables a man to better understand the world he lives in (0.450).

5.5. I feel the most important thing in mathematics is getting the correct answer (0.438).

5.6. Mathematics is continuously evolving. New things are still being discovered (0.416).

BELIEFS QUESTIONNAIRE 1

STUDENT ID:

Please rate the following statements of beliefs about mathematics education and the self as a mathematician according to the following rating scale, indicating your degree of agreement or disagreement by circling your rating :

Strongly agree	5
Agree	4
Uncertain	3
Disagree	2
Strongly disagree	1

	Strongly agree	Agree	Uncertain	Disagree	Strongly disagree
1. I am hard working by nature	5	4	3	2	1
2. I always prepare myself carefully for exam	5	4	3	2	1
3. I prefer mathematics tasks for which I have to work or think hard inorder to find a solution	5	4	3	2	1
4. I am sure that I can learn mathematics	5	4	3	2	1
5. I know I can do well in mathematics	5	4	3	2	1
6. I think I could handle more difficult math	5	4	3	2	1
7. Considering the difficulty of this course, the teacher, and my skills, I think I will do well in mathematics	5	4	3	2	1
8. I can understand the course material in mathematics	5	4	3	2	1
9. Anyone can learn mathematics	5	4	3	2	1
10. I feel confident in my ability to solve mathematics problem	5	4	3	2	1
11. It is my own fault if I don't learn the material in this course	5	4	3	2	1

12. Mathematics is my favourite subject	5	4	3	2	1
13. I enjoy pondering mathematical exercises	5	4	3	2	1
14. I like doing mathematics	5	4	3	2	1
15. To me mathematics is an important subject	5	4	3	2	1
16. I am very interested in mathematics	5	4	3	2	1
17. I think I will be able to use what I learn in mathematics also in other courses	5	4	3	2	1
18. Mathematics enables a man to better understand the world he lives in	5	4	3	2	1
19. Mathematics is used by a lot of people in their daily life	5	4	3	2	1
20. I need mathematics in order to study what I would like after I finish high school	5	4	3	2	1
21. Learning mathematics requires a lot of effort	5	4	3	2	1
22. If I try hard enough, then I will understand the course material of the mathematics class	5	4	3	2	1
23. Solving a mathematics problem is demanding and requires thinking, also from smart students	5	4	3	2	1
24. One learns mathematics through doing exercises	5	4	3	2	1
25. Group work facilitates the learning of mathematics	5	4	3	2	1
26. Mathematics is continuously evolving. New things are still being discovered	5	4	3	2	1
27. There are several ways to find the correct solution of a mathematics problem	5	4	3	2	1
28. Making mistakes is part of learning mathematics	5	4	3	2	1
29. When I have the opportunity I choose mathematical assignments that I can learn from even if I'm not at all sure of getting a good grade	5	4	3	2	1
30. For me the most important thing in learning mathematics is to understand	5	4	3	2	1
31. If I study in appropriate ways, then I will be able to learn the material in this course	5	4	3	2	1
32. I ask myself questions to make sure I understand the material I have been studying in this class	5	4	3	2	1
33. I try to relate ideas in this subject to those in other courses whenever possible.	5	4	3	2	1

	5	4	3	2	1
34. I try to play around with ideas of my own and relate them to what I am learning in this course	5	4	3	2	1
35. When I can't understand the material in this course, I ask another student or my teacher for help	5	4	3	2	1
36. Whenever I read or hear an assertion or conclusion in this class, I think about possible alternatives	5	4	3	2	1
37. I try to understand the material in this class by making connections between the readings and the concepts from the lesson	5	4	3	2	1
38. Mathematics is about solving problems	5	4	3	2	1
39. When I have finished working on the problem, I look back to see whether my answer makes sense.	5	4	3	2	1
40. I try a different approach when my first attempt fails	5	4	3	2	1
41. When I find a solution, I always look for other ways of solving the problem.	5	4	3	2	1

BELIEFS QUESTIONNAIRE 2

STUDENT ID:

Please rate the following statements of beliefs about mathematics education and the self as a mathematician according to the following rating scale, indicating your degree of agreement or disagreement by circling your rating :

Strongly agree   1
Agree   2
Uncertain   3
Disagree   4
Strongly disagree   5

	Strongly agree	Agree	Uncertain	Disagree	Strongly disagree
1. I am not good in mathematics	1	2	3	4	5
2. Mathematics has always been my worst subject	1	2	3	4	5
3. I am not the type to do well in mathematics	1	2	3	4	5
4. It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem	1	2	3	4	5
5. Those who are good in mathematics can solve any problem in a few minutes	1	2	3	4	5
6. The teacher must always show me which method to solve a given mathematics problem	1	2	3	4	5
7. Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone.	1	2	3	4	5
8. I rarely find time to review my notes before an exam	1	2	3	4	5
9. I am only satisfied when I get a good grade in mathematics	1	2	3	4	5
10. I usually understand a new idea in mathematics quickly	1	2	3	4	5
11. Mathematics is a mechanical and boring subject	1	2	3	4	5

12. The problems we work on in mathematics class have no relationship to daily life	1	2	3	4	5
13. Mathematics is difficult	1	2	3	4	5
14. Mathematics is numbers and calculations	1	2	3	4	5
15. One learns mathematics best by memorizing facts and procedures	1	2	3	4	5
16. Mathematics is a set of rules and techniques	1	2	3	4	5
17. The problem solving process is linear: you advance directly towards the solution	1	2	3	4	5
18. There are always numbers in formulations of mathematics problems	1	2	3	4	5
19. There is only one way to find the correct solution of a mathematics problem	1	2	3	4	5
20. Mathematics is a static and rigid body of knowledge: No new things about mathematics are yet to be discovered.	1	2	3	4	5
21. I feel the most important thing in mathematics is getting the correct answer	1	2	3	4	5
22. It is not important to understand why a mathematical procedure works as long as it gives the correct answer	1	2	3	4	5

Appendix C

1. What is mathematics?

2. In your own words, describe what a mathematics problem is.

3. What do you like or dislike about mathematics as a discipline or subject?

4. Describe the type of questions you usually encounter in mathematics problems.

5. What is solving a mathematical problem?

6. Give examples of mathematical problems, exercises and other activities you enjoy solving or doing most. May you state reasons why you enjoy them most?

7. Given the following two mathematical problems, which one will you like to answer?(Don't try to solve it). Give reasons for your choice.

(a) Find the value of x in: $\frac{1}{4}x+5=25$.

(b) Mpho spends a quarter of her money on chips and R5 on soft drinks.

Together she had spent R25. How much did she have initially?

Mathematics Problem solving test

Time:

Instructions

- Answer all questions

- Show all your working on the answer sheets provided

- Show any rough work done that contributes to the solution on the spaces provided.

- You may show any other alternative solutions to each problem.

- You may use calculators where necessary

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

    (Adopted from Muis, 2004, p. 114)

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

    (Adopted from Greenes et al., 1986, p. 12)

3. A game management scientist found that 90% of the male calves born and 95% of the female calves born survive their first year. If 50% of the calves that are born are male, how many calves must be born to guarantee that 100 survive the first year?

    (Adopted from Greenes et al., 1986, p. 41)

4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and cakes cost R 5.00 each. Everyone had the same number of cups and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had?

    (Adopted from Burton, 1984, p. 80)

5. There are some rabbits and some rabbit hutches. If seven rabbits are put in each rabbit hutch, one rabbit is left over. If nine rabbits are put in each rabbit hutch, one hutch is left empty.

Can you find how many rabbit hutches and how many rabbits there are?

(Adopted from Burton, 1984, p. 64)

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.

- Annah and Refilwe together caught as many as Joel and Thabo

- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth?

(Adapted from Callejo & Vila, 2009, p. 115)

Suggested solutions to Problem Test

1.
	T1	T2	T3		Tn
Thabang :	100	110	120.....		$100+(n-1)10$
Mpho :	40	65	90		$40 + (n-1)25$

Let the number of days be n

$$100+(n-1)10 = 40+(n-1)25$$
$$90+10n = 15+25n$$
$$75 = 15n$$
$$n = 5$$

OR listing some possible values

	T1	T2	T3	T4	**T5**
Thabang :	100	110	120	130	**140 . . . .**
Mpho :	40	65	90	115	**140 . . . .**

$$\therefore n = 5.$$

_____

2. Old car : $16\text{km} \rightarrow 1 \text{ gallon}$

$$15000\text{km} \rightarrow \frac{15000}{16} \text{ gallons}$$

cost of travelling 15 000 km $= \dfrac{15000}{16} \times \text{R}2.00$

$$= \text{R}1875.00$$

New car:

$26\text{km} \rightarrow 1 \text{ gallon}$

$$15\,000\text{km} \rightarrow \frac{15000}{26} \text{ gallons}$$

cost of travelling 15 000km $= \dfrac{15000}{26} \times \text{R}2.00$

$$= \text{R}1\,153.85$$

$\therefore$ Saving $=$ R1 875 - R1 153.85

$$= \text{R}721.15$$

3. Given 50 % of calves born are males, 50 % of calves born are females.

         Let x be the number of calves born.

    Male calves = female calves = 0.5 x

    If 100 calves should survive, then

$$0,9(0,5x)+0,95(0,5x) >100$$

$$0,925 >100$$

$$x >108,1$$

$$\therefore x=110 \text{ is the least possible value.}$$

_____

4. Note: The bill for each person must be a divisor of 133. So must be : 1; 7; 19 or 133
    (number of people) x (bill of one person) = 133

    The bill cannot be R1 or R7, because R 3 and R 5 are the only prices. The bill
    cannot be R 133 for each person because we assume there is more than one
    person. Therefore the only possible value is 19.
    Since 19 = (2x5) + (3x3), the answer is 3 cups of tea each

    OR
    Using systematic trial and error

# of people	Each person's bill	Is this possible?
1	133	No
2	66,5	No- why?
3	44.5	No-Why?
4		
5		
6		
7	19	Yes

Note: We expect the bill for each person to be a whole number divisor of 133 if R3 and
R5 are the only prices.

5. Let h = number of hutches
   r = number of rabbits

   By systematic trial and error

   If seven rabbits are put in a rabbit hutch, one rabbit is left over, then $r = 7h + 1$.
   If nine rabbits are put in each hutch, one hutch is left empty, then $r = 9(h-1)$.

h	1	2	3	4	5	6
$r = 7h+1$	8	15	22	29	**36**	43
$r = 9(h-1)$	0	9	18	27	**36**	45

   Therefore there are 36 rabbits and 5 hutches

   OR **By Algebra**

$7h + 1 = 9h - 9$	Subst 5 for h in $r = 7h + 1$
$2h = 10$	$r = 7(5) + 1 = 36$
$h = 5$	$\therefore\ h = 5$ and $r = 36$

_____

6. Let A stand for Annah, R for Refilwe, J for Joel and T for Thabo
   (a) T > J (Thabo caught more than Joel)
   (b) A+ R= J+T
   (c) A+ T < R+J
       From (2): J= A + R –T

Subst in (3): A+T< R+ A+R- T
       2 T < 2 R
       T< R
       From (2): R = J + T – A

Subst in (3): A+T< J+ J+T- A
       2A < 2J
       A < J
       $\therefore$ R > T >J >A
$\therefore$Refilwe caught most
Thabo – second
Joel – third
Annah – fourth

_____

Retrospective Questionnaire 1

This survey requires you to reflect on your responses to the problem test. You may refer back to the copy of your written work, if there is need.

Question 1

Reflect on how you answered question 1, and answer the following questions:

1.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

1) Absolutely sure that I could do it correctly.

2) Quite sure that I could do it correctly

3) Not sure, I didn't know how correctly I could do it.

4) Really not sure, I thought probably that I could not succeed.

5) Absolutely sure that I could not do it correctly.

1.2. Think over what you have done and try to describe what kind of strategies you used to solve the task. (monitoring)

_____
_____
_____
_____

1.3. State any difficulties or obstacles you met on solving the problem. (monitoring)

_____
_____
_____
_____

1.4. Now that you have answered the question, try to say if you are sure that you have done it correctly: (evaluation)

(Circle your response)

1. You are absolutely sure that you have done it in the right way.

2. You are quite sure that you have done it the right way.

3. You are not sure; you don't know how correctly you have done it.

4. You are really not sure, and you think you have probably made a mistake.

5. You know that you have made a mistake.

Retrospective Questionnaire 2

Question 2

Reflect on how you answered question 2, and answer the following questions:

2.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

1. Absolutely sure that I could do it correctly.

2. Quite sure that I could do it correctly

3. Not sure, I didn't know how correctly I could do it.

4. Really not sure, I thought probably that I could not succeed.

5. Absolutely sure that I could not do it correctly.

2.2. Think over what you have done and try to describe what kind of strategies you used to solve the task. (monitoring)

_____

_____

_____

2.3. State any difficulties or obstacles you met on solving the problem. (monitoring)

_____
_____
_____

2.4. Now that you have answered the question, try to say if you are sure that you have done it correctly. (evaluation)

(Circle your response)

1. You are absolutely sure that you have done it in the right way.

2. You are quite sure that you have done it the right way.

3. You are not sure, you don't know how correctly you have done it.

4. You are really not sure, and you think you have probably made a mistake.

5. You know that you have made a mistake.

Retrospective Questionnaire 3

Question 3

Reflect on how you answered question 3, and answer the following questions:

3.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

    1. Absolutely sure that I could do it correctly.

    2. Quite sure that I could do it correctly

    3. Not sure, I didn't know how correctly I could do it.

    4. Really not sure, I thought probably that I could not succeed.

    5. Absolutely sure that I could not do it correctly.

3.2. Think over what you have done and try to describe what kind of strategies you used to solve the task. (monitoring)

_____
_____
_____

3.2. State any difficulties or obstacles you met on solving the problem. (monitoring)

_____
_____
_____

3.3. Now that you have answered the question, try to say if you are sure that you have done it correctly: (evaluation)

(Circle your response)

    1.   You are absolutely sure that you have done it in the right way.

    2.   You are quite sure that you have done it the right way.

    3.   You are not sure; you don't know how correctly you have done it.

    4.   You are really not sure, and you think you have probably made a mistake.

    5.   You know that you have made a mistake.

Retrospective Questionnaire 4

Question 4

Reflect on how you answered question 4, and answer the following questions:

4.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

1. Absolutely sure that I could do it correctly.

2. Quite sure that I could do it correctly

3. Not sure, I didn't know how correctly I could do it.

4. Really not sure, I thought probably that I could not succeed.

5. Absolutely sure that I could not do it correctly.

4.2. Think over what you have done and try to describe what kind of strategies you used to solve the task.
(monitoring)

_____
_____
_____

4.3. State any difficulties or obstacles you met on solving the problem.
(monitoring)

_____
_____
_____

4.4. Now that you have answered the question, try to say if you are sure that you have done it correctly.
(evaluation)

(Circle your response)

1. You are absolutely sure that you have done it in the right way.

2. You are quite sure that you have done it the right way.

3. You are not sure; you don't know how correctly you have done it.

4. You are really not sure, and you think you have probably made a mistake.

5. You know that you have made a mistake.

Retrospective Questionnaire 5

Question 5

Reflect on how you answered question 5, and answer the following questions:

5.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

1. Absolutely sure that I could do it correctly.

2. Quite sure that I could do it correctly

3. Not sure, I didn't know how correctly I could do it.

4. Really not sure, I thought probably that I could not succeed.

5. Absolutely sure that I could not do it correctly.

5.2. Think over what you have done and try to describe what kind of strategies you used to solve the task. (monitoring)

_____
_____
_____

5.3. State any difficulties or obstacles you met on solving the problem. (monitoring)

_____
_____
_____

5.4. Now that you have answered the question, try to say if you are sure that you have done it correctly:

(evaluation)

(Circle your response)

1. You are absolutely sure that you have done it in the right way.

2. You are quite sure that you have done it the right way.

3. You are not sure; you don't know how correctly you have done it.

4. You are really not sure, and you think you have probably made a mistake.

5. You know that you have made a mistake.

Retrospective Questionnaire 6

Question 6

Reflect on how you answered question 6, and answer the following questions:

6.1. How confident were you that you could solve it correctly? (prediction)

(Circle your response)

1. Absolutely sure that I could do it correctly.

2. Quite sure that I could do it correctly

3. Not sure, I didn't know how correctly I could do it.

4. Really not sure, I thought probably that I could not succeed.

5. Absolutely sure that I could not do it correctly.

6.2. Think over what you have done and try to describe what kind of strategies you used to solve the task. (monitoring)

_____
_____
_____

6.3. State any difficulties or obstacles you met on solving the problem. (monitoring)

_____
_____
_____

6.4. Now that you have answered the question, try to say if you are sure that you have done it correctly. (evaluation)

(Circle your response)

1. You are absolutely sure that you have done it in the right way.

2. You are quite sure that you have done it the right way.

3. You are not sure; you don't know how correctly you have done it.

4. You are really not sure, and you think you have probably made a mistake.

5. You know that you have made a mistake.

[Adapted from Lucangeli &Cabrele, 2006, p. 130]

Appendix F

Interview Schedule 1

Could you tell me why you rated this item 1(or 2, 3, 4, 5):

1. It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.

2. Even if I have trouble learning the material in this class, I try to do the work on my own, without help from anyone.

3. Anyone can learn mathematics.

4. To me mathematics is an important subject.

5. One learns mathematics best by memorizing facts and procedures

6. Making mistakes is part of learning mathematics.

7. It is not important to understand why a mathematical procedure works as long as it gives the correct answer.

Interview Schedule 2

1. What is the best way you think you can learn mathematics?

2. Can you state, in brief, the strategies you employ in studying or learning mathematics.

3. What is your main goal in learning mathematics?

4. Why do you think mathematics is or is not important to you?

5. What do you gain from learning mathematics?

6. How much comfortable are you in learning new material?

How do you feel in face of challenging material?

7. Which set of problems do you enjoy solving between (a) problems with known algorithms or (b) problems which can be best solved by using logic or reasoning? Give a reason for your choice.

8. What is your comment on the statement: "Mathematics is a static and rigid body of knowledge. No new things about mathematics are yet to be discovered".

Cluster
Membership

Case	2 Clusters
1:A2	1
2:A3	2
3:A4	1
4:A6	1
5:A7	1
6:A8	1
7:A10	2
8:A11	2
9:A12	2
10:A13	1
11:A14	1
12:A15	1
13:A17	1
14:A19	1
15:A20	1
16:A21	1
17:A22	1
18:A23	1
19:A24	1
20:A25	2
21:A26	1
22:A31	1
23:A33	2
24:A34	1
25:A35	1
26:A36	1
27:A37	2
28:A41	1
29:A43	1
30:A44	1
31:A45	1
32:B1	1
33:B3	1
34:B4	1

Case	2 Clusters
35:B6	1
36:B7	1
37:B8	2
38:B9	1
39:B11	1
40:B12	2
41:B14	1
42:B16	1
43:B17	1
44:B19	1
45:B20	1
46:B21	2
47:B22	1
48:B23	1
49:B24	1
50:B25	1
51:B26	1
52:B27	1
53:B31	2
54:B32	1
55:B33	1
56:B37	1
57:B39	1
58:B41	1
59:B43	2
60:B44	1
61:B45	1
62:B46	1
63:B47	1
64:B48	1
65:B49	1
66:B50	1
67:B51	1
68:B52	1
69:B53	2
70:B57	1

Case	2 Clusters
71:B59	1
72:B61	2
73:B62	1
74:B63	1
75:B64	1
76:B65	1
77:C1	1
78:C5	1
79:C6	1
80:C8	1
81:C10	1
82:C11	1
83:C12	1
84:C13	2
85:C14	1
86:C15	1
87:C16	1
88:C17	1
89:C19	1
90:C20	2
91:C21	1
92:C22	1
93:C23	2
94:C25	1
95:C27	2
96:C28	1
97:C31	1
98:C32	1
99:C33	1
100:C34	2
101:C36	1
102:C37	1
103:C38	1
104:C39	1
105:D3	1
106:D4	1

107:D5	2	151:F3	1	195:G25	2
108:D8	1	152:F5	2	196:G26	1
109:D11	2	153:F7	2	197:G31	2
110:D12	1	154:F8	1	198:G32	1
111:D14	2	155:F9	2	199:G35	1
112:D15	1	156:F12	1	200:G40	2
113:D16	1	157:F13	1	201:G42	1
114:D19	1	158:F14	1	202:G43	1
115:D24	1	159:F17	1	203:G46	1
116:D28	2	160:F21	1	204:G47	1
117:D29	1	161:F22	1	205:G50	1
118:D30	1	162:F24	2	206:H1	1
119:D32	2	163:F27	1	207:H2	1
120:D35	1	164:F28	1	208:H3	1
121:D41	2	165:F30	1	209:H4	1
122:D42	1	166:F31	1	210:H5	1
123:D43	2	167:F35	1	211:H9	1
124:D44	1	168:F36	1	212:H10	1
125:E3	2	169:F37	1	213:H11	2
126:E5	1	170:F38	1	214:H12	2
127:E8	2	171:F39	2	215:H15	1
128:E11	1	172:F40	1	216:H17	1
129:E12	1	173:F41	1	217:H19	1
130:E18	1	174:F43	2	218:H21	2
131:E19	2	175:F44	2	219:H22	1
132:E20	1	176:F45	1	220:H23	2
133:E25	2	177:F47	1	221:H26	1
134:E26	2	178:F48	2	222:H27	2
135:E28	1	179:F49	1	223:H29	1
136:E32	2	180:F53	1	224:H31	1
137:E33	1	181:F54	2	225:H35	1
138:E34	1	182:F55	1	226:H39	2
139:E36	1	183:F59	1	227:H40	1
140:E37	2	184:G3	1	228:H42	2
141:E38	1	185:G4	1	229:H46	2
142:E40	1	186:G5	2	230:H47	2
143:E42	1	187:G7	1	231:H49	2
144:E43	1	188:G8	1	232:I1	2
145:E44	1	189:G11	1	233:I6	2
146:E45	1	190:G12	1	234:I13	1
147:E46	2	191:G14	1	235:I15	2
148:E49	1	192:G18	2	236:I17	1
149:F1	1	193:G23	1	237:I18	2
150:F2	1	194:G24	2	238:I19	2

239:I20	2
240:I21	1
241:I26	2
242:I27	2
243:I30	1
244:I35	1
245:I38	2
246:I39	1

247:I40	2
248:I44	2
249:I47	2
250:I48	1
251:J1	2
252:J2	2
253:J3	2
254:J4	2

255:J5	2
256:J7	1
257:J8	2
258:J9	1
259:J10	2
260:J14	1
261:J17	1

Dendrogram

*H I E R A R C H I C A L   C L U S T E R   A N A L Y S I S *

Dendrogram using Complete Linkage

```
                    Rescaled Distance Cluster Combine

      C A S E         0         5        10        15        20        25
    Label    Num      +---------+---------+---------+---------+---------+

    J7       256      
    J9       258      
    F13      157      
    E33      137      
    A17       13      
    C37      102      
    F21      160      
    B47       63      
    F27      163      
    E36      139      
    I13      234      
    B14       41      
    B49       65      
    F12      156      
    H5       210      
    A24       19      
    D42      122      
    B52       68      
    E45      146      
    F41      173      
    B17       43      
    D19      114      
    J14      260      
    B33       55      
    G43      202      
    A26       21      
    G46      203      
    A15       12      
    H4       209      
    C5        78      
    A6         4      
    A22       17      
    B25       50      
    B39       57      
    C17       88      
    F30      165      
    C25       94      
    F3       151      
    E42      143      
    H22      219      
    H17      216      
    B27       52      
    D30      118      
    B62       73      
    D16      113      
```
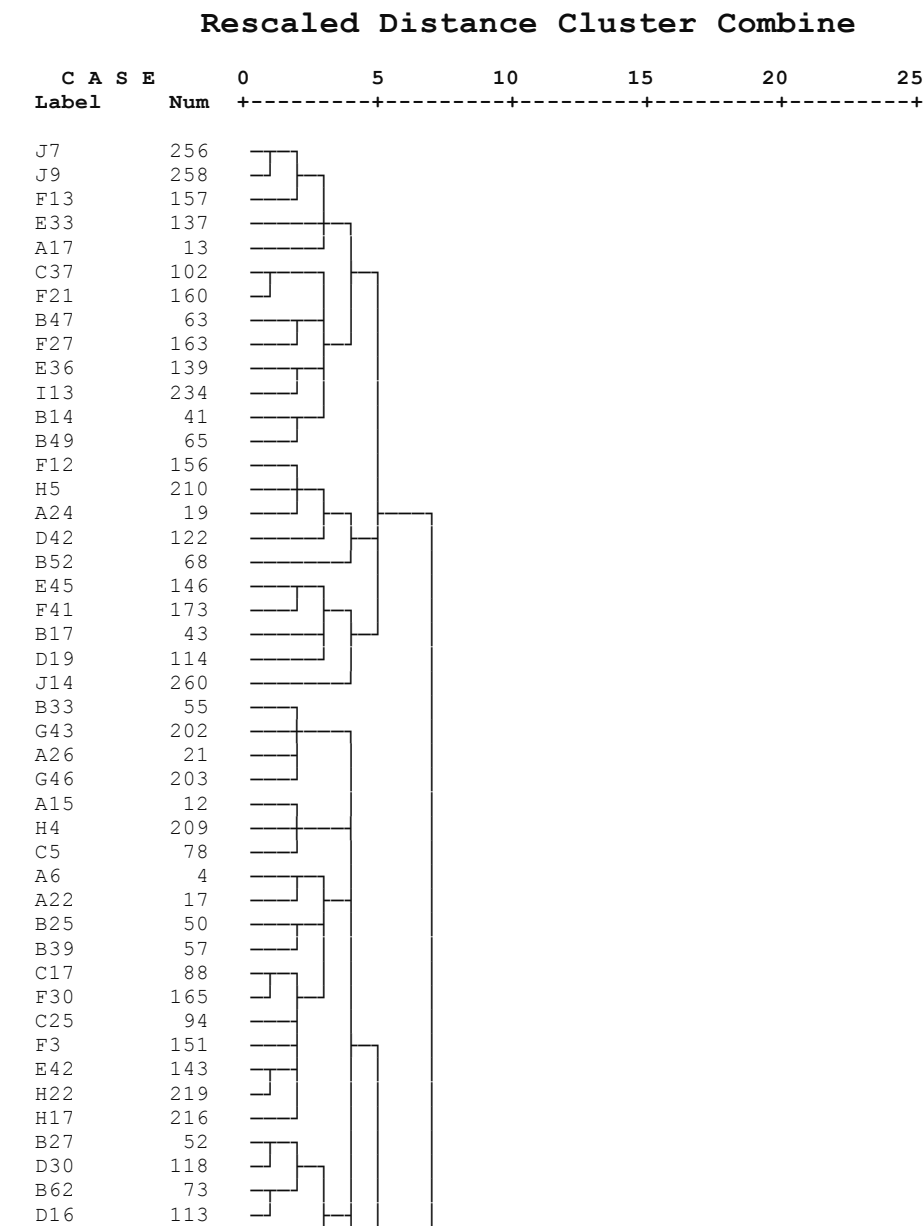
```
E20    132  ─┐
E40    142  ─┤
B48     64  ─┤
B6      35  ─┐
G32    198  ─┤
H35    225  ─┤
B1      32  ─┤
B63     74  ─┐
F17    159  ─┤
B7      36  ─┤
B50     66  ─┤
F35    167  ─┐
G14    191  ─┤
G7     187  ─┤
H26    221  ─┤
E49    148  ─┐
G12    190  ─┤
H1     206  ─┤
I30    243  ─┤
A21     16  ─┤
C15     86  ─┤
E5     126  ─┤
F1     149  ─┐
F28    164  ─┤
C38    103  ─┤
H19    217  ─┤
B9      38  ─┐
D24    115  ─┤
B19     44  ─┤
B44     60  ─┤
F53    180  ─┤
G3     184  ─┤
C8      80  ─┐
C39    104  ─┤
C11     82  ─┤
A44     30  ─┐
C21     91  ─┤
C1      77  ─┤
B22     47  ─┤
H3     208  ─┤
A31     22  ─┐
B51     67  ─┤
C10     81  ─┤
H40    227  ─┤
E34    138  ─┤
E44    145  ─┤
E12    129  ─┤
H2     207  ─┤
C6      79  ─┐
C28     96  ─┤
A36     26  ─┤
A35     25  ─┐
J17    261  ─┤
E11    128  ─┤
A41     28  ─┤
C12     83  ─┤
A34     24  ─┤
F55    182  ─┤
B26     51  ─┤
F37    169  ─┤
A43     29  ─┤
G23    193  ─┤
D29    117  ─┤
B23     48  ─┤
G26    196  ─┤
C32     98  ─┤
I17    236  ─┤
B20     45  ─┤
A8       6  ─┤
B3      33  ─┤
A14     11  ─┤
```

F45   176
D35   120
I48   250
F49   179
B41   58
F8   154
F2   150
F40   172
C31   97
F47   177
B57   70
B65   76
E18   130
D3   105
D4   106
A13   10
H31   224
G42   201
A23   18
D12   110
G35   199
C22   92
C33   99
E38   141
D8   108
B32   54
B64   75
F59   183
C14   85
F38   170
B45   61
A20   15
B16   42
F36   168
C19   89
F22   161
A2   1
E28   135
B24   49
H15   215
E43   144
H10   212
C16   87
G50   205
D15   112
G11   189
G47   204
B59   71
H29   223
F31   166
A7   5
A19   14
I21   240
A4   3
I35   244
C36   101
D44   124
H9   211
I39   246
B4   34
B11   39
A45   31
B37   56
B46   62
G4   185
G8   188
F14   158
D14   111
H49   231
C34   100
F48   178

```
E19      131    ──────────┐      ┌──┐
D11      109    ──────────────┘  │
I6       233    ────────────────────┘
```

APPENDIX H: Extract of matrix of inter-correlations between variables

		I am hard working by nature	I always prepare myself carefully for exam	I prefer maths tasks for which I have to work or think hard inorder to find a solution	I am sure that I can learn maths
Correlation	I am hard working by nature	1.000	.297	.228	.306
	I always prepare myself carefully for exam	.297	1.000	.270	.187
	I prefer maths tasks for which I have to work or think hard inorder to find a solution	.228	.270	1.000	.273
	I am sure that I can learn maths	.306	.187	.273	1.000
	I know I can do well in math	.284	.145	.275	.581
	I think I could handle more difficult math	.231	.159	.275	.423
	Considering the difficulty of this course, the teacher, and my skills, I think I will do well in maths	.190	.162	.249	.342
	I can understand the course material in maths	.263	.164	.274	.423
	Anyone can learn maths	.042	.096	.102	.061
	I feel confident in my ability to solve maths problem	.300	.278	.291	.305
	It is my own fault if I don't learn the material in this course	.111	.086	.201	.281
	Math is my favourite subject	.243	.222	.332	.366
	I enjoy pondering mathematical exercises	.301	.291	.316	.306
	I like doing maths	.228	.194	.290	.364
	To me maths is an important subject	.127	.101	.188	.249
	I am very interested in maths	.215	.195	.339	.459
	I think I will be able to use what I learn in maths also in other courses	.166	.062	.229	.282
	Maths enables a man to better understand the world he lives in	.206	.076	.148	.129
	Maths is used by a lot of people in their daily life	.156	.057	.092	.163
	I need maths in order to study what I would like after I finish high school	.207	.132	.156	.266

APPENDIX I: CONSENT LETTERS

(1) Request to conduct research at the school

Mr Chirove Munyaradzi
4910 Unit D Ext 4
Temba
0407

Cell: 078 620 5990
E-mail: **munyaradzi.chirove25@gmail.com**

Dear School Management Team/ Principal

Re: REQUEST TO CONDUCT A RESEARCH STUDY AT YOUR SCHOOL

I am Mr Chirove Munyaradzi. I am a PhD student at UNISA. I am studying the relationship between high school learners' mathematics-related belief systems and their approach to mathematical non-routine problem-solving. The study entails collection of data from grades 10, 11 and 12 learners at schools. In respect of the study, I kindly request for permission to conduct the research study at your school.

The data collection will be done after school contact time. The data are to be used for research purposes only. The name of the school and all the participants in the research study are to be kept anonymous. The results of the research may inform both policy and practice. Your participation in the study, as a school, is voluntary and you are free to withdraw from the research at any time you wish to do so.

I am looking forward to hearing from you.

Yours faithfully

Chirove Munyaradzi

(2) Request for learner participation

Mr Chirove Munyaradzi
4910 Unit D Ext 4
Temba
0407

Cell: 078 620 5990
E-mail: munyaradzi.chirove25@gmail.com

Dear Parent/ Guardian

Re: REQUEST FOR YOUR CHILD TO PARTICIPATE IN RESEARCH STUDY

I am Mr Chirove Munyaradzi. I am a PhD student at UNISA. I am studying the relationship between high school learners' mathematics-related belief systems and their approach to mathematical non-routine problem-solving. The study entails collection of data from grades 10, 11 and 12 learners at schools. Learners, who are willing to participate in the research study, will complete questionnaires, attempt a mathematics problem-solving test, and attend interviews after school contact time. Your child is free to withdraw from the research at any time he/ she wishes to do so.

The data are to be used for research purposes only. The names of the participating learners are to be kept anonymous. The results of the research may inform both policy and practice. I kindly request you to grand your child permission to participate in the research study.

I am looking forward to hearing from you.

Yours faithfully

Chirove Munyaradzi

_____

Please sign and return the bottom portion of this consent form

I, the parent/ legal guardian of_____, acknowledge receipt of a letter explaining the purpose of the research, the nature of participation of my child and the confidentiality of the data collected. I, unreservedly, consent to my child's participation in the research study conducted by Mr Chirove Munyaradzi.

Name of learner:_____

Signature of parent/ legal guardian: _____

Date: _____

_____

(3) Informed Consent Form

Purpose of Study

The questionnaires, mathematics problem test and interview schedules to be administered will provide me with data on learners' mathematics-related beliefs and their approaches to mathematics problem solving. The data are to be used to identify the possible relationship between learners' mathematics-related beliefs and their approach to problem solving. The results of this study will raise both educators and learners' awareness of the consequences of their beliefs to their teaching and learning of mathematics.

Nature of participation

Participation in this study is voluntary and participants are free to withdraw from the research at any time without any negative or undesirable consequences to themselves.

Privacy, anonymity and confidentiality

The participants' responses will be treated confidentially and used for research purposes only. Participants are to be kept anonymous. They will not be allowed to disclose their identity.

Name of researcher: Chirove M

Phone/ cell number: 078 720 5990

Research supervisor: Prof. Mogari D

Phone/ cell number: 012 429 3904

Institution: UNISA

I_____ (please print your name in full) have read and understood the contents of the 'Informed Consent Form'. I agree to participate in the research study.

Signature:_____

(Principal/HOD/Learner).... (tick the applicable)

APPENDIX J: <u>CLEARANCE LETTER FROM DEPARTMENT OF EDUCATION</u>

education

Department: Education
GAUTENG PROVINCE

For administrative use:
Reference no. D2013/168

GDE RESEARCH APPROVAL LETTER

Date:	**20 August 2012**
Validity of Research Approval:	**20 August 2012 to 30 September 2012**
Name of Researcher:	**Chirove M.**
Address of Researcher:	**4600 Unit D**
	Extension 1
	Temba
	Hammanskraal
	0407
Telephone Number:	**078 620 5990**
Email address:	**munyaradzi.chirove@gmail.com**
Research Topic:	**Relationship between learners' mathematically-related belief system and their approach to mathematical problem-solving: A case study of three high schools in Tshwane North District, South Africa**
Number and type of schools:	**THREE Secondary Schools**
District/s/HO	**Tshwane North**

<u>Re: Approval in Respect of Request to Conduct Research</u>

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

Making education a societal priority

Office of the Director: Knowledge Management and Research
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

1. The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.
2. The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.
3. A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.
4. A letter / document that outlines the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.
5. The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.
6. Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.
7. Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year.
8. Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.
9. It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.
10. The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.
11. The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.
12. On completion of the study the researcher must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.
13. The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.
14. Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

[signature]
....

Dr David Makhado

Director: Knowledge Management and Research

DATE: 2012 / 08 / 21
....

Appendix K: Declaration by supervisor

DECLARATION BY SUPERVISOR / PROMOTER / LECTURER	
I declare that: (Name of Researcher) Chirove Munyaradzi	
1. is enrolled at the institution	

2. The questionnaires / structured interviews / tests meet the criteria of:
    • Educational Accountability
    • Proper Research Design
    • Sensitivity towards Participants
    • Correct Content and Terminology
    • Acceptable Grammar
    • Absence of Non-essential / Superfluous items	

Surname:	Mogari
First Name/s:	David
Institution / Organisation:	UNISA
Faculty / Department (where relevant):	ISTE
Telephone:	012 429 3904
Fax:	012 429 8690
E-mail:	mogarld@unisa.ac.za
Signature:	
Date:	August 20, 2012

N.B. This form (and all other relevant documentation where available) may be completed and forwarded electronically to Diane.Buntting@gauteng.gov.za The last 2 pages of this document must however contain the original signatures of both the researcher and his/her supervisor or promoter. These pages may be faxed to (086 594 1781) or hand delivered (in sealed envelope) to Diane Buntting, Room 509, 111 Commissioner Street, Johannesburg. All enquiries pertaining to the status of research requests can be directed to Diane Buntting on tel. no. 011 843 6503.

Appendix L: Letter of transmittal

UNISA | university of south africa

August 01, 2012

To whom it may concern

I write to confirm that Mr Munyaradzi Chirove is a registered PhD (Mathematics Education) student at our university. As part of his studies, he needs to carry out a research project and put together a thesis. The research project entails gathering data in high schools. I therefore ask that he be allowed to use the schools he has identified so that he can collect all the required data.

Thanking you in anticipation of your much appreciated co-operation and invaluable assistance

Yours faithfully

Prof LD Mogari
Supervisor