

ABSTRACT

Title of dissertation: ANALYSIS OF SELF-ORGANIZATION

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The dissertation is devoted to the study of problems in calculus of variation, free boundary problems and gradient flows with respect to the Wasserstein metric.

More concretely, we consider the problem of characterizing the regularity of minimizers to a certain interaction energy. Minimizers of the interaction energy have a somewhat surprising relationship with solutions to obstacle problems. Here we prove and exploit this relationship to obtain novel regularity results.

Another problem we tackle is describing the asymptotic behavior of the Cahn-Hilliard equation with degenerate mobility. By framing the Cahn-Hilliard equation with degenerate mobility as a gradient flow in Wasserstein metric, in one space dimension, we prove its convergence to a degenerate parabolic equation under the framework recently developed by Sandier-Serfaty.

ANALYSIS OF SELF-ORGANIZATION

by

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Table of Contents

1	Introduction	1
1.1	Motivation: Self Organization and Free Boundary Problems	1
1.2	Math Preliminaries	3
1.2.1	Mass transportation	3
1.2.2	Gradient Flows	6
1.2.2.1	General Metric Spaces	6
1.2.2.2	Differential Structure of the L^2 -Wasserstein metric	9
2	Regularity of local minimizers of the interaction energy via obstacle problems	13
2.1	Overview	13
2.2	Main results and strategy	19
2.2.1	Hypothesis	19
2.2.2	Euler-Lagrange condition in d_2 and d_∞	21
2.2.3	Continuity of the potential function ψ and the obstacle problem	23
2.2.4	Regularity of ψ and μ	27
2.2.4.1	Cases $s > 1$	28
2.2.4.2	Case $s = 1$	29
2.2.4.3	Cases $s \in (0, 1)$	31
2.2.5	A uniqueness result	33
2.3	Proof of the Continuity of the potential function ψ	36
2.4	Proofs of the regularity of μ	43
2.4.1	Proof of Proposition 2.2.15 and Theorem 2.2.13	43
2.4.2	Proof of Theorem 2.2.16	46
2.4.3	Proof of Theorem 2.2.18	48
2.4.4	Proof of Theorem 2.2.19	50
3	Cahn-Hilliard Equation	53
3.1	Overview	53
3.1.1	Motivation	58
3.2	Notation and Assumptions	62
3.2.1	Assumptions on F	62
3.2.2	Functionals $\mathcal{F}^\varepsilon, \mathcal{F}^{**}, \mathcal{G}^\varepsilon, \nabla \mathcal{F}^{**} $	63

3.2.3	Existence of ν^ε and ν_0	66
3.3	Statement of the Result	67
3.4	Preliminary to the proof of Theorem 3.3.2	69
3.4.1	Uniform L^∞ estimate	69
3.4.2	Control of the oscillations in the good set	70
3.5	H^1 estimate in the good set Ω	81
3.6	Proof of Theorem 3.3.2	83
3.7	Proof of Theorem 3.3.1	86
3.8	Lower semi-continuity of \mathcal{G}^ε	88
	Bibliography	97

Chapter 1: Introduction

1.1 Motivation: Self Organization and Free Boundary Problems

Self-organization is the phenomena by which some form of order or coordination arises out of the local interactions of an initially disordered system. Self-organizing systems are adaptive and robust. They can reconfigure themselves and thus keep on functioning even if they are perturbed. Self-organization can be observed in nature and social interaction; e.g. swarming of animals, arrangements of molecules in materials, pattern formation in traffic, etcetera.

My research has taken me to analyze two different types of attractive-repulsive models, that exhibit self-organization behavior. When dealing with a discrete number of particles, attractive-repulsive models hypothesize that particles are attracted towards each other, but at the same time there is a mechanism that prevents overcrowding. This framework is very general and could be modeling atoms, penguins or humans in a crowd; even the Cahn-Hilliard equation, modeling phase separation of a binary fluid, lies under this spectrum (see Section 3.1.1).

These type of models are often equipped with an energy functional that is dissipated by the dynamics. Self-Organization can be understood mathematically in these models as the minimizer(s) of the energy being attractor(s) for the dynam-

ics. Therefore, starting from any random initial configuration, the outcome of the evolution can, in some ways, be predicted.

In many interesting cases, the evolution is described by a gradient flow of this energy functional, namely the solutions flow in the direction of the steepest descent of the energy. This led me to study deeply the theory of gradient flows. The main reference for this is "Gradient flows on metric spaces" by Ambrosio, Gigli and Savare [1] that proves that the notion of gradient flow in metric spaces is well-posed for lambda-convex (a generalization of convex) functionals (see Section 1.2.2). Of course, the interesting energies are rarely convex, in any metric, since the hypothesized rules of attraction-repulsion clash with each other, making these problems interesting.

The two particular models of self organization studied in this dissertation can be thought as gradient flows of Energies that can be viewed as the difference of two Sobolev norms. In Chapter 2, we consider the Interaction Energy, which in the simplest case is of the form

$$E[\mu] = \|\mu\|_{H^{-s_1}(\mathbb{R}^N)}^2 - \|\mu\|_{H^{-s_2}(\mathbb{R}^N)}^2,$$

with $0 < s_1 < s_2 < \frac{N}{2}$. Whereas in Chapter 3, we consider the Modica-Mortola functional, which is a regularization of Energies of the form

$$\mathcal{F}[\rho] = \|\rho\|_{L^{p_1}(\mathbb{T})}^{p_1} - \|\rho\|_{L^{p_2}(\mathbb{T})}^{p_2},$$

with $0 < p_2 < p_1 < \infty$. The regularization

$$\mathcal{F}^\varepsilon[\rho] = \mathcal{F}[\rho] + \frac{\varepsilon}{2} \|\rho\|_{H^1(\mathbb{T})}^2,$$

is needed as the Sobolev semi-norms involved in the definition of \mathcal{F} are of the same order which creates uncontrolled oscillations and make the gradient flow ill-posed.

Chapter 2 is mostly devoted to proving and exploiting the somewhat surprising relationship between minimizers of the interaction energy and solutions to obstacle problems. This relationship yields the best known results on the regularity of minimizers. Chapter 2 also addresses some particular cases of the big open problem which is the uniqueness of minimizers.

Chapter 3 frames the Cahn-Hilliard equation with degenerate mobility as a gradient flow in the L^2 -Wasserstein metric in one space dimension and proves its convergence to a degenerate parabolic equation under the framework recently developed by Sandier-Serfaty (see [2]) for the convergence of gradient flows.

1.2 Math Preliminaries

1.2.1 Mass transportation

Given a complete separable metric space X , we consider the set $\mathcal{P}(X)$ the Borel probability measures on X . In this work, we will consider the case $X = \mathbb{R}^N$ the N -dimensional Euclidean space and the case $X = \mathbb{T}$ the 1-dimensional Torus. In this section, we define the family of L^p -Wasserstein distances, d_p , with $1 \leq p \leq \infty$ on $\mathcal{P}(X)$. To be more precise, d_p is a family of pseudo-distances as they can take the value of plus infinity. The case of $p = 2$ is interesting due to its differential structure, that allows to consider $\mathcal{P}(X)$ as an infinite dimensional Riemannian manifold. We give a short introduction to this interpretation in Section 1.2.2.2. The case $p = \infty$ is

interesting as it induces the coarsest of the topologies and its salient feature is that it controls the Hausdorff distance of the supports considered as sets; this makes it interesting as novel modelling tool. We start with a couple of auxiliary definitions.

Definition 1.2.1. *Given $\mu, \nu \in \mathcal{P}(X)$, we call $\pi \in \mathcal{P}(X \times X)$ a transference plan if*

$$\pi(A \times X) = \mu(A) \quad \text{and} \quad \pi(X \times A) = \nu(A),$$

for every Borel set $A \subset X$.

We denote the set of transference plans from μ to ν as $\Pi(\mu, \nu)$.

Heuristically, a transference plan $\pi \in \Pi(\mu, \nu)$ encodes a way of re-arranging the mass from μ to ν . In particular, $\pi(A \times B)$ can be interpreted as the amount of mass that is transported from A to B . Moreover, $\Pi(\mu, \nu)$ is never an empty set, as $\mu \times \nu \in \Pi(\mu, \nu)$. In terms of our interpretation, $\mu \times \nu$ represents spreading mass evenly.

We also recall the definition of support; the smallest closed set that has full measure.

Definition 1.2.2. *The support of a measure $\mu \in \mathcal{P}(X)$ is the closed set defined by*

$$\text{supp}(\mu) := \{x \in X : \mu(B_\varepsilon(x)) > 0 \text{ for all } \varepsilon > 0\},$$

where $B_\varepsilon(x)$ is the ball of radius ε around x .

With the concept of transference plans, we can define the distances d_p :

Definition 1.2.3. *Given $\mu, \nu \in \mathcal{P}(X)$, we define their L^p -Wasserstein distance as*

$$d_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{X \times X} \text{dist}(x_1, x_2)^p d\pi(x_1, x_2) \right\} \right)^{\frac{1}{p}}.$$

Moreover, for $p = \infty$ we get the distinguished distance d_∞

Definition 1.2.4. Given $\mu, \nu \in \mathcal{P}(X)$, we define their L^∞ -Wasserstein distance as

$$d_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \sup_{(x_1, x_2) \in \text{supp}(\pi)} \text{dist}(x_1, x_2).$$

Remark 1.2.5. We have the inequality

$$d_H(\text{supp}(\mu), \text{supp}(\nu)) \leq d_\infty(\mu, \nu),$$

where d_H : is the Hausdorff distance of sets given by

$$d_H(A, B) = \max\left\{ \sup_{x_1 \in A} \inf_{x_2 \in B} \text{dist}(x_1, x_2), \sup_{x_2 \in B} \inf_{x_1 \in A} \text{dist}(x_1, x_2) \right\}$$

for any $A, B \subset X$.

Remark 1.2.6. By an application of Hölder's inequality, we know that d_p distances are ordered. Given $\pi \in \Pi(\mu, \nu)$

$$\begin{aligned} \left(\int_{X \times X} \text{dist}(x_1, x_2)^p d\pi(x_1, x_2) \right)^{\frac{1}{p}} &\leq \left(\int_{X \times X} d\pi(x_1, x_2) \right)^{1 - \frac{p}{q}} \left(\int_{X \times X} \text{dist}(x_1, x_2)^q d\pi(x_1, x_2) \right)^{\frac{1}{q}} \\ &= \left(\int_{X \times X} \text{dist}(x_1, x_2)^q d\pi(x_1, x_2) \right)^{\frac{1}{q}}, \end{aligned}$$

for any $q > p$. So,

$$d_p(\mu, \nu) \leq d_q(\mu, \nu) \quad \text{for any } q > p.$$

By this monotonicity, we have an alternative definition of d_∞ :

$$\lim_{p \rightarrow \infty} d_p(\mu, \nu) = d_\infty(\mu, \nu).$$

The distances d_p can take infinite values in general, but they are obviously finite for measures with bounded support. Moreover, these distances d_p induce

complete metric structure restricted to the set of probability measure with finite p moment

$$\mathcal{P}_p(X) = \left\{ \mu \in \mathcal{P}(X) : \text{for any } x_0 \in X \int_X \text{dist}(x, x_0)^p d\mu(x) < \infty \right\}.$$

In particular, d_∞ induces a complete metric structure on

$$\mathcal{P}_\infty(X) = \bigcap_{1 \leq p < \infty} \mathcal{P}_p(X)$$

as proven in [3].

1.2.2 Gradient Flows

In the spirit of being self-contained, we briefly review some important Definitions and Theorems of "Gradient flows: in metric spaces and in the space of probability measures" by Ambrosio, Gigli and Savare [1]. We try to outline all of the tools and terminology used in this work, but it is in no way complete and the interested reader should definitely take the time to read [1].

1.2.2.1 General Metric Spaces

We start with some notions defined for a general complete metric space (X, d) , which we later analyze in the $\mathcal{W}^2(\mathbb{T}) = (\mathcal{P}(\mathbb{T}), d_2)$ case. We begin with the notion of an absolutely continuous curve:

Definition 1.2.7. *Let $v : (0, 1) \rightarrow X$ be a curve, we say that $v \in AC^p(a, b; X)$ with $p \in [1, \infty)$, if there exists $m \in L^p(a, b)$ such that*

$$d(v(s), v(t)) \leq \int_s^t m(r) dr \quad \forall a < s < t < b. \quad (2.1)$$

If $p = 1$, we suppress the superscript and just denote it by AC .

Absolutely continuity is enough to define the size of a derivative at almost every point, this is the subject of the next theorem.

Theorem 1.2.8. *Let $v \in AC^p(a, b; X)$, then the limit*

$$|v'(t)| := \lim_{h \rightarrow 0} \frac{d(v(t+h) - v(t))}{h}$$

exists a.e. in (a, b) and $|v'| \in L^p(a, b)$. Moreover, it is minimal in the sense that it holds (2.1), and if

$$d(v(s), v(t)) \leq \int_s^t m(r) dr \quad \forall a < s < t < b,$$

then $|v'(t)| \leq m(t)$ a.e. in (a, b) .

Now that we have the concept of the size of the derivative of a curve, we can give a notion the size of gradients for functionals defined in X . From now on, ϕ is a lower semi-continuous real-valued function on X .

Definition 1.2.9. *A function $g : X \rightarrow [0, +\infty]$ is a strong upper gradient for ϕ if for any $v \in AC(a, b; X)$, the function $g \circ v$ is borel and*

$$|\phi(v(t)) - \phi(v(s))| \leq \int_s^t g \circ v(r) |v'(r)| dr \quad \forall a < s < t < b.$$

In particular, if $g \circ v(r) |v'(r)| \in L^1(a, b)$, then $\phi \circ v$ is absolutely continuous and

$$|(\phi \circ v)'(t)| \leq g \circ v(r) |v'(r)| \quad \text{a.e. in } (a, b).$$

The most natural candidate to satisfy the definition above is the slope of ϕ .

Definition 1.2.10. *The slope at ϕ at v is defined by*

$$|\partial\phi(v)| := \limsup_{w \rightarrow v} \frac{(\phi(w) - \phi(v))^+}{d(w, v)}.$$

To be able to relate the two definitions we need to consider a more restrictive set of functionals, for instance λ -convex functionals.

Definition 1.2.11. *Given $\lambda \in \mathbb{R}$, we say that ϕ is λ -convex with respect to the geodesics, if for every $\gamma_t : [0, 1] \rightarrow X$ constant speed geodesic, we have that*

$$\phi(\gamma_t) \leq (1 - t)\phi(\gamma_0) + t\phi(\gamma_1) - \frac{1}{2}\lambda t(1 - t)(d(\gamma_0, \gamma_1))^2.$$

With this definition we can write the following Theorem.

Theorem 1.2.12. *Suppose that ϕ is λ convex with respect to the geodesics, then $|\partial\phi|$ is a strong upper gradient.*

Proof. See Corollary 2.4.10 in [1]. □

Now, we are ready to define the curves of maximal slope for λ -convex functionals:

Definition 1.2.13. *We say that the locally absolutely continuous map $u : (a, b) \rightarrow X$ is a curve of maximal slope of ϕ with respect to its upper gradient $|\partial\phi|$ if*

$$\phi(u(t)) - \phi(u(s)) \geq \int_s^t \frac{|u'(r)|^2}{2} + \frac{|\partial\phi(u(r))|^2}{2} dr. \quad (2.2)$$

Remark 1.2.14. *If (X, d) is a Hilbert space, and ϕ is λ -convex, then $|\partial\phi(v)|$ is actually the norm of the minimal selection in the sub-differential at v . Moreover, $u(\cdot)$ is a curve of maximal slope, if and only if, $u(\cdot)$ is a gradient flow. This follows from an application of the Cauchy-Schwartz and Young's inequality.*

1.2.2.2 Differential Structure of the L^2 -Wasserstein metric

The distance d_2 has been extensively studied in the literature and we recommend [4], which also contains a pedagogical introduction to the gradient flow theory. In this work, we are mostly interested on its differential structure.

Theorem 1.2.15. *Let the curve $\mu_t : I \rightarrow \mathcal{P}(\mathbb{T})$ be absolutely continuous with respect to d_2 and let $|\mu'| \in L^1(I)$ be its metric derivative, then there exists a Borel vector field v such that*

$$\|v(\cdot, t)\|_{L^2_{\mu_t}} \leq |\mu'(t)| \quad \text{a.e. } t \in I$$

and the continuity equation

$$\partial_t \mu_t + \nabla \cdot (v(\cdot, t) \mu_t) = 0 \tag{2.3}$$

is solved in the sense of distributions.

Conversely, if $\mu_t : I \rightarrow \mathcal{P}(\mathbb{T})$ is continuous with respect to d_2 and satisfies the continuity equation (2.3) for some Borel velocity field v with $\|v(\cdot, t)\|_{L^2_{\mu_t}} \in L^1(I)$, then μ_t is absolutely continuous and $|\mu'(t)| \leq \|v(\cdot, t)\|_{L^2_{\mu_t}}$ a.e. $t \in I$.

Proof. See Theorem 8.3.1. [1]. □

Heuristically with Theorem 1.2.15, we can try to comprehend $\mathcal{P}(\mathbb{T})$ with d_2 as an infinite dimensional Riemmanian manifold.

One could consider the vector space $L^2(\mathbb{T}, d\mu)$ as the tangent space at $\mu \in \mathcal{P}(\mathbb{T})$, though we would be missing an extra condition to uniquely determine the vector field v . We can always perturb by a field w such that $\nabla \cdot (w \mu_t) = 0$, without

changing the continuity equation (2.3). In fact, up to making the quotient over the divergence free fields, we can uniquely determine a tangent direction. By Hodge's decomposition of $L^2(\mathbb{T}, d\mu)$, this tangent space can be represented as:

Definition 1.2.16. *Let $\mu \in \mathcal{P}(\mathbb{T})$, we define*

$$\text{Tan}_\mu \mathcal{P}(\mathbb{T}) = \text{cl}(\{\nabla \phi : \phi \in C^\infty(\mathbb{T})\}),$$

where cl denotes the closure with respect to the L^2_μ topology.

Moreover, the metric in the tangent space is the one induced by the $L^2(\mathbb{T}, d\mu)$ inner product.

This heuristic discussion is justified in the following Theorem:

Theorem 1.2.17. *Let $\mu_t : I \rightarrow \mathcal{P}(\mathbb{T})$ be an absolutely continuous curve and let v be such that the continuity equation (2.3) is satisfied. Then, $|\mu'(t)| = \|v(\cdot, t)\|_{L^2_{\mu_t}}$ a.e. $t \in I$, if and only if, $v \in \text{Tan}_{\mu_t} \mathcal{P}(\mathbb{T})$ a.e. $t \in I$.*

Moreover, the vector field v is a.e. uniquely determined by these conditions.

Proof. See Theorem 8.3.1. [1]. □

Exploiting the inner product structure in L^2_μ , we are able to define the subdifferential of a λ -convex functional

Definition 1.2.18. *We say that $\zeta \in L^2_\mu(\mathbb{T})$ is a strong subdifferential of ϕ at μ , denoted by $\partial\phi(\mu)$, if*

$$\phi(H\#\mu) - \phi(\mu) \leq \int_{\mathbb{T}} \langle \zeta(x), H(x) - x \rangle d\mu(x) + o(\|H - I\|_{L^2_\mu(\mathbb{T})}),$$

where H is a Borel vector field and the push-forward $H\#\mu$ is defined by the condition $H\#\mu(A) = \mu(H^{-1}(A))$ for every Borel set A .

As a next step, we characterize the strong subdifferentials of functionals that are classical to the literature of Calculus of Variations.

$$\mathcal{F}[\mu] = \begin{cases} \int_{\mathbb{T}} F(x, \rho(x), \nabla \rho(x)) dx & \text{if } \mu = \rho d\mathcal{L} \text{ and } \rho \in C^1(\mathbb{T}) \\ +\infty & \text{otherwise,} \end{cases}$$

where $d\mathcal{L}$ is the Lebesgue measure in \mathbb{T} . We denote $(x, z, p) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}$ the variables of F . To simplify, we ask that $F \in C^2$ and that $F(x, 0, p) = 0$ for every x and p .

Lemma 1.2.19. *If $\mu = \rho d\mathcal{L} \in \mathcal{P}(T)$, with $\rho \in C^1$, satisfies $\mathcal{F}[\mu] < \infty$, then any strong subdifferential of \mathcal{F} at μ is μ -a.e. equal to*

$$\nabla \frac{\delta \mathcal{F}}{\delta \rho} = \nabla (F_z(x, \rho(x), \nabla \rho(x)) - \nabla \cdot F_p(x, \rho(x), \nabla \rho(x))). \quad (2.4)$$

Proof. See Lemma 10.4.1. in [1]. □

Remark 1.2.20. *Unfortunately, currently there is a lack of a complete understanding of the curves of maximal slope of these type of functionals that involve derivatives under the d_2 metric. One should remark that the easy cases of the classical calculus of variation, namely linearly convex functionals lie outside of the current well-posedness theory for gradient flows in the L^2 -Wasserstein metric. This is an exciting limitation to try to overcome in the future, perhaps by somehow marrying the concepts of linear convexity with displacement convexity.*

Now we can define the notion of gradient flow for a functional ϕ .

Definition 1.2.21. *We say that a map $\mu_t \in AC^2((0, \infty), \mathcal{P}(\mathbb{T}))$ is a solution to the gradient flow equation, if the vector field v from Theorem 1.2.17 satisfies*

$$v(\cdot, t) \in \partial \phi(\mu_t) \quad \forall t > 0.$$

In the λ -convex case, we can make the connection between gradient flows and curves of maximal slope.

Theorem 1.2.22. *If ϕ is λ -convex, then μ_t is a curve of maximal slope with respect to $|\partial\phi|$, if and only if, μ_t is a gradient flow and $\phi(\mu_t)$ is equal a.e. to a function of bounded variation.*

Moreover, given two gradient flows μ_t^1 and μ_t^2 , such that $\mu_t^1 \rightarrow \mu_1$ and $\mu_t^2 \rightarrow \mu_2$ as $t \rightarrow 0$, then

$$d_2(\mu_t^1, \mu_t^2) \leq e^{-\lambda t} d_2(\mu_1, \mu_2).$$

In particular, there is a unique gradient flow μ_t with initial condition μ_0 and it satisfies the maximal slope condition (2.2) with equality.

Proof. See Theorem 11.1.3 and Theorem 11.1.4 in [1]. □

Chapter 2: Regularity of local minimizers of the interaction energy via obstacle problems

2.1 Overview

Given a pointwise defined function $W : \mathbb{R}^N \rightarrow (-\infty, +\infty]$, we define the interaction energy of a probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$ by

$$E[\mu] := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) d\mu(x) d\mu(y). \quad (1.1)$$

Here, $\mathcal{P}(\mathbb{R}^N)$ denotes the space of Borel probability measures, and throughout the paper, we will always assume that the interaction potential W is a non-negative lower semi-continuous function in $L^1_{loc}(\mathbb{R}^N)$.

Under this assumption, the energy $E[\mu]$ is well defined for all $\mu \in \mathcal{P}(\mathbb{R}^N)$, with $E[\mu] \in [0, +\infty]$. Local integrability of the potential avoids too singular potentials for which the interaction energy is infinite for many smooth densities. These very singular potentials lead to very interesting questions in crystallization [5], whose study is outside the scope of this work. Also, under these assumptions, the potential function ψ associated to a given measure μ :

$$\psi(x) := W * \mu(x) = \int_{\mathbb{R}^N} W(x-y) d\mu(y)$$

can be defined pointwise in \mathbb{R}^N , and a simple application of Fatou's lemma implies that ψ is a lower semi-continuous function, see [6, Lemma 2].

The goal of this work is to investigate the regularity properties of the local minimizers of the interaction energy (1.1). For this, the keystone of this paper will be to show that the potential function $\psi(x)$ associated to a local minimizer solves an obstacle problem. This fact comes out naturally of the Euler-Lagrange conditions derived in [6], here we prove the continuity of the potential function, to make this relationship rigorous.

Note that in order to define precisely the notion of local minimizers, we need to specify a topology on the set of probability measure. We use here the framework developed in [6], where the authors consider local minimizers of the energy (1.1) with respect to the optimal transport distance d_∞ .

Lots of numerical results [6–14] show the rich structure and variety of local/global minimizers of the interaction energy by using different numerical approaches such as particle approximations, DG schemes for the gradient flow equation associated to the energy (1.1), direct resolution of the associated steady equations, radial coordinates, and so on. The interaction potentials used in most of these numerical experiments are repulsive near the origin and attractive at large distances. Typical choices are radial potentials with a unique minimum L for $r > 0$, decreasing (repulsive) before and increasing (attractive) after. In particular, for a system of two identical particles, the discrete energy would then be minimized when they are located at distance L from each other. Particular relevant examples are Morse potentials [15–17] and power-laws [8, 10, 18].

These repulsive/attractive interaction potentials emanated from applications in self-similar solutions for granular media models [19–21], collective behavior of animals (swarming) [8, 9, 15–17, 22, 23], and self-assembly of nanoparticles [24–26]. Let us mention that local minimizers of the interaction energy can be seen as steady states of the aggregation equation that have been studied thoroughly for fully attractive potentials [27, 28] and repulsive/attractive potentials [6, 8–11, 29–33], analysing qualitative properties of the evolution in different cases: finite time blow-up, stabilization towards equilibria, confinement of solutions and so on.

The natural result shown in [6], corroborated by the cited numerical studies, is that the support of local minimizers of the interaction energy increases as the repulsion at the origin gets stronger. In other words, concentration of particles is not allowed on small dimensional sets when the repulsion is large enough. Geometric measure theory techniques [34] were crucial to get the estimate on the dimension of the support based on the Euler-Lagrange conditions for local minimizers in transport distances. In this work, we give an alternative proof, to the result of dimensionality of [6]. This result follows naturally from the regularity of solutions to the obstacle problem.

To be able to prove the continuity of the potential function, we restrict ourselves to potentials that behave like power laws around the origin

$$W(x) \sim \frac{1}{|x|^{N-2s}}, \quad \text{as } x \rightarrow 0, \text{ for some } s \in (0, \frac{N}{2}) \text{ and } N \geq 2, \quad (1.2)$$

and are smooth enough outside the origin. We also consider the particular case $W(x) \sim -\log|x|$ if $s = 1$ and $N = 2$. More precise hypothesis are given below and

in Section 2.2. For the cases $s > N/2$, W is already continuous, so the continuity of the potential function is trivially true.

In the literature, the case $s = 1$ in (1.2) is of particular interest. It corresponds to Newtonian repulsion and it has received considerable attention due to its various applications. A repetitively rediscovered result in this classical case is that the global minimizer of the interaction energy for the potential

$$W(x) = \frac{1}{|x|^{N-2}} + \frac{|x|^2}{2},$$

is the characteristic function of an euclidean ball. This classical result, using potential theory and capacities, was proved by Frostman [35] (but in a bounded domain instead of confinement by quadratic potentials), and it has connections with the eigenvalue distribution of random matrices [36,37]. This precise result can be found for instance in [38, Proposition 2.13]. In [39], the authors show that the uniform distribution in a ball is the asymptotic behavior of the corresponding gradient flow evolution. The uniqueness of the global minimizer for more singular than Newtonian repulsion, i.e.,

$$W(x) = \frac{1}{|x|^{N-2s}} + \frac{|x|^2}{2},$$

with $0 < s < 1$, was obtained by Caffarelli and Vázquez via the connection to a classical obstacle problem in [40], and this strategy was also used in [41] to treat again the case $s = 1$ for the evolution problem as in [39].

One should note that the case when the attractive potentials is $|x|^2$ is atypical. An observation, that partially explains this claim, is that when

$$W(x) = \frac{1}{|x|^{N-2s}} + \frac{|x|^2}{2},$$

with $s \in (0, \frac{N}{2})$, the Energy (1.1) associated to W can be re-written as

$$E(\mu) = \|\mu\|_{H^{-s}} + 2 \left(\int_{\mathbb{R}^N} |x|^2 d\mu(x) - \left(\int_{\mathbb{R}^N} x d\mu(x) \right)^2 \right).$$

So, it becomes clear that E is linearly convex when restricted to probability measures with a fixed center of mass, which implies that E has, up to translation, a unique critical point on the set of probability measures. Similar considerations apply for the attractive potential $|x|^4$. The attractive potentials $|x|^{2n}$ with $n \in \mathbb{N}$, have a similar decomposition, but the Energy is not linearly convex when restricted to probability measures with a fixed center of mass.

It is also worth mentioning, the case of the potential

$$W(x) = \frac{1}{|x|^{N-2}} + \frac{|x|^a}{a},$$

with $a > 2$ or $2-d < a < 2$ that has been analysed in [8,9] showing the existence and uniqueness of compactly supported radial critical points of the interaction energy. Moreover, they show that these critical points are monotone, bounded, and smooth functions inside their support. The monotonicity of the critical points is rather remarkable and hints to the possibility of using rearrangement techniques to prove uniqueness of absolute minimizers. Boundedness and smoothness inside the support of radial compactly supported minimizers was also proved for the so-called Quasi-Morse potentials in [13]. These Quasi-Morse potentials behave at the origin as Newtonian potentials while they exhibit similar properties to Morse potentials in terms of existence of compactly supported radial minimizers. This particular case allow for explicit computations leading to analytic expressions for these minimizers.

The main result of this chapter is that for kernels satisfying (1.2), with $s \in (0, 1]$, and under mild assumptions on $W_a(x) = W(x) - |x|^{2s-N}$, local minimizers μ of the interaction energy (1.1) are absolutely continuous with respect to the Lebesgue measure, and their density function lies in $L_{loc}^\infty(\mathbb{R}^N)$ when $s = 1$ and in $C_{loc}^\alpha(\mathbb{R}^N)$ when $s \in (0, 1)$. Furthermore, we show that for $s = 1$ the density function is in $BV_{loc}(\mathbb{R}^N)$ and that the support of these local minimizers is a set with locally finite perimeter. In particular, our results will apply to potentials of the form

$$W(x) = \frac{1}{|x|^{N-2s}} + \frac{|x|^q}{q}, \quad \text{for } q > N - 2s.$$

These results will be obtained by exploiting the connection between the Euler-Lagrange conditions for local minimizers and classical obstacle problems [42].

In fact, we show that the potential functions of local minimizers are locally solutions of some obstacle problems. It is by using the regularity theory for the solutions of such obstacle problems [43, 44] that we will derive our main results on the regularity of local minimizers. Note that Newtonian repulsion ($s = 1$) will lead to the classical obstacle problem with the Laplace operator, while stronger repulsion ($s \in (0, 1)$) will lead to fractional obstacle problems (with fractional power of the Laplace operator) which have been more recently studied, in particular in [45, 46] (we will also use some results of [40, 47] where these obstacle problems arise in the study of fractional-diffusion versions of the porous medium equation). For potentials that are less repulsive than Newtonian ($s > 1$), we also show that the potential function solves an obstacle problem. However, these involve elliptic operators of higher order. A prototypical example is the case where $W(x) \sim -|x|$ in dimension $N = 3$, which

leads to a biharmonic obstacle problem. The regularity theory for these higher order obstacle problems is different, and far less developed. In these cases, our only regularity result, which matches with the dimensionality result of [6], is $\mu \in H^{-s+1}$.

Let us finally comment that some of our results require some additional uniformity assumptions on the potential W_a at infinity if the support of the local minimizer is not compact. In fact, the existence of compactly supported global minimizers for the interaction energy is a very interesting question by itself connected to statistical mechanics [48]. This property has recently been shown [49, 50] under natural conditions on the interaction potential W related to non H -stability as defined in [15, 48].

The plan of this chapter is as follows. Section 2 is devoted to describing the notion of local minimizers used in this paper and gives the precise statements of the main results of this chapter. Section 3 has the proofs of the continuity of the potential function. Section 4 has the proofs of the regularity of minimizers.

2.2 Main results and strategy

2.2.1 Hypothesis

(H1) W is a non-negative lower semi-continuous function in $L^1_{loc}(\mathbb{R}^N)$ and $W \in C(\mathbb{R}^N \setminus \{0\})$.

(H2) There exists $s \in (0, \frac{N}{2})$ and $\alpha > 0$, such that, up to re-normalizing W we have:

$$\limsup_{|x| \rightarrow 0} V_{N,s}(x)W(x) = 1 \quad \text{and} \quad \liminf_{|x| \rightarrow 0} V_{N,s}(x)W(x) = \alpha, \quad (2.1)$$

where

$$V_{N,s}(x) = \frac{C_{N,s}}{|x|^{N-2s}}$$

is the fundamental solutions of $(-\Delta)^s$. Namely,

$$(-\Delta)^s V_{N,s} = \delta.$$

If $N = 2$, all the results also include the case $s = 1$, $V_{2,1}(x) = -C \log(|x|)$.

Remark 2.2.1. *For the cases $s > \frac{N}{2}$, the potential $V_{N,s}$ is continuous.*

We define $W_a = W(x) - V_{N,s}$, by (H1) $W_a \in C(\mathbb{R}^d \setminus 0)$. For W_a , we consider that either one of the following holds:

(H3a) The support of μ , $\text{supp}(\mu)$, is compact in \mathbb{R}^N and $(-\Delta)^s W_a \in L^1_{loc}(\mathbb{R}^d)$.

or

(H3b) Given $\delta > 0$, W_a is uniformly continuous in $\mathbb{R}^d \setminus B_\delta(0)$ and $(-\Delta)^s W_a \in L^1(\mathbb{R}^d)$.

The motivation behind (H3a) or (H3b) is that, up to cutting off W_a around zero, $W_a * \mu$ is continuous. We note that (H3b) holds typically for potential that do not grow too much at ∞ , while it is expected that for potentials that grow fast enough at ∞ , local minimizers of the energy have compact support, i.e. (H3a) should hold (this last fact remains to be proved though). So conditions (H3a) and (H3b) should be seen as complementary. We recall also that the existence of compactly supported global minimizers of the interaction energy E has recently been proved in [49, 50] under natural conditions on the interaction potential related to non H -stability as defined in [15, 48]. Thus, relevant minimizers, in applications such as swarming [6, 8–10, 13, 15, 16], are typically compactly supported.

When (H3a) holds, we use the following trick: Because $\text{supp}(\mu)$ is compact, we find $R > 0$ such that $\text{supp}(\mu) \subset B_R(0)$. Then, we can cut-off the kernel W in a smooth way outside the ball $B_{4R}(0)$. The density μ will still be an ϵ -minimizer of the energy E and its potential ψ will be unchanged in the ball $B_{2R}(0)$. So whenever assuming (H3a), it is possible to assume (H3b) as well.

The need for $(-\Delta)^s W_a \in L^1(\mathbb{R}^N)$ is to assure that $(-\Delta)^s W_a * \mu \in H^{-s}(\mathbb{R}^N)$. This follows from Young's inequality for the convolution and the fact that $\mu \in H^{-s}(\mathbb{R}^N)$, which follows from the continuity of ψ .

Finally, to be able to derive interesting regularity results we need that W_a has slightly better regularity.

(H4) There exists $\delta > 0$, such that $(-\Delta)^{s+\delta} W_a \in L^1(\mathbb{R}^N)$.

Remark 2.2.2. *It is worth noticing that the hypothesis (H1)-(H4) are satisfied for Power-Law potentials of the form*

$$W = -\frac{|x|^a}{a} + \frac{|x|^b}{b}$$

if $a \in (-N, 0)$ and $a < b < \infty$, and compactly supported minimizers.

2.2.2 Euler-Lagrange condition in d_2 and d_∞

We consider the following concept of local minimizers in d_∞ and d_2 :

Definition 2.2.3. *We say that μ is an ϵ -local minimizer (or simply ϵ -minimizer) for the energy E with respect to d_∞ (d_2), if $E[\mu] < \infty$ and*

$$E[\mu] \leq E[\nu]$$

for all $\nu \in \mathcal{P}(\mathbb{R}^N)$ such that $d_\infty(\mu, \nu) < \varepsilon$ ($d_2(\mu, \nu) < \varepsilon$).

Let's try to describe the idea behind the Euler-Lagrange condition. Heuristically, by visualizing

$$E[\mu] = \int_{\mathbb{R}^N} \psi d\mu,$$

with $\psi = W * \mu$, one expects that if μ is a minimizer, then the support of μ is contained in the set of local minima of the associated potential ψ . Indeed, if the support is not contained in the local minima of ψ , one can prove that transferring mass to the set where ψ is smaller decreases the energy. In particular, for local d_2 minimizers the Euler-Lagrange conditions were derived in [6]. The d_2 Euler-Lagrange condition reads:

$$\begin{cases} \psi(x) = 2E[\mu] & \mu\text{-a.e.} \\ \psi(x) \geq 2E[\mu] & \text{a.e.} \end{cases} \quad (2.2)$$

Also in [6], a partial d_∞ Euler-Lagrange condition was derived and it is a local version of the second point of (2.2):

Proposition 2.2.4 ([6, Proposition 1]). *Assume that W satisfies (H1) and let μ be an ε -minimizer of the energy $E[\mu]$ in the sense of Definition 2.2.3. Then any point $x_0 \in \text{supp}(\mu)$ is a local minimum of $\psi = W * \mu$ in the sense that*

$$\psi(x_0) \leq \psi(x) \text{ for a.e. } x \in B_\varepsilon(x_0). \quad (2.3)$$

Remark 2.2.5. *An attentive reading of the proof of [6, Proposition 1] leads to the important observation that the ε appearing in (2.4) is the same as the ε appearing in Definition 2.2.3. In particular, it is independent of the point x_0 . Moreover, only*

local integrability of the interaction potential is needed for that proof, i.e., there is no need of uniform local integrability of W in the proof of [6, Proposition 1].

Using a similar method of proof employed in this last Proposition 2.2.4, we can refine the result. We actually obtain a local version of the first condition in (2.2).

Proposition 2.2.6. *Assume that W satisfies (H1) and let μ be a ε -minimizer of the energy $E[\mu]$ in the sense of Definition 2.2.3. Then any point $x_0 \in \text{supp}(\mu)$ is a local minimum μ -a.e. of $\psi = W * \mu$ in the sense that*

$$\psi(x_0) \leq \psi(x) \quad x \in B_\varepsilon(x_0) \quad \mu - a.e.. \quad (2.4)$$

The proof can be found in Section 2.3.

2.2.3 Continuity of the potential function ψ and the obstacle problem

As mentioned earlier, the keystone of this chapter is the observation that the potential function ψ solves (locally) an obstacle problem. In order to make this fact rigorous, we first prove that ψ is a continuous function. Heuristically, when we assume that the potential is well behaved away from zero the continuity should follow from continuity in $\text{supp}(\mu)$. In fact, assuming (H2), and (H3a) or (H3b), we can borrow arguments from potential theory (see [51]) to first prove continuity in $\text{supp}(\mu)$ and then continuity in \mathbb{R}^N .

The continuity of the potential function in $\text{supp}(\mu)$ follows from the Euler-Lagrange conditions and the following Lemma.

Lemma 2.2.7. *Given $s \in (0, \frac{N}{2})$, assume that W satisfies (H1), (H2), and either (H3a) or (H3b). Let μ be a ε -minimizer of the energy $E[\mu]$ in the sense of Definition 2.2.3. Then, for any $x \in \mathbb{R}^N$, we have*

$$\psi(x) = \lim_{r \rightarrow 0} \frac{1}{|B_r|} \int_{B_r(x)} \psi(y) dy.$$

The Lemma is proved in Section 2.3. With Lemma 2.2.7, proving the continuity in the support becomes an immediate consequence of the the Euler-Lagrange condition.

Corollary 2.2.8. *Given $s \in (0, \frac{N}{2})$, assume that W satisfies (H1), (H2), and either (H3a) or (H3b) hold. Let μ be a ε -minimizer of the energy $E[\mu]$ in the sense of Definition 2.2.3. Then, $\psi = W * \mu$ is continuous in $\text{supp}(\mu)$.*

Proof of Corollary 2.2.8. Given $z_0 \in \text{supp}(\mu)$, we know, by Proposition 2.2.6, that

$$\psi(x) \geq \psi(z_0) \quad \text{a.e. in } B_\varepsilon(z_0).$$

Then, by Lemma 2.2.7, we know that for any $z_1 \in \text{supp}(\mu) \cap B_\varepsilon(z_0)$, we have $\psi(z_1) \geq \psi(z_0)$. Reversing the roles of z_0 and z_1 , we obtain $\psi(z_0) = \psi(z_1)$. \square

With the previous Corollary we can emulate the continuity result from potential theory.

Proposition 2.2.9 (Continuity of the potential function). *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (0, \frac{N}{2})$, (H1), (H2) and either (H3a) or (H3b) hold. Then the potential function $\psi(x) := W * \mu(x)$ associated to μ is a continuous function in \mathbb{R}^N .*

This proposition will be proved in Section 2.3. As an incidental result we have the first regularity result for μ .

Corollary 2.2.10. *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (0, \frac{N}{2})$ (H1), (H2) and either (H3a) or (H3b) hold. Then, $\mu \in H^{-s}(\mathbb{R}^N)$ and moreover $(-\Delta)^s W_a * \mu \in H^{-s}(\mathbb{R}^N)$. If (H3a) holds, then we also know that $W_a * \mu$ is continuous.*

Moreover, we can now strengthen the Euler-Lagrange conditions:

Corollary 2.2.11. *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (0, \frac{N}{2})$ (H1), (H2) and either (H3a) or (H3b) hold. Then, given $x_0 \in \text{supp}(\mu)$, ψ satisfies*

$$\begin{cases} \psi(x) = \psi(x_0) & \text{on } B_\varepsilon(x_0) \cap \text{supp}(\mu) \\ \psi(x) \geq \psi(x_0) & \text{on } B_\varepsilon(x_0). \end{cases} \quad (2.5)$$

With the continuity Proposition 2.2.9, we can make the fact that ψ satisfies an obstacle problem rigorous. First, we observe that (H3) implies

$$(-\Delta)^s \psi = \mu + (-\Delta)^s W_a * \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

In particular, since μ is a non-negative measure, we deduce

$$(-\Delta)^s \psi \geq (-\Delta W_a)^s * \mu \quad \text{in } B_\varepsilon(x_0).$$

Second, if $x \in B_\varepsilon(x_0)$ is such that $\psi(x) > \psi(x_0)$, (2.5) implies that $x \notin \text{supp}(\mu)$, and so (by definition of $\text{supp}(\mu)$), $\mu(B_r(x)) = 0$ for some small $r > 0$. We deduce

$$(-\Delta)^s \psi = (-\Delta W_a)^s * \mu \quad \text{in } \mathcal{D}'(B_\varepsilon(x_0) \cap \{\psi > \psi(x_0)\}).$$

Combining Corollary 2.2.11 with the previous discussion, we have the following proposition:

Proposition 2.2.12. *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (0, \frac{N}{2})$ (H1), (H2) and either (H3a) or (H3b) hold. Then, for any $x_0 \in \text{supp}(\mu)$, the potential function ψ is equal, in $B_\varepsilon(x_0)$, to the unique solution of the obstacle problem*

$$\left\{ \begin{array}{ll} \varphi \geq C_0, & \text{in } B_\varepsilon(x_0) \\ (-\Delta)^s \varphi \geq -F(x), & \text{in } B_\varepsilon(x_0) \\ (-\Delta)^s \varphi = -F(x), & \text{in } B_\varepsilon(x_0) \cap \{\varphi > C_0\} \\ \varphi = \psi, & \text{on } B_\varepsilon^c(x_0), \end{array} \right. \quad (2.6)$$

where $C_0 = \psi(x_0)$ and $F = (-\Delta)^s W_a * \mu \in H^{-s}(\mathbb{R}^N)$. Furthermore, the measure μ is given by

$$\mu = (-\Delta)^s \psi + F. \quad (2.7)$$

Proof. The only point that is not immediate from the previous discussion is uniqueness. In fact, any solution to (2.6) is a linear critical point of the energy

$$J(\varphi) = |\varphi|_{H^s} + \langle F, \varphi \rangle, \quad (2.8)$$

in the set $K = \{\varphi \in H^s(\mathbb{R}^d) \text{ s.t. } \varphi = \psi \text{ in } B_\varepsilon(x_0) \text{ and } \varphi = \psi \text{ in } B_\varepsilon(x_0)^c\}$. Uniqueness follows from the fact that J is strictly convex in the convex set K . \square

Since F depends on μ itself, it seems difficult to exploit (2.6) to identify local minimizers or to prove global properties such as uniqueness or radial symmetry. However, because F is more regular than μ we are able to use (2.6) in the cases

$s \in (0, 1]$ to derive sharp regularity results of these local minimizers. For the cases $s > 1$, we obtain some regularity results that are probably not sharp. The difficulty of the cases $s > 1$, stems directly from the lack of a maximum principle for higher order elliptic operators.

We also insist here on the fact that in general the constant C_0 might depend on the choice of $x_0 \in \text{supp}(\mu)$. On the other hand, for global minimizers, as well as for local d_2 minimizers (see (2.2)), the constant can be fixed to be $2E[\mu]$.

Equation (2.7) suggests that there is a relation between the support of μ and the coincidence set $\psi = \psi(x_0)$. In fact, it is easy to check that $\mu = 0$ in the open set $\{\psi > \psi(x_0)\} \cap B_\varepsilon(x_0)$ in the sense that $\mu(\{\psi > \psi(x_0)\} \cap B_\varepsilon(x_0)) = 0$. We thus deduce using the continuity of ψ that

$$\text{supp}(\mu) \cap B_\varepsilon(x_0) \subset \{\psi = \psi(x_0)\} \cap B_\varepsilon(x_0).$$

But it is not obvious that these two sets should be equal. Nevertheless, in the case $s = 1$ we shall later see that, under a non-degeneracy condition on F , they are equal up to a set of measure zero.

2.2.4 Regularity of ψ and μ

In this Section, we state the regularity results for ψ and μ . All of them follow from regularity results for the obstacle problem (2.6).

2.2.4.1 Cases $s > 1$

For the cases $s > 1$ the obstacle problem is not fully understood. One of the few regularity results available, is due to Frehse (see [52] and [53]). In the framework of (2.6), it can be paraphrased as follows: if $s \in \mathbb{N}$, and $F \in H^{-s+1}(\mathbb{R}^N)$, then $\varphi \in H_{loc}^{s+1}(B_\varepsilon)$, which implies $\mu \in H_{loc}^{-s+1}(B_\varepsilon)$. Here we also prove that $\mu \in H_{loc}^{-s+1}(\mathbb{R}^d)$, by generalizing Frehse's result to any $s \in (1, \infty)$ and $F \in H^{-s+l}$ for any $l \in (0, 1]$.

Theorem 2.2.13. *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (1, \frac{N}{2})$ (H1), (H2), (H3a) and (H4) hold. Then $\mu \in H^{-s+1}$.*

Using this regularity result, we can now recover the results on the dimensionality of minimizers found in [6] with different hypothesis:

Corollary 2.2.14. *Let μ be an ε -minimizer of E in the sense of Definition 2.2.3, and assume that given $s \in (1, \frac{N}{2})$ (H1), (H2), (H3a) and (H4) hold. Then, given a Borel set $A \subset \mathbb{R}^d$ such that $\text{Hauss}^{N-2(s-1)}(A) < \infty$, we have $\mu(A) = 0$. Where $\text{Hauss}^{N-2(s-1)}$ is the Hausdorff measure of dimension $N - 2(s - 1)$.*

Proof. Follows from the same reference used in [6], [34, Theorem 4.13]. \square

Theorem 2.2.13 follows from a bootstrap argument and the following Proposition, which generalizes Frehse's result.

Proposition 2.2.15. *Let φ be a solution to (2.6) with $s > 1$ and the external condition ψ bounded in \mathbb{R}^N . Then, given $l \in [0, 1]$, if $F \in H^{-s+l}(\mathbb{R}^d)$, then $\varphi \in H_{loc}^{s+l}(B_\varepsilon)$.*

The proof of Proposition 2.2.15 and Theorem 2.2.13 can be found in Section 2.4.1

2.2.4.2 Case $s = 1$

The case $s = 1$ corresponds to the Harmonic Obstacle problem, which is by far the best understood. We use classical regularity results for the harmonic obstacle problem to study the properties of ε -minimizers of E . Our first result is the following:

Theorem 2.2.16 (L^∞ regularity of μ). *Assume W satisfies (H1), (H2), (H3a) and (H4), with $s = 1$. Let μ be a compactly supported ε -minimizer in the sense of Definition 2.2.3*

*Then, the potential function ψ is in $C^{1,1}(\mathbb{R}^N)$. In particular, the measure μ is absolutely continuous with respect to the Lebesgue measure and there exists a function $\rho \in L^\infty(\mathbb{R}^N)$ such that $\mu = \rho(x)d\mathcal{L}^N$. Finally, we have $\rho = \Delta W_a * \rho$ in the interior of $\text{supp}(\mu)$.*

The proof of this proposition will be given in Section 2.4.

Remark 2.2.17. *Under the conditions of Theorem 2.2.16, we can show that local minimizers are actually stationary states of the Wasserstein gradient flow associated to (1.1). Indeed, since $\nabla\psi \in C^{0,1}(\mathbb{R}^N)$ and $\rho \in L^\infty(\mathbb{R}^N)$ we have $\rho\nabla\psi \in L^\infty(\mathbb{R}^N)$. Moreover, since $\nabla\psi = 0$ in the interior of $\text{supp}(\rho)$, then $\rho\nabla\psi = 0$ a.e. in \mathbb{R}^N , and thus ρ satisfies*

$$|\partial_{W^2}E(\rho)|^2 = \int_{\mathbb{R}^N} |\nabla\psi(x)|^2 \rho(x) dx = 0.$$

In general, we cannot expect better regularity for ρ in \mathbb{R}^N . For instance if $\Delta W_a > 0$, then $\Delta W_a * \rho > 0$ on $\partial(\text{supp}(\mu))$ and so we expect ρ to be discontinuous in \mathbb{R}^N . Obviously, if W_a is smooth in \mathbb{R}^N , then ρ will be smooth in the interior of $\text{supp}(\mu)$. But it is not very difficult to prove (using a bootstrapping argument) that if W_a is a power like interaction potential, then ρ will be smooth in the interior of $\text{supp}(\mu)$.

Finally, under the assumptions of Theorem 2.2.16, we note that since $\psi \in W^{2,\infty}$, we have

$$\Delta\psi = -\rho + \Delta W_a * \rho = 0 \quad \text{a.e. in } \{\psi = \psi(x_0)\}.$$

Again, if we assume that $\Delta W_a * \rho > 0$ in $B_\varepsilon(x_0)$, then we have $\rho(x) > 0$ a.e. in $\{\psi = \psi(x_0)\}$ and thus

$$\text{meas}(\{\psi = \psi(x_0)\} \cap B_\varepsilon(x_0) \setminus \text{supp}(\mu)) = 0, \quad (2.9)$$

in other words, the support of μ and the coincidence set $\{\psi = \psi(x_0)\}$ are the same up to a set of measure zero.

As noted above, ρ is expected to be a discontinuous function and so does not belong, in general, to $W_{loc}^{1,1}$. However, under appropriate regularity assumption on ΔW_a , we can prove that ρ is in $BV_{loc}(\mathbb{R}^N)$:

Theorem 2.2.18 (Regularity of $\text{supp}(\mu)$). *Under the assumptions of Theorem 2.2.16, assume further that*

$$\Delta W_a \in W_{loc}^{1,1}(\mathbb{R}^N).$$

Then the density ρ lies in $BV_{loc}(\mathbb{R}^N)$. Furthermore, if $\Delta W_a * \rho > 0$ in a neighborhood of $\partial(\text{supp}(\mu))$, then $\text{supp}(\mu)$ is a set with locally finite perimeter.

Note that the condition that $\Delta W_a * \rho > 0$ in a neighborhood of $\partial(\text{supp}(\mu))$ is in particular satisfied if $\Delta W_a(x)$ is non-negative for all x and not identically zero (which is the case when $W_a(x) = |x|^q/q$ with $q > 2 - N$). This condition implies that ρ has a nonzero continuous extension on $\partial(\text{supp}(\mu))$ from the interior of $\text{supp}(\mu)$. In particular, ρ has a jump discontinuity at the boundary of its support, and the BV regularity is thus optimal in that sense.

Finally, let us point out that there are numerous results in the literature concerning further regularity of the free boundary $\partial(\text{supp}(\mu))$ for the obstacle problem, always under the same non-degeneracy requirement that $\Delta W_a * \mu > 0$ in a neighborhood of the free boundary, see [43, 54]. Clearly many of these results could be used here, but we will not pursue this direction, as we are mainly interested in the regularity of the measure μ itself.

2.2.4.3 Cases $s \in (0, 1)$

The obstacle problem (2.6) for $s \in (0, 1)$ has been studied by numerous authors in recent years, in particular by Silvestre [45]. However, some aspects of the theory for this fractional obstacle problem are different, or not as developed yet, as that of the regular obstacle problem. The only regularity result we prove is that the density μ is Hölder continuous:

Theorem 2.2.19. *Assume that W satisfies (H1), (H2), (H3a) ($\text{supp}(\mu)$ is compact)*

and (H4) with a fixed $s \in (0, 1)$, and let μ be an ε -minimizer in the sense of Definition 2.2.3.

Then the potential function ψ is in $C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma < s$. Furthermore, the measure μ is absolutely continuous with respect to the Lebesgue measure and there exists a function $\rho \in C^\alpha(\mathbb{R}^N)$ for all $\alpha < 1 - s$, such that $\mu = \rho(x)d\mathcal{L}^N$.

Remark 2.2.20. The optimal regularity for the potential function in the fractional obstacle problem is $C^{1,s}(\mathbb{R}^N)$, see [46], but it requires $W_a * \mu \in C^{2,1}(\mathbb{R}^N)$.

Remark 2.2.21. Again, as in the case $s = 1$, we can claim that if μ is an ε -minimizer in the sense of Definition 2.2.3, it is also classical steady state for the Wasserstein-2 gradient flow associated to (1.1).

With regards to the regularity of the free boundary, there is an analogous result to 2.2.18. In [55], the authors prove for the fractional obstacle problem that under a non-degeneracy condition the free boundary has locally finite $N - 1$ Hausdorff measure.

Theorem 2.2.22 (Regularity of $\text{supp}(\mu)$). *Under the assumptions of Theorem 2.2.19, assume further that for some $\gamma > 0$*

$$W_a * \rho \in C^{3,\gamma}$$

*and $\Delta W_a * \rho > 0$ in a neighborhood of $\partial(\text{supp}(\mu))$, then $\text{supp}(\mu)$ is a set with locally finite perimeter.*

Proof. See [55, Theorem 1.2] □

2.2.5 A uniqueness result

We end this section with some uniqueness results for d_2 -local minimizers and d_∞ local minimizers for the very particular cases of quadratic confinement $W_a(x) = K|x|^2$ or quartic confinement $W_a(x) = K|x|^4$. Both of these cases are particular, because if we restrict ourselves to

$$\mathcal{P}_0(\mathbb{R}^N) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^N) \text{ s.t. } \int_{\mathbb{R}^d} y \, d\mu(y) = 0 \right\}$$

we can re-write the attractive part of the energy as

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^4 \, d\mu(x) d\mu(y) &= 2 \int_{\mathbb{R}^N} |y|^4 \, d\mu(y) + 2 \left(\int_{\mathbb{R}^N} |y|^2 \, d\mu(y) \right)^2 \\ &\quad + 4 \sum_{i,j=1}^N \left(\int_{\mathbb{R}^N} y_i y_j \, d\mu(y) \right)^2. \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^2 \, d\mu(x) d\mu(y) = 2 \int_{\mathbb{R}^N} |y|^2 \, d\mu(y),$$

which are linearly convex. The case of $W_a = |x|^4$ is linearly convex as it can be viewed as a sum of a linear part and the square of a linear functional, which is convex. The observation for the case $W_a = |x|^2$ was first made in [18]. The cases $W_a = |x|^{2n}$ with $n \in \mathbb{N}$ also have a similar decomposition of the energy, but one has to restrict the space even further for the energy to be convex.

After making these observations, we also note that local d_2 minimizers are also linear critical points. Given $\mu \in \mathcal{P}(\mathbb{R}^N)$ with finite energy and $\nu \in \mathcal{P}(\mathbb{R}^N)$, then

$$\left. \frac{dE((1-t)\mu + t\nu)}{dt} \right|_{t=0} = \begin{cases} \int_{\mathbb{R}^N} \psi(x) d\nu(x) - E(\mu) & \text{if } E(\nu) < \infty \\ +\infty & \text{if } E(\nu) = \infty \end{cases}$$

Therefore, if μ is d_2 local minimizer, if we assume that its associated potential function ψ is continuous then by the Euler-Lagrange condition (see Corollary 2.2.11) we know that $\left. \frac{dE((1-t)\mu+t\nu)}{dt} \right|_{t=0} \geq 0$ for any $\nu \in \mathcal{P}(\mathbb{R}^N)$. So, if we also know that E is strictly linearly convex in $\mathcal{P}_0(\mathbb{R}^N)$ and μ is a d_2 local minimizer with mean zero we obtain

$$E(\nu) > E(\mu) + \left. \frac{dE((1-t)\mu+t\nu)}{dt} \right|_{t=0} \geq E(\mu) \quad \text{for any } \nu \in \mathcal{P}_0(\mathbb{R}^N) \text{ and } \nu \neq \mu.$$

Therefore, μ is equal to the unique minimizer of E in $\mathcal{P}_0(\mathbb{R}^N)$.

Let us remark that the results in [49] show the existence of global minimizers for $W(x) = \frac{1}{|x|^{n-2s}} + K|x|^2$ or $W(x) = \frac{1}{|x|^{n-2s}} + K|x|^4$ for $s \in (0, \frac{N}{2})$, see [49, Section 3]. Moreover, all global minimizers must be compactly supported and an attentive reading of Lemmas 2.6 and 2.7 in [49] yields that any d_2 -local minimizer is compactly supported in this particular case, since $W(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Parsing together this observations we have the following result:

Theorem 2.2.23 (Uniqueness of d_2 minimizer). *Assume that $W(x) = \frac{1}{|x|^{N-2s}} + W_a(x)$ for some $s \in (0, \frac{N}{2})$ and either $W_a(x) = K|x|^2$ or $W_a = K|x|^4$, where K is a constant. Then there exists a unique (up to translation) d_2 -local minimizer $\mu_0 \in \mathcal{P}_2(\mathbb{R}^N)$, which is also the unique global minimizer of E in $\mathcal{P}_2(\mathbb{R}^N)$. Furthermore, μ_0 is compactly supported and radial symmetric.*

Proof. Given μ_0 a d_2 local minimizer in $\mathcal{P}_0(\mathbb{R}^N)$, we know from [49, Section 3] that μ_0 has compact support. Therefore, by Proposition 2.2.9 we know that the associated potential function ψ is continuous, then the previous discussion applies, because

$$E[\mu] = \|\mu\|_{H^{-s}(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W_a(x-y) d\mu(x)d\mu(y),$$

which is strictly linearly convex in $\mathcal{P}_0(\mathbb{R}^N)$. Therefore, μ_0 is the unique minimizer of E in $\mathcal{P}_0(\mathbb{R}^N)$. \square

The case of d_∞ local minimizers is a little bit more subtle. To be able to relate d_∞ local minimizers to linear critical points, we would need to prove that the support is connected. Although a result like this sounds really intuitive, it is not yet available in the literature. Here we give the only known result of uniqueness of d_∞ local minimizers, by combining the regularity that we have proven with a result by Caffarelli and Vazquez on the fractional Porous medium equation.

Theorem 2.2.24. *[47, Section 6] Let $W = \frac{1}{|x|^{N-2s}} + K|x|^2$ with $s \in (0, 1)$ and $\mu \in C(\mathbb{R}^N) \cap \mathcal{P}_0(\mathbb{R}^N)$ be a critical point of the energy in the sense that*

$$|\partial_{W^2} E(\mu)|^2 = \int_{\mathbb{R}^N} |\nabla \psi|^2 d\mu = 0.$$

Then, μ is unique and is the solution to the associated obstacle problem in the whole space.

Remark 2.2.25. *In fact, Theorem 2.2.24 still applies in the case $s = 1$.*

Remark 2.2.26. *Unfortunately, the proof of Theorem 2.2.24 relies on the maximum principle, therefore it can not be extended to the cases $s > 1$.*

Using the previous result, the conclusion that d_∞ local minimizers with $W_a = K|x|^2$ are unique follows from the regularity we have proven in the cases $s \in (0, 1]$.

Theorem 2.2.27 (Uniqueness of d_∞ minimizer). *Assume that $W(x) = \frac{1}{|x|^{N-2s}} + K|x|^2$ for some $s \in (0, 1]$, where K is a constant. Then there exists a unique (up*

to translation) d_∞ -local minimizer $\mu_0 \in \mathcal{P}_2(\mathbb{R}^N)$, which is also the unique global minimizer of E in $\mathcal{P}_2(\mathbb{R}^N)$. Furthermore, μ_0 is radially symmetric.

Proof. We only need to combine Remark 2.2.17 and Remark 2.2.21 with the previous Theorem 2.2.24.

We should note that we dropped the assumption of μ_0 being compactly supported used throughout 2.2.4. This is because $W_a * \mu_0$ is a polynomial, which automatically smooth. □

2.3 Proof of the Continuity of the potential function ψ

We start with the proof of Proposition 2.2.6:

Proof of Proposition 2.2.6. We prove the proposition by contradiction. So, we assume there exists $x_0 \in \text{supp}(\mu)$, $A \subset B_\varepsilon(x_0)$ and $\gamma > 0$, such that

$$\psi(x_0) > \psi(y) + \gamma \quad \text{for all } y \in A. \quad (3.1)$$

By the lower-semicontinuity of ψ , we know there pick a small enough $\varepsilon_0 > 0$, such that $\varepsilon < \varepsilon_0$ and

$$\psi(x) > \psi(x_0) - \frac{\gamma}{2} \quad \text{for all } x \in B_{\varepsilon_0}(x_0). \quad (3.2)$$

Combining both inequalities we get

$$\inf_{x \in B_{\varepsilon_0}(x_0)} \psi(x) > \sup_{y \in A} \psi(y) + \frac{\gamma}{2}. \quad (3.3)$$

Using only this information we are going to construct μ_{t_0} , such that $d_\infty(\mu, \mu_{t_0}) < \varepsilon$ and contradicts the optimality of μ . With this in mind, we define probability measures μ_{ε_0} and μ_A , by appropriately rescaling the restriction onto the sets B_{ε_0} and A ,

respectively. For instance, μ_{ε_0} is defined by

$$\mu_{\varepsilon_0}(B) = \frac{\mu(B \cap B_{\varepsilon_0})}{\mu(B_{\varepsilon_0})}.$$

We can think of μ as being the convex combination between μ_{ε_0} and another probability measure μ_1 :

$$\mu = (1 - \mu(B_{\varepsilon_0}))\mu_1 + \mu(B_{\varepsilon_0})\mu_{\varepsilon_0}.$$

Using this decomposition we consider a curve of probabilities parametrized by t , given by

$$\mu_t = (1 - \mu(B_{\varepsilon_0}))\mu_1 + (\mu(B_{\varepsilon_0}) - t)\mu_{\varepsilon_0} + t\mu_A.$$

Because, both A and B_{ε_0} are subsets of B_ε , we know that $d_\infty(\mu, \mu_t) < \varepsilon$ for any t .

Now, we are going to check that there exists t_0 small enough, such that $E(\mu_{t_0}) < E(\mu)$.

A direct computation shows that

$$E(\mu_t) = E(\mu) + 2t(B[\mu_A, \mu] - B[\mu_{\varepsilon_0}, \mu]) + t^2(E(\mu_{\varepsilon_0}) + E(\mu_A) - 2B[\mu_A, \mu]), \quad (3.4)$$

where $B[\cdot, \cdot]$ is a symmetric bi-linear form given by

$$B[\nu_1, \nu_2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) d\nu_1(x) d\nu_2(y).$$

One can check that all the terms in (3.4) are finite, as they can be bound by a multiple of $E[\mu]$.

Finally, we realize that using (3.3) we can conclude

$$\left. \frac{dE(\mu_t)}{dt} \right|_{t=0} = B[\mu_A, \mu] - B[\mu_{\varepsilon_0}, \mu] \leq \sup_{y \in A} \psi(y) - \inf_{x \in B_{\varepsilon_0}(x_0)} \psi(x) < -\frac{\gamma}{2},$$

Therefore, taking t_0 small enough, we get the desired contradiction $E(\mu_{t_0}) < E(\mu)$.

□

Having the continuity of ψ μ -a.e. we are ready to prove the full continuity of ψ in $\text{supp}(\mu)$.

Proof of Lemma 2.2.7. As in (H3), up to constants, we can decompose $W(x) = \frac{1}{|x|^{N-2s}} + W_a(x)$. By (H2), we know that there exists δ_0 such that

$$\left(\frac{\alpha}{2} - 1\right) \frac{1}{|x|^{N-2s}} \leq W_a(x) \leq \frac{1}{2|x|^{N-2s}} \quad \text{for all } x \in B_{\delta_0}.$$

Therefore, setting $\gamma = \max\left(\frac{1}{2}, 1 - \frac{\alpha}{2}\right)$ we get

$$|W_a(x)| \leq \gamma \frac{1}{|x|^{N-2s}} \quad \text{for all } x \in B_{\delta_0}, \quad (3.5)$$

with $0 < \gamma < 1$.

We consider a smooth cutoff function $\eta : \mathbb{R}_+ \rightarrow [0, 1]$ which is decreasing and $\eta(t) = 1$ for $t \in [0, 1/2)$ and $\text{supp}(\eta) \subset [0, 1)$. We define

$$W_r(x) = \frac{1}{|x|^{N-2s}} + \eta\left(\frac{|x|}{\delta_0}\right) W_a(x);$$

by (3.5) we know that $\frac{1-\gamma}{|x|^{N-2s}} \leq W_r(x) \leq \frac{1+\gamma}{|x|^{N-2s}}$.

Using the definition of W_r together with (H3a) or (H3b), we realize that $\psi - W_r * \mu = ((1 - \eta)W) * \mu$ is continuous everywhere. Therefore, the conclusion of the Lemma follows, if and only if, we can prove it for $W_r * \mu$. This follows from [51, Theorem 1.11].

For any given positive measure ν , there exists a finite number A such that

$$\frac{1}{|B_r|} \int_{B_r(x)} W_r * \nu(y) dy \leq \left(\frac{1+\gamma}{1-\gamma} A\right) W_r * \nu(x). \quad (3.6)$$

This follows directly from the fact that we can bound uniformly in r

$$\frac{1}{|B_r|} \int_{B_r(x)} \frac{|x-y|^{N-2s}}{|z-y|^{N-2s}} dz \leq A.$$

By Proposition 2.2.6, we know that $W_r * \mu$ is finite for any $x \in \text{supp}(\mu)$. Therefore, for a fixed $x \in \text{supp}(\mu)$, we have that for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$W_r * \nu \leq \varepsilon$$

where ν is the restriction of μ to $B_\delta(x)$.

Fixing $x \in \text{supp}(\mu)$ and $\varepsilon > 0$, we use the associated ν from above. We observe that $W_r * (\mu - \nu)$ is continuous at x , therefore

$$\begin{aligned} W_r * \mu(x) &= W_r * (\mu - \nu)(x) + W_r * \nu(x) = \lim_{r \rightarrow 0} \frac{1}{|B_r|} \int_{B_r(x)} W_r * (\mu - \nu)(y) dy + W_r * \nu(x) \\ &\leq \liminf_{r \rightarrow 0} \frac{1}{|B_r|} \int_{B_r(x)} W_r * \mu(y) dy + \varepsilon. \end{aligned}$$

To have the other inequality we observe

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{1}{|B_r|} \int_{B_r(x)} W_r * \mu(y) dy &\leq \limsup_{r \rightarrow 0} \frac{1}{|B_r|} \int_{B_r(x)} W_r * (\mu - \nu)(y) dy + A_\gamma W_r * \nu(x) \\ &= W_r * \mu(x) + (A_\gamma + 1)W_r * \nu(x) \leq W_r * \mu(x) + (A_\gamma + 1)\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we have prove the desired convergence for any $x \in \text{supp}(\mu)$. For $x \notin \text{supp}(\mu)$, the Lemma follows from the continuity of $W_r * \mu$ around x . □

Proof of Proposition 2.2.9. The ideas of this proof come from potential theory, see [51, Theorem 1.7]. We follow the decomposition and notation from the Proof of Lemma 2.2.7.

Using the definition of W_r together with (H3a) or (H3b), we realize that $\psi - W_r * \mu = ((1 - \eta)W) * \mu$ is continuous everywhere. Therefore, ψ is continuous, if and only if, W_r is continuous.

To prove the desired continuity for W_r , we start by mimicking the maximum principle from potential theory [51, Theorem 1.5] for W_r :

CLAIM: Given any positive measure ν that satisfies $W_r * \nu \leq M$ ν -a.e., then we obtain $W_r * \nu \leq \frac{1+\gamma}{1-\gamma} 2^{N-2s} M$ in \mathbb{R}^d .

Proof of the CLAIM: We first observe that because $W_r * \nu$ is lower semi-continuous, from the hypothesis $W_r * \nu \leq M$ ν -a.e. we can conclude that $W_r * \nu \leq M$ everywhere in $\text{supp}(\nu)$.

Consider $x \in \mathbb{R}^d \setminus \text{supp}(\nu)$, and x' is the point at $\text{supp}(\nu)$ closest to x . For every $y \in \text{supp}(\nu)$ we have

$$|y - x'| \leq |y - x| + |x - x'| \leq 2|y - x|.$$

We note that by (3.5)

$$W_r(y - x') \geq \frac{1 - \gamma}{|y - x'|^{N-2s}} \geq \frac{1 - \gamma}{(2|y - x|)^{N-2s}} \geq \frac{1 - \gamma}{1 + \gamma} 2^{2s-N} W_r(x - y);$$

here we are using the inequalities

$$(1 - \gamma) \frac{1}{|z|^{N-2s}} \leq W_r(z) = \frac{1}{|z|^{N-2s}} + W_a(z) \leq (1 + \gamma) \frac{1}{|z|^{N-2s}}.$$

Therefore, by the first observation we have

$$M \geq W_r * \nu(x') \geq \frac{1 - \gamma}{1 + \gamma} 2^{2s-N} W_r * \nu(x).$$

This finishes the proof of the **CLAIM**.

Now we turn back to prove the continuity of $W_r * \mu$. As W_r is uniformly continuous in any set bounded away from zero, we know that for any $z \notin \text{supp}(\mu)$, $W_r * \mu$ is continuous at z .

Therefore, we are only missing to prove that if $z \in \text{supp}(\mu)$, then $W_r * \mu$ is continuous at z . We want to prove that for every $\omega > 0$, there exists λ , such that if $|x - z| < \lambda$, then $|W_r * \mu(x) - W_r * \mu(z)| \leq C\omega$, where C is fixed constant.

Given $\theta > 0$, to be chosen, we decompose our measure into

$$\mu = \mu|_{B_\theta} + \mu|_{B_\theta^c} = \mu_\theta + \mu_\theta^c.$$

Using this decomposition, we have the bound

$$|W_r * \mu(x) - W_r * \mu(z)| \leq W_r * \mu_\theta(x) + W_r * \mu_\theta(z) + |W_r * \mu_\theta^c(x) - W_r * \mu_\theta^c(z)|.$$

The idea is to pick θ in such a way that the first two terms are small, independently of x and z . Because $W_r * \mu$ is finite at z , there exists θ_1 , such that

$$W_r * \mu_{\theta_1}(z) \leq \omega,$$

Using that $W_r * \mu$ is continuous in $\text{supp}(\mu)$, there exists θ_0 , such that

$$W_r * \mu_{\theta_1}(y) \leq 2\omega \quad \text{in } \text{supp}(\mu) \cap B_{\theta_0}.$$

We fix $\theta_2 = \min(\theta_1, \theta_0)$.

Because W_r is positive, we know that $W_r * \mu_\theta$ is decreasing in θ , therefore

$$W_r * \mu_{\theta_2}(y) \leq 2\omega \quad \text{in } \text{supp}(\mu) \cap B_{\theta_2}.$$

By the **CLAIM** we can assure that

$$W_r * \mu_{\theta_2}(x) \leq 2C_{\gamma,s}\omega \quad \forall x \in \mathbb{R}^d.$$

Coming back to proving the continuity, we have the bound

$$|W_r * \mu(x) - W_r * \mu(z)| \leq W_r * \mu_{\theta_2}(x) + W_r * \mu_{\theta_2}(z) + |W_r * \mu_{\theta_2}^c(x) - W_r * \mu_{\theta_2}^c(z)|$$

$$\leq 2C_{\gamma,s}\omega + |W_r * \mu_{\theta_2}^c(x) - W_r * \mu_{\theta_2}^c(z)|.$$

We know that $W_r * \mu_{\theta_2}^c$ is continuous around z , because $z \notin \text{supp}(\mu_{\theta_2}^c)$, therefore there exists a $\lambda > 0$, such that $|x - z| < \lambda$ implies

$$|W_r * \mu_{\theta_2}^c(x) - W_r * \mu_{\theta_2}^c(z)| \leq \omega$$

Therefore, if $|x - z| < \lambda$, then

$$|W_r * \mu(x) - W_r * \mu(z)| \leq C\omega,$$

which proves the desired continuity. \square

Now we can prove Corollary 2.2.10.

Proof of Corollary 2.2.10. Because we are assuming that $W > 0$ and by definition $W_a = W - V_{N,s}$, then $W_a \geq -\frac{C}{\delta^{N-2s}}$ on $B_{\frac{\delta}{2}}^c$. Therefore, using the cutoff η and W_r from the proof of Proposition 2.2.9, we know that

$$\begin{aligned} c\|\mu\|_{H^{-s}} &= c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2s}} d\mu(x)d\mu(y) \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W_r(x-y) d\mu(x)d\mu(y) \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) - (1 - \eta(x-y))W_a(x-y) d\mu(x)d\mu(y) < \infty. \end{aligned}$$

If (H3b) holds, then $(-\Delta)^s W_a * \mu \in H^{-s}(\mathbb{R}^N)$, as $(-\Delta)^s W_a \in L^1(\mathbb{R}^N)$.

If (H3a), holds we are in the case that μ has compact support, we can always cut off the potential W_a to be compactly supported, then by (H2) we can claim that $|W_a| \leq \frac{C}{|x|^{N-2s}}$. From the arguments in the proof of Proposition 2.2.9 we know that $\frac{1}{|x|^{N-2s}} * \mu$ is bounded, therefore $W_a * \mu$ is also bounded, which implies $\psi = W * \mu$ is bounded. Now, we observe that

$$(-\Delta)^s \psi = \mu + (-\Delta)^s W_a * \mu,$$

using this identity we can re-write

$$W_a * \mu = W_a * (-\Delta)^s \psi - (-\Delta)^s W_a * [W_a * \mu] = (-\Delta)^s W_a * [\psi - W_a * \mu].$$

Therefore, $W_a * \mu$ is continuous because it is convolution of a function in L^1 and a bounded function. \square

2.4 Proofs of the regularity of μ

In this section, we prove the regularity results for ψ and μ stated in Section 2.2.4. For that, we are going back and forth between the regularity of the solution of the obstacle problem (2.6) and the regularity of $F = \Delta W_a * \mu$.

2.4.1 Proof of Proposition 2.2.15 and Theorem 2.2.13

To prove Proposition 2.2.15, we need a fine version of the mean value formula for the fractional Laplacian.

Lemma 2.4.1. *Given $l \in (0, 1]$, there exists a bounded function γ^l that is positive with integral one, which is continuous if $l < 1$, and a bounded function g^l that is positive, compactly supported and also with integral one. Such that given any $f \in L^1_{loc}(\mathbb{R}^N)$, that satisfies*

$$\int_{\mathbb{R}^N} \frac{f(x)}{1 + |x|^{N+2l}} dx < \infty,$$

we can write

$$(-\Delta)^l f = \lim_{\lambda \rightarrow 0} \frac{f - \gamma_\lambda^l * f}{\lambda^{2l}} = \lim_{\lambda \rightarrow 0} g_\lambda^l * (-\Delta)^l f, \quad (4.1)$$

where $\gamma_\lambda^l = \frac{1}{\lambda^N} \gamma^l\left(\frac{x}{\lambda}\right)$ and $g_\lambda^l = \frac{1}{\lambda^N} g^l\left(\frac{x}{\lambda}\right)$.

Proof. The construction of γ^l and g^l can be found in [45, Section 2]. For $l = 1$, γ_1 is just the normalized indicator of the ball of radius 1. \square

Proof of Proposition 2.2.15. By (2.6), we know that φ has a minimum in the B_ε for any $x_0 \in \text{supp}(\mu)$, where $\mu = (-\Delta)^s \varphi - F$. Therefore, we can construct a smooth bounded η_ε , which depends on ε and $\|\varphi\|_\infty$, such that

$$\varphi(x) + \eta_\varepsilon(x) \quad \text{has a global minimum at any } x_0 \in \text{supp}(\mu) \cap B_{\frac{\varepsilon}{2}}. \quad (4.2)$$

Taking γ^l from Lemma 2.4.1, we can combine the fact that γ^l is positive with integral one with (4.2) to derive

$$\frac{\varphi(x_0) + \eta_\varepsilon(x_0) - (\gamma_\lambda^l * (\varphi + \eta_\varepsilon))(x_0)}{\lambda^{2l}} \leq 0 \quad \text{for any } x_0 \in \text{supp}(\mu) \cap B_{\frac{\varepsilon}{2}}.$$

We define μ_ε to be the restriction of μ to the ball of radius $\frac{\varepsilon}{2}$. By the previous inequality, we know

$$\int_{\mathbb{R}^N} \frac{\varphi(x) + \eta_\varepsilon(x) - (\gamma_\lambda^l * (\varphi + \eta_\varepsilon))(x)}{\lambda^{2l}} d\mu_\varepsilon(x) \leq 0.$$

Using Lemma 2.4.1 and that η_ε is smooth, we can re-write this as

$$\int_{\mathbb{R}^N} g_\lambda^l * (-\Delta)^l \varphi(x) d\mu_\varepsilon(x) \leq C_\varepsilon.$$

Using that $\mu = (-\Delta)^s \varphi - F$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} (g_\lambda^l * (\mu - F))(x) (-\Delta)^{-s+l} \mu_\varepsilon(x) dx \leq C_\varepsilon$$

Now, using the hypothesis $F \in H^{-s+l}(\mathbb{R}^N)$ and Young's inequality we obtain

$$\int_{\mathbb{R}^N} (g_\lambda^l * \mu)(x) (-\Delta)^{-s+l} \mu_\varepsilon(x) - \frac{1}{2} \|g_\lambda^l * F\|_{H^{-s+l}(\mathbb{R}^N)}^2 - \frac{1}{2} \|(-\Delta)^{-s+l} \mu_\varepsilon\|_{H^{s-l}(\mathbb{R}^N)}^2 dx \leq C_\varepsilon$$

By taking $\lambda \rightarrow 0$, using that $g_\lambda^l \in L^1$ and using Fatou's Lemma we obtain

$$\begin{aligned}
\frac{1}{2} \|\mu_\varepsilon\|_{H^{-s+l}(\mathbb{R}^N)} &= \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^{-s+l} \mu_\varepsilon(x) d\mu_\varepsilon(x) \\
&\leq \int_{\mathbb{R}^N} (-\Delta)^{-s+l} \mu(x) d\mu_\varepsilon(x) - \frac{1}{2} \|\mu_\varepsilon\|_{H^{-s+l}(\mathbb{R}^N)} \quad (4.3) \\
&\leq \frac{1}{2} \|F\|_{H^{-s+l}(\mathbb{R}^N)}^2 + C_\varepsilon.
\end{aligned}$$

Finally, using that $(-\Delta)^s \varphi = \mu + F \in H^{-s+l}(B_{\frac{\varepsilon}{2}})$, we obtain that $\varphi \in H^{s+l}(B_{\frac{\varepsilon}{2}})$. \square

By using (H3a), we can always truncate W_a , so that we can assume that $\psi(x) \rightarrow 0$ as $x \rightarrow \infty$. Then, if we assume (H4) we can apply the previous Proposition 2.2.15 iteratively to obtain regularity for μ .

Proof of Theorem 2.2.13. By (H4), $F = (-\Delta)^s W_a * \mu \in H^{-s+\delta}$ which means we can apply to Proposition 2.2.15. Therefore, for any $x_0 \in \text{supp}(\mu)$, μ_ε , the restriction of μ to the ball of radius $\frac{\varepsilon}{2}$ around x_0 , has its $H^{-s+\delta}(\mathbb{R}^N)$ norm bounded by a universal constant depending on $\|\psi\|_\infty$ and ε . In fact, by (4.3) we know that

$$\int_{\mathbb{R}^N} \frac{\mu(x) d\mu_\varepsilon(x)}{|x-y|^{n-2(s-l)}} < C(\varepsilon, \|\psi\|_\infty). \quad (4.4)$$

As we are assuming that the support of μ is compact, we can cover its support by finitely many balls of radius $\frac{\varepsilon}{2}$. Therefore, by summing (4.4) a finite number of times, we obtain that $\mu \in H^{-s+\delta}(\mathbb{R}^N)$, which implies $F = (-\Delta)^s W_a * \mu \in H^{-s+2\delta}(\mathbb{R}^N)$. Applying Proposition 2.2.15 and (4.3) again, we obtain $\mu \in H^{-s+2\delta}(\mathbb{R}^N)$. This procedure can be bootstrapped all the way up to $\mu \in H^{-s+1}(\mathbb{R}^N)$. \square

2.4.2 Proof of Theorem 2.2.16

Since we are assuming that $\text{supp}(\mu) \subset B_R(0)$, and we are only interested in the properties of ψ and μ in a neighborhood of $\text{supp}(\mu)$, it is possible to modify the values of W_a outside a ball $B_{2R}(0)$ without changing the values of μ and ψ in B_R as discussed in Section 3. We can thus assume that ΔW_a has compact support and that

$$(-\Delta)^{1+\delta}W_a \in L^1(\mathbb{R}^N) \text{ for some } \delta > 0.$$

We will then use the following lemma (with $K = \Delta W_a$):

Lemma 2.4.2. *Given $K \in L^1(\mathbb{R}^N)$ with $(-\Delta)^\delta K \in L^1(\mathbb{R}^N)$, we have the following*

- (a) *If $\varphi \in L^\infty(\mathbb{R}^N)$, then $K * \varphi \in C^\beta(\mathbb{R}^N)$ for any $\beta < 2\delta$.*
- (b) *If $\varphi \in L^\infty(\mathbb{R}^N) \cap C^\alpha(\mathbb{R}^N)$, then $K * \varphi \in C^{2\delta+\alpha}(\mathbb{R}^N)$.*

Proof of Lemma 2.4.2. We note that $K * \varphi \in L^\infty$ and

$$(-\Delta)^\delta(K * \varphi) = [(-\Delta)^\delta K] * \varphi \in L^\infty(\mathbb{R}^N),$$

for case (a). In case (b), we also have that $(-\Delta)^\delta(K * \varphi) \in C^\alpha$. By standard regularity results for fractional elliptic equation (see [45, Proposition 2.8 & Proposition 2.9]) we know that $K * \varphi \in C^\beta(\mathbb{R}^N)$ for any $\beta < 2\delta$ in case (a) and $K * \varphi \in C^{2\delta+\alpha}(\mathbb{R}^N)$ in case (b). □

We then rely on the following important result for the regularity of the solution of the obstacle problem in [43]:

Proposition 2.4.3. *Let ψ be the solution of the obstacle problem (2.6). Then up to $C^{1,1}(\mathbb{R}^N)$ the function ψ is as regular as $W_a * \mu$. More precisely, we have*

- *If $W_a * \mu$ has a modulus of continuity $\sigma(r)$, then ψ has a modulus of continuity $C\sigma(2r)$.*
- *If $\nabla W_a * \mu$ has a modulus of continuity of $\sigma(r)$, then $\nabla\psi$ has a modulus of continuity $C\sigma(2r)$.*

Using a bootstrap argument and Lemma 2.4.2, we can now prove Theorem 2.2.16:

Proof of Theorem 2.2.16. Up to chopping W_a , we know can use the continuity of ψ and Corollary 2.2.10 to claim that both ψ and $W_a * \mu$ are continuous and bounded.

Since $\mu = -\Delta\psi + \Delta W_a * \mu$, we can write

$$W_a * \mu = -W_a * \Delta\psi + W_a * (\Delta W_a * \mu) = -\Delta W_a * (\psi + W_a * \mu).$$

By (H4) $(-\Delta)^\delta [(-\Delta)W_a] \in L^1(\mathbb{R}^N)$, we can use Lemma 2.4.2 to show that

$$W_a * \mu \in \mathcal{C}^\beta(\mathbb{R}^N), \quad \text{for all } \beta < 2\delta.$$

Then, by Proposition 2.4.3, we that $\psi \in C^\beta$ for all $\beta < 2\delta$.

Then using the same arguments as above and applying Lemma 2.4.2 again we obtain that both $W_a * \mu$ and ψ belong to $C^{\beta + 2\delta}$ for all $\beta < 2\delta$.

A simple bootstrap argument yields that $W_a * \mu$ and ψ are both in $\mathcal{C}^{1,1}(\mathbb{R}^N)$, and thus that μ has density $\rho \in L^\infty(\mathbb{R}^N)$. □

2.4.3 Proof of Theorem 2.2.18

We now prove Theorem 2.2.18. We note that under the assumption of Theorem 2.2.18, we have $F \in W_{loc}^{1,1}(\mathbb{R}^N)$ and, for the second part of the statement, we get $F(x) \neq 0$ in a bounded open neighborhood of $\text{supp}(\mu)$.

Under these assumptions, the regularity in $\text{BV}_{loc}(\mathbb{R}^N)$ of $\Delta\psi$, where ψ solves the obstacle problem (2.6) is a classical result, which implies Theorem 2.2.18. We will sketch the proof of this result for the reader's sake. The proof that we give below was first proposed by Brezis and Kinderlehrer in [56].

Proof of Theorem 2.2.18. First, we recall that the solution of the obstacle problem (2.6) can be approximated by the solutions ψ_δ of the nonlinear equation

$$\begin{aligned} -\Delta\psi_\delta + \beta_\delta(\psi_\delta - C_0) &= -F \quad \text{in } \Omega \\ \psi_\delta &= \psi \quad \text{on } \partial\Omega \end{aligned} \tag{4.5}$$

where $\Omega = B_\varepsilon(x_0)$ with $x_0 \in \text{supp}(\mu)$ and β_δ is an increasing function satisfying $s\beta_\delta(s) \geq 0$ for all s and such that

$$\beta_\delta(s) \longrightarrow \begin{cases} 0 & \text{when } s > 0 \\ -\infty & \text{when } s < 0 \end{cases} \quad \text{as } \delta \rightarrow 0.$$

Here, $\Omega = B_\varepsilon(x_0)$ for any point $x_0 \in \text{supp}(\mu)$. It is a classical result, see [56] for instance, that ψ_δ converges to ψ locally uniformly in $C^{1,\gamma}(\Omega)$ provided F is in $L^\infty(\Omega)$ (which we proved in Theorem 2.2.16).

Let now $\frac{\partial}{\partial \xi}$ denote any directional derivative, we are going to show that for any compact set $K \subset \subset \Omega$, we have

$$\int_K \left| \frac{\partial}{\partial x_i} \Delta\psi_\delta \right| \leq C. \tag{4.6}$$

where C does not depend on δ . Taking the limit $\delta \rightarrow 0$ and using the l.s.c. of the total variation, we deduce that $\Delta\psi \in BV_{loc}(\Omega)$, which gives the result.

In order to prove (4.6), we differentiate (4.5):

$$-\Delta\partial_\xi\psi_\delta + \beta'_\delta(\psi_\delta - C_0)\partial_\xi\psi_\delta = -\partial_\xi F \quad (4.7)$$

Let now χ be a test function in $\mathcal{D}(\Omega)$ such that $\chi \geq 0$ in Ω and $\chi = 1$ in K . We multiply (4.7) by $\chi\text{sign}(\partial_\xi\psi_\delta)$ and integrate over Ω to deduce

$$-\int_\Omega \chi\text{sign}(\partial_\xi\psi_\delta)\Delta\partial_\xi\psi_\delta dx + \int_\Omega \beta'_\delta(\psi_\delta - C_0)|\partial_\xi\psi_\delta|\chi dx = -\int_\Omega \partial_\xi F\chi\text{sign}(\partial_\xi\psi_\delta) dx.$$

Integrating by parts the left hand side yields

$$\begin{aligned} & \int_\Omega \chi\text{sign}'(\partial_\xi\psi_\delta)|\nabla\frac{\partial}{\partial\xi}\psi_\delta|^2 dx + \int_\Omega \beta'_\delta(\psi_\delta - C_0)|\partial_\xi\psi_\delta|\chi dx \\ &= -\int_\Omega \nabla\chi\nabla\partial_\xi\psi_\delta\text{sign}(\partial_\xi\psi_\delta) dx - \int_\Omega \partial_\xi F\chi\text{sign}(\partial_\xi\psi_\delta) dx. \end{aligned}$$

Using the fact that $\text{sign}'(s) \geq 0$ for all s , we deduce

$$\begin{aligned} \int_\Omega \beta'_\delta(\psi_\delta - C_0)|\partial_\xi\psi_\delta|\chi dx &\leq -\int_\Omega \nabla\chi\nabla|\partial_\xi\psi_\delta| dx - \int_\Omega \partial_\xi F\chi\text{sign}(\partial_\xi\psi_\delta) dx \\ &\leq \int_\Omega \Delta\chi|\partial_\xi\psi_\delta| dx + \int_\Omega |\partial_\xi F|\chi dx. \end{aligned}$$

Furthermore, multiplying (4.5) by $(\psi_\delta - C_0)\chi$, it is easy to show that

$$\int_K |\partial_\xi\psi_\delta|^2 dx \leq C(K)$$

for some constant depending on K but not on δ (using the regularity of F and the fact that ψ_δ converges locally uniformly to ψ). We conclude that

$$\int_K \beta'_\delta(\psi_\delta - C_0)|\partial_\xi\psi_\delta| dx \leq C$$

with C independent of δ . Finally, going back to (4.7), we get

$$\int_K |\Delta \partial_\xi \psi_\delta| dx \leq \int_K \beta'_\delta(\psi_\delta - C_0) |\partial_\xi \psi_\delta| dx + \int_K |\partial_\xi F| dx \leq C$$

and the result follows.

To prove the second part of the Theorem, we note that the function

$$\frac{-\Delta \psi + F}{F} = \frac{\rho}{F}$$

is almost everywhere equal to the indicator function of the set $\{\psi = \psi(x_0)\} \cap B_\varepsilon(x_0)$.

If F is never zero, we deduce that this function is in BV_{loc} , thus proving that

$\{\psi = \psi(x_0)\} \cap B_\varepsilon(x_0)$ and $\text{supp}(\mu) \cap B_\varepsilon(x_0)$ have finite perimeter. Here, we use

(2.9) and, more generally, the fact that if E is a subset of G and $|G \setminus E| = 0$, then

E and G have the same perimeter. □

2.4.4 Proof of Theorem 2.2.19

In order to apply known regularity results for the fractional obstacle problem (as found, for instance, in [45]), we need to show that ψ solves a fractional obstacle problem in the whole of \mathbb{R}^N .

It is possible to do this as follows: The set $\text{supp}(\mu) + B_{\varepsilon/4} = \{x + y; x \in \text{supp}(\mu), y \in B_{\varepsilon/4}(0)\}$ is an open set in $B_{R+1}(0)$. In particular, it is the countable union of its connected components A_i . Furthermore, since $\text{supp}(\mu)$ is compact, there are only finitely many A_i .

For all i , any two points x_1, x_2 in $\text{supp}(\mu) \cap A_i$ will satisfy $\psi(x_1) = \psi(x_2)$, by the minimality of the connected component and Corollary 2.2.11. We define

$D_i = \text{supp}(\mu) \cap A_i$, we denote $C_i = \psi|_{D_i}$ and we consider a smooth function f such that

$$f \leq C_i \text{ in } A_i$$

$$f = C_i \text{ on } D_i + B_{\frac{\varepsilon}{16}}$$

$$f = \inf W \text{ outside } \cup_i D_i + B_{\varepsilon/8}.$$

We can find such smooth function, because D_i are at least separated $\varepsilon/4$ from each other, and if they are closer than ε then the constant C_i has to match because of Corollary 2.2.11. The potential function ψ then solves the following obstacle problem in \mathbb{R}^N :

$$\begin{cases} \psi \geq f, & (-\Delta)^s \psi \geq -F(x) & \text{in } \mathbb{R}^N \\ & -(\Delta)^s \psi = -F(x), & \text{in } \{\varphi > f\} \end{cases} \quad (4.8)$$

where $F = -(-\Delta)^s W_a * \mu$.

Using this obstacle problem formulation, we can use the following proposition which is the fractional analog of Proposition 2.4.3 (See L. Silvestre [45]):

Proposition 2.4.4. *Let ψ be the solution of the obstacle problem (4.8). If $f \in C^2(\mathbb{R}^N)$ and $W_a * \mu$ is in $C^\beta(\mathbb{R}^N)$ with $\beta > 0$. Then $\psi \in C^\alpha(\mathbb{R}^N)$ for every $\alpha < \min(\beta, 1 + s)$ (with the notation $C^\alpha = C^{1, \alpha-1}$ if $\alpha > 1$).*

Proof of Theorem 2.2.19. We can now prove our main result by proceeding as in the proof of Theorem 2.2.16:

Up to chopping W_a , we know can use the continuity of ψ and Corollary 2.2.10 to claim that both ψ and $W_a * \mu$ are continuous and bounded.

Since $\mu = (-\Delta)^s \psi - (-\Delta)^s W_a * \mu$, we can write

$$W_a * \mu = -W_a * (-\Delta)^s \psi - W_a * ((-\Delta)^s W_a * \mu) = (-\Delta)^s W_a * (\psi - W_a * \mu).$$

By (H4) $(-\Delta)\delta[(-\Delta)W_a] \in L^1(\mathbb{R}^N)$, we can use Lemma 2.4.2 to show that

$$W_a * \mu \in \mathcal{C}^\beta(\mathbb{R}^N), \quad \text{for all } \beta < 2\delta.$$

Then, by Proposition 2.4.3, we that $\psi \in C^\beta$ for all $\beta < 2\delta$.

Then using the same arguments as above and applying Lemma 2.4.2 again we obtain that both $W_a * \mu$ and ψ belong to $C^\beta + 2\delta$ for all $\beta < 2\delta$.

A simple bootstrap argument yields that $W_a * \mu$ and ψ are both in $\mathcal{C}^{1,\alpha}(\mathbb{R}^N)$, for any $\gamma < s$ and thus that μ has density $\rho \in C^\alpha(\mathbb{R}^N)$ for all $\alpha < 1 - s$. \square

Chapter 3: Cahn-Hilliard Equation

3.1 Overview

Given a smooth non-convex potential $F : \mathbb{R}_+ \rightarrow \mathbb{R}$, we are interested in the properties of solutions ν^ε to

$$\begin{cases} \partial_t \nu &= (\nu(F'(\nu) - \varepsilon^2 \nu_{xx})_x)_x & \text{in } (0, \infty) \times \mathbb{T} \\ \nu(0) &= \nu_i^\varepsilon & \text{on } \{0\} \times \mathbb{T}. \end{cases} \quad (1.1)$$

and more specifically, in their behavior as $\varepsilon \rightarrow 0^+$, where \mathbb{T} denotes the one-dimension flat torus \mathbb{R}/\mathbb{Z} .

Equation (1.1) is known in the literature as the Cahn-Hilliard equation ([57], [58], [59]). The function ν^ε models the concentration of one of two phases in a system undergoing phase separation. Mathematically, this equation could be considered as a fourth order regularization of a forward-backward parabolic equation, by the fourth order term $-\varepsilon^2(\nu \nu_{xxx})_x$. In the case where F vanishes identically, we are left with a fourth order parabolic equation

$$\partial_t \nu = (m(\nu) \nu_{xxx})_x$$

known as the Thin-film equation, with mobility $m(\nu) = \nu$, which is interesting on its own (see for instance [60], [61], [62]). Note that, the Dirichlet Energy is a Lyapunov

functional and that when $m(\nu) = \nu$, the equation is formally the gradient flow of this Energy under the $\mathcal{W}^2(\mathbb{T})$ metric. This observation was made in the seminal paper by Otto [63] and has been exploited for some generalizations in [64], [65] and [59].

The main result of this chapter is the fact that, under some assumptions (see Theorem 3.3.1), ν^ε converges, as $\varepsilon \rightarrow 0$, to the unique solution ν_0 of the following well-posed degenerate parabolic equation

$$\begin{cases} \partial_t \nu &= (\nu(F^{**'}(\nu)))_x \\ \nu(0) &= \nu_i, \end{cases} \quad (1.2)$$

where F^{**} denotes the convex envelope of F .

The mathematical intuition behind this convergence comes from the fact that, formally at least, we know that ν^ε is the gradient flow of

$$\mathcal{F}^\varepsilon[\mu] = \int_{\mathbb{T}} \frac{\varepsilon^2}{2} |\mu_x|^2 + F(\mu) \, dx, \quad (1.3)$$

while ν_0 is the gradient flow of

$$\mathcal{F}^{**}[\mu] = \int_{\mathbb{T}} F^{**}(\mu) \, dx,$$

with respect to $\mathcal{W}^2(\mathbb{T})$, and it is somewhat classical that the energy \mathcal{F}^ε Γ -converges to \mathcal{F}^{**} in $\mathcal{W}^2(\mathbb{T})$. Unfortunately, it is well known that the Γ -convergence of the energy is not enough to prove the convergence of the gradient flows.

Indeed, to be able to prove the convergence of the gradient flows we need an additional condition on the gradient of the energy. A sufficient condition for Hilbert spaces was given in the paper by Sandier and Serfaty [66], which was later extended to metric spaces by Serfaty in [2]. This additional condition is usually written as

follows:

$$\Gamma - \liminf_{\varepsilon \rightarrow 0^+} |\nabla \mathcal{F}^\varepsilon| \geq |\nabla \mathcal{F}^{**}| \quad (\text{see Section 3.2 for definitions}), \quad (1.4)$$

and proving this inequality is always the hard part of the Sandier-Serfaty approach. However, in our case $|\nabla \mathcal{F}^\varepsilon|$ is not well understood, so we need to introduce a different quantity for which we prove a condition similar to (1.4) (see Theorem 3.3.2).

The framework of Sandier-Serfaty has been applied to an array of diverse problems. To name a few we have: Allen-Cahn [67], Cahn-Hilliard [68], [69], non-local interactions energies [70], TV flow [71] and Fokker Plank [72]. The most relevant reference for this paper was written by Belletini, Bertini, Mariani and Novaga [69], where they consider the convergence of the one dimensional Cahn-Hilliard equation on the Torus with mobility equal to one:

$$\partial_t \nu = (F'(\nu) - \epsilon^2 \nu_{xx})_{xx}. \quad (1.5)$$

We actually borrow some of the notations and the ideas on how to track the oscillations of the solution. The main difference between [69] and our work is that (1.5) is a gradient flow of (1.3) in the Hilbert space $H^{-1}(\mathbb{T})$, instead of the metric space $\mathcal{W}^2(\mathbb{T})$. Besides bringing some non-trivial technical issues, working with a degenerate mobility coefficient also means that the estimates degenerate when the solution is near zero; this actually turns out to be a major issue that keeps showing up in the Thin Film equation literature as well.

A word of caution is that the framework developed by Ambrosio, Gigli and Savare in [1] can not be applied to the functional \mathcal{F}^ε , as it is neither λ -convex in the sense given at [1] (or the relaxed notion [73]), nor regular (see Remark 1.2.20). In

fact, the subdifferential of \mathcal{F}^ε is not really well understood; no matter how regular the measure is, if it vanishes at some point, it has not been proven that the natural candidate is indeed a subdifferential.

We deal with this setback by considering Otto's approach in [63], which constructs solutions to the equation as the limit of Minimizing Movements, an idea that was originated by De Giorgi. In the case of \mathcal{F}^ε , this has been made rigorous in [59], where the authors are even able to prove a uniform $L_t^2(H_x^2)$ estimate for the constant interpolant of the discrete approximations, by using a discrete version of the entropy dissipation inequality (for the continuum case see [62]). In this paper we go a bit further and we obtain an Energy inequality (see (2.25)), by defining a non-standard functional \mathcal{G}^ε (see Section 3.2), which we prove to be lower semicontinuous in H^2 (see Lemma 3.8.1) and which agrees with the size of the subdifferential, when we know \mathcal{F}^ε to be strongly subdifferentiable and μ to be regular enough. To our knowledge, this is a completely novel result in the literature and gives a starting point to understand the $\mathcal{W}^2(\mathbb{T})$ gradient flows of energies involving derivatives. Shedding some light onto this topic will be part of the author's upcoming work.

Once we are able to prove the existence of an appropriate solution to our equation, the main obstacle we encounter, when we try to prove the convergence, is oscillatory behavior, known as the wrinkling phenomenon. Numerical simulations show that the functions ν^ε tend to oscillate quickly in the whole of the unstable set

$$\Sigma = cl(\{F > F^{**}\}).$$

However, in this paper we only prove that the wrinkling phenomenon occurs in a

subset of Σ and we do not explore further if it can be proven analytically that when oscillations occur, they actually encompass the whole of Σ .

We prove that oscillations only occur inside of Σ by proving that $d(\nu^\varepsilon, \Sigma)$ is uniformly lower-semicontinuous in ε (see Corollary 3.4.4), which allows us to derive a uniform H^1_{loc} estimate away from the unstable set (see Proposition 3.5.2). The degenerate diffusion at $\{\nu^\varepsilon = 0\}$ makes the control near zero very subtle. Only a careful study of the behavior of the solution near zero can rule out uncontrolled jumps (see proof of Theorem 3.4.3).

It is the intention of this chapter that the proofs make a clear connection between where the oscillations can occur and the tangent lines of the graph of F . In short, in the regions where the tangent lines do not cross the graph of F , the function cannot have large oscillations (see (4.39)). In this way, the function F^{**} appears naturally and does not seem to be only a mathematical artifact of Γ -convergence.

As usual with the framework of [66], [2], we have to make an assumption on the initial data being well prepared with respect to the energy, meaning that

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{F}^\varepsilon[\nu_i^\varepsilon] = \mathcal{F}^{**}[\nu_i].$$

In our case, the well preparedness can be interpreted as the fact that the approximations ν_i^ε stays away from Σ , so the convergence we prove only tells us that asymptotically the dynamic keeps it that way. With this assumption, we are missing how the wrinkling phenomenon is actually affecting the dynamic in the limit, which is a really interesting question on its own, but needs to be analyzed more carefully with other types of techniques.

The chapter is organized as follows: The rest of this Section deals with motivation. Section 3.2 provides the definitions and hypothesis of the objects we work with, and introduces a suitable notion of solution to (1.1). Section 3.3 contains the statements of the main result of the convergence (see Theorem 3.3.1) and the main auxiliary result of the lower semicontinuity of the size of the gradients (see Theorem 3.3.2). Section 3.4 presents and proves the result on where can oscillations occur (see Theorem 3.4.3). Section 3.5 proves that away from Σ the functions are in H^1 (see Proposition 3.5.2). Section 3.6 proves Theorem 3.3.2. Section 3.7 proves Theorem 3.3.1. Appendix 1.2.2 gives the necessary background of gradient flows in $\mathcal{W}^2(\mathbb{T})$. Section 3.8 finishes the proof of the existence of an appropriate solution to (1.1) and proves the lower semicontinuity of \mathcal{G}^ε (see Lemma 3.8.1).

3.1.1 Motivation

Our original motivation for studying (1.1) came from a model for biological aggregation introduced in [74] which we describe now:

We consider $\nu(x, t)$ a population density that moves with velocity $v(x, t)$, where $x, v \in \mathbb{R}^n$, $t \geq 0$. Then, ν satisfies the standard conservation equation, with initial population ν_0

$$\begin{cases} \partial_t \nu + \nabla \cdot (v\nu) = 0 \\ \nu(x, 0) = \nu_0. \end{cases} \quad (1.6)$$

The model assumes that the velocity depends only on properties of ν at the current time and can be written as the sum of an aggregation and a dispersal term:

$$v = v_a + v_d. \quad (1.7)$$

For aggregation, a sensing mechanism that degrades over distance, is hypothesized on the organisms. In the simplest case, the sensing function associated with an individual at position x is given by

$$s(x) = \int_{\mathbb{R}^n} K(x-y)\nu(y) dy = K * \nu(x),$$

where the kernel K is typically radially symmetric, compactly supported and of unit mass. Individuals aggregate by climbing gradients of the sensing function, so that the attractive velocity is given by:

$$v_a = \nabla K * \nu(x). \tag{1.8}$$

Dispersal is assumed to arise as an anti-crowding mechanism and operates over a much shorter length scale. It is considered to be local, go in the opposite direction of population gradients and increase with density. For example we can take the dispersive velocity given by:

$$v_d = -\nu \nabla \nu \tag{1.9}$$

(more generally $v_d = -f(\nu)\nabla \nu$).

Combining (1.6), (1.7), (1.8) and (1.9), we obtain the equation

$$\begin{cases} \partial_t \nu + \nabla \cdot (\nu(\nabla K * \nu - \nu \nabla \nu)) = 0 \\ \nu(x, 0) = \nu_0. \end{cases} \tag{1.10}$$

Now, by re-scaling, we want to consider what happens to a large population as we zoom out, over a large period of time. We thus set

$$\int_{\mathbb{R}^n} \nu_0 dx = \varepsilon^{-n},$$

for some $\varepsilon \ll 1$ and we re-scale time and space as follows:

$$\nu^\varepsilon(x, t) = \nu\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \quad (1.11)$$

the scaling in x is chosen such that $\int \nu_0^\varepsilon = 1$. Using (1.10), we obtain the following equation for ν^ε :

$$\begin{cases} \partial_t \nu^\varepsilon + \nabla \cdot (\nu^\varepsilon (\nabla K^\varepsilon * \nu^\varepsilon - \nu^\varepsilon \nabla \nu^\varepsilon)) = 0 \\ \nu^\varepsilon(x, 0) = \nu_0^\varepsilon, \end{cases} \quad (1.12)$$

where $K^\varepsilon = \frac{1}{\varepsilon^n} K\left(\frac{x}{\varepsilon}\right)$ is an approximation of the δ measure.

Adding and subtracting $\nabla \cdot (\nu^\varepsilon \nabla \nu^\varepsilon)$, we can rewrite (1.12) as

$$\partial_t \nu^\varepsilon + \nabla \cdot (\nu^\varepsilon (\nabla \nu^\varepsilon - \nu^\varepsilon \nabla \nu^\varepsilon + (\nabla K^\varepsilon * \nu^\varepsilon - \nabla \nu^\varepsilon))) = 0. \quad (1.13)$$

Assuming ν^ε to be smooth, and taking a Taylor expansion of ν^ε , we get that

$$K^\varepsilon * \nu^\varepsilon(x) - \nu^\varepsilon(x) = \varepsilon^2 k_0 \Delta \nu^\varepsilon(x) + \mathcal{O}(\varepsilon^4), \quad (1.14)$$

where

$$k_0 = \int |x|^2 K(x) dx.$$

Replacing (1.14) in (1.13), disregarding the $\mathcal{O}(\varepsilon^4)$ term, we finally obtain (1.1):

$$\begin{cases} \partial_t \nu^\varepsilon + \nabla \cdot (\nu^\varepsilon (-\nabla F'(\nu^\varepsilon) + \varepsilon^2 k_0 \nabla \Delta \nu^\varepsilon)) = 0 \\ \nu^\varepsilon(x, 0) = \nu_0^\varepsilon, \end{cases}$$

where $F'(x) = \frac{x^2}{2} - x$.

The Cahn-Hilliard equation we are studying in this chapter is thus an approximation of the non-local equation (1.12). Unfortunately, the techniques used in this

chapter to control the oscillations of solutions to (1.1) could not be generalized to deal with solutions of (1.12). The main issue being that the non-locality does not allow us to integrate exactly against the derivative of the solution. We are thus unable, at the present time, to fully describe the behavior of the solutions of (1.12) as $\varepsilon \rightarrow 0$. The only result that carries through is a uniform in ε , L^∞ estimate for the solutions of (1.13), which follows almost exactly as Lemma 3.4.1.

Remark 3.1.1. *It is worth noticing that (1.13) is the gradient flow of*

$$F^\varepsilon[\nu] = \int \frac{\nu^3(x)}{6} - \frac{1}{2} K^\varepsilon * \nu(x)\nu(x) dx,$$

with respect to the metric induced by the \mathcal{W}^2 distance. By adding and subtracting $\frac{\nu^2(x)}{2}$, in the expression, we obtain, after some calculations,

$$F^\varepsilon[\nu] = \int F(\nu) dx + \frac{1}{4} \int \int K^\varepsilon(x-y)(\nu(x) - \nu(y))^2 dx dy, \quad (1.15)$$

with $F(x) = \frac{x^3}{6} - \frac{x^2}{2}$.

The semi-norm

$$\frac{1}{4} \int \int K^\varepsilon(x-y)(\nu(x) - \nu(y))^2 dx dy,$$

is, up to a constant, a smooth non-local approximation of

$$\frac{\varepsilon^2}{2} \int |\nabla \nu|^2,$$

therefore (1.15) can be considered as a smooth non-local approximation of (1.3).

Remark 3.1.2. *Different scalings of time in (1.11) can be considered. The case of $\frac{t}{\varepsilon}$ is related, in the limit $\varepsilon \rightarrow 0$, to motion by mean curvature (see [68]).*

Remark 3.1.3. *A similar heuristic relationship between (1.1) and the non-local model in [74] has been drawn independently in [75].*

3.2 Notation and Assumptions

Throughout the chapter, we always consider measures $\mu \in \mathcal{P}(\mathbb{T})$ that are absolutely continuous with respect to the Lebesgue measure, we do not make any distinction between the measure and its density.

Also, we use the term sequence loosely: it may denote family of measures labeled by the continuous parameter ε .

3.2.1 Assumptions on F

We assume that F is in $C^2([0, \infty), [0, \infty))$; we denote by F^{**} its convex envelope. We define the auxiliary function Q , which is usually referred in the literature as pressure, such that

$$Q'(y) = yF'(y) - F(y), \quad (2.16)$$

we use the notation with a prime, because its derivative is related with the second derivative of F , namely

$$Q''(y) = yF''(y). \quad (2.17)$$

Moreover, we assume that F has the following properties:

- (H1) There exists a constant $C > 0$ such that for every $y \in \mathbb{R}$

$$|Q'(y)| \leq C(1 + F(y)). \quad (2.18)$$

and

$$|F'(y)| \leq C(1 + F(y)). \quad (2.19)$$

- (H2) $\lim_{y \rightarrow +\infty} Q'(y) = +\infty$
- (H3) The unstable set $\Sigma = cl(\{F > F^{**}\} \cup \{0\}) = \cup_{i=1}^p \Sigma_i$, where $p \in \mathbb{N}$ and $\Sigma_i = [a_i, b_i]$, with $a_{i+1} > b_i$.

The first interval could be degenerate in the sense of $a_1 = b_1 = 0$. As the dynamics near zero will be special, we will distinguish a value

$$m_0 = \begin{cases} b_1 + 1 & \text{if } p = 1 \\ \frac{b_1 + a_2}{2} & \text{if } p \geq 2. \end{cases} \quad (2.20)$$

- (H4) Given $K \subset \Sigma^c$ compact, $\inf_{A \in K} F''(A) > 0$. Equivalently, $F''(p) > 0$ for any $p \in \Sigma^c$.

Remark 3.2.1. Equation (1.1) and (1.2) are not affected by adding an affine function to F , so without loss of generality we will consider the case $F(0) = 0$ and $F'(0) = 0$.

3.2.2 Functionals \mathcal{F}^ε , \mathcal{F}^{**} , \mathcal{G}^ε , $|\nabla \mathcal{F}^{**}|$

For any $\varepsilon > 0$, we define

$$\mathcal{F}^\varepsilon : \mathcal{P}(\mathbb{T}) \rightarrow [0, +\infty]$$

the functional

$$\mathcal{F}^\varepsilon[\nu] = \begin{cases} \int_{\mathbb{T}} \frac{\varepsilon^2}{2} |\nu_x|^2 + F(\nu) dx & \text{if } \nu \in H^1(\mathbb{T}) \\ +\infty & \text{elsewhere.} \end{cases}$$

Formally, the subdifferential of \mathcal{F}^ε at μ , with respect to \mathcal{W}^2 is given by

$$\partial_{\mathcal{W}^2} \mathcal{F}^\varepsilon[\mu] = \nabla(F'(\mu) - \varepsilon^2 \Delta \mu).$$

and we have

$$|\partial_{\mathcal{W}^2} \mathcal{F}^\varepsilon|(\mu) = \int_{\mathbb{T}} \mu |\nabla(F'(\mu) - \varepsilon^2 \Delta \mu)|^2 dx.$$

However, to our knowledge, unless μ is assumed to be strictly positive, nobody has proven that \mathcal{F}^ε are actually sub-differentiable at μ , no matter how regular μ is.

For this reason, we introduce a functional

$$\mathcal{G}^\varepsilon(\cdot) : \mathcal{P}(\mathbb{T}) \rightarrow [0, +\infty],$$

which will play the role of $|\partial \mathcal{F}^\varepsilon|$; we define it, using an auxiliary map G^ε and an auxiliary set \mathcal{T}^ε . For $\mu \in H^1(\mathbb{T})$, we define

$$G^\varepsilon(\mu) = \mu F'(\mu) - F(\mu) + \frac{3\varepsilon^2}{2} |\mu_x|^2 - \varepsilon^2 (\mu \mu_x)_x \quad (2.21)$$

(formally at least, we have $G^\varepsilon(\mu)_x = \mu(F'(\mu) - \varepsilon^2 \mu_{xx})_x$) and

$$\mathcal{T}^\varepsilon(\mu) = \{g \in L^2(\mathbb{T}) : G^\varepsilon(\mu)_x = \sqrt{\mu}g\}$$

(possibly empty). We then set

$$\mathcal{G}^\varepsilon(\mu) = \begin{cases} \inf_{g \in \mathcal{T}^\varepsilon(\mu)} \|g\|_2 & \text{if } \mathcal{T}^\varepsilon(\mu) \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases} \quad (2.22)$$

Remark 3.2.2. *We always have the inequality*

$$\mathcal{G}^\varepsilon(\mu) \leq \int_{\mathbb{T}} \mu |(F'(\mu) - \varepsilon^2 \mu_{xx})_x|^2 dx.$$

Indeed, if the right hand side is infinite, there is nothing to prove, and if the right hand side is finite, then $\sqrt{\mu}(F'(\mu) - \varepsilon^2 \mu_{xx})_x \in \mathcal{T}^\varepsilon(\mu)$ and the inequality clearly holds.

Remark 3.2.3. *The idea, behind this cumbersome definition, is that*

$$\|G^\varepsilon(\mu)_x\|_1 \leq \mathcal{G}^\varepsilon(\mu)$$

and when μ is regular in $\{\mu > 0\}$, then

$$\int_{\mu > 0} \mu |(F'(\mu) - \varepsilon^2 \mu_{xx})_x|^2 dx \leq \mathcal{G}(\mu).$$

The fact that the integral is only on the set $\{\mu > 0\}$ is a standard inconvenience in the thin film equation literature and is the source of many difficulties in proving the existence of curves of maximal slope of \mathcal{F}^ε .

We also define

$$\mathcal{F}^{**} : \mathcal{P}(\mathbb{T}) \rightarrow [0, +\infty]$$

the functional

$$\mathcal{F}^{**}[\nu] = \begin{cases} \int_{\mathbb{T}} F^{**}(\nu) dx & \text{if } \nu \in L^1(\mathbb{T}) \\ +\infty & \text{elsewhere.} \end{cases}$$

\mathcal{F}^{**} is convex (see [76]) and its subdifferential is given by

$$\partial_{\mathcal{W}^2} \mathcal{F}^{**}[\mu] = \nabla F^{***}(\mu).$$

Therefore, we define the functional

$$|\nabla \mathcal{F}^{**}| : \mathcal{P}(\mathbb{T}) \rightarrow [0, +\infty] \tag{2.23}$$

by

$$|\nabla \mathcal{F}^{**}(\mu)| = \begin{cases} \left(\int_{\mathbb{T}} \mu |(F^{***}(\mu))_x|^2 dx \right)^{\frac{1}{2}} & \text{if } (Q^{***}(\mu)) \in W^{1,1}(\mathbb{T}) \\ +\infty & \text{elsewhere,} \end{cases}$$

where $Q^{***}(z) = zF^{***}(z) - F^{**}(z)$. For more details, see Section 10.4.3 in [1].

Remark 3.2.4. *The subtlety of the doubling condition on F^{**} is omitted, because we deal with measures that are bounded.*

Remark 3.2.5. *Because \mathcal{F}^{**} is convex, we have that $|\nabla \mathcal{F}^{**}|$ is a strong upper gradient. (See Definition 1.2.11 and Definition 1.2.9)*

3.2.3 Existence of ν^ε and ν_0

Given $\varepsilon > 0$ and an initial condition ν_i^ε , such that

$$\mathcal{F}^\varepsilon[\nu_i^\varepsilon] < +\infty,$$

we consider $\nu^\varepsilon(x, t)$ solution of equation (1.1) given by the following proposition:

Proposition 3.2.6. *Given $\nu_i^\varepsilon \in \mathcal{P}_2(\mathbb{T})$, such that $\mathcal{F}^\varepsilon[\nu_i^\varepsilon] < \infty$, then there exists $\nu^\varepsilon \in L^\infty((0, \infty); H^1(\mathbb{T})) \cap L^2_{loc}((0, \infty); H^2(\mathbb{T})) \cap C^{1,4}_{loc}(\{\nu^\varepsilon > 0\})$ such that*

$$\int_0^\infty \int_{\mathbb{T}} \nu^\varepsilon \phi_t \, dx dt - \int_0^\infty \int_{\mathbb{T}} (\varepsilon^2 \nu_{xx}^\varepsilon - F'(\nu^\varepsilon)) (\nu^\varepsilon \phi_x)_x \, dx dt = 0, \quad (2.24)$$

for every $\phi \in C_c^\infty((0, \infty) \times \mathbb{T})$.

Moreover,

$$\mathcal{F}^\varepsilon[\nu^\varepsilon(t)] + \frac{1}{2} \int_0^t \mathcal{G}(\nu^\varepsilon)^2 \, ds + \frac{1}{2} \int_0^t |\nu^{\varepsilon'}|^2 \, ds \leq \mathcal{F}^\varepsilon[\nu_i^\varepsilon] \quad \forall t > 0, \quad (2.25)$$

where $|\nu^{\varepsilon'}|$ is the size of the metric derivative of ν^ε with respect to $\mathcal{W}^2(\mathbb{T})$ (See Definition 1.2.8).

Remark 3.2.7. *We cannot claim that ν^ε is a curve of maximal slope, as defined in [1], since we do not prove that \mathcal{G}^ε is an upper gradient of \mathcal{F}^ε (See Definition 1.2.9).*

Remark 3.2.8. From the inclusion $H^2 \subset C^{1, \frac{1}{2}}$, we get that for almost every t , $\nu^\varepsilon(t) \in C^{1, \frac{1}{2}}$.

Proof. The existence of $\nu^\varepsilon \in L^\infty((0, \infty); H^1(\mathbb{T})) \cap L^2_{loc}((0, \infty); H^2(\mathbb{T}))$, that satisfies (2.24) is a particular case of Theorem 1 in [59]. More precisely, ν^ε is constructed as any accumulation point of the discrete interpolation of the solutions of the appropriate JKO scheme. The fact that $\nu^\varepsilon \in C^{1,4}_{loc}(\{\nu^\varepsilon > 0\})$ follows from Schauder estimates (see [77]).

The proof of (2.25) which plays a central role in the proof of our main result is somewhat more technical, and is detailed in Section 3.8. \square

As the functional \mathcal{F}^{**} is convex then, using Theorem 1.2.22, we denote by ν_0 the unique gradient flow of \mathcal{F}^{**} emanating from ν_i . Moreover, ν_0 is also the unique distributional solution to

$$\begin{cases} \partial_t \nu &= ((F^{**'}(\nu))_x \nu)_x \\ \nu(0) &= \nu_i. \end{cases} \quad (2.26)$$

It can be characterized by either the Energy inequality, also known as the maximal slope condition

$$\mathcal{F}^{**}[\nu(t)] + \frac{1}{2} \int_0^t |\nabla \mathcal{F}^{**}(\nu)|^2 ds + \frac{1}{2} \int_0^t |\nu'|^2 ds \leq \mathcal{F}^{**}[\nu_i] \quad \forall t > 0, \quad (2.27)$$

or the Energy equality

$$\mathcal{F}^{**}[\nu(t)] + \frac{1}{2} \int_0^t |\nabla \mathcal{F}^{**}(\nu)|^2 ds + \frac{1}{2} \int_0^t |\nu'|^2 ds = \mathcal{F}^{**}[\nu_i] \quad \forall t > 0. \quad (2.28)$$

3.3 Statement of the Result

The main result of this paper is the following:

Theorem 3.3.1. *Let $\{\nu_i^\varepsilon\}_\varepsilon$, $\nu_i \in \mathcal{P}(\mathbb{T})$ be such that*

$$\mathcal{F}^\varepsilon[\nu_i^\varepsilon] < +\infty \quad \text{and} \quad \mathcal{F}^{**}[\nu_i] < +\infty.$$

Suppose that,

$$\lim_{\varepsilon \rightarrow 0^+} \nu_i^\varepsilon = \nu_i \quad \text{in } \mathcal{W}^2(\mathbb{T}) \tag{3.29}$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{F}^\varepsilon[\nu_i^\varepsilon] = \mathcal{F}^{**}[\nu_i]. \tag{3.30}$$

Then, for any $T > 0$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \nu^\varepsilon &= \nu_0 \quad \text{in } C^0([0, T]; \mathcal{W}^2(\mathbb{T})), \\ \lim_{\varepsilon \rightarrow 0^+} \int_0^T (\mathcal{G}^\varepsilon(\nu^\varepsilon(t)) - |\nabla \mathcal{F}^{**}(\nu_0(t))|)^2 dt &= 0 \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{F}^\varepsilon[\nu^\varepsilon(t)] = \mathcal{F}^{**}[\nu_0(t)] \quad \forall t \geq 0,$$

*where ν^ε is the solution of (1.1) given by Proposition 3.2.6 with initial condition ν_i^ε and ν_0 is the unique Gradient flow of \mathcal{F}^{**} (solution of (2.26)) with initial condition ν_i , with respect to the metric $\mathcal{W}^2(\mathbb{T})$.*

As in [67], [68], [69], [70], [71] and [72] the key step in the proof of Theorem 3.3.1 is to prove the lower-semicontinuity in the convergence of \mathcal{G}^ε to $|\nabla \mathcal{F}^{**}|$, more specifically we need to prove:

Theorem 3.3.2. *Let $\{\rho^\varepsilon\}_{\varepsilon > 0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\rho^\varepsilon \in C^1(\mathbb{T}) \cap C_{loc}^4\{\rho^\varepsilon > 0\}$, $\rho^\varepsilon \rightarrow \rho_0$ in $\mathcal{W}^2(\mathbb{T})$ and $\sup_\varepsilon \mathcal{F}^\varepsilon(\rho^\varepsilon) < \infty$, then*

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{G}^\varepsilon(\rho^\varepsilon) \geq |\nabla \mathcal{F}^{**}|(\rho_0). \tag{3.31}$$

The next two section is devoted to some preliminary compactness results, which are used in the proof of Theorem 3.3.2 that can be found in Section 3.6. The proof of Theorem 3.3.1 can be found in Section 3.7.

3.4 Preliminary to the proof of Theorem 3.3.2

3.4.1 Uniform L^∞ estimate

The first step is to prove a uniform L^∞ estimate.

Proposition 3.4.1. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$, then*

$$\sup_\varepsilon \|\rho^\varepsilon\|_\infty \leq M < \infty.$$

Moreover, up to a subsequence,

$$\rho^\varepsilon \rightharpoonup \rho^0 \quad \text{weak-* } L^\infty.$$

Proof. Consider $G^\varepsilon(\rho^\varepsilon)$ as in (2.21):

$$G^\varepsilon(\rho^\varepsilon) = -\varepsilon^2 \rho^\varepsilon \rho_{xx}^\varepsilon + \frac{\varepsilon^2}{2} (\rho_x^\varepsilon)^2 + Q'(\rho^\varepsilon),$$

with Q' defined by (2.16). By Remark 3.2.3, we know that

$$\|G^\varepsilon(\rho^\varepsilon)_x\|_1 \leq \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C.$$

Moreover,

$$\int_{\mathbb{T}} G^\varepsilon(\rho^\varepsilon) dx = \int_{\mathbb{T}} \frac{3}{2} \varepsilon^2 (\rho_x^\varepsilon)^2 + Q'(\rho^\varepsilon) dx \leq 3\mathcal{F}^\varepsilon[\rho^\varepsilon] + D \int_{\mathbb{T}} (F(\rho^\varepsilon) + 1) dx,$$

by (H1). Therefore, $G^\varepsilon(\rho^\varepsilon)$ is uniformly in $W^{1,1}(\mathbb{T})$, which implies

$$\sup_{\varepsilon} \|G^\varepsilon(\rho^\varepsilon)\|_\infty < \infty.$$

Now, let's prove that ρ^ε is uniformly in L^∞ : take x_0 , such that $\rho^\varepsilon(x_0) = \|\rho^\varepsilon\|_\infty$, then $\rho_x(x_0) = 0$ and $\rho_{xx}(x_0) \leq 0$. We should note that, because $\mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$, then $\rho^\varepsilon \in C_{loc}^{2, \frac{1}{2}}(\{\rho^\varepsilon > 0\})$ and $\rho_{xx}(x_0)$ has a well defined value.

Now, we have the bound

$$\|G^\varepsilon(\rho^\varepsilon)\|_\infty \geq G^\varepsilon(\rho^\varepsilon)(x_0) \geq Q'(\rho^\varepsilon(x_0)),$$

which, by assumption (H2), gives a bound for

$$\sup_{\varepsilon} \|\rho^\varepsilon\|_\infty.$$

□

Corollary 3.4.2. *Under the assumptions of Proposition 3.4.1, $G^\varepsilon(\rho^\varepsilon)$ is bounded in $H^1(\mathbb{T})$ uniformly in ε . More precisely, we have the bound*

$$\|G^\varepsilon(\rho^\varepsilon)_x\|_{L^2} \leq \|\rho^\varepsilon\|_\infty \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C.$$

Therefore, $G^\varepsilon(\rho^\varepsilon) \in C^{\frac{1}{2}}(\mathbb{T})$ uniformly in ε .

3.4.2 Control of the oscillations in the good set

The key in the proof of Theorem 3.3.2 is to control the size of the oscillations of ρ^ε in the good sets. This will be given by the following Theorem:

Theorem 3.4.3. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_{\varepsilon} \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$, then, for any $L \geq 0$ there exists $\delta(\eta, C) > 0$, independent of ε , such that for any $\varepsilon < \varepsilon_0(\eta, C, L)$ and any pair of sequences $x_\varepsilon, y_\varepsilon$ satisfying:*

- $0 < y_\varepsilon - x_\varepsilon < \delta$,
- $|\rho_x^\varepsilon(x_\varepsilon)| < L$ and $|\rho_x^\varepsilon(y_\varepsilon)| < L$,

we have either

$$d(\rho^\varepsilon(z), \Sigma) < \eta \quad \forall z \in [x_\varepsilon, y_\varepsilon]$$

or

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(y_\varepsilon)| < \eta.$$

Theorem 3.4.3 is similar to Lemma 5.5 in the paper by Belletini et al. [69].

The main difference in the proof is that in [69] they have control of the H^1 norm of

$$e^\varepsilon(\rho^\varepsilon) = F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon, \tag{4.32}$$

while we only have control on

$$\int_{\mathbb{T}} |e^\varepsilon(\rho^\varepsilon)_x|^2 \rho^\varepsilon dx, \tag{4.33}$$

which is degenerate near $\{\rho^\varepsilon = 0\}$.

Theorem 3.4.3 can be interpreted as a uniform lower semi-continuity for $d(\rho^\varepsilon, \Sigma)$:

Corollary 3.4.4. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$ and that $\rho^\varepsilon \rightarrow \rho_0$ in $\mathcal{W}^2(\mathbb{T})$, then for x , any Lebesgue point of ρ_0 , there exists ε_x and $\delta' = \delta'(d(\rho_0(x), \Sigma))$ such that for every $\varepsilon < \varepsilon_x$ and $y \in (x - \delta', x + \delta')$, we have*

$$d(\rho_0(y), \Sigma) > \frac{d(\rho_0(x), \Sigma)}{2}.$$

Moreover, $\Omega := \{\rho_0 \notin \Sigma\}$ has an open representative.

Proof of Corollary 3.4.4. We start with the following claim:

Claim: For any $\beta > 0$, we define $\delta(\beta) = \frac{\delta(\beta, C)}{4}$ and $\varepsilon(\beta) = \varepsilon(C, \beta, \frac{4M}{\delta})$

(Given by Theorem 3.4.3). If for some $\varepsilon \in (0, \varepsilon_\beta)$, we have that $d(\rho^\varepsilon(x), \Sigma) > 2\beta$, then $d(\rho^\varepsilon(y), \Sigma) > \beta$ for all $y \in (x - \delta_\beta, x + \delta_\beta)$.

Proof of the Claim: We take $\delta = \delta(\eta, C)$ given by Theorem 3.4.3. Because we know that $\sup_\varepsilon |\rho^\varepsilon| \leq M$, we have $\text{osc}_{(x+\frac{\delta}{4}, x+\frac{\delta}{2})} \rho^\varepsilon \leq M$. Therefore, there exists $x_1 \in (x + \frac{\delta}{4}, x + \frac{\delta}{2})$ such that

$$|\rho_x^\varepsilon(x_1)| \leq \frac{4M}{\delta}.$$

Similarly, there exists $x_2 \in (x - \frac{\delta}{2}, x - \frac{\delta}{4})$ such that $|\rho_x^\varepsilon(x_2)| \leq \frac{4M}{\delta}$.

If $\varepsilon < \varepsilon(C, \eta, \frac{4M}{\delta})$ given by Theorem 3.4.3, then we can estimate the difference between the maximum and the minimum in $[x_2, x_1]$ (they are either a critical point or a boundary point). Therefore, we know that

$$\text{osc}_{(x_2, x_1)} \rho^\varepsilon < \eta,$$

by taking $\eta = \beta$ the **Claim** follows.

Because x is a Lebesgue point, for all r small enough, we know that

$$\left| \frac{1}{2r} \int_{x-r}^{x+r} \rho_0(y) dy - \rho_0(x) \right| < \frac{d(\rho_0(x), \Sigma)}{6} \quad (4.34)$$

We will fix $r_x < \frac{1}{2}\delta \left(\frac{d(\rho_0(x), \Sigma)}{3} \right)$ such that (4.34) holds.

By Proposition 3.4.1, we know that $\rho^\varepsilon \rightarrow \rho_0$ weak- $*$ L^∞ ; therefore, there exists ε_x such that for all $\varepsilon < \varepsilon_x$

$$\left| \frac{1}{2r_x} \int_{x-r_x}^{x+r_x} \rho^\varepsilon(y) dy - \frac{1}{2r_x} \int_{x-r_x}^{x+r_x} \rho_0(y) dy \right| < \frac{d(\rho_0(x), \Sigma)}{6}.$$

So, if $\varepsilon < \varepsilon_x$, there exists $x_\varepsilon \in (x - r_x, x + r_x)$, such that

$$|\rho^\varepsilon(x_\varepsilon) - \rho_0(x)| < \frac{d(\rho_0(x), \Sigma)}{3},$$

hence

$$d(\rho^\varepsilon(x_\varepsilon), \Sigma) > \frac{2}{3}d(\rho_0(x), \Sigma).$$

By the **Claim**, if ε is small enough, it follows that $d(\rho^\varepsilon(y), \Sigma) > \frac{d(\rho_0(x), \Sigma)}{3}$ for all $y \in (x_\varepsilon - \delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right), x_\varepsilon + \delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right))$. The result follows, because

$$\left(x - \frac{1}{2}\delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right), x + \frac{1}{2}\delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right)\right) \subset \left(x_\varepsilon - \delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right), x_\varepsilon + \delta \left(\frac{d(\rho_0(x), \Sigma)}{3}\right)\right).$$

□

To prove Theorem 3.4.3 we start by looking at the behavior of ρ^ε on the set $\{\rho^\varepsilon > h\}$ with $h > 0$. This case follows exactly as Lemma 5.5 in [69]; our main contribution here is to give a different proof in a simple case that makes the set $\Sigma = cl\{F > F^{**}\}$ appear more naturally.

Lemma 3.4.5. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$, then for any $h > 0$ and $L \geq 0$ there exists $\delta(\eta, C, h) > 0$, independent of ε , such that for any $\varepsilon < \varepsilon_0(\eta, C, h, L)$ and any pair of sequences $x_\varepsilon, y_\varepsilon$ satisfying:*

- $0 < y_\varepsilon - x_\varepsilon < \delta$,
- $|\rho_x^\varepsilon(x_\varepsilon)| < L$ and $|\rho_x^\varepsilon(y_\varepsilon)| < L$,
- $\rho^\varepsilon(z) > h \quad \forall z \in [x_\varepsilon, y_\varepsilon]$

then we have either

$$d(\rho^\varepsilon(z), \Sigma) < \eta \quad \forall z \in [x_\varepsilon, y_\varepsilon]$$

or

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(y_\varepsilon)| < \eta.$$

Proof. We only give a sketch of the proof, which shows why the set Σ appears naturally. For a complete proof see Lemma 5.5 in [69].

Since $\rho^\varepsilon(z) > h$ for every $z \in (x_\varepsilon, y_\varepsilon)$ and ε , then

$$(e^\varepsilon(\rho^\varepsilon))_x = (F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x$$

is bounded in $L^2(x_\varepsilon, y_\varepsilon)$ uniformly in ε (see (4.32), (4.33) and Remark 3.2.3). Moreover, we have

$$\int_{x_\varepsilon}^{y_\varepsilon} e^\varepsilon(\rho^\varepsilon)(z) dz = \int_{x_\varepsilon}^{y_\varepsilon} F'(\rho^\varepsilon(z)) - \varepsilon^2 \rho_{xx}^\varepsilon(z) dz \leq C + 2\varepsilon^2 L. \quad (4.35)$$

Sobolev's Embedding Theorem implies that $e^\varepsilon(\rho^\varepsilon)$ is also uniformly bounded in $C^{\frac{1}{2}}$.

Without loss of generality, we will assume that $\rho^\varepsilon(x_\varepsilon) \leq \rho^\varepsilon(y_\varepsilon)$, and that $\rho_x^\varepsilon(x_\varepsilon), \rho_x^\varepsilon(y_\varepsilon) \geq 0$, if not we work with the closest minimum to x_ε and the closest maximum to y_ε , inside the interval. We will also assume $\rho_{xx}^\varepsilon(x_\varepsilon) \geq 0$ and $\rho_{xx}^\varepsilon(y_\varepsilon) \leq 0$; if this condition is not satisfied, we can take

$$\tilde{x}_\varepsilon = \inf\{z : z \in (x_\varepsilon, y_\varepsilon) \cap \rho_{xx}^\varepsilon(z) < 0 \cap \rho_x^\varepsilon \geq 0\}.$$

Then, we obtain

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(\tilde{x}_\varepsilon)| \leq L\delta.$$

If $\rho^\varepsilon(\tilde{x}_\varepsilon)_x > 0$, then $\rho^\varepsilon(\tilde{x}_\varepsilon)_{xx} > 0$. If $\rho^\varepsilon(\tilde{x}_\varepsilon)_x = 0$ and $\rho^\varepsilon(\tilde{x}_\varepsilon)_{xx} < 0$, then $\rho^\varepsilon(\tilde{x}_\varepsilon)$ is a maximum. If this happens, we consider \tilde{z}_ε the closest minimum to \tilde{x}_ε , so that we get $\rho^\varepsilon(\tilde{z}_\varepsilon)_{xx} \geq 0$. We split the interval in three, $(x_\varepsilon, \tilde{x}_\varepsilon)$, $(\tilde{x}_\varepsilon, \tilde{z}_\varepsilon)$ and $(\tilde{z}_\varepsilon, y_\varepsilon)$, and we control each of the pieces separately.

If $\rho_{xx}^\varepsilon(y_\varepsilon) > 0$, we can repeat the same arguments.

Multiplying $e^\varepsilon(\rho^\varepsilon)$ by $\rho_x^\varepsilon(z)$ and integrating between x_ε and y_ε we get the following

$$\begin{aligned} \int_{x_\varepsilon}^{y_\varepsilon} e^\varepsilon(\rho^\varepsilon) \rho_x^\varepsilon dz &= \frac{\varepsilon^2}{2} (|\rho_x^\varepsilon(y_\varepsilon)|^2 - |\rho_x^\varepsilon(x_\varepsilon)|^2) + F(\rho^\varepsilon(y_\varepsilon)) - F(\rho^\varepsilon(x_\varepsilon)) \\ &\leq \frac{\varepsilon^2}{2} L^2 + F(\rho^\varepsilon(y_\varepsilon)) - F(\rho^\varepsilon(x_\varepsilon)). \end{aligned}$$

On the other hand, integrating by parts we also find

$$\int_{x_\varepsilon}^{y_\varepsilon} e^\varepsilon(\rho^\varepsilon) \rho_x^\varepsilon dz = - \int_{x_\varepsilon}^{y_\varepsilon} e_x^\varepsilon(\rho^\varepsilon) \rho^\varepsilon dz + [e^\varepsilon(\rho^\varepsilon) \rho^\varepsilon]_{x_\varepsilon}^{y_\varepsilon}.$$

Because e_x^ε is uniformly in L^2 and ρ^ε uniformly in L^∞ , we have

$$\left| \int_{x_\varepsilon}^{y_\varepsilon} e_x^\varepsilon(\rho^\varepsilon) \rho^\varepsilon dz \right| \leq C(y_\varepsilon - x_\varepsilon)^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}.$$

We decompose

$$[e^\varepsilon(\rho^\varepsilon) \rho^\varepsilon]_{x_\varepsilon}^{y_\varepsilon} = e^\varepsilon(\rho^\varepsilon)(x_\varepsilon) [\rho^\varepsilon(y_\varepsilon) - \rho^\varepsilon(x_\varepsilon)] + [e^\varepsilon(\rho^\varepsilon)]_{x_\varepsilon}^{y_\varepsilon} \rho^\varepsilon(y_\varepsilon); \quad (4.36)$$

using that e^ε is uniformly in $C^{\frac{1}{2}}$, we see that

$$|[e^\varepsilon(\rho^\varepsilon)]_{x_\varepsilon}^{y_\varepsilon} \rho^\varepsilon| \leq C(y_\varepsilon - x_\varepsilon)^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}.$$

Combining the five equations above, we see that given any $\lambda > 0$, we can choose ε and δ small enough, such that

$$F(\rho^\varepsilon(x_\varepsilon)) + e^\varepsilon(\rho^\varepsilon)(x_\varepsilon) (\rho^\varepsilon(y_\varepsilon) - \rho^\varepsilon(x_\varepsilon)) + \lambda \geq F(\rho^\varepsilon(y_\varepsilon)).$$

Using the assumption $\rho^\varepsilon(y_\varepsilon) \geq \rho^\varepsilon(x_\varepsilon)$, the definition of $e^\varepsilon(\rho^\varepsilon)$ and the fact that $\rho_{xx}^\varepsilon(x_\varepsilon) \geq 0$, we obtain

$$F(\rho^\varepsilon(x_\varepsilon)) + F'(\rho^\varepsilon(x_\varepsilon)) (\rho^\varepsilon(y_\varepsilon) - \rho^\varepsilon(x_\varepsilon)) + \lambda \geq F(\rho^\varepsilon(y_\varepsilon)). \quad (4.37)$$

Exchanging the roles of x_ε and y_ε in (4.36) and using the fact that $\rho_{xx}^\varepsilon(y_\varepsilon) \leq 0$, we obtain similarly

$$F(\rho^\varepsilon(y_\varepsilon)) - F'(\rho^\varepsilon(y_\varepsilon))(\rho^\varepsilon(y_\varepsilon) - \rho^\varepsilon(x_\varepsilon)) + \lambda \geq F(\rho^\varepsilon(x_\varepsilon)). \quad (4.38)$$

To use these conditions analytically, we define the sets

$$U_\lambda^F(A) = \{B \in \mathbb{R}_+ : F(A) + F'(A)(B - A) + \lambda \geq F(B)\},$$

so conditions (4.37) and (4.38) can be reformulated as:

$$\rho^\varepsilon(y_\varepsilon) \in U_\lambda^F(\rho^\varepsilon(x_\varepsilon)) \text{ and } \rho^\varepsilon(x_\varepsilon) \in U_\lambda^F(\rho^\varepsilon(y_\varepsilon)). \quad (4.39)$$

We now finish the proof under the extra assumption that $\Sigma \cap (h, \infty)$ contains only one interval:

By (H4), we know that for any fixed $\eta > 0$, F is uniformly convex in $\{p \in \mathbb{R}^+ : d(p, \Sigma) \geq \eta\}$; therefore we can choose λ_0 such that for all $\lambda < \lambda_0$, we have

$$U_\lambda^F(A) \subset (A - \eta, A + \eta) \text{ for all } A \in \{p \in \mathbb{R}^+ : p > h \cap d(p, \Sigma) \geq \eta\}. \quad (4.40)$$

If $d(\rho^\varepsilon(x_\varepsilon), \Sigma) > \eta$, then, with $A = \rho^\varepsilon(x_\varepsilon)$, (4.39) and (4.40) imply

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(y_\varepsilon)| < \eta.$$

The same holds for $d(\rho^\varepsilon(y_\varepsilon), \Sigma) > \eta$.

On the other hand, if $d(\rho^\varepsilon(x_\varepsilon), \Sigma) < \eta$ and $d(\rho^\varepsilon(y_\varepsilon), \Sigma) < \eta$, we take

$$z = \operatorname{argmax}_{t \in [x_\varepsilon, y_\varepsilon]} d(\rho^\varepsilon(t), \Sigma);$$

if $d(\rho^\varepsilon(z), \Sigma) < \eta$, we are done. If $d(\rho^\varepsilon(z), \Sigma) > \eta$, because we assume that $\Sigma \cap (h, \infty)$ is an interval, we know that $\rho_x^\varepsilon(z) = 0$. Therefore, the intervals (x_ε, z) and (z, y_ε)

satisfy the hypothesis of the Lemma. Then, arguing as before, because $d(\rho^\varepsilon(z), \Sigma) > \eta$, we have

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(z)| < \eta \text{ and } |\rho^\varepsilon(y_\varepsilon) - \rho^\varepsilon(z)| < \eta.$$

By our definition of z , we can conclude

$$|\rho^\varepsilon(x_\varepsilon) - \rho^\varepsilon(y_\varepsilon)| < \eta,$$

which proves the Lemma with the extra assumption of $\Sigma \cap (h, \infty)$ contains one interval.

A more convoluted argument, as the one in the proof of Theorem 3.4.3 (below), can be made for the cases when $\Sigma \cap (h, \infty)$ contains more than one interval. It is not included here, as a proof of this Lemma can already be found in [69] and the ideas of the argument can be found in the proof below. \square

We now turn to the proof of Theorem 3.4.3:

Proof of Theorem 3.4.3. We prove Theorem 3.4.3 by contradiction. Due to Lemma 3.4.5, we know that the theorem would be proven if we can prove that there is no $\eta_0 > 0$, such that there exist a sequence of $\varepsilon_i \rightarrow 0$ and sequences of points $\{x_i\}$, $\{y_i\}$, that satisfy for all i

- $|y_i - x_i| < \frac{1}{i}$
- $\max(|\rho_x^{\varepsilon_i}(x_i)|, |\rho_x^{\varepsilon_i}(y_i)|) < L$
- $\rho^{\varepsilon_i}(x_i) \rightarrow 0$
- $\rho^{\varepsilon_i}(y_i) > b_1 + \eta_0$ (See (H3)).

So, let's assume that such η_0 exists and derive a contradiction.

Without loss of generality, we assume, as in the proof of Lemma 3.4.5, that $\rho_x^{\varepsilon_i}(x_i) \geq 0$, $\rho_x^{\varepsilon_i}(y_i) \geq 0$, $\rho_{xx}^{\varepsilon_i}(x_i) \geq 0$ and $\rho_{xx}^{\varepsilon_i}(y_i) \leq 0$.

From the proof of Proposition 3.4.1 we know that the function

$$G^\varepsilon(\rho^\varepsilon) = -\varepsilon^2 \rho^\varepsilon \rho_{xx}^\varepsilon + \varepsilon^2 \rho_x^{\varepsilon^2} + Q'(\rho^\varepsilon)$$

is uniformly in $C^{\frac{1}{2}}$, using the fact that $\rho_{xx}^{\varepsilon_i}(x_i) \geq 0$ and $\rho_{xx}^{\varepsilon_i}(y_i) \leq 0$, we can conclude that

$$Q'(\rho^{\varepsilon_i}(y_i)) \leq Q'(\rho^{\varepsilon_i}(x_i)) + C \frac{1}{i^{\frac{1}{2}}} + \frac{\varepsilon_i^2}{2} L^2.$$

Moreover, since $\rho^\varepsilon(x_i) \rightarrow 0$, for every $\kappa > 0$, there exists i_0 such that

$$Q'(\rho^{\varepsilon_i}(x_i)) + C \frac{1}{i^{\frac{1}{2}}} + \frac{\varepsilon_i^2}{2} L^2 < \kappa,$$

for all $i > i_0$. This implies that $Q'(\rho^{\varepsilon_i}(y_i)) < \kappa$, for all $i > i_0$.

The first observation is that $Q'(b_1) = b_1 F'(b_1) - F(b_1) = 0$. This is just saying that the tangent of F at b_1 intersects the origin, which is satisfied because $b_1 = \inf_{t>0} \{F(t) = F^{**}(t)\}$ (for a picture see Figure 1).

The second observation is that by (H4), we have $\int_{s_1}^{s_2} t F''(t) = \int_{s_1}^{s_2} Q''(t) > 0$ for $s_1, s_2 \in (b_1, m_0)$, where m_0 is defined in (2.20). We deduce that, for every $\eta > 0$, there exists κ_0 , such that if $A \in (b_1, m_0)$ and $Q'(A) < \kappa_0$, then $A - b_1 < \eta$. Therefore, we get will get a contradiction, if we show that $\rho^{\varepsilon_i}(y_i) < m_0$.

Claim I: *If i is large enough, then $\rho^{\varepsilon_i}(y_i) < m_0$.*

Again, we will prove **Claim I** by contradiction; if $\rho^\varepsilon(y_i) \geq m_0$, then there exists $z_0^i \in [x_i, y_i]$ such that $\rho^i(z_0^i) = m_0$. If we assume also that $\rho_{xx}^{\varepsilon_i}(z_0^i) \leq 0$, then

proceeding as above, we get

$$0 < Q'(m_0) \leq Q'(\rho^{\varepsilon_i}(x_i)) + C \frac{1}{i^{\frac{1}{2}}} + \frac{\varepsilon_i^2}{2} L^2,$$

and taking i large enough it yields our desired contradiction. Therefore, we want to prove that we can indeed assure that $\rho_{xx}^{\varepsilon_i}(z_0) \leq 0$, for i large enough:

Claim II: *Let*

$$z_0 = \sup\{t \in (x_i, y_i) : \rho^{\varepsilon_i}(t) = m_0\}. \quad (4.41)$$

If

$$F(m_0) + F'(m_0)(\rho^{\varepsilon_i}(y_i) - m_0) < F(\rho^{\varepsilon_i}(y_i)) - C(y_i - z_0)^{\frac{1}{2}}(\rho^{\varepsilon_i}(y_i) - m_0), \quad (4.42)$$

for some C independent of i , then for all i big enough

$$\rho_{xx}^{\varepsilon_i}(z_0) \leq 0.$$

Proof of Claim II:

Due to the assumption on z_0 , we know that $\rho^{\varepsilon_i} > m_0$ in (z_0, y_i) . Therefore, we know that $e^{\varepsilon_i}(t) = F'(\rho^{\varepsilon_i}(t)) - \varepsilon^2 \rho_{xx}^{\varepsilon_i}(t)$ is uniformly in $H^1(z_0, y_i)$ (see (4.35)). We perform the following calculation

$$\begin{aligned} \int_{z_0}^{y_i} e^{\varepsilon_i}(t) \rho_x^{\varepsilon_i}(t) dt &= F(\rho^{\varepsilon_i}(y_i)) - F(\rho^{\varepsilon_i}(z_0)) - \frac{\varepsilon_i^2}{2} |\rho_x^{\varepsilon_i}(y_i)|^2 + \frac{\varepsilon_i^2}{2} |\rho_x^{\varepsilon_i}(z_0)|^2 \\ &\geq F(\rho^{\varepsilon_i}(y_i)) - F(m_0) - \frac{\varepsilon_i^2}{2} L^2. \end{aligned}$$

Using the same arguments used to derive (4.37) in the proof of Lemma 3.4.5, we get

$$e^{\varepsilon_i}(z_0)(\rho^{\varepsilon_i}(y_i) - \rho^{\varepsilon_i}(z_0)) + C(y_i - z_0)^{\frac{1}{2}}(\rho^{\varepsilon_i}(y_i) - \rho^{\varepsilon_i}(z_0)) \geq F(\rho^{\varepsilon_i}(y_i)) - F(\rho^{\varepsilon_i}(z_0)) - \frac{\varepsilon_i^2}{2} L^2.$$

If $\rho_{xx}^{\varepsilon_i}(z_0) \geq 0$, then $F'(\rho^{\varepsilon_i}(z_0)) \geq e^{\varepsilon_i}(z_0)$, and so

$$F(\rho^{\varepsilon_i}(z_0)) + F'(\rho^{\varepsilon_i}(z_0))(\rho^{\varepsilon_i}(y_i) - \rho^{\varepsilon_i}(z_0)) \geq F(\rho^{\varepsilon_i}(y_i)) - C(y_i - z_0)^{\frac{1}{2}}(\rho^{\varepsilon_i}(y_i) - \rho^{\varepsilon_i}(z_0)) - \frac{\varepsilon_i^2}{2}L^2.$$

Since $\frac{\varepsilon_i^2}{2}L^2 \rightarrow 0$, if i is large enough this contradicts (4.42), and thus proves **Claim**

II.

To finish the proof of **Claim I**, we have to show that if z_0 defined by (4.41) exist (in particular, if $\rho^{\varepsilon_i}(y_i) \geq m_0$), then (4.42) holds. First, we note that

$$F(m_0) + F'(m_0)(t - m_0) < F(t) \quad \forall t \neq m_0,$$

due to (H4). Therefore, there exists $\kappa_0 > 0$ such that

$$F(m_0) + F'(m_0)(t - m_0) < F(t) - \kappa_0 \quad \text{for every } t \text{ s.t. } Q'(t) < \frac{Q'(m_0)}{2}$$

(the choice of $\frac{Q'(m_0)}{2}$ is arbitrary).

Recalling that

$$\limsup_{i \rightarrow \infty} Q'(\rho^{\varepsilon_i}(y_i)) \leq 0,$$

we can take i large enough such that

$$Q'(\rho^{\varepsilon_i}(y_i)) < \frac{Q'(m_0)}{2}$$

and

$$\frac{C}{i^{\frac{1}{2}}}(\rho^{\varepsilon_i}(y_i) - \rho^{\varepsilon_i}(z_0)) < \kappa_0,$$

so we can conclude that (4.42) holds and **Claim II** yields

$$\rho_{xx}^{\varepsilon_i}(z_0^i) \leq 0.$$

This completes the proof of **Claim I** and of the Theorem. □

3.5 H^1 estimate in the good set Ω

We want to show that ρ^ε is bounded in $H_{loc}^1(\Omega)$ uniformly in ε , with $\Omega = \{\rho_0 \notin \Sigma\}$, in other words, that ρ^ε does not oscillate in the "good" limiting set. We start with the following proposition:

Proposition 3.5.1. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$ and $\rho^\varepsilon \rightarrow \rho_0$ in $\mathcal{W}^2(\mathbb{T})$, given $\phi \in \mathcal{D}(\Omega)$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, we have $F''(\rho^\varepsilon) \geq \lambda_\phi$ and $\rho^\varepsilon \geq \lambda_\phi$ in the support of ϕ , for some constant $\lambda_\phi > 0$ independent of ε .*

Proof. By assumption, if $x \in \Omega$, then $d(\rho_0(x), \Sigma(F)) > 0$. By Corollary 3.4.4, for any Lebesgue point $x \in \Omega$ there exists ε_x and δ_x such that for every $\varepsilon < \varepsilon_x$ and every $z \in (x - \delta_x, x + \delta_x)$

$$d(\rho^\varepsilon(z), \Sigma) > \frac{d(\rho_0(x), \Sigma)}{3}.$$

Now, the family of intervals $\{(x - \delta_x, x + \delta_x)\}_{x \in \hat{\Omega}}$, where $\hat{\Omega}$ is the Lebesgue points of Ω , is an open covering of the support of ϕ , therefore by compactness there exists a finite sub-covering, which proves the proposition. \square

Using Proposition 3.5.1 we can now prove:

Proposition 3.5.2. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that $\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$ and $\rho^\varepsilon \rightarrow \rho_0$ in $\mathcal{W}^2(\mathbb{T})$, for any $K \subset \Omega$ compact, there exists C and $\varepsilon_K > 0$ such that*

$$\int_K |\rho_x^\varepsilon|^2 dx < C \quad \forall \varepsilon < \varepsilon_K.$$

Therefore, up to a subsequence, ρ^ε converges pointwise to ρ_0 a.e. in Ω .

Proof. Take $\phi \in D(\Omega)$, with $\phi \geq \chi_K$. We start with the following computation,

$$\int_{\mathbb{T}} \rho_x^\varepsilon \partial_x [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon] \phi \, dx = \int_{\mathbb{T}} F''(\rho^\varepsilon) |\rho_x^\varepsilon|^2 \phi \, dx + \varepsilon^2 \int_{\mathbb{T}} |\rho_{xx}^\varepsilon|^2 \phi \, dx + \varepsilon^2 \int_{\mathbb{T}} \rho_x^\varepsilon \rho_{xx}^\varepsilon \phi_x \, dx,$$

from which we deduce

$$\begin{aligned} \int_{\mathbb{T}} F''(\rho^\varepsilon) |\rho_x^\varepsilon|^2 \phi \, dx + \varepsilon^2 \int_{\mathbb{T}} |\rho_{xx}^\varepsilon|^2 \phi \, dx &= -\varepsilon^2 \int_{\mathbb{T}} \rho_x^\varepsilon \rho_{xx}^\varepsilon \phi_x \, dx + \int_{\mathbb{T}} \rho_x^\varepsilon \partial_x [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon] \phi \, dx \\ &= \frac{\varepsilon^2}{2} \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi_{xx} \, dx + \int_{\mathbb{T}} \rho_x^\varepsilon \partial_{xx} [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon] \phi \, dx \\ &\leq C(\phi_{xx}) \varepsilon^2 \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \, dx + \left(\int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}} \phi |\partial_x [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon]|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C \varepsilon^2 \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \, dx + \frac{\lambda_\phi}{2} \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi \, dx + C(\lambda_\phi) \int_{\mathbb{T}} \phi |\partial_x [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon]|^2 \, dx, \end{aligned}$$

with the constant λ_ϕ given by Proposition 3.5.1. Therefore, we get:

$$\begin{aligned} \left(\inf_{x \in \text{supp}\{\phi\}} F''(\rho^\varepsilon(x)) - \frac{\lambda_\phi}{2} \right) \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi \, dx &\leq C \varepsilon^2 \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \, dx \\ &\quad + C(\lambda_\phi) \int_{\text{supp}\{\phi\}} |\partial_x [F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon]|^2 \, dx \end{aligned}$$

Using Proposition 3.5.1 we can conclude that in the support of ϕ we have that

$F''(\rho^\varepsilon) > \lambda_\phi$ and that $\rho^\varepsilon > \lambda_\phi$, for $\varepsilon < \varepsilon_0$, so we deduce

$$\int_K |\rho_x^\varepsilon|^2 \, dx \leq \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi \, dx \leq C(\phi) \mathcal{F}^\varepsilon[\rho^\varepsilon] + \frac{C(\lambda_\phi)}{\lambda_\phi} \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C \quad \forall \varepsilon < \varepsilon_0.$$

□

Proposition 3.5.3. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $\mathcal{P}(\mathbb{T})$ such that*

$\sup_\varepsilon \mathcal{F}^\varepsilon[\rho^\varepsilon] + \mathcal{G}^\varepsilon(\rho^\varepsilon) \leq C$ and $\rho^\varepsilon \rightarrow \rho_0$ in $\mathcal{W}^2(\mathbb{T})$, then

$$(F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x \rho^\varepsilon \rightarrow Q'(\rho_0)_x \quad \text{in } D'(\Omega).$$

Proof. Fix $\phi \in D(\Omega)$, then using that $\rho F''(\rho) = Q''(\rho)$ with an integration by parts,

we get

$$\int_{\mathbb{T}} (F'(\rho^\varepsilon)_x - \varepsilon^2 \rho_{xxx}^\varepsilon) \rho^\varepsilon \phi \, dx = - \int_{\mathbb{T}} Q'(\rho^\varepsilon) \phi_x \, dx + \varepsilon^2 \int_{\mathbb{T}} \rho_{xx}^\varepsilon \rho_x^\varepsilon \phi \, dx + \varepsilon^2 \int_{\mathbb{T}} \rho_{xx}^\varepsilon \rho^\varepsilon \phi_x \, dx \quad (5.43)$$

The first term converges to what we are looking for

$$\lim_{\varepsilon \rightarrow 0} - \int_{\mathbb{T}} Q'(\rho^\varepsilon) \phi_x dx = - \int_{\mathbb{T}} Q'(\rho_0) \phi_x dx,$$

by Lebesgue Dominated convergence and Proposition 3.5.2.

It remains to show that the last two terms in (5.43) go to zero. Integrating by parts again, we get

$$\varepsilon^2 \int_{\mathbb{T}} \rho_{xx}^\varepsilon \rho_x^\varepsilon \phi dx + \varepsilon^2 \int_{\mathbb{T}} \rho_{xx}^\varepsilon \rho^\varepsilon \phi_x dx = -\frac{3}{2} \varepsilon^2 \int_{\mathbb{T}} |\rho_x^\varepsilon|^2 \phi_x dx - \varepsilon^2 \int_{\mathbb{T}} \rho_x^\varepsilon \rho^\varepsilon \phi_{xx} dx$$

The first term goes to zero, by applying Proposition 3.5.2. The second term can be re-written as

$$\frac{1}{2} \varepsilon^2 \int_{\mathbb{T}} |\rho^\varepsilon|^2 \phi_{xxx} dx,$$

which goes to zero, because ρ^ε is in L^∞ uniformly. \square

3.6 Proof of Theorem 3.3.2

Proof. To begin with, we assume that $\liminf \mathcal{G}^\varepsilon(\rho^\varepsilon) < \infty$, otherwise there is nothing to prove. Therefore, up to relabeling, we can consider ρ^ε such that

$$\sup_{\varepsilon} \mathcal{F}^\varepsilon(\rho^\varepsilon) + \mathcal{G}^\varepsilon(\rho^\varepsilon) < \infty.$$

By Proposition 3.4.1, we know that, up to subsequence, $\rho^\varepsilon \rightharpoonup \rho_0$ weak-* L^∞ ; we define $\Omega = \{\rho_0 \notin \Sigma\}$. We start with the following bound: using Proposition 3.5.1, for any $K \subset \Omega$ compact, we have, for all ε small enough

$$\mathcal{G}^\varepsilon(\rho^\varepsilon)^2 \geq \int_{\{\rho^\varepsilon > 0\}} |(F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)|^2 \rho^\varepsilon dx \geq \int_K |(F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x|^2 \rho^\varepsilon dx.$$

Furthermore, we have

$$\int_K |(F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x|^2 \rho^\varepsilon dx \geq 2 \int_{\mathbb{T}} (F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x \phi \rho^\varepsilon dx - \int_{\mathbb{T}} \phi^2 \rho^\varepsilon dx \quad \text{for all } \phi \in \mathcal{D}(K),$$

which implies

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{G}^\varepsilon(\rho^\varepsilon)^2 \geq \liminf_{\varepsilon \rightarrow 0} \left[2 \int_{\mathbb{T}} (F'(\rho^\varepsilon) - \varepsilon^2 \rho_{xx}^\varepsilon)_x \phi \rho^\varepsilon dx - \int_{\mathbb{T}} \phi^2 \rho^\varepsilon dx \right] \quad \text{for all } \phi \in \mathcal{D}(K).$$

By Proposition 3.5.3, we deduce

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{G}^\varepsilon(\rho^\varepsilon)^2 \geq \sup_{\phi \in \mathcal{D}(K)} -2 \int_{\mathbb{T}} Q'(\rho_0) \phi_x dx - \int_{\mathbb{T}} \phi^2 \rho_0 dx.$$

By Proposition 3.5.2, and the lower semi continuity of the H^1 seminorm we also know that ρ_0 is in $H_{loc}^1(\Omega)$, so we can integrate by parts

$$\begin{aligned} \lim_\varepsilon \mathcal{G}^\varepsilon(\rho^\varepsilon)^2 &\geq \sup_{\phi \in \mathcal{D}(K)} 2 \int_{\mathbb{T}} Q'(\rho_0)_x \phi dx - \int_{\mathbb{T}} \phi^2 d\rho_0 dx \\ &= \sup_{\phi \in \mathcal{D}(K)} 2 \int_{\mathbb{T}} F'(\rho_0)_x \rho_0 \phi dx - \int_{\mathbb{T}} \phi^2 d\rho_0 dx \\ &= \|F'(\rho_0)_x\|_{L_{\rho_0}^2(K)}^2. \end{aligned}$$

Taking $K \rightarrow \Omega$ we obtain

$$\lim_\varepsilon |\mathcal{G}^\varepsilon|^2(\rho^\varepsilon) \geq \|F'(\rho_0)_x\|_{L_{\rho_0}^2(\Omega)}^2.$$

The rest of the proof is devoted to proving

$$\|F'(\rho_0)_x\|_{L_{\rho_0}^2(\Omega)}^2 = |\nabla \mathcal{F}^{**}|(\rho_0).$$

First, to have $\|F^{**'}(\rho_0)_x\|_{L_{\rho_0}^2(\mathbb{T})}^2 = |\nabla \mathcal{F}^{**}|(\rho_0)$, we need to prove that $Q^{**'} \in W^{1,1}$ (see (2.22)). We prove this by proving that

$$Q^{**'}(\rho_0)_x = Q^{**'}(\rho_0)_x 1_\Omega \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Since ρ^ε is continuous, if $\rho^\varepsilon(x) \in \Sigma_i$ and $\rho^\varepsilon(y) \in \Sigma_j$, there exists $z \in (x, y)$ such that $d(\rho^\varepsilon(z), \Sigma) \geq \inf_{i \neq j} \frac{d(\Sigma_i, \Sigma_j)}{2}$. By Corollary 3.4.4, we know that $d(\rho^\varepsilon, \Sigma)$ is uniformly lower semi continuous, therefore there exists δ_0 , independent of ε , such that $d(\rho^\varepsilon(t), \Sigma) > 0$, for any $t \in (z_0 - \delta_0, z_0 + \delta_0)$, then $|x - y| > 2\delta_0$. This implies that the sets $C_i = \{\rho_0 \in \Sigma_i\}$ are at a non zero distance from each other.

We define, as an auxiliary function, w in Σ by

$$w(x) = Q^{**'}(\Sigma_i) \text{ if } x \in C_i,$$

and we extend it to the whole of \mathbb{T} by linear interpolation. Since the sets C_i are separated, the function w is Lipschitz. Moreover, $Q^{**'}(\rho_0) = w$ in Ω^c , then for every $\phi \in \mathcal{D}(\mathbb{T})$

$$\int_{\mathbb{T}} (Q^{**'}(\rho_0) - w)\phi_x dx = \int_{\Omega} (Q^{**'}(\rho_0) - w)\phi_x dx.$$

Integrating by parts we have no boundary term, and so

$$\int_{\mathbb{T}} (Q^{**'}(\rho_0) - w)\phi_x dx = - \int_{\Omega} (Q^{**'}(\rho_0) - w)_x \phi dx.$$

Because w is Lipschitz and $w_x = 0$ in Σ , then

$$\int_{\Omega} w_x \phi dx = \int_{\mathbb{T}} w_x \phi dx = - \int_{\mathbb{T}} w \phi_x dx.$$

Therefore, we obtain that

$$\int_{\mathbb{T}} Q^{**'}(\rho_0)\phi_x dx = - \int_{\Omega} Q^{**'}(\rho_0)_x \phi dx,$$

for every $\phi \in \mathcal{D}(\mathbb{T})$.

Similarly, we can prove that $F^{**'}(\rho_0)_x = F'(\rho_0)_x \mathbb{1}_{\Omega^c}$, so we obtain the desired equality:

$$\|F'(\rho_0)_x\|_{L^2_{\rho_0}(\Omega)}^2 = |\nabla \mathcal{F}^{**}|(\rho_0) = \|F^{**'}(\rho_0)_x\|_{L^2_{\rho_0}(\mathbb{T})}^2.$$

□

3.7 Proof of Theorem 3.3.1

Proof. To be able to apply the framework developed by Sandier-Serfaty [66], we have to prove the compactness of the family ν^ε with respect to time.

Because the diameter of \mathbb{T} is finite we know that the diameter of $\mathcal{W}^2(\mathbb{T})$ is also finite, then

$$\nu^\varepsilon \in L^\infty([0, T]; \mathcal{W}^2(\mathbb{T})).$$

By the energy inequality (2.25), we know that

$$\int_0^T |\nu^{\varepsilon'}(t)|^2 dt$$

is uniformly bounded and therefore we know that

$$\nu^\varepsilon \text{ is uniformly bounded in } H^1((0, T); \mathcal{W}^2(\mathbb{T})).$$

By [78], we deduce that ν^ε is precompact in $L^2([0, T]; \mathcal{W}^2(\mathbb{T}))$, so up to a subsequence

$$\nu^\varepsilon \rightarrow \mu \text{ in } L^2([0, T]; \mathcal{W}^2(\mathbb{T})).$$

Also,

$$\int_0^T |\nu^{\varepsilon'}(t)|^2 dt = \sup_{h \in (0, T)} \int_0^{T-h} \frac{d_2(\nu^\varepsilon(t), \nu^\varepsilon(t+h))}{h} dt$$

is lower-semicontinuous with respect to the convergence in $L^2([0, T]; \mathcal{W}^2(\mathbb{T}))$, hence

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T |\nu^{\varepsilon'}(s)|^2 ds \geq \int_0^T |\mu'(s)|^2 ds. \quad (7.44)$$

Furthermore, by Arselà-Ascoli, we also know that up to a further subsequence,

$$\nu^\varepsilon \rightarrow \mu \text{ in } C^0([0, T]; \mathcal{W}^2(\mathbb{T})).$$

In particular,

$$\nu_i^\varepsilon \rightarrow \nu_i = \mu(0).$$

Now, we only have to follow the proof in [66] and obtain that μ is the gradient flow of \mathcal{F}^{**} with initial condition ν_i :

By equation (2.25), we know that

$$\mathcal{F}^\varepsilon[\nu_i^\varepsilon] - \mathcal{F}^\varepsilon[\nu^\varepsilon(t)] \geq \frac{1}{2} \int_0^t \mathcal{G}^\varepsilon(\nu^\varepsilon)^2 ds + \frac{1}{2} \int_0^t |\nu^{\varepsilon'}|^2 ds$$

Taking the limit $\varepsilon \rightarrow 0$, using Fatou's Lemma, Theorem 3.3.2 and (7.44) we get that

$$\liminf_{\varepsilon \rightarrow 0^+} (\mathcal{F}^\varepsilon[\nu_i^\varepsilon] - \mathcal{F}^\varepsilon[\nu^\varepsilon(t)]) \geq \frac{1}{2} \int_0^t |\nabla \mathcal{F}^{**}(\mu)|^2 ds + \frac{1}{2} \int_0^t |\mu'|^2 ds. \quad (7.45)$$

By Young's inequality, we know that

$$\frac{1}{2} \int_0^t |\nabla \mathcal{F}^{**}(\mu)|^2 ds + \frac{1}{2} \int_0^t |\mu'|^2 ds \geq \int_0^t |\nabla \mathcal{F}^{**}(\mu)| |\mu'| ds. \quad (7.46)$$

Because \mathcal{F}^{**} is convex with respect to the geodesics in $\mathcal{W}^2(\mathbb{T})$, we can apply Theorem 1.2.12 to obtain

$$\int_0^t |\nabla \mathcal{F}^{**}(\mu)| |\mu'| ds \geq \mathcal{F}^{**}[\mu(0)] - \mathcal{F}^{**}[\mu(t)]. \quad (7.47)$$

Since $\lim_\varepsilon \mathcal{F}^\varepsilon[\nu_i^\varepsilon] = \mathcal{F}^{**}[\nu_i] = \mathcal{F}^{**}[\mu(0)]$ by the well preparedness assumption,

(7.45), (7.46) (7.47) imply

$$\limsup \mathcal{F}^\varepsilon[\nu^\varepsilon(t)] \leq \mathcal{F}^{**}[\mu(t)].$$

The reverse inequality comes from the Γ -convergence of \mathcal{F}^ε to \mathcal{F}^{**} , so we have proven that

$$\lim \mathcal{F}^\varepsilon[\nu^\varepsilon(t)] = \mathcal{F}^{**}[\mu(t)],$$

and the inequalities (7.45), (7.46) (7.47) are in fact equalities. In particular (7.45) yields

$$\mathcal{F}^{**}[\nu_i] - \mathcal{F}^{**}[\mu(t)] = \frac{1}{2} \int_0^t |\nabla \mathcal{F}^{**}(\mu)|^2 ds + \frac{1}{2} \int_0^t |\mu'|^2 ds$$

for all $t > 0$. Finally, by Theorem 1.2.22, we deduce that $\nu_0 = \mu$. \square

3.8 Lower semi-continuity of \mathcal{G}^ε

In this section, we complete the proof of Proposition 3.2.6

Proof of Proposition 3.2.6(continuation). To lighten the notations, we give a proof for the case $\varepsilon = 1$, and drop the ε dependence.

The existence of solutions to (2.24) is proved by considering the uniform JKO scheme starting from ν_i , with step $\tau > 0$. Namely, we define inductively

$$\mu_\tau^0 = \nu_i^1, \quad \mu_\tau^{n+1} = \operatorname{argmin}_{\rho \in \mathcal{P}(\mathbb{T})} \{d_2^2(\mu_\tau^n, \rho) + 2\tau \mathcal{F}(\rho)\}. \quad (8.48)$$

Following the arguments made in Section 7.4 in [79], we know that the a part of the Euler-Lagrange condition associated to the minimization at every step of (8.48) is given by

$$\tau(-\Delta \mu_\tau^{n+1}) = \psi + C \quad \text{on } \{\mu_\tau^{n+1} > 0\},$$

where ψ is the associated optimal Kantorovich potential to the dual problem of optimal transport from μ_τ^{n+1} to μ_τ^n . From the classical theory of optimal transport, we know that $\frac{|x|^2}{2} + \psi(x)$ is convex, which means ψ is Lipschitz. Therefore, $\mu_\tau^{n+1} \in W_{loc}^{2,\infty}(\{\mu_\tau^{n+1} > 0\})$

The existence of a solution to (2.24) follows from Theorem 1 in [59], proven by defining ν as any accumulation point of the piecewise constant interpolation of $\{\mu_\tau^n\}_{n=0}^\infty$. Note that ν may not be unique, so we fix such a ν and a corresponding sequence of $\tau \rightarrow 0$, for which the constant interpolation of $\{\mu_\tau^n\}_{n=0}^\infty$ converges to ν .

Subsequently, we define the De Giorgi variational interpolation by

$$\bar{\mu}_\tau(t) = \operatorname{argmin}_{\rho \in \mathcal{P}(\mathbb{T})} \{d_2^2(\mu_\tau^n, \rho) + 2(t - (n-1)\tau)\mathcal{F}(\rho)\} \quad \text{when } t \in ((n-1)\tau, n\tau).$$

By Lemma 3.1.3 and 3.2.2 in [1], we know that \mathcal{F} is strongly subdifferentiable at $\bar{\mu}_\tau(t)$ for every $t > 0$ (see Definition 1.2.18), that $\bar{\mu}_\tau(t) \rightarrow \nu(t)$ for all $t \geq 0$, and that for every $n \in \mathbb{N}$,

$$\mathcal{F}(\mu_\tau^n) + \frac{1}{2\tau} \sum_{k=1}^n d_2^2(\mu_\tau^k, \mu_\tau^{k-1}) + \frac{1}{2} \int_0^{n\tau} |\partial\mathcal{F}(\bar{\mu}_\tau(t))|^2 dt \leq \mathcal{F}(\mu_\tau^0) = \mathcal{F}(\nu_i).$$

Moreover, by Proposition 4.1 in [59], we know that $\bar{\mu}_\tau(t) \in H^2$ for every $t > 0$. Therefore, because of the strong subdifferentiability we know that by Lemma 1.2.19 and Remark 3.2.2

$$\mathcal{G}(\bar{\mu}_\tau(t))^2 \leq |\partial\mathcal{F}(\bar{\mu}_\tau(t))|^2 = \int_{\mathbb{T}} |(F'(\bar{\mu}_\tau(t)) - \bar{\mu}_{\tau xx}(t))_x|^2 \bar{\mu}_\tau(t) dx \quad \forall t > 0.$$

We deduce

$$\mathcal{F}(\mu_\tau^n) + \frac{1}{2\tau} \sum_{k=1}^n d_2^2(\mu_\tau^k, \mu_\tau^{k-1}) + \frac{1}{2} \int_0^{n\tau} \mathcal{G}(\bar{\mu}_\tau(t))^2 dt \leq \mathcal{F}(\nu_i).$$

and the Energy inequality (2.25) follows by taking the limit $\tau \rightarrow 0$. More precisely, the metric derivative term in the Energy inequality (2.25) follows exactly as in the proof of Theorem 2.3.3 in [1]. The term involving \mathcal{G} follows from Fatou's Lemma and from the lower semicontinuity proven in Lemma 3.8.1 below, using that $\bar{\mu}_\tau(t) \in H^2(\mathbb{T}) \cap W_{loc}^{2,\infty}(\{\bar{\mu}_\tau(t) > 0\})$ for every $t > 0$ and that $\bar{\mu}_\tau(t) \rightarrow \nu(t)$ for every $t \geq 0$. \square

Lemma 3.8.1. *Given $\{\mu_n\}_{n \in \mathbb{N}}$, such that $\mu_n \in H^2 \cap W_{loc}^{2,\infty} \{\mu_n > 0\}$, $\sup_{n \in \mathbb{N}} |\mu_n|_{H^1} < C$ and that $\mu_n \rightarrow \mu$ in $\mathcal{W}^2(\mathbb{T})$, then*

$$\liminf_{n \rightarrow \infty} \mathcal{G}^\varepsilon(\mu_n) \geq \mathcal{G}^\varepsilon(\mu)$$

Proof of Lemma 3.8.1. Without loss of generality, we will assume that the potential $F = 0$ and $\varepsilon = 1$ and that $\liminf_{n \rightarrow \infty} \mathcal{G}(\mu_n) < \infty$. We can always take μ_{n_i} , such that

$$\lim_{i \rightarrow \infty} \mathcal{G}(\mu_{n_i}) = \liminf_{n \rightarrow \infty} \mathcal{G}(\mu_n) \text{ and } \sup_i \mathcal{G}(\mu_{n_i}) \leq C \text{ for some } C.$$

From now on, we drop the dependence on i .

Because the μ_n are probability measures, which are uniformly bounded in H^1 , we know that

$$\sup_n \|\mu_n\|_\infty < C.$$

Let $g_n \in \mathcal{T}(\mu_n)$ be such that $\|g_n\|_2 = \mathcal{G}(\mu_n)$ (see (2.22)) (we can always find such a g_n , because $\mathcal{T}(\mu_n)$ is closed) and by definition,

$$\int |G(\mu_n)_x|^2 \leq \|\mu_n\|_\infty \|g_n\|_2^2 = \|\mu_n\|_\infty \mathcal{G}(\mu_n)^2 \leq C.$$

Moreover, as

$$\int G(\mu_n) = \frac{3}{2} |\mu_n|_{H^1}^2 < C,$$

we conclude that

$$\sup_n \|G(\mu_n)\|_{H^1(\mathbb{T})} < C,$$

in particular $G(\mu_n)$ is bounded in L^∞ and in $C^{\frac{1}{2}}(\mathbb{T})$ uniformly in n .

Now, as $\mu_n \in H^2(\mathbb{T}) \subset C^{1,\frac{1}{2}}(\mathbb{T})$, we know that if $\mu_{nx}(x_0) \neq 0$, then $\mu_n(x_0) > 0$.

So, if x_0 is a max of $|\mu_{nx}|$, then, by the hypothesis that $\mu_n \in W_{loc}^{2,\infty} \{\mu_n > 0\}$, we

have enough regularity to assure that $\mu_{nxx}(x_0) = 0$. Then, we can bound

$$\|\mu_{nx}\|_\infty^2 = |\mu_{nx}(x_0)|^2 = 2G^\lambda(x_0) \leq 2\|G^\lambda\|_\infty \leq C.$$

Therefore,

$$\sup_n \|\mu_{nx}\|_\infty \leq C,$$

so we can conclude that μ is a Lipschitz function and

$$\|\mu\|_{Lip} \leq C.$$

Up to subsequence, we know that

$$G(\mu_n) \rightarrow H \text{ in } C^\alpha \text{ for all } \alpha < \frac{1}{2},$$

and

$$g_n \rightarrow g \text{ weakly in } L^2.$$

Because

$$G(\mu_n)_x = \sqrt{\mu_n} g_n$$

and $\mu_n \rightarrow \mu$ uniformly, we can pass to the limit in the sense of distributions to get

$$H_x = \sqrt{\mu} g.$$

Moreover, we know that

$$\|g\|_2 \leq \liminf \|g_n\|_2,$$

so it only remains to prove that

$$g \in \mathcal{T}(\mu),$$

or, equivalently, that

$$H = G(\mu).$$

The rest of the proof is devoted to proving this equality.

Because $\mu_n \in C_{loc}^3(\{\mu_n > 0\})$, we can use Remark 3.2.3 to obtain

$$\int_{\mu_n > 0} \mu_n |\mu_{nxxx}|^2 dx \leq \mathcal{G}(\mu_n),$$

then we have, up to subsequence,

$$\mu_n \rightarrow \mu \in C^2(\{\mu > \lambda\}).$$

Therefore,

$$G(\mu) = H \text{ in } \{\mu > 0\}.$$

Of course, the set $\{\mu = 0\}$ requires a more delicate argument.

First, we prove that $G(\mu) = 0$ a.e. in $\{\mu = 0\}$. By rewriting

$$G(\mu_n) = - \left(\frac{\mu_n^2}{2} \right)_{xx} + \frac{3}{2} |\mu_{nx}|^2, \tag{8.49}$$

and using the fact that μ_n is uniformly Lipschitz, then we can say that

$$\sup_n \|\mu_n^2\|_{W^{2,\infty}} < C,$$

therefore

$$\mu^2 \in W^{2,\infty}.$$

Stampacchia's lemma states that if $f \in W^{1,p}$, then $f_x = 0$ a.e. in $\{f = 0\}$ (See Lemma A.4. Chapter II [60]), hence $G(\mu) = 0$ a.e. in $\{\mu = 0\}$.

Therefore, we only need to show that $H = 0$ a.e. in $\{\mu = 0\}$. Instead, we prove something seemingly stronger, more specifically, we prove that if x_0 is such

that $|H(x_0)| = \delta \neq 0$ and $\mu(x_0) = 0$, then there exists a non-trivial interval (a_0, b_0) around x_0 such that $H = G(\mu)$ in (a_0, b_0) . The rest of the proof is devoted to proving this last statement.

Let x_0 be such that $|H(x_0)| = \delta \neq 0$ and $\mu_0(x_0) = 0$, then since $G(\mu_n)$ converges to H uniformly there exists n_0 , such that $n > n_0$ implies

$$|G(\mu_n)(x_0)| \geq \frac{\delta}{2}.$$

Given $\beta > 0$, to be chosen later, we consider the open sets

$$A_\beta^n = \{x : \mu_n < \beta\},$$

and

$$A_\beta^\infty = \{x : \mu < \beta\} = \cup_i (a_i^\beta, b_i^\beta),$$

written as the union of its connected components. From now on, we suppress the dependence on β on the end points of the intervals.

Since $x_0 \in A_\beta^\infty$, there exists a unique i_0 which we take to be 0, such that

$$x_0 \in (a_0, b_0),$$

and

$$\mu(a_0) = \mu(b_0) = \beta.$$

As $\mu_n \rightarrow \mu$ uniformly, then for all n big enough

$$(a_0, b_0) \subset A_{2\beta}^n.$$

and

$$\mu_n(a_0) > \frac{\beta}{2}, \quad \mu_n(b_0) > \frac{\beta}{2}.$$

Using the definition of $\mathcal{G}(\mu_n)$, we can bound the oscillations of $G(\mu_n)$ in the set

$$\text{osc}_{(a_0, b_0)} G(\mu_n) \leq \int_{a_0}^{b_0} |G(\mu_n)_x| \leq \mathcal{G}(\mu_n)^{\frac{1}{2}} \left(\int_{a_0}^{b_0} \mu_n \right)^{\frac{1}{2}} \leq C\sqrt{2\beta}.$$

Then,

$$|G(\mu_n)| \geq G(\mu_n)(x_0) - C\sqrt{2\beta} \quad \text{in } (a_0, b_0).$$

By taking β small enough, we deduce that

$$|G(\mu_n)| \geq \kappa > 0 \quad \text{in } (a_0, b_0).$$

If μ_n would vanish at any point in (a_0, b_0) , it would contradict the hypothesis that $\mu_n \in H^2$. We prove this by contradiction, if assume that μ_n vanishes at $y_0 \in (a_0, b_0)$, then, because $\mu_n \in C^{1, \frac{1}{2}}$, $\mu_{nx}(y_0) = 0$ and there exists ε_0 such that

$$|\mu_{nx}(x)|^2 < \kappa \quad \text{for every } x \in (y_0 - \varepsilon_0, y_0 + \varepsilon_0).$$

Therefore,

$$|\mu_n(x)\mu_{nxx}(x)| \geq |G(\mu_n)| - \frac{|\mu_{nx}|^2}{2} \geq \frac{\kappa}{2} \quad \text{for every } x \in (y_0 - \varepsilon_0, y_0 + \varepsilon_0).$$

Finally, using that μ_n is Lipschitz and $\mu_n(y_0) = 0$ we know that

$$\mu_n(x) \leq C(x - y_0).$$

We deduce that

$$|\mu_{nxx}(x)|^2 \geq \frac{C}{(x - y_0)^2} \quad \text{for every } x \in (y_0 - \varepsilon_0, y_0 + \varepsilon_0),$$

which is not integrable at y_0 and thus contradicts the fact that μ_n is in H^2 . So, we can conclude that

$$\mu_n > 0 \quad \text{in } (a_0, b_0).$$

Now we apply Theorem 4.3 of Bernis' integral inequalities [80], which proves that

$$|v|_{W^{2,3}(a_0,b_0)} \leq \int_{a_0}^{b_0} v |v_{xxx}|^2 dx$$

for $v \in C^3$, with $v > 0$ in (a_0, b_0) , such that $v'(a_0) = v'(b_0) = 0$.

Note that we cannot use this result directly because we do not know that $\mu'_n(a_0) = \mu'_n(b_0) = 0$. So, to be able to apply the Theorem, we consider ϕ_n smooth, such that

- $\phi_n(a_0) = \phi_n(b_0) = \frac{\beta}{4}$.
- $\phi'_n(a_0) = \mu_n(a_0)$, $\phi'_n(b_0) = \mu_n(b_0)$.
- $\phi_n < \max\left(0, \frac{\beta}{2} - |\mu_n|_{lip}(x - a_0), \frac{\beta}{2} - |\mu_n|_{lip}(b_0 - x)\right) < \mu_n$.

Because $\|\mu_{nx}\|_\infty$ is uniformly bounded, we get that ϕ_n is uniformly bounded in $W^{2,3}$ and H^3 . Moreover, $v_n = \mu_n - \phi_n$ satisfies the hypothesis of [80], then

$$|v_n|_{W^{2,3}(a_0,b_0)} \leq \int_{a_0}^{b_0} (\mu_n - \phi_n) |(\mu_n - \phi_n)_{xxx}|^2 dx \leq \int_{a_0}^{b_0} \mu_n |\mu_{nxxx}|^2 dx + |\mu_n|_\infty |\phi_n|_{H^3}.$$

Also,

$$|\mu_n|_{W^{2,3}(a_0,b_0)} \leq |v_n|_{W^{2,3}(a_0,b_0)} + |\phi_n|_{W^{2,3}(a_0,b_0)},$$

so we finally deduce the following uniform bound for μ_n in the interval (a_0, b_0) :

$$\sup_n \|\mu_n\|_{W^{2,3}(a_0,b_0)} < C.$$

This implies, in particular, that up to subsequence, μ_n converges to μ uniformly in $C^{1,\alpha}$ and so

$$\mu_{nx} \rightarrow \mu_x \text{ uniformly in } (a_0, b_0),$$

which combined with the fact that

$$(\mu_n^2)_{xx} \rightarrow (\mu^2)_{xx} \text{ in } \mathcal{D}',$$

yields, by passing to the limit in (8.49)

$$G(\mu_n) \rightarrow G(\mu) \text{ in } (a_0, b_0).$$

So, in particular

$$H = G(\mu) \text{ in } (a_0, b_0).$$

□

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