Analysis of Asymptotic Solutions for Cusp Problems in Capillarity

by

Yasunori Aoki

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, Canada, 2007

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Yasunori Aoki

Abstract

The capillary surface $u(x,y)$ near a cusp region satisfies the boundary value problem:

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \kappa u \qquad \text{in } \{(x,y) : 0 < x, f_2(x) < y < f_1(x)\}\,, \tag{1}$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma_1 \qquad \text{on } y = f_1(x)\,, \tag{2}$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma_2 \qquad \text{on } y = f_2(x)\,, \tag{3}$$

where $\lim_{x\to 0} f_1(x), f_2(x) = 0$, $\lim_{x\to 0} f_1'(x), f_2'(x) = 0$.

It is shown that the capillary surface is unbounded at the cusp and satisfies $u(x,y) = O\left(\frac{1}{f_1(x)-f_2(x)}\right)$, even for types of cusp not investigated previously (e.g. exponential cusps).

By using a tangent cylinder coordinate system, we show that the exact solution $v(x,y)$ of the boundary value problem:

$$\nabla \cdot \frac{\nabla v}{|\nabla v|} = \kappa v \qquad \text{in } \{(x,y) : 0 < x, f_2(x) < y < f_1(x)\}\,, \tag{4}$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|} = \cos\gamma_1 \qquad \text{on } y = f_1(x)\,, \tag{5}$$

$$\nu \cdot \frac{\nabla v}{|\nabla v|} = \cos\gamma_2 \qquad \text{on } y = f_2(x)\,, \tag{6}$$

exhibits sixth order asymptotic accuracy to the capillary equations (1)$-$(3) near a circular cusp.

Finally, we show that the solution is bounded and can be defined to be continuous at a symmetric cusp ($f_1(x) = -f_2(x)$) with the supplementary contact angles ($\gamma_2 = \pi - \gamma_1$). Also it is shown that the solution surface is of the order $O\left(f_1(x)\right)$, and moreover, the formal asymptotic series for a symmetric circular cusp region is derived.

Acknowledgements

I would like to thank my thesis supervisor Dr. David Siegel. He has given me much useful advice and a lot of encouragement. He has shown me the way mathematicians are expanding the knowledge of mankind and taught me to think like a mathematician. I have learned from him not only mathematics but also have learned many important qualities as a person, such as the importance of patience and hard work.

I would like to thank Dr. John Wainwright. He has gone out of his way to help me reorganize my thesis and made it significantly more readable and more understandable. Also I will never forget the fact he has initiated my interest in the area of partial differential equations and encouraged me to pursue an academic career.

I would like to thank Dr. Marek Stastna for being on the thesis examination committee.

I would like to thank the University of Waterloo, and the Department of Applied Mathematics for support during both graduate and undergraduate studies.

I would like to thank many anonymous individuals who have helped and supported me throughout my life and my academic career.

I would like to thank my father for giving me support and encouragement in every aspect of my life even after his passing.

I would like to thank my mother and my sister for their encouragement and for understanding my desire to stay in school for much longer than "usual".

To my father, Masanori Aoki.

Contents

List of Figures

Chapter 1

Introduction and Background

1.1 Introduction

Even a glass of water can amuse a person with eyes for careful observation. Although we often assume the liquid surface to be flat, this is rarely the case. The curling up of the water near the glass is an example of a capillary phenomenon. In fact, wherever there is a liquid and a solid boundary, we will find an interesting capillary surface. The capillary problem is one of the many examples of everyday phenomena, which through careful analysis leads us to very interesting mathematical and physical discoveries. Initial interest and the origin of word “capillarity” came from a discovery of rise of water in “capillaris” (hair-like in Latin) tube (see Figure 1.1). Careful observation of this phenomenon by Laplace and Young led to the second order non-linear elliptic partial differential boundary value problem. This mathematical model has been shown to be not only valid as a model of the height of the liquid in capillary tubes, but also of many other geometrical boundaries. Some people may claim that the experimental method is sufficient to understand capillary surfaces, however, in the same way as Finn’s careful mathematical analysis of the capillary surfaces near a corner has lead to the discovery of a critical angle for the corner region (refer to section 1.4.1), careful

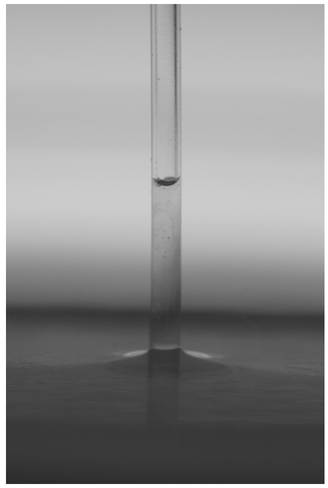

Figure 1.1: Capillary Tube Experiment (inner radius approximately 0.35mm)

analysis of the mathematical models can suggest discoveries of unknown experimental phenomena. Also, as our technology evolves, our scale of precision increases from milli-meter to micro-meter to even nano-meter, for example in the process of metal plating semiconductor chips. In the same way, extremely precise understanding of the height of a liquid surface becomes critical in these fields. On a much larger scale, surface tension forces (the cause of capillary phenomena) are significantly less than the gravitational force, and as a result we often see very flat liquid surfaces. As the gravitational force decreases, however, surface tension forces become more significant, and hence capillary phenomena become more dramatic. As Concus and Finn began their interests in capillarity in the design of a fuel tank for a spaceship, knowledge in capillarity has significant importance even in space science.

Capillarity problems are not only of interest to physicists but also to mathematicians. The capillary boundary value problem leads to one of the simplest second order non-linear elliptic partial differential equations with meaningful physical interpretation. Simple does not necessary mean easy, especially in the field of mathematics. Indeed, it has proved very difficult to get an exact solution of a capillarity problem. As a matter of a fact "[t]here is only one explicitly known solution"([8], page 480). Only the one dimensional capillarity problem has a known exact solution, "[it] can be interpreted physically as the height of a capillary surface on one side of an infinite vertical plate"([8], page 480). Hence careful study of this equation especially the development of new approximation techniques may later become useful to partial differential equations in the other fields.

As the wedge effect (refer to section 1.4.1) has initiated my interest in capillarity, my research is focused on the local analysis of capillary surfaces near cusp domains. Following the path developed by Finn, Miersemann, and Scholz, I was able to make a modest addition to the field, by extending the results of Scholz, finding a new technique to acquire the leading order term, and providing some answers to an open problem.

1.2 The Capillary Boundary Value Problem

In this section we will derive the mathematical model for a capillary surface. This model can be categorized as a boundary value problem for a second order non-linear elliptic partial differential equation.

1.2.1 Derivation of the Mathematical Model

We derive the mathematical model for a non-parametric capillary surface using the so called energy method, following from [1] (see pages 4-6). In this derivation, we consider three different energies associated with a capillary surface in a region bounded with vertical solid boundary walls. These three energies are "Free Surface Energy", "Wetting Energy" and "Gravitational Energy". The Free Surface Energy is associated with the interface between the liquid and the air. The Wetting Energy is associated with the interface between the air and the solid boundary, and the interface between the

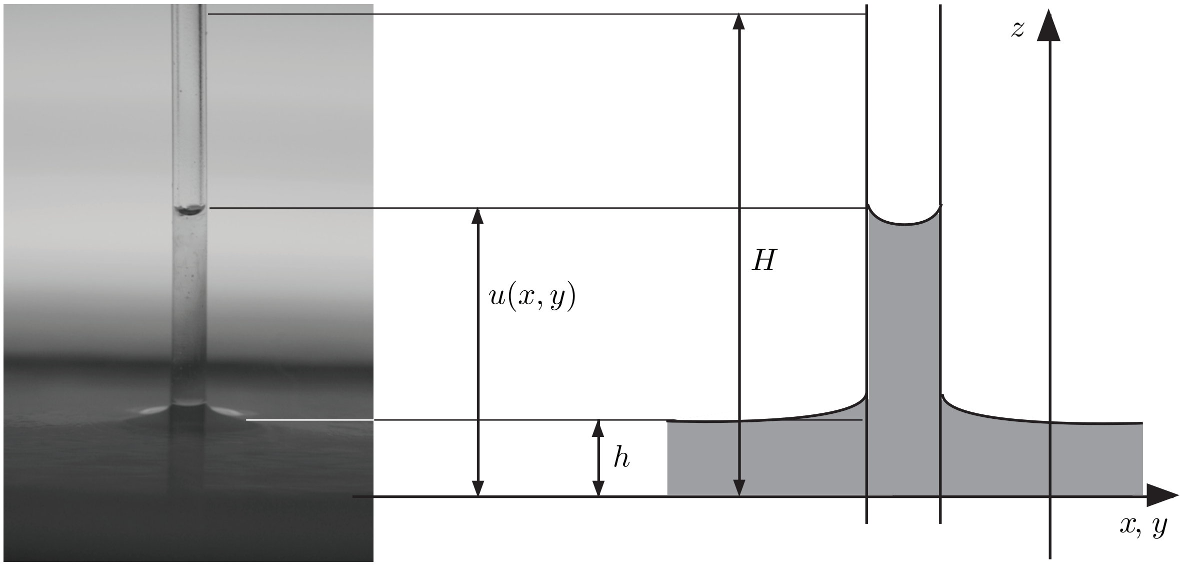

Figure 1.2: Derivation of the Capillary Equation.

liquid and the solid boundary. The Gravitational Energy is associated with the density difference between the liquid and the air and the height of the liquid. By minimizing the sum of these energies we aim to find a stable equilibrium capillary surface. Let $u(x, y)$ be the height of the liquid surface in the domain Ω. Also define the lower reference height to be h and upper reference height to be H, as labeled in Figure 1.2. Assume constant density of liquid (ρ_{liq}) and air (ρ_{air}), and the liquid to have higher density than the air ($\rho_{air} < \rho_{liq}$). Also assume the gravitational acceleration g to act in the downward direction and to be constant.

The energies associated with this physical configuration are then given by the following expressions.

Gravitational Energy:

$$\begin{aligned} E_{grav}(u) &= \int\int_{\Omega} \left(\int_h^u \rho_{liq} g z dz \right) dxdy + \int\int_{\Omega} \left(\int_u^H \rho_{air} g z dz \right) dxdy \\ &= \rho_{liq} g \int\int_{\Omega} \left(\frac{u^2 - h^2}{2} \right) dxdy + \rho_{air} g \int\int_{\Omega} \left(\frac{H^2 - u^2}{2} \right) dxdy \end{aligned} \quad (1.1)$$

Wetting Energy (between air and solid):

$$E_{as}(u) \quad = \quad \sigma_{as} \int_{\partial\Omega} (H - u) ds \tag{1.2}$$

Wetting Energy (between liquid and solid):

$$E_{ls}(u) \quad = \quad \sigma_{ls} \int_{\partial\Omega} (u - h) ds \tag{1.3}$$

Free Surface Energy (between air and liquid):

$$E_{al}(u) \quad = \quad \sigma_{al} \int \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx dy \tag{1.4}$$

where the σ's are areal surface energy densities.

The total energy (E) is then given by

$$E(u) \quad = \quad E_{grav} + E_{as} + E_{ls} + E_{al} \,. \tag{1.5}$$

We aim to find the height of the liquid surface $u(x, y)$ such that the energy $E(u)$ is a minimum, i.e. find $u(x, y)$ such that*

$$\frac{\partial}{\partial \epsilon} E(u(x, y) + \epsilon \eta(x, y)) \bigg|_{\epsilon=0} \quad = \quad 0 \,, \tag{1.6}$$

$$\frac{\partial^2}{\partial \epsilon^2} E(u(x, y) + \epsilon \eta(x, y)) \bigg|_{\epsilon=0} \quad > \quad 0 \,, \tag{1.7}$$

*Second variation is not required for the derivation of the PDE nor BC. We have kept this condition to show that we are actually finding local minimum and not maximum.

where $\eta(x,y)$ is an arbitrary function. Now take the derivative of each term with respect to ϵ and set $\epsilon = 0$:

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} E_{grav}(u+\epsilon\eta) &= \frac{\partial}{\partial \epsilon}\left(\rho_{liq} g \int\int_{\Omega} \left(\frac{(u+\epsilon\eta)^2 - h^2}{2}\right) dxdy \right. && (1.8)\\
&\quad \left. + \rho_{air} g \int\int_{\Omega} \left(\frac{H^2 - (u+\epsilon\eta)^2}{2}\right) dxdy \right) \\
&= \rho_{liq} g \int\int_{\Omega} \left((u+\epsilon\eta)\,\eta\right) dxdy + \rho_{air} g \int\int_{\Omega} \left(-(u+\epsilon\eta)\,\eta\right) dxdy \\
\left.\frac{\partial}{\partial \epsilon} E_{grav}(u+\epsilon\eta)\right|_{\epsilon=0} &= (\rho_{liq} - \rho_{air}) g \int\int_{\Omega} u\eta dxdy && (1.9)
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial}{\partial \epsilon}\left(E_{as} + E_{ls}\right)\right|_{\epsilon=0} &= \left.\frac{\partial}{\partial \epsilon}\left(\sigma_{as}\int_{\partial\Omega}(H - u - \epsilon\eta)dS + \sigma_{ls}\int_{\partial\Omega}(u + \epsilon\eta - h)dS\right)\right|_{\epsilon=0} \\
&= (\sigma_{ls} - \sigma_{as})\int_{\partial\Omega} \eta dS && (1.10)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} E_{al}(u+\epsilon\eta) &= \frac{\partial}{\partial \epsilon}\left(\sigma_{al}\int\int_{\Omega}\sqrt{1 + |\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\eta + \epsilon^2|\nabla\eta|^2}dxdy\right) \\
&= \sigma_{al}\int\int_{\Omega}\frac{\nabla u\cdot\nabla\eta + \epsilon|\nabla\eta|^2}{\sqrt{1 + |\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\eta + \epsilon^2|\nabla\eta|^2}}dxdy && (1.11)\\
\left.\frac{\partial}{\partial \epsilon} E_{al}(u+\epsilon\eta)\right|_{\epsilon=0} &= \sigma_{al}\int\int_{\Omega}\frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}dxdy && (1.12)\\
&= \sigma_{al}\int\int_{\Omega} Tu\cdot\nabla\eta\, dxdy && (1.13)
\end{aligned}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,. \qquad (1.14)$$

Consider

$$\begin{aligned}
\nabla\cdot(\eta Tu) &= \eta\nabla\cdot Tu + \nabla\eta\cdot Tu && (1.15)\\
\nabla\eta\cdot Tu &= \nabla\cdot(\eta Tu) - \eta\nabla\cdot Tu && (1.16)
\end{aligned}$$

$$\frac{\partial}{\partial \epsilon} E_{al}(u+\epsilon\eta)\bigg|_{\epsilon=0} = \sigma_{al} \int\int_{\Omega} \left(\nabla \cdot (\eta T u) - \eta \nabla \cdot T u\right) dxdy \tag{1.17}$$

$$= \sigma_{al} \int_{\partial\Omega} \eta T u \cdot \nu ds - \sigma_{al} \int\int_{\Omega} \eta \nabla \cdot T u dxdy\,, \tag{1.18}$$

where ν is the unit outward normal to the boundary of the region Ω.

Hence by equation (1.6),

$$\begin{aligned}\frac{\partial}{\partial \epsilon} E(u+\epsilon\eta)\bigg|_{\epsilon=0} &= (\rho_{liq} - \rho_{air}) g \int\int_{\Omega} u\eta dxdy + (\sigma_{ls} - \sigma_{as}) \int_{\partial\Omega} \eta ds \\ &\quad + \sigma_{al} \int_{\partial\Omega} \eta T u \cdot \nu ds - \sigma_{al} \int\int_{\Omega} \eta \nabla \cdot T u dxdy \\ &= 0\,. \end{aligned} \tag{1.19}$$

Also we can verify that $u(x,y)$ is minimum by calculating the second derivative of $E(u+\epsilon\eta)$ with respect to ϵ.

$$\frac{\partial^2}{\partial \epsilon^2} E_{grav}(u+\epsilon\eta) = (\rho_{liq} - \rho_{air})\, g \int\int_{\Omega} \eta^2 dxdy \tag{1.20}$$

$$> 0 \tag{1.21}$$

$$\frac{\partial^2}{\partial \epsilon^2} E_{as}(u+\epsilon\eta) = 0 \tag{1.22}$$

$$\frac{\partial^2}{\partial \epsilon^2} E_{ls}(u+\epsilon\eta) = 0 \tag{1.23}$$

$$\begin{aligned}\frac{\partial^2}{\partial \epsilon^2} E_{al}(u+\epsilon\eta) &= \sigma_{al} \int\int_{\Omega} \frac{|\nabla\eta|^2}{\sqrt{1+|\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\eta + \epsilon^2|\nabla\eta|^2}} dxdy \\ &\quad - \sigma_{al} \int\int_{\Omega} \frac{\left(\nabla u\cdot\nabla\eta + \epsilon|\nabla\eta|^2\right)^2}{\left(1+|\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\eta + \epsilon^2|\nabla\eta|^2\right)^{3/2}} dxdy \\ &= \sigma_{al} \int\int_{\Omega} \frac{|\nabla\eta|^2 + |\nabla\eta|^2\,|\nabla u|^2 - |\nabla u\cdot\nabla\eta|^2}{\left(1+|\nabla u|^2 + 2\epsilon\nabla u\cdot\nabla\eta + \epsilon^2|\nabla\eta|^2\right)^{3/2}} dxdy \end{aligned} \tag{1.24}$$

$$> 0 \tag{1.25}$$

Since $\eta(x, y)$ is an arbitrary function, and equation (1.19) has to be satisfied for any $\eta(x, y)$, we can choose $\eta = 0$ on $\partial\Omega$, obtaining

$$\int\int_{\Omega} [(\rho_{liq} - \rho_{air})gu - \sigma_{al}\nabla \cdot Tu]\, \eta\, dxdy \quad = \quad 0\,. \tag{1.26}$$

Equation (1.26) has to be satisfied for any η, which satisfies the condition $\eta = 0$ on $\partial\Omega$. This implies

$$(\rho_{liq} - \rho_{air})gu - \sigma_{al}\nabla \cdot Tu \quad = \quad 0\,. \tag{1.27}$$

We rewrite this equation in the form

$$\nabla \cdot Tu \quad = \quad \kappa u\,, \tag{1.28}$$

where

$$\kappa = \frac{\rho_{liq} - \rho_{air}}{\sigma_{al}} g\,, \tag{1.29}$$

and T is given by

$$Tu \quad = \quad \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\,. \tag{1.30}$$

The partial differential equation (1.28) is called the *capillary PDE* and constant κ is called the *capillary constant*.

Now assume equation (1.28) holds, and consider $\eta(x, y)$ without the restriction on

$\partial\Omega$. By equation (1.19)

$$(\sigma_{ls} - \sigma_{as}) \int_{\partial\Omega} \eta ds + \sigma_{al} \int_{\partial\Omega} \eta Tu \cdot \nu ds \quad = \quad 0 \tag{1.31}$$

$$\int_{\partial\Omega} \eta \left((\sigma_{ls} - \sigma_{as}) + \sigma_{al} Tu \cdot \nu \right) ds \quad = \quad 0 \tag{1.32}$$

for any η. This implies

$$Tu \cdot \nu \quad = \quad -\frac{(\sigma_{ls} - \sigma_{as})}{\sigma_{al}} \tag{1.33}$$

Define the downward unit normal of the capillary surface $u(x, y)$ (see Figure 1.3) to be ξ, i.e.

$$\xi \quad = \quad \frac{(u_x, u_y, -1)}{\sqrt{1 + |\nabla u|^2}}. \tag{1.34}$$

It follows that

$$\xi \cdot (\nu, 0) \quad = \quad \frac{(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})}{\sqrt{1 + |\nabla u|^2}} \cdot \nu \tag{1.35}$$

$$= \quad Tu \cdot \nu \tag{1.36}$$

$$= \quad -\frac{(\sigma_{ls} - \sigma_{as})}{\sigma_{al}}. \tag{1.37}$$

Since both ξ and $(\nu, 0)$ are unit vectors, $\xi \cdot (\nu, 0) = \cos\gamma$, where γ is the angle between two vectors. By equation (1.37)

$$\cos\gamma \quad = \quad -\frac{\sigma_{ls} - \sigma_{as}}{\sigma_{al}}. \tag{1.38}$$

We shall refer to γ as the *contact angle*. Note that γ only depends on the σ's, which depend only on the physical characteristics of the solid boundary, the liquid and the

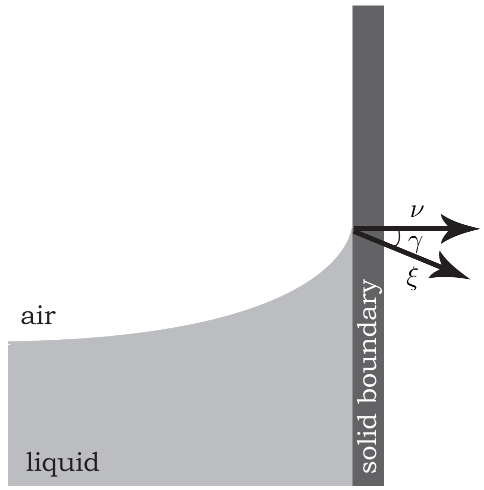

Figure 1.3: Contact Angle Boundary Condition

air. It follows from equations (1.33) and (1.38) that

$$Tu \cdot \nu = \cos\gamma \qquad \text{on } \partial\Omega\,, \tag{1.39}$$

which is called the *contact angle boundary condition*.

1.2.2 Statement of the Boundary Value Problem

We now summarize the discussion of the previous section.

The height $u(x, y)$ of a non-parametric capillary surface in a region Ω satisfies

$$\nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \kappa u \qquad \text{for } (x,y) \in \Omega\,, \tag{1.40}$$

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\gamma \qquad \text{for } (x,y) \in \partial\Omega\,. \tag{1.41}$$

<u>Remarks:</u>

1) for simplicity, we often use the differential operator T, which is defined as

$$Tu := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} . \tag{1.42}$$

2) By re-scaling the variables according to $u = \frac{1}{\sqrt{\kappa}}\tilde{u}$, $x = \frac{1}{\sqrt{\kappa}}\tilde{x}$, $y = \frac{1}{\sqrt{\kappa}}\tilde{y}$ we can normalize the capillary constant to be unity. We can now state the final form of the Capillary BVP, which we will use in the rest of this thesis.

<u>Capillary BVP</u>

The height $u(x, y)$ of a non-parametric capillary surface in a region Ω satisfies

$$\nabla \cdot Tu = u \qquad \text{for } (x,y) \in \Omega , \tag{1.43}$$

$$\nu \cdot Tu = \cos\gamma \qquad \text{for } (x,y) \in \partial\Omega , \tag{1.44}$$

where

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} ,$$

γ is the contact angle,

ν is the unit outward normal vector on $\partial\Omega$.

3) For further use we note that if u is a solution of the capillary BVP (1.43)-(1.44), then $\bar{u}(x,y) = -u(x,y)$ is a solution of the BVP[†]

$$\nabla \cdot T\bar{u} = \bar{u}\,, \tag{1.45}$$

$$\nu \cdot T\bar{u} = -\cos\gamma\,. \tag{1.46}$$

1.3 The Comparison Principle

Concus Finn Comparison Principle is one of the most useful tools in capillarity. As the name suggests this principle can be used to prove one capillary surface is above or below an another capillary surface.

Theorem 1.1 (The Comparison Principle) *Consider a domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. Suppose $\partial\Omega$ admits a decomposition*

$$\partial\Omega = \Sigma^0 \cup \Sigma^\alpha \cup \Sigma^\beta\,, \tag{1.47}$$

where

Σ^0 *is an union of finite number of points on the boundary,*

Σ^α *is an union of finite number of continuous curves on the boundary,*

Σ^β *is an union of finite number of C^1 curves on the boundary.*

[†]Even though T is a nonlinear operator, it satisfies $T(-u) = -T(u)$, as follows immediately from equation (1.42).

If $u, v \in C^2(\Omega)$ *satisfies*

$$\nabla \cdot Tu - \kappa u \geq \nabla \cdot Tv - \kappa v \qquad \text{for } (x,y) \in \Omega\,, \tag{1.48}$$

$$u \leq v \qquad \text{on } \Sigma^\alpha\,, \tag{1.49}$$

$$\nu \cdot Tu \leq \nu \cdot Tv \qquad \text{on } \Sigma^\beta\,, \tag{1.50}$$

then

$$u(x,y) \leq v(x,y) \qquad \text{for } (x,y) \in \Omega\,. \tag{1.51}$$

Proof: See [1], page 110 for the case where Ω is bounded, and Finn and Hwang [2] for the proof for the case Ω unbounded.

Remark: It follows immediately from the comparison principle that the capillary BVP has a unique solution. Since Finn and Hwang have extended the comparison principle to unbounded domains, uniqueness can be proven even for infinite domains. This result is surprising, because in order to ensure that a BVP for a second order elliptic PDE in an infinite domain has a unique solution one usually has to impose a growth condition at infinity.

We now apply the comparison principle to a domain with a corner or a cusp. This result can be directly applied to prove that a function is a sub-solution or a super-solution for the capillary BVP, and will be used many times in the later chapters.

Proposition 1.1 (Application of the Comparison Principle) *We consider a region* Ω *given by*

$$\Omega = \{(x,y) : 0 < x, f_1(x) < y < f_2(x)\}\,, \tag{1.52}$$

where $f_1(0) = f_2(0) = 0$, $f_1(x), f_2(x) \in C^1$. *Assume this infinite domain has vertical* *cylindrical walls with contact angle* γ *on the boundary* $\partial\Omega$. *Let* $u(x,y)$ *be the solution to the capillary BVP in* Ω, *and* $v(x,y)$ *be a comparison function. If*

$$\begin{aligned}
\nabla \cdot Tv - v &\gtreqless 0 \qquad \textit{for } (x,y) \in \Omega_{x_0}\,, && (1.53)\\
\frac{(-f_1'(x),1)}{\sqrt{1+f_1'(x)^2}} \cdot Tv(x,f_1(x)) &\lesseqgtr \cos\gamma \qquad \textit{for } 0 < x < x_0\,, && (1.54)\\
-\frac{(-f_2'(x),1)}{\sqrt{1+f_2'(x)^2}} \cdot Tv(x,f_2(x)) &\lesseqgtr \cos\gamma \qquad \textit{for } 0 < x < x_0\,, && (1.55)\\
v(x_0,y) &\lesseqgtr u(x_0,y) \qquad \textit{for } f_2(x_0) < y < f_2(x_0)\,, && (1.56)
\end{aligned}$$

where

$$\Omega_{x_0} = \{(x,y) : 0 < x < x_0, f_1(x) < y < f_2(x)\}\,, \qquad (1.57)$$

then

$$v(x,y) \lesseqgtr u(x,y) \qquad \textit{in } \Omega_{x_0}. \qquad (1.58)$$

Remark: We shall refer to the comparison function $v(x,y)$ as $\left\{ \begin{array}{c} \text{a sub-solution} \\ \text{a super-solution} \end{array} \right\}$ of the capillary BVP.

Proof: We apply the comparison principle with

$$\begin{aligned}
\Sigma^\alpha &= \{(x_0,y)\} \qquad \text{for } f_1(x_0) < y < f_2(x_0)\,, && (1.59)\\
\Sigma^\beta &= \{(x,f_1(x)) \cup (x,f_2(x))\} \qquad \text{for } 0 < x < x_0\,, && (1.60)\\
\Sigma^0 &= \{(0,0) \cup (x_0,f_1(x_0)) \cup (x_0,f_2(x_0))\}\,. && (1.61)
\end{aligned}$$

It follows from equations (1.53)-(1.56) that

$$\nabla \cdot Tv - v \;\geqslant\; 0 \;=\; \nabla Tu - u \qquad (x,y) \in \Omega_{x_0}\,, \tag{1.62}$$

$$\frac{(-f_1'(x),1)}{\sqrt{1+f_1'(x)^2}} \cdot Tv(x,f_1(x)) \;\leqslant\; \cos\gamma \;=\; \nu \cdot Tu \qquad \text{for } 0<x<x_0\,, \tag{1.63}$$

$$-\frac{(-f_2'(x),1)}{\sqrt{1+f_2'(x)^2}} \cdot Tv(x,f_2(x)) \;\leqslant\; \cos\gamma \;=\; \nu \cdot Tu \qquad \text{for } 0<x<x_0\,, \tag{1.64}$$

$$v(x_0,y) \;\leqslant\; u(x_0,y) \qquad \text{for } f_2(x_0)<y<f_2(x_0)\,. \tag{1.65}$$

Hence by the comparison principle,

$$v(x,y) \leqslant u(x,y) \qquad \text{for } (x,y) \in \Omega_{x_0}\,. \tag{1.66}$$

Theorem 1.2 (Upper-bound Principle) *Let $u(x,y)$ be a capillary BVP in Ω. Let $B_\delta(x_0,y_0)$ be a disk of radius δ and center (x_0,y_0). If $B_\delta(x_0,y_0) \subset \Omega$, then*

$$u(x,y) \;<\; \frac{2}{\delta} + \delta \qquad \textit{in } B_\delta. \tag{1.67}$$

Proof:

Choose δ', $0<\delta'<\delta$, let $B_{\delta'}$ be a disk of radius δ', concentric to B_δ, and let $[z = v'(x,y)]$ denote a lower hemisphere over $B_{\delta'}$, whose lowest point has the height $v_0' = 2/\delta'$ ([1], page 114).

$$v'(x,y) \;=\; -\sqrt{\delta'^2 - ((x-x_0)^2+(y-y_0)^2)} + \frac{2}{\delta'} + \delta'\,. \tag{1.68}$$

It follows immediately from equation (1.68) that

$$\frac{2}{\delta'} \;\le\; v'(x,y) \;\le\; \frac{2}{\delta'} + \delta' \qquad \text{for } (x,y) \in B_\delta'(x_0,y_0)\,. \tag{1.69}$$

After some calculation we get

$$\begin{aligned} \nabla \cdot Tv &= \frac{2}{\delta'} \qquad \text{for } (x,y) \in B_{\delta'}(x_0,y_0)\,, && (1.70)\\ \Rightarrow \qquad \nabla \cdot Tv - \kappa v &\le 0 = \nabla \cdot Tu\,, && (1.71)\\ \nu \cdot Tv &= 1 \qquad \text{for } (x,y) \in \partial B_{\delta'}(x_0,y_0)\,, && (1.72)\\ &> \nu \cdot Tu\,. && (1.73) \end{aligned}$$

We now apply the comparison principle (Theorem 1.1) with

$$\begin{aligned} \Sigma^0 &= \emptyset\,, && (1.74)\\ \Sigma^\alpha &= \emptyset\,, && (1.75)\\ \Sigma^\beta &= \partial\Omega\,. && (1.76) \end{aligned}$$

Hence

$$u(x,y) < v'(x,y) \le \frac{2}{\delta'} + \delta'\,. \qquad (1.77)$$

By letting $\delta' \to \delta$, we obtain

$$u(x,y) < \frac{2}{\delta} + \delta\,. \qquad (1.78)$$

■

Corollary 1.1 *Let $u(x,y)$ be the solution of the capillary BVP in Ω. Let B_δ be a disk of radius δ, with $B_\delta \subset \Omega$. If $\delta > \sqrt{2}$, then*

$$u(x,y) < 2\sqrt{2} \qquad (1.79)$$

where $(x, y) \in B_\delta$.

Proof: Given $\delta > \sqrt{2}$, for any point $(x_0, y_0) \in B_\delta$ we can construct a disk of radius $\sqrt{2}$ such that

$$(x_0, y_0) \in B_{\sqrt{2}} \subset B_\delta \,. \tag{1.80}$$

By Theorem 1.2,

$$u(x_0, y_0) \quad < \quad 2\sqrt{2} \,. \tag{1.81}$$

■

Note: $\delta = \sqrt{2}$ gives the smallest upper-bound for the height, i.e.

$$\min_{0<\delta} \left(\frac{2}{\delta} + \delta \right) \quad = \quad 2\sqrt{2} \,. \tag{1.82}$$

Theorem 1.3 (Existence Theorem) *Let* Ω *be a domain with a piecewise smooth boundary* Σ*. Let* $\Sigma = \Sigma^0 \cup \Sigma^*$*, where* Σ^0 *has one dimensional measure zero,* Σ^* *is open in* Σ *and* $\Sigma^* \in C^4$*. Suppose that on* Σ^**, the contact angle* γ *is piecewise smooth and satisfies* $0 < \gamma < \pi$*. Then there exists a solution* $u(x, y) \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma^*)$ *to the capillary BVP. (see[8], page 475)*

Note: The domain Ω can be bounded or unbounded.

1.4 Corners and Cusps

In this section I briefly mention previously known results concerning capillary problems near corners and cusps. Refer to each referenced paper and book for a detailed discussion and proof.

1.4.1 Finn's Critical Angle Condition and Leading Order Behaviour at a Corner Singularity

Finn's analysis of capillary surface near corners were the first attempt at a local analysis near a corner region (see [1], pages 115-122). It led to a very unexpected result: "Capillary surfaces in a domain with corner depend discontinuously on the boundary data" ([1], page 119).

Theorem 1.4 (Critical angle for a wedge problem) *Let $u(x, y)$ be a solution to the capillary BVP with capillary constant κ and contact angle γ in a wedge region Ω with opening angle α, given by*

$$\Omega = \{(r, \theta) : 0 < r, -\alpha < \theta < \alpha\}, \tag{1.83}$$

in polar coordinates.

i) If $\alpha + \gamma \geq \pi/2$, then

$$u(r, \theta) < \frac{2}{\kappa\delta} + \delta \qquad \text{for } (r, \theta) \in \Omega \cap B_\delta(\delta, 0). \tag{1.84}$$

ii) If $\alpha + \gamma < \pi/2$, then there exists constants A and r_0 such that

$$\left| u(r, \theta) - \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{kr} \right| < A \qquad (r, \theta) \in \Omega_{r_0}, \tag{1.85}$$

where $k = \sin\alpha / \cos\gamma$, $\Omega_{r_0} = \{(r, \theta) : 0 < r < r_0, -\alpha < \theta < \alpha\}$.

<u>Comments:</u>

i) According to this theorem, the angle $\alpha = \pi/2 - \gamma$ is a critical angle such that the behaviour of the capillary surface changes dramatically depending on the opening angle of the wedge. For $\alpha < \pi/2 - \gamma$ capillary surface is unbounded, and for $\alpha \geq \pi/2 - \gamma$

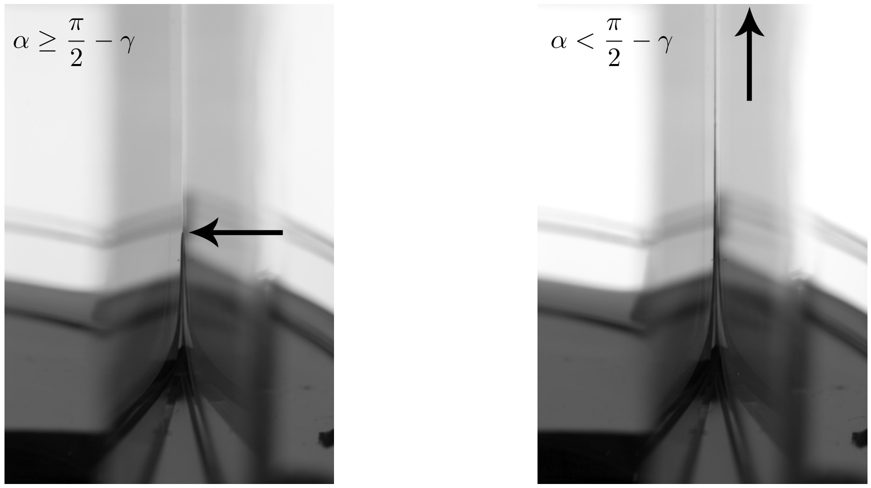

Figure 1.4: Wedge Experiment: a bounded capillary surface v.s. an unbounded capillary surface

it is bounded. This phenomenon has been demonstrated with an experiment. See the photograph in Figure 1.4 for an illustration of the experiment.

ii) The proof of the unboundedness for $\alpha + \gamma < \pi/2$ can be modified and used to prove the unboundedness of the capillary surface at a cusp (see section 2.1).

1.4.2 Miersemann's Asymptotic Series at a Corner Singularity

Following from Finn's leading order approximation of the capillary surface near a corner with $\alpha < \pi/2 - \gamma$, Miersemann has found a complete asymptotic series for this problem (see [4]).

Theorem 1.5 *Let $u(r, \theta)$ be the solution to the capillary BVP with constant contact angle γ, in a wedge region Ω with opening angle α, given by*

$$\Omega = \{(r, \theta) : 0 < r, -\alpha < \theta < \alpha\}, \tag{1.86}$$

where $\alpha < \pi/2 - \gamma$. Then

for a given non negative integer m there exist positive constants r_0, A and

$m+1$ functions $h_{4l-1}(\theta)$, $l = 0, ..., m$, analytic on $(-\alpha, \alpha)$ and bounded on $[-\alpha, \alpha]$, such that

$$\left| u(r,\theta) - \sum_{l=0}^{m} h_{4l-1}(\theta) r^{4l-1} \right| \leq A r^{4m+3} \qquad \text{in } \Omega_{r_0}, \tag{1.87}$$

where $\Omega_{r_0} = \{(r,\theta) : 0 < r < r_0, -\alpha < \theta < \alpha\}$ ([4], page 97).

1.4.3 Scholz's Asymptotic Series at a Cusp Singularity

Following Miersemann's work, his student Scholz applied a similar technique to find a complete asymptotic series solution in a cusp region (see [6] and [7]). The complete asymptotic series solution was found in a region Ω given by

$$\Omega = \{(x,y) : 0 < x, f_2(x) < y < f_1(x)\}, \tag{1.88}$$

where

$$f_1(x) = a_1 x^n + b_1 x^{n+1} + O(x^{n+2}), \tag{1.89}$$

$$f_2(x) = a_2 x^n + b_2 x^{n+1} + O(x^{n+2}). \tag{1.90}$$

Here $n \in \mathbb{N}$, $n > 1$, $a_i, b_i \in \mathbb{R}$. Also he found the leading order term for a more general cusp region of the form equation (1.88), where

$$f_1(x) = a_1 x^\alpha, \tag{1.91}$$

$$f_2(x) = a_2 x^\alpha, \tag{1.92}$$

with $\alpha > 1$, $a_i, b_i \in \mathbb{R}$, $a_1 > a_2$.

Chapter 2

Capillary Surface near a General Cusp

In this chapter, we consider capillary surfaces in regions with two cylindrical walls, which form a general shape. We aim to allow these domains to be as general as possible. However, some results in this chapter require minor restrictions on the shape of the domain.

We consider an unbounded region Ω with two boundary walls forming a cusp. We

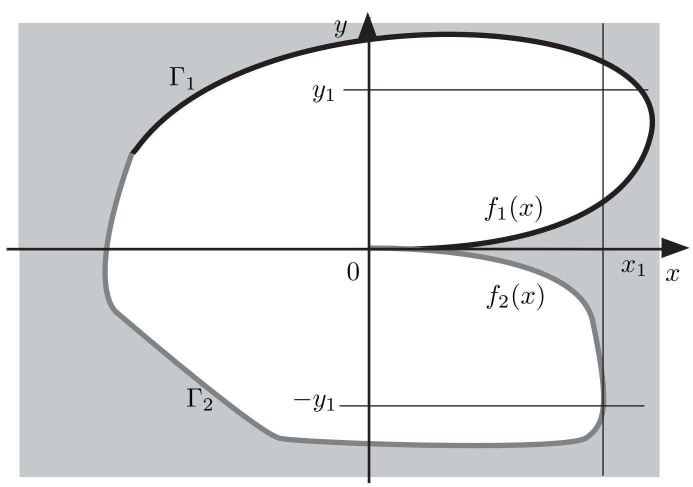

Figure 2.1: Region Ω

shall refer to each boundary as Γ_1 and Γ_2 with

$$\partial\Omega = \Sigma^0 \cup \Gamma_1 \cup \Gamma_2 \,, \tag{2.1}$$

where Σ^0 is one dimensional measure zero, $\Gamma_1, \Gamma_2 \in C^4$. We assume there is a cusp at the origin opening in the positive x-direction as in Figure 2.1. Consider a rectangle

$$R = \{(x,y) \in \mathbb{R}^2 : 0 < x < x_1,\, -y_1 < y < y_1\} \,, \tag{2.2}$$

and let

$$\Omega_1 = \Omega \cap R \,. \tag{2.3}$$

We choose x_1 and y_1 small enough that the subset of the boundary in Ω_1, i.e. $\Gamma_1 \cap R$ and $\Gamma_2 \cap R$ are given by

$$y = f_1(x) \qquad \text{for } 0 < x < x_1 \,, \tag{2.4}$$

$$y = f_2(x) \qquad \text{for } 0 < x < x_1 \,, \tag{2.5}$$

respectively, where $f_1(x)$ and $f_2(x)$ are smooth functions such that

$$\lim_{x \to 0^+} f_1(x) = 0 \,, \tag{2.6}$$

$$\lim_{x \to 0^+} f_2(x) = 0 \,, \tag{2.7}$$

$$\lim_{x \to 0^+} f_1'(x) = 0 \,, \tag{2.8}$$

$$\lim_{x \to 0^+} f_2'(x) = 0 \,. \tag{2.9}$$

We shall refer to Ω_1 as a *near cusp region.* In this thesis we assume homogeneity of the material constructing each wall. Let the contact angle of the wall Γ_1 be γ_1 and the contact angle of the wall Γ_2 be γ_2 $(0 < \gamma_1, \gamma_2 < \pi)$, then the boundary condition of the capillary BVP assumes the form

$$\nu \cdot Tu|_{y=f_1(x)} = \cos\gamma_1 \qquad \text{for } 0 < x < x_1\,, \tag{2.10}$$

$$\nu \cdot Tu|_{y=f_2(x)} = \cos\gamma_2 \qquad \text{for } 0 < x < x_1\,. \tag{2.11}$$

Also in this chapter we assume,

$$\gamma_1 \neq \pi - \gamma_2\,, \tag{2.12}$$

i.e.

$$\cos\gamma_1 + \cos\gamma_2 \neq 0\,. \tag{2.13}$$

We will address the case $\gamma_1 = \pi - \gamma_2$, i.e. $\cos\gamma_1 + \cos\gamma_2 = 0$, in Chapter 4.

Example 2.1 (Cusp) *The following regions Ω_1 and Ω_2 are cusp regions :*

$$\Omega_1 = \left\{(x,y) \in \mathbb{R}^2 \setminus (B_1(0,1) \cup B_2(0,-2))\right\}\,, \tag{2.14}$$

$$\Omega_2 = \left\{(x,y) : 0 < x < \infty,\ -be^{-\frac{1}{x^2}} < y < ae^{-\frac{1}{x^2}}\right\}\,, \tag{2.15}$$

where a and b are constants.

Note: The capillary surface in region Ω_1 can be asymptotically approximated using a result of Scholz [7]. However, this result cannot be applied to a capillary surface in a region such as Ω_2.

2.1 Unboundedness at the cusp

Theorem 2.1 (Unboundedness of the capillary surface at a cusp) *Let* $u(x, y)$ *be the solution to the capillary BVP in region* Ω*, with boundary conditions* (2.10) *and* (2.11)*, subject to* $\gamma_1 \neq \pi - \gamma_2$*, then the capillary surface* $u(x, y)$ *is unbounded at the cusp,* $(x, y) = (0, 0)$.

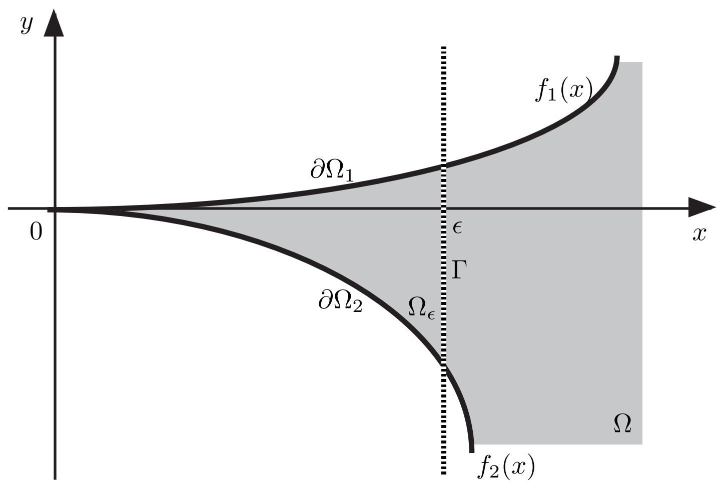

Figure 2.2: Region Near a Cusp

Proof: First consider the case $\cos\gamma_1 + \cos\gamma_2 > 0$.

We prove the result by contradiction. Assume there exists a constant $M > 0$ such that

$$u(x, y) \;<\; M \qquad \text{for } (x, y) \in \Omega\,. \tag{2.16}$$

Since $u(x, y)$ is a capillary surface, it satisfies the capillary PDE (1.43),

$$\nabla \cdot Tu \;=\; u\,. \tag{2.17}$$

Integrate both sides of equation (2.17) in a region $\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < x < \epsilon \leq$

$x_1, f_2(x) < y < f_1(x)\}$, where ϵ is a constant, to obtain

$$\int_{\Omega_\epsilon} \nabla \cdot Tu \, dxdy \;=\; \int_{\Omega_\epsilon} u \, dxdy \,. \tag{2.18}$$

By the divergence theorem,

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \;=\; \int_0^\epsilon \int_{f_2(x)}^{f_1(x)} u \, dydx \,, \tag{2.19}$$

where $\partial\Omega_\epsilon$ is the boundary of the region Ω_ϵ, and ν is the unit outward normal at the boundary. The boundary $\partial\Omega$ can be divided into three subsets such that

$$\partial\Omega_\epsilon \;=\; \partial\Omega_1 \cup \partial\Omega_2 \cup \Gamma \,, \tag{2.20}$$

where

$$\partial\Omega_1 \;=\; \{(x,y) \in \mathbb{R}^2 : 0 < x < \epsilon, y = f_1(x)\} \,, \tag{2.21}$$

$$\partial\Omega_2 \;=\; \{(x,y) \in \mathbb{R}^2 : 0 < x < \epsilon, y = f_2(x)\} \,, \tag{2.22}$$

$$\Gamma \;=\; \{(x,y) \in \mathbb{R}^2 : x = \epsilon, f_2(\epsilon) < y < f_1(\epsilon)\} \,. \tag{2.23}$$

So the left-hand-side of equation (2.19) can be written

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \;=\; \int_{\partial\Omega_1} Tu \cdot \nu \, ds + \int_{\partial\Omega_2} Tu \cdot \nu \, ds + \int_\Gamma Tu \cdot \nu \, ds \,. \tag{2.24}$$

By the boundary conditions (2.10) and (2.11),

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \;=\; \int_{\partial\Omega_1} \cos\gamma_1 \, ds + \int_{\partial\Omega_2} \cos\gamma_2 \, ds + \int_\Gamma Tu \cdot \nu \, ds \,. \tag{2.25}$$

Since

$$|Tu \cdot \nu| \leq |Tu|\,|\nu| = |Tu| \,, \tag{2.26}$$

$$= \frac{\sqrt{u_x^2 + u_y^2}}{\sqrt{1 + u_x^2 + u_y^2}} < 1 \,, \tag{2.27}$$

we have

$$-1 < Tu \cdot \nu < 1 \,. \tag{2.28}$$

Hence

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds > \cos\gamma_1 \int_{\partial\Omega_1} ds + \cos\gamma_2 \int_{\partial\Omega_2} ds - \int_{\Gamma} ds \,, \tag{2.29}$$

$$= \cos\gamma_1 |\partial\Omega_1| + \cos\gamma_2 |\partial\Omega_2| - \left(f_1(\epsilon) - f_2(\epsilon)\right) \,, \tag{2.30}$$

where $|\partial\Omega_1|$ and $|\partial\Omega_2|$ are the lengths of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. These can be calculated using the Cauchy Mean Value Theorem as

$$|\partial\Omega_1| = \int_0^\epsilon \sqrt{1 + f_1'^2(x)}\, dx \,, \tag{2.31}$$

$$= \sqrt{1 + f_1'(c)^2}\, \epsilon \qquad 0 < c < \epsilon \,, \tag{2.32}$$

$$|\partial\Omega_2| = \int_0^\epsilon \sqrt{1 + f_2'^2(x)}\, dx \,, \tag{2.33}$$

$$= \sqrt{1 + f_2'(d)^2}\, \epsilon \qquad 0 < d < \epsilon \,. \tag{2.34}$$

Substitute these expressions into equation (2.30) gives

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, dS > \cos\gamma_1 \sqrt{1 + f_1'(c)^2}\, \epsilon + \cos\gamma_2 \sqrt{1 + f_2'(d)^2}\, \epsilon - \left(f_1(\epsilon) - f_2(\epsilon)\right) \,. \tag{2.35}$$

Now consider the right-hand-side of equation (2.19). By the assumption (2.16),

$$\int_0^{\epsilon}\int_{f_2(x)}^{f_1(x)} u\,dydx \;<\; \int_0^{\epsilon}\int_{f_2(x)}^{f_1(x)} M\,dydx\,, \tag{2.36}$$

$$= M\int_0^{\epsilon}\int_{f_2(x)}^{f_1(x)} dydx\,, \tag{2.37}$$

$$\leq M\int_0^{\epsilon}\max_{0<x\leq\epsilon}\left(f_1(x)-f_2(x)\right)dx\,, \tag{2.38}$$

$$= \epsilon M\max_{0<x\leq\epsilon}\left(f_1(x)-f_2(x)\right) \tag{2.39}$$

Now substitute equations (2.35) and (2.39) into equation (2.19), and rearrange to obtain

$$\cos\gamma_1\sqrt{1+f_1'(c)^2}+\cos\gamma_2\sqrt{1+f_2'(d)^2}$$
$$<\; M\max_{0<x\leq\epsilon}\left(f_1(x)-f_2(x)\right)+\frac{\left(f_1(\epsilon)-f_2(\epsilon)\right)}{\epsilon}\,. \tag{2.40}$$

Taking the limit as $\epsilon\to 0$ of both sides, using equation (2.6)-(2.9) gives

$$\cos\gamma_1+\cos\gamma_2 \;<\; 0\,. \tag{2.41}$$

Since we are considering the case $\cos\gamma_1+\cos\gamma_2>0$ the assumption (2.16) is contradicted. Hence the solution surface $u(x,y)$ is not bounded above.

Next consider the case

$$\cos\gamma_1+\cos\gamma_2 \;<\; 0\,. \tag{2.42}$$

Similarly we can deduce a contradiction, by assuming there exists a constant $m<0$ such that,

$$m \;<\; u(x,y)\,. \tag{2.43}$$

Thus we have shown that, if $\cos\gamma_1 + \cos\gamma_2 > 0$, we cannot bound the solution surface $u(x, y)$ from above, and if $\cos\gamma_1 + \cos\gamma_2 < 0$ we cannot bound the solution surface $u(x, y)$ from below. Hence the solution surface $u(x, y)$ is unbounded if $\gamma_1 \neq \pi - \gamma_2$.

■

2.2 Growth Order and Formal Asymptotic Series

We now aim to give an approximation for the capillary surface in a region Ω. We would like to first prove the growth order of the solution using the comparison principle and then derive a formal asymptotic series, which satisfies the capillary equation and the boundary conditions asymptotically.

Motivation

By using the formulae from the proof of Theorem 2.1, we aim to estimate $u(x, y)$. We base this estimate on the assumption that there is a single variable function $v(x)$ such that the limit of the ratio between $u(x, y)$ and $v(x)$ equals one, i.e.

$$u(x, y) \quad \sim \quad v(x) \qquad \text{as } x \to 0. \tag{2.44}$$

From equation (2.19)

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \quad \sim \quad \int_0^\epsilon \int_{f_2(x)}^{f_1(x)} v(x)\, dy dx\,, \tag{2.45}$$

$$= \quad \int_0^\epsilon (f_1(x) - f_2(x))\, v(x) dx\,. \tag{2.46}$$

From equation (2.25)

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \quad = \quad \cos\gamma_1|\partial\Omega_1| + \cos\gamma_2|\partial\Omega_2| + \int_\Gamma Tu \cdot \nu \, dS\,. \tag{2.47}$$

As in the derivation of equation (2.30)

$$\left| \int_{\Gamma} Tu \cdot \nu \, dS \right| \quad < \quad f_1(\epsilon) - f_2(\epsilon) \, . \tag{2.48}$$

Assuming that $\cos\gamma_1 \, |\partial\Omega_1| + \cos\gamma_2 \, |\partial\Omega_2| \neq 0$, it follows from equations (2.6)-(2.9)

$$\frac{|f_1(\epsilon) - f_2(\epsilon)|}{\cos\gamma_1 \, |\partial\Omega_1| + \cos\gamma_2 \, |\partial\Omega_2|} \quad \to \quad 0 \qquad \text{as } \epsilon \to 0 \, , \tag{2.49}$$

$$\Rightarrow \qquad |\cos\gamma_1|\partial\Omega_1| + \cos\gamma_2|\partial\Omega_2|| \quad >> \quad \left| \int_{\Gamma} Tu \cdot \nu \, dS \right| \qquad \text{as } \epsilon \to 0 \, .$$

Hence it follows from equation (2.47) that

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \quad \sim \quad \cos\gamma_1 |\partial\Omega_1| + \cos\gamma_2 |\partial\Omega_2|. \tag{2.50}$$

Substituting equations (2.31) and (2.33) into equation (2.50) yields

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \quad \sim \quad \cos\gamma_1 \int_0^\epsilon \sqrt{1 + f_1'^2(x)} \, dx + \cos\gamma_2 \int_0^\epsilon \sqrt{1 + f_2'^2(x)} \, dx \, , \tag{2.51}$$

$$= \quad \int_0^\epsilon \left[\cos\gamma_1 \sqrt{1 + f_1'^2(x)} + \cos\gamma_2 \sqrt{1 + f_2'^2(x)} \right] \, dx \, . \tag{2.52}$$

Using Taylor series and equations (2.6)-(2.9),

$$\int_{\partial\Omega_\epsilon} Tu \cdot \nu \, ds \quad \sim \quad \int_0^\epsilon \left[\cos\gamma_1 \left(1 + \frac{1}{2} f_1'^2(x) \right) + \cos\gamma_2 \left(1 + \frac{1}{2} f_2'^2(x) \right) \right] \, dx \, , \tag{2.53}$$

$$\sim \quad \int_0^\epsilon \left(\cos\gamma_1 + \cos\gamma_2 \right) \, dx \, . \tag{2.54}$$

Substituting this result into equation (2.46) gives

$$\int_0^\epsilon \cos\gamma_1 + \cos\gamma_2 \, dx \quad \sim \quad \int_0^\epsilon \left(f_1(x) - f_2(x) \right) v(x) dx \, .$$

Since ϵ is an arbitrarily chosen small constant, it follows that

$$\begin{aligned} \cos\gamma_1 + \cos\gamma_2 &\sim (f_1(x) - f_2(x))\, v(x)\,, \\ \frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)} &\sim v(x)\,. \end{aligned} \tag{2.55}$$

Hence on recalling equation (2.44), we postulate the leading order of the asymptotic solution to the capillary BVP in Ω to be

$$\frac{\cos\gamma_1 + \cos\gamma_2}{f_1(x) - f_2(x)}\,. \tag{2.56}$$

Note: The analysis leading to equation (2.56) is heuristic in nature.

We now give a proof using the comparison principle to show that this approximation does in fact have the correct asymptotic order. We now give a formal statement of the result.

Theorem 2.2 (Growth Order of a capillary surface near a cusp) *Let $u(x, y)$ be a solution of the capillary BVP in Ω:*

$$\nabla \cdot Tu = u \qquad in\ \Omega\,, \tag{2.57}$$

$$\nu \cdot Tu = \cos\gamma_1 \qquad on\ \Gamma_1\,, \tag{2.58}$$

$$\nu \cdot Tu = \cos\gamma_2 \qquad on\ \Gamma_2\,. \tag{2.59}$$

If $f_1(x)$ *and* $f_2(x)$ *as defined in equations* (2.4) *and* (2.5) *satisfy*

$$f_1(x) - f_2(x) = o(f_1'(x) - f_2'(x)) \qquad \text{as } x \to 0\,, \tag{2.60}$$

$$f_1''(x) - f_2''(x) = o\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right) \qquad \text{as } x \to 0\,, \tag{2.61}$$

$$f_1'''(x) - f_2'''(x) = o\left(\frac{f_1'(x) - f_2'(x)}{(f_1(x) - f_2(x))^2}\right) \qquad \text{as } x \to 0\,, \tag{2.62}$$

then there exist constants A_+, A_- *and* $x_0 < x_1$ *such that*

$$\frac{A_-}{f_1(x) - f_2(x)} < u(x,y) < \frac{A_+}{f_1(x) - f_2(x)} \qquad \text{for } x < x_0, \tag{2.63}$$

where

$$A_-, A_+ > 0 \qquad \text{if} \quad \cos\gamma_1 + \cos\gamma_2 > 0\,, \tag{2.64}$$

$$A_-, A_+ < 0 \qquad \text{if} \quad \cos\gamma_1 + \cos\gamma_2 < 0\,. \tag{2.65}$$

Proof: Introduce new coordinates s and t according to

$$s := x\,, \tag{2.66}$$

$$t := \frac{2y - (f_1(x) + f_2(x))}{f_1(x) - f_2(x)}\,, \tag{2.67}$$

then the domain Ω_1 as defined in equation (2.3) becomes

$$\Omega_1 = \{(s,t) : 0 < s < x_1\,, -1 < t < 1\}\,. \tag{2.68}$$

We first assume $\cos\gamma_1 + \cos\gamma_2 > 0$. Let

$$v(s,t;A,g(t),C_0) = \frac{A}{f_1(s) - f_2(s)} + g(t)\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + C_0\,, \tag{2.69}$$

where A and C_0 are constants and $g(t)$ is a C^2 function. After lengthy calculations (see Appendix A.1) we obtain

$$\nabla\cdot Tv - v = \left(\frac{4g''(t)A^2}{\left(A^2+4g'^2(t)\right)^{3/2}} - A\right)\frac{1}{f_1(s)-f_2(s)} - C_0 + R(s,t;A,g(t))\,, \tag{2.70}$$

$$\nu\cdot Tv|_{t=1} = \frac{2g'(1)}{\sqrt{A^2+4g'^2(1)}} + o(1) \qquad \text{as } s\to 0, \tag{2.71}$$

$$\nu\cdot Tv|_{t=-1} = -\frac{2g'(-1)}{\sqrt{A^2+4g'^2(-1)}} + o(1) \qquad \text{as } s\to 0, \tag{2.72}$$

where $R(s,t) = o\left(\frac{1}{f_1(s)-f_2(s)}\right)$ as $s \to 0$. We now aim to construct a super-solution $v_+(s,t;A,g(t))$ and a sub-solution $v_-(s,t;A,g(t))$ by choosing the constant A and the function $g(t)$ to satisfy the following conditions:

$$\frac{4g''(t)A^2}{\left(A^2+4g'^2(t)\right)^{3/2}} - A = -KA\,, \tag{2.73}$$

$$\frac{2g'(1)}{\sqrt{A^2+4g'^2(1)}} = \cos\gamma_1 + K\,, \tag{2.74}$$

$$-\frac{2g'(-1)}{\sqrt{A^2+4g'^2(-1)}} = \cos\gamma_2 + K\,, \tag{2.75}$$

where K is an arbitrary constant. Substituting equations (2.73)-(2.75) into equations (2.70)-(2.72) gives

$$\nabla\cdot Tv - v = -\frac{KA}{f_1(s)-f_2(s)} - C_0 + R(s,t;A,g(t))\,, \tag{2.76}$$

$$\nu\cdot Tv|_{t=1} = \cos\gamma_1 + K + o(1) \qquad \text{as } s\to 0, \tag{2.77}$$

$$\nu\cdot Tv|_{t=-1} = \cos\gamma_2 + K + o(1) \qquad \text{as } s\to 0. \tag{2.78}$$

We can solve equation (2.73) by separation of variables obtaining

$$\frac{4g'(t)}{A\sqrt{A^2+4g'^2(t)}} = (1-K)t+k_1\,, \tag{2.79}$$

where k_1 is an arbitrary constant of integration. Substituting equation (2.79) into equations (2.74) and (2.75) gives

$$\frac{2g'(1)}{\sqrt{A^2+4g'^2(1)}} = \frac{A}{2}\left((1-K)(1)+k_1\right) = \cos\gamma_1+K\,, \tag{2.80}$$

$$\frac{2g'(-1)}{\sqrt{A^2+4g'^2(-1)}} = \frac{A}{2}\left((1-K)(-1)+k_1\right) = -\cos\gamma_2-K\,. \tag{2.81}$$

Solving these two equations for k_1 and A gives

$$A = \frac{\cos\gamma_1+\cos\gamma_2+2K}{1-K}\,, \tag{2.82}$$

$$k_1 = \frac{(1-K)(\cos\gamma_1-\cos\gamma_2)}{\cos\gamma_1+\cos\gamma_2+2K}\,. \tag{2.83}$$

In order to have $A>0$, we restrict the choice of K by

$$-\frac{\cos\gamma_1+\cos\gamma_2}{2} < K < 1\,. \tag{2.84}$$

Solving equation (2.79) for $g'(s)$ gives

$$g'(t) = \frac{A^2((1-K)t+k_1)}{2\sqrt{4-A^2((1-K)t+k_1)^2}}\,, \tag{2.85}$$

$$g(t) = -\frac{\sqrt{4-A^2((1-K)t+k_1)^2}}{2(1-K)}+C_1\,. \tag{2.86}$$

Substituting equation (2.82) and (2.83) into equation (2.86) gives

$$g(t) = -\frac{\sqrt{4-((\cos\gamma_1+\cos\gamma_2+2K)t+(\cos\gamma_1-\cos\gamma_2))^2}}{2(1-K)}+C_1 \tag{2.87}$$

where C_1 is an arbitrary constant of integration. Here we choose $C_1 = 0$. In order to ensure that $g(t)$ is C^2 on $-1 < t < 1$, K must be chosen so that the expression inside the square root is non-negative, i.e.

$$((\cos\gamma_1 + \cos\gamma_2 + 2K)t + (\cos\gamma_1 - \cos\gamma_2))^2 \leq 4\,, \qquad \text{for } -1 \leq t \leq 1\,. \tag{2.88}$$

Since $(\cos\gamma_1 + \cos\gamma_2 + 2K)t + (\cos\gamma_1 - \cos\gamma_2)$ is linear, if equation (2.88)is satisfied at both $t = -1$ and $t = 1$, then it will be satisfied in the whole domain. Hence the required restriction on K becomes

$$|\cos\gamma_1 + K| \leq 1\,, \tag{2.89}$$

$$|\cos\gamma_2 + K| \leq 1\,. \tag{2.90}$$

Solving these inequalities give

$$-1 - \cos\gamma_1 \leq K \leq 1 - \cos\gamma_1\,, \tag{2.91}$$

$$-1 - \cos\gamma_2 \leq K \leq 1 - \cos\gamma_2\,. \tag{2.92}$$

Choosing K to satisfy

$$-\min\{1 - |\cos\gamma_1|, 1 - |\cos\gamma_2|\} \leq K \leq \min\{1 - |\cos\gamma_1|, 1 - |\cos\gamma_2|\}\,, \tag{2.93}$$

will satisfy equations (2.91) and (2.92).

Now define the super-solution as

$$\begin{aligned} v_+(s,t;C_+) &= \frac{\cos\gamma_1+\cos\gamma_2+2K_+}{1-K_+}\frac{1}{f_1(s)-f_2(s)} \\ &\quad -\frac{\sqrt{4-((\cos\gamma_1+\cos\gamma_2+2K_+)t+(\cos\gamma_1-\cos\gamma_2))^2}}{2(1-K_+)}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\ &\quad +C_+\,, \end{aligned} \tag{2.94}$$

where $K_+=\min\{1-|\cos\gamma_1|,1-|\cos\gamma_2|\}$, and C_+ is an unknown positive constant. It follows from equations (2.76)-(2.78) that

$$\nu\cdot Tv\big|_{t=1} = \cos\gamma_1+K_++o(1)\,, \tag{2.95}$$

$$\nu\cdot Tv\big|_{t=-1} = \cos\gamma_2+K_++o(1)\,, \tag{2.96}$$

$$\nabla\cdot Tv-v = -K_+\frac{\cos\gamma_1+\cos\gamma_2+2K_+}{1-K_+}\frac{1}{f_1(s)-f_2(s)}-C_++R_+(s,t)\,, \tag{2.97}$$

where $R_+(s,t)=o\left(\frac{1}{f_1(s)-f_2(s)}\right)$. Note that$R_+(s,t)$ is independent of C_+.
Since $C_+>0$

$$\nabla\cdot Tv-v < -K_+\frac{\cos\gamma_1+\cos\gamma_2+2K_+}{1-K_+}\frac{1}{f_1(s)-f_2(s)}+R_+(s,t)\,. \tag{2.98}$$

Since $0<K_+<1$, there exists a sufficiently small constant s_+ such that

$$\nu\cdot Tv\big|_{t=1} > \cos\gamma_1 = \nu\cdot Tu\big|_{t=1} \qquad \text{for } s<s_+\,, \tag{2.99}$$

$$\nu\cdot Tv\big|_{t=-1} > \cos\gamma_2 = \nu\cdot Tu\big|_{t=-1} \qquad \text{for } s<s_+\,, \tag{2.100}$$

$$\nabla\cdot Tv-v < 0 = \nabla\cdot Tu-u \qquad \text{for } s<s_+\,, \tag{2.101}$$

where s_+ depends on K_+ but not on C_+. Now, choose C_+ sufficiently large so that

$$v_+(s_+,t) \;>\; u(s_+,t)\,. \tag{2.102}$$

Thus by the comparison principle (Theorem 1.1),

$$v_+(s,t) \;>\; u(s,t) \qquad \text{for } s < s_+. \tag{2.103}$$

Similarly, we define a sub-solution according to

$$\begin{aligned} v_-(s,t;C_-) \;=\; & \frac{\cos\gamma_1+\cos\gamma_2+2K_-}{1-K_-}\frac{1}{f_1(s)-f_2(s)} \\ & -\frac{\sqrt{4-((\cos\gamma_1+\cos\gamma_2+2K_-)t+(\cos\gamma_1-\cos\gamma_2))^2}}{2(1-K_-)}\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)} \\ & +C_-\,, \end{aligned} \tag{2.104}$$

where $K_- = -\min\{1-|\cos\gamma_1|, 1-|\cos\gamma_2|, \frac{\cos\gamma_1+\cos\gamma_2}{3}\}$, and C_- is an unknown negative constant. By following a similar argument, we can define s_- and C_-, and then use the comparison principle to conclude that

$$v_-(s,t) \;<\; u(s,t) \qquad \text{for } s < s_-. \tag{2.105}$$

Hence

$$v_-(s,t) \;<\; u(s,t) \;<\; v_+(s,t) \qquad \text{for } s < s_0, \tag{2.106}$$

where $s_0 = \min\{s_-, s_+\}$. Since

$$v_-(s,t) \sim \frac{\cos\gamma_1 + \cos\gamma_2 + 2K_-}{1 - K_-} \frac{1}{f_1(s) - f_2(s)} \qquad \text{as } s \to 0\,, \tag{2.107}$$

$$v_+(s,t) \sim \frac{\cos\gamma_1 + \cos\gamma_2 + 2K_+}{1 - K_+} \frac{1}{f_1(s) - f_2(s)} \qquad \text{as } s \to 0\,, \tag{2.108}$$

there exist constants A_- and A_+ such that

$$0 < \frac{A_-}{f_1(s) - f_2(s)} < v_-(s,t)\,, \tag{2.109}$$

$$\frac{A_+}{f_1(s) - f_2(s)} > v_+(s,t)\,, \tag{2.110}$$

for sufficiently small s. Thus we conclude there exists a constant x_0 such that

$$0 < \frac{A_-}{f_1(x) - f_2(x)} < u(x,y) < \frac{A_+}{f_1(x) - f_2(x)} \qquad \text{for } x < x_0. \tag{2.111}$$

For $\cos\gamma_1 + \cos\gamma_2 < 0$, by following a similar procedure, there exist constants $\tilde{A}_-$ $\tilde{A}_+$ and $\tilde{x}_0$ such that

$$-\frac{\tilde{A}_+}{f_1(x) - f_2(x)} < u(x,y) < -\frac{\tilde{A}_-}{f_1(x) - f_2(x)} < 0 \qquad \text{for } x < \tilde{x}_0. \tag{2.112}$$

■

<u>Note:</u> This result strengthens the statement of Scholz, "The solution rises with the same order like the order of contact of the two arcs, which form the cusps" ([7], page 234), since his proof of this statement only applies to the case where $f_1(x) \sim x^\alpha$, $f_2(x) \sim x^\beta$, for $\alpha, \beta > 1$, as $x \to 0$.

Corollary 2.1 (Formal Asymptotic Expansion) *Let*

$$\begin{aligned} v(x,y) &= \frac{\cos\gamma_1+\cos\gamma_2}{f_1(x)-f_2(x)} \\ &\quad -\frac{\sqrt{4-\left(\frac{2y-(f_1(x)+f_2(x))}{f_1(x)-f_2(x)}(\cos\gamma_1+\cos\gamma_2)+(\cos\gamma_1-\cos\gamma_2)\right)^2}}{2}\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)} \\ &\quad +C_0\,, \end{aligned} \tag{2.113}$$

where C_0 *is an arbitrary constant.*

If

$$f_1(x)-f_2(x) = o(f_1'(x)-f_2'(x)) \qquad \text{as } x\to 0\,, \tag{2.114}$$

$$f_1''(x)-f_2''(x) = o\left(\frac{f_1'(x)-f_2'(x)}{f_1(x)-f_2(x)}\right) \qquad \text{as } x\to 0\,, \tag{2.115}$$

$$f_1'''(x)-f_2'''(x) = o\left(\frac{f_1'(x)-f_2'(x)}{(f_1(x)-f_2(x))^2}\right) \qquad \text{as } x\to 0\,, \tag{2.116}$$

then $v(x,y)$ *satisfies the partial differential equation* (1.43) *in the region* Ω_1 *given by equation* (2.3)*, and boundary conditions* (2.10) (2.11) *asymptotically, i.e.*

$$\nabla\cdot Tv \sim v \qquad \text{in } \Omega\,, \tag{2.117}$$

$$\nu\cdot Tv \sim \cos\gamma_1 \qquad \text{on } y=f_1(x)\,, \tag{2.118}$$

$$\nu\cdot Tv \sim \cos\gamma_2 \qquad \text{on } y=f_2(x)\,, \tag{2.119}$$

as $x\to 0$.

Proof: Immediately follows from the proof of Theorem 2.2, by letting $K_+ = 0$ in equations (2.95) to (2.97).

Example 2.2 (Exponential Cusp) *Consider a cusp region given by*

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < \infty,\ -be^{-\frac{1}{x^2}} < y < ae^{-\frac{1}{x^2}}\right\}. \tag{2.120}$$

We verify that this type of cusp satisfies conditions (2.60) to (2.62). A straight-forward calculation with $f_1(x) = ae^{-\frac{1}{x^2}}$ and $f_2(x) = be^{-\frac{1}{x^2}}$ yields

$$f_1(x) - f_2(x) = (a+b)e^{-\frac{1}{x^2}}\,, \tag{2.121}$$

$$f_1'(x) - f_2'(x) = (a+b)\frac{2}{x^3}e^{-\frac{1}{x^2}}\,, \tag{2.122}$$

$$f_1''(x) - f_2''(x) = (a+b)\left(-\frac{6}{x^4}e^{-\frac{1}{x^2}} + \frac{4}{x^6}e^{-\frac{1}{x^2}}\right)\,, \tag{2.123}$$

$$f_1'''(x) - f_2'''(x) = (a+b)\left(\frac{24}{x^5} + \frac{36}{x^8} - \frac{24}{x^7} + \frac{16}{x^{12}}\right)e^{-\frac{1}{x^2}}\,. \tag{2.124}$$

It follows that

$$\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} = \frac{2}{x^3} >> 1\,, \tag{2.125}$$

$$\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)} = \frac{2}{x^3} >> (a+b)\left(-\frac{6}{x^4} + \frac{4}{x^6}\right)e^{-\frac{1}{x^2}} = f_1''(x) - f_2''(x)\,, \tag{2.126}$$

$$\frac{f_1'(x) - f_2'(x)}{(f_1(x) - f_2(x))^2} = \frac{2}{(a+b)x^3}e^{\frac{1}{x^2}} >> (a+b)\left(\frac{24}{x^5} + \frac{36}{x^8} - \frac{24}{x^7} + \frac{16}{x^{12}}\right)e^{-\frac{1}{x^2}} = f_1'''(x) - f_2'''(x)\,,$$

for sufficiently small x. Thus by Theorem 2.2 there exist constants A_- and A_+ such that

$$A_-\, e^{\frac{1}{x^2}} < u(x,y) < A_+\, e^{\frac{1}{x^2}} \qquad \text{for } x < x_0\,, \tag{2.127}$$

for sufficiently small x_0. Also by Corollary 2.1, we can find a formal asymptotic expansion $v(x, y)$ of the form

$$v(x,y) = \frac{\cos\gamma_1+\cos\gamma_2}{a+b}e^{\frac{1}{x^2}} - \frac{\sqrt{4-\left(\frac{2y-(a-b)e^{-\frac{1}{x^2}}}{(a+b)e^{-\frac{1}{x^2}}}(\cos\gamma_1+\cos\gamma_2)+(\cos\gamma_1-\cos\gamma_2)\right)^2}}{2}\frac{2}{x^3}.$$

Note: The leading order is significantly larger than the second order term. Also notice this type of cusp cannot be approximated using the result of Scholz [7].

Chapter 3

Capillary Surface near a Circular Cusp

In this chapter we consider a capillary surface near a specific cusp named the "Circular Cusp". We define a circular cusp as a domain, which is bounded by two circular cylindrical walls tangent to each other or one cylindrical wall and one planar wall tangent to each other. There are three possible types of domain given as

$$\Omega_1 = \left\{(x,y) \in \mathbb{R}^2 \setminus \left(B_{\frac{1}{2a}}\left(0,\frac{1}{2a}\right) \cup B_{-\frac{1}{2b}}\left(0,\frac{1}{2b}\right)\right)\right\} \qquad \text{for } b < 0\,, \tag{3.1}$$

$$\Omega_2 = \left\{(x,y) \in \left(\mathbb{R}^2 : y > 0\right) \setminus B_{\frac{1}{2a}}\left(0,\frac{1}{2a}\right)\right\} \qquad \text{for } b = 0\,, \tag{3.2}$$

$$\Omega_3 = \left\{(x,y) \in B_{\left|\frac{1}{2b}\right|}\left(0,\frac{1}{2b}\right) \setminus B_{\frac{1}{2a}}\left(0,\frac{1}{2a}\right)\right\} \qquad \text{for } b > 0\,, \tag{3.3}$$

where $a > 0$, $a > b$ (see Figure 3.1).

Although this is a stronger restriction on the shape of the cusp, circular cusps are likely to be the most common shape in engineering applications.

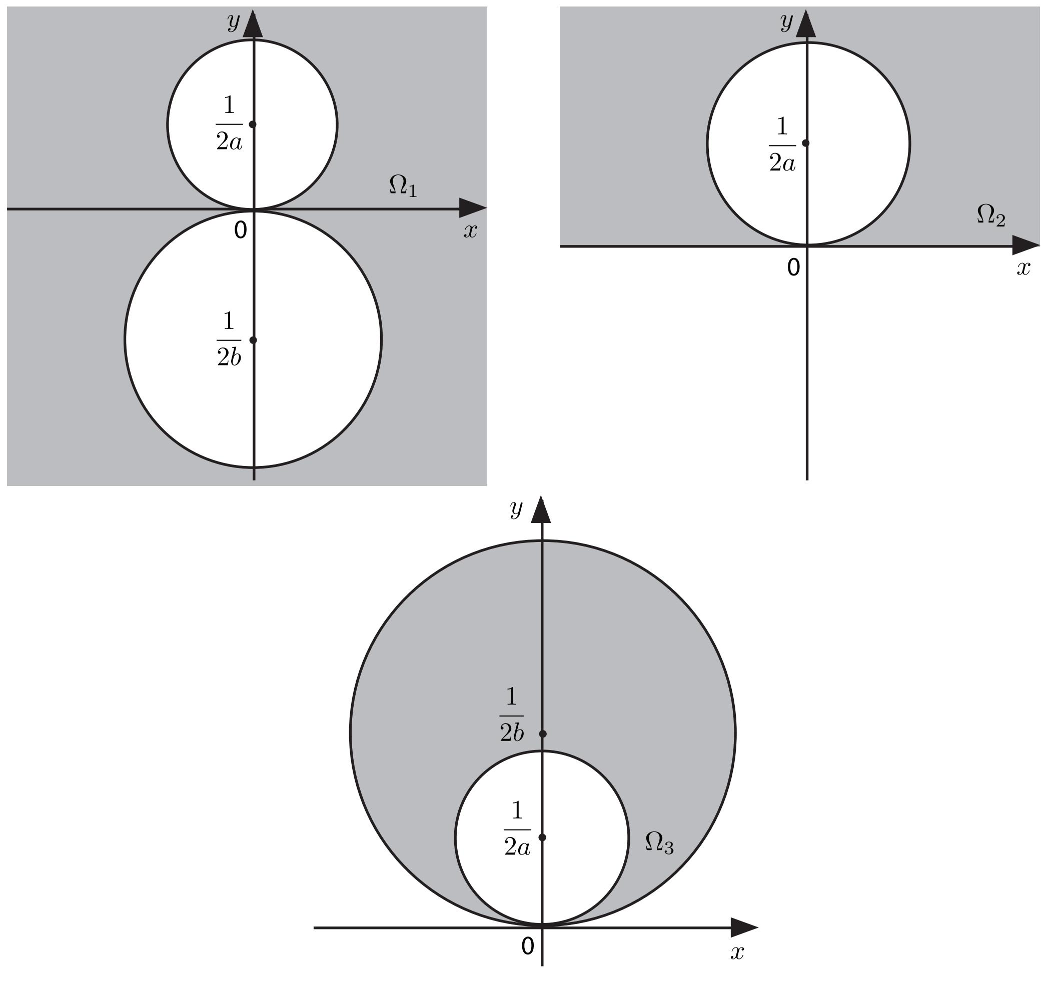

Figure 3.1: A circular cusp Ω

3.1 Tangent Cylinder Coordinate System

We now introduce the coordinate system that is most "natural" to work with in a circular cusp region. This coordinate system is the Tangent Cylinder coordinate system ([5], page 56). New coordinates p and q are defined by

$$p := \frac{x}{x^2 + y^2}, \tag{3.4}$$

$$q := \frac{y}{x^2 + y^2}. \tag{3.5}$$

The inverse of these inverse of these equations are

$$x = \frac{p}{p^2 + q^2}, \tag{3.6}$$

$$y = \frac{q}{p^2 + q^2}. \tag{3.7}$$

The region Ω's in equations (3.1)-(3.3) is now given by

$$\Omega = \left\{ (p, q) \in \mathbb{R}^2 : b < q < a, 0 < p < \infty \right\}, \tag{3.8}$$

with $a > 0$.

Note: The cusp at $(x, y) = (0, 0)$ is mapped to $p = \infty$. Thus in this coordinate system, we use an asymptotic series as $p \to \infty$ to perform the asymptotic analysis near the cusp.

The Capillary BVP in Tangent Cylinder Coordinate System

Since the tangent cylinder coordinate system is on orthogonal curvilinear coordinate

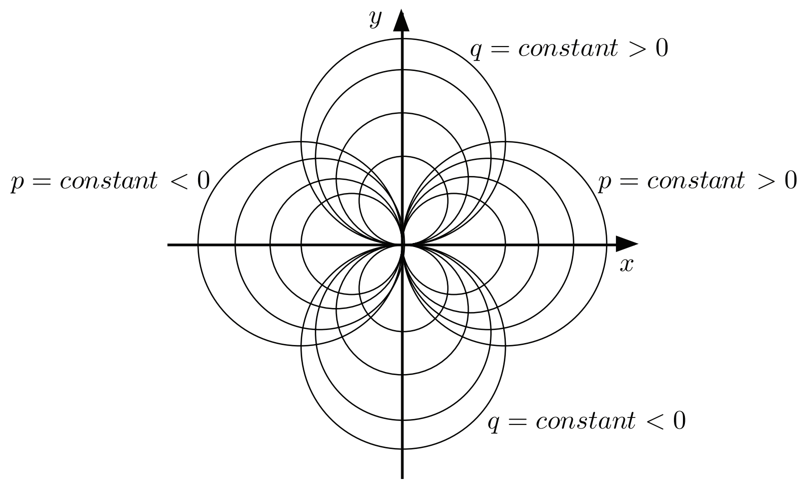

Figure 3.2: Coordinate Curves of Tangent Cylinder Coordinate System

system, we can transform the gradient and divergence operators as follows:

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial p}\hat{p} + \frac{1}{h_2}\frac{\partial f}{\partial q}\hat{q}\,, \tag{3.9}$$

$$\nabla \cdot \vec{F} = \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial p}F_p h_2 + \frac{\partial}{\partial q}F_q h_1\right)\,, \tag{3.10}$$

where

$$h_1 = \frac{1}{|\nabla p|} = \frac{1}{p^2+q^2}\,, \tag{3.11}$$

$$h_2 = \frac{1}{|\nabla q|} = \frac{1}{p^2+q^2}\,. \tag{3.12}$$

It follows that

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,, \tag{3.13}$$

$$= \frac{(p^2+q^2)\left(u_p\hat{p}+u_q\hat{q}\right)}{\sqrt{1+(p^2+q^2)^2\left(u_p^2+u_q^2\right)}}\,. \tag{3.14}$$

Hence the partial differential equation (1.43) and the boundary condition (1.44) for the region (3.8) becomes

$$\nabla \cdot Tu = (p^2+q^2)^2 \frac{\partial}{\partial p} \frac{u_p}{\sqrt{1+(p^2+q^2)^2\left(u_p^2+u_q^2\right)}} + (p^2+q^2)^2 \frac{\partial}{\partial q} \frac{u_q}{\sqrt{1+(p^2+q^2)^2\left(u_p^2+u_q^2\right)}} = u \qquad \text{in } \Omega\,, \tag{3.15}$$

$$\hat{q} \cdot Tu\big|_{q=a} = \left. \frac{(p^2+q^2)u_q}{\sqrt{1+(p^2+q^2)^2\left(u_p^2+u_q^2\right)}} \right|_{q=a} = \cos\gamma_1\,, \tag{3.16}$$

$$-\hat{q} \cdot Tu\big|_{q=b} = -\left. \frac{(p^2+q^2)u_q}{\sqrt{1+(p^2+q^2)^2\left(u_p^2+u_q^2\right)}} \right|_{q=b} = \cos\gamma_2\,. \tag{3.17}$$

In the remainder of this chapter we will consider the capillary BVP for a circular cusp, in terms of tangent cylinder coordinates p and q. The domain Ω is given by equation (3.8), PDE has the form (3.15), and BCs are given by equations (3.16)-(3.17). We will refer to equations (3.15)-(3.17) as the *capillary BVP in a circular cusp region.*

3.2 Upper Bound for a Capillary Surface near a Circular Cusp

First we obtain an upper bound for a capillary surface in a circular cusp region using Theorem 1.2.

Theorem 3.1 (Upper Bound for a Capillary Surface near a Circular Cusp) *Let u(x,y) be a solution of the capillary BVP in a circular cusp region.*

If $b \geq \frac{1}{2\sqrt{2}+\frac{1}{a}}$*, then*

$$u(p,q) \quad < \quad 4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2}\,. \tag{3.18}$$

If $b < \frac{1}{2\sqrt{2}+\frac{1}{a}}$*, then*

$$u(p,q) \quad < \quad \begin{cases} 4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2}\,, & \textit{for } p > \sqrt{\frac{a-b}{2\sqrt{2}} - ab}\,, \\ 2\sqrt{2}\,, & \textit{otherwise}\,. \end{cases} \tag{3.19}$$

Proof: First consider the case $b \neq 0$. Use the law of cosines for the triangle described in Figure 3.3 (refer to Appendix A.2 for the justification of this figure),

$$\left(\frac{1}{2a} - \frac{1}{2b}\right)^2 + \left(\delta + \frac{1}{2a}\right)^2 - 2\left(\frac{1}{2a} - \frac{1}{2b}\right)\left(\delta + \frac{1}{2a}\right)\cos 2\phi \quad = \quad \left(\delta - \frac{1}{2b}\right)^2\,, \tag{3.20}$$

where

$$\phi \quad = \quad \tan^{-1}\frac{a}{p}\,. \tag{3.21}$$

From trigonometric identities and geometry we obtain

$$\cos 2\phi \quad = \quad \cos^2\phi - \sin^2\phi\,, \tag{3.22}$$

$$\cos\phi \quad = \quad \frac{p}{\sqrt{a^2+p^2}}\,, \tag{3.23}$$

$$\sin\phi \quad = \quad \frac{a}{\sqrt{p^2+q^2}}\,, \tag{3.24}$$

$$\cos 2\phi \quad = \quad \frac{p^2-a^2}{a^2+p^2}\,. \tag{3.25}$$

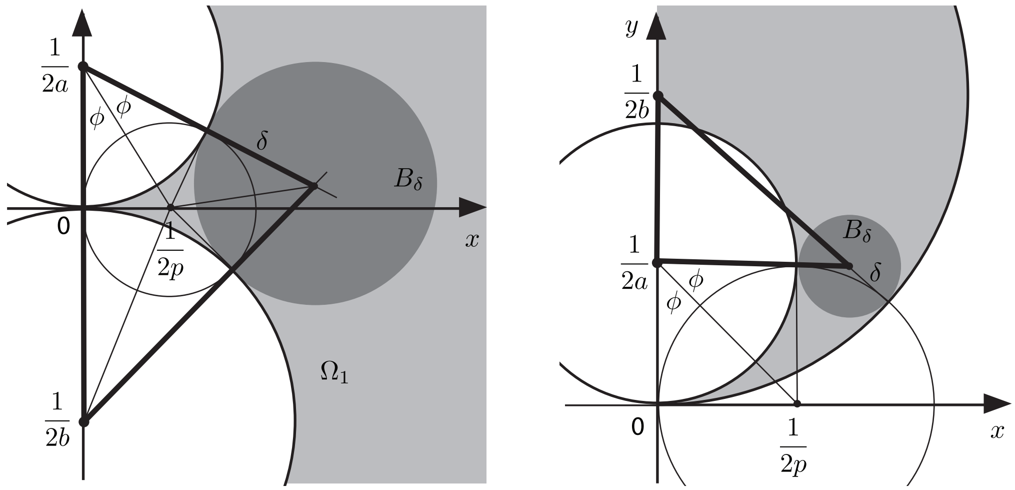

Figure 3.3: A Disk of Radius δ in Region Ω: $b \neq 0$

Substituting equation (3.25) into equation (3.20) and solving for δ gives

$$\delta(p) \quad = \quad \frac{1}{2}\frac{a-b}{ab+p^2}\,, \tag{3.26}$$

for

$$p \quad > \quad \begin{cases} \sqrt{-ab}, & \text{for } b < 0\,, \\ 0, & \text{for } b > 0\,. \end{cases} \tag{3.27}$$

Now consider the case $b = 0$. Applying the Pythagorean Theorem to the triangle described in Figure 3.4 gives

$$\left(\frac{1}{2a} + \delta\right)^2 \quad = \quad \left(\frac{1}{2a} - \delta\right)^2 + \left(\frac{1}{p}\right)^2\,, \tag{3.28}$$

$$\Rightarrow \qquad \delta(p) \quad = \quad \frac{a}{2p^2}\,. \tag{3.29}$$

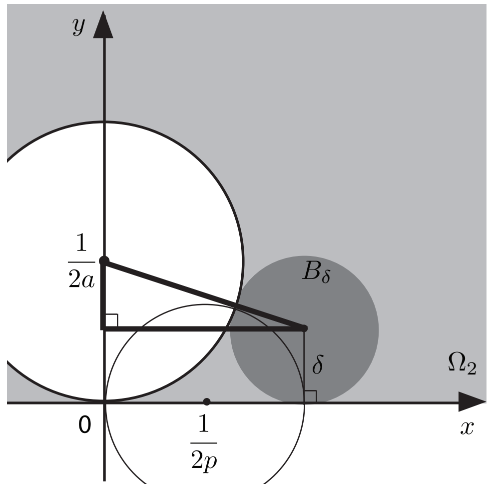

Figure 3.4: A Disk of Radius δ in Region Ω: $b = 0$

Hence we can combine equations (3.26) and (3.27) with equation (3.29) to obtain

$$\delta(p) = \frac{1}{2}\frac{a-b}{ab+p^2} \qquad \text{for } p > \begin{cases} \sqrt{-ab}, & \text{for } b < 0\,, \\ 0, & \text{for } b \geq 0\,. \end{cases} \tag{3.30}$$

It follows from Theorem 1.2 that

$$u(p,q) < \frac{2}{\delta} + \delta\,, \tag{3.31}$$

$$= 4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2}\,. \tag{3.32}$$

By Corollary 1.1 if $b \geq \frac{1}{2\sqrt{2}+\frac{1}{a}}$, then

$$u(p,q) < 4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2}\,. \tag{3.33}$$

If $b < \frac{1}{2\sqrt{2}+\frac{1}{a}}$, then

$$u(p,q) \quad < \quad \begin{cases} 4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2}\,, & \text{for } p > \sqrt{\frac{a-b}{2\sqrt{2}} - ab}\,, \\ 2\sqrt{2}\,, & \text{otherwise}\,. \end{cases}$$

■

<u>Note 1:</u> Similarly we can find a lower bound for the capillary BVP in a circular cusp region.

<u>Note 2:</u> The asymptotic order of this upper bound can be written as

$$4\frac{ab+p^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-p^2} \quad \sim \quad \frac{4p^2}{a-b} \qquad \text{as } p \to \infty\,, \tag{3.34}$$

$$= \quad O\left(p^2\right) \;=\; O\left(\frac{1}{x^2}\right)\,. \tag{3.35}$$

This result is consistent with Theorem 2.2.

<u>Note 3:</u> Unlike Theorem 2.2, this result can be applied globally even in a region away from the cusp.

3.3 Determining the Possible Leading Order Term

In this section, we will present a new way to find a possible leading order term to the asymptotic solution to the capillary BVP in a circular cusp region. The argument is heuristic and serves as motivation. In section 3.5.5 we will prove the validity of this asymptotic solution. Let $u(p,q)$ be a a solution to the capillary BVP in a circular cusp region. Without loss of generality, assume $\cos\gamma_1 + \cos\gamma_2 > 0$. It follows from Theorem 2.1 that the solution goes off to infinity at the cusp. Hence we assume that

$|\nabla u|^2$ becomes significantly larger than 1 near the cusp, i.e.

$$(p^2+q^2)^2\left(u_p^2+u_q^2\right) \quad >> \quad 1 \qquad \text{for sufficiently large } p\,. \tag{3.36}$$

The idea is to approximate $u(p,q)$ by $v(p,q)$ defined to be a solution of the following PDE and BCs:

$$(p^2+q^2)^2\left(\frac{\partial}{\partial p}\frac{v_p}{\sqrt{(p^2+q^2)^2\left(v_p^2+v_q^2\right)}}+\frac{\partial}{\partial q}\frac{v_q}{\sqrt{(p^2+q^2)^2\left(v_p^2+v_q^2\right)}}\right) \quad = \quad v(p,q)\,, \tag{3.37}$$

$$\left.\frac{v_q}{\sqrt{v_p^2+v_q^2}}\right|_{q=a} \quad = \quad \cos\gamma_1\,, \tag{3.38}$$

$$-\left.\frac{v_q}{\sqrt{v_p^2+v_q^2}}\right|_{q=b} \quad = \quad \cos\gamma_2\,. \tag{3.39}$$

This PDE and BCs are obtained from equations (3.15), (3.16) and (3.17) by dropping the additive term 1 in the denominator. We cancel the term p^2+q^2 in the numerator and denominator of the BCs. We shall refer this BVP as the *approximated BVP in a circular cusp region*. Based on the assumption (3.36), we expect $v(p,q)$ will satisfy the partial differential equation (3.15) and the boundary conditions (3.16) and (3.17) asymptotically in the region Ω.

Note: The uniqueness of this BVP can be proven in a very similar way as the uniqueness of the capillary BVP (refer to Finn and Hwang [2]).

In order to determine $v(p,q)$ we now make the key assumption that $\frac{v_q}{\sqrt{v_p^2+v_q^2}}$ is independent of p, i.e.

$$\frac{v_q}{\sqrt{v_p^2+v_q^2}} \quad = \quad g(q)\,, \tag{3.40}$$

where $g(q) \in C^2$, $g(a) = \cos\gamma_1$ and $g(b) = -\cos\gamma_2$. Assuming $v_p > 0$, we solve equation (3.40) to obtain

$$\frac{\sqrt{1-g^2(q)}}{g(q)}v_q = v_p\,. \tag{3.41}$$

By the method of characteristics, we can solve this partial differential equation,

$$v(p,q) = f\left(p + \int_b^q \frac{g(r)}{\sqrt{1-g^2(r)}}dr + K\right)\,, \tag{3.42}$$

where $f(\cdot)$ is an arbitrary function of one variable, and K is an arbitrary constant, kept for convenience. Differentiating equation (3.42) with respect to p and q gives

$$v_p(p,q) = f'\,, \tag{3.43}$$

$$v_q(p,q) = f'\frac{g(q)}{\sqrt{1-g^2(q)}}\,. \tag{3.44}$$

Substituting equations (3.43) and (3.44) into equation (3.37), after some simplification we get

$$v(p,q) = -2p\sqrt{1-g^2(q)} + (p^2+q^2)g'(q) - 2qg(q)\,. \tag{3.45}$$

Differentiate equation (3.45) with respect to p and q:

$$v_p(p,q) = -2\sqrt{1-g^2(q)} + 2pg'(q)\,, \tag{3.46}$$

$$v_q(p,q) = \frac{2pg(q)g'(q)}{\sqrt{1-g^2(q)}} + (p^2+q^2)g''(q) - 2g(q)\,. \tag{3.47}$$

Equating equations (3.43) and (3.46) gives

$$f' = -2\sqrt{1-g^2(q)} + 2pg'(q)\,. \tag{3.48}$$

Substitute equation (3.48) into equation (3.44) and equating with equation (3.47) gives

$$\frac{2pg(q)g'(q)}{\sqrt{1-g^2(q)}} + (p^2+q^2)g''(q) - 2g(q) \;=\; \left(-2\sqrt{1-g^2(q)} + 2pg'(q)\right)\frac{g(q)}{\sqrt{1-g^2(q)}}\,,$$

which reduces to

$$g''(q) \;=\; 0\,. \tag{3.49}$$

By solving equation (3.49) with the boundary conditions $g(a) = \cos\gamma_1$ and $g(b) = -\cos\gamma_2$, we get

$$g(q) \;=\; \frac{\cos\gamma_1 + \cos\gamma_2}{a-b}\left(q - \frac{b\cos\gamma_1 + a\cos\gamma_2}{\cos\gamma_1+\cos\gamma_2}\right). \tag{3.50}$$

For simplicity, introduce new constants

$$A \;=\; \frac{\cos\gamma_1+\cos\gamma_2}{a-b}\,, \tag{3.51}$$

$$q_0 \;=\; \frac{b\cos\gamma_1 + a\cos\gamma_2}{\cos\gamma_1+\cos\gamma_2}\,, \tag{3.52}$$

so that

$$g(q) \;=\; A(q-q_0)\,. \tag{3.53}$$

By substituting equation (3.53) into equation (3.45) we get

$$v(p,q) \;=\; Ap^2 - 2p\sqrt{1-A^2(q-q_0)^2} - A(q-q_0)^2 + Aq_0^2\,. \tag{3.54}$$

We conjecture that $v(p,q)$ as given by equation (3.54) is the leading order term of the asymptotic solution as $p\to\infty$ of the capillary BVP in a circular cusp region. We will prove this conjecture in section 3.5.5.

3.4 Asymptotic Order of the Term After the Leading Order

In order to construct a comparison function to justify the possible leading order term equation (3.54) found in section 3.3, we need to find the asymptotic order of the term after the leading order term. So we let $u(p,q) = v(p,q) + w(p,q)$, where $v(p,q)$ is given by equation (3.54), and substitute this into the original capillary BVP in a circular cusp region and aim to find the asymptotic order of $w(p,q)$ as $p \to \infty$.

First approximate the differential operator T (equation (3.14)) using a binomial series expansion,

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \tag{3.55}$$

$$\begin{aligned} &= \frac{\left(p^2+q^2\right)\left(u_p\hat{p}+u_q\hat{q}\right)}{\sqrt{1+\left(p^2+q^2\right)^2\left(u_p^2+u_q^2\right)}}, \\ &= \frac{u_p\hat{p}+u_q\hat{q}}{\sqrt{u_p^2+u_q^2}} \\ &\quad \cdot \left(1-\frac{1}{2}\left(\frac{1}{\left(p^2+q^2\right)^2\left(u_p^2+u_q^2\right)}\right)+\frac{3}{4\cdot 2}\left(\frac{1}{\left(p^2+q^2\right)^2\left(u_p^2+u_q^2\right)}\right)^2+...\right). \end{aligned} \tag{3.56}$$

On account of the assumption in equation (3.36), this series will converge for sufficiently large p.

We can approximate the boundary conditions (3.16) and (3.17) as follows:

$$\hat{q}\cdot Tu \sim \frac{u_q}{\sqrt{u_p^2+u_q^2}}\left(1-\frac{1}{2}\left(\frac{1}{\left(p^2+q^2\right)^2\left(u_p^2+u_q^2\right)}\right)\right). \tag{3.57}$$

Now we let

$$u(p,q) \quad = \quad v(p,q) + w(p,q)\,, \tag{3.58}$$

where we assume $w(p,q) = o(v(p,q))$ as $p \to \infty$. Substituting equation (3.58) into equation (3.57) gives

$$\hat{q}\cdot Tu \quad \sim \quad \frac{v_q + w_q}{\sqrt{(v_p + v_p)^2 + (v_q + w_q)^2}} \left(1 - \frac{1}{2}\frac{1}{(p^2+q^2)^2\left(v_p^2 + v_q^2\right)}\right)\,, \tag{3.59}$$

$$\sim \quad \frac{v_q}{\sqrt{v_p^2 + v_q^2}} \left(1 - \frac{w_p v_p + w_q v_q}{v_p^2 + v_q^2}\right)\left(1 - \frac{1}{2}\frac{1}{(p^2+q^2)^2\left(v_p^2 + v_q^2\right)}\right)\,. \tag{3.60}$$

Substituting equations (3.40) and (3.42) into equation (3.60) gives

$$\hat{q}\cdot Tu \quad \sim \quad g(q)\left(1 - \frac{\left(w_p + w_q\frac{g(q)}{\sqrt{1-g^2(q)}}\right)f'(\xi)}{\left(1 + \frac{g^2(q)}{1-g^2(q)}\right)f'(\xi)^2} - \frac{1}{2(p^2+q^2)^2\left(1 + \frac{g^2(q)}{1-g^2(q)}\right)f'(\xi)^2}\right)\,, \tag{3.61}$$

$$= \quad g(q)\left(1 - \frac{\left(w_p + w_q\frac{g(q)}{\sqrt{1-g^2(q)}}\right)f'(\xi) + \frac{1}{2(p^2+q^2)^2}}{\left(1 + \frac{g^2(q)}{1-g^2(q)}\right)f'(\xi)^2}\right)\,. \tag{3.62}$$

We now impose the BCs (equations (3.16) and (3.17)) and get

$$\begin{aligned} \hat{q}\cdot Tu\big|_{q=a} &= \cos\gamma_1 && (3.63)\\ &\sim \cos\gamma_1\left(1-\frac{\left(w_p+w_q\frac{g(a)}{\sqrt{1-g^2(a)}}\right)f'(\xi)+\frac{1}{2(p^2+a^2)^2}}{\left(1+\frac{g^2(a)}{1-g^2(a)}\right)f'(\xi)^2}\right). \\ -\hat{q}\cdot Tu\big|_{q=b} &= \cos\gamma_2 && (3.64)\\ &\sim \cos\gamma_2\left(1-\frac{\left(w_p+w_q\frac{g(b)}{\sqrt{1-g^2(b)}}\right)f'(\xi)+\frac{1}{2(p^2+b^2)^2}}{\left(1+\frac{g^2(b)}{1-g^2(b)}\right)f'(\xi)^2}\right). && (3.65) \end{aligned}$$

In order to ensure that $u(p,q)=v(p,q)+w(p,q)$ to satisfy the BCs, we require

$$\left(w_p+w_q\frac{g(a)}{\sqrt{1-g^2(a)}}\right)f'(\xi) \sim -\frac{1}{2(p^2+a^2)^2}\,, \quad (3.66)$$

$$\left(w_p+w_q\frac{g(b)}{\sqrt{1-g^2(b)}}\right)f'(\xi) \sim -\frac{1}{2(p^2+b^2)^2}\,. \quad (3.67)$$

We now assume $w(p,q)$ is in the form

$$w(p,q) \sim Kp^{\alpha}+H(q)p^{\alpha-1} \qquad p\to\infty\,, \quad (3.68)$$

where K is a constant and $H(q)$ is a function of q, then it is suitable to assume

$$w_p \sim \alpha K p^{\alpha-1} \quad (3.69)$$

$$w_q \sim H'(q)p^{\alpha-1} \qquad \text{as } p\to\infty\,. \quad (3.70)$$

Substituting equations (3.48), (3.69) and (3.70) into equations (3.66) and (3.67)

$$2g'(q)\left(\alpha K + H'(q)\frac{g(q)}{\sqrt{1-g^2(q)}}\right)p^{\alpha} \sim \frac{1}{2(p^2+q^2)^2}\,, \tag{3.71}$$

$$\Rightarrow \quad \alpha = -4\,. \tag{3.72}$$

Hence we postulate that $w(p,q)$ needs to be at least of order $O\left(p^{-4}\right)$ in order to satisfy the BCs.

Similarly we can asymptotically approximate the capillary PDE in a circular cusp region (equation (3.15)), by assuming the differentiability of each term,

$$\begin{aligned}
&\left(\frac{\partial}{\partial p}\frac{u_p}{\sqrt{u_p^2+u_q^2}}\right)+\left(\frac{\partial}{\partial q}\frac{u_q}{\sqrt{u_p^2+u_q^2}}\right)\\
&-\left(\left(\frac{\partial}{\partial p}\frac{u_p}{\sqrt{u_p^2+u_q^2}}\right)+\left(\frac{\partial}{\partial q}\frac{u_q}{\sqrt{u_p^2+u_q^2}}\right)\right)\frac{1}{2}\left(\frac{1}{(p^2+q^2)^2\left(u_p^2+u_q^2\right)}\right)\\
&+\frac{u_p}{\sqrt{u_p^2+u_q^2}}\left(-\frac{1}{2}\frac{\partial}{\partial p}\left(\frac{1}{(p^2+q^2)^2\left(u_p^2+u_q^2\right)}\right)\right)\\
&+\frac{u_q}{\sqrt{u_p^2+u_q^2}}\left(-\frac{1}{2}\frac{\partial}{\partial q}\left(\frac{1}{(p^2+q^2)^2\left(u_p^2+u_q^2\right)}\right)\right)\\
\sim\ &\frac{u}{(p^2+q^2)^2}\,.
\end{aligned} \tag{3.73}$$

Substituting equation (3.58) into equation (3.73), and assuming $w_p = o\left(v_p\right)$ and $w_q = o\left(v_q\right)$, as $p \to \infty$, we obtain

$$\frac{u_p}{\sqrt{u_p^2+u_q^2}} = \frac{v_p}{\sqrt{v_p^2+v_q^2}}\left(1-\frac{w_pv_p+w_qv_q}{v_p^2+v_q^2}+\ldots\right)\,. \tag{3.74}$$

Using above and expanding equation (3.58) we get

$$
\begin{aligned}
& \frac{\partial}{\partial p}\left(\frac{v_p}{\sqrt{v_p^2+v_q^2}}\left(1-\frac{w_p v_p+w_q v_q}{v_p^2+v_q^2}\right)\right)+\frac{\partial}{\partial q}\left(\frac{v_q}{\sqrt{v_p^2+v_q^2}}\left(1-\frac{w_p v_p+w_q v_q}{v_p^2+v_q^2}\right)\right) \\
& -\left(\left(\frac{\partial}{\partial p}\frac{v_p}{\sqrt{v_p^2+v_q^2}}\right)+\left(\frac{\partial}{\partial q}\frac{v_q}{\sqrt{v_p^2+v_q^2}}\right)\right)\frac{1}{2}\left(\frac{1}{(p^2+q^2)^2\left(v_p^2+v_q^2\right)}\right. \\
& +\frac{v_p}{\sqrt{v_p^2+v_q^2}}\left(-\frac{1}{2}\frac{\partial}{\partial p}\left(\frac{1}{(p^2+q^2)^2\left(v_p^2+v_q^2\right)}\right)\right) \\
& +\frac{v_q}{\sqrt{v_p^2+v_q^2}}\left(-\frac{1}{2}\frac{\partial}{\partial q}\left(\frac{1}{(p^2+q^2)^2\left(v_p^2+v_q^2\right)}\right)\right) \\
\sim\ & \frac{v+w}{(p^2+q^2)^2} \qquad \text{as } p \to \infty. \qquad (3.75)
\end{aligned}
$$

After some more calculations and using equations (3.37) and (3.54), equation (3.75) becomes

$$
\begin{aligned}
& -\frac{\partial}{\partial p}\left(\frac{2Ap}{\left(\frac{4A^2}{1-A^2(q-q_0)^2}p^2\right)^{3/2}}\left(w_p 2Ap+w_q\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p\right)\right) \\
& -\frac{\partial}{\partial q}\left(\frac{\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p}{\left(\frac{4A^2}{1-A^2(q-q_0)^2}p^2\right)^{3/2}}\left(w_p 2Ap+w_q\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p\right)\right) \\
& -\left(\frac{Ap^2}{(p^2+q^2)^2}\right)\frac{1}{2}\left(\frac{1}{(p^2+q^2)^2\left(\frac{4A^2}{1-A^2(q-q_0)^2}p^2\right)}\right) \\
& +\sqrt{1-A^2(q-q_0)^2}\left(-\frac{1}{2}\frac{\partial}{\partial p}\left(\frac{1-A^2(q-q_0)^2}{(p^2+q^2)^2\frac{4A^2}{1-A^2(q-q_0)^2}p^2}\right)\right) \\
& +A(q-q_0)\left(-\frac{1}{2}\frac{\partial}{\partial q}\left(\frac{1-A^2(q-q_0)^2}{(p^2+q^2)^2\frac{4A^2}{1-A^2(q-q_0)^2}p^2}\right)\right) \\
\sim\ & \frac{w}{(p^2+q^2)^2} \qquad \text{as } p \to \infty. \qquad (3.76)
\end{aligned}
$$

Following from equation (3.68), we assume

$$
\begin{aligned}
w_{pp} &\sim \alpha(\alpha-1)Kp^{\alpha-2}\,, && (3.77)\\
w_{pq} &\sim \alpha H'(q)p^{\alpha-2}\,, && (3.78)\\
w_{qq} &\sim H''(q)p^{\alpha-1} \qquad \text{as } p\to\infty\,. && (3.79)
\end{aligned}
$$

Substituting equations (3.69), (3.70), (3.77)-(3.79) into equation (3.76), we obtain

$$
\begin{aligned}
&-\frac{\partial}{\partial q}\left(\frac{q-q_0}{\frac{4A^2}{1-A^2(q-q_0)^2}}\left(2A\alpha K + H'(q)\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}\right)p^{\alpha-2}\right)\\
&+A(q-q_0)\left(-\frac{1}{2}\frac{\partial}{\partial q}\left(\frac{1}{4A^2(p^2+q^2)^2p^2}\right)\right)\\
\sim\;& \frac{Kp^{\alpha}}{(p^2+q^2)^2} \qquad \text{as } p\to\infty. \qquad (3.80)
\end{aligned}
$$

After some simplification of equation (3.80) we obtain $\alpha = -4$. Hence in order to ensure $u(p,q) = v(p,q) + w(p,q)$ to be the solution of the capillary BVP in a circular cusp region, $w(p,q)$ needs to be at least of order

$$O(w) \;=\; O\left(p^{-4}\right)\,. \qquad (3.81)$$

The preceding heuristic calculations suggest that

$$u(p,q) \;\sim\; Ap^2 - 2p\sqrt{1-A^2(q-q_0)^2} - A(q-q_0)^2 + Aq_0^2 + Kp^{-4} + H(q)p^{-5} \qquad (3.82)$$

is the asymptotic solution for the capillary BVP in a circular cusp region. It is of considerable interest that the coefficients of p^{-1}, p^{-2} and p^{-3} are zero. In the next section, we prove that equation (3.82) is in fact the correct asymptotic solution.

3.5 The Complete Asymptotic Series

In this section we derive the complete asymptotic series for the capillary BVP in a circular cusp region.

3.5.1 Statement of the Main Theorem

We consider the capillary BVP in a circular cusp region as given by equations (3.15)-(3.17). The terms $u_n(p,q)$ of the asymptotic series for this problem defined as follows:

$$\begin{aligned} u_n(p,q) &= Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 \\ &\quad + \sum_{i=1}^{n}\left(K_i p^{1-i} + f_i(q)p^{-i}\right) , \end{aligned} \tag{3.83}$$

with

$$A = \frac{\cos\gamma_1 + \cos\gamma_2}{a-b} , \tag{3.84}$$

$$q_0 = \frac{b\cos\gamma_1 + a\cos\gamma_2}{\cos\gamma_1 + \cos\gamma_2} . \tag{3.85}$$

Here K_i and $f_i(q)$ are determined recursively according to

$$K_i = \frac{\int_b^a g_i(s)ds - (h_i(a) - h_i(b))}{a-b} , \tag{3.86}$$

$$\begin{aligned} f_i(q) &= -\frac{K_i(1-i)\sqrt{1-A^2(q-q_0)^2}}{A} \\ &\quad - \int_{q_0}^{q} \frac{2A\left(h_i(b) - \int_b^t g_i(s)\,ds + K_i\,(t-b)\right)}{\left(1-A^2(t-q_0)^2\right)^{3/2}} dt , \end{aligned} \tag{3.87}$$

where $g_i(q)$, $h_i(a)$ and $h_i(b)$ are defined by

$$g_i(q) = \lim_{p\to\infty} \frac{\nabla \cdot Tu_{i-1} - u_{i-1}}{p^{-i+1}}, \tag{3.88}$$

$$h_i(a) = \lim_{p\to\infty} \frac{\hat{q} \cdot Tu_{i-1}\big|_{q=a} - \cos\gamma_1}{p^{-i-1}}, \tag{3.89}$$

$$h_i(b) = \lim_{p\to\infty} \frac{\hat{q} \cdot Tu_{i-1}\big|_{q=b} + \cos\gamma_2}{p^{-i-1}}. \tag{3.90}$$

Theorem 3.2 (The Complete Asymptotic Series) *Let $u(p,q)$ be the solution of the capillary BVP in a circular cusp region in tangent cylinder coordinates p and q. Then there exist constants L_{n+1} and p_{n+1} such that*

$$\left|u(p,q) - \big(u_n(p,q) - f_n(q)p^{-n}\big)\right| < \frac{L_{n+1}}{p^n}, \qquad \text{for } p > p_{n+1}. \tag{3.91}$$

3.5.2 The Formal Asymptotic Series

In this section we derive the series $u_n(p,q)$ as defined in equation (3.83).

Lemma 3.1 (The Formal Asymptotic Series) *The function $u_n(p,q)$ defined in equation* (3.83) *constitute a formal asymptotic series of the capillary BVP in a circular cusp region, i.e.*

$$\nabla \cdot Tu_n = u_n + O\left(p^{-n}\right), \tag{3.92}$$

$$\nu \cdot Tu_n\big|_{q=a} = \cos\gamma_1 + O\left(p^{-n-2}\right), \tag{3.93}$$

$$\nu \cdot Tu_n\big|_{q=b} = \cos\gamma_2 + O\left(p^{-n-2}\right), \tag{3.94}$$

for sufficiently large p, with $n \in \mathbb{N}$.

Proof: Prove this by mathematical induction.

Base case (u_0):

$$u_0(p,q) \;=\; Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 \tag{3.95}$$

It is immediately obvious that $u_0(p,q)$ is C^∞ function in Ω. It follows* from the calculation of Appendix A.3 that

$$\hat{q}\cdot Tu_0 \;=\; A(q-q_0) - \frac{(q-q_0)(1-A^2(q-q_0)^2)}{8Ap^6} + O(p^{-7})\,, \tag{3.96}$$

$$\nabla\cdot Tu_0 \;=\; Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 - \frac{1-3A^2(q-q_0)^2}{8Ap^4} + O(p^{-5})\,, \tag{3.97}$$

for sufficiently large p. Evaluating equation (3.96) at $q=a$ and $q=b$ gives

$$\nu\cdot Tu_0\big|_{q=a} \;=\; \hat{q}\cdot Tu_0\big|_{q=a} \;=\; \cos\gamma_1 - \frac{\cos\gamma_1(1-\cos^2\gamma_1)}{8A^2p^6} + O(p^{-7})\,, \tag{3.98}$$

$$\nu\cdot Tu_0\big|_{q=b} \;=\; -\,\hat{q}\cdot Tu_0\big|_{q=b} \;=\; \cos\gamma_2 - \frac{\cos\gamma_2(1-\cos^2\gamma_2)}{8A^2p^6} + O(p^{-7})\,. \tag{3.99}$$

It follows immediately from equation (3.97)-(3.99) that (3.92)-(3.94) satisfies for $n=0$.

Inductive step (u_m):

We are given that

$$u_{m-1}(p,q) \;=\; Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 + \sum_{i=1}^{m-1}\left(K_ip^{1-i} + f_i(q)p^{-i}\right)\,, \tag{3.100}$$

*Note $u_0(p,q)$ is equal to $v(p,q)$ (in equation (A.44)) evaluated at $L_1=0$ and $h(q)=0$.

with $m = 1, 2, 3, 4, ...,$ and

$$\nabla \cdot Tu_{m-1} = u_{m-1} + O\left(p^{-m+1}\right) , \tag{3.101}$$

$$\nu \cdot Tu_{m-1}\big|_{q=a} = \cos\gamma_1 + O\left(p^{-m-1}\right) , \tag{3.102}$$

$$\nu \cdot Tu_{m-1}\big|_{q=b} = \cos\gamma_2 + O\left(p^{-m-1}\right) , \tag{3.103}$$

for sufficiently large p. The big O terms of equations (3.101)-(3.103) are power series in p. It follows from this that the following limits exist:

$$g_m(q) = \lim_{p\to\infty} \frac{\nabla \cdot Tu_{m-1} - u_{m-1}}{p^{-m+1}} , \tag{3.104}$$

$$h_m(a) = \lim_{p\to\infty} \frac{\hat{q} \cdot Tu_{m-1}\big|_{q=a} - \cos\gamma_1}{p^{-m-1}} , \tag{3.105}$$

$$h_m(b) = \lim_{p\to\infty} \frac{\hat{q} \cdot Tu_{m-1}\big|_{q=b} + \cos\gamma_2}{p^{-m-1}} . \tag{3.106}$$

Also it is given that u_{m-1} is C^∞ following from this it can be shown that $g_m(q)$ is C^∞ function.[†] Now consider u_m defined as

$$u_m = u_{m-1} + K_m p^{1-m} + f_m(q) p^{-m} . \tag{3.107}$$

Assume $f_m(q)$ to be a C^∞ function and K_m is a constant.

We now aim to choose a constant K_m and a function $f_m(q)$ such that to satisfy

$$\lim_{p\to\infty} \frac{\nabla \cdot Tu_m - u_m}{p^{-m+1}} = 0 , \tag{3.108}$$

$$\lim_{p\to\infty} \frac{\hat{q} \cdot Tu_m\big|_{q=a} - \cos\gamma_1}{p^{-m-1}} = 0 , \tag{3.109}$$

$$\lim_{p\to\infty} \frac{\hat{q} \cdot Tu_m\big|_{q=b} + \cos\gamma_2}{p^{-m-1}} = 0 . \tag{3.110}$$

[†]This statement can be justified by expanding $\nabla \cdot Tu_{m-1}$ in series. Then show that with all the differentiability conditions, we can expand $\nabla \cdot Tu_{m-1}$ in a power series in p.

Firstly, expand Tu_m asymptotically. Applying ∇ to equation (3.107) and using the definition of T (equation (1.42)), gives

$$Tu_m = \frac{\nabla u_m}{\sqrt{1+|\nabla u_m|^2}} \tag{3.111}$$

$$= Tu_{m-1}\frac{\sqrt{1+|\nabla u_{m-1}|^2}}{\sqrt{1+|\nabla u_m|^2}} + \frac{\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)}{\sqrt{1+|\nabla u_m|^2}} \tag{3.112}$$

We apply ∇ to equation (3.107) and rearrange to obtain

$$\begin{aligned}
&\frac{\sqrt{1+|\nabla u_{m-1}|^2}}{\sqrt{1+|\nabla u_m|^2}} \\
&= \sqrt{\frac{1+|\nabla u_{m-1}|^2}{1+\left|\nabla u_{m-1}+\nabla\left(K_m p^{1-m}+f_m(q)p^{-m}\right)\right|^2}} \\
&= \sqrt{\frac{1}{\frac{1+|\nabla u_{m-1}|^2+2\nabla(K_m p^{1-m}+f_m(q)p^{-m})\cdot\nabla u_{m-1}+|\nabla(K_m p^{1-m}+f_m(q)p^{-m})|^2}{1+|\nabla u_{m-1}|^2}}} \\
&= \frac{1}{\sqrt{1+\frac{2\nabla(K_m p^{1-m}+f_m(q)p^{-m})\cdot\nabla u_{m-1}+|\nabla(K_m p^{1-m}+f_m(q)p^{-m})|^2}{1+|\nabla u_{m-1}|^2}}}\,.
\end{aligned} \tag{3.113}$$

Applying the binomial series expansion‡ to equation (3.113) gives

$$\begin{aligned}
&\frac{\sqrt{1+|\nabla u_{m-1}|^2}}{\sqrt{1+|\nabla u_m|^2}} \\
&= 1-\frac{1}{2}\left(\frac{2\nabla\left(K_m p^{1-m}+f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}+\left|\nabla\left(K_m p^{1-m}+f_m(q)p^{-m}\right)\right|^2}{1+|\nabla u_{m-1}|^2}\right) \\
&\quad +O(p^{-2m-2})\,.
\end{aligned} \tag{3.114}$$

Since

$$\left|\frac{2\nabla\left(K_m p^{1-m}+f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}+\left|\nabla\left(K_m p^{1-m}+f_m(q)p^{-m}\right)\right|^2}{1+|\nabla u_{m-1}|^2}\right| < 1 \tag{3.115}$$

‡ $\frac{1}{\sqrt{1+\xi}} = 1-\frac{1}{2}\xi + O\left(\xi^2\right) \qquad \text{for } |\xi|<1.$

for sufficiently large p, this binomial series converges for sufficiently large p. Also we can show that

$$\left|\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\right|^2 << \left|2\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}\right| \qquad (3.116)$$

for sufficiently large p. Thus by neglecting $\left|\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\right|^2$ in equation (3.114) we obtain

$$\frac{\sqrt{1+|\nabla u_{m-1}|^2}}{\sqrt{1+|\nabla u_m|^2}} = 1 - \frac{1}{2}\left(\frac{2\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}}{1+|\nabla u_{m-1}|^2}\right) + O(p^{-2m-2})\,. \qquad (3.117)$$

Again using binomial series expansion on $\frac{1}{1+|\nabla u_m|^2}$, we obtain

$$\begin{aligned} &\frac{\sqrt{1+|\nabla u_{m-1}|^2}}{\sqrt{1+|\nabla u_m|^2}} \\ = \;& 1 - \frac{1}{2}\left(\frac{2\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}}{|\nabla u_{m-1}|^2}\left(1 - \frac{1}{|\nabla u_{m-1}|^2} + \ldots\right)\right) \\ & + O\left(p^{-2m-2}\right)\,. \end{aligned} \qquad (3.118)$$

Since

$$\frac{1}{|\nabla u_{m-1}|} < 1 \qquad (3.119)$$

for sufficiently large p, this series converges for sufficiently large p. Substitute equation (3.118) into equation (3.112) gives

$$\begin{aligned} Tu_m = \;& Tu_{m-1}\left(1 - \frac{1}{2}\left(\frac{2\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)\cdot\nabla u_{m-1}}{|\nabla u_{m-1}|^2}\left(1 - \frac{1}{|\nabla u_{m-1}|^2}\right)\right)\right) \\ & + \frac{\nabla\left(K_m p^{1-m} + f_m(q)p^{-m}\right)}{\sqrt{1+|\nabla u_m|^2}} + O\left(p^{-2m-2}\right)\,. \end{aligned} \qquad (3.120)$$

Substituting equation (3.100) into equation (3.120) and simplifying it gives

$$\begin{aligned} Tu_m &= Tu_{m-1}\left(1-\frac{2AK_m(1-m)+\frac{2A^2(q-q_0)f'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{\frac{4A^2}{1-A^2(q-q_0)^2}}p^{-m-1}\right) \\ &\quad +\frac{(K_m(1-m))\,\hat{p}+(f_m'(q))\,\hat{q}}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m-1}+O\left(p^{-2m-2}\right). \end{aligned} \tag{3.121}$$

Equation (3.121) is the desired asymptotic expansion for Tu_m. It follows immediately from the binomial expansion and definition of u_m (equation (3.100)) that we can write Tu_m as an asymptotic series in a form

$$Tu_m = \sum_{0}^{\infty} \vec{F}(q)p^{-i}. \tag{3.122}$$

Now consider the left hand side of the capillary PDE (3.15), i.e. $\nabla \cdot Tu_m$. Given the differentiability of each term in a binomial series, we can differentiate the binomial series while maintaining the convergence of the series. As we have extensively used the idea of binomial series expansion to derive equation (3.121), we can differentiate it while maintaining the asymptotic relation.[§] Hence it follows from equation (3.121) that

$$\begin{aligned} \nabla \cdot Tu_m &= \nabla \cdot \left(Tu_{m-1}\left(1-\frac{2AK_m(1-m)+\frac{2A^2(q-q_0)f'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{\frac{4A^2}{1-A^2(q-q_0)^2}}p^{-m-1}\right)\right) \\ &\quad +\nabla \cdot \left(\frac{(K_m(1-m))\,\hat{p}+(f_m'(q))\,\hat{q}}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m-1}+O\left(p^{-2m-2}\right)\right). \end{aligned} \tag{3.123}$$

[§]Because of the differentiability of $f(q)$ and u_{m-1}, each term of the binomial series we have used is differentiable.

Simplifying equation (3.123) gives

$$\begin{aligned}\nabla\cdot Tu_m &= \nabla\cdot Tu_{m-1}\left(1-G_m p^{-m-1}\right)-(Tu_{m-1}\cdot\hat{q})\,\frac{\partial}{\partial q}G_m p^{-m+1}\\ &\quad+\frac{\partial}{\partial q}\frac{f_m'(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m+1}+O(p^{-m})\,,\end{aligned}\qquad(3.124)$$

where

$$G_m(q)=\frac{K_m(1-m)\left(1-A^2(q-q_0)^2\right)}{2A}+\frac{f_m'(q)(q-q_0)\sqrt{1-A^2(q-q_0)^2}}{2}\,.\qquad(3.125)$$

It follows from equation (3.96) and (3.121) that

$$\hat{q}\cdot Tu_{m-1} \;=\; A(q-q_0)+O\left(p^{-6}\right)\,.\qquad(3.126)$$

Substituting equation (3.126) into equation (3.124) gives

$$\begin{aligned}\nabla\cdot Tu_m &= \nabla\cdot Tu_{m-1}\left(1-G_m p^{-m-1}\right)-A(q-q_0)\,\frac{\partial}{\partial q}G_m p^{-m+1}\\ &\quad+\frac{\partial}{\partial q}\frac{f_m'(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m+1}+O(p^{-m})\,,\end{aligned}\qquad(3.127)$$

Imposing equation (3.108) gives an equality

$$\begin{aligned}\lim_{p\to\infty}\frac{\nabla\cdot Tu_m-u_m}{p^{-m+1}} &= \lim_{p\to\infty}\left(\frac{\nabla\cdot Tu_{m-1}}{p^{-m+1}}\left(1-G_m p^{-m-1}\right)-A(q-q_0)\,\frac{\partial}{\partial q}G_m\right)\\ &\quad+\lim_{p\to\infty}\left(\frac{\partial}{\partial q}\frac{f_m'(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}+O(p^{-1})-\frac{u_m}{p^{-m+1}}\right)\\ &= 0\,.\end{aligned}\qquad(3.128)$$

Substituting equation (3.104) into equation (3.128) and expanding it gives

$$\begin{aligned}
& \lim_{p\to\infty}\left(\left(g_m(q)+\frac{u_{m-1}}{p^{-m+1}}-u_{m-1}G_m p^{-2}-g_m(q)G_m p^{-m-1}\right)-A(q-q_0)\frac{\partial}{\partial q}G_m\right)\\
& +\lim_{p\to\infty}\left(\frac{\partial}{\partial q}\frac{f_m'(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}+O(p^{-1})-\frac{u_{m-1}}{p^{-m+1}}-K_m\right)\\
= \; & 0\,. \qquad (3.129)
\end{aligned}$$

Following from equation (3.100) we have $u_{m-1}(p,q)=Ap^2+O(p)$. Substituting this into equation (3.129) and evaluating the limit gives

$$g_m(q)-K_m-AG_m-A(q-q_0)\frac{\partial}{\partial q}G_m+\frac{\partial}{\partial q}\left(\frac{f_m'(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}\right) = 0\,. \qquad (3.130)$$

We rewrite equation (3.130) in a form

$$g_m(q)-K_m-\frac{\partial}{\partial q}\left(A(q-q_0)G_m-\frac{f_m'(q)\sqrt{1-A^2(q-q_0)^2}}{2A}\right) = 0\,. \qquad (3.131)$$

Substituting equation (3.125) back into equation (3.131) gives

$$\begin{aligned}
& g_m(q)-K_m\\
= \; & \frac{\partial}{\partial q}\left(\frac{K_m(1-m)A(q-q_0)(1-A^2(q-q_0)^2)}{2A}-\frac{f_m'(q)(1-A^2(q-q_0)^2)^{3/2}}{2A}\right)\,. \qquad (3.132)
\end{aligned}$$

Equation (3.132) is a first order ODE of $f_m'(q)$.

Now consider the boundary conditions, using equation (3.121) we can asymptotically expand $\hat{q} \cdot Tu_m$ as

$$\begin{aligned} \hat{q} \cdot Tu_m &= \hat{q} \cdot Tu_{m-1} \left(1 - \frac{2AK_m(1-m) + \frac{2A^2(q-q_0)f'_m(q)}{\sqrt{1-A^2(q-q_0)^2}}}{\frac{4A^2}{1-A^2(q-q_0)^2}} p^{-m-1} \right) \\ &\quad + \frac{f'_m(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}} p^{-m-1} + O\left(p^{-2m-2}\right) . \end{aligned} \tag{3.133}$$

We now impose equation (3.109) and obtain an equality such as

$$\begin{aligned} & \lim_{p\to\infty} \frac{\hat{q} \cdot Tu_m|_{q=a} - \cos\gamma_1}{p^{-m-1}} \\ = & \lim_{p\to\infty} \left[\frac{\hat{q} \cdot Tu_{m-1}|_{q=a}}{p^{-m-1}} \left(1 - \frac{2AK_m(1-m) + \frac{2\cos^2\gamma_1 f'_m(a)}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}} p^{-m-1} \right) \right] \\ & + \lim_{p\to\infty} \left(\frac{f'_m(a)}{\frac{2A}{\sqrt{1-\cos^2\gamma_1}}} + O\left(p^{-1}\right) - \frac{\cos\gamma_1}{p^{-m-1}} \right) \\ = & \ 0 . \end{aligned} \tag{3.134}$$

Substituting equation (3.105) into equation (3.134) we obtain

$$\begin{aligned} & \lim_{p\to\infty} \left[\left(\frac{\cos\gamma_1}{p^{-m-1}} + h_m(a) \right) \left(1 - \frac{2AK_m(1-m) + \frac{2\cos^2\gamma_1 f'_m(a)}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}} p^{-m-1} \right) \right] \\ & + \lim_{p\to\infty} \left(\frac{f'_m(a)}{\frac{2A}{\sqrt{1-\cos^2\gamma_1}}} + O\left(p^{-1}\right) - \frac{\cos\gamma_1}{p^{-m-1}} \right) \\ = & \ 0 . \end{aligned} \tag{3.135}$$

By evaluating the limit of equation (3.135) we obtain

$$h_m(a) - \cos\gamma_1 \frac{2AK_m(1-m) + \frac{2\cos^2\gamma_1 f'_m(a)}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}} + \frac{f'_m(a)}{\frac{2A}{\sqrt{1-\cos^2\gamma_1}}} = 0 . \tag{3.136}$$

After some simplification of equation (3.136) we get

$$h_m(a) - \frac{K_m(1-m)\cos\gamma_1(1-\cos^2\gamma_1)}{2A} + \frac{f'(a)\,(1-\cos^2\gamma_1)^{3/2}}{2A} = 0\,. \quad (3.137)$$

Similarly by imposing equation (3.110) and after some calculation we obtain

$$h_m(b) - \frac{K_m(1-m)\cos\gamma_2(1-\cos^2\gamma_2)}{2A} + \frac{f'(b)\,(1-\cos^2\gamma_2)^{3/2}}{2A} = 0\,. \quad (3.138)$$

Solving equations (3.137) and (3.138) for $f'_m(a)$ and $f'_m(b)$ gives

$$f'_m(a) = \frac{K_m(1-m)\cos\gamma_1}{\sqrt{1-\cos^2\gamma_1}} - \frac{2Ah_m(a)}{(1-\cos^2\gamma_1)^{3/2}}\,, \quad (3.139)$$

$$f'_m(b) = \frac{-K_m(1-m)\cos\gamma_2}{\sqrt{1-\cos^2\gamma_2}} - \frac{2Ah_m(b)}{(1-\cos^2\gamma_2)^{3/2}}\,. \quad (3.140)$$

These become the boundary conditions for the ODE (3.132).

Now we solve for the boundary value problem given by the ODE (3.132), i.e.

$$g_m(q) - K_m = \frac{\partial}{\partial q}\left(\frac{K_m(1-m)A(q-q_0)(1-A^2(q-q_0)^2)}{2A} - \frac{f'_m(q)(1-A^2(q-q_0)^2)^{3/2}}{2A}\right)$$

and BCs (3.139)-(3.140). We first have to choose K_m so that the ODE (3.132) satisfies the two boundary conditions (3.139)-(3.140). Integrating both sides of the ODE (3.132) from b to a gives

$$\int_b^a g_m(s)ds - K_m(a-b) = \left(\frac{K_m(1-m)\cos\gamma_1(1-\cos^2\gamma_1)}{2A} - \frac{f'_m(a)(1-\cos^2\gamma_1)^{3/2}}{2A}\right) - \left(\frac{-K_m(1-m)\cos\gamma_2(1-\cos^2\gamma_2)}{2A} - \frac{f'_m(b)(1-\cos^2\gamma_2)^{3/2}}{2A}\right)\,. \quad (3.141)$$

Substituting BCs (3.139)-(3.140) into equation (3.141) gives

$$K_m = \frac{\int_b^a g_m(s)ds - (h_m(a) - h_m(b))}{a-b}\,. \tag{3.142}$$

We now integrate both sides of ODE (3.132) from b to q and obtain

$$\begin{aligned}&\int_b^q g_m(s)ds - K_m(q-b)\\ =& \left(\frac{K_m(1-m)A(q-q_0)(1-A^2(q-q_0)^2)}{2A} - \frac{f'_m(q)(1-A^2(q-q_0)^2)^{3/2}}{2A}\right)\\ &-\left(\frac{-K_m(1-m)\cos\gamma_2(1-\cos^2\gamma_2)}{2A} - \frac{f'_m(b)(1-\cos^2\gamma_2)^{3/2}}{2A}\right)\,.\end{aligned} \tag{3.143}$$

Substituting equation (3.140) into equation (3.143) gives

$$\begin{aligned}&\int_b^q g_m(s)ds - K_m(q-b)\\ =& \left(\frac{K_m(1-m)A(q-q_0)(1-A^2(q-q_0)^2)}{2A} - \frac{f'_m(q)(1-A^2(q-q_0)^2)^{3/2}}{2A}\right) - h_m(b)\,.\end{aligned} \tag{3.144}$$

Solving equation (3.144) for $f'_m(q)$ gives

$$f'_m(q) = K_m(1-m)\frac{A(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}} - \frac{2A\left(h_m(b) + \int_b^q g_m(s)\,ds - K_m\,(q-b)\right)}{(1-A^2(q-q_0)^2)^{3/2}}\,. \tag{3.145}$$

Integrating above gives

$$\begin{aligned} f_m(q) &= -\frac{K_m(1-m)\sqrt{1-A^2(q-q_0)^2}}{A} + \frac{K_m(1-m)}{A} \\ &\quad - \int_{q_0}^{q} \frac{2A\left(h_m(b) + \int_b^t g_m(s)\,ds - K_m\,(t-b)\right)}{\left(1-A^2(t-q_0)^2\right)^{3/2}} dt + C_2\,, \end{aligned} \tag{3.146}$$

where C_2 is an arbitrary constant of integration. For simplicity we choose $C_2 = -\frac{2K_m(1-m)}{A}$, i.e.

$$\begin{aligned} f_m(q) &= -\frac{K_m(1-m)\sqrt{1-A^2(q-q_0)^2}}{A} \\ &\quad - \int_{q_0}^{q} \frac{2A\left(h_m(b) + \int_b^t g_m(s)\,ds - K_m\,(t-b)\right)}{\left(1-A^2(t-q_0)^2\right)^{3/2}} dt\,. \end{aligned} \tag{3.147}$$

Choosing $f_m(q)$ and K_m as equations (3.147) and equation (3.142) gives equations (3.108)-(3.110), i.e.

$$\begin{aligned} \lim_{p\to\infty} \frac{\nabla\cdot Tu_m - u_m}{p^{-m+1}} &= 0\,, \\ \lim_{p\to\infty} \frac{\hat{q}\cdot Tu_m|_{q=a} - \cos\gamma_1}{p^{-m-1}} &= 0\,, \\ \lim_{p\to\infty} \frac{\hat{q}\cdot Tu_m|_{q=b} + \cos\gamma_2}{p^{-m-1}} &= 0\,. \end{aligned}$$

Since Tu_m can be written in the form of equation (3.121), we can show that both $\hat{q}\cdot Tu_m$ and $\nabla\cdot Tu_m - u_m$ can be written as a power series of p. Hence equations (3.108)-(3.110) implies

$$\nabla\cdot Tu_m = u_m + O\left(p^{-m}\right)\,, \tag{3.148}$$

$$\nu\cdot Tu_m|_{q=a} = \cos\gamma_1 + O\left(p^{-m-2}\right)\,, \tag{3.149}$$

$$\nu\cdot Tu_m|_{q=b} = \cos\gamma_2 + O\left(p^{-m-2}\right)\,. \tag{3.150}$$

Hence by mathematical induction,

$$\nabla \cdot Tu_n = u_n + O\left(p^{-n}\right) , \quad (3.151)$$

$$\nu \cdot Tu_n\big|_{q=a} = \cos\gamma_1 + O\left(p^{-n-2}\right) , \quad (3.152)$$

$$\nu \cdot Tu_n\big|_{q=b} = \cos\gamma_2 + O\left(p^{-n-2}\right) , \quad (3.153)$$

for sufficiently large p, for any $n \in \mathbb{Z}^+$.

■

3.5.3 The Lowest Order Approximation

We now aim to prove that the formal asymptotic series derived at section 3.1 has p^0th order accuracy. Similarly to the proof of Theorem 2.2, we will construct sub-solution and super-solution using the Comparison Principle (as discussed in section 1.3) to prove that the error between the real solution and the formal asymptotic series can be bounded by a constant.

Lemma 3.2 *Let $u(p,q)$ be the solution to the capillary BVP in a circular cusp region. Then there exist constants L_0 and p_0 such that*

$$\left|u(p,q) - \left(Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}p - A(q-q_0)^2 + Aq_0^2\right)\right| < L_0 \,, \quad (3.154)$$

for $p > p_0$.

Proof: Without loss of generality we assume $\cos\gamma_1, \cos\gamma_2 > 0$ so that $A > 0$. Other cases can be proven similarly. As motivated in Section 3.3 (equation (3.82))[¶], we choose

[¶]We have modified equation (3.82) by letting $K = 0$. This modification was motivated by the fact we have an additive constant l_0 in equation (3.155), which will serve a similar purpose as K in equation (3.82).

a comparison function $v(p,q)$ to be

$$v(p,q) = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}p - A(q-q_0)^2 + Aq_0^2 + l_0 + \frac{h(q)}{p^5}, \quad (3.155)$$

where $h(q)$ is a C^2 function. After some calculation (refer to Appendix A.3) we get

$$\begin{aligned}\hat{q}\cdot Tv &= A(q-q_0) + \left(4h'(q)\sqrt{1-A^2(q-q_0)^2} - (q-q_0)\right)\frac{1-A^2(q-q_0)^2}{8Ap^6} \\ &\quad + R_1(p,q;h(q)), \qquad (3.156)\\ \nabla\cdot Tv &= Ap^2 - 2p\sqrt{1-A^2(q-q_0)^2} - A(q-q_0)^2 + Aq_0^2 \\ &\quad + \frac{12\sqrt{1-A^2(q-q_0)^2}(-A^2(q-q_0))h'(q) + 4(1-A^2(q-q_0)^2)^{3/2}h''(q)}{8Ap^4} \\ &\quad - \frac{(1-3A^2(q-q_0)^2)}{8Ap^4} + R_2(p,q;h(q)), \qquad (3.157)\end{aligned}$$

where $R_1(p,q;h(q)) = O(p^{-7})$, $R_2(p,q;h(q)) = O(p^{-5})$ and are independent of l_0. We now require that $h(q)$ satisfies

$$4h'(q)\sqrt{1-A^2(q-q_0)^2} - (q-q_0) = C_1A(q-q_0), \quad (3.158)$$

where $-2 < C_1 < 1$ is a parameter we later choose to construct sub-solution and super-solutions. Following from equation (3.158) we obtain

$$\begin{aligned}h'(q) &= \frac{(C_1A+1)(q-q_0)}{4\sqrt{1-A^2(q-q_0)^2}}, \qquad (3.159)\\ h''(q) &= \frac{C_1A+1}{4\left(1-A^2(q-q_0)^2\right)^{3/2}}. \qquad (3.160)\end{aligned}$$

Substituting equations (3.159) and (3.160) into equations (3.156) and (3.157) gives

$$\hat{q} \cdot Tv = A(q-q_0) + C_1 A(q-q_0)\frac{1-A^2(q-q_0)^2}{8Ap^6} + R_1(p,q;h(q)), \quad (3.161)$$

$$\nabla \cdot Tv = Ap^2 - 2p\sqrt{1-A^2(q-q_0)^2} - A(q-q_0)^2 + Aq_0^2 + C_1\frac{1-3A^2(q-q_0)^2}{8p^4} + R_2(p,q;h(q)). \quad (3.162)$$

Following from equation (3.162) we have

$$\nabla \cdot Tv - v = -l_0 + C_1\frac{1-3A^2(q-q_0)^2}{8p^4} + R_3(p,q;C_1), \quad (3.163)$$

where $R_3(p,q;C_1) = O(p^{-5})$ and is independent of l_0. Integrating equation (3.159) gives

$$h(q) = -\frac{(C_1 A+1)\sqrt{1-A^2(q-q_0)^2}}{4A^2} + C_0, \quad (3.164)$$

where C_0 is an arbitrary constant of integration. We choose $C_0 = 0$ for simplicity. Substituting equation (3.164) into equation (3.155) gives

$$v(p,q) = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 + l_0 - \frac{(C_1 A+1)\sqrt{1-A^2(q-q_0)^2}}{4A^2p^5}. \quad (3.165)$$

We now aim to choose constants C_1 and l_0 so that $v(p,q)$ becomes a super-solution. By re-naming constants C_1 and l_0 we define a super-solution $v_+(p,q)$ as

$$v_+(p,q) = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 + l_0^+ - \frac{(C_+ A+1)\sqrt{1-A^2(q-q_0)^2}}{4A^2p^5}. \quad (3.166)$$

We now arbitrary choose a constant C_+ that satisfies $0 < C_+ < 1$. Following from equation (3.161) and evaluating it at $q = a$ and $q = b$[||] we obtain

$$\hat{q} \cdot Tv_+|_{q=a} = \cos\gamma_1 + C_+ \cos\gamma_1 \frac{1-\cos^2\gamma_1}{8Ap^6} + \tilde{R}_1(p,a)\,, \tag{3.167}$$

$$\hat{q} \cdot Tv_+|_{q=b} = -\cos\gamma_2 - C_+ \cos\gamma_2 \frac{1-\cos^2\gamma_2}{8Ap^6} + \tilde{R}_1(p,b)\,, \tag{3.168}$$

where $\tilde{R}_1(p,a), \tilde{R}_1(p,b)$ are $O(p^{-7})$. Since $C_+ > 0$, $A > 0$, there exists a sufficiently large p_0^+ such that[**]

$$\hat{q} \cdot Tv_+|_{q=a} > \cos\gamma_1 = \hat{q} \cdot Tu|_{q=a}\,, \qquad \text{for } p > p_0^+, \tag{3.169}$$

$$-\hat{q} \cdot Tv_+|_{q=b} > \cos\gamma_2 = -\hat{q} \cdot Tu|_{q=b}\,, \qquad \text{for } p > p_0^+. \tag{3.170}$$

Equations (3.169) and (3.169) reduces to

$$\nu \cdot Tv_+ > \nu \cdot Tu\,, \qquad \text{for } p > p_0^+. \tag{3.171}$$

It follows from equation (3.163) that

$$\nabla \cdot Tv_+ - v_+ = -l_0^+ + C_+ \frac{1-3A^2(q-q_0)^2}{8p^4} + \tilde{R}_3(p,q;C_+)\,, \tag{3.172}$$

where $\tilde{R}_3(p,q;C_+) = O(p^{-5})$. We now choose $\tilde{l}_0^+$ such that

$$\tilde{l}_0^+ \geq \max\left[\max_{b<q<a}\left\{C_+ \frac{1-3A^2(q-q_0)^2}{8{p_0^+}^4} + \tilde{R}_3(p_0^+,q;C_+)\right\}, 0\right]. \tag{3.173}$$

[||]It follows from equations (3.51) and (3.52) that $A(a-q_0) = \cos\gamma_1$ and $A(b-q_0) = -\cos\gamma_2$

[**]Following from the BCs of the Capillary BVP in a circular cusp region (equations (3.16) and (3.17)), $\hat{q} \cdot Tu|_{q=a} = \cos\gamma_1$, $-\hat{q} \cdot Tu|_{q=b} = \cos\gamma_2$.

It follows immediately from equation (3.173) that

$$\tilde{l}_0^+ > C_+ \frac{1 - 3A^2(q-q_0)^2}{8p^4} + \tilde{R}_3(p,q;C_+)\,, \qquad \text{for } p > p_0^+\,. \tag{3.174}$$

Following from the capillary PDE (3.15) and equations (3.172)-(3.174) we obtain[††]

$$\nabla \cdot Tv_+ - v_+ \;<\; 0 \;=\; \nabla \cdot Tu - u\,, \qquad \text{for } l_0^+ \geq \tilde{l}_0^+, p > p_0^+\,. \tag{3.175}$$

By Theorem 3.1, we know that the solution $u(p,q)$ is bounded above at $p = p_0^+$. It follows immediately from equations (3.18) and (3.19) that
if $b \geq \frac{1}{2\sqrt{2}+\frac{1}{a}}$, then

$$u(p_0^+, q) \;<\; 4\frac{ab + {p_0^+}^2}{a-b} + \frac{1}{2}\frac{a-b}{ab - {p_0^+}^2}\,,$$

if $b < \frac{1}{2\sqrt{2}+\frac{1}{a}}$, then

$$u(p_0^+, q) \;<\; \begin{cases} 4\frac{ab+{p_0^+}^2}{a-b} + \frac{1}{2}\frac{a-b}{ab-{p_0^+}^2}\,, & \text{for } p_0^+ > \sqrt{\frac{a-b}{2\sqrt{2}} - ab}\,, \\ 2\sqrt{2}\,, & \text{otherwise}\,. \end{cases}$$

Hence there exists a sufficiently large constant $l_0^+ \geq \tilde{l}_0^+$ such that

$$v_+(p_0^+, q) \;>\; u(p_0^+, q)\,. \tag{3.176}$$

By the comparison principle (Theorem 1.1) with equations (3.171), (3.175) and (3.176) we conclude that,

$$v_+(p,q) > u(p,q) \qquad \text{for } (p,q) \in \{p_0^+ < p, b < q < a\}\,. \tag{3.177}$$

[††]Since $\tilde{R}_3(p,q;C_+)$ is independent of l_0^+ this holds for all $l_0^+ \geq \tilde{l}_0^+$.

Similarly we can define a sub-solution

$$v_-(p,q) = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2 + l_0^- - \frac{(C_-A+1)\sqrt{1-A^2(q-q_0)^2}}{4A^2p^5}, \tag{3.178}$$

where $-2 < C_- < -1$ and choose constants l_0^- and p_0^- as we have done for the super-solution, and conclude that

$$v_-(p,q) < u(p,q) \qquad \text{for } (p,q) \in \{p_0^- < p, b < q < a\}. \tag{3.179}$$

We now choose

$$p_0 = \max\{p_0^+, p_0^-\} \tag{3.180}$$

$$L_0 = \max\left\{\left|l_0^-\right|, \left|l_0^+\right|\right\}. \tag{3.181}$$

It follows from equations (3.166), (3.177), (3.178) and (3.179) that

$$\left|u(p,q) - \left(Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2\right)\right| < L_0 \qquad \text{for } p > p_0. \tag{3.182}$$

■

3.5.4 The Error Estimate

We now prove the error estimates for each partial sum of the asymptotic series as described in Theorem 3.2.

Proof of Theorem 3.2: Without loss of generality we assume $\cos\gamma_1 + \cos\gamma_2 > 0$ so that $A > 0$, $\cos\gamma_1 + \cos\gamma_2 < 0$ case can be proven similarly.

Base case (u_0):

Following from equation (3.83) we have

$$u_0 = Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}p - A(q-q_0)^2 + Aq_0^2 \,. \tag{3.183}$$

It follows immediately from Lemma 3.2 (equation (3.154)) that there exist constants L_0 and p_0 such that

$$|u(p,q) - u_0(p,q)| < L_0 < \frac{L_0}{p_0}p \qquad \text{for } p > p_0. \tag{3.184}$$

Hence equation (3.91) is satisfied for $n = 0$.

Inductive step (u_m) :

We are given that there exist constants L_{m-1}, p_{m-1} such that

$$|u(p,q) - u_{m-1}(p,q)| < L_{m-1}p^{-m+2} \qquad \text{for } p > p_{m-1},\ m = 1,2,3,\ldots. \tag{3.185}$$

Now we construct a comparison function, $v_m(p,q)$ such that

$$v_m(p,q;l_m) = u_{m-1} + (l_m + K_m)p^{-m+1} + (U_m(q;l_m) + f_m(q))\,p^{-m}\,, \tag{3.186}$$

$$= u_m + l_m p^{-m+1} + U_m(q;l_m)p^{-m}\,, \tag{3.187}$$

where

$$U_m(q;l_m) = \frac{(m-1)\,l_m}{A}\sqrt{1-A^2(q-q_0)^2} + A\,l_m\frac{1-A^2(q-q_0)\left(\frac{a+b}{2}+q_0\right)}{A^2\sqrt{1-A^2(q-q_0)^2}}\,. \tag{3.188}$$

We choose l_m to construct a sub-solution and a super-solution.

Here we show how to construct and prove the super-solution. An identical method can be applied to the sub-solution.

We now aim to find l_m^+ such that to satisfy

$$u_{m-1}(\tilde{p_m}, q) + L_{m-1}\tilde{p_m}^{-m+1} \leq v_m(\tilde{p_m}, q; l_m^+) , \qquad \text{for } b < q < a , \tag{3.189}$$

where $\tilde{p_m}$ is an arbitrary constant such satisfy $\tilde{p_m} > p_{m-1}$. l_m^+ is dependent on $\tilde{p_m}$ and is independent of p and q. Substituting equation (3.186) into equation (3.189) gives

$$L_{m-1}\tilde{p_m}^{-m+1} \leq (l_m^+ + K_m)\tilde{p_m}^{-m+1} + \left(U_m(q; l_m^+) + f_m(q)\right) \tilde{p_m}^{-m} , \tag{3.190}$$

for $b < q < a$. Simplifying equation (3.190) and substituting equation (3.188) into it gives

$$\begin{aligned} &(L_{m-1} - K_m)\tilde{p_m} - f_m(q) \\ \leq\ & l_m^+ \left(\tilde{p_m} + \frac{m-1}{A}\sqrt{1 - A^2(q-q_0)^2} + A\frac{1 - A^2(q-q_0)\left(\frac{a+b}{2} + q_0\right)}{A^2\sqrt{1 - A^2(q-q_0)^2}} \right) , \end{aligned} \tag{3.191}$$

for $b < q < a$. Following from equation (3.191) we obtain

$$\frac{(L_{m-1} - K_m)\tilde{p_m}^{+1} - f_m(q)}{\tilde{p_m} + \frac{m-1}{A}\sqrt{1 - A^2(q-q_0)^2} + A\frac{1 - A^2(q-q_0)\left(\frac{a+b}{2} + q_0\right)}{A^2\sqrt{1 - A^2(q-q_0)^2}}} \leq l_m^+ \qquad \text{for } b < q < a. \tag{3.192}$$

We now choose $l_m^+(\tilde{p}_m)$ such as

$$l_m^+(\tilde{p_m}) = \max\left\{ \max_{b<q<a} \left(\frac{(L_{m-1} - K_m)\tilde{p_m} - f_m(q)}{\tilde{p_m} + \frac{m-1}{A}\sqrt{1 - A^2(q-q_0)^2} + A\frac{1 - A^2(q-q_0)\left(\frac{a+b}{2} + q_0\right)}{A^2\sqrt{1 - A^2(q-q_0)^2}}} \right), 0 \right\} . \tag{3.193}$$

From equations (3.185) and (3.189), we now obtain

$$u(\tilde{p_m}, q) \quad < \quad v_m(\tilde{p_m}, q; l_m^+)\,, \tag{3.194}$$

where $b < q < a <, \tilde{p_m} > p_{m-1}$.

Employing the similar method as in the proof of Lemma 3.1 (as described in equations (3.112)-(3.121)), we now aim to expand Tv_m asymptotically. In order to apply this method, we require that

$$\left| \frac{2\nabla\left((K_m + l_m^+)p^{1-m} + (U_m + f_m(q))p^{-m}\right)\cdot\nabla u_{m-1}}{1 + |\nabla u_{m-1}|^2} \right. \\ \left. + \frac{|\nabla\left((K_m + l_m^+)p^{1-m} + (U_m + f_m(q))p^{-m}\right)|^2}{1 + |\nabla u_{m-1}|^2} \right| < 1\,, \tag{3.195}$$

$$\left|\nabla\left((K_m + l_m^+)p^{1-m} + (U_m + f_m(q))p^{-m}\right)\right|^2 \\ << \left|2\nabla\left((K_m + l_m^+)p^{1-m} + (U_m + f_m(q))p^{-m}\right)\cdot\nabla u_{m-1}\right|\,, \tag{3.196}$$

$$\frac{1}{|\nabla u_{m-1}|} < 1\,, \tag{3.197}$$

satisfy. It follows immediately from equation (3.193) that

$$l_m^+ \quad = \quad O(\tilde{p_m}^0)\,, \qquad \text{for sufficiently large } \tilde{p_m}. \tag{3.198}$$

Hence it follows from equations (3.188) and (3.198) that

$$U_m(q; l_m^+) \quad = \quad O(\tilde{p_m}^0)\,, \qquad \text{for sufficiently large } \tilde{p_m}. \tag{3.199}$$

Equations (3.198) and (3.199) imply that there exists a constant $\tilde{p_m}$ such that equations (3.195)-(3.197) holds for any $p \geq \tilde{p_m}$. Hence we can derive the asymptotic expansion of Tv_m similarly to the way we have derived equation (3.121), as discussed in detail in equations (3.112)-(3.121). Following these discussions we obtain

$$Tv_m(p,q;l_m^+) = Tu_{m-1}\left(1 - \frac{2A(l_m^+ + K_m)(1-m) + \frac{2A^2(q-q_0)(f'(q)+U_m'(q;l_m^+))}{\sqrt{1-A^2(q-q_0)^2}}}{\frac{4A^2}{1-A^2(q-q_0)^2}}p^{-m-1}\right)$$
$$+\frac{((l_m^+ + K_m)(1-m))\,\hat{p} + (f_m'(q) + U_m'(q;l_m))\,\hat{q}}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m-1} + O\left(p^{-2m-2}\right). \tag{3.200}$$

For simplicity of writing, we shall refer to $v_m(p,q;l_m^+)$ as v_m^+. We now consider the left hand side of the capillary PDE, i.e. $\nabla \cdot Tv_m$. Given the differentiability of each term in a binomial series, we can differentiate the binomial series while maintaining the convergence of the series. As we have extensively used the idea of binomial series expansion to derive equation (3.200), we can differentiate it while maintaining the asymptotic relation.[‡‡] Hence it follows from equation (3.200) that

$$\begin{aligned}\nabla \cdot Tv_m^+ &= \nabla \cdot Tu_{m-1}\left(1 - \tilde{G_m}p^{-m-1}\right) - (Tu_{m-1}\cdot\hat{q})\,\frac{\partial}{\partial q}\tilde{G_m}p^{-m+1} \\ &\quad + \frac{\partial}{\partial q}\frac{f_m'(q) + U_m'(q;l_m)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m+1} + O(p^{-m}),\end{aligned} \tag{3.201}$$

where

$$\begin{aligned}\tilde{G_m}(q) &= G_m(q) + \frac{l_m^+(1-m)\,(1-A^2(q-q_0)^2)}{2A} \\ &\quad + \frac{U_m'(q;l_m^+)(q-q_0)\sqrt{1-A^2(q-q_0)^2}}{2}.\end{aligned} \tag{3.202}$$

[‡‡]Because of the differentiability of $f(q)$, $U_m(q;l_m^+)$ and u_{m-1}, each term of the binomial series we have used is differentiable.

It follows from equation (3.188) that

$$U'_m(q;l_m) \quad = \quad -\frac{(m-1)\,l_m A(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}} + l_m A \frac{q-\frac{a+b}{2}}{\left(1-A^2(q-q_0)^2\right)^{3/2}}\,. \qquad (3.203)$$

Substituting equation (3.203) into equation (3.202) gives

$$\tilde{G_m}(q) \quad = \quad G_m(q) + \frac{l_m^+}{2A}\left((1-m) + \frac{A\left(q-\frac{a+b}{2}\right)A(q-q_0)}{1-A^2(q-q_0)^2}\right). \qquad (3.204)$$

As we have chosen $f_m(q)$ and K_m in section 3.5.2 such that to satisfy $\nabla \cdot Tu_m = u_m + O\left(p^{-m}\right)$, following from equation (3.124) we obtain

$$\begin{aligned} \nabla \cdot Tu_m \quad &= \quad \nabla \cdot Tu_{m-1}\left(1 - G_m p^{-m-1}\right) - \left(Tu_{m-1}\cdot\hat{q}\right)\frac{\partial}{\partial q} G_m p^{-m+1} \\ &\quad + \frac{\partial}{\partial q}\frac{f'_m(q)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}} p^{-m+1} + O(p^{-m})\,, \qquad (3.205) \\ &= \quad u_m + O\left(p^{-m}\right)\,. \qquad (3.206) \end{aligned}$$

Substituting equations (3.204) into equation (3.201) and then substituting (3.206) into it gives

$$\begin{aligned} \nabla \cdot Tv_m^+ \quad &= \quad u_m - \nabla \cdot Tu_{m-1}\left(\frac{l_m^+}{2A}\left((1-m) + \frac{A\left(q-\frac{a+b}{2}\right)A(q-q_0)}{1-A^2(q-q_0)^2}\right)p^{-m-1}\right) \\ &\quad - \left(Tu_{m-1}\cdot\hat{q}\right)\frac{\partial}{\partial q}\frac{l_m^+}{2A}\left((1-m) + \frac{A\left(q-\frac{a+b}{2}\right)A(q-q_0)}{1-A^2(q-q_0)^2}\right)p^{-m+1} \\ &\quad + \frac{\partial}{\partial q}\frac{U'_m(q;l_m)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}} p^{-m+1} + O(p^{-m})\,. \qquad (3.207) \end{aligned}$$

Following from equation (3.92) we have

$$\nabla \cdot Tu_{m-1} \quad = \quad Ap^2 + O(p)\,. \qquad (3.208)$$

Substituting equations (3.126) (3.208) into (3.207) and simplifying it gives

$$\begin{aligned}\nabla \cdot T v_m^+ &= u_m - \frac{l_m^+}{2A}\frac{\partial}{\partial q}\left(A(q-q_0)\left((1-m) + \frac{A\left(q-\frac{a+b}{2}\right)A(q-q_0)}{1-A^2(q-q_0)^2}\right)p^{-m+1}\right) \\ &\quad + \frac{\partial}{\partial q}\frac{U_m'(q;l_m)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}}p^{-m+1} + O(p^{-m})\,. \end{aligned} \tag{3.209}$$

Substituting equation (3.203) into (3.209) gives

$$\begin{aligned}\nabla \cdot T v_m^+ &= u_m - \frac{l_m^+}{2A}\frac{\partial}{\partial q}\left(A(q-q_0)\left((1-m) + \frac{A\left(q-\frac{a+b}{2}\right)A(q-q_0)}{1-A^2(q-q_0)^2}\right)\right)p^{-m+1} \\ &\quad - \frac{l_m^+}{2A}\frac{\partial}{\partial q}\left((m-1)\,A(q-q_0) - A\frac{q-\frac{a+b}{2}}{(1-A^2(q-q_0)^2)}\right)p^{-m+1} + O(p^{-m})\,. \end{aligned} \tag{3.210}$$

Simplifying equation (3.210) by canceling terms inside of the derivatives gives

$$\nabla \cdot T v_m^+ = u_m + \frac{l_m^+}{2}\frac{\partial}{\partial q}\left(q - \frac{a+b}{2}\right)p^{-m+1} + O(p^{-m})\,. \tag{3.211}$$

Evaluating the derivative of equation (3.211) and subtracting v_m^+ (defined in equation (3.187)) from both sides gives

$$\nabla \cdot T v_m^+ - v_m^+ = \left(-l_m^+ + \frac{l_m^+}{2}\right)p^{-m+1} + O(p^{-m})\,, \tag{3.212}$$

$$= -\frac{l_m^+}{2}p^{-m+1} + O(p^{-m})\,. \tag{3.213}$$

Thus it follows from equation (3.213) that there exists a sufficiently large $\tilde{p_m}$ such that

$$\nabla \cdot T v_m^+ - v_m^+ < 0 \qquad \text{for } p > \tilde{p_m}\,. \tag{3.214}$$

We now consider boundary conditions $\hat{q} \cdot Tv_m^+$. Following from equation (3.200) we obtain

$$\begin{aligned}\hat{q} \cdot Tv_m^+ &= \hat{q} \cdot Tu_{m-1} \left(1 - \frac{2A(l_m^+ + K_m)(1-m) + \frac{2A^2(q-q_0)(f'(q)+U_m'(q;l_m^+))}{\sqrt{1-A^2(q-q_0)^2}}}{\frac{4A^2}{1-A^2(q-q_0)^2}} p^{-m-1} \right) \\ &\quad + \frac{f_m'(q) + U_m'(q;l_m)}{\frac{2A}{\sqrt{1-A^2(q-q_0)^2}}} p^{-m-1} + O\left(p^{-2m-2}\right) . \qquad (3.215)\end{aligned}$$

Evaluating equation (3.215) at $q = a$ gives

$$\begin{aligned}&\hat{q} \cdot Tv_m^+\big|_{q=a} \\ &= \hat{q} \cdot Tu_{m-1}\big|_{q=a} \left(1 - \frac{2A(l_m^+ + K_m)(1-m) + \frac{2A\cos\gamma_1(f'(a)+U_m'(a;l_m^+))}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}} p^{-m-1} \right) \\ &\quad + \frac{f_m'(a) + U_m'(a;l_m)}{\frac{2A}{\sqrt{1-\cos^2\gamma_1}}} p^{-m-1} + O\left(p^{-2m-2}\right) . \qquad (3.216)\end{aligned}$$

As we have chosen $f_m(q)$ and K_m in section 3.5.2 such that to satisfy $\hat{q} \cdot Tu_m\big|_{q=a} = \cos\gamma_1 + O\left(p^{-m-2}\right)$, following from equation (3.133) we obtain

$$\begin{aligned}\hat{q} \cdot Tu_m\big|_{q=a} &= \hat{q} \cdot Tu_{m-1}\big|_{q=a} \left(1 - \frac{2AK_m(1-m) + \frac{2A\cos\gamma_1 f_m'(a)}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}} p^{-m-1} \right) \\ &\quad + \frac{f_m'(a)}{\frac{2A}{\sqrt{1-\cos\gamma_1}}} p^{-m-1} + O\left(p^{-2m-2}\right) , \qquad (3.217) \\ &= \cos\gamma_1 + O\left(p^{-m-2}\right) \qquad (3.218)\end{aligned}$$

Substituting equation (3.218) into equation (3.216) gives

$$\hat{q}\cdot Tv_m^+\big|_{q=a} = \cos\gamma_1 + \hat{q}\cdot Tu_{m-1}\big|_{q=a}\left(-\frac{2A(l_m^+)(1-m)+\frac{2A\cos\gamma_1(U_m'(a;l_m^+))}{\sqrt{1-\cos^2\gamma_1}}}{\frac{4A^2}{1-\cos^2\gamma_1}}p^{-m-1}\right) + \frac{U_m'(a;l_m)}{\frac{2A}{\sqrt{1-\cos^2\gamma_1}}}p^{-m-1} + O\left(p^{-m-2}\right) . \tag{3.219}$$

Following from equation (3.203)

$$U_m'(a;l_m) = -\frac{(m-1)\,l_m\cos\gamma_1}{\sqrt{1-\cos^2\gamma_1}} + \frac{l_m A}{2}\frac{a-b}{(1-\cos^2\gamma_1)^{3/2}} . \tag{3.220}$$

Substituting equation (3.220) into equation (3.219) gives

$$\hat{q}\cdot Tv_m^+\big|_{q=a} = \cos\gamma_1 + \hat{q}\cdot Tu_{m-1}\big|_{q=a} \left(-\frac{2A(l_m^+)(1-m) - \frac{2A\cos\gamma_1\left((m-1)l_m^+\cos\gamma_1\right)}{1-\cos^2\gamma_1} + \frac{l_m^+A^2\cos\gamma_1(a-b)}{(1-\cos^2\gamma_1)^2}}{\frac{4A^2}{1-\cos^2\gamma_1}}\right)p^{-m-1} - \left(\frac{(m-1)\,l_m^+\cos\gamma_1}{2A} - l_m^+\frac{a-b}{4(1-\cos^2\gamma_1)}\right)p^{-m-1} + O\left(p^{-m-2}\right) . \tag{3.221}$$

Simplifying equation (3.221) gives

$$\hat{q}\cdot Tv_m^+\big|_{q=a} = \cos\gamma_1 + \hat{q}\cdot Tu_{m-1}\big|_{q=a}\left(-\frac{l_m^+(1-m)}{2A} - l_m^+\frac{\cos\gamma_1(a-b)}{4(1-\cos^2\gamma_1)}\right)p^{-m-1} - \left(\frac{(m-1)\,l_m^+\cos\gamma_1}{2A} - l_m^+\frac{a-b}{4(1-\cos^2\gamma_1)}\right)p^{-m-1} + O\left(p^{-m-2}\right) . \tag{3.222}$$

From equation (3.93) we have

$$\hat{q}\cdot Tu_{m-1}\big|_{q=a} = \cos\gamma_1 + O\left(p^{-m-1}\right) . \tag{3.223}$$

Substituting equation (3.223) into equation (3.221) gives

$$\begin{aligned}\hat{q}\cdot Tv_m^+\big|_{q=a} &= \cos\gamma_1 + \cos\gamma_1\left(-\frac{l_m^+(1-m)}{2A} - l_m^+\frac{\cos\gamma_1(a-b)}{4(1-\cos^2\gamma_1)}\right)p^{-m-1} \\ &\quad -\left(\frac{(m-1)\,l_m^+\cos\gamma_1}{2A} - l_m^+\frac{a-b}{4(1-\cos^2\gamma_1)}\right)p^{-m-1} + O\left(p^{-m-2}\right)\,.\end{aligned} \tag{3.224}$$

Simplifying equation (3.224) gives

$$\hat{q}\cdot Tv_m^+\big|_{q=a} = \cos\gamma_1 + l_m^+\frac{a-b}{4}p^{-m-1} + O\left(p^{-m-2}\right)\,. \tag{3.225}$$

Similarly evaluating equation (3.215) at $q=b$ gives

$$\hat{q}\cdot Tv_m^+\big|_{q=b} = -\cos\gamma_2 + l_m^+\frac{b-a}{4}p^{-m-1} + O\left(p^{-m-2}\right)\,. \tag{3.226}$$

Given $a>b$, following from equations (3.225) and (3.226) we can show that there exists a constant $\tilde{p_m}$ such that

$$\hat{q}\cdot Tv_m^+\big|_{q=a} > \cos\gamma_1\,, \quad \text{for } p>\tilde{p_m}\,, \tag{3.227}$$

$$-\hat{q}\cdot Tv_m^+\big|_{q=b} > \cos\gamma_2\,, \quad \text{for } p>\tilde{p_m}\,. \tag{3.228}$$

Following from equations (3.194), (3.214), (3.227) and (3.228) there exists a constant $\tilde{p_m}$ such satisfies

$$u(\tilde{p_m},q) < v_m(\tilde{p_m},q;l_m^+)\,, \tag{3.229}$$

$$\nabla\cdot Tv_m^+ - v_m^+ < 0 = \nabla\cdot Tu - u\,, \qquad \text{for } p>\tilde{p_m}\,, \tag{3.230}$$

$$\hat{q}\cdot Tv_m^+\big|_{q=a} > \cos\gamma_1 = \nu\cdot Tu\big|_{q=a}\,, \qquad \text{for } p>\tilde{p_m}\,, \tag{3.231}$$

$$-\hat{q}\cdot Tv_m^+\big|_{q=b} > \cos\gamma_2 = \nu\cdot Tu\big|_{q=b}\,, \qquad \text{for } p>\tilde{p_m}\,. \tag{3.232}$$

By the comparison principle we obtain

$$u(p,q) \;<\; v_m^+(p,q) \;=\; u_m + l_m^+ p^{-m+1} + U_m(q;l_m^+)p^{-m} \qquad \text{for } p > \tilde{p_m}\,. \quad (3.233)$$

Following from equation (3.233) we obtain

$$u(p,q) - \; u_m \;<\; l_m^+ p^{-m+1} + U_m(q;l_m^+)p^{-m} \qquad \text{for } p > \tilde{p_m}\,. \quad (3.234)$$

Similarly we can construct a sub-solution v_m^- and show that there exists a constant $\tilde{p_m}$ such satisfies

$$u(p,q) - \; u_m \;>\; l_m^- p^{-m+1} + U_m(q;l_m^-)p^{-m} \qquad \text{for } p > \tilde{p_m}\,. \quad (3.235)$$

Thus we can conclude that there exist constants L_m and p_m such satisfy

$$|u(p,q) - u_m(p,q)| \;<\; L_m p^{-m+1} \qquad \text{for } p > p_m\;. \quad (3.236)$$

Hence by mathematical induction there exist constants L_n and p_n such satisfy

$$|u(p,q) - u_n(p,q)| \;<\; L_n p^{-n+1} \qquad \text{for } p > p_n,\; n \in \mathbb{Z}^+\,. \quad (3.237)$$

Following from equation (3.83), we can re-write equation (3.237) as

$$\Big|u(p,q) - \big(u_{n-2}(p,q) + K_{n-1}p^{-n+2} + f_{n-1}p^{-n+1} + K_n p^{-n+1} + f_n p^{-n}\big)\Big| \;<\; L_n p^{-n+1}\,, \quad (3.238)$$

for $p > p_n$, $n = 2, 3, 4...$. Following from equation (3.87) we can show that $f_{n-1}(q)$ and $f_n(q)$ is bounded in the domain. This implies that there exist constants L_n and p_n such that

$$\left|u(p,q) - \left(u_{n-2}(p,q) + K_{n-1}p^{-n+2}\right)\right| \quad < \quad L_n p^{-n+1} \qquad \text{for } p > p_n,\ n = 2,3,4\ldots. \tag{3.239}$$

It follows from equations (3.239) that

$$\left|u(p,q) - \left(u_{n-1}(p,q) + K_n p^{-n+1}\right)\right| \quad < \quad L_{n+1} p^{-n} \qquad \text{for } p > p_{n+1},\ n = 1,2,3\ldots. \tag{3.240}$$

Following from equation (3.83) we have $u_{n-1}(p,q) + K_n p^{-n+1} = u_n(p,q) - f_n(q)p^{-n}$ for $n = 1,2,3\ldots.$ Substituting this into equation (3.240) gives

$$\left|u(p,q) - \left(u_n(p,q) - f_n(q)p^{-n}\right)\right| < \frac{L_{n+1}}{p^n}\,, \qquad \text{for } p > p_{n+1},\ n \in \mathbb{N}\,. \tag{3.241}$$

■

3.5.5 Accuracy of the Approximation of Section 3.3

In this section we will prove the accuracy of the approximate solution, which we have obtained in section 3.3 (equation (3.54)).

Theorem 3.3 *Let $u(p,q)$ be a solution to the capillary BVP in a circular cusp region. Then there exist constants L_6 and p_6 such that*

$$\left|u(p,q) - \left(Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2\right)\right| < \frac{L_6}{p^5}\,, \tag{3.242}$$

for $p > p_6$.

Proof:

We will apply Theorem 3.2 to prove this theorem. Let,

$$u_0(p,q) \;=\; Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - A(q-q_0)^2 + Aq_0^2\,. \tag{3.243}$$

Following from equations (3.96) and (3.97) we obtain

$$\nabla\cdot Tu_0 - u_0 \;=\; -\frac{1-3A^2(q-q_0)^2}{8Ap^4} + O(p^{-5})\,, \tag{3.244}$$

$$\hat{q}\cdot Tu_0 \;=\; A(q-q_0) - \frac{(q-q_0)(1-A^2(q-q_0)^2)}{8Ap^6} + O(p^{-7})\,. \tag{3.245}$$

Hence we have:

$$g_2(q),\, g_3(q),\, g_4(q) \;=\; 0\,, \tag{3.246}$$

$$g_5(q) \;=\; -\frac{1-3A^2(q-q_0)^2}{8A}\,, \tag{3.247}$$

$$h_2(a),\, h_3(a),\, h_4(a) \;=\; 0\,, \tag{3.248}$$

$$h_5(a) \;=\; -\frac{(a-q_0)(1-\cos^2\gamma_1)}{8A}\,, \tag{3.249}$$

$$h_2(b),\, h_3(b),\, h_4(b) \;=\; 0\,, \tag{3.250}$$

$$h_5(b) \;=\; -\frac{(b-q_0)(1-\cos^2\gamma_2)}{8A}\,. \tag{3.251}$$

Thus by Theorem 3.2 we can calculate K's. Following from equation (3.86), by inspection we obtain

$$K_2,\, K_3,\, K_4, \;=\; 0\,. \tag{3.252}$$

Also after some calculation we get

$$K_5 = 0\,. \tag{3.253}$$

Again from Theorem 3.2 we can calculate f's. Following from equation (3.87), by inspection we obtain

$$f_2(q),\ f_3(q),\ f_4(q) \;=\; 0\,. \tag{3.254}$$

After some calculation we get

$$f_5(q) \;=\; -\frac{\sqrt{1-A^2(q-q_0)^2}}{4A^2}\,. \tag{3.255}$$

This implies there are four zero-terms in asymptotic solution such that

$$\begin{aligned} u_5(p,q) \;=\;& Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - \left(A(q-q_0)^2 - Aq_0^2\right) + 0\frac{1}{p} + 0\frac{1}{p^2} + 0\frac{1}{p^3} \\ & +0\frac{1}{p^4} - \frac{\sqrt{1-A^2(q-q_0)^2}}{4A^2}\frac{1}{p^5}\,. \end{aligned} \tag{3.256}$$

Following from equation (3.256) we have

$$u_5(p,q) - f_5(q)p^{-5} \;=\; u_5(p,q) - \left(-\frac{\sqrt{1-A^2(q-q_0)^2}}{4A^2}\frac{1}{p^5}\right), \tag{3.257}$$

$$\;=\; Ap^2 - 2\sqrt{1-A^2(q-q_0)^2}\,p - \left(A(q-q_0)^2 - Aq_0^2\right) \tag{3.258}$$

Thus by Theorem 3.2 we conclude equation (3.242) holds.

■

Theorem 3.5.5 implies that the solution of the approximated BVP (as defined in equations (3.37)-(3.39)) is in fact first three terms of the asymptotic series solution. Also this first three terms of the asymptotic solution possesses the accuracy equivalent to the first seven terms of the asymptotic series solution. This unexpectedly accurate approximation is due to the fact the approximated BVP is solvable exactly. Solvability of the approximated BVP depends on the choice of coordinate system, and it turns out that the circular cylinder coordinate system is a suitable choice for this problem. (So far we have observed only one other case, which the approximated BVP were exactly solvable. As we have discussed in Appendix B the approximated BVP for a wedge problem was exactly solvable in polar coordinate system, and it turns out to be a very accurate solution to the original problem.) Also the approximated BVP cannot be solved exactly in the coordinate system Scholz has used in his paper [7]. This approximation technique and the accurate approximation is our original result.

Chapter 4

Odd Boundary Conditions: Supplementary Contact Angles

In this chapter we will discuss the case $\cos\gamma_1 + \cos\gamma_2 = 0$, which we did not consider in previous chapters. Not many results for this type of contact angle conditions are known. It was stated in Scholz' paper that for a cusp region "[t]he case $\gamma_1 = \pi - \gamma_2$ keeps an open question" ([7], page 234).* In this chapter we aim to address some aspects of this open question. We shall refer such contact angle conditions as *supplementary contact angles*.

4.1 Odd Boundary Condition

We consider a region Ω that is symmetric about the x-axis, i.e.

$$\Omega = \Omega_+ \cup \Omega_- , \tag{4.1}$$

*$\cos\gamma_1 + \cos\gamma_2 = 0$ for $0 < \gamma_2 < \frac{\pi}{2}$ implies $\gamma_1 = \pi - \gamma_2$.

where

$$\Omega_+ \;=\; \Omega \cap \{(x,y) \in \mathbb{R}^2 : y \geq 0\}\,, \tag{4.2}$$

$$\Omega_- \;=\; \{(x,-y) : (x,y) \in \Omega_+\}\,. \tag{4.3}$$

The upper and lower boundaries Γ_+ and Γ_- are given by

$$\Gamma_+ \;=\; \partial\Omega \cap \{(x,y) \in \mathbb{R}^2 : y > 0\}\,, \tag{4.4}$$

$$\Gamma_- \;=\; \partial\Omega \cap \{(x,y) \in \mathbb{R}^2 : y < 0\}\,. \tag{4.5}$$

We assume there is a cusp/corner at the origin opening in the positive x-direction as in Figure 4.1. Consider a rectangle

$$R \;=\; \{(x,y) :\in \mathbb{R}^2 : 0 < x < x_1, 0 < y < y_1\}, \tag{4.6}$$

and let

$$\Omega_1^+ \;=\; \Omega_+ \cap R\,. \tag{4.7}$$

We choose x_1 and y_1 small enough[†] that the subset of the boundary in Ω_1^+, i.e. $\Gamma_+ \cap R$, is given by

$$y = f(x) \qquad 0 < x < x_1\,, \tag{4.8}$$

where $f(x)$ is a piecewise smooth function such that

$$\lim_{x \to 0^+} f(x) \;=\; 0\,. \tag{4.9}$$

[†]Note x_1 may be finite or infinite.

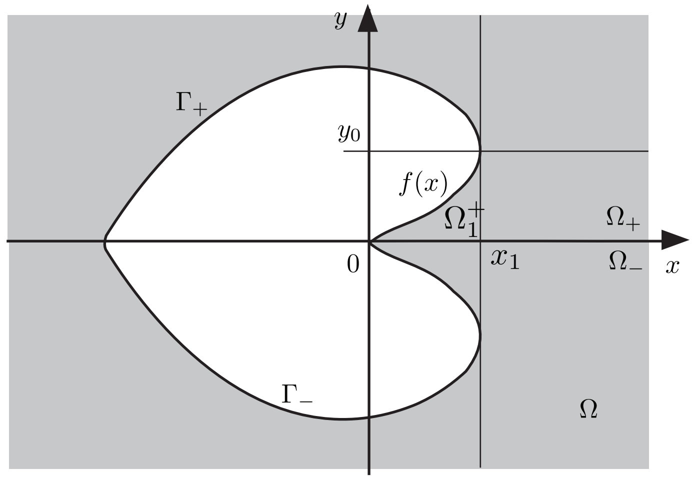

Figure 4.1: Symmetric Region with Boundaries.

The derivative will satisfy

$$\lim_{x\to 0^+} f'(x) \;=\; \begin{cases} 0 & \text{for a cusp region}\,, \\ \textit{constant} & \text{for a corner region}\,. \end{cases} \tag{4.10}$$

We shall refer to region Ω_1^+ as a *near cusp/corner region.*

In the present situation with the restriction $\cos\gamma_1 + \cos\gamma_2 = 0$, the capillary BVP assumes the form[‡]

$$\nabla \cdot Tu = u \qquad \text{in } \Omega\,, \tag{4.11}$$

$$\nu \cdot Tu = \cos\gamma \qquad \text{on } \Gamma_+\,, \tag{4.12}$$

$$\nu \cdot Tu = -\cos\gamma \qquad \text{on } \Gamma_-\,. \tag{4.13}$$

We shall refer to this boundary value problem as the *capillary BVP with odd BCs.*

In this chapter we assume $\cos\gamma > 0$. Similar results for $\cos\gamma < 0$ cases can be derived using a negative solution (see remark 3 in section 1.2.2).

[‡]The BCs (4.12) and (4.13) correspond to the BCs (2.10) and (2.11) when the restriction $\cos\gamma_1 + \cos\gamma_2 = 0$ is imposed.

First we show how we can apply the comparison principle (Theorem 1.1) to the capillary BVP with odd BCs.

Theorem 4.1 (the Comparison Principle for Odd BCs) *Let $u(x, y)$ be a solution of the capillary BVP with odd BCs.*
Case A: If

$$\nabla \cdot Tv - v \gtreqless 0 \qquad in\ \Omega_+\,, \tag{4.14}$$

$$\nu \cdot Tv \lesseqgtr \cos\gamma \qquad on\ \Gamma_+\,, \tag{4.15}$$

$$v = 0 \qquad on\ y = 0\,, \tag{4.16}$$

then

$$v(x, y) \quad \lesseqgtr \quad u(x, y) \qquad in\ \Omega_+\,, \tag{4.17}$$

$$-v(x, -y) \quad \gtreqless \quad u(x, y) \qquad in\ \Omega_-\,. \tag{4.18}$$

Case B: In this case we consider a near cusp/corner region. Let Ω_0^+ be defined as

$$\Omega_0^+ \quad = \quad \{(x, y) \in \mathbb{R}^2 : 0 < x < x_0, 0 \ < y < f(x)\}\,, \tag{4.19}$$

where $x_0 < x_1$. If

$$\nabla \cdot Tv - v \gtreqless 0 \qquad in\ \Omega_0^+\,, \tag{4.20}$$

$$\nu \cdot Tv \lesseqgtr \cos\gamma \qquad on\ \{(x, y) : 0 < x < x_0, y = f(x)\}\,, \tag{4.21}$$

$$v = 0 \qquad on\ y = 0\,, \tag{4.22}$$

$$v \lesseqgtr u \qquad on\ \{(x, y) : x = x_0, 0 < y < f(x)\}\,, \tag{4.23}$$

then

$$\begin{aligned} v(x,y) &\leqslant u(x,y) \qquad \text{in } \Omega_+\,, && (4.24)\\ -v(x,-y) &\geqslant u(x,y) \qquad \text{in } \Omega_-\,. && (4.25) \end{aligned}$$

In order to prove this theorem, it is necessary to show that the solution surface $u(x,y)$ is odd with respect to the x-axis.

Lemma 4.1 *Let $u(x,y)$ be a solution to the capillarity BVP with odd BCs, then $u(x,y)$ is an odd function with respect to y, i.e.*

$$u(x,y) = -u(x,-y)\,. \tag{4.26}$$

Proof:

Let

$$\tilde{u}(x,y) = -u(x,-y)\,. \tag{4.27}$$

It follows from equation (4.27) that

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial x}(x,y) &= -\frac{\partial u}{\partial x}(x,-y)\,, && (4.28)\\ \frac{\partial \tilde{u}}{\partial y}(x,y) &= \frac{\partial u}{\partial y}(x,-y)\,, && (4.29)\\ \frac{\partial^2 \tilde{u}}{\partial x^2}(x,y) &= -\frac{\partial^2 u}{\partial x^2}(x,-y)\,, && (4.30)\\ \frac{\partial^2 \tilde{u}}{\partial^2 y}(x,y) &= -\frac{\partial^2 u}{\partial y^2}(x,-y)\,, && (4.31)\\ \frac{\partial^2 \tilde{u}}{\partial x \partial y}(x,y) &= \frac{\partial^2 u}{\partial x \partial y}(x,-y)\,. && (4.32) \end{aligned}$$

Expanding the left hand side of the PDE using the definition of T in equation (1.42) gives

$$\nabla \cdot T\tilde{u}(x,y) = \frac{\tilde{u}_{xx}(x,y)\left(1+\tilde{u}_y^2(x,y)\right)+\tilde{u}_{yy}(x,y)\left(1+\tilde{u}_x^2(x,y)\right)}{\left(1+\tilde{u}_x^2(x,y)+\tilde{u}_y^2(x,y)\right)^{3/2}} - \frac{2\tilde{u}_x(x,y)\tilde{u}_y(x,y)\tilde{u}_{xy}(x,y)}{\left(1+\tilde{u}_x^2(x,y)+\tilde{u}_y^2(x,y)\right)^{3/2}}. \tag{4.33}$$

Substituting equations (4.28)-(4.32) into equation (4.33) gives

$$\nabla \cdot T\tilde{u}(x,y) = \frac{-u_{xx}(x,-y)\left(1+u_y^2(x,-y)\right)-u_{yy}(x,-y)\left(1+u_x^2(x,-y)\right)}{\left(1+u_x^2(x,-y)+u_y^2(x,-y)\right)^{3/2}} + \frac{2u_x(x,-y)u_y(x,-y)u_{xy}(x,-y)}{\left(1+u_x^2(x,-y)+u_y^2(x,-y)\right)^{3/2}}. \tag{4.34}$$

By comparing the right hand side of equation (4.34) to equation (4.33), we observe that equation (4.34) can be written as

$$\nabla \cdot T\tilde{u}(x,y) = -\nabla \cdot Tu(x,-y). \tag{4.35}$$

Replacing y by $-y$ in equation (4.11) gives

$$\nabla \cdot Tu(x,-y) = u(x,-y). \tag{4.36}$$

It now follows from equations (4.27), (4.35) and (4.36) that

$$\nabla \cdot T\tilde{u}(x,y) = \tilde{u}(x,y). \tag{4.37}$$

Hence $\tilde{u}$ satisfies the capillary PDE (4.11).

Let ν_+ and ν_- be the unit outward normal vectors on the boundaries Γ_+ and Γ_-, respectively. Also let ν_x and ν_y be the x and y components of unit outward normal vector ν_+. By the symmetry, the unit outward normal vector on Γ_- can be written as $\nu_- = (\nu_x, -\nu_y)$. Also it follows from equations(4.1)-(4.5) that

$$(x, y) \in \Gamma_+ \quad \Longleftrightarrow \quad (x, -y) \in \Gamma_- \,, \tag{4.38}$$

$$(x, -y) \in \Gamma_+ \quad \Longleftrightarrow \quad (x, y) \in \Gamma_- \,. \tag{4.39}$$

Following from the definition of T (equation (1.42)), BCs (4.12) and (4.13) can be expanded as

$$\nu_+ \cdot Tu(x,y)\big|_{(x,y)\in\Gamma_+} = (\nu_x, \nu_y) \cdot \left.\frac{u_x\hat{x} + u_y\hat{y}}{\sqrt{1+u_x^2+u_y^2}}\right|_{(x,y)\in\Gamma_+} = \cos\gamma \,, \tag{4.40}$$

$$\Rightarrow \quad \left.\frac{\nu_x u_x + \nu_y u_y}{\sqrt{1+u_x^2+u_y^2}}\right|_{(x,y)\in\Gamma_+} = \cos\gamma \,, \tag{4.41}$$

$$\nu_- \cdot Tu(x,y)\big|_{(x,y)\in\Gamma_-} = (\nu_x, -\nu_y) \cdot \left.\frac{u_x\hat{x} + u_y\hat{y}}{\sqrt{1+u_x^2+u_y^2}}\right|_{(x,y)\in\Gamma_-} = -\cos\gamma \,, \tag{4.42}$$

$$\Rightarrow \quad \left.\frac{\nu_x u_x - \nu_y u_y}{\sqrt{1+u_x^2+u_y^2}}\right|_{(x,y)\in\Gamma_-} = -\cos\gamma \,. \tag{4.43}$$

Expanding $\nu \cdot T\tilde{u}$ gives

$$\nu_+ \cdot T\tilde{u}(x,y)\big|_{(x,y)\in\Gamma_+} = \left.\frac{\nu_x \tilde{u}_x(x,y) + \nu_y \tilde{u}_y(x,y)}{\sqrt{1+\tilde{u}_x^2(x,-y)+\tilde{u}_y^2(x,-y)}}\right|_{(x,y)\in\Gamma_+} . \tag{4.44}$$

Substituting equations (4.28)-(4.32) into equation (4.44) gives

$$\nu_+ \cdot T\tilde{u}(x,y)\big|_{(x,y)\in\Gamma_+} = -\left.\frac{\nu_x u_x(x,-y) - \nu_y u_y(x,-y)}{\sqrt{1+u_x^2(x,-y)+u_y^2(x,-y)}}\right|_{(x,y)\in\Gamma_+} . \tag{4.45}$$

Using equation (4.38) it follows from equation (4.45) that

$$\nu_+ \cdot T\tilde{u}(x,y)\big|_{(x,y)\in\Gamma_+} = -\left.\frac{\nu_x u_x(x,-y) - \nu_y u_y(x,-y)}{\sqrt{1+u_x^2(x,-y)+u_y^2(x,-y)}}\right|_{(x,-y)\in\Gamma_-} . \quad (4.46)$$

Substituting equation (4.43) into gives

$$\nu_+ \cdot T\tilde{u}(x,y)\big|_{(x,y)\in\Gamma_+} = -(-\cos\gamma) = \cos\gamma\,. \quad (4.47)$$

Similarly we can show that

$$\nu_- \cdot T\tilde{u}(x,y)\big|_{(x,y)\in\Gamma_-} = -\cos\gamma\,. \quad (4.48)$$

Hence $\tilde{u}$ satisfies the boundary conditions (4.12) and (4.13). Following from the uniqueness of the solution to the capillary BVP we have $u = \tilde{u}$, i.e.

$$u(x,y) = -u(x,-y)\,. \quad (4.49)$$

$\blacksquare$

Proof of Theorem 4.1:

Following from Lemma 4.1 we have

$$u(x,0) = -u(x,-0)\,, \quad (4.50)$$

$$\Rightarrow \quad u(x,0) = 0\,. \quad (4.51)$$

Case A: Substituting equations (4.11)-(4.12) into equations (4.14)-(4.15) gives

$$\nabla \cdot Tv - v \gtrless \nabla \cdot Tu - u \qquad \text{in } \Omega_+ \,, \tag{4.52}$$

$$\nu \cdot Tv \lessgtr \nu \cdot Tu \qquad \text{on } \Gamma_+ \,. \tag{4.53}$$

Also substituting equation (4.51) into equation (4.16) gives

$$v(x,0) = u(x,0)\,. \tag{4.54}$$

Apply the comparison principle (Theorem 1.1) to the region Ω_+ with the boundary $\partial\Omega_+ = \Sigma^\alpha \cup \Sigma^\beta \cup \Sigma^0$, where

$$\Sigma^\alpha = \{y = 0\}\,, \tag{4.55}$$

$$\Sigma^\beta = \Gamma_+\,, \tag{4.56}$$

$$\Sigma^0 = \{(0,0)\}\,. \tag{4.57}$$

By the comparison principle

$$v(x,y) \lessgtr u(x,y)\,, \qquad \text{in } \Omega_+. \tag{4.58}$$

Now we consider the region Ω_-. Multiplying -1 to equation (4.58) gives

$$-v(x,y) \gtrless -u(x,y)\,, \qquad \text{in } \Omega_+. \tag{4.59}$$

Substituting equation (4.49) into equation (4.59) gives

$$-v(x,y) \gtrless u(x,-y) \qquad \text{for } (x,y) \in \Omega_+\,. \tag{4.60}$$

$$\Rightarrow \qquad -v(x,y) \gtrless u(x,-y) \qquad \text{for } (x,-y) \in \Omega_+\,. \tag{4.61}$$

Following from equation (4.5) we have $(x,-y) \in \Omega_+ \iff (x,y) \in \Omega_-$. It follows that

$$-v(x,-y) \;\gtreqless\; u(x,y) \qquad \text{for } (x,y) \in \Omega_- \,. \tag{4.62}$$

<u>Case B:</u> Similarly we are given equations (4.20)-(4.23). It follows that

$$\begin{aligned}
\nabla \cdot Tv - v \gtreqless 0 \;=\; \nabla \cdot Tu - u \qquad & \text{in } \Omega_0^+ \,, & (4.63)\\
\nu \cdot Tv \lesseqgtr \cos\gamma \;=\; \nu \cdot Tu \qquad & \text{on } f(x) \,, & (4.64)\\
v \;=\; 0 \;=\; u \qquad & \text{on } y = 0 \,, & (4.65)\\
v \lesseqgtr u \qquad & \text{on } \{(x,y) : x = x_0, 0 < y < f(x_0)\} \,. & (4.66)
\end{aligned}$$

We now apply the comparison principle (Theorem 1.1) in the region Ω_0^+ with the boundary $\partial\Omega_+ = \Sigma^\alpha \cup \Sigma^\beta \cup \Sigma^0$, where

$$\begin{aligned}
\Sigma^\alpha &= \{(x,y) : 0 < x < x_0, y = 0\} \cup \{(x,y) : x = x_0, 0 < y < f(x_0)\} \,, & (4.67)\\
\Sigma^\beta &= \{(x,y) : 0 < x < x_0, y = f(x)\} \,, & (4.68)\\
\Sigma^0 &= \{(0,0)\} \cup \{(x_0,0)\} \cup \{(x_0, f(x_0))\} \,. & (4.69)
\end{aligned}$$

By the comparison principle,

$$v(x,y) \;\lesseqgtr\; u(x,y) \,, \qquad \text{for } (x,y) \in \Omega_0^+ \,. \tag{4.70}$$

Following the same argument as equations (4.59)-(4.62) we obtain

$$\begin{aligned}
& -v(x,y) \;\gtreqless\; -u(x,y) \qquad \text{for } (x,y) \in \Omega_0^+ \,, & (4.71)\\
\Rightarrow \quad & -v(x,y) \;\gtreqless\; u(x,-y) \qquad \text{for } (x,y) \in \Omega_0^+ \,, & (4.72)\\
\Rightarrow \quad & -v(x,-y) \;\gtreqless\; u(x,y) \qquad \text{for } (x,y) \in \Omega_0^- \,. & (4.73)
\end{aligned}$$

■

4.2 Bounded Solution at a Cusp and a Corner

In this section we aim to explain the qualitative behaviour of a solution to the capillary BVP with odd BCs.

Theorem 4.2 (Bounded Solution at a Cusp and a Corner) *Let $u(x,y)$ be a solution to the capillary BVP with odd BCs. If the function $f(x)$, which is defined by equation (4.8), satisfies the following conditions*

$$\sup_{0<x<x_0} |f'(x)| \; < \; \tan\gamma \,, \qquad \textit{where } x_0 \le x_1 \,, \tag{4.74}$$

$$f(x) \; \in \; C^1 \cap \textit{piecewise } C^4 \,, \qquad \textit{in } 0 < x < x_0 \,, \tag{4.75}$$

then there exists a constant K such satisfies

$$0 \; < \; u(x,y) \; < \; Ky \qquad \textit{for } 0 < y < f(x) \,, \tag{4.76}$$

$$0 \; > \; u(x,y) \; > \; Ky \qquad \textit{for } -f(x) < y < 0 \,, \tag{4.77}$$

with $0 < x < x_0$. Here x_0 may be finite or infinite.

Proof:

Let the comparison function $v(x,y)$ be

$$v(x,y) \;\; = \;\; K_1 y \,, \tag{4.78}$$

where $K_1 > 0$ is a constant. We now compare this function to $u(x, y)$ in the region Ω_0^+ given by

$$\Omega_0^+ \quad = \quad \left\{(x, y) \in \mathbb{R}^2 : 0 < x < x_0, 0 < y < f(x)\right\} . \tag{4.79}$$

By inspection we get

$$v(x, 0) \quad = \quad 0\,, \qquad \text{for } 0 < x < x_0\,, \tag{4.80}$$

$$Tv(x, y) \quad = \quad \frac{K_1 \hat{y}}{\sqrt{1 + K_2^2}}\,. \tag{4.81}$$

It follows from equation (4.81) that

$$\nabla \cdot Tv - v \quad = \quad -K_1 y\,, \qquad \text{for } (x, y) \in \Omega_0^+\,. \tag{4.82}$$

Following from equation (4.82) and substituting the capillary PDE (4.11) gives

$$Tv - v \quad < \quad 0 \; = \; \nabla \cdot Tu - u\,, \qquad \text{for } (x, y) \in \Omega_0^+\,. \tag{4.83}$$

Unit outward normal vector ν on the boundary $y = f(x)$ is given by

$$\nu \quad = \quad \frac{-f'(x)\hat{x} + \hat{y}}{\sqrt{1 + (f'(x))^2}}\,. \tag{4.84}$$

It follows from equations (4.81) and (4.84) that

$$\nu \cdot Tv \quad = \quad \frac{1}{\sqrt{1 + (f'(x))^2}} \frac{K_1}{\sqrt{1 + K_1^2}}\,. \tag{4.85}$$

Since $\frac{K_1}{\sqrt{1+K_1^2}}$ is a monotonically increasing function such that

$$0 < \frac{K_1}{\sqrt{1+K_1^2}} < 1\,, \tag{4.86}$$

if

$$\frac{1}{\sqrt{1+(f'(x))^2}} \quad > \quad \cos\gamma\,, \tag{4.87}$$

then there exists K_2 such satisfies

$$\frac{1}{\sqrt{1+(f'(x))^2}}\frac{K_1}{\sqrt{1+K_1^2}} \quad > \quad \cos\gamma\,, \qquad \text{for all } K_1 \geq K_2. \tag{4.88}$$

<u>Case A:</u> Suppose that $x_0 = \infty$, following from this we have

$$\sup_{0<x} |f'(x)| \quad < \quad \tan\gamma\,, \tag{4.89}$$

then

$$\frac{1}{\sqrt{1+(f'(x))^2}}\frac{K_2}{\sqrt{1+K_2^2}} \quad > \quad \cos\gamma\,. \tag{4.90}$$

Thus by Theorem 4.1 Case A

$$u(x,y) \quad < \quad K_2 y \qquad \text{for } 0 < y < f(x)\,, \tag{4.91}$$

$$u(x,y) \quad > \quad K_2 y \qquad \text{for } -f(x) < y < 0\,. \tag{4.92}$$

Case B: Suppose that x_0 is finite. By Theorem 1.3 $u(x,y) \in C^2$, and by Lemma 4.1 $u(x,0) = 0$, we can show that there exists a constant K_3 sufficiently large such satisfies

$$K_1 y \;>\; u(x_0, y)\,, \qquad \text{for } K_1 \geq K_3 \geq K_2. \tag{4.93}$$

By Theorem 4.1 Case B we conclude

$$u(x,y) \;<\; K_3 y \qquad \text{for } 0 < y < f(x)\,, \tag{4.94}$$

$$u(x,y) \;>\; K_3 y \qquad \text{for } -f(x) < y < 0\,, \tag{4.95}$$

with $0 < x < x_0$. We can also prove the lower bound of $u(x,y)$ in Ω^+. By letting $K = 0$, i.e. $v(x,y) = 0$, we obtain

$$\nabla \cdot Tv - v \;=\; 0 \;=\; \nabla \cdot Tu - u, \qquad \text{for } (x,y) \in \Omega^+\,, \tag{4.96}$$

$$\nu \cdot Tv \;=\; 0 \;<\; \cos\gamma \;=\; \nu \cdot Tu, \qquad \text{for } (x,y) \in \Gamma_+\,, \tag{4.97}$$

$$v(x,0) \;=\; 0\,. \tag{4.98}$$

By Theorem 4.1 Case A we conclude

$$0 \;<\; u(x,y) \qquad \text{for } 0 < y < f(x)\,, \tag{4.99}$$

$$0 \;>\; u(x,y) \qquad \text{for } -f(x) < y < 0\,. \tag{4.100}$$

■

Theorem 4.2 gives that

$$\lim_{(x,y)\to(0,0)} u(x,y) \;=\; 0 \tag{4.101}$$

in any path from inside of domain Ω. Thus we know that $u(x, y)$ is not only bounded at $(0, 0)$ but can be defined to be continuous at $(0, 0)$.

Since we have $\lim_{x \to 0^+} f'(x) = 0$ for a cusp region, there exists a constant x_0 such that to satisfy $\sup_{0<x<x_0} |f'(x)| < \tan \gamma$. Thus a capillary surface in a cusp region with odd BCs is bounded at the cusp and also can be defined to be continuous at the cusp.

In a corner region with opening angle less than 2γ, the condition $\sup_{0<x<x_0} |f'(x)| < \tan \gamma$ satisfies. Thus a capillary surface in a corner region with opening angle less than 2γ with odd BCs is bounded at the corner and also can be defined to be continuous at the corner. The continuity at the corner with odd BCs is also proved by Lancaster and Siegel [3]. However for the case of opening angle greater than 2γ is still an open question.

We can write upper and lower bound of the solution function as

$$-Kf(x) < u(x, y) < Kf(x) \,, \qquad \text{for } 0 < x < x_0 \,, \; -f(x) < y < f(x) \,. \tag{4.102}$$

As shown in Chapter 2 in the case of $\gamma_2 \neq \pi - \gamma_1$, the capillary surface is unbounded at a cusp. However, as shown in this chapter, the capillary surface at a symmetric cusp is bounded in the case of $\gamma_2 = \pi - \gamma_1$. Hence this is an another example of capillary surfaces depending discontinuously on the boundary data.

4.3 Circular Cusp Region with Odd Boundary Conditions

In this section we consider a capillary surface in an infinite bath of water with two equi-radii cylinders with supplementary contact angles tangent to each other. As in Chapter 3 we make use of the tangent cylindrical coordinate system. Let $u(p, q)$ be a function for height of a capillary surface in region Ω, then it satisfies

$$\nabla \cdot Tu - u = 0 \qquad \text{in } \Omega\,, \tag{4.103}$$

$$\hat{q} \cdot Tu = \cos\gamma \qquad \text{on } q = a\,, \tag{4.104}$$

$$-\hat{q} \cdot Tu = -\cos\gamma \qquad \text{on } q = -a\,. \tag{4.105}$$

where

$$\Omega = \{(p,q) : 0 < p < \infty, -a < q < a\}\ . \tag{4.106}$$

We shall refer to this BVP as the *capillary BVP in a circular cusp region with odd BCs.*

Theorem 4.3 (Global Bound) *Let $u(p,q)$ be a solution to the capillary BVP in a circular cusp region with odd BCs, then*

$$0 < u(p,q) < \frac{\cot\gamma}{a^2} q \qquad \textit{for } q > 0 \tag{4.107}$$

$$0 > u(p,q) > \frac{\cot\gamma}{a^2} q \qquad \textit{for } q < 0 \tag{4.108}$$

in $(p,q) \in \Omega$.

Proof:

Let the comparison function $v(p,q)$ be

$$v(p,q) = \frac{\cot\gamma}{a^2} q\,. \tag{4.109}$$

After some calculation we get

$$\hat{q} \cdot Tv = (p^2 + q^2) \frac{\frac{\cot\gamma}{a^2}}{\sqrt{1 + (p^2+q^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2}}\,, \tag{4.110}$$

$$\Rightarrow \qquad \nu \cdot Tv\big|_{q=a} = \frac{(p^2+a^2)\frac{\cot\gamma}{a^2}}{\sqrt{1 + (p^2+a^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2}}\,. \tag{4.111}$$

Following from the fact $\frac{\xi}{\sqrt{1+\xi^2}}$ is a monotonically increasing function with respect to ξ and $(p^2+a^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2 > (a^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2$, we obtain

$$\nu \cdot Tv\big|_{q=a} > \frac{(a^2)\frac{\cot\gamma}{a^2}}{\sqrt{1 + (a^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2}} \tag{4.112}$$

$$= \cos\gamma\,. \tag{4.113}$$

Also after some calculation we obtain

$$\nabla \cdot Tv_{-1} - v_{-1} = -2 \frac{\left(\frac{\cot\gamma}{a^2}\right)^3 (p^2+q^2)^3 q}{\left(1 + (p^2+q^2)^2 \left(\frac{\cot\gamma}{a^2}\right)^2\right)^{3/2}} - \frac{\cot\gamma}{a^2} q\,, \tag{4.114}$$

$$< 0 \qquad \text{for } q > 0. \tag{4.115}$$

By Theorem 4.1 Case A, it follows that

$$u(p,q) < \frac{\cot\gamma}{a^2} q \qquad \text{for } q > 0 \tag{4.116}$$

$$u(p,q) > \frac{\cot\gamma}{a^2} q \qquad \text{for } q < 0\,. \tag{4.117}$$

By inspection, applying Theorem 4.1 Case A to $v(p,q) = 0$ gives

$$0 < u(p,q) \qquad \text{for } q > 0\,, \tag{4.118}$$

$$0 > u(p,q) \qquad \text{for } q < 0\,. \tag{4.119}$$

Thus we conclude

$$0 < u(p,q) < \frac{\cot\gamma}{a^2} q \qquad \text{for } q > 0\,, \tag{4.120}$$
$$0 > u(p,q) > \frac{\cot\gamma}{a^2} q \qquad \text{for } q < 0\,, \tag{4.121}$$

in $(p,q) \in \Omega$.

■

<u>Note:</u> This bound is valid in the entire region Ω.

Now we give a formal asymptotic series approximation to a solution of the capillary BVP in a circular cusp region with odd BCs.

Theorem 4.4 (The Formal Asymptotic Series) *Let $u_n(p,q)$ be*

$$u_n(p,q) = \sum_{j=0}^{n} \sum_{i=0}^{j} A_{ji} q^{2i+1} p^{-2j-2}\,, \tag{4.122}$$

where constants A_{ji} are defined recursively as

$$A_{00} = \cot\gamma \tag{4.123}$$
$$A_{j0} = = -\frac{\sum_{i=1}^{j+1} b_{j+1,i} a^{2i-2} - \sum_{i=1}^{j} \frac{c_{ji}}{2i} a^{2i}}{\sin^3\gamma}\,, \tag{4.124}$$
$$A_{ji} = -\frac{c_{ji}}{\sin^3\gamma(2i+1)2i} \qquad \textit{for } i = 1,2,3,...,j\ , \tag{4.125}$$

with b_{ji} and c_{ji} given as

$$\nu \cdot Tu_{j-1}\big|_{q=a} = \cos\gamma + \left(\sum_{i=1}^{j+1} b_{ji} a^{2i-2}\right) p^{-2j} + O(p^{-2(j+1)})\,, \tag{4.126}$$
$$\nabla \cdot Tu_{j-1} - u_{j-1} = \sum_{i=1}^{j} c_{ji} q^{2i-1} p^{-2j+2} + O(p^{-2j})\,, \tag{4.127}$$

then

$$\nabla \cdot T u_n = u_n + O\left(p^{-2n-2}\right) \qquad \textit{in } \Omega \,, \tag{4.128}$$

$$\hat{q} \cdot T u_n = \cos\gamma + O\left(p^{-2n}\right) \qquad \textit{for } q = a \,, \tag{4.129}$$

$$-\hat{q} \cdot T u_n = -\cos\gamma + O\left(p^{-2n}\right) \qquad \textit{for } q = b \,, \tag{4.130}$$

for all $n \in \mathbb{Z}^+$ *for sufficiently large* p.

Proof: Prove this by mathematical induction.

Base case (u_0): Following from equation (4.122) we have

$$u_0 = \frac{\cot \gamma q}{p^2} \,. \tag{4.131}$$

After some calculation we get

$$\nabla \cdot T u_0 \sim 6 \sin^2 \gamma \cos \gamma q \,, \qquad \text{as } p \to \infty \,, \tag{4.132}$$

$$\hat{q} \cdot T u_0\big|_{q=a} \sim \cos\gamma + a^2 \cos\gamma \frac{1 - 3\cos^2\gamma}{p^2} \,, \qquad \text{as } p \to \infty \,. \tag{4.133}$$

Hence equations (4.128) and (4.129) are satisfied for $n = 0$.

Inductive Step (u_m):

We are given that

$$u_{m-1} = \sum_{j=0}^{m-1} \sum_{i=0}^{j} A_{ji} q^{2i+1} p^{-2j-2} \,, \tag{4.134}$$

$$\hat{q} \cdot T u_{m-1}\big|_{q=a} \sim \cos\gamma + f_m(a) p^{-2m} \,, \tag{4.135}$$

$$\nabla \cdot T u_{m-1} - u_{m-1} \sim g_m(q) p^{-2m+2} \,, \tag{4.136}$$

as $p \to \infty$. Given equation (4.134) expand Tu_{m-1}

$$\begin{aligned}
& Tu_{m-1} \\
= \quad & \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q}\right)}{\sqrt{1+(p^2+q^2)^2\left(\left(\frac{\partial u_{m-1}}{\partial p}\right)^2 + \left(\frac{\partial u_{m-1}}{\partial q}\right)^2\right)}}\,, \qquad (4.137) \\
= \quad & \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q}\right)}{\sqrt{1+(p^4+2p^2q^2+q^4)\left(\left(\frac{\partial u_{m-1}}{\partial p}\right)^2 + \left(\frac{\partial u_0}{\partial q}\right)^2 + 2\frac{\partial u_0}{\partial q}\frac{\partial u_{m-1}-u_0}{\partial q} + \left(\frac{\partial u_{m-1}-u_0}{\partial q}\right)^2\right)}}\,, \\
= \quad & \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q}\right)}{\sqrt{1+p^4\left(\frac{\partial u_0}{\partial p}\right)^2 + \frac{F(p,q)}{\sin^2\gamma}}}\,, \qquad (4.138)
\end{aligned}$$

where

$$\begin{aligned}
F(p,q) \quad = \quad & \left((2p^2q^2+q^4)\left(\frac{\partial u_{m-1}}{\partial p}^2 + \frac{\partial u_0}{\partial q}^2 + 2\frac{\partial u_0}{\partial q}\frac{\partial u_{m-1}-u_0}{\partial q} + \frac{\partial u_{m-1}-u_0}{\partial q}^2\right)\right. \\
& \left. + p^4\left(\frac{\partial u_{m-1}}{\partial p}^2 + 2\frac{\partial u_0}{\partial q}\frac{\partial u_{m-1}-u_0}{\partial q} + \frac{\partial u_{m-1}-u_0}{\partial q}^2\right)\right)\sin^2\gamma\,. \qquad (4.139)
\end{aligned}$$

Following from equation (4.122) we have $1+p^4\left(\frac{\partial u_0}{\partial p}\right)^2 = \frac{1}{\sin^2\gamma}$. Substituting this into equation (4.138) gives

$$\begin{aligned}
Tu_{m-1} \quad = \quad & \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q}\right)}{\sqrt{\frac{1}{\sin^2\gamma} + \frac{F(p,q)}{\sin^2\gamma}}}\,, \qquad (4.140) \\
= \quad & \sin\gamma\left((p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q}\right)\right)\frac{1}{\sqrt{1+F(p,q)}}\,. \qquad (4.141)
\end{aligned}$$

Expanding $\frac{1}{\sqrt{1+F(p,q)}}$ using binomial series gives

$$Tu_{m-1} = \sin\gamma \left((p^2+q^2) \left(\frac{\partial u_{m-1}}{\partial p}\hat{p} + \frac{\partial u_{m-1}}{\partial q}\hat{q} \right) \right) \left(1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} F^k \right). \tag{4.142}$$

By equation (4.134) we can show that

$$\frac{\partial u_{m-1}}{\partial p} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} A_{ji} q^{2i+1}(-2j-2)p^{-2j-3}, \tag{4.143}$$

$$\frac{\partial^2 u_{m-1}}{\partial p^2} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} A_{ji} q^{2i+1}(-2j-2)(-2j-3)p^{-2j-4}, \tag{4.144}$$

$$\frac{\partial u_{m-1}}{\partial q} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} A_{ji}(2i+1) q^{2i} p^{-2j-2}, \tag{4.145}$$

$$\frac{\partial^2 u_{m-1}}{\partial q^2} = \sum_{j=1}^{m-1}\sum_{i=1}^{j} A_{ji}(2i+1)(2i) q^{2i-1} p^{-2j-2}, \tag{4.146}$$

$$\frac{\partial^2 u_{m-1}}{\partial p \partial q} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} A_{ji}(2i+1) q^{2i}(-2j-2) p^{-2j-3}. \tag{4.147}$$

Following from equations (4.143)-(4.147), we can show that $F(p,q) < 1$ for sufficiently large p. Hence equation (4.142) is a convergent series for sufficiently large p.

We now consider the boundary condition. It follows from equation (4.142) that

$$\hat{q} \cdot Tu_{m-1} = \sin\gamma \left((p^2+q^2) \left(\frac{\partial u_{m-1}}{\partial q} \right) \right) \left(1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} F^k \right). \tag{4.148}$$

Substituting equations (4.143)-(4.147) into equation (4.139) implies that there exists a set of constants K_{ji} given by

$$F(p,q) = \sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} . \tag{4.149}$$

Substituting equation (4.149) into equation (4.148) gives

$$\begin{aligned} \hat{q} \cdot Tu_{m-1} &= \sin\gamma \left((p^2+q^2) \left(\sum_{j=0}^{m-1} \sum_{i=0}^{j} A_{ji}(2i+1) q^{2i} p^{-2j-2} \right) \right) \\ &\cdot \left(1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} \left(\sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} \right)^k \right) . \end{aligned} \tag{4.150}$$

It follows from equation (4.150) that there exists a set of constants b_{ji} given by

$$\hat{q} \cdot Tu_{m-1} = \sum_{j=1}^{\infty} \sum_{i=1}^{j} b_{ji} q^{2i-2} p^{-2j+2} . \tag{4.151}$$

Equating equation (4.135) and (4.151) gives

$$b_{11} = \cos\gamma , \tag{4.152}$$

$$f_m(a) = \sum_{i=1}^{m+1} b_{m+1,i} a^{2i-2} . \tag{4.153}$$

Also following from equation (4.153) we obtain

$$\nu \cdot Tu_{m-1} - \cos\gamma - \sum_{i=1}^{m+1} b_{m+1,i} q^{2i-2} p^{-2m} = O\left(p^{-2m-2}\right) . \tag{4.154}$$

Similarly we aim to expand the left hand side of the capillary PDE, i.e. $\nabla T u_{m-1}$. Since equation (4.142) is a power series in p and q we can differentiate the series. Thus we obtain

$$\begin{aligned}
\nabla\cdot Tu_{m-1} &= (p^2+q^2)^2\frac{\partial}{\partial p}\sin\gamma\left(\frac{\partial u_{m-1}}{\partial p}\right)\left(1+\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}F^k\right)\\
&\quad+(p^2+q^2)^2\frac{\partial}{\partial q}\sin\gamma\left(\frac{\partial u_{m-1}}{\partial q}\hat{q}\right)\left(1+\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}F^k\right)\\
&= (p^2+q^2)^2\sin\gamma\left(\left(\frac{\partial^2 u_{m-1}}{\partial p^2}\right)\left(1+\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}F^k\right)\right.\\
&\quad+\left(\frac{\partial u_{m-1}}{\partial p}\right)\left(\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}kF^{k-1}\frac{\partial F}{\partial p}\right)\\
&\quad+\left(\frac{\partial^2 u_{m-1}}{\partial q^2}\right)\left(1+\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}F^k\right)\\
&\quad\left.+\left(\frac{\partial u_{m-1}}{\partial q}\right)\left(\sum_{k=1}^{\infty}\frac{\prod_{l=1}^{k}\left(-\frac{2l-1}{2}\right)}{k!}kF^{k-1}\frac{\partial F}{\partial q}\right)\right)
\end{aligned} \tag{4.155}$$

Following from equation (4.149) we have

$$\frac{\partial F}{\partial p}(p,q) = \sum_{j=0}^{2m+1}\sum_{i=0}^{j}-2jK_{ji}q^{2j}p^{-2j-1} \tag{4.156}$$

$$\frac{\partial F}{\partial q}(p,q) = \sum_{j=0}^{2m+1}\sum_{i=0}^{j}2jK_{ji}q^{2j-1}p^{-2j}\,. \tag{4.157}$$

Substituting equation (4.156) and (4.157) into equation (4.155) gives

$$
\begin{aligned}
\nabla \cdot T u_{m-1} \;=\;& (p^2+q^2)^2 \sin\gamma \Bigg(\left(\sum_{j=0}^{m-1} \sum_{i=0}^{j} A_{ji} q^{2i+1} (-2j-2)(-2j-3) p^{-2j-4} \right) \\
& \cdot \left(1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} \left(\sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} \right)^{k} \right) \\
& + \left(\sum_{j=0}^{m-1} \sum_{i=0}^{j} A_{ji} q^{2i+1} (-2j-2) p^{-2j-3} \right) \\
& \cdot \left(\sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} k \left(\sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} \right)^{k-1} \right. \\
& \left. \cdot \sum_{j=0}^{2m+1} \sum_{i=0}^{j} -2j K_{ji} q^{2j} p^{-2j-1} \right) \\
& + \left(\sum_{j=1}^{m-1} \sum_{i=1}^{j} A_{ji} (2i+1)(2i) q^{2i-1} p^{-2j-2} \right) \\
& \cdot \left(1 + \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} \left(\sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} \right)^{k} \right) \\
& + \left(\sum_{j=0}^{m-1} \sum_{i=0}^{j} A_{ji} (2i+1) q^{2i} p^{-2j-2} \right) \\
& \cdot \left(\sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k} \left(-\frac{2l-1}{2}\right)}{k!} k \left(\sum_{j=0}^{2m+1} \sum_{i=0}^{j} K_{ji} q^{2j} p^{-2j} \right)^{k-1} \right. \\
& \left. \cdot \sum_{j=0}^{2m+1} \sum_{i=0}^{j} 2j K_{ji} q^{2j-1} p^{-2j} \right) \Bigg)
\end{aligned}
\tag{4.158}
$$

Following from equation (4.158) we can show that there exists a set of constants C_{ji} given by

$$
\nabla \cdot T u_{m-1} \;=\; \sum_{j=1}^{\infty} \sum_{i=1}^{j} C_{ji} q^{2i-1} p^{-2j+2} \,. \tag{4.159}
$$

It follows from equation (4.134) and equation (4.159) that there exists a set of constants c_{ji} given by

$$\nabla \cdot T u_{m-1} - u_{m-1} = \sum_{j=1}^{\infty} \sum_{i=1}^{j} c_{ji} q^{2i-1} p^{-2j+2} \,. \tag{4.160}$$

Hence by equating equation (4.136) and (4.160) we obtain $g_m(q)$ to be in a form

$$g_m(q) = \sum_{i=1}^{m} c_{mi} q^{2i-1}. \tag{4.161}$$

Also from equation (4.160) we obtain

$$\nabla \cdot T u_{m-1} - u_{m-1} - \sum_{i=1}^{m} c_{mi} q^{2i-1} p^{-2m+2} = O\left(p^{-2m}\right) \qquad \text{for sufficiently large } p. \tag{4.162}$$

We now let the next order term u_m to be

$$u_m = u_{m-1} + \sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2(m+1)} \,. \tag{4.163}$$

After some calculation (refer to Appendix A.4) we obtain

$$\begin{aligned} \nabla \cdot T u_m &= \nabla \cdot T u_{m-1} + \sin^3\gamma \sum_{i=1}^{m} A_{mi}(2i+1)2iq^{2i-1} p^{-2m+2} \\ &\quad + \sum_{i=1}^{m+1} D_{mi} q^{2i-1} p^{-2m} + O(p^{-2m-2}) \,. \end{aligned} \tag{4.164}$$

It follows from equation (4.164) that

$$\begin{aligned}\nabla \cdot T u_m - u_m \;&=\; \sin^3\gamma \sum_{i=1}^{m} A_{mi}(2i+1)2iq^{2i-1}p^{-2m+2} \\ &\quad + \sum_{i=1}^{m+1} D_{mi}q^{2i-1}p^{-2m} + O(p^{-2m-2}) \\ &\quad + \sum_{i=1}^{m} c_{mi}q^{2i-1}p^{-2m+2} - \sum_{i=0}^{m} A_{mi}q^{2i+1}p^{-2(m+1)} + O(p^{-2m})\,.\end{aligned} \tag{4.165}$$

Simplifying equation (4.165) gives

$$\nabla \cdot T u_m - u_m \;=\; \sum_{i=1}^{m} \left(\sin^3\gamma A_{mi}(2i+1)2i + c_{mi}\right) q^{2i-1}p^{-2m+2}\,. \tag{4.166}$$

Now now choose A_{mi} so that

$$\sin^3\gamma A_{mi}(2i+1)2i + c_{mi} \;=\; 0\,. \tag{4.167}$$

Solving equation (4.167) for A_{mi} gives

$$A_{mi} \;=\; -\frac{c_{mi}}{\sin^3\gamma(2i+1)2i}\,, \qquad \text{for } i = 1, 2, 3 \ldots m\,. \tag{4.168}$$

Choosing A_{mi} as in equation (4.168) gives

$$\nabla \cdot T u_m - u_m = O(p^{-2m})\,. \tag{4.169}$$

After some calculation (refer to Appendix A.4)

$$
\begin{aligned}
\hat{q}\cdot Tu_m\big|_{q=a} \sim\ & \cos\gamma + \left(\sum_{i=1}^{m+1} b_{m+1,i}a^{2i-2} + \sin^3\gamma \sum_{i=0}^{m} A_{mi}(2i+1)a^{2i}\right) p^{-2m} \\
& + \left(\sin\gamma a^2 \sum_{i=0}^{m} A_{mi}(2i+1)a^{2i} - \frac{\sin^2\gamma\cos\gamma}{2}\left(C + 2a^2 B\right)\right. \\
& \left. -2a^2\sin\gamma \sum_{i=0}^{m} A_{mi}(2i+1)a^{2i} + \frac{4a^2 B\sin^2\gamma\cos\gamma}{2}\right) p^{-2(m+1)}\,,
\end{aligned} \tag{4.170}
$$

as $p \to \infty$. Following from equation (4.170) we obtain

$$
\begin{aligned}
\hat{q}\cdot Tu_m\big|_{q=a} =\ & \cos\gamma + \left(\sum_{i=1}^{m+1} b_{m+1,i}a^{2i-2} + \sin^3\gamma \sum_{i=0}^{m} A_{mi}(2i+1)a^{2i}\right) p^{-2m} \\
& + O\left(p^{-2m-2}\right)\,, \qquad \text{for sufficiently large } p\,.
\end{aligned} \tag{4.171}
$$

Now we choose A_{m0} such that

$$
\sum_{i=1}^{m+1} b_{m+1,i}a^{2i-2} + \sin^3\gamma \sum_{i=0}^{m} A_{mi}(2i+1)a^{2i} = 0\,. \tag{4.172}
$$

Solving equation (4.172) for A_{m0} gives

$$
A_{m0} = -\frac{\sum_{i=1}^{m+1} b_{m+1,i}a^{2i-2} + \sin^3\gamma \sum_{i=1}^{m} A_{mi}(2i+1)a^{2i}}{\sin^3\gamma}\,. \tag{4.173}
$$

Substituting equation (4.168) into equation (4.173) gives

$$
A_{m0} = -\frac{\sum_{i=1}^{m+1} b_{m+1,i}a^{2i-2} - \sum_{i=1}^{m} \frac{c_{mi}}{2i}a^{2i}}{\sin^3\gamma}\,. \tag{4.174}
$$

Choosing A_{m0} as in equation (4.174) gives

$$\hat{q} \cdot Tu_m\big|_{q=a} = \cos\gamma + O(p^{-2m-2}) \,. \tag{4.175}$$

Hence we have

$$u_m = \sum_{j=0}^{m}\sum_{i=0}^{j} A_{ji} q^{2i+1} p^{-2j-2} \,, \tag{4.176}$$

where

$$A_{m0} = -\frac{\sum_{i=1}^{m+1} b_{m+1,i} a^{2i-2} - \sum_{i=1}^{m} \frac{c_{mi}}{2i} a^{2i}}{\sin^3\gamma} \,, \tag{4.177}$$

$$A_{mi} = -\frac{c_{mi}}{\sin^3\gamma(2i+1)2i} \,, \qquad \text{for } i = 1, 2, 3 \ldots m \,. \tag{4.178}$$

$$\hat{q} \cdot Tu_m\big|_{q=a} \sim \cos\gamma + f_{m+1}(a) p^{-2m-2} \,, \qquad \text{as } p \to \infty \,, \tag{4.179}$$

$$\nabla \cdot Tu_m - u_m \sim g_{m+1}(q) p^{-2m} \,, \qquad \text{as } p \to \infty \,. \tag{4.180}$$

Hence equations (4.128) and (4.129) are satisfied for $n = m$. By Mathematical Induction, we have proved that

$$u_n = \sum_{j=0}^{n}\sum_{i=0}^{j} A_{ji} q^{2i+1} p^{-2j-2} \,, \tag{4.181}$$

$$\hat{q} \cdot Tu_n\big|_{q=a} = \cos\gamma + O\left(p^{-2n-2}\right) \,, \tag{4.182}$$

$$\nabla \cdot Tu_n = u_n + O\left(p^{-2n}\right) \,, \tag{4.183}$$

for sufficiently large p, for all $n \in \mathbb{Z}^+$. Following from the fact equation (4.181) is odd with respect to q we have

$$-\,\hat{q}\cdot T u_n\big|_{q=-a} \;=\; -\cos\gamma + O\left(p^{-2n-2}\right) . \qquad (4.184)$$

Hence equations (4.128) and (4.129) are satisfied for all $n \in \mathbb{Z}^+$.

■

Chapter 5

Future Work

Currently, Theorem 2.2 can only be applied to cusp regions, which satisfy equations (2.60)-(2.62). However, we suspect these conditions can be reduced because we have been unable to find any cusps that did not satisfy these equations. We would also like to prove that the leading order term of the formal asymptotic expansion, equation (2.113), is actually the leading order of the asymptotic solution. This may be proven by finding the third order formal asymptotic expansion and by using the comparison principle. The technique we introduced in Section 3.3 gives an accurate approximation to the capillary BVP in circular cusp regions. We would like to expand the application of this technique from the wedge problem (see Appendix B) to other types of cusps as well. Finally, as in Section 3.5, we would like to use the comparison principle to prove that the formal asymptotic series found in Theorem 4.4 is also the complete asymptotic series.

As a course project, I have used an iterative finite difference method to numerically compute solutions to the capillary BVPs. These numerical simulations were only successful for solutions without singularities and not for those with singularities. We would like to demonstrate that the solutions to the capillary BVPs both with and without singularities may be obtained by using a least square finite element method.

Appendix A

Calculations and Justifications

A.1 Calculation for Theorem 2.2

We are given that

$$
\begin{aligned}
f_1(x) - f_2(x) &= o(f_1'(x) - f_2'(x))\,, && \text{(A.1)}\\
f_1''(x) - f_2''(x) &= o\left(\frac{f_1'(x) - f_2'(x)}{f_1(x) - f_2(x)}\right)\,, && \text{(A.2)}\\
f_1'''(x) - f_2'''(x) &= o\left(\frac{f_1'(x) - f_2'(x)}{(f_1(x) - f_2(x))^2}\right)\,, && \text{(A.3)}
\end{aligned}
$$

as $x \to 0$. We now aim to calculate

$$\nu \cdot Tv|_{t=1}\,, \tag{A.4}$$

$$\nu \cdot Tv|_{t=-1}\,, \tag{A.5}$$

$$\nabla \cdot Tv - v\,, \tag{A.6}$$

where

$$v = \frac{A}{f_1(s) - f_2(s)} + g(t)\frac{f_1'(s) - f_2'(s)}{f_1(s) - f_2(s)} + C_0\,, \tag{A.7}$$

with

$$s := x\,, \tag{A.8}$$

$$t := \frac{2y-(f_1(x)+f_2(x))}{f_1(x)-f_2(x)}\,. \tag{A.9}$$

Equations (A.4) and (A.5) can be written as

$$\nu\cdot Tv|_{t=1} = \frac{(-f_1'(s),1)}{\sqrt{1+f_1'^2(s)}}\cdot\frac{(v_x,v_y)}{\sqrt{1+v_x^2+v_y^2}}\,, \tag{A.10}$$

$$\nu\cdot Tv|_{t=-1} = \frac{(f_2'(s),-1)}{\sqrt{1+f_1'^2(s)}}\cdot\frac{(v_x,v_y)}{\sqrt{1+v_x^2+v_y^2}}\,. \tag{A.11}$$

By the chain rule, we re-write the above formula in terms of new coordinate variables s and t:

$$v_x = \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x}\,, \tag{A.12}$$

$$v_y = \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y}\,. \tag{A.13}$$

Following from equation (A.7) we have

$$\frac{\partial v}{\partial s} = -\frac{A(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}+g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}-g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\,, \tag{A.14}$$

$$\sim -\frac{A(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}+g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}\,, \tag{A.15}$$

$$\frac{\partial v}{\partial t} = g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}\,. \tag{A.16}$$

Following from equation (A.9) we obtain

$$\frac{\partial t}{\partial x} = -\frac{t(f_1'(s)-f_2'(s))+(f_1'(s)+f_2'(s))}{f_1(s)-f_2(s)}\,, \tag{A.17}$$

$$\frac{\partial t}{\partial y} = \frac{2}{f_1(s)-f_2(s)}\,. \tag{A.18}$$

Substituting equations (A.17) and (A.18) into equations (A.12) and (A.13) gives

$$\begin{aligned}
v_x &\sim -g'(t)\frac{f_1'(s)-f_2'(s)}{f_1(s)-f_2(s)}\frac{t(f_1'(s)-f_2'(s))+(f_1'(s)+f_2'(s))}{f_1(s)-f_2(s)} \\
&\quad -\frac{A(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}+g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}-g(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^2}\,, \qquad \text{(A.19)} \\
&\sim -\frac{A(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}+g(t)\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}\,, \qquad \text{(A.20)} \\
&\quad \left(\text{Assume: } \left|\frac{(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}\right| >> \left|\frac{f_1''(s)-f_2''(s)}{f_1(s)-f_2(s)}\right|\right) \\
&\sim -A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\,, \qquad \text{(A.21)} \\
v_y &= 2g'(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\,, \qquad \text{(A.22)}
\end{aligned}$$

then substituting equations (A.21) and (A.22) into equations (A.10) and (A.11) gives

$$\begin{aligned}
\nu\cdot Tv|_{t=1} &\sim \frac{(-f_1'(s),1)}{\sqrt{1+f_1'^2(s)}}\cdot\frac{\left(-A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2},\,2g'(1)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)}{\sqrt{1+\left(-A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)^2+\left(2g'(1)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)^2}}\,, \qquad \text{(A.23)} \\
&\quad \left(\text{Assume: } \left|\frac{(f_1'(s)-f_2'(s))}{(f_1(s)-f_2(s))^2}\right| >> 1\right) \\
&\sim \frac{Af_1'(s)+2g'(1)}{\sqrt{A^2+4g'^2(1)}}\,, \qquad \text{(A.24)} \\
\nu\cdot Tv|_{t=-1} &\sim \frac{(f_2'(s),-1)}{\sqrt{1+f_1'^2(s)}}\cdot\frac{\left(-A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2},\,2g'(-1)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)}{\sqrt{1+\left(-A\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)^2+\left(2g'(-1)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^2}\right)^2}}\,, \\
&\sim -\frac{Af_2'(s)+2g'(-1)}{\sqrt{A^2+4g'^2(-1)}}\,. \qquad \text{(A.25)}
\end{aligned}$$

We now consider the left hand side of the PDE (A.6). By the definition of T we can expand $\nabla \cdot Tv$ as

$$\begin{aligned}
\nabla \cdot Tv &= \frac{\partial}{\partial x}\frac{v_x}{\sqrt{1+v_x^2+v_y^2}} + \frac{\partial}{\partial y}\frac{v_y}{\sqrt{1+v_x^2+v_y^2}}\,, && \text{(A.26)}\\
&= \frac{v_{xx}}{\sqrt{1+v_x^2+v_y^2}} - \frac{v_x\left(v_x v_{xx} + v_y v_{xy}\right)}{\left(1+v_x^2+v_y^2\right)^{3/2}} && \text{(A.27)}\\
&\quad + \frac{v_{yy}}{\sqrt{1+v_x^2+v_y^2}} - \frac{v_y\left(v_x v_{xy} + v_y v_{yy}\right)}{\left(1+v_x^2+v_y^2\right)^{3/2}}\,, && \text{(A.28)}\\
&= \frac{v_{xx} + v_{xx}v_x^2 + v_{xx}v_y^2 - v_x\left(v_x v_{xx} + v_y v_{xy}\right)}{\left(1+v_x^2+v_y^2\right)^{3/2}} && \text{(A.29)}\\
&\quad + \frac{v_{yy} + v_{yy}v_x^2 + v_{yy}v_y^2 - v_y\left(v_x v_{xy} + v_y v_{yy}\right)}{\left(1+v_x^2+v_y^2\right)^{3/2}}\,, && \text{(A.30)}\\
&= \frac{v_{xx}\left(1+v_y^2\right) + v_{yy}\left(1+v_x^2\right) - 2v_x v_y v_{xy}}{\left(1+v_x^2+v_y^2\right)^{3/2}}\,. && \text{(A.31)}
\end{aligned}$$

By the chain rule each second derivative of v becomes:

$$
\begin{aligned}
v_{xx} &= \frac{\partial}{\partial s}(v_x)\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}(v_x)\frac{\partial t}{\partial x}, \qquad (A.32)\\
&= \frac{\partial}{\partial s}\Bigg(-tg'(t)\frac{(f_1'-f_2')^2}{(f_1-f_2)^2} - g'(t)\frac{(f_1'-f_2')(f_1'+f_2')}{(f_1-f_2)^2}\\
&\quad -\frac{A(f_1'-f_2')}{(f_1-f_2)^2} + g(t)\frac{f_1''+f_2''}{f_1-f_2} - g(t)\frac{(f_1'-f_2')^2}{(f_1-f_2)^2}\Bigg)\\
&\quad +\frac{\partial}{\partial t}\Bigg(-tg'(t)\frac{(f_1'-f_2')^2}{(f_1-f_2)^2} - g'(t)\frac{(f_1'-f_2')(f_1'+f_2')}{(f_1-f_2)^2} - \frac{A(f_1'-f_2')}{(f_1-f_2)^2}\\
&\quad +g(t)\frac{f_1''+f_2''}{f_1-f_2} - g(t)\frac{(f_1'-f_2')^2}{(f_1-f_2)^2}\Bigg)\cdot\left(-\frac{t(f_1'(s)-f_2'(s))+(f_1'(s)+f_2'(s))}{f_1(s)-f_2(s)}\right),\\
&= -(tg'(t)+g(t))\left(\frac{2(f_1'-f_2')(f_1''-f_2'')}{(f_1-f_2)^2} - 2\frac{(f_1'-f_2')^3}{(f_1-f_2)^3}\right)\\
&\quad -g'(t)\left(\frac{(f_1''-f_2'')(f_1'+f_2')+(f_1'-f_2')(f_1''+f_2'')}{(f_1-f_2)^2} - 2\frac{(f_1'-f_2')^2(f_1'+f_2')}{(f_1-f_2)^3}\right)\\
&\quad -\frac{A(f_1''-f_2'')}{(f_1-f_2)^2} + 2\frac{A(f_1'-f_2')^2}{(f_1-f_2)^3} + g(t)\frac{f_1'''+f_2'''}{f_1-f_2} - g(t)\frac{(f_1''+f_2'')(f_1'-f_2')}{(f_1-f_2)^2}\\
&\quad \Bigg(-2(tg''(t)+2g'(t))\frac{(f_1'-f_2')^2}{(f_1-f_2)^3} - 2g''(t)\frac{(f_1'-f_2')(f_1'+f_2')}{(f_1-f_2)^3}\\
&\quad +2g'(t)\frac{f_1''+f_2''}{(f_1-f_2)^2}\Bigg)\left(-t(f_1'(s)-f_2'(s))+(f_1'(s)+f_2'(s))\right), \qquad (A.33)\\
&\sim -\frac{A(f_1''(s)-f_2''(s))}{(f_1(s)-f_2(s))^2} + 2\frac{A(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3} + g(t)\frac{f_1'''(s)+f_2'''(s)}{f_1(s)-f_2(s)}, \qquad (A.34)\\
v_{xy} &= \frac{\partial}{\partial s}(v_x)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}(v_x)\frac{\partial t}{\partial y}, \qquad (A.35)\\
&\sim -2g''(t)\frac{(f_1'(s)-f_2'(s))\left(t(f_1'(s)-f_2'(s))+(f_1'(s)+f_2'(s))\right)}{(f_1(s)-f_2(s))^3}\\
&\quad -4g'(t)\frac{(f_1'(s)-f_2'(s))^2}{(f_1(s)-f_2(s))^3} + 2g'(t)\frac{f_1''(s)-f_2''(s)}{(f_1(s)-f_2(s))^2}, \qquad (A.36)\\
v_{yy} &= \frac{\partial}{\partial s}(v_y)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}(v_y)\frac{\partial t}{\partial y}, \qquad (A.37)\\
&= 4g''(t)\frac{f_1'(s)-f_2'(s)}{(f_1(s)-f_2(s))^3}. \qquad (A.38)
\end{aligned}
$$

Substituting equations (A.34)-(A.38) into equation (A.31) gives

$$
\begin{aligned}
\nabla \cdot Tv \;\sim\; & \Bigg(\Bigg(2\frac{A(f_1'-f_2')^2}{(f_1-f_2)^3} - \frac{A(f_1''-f_2'')}{(f_1-f_2)^2} + g(t)\frac{f_1'''-f_2'''}{f_1-f_2}\Bigg) \\
& \Bigg(1 + \Bigg(2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg)^2\Bigg) + 4g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^3} \\
& + \Bigg(4g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^3}\Bigg)\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg)^2 \\
& -2\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg) 2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg(2g'(t)\frac{f_1''+f_2''}{(f_1-f_2)^2} - 4g'(t)\frac{(f_1'-f_2')^2}{(f_1-f_2)^3} \\
& 2g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg(-\frac{t(f_1'(x)-f_2'(x))}{f_1(x)-f_2(x)} - \frac{f_1'(x)+f_2'(x)}{f_1(x)-f_2(x)}\Bigg)\Bigg)\Bigg) \\
& \Bigg/ \Bigg(1 + \Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg)^2 + \Bigg(2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg)^2\Bigg)^{3/2} \qquad \text{(A.39)} \\
\sim\; & \Bigg(\Bigg(-\frac{A(f_1''-f_2'')}{(f_1-f_2)^2} + g(t)\frac{f_1'''-f_2'''}{f_1-f_2}\Bigg)\Bigg(2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg)^2 \\
& +4g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^3} \\
& + \Bigg(4g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^3}\Bigg)\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg)^2 \\
& -2\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg) 2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg(2g'(t)\frac{f_1''+f_2''}{(f_1-f_2)^2}\Bigg)\Bigg) \\
& \Bigg/ \Bigg(\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg)^2 + \Bigg(2g'(t)\frac{f_1'-f_2'}{(f_1-f_2)^2}\Bigg)^2\Bigg)^{3/2} \qquad \text{(A.40)} \\
\sim\; & \Bigg(\Bigg(2g'(t)g(t)\frac{(f_1'''-f_2''')(f_1'-f_2')^2}{(f_1-f_2)^5}\Bigg) \\
& + \Bigg(4g''(t)\frac{f_1'-f_2'}{(f_1-f_2)^3}\Bigg)\Bigg(-\frac{A(f_1'-f_2')}{(f_1-f_2)^2}\Bigg)^2 \\
& -2\Bigg(-\frac{A(f_1'-f_2')^2}{(f_1-f_2)^2}\Bigg)\Bigg(4g'^2(t)\frac{f_1''+f_2''}{(f_1-f_2)^4}\Bigg)\Bigg) \\
& \Bigg/ \big(A^2 + 4g'^2(t)\big)^{3/2}\,\frac{(f_1'-f_2')^3}{(f_1-f_2)^6}\;. \qquad \text{(A.41)}
\end{aligned}
$$

We can further simplify equation (A.41) and obtain

$$
\begin{aligned}
&\nabla\cdot Tv \\
\sim\quad & \frac{\left(2g'(t)g(t)\frac{(f_1'''-f_2''')(f_1-f_2)}{(f_1'-f_2')}\right)+(4g''(t))\left(\frac{A^2}{(f_1-f_2)}\right)-2\left(-\frac{A}{f_1'-f_2'}\right)\left(4g'^2(t)(f_1''+f_2'')\right)}{\left(A^2+4g'^2(t)\right)^{3/2}} \\
&\left(\text{Assuming}:\left|\frac{(f_1'''-f_2''')(f_1-f_2)}{f_1'-f_2'}\right|<<\left|\frac{1}{f_1-f_2}\right|\right) \\
\sim\quad & \frac{4g''(t)A^2}{\left(A^2+4g'^2(t)\right)^{3/2}}\frac{1}{(f_1-f_2)}\,.
\end{aligned}
\tag{A.42}
$$

Following from equation (A.7) and (A.42) we obtain

$$
\nabla\cdot Tv - v \quad\sim\quad \frac{4g''(t)A^2}{\left(A^2+4g'^2(t)\right)^{3/2}}\frac{1}{f_1(s)-f_2(s)}-\frac{A}{f_1(s)-f_2(s)}-C_0 \qquad \text{as } s\to 0\,. \tag{A.43}
$$

A.2 Justification of Figure 3.3

In this section, we justify that there exists a unique triangle for each c such described in Figure 3.3.

First consider the case for $b<0$ (see Figure A.1). In a region Ω, draw a circle of radius c centered at $(c,0)$. Name each intersection of the circle with the upper and lower boundary "d" and "f", respectively. Draw a line between the center of the upper boundary (D) and d and also draw a line between the center of the lower boundary (E) and f and extend them and name the intersection of these lines to be F. Since the line segment $c-d$ and $c-0$ are the radii of a circle, they have the same length, also length of line segments $D-O$ and $D-d$ are the same. Thus $\triangle DOc$ and $\triangle Ddc$ are congruent. Since the point c is on x-axis, $\angle DOc$ is right angle, this implies $\angle Ddc$ is also right angle. Similarly, $\angle Efc$ is right angle. Hence, line segment $D-F$, $D-E$ and $F-E$ are tangential to the circle. That is to say, $\triangle DFE$ inscribes a circle of radius

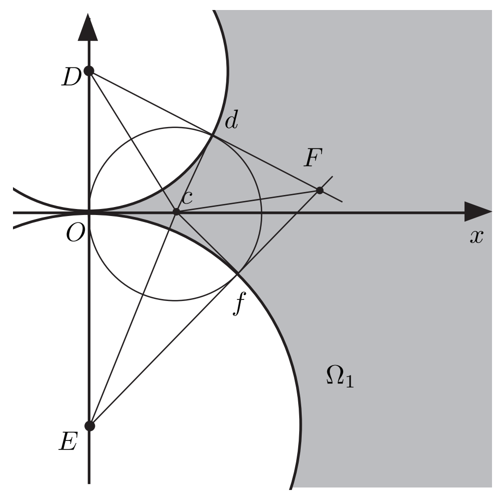

Figure A.1: Upper bound Approximation near a Circular Cusp: $b < 0$ case

c centered at $(c, 0)$. By the property of triangle inscribing a circle, the length of line segments $d - F$ and $f - F$ are the same. So we can draw a circle centered at F and pass through points d and f. Since both $d - F$ and $f - F$ are radii of a circle, tangent line segments of a circle at point d and f are orthogonal to $d - F$ and $f - F$, hence $c - d$ and $c - f$ are tangential to the circle.

Thus, the circle tangential to the upper and lower boundaries at points d and f has a center at the point F.

Similarly for the case $b > 0$ (see Figure A.2). Since both $E - O$ and $E - f$ are the radii of a circle, length of $E - O$ and $E - f$ are the same. Similarly length of $c - O$ and $c - f$ are the same. Hence $\triangle OEc$ and $\triangle fEc$ are congruent. Since c is on x-axis, $\angle cOE$ is right angle, this gives $\angle Efc$ to be right angle. Similarly we can show $\angle cfE$ is right angle. This gives line segments $d - F$ and $f - F$ to be tangential to the circle centered at c and this gives the length of line segments $d - F$ and $f - F$ to be the same. Thus the center of a circle which is tangential to the upper and lower boundary at points f and d has a center at point F.

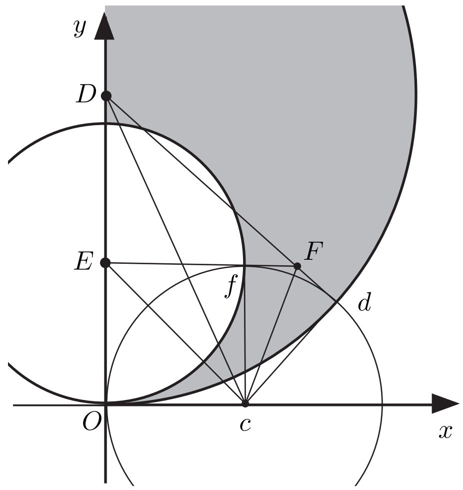

Figure A.2: Upperbound Approximation near a Circular Cusp: $b > 0$ case

A.3 Calculation for Lemma 3.2

We would like to expand $\hat{q} \cdot Tv$ and $\nabla \cdot Tv$ asymptotically as $p \to \infty$.

$v(p,q)$ is defined as:

$$v(p,q) = Ap^2 - 2\sqrt{1 - A^2(q-q_0)^2}p - A(q-q_0)^2 + Aq_0^2 + L_1 + \frac{h(q)}{p^5}\,. \quad \text{(A.44)}$$

Each derivative can be calculated as

$$\frac{\partial v}{\partial p} = 2Ap - 2\sqrt{1 - A^2(q-q_0)^2} - 5\frac{h(q)}{p^6}\,, \quad \text{(A.45)}$$

$$\frac{\partial v}{\partial q} = \frac{2A^2(q-q_0)}{\sqrt{1 - A^2(q-q_0)^2}}p - 2A(q-q_0) + \frac{h'(q)}{p^5}\,, \quad \text{(A.46)}$$

$$\frac{\partial^2 v}{\partial p^2} = 2A + \frac{30h(q)}{p^7}\,, \quad \text{(A.47)}$$

$$\frac{\partial^2 v}{\partial q^2} = \frac{2A^4(q-q_0)^2}{(1 - A^2(q-q_0)^2)^{3/2}}p + \frac{2pA^2}{\sqrt{1 - A^2(q-q_0)^2}} - 2A + \frac{h''(q)}{p^5}\,, \quad \text{(A.48)}$$

$$\frac{\partial^2 v}{\partial p \partial q} = \frac{2A^2(q-q_0)}{\sqrt{1 - A^2(q-q_0)^2}} - \frac{5h'(q)}{p^6}\,. \quad \text{(A.49)}$$

Following from equations (A.45) and (A.46) we can calculate $\frac{\partial v}{\partial p}^2 + \frac{\partial v}{\partial q}^2$ as

$$\begin{aligned}
& \frac{\partial v}{\partial p}^2 + \frac{\partial v}{\partial q}^2 \\
= \; & \left(2Ap - 2\sqrt{1 - A^2(q-q_0)^2} - 5\frac{h(q)}{p^6}\right)^2 \\
& + \left(\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p - 2A(q-q_0) + \frac{h'(q)}{p^5}\right)^2 \\
\sim \; & 4A^2p^2 + 4\left(1 - A^2(q-q_0)^2\right) - 8A\sqrt{1-A^2(q-q_0)^2}p + \frac{4A^4(q-q_0)^2}{1-A^2(q-q_0)^2}p^2 \\
& +4A^2(q-q_0)^2 - \frac{8A^3(q-q_0)^2}{\sqrt{1-A^2(q-q_0)^2}}p + \frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}p^4}\,, \qquad \text{(A.50)} \\
= \; & 4 + \frac{4A^2}{1-A^2(q-q_0)^2}p^2 - \frac{8A}{\sqrt{1-A^2(q-q_0)^2}}p + \frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}p^{-4}\,, \\
= \; & 4\frac{(Ap - \sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2} + \frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}p^{-4}\,. \qquad \text{(A.51)}
\end{aligned}$$

First perform an asymptotic expansion to $\hat{q} \cdot Tv$,

$$\begin{aligned}
\hat{q} \cdot Tv &= \frac{(p^2+q^2)\frac{\partial v}{\partial q}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial v}{\partial p}^2+\frac{\partial v}{\partial q}^2\right)}} && \text{(A.52)}\\
&\sim \frac{(p^2+q^2)\left(\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p-2A(q-q_0)+\frac{h'(q)}{p^5}\right)}{\sqrt{1+(p^2+q^2)^2\left(4\frac{(Ap-\sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2}+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}p^{-4}\right)}} && \text{(A.53)}\\
&\sim \frac{(p^2+q^2)\left(\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p-2A(q-q_0)+\frac{h'(q)}{p^5}\right)}{\sqrt{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}+(p^2+q^2)^2\left(4\frac{(Ap-\sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2}\right)}}, && \text{(A.54)}\\
&= \frac{(p^2+q^2)\left(\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p-2A(q-q_0)+\frac{h'(q)}{p^5}\right)}{\sqrt{(p^2+q^2)^2\left(4\frac{(Ap-\sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2}\right)}}\\
&\quad \frac{1}{\sqrt{1+\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{(Ap-\sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2}\right)}}}. && \text{(A.55)}
\end{aligned}$$

Apply binomial series expansion to $\frac{1}{\sqrt{1+\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{(Ap-\sqrt{1-A^2(q-q_0)})^2}{1-A^2(q-q_0)^2}\right)}}}$. Since

$$\left|\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}\right| < 1\,, \qquad \text{(A.56)}$$

for sufficiently large p, the series solution converges. Hence we obtain

$$
\begin{aligned}
\hat{q}\cdot Tv &\sim \frac{\left(\frac{2A^2(q-q_0)}{\sqrt{1-A^2(q-q_0)^2}}p - 2A(q-q_0) + \frac{h'(q)}{p^5}\right)}{2\frac{Ap-\sqrt{1-A^2(q-q_0)^2}}{\sqrt{1-A^2(q-q_0)^2}}} \\
&\quad \cdot\left(1-\frac{1}{2}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}+\ldots\right) \\
&= \left(A(q-q_0) + \frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5}\right) \\
&\quad \cdot\left(1-\frac{1}{2}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}+\ldots\right) \\
&\sim A(q-q_0) + \frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \\
&\quad -\frac{1}{2}A(q-q_0)\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)} \\
&\sim A(q-q_0) + \frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2Ap^6} \\
&\quad -\frac{A(q-q_0)}{2}\frac{(1-A^2(q-q_0)^2)+4A^2(q-q_0)h'(q)\sqrt{1-A^2(q-q_0)^2}}{4A^2p^6} \\
&\sim A(q-q_0) + \frac{(1-A^2(q-q_0)^2)^{3/2}h'(q)}{2Ap^6} - \frac{(q-q_0)(1-A^2(q-q_0)^2)}{8Ap^6}
\end{aligned}
$$

as $p\to\infty$. This implies that

$$
\begin{aligned}
\hat{q}\cdot Tv &= A(q-q_0) + \frac{(1-A^2(q-q_0)^2)^{3/2}h'(q)}{2Ap^6} - \frac{(q-q_0)(1-A^2(q-q_0)^2)}{8Ap^6} \\
&\quad + O\left(p^{-7}\right)\,. \qquad \text{(A.57)}
\end{aligned}
$$

Similarly we calculate the asymptotic expansion of $\hat{p} \cdot Tv$ as

$$\begin{aligned}
& \frac{(p^2+q^2)\frac{\partial v}{\partial p}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial v}{\partial p}^2+\frac{\partial v}{\partial q}^2\right)}} \\
\sim\ & \frac{2Ap-2\sqrt{1-A^2(q-q_0)^2}-5\frac{h(q)}{p^6}}{2\frac{Ap-\sqrt{1-A^2(q-q_0)^2}}{\sqrt{1-A^2(q-q_0)^2}}} \\
& \cdot\left(1-\frac{1}{2}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}+...\right) \\
=\ & \left(\sqrt{1-A^2(q-q_0)^2}-\frac{5}{2}\frac{h(q)\sqrt{1-A^2(q-q_0)^2}}{(Ap-\sqrt{1-A^2(q-q_0)^2})p^6}\right) \\
& \cdot\left(1-\frac{1}{2}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}+...\right) \\
\sim\ & \left(\sqrt{1-A^2(q-q_0)^2}\right)\left(1-\frac{1}{2}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}+...\right)
\end{aligned}$$

Since $h(q)$ is C^2 function and $\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)}$ is differentiable we can differentiate Tv without losing the asymptotic relationship. Hence we can calculate the left hand side of the PDE, i.e. $\nabla \cdot Tv$ as

$$\begin{aligned}
\nabla \cdot Tv &= (p^2+q^2)^2 \\
&\cdot \left(\frac{\partial}{\partial p} \frac{\frac{\partial v}{\partial p}}{\sqrt{1+(p^2+q^2)^2 \left(\frac{\partial v}{\partial p}^2 + \frac{\partial v}{\partial q}^2 \right)}} + \frac{\partial}{\partial q} \frac{\frac{\partial v}{\partial q}}{\sqrt{1+(p^2+q^2)^2 \left(\frac{\partial v}{\partial p}^2 + \frac{\partial v}{\partial q}^2 \right)}} \right) \\
&\sim (p^2+q^2)^2 \\
&\cdot \left(\frac{\partial}{\partial p} \left(\frac{\sqrt{1-A^2(q-q_0)^2}}{(p^2+q^2)} \left(1 - \frac{1}{2} \frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2 \left(4 \frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2} \right)} \right) \right) \right) \\
&+ \frac{\partial}{\partial q} \left(\frac{A(q-q_0)}{(p^2+q^2)} + \frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2(p^2+q^2)\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \right) \\
&- \frac{\partial}{\partial q} \left(\frac{1}{2} \frac{A(q-q_0)\left(1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}\right)}{(p^2+q^2)^3 \left(4 \frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2} \right)} \right) \Bigg) \qquad (A.58) \\
&= (p^2+q^2)^2 \\
&\cdot \left(-\frac{2p\sqrt{1-A^2(q-q_0)^2}}{(p^2+q^2)^2} \left(1 - \frac{1}{2} \frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2 \left(4 \frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2} \right)} \right) \right) \\
&- \frac{1}{2} \frac{\sqrt{1-A^2(q-q_0)^2}}{(p^2+q^2)} \frac{\partial}{\partial p} \frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2 \left(4 \frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2} \right)} + \frac{A}{(p^2+q^2)} \\
&- \frac{2A(q-q_0)q}{(p^2+q^2)^2} + \frac{\partial}{\partial q} \frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2(p^2+q^2)\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \\
&- \frac{1}{2} \frac{\partial}{\partial q} \frac{A(q-q_0)\left(1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}\right)}{(p^2+q^2)^3 \left(4 \frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2} \right)} \Bigg) . \qquad (A.59)
\end{aligned}$$

It follows from equation (A.59) that

$$
\begin{aligned}
\nabla \cdot Tv \;\sim\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& (p^2+q^2)^2 \left(-\frac{1}{2}\frac{\sqrt{1-A^2(q-q_0)^2}}{(p^2+q^2)}\frac{\partial}{\partial p}\frac{1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}}{(p^2+q^2)^2\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)} \right. \\
& + \frac{\partial}{\partial q}\frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2(p^2+q^2)\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \\
& \left. -\frac{1}{2}\frac{\partial}{\partial q}\frac{A(q-q_0)\left(1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}\right)}{(p^2+q^2)^3\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)} \right) \\
\sim\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& +(p^2+q^2)^2 \left(\frac{\partial}{\partial q}\frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2(p^2+q^2)\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \right. \\
& \left. -\frac{1}{2}\frac{\partial}{\partial q}\frac{A(q-q_0)\left(1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}\right)}{(p^2+q^2)^3\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)} \right) \\
\sim\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& +(p^2+q^2)\frac{\partial}{\partial q}\frac{h'(q)\sqrt{1-A^2(q-q_0)^2}}{2\left(Ap-\sqrt{1-A^2(q-q_0)^2}\right)p^5} \\
& -\frac{1}{2}\frac{\partial}{\partial q}\frac{A(q-q_0)\left(1+\frac{4A^2(q-q_0)h'(q)}{\sqrt{1-A^2(q-q_0)^2}}\right)}{(p^2+q^2)\left(4\frac{\left(Ap-\sqrt{1-A^2(q-q_0)}\right)^2}{1-A^2(q-q_0)^2}\right)} \\
\sim\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& +\frac{\partial}{\partial q}\left(\frac{h'(q)}{2Ap^4} - \frac{A(q-q_0)\left(\sqrt{1-A^2(q-q_0)^2}+4A^2(q-q_0)h'(q)\right)}{8A^2p^4}\right) \\
& \cdot\sqrt{1-A^2(q-q_0)^2}\,.
\end{aligned}
\tag{A.60}
$$

Further simplifying equation (A.60) gives

$$
\begin{aligned}
& \nabla \cdot Tv \\
\sim\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& + \frac{\partial}{\partial q}\left(\frac{\left(1-A^2(q-q_0)^2\right)h'(q)}{2} - \frac{(q-q_0)\sqrt{1-A^2(q-q_0)^2}}{8} \right) \frac{\sqrt{1-A^2(q-q_0)^2}}{Ap^4} \\
=\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& + \frac{\partial}{\partial q}\left(\frac{\left(1-A^2(q-q_0)^2\right)\sqrt{1-A^2(q-q_0)^2}h'(q)}{2} \right. \\
& \left. - \frac{(q-q_0)\left(1-A^2(q-q_0)^2\right)}{8} \right) \frac{1}{Ap^4} \\
=\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& + \left(\frac{\left(-2A^2(q-q_0)\right)\sqrt{1-A^2(q-q_0)^2}h'(q)}{2} \right. \\
& + \frac{(1-A^2(q-q_0)^2)(-2A^2(q-q_0))h'(q)}{4\sqrt{1-A^2(q-q_0)^2}} \\
& + \frac{\left(1-A^2(q-q_0)^2\right)\sqrt{1-A^2(q-q_0)^2}h''(q)}{2} \\
& \left. - \frac{\left(1-A^2(q-q_0)^2\right)}{8} + \frac{A^2(q-c)^2}{4} \right) \frac{1}{Ap^4} \\
=\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& + \left(\frac{(3-3A^2(q-q_0)^2)(-2A^2(q-q_0))h'(q)}{4\sqrt{1-A^2(q-q_0)^2}} \right. \\
& \left. + \frac{\left(1-A^2(q-q_0)^2\right)\sqrt{1-A^2(q-q_0)^2}h''(q)}{2} - \frac{\left(1-3A^2(q-q_0)^2\right)}{8} \right) \frac{1}{Ap^4} \\
=\; & -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\
& + \frac{12\sqrt{1-A^2(q-q_0)^2}(-A^2(q-q_0))h'(q) + 4\left(1-A^2(q-q_0)^2\right)^{3/2}h''(q)}{8Ap^4} \\
& - \frac{\left(1-3A^2(q-q_0)^2\right)}{8Ap^4}\,, \text{as } p \to \infty\,.
\end{aligned} \tag{A.61}
$$

Following from equation (A.61) we obtain

$$\begin{aligned}\nabla \cdot Tv &= -2p\sqrt{1-A^2(q-q_0)^2} + A(p^2+q^2) - 2A(q-q_0)q \\ &+ \frac{12\sqrt{1-A^2(q-q_0)^2}(-A^2(q-q_0))h'(q) + 4\left(1-A^2(q-q_0)^2\right)^{3/2} h''(q)}{8Ap^4} \\ &- \frac{(1-3A^2(q-q_0)^2)}{8Ap^4} + O\left(p^{-5}\right) \text{, for sufficiently large } p. \qquad \text{(A.62)}\end{aligned}$$

A.4 Calculation for Theorem 4.4

We now aim to perform an asymptotic expansion of the following:

$$\nu \cdot Tu_m \,, \qquad \text{(A.63)}$$

$$\nabla \cdot Tu_m \,, \qquad \text{(A.64)}$$

where

$$u_m = u_{m-1} + \sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2(m+1)} \,, \qquad \text{(A.65)}$$

$$\sim \frac{\cot\gamma q}{p^2} + \cot\gamma \frac{2\frac{a^2 q}{\cos^2\gamma} - q^3}{p^4} \,. \qquad \text{(A.66)}$$

Following from equation (A.65), we can compute each derivative as

$$\frac{\partial u_m}{\partial p} = \frac{\partial u_{m-1}}{\partial p} - 2(m+1)\sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2(m+1)-1} \,, \qquad \text{(A.67)}$$

$$\sim -2\frac{\cot\gamma q}{p^3} - 4\cot\gamma \frac{2\frac{a^2 q}{\cos^2\gamma} - q^3}{p^5} \,, \qquad \text{(A.68)}$$

$$\frac{\partial u_m}{\partial q} = \frac{\partial u_{m-1}}{\partial q} + \sum_{i=0}^{m} A_{mi}(2i+1) q^{2i} p^{-2(m+1)} \,, \qquad \text{(A.69)}$$

$$\sim \frac{\cot\gamma}{p^2} + \cot\gamma \frac{2\frac{a^2}{\cos^2\gamma} - 3q^2}{p^4} \,. \qquad \text{(A.70)}$$

It follows from equations (A.67) and (A.69) that

$$\begin{aligned}
\frac{\partial u_m}{\partial p}^2 + \frac{\partial u_m}{\partial q}^2 &= \left(\frac{\partial u_{m-1}}{\partial p} - 2(m+1) \sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2(m+1)-1} \right)^2 \\
&\quad + \left(\frac{\partial u_{m-1}}{\partial q} + \sum_{i=0}^{m} A_{mi} (2i+1) q^{2i} p^{-2(m+1)} \right)^2 , \qquad \text{(A.71)} \\
&= \frac{\partial u_{m-1}}{\partial p}^2 + \frac{\partial u_{m-1}}{\partial q}^2 \\
&\quad +2 \left(-2 \frac{\cot \gamma q}{p^3} - 4 \cot \gamma \frac{2 \frac{a^2 q}{\cos^2 \gamma} - q^3}{p^5} \right) \\
&\quad \cdot \left(-2(m+1) \sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2(m+1)-1} \right) \\
&\quad +2 \left(\frac{\cot \gamma}{p^2} + \cot \gamma \frac{2 \frac{a^2}{\cos^2 \gamma} - 3q^2}{p^4} \right) \\
&\quad \cdot \left(\sum_{i=0}^{m} A_{mi} (2i+1) q^{2i} p^{-2(m+1)} \right) , \qquad \text{(A.72)} \\
&= \frac{\partial u_{m-1}}{\partial p}^2 + \frac{\partial u_{m-1}}{\partial q}^2 + B p^{-2(m+2)} + C p^{-2(m+3)} , \qquad \text{(A.73)}
\end{aligned}$$

where

$$B = 2 \sum_{i=0}^{m} \left(\cot \gamma A_{mi} (2i+1) q^{2i} \right) , \qquad \text{(A.74)}$$

$$C = 2 \cot \gamma \sum_{i=0}^{m} \left(4(m+1) A_{mi} q^{2i+2} + \left(2 \frac{a^2}{\cos^2 \gamma} - 3q^2 \right) A_{mi} (2i+1) q^{2i} \right) . \qquad \text{(A.75)}$$

By the definition of T we can expand $\hat{q} \cdot T u_m$ as

$$\hat{q} \cdot T u_m = \frac{(p^2 + q^2) \frac{\partial u_m}{\partial q}}{\sqrt{1 + (p^2+q^2)^2 \left(\frac{\partial u_m}{\partial p}^2 + \frac{\partial u_m}{\partial q}^2 \right)}} . \qquad \text{(A.76)}$$

Substituting equation (A.73) into (A.76) gives

$$\begin{aligned}\hat{q}\cdot Tu_m &= \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial q}+\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}\\ &\quad\cdot\sqrt{\frac{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2+Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}},\\ &= \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial q}+\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}\\ &\quad\cdot\sqrt{\frac{1}{1+\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}}\,. \end{aligned} \tag{A.77}$$

Applying binomial expansion to $\sqrt{\frac{1}{1+\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}}$ gives

$$\begin{aligned}\hat{q}\cdot Tu_m &= \frac{(p^2+q^2)\left(\frac{\partial u_{m-1}}{\partial q}+\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}\\ &\quad\cdot\left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right.\\ &\quad\left.+\frac{3}{8}\left(\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right)^2-\ldots\right)\,. \end{aligned} \tag{A.78}$$

It can be shown that

$$\left|\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right|<1\,,\qquad \text{for sufficiently large } p\text{, for } m>0. \tag{A.79}$$

This implies that the binomial series converges. Also note $\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}$ is differentiable. It follows from equation (A.78) that

$$\begin{aligned}
\hat{q}\cdot Tu_m \;\sim\; & \left(\cos\gamma+\sum_{i=1}^{m+1} b_{m+1,i}q^{2i-2}p^{-2m}\right) \\
& \cdot\left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right. \\
& \left.+\frac{3}{8}\left(\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right)^2-\ldots\right), \\
& +\frac{(p^2+q^2)\left(\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \\
& \cdot\left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right. \\
& \left.+\frac{3}{8}\left(\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right)^2-\ldots\right), \\
\sim\; & \left(\cos\gamma+\sum_{i=1}^{m+1} b_{m+1,i}q^{2i-2}p^{-2m}\right) \\
& \cdot\left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right) \\
& +\frac{(p^2+q^2)\left(\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \qquad \text{(A.80)} \\
\sim\; & \cos\gamma-\frac{\cos\gamma}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)} \\
& +\sum_{i=1}^{m} b_{mi}q^{2i-1}p^{-2m} \\
& +\frac{(p^2+q^2)\left(\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}}. \qquad \text{(A.81)}
\end{aligned}$$

Simplifying equation (A.81) further gives

$$
\begin{aligned}
& \hat{q}\cdot Tu_m \\
\sim\ & \cos\gamma - \frac{\cos\gamma}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\cot^2\gamma}{p^4}+\left(\frac{4a^2}{\sin^2\gamma}-2\cot^2\gamma q^2\right)\frac{1}{p^6}\right)} \\
& +\sum_{i=1}^{m+1} b_{m+1,i}q^{2i-2}p^{-2m} \\
& +\frac{(p^2+q^2)\left(\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)}{\sqrt{1+(p^2+q^2)^2\left(\frac{\cot^2\gamma}{p^4}+\left(\frac{4a^2}{\sin^2\gamma}-2\cot^2\gamma q^2\right)\frac{1}{p^6}\right)}} \\
\sim\ & \cos\gamma \\
& +\left(\sum_{i=1}^{m+1} b_{m+1,i}q^{2i-2}+\frac{\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}}{\sqrt{1+\cot^2\gamma+\frac{4a^2}{\sin^2\gamma}\frac{1}{p^2}}}-\frac{\cos\gamma}{2}\frac{B}{1+\cot^2\gamma+\frac{4a^2}{\sin^2\gamma}\frac{1}{p^2}}\right)p^{-2m} \\
& +\left(\frac{q^2\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}}{\sqrt{1+\cot^2\gamma}}-\frac{\cos\gamma}{2}\frac{C+2q^2B}{1+\cot^2\gamma}\right)p^{-2(m+1)} \qquad \text{(A.82)} \\
=\ & \cos\gamma \\
& +\left(\sum_{i=1}^{m} b_{mi}q^{2i-1}+\frac{\sin\gamma\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}}{\sqrt{1+\frac{4a^2}{p^2}}}-\frac{\sin^2\gamma\cos\gamma}{2}\frac{B}{1+\frac{4a^2}{p^2}}\right)p^{-2(m)} \\
& +\left(\sin\gamma q^2\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}-\frac{\sin^2\gamma\cos\gamma}{2}\left(C+2q^2B\right)\right)p^{-2(m+1)} \\
\sim\ & \cos\gamma \\
& +\left(\sum_{i=1}^{m+1} b_{m+1,i}q^{2i-2}+\sin\gamma\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}-\frac{B\sin^2\gamma\cos\gamma}{2}\right)p^{-2(m)} \\
& +\left(\sin\gamma q^2\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}-\frac{\sin^2\gamma\cos\gamma}{2}\left(C+2q^2B\right)\right. \\
& \left.-2a^2\sin\gamma\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}+\frac{4a^2B\sin^2\gamma\cos\gamma}{2}\right)p^{-2(m+1)}. \qquad \text{(A.83)}
\end{aligned}
$$

Hence equation (A.84) implies that

$$\begin{aligned}
&\hat{q} \cdot Tu_m \\
\sim\ & \cos\gamma + \left(\sum_{i=1}^{m+1} b_{m+1,i} q^{2i-2} + \sin^3\gamma \sum_{i=0}^{m} A_{mi}(2i+1) q^{2i} \right) p^{-2(m)} \\
& + \left(\sin\gamma q^2 \sum_{i=0}^{m} A_{mi}(2i+1) q^{2i} - \frac{\sin^2\gamma\cos\gamma}{2} \left(C + 2q^2 B \right) \right. \\
& \left. -2a^2 \sin\gamma \sum_{i=0}^{m} A_{mi}(2i+1) q^{2i} + \frac{4a^2 B \sin^2\gamma\cos\gamma}{2} \right) p^{-2(m+1)}, \qquad \text{as } p \to \infty.
\end{aligned} \tag{A.84}$$

We now consider the left hand side of the PDE, i.e. $\nabla \cdot Tu_m$. By the definition of T we can expand $\nabla \cdot Tu_m$ to be

$$\begin{aligned}
\nabla \cdot Tu_m \quad = \quad & (p^2+q^2)^2 \frac{\partial}{\partial p} \frac{\frac{\partial u_m}{\partial p}}{\sqrt{1 + (p^2+q^2)\left(\frac{\partial u_m}{\partial p}^2 + \frac{\partial u_m}{\partial q}^2 \right)}} \\
& + (p^2+q^2)^2 \frac{\partial}{\partial q} \frac{\frac{\partial u_m}{\partial q}}{\sqrt{1 + (p^2+q^2)\left(\frac{\partial u_m}{\partial p}^2 + \frac{\partial u_m}{\partial q}^2 \right)}}\,.
\end{aligned} \tag{A.85}$$

By applying binomial series expansion we can asymptotically expand equation (A.85) as

$$\begin{aligned}
\nabla \cdot T u_m \sim\ & (p^2+q^2)^2 \frac{\partial}{\partial p}\left(\frac{\frac{\partial u_{m-1}}{\partial p}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot \left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right) \\
& \left. +\frac{\sum_{i=0}^{m} A_{mi} q^{2i+1}(-2(m+1))p^{-2(m+1)-1}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right) \\
& +(p^2+q^2)^2 \frac{\partial}{\partial q}\left(\frac{\frac{\partial u_{m-1}}{\partial q}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot \left(1-\frac{1}{2}\frac{(p^2+q^2)^2\left(Bp^{-2(m+2)}+Cp^{-2(m+3)}\right)}{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}\right) \\
& \left. +\frac{\sum_{i=0}^{m} A_{mi}(2i+1) q^{2i} p^{-2(m+1)}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right). \qquad \text{(A.86)}
\end{aligned}$$

We can expand and simplify equation (A.86) as follows:

$$\begin{aligned}
\nabla \cdot T u_m \sim\; & (p^2+q^2)^2 \frac{\partial}{\partial p}\left(\frac{\frac{\partial u_{m-1}}{\partial p}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot\left(1-\frac{1}{2}\frac{(p^4+2p^2q^2)Bp^{-2(m+2)}}{1+\cot^2\gamma+\frac{4a^2}{p^2}}-\frac{\cos^2\gamma}{2}Cp^{-2(m+1)}\right) \\
& \left. +\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}q^{2i+1}(-2(m+1))p^{-2(m+1)-1}\right) \\
& +(p^2+q^2)^2 \frac{\partial}{\partial q}\left(\frac{\frac{\partial u_{m-1}}{\partial q}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot\left(1-\frac{1}{2}\frac{(p^4+2p^2q^2)Bp^{-2(m+2)}}{1+\cot^2\gamma+\frac{4a^2}{p^2}}-\frac{\cos^2\gamma}{2}Cp^{-2(m+1)}\right) \\
& \left. +\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)\,, \qquad \text{(A.87)} \\
\sim\; & (p^2+q^2)^2 \frac{\partial}{\partial p}\left(\frac{\frac{\partial u_{m-1}}{\partial p}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot\left(1-\frac{\sin^2\gamma}{2}Bp^{-2(m)}+\left(\sin^4\gamma B2a^2-\sin^2\gamma Bq^2-\frac{\cos^2\gamma}{2}C\right)p^{-2(m+1)}\right) \\
& \left. +\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}q^{2i+1}(-2(m+1))p^{-2(m+1)-1}\right) \\
& +(p^2+q^2)^2 \frac{\partial}{\partial q}\left(\frac{\frac{\partial u_{m-1}}{\partial q}}{\sqrt{1+(p^2+q^2)^2\left(\frac{\partial u_{m-1}}{\partial p}^2+\frac{\partial u_{m-1}}{\partial q}^2\right)}} \right. \\
& \cdot\left(1-\frac{\sin^2\gamma}{2}Bp^{-2(m)}+\left(\sin^4\gamma B2a^2-\sin^2\gamma Bq^2-\frac{\cos^2\gamma}{2}C\right)p^{-2(m+1)}\right) \\
& \left. +\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\right)\,, \qquad \text{(A.88)}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \nabla \cdot T u_{m-1} \\
& +(p^2+q^2)^2 \frac{\partial}{\partial p} \left(\frac{-2\frac{\cot\gamma q}{p^3} - 4\cot\gamma \frac{\frac{2a^2 q}{\cos^2\gamma} - q^3}{p^5}}{\sqrt{1+\cot^2\gamma + \frac{4a^2}{p^2}}} \right. \\
& \cdot \left(-\frac{\sin^2\gamma}{2} B p^{-2(m)} + \left(\sin^4\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2} C \right) p^{-2(m+1)} \right) \\
& \left. + \sin\gamma \left(1 - \frac{1}{2}\sin^2\gamma \frac{4a^2}{p^2} \right) \sum_{i=0}^{m} A_{mi} q^{2i+1} (-2(m+1)) p^{-2(m+1)-1} \right) \\
& +(p^2+q^2)^2 \frac{\partial}{\partial q} \left(\frac{\frac{\cot\gamma}{p^2} + \cot\gamma \frac{\frac{2a^2}{\cos^2\gamma} - 3q^2}{p^4}}{\sqrt{1+\cot^2\gamma + \frac{4a^2}{p^2}}} \right. \\
& \cdot \left(-\frac{\sin^2\gamma}{2} B p^{-2(m)} + \left(\sin^4\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2} C \right) p^{-2(m+1)} \right) \\
& \left. + \sin\gamma \left(1 - \frac{1}{2}\sin^2\gamma \frac{4a^2}{p^2} \right) \sum_{i=0}^{m} A_{mi} (2i+1) q^{2i} p^{-2(m+1)} \right), \qquad \text{(A.89)} \\
\sim \quad & \nabla \cdot T u_{m-1} \\
& +(p^2+q^2)^2 \frac{\partial}{\partial p} \left(\left(\left(\left(-2\frac{\cot\gamma q}{p^3} - 4\cot\gamma \frac{\frac{2a^2 q}{\cos^2\gamma} - q^3}{p^5} \right) \sin\gamma + 4a^2 \sin^3\gamma \frac{\cot\gamma q}{p^5} \right) \right. \right. \\
& \cdot \left(-\frac{\sin^2\gamma}{2} B p^{-2(m)} + \left(\sin^4\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2} C \right) p^{-2(m+1)} \right) \\
& + \sin\gamma \left(1 - \frac{1}{2}\sin^2\gamma \frac{4a^2}{p^2} \right) \sum_{i=0}^{m} A_{mi} q^{2i+1} (-2(m+1)) p^{-2(m+1)-1} \Bigg) \\
& +(p^2+q^2)^2 \frac{\partial}{\partial q} \left(\left(\left(\left(\frac{\cot\gamma}{p^2} + \cot\gamma \frac{\frac{2a^2}{\cos^2\gamma} - 3q^2}{p^4} \right) \sin\gamma - 2a^2 \sin^3\gamma \frac{\cot\gamma}{p^4} \right) \right. \right. \\
& \cdot \left(-\frac{\sin^2\gamma}{2} B p^{-2(m)} + \left(\sin^4\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2} C \right) p^{-2(m+1)} \right) \\
& + \sin\gamma \left(1 - \frac{1}{2}\sin^2\gamma \frac{4a^2}{p^2} \right) \sum_{i=0}^{m} A_{mi} (2i+1) q^{2i} p^{-2(m+1)} \Bigg), \qquad \text{(A.90)}
\end{aligned}
$$

$$
\begin{aligned}
&\sim \nabla \cdot T u_{m-1} \\
&+(p^2+q^2)^2 \\
&\cdot\frac{\partial}{\partial p}\left(-\frac{\sin^2\gamma}{2}Bp^{-2(m)}\left(\left(\left(-2\frac{\cot\gamma q}{p^3}-4\cot\gamma\frac{\frac{2a^2q}{\cos^2\gamma}-q^3}{p^5}\right)\sin\gamma+4a^2\sin^3\gamma\frac{\cot\gamma q}{p^5}\right)\right.\right. \\
&-2\frac{\cos\gamma q}{p^3}\left(\left(\sin^4\gamma B2a^2-\sin^2\gamma Bq^2-\frac{\cos^2\gamma}{2}C\right)p^{-2(m+1)}\right) \\
&+\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}q^{2i+1}(-2(m+1))p^{-2(m+1)-1}\Bigg) \\
&+(p^2+q^2)^2 \\
&\cdot\frac{\partial}{\partial q}\left(-\frac{\sin^2\gamma}{2}Bp^{-2(m)}\left(\left(\left(\frac{\cot\gamma}{p^2}+\cot\gamma\frac{\frac{2a^2}{\cos^2\gamma}-3q^2}{p^4}\right)\sin\gamma-2a^2\sin^3\gamma\frac{\cot\gamma}{p^4}\right)\right.\right. \\
&+\frac{\cos\gamma}{p^2}\left(\left(\sin^4\gamma B2a^2-\sin^2\gamma Bq^2-\frac{\cos^2\gamma}{2}C\right)p^{-2(m+1)}\right) \\
&+\sin\gamma\left(1-\frac{1}{2}\sin^2\gamma\frac{4a^2}{p^2}\right)\sum_{i=0}^{m}A_{mi}(2i+1)q^{2i}p^{-2(m+1)}\Bigg)\ ,
\end{aligned}
\tag{A.91}
$$

$$
\begin{aligned}
&\sim \nabla \cdot T u_{m-1} \\
&+(p^2+q^2)^2 \\
&\cdot \frac{\partial}{\partial p}\left(-\frac{\sin^3\gamma}{2}B\left(-2\cot\gamma q p^{-2(m)-3} - 4\cot\gamma\left(\frac{2a^2 q}{\cos^2\gamma} + a^2\sin^2\gamma q - q^3\right)p^{-2(m)-5}\right)\right. \\
&-2\cos\gamma q\left(\left(\sin^3\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2}C\right)p^{-2(m)-5}\right) \\
&\left.-2(m+1)\sin\gamma\sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2m-3} + 2(m+1)2a^2\sin^3\gamma\sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2m-5}\right) \\
&+(p^2+q^2)^2 \\
&\cdot \frac{\partial}{\partial q}\left(-\frac{2\sum_{i=0}^{m}\left(\cot\gamma A_{mi}(2i+1)q^{2i}\right)\sin^2\gamma\cos\gamma}{2}p^{-2m-2}\right. \\
&-\left(\frac{B\sin^2\gamma\cos\gamma}{2}\left(\frac{2a^2}{\cos^2\gamma} - 3q^2\right) - a^2\sin^5\gamma\cot\gamma\right)p^{-2m-4} \\
&+\cos\gamma\left(\left(\sin^4\gamma B 2a^2 - \sin^2\gamma B q^2 - \frac{\cos^2\gamma}{2}C\right)p^{-2(m)-4}\right) \\
&\left.+\sin\gamma\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2m-2} - 2a^2\sin^3\gamma\sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2m-4)}\right)\ ,
\end{aligned}
\tag{A.92}
$$

Since we can take derivative of binomial series we obtain

$$
\begin{aligned}
&\sim \nabla \cdot T u_{m-1} \\
&+(p^2+q^2)^2 \\
&\cdot \Bigg(-\sin^3\gamma \sum_{i=0}^{m} \left(\cot\gamma A_{mi}(2i+1)q^{2i}\right)\left(-2(-2m-3)\cot\gamma q p^{-2m-4}\right) \\
&+ (2m+3)2(m+1)\sin\gamma \sum_{i=0}^{m} A_{mi} q^{2i+1} p^{-2m-4} \Bigg) \\
&+(p^2+q^2)^2 \\
&\cdot \frac{\partial}{\partial q} \Bigg(-\sum_{i=0}^{m} \left(A_{mi}(2i+1)q^{2i}\right)\sin\gamma\cos^2\gamma p^{-2m-2} \\
&- \left(\sum_{i=0}^{m} \left(A_{mi}(2i+1)\right)\sin\gamma\cos^2\gamma \left(\frac{2a^2}{\cos^2\gamma} q^{2i} - 3q^{2i+2}\right) - a^2\sin^5\gamma\cot\gamma \right) p^{-2m-4} \\
&+ \left(\left(2\sin^3\gamma\cos^2\gamma a^2 - \sin\gamma\cos^2\gamma q^2\right) 2\sum_{i=0}^{m}\left(A_{mi}(2i+1)q^{2i}\right)\right) p^{-2(m)-4} \\
&- \sin\gamma\cos^2\gamma \sum_{i=0}^{m} \left(4(m+1)A_{mi}q^{2i+2} + \left(2\frac{a^2}{\cos^2\gamma} - 3q^2\right) A_{mi}(2i+1)q^{2i} \right) p^{-2m-4} \\
&+ \sin\gamma \sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2m-2} - 2a^2\sin^3\gamma \sum_{i=0}^{m} A_{mi}(2i+1)q^{2i}p^{-2m-4)} \Bigg) \ , \quad \text{(A.93)}
\end{aligned}
$$

$$
\begin{aligned}
\sim\quad & \nabla\cdot Tu_{m-1} \\
& +(p^2+q^2)^2 \\
& \cdot\left(-\sin^3\gamma\sum_{i=0}^{m}\left(\cot\gamma A_{mi}(2i+1)q^{2i}\right)\left(-2(-2(m)-3)\cot\gamma q p^{-2(m)-4}\right)\right. \\
& \left.+\,(2m+3)2(m+1)\sin\gamma\sum_{i=0}^{m}A_{mi}q^{2i+1}p^{-2m-4}\right) \\
& +(p^2+q^2)^2 \\
& \cdot\left(-\sum_{i=1}^{m}\left(A_{mi}(2i+1)2iq^{2i-1}\right)\sin\gamma\cos^2\gamma p^{-2m-2}\right. \\
& -\sum_{i=1}^{m}\left(A_{mi}(2i+1)\right)\sin\gamma\cos^2\gamma\frac{4ia^2}{\cos^2\gamma}q^{2i-1}p^{-2m-4} \\
& -3\sum_{i=0}^{m}\left(A_{mi}(2i+1)\right)\sin\gamma\cos^2\gamma(2i+2)q^{2i+1}p^{-2m-4} \\
& +\sum_{i=1}^{m}2\sin^3\gamma\cos^2\gamma 2ia^2q^{2i-1}2\left(A_{mi}(2i+1)\right)p^{-2(m)-4} \\
& -\sum_{i=0}^{m}\sin\gamma\cos^2\gamma(2i+2)q^{2i+1}2\left(A_{mi}(2i+1)\right)p^{-2(m)-4} \\
& -\sin\gamma\cos^2\gamma\sum_{i=0}^{m}\left(4(m+1)A_{mi}(2i+2)q^{2i+1}p^{-2m-4}\right. \\
& +\sum_{i=1}^{m}2\frac{a^2}{\cos^2\gamma}2iq^{2i-1}A_{mi}(2i+1)p^{-2m-4} \\
& +\sum_{i=0}^{m}-3(2i+2)q^{2i+1}A_{mi}(2i+1)p^{-2m-4} \\
& +\sin\gamma\sum_{i=1}^{m}A_{mi}(2i+1)2iq^{2i-1}p^{-2m-2} \\
& \left.-2a^2\sin^3\gamma\sum_{i=1}^{m}A_{mi}(2i+1)2iq^{2i-1}p^{-2m-4)}\right)\,,
\end{aligned}
\tag{A.94}
$$

$$
\begin{aligned}
&= \nabla \cdot T u_{m-1} - \sum_{i=1}^{m} \left(A_{mi}(2i+1)2iq^{2i-1} \right) \sin\gamma \cos^2\gamma p^{-2m+2} \\
&\quad + \sin\gamma \sum_{i=1}^{m} A_{mi}(2i+1)2iq^{2i-1}p^{-2m+2} + \sum_{i=1}^{m+1} D_{mi}q^{2i-1}p^{-2m} + O(p^{-2m-2}) \,, \\
&= \nabla \cdot T u_{m-1} + \sin^3\gamma \sum_{i=1}^{m} A_{mi}(2i+1)2iq^{2i-1}p^{-2m+2} + \sum_{i=1}^{m+1} D_{mi}q^{2i-1}p^{-2m} \\
&\quad + O(p^{-2m-2}) \,. \qquad \text{(A.95)}
\end{aligned}
$$

Appendix B

Analysis of Jumping of Asymptotic Orders at Corner Singularity

In this Appendix we will apply the method we have developed in sections 3.3 and 3.4 to the singular capillary surface near a corner. (i.e. wedge region with opening angle less than $\pi - 2\gamma$). The leading order of the asymptotic solution was originally motivated by Finn's geometrical argument:

> We seek a "near solution" in the form of a function whose level curves are circular arcs that meet [the boundary] in the angle γ (page 116 [1]),

We consider the capillary boundary value problem in polar coordinate system:

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{r u_r}{\sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}}\right)+\frac{\partial}{\partial \theta}\left(\frac{u_\theta}{r\sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}}\right)\right) = u\,, \tag{B.1}$$

$$\left.\frac{u_\theta}{r\sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}}\right|_{\theta=\alpha} = \cos\gamma\,, \tag{B.2}$$

$$-\left.\frac{u_\theta}{r\sqrt{1+u_r^2+\frac{1}{r^2}u_\theta^2}}\right|_{\theta=-\alpha} = \cos\gamma\,. \tag{B.3}$$

Assuming $u_r^2 + \frac{1}{r^2}u_\theta^2 >> 1$, now approximate $u(r,\theta)$ by $v(p,q)$ defined to be a solution of the following PDE and BCs:

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{r v_r}{\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}\right) + \frac{\partial}{\partial \theta}\left(\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}\right)\right) = v\,, \tag{B.4}$$

$$\left.\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}\right|_{\theta=\alpha} = \cos\gamma\,, \tag{B.5}$$

$$-\left.\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}\right|_{\theta=-\alpha} = \cos\gamma\,. \tag{B.6}$$

The uniqueness of this BVP can be proven. We make another key assumption such that $\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}$ only depends in θ,i.e.

$$\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}} = g(\theta)\,, \tag{B.7}$$

where $g(a) = \cos\gamma_1$ and $g(b) = -\cos\gamma_2$. We now aim to solve equation (B.7). First collect θ dependance term to the left of the equation and r dependence term to the right of the equation such as

$$v_\theta^2 = \left(r^2 v_r^2 + v_\theta^2\right) g^2(\theta)\,, \tag{B.8}$$

$$-\sqrt{1-g^2(\theta)}\,v_\theta = r g(\theta) v_r\,, \tag{B.9}$$

$$\frac{\sqrt{1-g^2(\theta)}}{g(\theta)} v_\theta = -r v_r\,. \tag{B.10}$$

With the method of characteristic, we solve equation (B.10) and obtain

$$v(r,\theta) = f\left(-\ln r + \int_0^\theta \frac{g(s)}{\sqrt{1-g^2(s)}}ds + L\right)\,, \tag{B.11}$$

where $f(\cdot)$ is an arbitrary function of one variable and L is an arbitrary constant of integration kept for later use. Each derivative can be written in terms of f as

$$v_r = -\frac{1}{r} f' \,, \tag{B.12}$$

$$v_\theta = \frac{g(\theta)}{\sqrt{1-g^2(\theta)}} f' \,. \tag{B.13}$$

Substituting equations (B.12) and (B.13) into equation (B.4) gives

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(-r\frac{f'}{\sqrt{f'^2\frac{1}{1-g^2(\theta)}}}\right)+\frac{\partial}{\partial \theta}\left(\frac{\frac{g(\theta)}{\sqrt{1-g^2(\theta)}}f'}{\sqrt{f'^2\frac{1}{1-g^2(\theta)}}}\right)\right) = v \,, \tag{B.14}$$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{\partial}{\partial r}\left(-r\sqrt{1-g^2(\theta)}\right)+\frac{\partial}{\partial \theta} g(\theta)\right) = v \,, \tag{B.15}$$

$$\Rightarrow \quad \frac{-\sqrt{1-g^2(\theta)}}{r}+\frac{g'(\theta)}{r} = v \,. \tag{B.16}$$

It follows from equation (B.16) each derivative can be calculated as

$$v_r = \frac{\sqrt{1-g^2(\theta)}}{r^2} - \frac{g'(\theta)}{r^2} \,, \tag{B.17}$$

$$v_\theta = \frac{g(\theta)g'(\theta)}{r\sqrt{1-g^2(\theta)}} + \frac{g''(\theta)}{r} \,. \tag{B.18}$$

Equating equations (B.12)-(B.13) and equations (B.17)-(B.18) gives

$$\frac{\sqrt{1-g^2(\theta)}}{r^2} - \frac{g'(\theta)}{r^2} = -\frac{1}{r} f' \,, \tag{B.19}$$

$$\frac{g(\theta)g'(\theta)}{r\sqrt{1-g^2(\theta)}} + \frac{g''(\theta)}{r} = \frac{g(\theta)}{\sqrt{1-g^2(\theta)}} f' \,. \tag{B.20}$$

Multiplying equation (B.19) by $r\frac{g(\theta)}{\sqrt{1-g^2(\theta)}}$ and adding it with equation (B.20) gives

$$\frac{g''(\theta)+g(\theta)}{r} = 0 \,. \tag{B.21}$$

Since $r > 0$ equation (B.21) implies that

$$g''(\theta) + g(\theta) \;=\; 0\,. \tag{B.22}$$

Hence the general solution to this ODE (B.22) can be found by inspection such that

$$g(\theta) \;=\; A\cos\theta + B\sin\theta\,. \tag{B.23}$$

Determine arbitrary constants A and B by the boundary conditions.

$$g(\alpha) \;=\; A\cos\alpha + B\sin\alpha \;=\; \cos\gamma\,, \tag{B.24}$$

$$g(-\alpha) \;=\; A\cos\alpha - B\sin\alpha \;=\; -\cos\gamma\,, \tag{B.25}$$

$$g(\theta) \;=\; \frac{\cos\gamma}{\sin\alpha}\sin\theta\,. \tag{B.26}$$

For simplicity of writing we let $K = \frac{\sin\alpha}{\cos\gamma}$. Substituting equation (B.26) into the integral $\int_0^\theta \frac{g(s)}{\sqrt{1-g^2(s)}}ds$, and evaluating it gives

$$\int_0^\theta \frac{\sin s}{K\sqrt{1-\frac{\sin^2}{K^2}s}}ds \;=\; \ln\left(\frac{\cos\theta}{K} - \sqrt{1-\frac{\sin^2\theta}{K^2}}\right) - \ln\left(\frac{1}{K}-1\right)\,. \tag{B.27}$$

We now choose $L = \ln\left(\frac{1}{K}-1\right)$ so that $v(r,\theta)$ can be written as

$$v(r,\theta) \;=\; f\left(-\ln r + \ln\left(\frac{\cos\theta}{K} - \sqrt{1-\frac{\sin^2\theta}{K^2}}\right)\right)\,. \tag{B.28}$$

Let $\xi = -\ln r + \ln\left(\frac{\cos\theta}{K} - \sqrt{1-\frac{\sin^2\theta}{K^2}}\right)$, we choose $f(\xi) = e^\xi$ so that

$$f(x) \;=\; v(r,\theta) \;=\; \frac{\cos\theta - \sqrt{K^2-\sin^2\theta}}{Kr}\,. \tag{B.29}$$

We now aim to determine the asymptotic order of the term after $v(r,\theta)$. Let

$$u(r,\theta) \quad = \quad v(r,\theta) + w(r,\theta)\,, \tag{B.30}$$

where $w(r,\theta) = o\left(v(r,\theta)\right)$. Assuming $1 > |\nabla u|^2$ for sufficiently small r, expanding $\frac{1}{\sqrt{1+|\nabla u|^2}}$ with binomial series gives

$$\frac{1}{\sqrt{1+|\nabla u|^2}} \quad = \quad \frac{1}{|\nabla u|}\left(1 - \frac{1}{2}\frac{1}{|\nabla u|^2} + \frac{3}{4\cdot 2}\frac{1}{|\nabla u|^4} + ...\right). \tag{B.31}$$

We now require $u(r,\theta)$ to satisfy equation (B.2). Asymptotically expanding the left hand side of equation (B.2) using equation (B.31) and substituting equation (B.30) into it gives

$$\left.\frac{v_\theta + w_\theta}{r\sqrt{v_r^2 + 2v_r w_r + \frac{1}{r^2}v_\theta^2 + \frac{2}{r^2}v_\theta w_\theta}}\right|_{\theta=\alpha} - \left.\frac{1}{2}\frac{v_\theta}{r\left(v_r^2 + \frac{1}{r^2}v_\theta^2\right)^{3/2}}\right|_{\theta=\alpha} \sim \cos\gamma\,. \tag{B.32}$$

Expanding the left hand side of equation (B.32) asymptotically gives

$$\begin{aligned} &\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}} + \frac{w_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}} - \frac{1}{2}\frac{v_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}}\frac{2w_r v_r + \frac{2}{r^2}v_\theta w_\theta}{v_r^2 + \frac{1}{r^2}v_\theta^2} \\ &\left. - \frac{1}{2}\frac{v_\theta}{r\left(v_r^2 + \frac{1}{r^2}v_\theta^2\right)^{3/2}}\right|_{\theta=\alpha} \sim \cos\gamma\,. \end{aligned} \tag{B.33}$$

Substituting equation (B.7) and subtracting $\cos\gamma$ from both sides gives

$$\frac{w_\theta}{r\sqrt{v_r^2 + \frac{1}{r^2}v_\theta^2}} - \frac{1}{2}\cos\gamma\frac{2w_r v_r + \frac{2}{r^2}v_\theta w_\theta}{v_r^2 + \frac{1}{r^2}v_\theta^2} - \frac{1}{2}\frac{v_\theta}{r\left(v_r^2 + \frac{1}{r^2}v_\theta^2\right)^{3/2}} \sim 0. \tag{B.34}$$

Assuming $w(r,\theta)$ to be in a form of

$$w(r,\theta) \sim H(\theta)r^{\alpha}, \qquad \text{as } r \to 0, \tag{B.35}$$

$$w_r(r,\theta) \sim \alpha H(\theta)r^{\alpha-1}, \qquad \text{as } r \to 0, \tag{B.36}$$

$$w_\theta(r,\theta) \sim H'(\theta)r^{\alpha}, \qquad \text{as } r \to 0. \tag{B.37}$$

Substituting equations (B.12)-(B.13) and (B.36)-(B.37) into equation (B.34) gives

$$\frac{H'(\theta)r^{\alpha}\sqrt{1-g(\theta)^2}}{f'(\xi)} - \cos\gamma\frac{-\alpha H(\theta)r^{\alpha}f'(\xi) + \frac{g(\theta)}{\sqrt{1-g^2(\theta)}}f'(\xi)H'(\theta)r^{\alpha}}{f'^2(\xi)}(1-g^2(\theta))$$
$$-\frac{r^2}{2}\frac{(1-g^2(\theta))g(\theta)}{f'^2(\xi)}\bigg|_{\theta=\alpha} \sim 0, \qquad \text{as } r \to 0. \tag{B.38}$$

Solving equation (B.38) for α gives $\alpha = 3$. Similarly we require $u(r,\theta)$ to satisfy equation (B.3) and show that $\alpha = 3$. Hence to satisfy the boundary condition we require,

$$O(w) = O(r^3). \tag{B.39}$$

We now require $u(r,\theta)$ to satisfy equation (B.1). Asymptotically expanding the left hand side of equation (B.1) using equation (B.31) and substituting equation (B.30) into it gives

$$
\begin{aligned}
&\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{rv_r}{\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}+\frac{rw_r}{\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}-\frac{1}{2}\frac{rv_r}{\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}\frac{2w_rv_r+\frac{2}{r^2}v_\theta w_\theta}{v_r^2+\frac{1}{r^2}v_\theta^2}\right.\right.\\
&\left.-\frac{1}{2}\frac{rv_r}{\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\\
&+\frac{\partial}{\partial\theta}\left(\frac{v_\theta}{r\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}+\frac{w_\theta}{r\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}-\frac{1}{2}\frac{v_\theta}{r\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}\frac{2w_rv_r+\frac{2}{r^2}v_\theta w_\theta}{v_r^2+\frac{1}{r^2}v_\theta^2}\right.\\
&\left.\left.-\frac{1}{2}\frac{v_\theta}{r\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\right)\\
&\sim v+w\,.
\end{aligned}
\tag{B.40}
$$

Substituting equation (B.4) into (B.40) and subtracting v from both sides gives

$$
\begin{aligned}
&\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{rw_r}{\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}-\frac{1}{2}\frac{rv_r}{\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}\frac{2w_rv_r+\frac{2}{r^2}v_\theta w_\theta}{v_r^2+\frac{1}{r^2}v_\theta^2}-\frac{1}{2}\frac{rv_r}{\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\right.\\
&\left.+\frac{\partial}{\partial\theta}\left(\frac{w_\theta}{r\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}-\frac{1}{2}\frac{v_\theta}{r\sqrt{v_r^2+\frac{1}{r^2}v_\theta^2}}\frac{2w_rv_r+\frac{2}{r^2}v_\theta w_\theta}{v_r^2+\frac{1}{r^2}v_\theta^2}-\frac{1}{2}\frac{v_\theta}{r\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\right)\\
&\sim w\,.
\end{aligned}
\tag{B.41}
$$

Simplifying equation (B.41) gives

$$
\begin{aligned}
&\frac{\partial}{\partial r}r\left(\frac{w_r\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)-v_r\left(w_rv_r+\frac{1}{r^2}v_\theta w_\theta\right)-\frac{1}{2}v_r}{\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\\
&+\frac{\partial}{\partial\theta}\left(\frac{w_\theta\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)-v_\theta\left(w_rv_r+\frac{1}{r^2}v_\theta w_\theta\right)-\frac{1}{2}v_\theta}{r\left(v_r^2+\frac{1}{r^2}v_\theta^2\right)^{3/2}}\right)\\
\sim\ & rw\,.
\end{aligned}
\tag{B.42}
$$

Substituting equations (B.12)-(B.13) and (B.36)-(B.37) into equation (B.42) gives

$$\frac{\partial}{\partial r} r \left(\frac{\alpha H(\theta) r^{\alpha-3} \left(\frac{f'^2(\xi)}{1-g^2(\theta)}\right) + \frac{f'(\xi)}{r}\left(-\alpha H(\theta) r^{\alpha-2} f'(\xi) + \frac{g(\theta) f'(\xi) H'(\theta) r^{\alpha-2}}{\sqrt{1-g^2(\theta)}}\right) + \frac{1}{2}\frac{f'(\xi)}{r}}{\frac{1}{r^2}\left(\frac{f'^2(\xi)}{1-g^2(\theta)}\right)^{3/2}} \right)$$
$$+\frac{\partial}{\partial \theta} \left(\frac{H'(\theta) r^{\alpha-2} \left(\frac{f'^2(\xi)}{1-g^2(\theta)}\right) - \frac{g(\theta)}{\sqrt{1-g^2(\theta)}} f'(\xi) \left(-\alpha H(\theta) r^{\alpha-1} \frac{f'(\xi)}{r} + \frac{g(\theta) f'(\xi) H'(\theta) r^{\alpha-2}}{\sqrt{1-g^2(\theta)}}\right)}{\frac{1}{r^2}\left(\frac{f'^2(\xi)}{1-g^2(\theta)}\right)^{3/2}} \right)$$
$$-\frac{\partial}{\partial \theta} \left(\frac{\frac{1}{2}\frac{g(\theta)}{\sqrt{1-g^2(\theta)}} f'(\xi)}{\frac{1}{r^2}\left(\frac{f'^2(\xi)}{1-g^2(\theta)}\right)^{3/2}} \right) \sim rw\,. \tag{B.43}$$

Solving equation (B.43) for α gives $\alpha = 3$. Thus the order of $w(r, \theta)$ should be at least of

$$O(w) = O(r^3)\,. \tag{B.43}$$

This result is consistent with Miersemann's result [4].

Bibliography

[1] Robert Finn. *Equilibrium Capillary Surfaces.* Springer-Verlag, 1986.

[2] Robert Finn and Jenn-Fang Hwang. On the comparison principle for capillary surfaces. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.*, (36):131–134, 1989.

[3] Kirk E. Lancaster and David Siegel. Existence and behavior of the radial limits of a bounded capillary surface at a corner. *Pacific Journal of Mathematics*, 176(1):165–194, 1996.

[4] Erich Miersemann. Asymptotic expansion at a corner for the capillary problem: the singular case. *Pacific Journal of Mathematics*, 157(1):95–107, 1993.

[5] P. Moon and D.E. Spencer. *Field Theory Handbook.* Springer-Verlag, 1970.

[6] Markus Scholz. Über das verhalten von kapillarflächen in spitzen. 2001.

[7] Markus Scholz. On the asymptotic behaviour of capillary surfaces in cusps. *Z. angew. Math. Phys.*, 55:216–234, 2003.

[8] David Siegel. Height estimates for capillary surfaces. *Pacific Journal of Mathematics*, 88(2):471–515, 1980.