ABSTRACT

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Inspired by [1, 2], modern general equilibria under uncertainty are modeled based on the recognition that all risks cannot be eliminated, perfect hedging is not possible, and some risk exposures must be tolerated. Therefore, we need to define the set of acceptable risks [2] as a primitive of the financial economy. This set will be a cone, hence the word conic. Such a conic perspective challenges classical economics by introducing finance into the economic models and enables us to rewrite major chapters of classical micro- and macro-economics textbooks [3,4]. The classical models dictate that economic players are able to trade the whole of their endowments at what is known as a market-clearing price and direct all proceeds to the consumption of goods and services. According to these models, the aggregate consumption does not exceed the total endowment, suggesting that finance is not a necessary component in the economy. This work proposes a case in which some gap occurs between the aggregate supply and demand whereby the financial primitives cover the aforementioned gap. This also generates a bid-ask spread at equilibrium depending on the cone of acceptable risks [2]. This work questions the

traditional law of one price and poses a direct challenge to Adam Smith's "invisible hand" theory. Since the housing crisis in 2008, economists and statisticians have questioned the law of one price (see e.g., [5]). The implications of this academic debate are sweeping and affect players at all levels of the economy. Though we spend little time on empirical applications, the perceived empirical failures of the standard complete markets general equilibrium model stimulated the development of this work. For example, the standard complete markets model has the following empirical problems: (1) there is too much correlation between individual income and consumption growth in micro data (see e.g., [6, 7]); (2) the equity premium is larger in the data than is implied by a representative agent asset-pricing model with reasonable risk-aversion parameter (see e.g., [8]); and (3) the risk-free interest rate is too low relative to the observed aggregate rate of consumption growth (see e.g., [9]). There have been numerous attempts to explain these puzzles by altering the preferences in the standard complete markets model (see e.g., [10]). Alternatively, one might as well abandon the complete markets assumption and replace it with some versions of either exogenously [11] or endogenously [12] incomplete markets. However, this work takes a totally different approach in the sense that the classical complete markets models will be sub-cases of our conic framework.

CONIC ECONOMICS

by

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Preface

The classical thinking is that all the financial system does is to operate as an intermediary, bringing borrowers and lenders together, but the size of the economy is not affected by the financial system. In a classical complete markets model, as a direct consequence of the fundamental theorems of welfare economics, the financial system is irrelevant to the actual economy. In fact, if all one wants to prove is that a free market system can deliver what a central planner can mathematically achieve, then one should operate under the rules of the central planner. Endowments, preferences, and the technology is the economy, and finance has to be irrelevant. Of course, what we show in this work is that the financial market configurations have real welfare effects and that a free market system can do better than a central planner because a conic economy is ready to take extra risk. Now, of course, you could get into the troublesome situation of being at a very high aggregate welfare level, like in 2007 when the financial system was so forgiving that 2008 came and knocked the whole system out. This happened because the financial market was simply too forgiving in 2007. A conic perspective towards the economy enables us to ask some very serious questions; How forgiving should the financial system be? In other words, how much risk taking should be permitted? Who or what determines the financial market configurations? In other words, is there a way to write an endogenous model of the financial market? What role can the government play in managing the financial market and therefore the size of the economy? Simply because of the implicit assumption of the irrelevance of the financial market to the actual economy, these fundamental questions have been totally ignored and are still open, yet to be answered. In fact, these are questions of managing the capital requirements and leverage of an economy and should be addressed by the Federal Reserve. Basically, in a free market environment, we say that we don't want humans, governments, or committees to make the decisions. All decisions should be made in the market, but the market has its own incentives. Perhaps an example from the airline industry can help clarify these incentives. The point is that the Federal Aviation Administration (FAA) decides the weight of a plane that can take off. United Airlines does not make a decision about the weight of its aircraft at take off. The rules for that should be set by the FAA, because airlines have an incentive to allow more weights. However, there is a science behind how much weight a plane can tolerate, and the FAA relies on that science. Similarly, for the economy in general and the Federal Reserve in particular to manage leverage and determine how much capital backing it needs for risky positions, we need to have a science of leverage. However, we can not have a science of leverage, if we assume that we live in complete markets and there is no risk. Because then we don't need capital and so we will never figure out leverage policy listening to classical economics since it has already assumed it away. Fortunately, the Basel Committee are making recommendations on capital requirements but this is all done outside of professional economics.

Dedication

To my family, Mahdieh, Parvaneh, Mehdi, Cyrus, and Dorsa.

Acknowledgments

I owe my gratitude to all the people who have made this dissertation possible. First and foremost, I would like to thank my advisor, Professor Dilip Madan, for being a constant source of inspiration throughout this work. It has been a great pleasure to work with and learn from such an extraordinary individual. Moreover, thanks are due to Professor Luminita Stevens, Professor Mark Loewenstein, Professor Emel Filiz-Ozbay, and Professor Eric Slud for agreeing to serve on my dissertation committee and for sparing their invaluable time reviewing the manuscript. I would also like to acknowledge the fact that this dissertation would have been a distant dream without the existence of a truly interdisciplinary program; i.e., AMSC (Applied Mathematics & Statistics, and Scientific Computation). In addition, I would like thank the Mathematics, Economics, and Finance community at UMD for warmly accepting me as an outsider and giving me the invaluable opportunity to learn from them.

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Chapter 1: Conic general equilibrium

Building upon [1, 13] and based on the recognition that all risks cannot be eliminated, we take a novel perspective towards modeling general equilibrium under uncertainty. General equilibrium under uncertainty (see [3], chapter 19) lies at the foundation of a very rich body of literature in economics and finance theory (see e.g., [14–17]). Our modeling approach relies on defining the *set of acceptable risks* [2] (see definition 1.1) as a primitive of the financial market. Such a set will be a cone (see figures 1.1 and 1.2), and that is why we use the word *conic*.

1.1 Conic financial market

Let us start by assuming that an exhaustive set S of states of the world is given to us. For simplicity, we take S to be a finite set. A typical element is denoted by $s \in S$. We suppose that there are two dates, $t = 0$ and $t = 1$, that there is no information whatsoever at $t = 0$, and that the uncertainty has resolved completely at $t = 1$. The probability of observing a particular event s is denoted by $\pi(s)$. The probability distribution π is therefore given by $\pi = \{\pi(s) : s \in S\}$. The financial market is characterized by two sets $\mathcal{M}$ and $\mathcal{N}$ of probability measures or "generalized scenarios" [1, 2] identifying the "acceptable" amount of individual and aggregate risks, respectively. Essentially, a risk is a random variable and is acceptable to the market if it has a positive valuation under all generalized scenarios. The more scenarios considered, the more conservative (ambiguity[1] or uncertainty averse [26]) is the financial market. We denote a typical scenario by $\widehat{\pi} = \{\widehat{\pi}(s) : s \in S\}$. We assume $\pi \in \mathcal{M} \subset \mathcal{N}$ which indicates that the physical measure π is one of the scenarios considered by the financial market and that the financial market is more strict towards aggregate risk. Using the sets $\mathcal{M}$ and $\mathcal{N}$ of generalized scenarios, we can define the notion of acceptable risks[2] .

Definition 1.1 (acceptable risks)**.** *Given a state price vector $q = \{q(s) : s \in S\}$, the set of acceptable individual $\mathcal{A}^{\mathcal{M}}$ (or aggregate $\mathcal{A}^{\mathcal{N}}$) risks $z = \{z(s) : s \in S\}$ to the financial market is defined by*

$$\mathcal{A}^{\mathcal{M}} \ (or\ \mathcal{A}^{\mathcal{N}}) := \left\{ z : \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z(s) \geq 0,\ \forall \widehat{\pi} \in \mathcal{M} \ \ (or\ \mathcal{N}) \right\}.$$

In other words, an individual or aggregate risk $z = \{z(s) : s \in S\}$ is acceptable to the market if it has a positive valuation

$$\sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z(s)$$

under all generalized scenarios considered by the market at the individual or aggregate level, respectively. As we proceed throughout this work, it is always a good

[1]There seems to be a close link between this work and the existing literature on general equilibrium under ambiguity (see e.g., [18–25]) where the consumers' beliefs are altered to reflect uncertainty aversion. In contrast, this work adopts standard preferences for the households and models the financial market as an ambiguity averse entity.

[2]For an axiomatic treatment of the notion of acceptable risks, the reader is strongly encouraged to refer to the paper by Artzner, Delbaen, Eber, and Heath [2] on *coherent measures of risk*.

exercise to keep in mind the extreme cases of $\mathcal{M} = \{\pi\}$ being a singleton and $\mathcal{N} = \{\widehat{\pi} : \sum_{s\in S} \widehat{\pi}(s) = 1\}$ being the set of all probability measures. In particular, if $\mathcal{M} = \{\pi\}$ is a singleton, a risk z is acceptable to the market at the individual level if it has a positive price in the traditional sense; i.e.,

$$\sum_{s\in S} q(s)z(s) \geq 0.$$

Similarly, if $\mathcal{N} = \{\widehat{\pi} : \sum_{s\in S} \widehat{\pi}(s) = 1\}$ is the set of all probability measures, then only random outcomes that have positive values in all states of the world are acceptable to the market at the aggregate level; i.e.,

$$z(s) \geq 0, \ \forall s \in S.$$

This is because the test measures $\{(1, 0, \ldots, 0), \ (0, 1, \ldots, 0), \ldots, \ (0, 0, \ldots, 1)\}$ are all included in $\mathcal{N}$. Furthermore, the notion of acceptable risks defined above (see definition 1.1) leads naturally to the definition of ask and bid prices. Basically, the price at which one can trade depends on the direction of the trade and there are typically different quotes at which one may sell or buy. The best price (bid) at which one may sell a random outcome z of cash flows is then the infimum or minimal valuation of the cash flow being priced under all generalized scenarios. Similarly the best price (ask) at which one may buy a random outcome of cash flows is the supremum or maximal valuation under all scenarios.

Definition 1.2 (ask and bid prices)**.** *Given a state price vector q, the ask and bid prices of a random payoff z are defined as*

$$\mathrm{ask}(z; \mathcal{M}, q) = \sup_{\widehat{\pi}\in\mathcal{M}} \sum_{s\in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s)z(s),$$

and

$$\text{bid}(z; \mathcal{M}, q) = \inf_{\widehat{\pi} \in \mathcal{M}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z(s),$$

respectively.

From the observation of equality between buying a random cash-flow z and selling its negative $-z$, one can simply deduce that $\text{ask}(z; \mathcal{M}, q) = -\text{bid}(-z; \mathcal{M}, q)$. By virtue of the infimum, the bid price is a concave function on the space of random outcomes and is suited to being maximized. The ask price on the other hand is a convex function of the random outcomes, suited to being minimized. Moreover, it is worth noting that if $\mathcal{M} = \{\pi\}$ is a singleton, then

$$\text{ask}(z; \mathcal{M}, q) = \text{bid}(z; \mathcal{M}, q) = \sum_{s \in S} q(s) z(s),$$

and we recover the classical law of one price. Having configured our financial market structure, in the following, we will introduce the conic equilibrium concept for a pure exchange economy with a single commodity.

1.2 Pure exchange economy – single commodity

Let us start with an exchange economy composed of $I > 0$ consumers and one commodity. Each consumer $i = 1, \ldots, I$ is characterized by a vector of initial endowments $y^i = \{y^i(s) : s \in S\}$ and a utility function $U(\cdot)$ over consumption plans $c^i = \{c^i(s) : s \in S\}$ given by

$$U(c^i) = \sum_{s \in S} u[c^i(s)] \pi(s).$$

Notice that we are imposing identical preference orderings across all individuals i that can be represented in terms of expected utility with common utility function $u(\cdot)$ and common probability distribution π. In the following, we will present a formal definition of the conic equilibrium concept. Later, we will elaborate on key features and consequences of such a modeling approach using concrete examples.

Definition 1.3 (conic equilibrium). *Given a financial economy*

$$\left(S, \pi, U(\cdot), \{y^i\}_{i=1}^{I}, \mathcal{M}, \mathcal{N}\right),$$

a collection formed by

- *a state price vector* $q = \{q(s) : s \in S\}$ *and,*
- *for every consumer* $i = 1, \ldots, I$*, trading plan* $z_*^i = \{z_*^i(s) : s \in S\}$ *at* $t = 0$ *and consumption plan* $c_*^i := \{c_*^i(s) : s \in S\}$ *at* $t = 1$*,*

constitutes a conic equilibrium if:

- *given the state price vector* q *and the financial market configuration* $\mathcal{M}$ *at the individual level, for every consumer* $i = 1, \ldots, I$*, the trading* z_*^i *and consumption* c_*^i *plans solve the household's problem*

$$\begin{aligned}
\max_{z^i, c^i} \quad & U(c^i) = \sum_{s \in S} u[c^i(s)]\pi(s) \\
s.t. \quad & c^i(s) \leq y^i(s) + z^i(s), \quad \forall s \in S, \\
& \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) \leq 0, \quad \forall \widehat{\pi} \in \mathcal{M},
\end{aligned} \tag{1.1}$$

- *and, given the financial market configuration $\mathcal{N}$ at the aggregate level, the state price vector q is chosen such that the markets clear; i.e.,*

$$\sum_{s\in S}\frac{\widehat{\pi}(s)}{\pi(s)}q(s)\sum_{i}z_{*}^{i}(s)\leq 0, \quad \forall\widehat{\pi}\in\mathcal{N}. \tag{1.2}$$

The budget constraint (1.1) can be equivalently expressed as $\mathrm{ask}(z^{i};\mathcal{M},q)\leq 0$, or $-z^{i}\in\mathcal{A}^{\mathcal{M}}$. The latter means that $-z^{i}$ should be acceptable to the market at the individual level. The market clearing condition (1.2), on the other hand, indicates that

$$-\sum_{i}z_{*}^{i}\in\mathcal{A}^{\mathcal{N}},$$

meaning that the aggregate risk of the economy $-\sum_{i}z_{*}^{i}$ should be acceptable to the market. Note that for the extreme cases of $\mathcal{M}=\{\pi\}$ and $\mathcal{N}=\left\{\widehat{\pi}:\sum_{s\in S}\widehat{\pi}(s)=1\right\}$, the conic equilibrium concept defined above simplifies to the traditional Radner equilibrium (see [3], chapter 19). Specifically, household i's budget constraint simplifies to

$$\sum_{s\in S}q(s)z^{i}(s)\leq 0,$$

while the market clearing condition matches the classical one; i.e.,

$$\sum_{i}z_{*}^{i}(s)\leq 0, \quad \forall s\in S.$$

This particular setup is exactly what is known as the complete markets model. The market clearing condition in a traditional complete markets model can be equivalently written as

$$\sum_{i}c_{*}^{i}(s)\leq\sum_{i}y^{i}(s), \quad \forall s\in S.$$

This indicates *positive excess supply* meaning that the aggregate endowment should exceed the aggregate consumption of the economy in all states of the world. This suggests that finance is not a necessary component of the economy. In fact, if we live in an economy in which no matter what happens the supply should always exceed demand, the economy has to close down and there should be no transactions. Basically, one can not actually ever conceive of any economy capable of guaranteeing positive excess supply in every state of the world. One major contribution of this work is its ability to relax the classical positive excess supply requirement by allowing $\mathcal{N}$ to be a proper subset of the set of all probability measures. In particular, the market clearing condition (1.2) allows the aggregate economy to consume more than its endowment in some states of the world. Moreover, it is also instructive to consider the case where $\mathcal{M} = \left\{\widehat{\pi} : \sum_{s\in S} \widehat{\pi}(s) = 1\right\}$. At this other extreme, no risk is tolerated by the financial market at the individual level, and households are left to consume their endowments; i.e., $c^i(s) \leq y^i(s)$, for all $s \in S$. Thus, our conic perspective towards general equilibrium under uncertainty provides us with a unifying framework to model a whole spectrum between incomplete and complete, illiquid and perfectly liquid markets, and beyond. In the following, we will elaborate more on the conic equilibrium concept using a common tool in general equilibrium analysis, namely, the Edgeworth box (see [3], chapter 15). This allows us to study the interaction of two individuals trading one commodity under uncertainty.

1.2.1 Conic Edgeworth box

Consider an economy in which there are two states of the world $S = \{s_1, s_2\}$ and two consumers with endowments $y^1 = \{y^1(s_1), y^1(s_2)\}$ and $y^2 = \{y^2(s_1), y^2(s_2)\}$. Let $(\pi, 1-\pi)$ denote the physical probability measure, where π is the probability of being in state s_1. Thus, the financial market can be characterized by two intervals $\mathcal{M} = [m_L, m_R]$ and $\mathcal{N} = [n_L, n_R]$ of generalized scenarios $\widehat{\pi}$. Given the state price vector[3] $(q, 1-q)$, consumer $i = 1, 2$ solves

$$\begin{aligned} \max_{c^i(s_1), c^i(s_2), z^i(s_1), z^i(s_2)} \quad & U(c^i) = \pi u[c^i(s_1)] + (1-\pi) u[c^i(s_2)] \\ \text{s.t.} \quad & c^i(s) \leq y^i(s) + z^i(s), \ \forall s \in S = \{s_1, s_2\}, \\ & \frac{\widehat{\pi}}{\pi} q z^i(s_1) + \frac{1-\widehat{\pi}}{1-\pi}(1-q) z^i(s_2) \leq 0, \\ & \qquad \forall \widehat{\pi} \in \mathcal{M} = [m_L, m_R] \subseteq [0,1]. \end{aligned} \tag{1.3}$$

Here, $m_L \leq \pi \leq m_R$. Since the budget constraint (1.3) is a linear function of $\widehat{\pi}$, its maximum is achieved at either m_L or m_R boundary of the interval $\mathcal{M}$. Therefore, this constraint can be equivalently expressed as

$$\begin{aligned} \frac{m_L}{\pi} q z^i(s_1) + \frac{1-m_L}{1-\pi}(1-q) z^i(s_2) \leq 0, \\ \frac{m_R}{\pi} q z^i(s_1) + \frac{1-m_R}{1-\pi}(1-q) z^i(s_2) \leq 0. \end{aligned}$$

Moreover, the corresponding set of acceptable individual risks $\mathcal{A}^{\mathcal{M}}$ to the financial market is therefore a cone and is depicted in figure 1.1. Note that if $m_L = m_R = \pi$,

[3] It should be noted that the state prices are determined upto a constant in equilibrium. Therefore, we seek a state price vector of the form $(q, 1-q)$.

these two constraints merge into the classical one; i.e.,

$$qz^i(s_1) + (1-q)z^i(s_2) \leq 0,$$

which corresponds to the usual budget constraint in the Radner equilibrium (see [3], chapter 19). Figure 1.2 depicts the corresponding cone (half space) of acceptable individual risks for the case of the Radner equilibrium. The first order conditions of consumer i's problem can be written as

$$\begin{aligned}
&\pi u'[c_*^i(s_1)] = \frac{m_L}{\pi} q\mu_L^i + \frac{m_R}{\pi} q\mu_R^i, \\
&(1-\pi)u'[c_*^i(s_2)] = \frac{1-m_L}{1-\pi}(1-q)\mu_L^i + \frac{1-m_R}{1-\pi}(1-q)\mu_R^i, \\
&\left[\frac{m_L}{\pi} q z_*^i(s_1) + \frac{1-m_L}{1-\pi}(1-q) z_*^i(s_2)\right] \mu_L^i = 0, \\
&\left[\frac{m_R}{\pi} q z_*^i(s_1) + \frac{1-m_R}{1-\pi}(1-q) z_*^i(s_2)\right] \mu_R^i = 0, \\
&c_*^i(s) = y^i(s) + z_*^i(s), \;\; \forall s \in \{s_1, s_2\}.
\end{aligned}$$

Here, μ_L^i and μ_R^i are the Lagrange multipliers on the household's budget constraints. It is worth observing that

$$\frac{\pi u'[c_*^i(s_1)]}{(1-\pi)u'[c_*^i(s_2)]} = \frac{q}{1-q}\left(\frac{m_L\mu_L^i + m_R\mu_R^i}{(1-m_L)\mu_L^i + (1-m_R)\mu_R^i}\frac{1-\pi}{\pi}\right),$$

which for the Radner equilibrium, i.e., $m_L = m_R = \pi$, simplifies to the classical relationship between the marginal utilities of consumption across states

$$\frac{\pi u'[c_*^i(s_1)]}{(1-\pi)u'[c_*^i(s_2)]} = \frac{q}{1-q}.$$

Furthermore, the market clearing condition (1.2) can be written as

$$\frac{\widehat{\pi}}{\pi} q[z_*^1(s_1) + z_*^2(s_1)] + \frac{1-\widehat{\pi}}{1-\pi}(1-q)[z_*^1(s_2) + z_*^2(s_2)] \leq 0, \;\; \forall \widehat{\pi} \in \mathcal{N} = [n_L, n_R] \subseteq [0,1],$$

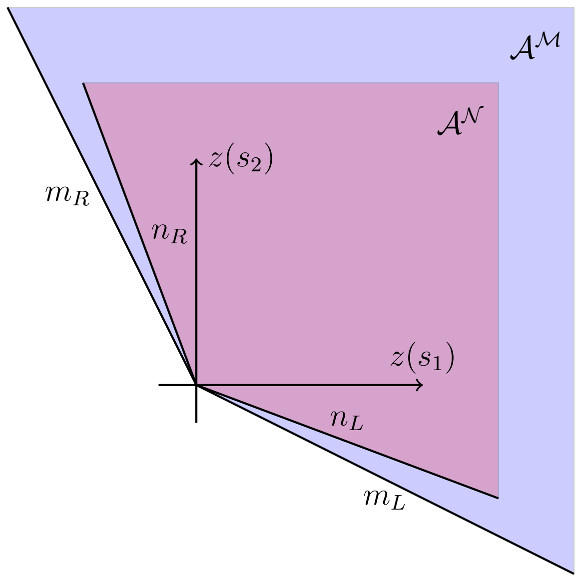

Figure 1.1: Cones of acceptable individual $\mathcal{A}^{\mathcal{M}}$ and aggregate $\mathcal{A}^{\mathcal{N}}$ risks.

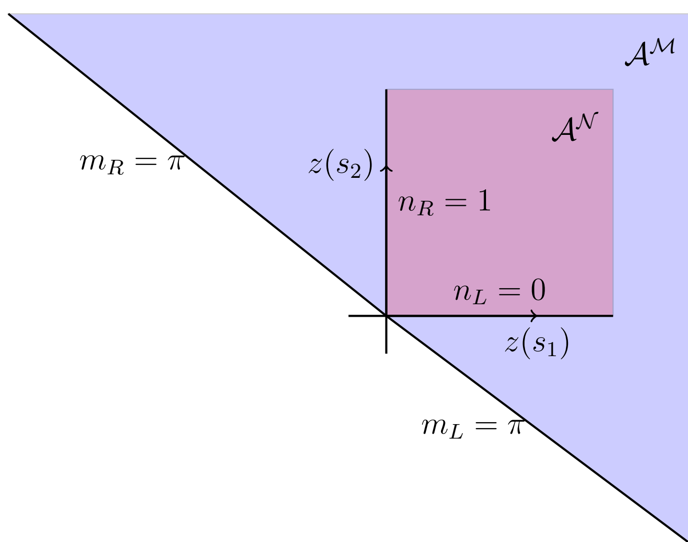

Figure 1.2: Cones of acceptable individual $\mathcal{A}^{\mathcal{M}}$ and aggregate $\mathcal{A}^{\mathcal{N}}$ risks for the case of Radner equilibrium.

or equivalently as

$$\frac{n_L}{\pi}q[z_*^1(s_1)+z_*^2(s_1)]+\frac{1-n_L}{1-\pi}(1-q)[z_*^1(s_2)+z_*^2(s_2)]\leq 0,$$
$$\frac{n_R}{\pi}q[z_*^1(s_1)+z_*^2(s_1)]+\frac{1-n_R}{1-\pi}(1-q)[z_*^1(s_2)+z_*^2(s_2)]\leq 0.$$

Therefore, the corresponding set of acceptable aggregate risks $\mathcal{A}^{\mathcal{N}}$ to the financial market is a cone and can be depicted as in figure 1.1. Here, $\mathcal{M}\subseteq\mathcal{N}$, i.e., $n_L\leq m_L\leq\pi\leq m_R\leq n_R$. Note that if $n_L=0$ and $n_R=1$, the market clearing condition simplifies to

$$z_*^1(s)+z_*^2(s)\leq 0,\ \forall s\in\{s_1,s_2\},$$

which is exactly the market clearing condition in the Radner equilibrium. Figure 1.2 depicts the corresponding cone of acceptable aggregate risks for the Radner equilibrium. Our equilibrium concept can be best illustrated by means of the *conic Edgeworth box* depicted in figure 1.3. The budget sets of the two consumers along with the aggregate cone offered by the financial market are depicted in distinct colors. Moreover, black filled circles denote optimal consumption allocations for the two consumers and the aggregate economy. The conic Edgeworth box reminds us of the metaphor "thinking out of the box" which means to think differently, unconventionally, or from a new perspective. In particular, in a conic economy, the aggregate consumption is allowed to go beyond the box defined by the aggregate endowments in some states of the world, whereby the financial primitives cover the resulting gap. Hence, the conic economy is a bigger economy because it is willing to absorb some aggregate risk. The economy hopes not to end up in a bad state of the

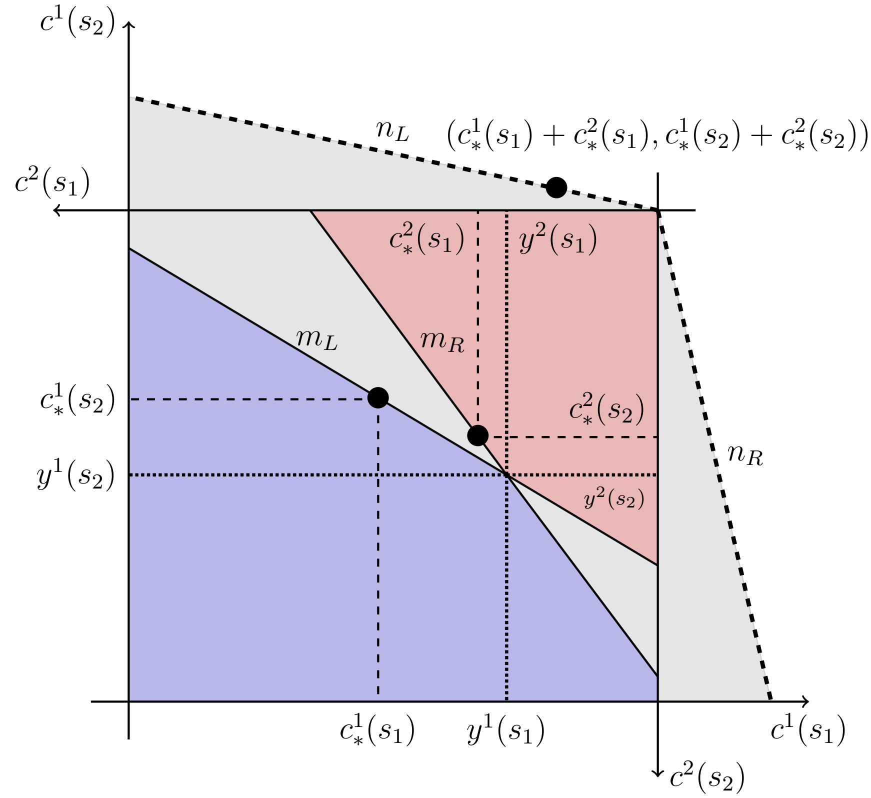

Figure 1.3: Conic Edgeworth box.

world but if it does, it is just like the Federal Reserve stepping in and covering the loss. The following numerical example adds more detail concerning what has been discussed so far.

1.2.2 Numerical example

Let $\pi = 0.5$, $u(c) = \log(c)$, and assume that $y^1 = (3, 1)$ and $y^2 = (1, 2)$ are the endowments of the two consumers. First, consider the classical case with $m_L = m_R = \pi$, $n_L = 0$, and $n_R = 1$. This case corresponds to the Radner equilibrium and is presented here for comparison reasons. The market clearing conditions yield

Table 1.1: Radner equilibrium – exchange economy.

i	$c_*^i(s_1)$	$c_*^i(s_2)$	$z_*^i(s_1)$	$z_*^i(s_2)$	μ_L^i	μ_R^i
1	2.16667	1.625	-0.833333	0.625	0.538462	0
2	1.83333	1.375	0.833333	-0.625	0	0.636364
Aggregate	4	3	0	0		

a state price $q = 3/7 \approx 0.428571$, and the corresponding trading and consumption plans are given in table 1.1. Next, let us consider the case with $m_L = 0.4$, $m_R = 0.6$, $n_L = 0.2$, and $n_R = 0.8$. In this case, relative to the classical economy, the market is less forgiving at the indivual level but more forgiving at the aggregate level. Solving the market clearing conditions for the state price gives $q = 0.478261$. Furthermore, we obtain the consumption and trading plans given in table 1.2. It is worth noting that $z_*^1(s_1) + z_*^2(s_1) < 0$ and $z_*^1(s_2) + z_*^2(s_2) > 0$ which means that the financial market is compensating for the shortcomings in the second state of the world. In contrast to the classical model (see table 1.1), where the aggregate consumption does not exceed the total endowment, as both table 1.2 and figure 1.3 illustrate, some gap is permitted to occur between the aggregate consumption and endowment in a conic environment. In the following, we will examine the welfare consequences of adopting a conic perspective towards general equilibrium under uncertainty.

1.2.3 Welfare analysis

Using the same setup as in the numerical example of section 1.2.2, the utility values resulting from different financial configurations are presented in table 1.3.

Table 1.2: Conic equilibrium – exchange economy.

i	$c_*^i(s_1)$	$c_*^i(s_2)$	$z_*^i(s_1)$	$z_*^i(s_2)$	μ_L^i	μ_R^i
1	2.31818	1.41667	-0.681818	0.416667	0.563725	0
2	1.22727	1.6875	0.227273	-0.3125	0	0.709877
Aggregate	3.54545	3.10417	-0.454545	0.104167		

From the first five rows of this table, one can infer that as the financial market becomes more strict towards aggregate risk, the overall welfare along with the utility of the richer consumer (i.e., consumer 1) decreases while the utility of the poorer consumer (i.e., consumer 2) increases. Moreover, the next five rows of this table show that as the financial market becomes less tolerant of individual risks, the overall welfare along with the utility of both consumers decreases. The highest total welfare corresponds to the case with $n_L = m_L = \pi = m_R = n_R = 0.5$. For this financial configuration, consumer 1 has the highest welfare possible and consumer 2 has the lowest. The last row of table 1.3 showcases a financial market configuration with a higher total welfare than that of a Radner equilibrium, without being as forgiving as the Radner equilibrium at the individual level. The overall message is that the financial market configurations have real effect on individual and aggregate welfare (happiness) of the economy and it can justify government intervention and regulation of markets in certain economic situations. The following utility analysis exercise will help us elaborate more on this message.

Specifically, let us fix the financial market configurations m_L, m_R, n_L, n_R and perform the following exercise. Given this configuration, we solve our example

Table 1.3: Welfare analysis for different financial configurations.

m_L	m_R	n_L	n_R	$U(c^1_*)$	$U(c^2_*)$	$U(c^1_*)+U(c^2_*)$
0.40	0.60	0.4	0.6	0.665027	0.346574	1.0116
0.40	0.60	0.3	0.7	0.608198	0.356883	0.965081
0.40	0.60	0.2	0.8	0.594545	0.364021	0.958566
0.40	0.60	0.1	0.9	0.588543	0.368143	0.956686
0.40	0.60	0.0	1.0	0.58519	0.37078	0.955969
0.50	0.50	0.0	1.0	0.629349	0.462295	1.091644
0.45	0.55	0.0	1.0	0.608427	0.405924	1.01435
0.40	0.60	0.0	1.0	0.58519	0.37078	0.955969
0.35	0.65	0.0	1.0	0.563129	0.352632	0.915761
0.30	0.70	0.0	1.0	0.549814	0.346713	0.896527
0.50	0.50	0.5	0.5	0.906189	0.346574	1.25276
0.49	0.51	0.487411	0.512589	0.80258	0.353031	1.15561

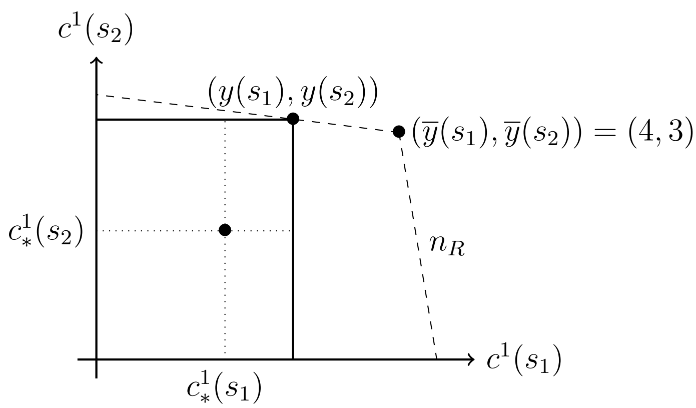

Figure 1.4: Box defined by equilibrium allocations.

problem (see section 1.2.2) for equilibrium allocations and prices. The equilibrium allocations $(c_*^1(s_1), c_*^1(s_2))$ and $(c_*^2(s_1), c_*^2(s_2))$ will give us a new box given by $(y(s_1), y(s_2)) = (c_*^1(s_1), c_*^1(s_2)) + (c_*^2(s_1), c_*^2(s_2))$ as depicted in figure 1.4. Given the box $(y(s_1), y(s_2))$, we redistribute initial endowments $(\overline{y}(s_1), \overline{y}(s_2)) = (4, 3)$ among agents in such a way that the resulting equilibrium allocations gives us the same box $(y(s_1), y(s_2))$. We then compute the utility values of such allocations and plot the resulting utility frontier. This procedure is detailed in the following. We first solve the market clearing condition for q. In this example, the market clearing condition is given by

$$\frac{n_L}{\pi} q[y(s_1) - \overline{y}(s_1)] + \frac{1 - n_L}{1 - \pi}(1 - q)[y(s_2) - \overline{y}(s_2)] = 0.$$

Choose $(c^1(s_1), c^1(s_2))$ in the box given by $(y(s_1), y(s_2))$. Let

$$(c^2(s_1), c^2(s_2)) = (y(s_1), y(s_2)) - (c^1(s_1), c^1(s_2))$$

and solve the following system of equations for $(y^1(s_1), y^1(s_2))$ and $(y^2(s_1), y^2(s_2))$.

$$\begin{aligned}
&\frac{m_L}{\pi} q[c^1(s_1) - y^1(s_1)] + \frac{1 - m_L}{1 - \pi}(1 - q)[c^1(s_2) - y^1(s_2)] = 0, \\
&\frac{m_R}{\pi} q[c^2(s_1) - y^2(s_1)] + \frac{1 - m_R}{1 - \pi}(1 - q)[c^2(s_2) - y^2(s_2)] = 0, \\
&y^1(s_1) + y^2(s_1) = \overline{y}(s_1), \\
&y^1(s_2) + y^2(s_2) = \overline{y}(s_2).
\end{aligned}$$

Now, fix $(n_L, n_R) = (0, 1)$ and change (m_L, m_R) from $(0.5, 0.5)$ to $(0.4, 0.6)$ in order to obtain the utility frontiers given in figure 1.5. This figure indicates that as the market becomes more strict towards individual risks, the utility frontier retracts.

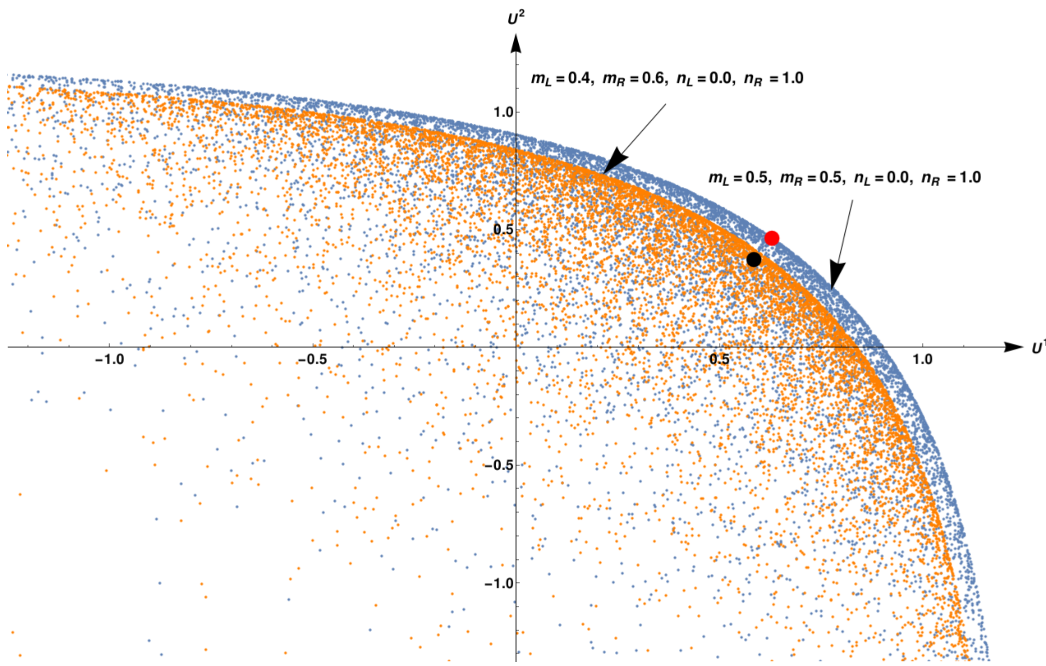

Figure 1.5: Utility frontier retracts as the market becomes more strict towards individual risks. This figure and the next one are obtained by plotting actual numerical utility values for different endowment redistributions resulted from a Monte-Carlo sampling procedure.

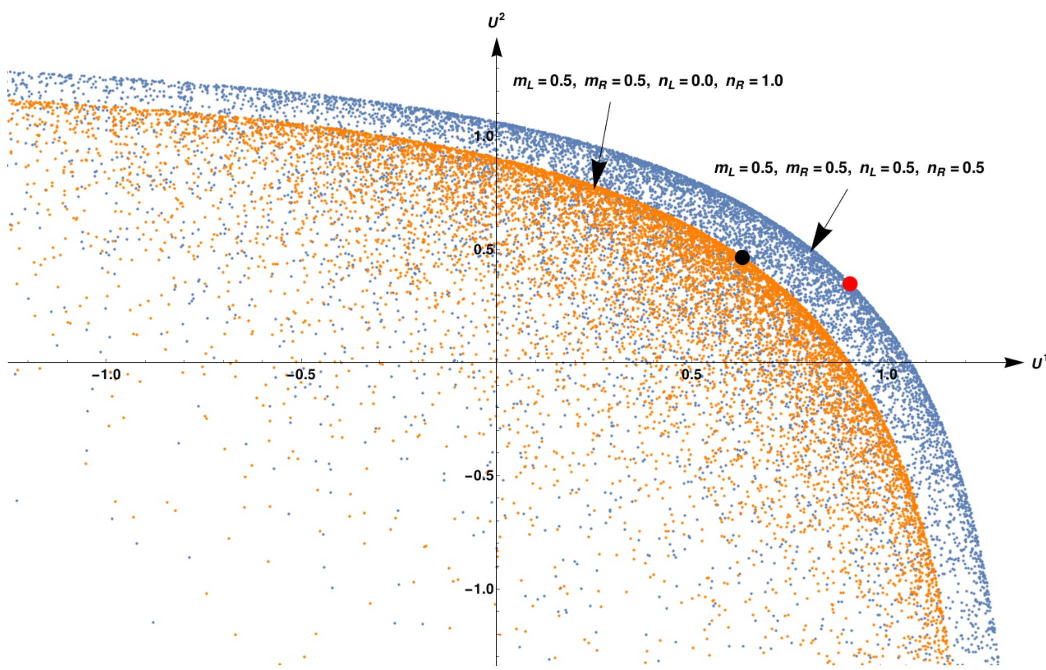

Figure 1.6: Utility frontier retracts as the market becomes more strict towards aggregate risks.

Moreover, fixing $(m_L, m_R) = (0.5, 0.5)$ and changing (n_L, n_R) from $(0, 1)$ to $(0.5, 0.5)$ yields the utility frontiers given in figure 1.6. This figure also indicates that as the market becomes less tolerant of aggregate risks, the utility frontier retracts. Figures 1.5 and 1.6, reinforce the previous message that the financial market configurations have real welfare effects in a conic economy.

1.3 Equilibrium prices and quantities

Let us go back to the more general structure of a conic equilibrium specified in definition 1.3, and assume that for some $\widehat{\pi}(z^i) = \{\widehat{\pi}(s; z^i) : s \in S\} \in \mathcal{M}$,

$$\sup_{\widehat{\pi} \in \mathcal{M}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) = \sum_{s \in S} \frac{\widehat{\pi}(s; z^i)}{\pi(s)} q(s) z^i(s).$$

This is a plausible assumption and indicates that the supremum is attainable. Thus, the household i's budget constraint (1.1) can be equivalently expressed as

$$\sum_{s \in S} \frac{\widehat{\pi}(s; z^i)}{\pi(s)} q(s) z^i(s) \leq 0.$$

Attach a Lagrange multiplier μ^i to this constraint, form the Lagrangian, and use the Envelope theorem [27], to obtain the following first order condition for the household i's problem; i.e.,

$$u'\left[c^i(s)\right] \pi(s) = \mu^i \frac{\widehat{\pi}(s; z^i)}{\pi(s)} q(s), \tag{1.4}$$

for all i and $s \in S$. This implies that

$$\frac{u'\left[c^i(s)\right]}{u'\left[c^j(s)\right]} = \frac{\mu^i}{\mu^j} \frac{\widehat{\pi}(s; z^i)}{\widehat{\pi}(s; z^j)}, \tag{1.5}$$

for all pairs (i, j). Note that if $\mathcal{M} = \{\pi\}$ is a singleton, then

$$\widehat{\pi}(s; z^i) = \widehat{\pi}(s; z^j) = \pi(s),$$

and consequently the ratios of marginal utilities between pairs of agents are constant across all states, i.e.,

$$\frac{u'\left[c^i(s)\right]}{u'\left[c^j(s)\right]} = \frac{\mu^i}{\mu^j}.$$

However, in general, this no longer holds (see equation (1.5)) in a conic economy with an arbitrary financial market configuration $\mathcal{M}$. Similarly, assuming that for some $\widehat{\pi}(Z) \in \mathcal{N}$, where $Z = \sum_i z^i$, we have

$$\sup_{\widehat{\pi} \in \mathcal{N}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) Z(s) = \sum_{s \in S} \frac{\widehat{\pi}(s; Z)}{\pi(s)} q(s) Z(s),$$

meaning that the supremum is attainable, the market clearing condition (1.2) can be written as

$$\sum_{s \in S} \frac{\widehat{\pi}(s; Z)}{\pi(s)} q(s) Z(s) = 0.$$

Using the first order condition (1.4) for any household i, we obtain the following form for the market clearing condition,

$$\sum_{s \in S} \frac{1}{\mu^i} \frac{\widehat{\pi}(s; Z)}{\widehat{\pi}(s; z^i)} u'\left[c^i(s)\right] \pi(s) Z(s) = 0.$$

To compute an equilibrium, we propose the following algorithm[4] which generalizes the *Negishi algorithm* (see [4], chapter 8).

1. Fix $\mu^1 = 1$, throughout the algorithm, and guess some positive initial values for the remaining μ^i, $i = 2, \ldots, I$.

[4] The non-linearities involved in the definition of a conic general equilibrium makes a convergence proof of the proposed algorithm non-trivial. A convergence proof would require imposing further restrictions on the sets $\mathcal{M}$ and $\mathcal{N}$ identifying the financial market configurations. This will be subject of future research.

2. Make initial guesses for $\widehat{\pi}(z^i)$, $i = 1, \ldots, I$, and $\widehat{\pi}(Z)$. A good initial guess is usually given by the actual physical measure π.

3. Solve the following system of equations, using the Levenberg-Marquardt [28, 29] method for instance, for candidate consumption $\{c^i\}_{i=1}^I$ and trading $\{z^i\}_{i=1}^I$ allocations;

$$
\begin{aligned}
& c^i(s) = y^i(s) + z^i(s), \ \forall i = 1, \ldots, I, \ \forall s \in S, \\
& \frac{u'\left[c^i(s)\right]}{u'\left[c^1(s)\right]} = \frac{\mu^i}{\mu^1} \frac{\widehat{\pi}(s; z^i)}{\widehat{\pi}(s; z^1)}, \ \forall i = 2, \ldots, I, \ \forall s \in S, \\
& \sum_{s \in S} \frac{1}{\mu^1} u'\left[c^1(s)\right] \pi(s) z^1(s) = 0, \\
& \sum_{s \in S} \frac{1}{\mu^1} \frac{\widehat{\pi}(s; Z)}{\widehat{\pi}(s; z^1)} u'\left[c^1(s)\right] \pi(s) \sum_i z^i(s) = 0.
\end{aligned}
$$

4. Use the following for household 1 to solve for the price system q.

$$
u'\left[c^1(s)\right] \pi(s) = \mu^1 \frac{\widehat{\pi}(s; z^1)}{\pi(s)} q(s).
$$

5. Check that the following requirements for households $i = 1, \ldots, I$ are satisfied

$$
\sup_{\widehat{\pi} \in \mathcal{M}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) = \sum_{s \in S} \frac{\widehat{\pi}(s; z^i)}{\pi(s)} q(s) z^i(s).
$$

Moreover, corresponding to the market clearing condition, check the validity of the following requirement; i.e.,

$$
\sup_{\widehat{\pi} \in \mathcal{N}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) Z(s) = \sum_{s \in S} \frac{\widehat{\pi}(s; Z)}{\pi(s)} q(s) Z(s).
$$

Update $\widehat{\pi}(z^i)$, for $i = 1, \ldots, I$, and $\widehat{\pi}(Z)$ accordingly. It is worth noting that this step boils down to maximizing linear objective functions.

6. Iterate on steps 3-5 until the requirements of step 5 are satisfied.

7. For $i = 2, \ldots, I$, check the budget constraint

$$\sum_{s \in S} \frac{\widehat{\pi}(s; z^i)}{\pi(s)} q(s) z^i(s) \leq 0.$$

   Increase μ^i for those i's that violate this constraint and decrease it for others.

8. Iterate to convergence on steps 2-7.

Applying the algorithm outlined above to the numerical settings of section 1.2.2, we obtain $\mu^2 = 1.2593$, $q(s_1) = 0.2696$, and $q(s_2) = 0.2941$. The resulting equilibrium allocations are the same as the ones given in table 1.2. Moreover,

$$\frac{q(s_1)}{q(s_1) + q(s_2)} = 0.4783,$$

which matches the value q of the state price vector $(q, 1-q)$ from the aforementioned section. In the following, we will demonstrate that the conic equilibrium concept is inevitably immune to some of the most empirically criticized conclusions of a complete markets model.

CRRA utility

In particular, suppose that the one-period utility function is of the constant relative risk-aversion (CRRA) form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \ \gamma > 0.$$

Then, equation (1.5) yields

$$c^i(s) = c^j(s) \left\{ \frac{\mu^i}{\mu^j} \frac{\widehat{\pi}(s; z^i)}{\widehat{\pi}(s; z^j)} \right\}^{-\frac{1}{\gamma}}. \tag{1.6}$$

In the classical case, where $\mathcal{M} = \{\pi\}$, this equation simplifies to

$$c^i(s) = c^j(s) \left\{ \frac{\mu^i}{\mu^j} \right\}^{-\frac{1}{\gamma}},$$

and states that the consumption allocations for two distinct agents are constant fractions of one another. Therefore,

$$\sum_i c^i(s) = c^j(s) \sum_i \left\{ \frac{\mu^i}{\mu^j} \right\}^{-\frac{1}{\gamma}}.$$

Combined with the classical market clearing condition

$$\sum_i c^i(s) = \sum_i y^i(s),$$

corresponding to $\mathcal{N} = \left\{ \widehat{\pi} : \sum_{s \in S} \widehat{\pi}(s) = 1 \right\}$, it says that individual consumption is perfectly correlated with the aggregate endowment or aggregate consumption of the economy; i.e.,

$$c^j(s) = \left(\sum_i \left\{ \frac{\mu^i}{\mu^j} \right\}^{-\frac{1}{\gamma}} \right)^{-1} \sum_i y^i(s).$$

This implies that the consumption $c^j(s)$ is independent of the household's individual endowment $y^j(s)$ in state s. However, all of these nice and oversimplified conclusions will evaporate as soon as we allow $\mathcal{M}$ to be more general than a singleton (see equation (1.6)), or if we let $\mathcal{N}$ to be a proper subset of the set of all probability measures.

1.4 Asset markets

The contingent commodities $z^i = \{z^i(s) : s \in S\}$ considered in the previous sections serve the purpose of transferring wealth across states of the world. They are, however, only theoretical constructs that rarely have exact counterparts in reality. Nevertheless, in reality there are *securities*, or *assets*, that to some extent perform the wealth-transferring role assigned to z^i. It is therefore important to study the functioning of these asset markets. We begin by letting $r_s = \{r_s(s') : s' \in S\}$ denote an Arrow-Debreu security with returns $r_s(s) = 1$ and $r_s(s') = 0$ if $s' \neq s$. Moreover, let $z^i(s)$ be the quantities demanded by consumer i for security r_s. The ask and bid prices (see definition 1.2) of an arbitrary Arrow-Debreu security with return vector r_s are given by

$$\mathrm{ask}(r_s; \mathcal{M}, q) = \sup_{\widehat{\pi} \in \mathcal{M}} \frac{\widehat{\pi}(s)}{\pi(s)} q(s), \text{ and } \mathrm{bid}(r_s; \mathcal{M}, q) = \inf_{\widehat{\pi} \in \mathcal{M}} \frac{\widehat{\pi}(s)}{\pi(s)} q(s),$$

respectively. Note that if $\mathcal{M} = \{\pi\}$ is a singleton, then

$$\mathrm{ask}(r_s; \mathcal{M}, q) = \mathrm{bid}(r_s; \mathcal{M}, q) = q(s).$$

Furthermore, if household i takes a position $z^i = \{z^i(s) : s \in S\}$ in these securities, its budget constraint should be written as

$$\sum_{s \in S} |z^i(s)| \, \mathrm{ask}\left(\frac{z^i(s)}{|z^i(s)|} r_s; \mathcal{M}, q\right) \leq 0.$$

Note that if for some s, $z^i(s) < 0$, then

$$|z^i(s)| \, \mathrm{ask}\left(\frac{z^i(s)}{|z^i(s)|} r_s; \mathcal{M}, q\right) = -z^i(s) \, \mathrm{ask}\left(-r_s; \mathcal{M}, q\right) = z^i(s) \, \mathrm{bid}(r_s; \mathcal{M}, q).$$

In other words, a negative position, i.e., $z^i(s) < 0$, means that household i is selling security r_s at the bid price, while a positive position, i.e., $z^i(s) > 0$, means that it is buying the asset at the ask price. Moreover, we have

$$\mathrm{ask}(z^i; \mathcal{M}, q) \leq \sum_{s \in S} \mathrm{ask}\left(z^i(s) r_s; \mathcal{M}, q\right) = \sum_{s \in S} |z^i(s)| \, \mathrm{ask}\left(\frac{z^i(s)}{|z^i(s)|} r_s; \mathcal{M}, q\right),$$

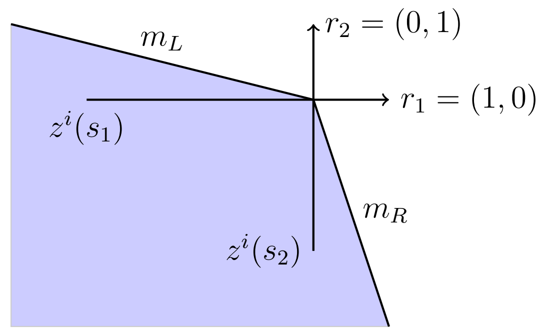

Figure 1.7: Arrow-Debreu securities.

since

$$\frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) \leq \text{ask}\left(z^i(s) r_s; \mathcal{M}, q\right), \ \forall \widehat{\pi} \in \mathcal{M}.$$

Therefore, the budget constraint $\text{ask}(z^i; \mathcal{M}, q) \leq 0$, or equivalently

$$\sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) \leq 0, \ \forall \widehat{\pi} \in \mathcal{M},$$

can be reinterpreted as a liquidity constraint and it limits the quantity of these Arrow-Debreu securities that can be demanded. In other words, the market for these securities is not perfectly liquid. When $S = \{s_1, s_2\}$, this interpretation can be illustrated using figure 1.7. It is worth mentioning that instead of Arrow-Debreu, we could use any other securities. For instance, when $S = \{s_1, s_2\}$, we could employ securities with return vectors $(1, 0)$ and $(1, 1)$.

1.5 Pure exchange economy – multiple commodities

It is straightforward, but necessary, to extend the single commodity framework developed so far to multiple commodities. Let us begin by assuming that there are

L commodities in the economy and that at $t = 0$ consumers have expectations regarding the spot prices prevailing at $t = 1$ for each of the L commodities. It is important to emphasize that the correct anticipation of future spot prices is a crucial assumption and is common practice in the literature (see [3], chapter 19). Let the endowment of consumer $i = 1, \dots, I$ be a contingent commodity vector

$$\boldsymbol{y}^i = \{\boldsymbol{y}^i(s) = (y_1^i(s), \dots, y_L^i(s)) : s \in S\}.$$

This means that if state s occurs then consumer i has endowment vector $\boldsymbol{y}^i(s) = (y_1^i(s), \dots, y_L^i(s))$.

Definition 1.4 (conic equilibrium – multiple commodities)**.** *Given a financial economy*

$$\left(S, \pi, U(\cdot), \{\boldsymbol{y}^i\}_{i=1}^I, \mathcal{M}, \mathcal{N}\right),$$

a collection formed by

- *a state price vector* $q = \{q(s) : s \in S\}$,
- *a spot price vector* $\boldsymbol{p}(s) = (p_1(s), \dots, p_L(s))$, *for every* $s \in S$, *and,*
- *for every consumer* i, *trading plan* $z_*^i = \{z_*^i(s) : s \in S\}$ *at* $t = 0$ *and consumption plan* $\boldsymbol{c}_*^i := \{\boldsymbol{c}_*^i(s) = (c_{*1}^i(s), \dots, c_{*L}^i(s)) : s \in S\}$ *at* $t = 1$,

constitutes a conic equilibrium if:

- *given the state price vector* q, *the spot price vector* $\boldsymbol{p}(s)$, *for every* $s \in S$, *and the financial market configuration* $\mathcal{M}$ *at the individual level, for every* *consumer* $i = 1, \ldots, I$*, the trading* z^i_* *and consumption* $\mathbf{c}^i_*$ *plans solve the household's problem*

$$\begin{aligned} \max_{z^i, \boldsymbol{c}^i} \quad & U(\boldsymbol{c}^i) = \sum_{s \in S} u[\boldsymbol{c}^i(s)]\pi(s) \\ s.t. \quad & \boldsymbol{p}(s) \cdot \boldsymbol{c}^i(s) \leq \boldsymbol{p}(s) \cdot \boldsymbol{y}^i(s) + p_1(s) z^i(s), \quad \forall s \in S, \\ & \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) z^i(s) \leq 0, \quad \forall \widehat{\pi} \in \mathcal{M}, \end{aligned}$$

- *and, given the financial market configuration* $\mathcal{N}$ *at the aggregate level, the state price vector* q *and the spot price vector* $\boldsymbol{p}(s)$*, for every* $s \in S$*, are chosen such that the markets clear; i.e.,*

$$\begin{aligned} & \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) \sum_i z^i_*(s) \leq 0, \quad \forall \widehat{\pi} \in \mathcal{N}, \\ & \sum_i \boldsymbol{c}^i_*(s) \leq \sum_i \boldsymbol{y}^i(s) + \left(\sum_i z^i_*(s), 0, \ldots, 0 \right), \quad \forall s \in S. \end{aligned}$$

It is worth emphasizing again that for $\mathcal{M} = \{\pi\}$ and $\mathcal{N} = \left\{\widehat{\pi} : \sum_{s \in S} \widehat{\pi}(s) = 1\right\}$, the conic equilibrium concept defined above simplifies to the Radner equilibrium (see [3], chapter 19). Furthermore, the market clearing conditions, the way we have defined them, indicate that only in one of the commodities (the numeraire, i.e., good 1) the economy is permitted to consume beyond its aggregate endowment. This is a simplifying assumption and in reality we don't have to have the markets for other commodities to be non-conic. However, it makes the notation more convenient and the analysis simpler to assume that only the markets for cash (good 1) are conic.

Numerical example

Consider an economy with two commodities in which there are two states of the world $S = \{s_1, s_2\}$ and two consumers with endowments

$$\begin{aligned}
\boldsymbol{y}^1 &= \{\boldsymbol{y}^1(s_1), \boldsymbol{y}^1(s_2)\} = (y_1^1(s_1), y_2^1(s_1), y_1^1(s_2), y_2^1(s_2)) = (3, 1, 1, 1), \\
\boldsymbol{y}^2 &= \{\boldsymbol{y}^2(s_1), \boldsymbol{y}^2(s_2)\} = (y_1^2(s_1), y_2^2(s_1), y_1^2(s_2), y_2^2(s_2)) = (1, 1, 2, 1).
\end{aligned}$$

Given the state price vector $(q, 1-q)$, consumer $i = 1, 2$ solves

$$\begin{aligned}
\max_{\boldsymbol{c}^i, z^i} \quad & U(\boldsymbol{c}^i) = \pi u[c_1^i(s_1), c_2^i(s_1)] + (1-\pi) u[c_1^i(s_2), c_2^i(s_2)] \\
\text{s.t.} \quad & c_1^i(s) + p(s) c_2^i(s) \leq y_1^i(s) + p(s) y_2^i(s) + z^i(s), \ \forall s \in \{s_1, s_2\} \\
& \frac{m_L}{p} q z^i(s_1) + \frac{1-m_L}{1-p}(1-q) z^i(s_2) \leq 0, \\
& \frac{m_R}{p} q z^i(s_1) + \frac{1-m_R}{1-p}(1-q) z^i(s_2) \leq 0.
\end{aligned}$$

The market clearing conditions can be written as

$$\begin{aligned}
& \frac{n_L}{p} q [z_*^1(s_1) + z_*^2(s_1)] + \frac{1-n_L}{1-p}(1-q)[z_*^1(s_2) + z_*^2(s_2)] \leq 0, \\
& \frac{n_R}{p} q [z_*^1(s_1) + z_*^2(s_1)] + \frac{1-n_R}{1-p}(1-q)[z_*^1(s_2) + z_*^2(s_2)] \leq 0, \\
& c_{*2}^1(s) + c_{*2}^2(s) \leq y_2^1(s) + y_2^2(s). \ \forall s \in \{s_1, s_2\}.
\end{aligned}$$

Assume $\pi = 0.5$, $U(\boldsymbol{c}^i) = \pi \log\left[(c_1^i(s_1))^\alpha (c_2^i(s_1))^{1-\alpha}\right] + (1-\pi) \log\left[(c_1^i(s_2))^\alpha (c_2^i(s_2))^{1-\alpha}\right]$, and $\alpha = 0.5$. Let us first consider the case where $m_L = m_R = \pi$ and $n_L = 0, n_R = 1$. In this case, we obtain $q = 3/7$, $p(s_1) = 2$, and $p(s_2) = 1.5$. Table 1.4 gives the consumption plans for both consumers. Let us now consider the case where $m_L = 0.4, m_R = 0.6, n_L = 0.2$, and $n_R = 0.8$. In this case, we obtain $q = 0.436104$,

Table 1.4: Radner equilibrium – multiple commidities.

i	$c^i_{*1}(s_1)$	$c^i_{*2}(s_1)$	$c^i_{*1}(s_2)$	$c^i_{*2}(s_2)$	$z^i_*(s_1)$	$z^i_*(s_2)$	μ^i_L	μ^i_R
1	2.08333	1.04167	1.5625	1.04167	-0.833333	0.625	0.28	0
2	1.91667	0.958333	1.4375	0.958333	0.833333	-0.625	0	0.304348
Aggregate	4	2	3	2	0	0		

Table 1.5: Conic equilibrium – multiple commodities.

i	$c^i_{*1}(s_1)$	$c^i_{*2}(s_1)$	$c^i_{*1}(s_2)$	$c^i_{*2}(s_2)$	$z^i_*(s_1)$	$z^i_*(s_2)$	μ^i_L	μ^i_R
1	2.45902	1.24224	1.26783	0.842993	-0.0614754	0.0316957	0.291406	0
2	1.5	0.757764	1.7401	1.15701	0.0204918	-0.0237718	0	0.318477
Aggregate	3.95902	2	3.00792	2	-0.0409836	0.00792393		

$p(s_1) = 1.97951$, and $p(s_2) = 1.50396$. The consumption plans are given in table 1.5. It is worth noting that $z^1_*(s_1) + z^2_*(s_1) < 0$ and $z^1_*(s_2) + z^2_*(s_2) > 0$ which means that the financial market is compensating for the shortcomings in the second state of the world.

1.6 Firm behavior

In previous sections, we have focused on the study of exchange economies. For once, this has not been just for simplicity. In fact, the consideration of production and firms is genuinely more difficult (see [30, 31]). As before, we consider a setting with two periods, $t = 0$ and $t = 1$, and S possible states at $t = 1$. There are L physical commodities traded in the spot markets of period $t = 1$. There is no consumption at $t = 0$. We introduce into our model a firm that produces a random amount of the numeraire (good 1) at date $t = 1$. The firm produces perhaps by means of inputs used at time $t = 0$, but we do not formalize this part explicitly. Let $d = \{d(s) : s \in S\}$ denote the state contingent levels of production of the firm. The shares $\alpha^i \geq 0$, with $\sum_i \alpha^i = 1$, give the proportion of the firm that belongs to consumer i. We take the natural point of view (see [3], chapter 19) that the firm is an asset with return vector $d = \{d(s) : s \in S\}$ whose shares are tradeable in the financial market at time $t = 0$. Suppose that the firm can actually choose its random production plan within a range D of possible choices of return vectors. We assume that the return vector is chosen before the opening of financial markets at time $t = 0$. Moreover, we assume that we are dealing with a small project relative to the size of the economy. This would justify that the equilibrium spot prices $\boldsymbol{p}(s)$ and state prices $q = \{q(s) : s \in S\}$ are constants independent of the particular production plan chosen by the firm. For the state prices $q = \{q(s) : s \in S\}$, the market value of any production plan $d = \{d(s) : s \in S\}$ should naturally be given by

$$\mathrm{bid}(d; \mathcal{M}, q) = \inf_{\widehat{\pi} \in \mathcal{M}} \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) d(s).$$

Taking the bid price as the market value of a production plan is well-justified because the firm resides on the supply side of the economy for assets. In fact, the firm should be sold at the bid price. By virtue of the infimum, the bid price is a concave and typically nonlinear function on the space of random outcomes and is suited to being maximized. On the demand side of the economy for assets, reside the households. In particular, given state prices $q = \{q(s) : s \in S\}$ and spot prices $\boldsymbol{p}(s)$, consumer i

solves

$$\begin{aligned}
\max_{\boldsymbol{c}^i, z^i, \alpha^i} \quad & U(\boldsymbol{c}^i) = \sum_{s \in S} u[\boldsymbol{c}^i(s)]\pi(s) \\
\text{s.t.} \quad & \boldsymbol{p}(s) \cdot \boldsymbol{c}^i(s) \leq \boldsymbol{p}(s) \cdot \boldsymbol{y}^i(s) + p_1(s) z^i(s), \quad \forall s \in S, \\
& \operatorname{ask}(z^i; \mathcal{M}, q) \leq \alpha^i \operatorname{bid}(d; \mathcal{M}, q). \qquad (1.7)
\end{aligned}$$

It follows from the form of the budget constraint (1.7) that the objective of market value $\operatorname{bid}(d; \mathcal{M}, q)$ maximization will be the unanimous desire of the firm's initial owners. Furthermore, the market clearing conditions are given by

$$\begin{aligned}
& \sum_{s \in S} \frac{\widehat{\pi}(s)}{\pi(s)} q(s) \left(\sum_i z^i(s) - d(s) \right) \leq 0, \quad \forall \widehat{\pi} \in \mathcal{N}, \qquad (1.8) \\
& \sum_i \boldsymbol{c}^i(s) \leq \sum_i \boldsymbol{y}^i(s) + \left(\sum_i z^i(s) - d(s), 0, \ldots, 0 \right), \quad \forall s \in S.
\end{aligned}$$

Basically, if $\mathcal{N}$ is the set of all probability measures, then we recover the classical market clearing conditions

$$\begin{aligned}
& \sum_i z^i(s) - d(s) \leq 0, \ \forall s \in S, \\
& \sum_i \boldsymbol{c}^i(s) \leq \sum_i \boldsymbol{y}^i(s), \ \forall s \in S.
\end{aligned}$$

The one-consumer, one-producer economy

Consider again an economy in which there are two states of the world $S = \{s_1, s_2\}$ and one commodity. The economy consists of one consumer with endowment $y = (y(s_1), y(s_2)) = (3, 1)$ and a firm. Let us assume that the firm chooses its production plan $d = (d(s_1), d(s_2))$ from the set

$$D = \{d : \ d(s_1) \geq 0, \ d(s_2) \geq 0, \ d(s_1)^2 + d(s_2)^2 \leq 1\}.$$

Given the state price vector $(q, 1-q)$, the firm maximizes its bid value subject to its production capacity, i.e.,

$$\begin{aligned} \max_{d} \quad & \mathrm{bid}(d; \mathcal{M}, q) \\ \text{s.t.} \quad & d \in D. \end{aligned}$$

Thus, the firm's problem can be written as

$$\begin{aligned} \max_{d(s_1), d(s_2)} \quad & \inf_{\widehat{\pi} \in [m_L, m_R]} \frac{\widehat{\pi}}{\pi} q d(s_1) + \frac{1-\widehat{\pi}}{1-\pi}(1-q) d(s_2) \\ \text{s.t.} \quad & d(s_1)^2 + d(s_2)^2 \leq 1. \end{aligned}$$

Since the consumer is the owner of the firm and has less endowment in the second state of the world, it is plausible to assume that

$$\frac{q}{\pi} d(s_1) \leq \frac{1-q}{1-\pi} d(s_2).$$

Under this assumption, the firm's objective simplifies to

$$\begin{aligned} \max_{d(s_1), d(s_2)} \quad & \frac{m_R}{\pi} q d(s_1) + \frac{1-m_R}{1-\pi}(1-q) d(s_2) \\ \text{s.t.} \quad & d(s_1)^2 + d(s_2)^2 \leq 1. \end{aligned}$$

Moreover, given the state price vector $(q, 1-q)$ and the firm's market value of $\mathrm{bid}(d_*; \mathcal{M}, q)$, the household solves

$$\begin{aligned} \max_{c(s_1), c(s_2), z(s_1), z(s_2)} \quad & \pi u[c(s_1)] + (1-\pi) u[c(s_2)] \\ \text{s.t.} \quad & c(s) \leq y(s) + z(s), \ \forall s \in \{s_1, s_2\}, \\ & \frac{m_L}{\pi} q z(s_1) + \frac{1-m_L}{1-\pi}(1-q) z(s_2) \leq \mathrm{bid}(d_*; \mathcal{M}, q), \\ & \frac{m_R}{\pi} q z(s_1) + \frac{1-m_R}{1-\pi}(1-q) z(s_2) \leq \mathrm{bid}(d_*; \mathcal{M}, q). \end{aligned}$$

Table 1.6: Classical equilibrium – production economy.

$c_*(s_1)$	$c_*(s_2)$	$z_*(s_1)$	$z_*(s_2)$	μ_L	μ_R	$d_*(s_1)$	$d_*(s_2)$
3.47569	1.87961	0.475687	0.879615	0.409868	0	0.475687	0.879615

Table 1.7: Conic equilibrium – production economy.

$c_*(s_1)$	$c_*(s_2)$	$z_*(s_1)$	$z_*(s_2)$	μ_L	μ_R	$d_*(s_1)$	$d_*(s_2)$
3.56475	1.71	0.564754	0.710002	0.418992	0	0.733547	0.679639

Let $\pi = 0.5$ and $u(c) = \log(c)$. If $m_L = m_R = \pi$, $n_L = 0$, and $n_R = 1$, we get the following classical results. The market clearing condition

$$\frac{\widehat{\pi}}{\pi}q[z(s_1) - d(s_1)] + \frac{1-\widehat{\pi}}{1-\pi}(1-q)[z(s_2) - d(s_2)] \leq 0, \quad \forall \widehat{\pi} \in \mathcal{N} = [n_L, n_R],$$

gives $q = 0.350982$. Moreover, we obtain the consumption and production plans given in table 1.6. Note that $z_* - d_* = 0$, indicating positive excess supply in both states of the world. The utility value resulting from this financial configuration is given by 0.938429. Furthermore, if $m_L = 0.4, m_R = 0.6, n_L = 0.2$, and $n_R = 0.8$, we get $q = 0.418451$ and the consumption and production plans given in table 1.7. It should be noted that $z_* - d_* = (-0.168793, 0.0303636)$ which means that the financial market is compensating for the shortcomings in the second state of the world. Moreover, the utility value resulting from this financial configuration is given by 0.903795.

1.7 Concluding remarks

In summary, we have presented a modern perspective towards modeling general equilibrium under uncertainty. We have relied upon the definition of acceptable risks in order to generalize the classical complete markets model. Essentially, a risk is acceptable to the market, if it has a positive valuation under all scenarios considered by the financial system. Our modeling framework leads naturally to a two-price (conic) economy and relaxes the traditional positive excess supply assumption. In fact, in a conic environment, the aggregate economy is permitted to consume more than its endowment in some states of the world. Therefore, a conic economy is a bigger economy, relative to the classical models, because it is willing to tolerate more risk at the aggregate level. Our conic perspective towards modeling general equilibrium under uncertainty provides us with a unifying framework to model a whole spectrum between incomplete and complete, illiquid and perfectly liquid markets, and beyond. Moreover, we have managed to generalize a common tool in general equilibrium analysis, namely, the Edgeworth box. We have also preformed some welfare analysis exercises to draw the conclusion that the financial market configurations have real effect on individual and aggregate welfare of the economy. This can justify government intervention and regulation of markets in certain economic situations. Our conic perspective towards the economy enables us to ask some very serious questions; How forgiving should the financial system be? Put differently, how much risk taking should be permitted? Who or what determines the financial market configurations $\mathcal{M}$ and $\mathcal{N}$? In other words, is there a way to write an endogenous model of the financial market? What role can the government play in managing $\mathcal{M}$ and $\mathcal{N}$, and therefore the size of the economy? These are questions of managing the capital requirements and leverage of an economy and should be addressed by the Federal Reserve. These fundamental questions have been totally ignored because of the complete markets assumptions and as a consequence of the fundamental theorems of welfare economics. We have therefore demonstrated the importance of developing a science of leverage. Furthermore, we have briefly mentioned that a conic equilibrium model is inevitably immune to some of the most empirically criticized conclusions of a complete markets model. We have also addressed, from a conic perspective, the more difficult problem of modeling the firm's behavior in a general equilibrium under uncertainty. Throughout this chapter we have tried to maintain a normative rather than positive outlook. In particular, we have spent very little time addressing question of the form: Does an equilibrium exist? Are the equilibria typically isolated? Is the equilibrium unique? Is it stable? What are the effects of shocks? These are very important theoretical and methodological questions that are of relevance to any theory of equilibrium. Hence, these questions must be subject of future research, should the conic perspective towards the economy prove useful from a normative point of view.

Chapter 2: Dynamic conic general equilibrium

This chapter introduces dynamic conic equilibria for an infinite horizon pure exchange economy with stochastic endowments (see [4], chapter 8). These are useful for studying consumption, risk sharing, and asset pricing (see chapter 4). We consider two types of financial markets entailing different assets and timings of trades: a time 0 trading arrangement, and a sequential-trading structure. We will explain how to formulate a recursive structure within such an exchange economy. Recursive representations are very important in the analysis of dynamic systems in macroeconomics (see [4]) and therefore it will be of particular interest to learn how to devise a recursive representation of our dynamic conic equilibrium concept. This will be achieved by finding an appropriate formulation of a state vector in terms of which to cast the conic equilibrium.

2.1 Physical setting

Let us start by assuming that in each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. Let $s^t = [s_0, s_1, \ldots, s_t]$ denote the history of events up and until time t. The unconditional probability of observing a particular sequence of events s^t is given by a probability measure $\pi_t(s^t)$. For $\tau > t$, the probability of observing s^τ conditional on the realization of s^t will be denoted by $\pi_\tau^t(s^\tau)$. There are I agents in the economy labeled $i = 1, \dots, I$ with stochastic endowments $y^i = \{y_t^i(s^t)\}_{t=0}^\infty$ of one good. The history s^t is publicly observable. Household i orders consumption streams $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$ by

$$U(c^i) = \sum_{t=0}^\infty \sum_{s^t} \beta^t u\left[c_t^i(s^t)\right] \pi_t(s^t).$$

Notice that we are imposing identical preference orderings across all individuals i that can be represented in terms of discounted expected utility with common discount factor β, common utility function $u(\cdot)$, and common probability distributions $\pi_t(s^t)$. Here $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption $c \geq 0$ of one good. The utility function satisfies the usual Inada condition $\lim_{c \to 0} u'(c) = +\infty$. One role for this condition is to make sure that the consumption of each agent is strictly positive in every date-history pair. Another related role of the Inada condition is to deliver a state-by-state borrowing limit to impose in economies with sequential trading. Before trading, the situation of household i at time t depends on the history s^t. A natural measure of household i's "luck" in life is $\{y_0^i(s^0), y_1^i(s^1), \dots, y_t^i(s^t)\}$, which evidently in general depends on the history s^t. A remarkable and empirically questionable result in the classical complete markets models is that the consumption allocation at time t depends only on the aggregate endowment realization at time t and some time-invariant parameters that describe the time 0 initial distribution of wealth (see [4], chapter 8). The conic market structure of this chapter will break this result and will introduce history dependence into equilibrium allocations.

2.2 Time 0 trading

Furthermore, let us assume that trading occurs at time 0 and after observing s_0. In fact, for the initially given value of $s_0 = s^0$, we have $\pi_0(s_0) = \pi_0^0(s^0) = 1$. Similarly, we set $\pi_t(s^t) = \pi_t^0(s^t)$. Let us use $q^0 = \{q_t^0(s^t)\}_{t=0}^{\infty}$ to denote the state prices as of time $t = 0$ and after observing s^0. The superscript 0 in state price $q_t^0(s^t)$ refers to the date at which trades occur, while the subscript t refers to the date that deliveries are to be made. Similar to chapter 1, the financial market at time 0 and after observing state s^0 is characterized by two sets $\mathcal{M}^0$ and $\mathcal{N}^0$ of probability measures or "generalized scenarios". The more scenarios considered, the more conservative is the financial market. We denote a typical scenario by $\widehat{\pi}^0 = \{\widehat{\pi}_t^0(s^t)\}_{t=0}^{\infty}$. We assume $\pi^0 = \{\pi_t^0(s^t)\}_{t=0}^{\infty} \in \mathcal{M}^0 \subset \mathcal{N}^0$ which indicates that the physical measure $\pi = \pi^0$ is one of the scenarios considered by the financial market and that the financial market is more strict towards aggregate risk. In the following, we will present the formal definition of the dynamic conic equilibrium concept.

Definition 2.1 (dynamic conic equilibrium – time 0 trading)**.** *Given a financial economy*

$$\left(S, \pi^0, U(\cdot), \{y^i\}_{i=1}^{I}, \mathcal{M}^0, \mathcal{N}^0\right),$$

a collection formed by

- *state prices* $q^0 = \{q_t^0(s^t)\}_{t=0}^{\infty}$ *and,*
- *for every consumer* i*, trading* $z_*^i = \{z_{*t}^i(s^t)\}_{t=0}^{\infty}$ *and consumption* $c_*^i = \{c_{*t}^i(s^t)\}_{t=0}^{\infty}$

plans,

constitutes a dynamic conic equilibrium with time 0 trading if:

- *given state prices* q^0 *and the financial market configuration* $\mathcal{M}^0$ *at the individual level, for every consumer* $i = 1, \ldots, I$, *the trading* z_*^i *and consumption* c_*^i *plans solve the household's problem;*

$$\begin{aligned} \max_{z^i, c^i} \quad & U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u \left[c_t^i(s^t) \right] \pi_t(s^t) \\ \text{s.t.} \quad & c_t^i(s^t) \leq y_t^i(s^t) + z_t^i(s^t), \quad \forall t, s^t, \\ & \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t) \leq 0, \ \forall \widehat{\pi}^0 \in \mathcal{M}^0, \end{aligned} \tag{2.1}$$

- *and, given the financial market configuration* $\mathcal{N}^0$ *at the aggregate level, the state prices* $q^0 = \{q_t^0(s^t)\}_{t=0}^{\infty}$ *are chosen such that the markets clear; i.e.,*

$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t) \leq 0, \ \forall \widehat{\pi}^0 \in \mathcal{N}^0, \tag{2.2}$$

where $Z_t(s^t) := \sum_i z_t^i(s^t)$.

It is worth emphasizing that if $\mathcal{M}^0 = \{\pi^0\}$ is a singleton, then household i's budget constraint (2.1) simplifies to the classical one (see [4], chapter 8); i.e.,

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) z_t^i(s^t) \leq 0.$$

Moreover, if $\mathcal{N}^0 = \{\widehat{\pi}^0 : \sum_{s^t} \widehat{\pi}_t^0(s^t) = 1, \text{ for some } t\}$, the market clearing condition (2.2) yields the classical one, i.e.,

$$Z_t(s^t) \leq 0, \quad \forall t, s^t.$$

In contrast to the classical economy, the market clearing condition (2.2) allows the aggregate economy to transfer wealth through time and histories. Similar to chapter 1, we can define the notions of ask and bid prices.

Definition 2.2 (bid and ask prices). *Given state prices $q^0 = \{q_t^0(s^t)\}_{t=0}^{\infty}$, the ask and bid prices of a random cash-flow $z = \{z_t(s^t)\}_{t=0}^{\infty}$ are defined as*

$$\text{ask}\left(z; \mathcal{M}^0, q^0\right) = \sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t),$$

and

$$\text{bid}\left(z; \mathcal{M}^0, q^0\right) = \inf_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t),$$

respectively.

One can simply observe that $\text{ask}\,(z; \mathcal{M}^0, q^0) = -\,\text{bid}\,(-z; \mathcal{M}^0, q^0)$. Moreover, it is worth noting that if $\mathcal{M}^0 = \{\pi^0\}$ is a singleton, then

$$\text{ask}\left(z; \mathcal{M}^0, q^0\right) = \text{bid}\left(z; \mathcal{M}^0, q^0\right) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) z_t(s^t),$$

and we recover the classical law of one price.

Equilibrium prices and quantities

Assuming that for some $\widehat{\pi}^0(z^i) = \{\widehat{\pi}_t^0(s^t; z^i)\}_{t=0}^{\infty} \in \mathcal{M}^0$,

$$\sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; z^i)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t),$$

which indicates that the supremum is attainable, household i's budget constraint can be written as

$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; z^i)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t) \leq 0.$$

Attach a Lagrange multiplier μ^i to this constraint and use the Envelope theorem [27], to obtain the first order conditions

$$\beta^t u' \left[c_t^i(s^t)\right] \pi_t^0(s^t) = \mu^i \frac{\widehat{\pi}_t^0(s^t; z^i)}{\pi_t^0(s^t)} q_t^0(s^t), \tag{2.3}$$

for all i, t, s^t. This implies that

$$\frac{u' \left[c_t^i(s^t)\right]}{u' \left[c_t^j(s^t)\right]} = \frac{\mu^i}{\mu^j} \frac{\widehat{\pi}_t^0(s^t; z^i)}{\widehat{\pi}_t^0(s^t; z^j)}, \tag{2.4}$$

for all pairs (i, j). We can make the natural assumptions that $\widehat{\pi}_0^0(s^0) = q_0^0(s^0) = 1$, to obtain

$$\mu^i = u' \left[c_0^i(s_0)\right].$$

Note that if $\mathcal{M}^0 = \{\pi^0\}$ is a singleton, then

$$\widehat{\pi}_t^0(s^t; z^i) = \widehat{\pi}_t^0(s^t; z^j) = \pi_t^0(s^t),$$

and the ratios of marginal utilities between pairs of agents will be constant across all histories and dates, i.e.,

$$\frac{u' \left[c_t^i(s^t)\right]}{u' \left[c_t^j(s^t)\right]} = \frac{\mu^i}{\mu^j}.$$

However, in general, this no longer holds (see equation (2.4)) in a conic economy with an arbitrary financial market configuration $\mathcal{M}^0$. Similarly, assuming that for some $\widehat{\pi}^0(Z) \in \mathcal{N}^0$, where $Z = \sum_i z^i$, we have

$$\sup_{\widehat{\pi}^0 \in \mathcal{N}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; Z)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t),$$

meaning that the supremum is attainable, the market clearing condition (2.2) can be written as

$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; Z)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t) = 0.$$

Using the first order condition (2.3) for any household i, we obtain the following form for the market clearing condition,

$$\sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t}{\mu^i}\frac{\widehat{\pi}_t^0(s^t;Z)}{\widehat{\pi}_t^0(s^t;z^i)}u'\left[c_t^i(s^t)\right]\pi_t^0(s^t)Z_t(s^t)=0.$$

Generalized Negishi algorithm

To compute an equilibrium, we propose to modify[1] the *Negishi algorithm* (see [4], chapter 8). The following accomplishes this modification.

1. Fix $\mu^1=1$, throughout the algorithm, and guess some positive initial values for the remaining μ^i, $i=2,\ldots,I$.

2. Make initial guesses for $\widehat{\pi}^0(z^i)$, $i=1,\ldots,I$, and $\widehat{\pi}^0(Z)$. A good initial guess is usually given by the actual physical measure π^0.

3. Solve the following equations for candidate consumption $\{c^i\}_{i=1}^I$ and trading $\{z^i\}_{i=1}^I$ allocations:

$$\begin{aligned}
&c_t^i(s^t)=y_t^i(s^t)+z_t^i(s^t),\ \forall i=1,\ldots,I,\ \forall t\geq 0,\ \forall s^t,\\
&\frac{u'\left[c_t^i(s^t)\right]}{u'\left[c_t^1(s^t)\right]}=\frac{\mu^i}{\mu^1}\frac{\widehat{\pi}_t^0(s^t;z^i)}{\widehat{\pi}_t^0(s^t;z^1)},\ \forall i=2,\ldots,I,\ \forall t\geq 0,\ \forall s^t,\\
&\sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t}{\mu^1}u'\left[c_t^1(s^t)\right]\pi_t^0(s^t)z_t^1(s^t)=0,\\
&\sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t}{\mu^1}\frac{\widehat{\pi}_t^0(s^t;Z)}{\widehat{\pi}_t^0(s^t;z^1)}u'\left[c_t^1(s^t)\right]\pi_t^0(s^t)Z_t(s^t)=0.
\end{aligned}$$

[1] The non-linearities involved in the definition of a conic general equilibrium makes a convergence proof of the proposed algorithm non-trivial. A convergence proof would require imposing further restrictions on the sets $\mathcal{M}^0$ and $\mathcal{N}^0$ identifying the financial market configurations. This will be subject of future research.

4. Use the following for household 1 to solve for the price system q^0.

$$\beta^t u'\left[c_t^1(s^t)\right]\pi_t^0(s^t) = \mu^1 \frac{\widehat{\pi}_t^0(s^t;z^1)}{\pi_t^0(s^t)} q_t^0(s^t).$$

5. Check that the following requirements for households $i = 1, \ldots, I$ are satisfied

$$\sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t;z^i)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t).$$

   Moreover, corresponding to the market clearing condition, check the validity of the following requirement

$$\sup_{\widehat{\pi}^0 \in \mathcal{N}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t;Z)}{\pi_t^0(s^t)} q_t^0(s^t) Z_t(s^t).$$

   Update $\widehat{\pi}^0(z^i)$, for $i = 1, \ldots, I$, and $\widehat{\pi}^0(Z)$ accordingly. It is worth noting that this step boils down to maximizing linear objective functions.

6. Iterate on steps 3-5 until the requirements of step 5 are satisfied.

7. For $i = 2, \ldots, I$, check the budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t;z^i)}{\pi_t^0(s^t)} q_t^0(s^t) z_t^i(s^t) \leq 0.$$

   Increase μ^i for those i's that violate this constraint and decrease it for others.

8. Iterate to convergence on steps 2-7.

CRRA utility

Suppose that the one-period utility function is of the constant relative risk-aversion (CRRA) form

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \ \gamma > 0.$$

Then, equation (2.4) yields

$$c_t^i(s^t) = c_t^j(s^t) \left\{ \frac{\mu^i}{\mu^j} \frac{\widehat{\pi}_t^0(s^t; z^i)}{\widehat{\pi}_t^0(s^t; z^j)} \right\}^{-\frac{1}{\gamma}}. \tag{2.5}$$

In the classical case, where $\mathcal{M}^0 = \{\pi^0\}$, this equation simplifies to

$$c_t^i(s^t) = c_t^j(s^t) \left\{ \frac{\mu^i}{\mu^j} \right\}^{-\frac{1}{\gamma}},$$

and states that time t elements of consumption allocations for two distinct agents are constant fractions of one another. Combined with the classical market clearing condition

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \ \forall t, \ \forall s^t,$$

corresponding to $\mathcal{N}^0 = \{\widehat{\pi}^0 : \sum_{s^t} \widehat{\pi}_t^0(s^t) = 1, \text{ for some } t\}$, it says that individual consumption is perfectly correlated with the aggregate endowment or aggregate consumption. This implies that conditional on the history s^t, time t consumption $c_t^i(s^t)$ is independent of the household's individual endowment at t, s^t, $y_t^i(s^t)$. Mace [32], Cochrane [6], and Townsend [33] have tested and rejected versions of this conditional independence hypothesis. As is evident from equation (2.5), our dynamic conic equilibrium concept is capable of explaining[2] these rejections. In fact, for arbitrary financial market configurations $\mathcal{M}^0$, the consumption allocations ratio (2.5) for two distinct agents depends on time t, history s^t, and consumption allocations are not necessarily constant fractions of one another.

[2]As a matter of fact, we have not shown that our framework "explains" the empirical results seen in rejecting this conditional independence hypothesis, only that our conic framework is compatible with it not holding.

2.3 Sequential trading

Building upon the insight of Arrow [34] that one-period securities are enough to implement complete markets, this section introduces an alternative financial market structure, i.e., sequential trading. In order to construct such a sequential trading arrangement, we need to identify a variable to serve as the state in a value function for the household at date t. We begin by asking the following question. In the dynamic conic equilibrium with time 0 trading, what is the implied continuation wealth of household i at time t after history s^t, but before adding in its time t, history s^t endowment $y_t^i(s^t)$? This question can be answered by defining the household's continuation wealth or *financial wealth*, expressed in terms of the date t, history s^t consumption good, to be denoted by $a_t^i(s^t)$ and to be given by

$$a_t^i(s^t) = \sup_{\widehat{\pi}^t \in \mathcal{M}^t} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) z_\tau^i(s^\tau), \tag{2.6}$$

where

$$\widehat{\pi}_\tau^t(s^\tau) := \frac{\widehat{\pi}_\tau^0(s^\tau)}{\widehat{\pi}_t^0(s^t)}, \; q_\tau^t(s^\tau) := \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)}, \text{ and } \widehat{\pi}^t := \left\{\widehat{\pi}_\tau^t(s^\tau)\right\}_{\tau=t}^{\infty} \in \mathcal{M}^t.$$

Here, $\mathcal{M}^t$ denotes the set of generalized scenarios considered by the financial market as of time t and after observing history s^t. The above definition (2.6) for financial wealth can be further justified if we rewrite it in the following form;

$$\sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) z_\tau^i(s^\tau) \le a_t^i(s^t), \; \forall \widehat{\pi}^t \in \mathcal{M}^t.$$

This means that household i's future capabilities $\{z_\tau^i(s^\tau)\}_{\tau=t}^{\infty}$ in transferring wealth across time and histories, as of time t after history s^t, is limited by its financial wealth $a_t^i(s^t)$. At time 0, the household's budget constraint (2.1) in the dynamic conic equilibrium with time 0 trading gives $a_0^i(s^0) = 0$ for all $i = 1, \ldots, I$. However, at time $t > 0$, the financial wealth $a_t^i(s^t)$ typically differs from zero for individual i. Furthermore, we obtain the following proposition.

Proposition 2.1. *The financial wealth of consumer i satisfies the following recursive inequality*

$$a_t^i(s^t) \leq \tilde{a}_t^i(s^t)$$

where

$$\tilde{a}_t^i(s^t) := z_t^i(s^t) + \sup_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t),$$

and

$$\widehat{\pi}_{t+1}^t = \left\{ \widehat{\pi}_{t+1}^t(s_{t+1}) : s_{t+1} \in S \right\} \in \mathcal{M}_{t+1}^t.$$

Here, $\mathcal{M}_{t+1}^t$ represents the one-period-ahead financial market at time t and in history s^t.

Proof. Let us start by defining

$$x_t^i\left(s^t; \widehat{\pi}^t\right) := \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) z_\tau^i(s^\tau),$$

and observing that

$$x_t^i\left(s^t; \widehat{\pi}^t\right) = z_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) x_{t+1}^i\left(s_{t+1}, s^t; \widehat{\pi}^{t+1}\right),$$

since $\pi_\tau^t(s^\tau) = \pi_{t+1}^t(s_{t+1}) \pi_\tau^{t+1}(s^\tau)$, $\widehat{\pi}_\tau^t(s^\tau) = \widehat{\pi}_{t+1}^t(s_{t+1}) \widehat{\pi}_\tau^{t+1}(s^\tau)$, and

$$q_\tau^t(s^\tau) = q_{t+1}^t(s_{t+1}) q_\tau^{t+1}(s^\tau).$$

Moreover,

$$\begin{aligned} a_t^i(s^t) &= \sup_{\widehat{\pi}^t \in \mathcal{M}^t} x_t^i\left(s^t; \widehat{\pi}^t\right), \\ a_{t+1}^i(s_{t+1}, s^t) &= \sup_{\widehat{\pi}^{t+1} \in \mathcal{M}^{t+1}} x_{t+1}^i\left(s_{t+1}, s^t; \widehat{\pi}^{t+1}\right), \end{aligned}$$

and consequently

$$x_{t+1}^i\left(s_{t+1}, s^t; \widehat{\pi}^{t+1}\right) \leq a_{t+1}^i(s_{t+1}, s^t), \quad \forall \widehat{\pi}^{t+1} \in \mathcal{M}^{t+1}.$$

We want to show that $a_t^i(s^t) \leq \tilde{a}_t^i(s^t)$. For all $\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t$, we have

$$z_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t) \leq \tilde{a}_t^i(s^t).$$

Therefore, for every $\widehat{\pi}^{t+1} \in \mathcal{M}^{t+1}$ and $\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t$, we obtain

$$z_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) x_{t+1}^i\left(s_{t+1}, s^t; \widehat{\pi}^{t+1}\right) \leq \tilde{a}_t^i(s^t).$$

Thus, for every $\widehat{\pi}^t \in \mathcal{M}^t$,

$$x_t^i\left(s^t; \widehat{\pi}^t\right) \leq \tilde{a}_t^i(s^t),$$

which yields $a_t^i(s^t) \leq \tilde{a}_t^i(s^t)$. $\square$

In proposition 2.1, we have established that $a_t^i(s^t) \leq \tilde{a}_t^i(s^t)$. We can obtain the actual equality $a_t^i(s^t) = \tilde{a}_t^i(s^t)$, if we assume that for some $\widehat{\pi}^{t+1}(z^i) \in \mathcal{M}^{t+1}$,

$$a_{t+1}^i(s_{t+1}, s^t) = x_{t+1}^i\left(s_{t+1}, s^t; \widehat{\pi}^{t+1}(z^i)\right),$$

indicating that the supremum is attainable. Let us perform a proof by contradiction and falsely assume that $a_t^i(s^t) < \tilde{a}_t^i(s^t)$. Therefore,

$$a_t^i(s^t) < z_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t),$$

for some $\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t$. Thus,

$$a_t^i(s^t) < z_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) x_{t+1}^i \left(s_{t+1}, s^t; \widehat{\pi}^{t+1}(z^i)\right),$$

and we arrive at the contradiction that

$$a_t^i(s^t) < x_t^i \left(s^t; \widehat{\pi}^t\right) \le a_t^i(s^t),$$

for some $\widehat{\pi}^t \in \mathcal{M}^t$ given explicitly by $\widehat{\pi}_\tau^t(s^\tau) = \widehat{\pi}_{t+1}^t(s_{t+1})\widehat{\pi}_\tau^{t+1}(s^\tau; z^i)$.

In moving from the economy with time 0 trading to one with sequential trading, we propose to match the time t, history s^t wealth of the household in the sequential economy with the equilibrium tail wealth $a_t^i(s^t)$ from the dynamic conic equilibrium with time 0 trading. However, before we give the definition of conic equilibrium with sequential trading, we have to say something about debt limits, a feature that was only implicit in the time 0 budget constraint in the economy with time 0 trading. Hence, we define the *natural debt limit* $b_t^i(s^t)$ of consumer i at time t and in history s^t to be given by

$$b_t^i(s^t) \quad = \quad \inf_{\widehat{\pi}^t \in \mathcal{M}^t} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) y_\tau^i(s^\tau) =: \text{bid}\left(y^i; \mathcal{M}^t, q^t\right).$$

It is the maximal value that household i can repay starting from that period, assuming that his consumption is zero always. To rule out Ponzi schemes, we impose the state-by-state borrowing constraints

$$-a_{t+1}^i(s_{t+1}, s^t) \le b_{t+1}^i(s_{t+1}, s^t), \ \forall s_{t+1} \in S.$$

Similar to proposition (2.1) and the short argument following it, we obtain

$$b_t^i(s^t) = y_t^i(s^t) + \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) b_{t+1}^i(s_{t+1}, s^t),$$

We are now well-equipped to define the dynamic conic equilibrium concept with sequential trading.

Definition 2.3 (dynamic conic equilibrium – sequential trading)**.** *Given*

- *the initial state of the economy* $s_0 \in S$*,*
- *an initial distribution of wealth* $\{a_0^i(s_0)\}_{i=1}^I$*,*
- *and a sequence of one-period-ahead conic financial markets*

$$\left\{\mathcal{M}_{t+1}^t, \mathcal{N}_{t+1}^t\right\}_{t=0}^{\infty},$$

a collection formed by

- *pricing kernels* $q_{t+1}^t = \left\{q_{t+1}^t(s_{t+1}) : s_{t+1} \in S\right\}$*, for all* $t \geq 0$ *and* s^t*,*
- *and, for every consumer* i*, borrowing limit* $b^i = \{b_t^i(s^t)\}_{t=0}^{\infty}$*, trading plan* $a_*^i = \{a_{*t}^i(s^t)\}_{t=0}^{\infty}$*, and consumption stream* $c_*^i = \{c_{*t}^i(s^t)\}_{t=0}^{\infty}$*,*

constitutes a dynamic conic equilibrium with sequential trading if:

- *for every consumer* $i = 1, \ldots, I$*, the borrowing limit satisfies the recursion*

$$b_t^i(s^t) = y_t^i(s^t) + \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) b_{t+1}^i(s_{t+1}, s^t),$$

- *for every consumer* $i = 1, \ldots, I$*, given* $a_0^i(s_0)$*, the borrowing limit* $b^i = \{b_t^i(s^t)\}_{t=0}^{\infty}$*, the pricing kernels* $\left\{q_{t+1}^t\right\}_{t=0}^{\infty}$*, and the financial market primitives, the trading* *and consumption plans solve the problem*

$$\begin{aligned}
\max_{a^i, c^i} \quad & U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u \left[c_t^i(s^t) \right] \pi_t(s^t) \\
\text{s.t.} \quad & c_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t), \\
& \qquad \forall t, \ \forall s^t, \ \forall \widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t, \qquad (2.7) \\
& -a_{t+1}^i(s_{t+1}, s^t) \leq b_{t+1}^i(s_{t+1}, s^t), \ \forall t, \ \forall s^t, \ \forall s_{t+1} \in S,
\end{aligned}$$

- *and the pricing kernels* $q_{t+1}^t = \left\{ q_{t+1}^t(s_{t+1}) : s_{t+1} \in S \right\}$ *are chosen such that the markets clear; i.e.,*

$$\sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) \sum_i a_{t+1}^i(s_{t+1}, s^t) \leq \sum_i a_t^i(s^t), \qquad (2.8)$$

$$\forall t, \ \forall s^t, \ \forall \widehat{\pi}_{t+1}^t \in \mathcal{N}_{t+1}^t.$$

Here, to be consistent with the time 0 trading setup, we set $a_0^i(s_0) = 0$ for all $i = 1, \ldots, I$. It should be noted that the budget constraint of household i in the conic equilibrium with sequential trading is well justified in light of proposition 2.1 and the short argument following it. Moreover, it is worth noting that if $\mathcal{M}_{t+1}^t = \left\{ \pi_{t+1}^t \right\}$ is a singleton, then household i's budget constraint (2.7) simplifies to the classical one (see [4], chapter 8); i.e.,

$$c_t^i(s^t) + \sum_{s_{t+1} \in S} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t).$$

Moreover, if $\mathcal{N}_{t+1}^t = \left\{ \widehat{\pi}_{t+1}^t : \sum_{s_{t+1} \in S} \widehat{\pi}_{t+1}^t(s_{t+1}) = 1 \right\}$, the market clearing condition (2.8) simplifies to

$$\frac{q_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} \sum_i a_{t+1}^i(s_{t+1}, s^t) \leq \sum_i a_t^i(s^t), \quad \forall s_{t+1} \in S.$$

This, along with $a_0^i(s_0) = 0$, yields the classical market clearing condition (see [4], chapter 8); i.e.,

$$\sum_i a_{t+1}^i(s_{t+1}, s^t) \leq 0, \quad \forall t, \ \forall s^t, \ \forall s_{t+1} \in S.$$

Equilibrium prices and quantities

Regarding consumer $i = 1, \ldots, I$, let us assume that for some generalized scenario $\widehat{\pi}_{t+1}^t(a_{t+1}^i) \in \mathcal{M}_{t+1}^t$, we have

$$\begin{aligned} &\sup_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t) \\ &\qquad = \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1}; a_{t+1}^i)}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t). \end{aligned} \tag{2.9}$$

Then, household i's budget constraint (2.7) at time t in history s^t can be written as

$$c_t^i(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1}; a_{t+1}^i)}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t).$$

Let $\eta_t^i(s^t)$ and $\nu_{t+1}^i(s_{t+1}, s^t)$ be the non-negative Lagrange multipliers on the budget constraint and the borrowing constraint, respectively. Forming the Lagrangian and using the Envelope theorem [27], we obtain the following first order conditions:

$$\begin{aligned} &\beta^t u'\left[c_t^i(s^t)\right] \pi_t^0(s^t) - \eta_t^i(s^t) = 0, \\ &-\eta_t^i(s^t) \frac{\widehat{\pi}_{t+1}^t(s_{t+1}; a_{t+1}^i)}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) + \nu_{t+1}^i(s_{t+1}, s^t) + \eta_{t+1}^i(s_{t+1}, s^t) = 0, \end{aligned}$$

for all s_{t+1}, t, s^t. In the optimal solution to this problem, the natural debt limit will not be binding, and hence the Lagrange multipliers $\nu_{t+1}^i(s_{t+1}, s^t)$ all equal zero for the following reason (see [4], chapter 8): if there were any history s^{t+1} leading to a binding natural debt limit, the household would from then on have to set consumption equal to zero in order to honor its debt. Because the household's utility function satisfies the Inada condition $\lim_{c \to 0} u'(c) = +\infty$, that would mean that all future marginal utilities would be infinite. Thus, it would be easy to find alternative affordable allocations that yield higher expected utility by postponing earlier consumption to periods after such a binding constraint. After setting those multipliers equal to zero, the first-order conditions imply the following restrictions on the optimal choices of consumption:

$$\frac{\widehat{\pi}^t_{t+1}(s_{t+1}; a^i_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) = \beta \frac{u'\left[c^i_{t+1}(s_{t+1}, s^t)\right]}{u'\left[c^i_t(s^t)\right]} \pi^t_{t+1}(s_{t+1}), \tag{2.10}$$

for all s_{t+1}, t, s^t. Now, for comparison, take household i's first order condition (2.3) for the dynamic conic economy with time 0 trading from two successive periods and divide one by the other to get

$$\frac{\widehat{\pi}^t_{t+1}(s_{t+1}; z^i)}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) = \beta \frac{u'\left[c^i_{t+1}(s_{t+1}, s^t)\right]}{u'\left[c^i_t(s^t)\right]} \pi^t_{t+1}(s_{t+1}).$$

This shows that the dynamic conic equilibrium with sequential trading is consistent[3] with the financial economy with time 0 trading.

2.4 Recursive formulation

At this level of generality, the one-period-ahead financial market configurations $\mathcal{M}^t_{t+1}$, $\mathcal{N}^t_{t+1}$, the pricing kernels q^t_{t+1}, and the wealth distributions $a^i_t(s^t)$ in the

[3]Establishing that equilibrium allocations are exactly the same in the economy with time 0 trading and in a sequential-trading arrangement is non-trivial due to the non-linearities involved in a Conic framework and requires further investigations.

sequential trading conic economy all depend on the history s^t, so all are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$. This makes it difficult to confront our sequential trading dynamic conic economy with empirical observations. Following much of the literature in macroeconomics and econometrics, we prefer a model in which economic outcomes are functions of a limited number of "state variables" that summarize the effects of past events and current information. This leads us to make a few specializations of the exogenous processes that facilitates a recursive formulation of the dynamic conic equilibrium with sequential trading. Let $\pi(s'|s)$ be a Markov chain with given initial distribution $\pi_0(s)$ and state space $s \in S$. It means that $\text{Prob}(s_{t+1} = s'|s_t = s) = \pi(s'|s)$ and $\text{Prob}(s_0 = s) = \pi_0(s)$. The chain induces a sequence of probability measures $\pi_t(s^t)$ on histories s^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_0)\pi_0(s_0).$$

In this chapter, we have assumed that trading occurs after the initially given value of s_0 has been observed, which we capture by setting $\pi_0(s_0) = 1$. Because of the Markov property, for $\tau > t$, the conditional probability $\pi_\tau^t(s^\tau)$ depends only on the state s_t at time t and does not depend on the history before t,

$$\pi_\tau^t(s^\tau) = \pi(s_\tau|s_{\tau-1})\pi(s_{\tau-1}|s_{\tau-2})\dots\pi(s_{t+1}|s_t).$$

Moreover, we assume that households' endowments in period t are time invariant measurable functions of s_t, $y_t^i(s^t) = y^i(s_t)$ for each i. This along with the Markov assumption for s_t imposes further structure on our dynamic conic equilibrium. This motivates us to seek the following recursive formulation.

Definition 2.4 (recursive conic equilibrium)**.** *Given*

- *the initial state of the economy* $s_0 \in S$,

- *an initial distribution of wealth* $\{a_0^i(s_0)\}_{i=1}^I$,

- *and a one-period-ahead conic financial market*

$$\{(\mathcal{M}(s), \mathcal{N}(s)) : s \in S\},$$

a collection formed by

- *a pricing kernel* $q(\cdot|s) = \{q(s'|s) : s' \in S\}$ *and,*

- *for every consumer* i*, borrowing limit* $b^i(s)$*, value function* $v^i(a^i, s)$*, and decision rules* $c^i(a^i, s)$ *and* $\widetilde{a}^i(s', a^i, s)$*,*

constitutes a recursive conic equilibrium if:

- *for every consumer* $i = 1, \ldots, I$*, the state-by-state borrowing limit satisfies the recursion*

$$b^i(s) = y^i(s) + \inf_{\widehat{\pi}(\cdot|s) \in \mathcal{M}(s)} \sum_{s' \in S} \frac{\widehat{\pi}(s'|s)}{\pi(s'|s)} q(s'|s) b^i(s', s),$$

- *for every consumer* $i = 1, \ldots, I$*, given* $a_0^i(s_0)$*, the debt limits* $b^i(s)$*, the pricing kernel* $q(\cdot|s)$*, and the financial market configuration* $\mathcal{M}(s)$*, the value function* $v^i(a^i, s)$ *and decision rules* $c^i(a^i, s)$ *and* $\widetilde{a}^i(s', a^i, s)$ *solve the Bellman equation*

$$v^i(a^i, s) = \max_{c^i, \{\widetilde{a}^i(s') :\ s' \in S\}} u(c^i) + \beta \sum_{s' \in S} v^i\left(\widetilde{a}^i(s'), s'\right) \pi(s'|s),$$

subject to

$$c^i + \sum_{s' \in S} \frac{\widehat{\pi}(s'|s)}{\pi(s'|s)} q(s'|s) \widetilde{a}^i(s') \leq y^i(s) + a^i, \ \forall \widehat{\pi}(\cdot|s) \in \mathcal{M}(s), \quad (2.11)$$

$$-\widetilde{a}^i(s') \leq b^i(s', s), \ \forall s' \in S,$$

- *For any realizations of* $\{s_t\}_{t=0}^{\infty}$*, the corresponding consumption and asset portfolios of consumer* i *implied by the decision rules are given by*

$$\begin{aligned}
c_t^i &= c^i(a_t^i, s_t), \\
a_{t+1}^i(s_{t+1}) &= \widetilde{a}^i(s_{t+1}, a_t^i, s_t), \\
c_{t+1}^i &= c^i(a_{t+1}^i(s_{t+1}), s_{t+1}) = c^i(\widetilde{a}^i(s_{t+1}, a_t^i, s_t), s_{t+1}),
\end{aligned}$$

- *and given the financial market configuration* $\mathcal{N}(s)$*, the pricing kernel* $q(\cdot|s) = \{q(s'|s) : s' \in S\}$ *is chosen such that the markets clear. Specifically, for any realizations of* $\{s_t\}_{t=0}^{\infty}$*, the asset portfolios implied by the decision rules satisfy*

$$\begin{aligned}
\sum_{s_{t+1} \in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) \sum_i \widetilde{a}^i(s_{t+1}, a_t^i, s_t) & \\
\leq \sum_i a_t^i, \ \forall \widehat{\pi}(\cdot|s_t) \in \mathcal{N}(s_t).
\end{aligned} \tag{2.12}$$

It is worth noting that if $\mathcal{M}(s) = \{\pi(\cdot|s)\}$ is a singleton, then household i's budget constraint (2.11) simplifies to the classical one, i.e.,

$$c^i + \sum_{s' \in S} q(s'|s)\widetilde{a}^i(s') \leq y^i(s) + a^i.$$

Moreover, if $\mathcal{N}(s) = \left\{\widehat{\pi}(\cdot|s) : \sum_{s' \in S} \widehat{\pi}(s'|s) = 1\right\}$, the market clearing condition (2.12) simplifies to

$$\frac{q(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} \sum_i \widetilde{a}^i(s_{t+1}, a_t^i, s_t) \leq \sum_i a_t^i, \quad \forall s_{t+1} \in S.$$

This, along with $a_0^i(s_0) = 0$, yields the classical market clearing condition

$$\sum_i \widetilde{a}^i(s_{t+1}, a_t^i, s_t) \leq 0, \quad \forall s_{t+1} \in S.$$

Equilibrium prices and quantities

Regarding consumer $i = 1, \ldots, I$, let us assume that for some generalized scenario $\widehat{\pi}(\cdot|s;\widetilde{a}^i) \in \mathcal{M}(s)$, we have

$$\sup_{\widehat{\pi}(\cdot|s)\in\mathcal{M}(s)} \sum_{s'\in S} \frac{\widehat{\pi}(s'|s)}{\pi(s'|s)} q(s'|s)\widetilde{a}^i(s') = \sum_{s'\in S} \frac{\widehat{\pi}(s'|s;\widetilde{a}^i)}{\pi(s'|s)} q(s'|s)\widetilde{a}^i(s').$$

Then, household i's budget constraint (2.11) can be written as

$$c^i + \sum_{s'\in S} \frac{\widehat{\pi}(s'|s;\widetilde{a}^i)}{\pi(s'|s)} q(s'|s)\widetilde{a}^i(s') \le y^i(s) + a^i.$$

Let $\eta^i(a^i, s)$ and $\nu^i(s', a^i, s)$ be the non-negative Lagrange multipliers on the budget constraint and the borrowing constraint, respectively. Forming the Lagrangian and using the Envelope theorem [27], we obtain the following first order conditions:

$$\begin{aligned}
&u'\left[c^i(a^i, s)\right] - \eta^i(a^i, s) = 0, \\
&\beta v_a^i\left(\widetilde{a}^i(s', a^i, s), s'\right)\pi(s'|s) - \eta^i(a^i, s)\frac{\widehat{\pi}(s'|s;\widetilde{a}^i)}{\pi(s'|s)} q(s'|s) + \nu^i(s', a^i, s) = 0,
\end{aligned}$$

for all s', a^i, s. Using the Envelope theorem another time, we obtain

$$v_a^i(a^i, s) = \eta^i(a^i, s).$$

Moreover, in the optimal solution to this problem, the natural debt limit will not be binding (see [4], chapter 8), and hence the Lagrange multipliers $\nu^i(s', a^i, s)$ all equal zero. After setting those multipliers equal to zero, the first-order conditions imply the following restrictions on the optimal choices of consumption; i.e.,

$$\frac{\widehat{\pi}(s'|s;\widetilde{a}^i)}{\pi(s'|s)} q(s'|s) = \beta \frac{u'\left[c^i(\widetilde{a}^i(s', a^i, s), s')\right]}{u'\left[c^i(a^i, s)\right]}\pi(s'|s),\ \forall s',\ \forall a^i,\ \forall s.$$

2.5 Concluding remarks

The dynamic conic framework of this chapter could serve much of macroeconomics as foundation. We briefly mentioned how this modeling framework can be applied to risk sharing. This approach could also help explain a variety of empirical observations that seem to be inconsistent with complete markets models. For instance, in chapter 4, we will refer to the equity premium puzzle [8] and will describe how a conic perspective could potentially explain the puzzle. Furthermore, to take monetary theory as another example, complete markets models dispose of any need for money or any other medium of exchange. This is because complete markets contain an efficient multilateral trading mechanism. Any modern model of money (see e.g., the cash-in-advance model of Lucas [35], the shopping time model [36], the Townsend turnpike model [37], or the Kiyotaki-Wright search model [38]) introduces frictions that impede complete markets. Along exactly the same lines, the conic perspective adopted in this work is capable of introducing the required impediments to complete markets. However, these conjectures require further investigations that are beyond the scope of this work and could be subject of future research. Moreover, the issues of *dynamic consistency*, related to ensuring that trading plans acceptable at $t + 1$ are also automatically acceptable at t, are in the background of our dynamic formulation. This requires valuation functionals, i.e., ask and bid prices, to be nonlinear expectations related to solutions of backward stochastic differential equations (BSDE's). Although we didn't go into much details, recent years have seen the development of the theory of nonlinear conditional expectations that keep all the properties of conditional expectations excepting the linearity. Peng [39–41], Bion-Nadal [42–44], Jobert and Rogers [45], Cohen and Elliott [46], and El Karoui, Peng and Quenez [47], have important contributions in this direction.

Chapter 3: Conic real business cycle model

In this chapter, we shall focus on the stochastic growth model (see [4], chapter 12). The stochastic growth model was formulated and fully analyzed by Brock and Mirman [48] and is a workhorse for studying macroeconomic fluctuations. Kydland and Prescott [49] used the framework to study quantitatively the importance of persistent technology shocks for business cycle fluctuations. Many other authors have used either a stochastic or nonstochastic version of the growth model to approximate features of the business cycle. To name a few prominent works in this direction, one could mention the papers by Lucas [50], Prescott [51], Ingram, Kocherlakota, and Savin [52], Hall [53], Wen [54], Otrok [55], Christiano, Eichenbaum, and Evans [56], Christiano, Motto, and Rostagno [57], Greenwood, Hercowitz, and Krusell [58], Jonas Fisher [59], Davig, Leeper, and Walker [60], Schmitt-Grohe and Uribe [61], and Kim and Kim [62]. This chapter is in the spirit of the papers by Lucas and Prescott [63] and Mehra and Prescott [8], but differs substantially in its financial market configurations. We introduce, to the stochastic growth model, alternative ways of representing dynamic conic equilibria. In particular, similar to chapter 2, we consider two types of conic financial markets entailing different assets and timings of trades: a time 0 trading arrangement, and a sequential-trading structure. We are interested in formulating recursive representations, however, because there are endogenous state variables in the growth model, we shall have to extend the method used in chapter 2.

3.1 Physical setting

Let us first spell out the basic ingredients of the stochastic growth model (see [4], chapter 12): preferences, endowment, technology, and information. In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. Let $s^t = [s_0, s_1, \ldots, s_t]$ denote the history of events up and until time t. The unconditional probability of observing a particular sequence of events s^t is given by a probability measure $\pi_t(s^t)$. For $\tau > t$, the probability of observing s^τ conditional on the realization of s^t can be written as $\pi^t_\tau(s^\tau)$. We use s^t as a commodity space in which goods are differentiated by histories. A representative household has preferences over non-negative streams of consumption $c = \{c_t(s^t)\}_{t=0}^\infty$ and leisure $\ell = \{\ell_t(s^t)\}_{t=0}^\infty$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t(s^t), \ell_t(s^t)\right] \pi_t(s^t),$$

where $0 < \beta < 1$ and u is strictly increasing in its two arguments, twice continuously differentiable, strictly concave, and satisfies the Inada conditions

$$\lim_{c \to 0} u_c(c, \ell) = \lim_{\ell \to 0} u_\ell(c, \ell) = \infty.$$

In each period, the representative household is endowed with one unit of time that can be devoted to leisure $\ell_t(s^t)$ or labor $n_t(s^t)$, i.e.,

$$1 = \ell_t(s^t) + n_t(s^t).$$

The household is also endowed with a capital stock k_0 at the beginning of period 0. The technology is

$$\begin{aligned} c_t(s^t) + x_t(s^t) &\leq A_t(s^t)F\left(k_t(s^{t-1}), n_t(s^t)\right), && (3.1) \\ k_{t+1}(s^t) &= (1-\delta)k_t(s^{t-1}) + x_t(s^t), && (3.2) \end{aligned}$$

where F is a twice continuously differentiable, constant-returns-to-scale production function with capital $k_t(s^{t-1})$ and labor $n_t(s^t)$ as inputs. Here, $A_t(s^t)$ is a stochastic process of technology shocks. Outputs are the consumption $c_t(s^t)$ and investment $x_t(s^t)$ goods. In (3.2), the investment good augments a capital stock that is depreciating at the rate δ. Negative values for $x_t(s^t)$ are permitted and mean that the capital stock can be converted back into the consumption good. We assume that the production function satisfies the standard assumptions of positive but diminishing marginal products,

$$F_j(k,n) > 0, \quad F_{jj}(k,n) < 0, \quad \text{for } j = k, n;$$

and the usual Inada conditions,

$$\begin{aligned} \lim_{k\to 0} F_k(k,n) &= \lim_{n\to 0} F_n(k,n) = \infty, \\ \lim_{k\to 0} F_k(k,n) &= \lim_{n\to 0} F_n(k,n) = \infty. \end{aligned}$$

3.2 Time 0 trading

Trades occur among a representative household and two types of representative firms. Let us first assume that trading occurs at time 0 and after observing s_0. In fact, for the initially given value of $s_0 = s^0$, we have $\pi_0(s_0) = \pi_0^0(s^0) = 1$. Similarly, we set $\pi_t(s^t) = \pi_t^0(s^t)$. Let us use $q^0 = \{q_t^0(s^t)\}_{t=0}^{\infty}$ to denote the state prices as of time $t = 0$ and after observing s^0. The superscript 0 in the state price $q_t^0(s^t)$ refers to the date at which trades occur, while the subscript t refers to the date that deliveries are to be made. Similar to chapters 1 and 2, the financial market at time 0 and after observing state s^0 is characterized by two sets $\mathcal{M}^0$ and $\mathcal{N}^0$ of probability measures or "generalized scenarios". The more scenarios considered, the more conservative is the financial market. We denote a typical scenario by $\widehat{\pi}^0 = \{\widehat{\pi}_t^0(s^t)\}_{t=0}^{\infty}$. We assume $\pi^0 = \{\pi_t^0(s^t)\}_{t=0}^{\infty} \in \mathcal{M}^0 \subset \mathcal{N}^0$ which indicates that the physical measure $\pi = \pi^0$ is one of the scenarios considered by the financial market and that the financial market is more strict towards aggregate risk. Before we describe the problems of the representative household and the two types of firms in the production economy with time 0 trading, it must be emphasized that in the economy we include spot markets for both labor and capital services that reopen in each period. Hence, it is important to distinguish between the spot markets and the financial market.

Household

In the spot market for labor, the household sells labor services to the type I firm that operates the production technology (3.1). Let $w_t(s^t)$ denote the spot price of labor at time t and in history s^t. Moreover, the household owns the initial capital stock k_0 and, in the spot market for capital at date 0, sells it to the type II firm that operates the capital storage technology (3.2). Let p_{k_0} be the unit price of the initial capital stock. Therefore, the household maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u \left[c_t(s^t), 1 - n_t(s^t)\right] \pi_t(s^t),$$

subject to

$$c_t(s^t) \leq w_t(s^t) n_t(s^t) + z_t(s^t), \quad \forall t, s^t,$$
$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t) \leq p_{k_0} k_0, \quad \forall \widehat{\pi}^0 \in \mathcal{M}^0. \tag{3.3}$$

Notice how the household's budget constraints emphasize the distinction between spot markets and the financial market. Assuming that for some $\widehat{\pi}^0(z) \in \mathcal{M}^0$,

$$\sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; z)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t),$$

in other words the supremum is attainable, the household's budget constraint (3.3) can be equivalently written as

$$\sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; z)}{\pi_t^0(s^t)} q_t^0(s^t) z_t(s^t) \leq p_{k_0} k_0.$$

Employing the Envelope theorem [27], the first order conditions with respect to $z_t(s^t)$ and $n_t(s^t)$, respectively, are

$$\beta^t u_c \left[c_t(s^t), 1 - n_t(s^t)\right] \pi_t^0(s^t) = \eta \frac{\widehat{\pi}_t^0(s^t; z)}{\pi_t^0(s^t)} q_t^0(s^t), \tag{3.4}$$
$$\beta^t u_\ell \left[c_t(s^t), 1 - n_t(s^t)\right] \pi_t^0(s^t) = \eta \frac{\widehat{\pi}_t^0(s^t; z)}{\pi_t^0(s^t)} q_t^0(s^t) w_t(s^t), \tag{3.5}$$

where $\eta > 0$ is a multiplier on the budget constraint. It is natural to set $\widehat{\pi}_0^0(s^0) = q_0^0(s^0) = 1$ and obtain $\eta = u_c\left[c_0(s_0), 1 - n_0(s_0)\right]$.

Type I firm

At each date $t \geq 0$ after history s^t, the type I firm is a production firm that operates the production technology (3.1) and solves a static optimization problem. In particular, the type I firm seeks to maximize

$$A_t(s^t)F(k_t^I(s^t), n_t(s^t)) - r_t(s^t)k_t^I(s^t) - w_t(s^t)n_t(s^t).$$

In fact, in the spot markets at time t and in history s^t, the type I firm rents capital $k_t^I(s^t)$ from the type II firm and labor $n_t(s^t)$ from the household at rental prices $r_t(s^t)$ and $w_t(s^t)$, respectively. The first order conditions are

$$r_t(s^t) = A_t(s^t)F_k\left(k_t^I(s^t), n_t(s^t)\right), \tag{3.6}$$

$$w_t(s^t) = A_t(s^t)F_n\left(k_t^I(s^t), n_t(s^t)\right). \tag{3.7}$$

If these conditions are satisfied, the firm makes zero profits and its size is indeterminate. The firm of type I is willing to produce any quantities of $c_t(s^t)$ and $x_t(s^t)$ that the market demands, provided that these zero profit conditions are satisfied. According to these equilibrium conditions, each input in the production technology is paid its marginal product, and hence profit maximization of the type I firm ensures an efficient allocation of labor services and capital. Moreover, since the production function has constant returns to scale, we can define

$$F(k, n) =: f(\overline{k})n,$$

where $\overline{k} := k/n$. Another property of a linearly homogeneous function $F(k, n)$ is that its first derivatives are homogeneous of degree 0, and thus the first derivatives are functions only of the ratio $\overline{k}$. In particular, we have

$$\begin{aligned}
F_k(k,n) &= \frac{\partial f(k/n)n}{\partial k} = f'(\overline{k}), \\
F_n(k,n) &= \frac{\partial f(k/n)n}{\partial n} = f(\overline{k}) - f'(\overline{k})\overline{k}.
\end{aligned}$$

Therefore, the first order conditions (3.6) and (3.7) simplify to

$$\begin{aligned}
r_t(s^t) &= A_t(s^t) f'(\overline{k}_t^I(s^t)), \\
w_t(s^t) &= A_t(s^t) \left(f(\overline{k}_t^I(s^t)) - f'(\overline{k}_t^I(s^t))\overline{k}_t^I(s^t) \right).
\end{aligned}$$

This leads to the observation that

$$w_t(s^t)n_t(s^t) + r_t(s^t)k_t^I(s^t) = A_t(s^t)F\left(k_t^I(s^t), n_t(s^t)\right),$$

since

$$\begin{aligned}
& w_t(s^t)n_t(s^t) + r_t(s^t)k_t^I(s^t) \\
&= \left[A_t(s^t)\left(f(\overline{k}_t^I(s^t)) - f'(\overline{k}_t^I(s^t))\overline{k}_t^I(s^t) \right) + A_t(s^t) f'(\overline{k}_t^I(s^t))\overline{k}_t^I(s^t) \right] n_t(s^t) \\
&= A_t(s^t) f(\overline{k}_t^I(s^t)) n_t(s^t) = A_t(s^t) F\left(k_t^I(s^t), n_t(s^t)\right).
\end{aligned}$$

Type II firm

The representative firm of type II operates technology (3.2) to transform output into capital. The type II firm purchases capital at time 0 from the household sector and thereafter invests in new capital, earning revenues by renting capital to the type I firm. Thus, the type II firm maximizes

$$-p_{k_0}k_0^{II} + \inf_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) y_t(s^t),$$

subject to

$$x_t(s^t) + y_t(s^t) \le r_t(s^t) k_t^{II}(s^{t-1}),$$

$$k_{t+1}^{II}(s^t) = (1-\delta) k_t^{II}(s^{t-1}) + x_t(s^t).$$

As in chapter 1, we take the natural point of view that the firm is an asset $y = \{y_t(s^t)\}_{t=0}^{\infty}$ whose shares are tradeable in the financial market at time $t = 0$. In fact, the firm's profit can be compactly expressed as

$$-p_{k_0} k_0^{II} + \text{bid}\left(y; \mathcal{M}^0, q^0\right).$$

Note that the firm's capital stock k_0^{II} in period 0 is bought without any uncertainty about the rental price in that period. However, the investment in capital $k_{t+1}^{II}(s^t)$ for a future period $t+1$ is conditioned on the realized history s^t. Thus, the type II firm manages the risk associated with technology constraint (3.2). In particular, the capital storage technology (3.2) states that capital must be assembled one period prior to becoming an input for production. In contrast, the type I firm of the previous subsection can choose how much capital $k_t^I(s^t)$ to rent in period t conditioned on history s^t. Note that the firm's profit in general is a non-linear and concave function in y suited to be maximized. Assuming that for some $\widehat{\pi}^0(y) \in \mathcal{M}^0$

$$\inf_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) y_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t; y)}{\pi_t^0(s^t)} q_t^0(s^t) y_t(s^t),$$

in other words the infimum is attainable, the firm's objective function can be equivalently written as

$$k_0^{II}\left[-p_{k_0}+r_0(s^0)+(1-\delta)\right]+\sum_{t=0}^{\infty}\sum_{s^t}k_{t+1}^{II}(s^t)\Big[-\frac{\widehat{\pi}_t^0(s^t;y)}{\pi_t^0(s^t)}q_t^0(s^t)$$
$$+\sum_{s^{t+1}|s^t}\frac{\widehat{\pi}_{t+1}^0(s^{t+1};y)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1})\left(r_{t+1}(s^{t+1})+(1-\delta)\right)\Big].$$

Here, we are making the natural assumptions that $\widehat{\pi}_0^0(s^0)=q_0^0(s^0)=1$. Using the Envelope theorem [27], the first order conditions can be written as

$$p_{k_0}=r_0(s^0)+(1-\delta), \tag{3.8}$$
$$\frac{\widehat{\pi}_t^0(s^t;y)}{\pi_t^0(s^t)}q_t^0(s^t)=\sum_{s^{t+1}|s^t}\frac{\widehat{\pi}_{t+1}^0(s^{t+1};y)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1})\left(r_{t+1}(s^{t+1})+(1-\delta)\right). \tag{3.9}$$

These conditions impose no-arbitrage restrictions across prices.

Market clearing

The market clearing conditions are given by $k_t^{II}(s^{t-1})=k_t^I(s^t)$ and

$$\sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)}q_t^0(s^t)\left[z_t(s^t)-y_t(s^t)\right]\leq 0,\ \forall\widehat{\pi}^0\in\mathcal{N}^0.$$

This can be equivalently written as

$$\sup_{\widehat{\pi}^0\in\mathcal{N}^0}\sum_{t=0}^{\infty}\sum_{s^t}\Bigg\{\frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)}q_t^0(s^t)$$
$$\left[c_t(s^t)+k_{t+1}^{II}(s^t)-w_t(s^t)n_t(s^t)-\left(r_t(s^t)+(1-\delta)\right)k_t^{II}(s^{t-1})\right]\Bigg\}\leq 0.$$

Assuming that the supremum is attained at some $\widehat{\pi}^0(z,y)\in\mathcal{N}^0$, we obtain

$$\sum_{t=0}^{\infty}\sum_{s^t}\Bigg\{\frac{\widehat{\pi}_t^0(s^t;z,y)}{\pi_t^0(s^t)}q_t^0(s^t) \tag{3.10}$$
$$\left[c_t(s^t)+k_{t+1}^{II}(s^t)-w_t(s^t)n_t(s^t)-\left(r_t(s^t)+(1-\delta)\right)k_t^{II}(s^{t-1})\right]\Bigg\}\leq 0.$$

Alternatively, we can write

$$\sum_{t=0}^{\infty}\sum_{s^t}\left\{\frac{\widehat{\pi}_t^0(s^t;z,y)}{\pi_t^0(s^t)}q_t^0(s^t)\left[c_t(s^t)+k_{t+1}^{II}(s^t)-w_t(s^t)n_t(s^t)\right]\right.$$
$$\left.-k_{t+1}^{II}(s^t)\sum_{s^{t+1}|s^t}\frac{\widehat{\pi}_{t+1}^0(s^{t+1};z,y)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1})\left(r_{t+1}(s^{t+1})+(1-\delta)\right)\right\}$$
$$\leq k_0^{II}\left[r_0(s_0)+(1-\delta)\right].$$

Equilibrium prices and quantities

Combining the first order conditions of the household (3.5) and the type I firm (3.7), we obtain

$$\beta^t u_\ell\left[c_t(s^t),1-n_t(s^t)\right]\pi_t^0(s^t)=\eta\frac{\widehat{\pi}_t^0(s^t;z)}{\pi_t^0(s^t)}q_t^0(s^t)A_t(s^t)F_n\left(k_t^I(s^t),n_t(s^t)\right).$$

Moreover, the first order conditions of the household (3.4), type II (3.9), and type I (3.6) firms, combined together, give

$$\beta^t u_c\left[c_t(s^t),1-n_t(s^t)\right]\pi_t^0(s^t)=\eta\frac{\widehat{\pi}_t^0(s^t;z)}{\widehat{\pi}_t^0(s^t;y)}$$
$$\cdot\sum_{s^{t+1}|s^t}\frac{\widehat{\pi}_{t+1}^0(s^{t+1};y)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1})\left(r_{t+1}(s^{t+1})+(1-\delta)\right)$$
$$=\eta\frac{\widehat{\pi}_t^0(s^t;z)}{\widehat{\pi}_t^0(s^t;y)}\sum_{s^{t+1}|s^t}\left\{\frac{\widehat{\pi}_{t+1}^0(s^{t+1};y)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1})\right.$$
$$\left.\cdot\left[A_{t+1}(s^{t+1})F_k\left(k_{t+1}^I(s^{t+1}),n_{t+1}(s^{t+1})\right)+(1-\delta)\right]\right\}.$$

Now, using the household's first order condition (3.4),

$$\beta^t u_c\left[c_t(s^t),1-n_t(s^t)\right]\pi_t^0(s^t)=\eta\frac{\widehat{\pi}_t^0(s^t;z)}{\pi_t^0(s^t)}q_t^0(s^t),$$
$$\beta^{t+1}u_c\left[c_{t+1}(s^{t+1}),1-n_{t+1}(s^{t+1})\right]\pi_{t+1}^0(s^{t+1})=\eta\frac{\widehat{\pi}_{t+1}^0(s^{t+1};z)}{\pi_{t+1}^0(s^{t+1})}q_{t+1}^0(s^{t+1}),$$

we obtain

$$u_\ell\left[c_t(s^t),1-n_t(s^t)\right]=u_c\left[c_t(s^t),1-n_t(s^t)\right]A_t(s^t)F_n\left(k_t^I(s^t),n_t(s^t)\right),\qquad(3.11)$$

and

$$u_c\left[c_t(s^t), 1-n_t(s^t)\right]\pi_t^0(s^t) \tag{3.12}$$
$$= \beta\frac{\widehat{\pi}_t^0(s^t;z)}{\widehat{\pi}_t^0(s^t;y)}\sum_{s^{t+1}|s^t}\left\{\frac{\widehat{\pi}_{t+1}^0(s^{t+1};y)}{\widehat{\pi}_{t+1}^0(s^{t+1};z)}u_c\left[c_{t+1}(s^{t+1}), 1-n_{t+1}(s^{t+1})\right]\pi_{t+1}^0(s^{t+1})\right.$$
$$\left.\left[A_{t+1}(s^{t+1})F_k\left(k_{t+1}^I(s^{t+1}), n_{t+1}(s^{t+1})\right)+(1-\delta)\right]\right\}.$$

Furthermore, using the household's first order condition (3.4) along with the type I firm's zero profit conditions (3.6, 3.7), the market clearing (3.10) can be written as

$$\sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t}{\eta}\left\{\frac{\widehat{\pi}_t^0(s^t;z,y)}{\widehat{\pi}_t^0(s^t;z)}u_c\left[c_t(s^t), 1-n_t(s^t)\right]\pi_t^0(s^t)\right. \tag{3.13}$$
$$\left.\left[c_t(s^t)+k_{t+1}^{II}(s^t)-(1-\delta)k_t^{II}(s^{t-1})-A_t(s^t)F\left(k_t^{II}(s^{t-1}), n_t(s^t)\right)\right]\right\}=0.$$

Recall that $\eta = u_c\left[c_0(s_0), 1-n_0(s_0)\right]$. In summary, we propose to use the following algorithm[1] which generalizes the *Negishi algorithm* (see [4], chapter 8) to find equilibrium quantities and prices.

1. Make initial guesses for $\widehat{\pi}^0(z)$, $\widehat{\pi}^0(y)$, and $\widehat{\pi}^0(z,y)$. A good initial guess is usually given by the actual physical measure π^0.

2. Solve equations (3.11), (3.12), and (3.13) for candidate consumption c, labor n, and capital $k^I = k^{II}$.

3. Use the following (see equation (3.4)) to solve for the price system q^0.

$$\beta^t u_c\left[c_t(s^t), 1-n_t(s^t)\right]\pi_t^0(s^t) = \eta\frac{\widehat{\pi}_t^0(s^t;z)}{\pi_t^0(s^t)}q_t^0(s^t).$$

[1] The non-linearities involved in the definition of a conic general equilibrium makes a convergence proof of the proposed algorithm non-trivial. A convergence proof would require imposing further restrictions on the sets $\mathcal{M}^0$ and $\mathcal{N}^0$ identifying the financial market configurations. This will be subject of future research.

4. We can find the spot prices $r_t(s^t)$ and $w_t(s^t)$ for capital and labor, respectively, using the type I firm's zero profit conditions (3.6, 3.7), i.e.,

$$r_t(s^t) = A_t(s^t)F_k\left(k_t^I(s^t), n_t(s^t)\right),$$

$$w_t(s^t) = A_t(s^t)F_n\left(k_t^I(s^t), n_t(s^t)\right).$$

Moreover equation (3.8) gives the unit price of the initial capital stock, i.e.,

$$p_{k_0} = r_0(s_0) + (1-\delta).$$

5. The equilibrium demand z, and supply y for assets, along with capital investments x, can be obtained using

$$c_t(s^t) = w_t(s^t)n_t(s^t) + z_t(s^t),$$

$$x_t(s^t) + y_t(s^t) = r_t(s^t)k_t^{II}(s^{t-1}),$$

$$k_{t+1}^{II}(s^t) = (1-\delta)k_t^{II}(s^{t-1}) + x_t(s^t).$$

6. Check the validity of the following requirements and update $\widehat{\pi}^0(z)$, $\widehat{\pi}^0(y)$, and $\widehat{\pi}^0(z,y)$ accordingly.

$$\sup_{\widehat{\pi}^0\in\mathcal{M}^0}\sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)}q_t^0(s^t)z_t(s^t) = \sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t;z)}{\pi_t^0(s^t)}q_t^0(s^t)z_t(s^t),$$

$$\inf_{\widehat{\pi}^0\in\mathcal{M}^0}\sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)}q_t^0(s^t)y_t(s^t) = \sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t;y)}{\pi_t^0(s^t)}q_t^0(s^t)y_t(s^t),$$

$$\sup_{\widehat{\pi}^0\in\mathcal{N}^0}\sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)}q_t^0(s^t)\left(z_t(s^t)-y_t(s^t)\right)$$
$$= \sum_{t=0}^{\infty}\sum_{s^t}\frac{\widehat{\pi}_t^0(s^t;z,y)}{\pi_t^0(s^t)}q_t^0(s^t)\left(z_t(s^t)-y_t(s^t)\right).$$

It is worth noting that this step boils down to maximizing linear objective functions.

7. Iterate on steps 2-6 until the requirements of step 6 are satisfied.

3.3 Sequential trading

This section describes the production economy with an alternative financial market structure, i.e., sequential trading. As in chapter 2, we begin by asking the following question. In the conic production economy with time 0 trading, what is the implied continuation wealth of the household at time t after history s^t, but before adding in its time t, history s^t value of labor $w_t(s^t)n_t(s^t)$? Thus, the household's continuation wealth or *financial wealth* expressed in terms of the date t, history s^t consumption good is denoted by $a_t(s^t)$ and is given by

$$a_t(s^t) = \sup_{\widehat{\pi}^t \in \mathcal{M}^t} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) z_\tau(s^\tau),$$

where

$$\widehat{\pi}_\tau^t(s^\tau) := \frac{\widehat{\pi}_\tau^0(s^\tau)}{\widehat{\pi}_t^0(s^t)},\ q_\tau^t(s^\tau) := \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)},\ \text{and } \widehat{\pi}^t := \left\{\widehat{\pi}_\tau^t(s^\tau)\right\}_{\tau=t}^{\infty} \in \mathcal{M}^t.$$

Here, $\mathcal{M}^t$ denotes the set of generalized scenarios considered by the financial market as of time t and after observing history s^t. At time 0, the household's budget constraint (3.3) in the dynamic conic equilibrium with time 0 trading implies $a_0(s_0) = p_{k_0} k_0$. Similar to chapter 2, in moving from the economy with time 0 trading to one with sequential trading, we propose to match the time t, history s^t wealth of the household in the sequential economy with the equilibrium tail wealth $a_t(s^t)$ from the dynamic conic equilibrium with time 0 trading. In the following, we describe the problems of the representative household and the type II firm in the production economy with sequential trading. The representative firm of type I behaves as before.

Household

At each date $t \geq 0$ after history s^t, the representative household buys consumption goods $c_t(s^t)$ and sells labor services $n_t(s^t)$ in the corresponding spot markets. Moreover, in the financial market at time t and in history s^t, the household trades claims to date $t+1$ consumption, whose payment is contingent on the realization of s_{t+1}. Therefore, the household maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u \left[c_t(s^t), 1 - n_t(s^t) \right] \pi_t(s^t),$$

subject to the sequential budget constraint

$$c_t(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}^t_{t+1}(s_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) a_{t+1}(s_{t+1}, s^t) \leq w_t(s^t) n_t(s^t) + a_t(s^t),$$
$$\forall t,\ \forall s^t,\ \forall \widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}. \qquad (3.14)$$

Here, $a_t(s^t)$ denotes the claims to time t consumption that the household brings into time t in history s^t. To rule out Ponzi schemes, we must impose borrowing constraints on the household's asset position. We could follow the approach of chapter 2 and compute state-contingent *natural debt limits*. In particular, the counterpart to the earlier present value of the household's endowment stream would be the present value of the household's time endowment. Alternatively, we just impose that the household's indebtedness in any state next period, $-a_{t+1}(s_{t+1}, s^t)$, is bounded by some arbitrarily large constant. Such an arbitrary debt limit works well for the following reason (see [4], chapter 12). As long as the household is constrained so that it cannot run a true Ponzi scheme with an unbounded budget constraint, equilibrium forces will ensure that the representative household willingly holds the market portfolio. In the present setting, we can for example set that arbitrary debt limit equal to zero (see [4], chapter 12). Therefore, the borrowing constraint at time t in history s^t is given by

$$-a_{t+1}(s_{t+1}, s^t) \leq 0, \; \forall s_{t+1} \in S.$$

Let us assume that for some generalized scenario $\widehat{\pi}^t_{t+1}(a_{t+1}) \in \mathcal{M}^t_{t+1}$, we have

$$\begin{aligned} \sup_{\widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}} \sum_{s_{t+1} \in S} & \frac{\widehat{\pi}^t_{t+1}(s_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) a_{t+1}(s_{t+1}, s^t) \\ &= \sum_{s_{t+1} \in S} \frac{\widehat{\pi}^t_{t+1}(s_{t+1}; a_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) a_{t+1}(s_{t+1}, s^t). \end{aligned}$$

Then, the household's budget constraint at time t in history s^t can be written as

$$c_t(s^t) + \sum_{s_{t+1} \in S} \frac{\widehat{\pi}^t_{t+1}(s_{t+1}; a_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) a_{t+1}(s_{t+1}, s^t) \leq w_t(s^t) n_t(s^t) + a_t(s^t).$$

Let $\eta_t(s^t)$ and $\nu_{t+1}(s_{t+1}, s^t)$ be the non-negative Lagrange multipliers on the budget constraint and the borrowing constraint, respectively. Forming the Lagrangian and using the Envelope theorem [27], we obtain the following first order conditions;

$$\begin{aligned} &\beta^t u_c \left[c_t(s^t), 1 - n_t(s^t)\right] \pi^0_t(s^t) - \eta_t(s^t) = 0, \\ &-\beta^t u_\ell \left[c_t(s^t), 1 - n_t(s^t)\right] \pi^0_t(s^t) + \eta_t(s^t) w_t(s^t) = 0, \\ &-\eta_t(s^t) \frac{\widehat{\pi}^t_{t+1}(s_{t+1}; a_{t+1})}{\pi^t_{t+1}(s_{t+1})} q^t_{t+1}(s_{t+1}) + \nu_{t+1}(s_{t+1}, s^t) + \eta_{t+1}(s_{t+1}, s^t) = 0, \end{aligned}$$

for all s_{t+1}, t, s^t. We proceed under the conjecture (see [4], chapter 12) that the arbitrary debt limit of zero will not be binding, and hence the Lagrange multipliers $\nu_{t+1}(s_{t+1}, s^t)$ are all equal to zero. After setting those multipliers equal to zero, the first-order conditions imply the following conditions for the optimal choices of consumption and labor;

$$
\begin{aligned}
w_t(s^t) &= \frac{u_c\left[c_t(s^t), 1-n_t(s^t)\right]}{u_\ell\left[c_t(s^t), 1-n_t(s^t)\right]}, \\
\frac{\widehat{\pi}_{t+1}^t(s_{t+1}; a_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) &= \beta \frac{u_c\left[c_{t+1}(s_{t+1}, s^t), 1-n_{t+1}(s_{t+1}, s^t)\right]}{u_c\left[c_t(s^t), 1-n_t(s^t)\right]} \pi_{t+1}^t(s_{t+1}),
\end{aligned} \tag{3.15}
$$

for all s_{t+1}, t, s^t.

Type II firm

A type II firm transforms output into capital, stores capital, and earns its revenues by renting capital to the type I firm. Thus, at each date $t \geq 0$ after history s^t, the type II firm maximizes

$$
b_t(s^t) = y_t(s^t) + \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) b_{t+1}(s_{t+1}, s^t),
$$

subject to

$$
x_t(s^t) + y_t(s^t) \leq r_t(s^t) k_t^{II}(s^{t-1}),
$$

$$
k_{t+1}^{II}(s^t) = (1-\delta) k_t^{II}(s^{t-1}) + x_t(s^t).
$$

We take the natural point of view that the firm is an asset

$$
b_{t+1}(\cdot, s^t) = \left\{ b_{t+1}(s_{t+1}, s^t) : s_{t+1} \in S \right\}
$$

whose shares are tradeable in the financial market at time t after history s^t. Alternatively, the firm's profit can be equivalently written as

$$
\begin{aligned}
b_t(s^t) &= \left[r_t(s^t) + (1-\delta) \right] k_t^{II}(s^{t-1}) - k_{t+1}^{II}(s^t) \\
&+ \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) b_{t+1}(s_{t+1}, s^t).
\end{aligned}
$$

To be consistent with the conic equilibrium with time 0 trading we assume that $[r_0(s_0) + (1-\delta)]k_0 = a_0(s_0) = p_{k_0}k_0$. Because of the technological assumption that capital can be reconverted to the consumption good, we can without loss of generality consider a two-period optimization problem where the type II firm decides how much capital $k_{t+1}^{II}(s^t)$ to store at the end of period t after history s^t in order to earn a stochastic rental revenue $r_{t+1}(s_{t+1}, s^t)k_{t+1}^{II}(s^t)$ and liquidation value $(1-\delta)k_{t+1}^{II}(s^t)$ in the following period. Therefore, at each date $t \geq 0$ after history s^t, the type II firm chooses $k_{t+1}^{II}(s^t)$ to maximize

$$k_{t+1}^{II}(s^t)\left\{ -1 + \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1}\in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) \left[r_{t+1}(s_{t+1}, s^t) + (1-\delta)\right] \right\}.$$

The zero-profit condition is

$$1 = \inf_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1}\in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) \left[r_{t+1}(s_{t+1}, s^t) + (1-\delta)\right].$$

The size of the type II firm is indeterminate. Assuming that the infimum is attained at some $\widehat{\pi}_{t+1}^t(r_{t+1}) \in \mathcal{M}_{t+1}^t$, the zero-profit condition can be written as

$$1 = \sum_{s_{t+1}\in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1}; r_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) \left[r_{t+1}(s_{t+1}, s^t) + (1-\delta)\right]. \tag{3.16}$$

Market clearing

The market clearing conditions are given by $k_t^{II}(s^{t-1}) = k_t^I(s^t)$ and

$$\begin{aligned}\sum_{s_{t+1}\in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) \Big\{ a_{t+1}(s_{t+1}, s^t) - [r_{t+1}(s_{t+1}, s^t) + (1-\delta)]k_{t+1}^{II}(s^t) \Big\} \\ \leq a_t(s^t) - [r_t(s^t) + (1-\delta)]k_t^{II}(s^{t-1}), \quad \forall t,\ \forall s^t,\ \forall \widehat{\pi}_{t+1}^t \in \mathcal{N}_{t+1}^t.\end{aligned}$$

Note that $a_0(s_0) = [r_0(s_0) + (1-\delta)]k_0 = p_{k_0}k_0$.

Equilibrium prices and quantities

Combining the first order conditions of the household (3.15) and the type I firm (3.7), we obtain

$$u_c\left[c_t(s^t), 1-n_t(s^t)\right] = u_\ell\left[c_t(s^t), 1-n_t(s^t)\right] A_t(s^t) F_n\left(k_t^I(s^t), n_t(s^t)\right).$$

Moreover, the first order conditions of the household (3.15), type II (3.16), and type I (3.6) firms, combined together, give

$$1 = \beta \sum_{s^{t+1}|s^t} \Bigg\{ \frac{\widehat{\pi}_{t+1}^t(s_{t+1}; r_{t+1})}{\widehat{\pi}_{t+1}^t(s_{t+1}; a_{t+1})} \frac{u_c\left[c_{t+1}(s^{t+1}), 1-n_{t+1}(s^{t+1})\right]}{u_c\left[c_t(s^t), 1-n_t(s^t)\right]} \pi_{t+1}^t(s^{t+1})$$
$$\left[A_{t+1}(s^{t+1}) F_k\left(k_{t+1}^I(s^{t+1}), n_{t+1}(s^{t+1})\right) + (1-\delta)\right] \Bigg\}.$$

Comparing this with equation (3.12) shows that the dynamic conic equilibrium with sequential trading is consistent with the financial economy with time 0 trading.

3.4 Recursive formulation

Our findings so far hold for an arbitrary technology process $A_t(s^t)$, defined as a measurable function of the history of events s^t which in turn are governed by some arbitrary probability measure $\pi_t(s^t)$. At this level of generality, all prices $\{q_{t+1}^t, w_t(s^t), r_t(s^t)\}$, the financial market primitives $\mathcal{M}_{t+1}^t$, $\mathcal{N}_{t+1}^t$, and the capital stock $k_{t+1}(s^t)$ in the sequential-trading economy depend on the history s^t. That is, these objects are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$. This leads us to make a few specializations of the exogenous processes that facilitates a recursive formulation of the production economy with sequential trading. Similar to chapter 2, we let $\pi(s'|s)$ be a Markov chain with given initial distribution $\pi_0(s)$ and state space $s \in S$. It means that $\text{Prob}(s_{t+1} = s'|s_t = s) = \pi(s'|s)$ and $\text{Prob}(s_0 = s) = \pi_0(s)$. The chain induces a sequence of probability measures $\pi_t(s^t)$ on histories s^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \ldots \pi(s_1|s_0)\pi_0(s_0).$$

In this chapter, we have assumed that trading occurs after the initially given value of s_0 has been observed, which we capture by setting $\pi_0(s_0) = 1$. Because of the Markov property, for $\tau > t$, the conditional probability $\pi^t_\tau(s^\tau)$ depends only on the state s_t at time t and does not depend on the history before t,

$$\pi^t_\tau(s^\tau) = \pi(s_\tau|s_{\tau-1})\pi(s_{\tau-1}|s_{\tau-2}) \ldots \pi(s_{t+1}|s_t).$$

Next, we assume that the aggregate technology level $A_t(s^t)$ in period t is a time-invariant measurable function of its level in the last period and the current stochastic event s_t, i.e., $A_t(s^t) = A\left(A_{t-1}(s^{t-1}), s_t\right)$. For example, here we will proceed with the multiplicative version

$$A_t(s^t) = s_t A_{t-1}(s^{t-1}) = s_0 s_1 \ldots s_t A_{-1},$$

given the initial value A_{-1}. This specialization of the technology process enables us to explore recursive formulations of the sequential-trading equilibrium. Let k denote the beginning-of-period capital and employ the state vector $\xi := [k \ A \ s]$ to completely summarize the economy's current position. We specify *price functions* $r(\xi), w(\xi), q(\xi'|\xi)$ that represent, respectively, the rental price of capital, the wage rate for labor, and the price of a claim to one unit of consumption next period when next period's state is ξ' and this period's state is ξ. The prices are all measured in units of this period's consumption good. We also take as given an arbitrary candidate for the law of motion for k; i.e.,

$$k' = \kappa(\xi).$$

This equation along with $A' = As$ and a given transition density $\pi(s'|s)$ induces a transition density $\rho(\xi'|\xi)$ for the state ξ. For now κ is arbitrary. As in chapter 2, the one-period-ahead financial market in state $s \in S$ is characterized by two sets $\mathcal{M}(s)$ and $\mathcal{N}(s)$ of generalized scenarios $\widehat{\pi}(s'|s)$. Similar to the transition density $\rho(\xi'|\xi)$, we obtain the induced generalized scenarios $\widehat{\rho}(\xi'|\xi)$ and the induced financial market configurations $\mathcal{M}(\xi)$ and $\mathcal{N}(\xi)$.

Household problem

The Bellman equation of the household is

$$v(a,\xi) = \max_{c,n,a'(\cdot)} \left\{ u\left(c, 1-n\right) + \beta \sum_{\xi' \in X} v(a'(\xi'), \xi')\rho(\xi'|\xi) \right\}$$

subject to

$$c + \sup_{\widehat{\rho}(\cdot|\xi) \in \mathcal{M}(\xi)} \sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) a'(\xi') \leq w(\xi)n + a.$$

Here, $\mathcal{M}(\xi)$ denotes the set of *induced* generalized scenarios at the individual level considered by the financial market in state ξ. Moreover, it should be noted that

$$c = c(a,\xi),\ n = n(a,\xi),\ \text{and } a'(\xi') = a'(\xi', a, \xi).$$

Assuming that for some $\widehat{\rho}(\cdot|\xi; a') \in \mathcal{M}(\xi)$,

$$\sup_{\widehat{\rho}(\cdot|\xi) \in \mathcal{M}(\xi)} \sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) a'(\xi') = \sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi; a')}{\rho(\xi'|\xi)} q(\xi'|\xi) a'(\xi'),$$

and by evoking the Envelope theorem twice, we can represent the first-order conditions for the household's problem as

$$u_\ell\left[c(a,\xi), 1-n(a,\xi)\right] = u_c\left[c(a,\xi), 1-n(a,\xi)\right] w(\xi),$$
$$\frac{\widehat{\rho}(\cdot|\xi;a')}{\rho(\xi'|\xi)} q(\xi'|\xi) = \beta \frac{u_c\left[c(a'(\xi',a,\xi),\xi'), 1-n(a'(\xi',a,\xi),\xi')\right]}{u_c\left[c(a,\xi), 1-n(a,\xi)\right]} \rho(\xi'|\xi).$$

Type I firm

In the recursive formulation, the problem of a type I firm can be written as

$$\max_{c,x,k,n} \left\{c + x - r(\xi)k - w(\xi)n\right\},$$

subject to

$$c + x \leq AsF(k,n).$$

The zero-profit conditions are

$$r(\xi) = AsF_k(k,n),$$

$$w(\xi) = AsF_n(k,n).$$

Type II firm

At each state ξ, the type II firm maximizes

$$b(\xi) = y + \inf_{\widehat{\rho}(\cdot|\xi)\in\mathcal{M}(\xi)} \sum_{\xi'\in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) b(\xi',\xi)$$

subject to $x + y \leq r(\xi)k$ and $k' = (1-\delta)k + x$. We take the natural point of view that the firm is an asset

$$b(\cdot,\xi) = \{b(\xi',\xi) : \xi' \in X\}$$

whose shares are tradeable in the financial market in state ξ. Alternatively, the firm's problem can be equivalently written as

$$\max_{k'} k' \Big\{ -1 + \inf_{\widehat{\rho}(\cdot|\xi)\in\mathcal{M}(\xi)} \sum_{\xi'\in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi)\left[r(\xi',\xi) + (1-\delta)\right] \Big\}.$$

Assuming that for some $\widehat{\rho}(\cdot|\xi;r) \in \mathcal{M}(\xi)$ the infimum is attained, we obtain the zero-profit condition

$$1 = \sum_{\xi'\in X} \frac{\widehat{\rho}(\xi'|\xi;r)}{\rho(\xi'|\xi)} q(\xi'|\xi)\left[r(\xi',\xi) + (1-\delta)\right].$$

Market clearing

The market clearing condition is given by

$$\sup_{\widehat{\rho}(\cdot|\xi)\in\mathcal{N}(\xi)} \sum_{\xi'\in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi)\left[a'(\xi',a,\xi) - \left[r(\xi') + (1-\delta)\right]k'\right] \leq a - \left[r(\xi) + (1-\delta)\right]k.$$

Recursive equilibrium

We can summarize the preceding discussion in the following definition.

Definition 3.1 (recursive conic equilibrium – production economy)**.** *Given*

- *an initial state vector* $\xi_0 = [k_0\ A_0\ s_0]$, *where* k_0 *is an initial capital stock,* A_0 *is an initial aggregate technology level, and* $s_0 \in S$,
- *an initial wealth level* $a_0(s_0)$, *a one-period-ahead conic financial market*

$$\{(\mathcal{M}(s), \mathcal{N}(s)) : s \in S\},$$

- *a transition density* $\pi(s'|s)$, *and a law of motion* $A' = As$ *for the aggregate technology level,* *a collection formed by*

- *a pricing kernel* $q(\cdot|\xi) = \{q(\xi'|\xi) : s' \in S\}$, *where* $\xi = [k \;\; A \;\; s]$ *is the state vector,*

- *spot prices* $r(\xi)$ *and* $w(\xi)$,

- *a perceived law of motion* $k' = \kappa(\xi)$ *along with the associated induced transition density* $\rho(\xi'|\xi)$ *and the corresponding induced one-period-ahead conic financial market*

$$\{(\mathcal{M}(\xi), \mathcal{N}(\xi)) : \xi \in X\},$$

- *and a household value function* $v(a, \xi)$ *along with decision rules* $c(a, \xi)$, $n(a, \xi)$, *and* $a'(\xi', a, \xi)$,

constitutes a recursive conic equilibrium if

- *given initial wealth* $a_0(s_0)$, *wage* $w(\xi)$, *pricing kernel* $q(\cdot|\xi)$, *and the financial market configuration* $\mathcal{M}(\xi)$, *the value function* $v(a, \xi)$ *along with the decision rules* $c(a, \xi)$, $n(a, \xi)$, *and* $a'(\xi', a, \xi)$ *solve the Bellman equation*

$$v(a, \xi) = \max_{c,n,\{a'(\xi'):\ \xi' \in X\}} u(c, 1-n) + \beta \sum_{\xi' \in X} v\left(a'(\xi'), \xi'\right) \rho(\xi'|\xi),$$

subject to

$$c + \sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) a'(\xi') \le w(\xi)n + a, \ \forall \widehat{\rho}(\cdot|\xi) \in \mathcal{M}(\xi), \tag{3.17}$$
$$-a'(\xi') \le 0, \ \forall \xi' \in X,$$

- *for all* $\xi \in X$, *given* $r(\xi)$ *and* $w(\xi)$, *the type I firm solves*

$$\max_{c,x,k,n} \left\{c + x - r(\xi)k - w(\xi)n\right\},$$

subject to

$$c + x \leq AsF(k, n),$$

- *at each state* $\xi \in X$*, given a stochastic rental price* $r(\cdot, \xi)$*, pricing kernel* $q(\cdot|\xi)$*, and the financial market configuration* $\mathcal{M}(\xi)$*, the type II firm solves*

$$\max_{k'} k' \Big\{ -1 + \inf_{\widehat{\rho}(\cdot|\xi) \in \mathcal{M}(\xi)} \sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) \left[r(\xi', \xi) + (1 - \delta) \right] \Big\},$$

- *and the pricing kernel* $q(\cdot|\xi) = \{q(\xi'|\xi) : s' \in S\}$*, spot prices* $r(\xi)$ *and* $w(\xi)$*, and the perceived law of motion* $k' = \kappa(\xi)$ *are chosen such that the markets clear; i.e.,*

$$\sum_{\xi' \in X} \frac{\widehat{\rho}(\xi'|\xi)}{\rho(\xi'|\xi)} q(\xi'|\xi) \Big\{ a'(\xi', a, \xi) - [r(\xi', \xi) + (1 - \delta)] k' \Big\} \leq a - \left[r(\xi) + (1 - \delta) \right] k,$$

$$\forall \widehat{\rho}(\cdot|\xi) \in \mathcal{N}(\xi).$$

3.5 Concluding remarks

To motivate interest in the role of financial factors in business fluctuations it is no longer necessary to appeal either to the Great Depression or to the experiences of many emerging market economies. Indeed, the financial crisis of 2007–09 put a spotlight (see e.g., [5, 64, 65]) on the need for a unified framework that can help us organize our thinking about financial markets and aggregate economic activity. The recursive conic equilibrium concept of this chapter provides us with such a framework, and can help us address questions of the form: How disruptions in financial markets can induce a crisis that affects real activity? How various financial market interventions might work to mitigate the crisis?

Chapter 4: Primer on conic asset pricing

An equilibrium price system for an economy with conic financial markets (see chapter 2) can be used to price any redundant assets. An asset is redundant if it offers a bundle of history-contingent dated claims whose payoff could be synthesized as a measurable function of the economy's state.

4.1 Pricing redundant assets

Let $\{d_t(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t, history s^t consumption, where $d_t(s^t)$ is a measurable function of s^t. The ask and bid prices of an asset entitling the owner to this stream is given by

$$\text{ask}_0^0(s^0) = \text{ask}\left(d; \mathcal{M}^0; q^0\right) = \sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) d_t(s^t),$$

and

$$\text{bid}_0^0(s^0) = \text{bid}\left(d; \mathcal{M}^0; q^0\right) = \inf_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t) d_t(s^t).$$

Riskless consol

As an example, consider the price of a *riskless consol*, that is, an asset offering to pay one unit of consumption for sure each period. Then $d_t(s^t) = 1$ for all t and

s^t, and the ask and bid prices of this asset are given by

$$\sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t),$$

and

$$\inf_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{t=0}^{\infty} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t),$$

respectively.

Riskless strips

As another example, consider a sequence of *strips* of payoffs on the riskless consol. The time t strip is just the payoff process $d_\tau = 1$ if $\tau = t \geq 0$, and 0 otherwise. Thus, the owner of the strip is entitled to the time t coupon only. The ask and bid values of the time t strip at time 0 are

$$\sup_{\widehat{\pi}_t^0 \in \mathcal{M}_t^0} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t),$$

and

$$\inf_{\widehat{\pi}_t^0 \in \mathcal{M}_t^0} \sum_{s^t} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t),$$

respectively. We can think of the t-period riskless strip as a t-period zero-coupon bond.

Arrow-Debreu security

Consider an Arrow-Debreu security entitling the owner to one unit of consumption at t and in history s^t. The ask and bid values of the time t, history s^t

Arrow-Debreu security at time 0 are

$$\sup_{\widehat{\pi}_t^0 \in \mathcal{M}_t^0} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t), \text{ and } \inf_{\widehat{\pi}_t^0 \in \mathcal{M}_t^0} \frac{\widehat{\pi}_t^0(s^t)}{\pi_t^0(s^t)} q_t^0(s^t),$$

respectively.

Tail assets

Let $\text{ask}_t^0(s^t)$ be the time 0 ask price of an asset that entitles the owner to dividend stream $\{d_\tau(s^\tau)\}_{\tau=t}^\infty$ if history s^t is realized, i.e.,

$$\text{ask}_t^0(s^t) = \sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^0(s^\tau)}{\pi_\tau^0(s^\tau)} q_\tau^0(s^\tau) d_\tau(s^\tau).$$

When the units of the price are time 0, state s^0 goods, the normalization is $\widehat{\pi}_0^0(s^0) = q_0^0(s^0) = 1$. To convert the price into units of time t, history s^t consumption goods, divide by $q_t^0(s^t)$ to obtain

$$\frac{\text{ask}_t^0(s^t)}{q_t^0(s^t)} = \sup_{\widehat{\pi}^0 \in \mathcal{M}^0} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^0(s^\tau)}{\pi_\tau^0(s^\tau)} q_\tau^t(s^\tau) d_\tau(s^\tau).$$

Where

$$q_\tau^t(s^\tau) := \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)}$$

is the price of one unit of consumption delivered at time τ, history s^τ in terms of the date t, history s^t consumption good. Similarly, define

$$\widehat{\pi}_\tau^t(s^\tau) := \frac{\widehat{\pi}_\tau^0(s^\tau)}{\widehat{\pi}_t^0(s^t)}$$

and let $\mathcal{M}^t$ denote the set of generalized scenarios $\widehat{\pi}^t = \{\widehat{\pi}_\tau^t(s^\tau)\}_{\tau=t}^\infty$ considered by the financial market at time t, history s^t. Thus, the ask price at time t, history s^t

for the "tail asset" should be

$$\operatorname{ask}_t^t(s^t) = \sup_{\widehat{\pi}^t \in \mathcal{M}^t} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) d_\tau(s^\tau),$$

Similarly, the bid price at time t, history s^t for the tail asset is

$$\operatorname{bid}_t^t(s^t) = \inf_{\widehat{\pi}^t \in \mathcal{M}^t} \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{\widehat{\pi}_\tau^t(s^\tau)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) d_\tau(s^\tau).$$

One-period returns

Let us start by noting that equation (2.3) yields

$$\frac{\widehat{\pi}_\tau^t(s^\tau; z^i)}{\pi_\tau^t(s^\tau)} q_\tau^t(s^\tau) = \beta^{\tau-t} \frac{u'[c_\tau^i(s^\tau)]}{u'[c_t^i(s^t)]} \pi_\tau^t(s^\tau).$$

Therefore, we obtain the one-period *pricing kernel* q_{t+1}^t at time t in history s^t to be given by

$$\frac{\widehat{\pi}_{t+1}^t(s_{t+1}; z^i)}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) = \beta \frac{u'[c_{t+1}^i(s^{t+1})]}{u'[c_t^i(s^t)]} \pi_{t+1}^t(s_{t+1}).$$

If we want to find the ask price at time t in history s^t of a claim to a random payoff $w_{t+1} = \{w(s_{t+1}) : s_{t+1} \in S\}$, we use

$$\operatorname{ask}_t^t(s^t) = \sup_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \sum_{s_{t+1} \in S} \frac{\widehat{\pi}_{t+1}^t(s_{t+1})}{\pi_{t+1}^t(s_{t+1})} q_{t+1}^t(s_{t+1}) w_{t+1}(s_{t+1}),$$

or equivalently

$$\operatorname{ask}_t^t(s^t) = \sup_{\widehat{\pi}_{t+1}^t \in \mathcal{M}_{t+1}^t} \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{\widehat{\pi}_{t+1}^t}{\widehat{\pi}_{t+1}^t(z)} w_{t+1} \right],$$

where $\mathbb{E}_t$ is the conditional expectation operator. We removed the superscript i, since the above equality holds for every household i. Define $R_{t,t+1}^{\text{ask}}$ to be the one-period gross *return*, corresponding to the ask price, on the asset; i.e.,

$$R_{t,t+1}^{\text{ask}}(s_{t+1}) := \frac{w_{t+1}(s_{t+1})}{\operatorname{ask}_t^t(s^t)}.$$

Then, we obtain

$$1 = \sup_{\widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}} \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)} R^{\text{ask}}_{t,t+1} \right] =: \sup_{\widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}} \mathbb{E}_t \left[\frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)} m^t_{t+1} R^{\text{ask}}_{t,t+1} \right],$$

where the term

$$m^t_{t+1} := \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

functions as a *stochastic discount factor*. Similarly, we obtain

$$1 = \inf_{\widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}} \mathbb{E}_t \left[\frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)} m^t_{t+1} R^{\text{bid}}_{t,t+1} \right].$$

Therefore, we have the following restriction on the conditional moments of the returns and m^t_{t+1}:

$$\mathbb{E}_t \left[\frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)} m^t_{t+1} R^{\text{ask}}_{t,t+1} \right] \leq 1 \leq \mathbb{E}_t \left[\frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)} m^t_{t+1} R^{\text{bid}}_{t,t+1} \right], \ \forall \widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}. \tag{4.1}$$

Since this is true for every $\widehat{\pi}^t_{t+1} \in \mathcal{M}^t_{t+1}$, we arrive at the following fundamental inequality

$$\mathbb{E}_t \left[m^t_{t+1} R^{\text{ask}}_{t,t+1} \right] \leq 1 \leq \mathbb{E}_t \left[m^t_{t+1} R^{\text{bid}}_{t,t+1} \right]. \tag{4.2}$$

Note that if $\mathcal{M}^t_{t+1} = \{\pi^t_{t+1}\}$ is a singleton, then $R^{\text{bid}}_{t,t+1} = R^{\text{ask}}_{t,t+1} = R^t_{t+1}$ and we obtain the classical conditional moments restriction

$$1 = \mathbb{E}_t \left[m^t_{t+1} R^t_{t+1} \right]. \tag{4.3}$$

The above equation (4.3) summarizes in a nutshell most of the classical asset pricing theories (see [4], chapter 13). Empirically, for the stochastic discount factor

$$m^t_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma},$$

where γ is a coefficient of relative risk aversion, restriction (4.3) fails to work well when applied to data on returns of stocks and risk-free bonds. Mehra and Prescott [8] called this difficulty the *equity premium puzzle*. A substantial part of the problem is that with aggregate U.S. data for c_t and "reasonable" values for γ, the stochastic discount factor m^t_{t+1} is simply insufficiently volatile. For insightful reviews and lists of possible resolutions of the equity premium puzzle, see the papers by Aiyagari [66], Kocherlakota [67], and Cochrane [68]. In a conic framework, as equation (4.1) indicates, the additional factor

$$\frac{\widehat{\pi}^t_{t+1}}{\widehat{\pi}^t_{t+1}(z)}$$

can help increase the volatility and consequently explain the equity premium puzzle. Moreover, Hansen and Jagannathan [69] showed that a very weak theoretical restriction on prices, namely a "law of one price", is enough to imply that there exists a stochastic discount factor m that satisfies equation (4.3). Therefore, the law of one price is another major contributor to the equity premium puzzle. Given that a conic economy is indeed a two price economy (see equation (4.2)), it can help explain the puzzle from this perspective as well. Indeed, these observations need more investigations that are beyond the scope of this work.

4.2 j-step pricing kernel

The j-step pricing kernel, denoted by $q_j(s'|s)$, gives the price of one unit of consumption j periods ahead, contingent on the state in that future period being s', given that the current state is s. In particular, $q_1(s'|s)$ corresponds to the one-period-ahead pricing kernel $q(s'|s)$ when $j = 1$. With markets in all possible j-step-ahead contingent claims, the augmented version of constraint (2.11), the household's budget constraint at time t is given by

$$\begin{aligned} c_t^i \quad + \quad & \sum_{j=1}^{\infty} \sum_{s_{t+j} \in S} \frac{\widehat{\pi}_j(s_{t+j}|s_t)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) z_{t,j}^i(s_{t+j}) \leq y^i(s_t) + a_t^i, \\ & \forall \widehat{\pi}_j(\cdot|s_t) \in \mathcal{M}_j(s_t), \ \forall j \geq 1. \end{aligned}$$

Here, $\mathcal{M}_j(s_t)$ denotes the set of generalized scenarios considered by the financial market in state s_t for j-step-ahead contingent claims. Moreover, $z_{t,j}^i(s_{t+j})$ is household i's holdings, at the end of period t, of contingent claims that pay one unit of the consumption good j periods ahead at date $t+j$, contingent on the state at date $t+j$ being s_{t+j}. The household's wealth $a_{t+1}^i(s_{t+1})$ in the next period depends on the chosen asset portfolio and the realization of s_{t+1}, and is given by

$$\begin{aligned} z_{t,1}^i(s_{t+1}) \quad + \quad & \sum_{j=2}^{\infty} \sum_{s_{t+j} \in S} \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1})}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}^i(s_{t+j}) \leq a_{t+1}^i(s_{t+1}), \\ & \forall \widehat{\pi}_{j-1}(\cdot|s_{t+1}) \in \mathcal{M}_{j-1}(s_{t+1}), \ \forall j \geq 2. \end{aligned}$$

The realization of s_{t+1} determines which element of the vector of one-period-ahead claims $\{z_{t,1}^i(s_{t+1})\}$ pays off at time $t+1$, and also the capital gains and losses inflicted on the holdings of longer horizon claims implied by equilibrium state prices $q_j(s_{t+j+1}|s_{t+1})$ and the financial market configuration $\mathcal{M}_j(s_{t+1})$. Using the Envelope theorem, the first order conditions with respect to $z_{t,j}^i(s_{t+j})$ for $j \geq 2$ yield

$$\begin{aligned} & \frac{\widehat{\pi}_j(s_{t+j}|s_t;\star)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) = \\ & \qquad \sum_{s_{t+1}} \beta \frac{u'[c_{t+1}^i(s_{t+1})]}{u'(c_t^i)} \pi(s_{t+1}|s_t) \ \frac{\pi_{j-1}(s_{t+j}|s_{t+1};\star)}{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}). \end{aligned}$$

This expression, evaluated at the conic equilibrium consumption allocation, characterizes two adjacent pricing kernels. Here, $\widehat{\pi}_j(\cdot|s_t,\star) \in \mathcal{M}_j(s_t)$ and $\widehat{\pi}_{j-1}(\cdot|s_{t+1},\star) \in \mathcal{M}_{j-1}(s_{t+1})$ are such that

$$\sup_{\widehat{\pi}_j(\cdot|s_t)\in\mathcal{M}_j(s_t)} \sum_{s_{t+j}\in S} \frac{\widehat{\pi}_j(s_{t+j}|s_t)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) z^i_{t,j}(s_{t+j})$$
$$= \sum_{s_{t+j}\in S} \frac{\widehat{\pi}_j(s_{t+j}|s_t,\star)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) z^i_{t,j}(s_{t+j}),$$
$$\sup_{\widehat{\pi}_{j-1}(\cdot|s_{t+1})\in\mathcal{M}_{j-1}(s_{t+1})} \sum_{s_{t+j}\in S} \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1})}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) z^i_{t,j}(s_{t+j})$$
$$= \sum_{s_{t+j}\in S} \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1},\star)}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) z^i_{t,j}(s_{t+j}).$$

Furthermore, one can deduce that the kernels q_j, for $j \geq 2$, can be computed recursively, i.e.,

$$\frac{\widehat{\pi}_j(s_{t+j}|s_t,\star)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t)\ \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1},\star)}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}), \tag{4.4}$$

since

$$\frac{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) = \beta \frac{u'[c^i_{t+1}(s_{t+1})]}{u'(c^i_t)} \pi(s_{t+1}|s_t),$$

for some $\widehat{\pi}(\cdot|s_t;a^i_{t+1}) \in \mathcal{M}(s_t)$ as in equation (2.10). Note that if $M_j(s)$ is a singleton for every $j \geq 1$ and $s \in S$, we obtain the classical result (see [4], chapter 8); i.e.,

$$q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} q(s_{t+1}|s_t)\ q_{j-1}(s_{t+j}|s_{t+1}),\ \forall j \geq 2.$$

4.3 Arbitrage-free pricing

By manipulating budget sets with redundant assets, we will describe how arbitrage free pricing theory deduces restrictions on asset prices. We augment the trading opportunities in our conic economy by letting the consumer also trade an ex-dividend Lucas tree. The Lucas tree refers to a colorful interpretation of a dividend stream as "fruit" falling from a "tree" in a pure exchange economy studied by Lucas [70]. Assume that at time t, in addition to purchasing a quantity $z_{t,j}(s_{t+j})$ of j-step-ahead claims paying one unit of consumption at time $t+j$ if the state takes value s_{t+j} at time $t+j$, the consumer also purchases $N_t > 0$ units of a stock or Lucas tree. Let the ex-dividend ask price of the tree at time t be $\text{ask}(s_t)$. Next period, the tree pays a dividend $d(s_{t+1})$ depending on the state s_{t+1}. Ownership of the $N_t > 0$ units of the tree at the beginning of $t+1$ entitles the consumer to a claim on $N_t[\text{ask}(s_{t+1}) + d(s_{t+1})]$ units of time $t+1$ consumption. As before, let a_t be the wealth of the consumer, apart from his endowment, $y(s_t)$. In this setting, the consumer's budget constraint, if $N_t > 0$, is

$$c_t + \sum_{j=1}^{\infty} \sum_{s_{t+j} \in S} \frac{\widehat{\pi}_j(s_{t+j}|s_t)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) z_{t,j}(s_{t+j}) + \text{ask}(s_t) N_t \leq y(s_t) + a_t,$$
$$\forall \widehat{\pi}_j(\cdot|s_t) \in \mathcal{M}_j(s_t),\ \forall j \geq 1, \tag{4.5}$$

and

$$\begin{aligned} z_{t,1}(s_{t+1}) \quad &+ \quad [\text{ask}(s_{t+1}) + d(s_{t+1})] N_t, \qquad\qquad (4.6) \\ &+ \quad \sum_{j=2}^{\infty} \sum_{s_{t+j} \in S} \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1})}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}(s_{t+j}) \leq a_{t+1}(s_{t+1}), \\ &\qquad \forall \widehat{\pi}_{j-1}(\cdot|s_{t+1}) \in \mathcal{M}_{j-1}(s_{t+1}),\ \forall j \geq 2. \end{aligned}$$

Multiply equation (4.6) by

$$\frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t),$$

sum over s_{t+1}, and solve for

$$\sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) z_{t,1}(s_{t+1}),$$

to obtain:

$$\begin{aligned}
& \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) z_{t,1}(s_{t+1}) = \\
- & \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t)[\text{ask}(s_{t+1}) + d(s_{t+1})] N_t \\
- & \sum_{j=2}^{\infty} \sum_{s_{t+j} \in S} \left\{ \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) \ \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1}, \star)}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) \right\} z_{t,j}(s_{t+j}) \\
+ & \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) a_{t+1}(s_{t+1}).
\end{aligned}$$

Substituting this expression into (4.5) yields

$$\begin{aligned}
c_t \quad + & \quad \left\{ \text{ask}(s_t) - \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t)[\text{ask}(s_{t+1}) + d(s_{t+1})] \right\} N_t \\
+ & \quad \sum_{j=2}^{\infty} \sum_{s_{t+j} \in S} \left\{ \frac{\widehat{\pi}_j(s_{t+j}|s_t, \star)}{\pi_j(s_{t+j}|s_t)} q_j(s_{t+j}|s_t) \right. \\
- & \quad \left. \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) \ \frac{\widehat{\pi}_{j-1}(s_{t+j}|s_{t+1}, \star)}{\pi_{j-1}(s_{t+j}|s_{t+1})} q_{j-1}(s_{t+j}|s_{t+1}) \right\} z_{t,j}(s_{t+j}) \\
+ & \quad \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t) a_{t+1}(s_{t+1}) \leq y(s_t) + a_t.
\end{aligned}$$

Thus, we arrive at the following *arbitrage pricing formula*; i.e.,

$$\sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t)[\text{ask}(s_{t+1}) + d(s_{t+1})] \leq \text{ask}(s_t), \quad \forall \widehat{\pi}(\cdot|s_t) \in \mathcal{M}(s_t).$$

Otherwise, the consumer can attain unbounded consumption and future wealth. Using a similar argument for the bid price corresponding to case where $N < 0$, we obtain

$$\text{bid}(s_t) \leq \sum_{s_{t+1}} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\pi(s_{t+1}|s_t)} q(s_{t+1}|s_t)[\text{bid}(s_{t+1}) + d(s_{t+1})], \quad \forall \widehat{\pi}(\cdot|s_t) \in \mathcal{M}(s_t).$$

However, since $\widehat{\pi}_j(\cdot|s_t,\star)$ and $\widehat{\pi}_{j-1}(\cdot|s_{t+1},\star)$ are functions of the equilibrium outcome of the economy, the above no-arbitrage argument applied to $z_{t,j}(s_{t+j})$ is not able to tell us anything more than equation (4.4).

4.4 Equivalent martingale measure

Let us recall that the state s_t is assumed to evolve according to a Markov chain with transition probabilities $\pi(s_{t+1}|s_t)$. Moreover, let an asset pay a stream of dividends $d = \{d(s_t)\}_{t=0}^{\infty}$. Similar to proposition 2.1 and the short argument following it, the cum-dividend time t ask price of this asset can be expressed recursively as

$$\text{ask}(s_t) = d(s_t) + \sup_{\widehat{\pi}(\cdot|s_t)\in\mathcal{M}(s_t)} \beta \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \frac{u'[c^i_{t+1}(s_{t+1})]}{u'[c^i_t(s_t)]} \text{ask}(s_{t+1})\pi(s_{t+1}|s_t),$$

for some $\widehat{\pi}(\cdot|s_t;a^i_{t+1}) \in \mathcal{M}(s_t)$. To arrive at this expression, we are implicitly employing equation (2.10). Similarly, we have

$$\text{bid}(s_t) = d(s_t) + \inf_{\widehat{\pi}(\cdot|s_t)\in\mathcal{M}(s_t)} \beta \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \frac{u'[c^i_{t+1}(s_{t+1})]}{u'[c^i_t(s_t)]} \text{bid}(s_{t+1})\pi(s_{t+1}|s_t).$$

Therefore, for all $\widehat{\pi}(\cdot|s_t) \in \mathcal{M}(s_t)$,

$$\begin{aligned}
\text{bid}(s_t) &\leq d(s_t) + \beta \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \frac{u'[c^i_{t+1}(s_{t+1})]}{u'[c^i_t(s_t)]} \text{bid}(s_{t+1})\pi(s_{t+1}|s_t) \\
&\leq d(s_t) + \beta \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \frac{u'[c^i_{t+1}(s_{t+1})]}{u'[c^i_t(s_t)]} \text{ask}(s_{t+1})\pi(s_{t+1}|s_t) \leq \text{ask}(s_t).
\end{aligned}$$

This can be written as

$$\begin{aligned}
\text{bid}(s_t) &\leq d(s_t) + R_t^{-1} \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \text{bid}(s_{t+1})\widetilde{\pi}(s_{t+1}|s_t) \\
&\leq d(s_t) + R_t^{-1} \sum_{s_{t+1}\in S} \frac{\widehat{\pi}(s_{t+1}|s_t)}{\widehat{\pi}(s_{t+1}|s_t;a^i_{t+1})} \text{ask}(s_{t+1})\widetilde{\pi}(s_{t+1}|s_t) \leq \text{ask}(s_t),
\end{aligned}$$

where

$$
\begin{aligned}
R_t^{-1} = R_t^{-1}(s_t) &:= \beta \sum_{s_{t+1} \in S} \frac{u'[c_{t+1}^i(s_{t+1})]}{u'[c_t^i(s_t)]} \pi(s_{t+1}|s_t), \\
\widetilde{\pi}(s_{t+1}|s_t) &:= R_t \beta \frac{u'[c_{t+1}^i(s_{t+1})]}{u'[c_t^i(s_t)]} \pi(s_{t+1}|s_t).
\end{aligned}
$$

Equivalently, for all $\widehat{\pi}(\cdot|s_t) \in \mathcal{M}(s_t)$,

$$
\begin{aligned}
R_t \left[\mathrm{bid}(s_t) - d(s_t)\right] &\leq \widetilde{\mathbb{E}}_t \left[\frac{\widehat{\pi}(\cdot|s_t)}{\widehat{\pi}(\cdot|s_t; a_{t+1}^i)} \, \mathrm{bid}_{t+1} \right] \\
&\leq \widetilde{\mathbb{E}}_t \left[\frac{\widehat{\pi}(\cdot|s_t)}{\widehat{\pi}(\cdot|s_t; a_{t+1}^i)} \, \mathrm{ask}_{t+1} \right] \leq R_t \left[\mathrm{ask}(s_t) - d(s_t)\right],
\end{aligned}
$$

where $\widetilde{\mathbb{E}}_t$ is the mathematical expectation with respect to the distorted transition density $\widetilde{\pi}(s_{t+1}|s_t)$. Since the above inequalities hold for any $\widehat{\pi}(\cdot|s_t) \in \mathcal{M}(s_t)$, we obtain

$$
R_t \left[\mathrm{bid}(s_t) - d(s_t)\right] \leq \widetilde{\mathbb{E}}_t \left[\mathrm{bid}_{t+1}\right] \leq \widetilde{\mathbb{E}}_t \left[\mathrm{ask}_{t+1}\right] \leq R_t \left[\mathrm{ask}(s_t) - d(s_t)\right].
$$

The transformed or "twisted" transition measure $\widetilde{\pi}(s_{t+1}|s_t)$ can be used to define the twisted measure

$$
\widetilde{\pi}_t(s^t) = \widetilde{\pi}(s_t|s_{t-1}) \cdots \widetilde{\pi}(s_1|s_0) \widetilde{\pi}(s_0).
$$

The twisted measure $\widetilde{\pi}_t(s^t)$ is called an *equivalent martingale measure* (see [4], chapter 13). In fact, under the law of one price (i.e., $\mathcal{M}(s_t)$ being a singleton), the existence of an equivalent martingale measure implies both the existence of a positive stochastic discount factor [69, 71], and the absence of arbitrage opportunities [14]. Moreover, consider the particular case of an asset with dividend stream $d_T = \{d(s_T) : s_T \in S\}$ and $d_t = 0$ for $t \neq T$. The cum-dividend bid and ask prices of this asset can be expressed as

$$
\begin{aligned}
\mathrm{bid}_T(s_T) = d(s_T), \qquad & d(s_T) = \mathrm{ask}_T(s_T), \\
\mathrm{bid} \le R_{T-1}^{-1}\widetilde{\mathbb{E}}_{T-1}\left[\mathrm{bid}_T\right] \quad \le \quad & R_{T-1}^{-1}\widetilde{\mathbb{E}}_{T-1}\left[\mathrm{ask}_T\right] \le \mathrm{ask}_{T-1}(s_{T-1}), \\
& \vdots \\
\mathrm{bid}_t(s_t) \le R_t^{-1}\widetilde{\mathbb{E}}_t\left[R_{t+1}^{-1}\cdots R_{T-1}^{-1}\,\mathrm{bid}_T\right] \quad \le \quad & R_t^{-1}\widetilde{\mathbb{E}}_t\left[R_{t+1}^{-1}\cdots R_{T-1}^{-1}\,\mathrm{ask}_T\right] \le \mathrm{ask}_t(s_t),
\end{aligned}
$$

where $\widetilde{\mathbb{E}}_t$ denotes the conditional expectation under the equivalent martingale measure $\widetilde{\pi}$. Now fix $t < T$ and define the "deflated" or "interest-adjusted" asset price processes

$$
\overline{\mathrm{bid}}_{t,t+j} := \frac{\mathrm{bid}_{t+j}}{R_t R_{t+1}\cdots R_{t+j-1}}, \quad \overline{\mathrm{ask}}_{t,t+j} := \frac{\mathrm{ask}_{t+j}}{R_t R_{t+1}\cdots R_{t+j-1}},
$$

for $j = 1, \ldots, T-t$. It follows from the above arguments that

$$
\mathrm{bid}_t(s_t) =: \overline{\mathrm{bid}}_{t,t} \le \widetilde{\mathbb{E}}_t\overline{\mathrm{bid}}_{t,t+j} \le \widetilde{\mathbb{E}}_t\overline{\mathrm{ask}}_{t,t+j} \le \overline{\mathrm{ask}}_{t,t} := \mathrm{ask}_t(s_t)
$$

In other words, relative to the equivalent martingale measure $\widetilde{\pi}$, the interest-adjusted bid price is a sub-martingale while the deflated ask price is a super-martingale. Basically, using the equivalent martingale measure, the best prediction of the future interest-adjusted bid price is somewhere above its current bid value. Similarly, the best prediction of the future interest-adjusted ask price is somewhere below its current ask value. Alternatively, we can write the following equation,

$$
R_t\left[\mathrm{bid}_t(s_t) - d(s_t)\right] \le \widetilde{\mathbb{E}}\left[\mathrm{bid}_{t+1}\,|s_t\right] \le \widetilde{\mathbb{E}}\left[\mathrm{ask}_{t+1}\,|s_t\right] \le R_t\left[\mathrm{ask}_t(s_t) - d(s_t)\right],
$$

which is another way of stating that, after adjusting for risk-free interest and dividends, the bid and ask prices of the asset are sub- and supper-martingales relative to the equivalent martingale measure $\widetilde{\pi}$, respectively. One can proceed even further and obtain the following pricing formulas;

$$R_t \left[\mathrm{bid}_t(s_t) - d(s_t)\right] = \inf_{\theta_{t+1}\in\Theta} \widetilde{\mathbb{E}} \left[\theta_{t+1}\, \mathrm{bid}_{t+1} \,|s_t\right],$$

$$R_t \left[\mathrm{ask}_t(s_t) - d(s_t)\right] = \sup_{\theta_{t+1}\in\Theta} \widetilde{\mathbb{E}} \left[\theta_{t+1}\, \mathrm{ask}_{t+1} \,|s_t\right],$$

where Θ is some set of measure changes θ_{t+1} (see [72]). Therefore, a conic economy provides us with the natural means to extend the classical asset pricing theories. For instance, using a continuous-time specification of $\widetilde{\pi}$, one can obtain conic Black and Scholes [73] option pricing formulas (see [13]).

4.5 Concluding remarks

In this section, we have briefly described how the conic modeling framework of chapter 2 can be applied to extend the classical asset pricing theories and option pricing formulas. In addition, we have concisely alluded to the equity premium puzzle and have describe how a conic perspective towards the economy could potentially explain the puzzle. Indeed, these observations need more investigations that are beyond the scope of this work and could be subject of future research.

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