ABSTRACT

Title of Thesis:	DESIGN AND COMPARISON OF SAMPLE ALLOCATION SCHEMES FOR MULTI-ATTRIBUTE DECISION MAKING
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In case of multi-attribute decisions, when a decision maker has a limited budget for data collection, then the decision maker has to decide on the number of samples to observe from each alternative and its attributes. This allocation decision is of importance when the observation process is uncertain, such as with physical measurements. This thesis presents a sequential allocation approach in which measurements are conducted one at a time. Prior to making a measurement the decision-maker's current knowledge of the attribute values is used to identify the attribute and alternative pair to sample next using all these allocation procedures. The thesis discusses a simulations study that was performed to compare the *Sequential Allocation Approach*, *Proportional Allocation Approach* and *Uniform Allocation Approach.* We evaluated the frequency of selecting the true best alternative when the attribute value observations contain discrete random measurement error. The results indicate that the sequential approach is significantly better than the other approaches when the budget is small; as the budget increases, its advantage decreases.

DESIGN AND COMPARISON OF SAMPLE ALLOCATION SCHEMES FOR MULTI-ATTRIBUTE DECISION MAKING

by

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List of Abbreviations

PCS: Probability of Correct Selection

ePCS: Expected Probability of Correct Selection

fcs: Frequency of Correct Selection

Chapter 1: Introduction

1.1 Background

A multi-attribute decision problem is one in which a decision maker has to select an alternative from a finite set of alternatives with various attributes (common for each alternative) describing an alternative. In a selection process, the decision-maker (the person responsible for selecting the best alternative), must select an alternative from a large number of competing performances. When the true values of the attributes are unknown, measurements (or samples) of the attributes can provide valuable information, but, because the measurement process is inaccurate, the samples are imperfect information. Thus, the samples can reduce uncertainty about attribute values, but some uncertainty about the relative desirability of the alternatives will remain. When given a limited finite budget for sampling (sample is a measure of the value of an attribute of the alternative), the decision maker wants to determine which alternatives and attributes should be sampled in order to maximize the probability of selecting the correct alternative.

1.2 Motivating Examples

This section contains the two motivating examples behind this thesis research. The first example looks at substitution of materials in a manufacturing set up. This example considers replacing a material with a better substitute by analyzing various properties or attributes of the original and substituting materials.

1.2.1 Material Substitution in a manufacturing process

Consider a mechanical design engineer working for an automotive gear manufacturing company who wants to substitute an aluminum alloy for gray cast iron. When looking at material selection problem for design, substitution of a material requires knowledge of various attributes of the material to be substituted and the potential substitutes. As mentioned in the Dieter and Schmidt [14], while considering aluminum alloy as an alternative to gray cast iron, there are many variations of aluminum alloy available as shown in Table 1. There are various attributes of these different cast iron, like strength and corrosion resistance, that are paramount to the performance of these aluminum alloys. According to Dieter and Schmidt, cast iron has Valid strength (18 ksi), ultimate tensile strength (22ksi), shear strength (20 ksi) and elongation (0.5 inch). The corresponding strength properties of aluminum alloys are better than that of cast iron.

There is uncertainty in the performance of the alloys which comes from the uncertainty in the attributes of these alloys. The firm can take samples to reduce the uncertainty of the attributes considering a budget constraint.

Table 1: Mechanical properties of cast iron and alloys [Source: Dieter and Schmidt [14]

Material	Yield Strength, ksi	Ultimate Tensile Strength, ksi	Shear Strength, ksi	Elongation in 2 in, present
Gray Cast Iron	18	22	20	0.5
Alloy 356	15	26	18	3.5
Alloy 360	25	26	45	3.5
Alloy 356	28	38	38	5

The relative desirability of the different alloys depends upon the attributes or mechanical properties of yield strength, ultimate tensile strength, shear strength and elongation. It also depends upon the weights that reflect the importance of these attributes to the decision maker. This would result in a ranking and selection problem which forms the basis of this research. These factors and several others need to be considered in order to choose the best aluminum alloy for a particular application.

1.2.2 Product Design Selection accounting for Customer preference

Product selection by a company for its production workers is a selection problem. A company for instance wants to buy cordless screwdrivers for its production workers use. This example is based on information from Li and Azarm [18]

The selection of an alternative from various vendors would depend upon the following attributes of the product:

1. Maximum number of operations achieved with one charge of the battery.
2. Minimum time required for one operation.
3. Weight of the tool.
4. Cost (attribute with the least uncertainty due to quoted prices).

The major uncertainty about which alternative is best comes from the uncertainty in the attributes considered for each of the alternatives. The buyer can do sampling to reduce the uncertainty in the attribute values but the sampling procedure is subject to budget constraints.

1.3 Research Questions

As mentioned by Leber and Herrmann [19] when the decision maker has a limited finite budget for collecting information about multiple alternatives and their attributes for selection decision, the decision maker has to make the decision of how much of the budget present for information gathering to allot to each attribute of the alternative. There is a certain trade-off of gathering this information. If more samples are used to gather information about a certain attribute of an alternative, the value of that attribute becomes more certain but this leaves more uncertainty about the other attributes which could result in uncertainty in selecting the truly best attribute.

This thesis compares three sample allocation approaches: the sequential approach, the proportional approach, and the uniform approach.

The research described in this thesis seeks to answer the following research questions:

1. How does the relative performance of these sample allocation approaches vary as the total budget varies?
2. Which characteristics of decision instances significantly degrade the frequency of correct selection when using these sample allocation procedures?

For this research we are looking at discrete distributions for the measurements. The reason for using discrete values is that measurement devices have certain precision related to measurements they take. The output of the measurement from a device has a fixed precision depending upon the accuracy to which the device works. Thus, the value of measurement is a value on the discrete scale.

1.4 Thesis Overview

This section gives an overview of the organization of this thesis. Chapter 1 of this report gives a brief background of this research. It also presents the motivation behind this research and the research question that this thesis aims at answering. Chapter 2 of this report speaks about the literature reviewed for this research and the previous work done in this domain which forms the basis of this research. Chapter 3 explains the notations used in this report followed by explaining the uncertainty in decision making. This chapter also explains the assumptions behind this research and explains the sampling approach. Chapter 4 explains in detail the problem that is being addressed in this research and it also explains in detail each of the sampling approach in detail. Chapter 5 discusses in detail the simulations conducted to evaluate these allocation procedures, the design of experiments which defines how these simulations are designed, keeping in mind the factors that effect of design of these experiments of the simulation and the performance comparison of each of the allocation approaches. Chapter 6 discusses the conclusions drawn from the analysis done in the previous chapter and identifies the future scope of work that can be conducted based on this thesis.

Chapter 2: Literature Review

2.1 Decision Analysis

Decision theory (or the theory of choice) is the study of the reasoning underlying an agent's choices. Decision theory can be broken into three branches: normative decision theory, which gives advice on how to make the best decisions, given a set of uncertain beliefs and a set of values; descriptive decision theory, which analyzes how existing, possibly irrational agents actually make decisions; and prescriptive decision theory, which tries to guide or give procedures on how or what we should do in order to make best decisions in line with the normative theory.

For the certainty-risk-uncertainty, classification in Luce and Raiffa [6], if we assume that the choices are made between two actions, it is assumed to be in the realm of decision under:

- *Certainty* if each action is known to lead invariably to a specific outcome.
- *Risk* if each action leads to one of a set of specific outcomes, each outcome occurring with a known probability. The probabilities are assumed to be known to the decision maker.
- *Uncertainty* if either action or both has as its consequence a set of possible outcomes, but where the probabilities of these outcomes are completely unknown or are not even meaningful.

The idea of decision analysis with attribute value uncertainty is used by Leber and Herrmann [18] to present various approaches to several approaches to incorporate attribute value uncertainty into the decision analysis for choosing a roofing firm based on data.

Powell and Ryzhov [15] discussed the challenge of collecting information as efficiently as possible as because information gathering is time consuming and expensive. They studied the problem of making observations (measurements) to determine which choice is the best. The work focusses on a knowledge gradient strategy and its use in with a wide range of belief models, including lookup table and parametric and for online and offline problems. Loch et al. [12] explained the process of parallel and sequential testing of design alternatives. Although parallel testing is faster, it does not take into account the potential for learning as in serial testing. The paper derives an optimal strategy as a function of testing cost, prior knowledge, and testing lead time.

2.2 Ranking and Selection

The idea of ranking and selection is used in this thesis to rank a set of alternatives based on certain criteria and then the selection decision is made based on this ranking criterion. The criteria for these ranking are statistical parameters which are either experimentally determined or simulated using computer software.

A great and extensive review of the concept of ranking and selection is provided in Kim and Nelson [19]. Their work describes the principles of ranking and selection by defining an indifferent zone (IZ) allocation procedure for selection of the best alternative. The IZ talked about in this which does not have any limits on the number of observations. This procedure measures how often each alternative is observed or sampled and providing a probability of correct selection. Kim and Nelson [19] described four classes of comparisons as they relate to ranking and selection problems: selecting the alternative with the largest or smallest expected performance measure (selection of the best), comparing all alternatives against a standard (comparison with a standard), selecting the alternative with the largest probability of actually being the best performer (multinomial selection), and selecting the system with the largest probability of success (Bernoulli selection).

Although all the previous techniques consider the allocation of samples across multiple alternatives with a single performance measure, while our work is focused on the allocation of samples across both the multiple alternatives and the multiple attributes, the ranking and selection techniques used in this research is based on Leber [5] allocation of samples across both the multiple alternatives and the multiple attributes. Butler et al. [20] applies the IZ procedure to a multiple attribute decision problem using a multiple attribute value model and combined the multiple uncertain attribute values using a multiple attribute decision model to provide an alternative's overall performance measure to develop a sequential allocation approach. The ranking and selection approach used in this thesis research is based on this idea presented by Leber [5] in his research.

2.3 Experiment Design for Bayesian Estimation

Bayesian updating is the estimation technique in which the posterior distribution is calculated by adjusting the prior distribution or a *priori* in a way that is consistent with the new data obtained. In this thesis research we used a Bayesian estimation experiment design to estimate the value of attributes of alternatives from the beliefs the decision-maker.

In Chaloner and Verdinelli [4], a decision theory approach to experiment design is explained. The decision η must be chosen from a set *H*, and data y from a sample space *Y* will be observed. Based on y a decision *d* will be chosen from *D.* The decision is in two parts: first the selection, and then the choice of a terminal decision d. The unknown parameters are θ, and the parameter space is Θ. A utility function is of the form U (d, θ, η, y). For any decision η, let $U(\eta)$ be the expected utility. This can be determined as:

$$U(\eta) = \int_y \max_{d \in D} \int_\Theta U\ (d, \theta, \eta, y)\ p_\theta(\theta|y,\eta) p_y(y|\eta) d\theta dy \qquad (2.1)$$

where *p(.)* denotes the probability density function with respect to an appropriate measure. The Bayesian solution to the experimental design problem is provided by the experimental design η^* with the greatest utility. That is,

$$U(\eta) = \max_{\eta \in H} \int_y \max_{d \in D} \int_\Theta U\ (d, \theta, \eta, y)\ p(\theta|y,\eta) p(y|\eta) d\theta dy \qquad (2.2)$$

In the research by DasGupta [16], Bayesian formulation of a typical optimal design problem is explained in detail. According to the research, in a strictly Bayesian decision theoretic setup, one has a set of parameters with a prior distribution G, a specific likelihood function f(x|), and a loss function L(,). Given a design, there is an associated Bayes rule with respect to the trio(f,L,G); an optimal design should minimize over all designs the Bayes risk, i.e. the average loss of the Bayes estimate over all samples and the parameters.

In the design of this research, the probability of the sample attaining a certain value in the measurement that is taken is estimated using a prior distribution. This prior distribution is used to get a posterior distribution to estimate the value of the next sample, x_{ij} for alternative *i* and attribute j.

Chaloner and Verdinelli [4] defined the mathematics of Bayesian estimation design as generally the same as that in classical optimal design.

There are 3 main routes to obtaining an optimal design:

i. Use an equivalence theorem.
ii. In polynomial models use inherent symmetry (if there is such symmetry) in the problem and convexity of the criterion function in conjunction with Caratheodory type bounds on the cardinality of the support, and
iii. Use of arguments, which usually go by the name of Elfving geometry.

2.4 Review of Previous Work

The work done in this thesis research is based on previous work by Dennis Leber [5] on multi attribute decision making in a Gaussian set up. This section gives a brief overview of the previous work done and the extension of the work that is carried out as part of this research. The section also gives a brief description of the how the work done in this thesis is based on the previous work done.

2.4.1 Background

Leber and Herrmann [18] uses Bayesian updating to maximize the probability of selecting the true best alternative when the attribute value observations contain continuous Gaussian measurement error for three different allocation procedures. The Uniform allocation procedure is taken as the baseline for these measurements. The previous work considers a case in which the decision-maker has a finite budget for samples. This work covers a specific class of decisions tested which is limited to the Gaussian data for measurements and their corresponding errors. The research compares the performance of the three strategies: (i) *sequential* (ii) *proportional* and (ii) *uniform allocation procedures* and concludes that the sequential sampling approach performs better than the non-sequential approaches.

2.4.2 Limitations

The work done in the above-mentioned paper discusses a continuous Gaussian measurement error as part of the measurement process which is a very specific case. Leber [5] considers the decision-makers beliefs about the attribute's true value. To describe the decision-makers beliefs, only normally distributed Bayesian posterior distributions were considered.

The scope of this thesis is to explain sample allocation for a finite budget for a multi-attribute decision making process by considering that the true values of the attributes are unknown but their discrete error distributions are known to the decision-maker. This thesis also goes a step further and moves out of the domain of a Gaussian setup as presumed in the previous research. In this work, a general discrete distribution of the actual measurement and measurement error values. Although this thesis continues to consider a Bayesian setting for the decision-makers knowledge of the true best attribute for the general case, it considers the distribution to be an unknown and discrete and hence is an extension of the earlier work done.

Chapter 3: Decision Uncertainty in Alternative Selection

3.1 Notation

This section describes in detail the notation and abbreviations used in this thesis. This notation generally follows the notation used by Leber [5]. Associated with the decision values and attribute values are their true values and random variables that describe the decision-makers beliefs about these unknown values.

Table 2: Notation used in this research

a_i	An individual alternative, indexed by i, $i = 1, \ldots, m$
$\{a_1, \ldots, a_m\}$	Set of m alternatives
j	An individual attribute j , $j = 1, \ldots, k$
μ_{ij}	The true value of attribute j of alternative a_i
M_{ij}	A random variable that represents the unknown value of attribute j of alternative a_i
E_j	A random variable that represents the error of the measurement process for attribute j
X_{ij}	A random variable that represents the outcome of measuring (sampling) attribute j of alternative a_i
x_{ij}	A sample observation of the attribute j of alternative a_i
n_{ij}	Number of samples of attribute j of alternative a_i
λ_j	The decision weight associated with attribute j; $\lambda_j > 0$
W_i	Set of weights for all the attributes.
ξ_i	Decision value for alternative a_i
Z_i	Random variable that represents the unknown value of the decision value for alternative a_i
z_i	A possible value of Z_i
P_{ij}^{r}	Prior distribution that describes the decision-maker's beliefs about μ_{ij}.
P_{ij}^{st}	Posterior Distribution for estimation of attribute values

R_{ij}	Set of all possible values of μ_{ij}
${p^p}_i$	The probability that the alternative *i* has the maximum decision value
PCS^p	Probability of Correct Selection (PCS)

3.2 Problem Formulation

This section formulates the sample allocation problem. A finite set of m distinct alternatives $\{a_1, \ldots, a_m\}$ is provided. Each alternative is described by $k \geq 2$ attributes. The true but unknown value of attribute *j* of alternative a_i is μ_{ij}, *i* = 1, …, *m*, *j* = 1, …, *k*. The decision-maker must select one alternative. The decision-maker's preferences are modeled as a linear multi-attribute value function. Let ξ_i be the decision value of alternative *i*.

$$\xi_i = \sum_{j=1}^{k} \lambda_j \mu_{ij}$$

The decision-maker would like to select the alternative with the greatest value of ξ_i. Because the attribute values are unknown, the decision-maker is not sure which alternative has the greatest value. From the decision-maker's perspective, the attribute values and the decision values are uncertain and are modeled as random variables (M_{ij} and Z_i). The decision-maker will select the alternative that, given his beliefs about the attribute values, has the greatest probability of having the best (largest) decision value.

The decision-maker begins with prior distributions about the attribute values P_{ij}^r.

$$P_{ij}^r(v) = P\{M_{ij} = v\} \text{ for all } v \in R_{ij}$$

To reduce his uncertainty about the attribute values (and decision values), the decision-maker can obtain samples of the attributes by measuring them. One sample is the result of one measurement of one attribute for one alternative. The value of the sample is uncertain because the measurement process is imperfect; that is, the sample has an error. Let E_j be the random error of a measurement of attribute j (on any alternative), and let X_{ij} be the random variable that represents the sample of attribute j on alternative i:

$$X_{ij} = \mu_{ij} + \mathrm{E}_j$$

The probability distributions of the measurement errors are known. Due to time or cost constraints, the decision-maker can obtain at most B samples. After obtaining these samples, the decision-maker will update his beliefs about the attribute values (to create P_{ij}^{st}, his subjective posterior probability distributions for the M_{ij}, which also yield posterior probability distributions for the Z_i) and select the alternative that, given these beliefs, has the greatest probability of having the best (largest) decision value. In this thesis, the term "probability of correct selection" (PCS) is used to describe the probability that an alternative has the best (largest) decision value, and this is based on the decision-makers beliefs. The equation for calculating this is given in Chapter 4.

The decision-maker's problem is to allocate the B samples to the different attributes and alternatives in a way that maximizes the likelihood that his selection is the actual best alternative (the one with the greatest value of ξ_i). A sample allocation specifies the values of n_{ij}, the number of samples of attribute j of alternative a_i, such that the total equals *B*.

Clearly, if *B* were large enough, then the decision-maker could obtain enough samples of every attribute of every alternative to reduce the attribute value and decision value uncertainty enough that the identity of the actual best alternative would be certain. When *B* is not that large, however, the attribute value and decision value uncertainty create some risk that the selection is not the actual best alternative.

Chapter 4: Information Gathering

This chapter describes the three sample allocation procedures tested in this thesis and the method for updating the decision-makers beliefs. Other procedures have been developed and should be studied in future research (see the discussion in Chapter 6).

4.1 Uniform Sample Allocation Procedure

The uniform allocation approach uses very little information to make a sample allocation. Every attribute is sampled the same number of times, n_{ij} = $\frac{B}{km}$ for i = 1, …, m, j = 1, …, k.

If the value $\frac{B}{km}$ is not an integer, then some values of n_{ij} will be $\frac{B}{km}$ (the smallest integer greater than $\frac{B}{km}$), and other values of n_{ij} will be $\frac{B}{km}$ (the greatest integer less than $\frac{B}{km}$).

4.2 Proportional Sample Allocation Procedure

This section presents the proportional allocation approach to sample allocation. This procedure allocates more samples to the attributes that have the greatest weights in the value function. Let $\Lambda = \sum_{j=1}^{k} \lambda_j$, then the number of samples allotted to each attribute of an alternative, $n_{ij} = \frac{B\lambda_j}{m\lambda}$ for *i* = 1, …, *m*, *j* = 1, …, *k*.

Note that the total number of samples allocated to each alternative is $\frac{B}{m}$.

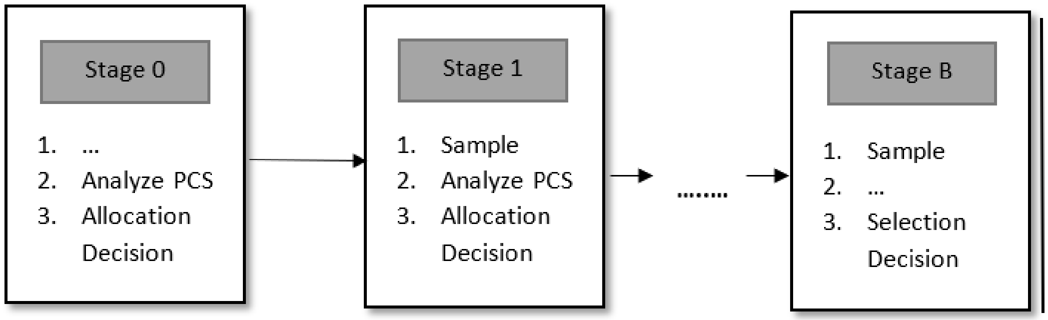
If any calculated value is not an integer, then it should be rounded down to the next integer. If the sum of the samples allocated this way is less than B, then the remaining samples are allocated using the uniform allocation approach.

4.3 Sequential Sample Allocation Procedure

This section describes the sequential allocation approach. In this approach, the decision-maker updates his beliefs about the attribute and decision values after each sample and uses the updated beliefs to determine which attribute and alternative should be sampled next. The approach includes $B+1$ stages. The first B stages determine, for each attribute and alternative, the expected impact of obtaining one more sample for that attribute and alternative. The impact is measured as the expected PCS. The attribute and alternative that yield the greatest expected PCS will be sampled in the next stage.

The stage-wise sequential sampling approach is graphically described in Figure 1. Stage 0 begins by evaluating the expected PCS for each possible new sample (the “Analyze PCS” step) and determining which one will yield the greatest expected PCS (the “Allocation decision” step). Stages 1 to $B-1$ obtain the desired sample and update the decision-makers beliefs about that attribute (the “Sample” step), repeat the Analyze PCS step with the updated beliefs, and make a new allocation decision. Stage B obtains the last sample and updates the decision-maker’s beliefs; then it determines the PCS for each alternative and selects the alternative with the greatest PCS (the “Selection decision” step).

Figure 1: Sequential sample Allocation



Bayesian estimation is used to define posterior distribution to describe the decision-maker's knowledge of the true attribute value, μ_{ij}, of attribute j of alternative a_i. Samples are allocated to attributes, the decision makers decision-maker's prior beliefs about μ_{ij} can be described by the prior distribution for the random variable M_{ij}. These random variables are independent. The random variable Z_i represents the uncertain value of ξ_i. $Z_i = \sum_{j=1}^{k} \lambda_j M_{ij}$. Let R_{ij} be the set of all possible values of M_{ij}. The prior probability distribution is represented by Equation (4.1).

$$P^r{}_{ij}\left(w_{ij}\right) = P\left\{M_{ij} = w_{ij}\right\} \forall\, w_{ij} \in R_{ij} \tag{4.1}$$

Let S_{ij} be the set of all possible values for Z_i, $z \in S_i$ iff $\exists$ $v_{i1} \in R_{i1}, v_{i2} \in R_{i2} \dots \dots v_{ik} \in R_{ik}$ such that $z = \sum_{j=1}^{k} \lambda_j\, v_{ij}$. The decision-maker's prior distribution for the decision value can be described by the distribution in Equation (4.2).

$$P^r{}_i(z) = P\{Z_i = z\} \tag{4.2}$$

$$P^r{}_{ij}(Z_i = z) = \sum_{w_{i1}\in R_{i1}} \sum_{w_{i2}\in R_{i2}} \ldots.. \sum_{w_{ik}\in R_{ik}} I(\sum_{j=1}^{k} \lambda_j w_{ij} = z) \prod_{j=1}^{k} P^r{}_{ij}(w_{ij})$$

(4.3)

Where $I()$ is the indicator function defined as $I(A) = I_A = I(A) = \begin{cases} 1 & if\ A\ is\ true \\ 0 & if\ A\ is\ false \end{cases}$ .The distribution is used in later to derive a selection rule and an allocation rule to maximize the probability that the decision-maker makes a correct selection.

The alternative with the largest decision value is the decision-maker's preferred alternative. Given the decision-maker's knowledge of each decision value (the prior equations described in Equation (4.4)), Equation (4.5) gives the probability, $\widehat{p_\iota}$, that alternative a_i has the largest decision value:

$$p_i^p = P(Z_i > Z_r \forall r = 1, \ldots, m; r \neq i) \tag{4.4}$$

If the decision-maker selects alternative a_i, let the *probability of correct selection (PCS)* be the probability that a_i has the largest decision value (Equation (4.5)).

$$PCS^p = p^p{}_i = P(\xi_i > \xi_r \forall r = 1, \ldots, m; r \neq i) \tag{4.5}$$

The decision maker who wants to maximize PCS^p will select a_i where $s = arg \max_i \hat{p}_i$. We refer to this procedure as *multinomial selection* because it is consistent with existing multinomial selection procedures Note that when developing OCBA, Chen and Lee [7] defined OCB in a manner similar to Equation (4.5) but suggested that alternative be selected according to their expected decision value

In stage $t = 0, \ldots, B - 1$, the approach analyzes the available information and identifies the alternative and attribute to sample in stage $t + 1$ (that is, the next sample is allocated to that alternative and attribute).

Let $\boldsymbol{x}_{ij}(t) = x_{ij1}, \ldots x_{ij(t)}$ be the data collected in stages $1, \ldots, t$ for alternative a_i and attribute j. Let $\boldsymbol{x_i}(t) = \boldsymbol{x}_{i1}(t), \ldots, \boldsymbol{x}_{ik}(t)$ and $\boldsymbol{X}(t) = \boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_m(t)$. We note that in any stage $t = 0, \ldots, B$, the probability, $p^p{}_i$, that alternative a_i has the largest decision value can be calculated (Equation (4.6)) and the alternative a_s where $p_s = \arg\max_i p_i^p$ and $p_s = \arg\max_i \widehat{p_\iota}$ identified. Thus, the PCS^p at stage t is as described in Equation (4.7).

$$p_s^p = \sum_{z_s \in S_s} P\{Z_s = z_s\} P\{Z_r \leq z_s, r = 1, \ldots, m; r \neq s\}$$

$$p_s^p = \sum_{z_s \in S_s} P\{Z_s = z_s\} \prod_{r \neq s} P\{Z_r \leq z_s\}$$

$$p_s^p = \textstyle\sum_{z_s \in S_s} P\{Z_s = z_s\} \prod_{r \neq s} \sum_{z_r \in S_r} P\{Z_r = z_r\} \qquad (4.6)$$

$$PCS^p(t) = \max_s p_s^p \qquad (4.7)$$

Because the probability distributions of the decision values are independent, the joint posterior probability distribution of $Z_1, \ldots, Z_m$ is the product of the individual marginal distributions.

To make the sample allocation decision at stage t, we note that the next sample, $x_{ij(t+1)}$, observed from alternative and attribute pair (a_i, j), will lead to a new PCS value. Although the value of the sample and the subsequent new PCS cannot be known until the observation is made, the probability distribution of each can be described based upon the decision-maker's current knowledge. The distribution of the new observation is described by its predictive distribution with density $P^o{}_{ij}(v|x(t)) = \sum_{\mu_{ij}=-\infty}^{+\infty} P_j(v|\mu_{ij}) P^r\left(\mu_{ij}|\boldsymbol{x}_{ij}(t\,)\right)$. For each of the mk alternative and attribute pairs, assuming that the selection is to be made using the multinomial selection approach, the expected PCS^p in stage $t\ +1$ if attribute j of alternative a_i is sampled can be calculated according to Equation (4.8)

$$E\left(PCS_{ij}^{p}(t+1)\right) = \sum_{v_{ij}\in R_{ij}} P_{ij}(v|x(t)) \left[\max_{s} \sum_{z_s\in S_s} P^{op}{}_{s}(z_s|X(t), v_{ij}) \prod_{r\neq s} \sum_{z_r\in S_r, z_r\leq z_s} P^{op}{}_{r}(z_r|X(t), v_{ij})\right] \tag{4.8}$$

The sequential allocation approach allocates the sample in stage t +1 to the alternative and attribute pair that yields the maximum $E\left(PCS_{ij}^{p}(t+1)\right)$. Upon collecting the final observation in stage B, the approach calculates the probability, p_i^p, that alternative a_i has the largest decision value according to Equation (4.3) and identifies the selected alternative, a_s , where $s\ =\ arg\ \max_i p_i^p$.

4.4 Summary

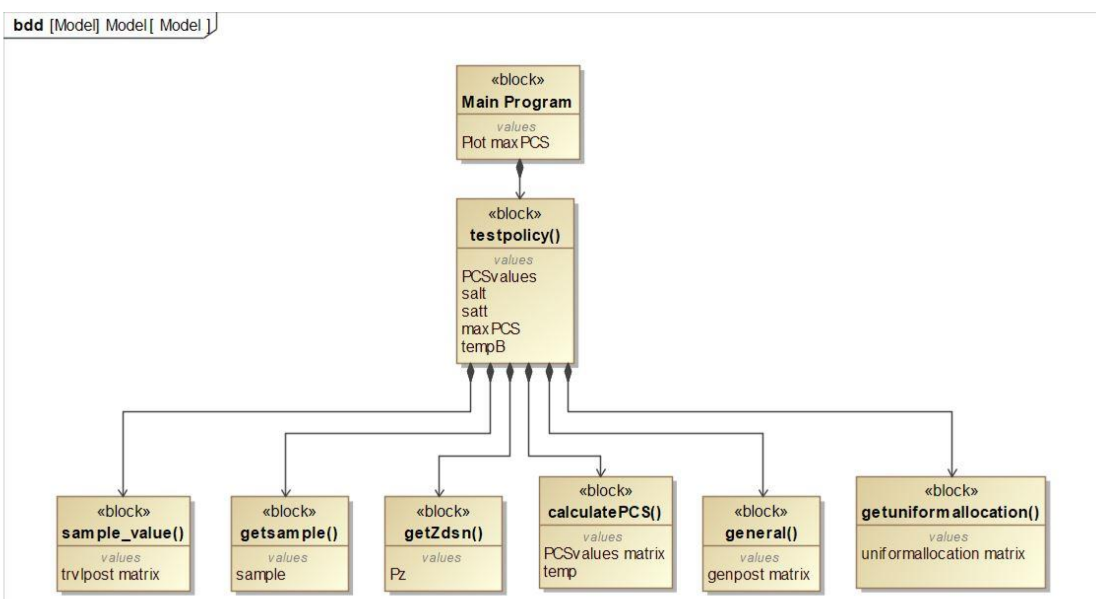
This chapter presented the sample allocation approaches that were tested in this study. Although the uniform and proportional allocation approaches are conceptually simple and are easy to implement, the sequential allocation approach requires extensive calculations at each stage because it must update the decision-maker's beliefs with each sample and then determine the expected outcome of the next sample. At each stage, this requires determining *mk* values of expected PCS. Each expected PCS calculation requires considering every possible value of that attribute and performing the updates and PCS calculations. Thus, the computational effort of the sequential allocation approach grows as the number of alternatives, number of attributes, and number of possible attribute values increase.

Chapter 5: Simulation Results

This chapter talks about the detailed simulations carried out for the purpose of this research. Section 5.1 explains the simulation approach including the *Design of Experiments (DOE)* carried for the purpose of this research. The results of the simulation study are explained in the Section 5.2 and results are summarized in Section 5.3.

The Block Definition Diagram (BDD) in Figure 2 shows the main program and how it is decomposed into its constituent functions. The diagram also shows the output coming out of each function as written in the MATLAB code. The code structure is such that the main function calls the *testpolicy* function which the executes the sample allocation according to the type of procedure selected by the decision-maker.

Figure 2: Simulation Block Definition Diagram



5.1 Simulation Approach

In this section we explain the approach used for simulating the allocation procedures using MATLAB. This section explains the detailed Design of Experiments carried for the purpose of this thesis research.

The experiment included four sets, each with 20 randomly-generated instances. Table 3 lists the factors that were varied across the sets. The following parameters were the same for each set: The set of possible values for every attribute was {1, …, 8}. The prior distributions for each attribute were uniform. The set of possible errors was {-2, -1, 0, 1, 2}. Within a set, the error distributions for different attributes were different, but the same error distributions were used in every set. (These are shown in Section 5.2.3.) The budget was set to B = 20, 50 and 100.

Generating a random instance required drawing values for every attribute for every alternative. For each one, a value was randomly chosen from the set {1, …, 8}. Every value was equally likely to be selected.

Table 3:Values of attributes over various Sets

Factors	Sets			
	s_1	s_2	s_3	s_4
Number of Alternatives (n_{alt})	3	5	3	5
Number of Attributes (n_{att})	2	2	5	5
Number of iterations (n_{itr}) (simulation runs)	100	100	10	10

W_iThe weight distributions for all the sets, is shown in the Table 4 and Table 5 respectively.

Table 4: Weight sets for 2 attribute sets

Weight set	λ_1	λ_2
W_1	5	5
W_2	9	1
W_3	1	9

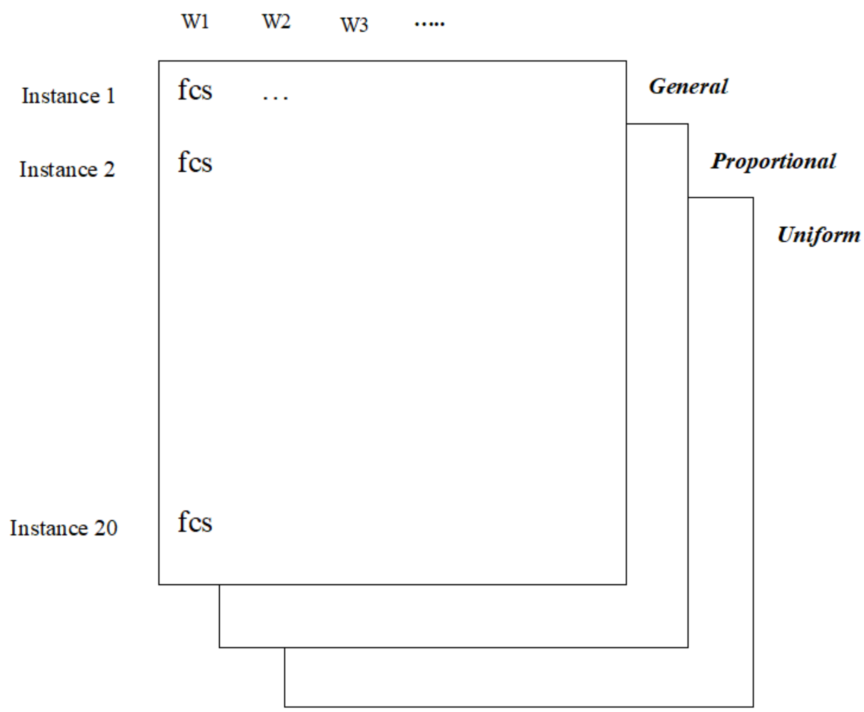
Table 5: Weight sets for 5 attribute sets

Weight set	λ_1	λ_2	λ_3	λ_4	λ_5
W_1	4	4	4	4	4
W_2	12	2	2	2	2
W_3	2	2	2	2	12
W_4	2	2	12	2	2
W_5	2	7	2	7	2

5.2 Results of Simulation

This chapter explains the results of simulations carried out for testing each of the allocation strategy. The simulations were carried out in a methodical way as shown in Figure 3.

Figure 3 ; Simulation study of frequency correct selection (fcs)



As shown in Figure 3, the simulations were developed for multiple instances for each set of values. Each combination set, s $(s_1, s_2, s_3 \dots s_n)$ is a combination value of values of the alternative-attribute pair. All these instances then were computed for a specific set of weights, $W\,(\lambda_1, \lambda_2, \dots .\lambda_n)$ for the attributes shown in Table 4 and Table 5. Each simulation is run over an instance of each s_i and then eventually carried out for all the instances of the set. The Probability of Correct Selection (PCS) was calculated for each of the alternative a_i for every stage from 0 to B. The *PCS* was plotted for each of the alternatives for every stage and for every instance for an instance of set s_i. For the first simulation study, the PCS was calculated for the range of input values as mentioned in Table 4. Through this study an evaluation of the variation of the PCS for each of the alternative for different true value combinations was developed. Another simulation deduction was to calculate the *frequency of correct selection (fcs)* for every separate value of μ for different decision cases. The results of these studies are discussed in the sections 5.2.1 and section 5.2.2 of this thesis.

5.2.1 Probability of Correct Selection (PCS)

The probability of correct selection is the probability that the true best alternative was selected at each stage of the allocation procedure. The probability is computed for a finite experimental budget which was B=10 and 50. The sample allocation was done using a technique mentioned in each of the three allocation procedures. For the simulation, the PCS values are calculated over each step of the budget, B starting for the initial stage 0 till stage B and the results are represented graphically for various λ_j (true values) for the attributes. The plots of the PCS demonstrated that the PCS over the stages goes up for the alternative which has the best decision value which is the sum over all the $\lambda_{ij} * \mu_{ij}$ over the alternatives. The different weight distributions W_i considered for the purpose of the simulation for 2 attribute case and 5 attribute case are mentioned in Appendix A.

However, there were certain cases which showed unexpected behavior. The alternative attribute pairs which had the equal decision values for the Sequential allocation are show in the Figure 4 and Figure 5.

Figure 4: PCS: Weight 1, Instance 6, Sequential Allocation

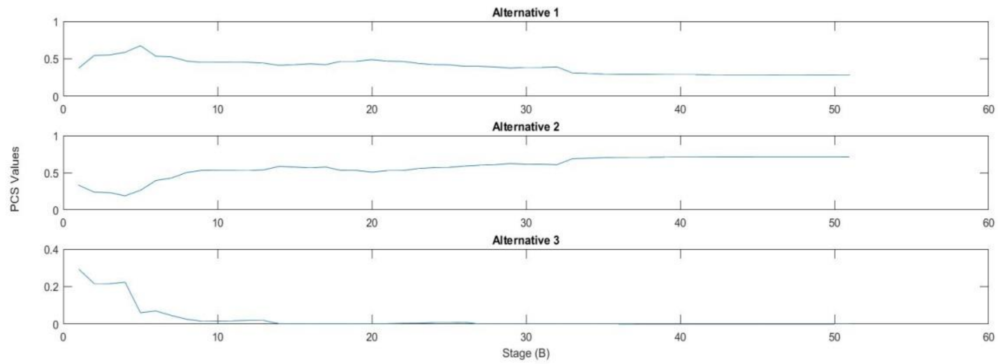


Figure 5: PCS: Weight 4, Instance 5, Sequential Allocation

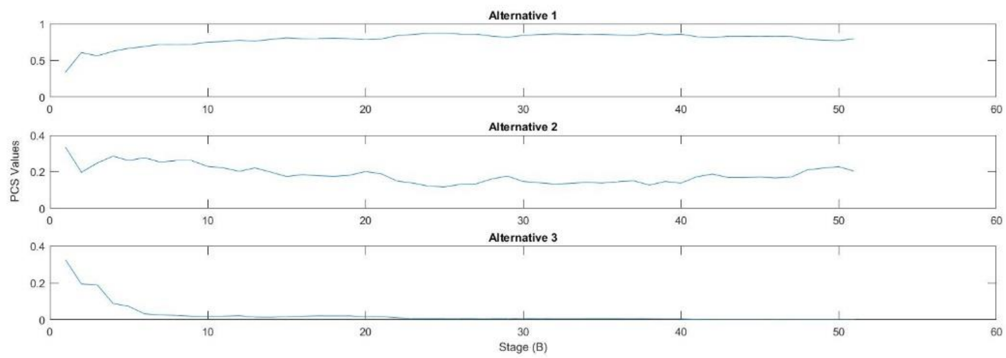


Figure 4 shows the plot of the Probability of Correct Selection (PCS) for Sequential sampling for $W_1 = (\lambda_1, \lambda_2) = (5,5)$. Figure 5 shows the results for the Sequential sampling for the 3 alternative-5 attribute case where $W_4 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (2,2,12,2,2)$. The decision values as created in the simulation for Figure 5 were (50, 50, 32) for the instance 5. Ideally the plots should converge to the same value for the alternatives having the same decision value as all of these alternatives will be the "true best" alternatives. But since we assume that the decision maker has to make a selection, the algorithm selects the earlies possible alternative as the true best alternatives for the sampling in the next stage. This explains the PCS of Alternative 1

(50) being a little higher in figure 6 as compared to the Alternative 2 (8), even though both the alternatives have the exact same true value.

Figure 6: max PCS for all the 20 instances for W1

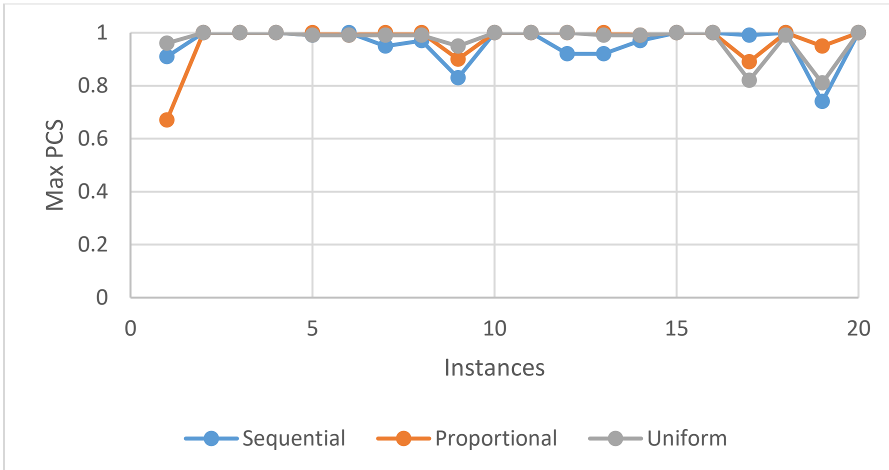


Figure 7: max PCS for all the 20 instances for W2

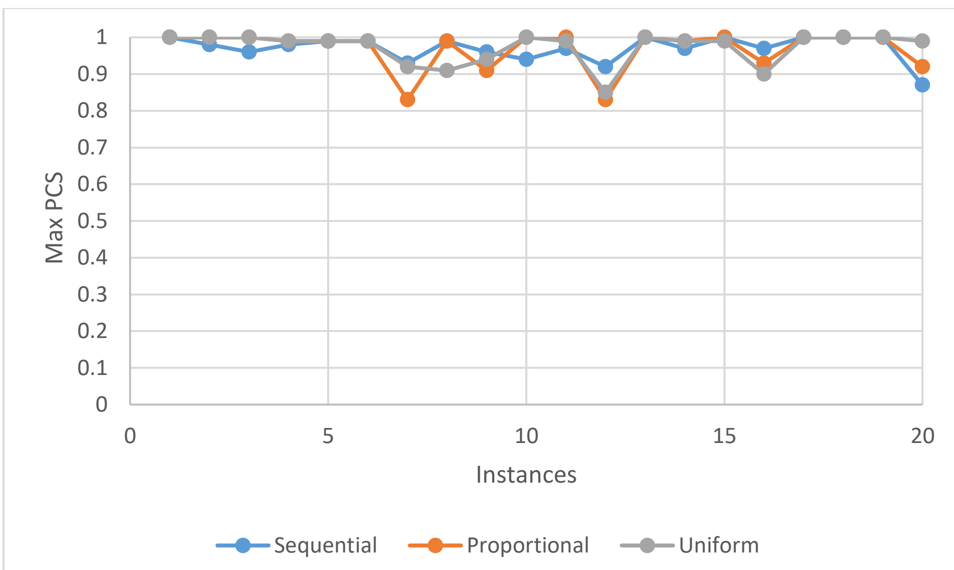
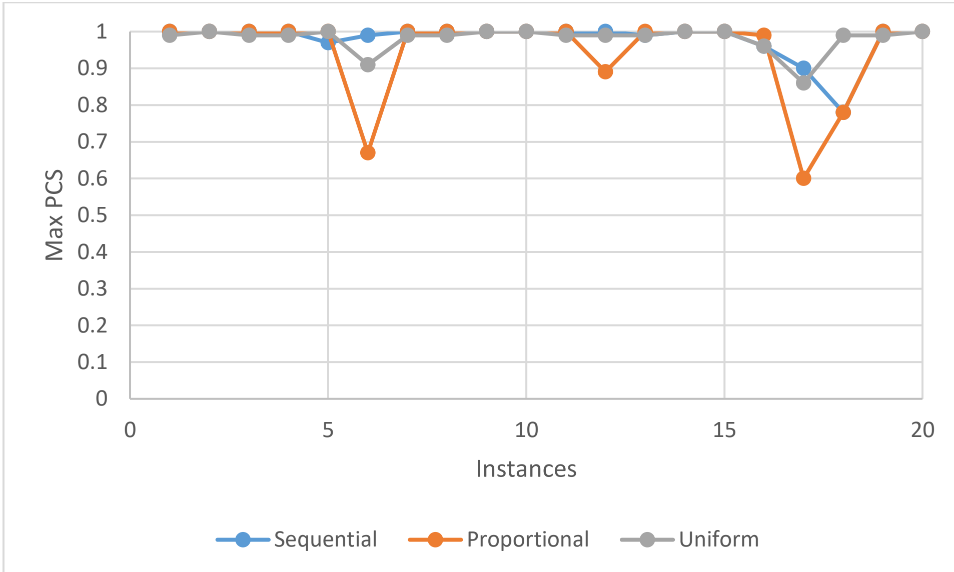


Figure 8: max PCS for all the 20 instances for W3



The analysis of the performance (PCS) of the three different strategies was done for all the instances and the PCS was examined for all the different weights of a 3 attribute 2 alternative set up. Although, we see that all the strategies perform in a very similar manner, the Sequential strategy has the best PCS values. The average PCS was calculated for four different sets of values:

- s_1 : 3 alternatives, 2 attributes
- s_2 : 5 alternatives, 2 attributes
- s_3 : 3 alternatives, 5 attributes
- s_4 : 5 alternatives, 5 attributes

Table 6: Average of max PCS values for different Sets

	Sets			
Policy	s_1	s_2	s_3	s_4
Sequential	0.97	0.96	0.97	0.96
Proportional	0.95	0.91	0.94	0.95
Uniform	0.94	0.92	0.90	0.88

5.2.2 Frequency of Correct Selection (*fcs*)

Frequency of correct selection (*fcs*) is the performance parameter of each of the allocation strategy for a finite allocation budget. The fcs, a fraction in the interval [0, 1], is the number of replications that the sample allocation approach led the decision-maker to select a true best alternative divided by the total number of replications. The fcs was calculated for each sample allocation approach, set of instances, and set of weights.

The *fcs* is calculated for each of the allocation procedures as mentioned in section 4. The results are plotted for these standard set of weights, *W*. The fcs give a view of the performance of each allocation procedure over the sets of instances, *Si*.

Figure 9: Average fcs over W_i (i = 1 to 3)

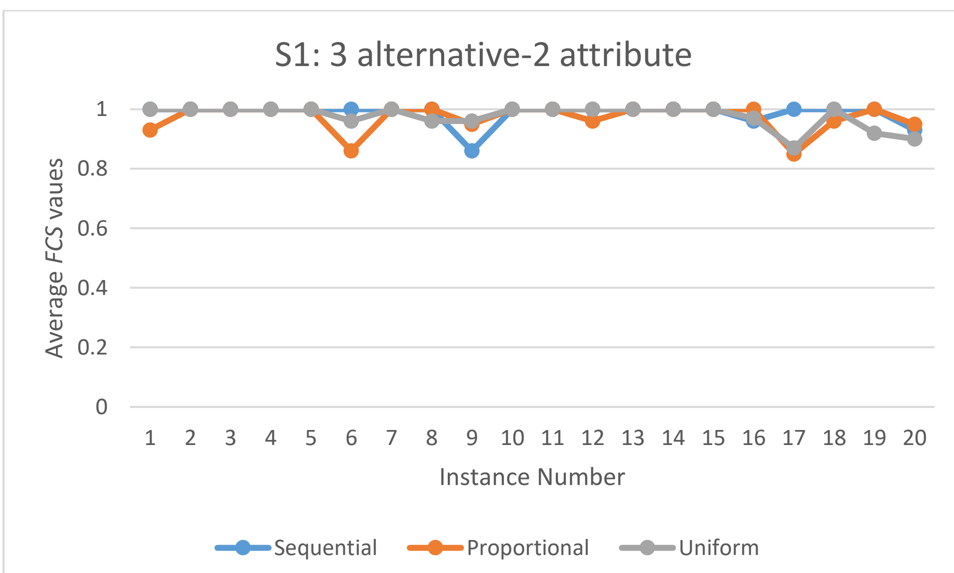


Figure 10: Average fcs over W_i (i = 1 to 5)

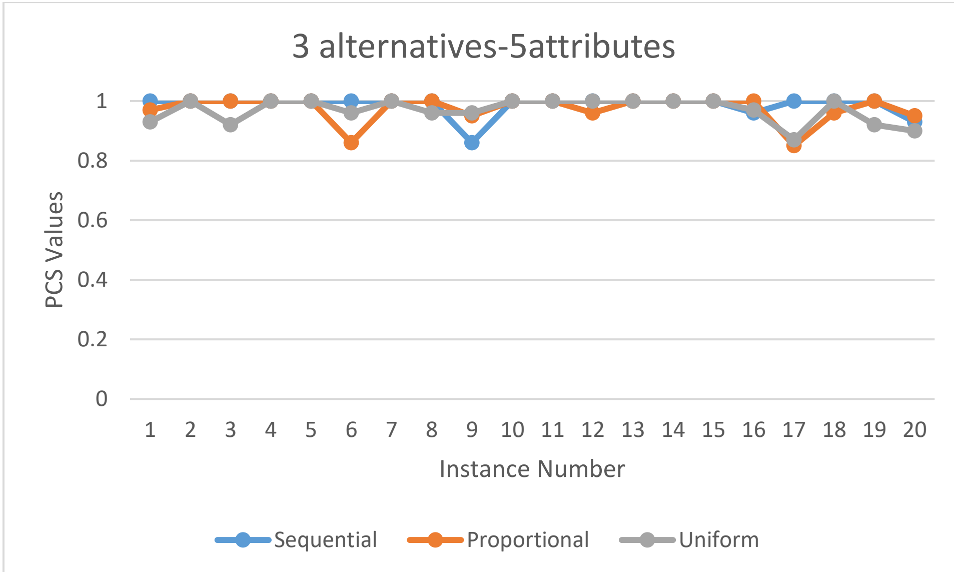


Figure 11: Average fcs over W_i (i = 1 to 5)

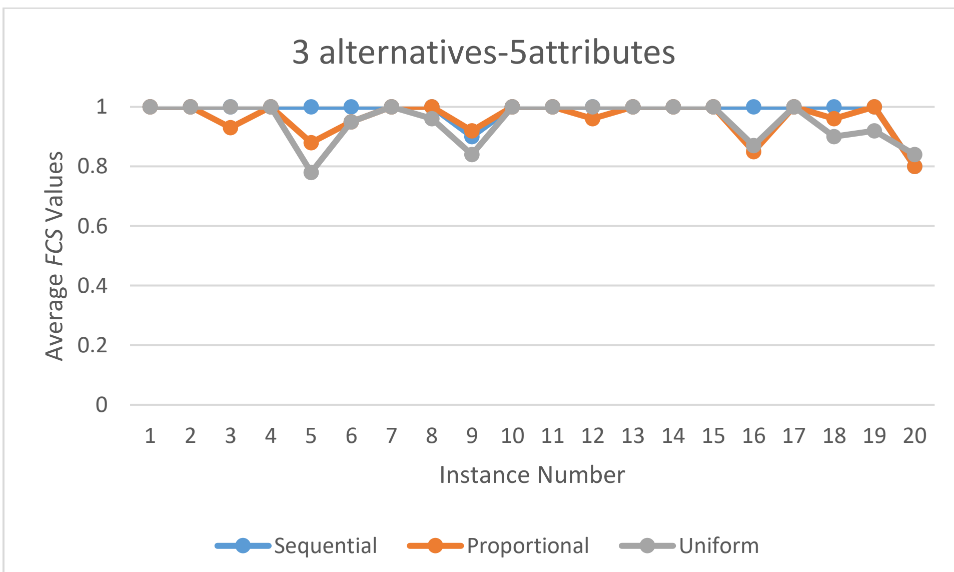
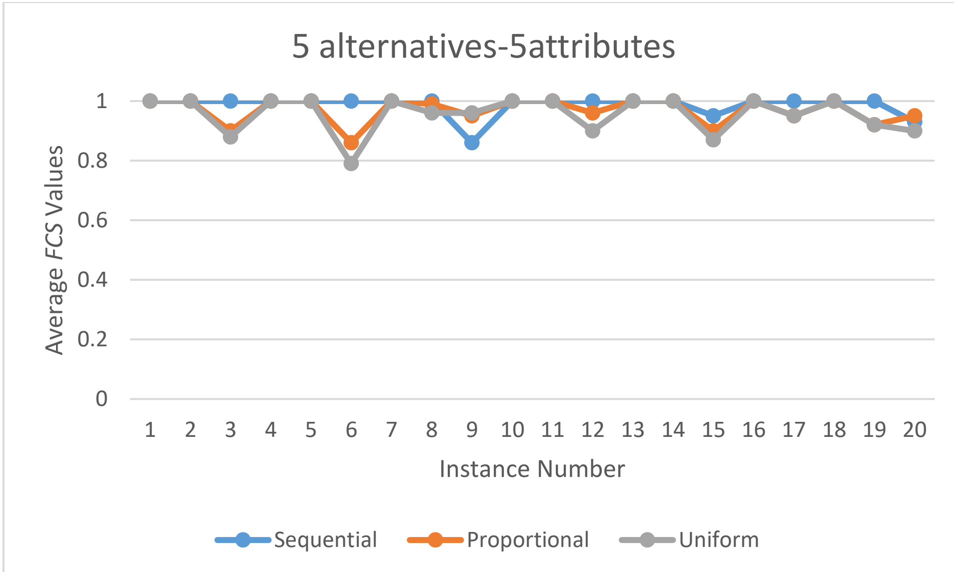


Figure 12: Average fcs over W_i (i = 1 to 5)



The fcs values for all the simulation cases: S_1, S_2, $S_3$and S_4 was averaged over all the iterations. These *fcs* values were averaged for each policy to get a final fcs for every policy for every case. Finally, the fcs for every policy for every case to get a fraction which is the *fcs* for every policy taking into consideration all the simulation cases and iterations for every simulation case. Tables 11, 12 and 13 shows the final fcs denoting the performance of sequential allocation as compared to the other allocation strategies for a 95% confidence interval (using the equations in Appendix D) for all the sets for B=20, 50 and 100 respectively. Due to computation difference, there were 6000 trials done for sets $s_1$and s_2 which had 2 attributes and 1000 total trials for sets $s_3$and s_4 which had 5 attributes which becomes evident on analyzing the confidence intervals for all the sets. Because the number of trials are greater for $s_1$and s_2, the confidence intervals have smaller values whereas the intervals are bigger for $s_3$and s_4.

Table 7: Average fcs for B=20 over all the simulation procedures

Allocation Strategy	**95% Confidence Interval *fcs***			
	s_1	s_2	s_3	s_4
Sequential	0.991 ± 0.0023	0.989 ± 0.0025	0.988 ± 0.0067	0.987 ± 0.0072
Proportional	0.944 ± 0.0058	0.951 ± 0.0054	0.954 ± 0.0128	0.953 ± 0.0131
Uniform	0.897 ± 0.0076	0.902 ± 0.0075	0.886 ± 0.0196	0.895 ± 0.0185

Table 8: Average fcs for B=50 over all the simulation procedures

Allocation Strategy	**95% Confidence Interval *fcs***			
	s_1	s_2	s_3	s_4
Sequential	0.987 ± 0.0028	0.979 ± 0.0036	0.980 ± 0.0112	0.983 ± 0.0102
Proportional	0.969 ± 0.0043	0.964 ± 0.0047	0.971 ± 0.0132	0.965 ± 0.0143
Uniform	0.947 ± 0.0056	0.952 ± 0.0054	0.948 ± 0.0177	0.955 ± 0.0165

Table 9: Average fcs for B=100 over all the simulation procedures

Allocation Strategy	**95% Confidence Interval *fcs***			
	s_1	s_2	s_3	s_4
Sequential	0.986 ± 0.0029	0.989 ± 0.0026	0.980 ± 0.0112	0.986 ± 0.0091
Proportional	0.982 ± 0.0033	0.983 ± 0.0032	0.978 ± 0.0116	0.982 ± 0.0102
Uniform	0.979 ± 0.0036	0.980 ± 0.0035	0.976 ± 0.0120	0.982 ± 0.0102

The difference in the accuracy of the intervals is due to the number of trials taken for each of the set. It was concluded that the sequential allocation procedure better than proportional and uniform with uniform allocation being the strategy with least *fcs* for a 95% confidence interval.

The variation in the *fcs* based performance is significantly reduced as the budget increases. Even the simpler sample allocation approaches are sufficient if the decision-maker has large enough Budget to allot significant number of samples for each attribute of all the alternatives.

5.2.3 Discrete Error Distributions

The error distributions are assumed to be discrete triangular distributions for the experiments carried out for this thesis research. The range of the error varies from -E_{max} to +E_{max} with the discrete values. The error distributions for the sets with 2 attributes and 5 attributes are shown in Figure 15 and Figure 16 respectively.

Figure 13: Discrete Error Distributions for 2 attribute sets

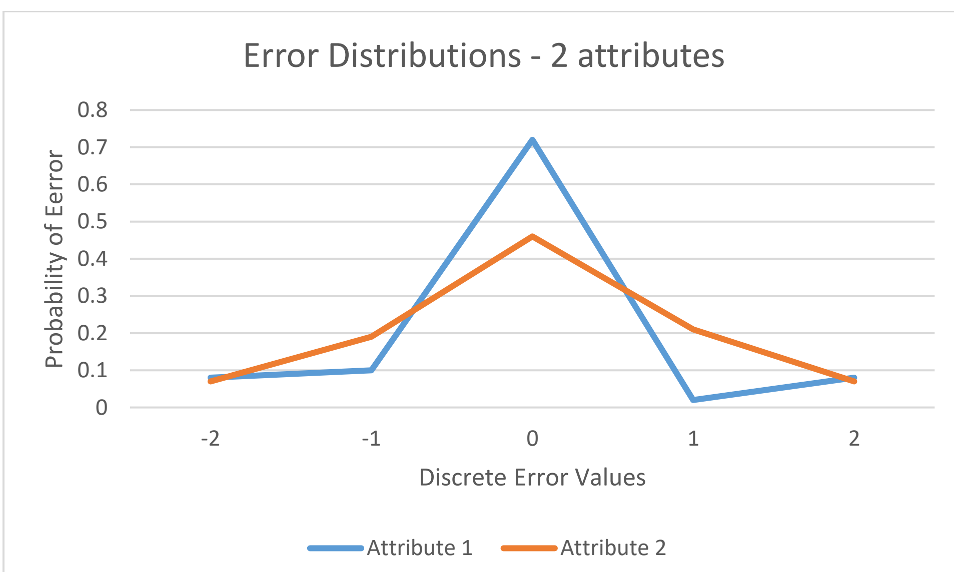
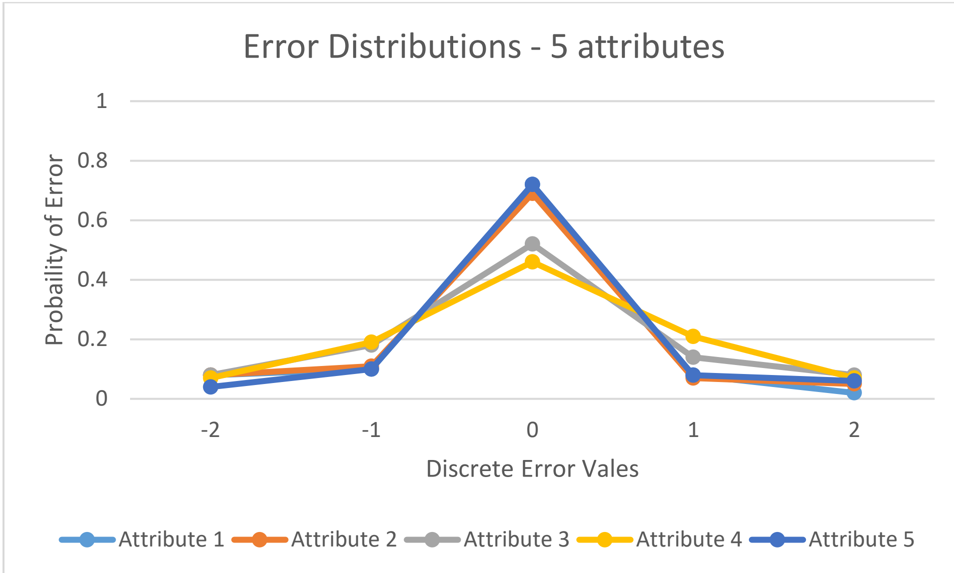


Figure 14: Error Distribution for 5 attribute sets



5.3 Discussion

The performance of each sample allocation approach was measured as the frequency of correct selection (how often using that approach led the decision-maker to select the true best alternative). The sequential allocation approach performed significantly better than the uniform and proportional allocation approaches when the budget was only 20 or 50 samples. When the budget increased to 100 samples, all three allocation approaches had the same performance. The performance of the sequential allocation approach remained the same as the budget increased, but the performance of the uniform and proportional allocation approaches improved as the budget increased.

There is certain trade-off in selecting these strategies for selecting an alternative. The Sequential allocation approach is a more reactive approach as the allocation done in each stage is dependent on the results of allocation of the previous stage. This makes this strategy computationally more strenuous, but it should perform better when the budget is small. The other two approaches are not so reactive are the allocation procedure is decided in advance based on the attribute weights and the order of sampling. The proportional and the uniform allocation strategies are better computationally and perform well when the budget is large.

Chapter 6: Conclusions, Contributions, and Future Work

This thesis studied the problem of gathering information for a multi-attribute decision problem. The study considers instances of varying sizes. The research focused on how the allocation under finite budget is done for various allocation procedures for a general discrete distribution. The sequential allocation procedure uses Bayesian estimation to update the decision-maker's beliefs, which are used to estimate the value of the next sample, which is used to determine which attribute to sample. The proportional allocation strategy allocates the samples based on the proportion of weight of each of the attribute of the alternative. The uniform allocation performs the allocation without any prior knowledge about the attributes and their value probabilities by evenly diving the total number of samples among the alternative-attribute pairing.

The contributions of this thesis research can be summarized as follow:

1. This research presented a sequential sampling approach for allocating samples to the alternative-attribute combinations based on the beliefs (prior distribution) of the decision-making. The attribute value and error measurement distributions were propagated as a set of discrete probabilities. It was also concluded by a set of statistical testing that on selecting the alternative that has the maximum probability, the sequential allocation strategy gives the best results for selecting the alternative with the best *Probability of Correct Selection (PCS)*. The baseline for analyzing the *Sequential Allocation Procedure* and *Proportional Allocation Procedure* strategies was the *Uniform allocation strategy*.

2. A set of allocation procedures were analyzed for different instances having randomly generated true values as the measures of the true values of the attributes. There was evidence provided for the performance of the three allocation procedures by comparing the *frequency of correct selection (fcs)*. The sequential allocation approach, which uses the samples obtained to determine the next allocation, performed better than the proportional and uniform allocation approaches.

Although these results have answered the research questions that motivated this study, much additional work is needed to determine the best ways to gather information in this domain.

As part of the future work, it would be interesting to streamline the computations required for the sequential allocation strategy in order to reduce its computational effort. A hybrid allocation approach could use the proportional approach to allocate some of the samples and the sequential approach to allocate the remaining ones. This would help the decision maker in having a smaller budget for making decisions and having a higher accuracy than spending the entire budget on either *Uniform* or *Proportional Allocation Procedure*.

Approaches for guiding information gathering activities in other domains could be adapted for the multi-attribute decision problem. Possible approaches include the knowledge gradient and expected improvement techniques used in optimization [15], the most-starving OCBA algorithm [8], and approaches from multiple-objective simulation optimization [22].

Appendix A: Attribute True Values Instances

Table 10: 20 instances for 3 alternatives 2 attribute case

Instance	Attribute values
1	7 6 5 2 1 2
2	4 7 2 4 7 8
3	7 5 5 1 2 8
4	7 3 4 1 5 7
5	7 6 6 5 7 1
6	7 6 5 2 1 2
7	7 3 2 1 4 6
8	4 7 4 6 2 1
9	7 4 4 8 1 5
10	5 3 6 8 4 8
11	1 8 8 4 6 4
12	8 3 8 3 4 4
13	2 1 1 2 8 6
14	3 1 7 6

	6 6
15	8 4 8 2 3 1
16	5 3 2 5 3 6
17	1 2 6 8 6 1
18	2 2 2 1 8 3
19	7 1 8 5 7 2
20	5 8 8 6 4 1

Table 11: 20 instances for 5 alternative 2 attribute case

Instance	Attribute values
1	5 4 5 3 4 4 8 1 3 7
2	8 4 2 8 4 7 1 2 7 3
3	1 6 1 6 7 4 3 8 2 8
4	2 8 3 3 5 3 8 5 1 3
5	5 7 3 8 2 4

	4 2 2 6
6	3 6 3 7 2 1 8 1 2 8
7	8 6 8 2 1 1 8 4 2 1
8	2 8 2 6 2 1 7 1 6 7
9	1 2 7 7 3 2 7 6 1 1
10	7 1 4 1 3 8 7 5 4 3
11	1 2 4 3 2 3 4 1 4 5
12	2 5 2 1 2 1 1 7 7 6
13	1 6 2 7 7 8 7 2 2 5
14	1 1 5 5 4 8 4 7

	1 8
15	2 6 3 4 7 6 1 2 5 2
16	7 5 1 7 2 6 7 7 4 6
17	4 6 1 1 3 7 2 6 6 8
18	2 4 2 3 5 6 4 6 5 3
19	1 2 6 1 4 5 3 1 1 2
20	3 5 1 4 8 6 6 3 6 1

Table 12: 20 instances for 5 alternative 5 attribute case

Instance	**Attribute values**
1	7 1 2 2 6 8 3 8 4 1 2 5 8 8 7 8 8 4 7 8 6 8 7 8 6
2	7 6 7 4 4 6 1 6 4 4 4 3 3 7 6 6 1 8 7 6 2 1 1 8 7
3	3 4 7 8 7 6 8 3 5 3

	6 3 5 2 7 2 5 6 2 2 1 2 8 3 8
4	2 5 4 7 1 7 3 1 5 8 3 6 2 8 1 5 6 8 1 7 2 6 2 4 7
5	7 4 2 7 1 1 8 7 5 2 4 2 5 3 1 3 3 5 5 2 7 2 2 4 2
6	7 4 2 7 1 1 8 7 5 2 4 2 5 3 1 3 3 5 5 2 7 2 2 4 2
7	4 4 7 2 2 1 3 4 8 3 8 8 2 8 7 8 3 4 5 1 4 1 1 1 1
8	2 5 2 8 3 6 3 3 7 5 6 6 6 4 5 6 2 7 4 7 4 6 1 4 7
9	6 8 2 2 3 4 8 3 2 8 7 5 4 2 4 5 5 2 2 2 3 5 7 4 8
10	8 5 1 1 6 4 3 3 3 4 1 5 3 7 5 3 6 4 1 2 4 2 5 8 4
11	8 5 1 3 1 5 6 8 3 6 5 4 8 6 4 2 3 7 2 7 4 8 1 6 6
12	8 1 5 2 2 8 6 5 2 8 3 5 7 8 6

	6 4 7 1 5 2 8 5 4 4
13	1 1 6 4 2 6 7 5 4 4 1 7 8 7 7 1 6 6 1 7 5 2 7 2 1
14	4 3 1 8 4 5 4 3 8 5 4 1 2 1 8 6 8 4 6 4 6 2 3 3 8
15	3 6 1 3 6 6 2 5 4 4 6 2 8 8 2 5 8 6 2 4 6 2 2 7 4
16	1 3 8 8 1 5 3 6 8 4 2 5 3 7 1 4 3 5 3 2 5 7 1 5 2
17	4 6 5 8 7 1 6 6 5 4 5 1 4 3 1 4 1 7 1 3 6 3 6 5 2
18	3 5 6 3 7 4 8 3 2 6 5 6 6 6 1 4 8 6 7 5 8 2 1 3 4
19	8 7 2 5 3 1 3 6 2 7 4 7 4 6 2 4 4 2 2 3 4 1 3 8 1
20	5 6 6 4 4 6 6 2 6 7 5 6 1 7 7 5 8 5 3 3 6 2 4 6 5

Table 13: 20 instances for 5 alternative 2 attribute case

Instance	Attribute Values
1	5 7 8 3 8 8 6 7 7 3
2	2 5 2 8 3 2 1 3 5 3
3	2 3 3 6 5 4 5 4 5 5
4	6 2 5 3 4 8 1 3 1 2
5	8 1 8 4 7 1 8 7 5 6
6	8 8 1 6 5 1 8 2 6 8
7	4 1 1 7 3 1 4 1 5 7
8	2 1 5 5 6 5 7 1 8 7
9	8 5 8 7 7 3 5 3

	7 8
10	1 4 8 4 6 6 7 6 8 1
11	8 1 8 3 5 2 2 4 3 5
12	8 2 1 1 4 7 1 2 1 2
13	8 3 2 5 7 6 1 3 7 5
14	3 8 2 1 8 7 1 4 2 8
15	6 3 8 3 2 4 6 5 2 5
16	6 1 2 2 4 2 6 1 6 6
17	6 5 3 1 2 1 2 7 8 5
18	6 8 8 7 8 5 2 6 8 4

19	2 1 6 8 4 4 3 3 6 7
20	3 5 5 2 6 4 5 4 3 4

***Note**: The rows denote different attributes and the columns represent different alternatives.*

Appendix B: Creating Z-distributions

Z values are the values that belong to the set S_{ij} of all the decision values (ξ_i) such that z ϵ S_i. The values are used to calculate the probability of occurance of each of the decision values in the set as given in the Equation A.1.

$$P^r{}_i(z) = P\{\xi_i = z\} \tag{A.1}$$

The code for this simulation which generated the P(z) was simulated in MATLAB using a function *getzdsn2()* is shown below:

```
%function to get Z distribution
function [Pz]= getzdsn2(n_att,n_alt,W,pa,Vmax,Zmax)

% getzdsn2 created 3-23-3018 by Jeffrey W. Herrmann

% this function calculates prob. dsn. for Z values for every alternative

% Z_i = sum W_j * a_ij

% INPUTS
% n_att = number of attributes
% n_alt = number of alternatives
% W = weights to combine attributes
% pa = prob. dsn. for attributes
% Vmax = max value of attribute (1 to Vmax)
% Zmax = max value of Z

Pz = zeros(n_alt,Zmax); %initializing  z matrix  P{Z_i = z}
npoints = Vmax^n_att;  % total number of combinations of attribute values
attrv = ones(n_att,1);  % all attributes start at 1

for np=1:npoints % loop over combinations of attribute values
    z = W*attrv; % evaluate z = sum W_j * a_ij
    if z <= Zmax % if z is feasible
        for j=1:n_alt %iterating over alternatives
            pa_temp=1;
            for k=1:n_att %iterating over attributes
                pa_temp = pa_temp*pa(j,attrv(k),k); %updating the pa matrix
            end
            Pz(j,z) = Pz(j,z)+pa_temp; %updating the z distribution matrix
        end % end of iterating over all the values of z
    end
    % go to next combination of attribute values
    for ai=n_att:-1:1  % loop over attributes from last to first
```

```
        if attrv(ai) < Vmax  % if this attribute < Vmax
            attrv(ai) = attrv(ai) + 1;  % increase
            break % exit loop over attributes
        else
            attrv(ai) = 1; % reset to 1 and go to next attribute
        end
    end
end % loop over combinations of attribute values
end
```

Appendix C: Calculation the Probability of Correct Selection (PCS)-distributions

For computing the PCS for the sequential allocation case, a MATLAB function was developed which the calculated the PCS equation written in Equation B.1:

$$PCS^p(t) = \ P(\xi_s > \xi_{\ r}, \ \forall_r = 1, \dots, m; r \neq s \mid \boldsymbol{X}(t)) = \tag{B.1}$$

$$\left[\max_s \sum_{z_s \in S_s} P^o{}_s(z_s|X(t)) \prod_{r \neq s} \sum_{z_r \in S_r, z_r \leq z_s} P^o{}_r(z_r|X(t))\right]$$

Where PCS^p being the probability of Correct Selection (PCS) for the Sequential Allocation Strategy. The MATLAB function that does this is shown below:

```
%function to get PCS values for alternatives
function [PCSvalues,Pz] = calculatePCS(n_alt,n_att,Zmax,W,pa,Vmax)
PCSvalues = zeros(n_alt,1);
Pz = getzdsn2(n_att,n_alt,W,pa,Vmax,Zmax);
Cdfz = zeros(n_alt,Zmax); %initializing the cdf matrix
Cdfz = cumsum(Pz,2);  % CDF for all the alternatives

% ASSUMPTION: in case of tie values, the smaller-numbered alternative is...
...selected.

% if all Z = 0, then alternative 1 is selected
PCSvalues(1) = prod(Cdfz(:,1));  % multiply all P{Za <= 1}
%Calculation of PCS for each alternative
temp = zeros(n_alt,Zmax);
for i=2:Zmax %iterating over the values of Z
    for j = 1:n_alt %iterating over alternatives
        Cdf_temp=1;
        for k = 1:n_alt %iterating over alternatives
            if k < j
                Cdf_temp = Cdf_temp*Cdfz(k,i-1); %the cummulative
distribution function for decision values for lower-numbered alternatives
            elseif k > j
                Cdf_temp = Cdf_temp*Cdfz(k,i); %the cummulative distribution
function for decision values including i for higher alternatives
            end %end of loop over alternatives
        end %end of loop over alternatives
        PCSvalues(j) = PCSvalues(j) + Pz(j,i)*Cdf_temp; %updating the
PCSvalues matrix
```

```
        temp(j,i) = PCSvalues(j); %updating temp to plot
    end
end
end
```

Appendix D: Approximate (100-α) Confidence Intervals

For a large number of runs (n), and "significance level" (α), the approximate confidence interval is given by the equation:

$$\frac{y}{n} \pm Z_{\alpha/2}\sqrt{\frac{(\frac{y}{n})\mathrm{x}(1-\frac{y}{n})}{n}} \qquad \text{(D.1)}$$

Where,

- $\boldsymbol{Z_{\alpha/2}}$: z-statistic value for $\alpha/2$ from the z-statistic table
- **y** : Number of successes
- **n** : Number of trials

For the calculations done in Table 11, α=0.05 for a 95%confidence interval. The Z statistic value, $Z_{\alpha/2}$ = 1.96 and *y* is the number of times the fcs was 1; *n* is the total number of times the fcs was calculated.

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