AN INVESTIGATION INTO THE USE OF PROBLEM-SOLVING HEURISTICS TO IMPROVE THE TEACHING AND LEARNING OF MATHEMATICS

by

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Declaration

I declare that the thesis "An investigation into the use of problem-solving heuristics to improve the teaching and learning of mathematics" is my own work and has not previously been submitted to any other institution of higher education. All sources cited or quoted are acknowledged by means of a comprehensive list of references.

23-08-2017

Daniel Ofori-Kusi **Date**

ABSTRACT

The aim of this study was to explore the effects of a problem-solving heuristic instructional method on Grade 6 learners' achievements in algebra. Two main theories inspired the design of this teaching method, namely the modelling and modelling perspective, and action, process, object, schema (APOS) theory. Modelling and modelling perspectives guided the development of modelling-eliciting activities used in the teaching method and the APOS theory guided the sequence of activities used to develop Grade 6 learners' conceptions in algebra.

The impact of the problem-solving heuristic instructional method was investigated with 198 Grade 6 learners from four different primary schools in the Zululand district of Kwazulu-Natal that were conveniently sampled. A mixed-method approach was used in this study and a hypothesis was formulated to investigate the effects of the teaching method on the learners' achievements in algebra. The qualitative component consisted of a pre-intervention class observation of mathematics lessons of all four mathematics educators in the schools used for this study. The design and implementation of the problem-solving heuristic instructional method and the quantitative component employed non-equivalent control group design with pre-test and post-test measure.

The main instruments for data collection were an observation schedule to document sequence of events in the classroom during the class observation, a standardized achievement test in algebra used to measure effects of the problem-solving heuristic instructional method and modelling-eliciting activities used as a medium of interaction between learners and the researcher during the implementation of the problem-solving heuristic instructional method.

Findings from the class observation indicated all four schools made use of comparable traditional methods of instruction. The implementation of the problem-solving instructional method gave insights into how a problem-solving heuristic instructional method can be developed and used in Grade 6 algebra lessons, and the factors that could influence learners' conceptual development in algebra. The findings from the quantitative component supported the initial hypothesis that improved scores in algebra are achieved through participation in the problem-solving heuristic instructional method. Quantitative data was analysed using the t-test, analysis of covariance, Johnson-Neyman (J-N) technique and the effect size.

Key words: Problem solving; problem-solving heuristic instruction; modelling and modelling perspective; modelling-eliciting activity; APOS theory; genetic decomposition; algebra lessons

DEDICATION

This thesis is dedicated to my late brother Henry Asante – Kusi and late parents, William Asante-Kusi and Felicia Kyeremaa.

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ABBREVIATIONS

ANA:	Annual National Assessment
ANCOVA:	analysis of covariance
APOS:	action process object schema
CAPS:	Curriculum Assessment Policy Statement
DBE:	Department of Basic Education
HOD:	Head of Department
HOS:	homogeneity of regression slopes
J-N:	Johnson-Neyman procedure
LO:	learning outcome
HSRC:	Human Sciences Research Council
MEA:	modelling-eliciting activity
NAEYC:	National Association for the Education of Young Children
NCTM:	National Council of Teachers of Mathematics
NCISLA:	National Centre for Improving Students' Learning and Achievement
NSC:	National Senior Certificate
KZN DOE:	KwaZulu-Natal Department of Education
RPK:	relevant previous knowledge
SPSS:	statistical package for the social sciences

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CHAPTER ONE
INTRODUCTION

1.1 BACKGROUND TO THE STUDY

The South African school system is broken down into four phases, namely Foundation Phase (FP), Intermediate Phase (IP), Senior Phase (SP) and Further Education and Training (FET) Phase. These four phases have 13 grades in total and in theory, the ages of learners in these grades ranges between 5 and 18 years. The FP consists of four grades from Grade R to 3 with learners' ages ranging between 5 and 8 years; the IP consists of 3 grades from Grade 4 to 6 with learners aged between 9 and 12 years. The SP consists of 3 grades from Grade 7 to 9 with learners' ages ranging between 13 and 15 years. The FET phase also consists of 3 grades, from Grade 10 to 12 with learners' ages ranging between 15 and 18 years. In practise these ages for the various phases may differ due to factors such as repetition of grades and late or early entry into the school system to mention only a few. (see DBE, 2011a, for details).

The South African Department of Basic Education (DBE) (DBE, 2011b) has prioritised the improvement of the quality and levels of educational outcomes in the school system with a view to, among others, improving learners' performance in mathematics with special emphasis placed on the FP, IP and SP phases. The extent to which these outcomes are achieved is determined and monitored through the administration of the Annual National Assessments (ANA), which involve standardised literacy and numeracy skills tests written by learners in Grades 1 to 6 and 9. The tests are managed by the schools themselves. The overarching goal, as per injunction of the President of the Republic of South Africa in the State of the Nation Address in 2009 when the ANA was introduced, was that by 2014 at least 60% of learners in Grades 3, 6 and 9 should have achieved acceptable levels of competency in mathematics (DBE, 2011b). The Human Sciences Research Council (HSRC) was commissioned to verify whether the 2014 ANA found consistencies with few exceptions. Among others, the results showed poor performance in mathematics in Grade 6. A percentage score of 41.8% was obtained at the national level, with only 32.4% of the learners scoring above 50% (HSRC, 2014).

Table 1.1 indicates the percentage of Grade 6 learners who scored more than 50% in the ANA in mathematics provincially and nationally from 2012 to 2014 and Table 1.2 shows the average marks obtained by Grade 6 learners in the ANA in mathematics from 2012 to 2014 as published by HSRC (2014).

Table 1.1: Percentage of learners achieving acceptable levels provincially and nationally from 2012 to 2014 in ANA examination

Province	**Acceptable achievement (≥50%)**			
	2012	**2013**	**2014**	**2014 Verification by HSRC**
EASTERN CAPE	8.1	16.2	23.3	22.0
FREE STATE	11.7	26.5	44	41.0
GAUTENG	16.4	38.4	51.7	53.1
KWAZULU NATAL	11.8	30.4	36.4	31.1
LIMPOPO	4.6	15.3	21.3	15.9
MPUMALANGA	5.7	16.1	27.0	27.1
NORTHERN CAPE	7.6	20.5	28.2	24.4
NORTH WEST	7.1	20.8	26.6	20.0
WESTERN CAPE	19.9	37.7	50.9	44.9
NATIONAL	**10.6**	**26.5**	**35.4**	**32.4**

Table 1.2: Average percentage marks in Grade 6 mathematics provincially and nationally from 2012 to 2014 in ANA examination

Province	Average mark (%)			
	2012	**2013**	**2014**	**2014 Verification by HSRC**
EASTERN CAPE	24.9	33.0	36.8	38.1
FREE STATE	28.4	40.0	47.7	48.2
GAUTENG	30.9	44.7	51.1	50.0
KWAZULU NATAL	28.1	41.2	43.8	40.1
LIMPOPO	21.4	32.9	35.3	32.8
MPUMALANGA	23.4	33.6	39.9	39.9
NORTHERN CAPE	23.8	36.6	39.3	41.6
NORTH WEST	23.6	36..5	38.8	36.5
WESTERN CAPE	32.7	44.9	50.9	41.8
NATIONAL	**26.7**	**39**	**43.1**	**41.8**

Despite an improvement in both the average percentage scores and the number of learners achieving acceptable marks between 2012 and 2014, as shown in Tables 1.1 and 1.2, the results indicate that the general performance was still falling below the set target (DBE, 2014). A substantial percentage of Grade 6 learners were still experiencing challenges in basic numeracy. The diagnostic report compiled by the HSRC indicated that the poorest performance was by quintile 1 schools[1]. Table 1.3 shows the average marks in the mathematics ANA examinations obtained by Grade 6 learners per quintile in 2014.

[1]A *quintile* is used to categorize South African schools, largely for purposes of the allocation of financial resources. Quintile one is the “poorest” quintile, while quintile five is the “least poor” (Grant, 2013).

Table 1.3: Average percentage mark in mathematics for Grade 6 learners by poverty quintile in 2014 ANA examination

Poverty quintile	Average mark
Quintile 1	38.1
Quintile 2	39.6
Quintile 3	40.4
Quintile 4	46.1
Quintile 5	60.3

The figures in Table 1.3 point to a direct correlation between the poverty quintile and the average marks obtained in the quintile; the lower the quintile the lower the average marks, and *vice versa*. There was a difference of 22.2 marks between the average marks obtained by quintile 1 and the average marks obtained by quintile 5 schools.

One of the most challenging topics identified in the ANA diagnostic report of 2014 for the intermediate and senior phases (i.e. Grades 6 and 9, respectively) in mathematics was algebra: "A high percentage of learners were unable to see the relationship between the input and output values given in a table" (DBE, 2014, p. 43).

The learning of mathematics is a complicated process because the pace at which learners acquire knowledge, skills and attitudes, and the context within which this takes place, are unique to each one. The educator's teaching techniques, though correct, may not be appropriate for all the topics for all the learners and at all times (Abonyi & Umeh, 2014).

Learners may have different starting points in their quest to learn mathematics and also may not all have the same interest and ability to learn the subject. Some may find it enjoyable while others may find it challenging. Some may find the theorems and results intriguing while others may find the formulae and rules bewildering (Curriculum Planning and Development Division of the Ministry of Education, 2012). It is argued that the mathematics curriculum has to provide differentiated pathways and choices to support every learner in order to maximise their potential (Curriculum Planning and Development Division of the Ministry of Education, 2013).

Therefore, there is a dire need to find ways and means to improve the performance of learners in mathematics, particularly in the primary schools, as the process of learning is more important than merely what is to be taught and remembered. Several education systems are beginning to rethink the nature of the mathematical experiences they should provide for their learners, in terms of the scope of the content covered, the approaches to the learners' learning, the ways of assessing the learners' learning, and the ways of increasing the learners' access to quality learning in mathematics (English & Watters, 2005). To this end, research supports the importance of developing conceptual understanding for future success in mathematics (National Council of Teachers of Mathematics, 2000). Research also demonstrates that instruction that focuses on conceptual understanding improves numerical reasoning and procedural fluency and accuracy (National Research Council, 1989).

There is widespread agreement that teaching through the problem-solving approach holds the promise of fostering the learners' conceptual understanding of mathematics (English & Sriraman, 2010; Schroeder & Lester, 1989). By solving problems, learners develop a rich understanding of the relationship between the elements in the problem and number facts (Baroody, 1998; Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Kilpatrick, Swafford & Findell, 2001; Reys, Lindquist, Lambdin, Suydam & Smith, 2001). Cai(2003, p.247) mentions that "While teaching through problem-solving starts with problems, only worthwhile problems give learners the chance both to solidify and extend what they know and to stimulate their learning".
According to English and Sriraman (2010), learners need a classroom mathematics experience that exposes them to problem situations, promotes the generation of important mathematical ideas and not merely the application of previously taught ideas, rules and procedures, which is the case in many mathematics classrooms. Kaur, Yeap and Kapur (2009) explain that an important component of improving mathematics learning through the problem-solving approach is for the educators concerned to be able to identify the types of mathematical problems that prompt learners' engagement, their thinking and the making of cognitive connections. The associated educator actions that support the use of these problems include addressing

the needs of individual learners. Lester (2013) shared the sentiments expressed by Kaur et al. (2009) by proposing that the success of a problem-solving instructional method depends on the consideration of a wide range of factors by the educator involved, which includes selection of problems, the type of problem- solving experiences to use, the stage at which problem-solving instruction should be used, the level of guidance to give to the learners and how to assess learners' progress. Educators must challenge learners with problems and at the same time offer support to those learners experiencing difficulty (Kaur et al., 2009). The problems educators challenge learners with must extend their thinking and reasoning in mathematics.

In the light of this, the current study explores the use of a problem-solving heuristic approach in the learning of algebra in grade 6. A problem-solving heuristic teaching method is an innovative approach to teaching and learning which is self-inviting and aids the self-directed development of the learner (Blinkston, 2000). The technique used in this teaching method is based on the learners' experience which aids problem-solving and discovery learning, and has its roots in the correct signal learners have with their immediate environment (Abonyi & Umeh, 2014). It advances the learners' external physical transformation from a lower learning plane to a higher one.

In this study the class of authentic real-life problems, known as modelling-eliciting activities (MEAs) are used to explore learner development in algebra. MEAs have been identified as an effective medium to foster critical mathematical thinking in learners (Skovsmose, 1994; Sriraman & Lesh, 2006). Two main theories are employed in the problem-solving heuristic instructional method, namely the modelling and modelling perspective and action, process, object, schema (APOS) theory. The modelling and modelling perspective informs the design of effective MEAs in which algebraic concepts are embedded, whereas the APOS theory explains the learning structure and mechanism the learners go through in developing conceptions in algebra. It focuses on non-lecture instructional strategies where the researcher-educator designs problem-solving activities and the learners work collaboratively and learn through experience. Algebra, which is defined by the DBE as a "language for reading mathematics" (DBE, 2011c) plays a crucial role in the South African mathematics curriculum and the learners' mathematical understanding in general.

1.2 PROBLEM OF THE STUDY

The study assesses the feasibility and effects of a heuristic teaching instruction on Grade 6 learners' achievements in algebra.

1.3 OBJECTIVES OF THE STUDY

- To develop a heuristic teaching sequence in the learning of algebra in Grade 6 in line with the modelling and modelling perspective and the APOS theory aimed at improving the teaching and learning of algebra at that level;
- To determine whether the heuristic teaching method has any effect on learners' achievement in algebra at the Grade 6 level.

1.4 RESEARCH QUESTIONS

In order to achieve the objectives of the study, the following research questions will be answered, namely:

- How can a problem-solving heuristic instructional method be developed and used in the teaching of algebra to Grade 6 learners?
- What is the impact of a problem-solving heuristic instructional method on learners' achievements in algebra at the level of Grade 6?

1.5 HYPOTHESIS OF THE STUDY

The hypothesis and the null hypothesis of the study were formulated as follows:

Hypothesis (H_A): There is a statistically significant improvement in the algebra test scores of the learners who participated in the problem-solving heuristic instructional method.

Null hypothesis (H_0): There is no statistically significant improvement in the algebra test scores of the learners who participated in the teaching treatment.

1.6 JUSTIFICATION FOR THE STUDY

Learners' engagement in problem-solving has been identified as an effective interaction media between the learner and the educator during the process of fostering the challenge of learning mathematics (see Kaur et al., 2009; Lester, 2013), but less is known about the actual mechanisms used by learners to learn and to make sense of mathematics through problem-solving (Schroeder & Lester, 1989). Even though some research has been conducted on how problem-solving has driven instruction (see English, 2009; Maclean, 2001), a host of researchers has cited a lack of studies that address problem-solving driven conceptual development as it occurs with problem-solving competencies (e.g. Cai, 2003; Lester & Charles, 2003; Schoen & Charles, 2003). Further research is required to shed more light on how concept development in mathematics can be achieved using problem-solving tools (English & Sriraman, 2010). Lester (2013) has also emphasised that further research is required in this regard, as the accumulation of knowledge in problem-solving instruction has been slow.

The current study will also highlight factors in instruction based on problem-solving that could foster young learners' conceptual understanding and interest in algebra and thereby encourage them to take up mathematics in the higher grades. This would add to ongoing perspectives on how a problem-solving instructional method could be used to improve the teaching and learning of algebra and open new research pathways on how the learning of mathematics in general could be improved using the problem-solving heuristic instructional method.

1.7 DEFINITION OF KEY TERMS

The following terms are defined for the purposes of clarity.

1.7.1 Problem

According to Newell and Simon (1972, p. 72),

> A person is confronted with a *problem* when he or she wants something and does not know immediately what series of actions he can perform to get it.

A problem arises when a task provides some form of blockage for the learners (Kroll & Miller, 1993) and the problem solver needs to develop a more productive way of dealing with the given situation (Lesh & Zawojewski, 2007). Drawing on these definitions, the study defines a problem as a task in which a learner does not have an immediate known solution, and which the educator must use as a medium to develop learners' conceptual understanding of a particular mathematical concept.

1.7.2 Problem-solving

Problem-solving is an activity requiring a learner to engage in a process of finding a solution to a problem using knowledge and skills.

1.7.3 Conception

A *conception* is a learner's individual understanding of a particular mathematical concept. According to Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros and Weller (2013, p. 19), "The depth and complexity of a learner's understanding of a concept depends on her or his ability to establish connections among the mental structures that constitute" the mathematical concept under consideration.

1.8 STRUCTURE OF THE REPORT

The study is organized into five chapters.

Chapter one presents the background, the problem statement, the aims, the research questions and justification for the study.

Chapter two is broken down into three main sections: the first section explains heuristics, heuristics teaching, the problem-solving processes and mathematical modelling; the second section gives an overview of algebra; and the last section deals with the theoretical framework that guided and informed the heuristic teaching experiment.

In the first section, the discussion on heuristics and problem-solving processes is linked to the discussion of mathematical modelling and its advantages and challenges when implemented with primary school learners. The second section gives an overview of algebra and explains the common conceptions and misconceptions learners have. This is then linked to a study of algebra in primary school in general, and a study of algebra in the intermediate phase and Grade 6 in the South African school system in particular. The last section of this chapter discusses the theoretical framework that informed and guided the problem-solving heuristic instructional method, namely the modelling and modelling perspective and the APOS theory and justifies the integration of the two theories in the teaching experiment.

Chapter three explains the research methodology that was used to access the impact of the heuristic teaching treatment which followed a mixed method approach. Among the issues discussed are the following: research paradigm; research design; sampling; pre-intervention classroom observation schedule; design and implementation of the problem-solving heuristic instructional method; measurement of its effects on learners' achievement in algebra; development of instruments, and data collection and analysis. The chapter concludes with the ethical issues that guided the researcher during the data collection phase.

Chapter four reports on the qualitative and quantitative findings and is broken down into four main sections. In the first section, the researcher reports on the standard of teaching and learning in the Grade 6 mathematics classrooms of the respective schools. In the second section, the researcher reports on how the problem-solving heuristic instructional method was implemented and gives a brief description of this implementation. Qualitative evidence is given on how learners can develop a conceptual understanding in algebra as they learn through the problem-solving heuristic instructional method. In the third section, the researcher reports on statistical changes in the learners' test scores in algebra after their participation or non-participation in the problem-solving heuristic instructional method. The analysed samples of answers learners gave in the pre-test and post-test were used to further support the statistical changes on the effects of the problem solving heuristic instructional method on learners' achievement in algebra as reported in section four of this chapter.

Chapter five presents a summary of the study and discusses the findings presented in chapter four of the study in the light of the research questions of the study. The conclusions and recommendations on how the theoretical framework that informed the teaching experience could be advanced in teaching and research and the limitations of the study are also presented in this chapter.

1.9 REFLECTING ON THE CHAPTER

Problem-solving instruction has been identified in this study as a viable pedagogical method of improving learners' conceptual understanding in mathematics. Against this backdrop the study investigates the viability of a problem-solving instructional method in the learning of algebra in Grade 6. In the light of this, the background of the study, statement of the problem, objectives, research questions, justification of the study, and the structure of the study were briefly discussed.

CHAPTER TWO

PROBLEM-SOLVING AND LEARNING OF ALGEBRA

In this chapter, the researcher first discusses heuristics, the problem-solving processes and mathematical modelling. Thereafter an overview is given of algebra in general, and in primary schools in particular. Lastly, the theoretical framework that informed and guided the heuristic teaching experiment in the study is discussed.

2.1 HEURISTICS, PROBLEM-SOLVING PROCESSES AND MATHEMATICAL MODELLING

2.1.1 Heuristics

The term *heuristic* is derived from the Greek, meaning to 'find' or to 'discover'. It is an adjective for experience-based techniques that help in problem-solving, learning and discovery (Jaszczolt, 2006). In mathematics, heuristics is a general way of solving problems, and is particularly used to come to a solution that is hoped to be close to the best possible solution of a mathematical problem (Abonyi & Umeh, 2014). Sickafus (2004) refers to *heuristics* as non-algorithmic tools, techniques and tricks that are general in nature and guide the search for a means of solving a problem.

In contrast to algorithms, which are fixed finite sequences of explicitly-given operations and decision-making capabilities at a given level when solving problems (Scandura, 1971), heuristics does not solve problems. It rather provides a way of looking at problems in different ways to find new insights (Sickafus, 2004). According to Polya (1945, p. 112), "The aim of heuristics is to study the methods and rules of discovery and invention". Problem-solvers use heuristics to 'seed' their subconscious minds during the search for new concepts (Sickafus, 2004); it enables them to select from a reduced set of alternative methods, and to order their solution-process steps (Lucas, 1972). Heuristics aids discovery, but rarely provides infallible guidance. It often works, but the results are variable and success is rarely guaranteed (Newell, Shaw & Simon, 1959).

Polya (1945, p. 113) stressed the fact that the aim of heuristics is "to study the methods of discovery and invention". Polya (1945) proposed a number of strategies, such as to work out a plan, to identify the 'givens' and the goals, to draw a picture, to work backwards, and to look for a similar problem. These strategies are seen as tools for the expert problem-solver.

Other heuristics strategies identified in the literature include finding a pattern, using analogies, considering extreme cases, modelling, systematic guessing and checking, and logical reasoning (Engel, 1998; Muis, 2004). Modern heuristics is built on experience in solving problems or in watching others solve problems, and aims at understanding the process of solving problems, especially the mental operations typically useful in the process (Polya, 1945).

"Heuristics plays a dominant role in the creative thinking involved in problem-solving… yet it is not nearly as generally accepted, as are algorithms in the scaling phase of problem-solving" (Sikafus, 2004, p. 16). According to Sikafus (2004), heuristics has not achieved the same status or acceptance as algorithms, which is backed by generations of research. It is against this backdrop that the current study is being pursued.

2.1.1.1 Using heuristics as teaching tool

Heuristic teaching makes use of one or more problem-solving techniques (Stone, 1983) and "aims to lead learners through well-chosen questions to discover facts, information, relationships and principles for themselves" (Butler & Wren, 1960, p. 167). Stone continued by explaining that heuristic teaching encourages the learners to seek new tricks and manipulations, and/or 'how' to arrive at a solution as opposed to 'why': "With heuristic teaching, an attempt is made to relate the logic of the teaching sequence to the logical (or psychological) patterns of problem-solving" (Stone, 1983, p. 9). Higgins (1971) highlights four characteristics of the heuristic teaching method as follows, namely it

1. approaches content through problems;
2. reflects on problem-solving techniques in the logical construction of instructional procedures;
3. demands flexibility for uncertainty and alternative approaches; and
4. seeks to maximize learners' actions and participation in the educator-learning process.

This study drew on the suggestions by Butler and Wren (1960), Higgins (1971) and Stone (1983) in the implementation of heuristic teaching instruction.

2.1.2 Problem-solving processes

Polya (1945) did seminal work on mathematical problem-solving and summarized a four-step solution to problem-solving, namely (i) understand the problem, (ii) devise a plan, (iii) carry out the plan, and, (iv) look back at the problem. The problem-solving heuristics approach proposed by Polya formed the basis for the development of other heuristics created thereafter. Polya's four-step model has impacted enormously on the teaching of problem-solving in schools over the past half-century. Lester (1980) commented that Polya's model should rather be seen as a proposal for teaching learners how to solve problems than as a description of how successful problem-solvers think. In other words, Polya's model provides a guide to organizing the instruction of problem-solving, but not a guide to identifying problem-solvers' difficulties in problem-solving or the mental processes involved in successful problem-solving (Joseph, 2011).

Other researchers have presented similar descriptions of cognitive activities used in the process of problem-solving based on the four-step, problem-solving model by Polya. This study discusses a number of such descriptions, for example, by Suydam (1980), Newman (1983), Burton (1984), Wilson, Fernandez and Hadaway (1993), and Joseph (2011).

Suydam (1980) identified the following steps: understand the problem, plan how to solve it, solve it, and finally, review the adequacy of the solution as carried out by effective problem-solvers in solving problems. Newman (1983) also developed a five-level hierarchy to summarize responses to verbal arithmetic problems presented in written form. According to Newman (1983), all respondents to the questions in their search to provide satisfactory solutions, would decode the problems, comprehend them, transform them from the written or verbal form to an appropriate mathematical form, apply the necessary mathematical process skills and encode the answer in a way that satisfies the original questions.

Similarly, Burton (1984) identified the following four phases in the problem-solving process, namely entry, attack, review and extension. Burton’s model explains that the problem-solver’s entry phase to problem-solving may generate curiosity from the problem-solver that is followed by the attack phase, leading to a solution for confident problem-solvers; accordingly, those who are not confident will withdraw at this stage. Burton (1984) suggests that those who find a solution to the problem will have a sense of achievement that fuels their looking-back and taking a review phase. Problem-solvers can then extend the skills gained to other mathematical problems.

In their work, Wilson et al. (1993) proposed a five-step problem-solving approach, including problem-posing, understanding the problem, making a plan, carrying out the plan, and looking back. Their model has been described as a framework for discussing various pedagogical, curricular, instructional, and learning issues involved with the goals of mathematical problem-solving in schools.

Joseph (2011) discussed an eight-step problem-solving framework that was adapted from the problem-solving model by Polya. Joseph (2011) explained that the steps, which were comprised of reading the question, highlighting the information, identifying the goal, identifying the information given, establishing the link and the relationship, planning, working it out and checking if the answer is correct, were meant to guide the learners in their thinking and problem-solving processes. Based on his eight-step problem-solving framework, Joseph (2011) concluded that the learners' problem-solving abilities would not simply follow from the development of general mathematical competence.

Although there are slight differences in the approaches, most problem-solving models recommend that problem-solvers should clarify the goals to be achieved when tasked to solve a particular problem, draw up a plan of attack, carry out the plan and revise it if necessary, review the processes employed, and check the solutions obtained (Clements & Ellerton, 1991).

2.1.2.1 Overcoming the difficulties learners face during problem-solving

In a study based on their first-person perspective on problem-solving and personal experience with the problems, McGinn and Boote (2003) identified the following four primary factors that affected the perceptions of the difficulty of a problem:

1. Categorization – the ability to recognize that a problem fits into an identifiable category of problems that run in a continuum from easily categorized to uncategorized;
2. Goal interpretation – figuring out how a solution would appear that runs in a continuum from well-defined to undefined;
3. Resource relevance – referring to how readily resources are recognized as relevant, from highly relevant to peripherally relevant; and
4. Complexity – performing a number of operations for a solution.

McGinn and Boote (2003) further suggested that the level of difficulty of a problem depended on the problem-solvers' perceptions of whether they had suitably categorized the situation, interpreted the intended goal, identified the relevant resources and executed adequate operations to lead toward a solution. This was also evident in Singaporean studies conducted by Kaur (1995) and Yeo (2009). Kaur (1995) indicated that Singaporean learners experienced problem-solving difficulties, such as (i) a lack of comprehension of the problem posed, (ii) a lack of strategy knowledge, and (iii) an inability to translate the problem into a mathematical form. Yeo (2009) explored the difficulties faced by 56 secondary (13 to 14-year-old) learners when solving problems. From the information obtained by means of interviews, Yeo (2009) identified the same difficulties as those identified by Kaur (1995), namely the factors that prevented secondary school learners from obtaining the correct solution to a problem.

Researchers (e.g., Kamii, 1989; Maher & Martino, 1996; Resnick, 1989) have investigated learners' mathematical thinking when solving problems. They indicated that the learners could explore problem situations and 'invent' ways to solve the problems. They also found that those learners who made use of invented strategies before they learned standard algorithms demonstrated better knowledge of base-ten number concepts and were more successful in extending their knowledge to new situations than were those learners who initially learned standard algorithms. Cai

(2003) also stressed that invented strategies during problem-solving could serve as a basis for the learners' understanding of mathematical ideas and procedures, but was quick to point out that this was based on their level of understanding, and that the learners should be guided to develop efficient strategies.

This study hypothesizes that one way of assisting learners to invent their own strategies is to develop problems that are familiar to them and with which they can easily identify.

2.1.2.2 Problem-solving in the real-world context

Solving problems in a real-world context often creates compelling and relevant need-to-know situations for learning that heighten the learners' interest and motivate them to invest in their learning, hence sustaining their cognitive engagement (Harris & Max, 2009). Thus problem-solving in the real-world context can serve as a motivation for learning (Lombardi, 2007). Motivation can essentially lead to increased cognitive engagement which improves learning (Blumenfeld, Kempler & Krajcik, 2006). Dindyal (2010, p. 97) proposes that "The opportunity to explore real-life applications makes mathematics more meaningful for learners and aids in the development of other skills".

Learners tend to lose interest and the desire to learn if the instruction or information is incomprehensible. Baroody and Ginsburg (1990, p. 55) argue that

> Learners do not merely absorb or make a mental copy of new information; they assimilate it. That is, learners filter and interpret new information in terms of their existing knowledge. Learners cannot assimilate new information that is completely unfamiliar.

According to Moodley (2007, p. 7), "Cognitive theories propose that it is a strenuous mental strain on learners to immediately consume and comprehend abstract information contained in the instruction". This author went further to explain that "Learners must be given the opportunity to assimilate the mathematics, which can be achieved by connecting the new information to their relatively personal, concrete and informal knowledge in their daily lives" (p. 13). The assimilation of thoughts and ideas about new information is possible if it is not completely unfamiliar to the learner. For this reason, mathematical problems in the current study are based on the natural environment of the learners.

In early childhood contexts, the learners' ability to 'mathematize' situations is highlighted as a desirable development (Fox, 2006). Mathematization is the transition of an authentic real-life problem into a mathematical form which bridges the gap between the mathematics that learners practise in their daily lives and the mathematics they learn in the classroom (See Madusise, 2013). Mathematization serves as a catalyst to translate learners' everyday mathematical thinking towards the more formal mathematics they practise in the classroom (Arcavi, 2002). According to Bonotto (2007, p. 187), "The process of bringing the real world into mathematics by starting from a learner's everyday life experience, is fundamental in school practice for the development of new mathematical knowledge". This objective in mathematics learning "can only be completely fulfilled if learners and educators can bring mathematics into reality which "can be implemented in a classroom by encouraging learners to analyse mathematical facts embedded in appropriate cultural artefacts" (Bonotto, 2007, pp. 187-188).

It was recommended by the National Association for the Education of Young Children (NAEYC) and the National Council of the Teachers of Mathematics (NCTM) (as cited in Fox, 2006, p. 5) that mathematical activities should "enhance learners' natural interest in mathematics and their disposition to use it to make sense of their physical and social worlds". Mathematics problem-solving in the real-world context enables learners to understand the world and make use of those understandings in their daily lives. In the National Curriculum Statement (DBE, 2003, p. 10) it is emphasized that, "An important purpose of mathematics is the establishment of proper connections between mathematics as a discipline and the application of mathematics in real-world contexts". The Standards of Excellence in Teaching Mathematics in Australian Schools indicates that excellent educators of mathematics need to "establish an environment that maximises learners' learning opportunities", empowering them "to become independent learners" by modelling "mathematical thinking and reasoning" and providing "purposeful and timely feedback" (The Australian Association of Mathematics Teachers Inc., 2006, Sec. 3).

The practice of activating learners' real-life experiences, together with their mathematical knowledge, has been backed by both theoretical deduction and empirical data (McNeil, Uttal, Jarvin & Sternberg, 2009). Theoretically and from a cognitive perspective, learning mathematics in the context of real-life situations should facilitate performance because it provides the catalyst to recall effective problem-solving strategies (Kotovsky, Hayes & Simon,1985; Schliemann & Carraher, 2002), minimizes the cognitive load of learning (Ericsson, Chase & Faloon, 1980), prevents the learners from mimicking problems presented to them as games that bear no resemblance to real life (Greer, 1997; Verschaffel, Greer & De Corte, 2000), and enables them to mentally stimulate and ground mathematical concepts that might be too abstract to understand (Glenberg, Gutierrez, Levin, Japuntich & Kaschak, 2004). Grigorenko, Jarvin and Sternberg (2002) also indicated that presenting learners with problems within the context of authentic real-life situations also benefits the non-cognitive aspects of their problem-solving performance, such as their motivation and interest in the task.

Empirically, learners have shown evidence of advanced mathematical reasoning as they solve problems in a real-world mathematical setting (DeFranco & Curcio, 1997; Guberman, 1996), and shown better academic performance in mathematics when the subject is presented in a setting that appeals to real-world knowledge (Verschaffel et al., 2000; Wyndhamn & Säljö, 1997).

A seminal study in this area was conducted by Carraher, Carraher and Schliemann (1985). In that study, young learners between the ages of 9 and 15 who were working as vendors in Brazil performed better in a vending context (for an example, I would like to have ten oranges, how much will it cost?), than on the same problem presented symbolically (e.g. 10 x 5) in class. Based on this study, the authors hypothesized that learners might benefit in their conceptual understanding of various mathematical topics if the lessons were designed in a context that activates their real-world knowledge.

Contradictory reports have also been indicated in other theoretical and empirical research. In a study by Baranes, Perry and Stigler (1989), learners were found to perform more poorly in word problems than in symbolic problems. Other research found that learners generally did not perform well when drawing on their real-world knowledge when solving word problems in school (Carpenter, Corbitt, Kepner, Lindquist & Reys, 1980; Reusser & Stebler, 1997; Verschaffel et al., 2000; Yoshida, Verschaffel & De Corte, 1997). They rather relied on the algorithms and constraints they had previously learnt, even if those constraints did not make sense when the given problem was to be interpreted in the real-world context (Reusser & Stebler,1997; Schoenfeld, 1989; Skemp, 1971; Verschaffel et al., 2000).

In the light of the above findings, researchers began to look at situations that would encourage learners to draw from their real-life experiences as they learn and develop their conceptual understanding of mathematics (De Bock, Verschaffel, Janssens, Van Dooren & Claes, 2003; Reusser & Stebler, 1997; Schliemann & Carraher, 2002; Yoshida et al., 1997).

McNeil et al. (2009) stated emphatically that changes that ought to be made to help learners reap the benefits of real-life contexts when solving word problems in a traditional mathematics classroom might be relatively minor and easy to implement. According to McNeil et al. (2009), a straightforward way of activating real-world knowledge when learners are solving word problems is to expose them to concrete objects that will reinforce the real-world scenarios as described in the real problem. McNeil et al. (2009) referred to the example (as cited in Baranes et al., 1989), namely that when a problem situation depicts items being purchased or money being divided among people, the actual money could be made available to the learners, for example "The use of such objects may also help learners perform to the best of their abilities by encouraging learners to consult their real-world knowledge" (McNeil et al., 2009, p.173). Guberman (1996) and Saxe (1988) also emphasised that the presence of concrete objects, such as real currency, helped the learners to solve mathematics problems successfully. There is therefore enough reason to believe that the learners will perform better in word problems when they have access to concrete objects that cue real knowledge, than if they do not have access to such objects (Hiebert & Wearne, 1996; Hiebert, Wearne & Taber, 1991). Using concrete materials in modelling problems in the classroom makes the objects highly realistic and "help[s] learners evoke real-world knowledge and thus, help[s] them perform better than they otherwise would on relevant problems" (McNeil et al., 2009, p. 173).

Contrary to the above findings by researchers on the success of concrete objects in aiding learners to solve mathematical problems, other researchers have had misgivings about using concrete objects that are highly realistic (Bassok & Holyoak, 1989; Goldstone & Sakamoto, 2003; Gravemeijer, 2002; Kaminski & Sloutsky, 2007; Schnotz & Bannert, 2003; Sloutsky, Kaminski & Heckler, 2005; Sweller, 1994; Uttal, Scudder & DeLoache, 1997). The findings of these researchers suggest that the transfer and generalization of knowledge can be hampered if the mathematical objects used in the mathematical problems are highly realistic. Highly realistic concrete objects in modelling problems may initially activate real-world knowledge and help the problem-solvers to construct contextually-relevant interpretations of a problem, but such objects may become redundant once the corresponding real-world knowledge has been activated, which then imposes an extraneous cognitive load on the learner, as redundant information such as colour, texture, size, etc. in problem-solving may be difficult for the learners to ignore (Kalyuga, Ayres, Chandler & Sweller, 2003).

Gravemeijer (2002) advocated for a gradual decontextualization of symbols used in the mathematics classrooms, which may lead to the learners' ability to identify the mathematics embedded in the problem situation (Goldstone, 2006). The use of highly realistic concrete objects may also hinder the learners' ability to learn mathematics, as it may require the learners to change their familiar representation of objects. Moreover, the process of changing their perception of familiar objects may be challenging when their old ways of representing these objects have already been established (Mack, 1995; McNeil & Alibali, 2005). Concrete objects do not necessarily need to be highly realistic in the perception of learners to help them solve problems; the contrary may sometimes be true (Cai, 1995; Stigler & Stevenson, 1992).

A promising method from research (e.g. Gravemeijer, 2002; Greer, 1997; Verschaffel et al., 2000) involved mathematical problems where the learners work collaboratively with the educator to understand the context, generate plausible contextual relevant approaches, and discuss the merits of these approaches. In spite of the use of this strategy, it takes an enormous amount of time and resources to implement, and requires educators to adapt to various changes in the classroom (Greer, 1997).

The above discussion has arguments for and against using highly realistic concrete objects when learners are solving mathematical problems in the classroom.

Mathematical modelling was employed as a problem-solving approach in this study. The concrete objects used were not available to the learners in the classroom but were physical objects they knew in their daily lives and which could be imagined in the classroom. The problems required from the learners to infer what they know about these concrete objects and reminded them of their experiences of the concrete objects in their daily lives. Since the construction of abstract concepts is known to be a difficult process, the use of modelling activities can provide the setting for learners to use their knowledge and to confront their new conceptual needs (Possani, Trigueros, Preciado & Lozano, 2010).

2.1.3 Mathematical modelling

Mathematical modelling can be defined as a mathematical process that involves observing a phenomenon, conjecturing relationships, applying mathematical analyses (equations, symbolic structures, etc.), obtaining mathematical results, and reinterpreting the mathematical models (Swetz & Hartzler, 1991). Ang (2009, p. 161) argued that mathematical modelling should be thought of "as a process in which there is a sequence of tasks carried out with a view to obtaining a reasonable mathematical representation of the real world". Some mathematics educators define mathematical modelling as the process of "using the power of mathematics to solve real-world problems" (Hebborn, Parramore & Stephens, 1997, p. 42). Despite many differences of opinion among researchers on the term 'mathematical modelling', one common feature that stands out among the diverse opinions is that mathematical modelling involves real-life problems (Kaur & Dindyal, 2010). To this end, Blum (2002, p. 273) writes that

> Modelling aims, among other things, at providing learners with a better apprehension of mathematical concepts, teaching them to formulate and to solve specific situation problems, awakening their critical and creative senses and shaping their attitude towards mathematics and their picture of it.

Mathematical modelling has been considered fundamental to the development of learner competencies and its integration into the mathematics curricula has been emphasized (Blum, Galbraith, Henn, & Niss, 2007). It is increasingly recognized that modelling provides learners with a "sense of agency" in appreciating the potential of mathematics as a critical tool for analysing important issues in their lives, their communities and in society in general (Greer, Verschaffel & Mukhopadhyay, 2007). A modelling approach to the teaching and learning of mathematical concepts focuses on the mathematization of realistic situations that are meaningful to learners.

Years of research have revealed that learning on the basis of mathematical modelling leads to improved performance in mathematics (Hickey, Moore & Pelligrino, 2001). In other words,

> During this process, meaningful mathematics is learned, through which reality can be understood, predicted and controlled… the context of a real-life problem is gradually stripped away and the question shaped into a mathematical problem (Wessels, 2014, p. 24).

Learners learn mathematics that is worthwhile when learning the subject through mathematical modelling, and their ability to apply mathematical knowledge is refined (Niss, Blum & Galbraith, 2007). According to Glas (2002), certain benefits are connected to the use of models and modelling in the classroom, namely that the learners not only develop a concept of the interconnectedness of the topics in mathematics, but also develop conceptions between topics outside of mathematics; they develop a realisation that different perspectives of knowledge domains exist; they become creative in their mathematical thinking; and they learn to see mathematics as practical and applicable to the world they live in. Learners engaging in mathematical modelling have the opportunity to discover new mathematical concepts while learning in a familiar context (Gravemeijer, 1997; Van den Heuvel-Panhuizen, 2003). Modelling tasks may be particularly important for learners from diverse backgrounds, especially those whose language and cultural backgrounds differ from the mainstream and who may not perceive relevant connections between the school and their everyday interests and lives (Moll, Amanti, Neff, & Gonzalez, 1992).

The class of problems that depict realistically complex situations, known as modelling-eliciting activities (MEAs), can be used to confront learners with the need to develop models through expressing, testing, and refining their mathematical thinking (Chan, 2008).

2.1.3.1 Modelling-eliciting activities (MEAs)

MEAs were created in the 1970s by a group of mathematics educators (Chamberlin, 2002; Lesh, Hoover, Hold, Kelly & Post, 2000; Lesh & Lamon, 1992). MEAs are mathematically-based activities designed for use with learners in Grades 4–12, with a special emphasis on Grades 5–8 (Chamberlin & Moon, 2005). The focus, however, in this study will be on Grade 6 learners.

According to Chamberlin and Moon (2005), any MEA must consist of four sections: the first two sections set up the context and parameters, and the last two present the problem. The first section of an MEA is an article about the problem; the second section is learners' readiness to answer questions about the preceding article; the third section is a data-collection section that can take the form of a diagram, map or a table; and the fourth section is the problem-solving task. Chamberlin and Moon (2005) continued to explain that the last two sections contain the most mathematics, which creates an ill-structured problem for the learners to solve, generating mathematical creativity and modelling. They explain that a unique characteristic of an MEA is that learners solve problems given to them, and then generalize their models to suit new situations.

MEAs are designed to engage learners in the process of the interpretation, analysis and mathematization of a real-life problem, the product of which is a mathematical model of the presented situation (Lesh & Doerr, 2003; Mousoulides, Christou & Sriraman, 2008; Mousoulides, Sriraman, Pittalis & Christou, 2007). The general purpose of MEAs often has less to do with helping learners to make effective use of exercising ways of thinking, but has more to do with helping them to overcome the debilitating characteristics associated with their current inadequate ways of thinking (Zawojewski, Lesh &English, 2003). MEAs are open mathematical tasks that enable learners to develop mathematical models and elicit creative applied mathematical knowledge (Chamberlin & Moon, 2005; Mousoulides et al., 2007). MEAs require learners to identify the variables of the problems and the interrelations among them, and describe the situation in mathematical terms (Doerr & English, 2003; Mousoulides et al., 2008). Problem-solving, in essence, means finding ways of mathematically interpreting meaningful situations through multiple modelling cycles of progressing from givens to goals (Lesh & Doerr, 2003). The iteration of trial procedures between the ‘givens’ and the goals, in order to find a successful solution, would see the problem-solvers move from ‘givens’ to goals to test their hypotheses, to refine their results and to improve their solutions (Lesh & Doerr, 2003):

> The use of MEAs holds promise in surfacing learners' mathematical thinking and problem-solving processes as well as in helping them move beyond primitive ways of thinking (Chan, 2008, p. 47).

MEAs foster and reveal the learners’ mathematical thinking, thus enabling educators to capitalise on the insights gained into their learners’ mathematical developments (English & Watters, 2005).

If we wish to establish situations of realistic mathematical modelling, in the sense of "both real-world based and quantitatively constrained sense-making" in problem-solving activities, we have to:

1. change the type of activity aimed at creating interplay between the real world and mathematics towards more realistic and less stereotyped problem situations;
2. change the learners' conceptions of beliefs and attitudes towards mathematics (this means changing the educators' conceptions, beliefs and attitudes as well); and
3. change the classroom culture by establishing new classroom socio-mathematical norms (Reusser & Stebler, 1997).

The solving of MEAs comprises a process of interaction between modelling *competencies* and the modelling *process* during which a *group* product is produced that is *new and useful* in a real-world context (Biccard, 2010; Lesh, Cramer, Doerr, Post & Zawojewski, 2003; Lesh & Doerr, 2003).

The solution to MEAs “requires a mathematical model that should be useful to the client that is identified in the problem. The learners should therefore clearly describe their thinking processes and supply convincing reasons for their solution to make it useful for the client” described in the MEA (Wessels, 2014, p. 4). “An MEA does not have only one solution, but the learners should try to find the optimal solution and usually need to change, improve, refine or adapt their first solution" (Wessels, 2014, p. 25).

There are two main reasons why MEAs should be developed and used (English, 2003; Lesh et al., 2000). Firstly, learners are given the opportunity, through the modelling of complex mathematical problems, to consolidate their existing mathematical knowledge and to build new knowledge. Secondly, educators are given the opportunity to study learners' mathematical thinking.

MEAs provide an important contribution to the development of mathematics which meets the individual abilities of many learners. The development of MEAs and the necessity to simplify complex reality enable learners to develop solutions by themselves, according to their capabilities (Kaiser & Maaβ, 2007). Wessels (2011) explains that MEAs offer learners the opportunity to be able to access and process complex mathematical problems at different levels of intellectual sophistication and solve these problems through the interaction between their informal and more formal mathematical knowledge:

> The paradigm shift from traditional teaching and learning of mathematics to a problem-centred approach and a mathematical modelling perspective represents a shift to a more equitable situation in mathematics education... Learners who are exposed to MEAs often change their beliefs about mathematics positively and enjoy these activities, resulting in a shift to positive dispositions (Wessels, 2011, p. 1).

Wessels further proposes:

> MEAs therefore, offer learners the opportunity to mathematize situations through reasoning, communication, justification, revision, and the refining and predicting of skills when they are engaged in solving problems. These activities help to develop divergent thinking, communication skills, fluency with representations, cognitive flexibility, creativity, and the ability to apply mathematical knowledge (Wessels, 2014, p. 25).

MEAs make the learners focus on mathematical understanding and develop an appreciation for the use of mathematics in a real-life context (Chamberlin & Moon, 2005; English, 2006). Through the use of MEAs learners develop their mathematical abilities and conceptual knowledge that go beyond the boundaries of the classroom (Lesh, 2001; Lesh & Lehrer, 2003). In contrast to traditional problem-solving, MEAs are initiated by complex, real-life situations that may have multiple interpretations and suitable solutions (English, 2003). These activities are espoused to foster mathematical reasoning processes (English & Watters, 2005).

MEAs promote both the procedural and the abstract skills of the learners with conceptual mathematical understanding as a result of the modelling process and the personal engagement with the problem (English, 2003; Lesh & Doerr, 2003; Lesh & Zawojewski, 2007; Mousoulides et al., 2007), challenging them to develop their mathematical thinking, that is, to describe, relate, transform and generalize information that goes beyond specific mathematical content or skills (Burton, 1984; Lesh & Zawojewski, 2007; Swan, Turner, Yoon & Muller, 2007).

Fox (2006) explains that MEAs contribute to the successful learning experiences of primary school learners, hence careful consideration needs to be given to their formation. According to Fox (2006, p. 225),

> The problem context will need to be meaningful and challenging to the learners. Key mathematical ideas presented in the task must be relevant to the learners' developmental levels. A variety of representation modes needs to be available for the learners to choose from. MEAs must be couched in authentic contexts that provide real situations in which learners can develop their mathematical thinking.

MEAs are structured to encourage the learners to build mathematical models based on what their group deems important, which may differ from those of another group. All learners have the potential to develop mathematical models from MEAs which can be identified and promoted (Ball, 1994).

Engaging learners in modelling experiences of this nature is not seen as simply finding a solution to a given isolated problem (English, 2004). Such an engagement would rather involve the learners in multiple activities where significant mathematical constructs are developed, explored, extended, and applied, and which results in a system or model that is reusable in a range of contexts (Doerr & English, 2003).

2.1.3.2 Collaborative learning through mathematical modelling

Numerous studies carried out late last century show that collaborative problem-solving in small groups is a key feature of MEAs. In particular, the studies have shown that small-group discussions and debates enhance higher-order thinking and promote shared knowledge construction (e.g. Blumenfeld, Marx, Soloway & Krajcik, 1996; Vye, Goldman, Voss, Hmelo & Williams, 1997). Talking, listening and negotiating are social reflections of "theoretical and empowerment spaces" of understanding, analysing, interpreting and applying knowledge (Dowling, 1998). They also "require classroom discourse and the organisation of learner groups that are orthogonal to independent learning or listening to lecture-style explanations" (Kaur & Dindyal, 2010, p. 14). MEAs support the use of peer-directed group-work (Webb, Nemer & Ing, 2006) and have "demonstrated the importance of implementing activities that inherently develop learners' discourse in cooperative groups" (English, 2007, p. 3). Collaborative learning is generally encouraged as the learners engage in MEAs to enable them to clarify their mathematical ideas and to develop their argumentation and communication skills in a mathematical context while working with peers (English, 2003; Lesh & Doerr, 2003; Swan et al., 2007). The learners' engaging in MEAs is a social experience (English, 2004; Zawojewski, Lesh & English, 2003).

According to Fox (2006, p. 226),

> A social constructivism perspective which underpins many early childhood curricula documents is informed by Vygotsky's learning theories. Modelling activities are designed for small-group work in which the learners develop effective communication and teamwork skills whilst the educator adopts a facilitator role. Interactions occur between the learner and other learners, the learner and the educator, and the learner and the problem.

Zawojewski et al. (2003, p. 343) also emphasize that "peer interaction has the potential to amplify the interest and motivation of the learners involved, increasing the potential mathematical power". "Underlying this pedagogical approach is a conception of mathematical understanding as learners construct mathematical relationships, reflect on and articulate those relationships" (Carpenter & Franke, 2001, p. 1). It thus means that for heuristic teaching to work effectively learners should work collaboratively on given tasks. But, they must be closely monitored to ensure that each learner is actively involved in the given tasks. Furthermore, educators should not just leave learners to work on their own with no support and guidance, otherwise, as suggested by the notion of Vygotsky's zone of proximal development, the noble idea of collaborative learning may be compromised.

2.1.3.3 Justification for implementing mathematical modelling in primary schools

Mathematical modelling is implemented in the secondary and tertiary levels in most education systems based on the notion that primary school learners do not have the ability to develop their own models and sense-making systems for dealing with complex situations (Greer et al., 2007). Yet there is evidence that primary school learners have shown their ability in dealing with mathematical situations that involve more than just simple counting and measuring (English, 2006; English & Watters, 2005). Mathematical modelling allows primary school learners to link the mathematics they learn at school to their everyday real-life experiences, thereby exposing them to the applications of the mathematics they learn (Stillman, 2010; Zbiek & Conner, 2006). Moreover, if the mathematics content taught in primary school is substantively based on real-life tasks it becomes highly relevant to what happens in the environment outside the school.

Exposing young learners to mathematical situations that could be related to their personal real-life experiences and that are more consistent with their mathematical sense-making, allows them to deepen and broaden their conceptual understanding of mathematics, as well as to develop new ways of thinking mathematically, supported by mathematizing real-life situations. In this way, more effective instructional techniques could be developed that give the learners a better chance of developing their conceptual understanding of mathematical concepts (Bonotto, 2007). Swan et al. (2007, p. 280) explained that learners who engage in modelling activities improve their competencies by using symbolic and formal mathematics systems that offer them powerful opportunities to strengthen their understanding of such systems by forging connections between contexts and the formal mathematical expressions related to those contexts, and motivating the study of application of abstract mathematical formulations. Fox (2006, p. 227) concluded that

> The impact that mathematical modelling can have on early mathematical development is statistically significant. We need to provide young learners with greater access to rich, challenging, mathematical experiences that will help them flourish in a global future-orientated world. Mathematical modelling experiences fit current early childhood learning perspectives and provide unbounded opportunities for learners' mathematical growth.

According to Swan et al. (2007, p. 281),

> Modelling is a powerful promoter of meaning and understanding in mathematics. When presented with problems set in some real-world context, learners formulate questions about the context and think about the usefulness of their mathematical knowledge to investigate the questions. They are immediately encouraged to connect their mathematical knowledge with the external context. Mathematical thinking is promoted and reasoning skills are exercised, as learners seek to make those connections.

Modelling develops the learners' understanding of a wide range of key mathematical concepts and "should be fostered at every age and grade . . . as a powerful way to accomplish learning with understanding in mathematics and science classrooms" (Romberg, Carpenter & Kwako, 2005, p. 10).

Mathematical modelling motivates the learners' learning of mathematics by giving direct cognitive support for the learners' mathematical conceptions (Blomhoj, 2004). English (2007, p. 276) also stresses that

> Mathematical modelling in primary school presents learners with a future-oriented approach to learning. The mathematics they experience differs from what is taught traditionally in the curriculum for their grade level, because different types of quantities and operations are needed to mathematize realistic situations.

This approach is supported by Kaiser and Maaβ (2007, p. 104) who state that "Learners at lower secondary level are able to develop modelling competencies, which include meta-knowledge of modelling processes". According to these authors, mathematical modelling provides an important educational contribution to mathematics learning. It is thus considered important for use in heuristic teaching.

2.1.3.4 Challenges to implementing mathematical modelling in the classroom

Challenges associated with the implementation of mathematical modelling date back several years. For example, Blum (1993) enumerated the obstacles to implementing mathematical modelling in the classroom from the point of view of instruction, the learners' point of view, and the educator's point of view, respectively. Blum (1993) notes some of the obstacles to implementation. For example, from the point of view of instruction, educators are often of the opinion that there is no time or space to include applications and modelling in an already overcrowded curriculum. Another obstacle is that some educators are not convinced that modelling applications and their connections to other subjects should belong to the instruction of mathematics at all. From the learners' point of view, Blum (1993) argues that working with modelling and their applications to other disciplines makes the mathematical classroom less predictable and far more demanding, which is manifested as a learner-inherent obstacle. Finally, from the educators' perspective, the introduction of modelling requires more from the educators than simply pure mathematical knowledge; "additional 'non-mathematical' qualifications are necessary" (Blum, 1993, p. 10). Ärlebäck (2010) contended that many educators felt uneasy and unable to deal with applied problems and examples originating from subjects and disciplines they had not studied themselves and that lie outside their field of expertise, which would make it difficult for them to assess the learners' progress and achievements.

Burkhardt (2006, pp. 190-193) discussed four types of *systemic barriers* that counteracted the larger-scale implementation of mathematical modelling in mathematics education. These systemic barriers include:

1. Systemic inertia barrier: This barrier relates to challenges in implementing any innovative teaching strategy in mainstream classroom practice in many countries. According to Burkhardtand Pollak (2006), "the EEE style of teaching (Explanation, worked Examples, imitative Exercises) still dominates, as does the focus on learnt facts, concepts and skills" (p. 190). Learning mathematics has to involve processes of high-level thinking in the form of mathematical modelling, and less direct teaching by educators.

2. Real-world barrier: The second barrier relates to the fact that educators are not comfortable in introducing mathematical modelling as a method of teaching mathematics in the classroom. They think that introducing real-world problems into the mathematics classroom will make an already tedious task more tedious, involved and complicated.
3. Limited professional development barrier: This barrier relates to the fact that many countries expect their educators to teach the curriculum based on the training they have received. "This approach may have worked well when the curriculum changed little during an educator's career; it is clearly inadequate now" (Burkhardt& Pollak 2006, p.191).
4. Role and nature of research and development in education: This barrier relates to the fact that research in education "is not well-organised for turning research insights into improved practice" (Burkhardt& Pollak, 2006, p. 192).

Kaiser and Maaβ (2007) claimed that the learners' beliefs in mathematics may even prevent a broad implementation of realistic tasks in everyday mathematics teaching. This study explores authentic real-life problems in which algebraic concepts are embedded. It hopes to give further insight into how some of the above-mentioned barriers can be arrested and possibly how to overcome them.

2.2 OVERVIEW OF ALGEBRA

2.2.1 Definition of algebra

Most learners have a limited understanding of the meaning of algebra, namely as "working with symbols, finding the unknown, simplifying and solving for x", which is not entirely strange since "the traditional image of algebra, based on more than a century of school algebra, is one of simplifying algebraic expressions, solving equations, learning the rules for manipulating symbols" (Kaput, 1999, p. 134). According to Vermeulen (2007), most learners have a very negative attitude towards algebra since they do not see why they should study it or where it is going to be used; while others regard it only as a boring and abstract subject. Vermeulen (2007) explains that this is partly due to the fact that the teaching and learning of algebra in schools has not been conceptually understood in the classroom, which contributes to the poor quality of the learning of algebra in schools.

In the Curriculum Assessment Policy Statement (DBE, 2011c) algebra for the intermediate phase is defined as a language for investigating and communicating most of mathematics, a generalised arithmetic, and it can be extended to the study of functions and other relationships between variables (DBE, 2011c).

According to Bednarz, Kieran and Lee (1996), algebra is the study of a language and its syntax; the study of solving procedures for certain classes of problems (algebra here is conceived not only as a tool for solving specific problems but also as a tool for expressing general solutions); the study of regularities governing numerical relations (a conception of algebra that centres on generalization and that can be widened by adding components of proof and validation); and the study of relations among quantities that vary.

The National Council of Teachers of Mathematics (NCTM) (as cited in Vermeulen, 2007) categorized algebra into four themes:

i. Functions and relations: Behind the equations, tables and graphs so common to algebra is the central mathematical concept of function. Functions, and the related concept of variable, give organised ways of thinking about an enormous variety of mathematical settings.
ii. Modelling: Many complex phenomena can be modelled by relatively simple algebraic relationships. Viewing algebraic relations in terms of the phenomena they model is an effective way of giving life to them and bringing to the study of algebra the richness of experience all learners carry.
iii. Structure: Through the efficient and compressed symbol systems of algebra, deep yet simple structures and patterns can be represented.
iv. Language and representation: Algebra can be seen as a language – with its various 'dialects' of literal symbols, graphs, and tables. For instance, algebra can be seen as the language for generalizing arithmetic.

The second category, described by the NCTM, was central to this study where the learners conceptualized algebra by representing it in a natural phenomenon they are familiar with.

Vermeulen (2007, p. 15) also describes algebra as follows, namely it is

- a mathematical language that enables us to express generalisations, to investigate and describe patterns, relationships and procedures, and to derive new relationships and procedures by appropriate manipulation;
- generalised arithmetic;
- a study of relationships between variables; and
- often through modelling, a tool used to solve problems.

2.2.2 Conceptions in algebra

Documented research exists on conceptions in algebra that discuss the structures of algebra and how they are linked to form a wider conception of algebra in line with its already existing development (Egodawatte, 2011; Usiskin, 1988; Vermeulen, 2007). Usiskin (1988) discusses four conceptions in algebra relevant to the study of algebra in the South African school system.

The first conception considers algebra as a generalization of arithmetic, and a variable is considered as a pattern generalizer. Relations are found between the numbers that we wish to describe mathematically. A key instruction for learners in the first conception is to 'translate' and to generalize. An example is the generalization of $3 * 5 = 15; 4 * 4 = 16; 8 * 6 = 48$ to $x * y = xy$**.**

In the second conception, known relations among variables are generalized, and a conception of algebra as a concept of procedure begins. An example of this is: 5 is added to 3 times a certain number and the sum is 40, find the number (Usiskin, 1988). Translating the problem into algebraic language would be as follows: $3x - 5 = 40$, which is 'simplified' to $3x = 45$ and 'solved' with a solution of $x = 15$. In this conception, the variables are seen as 'unknowns' or constants, and the key instructions are 'simplify' and 'solve'. According to Usiskin (1988), whilst solving these kinds of problems, the learners may experience difficulties while advancing from arithmetic to algebra. While the arithmetic solution ('solution in your head') involves subtracting 5 and dividing by 3, the algebraic form $3x + 5$ involves multiplying 3 and adding 5, which is an inverse operation to that of the arithmetic operation: "That is, to set up an equation, you must think precisely the opposite of the way you will solve it using arithmetic" (Usiskin, 1988, p. 10).

The third conception considers algebra as the study of relationships between known quantities. There is no feeling of an unknown, as there is nothing to solve. For example, a formula for the area of a rectangle is $A = LxB$. Where the conception of algebra as a study of relationships may begin with formulae, a crucial distinction between this and the previously-mentioned two conceptions is that the variables vary. An example is the following: "What happens to the value of $\frac{1}{x}$ as x gets bigger?" This problem does not ask that anything be found, hence x is not an unknown. It also does not ask the learner to generalize. Even though there is a pattern to generalize it, it is not an arithmetic pattern. The third conception is the only conception in which the notion of a dependent and an independent event exists.

The fourth conception recognises algebra as a study of structures. Consider the example, "Factorize the expression:$3x^2 + 4ax - 132a^2$". The conception of the variable in the fourth conception is different from the three conceptions already discussed. There is no function or relation and no equation to be solved; hence the variable cannot act as an unknown. There is also no arithmetic pattern to generalize. In this kind of problem faith is placed in the properties of the variables. The variable has become an arbitrary object in a structure related to certain properties. It is the view of variables found in abstract algebra. It should also be noted that algebra has use in solving problems in other areas of mathematics. It is for this reason that this study's interest is on finding ways to improve algebra learning.

2.2.3 Common misconceptions in learners' understanding of algebra

A great number of learner misconceptions of algebra exist in the literature (Usiskin, 1988; Vermeulen, 2007). This study identifies the findings of Vermeulen (2007) as the most relevant for this study. Vermeulen (2007) identified three major misconceptions learners have displayed in respect of algebra, including

i. Conjoining or closure: $3 + x = 3x$ or $x + y = xy$

ii. Over-generalisation of the distributive property: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ or $(x+y)^2 = x^2 + y^2$

iii. Incomplete application of the distributive property: $2(x+y)=2x+y$ or $\frac{4x+6}{2}=2x+6$

The above misconceptions indicate a clear lack of understanding of the major procedures used in algebra. The lack of procedural understanding can be attributed to a lack of conceptual understanding of the variables, algebraic expressions and equations the learners are confronted with. The learners' understanding of how a particular algebraic expression or equation comes about and why it should be simplified and solved enhances their ability to create, manipulate and solve algebraic equations/expressions correctly.

This study hypothesizes that if the learners are able to properly conceptualize the origins of algebraic expressions, they may be able to overcome some of these conceptions that may translate to an improvement in their conceptions in arithmetic.

2.2.4 Algebra in Grade 6

The CAPS statement (see DBE, 2011c) structures mathematics in Grade 6 into five learning outcomes (LO):

i. Numbers, operations and relationships (LO1)
ii. Patterns, functions and algebra (LO2)
iii. Space and shape (LO3)
iv. Measurement (LO4)
v. Data-handling (LO5).

Algebra is represented in various contexts in learning outcomes three, four and five. "However, the core of the development of a learner's knowledge and understanding of algebra, takes place with the first two learning outcomes" (Vermeulen, 2007, p. 20), which fall within the first two conceptions of algebra, as described by Usiskin (1988) in section 2.2.2. It categorizes algebra in Grade 6 into three main components:

1. Number sentences: Learners are expected to write number sentences with other representations such as mathematical problems and solve and complete

number sentences by inspection, trial and improvement, and check solutions by substitution;

2. Number patterns: Learners are expected to look for and describe relationships or rules in a number pattern, determine input and output values of a number pattern;
3. Determine equivalent descriptions of the same rule or relationships established and geometric patterns: Learners are expected to look for relationships or rules in a geometric pattern, determine input and output values of a geometric pattern, and find equivalent descriptions of the same relationship or rules established.

2.2.5 Justification for algebra to be considered as a major part of the primary school curriculum

Many curriculum developers have advocated for the learning of algebra to be a major part of early school mathematics, but the reality is that algebra is not considered a major part of primary school mathematics until the secondary school years, despite the following observation: “Nevertheless, there are many aspects of mathematics in the primary school that prepare learners for later algebra study” (Chick & Harris, 2007, p. 121). Research indicates that primary school learners are able to do more in school algebra than we expect of them (Becker & Rivera, 2006; Blanton & Kaput, 2004; Fujii & Stephens, 2001; Lins & Kaput, 2004; Warren, 2005).

Blanton and Kaput (2004) examined primary school learners’ ability to describe functional relationships and found evidence that young learners can keep track of how one variable changes with respect to another. Warren (2005) found that Grade 4 learners are capable of thinking functionally, and can describe in visual terms how a pattern is generated. Becker and Rivera (2006) came to similar conclusions in their research with six learners. They examined how figurative reasoning usually results in greater success than reasoning with numerical quantities alone.

The three basic components in primary school algebra, as described by Bednarz et al. (1996) are (i) the generalisation of patterns, (ii) the generalization of numerical laws, and (iii) functional situations. Kieran (1996) groups the components of primary school algebra under the heading ‘generational activities’, since each involves the production of some algebraic objects, namely an equation-relating quantity, a description or relation capturing the generality of a pattern, or a set of numbers that describes some general numeric behaviour (Kieran, 2004).

Kieran (2006) believes that young learners must have a conceptual understanding of algebra. She also noticed that structure, justification and proving are lacking in school algebra. Kieran (2006) explains that the early algebra of analysing relationships, generalising, noticing structure, and predicting is a way of thinking that is foundational for conventional letter-symbolic algebra.

The current study deemed it fit to explore whether a problem-solving oriented instruction can be used to develop learners' conceptual understanding of algebra as it is represented in all aspects of the five learning outcomes in the intermediate phase of the South African school system of which Grade 6 forms part.

2.3 THEORETICAL FRAMEWORK

Two main theories supported and guided the heuristic teaching experiment. These are the modelling and modelling perspective and the APOS theory. Arnon and Dubinsky (2000, p. 7) note that "a theory is neither correct nor incorrect rather a theory is a tool for doing research and improving teaching. Therefore, rather than its correctness we are interested in its effectiveness". The modelling and modelling perspective guided the design of an effective MEA, whereas the APOS theory guided the design of the instructional sequence used in this study, explaining the learners' mental structures and the mechanisms they need to achieve a conception in algebra.

2.3.1 Modelling and modelling perspective

Researchers and educators of mathematics have to follow certain guidelines in order to develop an MEA. These guidelines are referred to as the six principles of design, known as the modelling and modelling perspective (Chamberlin, 2004; Lesh et al., 2000). The modelling and modelling perspective are based on six principles that arose out of the work of a number of researchers and educators, but were subsequently refined by Lesh et al. (2000). According to Lesh, Amit and Schorr (1997, p. 2), these principles have been violated by mathematics problems seen in every major mathematics textbook and test, "therefore, in some sense, they are quite radical". Furthermore, "The principles ensure that each MEA will have the intended curricular and learning characteristics” (Chamberlin & Moon, 2005, p. 39). There are six principles.

2.3.1.1 Reality principle

The task focuses on problems with which learners are confronted in their lives where they are encouraged to make sense of a situation, based on extensions of their own personal knowledge and experiences. The activity must aid the learner to be able to interpret the problem given. Possani et al. (2010) emphasize that the context that motivates the MEAs must be motivational and realistic enough for the learners. At the same time, the mathematical elements embedded in the problem must not be compromised, and the specific mathematical concepts embedded in the context must be clearly outlined. Other researchers refer to the ‘reality principle’ as the meaningful principle, which is meant to increase the learners’ interest and stimulate the kind of activities in which mathematicians engage when solving problems (Chamberlin & Moon, 2005). They stress that the more realistic the problem, the more potential exists for creative solutions based on the learners’ familiarity with the problem.

2.3.1.2 Model-construction principle

The task involves the construction of a model where the learners will construct, explain, manipulate and predict a structurally significant system. MEAs must be designed to elicit creative behaviours and high-level thinking, especially at the level of synthesis (Chamberlin & Moon, 2005). The MEAs must be able to push the learner to explicitly describe and explain a given situation mathematically. The problem-setting must be authentic enough to “need mathematical concepts in the construction of a model” (Possani et al., 2010, p. 2128).

2.3.1.3 Self-evaluation principle

The design of the task makes it easier for the learners to assess the use of their responses and of those of others, and also be able to judge whether their responses are adequate. The activity must contain a criterion that enables learners themselves to revise and test their current way of thinking mathematically. The learners must be able to recognise the appropriateness and use of their model without input from the educator (Chamberlin & Moon, 2005).

2.3.1.4 Model-documentation principle

The mathematical concepts embedded in the task have to enable the learners to reveal how they understand the situation mathematically, and to reveal their mathematization processes as they work on the task. It ensures that while working on the activity the learners create some form of documentation to reveal their thinking of the problem situation. Lesh et al. (2000) refer to this principle as the 'taught revealing activity', in that it is able to reveal how the learners are thinking as they work on an MEA which can be documented by the educator. The model-documentation principle helps the educator who implements MEAs to focus on the thinking processes of the learners during problem-solving, as well as on their final model (Chamberlin & Moon, 2005).

2.3.1.5 Model-generalisation principle

This feature requires the learners to be able to produce sharable and re-usable solutions so that mathematical model(s) could be transferred and used in other real-life situations. If the model can be transferred to other parallel real-life situations requiring a similar model, then the learners' responses are deemed to be successful (Chamberlin & Moon, 2005).

2.3.1.6 Simple prototype principle

The task must be designed to elicit the creation of a model while still being as simple as possible. The activity must be as simple as possible, whilst at the same time being mathematically significant. According to Chamberlin and Moon (2005), the principle requires the model created by the learners to be easily interpreted by other learners. They emphasize the difference between this model-generalization principle and the simple prototype principle in that in the simple prototype principle the learners may use the prototype in a similar situation but not in a parallel situation.

Lesh et al. (1997, pp. 2-3) also articulated a number of questions to be asked regarding each of the principles as an MEA is being designed.

(a) Reality principle

Could this really happen in a real-life situation? Will the learners be encouraged to make sense of the situation based on extensions of their own personal knowledge and experiences? Will the learners' ideas be taken seriously, or will they be forced to conform to the educator's notion of the 'correct' way to think about the problem situation?

(b) Model-construction principle

Does the task create the need for a model to be constructed, or modified, or extended, or refined? Does the task involve constructing, explaining, manipulating, predicting, or controlling a structurally significant system? Is attention focused on underlying patterns and regularities rather than on surface-level characteristics?

(c) Self-evaluation principle

Are the criteria clear for assessing the usefulness of alternative responses? Will the learners be able to judge for themselves when their responses are good enough? For what purposes are the results required? By whom are they required? When?

(d) Model-documentation principle

Will the responses require the learners to explicitly reveal how they are thinking about the situation ('givens', goals, possible solution paths)? What kind of system (mathematical objects, relations, operations, patterns, regularities) are they thinking about?

(e) Model-generalization principle

Does the model that is constructed apply to only a particular situation, or can it be applied to a broader range of situations?

(f) Simple prototype principle

Is the situation as simple as possible while still creating the need for a significant model? Will the solution provide a useful prototype (or metaphor) for interpreting a variety of other structurally similar situations?

The study drew on these principles in the design of an effective MEA that served as a medium of instruction in developing learners' conceptual understanding in algebra at the level of Grade 6.

2.3.1.7 Characteristics of the MEA in the lens of the modelling and modelling perspective

Chamberlin and Moon (2005) explain that MEAs that are designed on the bases of the six principles of the modelling and modelling perspectives must have the following characteristics:

i. **Inter-disciplinary:** This enables educators to integrate other disciplines. In addition to mathematics literacy, which is the main goal of MEA, MEAs have a context related to social studies, science, physical education, etc. When the learners use knowledge from various subjects when solving MEAs it increases their ability to reason creatively.
ii. **Well-structured problems:** MEAs are well-structured problems in the sense that all the necessary information to solve them is within the problems or is readily available to the learner. The learner does not have to do any research in order to solve the problem.
iii. **Realistic problems:** MEAs must be realistic problems that are relevant in the lives of the learners (Lesh et al., 2000). According to Cooper and Harries (2003), realistic problems in MEAs are likely to promote learning mathematics with understanding compared to problems without context.
iv. **Meta-cognitive coaching:** MEAs are administered successfully when the educator acts as a meta-cognitive coach when the learners are solving MEAs, and pose questions to the learners rather than answering them.
v. **Explication of learner thinking:** MEAs provide the opportunity for educators to explore the learners' thinking as they work on the MEAs which can give much insight when the curriculum is being revised.

2.3.2 APOS theory

The APOS theory is a constructivist theory that arose out of an attempt to understand the mechanism of *reflective abstraction* introduced by Piaget to describe the development of logical thinking in young learners. This idea was extended to more advanced mathematical concepts (Dubinsky, 1991). The theory is used to model the way learners learn mathematics in order to design teaching sequences that can prove effective in terms of the learners' learning, and to analyse the knowledge that the learners display when solving a specific problem at a particular moment in time (Possani et al., 2010).

The APOS theory is a theory about how particular mathematical concepts can be learnt, and focuses on what might be going through the mind of a learner as he or she tries to learn a mathematical concept (Arnon et al., 2013). According to APOS theory, learning and understanding any mathematical concept starts with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized or internalized to form processes which are then encapsulated into objects. The processes and objects are then organized into schemas (Dubinsky, 2000). According to Dubinsky (2000), there is a misleading aspect of the APOS theory which is seen as a linear progression of action through a process to an object and finally to a schema. Dubinsky (2000) describes the APOS theory as 'dialectic', where there is not only a partial development of a particular mathematical concept at one level; it then moves on to a more sophisticated level of the same concept and turns back to the previous level, going back and forth. Dubinsky (2000, p. 5) explains that "The developments of each level influence the developments at both higher and lower levels". The APOS theory hypothesizes that a learner can learn and develop an understanding of any mathematical concept for which he or she has made the necessary mental constructions (Arnon et al., 2013). It can be applied from Grade R to Grade 12 with fractions and algebraic thinking (Weller, Clark, Dubinsky, Loch, McDonald & Merkovsky, 2003). The development and implementation of the heuristic teaching sequence to develop the appropriate mental structures and mechanisms required to develop the learners' algebraic thinking was based on the APOS theory, described by Dubinsky (2000) as follows.

Action: A transformation is initially conceived as an action when it is a reaction to stimuli which the learner perceives as external. The learner needs complete and understandable instructions, namely being given precise details on steps to take in connection with the concept. For example, a learner "who requires an explicit algebraic expression in order to think about the concept of function and can do little more than substitute a variable in the expression and manipulate it is considered to have an action understanding of functions" (Dubinsky, Weller, McDonald, & Brown, 2005, p. 338). According to APOS theory, the learners' inability to construct mathematical knowledge results from their inability to interiorize actions into processes or to encapsulate processes into objects:

> Even though an action conception is very limited, it is an important part to begin to understand a mathematical concept. Therefore, instructions should begin with activities designed to help learners to construct actions (Dubinsky, 2000, p. 6).

The learner at this level is able to conceptualize a rule to explain the goals of an MEA and to substitute input values to give output values in accordance with the goals of the MEA.

Process: As a learner repeats and reflects on an action, the action can be perceived as part of the individual, and he or she can establish control over it and interiorize it into a mental process. The process-conception of algebra enables a learner to conceptualize, say, an algebraic expression as a rule, that dynamically transforms one set of elements in a situation to another set of elements in the same situation without substituting any values. A learner "with a process understanding of function will construct a mental process for a given function and think in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs" (Dubinsky et al., 2005a, p. 339).

Object: When a learner reflects on the operations applied in a particular process he or she becomes aware of the process as a totality, realises that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the learner has encapsulated the process into a cognitive object. When an action or a process is performed on a cognitive object, it is always necessary to de-encapsulate the object back into the process from which the object was obtained in order for the learners to manipulate and reconstruct its properties. The encapsulation of processes into objects and the de-encapsulation of objects into processes enable the learners to modify, say, algebraic expressions or equations to fit a new situation in terms of its properties. When the learner is able to transform a rule by adding, subtracting, dividing or multiplying a constant to describe similar problem-situations, we say the learner has encapsulated the process conception into an object conception s*chema.*

> In general encapsulating processes into objects is seen to be extremely difficult and not very many pedagogical strategies have been effective in helping learners in developing a conceptual understanding of these situations in algebra (Dubinsky, 2000, p. 6).

Schema: Once constructed, objects and processes can be interconnected in various ways. A collection of actions, processes, and objects can be organized in a structured manner to form a schema which may also include previously constructed schemas (see Figure 2.1).

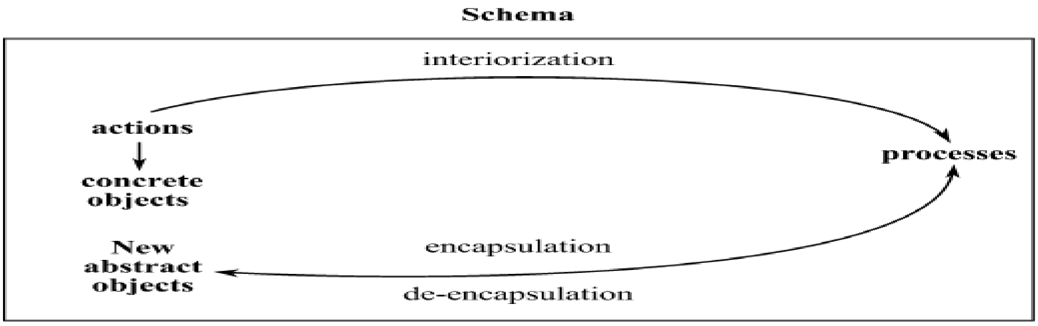


Figure 2.1: APOS theory schema (based on Arnon et al., 2013)

The structure of a schema has coherence, in the sense that the learner understands implicitly or explicitly which phenomena the schema can be used to deal with.

The APOS theory develops possible pedagogical strategies for learning a particular concept, known as the ‘genetic decomposition’. Data is gathered in the process to either validate the teaching pedagogy or to call for amendments to the teaching pedagogy (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996; Dubinsky, 1991; Dubinsky, 2000; Dubinsky & McDonald, 2002; Weller et al., 2003).

2.3.2.1 Genetic decomposition

According to Possani et al. (2010), The application of the APOS theory to describe particular constructions by learners requires researchers to develop a genetic decomposition – a description of specific mental constructions, a mental structure and a mechanism a learner may make in the process of understanding mathematical concepts and their relationships.

Researchers employing the APOS theory are expected to develop a generic decomposition. Asiala et al. (1996) also describe genetic decomposition as a concept of a structured set of mental constructions which may describe how the concept can develop in the mind of a learner. Genetic decomposition models a learner's conception of a particular mathematical concept that aligns with the communal understanding of the particular mathematical concept by a community of mathematicians (Arnon et al., 2013).

Genetic decomposition is hypothesized theoretically and tested empirically by the researcher which may help to unearth difficulties the learners may face in learning a particular mathematical topic (Arnon et al., 2013). The genetic decomposition formulated for the study to describe the learners' mental construction in learning algebra through authentic real-life problems is described in Section 3.5.3.1.

2.3.2.2 APOS paradigm in research and curriculum development

Kuhn (1962, p. 10) explains two characteristics of a 'paradigm' as follows, namely a theory powerful enough to "attract an enduring group of researchers", and a theory that provides enough open ends to sustain researchers in challenging mathematical topics that need to be learnt by the learners. In the light of the above characteristics, Kuhn (1962) and Arnon et al. (2013) argue that any research which is based on the APOS theory can be referred to as a 'paradigm', for four reasons: (i) it differs from most research on mathematics education in its theoretical approach, methodology, and types of results offered; (ii) it contains theoretical, methodological, and pedagogical components that are closely linked together; (iii) it continues to attract researchers who find it useful to answer questions related to the learning of numerous mathematical concepts; and (iv) it continues to supply open-ended questions to be answered by the research community (p. 93). According to Asiala et al. (1996), APOS-based research and/or curriculum development consists of three components, namely theoretical analysis, the design and implementation of instruction, and the collection and analysis of data. Figure 2.2 explains how these components are related.

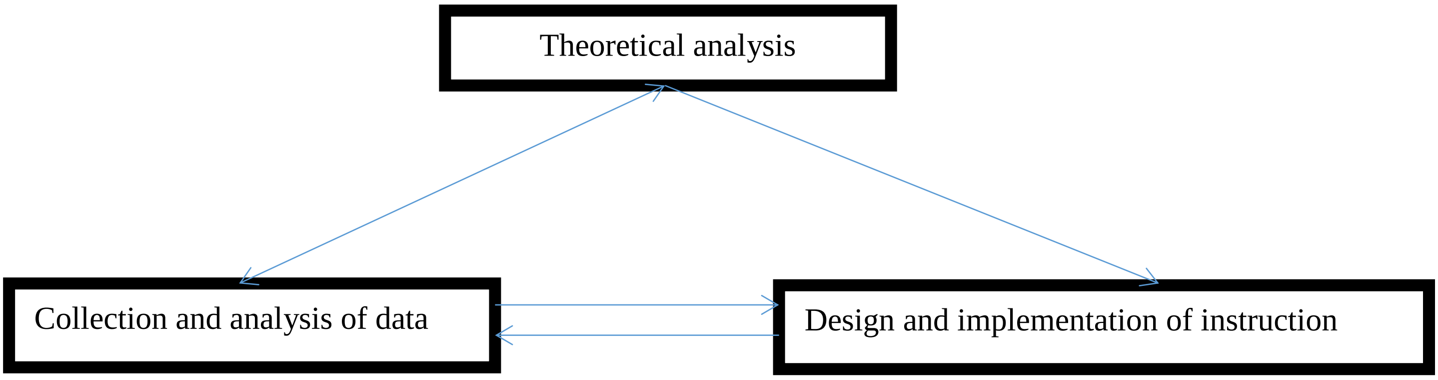


Figure 2.2: APOS-based research cycle (adapted from Asiala et al., 1996)

The APOS-based research paradigm starts with a theoretical analysis of the mathematical concept under consideration. This then gives rise to a preliminary genetic decomposition, which is a specific mental construction and mechanism a learner must make in constructing his or her understanding of the mathematical concept (algebra) under consideration. As indicated in Figure 2.2, the theoretical analysis drives the design and implementation of instruction through activities which are intended to foster the learners' mental constructions called for by the theoretical analysis. "These activities must be designed to help learners to construct actions, interiorize actions into processes, encapsulate processes into objects and coordinate two or more processes to construct new processes" (Arnon et al., 2013, p. 96). Arnon et al. (2013) explain that pedagogical strategies that can be effective in APOS-based research may include cooperative learning, small-group problem-solving, and lecturing.

The problem-solving heuristic instructional method employed in the current study explored the use of small-group problem-solving to theorize a mental construction that may enable the learners to develop sound conceptions in algebra.

Arnon et al. (2013) continued to explain that the implementation of a teaching pedagogy gives rise to the collection and analysis of data in the lens of the APOS theory and has to answer two questions, namely (i) Did the learners make mental constructions called for by the theoretical analysis? and (ii) How well did the learners learn the mathematical concept in question? According to Arnon et al. (2013), if at least one of the answers to the two questions is negative, it will be necessary to reconsider and revise the theoretical analysis which has to be cycled repeatedly until such time that the abovementioned two questions are answered positively and the researcher is satisfied that the learners have learnt the mathematical concept in question sufficiently well.

Arnon et al. (2013) also explain five kinds of instruments that are used to investigate the answers to the two questions mentioned above. These include:

i. Interviews: This instrument may be used to compare the learners' mathematical performances in the light of different instructional methods. The goal with interviews is to assess whether the learners have made mental constructions as set out by the preliminary genetic decomposition.

ii. Written questions: This instrument can be used when the research involves a large number of learners. The written questions can be administered in the form of a formal examination or a questionnaire, and provide basic information about the learners' mathematical performances in respect of the mathematical concept in question. Written questions can also be used to design interview questions since they could reveal the learners' problems with the mathematical concept being learnt.

iii. Classroom observation: This instrument may reveal interesting information about the learners' mathematical abilities if the pedagogical instruction is not based on elements related to the APOS theory or if the instructor has little or no experience with the APOS approach.

iv. Textbook analysis: The learners' textbooks can be analysed in support of the pedagogical instruction to be followed. This is done in the light of the elements in the APOS theory in respect of the mathematical concept in question to determine which results, rules and theorems make use of the mathematical concept under investigation, and to assess whether the notation being used may have an influence on the learners' understanding of the concept.

v. **Historical/Epistemological analysis:** This analysis enables the researcher to contextualize the learners' difficulties in terms of obstacles they may face through developing a mathematical concept, as well as to explain those difficulties in cognitive terms (Arnon et al., 2013, pp.95-104).

This study employed written questions to evaluate the impact of APOS-based teaching.

2.3.3 Motivation for integrating above-mentioned two theories in the implementation of the heuristic teaching experiment

The APOS theory has been applied successfully as a developmental and evaluative tool in various mathematical topics in both secondary and post-secondary mathematics (Brijlall & Ndlovu, 2013; Chimhande, 2013; De Castro, 2011; Jojo, Maharaj & Brijlall, 2013; Maharaj, 2010; Maharaj, 2013; Mulqueeny, 2012; Stalvey, 2014; Tabaghi, 2007). Brijhall and Ndlovu (2013) used the APOS theory to explore Grade 12 learners' mental construction when solving problems in calculus and found that most Grade 12 learners functioned effectively at the action level of the APOS theory when solving problems in calculus. Chimhande (2013) used the APOS theory through a generic research model to improve the teaching of functions in Grade 11 and found that although learners could not reach the intended schema of the APOS theory framework, they were able to progress smoothly through the various levels of the APOS theory. De Castro (2011) used the APOS theory framework to explore secondary school learners' conceptual understanding of limits and derivatives when utilizing specifically designed computational tools. The study gave insights into the effective design and use of computational tools in fostering conceptual understanding. Maharaj (2010) used the APOS theory framework to explore university learners' understanding of limits of functions. The findings of the study confirmed that the learners found it difficult to understand limits and was due to the fact that they did not have the appropriate mental structures at the process, object and schema level of the APOS theory framework. Maharaj (2013) used the APOS theory framework to investigate university learners' understanding of derivatives and their applications and came to the conclusion made earlier (Maharaj, 2010), namely that university students found it difficult and did not have the necessary mental structures at the process, object and schema level of the APOS theory framework. Mulqueeny (2012) used the APOS theory to investigate college learners' understanding of the logarithmic concepts. The results of the study suggested a framework that a learner might use to construct logarithmic concepts. Stalvey (2014) used the APOS theory to explore university learners' understanding of parametric functions and proposed and revised a genetic decomposition that learners could use to construct concepts' parametric functions. Lastly, Tabaghi (2007) also used the APOS theory framework to explore learners' understanding of logarithmic functions and found that the understanding of logarithmic functions by most learners did not go beyond process level.

There is rarely any evidence of the application of the APOS theory to develop the learners' understanding of mathematical concepts in the primary school. The only notable study found in the literature was the use of the APOS theory to teach Grade 4 learners the part-whole relationships of fractions (Arnon, 1998) and learners in Grade 5 the equivalent relationship of fractions (Arnon, Nesher & Nirenburg, 1999, 2001). According to Arnon et al. (2013, p. 104), "APOS is a cognitively oriented theory and as such provides a useful tool for modelling learner understanding of mathematical concepts".

This study deemed it fit to continue with their work by investigating how the APOS theory could be used to explain learners' mental structures and mechanisms in the learning of algebra in Grade 6 as it occurred during problem-solving.

Piaget (1975, 1976), on whose work (i.e. reflective abreaction) Dubinsky (1991) developed the APOS theory, found that secondary and post-secondary school learners are expected to be at the stage of formal operations. This means that mathematical objects on which actions are performed should be abstract objects. On the other hand, Piaget also found that primary school learners are at a stage of concrete operations and thus mathematical objects on which primary school learners perform actions need to be concrete objects, namely that they can be perceived by one's senses. Arnon et al. (2013, p. 151) indicate that"from the perspective of the APOS theory, the principal difference between the elementary and post-secondary mathematics classroom lies in the nature of the objects to which actions are applied".

Against the backdrop of Piaget's findings (1975, 1976), this study combined the effects of the APOS theory and the modelling and modelling perspective in its teaching treatment. The modelling and modelling perspective informed the development of the MEAs used for this study that placed the learners at the stage of concrete operations to develop abstract algebraic concepts as the result of the reflections upon the actions on perceived concrete objects. On the other hand, the APOS theory gave the researcher the theoretical basis to design and implement a learning sequence that describes the mental mechanisms and structures the learners may make when developing an understanding in algebra. The APOS theory may be considered as a developmental framework for the design and implementation of instructional materials and settings based on preliminary or revised genetic decomposition (Arnon et al., 2013). Integrating these two theories may give the researcher valuable insight into how learners can develop sound conceptual understandings in algebra.

2.4 REFLECTING ON CHAPTER TWO

In the first section of this chapter the researcher discussed a definition and the characteristics of heuristics and heuristics teaching, together with its use in teaching through problem-solving. The first section then explained a number of problem-solving processes as well as the difficulties the learners may face when solving problems, and how it motivated the selection of mathematical modelling as a suitable problem-solving technique. Finally, the researcher discussed the class of problems used in mathematical modelling, namely MEAs, together with the feasibility and challenges of implementing them in a Grade 6 classroom.

In the second section of the chapter the researcher gave a definition of algebra together with common conceptions and misconceptions learners may have in algebra. The researcher then discussed the state of algebra learning in the intermediate phase in the South African school system, and concluded with the general advantages of learning algebra in the primary school.

In the last section, the researcher discussed the two theories that underpin the study, namely the modelling and modelling perspective and the APOS theory that guided and informed the heuristic teaching experiment. The modelling and modelling perspective gave guidelines on the design of MEAs used in this study. It was found applicable when explaining why MEAs are useful in nurturing creativity in mathematics. The APOS theory explained the particular mental constructions and mechanisms a learner has to go through in order to achieve a conception in algebra in the light of solving MEAs. Justification was given on why the two theories were integrated into the teaching experiment.

In the next chapter the researcher discusses the methodology used to assess the impact of combining the effects of these two theories in the problem-solving heuristic instructional method in the learning of algebra in Grade 6.

CHAPTER THREE
RESEARCH METHODOLOGY

Research requires decisions with regard to sampling, instrumentation, data-collection and the methods of data analyses (McMillan & Schumacher, 2001). The decisions in respect of method made in this chapter are aimed at establishing the effects of the problem-solving heuristic instructional method designed for this study as will be explained by the preliminary genetic decomposition in section 3.5.3.1. This chapter is organized into 10 main sections. Section 3.1 explains the research paradigm, section 3.2 discusses the research design, section 3.3 discusses the population, sampling and sampling technique, section 3.4 explains the observation of regular mathematics educators during the problem-solving heuristic instructional method, section 3.5 explains the design and implementation of the heuristic teaching design, section 3.6 explains how the effects of the problem-solving heuristic instructional method on learners' achievements in algebra were measured, section 3.7 discusses the data collection instruments used for this study, section 3.8 discusses the data collection procedure used for this study, section 3.9 explains how data gathered was analysed, and section 3.10 discusses ethical consideration made before data collection.

3.1 RESEARCH PARADIGM

A research paradigm is "a basic set of beliefs that guide [the] action" of the researcher during research (Guba, 1990, p. 17). A research paradigm explains the researcher's theoretical lens which influences his or her research methods (Dobson, 2002). A study can be executed with significant achievements if a research paradigm that best suits the study is used (Flowers, 2009). This study is underpinned by the philosophy of pragmatism. Pragmatism, which arose out of the work of William James, John Dewey, and Charles Sanders Peirce (Cherryholmes, 1992), focuses on the research problem and uses pluralistic approaches with the hope of understanding the research problem (see Rossman & Wilson, 1985). The motivation for choosing a research paradigm based on pragmatism was to enable the study to explain fully the effects of a problem-solving heuristic instructional method and how it evolves using both qualitative and quantitative research methods in data collection and analysis. Through this approach the study hoped to report on the true nature of a problem-solving heuristic instructional method on Grade 6 learners' achievement in algebra. Pragmatism looks at what, and how, to research, and bases decisions on the consequences of the research problem. Hence it makes use of multiple approaches for collecting and analyzing data rather than subscribing to only one method (Cherryholmes, 1992; Creswell, 2009; Morgan, 2007). Pragmatism applies mixed-method research where the researcher draws liberally from both quantitative and qualitative research methods. In this way, the researcher freely chooses methods, techniques and procedures of research that best suit the aims and objectives of the research (Creswell, 2009).

3.2 RESEARCH DESIGN

The study followed a mixed-method approach to determine the effects of the problem-solving heuristic instructional method in learners' achievements in algebra. A mixed-method approach combines both qualitative and quantitative methods of data collection and analysis that can make the study answer the research questions as set out in the study (Cresswell, Klassen, Plano Clark and Smith, 2011). The qualitative component of the research design involved firstly a pre-intervention class observation of Grade 6 mathematics lesson to identify the teaching methods being used by the educators and secondly, the intervention which entailed the design and implementation of the problem-solving heuristic instructional method which helped explain how learners' knowledge in algebra evolves when they are taught algebra through the problem-solving heuristic instructional method. The quantitative component of the research design involved a non-equivalent control group quasi-experimental design with pre-test and post-test measure and was used to measure the effects of the problem-solving heuristic instructional method on learners' achievements in algebra.

3.2.1 Rationale for using a mixed-method approach

A research problem should be investigated holistically, based on the initial premise that human beings who formed the basis of this research are influenced by various factors in several ways. Therefore, one should not only consider the intellectual or psychosocial aspects of a person's being, but rather all aspects relating to the person-in-the-world (Vrey, 1984). A mixed method approach was chosen since "mixed methods procedures employ aspects of both quantitative methods and qualitative procedures" (Creswell, 2003, p. 17) and the two methods complement one another. In a way, the limitations inherent in either of the methods can be neutralized by combining the effects of both methods (Creswell, 2002). In this way, several challenges that would have arisen if only the quantitative or the qualitative method was used were overcome. Bias that developed in one method neutralizes and cancels bias in the other method, and *vice versa.* The use of multiple approaches gave deeper insight into the effects of the problem-solving heuristic instructional method. The rationale for the choice of the design was therefore to enable the researcher to determine the impact, if any, of the problem-solving heuristic instructional method on learners' achievements in algebra and explain how the instruction was used.

Combining both the qualitative and quantitative methods also corroborated the findings of both methods. The quantitative findings informed and supported the richness of the qualitative findings by providing statistical evidence. Hence a more comprehensive investigation of the problem at hand was evident.

3.2.2 Description of the research site and the participants

The research was done in four community quintile 1 schools in the Zululand district of KwaZulu-Natal. The Zululand district has a total land area of 14799km^{2}, and the four schools were at least 50 kilometres away from one another. Two schools represented the control group and the other two schools represented the experimental group. The study was conducted in an environment (classroom) that was familiar to the respondents (learners). This made the implementation of the investigation more convenient and easier to manage. The participants in the research were 198 Grade 6 learners from the four schools, the intermediate phase heads of departments (HODs) in all four schools and the four mathematics educators in all four schools. The two schools in the control group were made up of the two Grade 6 classes with a population of 51 and 55 respectively. The two groups of the experimental group were also made up of the two Grade 6 classes in the respective schools, with populations of 49 and 43 learners respectively. The two Grade 6 classes in school 1 and school 2, the control group, will henceforth be referred to as control group 1 and control group 2, whereas the two Grade 6 classes in school 3 and school 4 will be referred to as experimental group 1 and experimental group 2. Table 3.1 shows the participants at various stages of data collection and Table 3.2 shows the timelines of the various stages of the study.

Table 3.1: Number of participants at different stages of the study

Stages	**Groups involved**	**Number of learners**	**No. Grade 6 mathematics educators**	**No. intermediate phase HODs**	**Total number of participants**
Pre-test	Experimental and control group	198	4	4	206
Pre-intervention Class observation	Experimental and control group	198	4	4	206
Problem-solving heuristic instructional method	Experimental group	92	2		94
Post-test	Experimental and control group	198	4	4	206

Table 3.2: Timeline of data-collection

Period	Activity
July 2014	Pre-intervention observation by researcher
July 2014	Pre-tests with control and experimental groups
July–October 2014	Intervention with experimental group (exposing learners to the teaching treatment)
October 2014	Post-tests with control and experimental groups

3.3 POPULATION, SAMPLE AND SAMPLING

3.3.1 Population of the study

The population of the study was Grade 6 learners in quintile one schools in the Zululand district of Kwazulu-Natal. Schools from quintile one were chosen as the population for the study because this quintile has the poorest mathematical average in the ANA examination (see section 1.1). Most quintile one schools are in the deep rural areas where there are most commonly inadequate resources in terms of teaching and learning materials, infrastructure and a lack of qualified educators. The study chose Grade 6 learners because the foundational development of algebra in the South African school system begins in the intermediate phase of which Grade 6 is the last grade in that phase. Secondly, the drop in the number of both learners who choose to take mathematics in Grade 10 and the poor quality of mathematics grades in the Grade 12 National Certificate examination have their roots in the teaching of mathematics at the basic level where learners fail to acquire basic mathematical skills (see Campbell & Prew, 2014). According to Campbell and Prew (2014), there is an urgent need for attention to pedagogy and content at the upper primary and lower secondary level. Moreover, some studies have revealed that marked changes in learners' problem-solving skills are observed between the ages of 11 to 14 (see Proctor, 2010; Zhu & Fan, 2006). Yan (2000) also explains that the optimal age at which learners are able to develop their problem-solving skills is from 10 to 16 years. On this basis, the study deemed it fit to develop and test this problem-driven teaching method with Grade 6 learners.

3.3.2 Sample of the study

The sample for the study was intact Grade 6 classes from four schools in the Zululand district of Kwazulu-Natal; one school was selected from each of the four circuits in the district. Although the study took pains to ensure that the schools selected had comparable characteristics in terms of their teaching and learning materials, infrastructure and educators' qualifications, there were still some inherent differences in these factors among the schools chosen for this study.

3.3.3 Sampling techniques used in this study

Purposive sampling was used to sample the four schools used for this study. (Louis, Lawrence, & Keith (2007, pp. 114-115) explains that

> In purposive sampling, the researcher handpicks the cases to be included in the sample on the basis of their judgement of their typicality or possession of a particular characteristics being sought. In this way, they build up a sample that is satisfactory to their specific needs.

The motivation for choosing a purposeful sampling technique was to ensure that the schools selected were of the same quintile and at the same performance level in terms of their Grade 6 end-of-year results and that these results, taken over a period of time, were comparable. Furthermore, the distance between any two of the four schools measured at least 50 kilometres. This factor ensured that learners in the control and experimental group did not meet each other. Maintaining a long distance between participating schools "prevents diffusion, contamination, rivalry and demoralisation" (Gaigher, 2006, p. 37) as the measure of the true effects of the problem-solving heuristic instructional method may be compromised if the learners in the control and experimental group interact with each other during the intervention stage (Shea, Arnold & Mann, 2004). Contamination also has the propensity to reduce the statistical significance, as well as the observed differences, between the control and experimental groups caused by exposing learners in the control group to the intervention (Howe, Keogh-Brown, Miles & Bachmann, 2007). One study notes that **"**Potential extraneous variables should not prejudice the relationship between the independent and dependent variables" as it may "lead to ambiguous results" of that study (Tierney, 2008, p. 2). This made random sampling basically impractical.

3.4 PRE-INTERVENTION CLASS OBSERVATION OF MATHEMATICS LESSONS OF SCHOOLS

Pre-intervention classroom observation was used to investigate the teaching methods adopted in all four schools chosen for this study compared to the problem-solving heuristic instructional method, as well as to examine the similarities and differences in the quality of mathematics teaching among these four schools. This enabled the study to assess whether any of the four schools had an advantage over the others with regard to teaching and learning. Among issues compared were the educators' lesson plan on the activities they intended to use to develop learners' understanding in the concept being taught; effective communication between the educator and learners; level of learners' participation and enthusiasm in the learning process and knowledge creation; integration of authentic real-life problems into the teaching and learning process and whether the specific educator developed a particular mental construction to develop learners' understanding of the topic being taught. Classroom observation is an excellent instrument to understand the real picture of any social phenomenon (Mulhall, 2003). The behaviour of learners and educators and the interactions between them can best be studied through natural observation of their activities in the classroom (Gay, Mills & Airasian, 2006). Through class observation the study hoped to control some of the variables that could influence the outcome of the problem-solving heuristic instructional method or pose a threat to the internal validity of the study.

3.5 PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL METHOD

The teaching treatment was conducted from July to October 2014. The researcher negotiated with the school authorities of the two experimental schools to allocate one hour per week as it was impossible to conduct the study after school hours for operational reasons. In total 9 hours were used in teaching each of the Grade 6 classes in the two experimental schools.

3.5.1 Designing the problem-solving heuristic instructional method

The problem-solving heuristic instructional method is underpinned by two theories, namely the APOS theory and the modelling and modelling perspectives approach. Adopting them as guide enabled the researcher to develop a preliminary genetic decomposition, which is a specific mental construction learners may make as they develop a conception in algebra, and this was implemented flexibly. Problem-solving entailing MEAs informed the pedagogical approach used to teach Grade 6 learners algebra. The APOS theory was used as a framework to develop learners' understanding of algebra. The researcher developed MEAs with which learners are familiar and commonly experience in their daily lives. Most importantly, the modelling-eliciting activities featured components of all six principles of the modelling and modelling activity. The modelling-eliciting activities create the necessary environment for learners to develop a more comprehensive understanding in algebra. When learners create meaning from their own symbolic representation it could be hypothesized that meaningful learning (Ausubel, 1962) is promoted, as opposed to imposing a system of symbols and notations on learners (Chamberlin & Coxbill, 2012). The activity designed had components that guided the development of learners' conceptual understanding in algebra as it occurs through the mathematization of the modelling-eliciting activity.

3.5.2 Implementation of the problem-solving heuristic instruction approach

3.5.2.1 *Who implemented the problem-solving heuristic instructional method?*

The researcher took up the role of researcher-educator and implemented the intervention himself. There was a myriad of reasons why the researcher decided to implement the problem-solving heuristic instruction approach by himself. Firstly, there was a challenge in training the mathematics educators in the experimental schools on how to implement the problem-solving heuristic instructional method because these educators had their own professional schedule and the researcher could not find an appropriate time to provide the training. Pursuing data collection in this direction could have delayed the data collection process. Secondly, the study had to ensure there was only limited variation in the implementation of the problem-solving heuristic instructional method across the two experimental schools, in order for the true impact of this approach to be measured. In experimental research, variables that are exposed to the experimental group and which are likely to change the dependent variable must be similar (Gay, Mills & Airasian, 2011). Lastly, the researcher wanted to ensure that the problem-solving heuristic instructional method was implemented correctly as per its design. The study was wary of the effect that the researcher-educator role, played by the researcher, could have on the internal validity of the study. However, through the prolonged process of the researcher teaching the learners in the experimental group, they gradually became accustomed to the researcher as being one of their educators. The researcher was already an educator in the education department in the same district and understood all polices and rules of the KZN education department with regard to teaching and learning. Hence it was not a challenge for the researcher to accustom himself to the learners and their classroom situation.

3.5.2.2 What was the classroom setting in each experimental school?

Ninety-two learners in the two experimental group schools participated in the problem-solving heuristic instructional method. Experimental Group 1 was divided into 9 groups of five members each and one group of four members. Experimental Group 2 consisted of 8 groups of five learners each and 1 group of three learners. According to Wessels (2014, p. 4), “MEAs are solved in groups of three to five people”. The APOS theory has a social component that relies on cooperative learning, as the context of group-work is more likely to give rise to more explicit questions, doubts, and explanations by learners than what would typically transpire in individual contexts (Vidakovic, 1993). Arnon et al. (2013, p. 107) suggest that “The APOS theory functions under the premise that working in groups makes a difference in the affective domains of the individuals”.

3.5.3 Instruction and learning

The researcher used the MEAs as a medium of instruction to develop and explain concepts in algebra through the preliminary genetic decomposition as explained in section 3.5.3.1(see section 2.3.2.1). Learners responded positively to the problem-solving heuristic instructional method by developing and linking their understanding to the algebraic concepts being taught with the everyday situations they experience in their daily life, which translates to learners developing a deep conceptual understanding of the algebraic concepts being taught.

Learners were also given activities to do at home individually. The activities were based on selected algebra questions from Grade 6 learners' mathematics work book. This was to enable them to reflect on the various stages of the preliminary genetic decomposition. In the following teaching session learners in each group were supposed to discuss, reach agreement and then present a common answer to the whole class.

3.5.3.1 Preliminary genetic decomposition

i. *Action stage:* The learners were guided to reflect on their understanding of the MEA to formulate a rule or relationship among the elements in the MEA in line with their understanding of the MEA. The learners were guided to test their rule by substituting input values to give output values in line with the objectives of the MEA.

ii. *Process stage*: As the learners repeated and reflected on these actions several times, they gradually began to interiorise the actions into a mental process, where they could conceptualise a rule as an expression that dynamically transforms an input value into an output value without performing any extensive calculation. Once this has been established, learners are guided to manipulate the rule in order to be able to substitute output values to predict input values in accordance with the objectives of the MEA.

iii. *Object stage:* As the learners reflect on the operations applied in this process, they gradually encapsulate the process-conception into an object-conception where they conceptualize a rule or relationship as a static object that can itself be transformed. At this stage the learners are guided in thinking of parallel problems similar to the MEA, and modify and manipulate the original rule to explain the goals of that activity.

iv. *Schema*: The learners are given activities that will enable them to reflect and interconnect the various stages of actions, processes and object conceptions in a structured manner.

3.5.4 Evidence of performance after engaging in the problem-solving heuristic instructional method

The conceptual understanding learners gained through the implementation of the problem-solving heuristic instructional method translated to learners being able to tackle correctly algebra questions they have not encountered before.

3.6 MEASURING EFFECTS OF THE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL METHOD ON LEARNERS' ACHIEVEMENT IN ALGEBRA

3.6.1 Pre-test-post-test non-equivalent quasi-experimental design

A pre-test-post-test non-equivalent quasi-experimental design was employed to draw comparisons between the post-test and pre-test scores of the experimental group and control group.

3.6.1.1 Hypothesis and the null hypothesis stated

A hypothesis and null hypothesis were set, using the main problem statement as basis. The hypothesis and the null hypothesis were stated as follows:

Hypothesis (H_A): There is a statistically significant improvement in the algebra test scores of the learners who participated in the problem-solving heuristic instructional method.

Null hypothesis (H_0): There is no statistically significant improvement in the algebra test scores of the learners who participated in the teaching treatment.

3.6.1.2 Non-equivalent control group design

The non-equivalent control group design was used to study the effect of the independent or manipulated variable (problem-solving heuristic instructional method) on the dependent variable (post-test). The design indicated the pre-test and the post-test as within-subject factors; and the control group versus the experimental group as the between-subject factors.

Experimental group O_1 X_1 O_2

Control group O_1 X_2 O_2

Sample of the experimental group: 92

Sample of the control group: 106

Total sample (n): 198

O_1: Observation before any teaching

O_2: Observation after teaching

X_1: Heuristic teaching

X_2: Traditional teaching

3.6.2 Rationale for using a quasi-experimental design

The quasi-experimental design, which has all the advantages of a true experimental design, enabled the researcher to determine whether the statistically significant changes in the algebraic test scores of learners were due to the problem-solving heuristic instructional method.

Using this strategy required the manipulation of the independent variable (problem-solving heuristic instructional method), holding constant as many unrelated variables as possible, and randomizing the effects of any remaining extraneous variables across treatments (Bordens & Abbott, 2005). The link between the cause (problem-solving heuristic instructional method) and the effect (post-test scores) was demonstrated when the independent variable was manipulated to produce a change in the dependent variable (Charles, 1988). Although holding variables constant reduces the generality of findings, randomizing the effects of the variables across the intervention can produce error variances that obscure the effects of the independent variable (the problem-solving heuristic instructional method) (Bordens & Abbott, 2005).

3.7 DATA COLLECTION INSTRUMENTS

The instruments the researcher used for data collection were an observation schedule for the class observation, an achievement test used for the pre-test and post-test and MEAs used to develop learners' conceptual understanding in algebra during the problem-solving heuristic instructional method.

3.7.1 Development of observation schedule

Classroom observation was carried out using two main instruments; the first instrument was a classroom observation tool adopted from Kotoka (2012) used by the researcher (see Appendix D). The second instrument was an already completed classroom observation tool used by the heads of departments of the respective four schools used (see Appendix E).

Proficiency and competency of HODs Intermediate phase: The study trusted the judgements of the HODs as all four of them were professionally qualified educators with at least 10 years' experience in the teaching profession and had been in the position of HOD of the intermediate phase of the South African school system for at least five years. The HODs also had the approval from the DBE to carry out this observation on their behalf. The findings of the HODs were also collaborated by the researchers own findings during the researcher's class visits to the four schools used for this study.

3.7.2 Development of modelling-eliciting activities (MEAs)

The three-main MEAs (see Appendix A) used in this study were based on learners' background and interests in their school. It was developed based on the six principles of the modelling and modelling perspective by Lesh and Lamon (1992), as discussed in chapter two, section 2.3.1. The proposed modelling task for this study contained features which included real-world relevance, accessibility, feasibility, sustainability and alignment with the learning of algebra in Grade 6 of the South African education system. The essence of the real-world component embedded in the tasks was to enable tasks to have personal value and meaning for the learners, thus giving them a sense of purpose when engaging in the tasks and assisting them in learning important algebraic content, in developing algebraic skills, and enhancing their algebraic understanding. The learners were not expected to be able to solve the MEA, rather the MEA was supposed to create a blockage[2] for the learners, which the researcher-educator could use as a medium of interaction to develop the learners' conceptual understanding in algebra. When lessons are centred on sound and relevant problems, learners have opportunities for prolonged and deep engagement with the tasks (Kaur et al., 2009). Learners work on challenges in which they need support through peer collaboration or through the guidance of the educator.

3.7.2.1 Validation of the modelling-eliciting activity

The three MEAs were validated by an independent researcher to verify whether the six principles of the modelling and modelling perspective were featured in the task and

[2] Problems learners are not yet able to solve (Kroll & Miller, 1993)

to check whether the task was a meaningful medium to develop learners' conception in algebra at the level of Grade 6. MEAs were piloted by the researcher in a fifth school that did not form part of the main study to assess its feasibility and suitability for Grade 6 learners. These sessions were video- and-audio recorded.

3.7.2.2 Reliability of the modelling-eliciting activity

Triangulation was used during the pilot study to ensure agreement between the different sources and methods of information during the teaching treatment. According to Cohen, Manion and Morrison (2007, p. 141), "Triangulation may be defined as the use of two or more methods of data collection in the study of some aspect of human behaviour." It is a powerful way of demonstrating concurrent validity, particularly in qualitative research (Campbell & Fiske, 1959).To seek corroboration of data gathered during the problem-solving heuristic instructional method, the researcher compared and cross-checked data from the worksheets of the subgroups in the experimental group during the implementation of the problem – solving heuristic instructional method, field notes from the two assistants and from transcripts of the video- and audio-recordings during the pilot study for consistency.

3.7.3 Development of pre-test and post-test

The pre-test and post-test were designed to measure how the performance of Grade 6 learners changed after participation or non-participation in the problem-solving heuristic instructional method. The same test was used in both the pre-test and the post-test and was made up of selected questions in algebra from the standard Grade 6 CAPs-approved mathematics textbooks and the ANA examination for Grade 6 conducted from 2010 to 2013 (see Appendix B). Using the same test before and after intervention ensured that the cognitive demands of the algebra questions, which were a mixture of low- and high-cognitive level, were maintained. There were 20 questions in all, 10 were multiple choice questions and the remaining 10 were written questions. The mark total for the test was 40 marks, later converted to 100 percentage points.

3.7.3.1 Content validity of the pre-test and post-test

The questions used in the pre-test and post-test were based on algebra and selected from standard Grade 6 mathematics textbooks approved by the DOE and ANA examination for Grade 6 that was conducted from 2010 to 2013. A validity test confirms the appropriateness of a test (Gay et al., 2011). The test was content validated by experienced mathematics educators and mathematics subject advisors in the intermediate phase. Even though they were unanimous that the questions were suitable for Grade 6 learners and could measure the level of their algebraic reasoning, the subject advisors raised issues about the high cognitive level of four of the questions and suggested alternative questions. These suggestions were taken into consideration before the test was administered. Using expert advice to determine the content validity of a test is a common phenomenon in educational research (See Demircioğlu, Demircioğlu & Çalik, 2009; Donkor, 2010; Hattingh & Killen, 2003; Kasanda, Lubben, Gaoseb, Kandjeo-Marenga, Kapenda & Campbell, 2005).

3.7.3.2 Reliability of pre-test and post-test

The pre-test and the post-test were piloted with 30 Grade 6 learners in a school that did not form part of the study. The test was administered to the learners twice over a period of two months. A reliability test confirms the consistency of the scores produced in a test (Gay et al., 2011). The SPSS software package was used to calculate the Cronbach alpha coefficient which stood at 0.74, confirming the consistency of the test.

3.8 DATA COLLECTION AND DATA COLLECTION PROCEDURE

3.8.1 Administration of pre-intervention class observation

Classroom observation was carried out by the researcher in all four Grade 6 classes in the four schools used for this study in July 2014. A completed classroom observation tool by the HODS of the four schools was also used as an important data source to assess the nature and quality of teaching and learning of the four schools used for this study. On this point, one author notes that "The observation schedule sought to collect information on how the educator introduces the lessons, learner involvement in the lessons, educator's ability to teach the content" (Kotoka, 2012, p. 49).

3.8.1.1 Grade 6 mathematics classroom visits by researcher

The researcher twice attended Grade 6 mathematics classes of each of the four schools using a classroom observation tool (see Appendix D) adapted from the work by Kotoka (2012) and field notes to assess and evaluate the activities in the classrooms. Data was gathered by means of personal observation and recorded on the classroom monitoring instrument. The researcher documented and summarized the classroom activities with the adapted tool with the view of understanding the nature and quality of the teaching methods adopted by the educators of the respective schools. He requested a copy of the lesson preparation documents from the respective mathematics Grade 6 educators of the four schools before the commencement of the observation and limited his role to that of an observer, not interacting with any of the learners or the educator to reduce the possibility of influencing the outcome of the class observation. Before the observation, he had requested permission from the school authorities of the four schools involved and also discussed in detail with the educators concerned the purpose of the class observation, explaining that it was in no way meant to assess their pedagogical and content knowledge in the subject. Educators in the respective schools had also explained the purpose of the visit to the learners prior to the researcher's arrival so that they were aware of the reason for his presence. Hence his presence did not interrupt the lesson in any way. Observations were made after the pre-test had been administered.

3.8.1.2 Grade 6 mathematics classroom visits by HODs

The classroom monitoring instrument (see Appendix E) used by HODs was also an important source of data. All HODs in the various phases of the South African school system are mandated to visit the classes of the educators in their phase once a term, in all four terms of the academic year. They are required to document their findings by completing a classroom monitoring instrument as prescribed by the DBE, and to offer recommendations or remedial action on sections of the classroom monitoring instruments where the educators need improvement. The instrument assesses the effectiveness of teaching and learning in the classroom and it is mandatory for the HODs to request and study the lesson preparation of the lesson they plan to observe. The completed instrument gave the researcher additional detail of what transpires in all four mathematics classrooms in the respective schools used. The HODs' monitoring was made after the pre-test had been administered.

3.8.2 Collection of data during administration of problem-solving heuristic instructional method

Data was collected by means of observation, transcripts from audio- and video-recordings and written work from the group worksheets.

3.8.2.1 Observation

Observation was carried out by the researcher and the Grade 6 mathematics educators of the two experimental schools. Grade 6 mathematics educators in the two experimental schools were contracted to observe the learning episodes and took down field notes when the problem-solving heuristic instructional method was being applied. Observation enabled the researcher to capture information *in situ* and provided "a unique example of real people in real situations" (Cohen et al., 2007, p. 253). The two mathematics educators documented important elements of learner and researcher actions during the teaching treatment that happened at the blindside of the researcher.

Training of research assistants: Before observation began the researcher had two training sessions with the mathematics educators of the two experimental schools. The training focused on reviewing the purpose of the observation and what they were supposed to look out for and document during the teaching treatment. The issues observers were trained to observe include the following:

i. Observe how learners cope with the MEAs in terms of the observed changes in their algebraic knowledge;
ii. Observe whether enabling prompts and assistance from researcher-educator helped learners maintain positive engagement with the MEA;
iii. Observe whether learners' engagement with the activity encouraged them to consider how to address the difficulties they faced in the MEA;
iv. Observe whether the researcher-educator acknowledged and responded to learners' queries and attempts during the heuristic instructional process.

3.8.2.2 Transcripts from audio and video recording

The teaching episodes were video and audio recorded. The presence of video and audio recorders was unfamiliar to the learners, which distracted them in the first two to three sessions, but because of its very limited nature it was possible to partially conceal it, and the learners gradually began to ignore it. The audio and video recordings were carefully analysed and transcribed. This strategy gave insights into learner actions and interactions that happened unbeknown to the researcher-educator and the observers. The transcripts from the video and audio recordings also gave the researcher the opportunity to review some of the preliminary observations he had made during the teaching treatment.

3.8.2.3 Group worksheets

Answers produced by learners on their individual worksheets were also important data sources as they were used to validate the transcribed data from the video and audio recordings and the field notes taken down by the observers (see section 4.3).

3.8.3 Administration of pre-test and post-test

All 198 learners in both the experimental and the control groups wrote the pre-test and post-test. Data collection comprised a pre-test and a post-test. The pre-test was administered in July 2014 before the problem-solving heuristic instructional method was administered to the experimental group; the post-test was administered in October 2014 after the problem-solving heuristic instructional method was administered to the experimental group. The researcher was not directly involved in administering and marking the pre-test and post-test but was assisted by the Grade 6 mathematics educators of the schools used. After the marking of both the pre-test and the post-test, three of the scripts were sampled using a simple random sampling for moderation by the respective HODs of the four schools to assess the quality of the marking and the necessary corrections and adjustments were made.

The pre-test as well as the post-test for the four schools could not be held simultaneously as the researcher needed to be at the respective schools when both the pre-test and post-test was being administered. Hence arrangements were made with the four schools to have the pre-test administered on four consecutive days, one day for each of the four schools used in this study and the post-test on four consecutive days, one day for each of the four schools used in this study. The post-test was conducted at a period when all four schools had concluded the learning programme for the year and were preparing for the end-of-the-year examinations.

Learners were asked not to write their names on the answer script, but to rather indicate the script by “E” if they were in the experimental group and “C” if they were in the control group. Each learner was giving a unique code to enable the researcher to identify the pre-test and post-test for each learner.

3.9 DATA ANALYSIS

This section explains how data gathered from the research instruments was analysed.

3.9.1 Qualitative data analysis of pre-intervention classroom observation

Data from the class monitoring was used to identify the teaching method used by Grade 6 mathematics educators in the four schools used for this study as compared to the problem-solving heuristic instructional method. The study also compared whether there were any similarities or differences in the quality of learning achieved by the four schools used for this study. Data taken from class visits by HODs was triangulated with the researcher's own classroom observation gathered on the observation schedule and through the researcher's field notes. This was to ensure that findings from the classroom observation were not overly dependent on a single source of data. Since no single method of data collection is complete by itself, multiple methods were used in order to fill the gap(s) that might be left by the use of only one method. Data taken from the class observation schedules by the HODs added valuable information to the data gathered by the researcher, as the HOD, being the educator's superior, was in a better position to provide credible information on the teaching methods used by educators in the classroom. This approach enabled the researcher to identify consistencies and exceptions, if any, in the teaching methods adopted by the educators in these four schools. Class observation by HODs provides essential information on the reality of the day-to-day experience of learners and the effectiveness of teaching, thereby enabling schools to improve the education of their learners (Marriott, 2001). This stance is supported by other authors, for example in the following comment: "Observation is one way of getting information which can help make sense of educational situations, gauge the effectiveness of educational practices, and plan attempts for improvements" (Malderez, 2003, p. 179).

3.9.2 Analysis of factors that influence learners' achievements in algebra through participation in the problem-solving heuristic instructional method

The analysis was done using data gathered during the teaching treatment to assess the effectiveness of the implementation of the problem-solving instructional method through preliminary genetic decomposition, to explain what factors in a problem-solving heuristic instructional method influence learners' achievements in algebra at the level of Grade 6, and how this approach could be implemented by educators in the classroom. Data gathered through the heuristic instructional method was analysed though the lens of the two theoretical frameworks, namely the modelling and modelling perspective, and APOS theory that guided and informed the problem-solving heuristic instructional method. The researcher used the data gathered to explain how learners responded to modelling-eliciting activity and how they were able to gradually build on their understanding of this activity to develop a conceptual understanding of algebra at the Grade 6 level in the lens of the modelling and modelling perspective and the APOS theory.

3.9.3 Quantitative data analyses of the effects of the problem-solving heuristic instructional method on learners' achievement in algebra

The purpose of this analysis was to support the hypothesis of statistically significant improved algebra test scores after the learners' participation in the intervention. The pre-test and post-test scores of both the experimental and the control groups were compared before and after the problem-solving heuristic instructional method to establish whether there was any statistically significant difference in the test scores of the two groups. The independent variable was the participation of the experimental group in the problem-solving heuristic instructional method and the non-participation of the control group.

The dependent variable was the post-test score of respondents in both the control group and the experimental group, and the explanatory variable was the pre-test score. The quantitative analysis seeks to explain any statistically significant difference between the pre-test and post-test scores of the experimental group and the control group.

Descriptive and inferential statistical tools were used to analyse the quantitative data (see section 4.3). The level of significance was set at $\alpha = 0.05$ (Creswell, 2002). The descriptive statistics used were mean, standard deviation and graphical representation and the inferential statistics used were the t-test, analysis of covariance (ANCOVA), test for homogeneity of regression slopes (HOS), to satisfy a precondition of using ANCOVA, the Johnson-Neyman (J-N) technique and the effect size.

3.9.3.1 Justification for using t-test

The t-test enabled the study to measure the effectiveness of the problem-solving heuristic instructional method by measuring whether the pre-test (before intervention) and post-test (after intervention) between the control and experimental groups were statistically significant. The t-test is suitable when comparing the mean scores of two groups. It is used to investigate whether there are statistically significant differences between the means of two groups, namely the control and the experimental group (Cohen et al., 2007).

3.9.3.2 Justification for using ANCOVA

The pre-test was conducted before the problem-solving heuristic instructional method; it informed the researcher of learners' pre-knowledge in algebra, which could affect the validity of study. The study also speculated that other unforeseen extraneous variables could contaminate the treatment effects due to the non-randomization during the selection of the schools. ANCOVA treats any extraneous variable, such as the pre-test, that may contaminate the effects of the intervention, as a covariate. The ANCOVA as a statistical tool investigates the effects of the pre-test (covariate) on the post-test (dependent variable) scores. ANCOVA as an inferential interpretative tool explains the degree of change between the pre-test scores and the post-test scores of the two groups.

The F-test of ANCOVA was used to check whether the problem-solving heuristic instructional method had any effect on the learners' development in algebra. ANCOVA, therefore, seemed to be one of the most appropriate statistical tools for this study, supported by the following statement:

> An assumption underlying the correct usage of ANCOVA is that the population regression slopes associated with the treatment populations are equal (Huitema, 2011, p. 144).

According to Huitema (2011), adjusted means are inadequate descriptive measures of the outcome of a study if the size of the treatment effect on the post-test (i.e. the vertical distance between the regression lines) is not the same at different levels of the covariate.

> If the slopes are heterogeneous, the treatment effects differ at different levels of the covariate; consequently, the adjusted means can be misleading because they do not convey this important information (Huitema, 2011, p. 146).

The study performed a homogeneity test to ascertain the homogeneity of the regression slopes (HOS). This is discussed in the next section.

3.9.3.3 Homogeneity of regression slopes

A test of the HOS between the pre-test and post-test for the control and the experimental group was performed as it was a requirement to analyse the data using ANCOVA (Huitema, 2011, p. 144-148**).** If both the experimental and the control groups have similar slopes, then these slopes are considered homogeneous. According to Huitema (2011), "When the slopes are homogeneous, the adjusted means are adequate descriptive measures because the treatment effects are the same at different levels of the covariate" (p. 146). The heterogeneity of the regression slopes is said to occur when there are differences in the slopes of the two lines. When the difference is small and group sizes are equal, Keppel (1991) and others argue that this type of heterogeneity is usually not a significant problem and ANCOVA remains robust. A test for the HOS was therefore used to test whether the regression lines between the pre-test and the post-test for the control and experimental group was homogenous as shown in Table 3.3.

Table 3.3: Homogeneity slope test

Source	**SS**	**DF**	**MS**	**F**	**p-value**
Heterogeneity of slopes	641.69	1	641.69	5.01	0.026
Individual residuals (res_i)	24816.79	194	127.92		
Within residual (res_w)	25458.48	195			
H_o: $\beta_1^{Control} = \beta_1^{Experimental}$	Critical values		$F_{0.05, 1, 194)}$	3.88983904	
H_1: $\beta_1^{Control} \neq \beta_1^{Experimental}$					
Compare p-value (0.026) with α = 0.05	If p < *α* then we reject the null hypothesis				
p-value is much lower than α	Null hypothesis is rejected				
Compare critical value with F	$F_{0.05}$ = 3.89 with F = 5.01			Reject null hypothesis if the F value is greater than the critical F-value	
Null hypothesis is rejected since $F_{0.05}$ = 3.89 < F= 5.01					

At a 95% significance level the $F_{(0.05, 1, 194)} < F$, hence the null hypothesis was rejected, which indicated the regression slope between the control and the experimental group was not equal, implying convergence of the two regression slopes. Figure 3.1 further illustrates the convergence of the regression lines between the control group and the experimental group. (This graph is also inserted in chapter four for reference purposes).

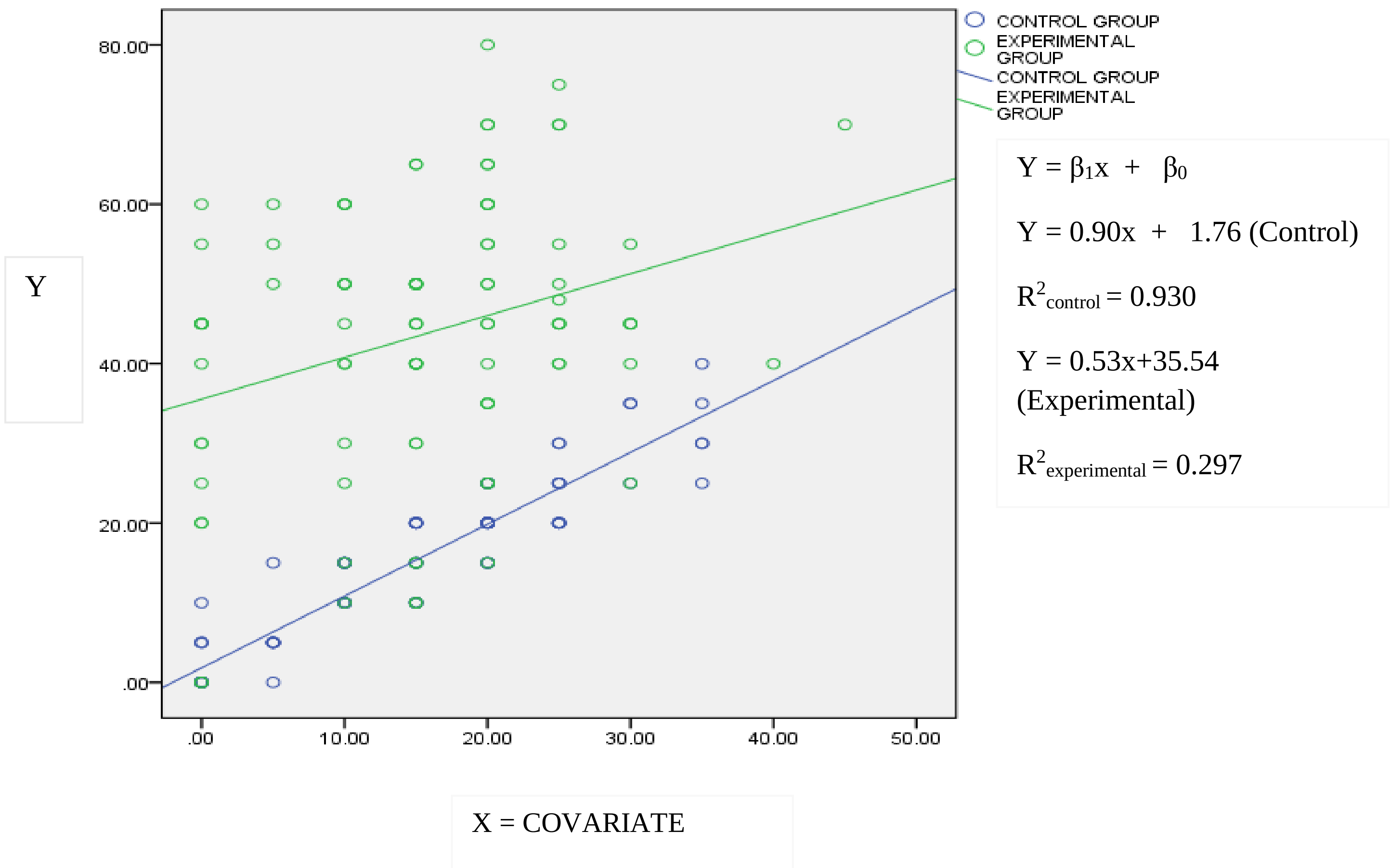


Figure 3.1: Convergence of regression slopes between control and experimental groups

There was sufficient evidence to claim a violation of the regression slopes, and the null hypothesis can be rejected (Huitema, 2011, pp. 144-157). In such cases the J-N technique is used to determine the values of the pre-test associated with the significant intervention effect, and the values of the pre-test associated with non-significant treatment effects.

3.9.3.4 Johnson-Neyman technique

The J-N technique was used as an inferential tool since homogeneity of regression slopes between the control group and the experimental group was not achieved (Maitland, 2010). The J-N technique was used to determine the following:

i. What marks in learners' pre-test on the scale of 0 to 100 are associated with the significant problem-solving heuristic instructional method effects?

ii. What marks in learners' pre-test on the scale of 0 to 100 are associated with the non-significant problem-solving heuristic instructional method effects?

Table 3.4 gives a summary of the statistics of the J-N technique. (This table is also inserted in chapter four for reference purposes)

Table 3.4: Summary of Johnson-Neyman technique with reference to the homogeneity of the regression slope lines of the scatter plot in Figure 3.1

Summary of statistics	**Control**	**Experimental**	
Sample size	106	92	
A	0.1031		
B	-12.9900		
C	1123.5		
X_{L1}	55.4439		**Bounded above by 100**
X_{L2}	196.545		

Where X_{L1} and X_{L2} are limits of the region of non-significance region

The learners whose pre-test marks fell within the limits of the area of non-significance were deemed to have improved their post-test scores through a natural variation and not necessarily because of their participation in the intervention.

The limits of region of non-significance on X were computed using X_{L1} as the lower bound and X_{L2} as the upper bound, as follows:

$$X_{L1} = \frac{-B - \sqrt{B^2 - AC}}{A} \quad \text{and} \quad X_{L2} = \frac{-B + \sqrt{B^2 - AC}}{A}$$

3.9.3.5 Justification for calculating effect size

The effect size was used in this study to quantify the effectiveness of the problem-solving heuristic instructional method to enable the reader to identify the educational significance of the study. Smith and Glass (1977), the originators of effect size, defined effect size as a statistic that gives both the direction and the strength of a difference between two means (i.e. control group means and experimental group means) in the following terms: “An effect size identifies the strength of the conclusions about group differences or the relationships among variables in quantitative studies” (Creswell, 2009, p. 131). The effect size for the study was calculated using Cohen’s d, as follows:

Cohen’s d = $\frac{\bar{x}_E - \bar{x}_c}{SD_{Pooled}}$

$\bar{x}_E$: Experimental group mean scores for post test

$\bar{x}_c$: Control group mean

SD_{Pooled} : Pooled estimate of the standard deviation

The following rubric as described by Cohen, Manion and Morisson (2007, p. 527) was used as a yardstick to quantify the effectiveness of the problem-solving heuristic instructional method:

0–0.20 = weak effect

0.21–0.50 = modest effect

0.51–1.00 = moderate effect

>1.00 = strong effect

The pooled estimate of the standard deviation was calculated as follows:

$$SD_{pooled} = \sqrt{\frac{(N_E - 1)SD_E^2 + (N_c - 1)SD_C^2}{N_E + N_C - 2}}$$

3.9.4 Analysis of pre-test and post-test answers in control and experimental group

The pre-test and post-test answers for learners in the experimental and control group were further analysed to verify the findings made in the descriptive and statistical tests. Two learners each were sampled from the control and experimental group and the specific answers they gave to sampled selected questions were analysed to assess the impact of the heuristic teaching treatment on learners’ achievements in algebra and to corroborate the quantitative findings made in section 3.9.3.

3.10 ETHICAL CONSIDERATIONS

Ethical issues in research concern beliefs about what is wrong and what is right from a moral perspective during the conducting of the research (McMillan & Schumacher, 2001). Research ethics, therefore, implies compliance with acceptable research norms, morals, standards and principles. To conform and comply with the University of South Africa's research ethical codes, guidelines, protocols and practices, the researcher applied for and was granted ethical approval by the Instructional Research Ethics Committee of the Institute of Science and Technology Education, University of South Africa (see Appendix K). Prior to that the researcher had applied for, and been granted, permission from the KZN Provincial Education Department to visit the schools selected for this research (see Appendix L).

In line with the principle of informed consent, the researcher explained honestly and openly in the form of a letter to the parents of the learners at the four schools the nature, aims, purpose and educational benefits of this study. The researcher explained to the learners' parents that their children's participation in this research was voluntary and that they could withdraw from the study at any time (see Appendix M). The learners' parents signed the Informed Consent and the Recording Consent forms. All these forms were translated into Zulu for the parents to clearly understand their content.

The researcher was also required to comply with ethical issues of confidentiality, anonymity and privacy. To ensure confidentiality, the names of the participants were not disclosed; the schools, educators, learners, and principals all remained anonymous and did not appear in the report. Instead, fictitious names were used throughout the study. The data gathered was solely and strictly used for the purpose of the research project. During and after the completion of the study, the research data was appropriately stored by the researcher under lock and key.

Such confidentiality initiatives and data-storage measures are all in the interest of ensuring and protecting the privacy and anonymity of the participants. McMillan and Schumacher (2001), and Neuman (2006) agree that guaranteeing privacy, anonymity and confidentiality means that access to the participants' responses, behaviour and information is restricted to the researcher and kept secret from the public. The researcher made the necessary effort and commitment to ensure and uphold the informants' privacy and the principles of research ethics during the fieldwork and in the compilation of this study.

3.11 SUMMARY AND CONCLUSION OF CHAPTER THREE

In chapter three the researcher described the research design used in this study to evaluate the effects of the problem-solving heuristic instructional method, as described in section 3.5. Among the issues explained were the research paradigm, the research design, sampling techniques, data collection procedures and analysis. The chapter concluded with an explanation of the ethical guidelines that the researcher used during the data collection procedures.

The next chapter explains the qualitative and quantitative findings of the study, namely findings from the pre-intervention class observation, design and implementation of the problem-solving heuristic instructional method and the effects of the problem-solving heuristic instructional method on learners' achievements in algebra.

CHAPTER FOUR
PRESENTATION OF THE FINDINGS

This chapter presents the findings from the empirical investigation. It is organized into four sections: section 4.1 reports on the findings from the classroom observation by the researcher and HODs of the four schools; section 4.2 gives a brief description of the design and implementation of the problem-solving heuristic instructional method and how it influences the learning of algebra ; section 4.3 reports on the effects of the heuristics teaching treatment on learners' achievements in algebra and section 4.4 analyses learners pre-test and post-test scripts to corroborate the effects of learners' achievements in algebra as reported in section 4.3.

4.1 PRE-INTERVENTION CLASSROOM OBSERVATIONS

All the schools chosen for this study were quintile one schools (see section 1.1), which had comparable characteristics. The study chose schools from the same quintile to ensure that all the four schools selected for this study where at a comparable level in terms of teaching, learning and resources used for teaching and learning.

The pre-intervention class observation enabled the study to compare the teaching methods adopted by the educators of these schools as compared to the problem-solving heuristic instructional method to understand whether there were some aspects of the problem-solving heuristic instructional method that featured in the teaching method used by educators in these four schools. Secondly, the pre- intervention class observation was used to measure if any of the four schools used for this study had an advantage over the others in terms of teaching and learning. In this way, the true effects of learners' achievements in algebra after their participation in the problem-solving heuristic instructional method could be reported**.** Lastly, the pre-intervention class observation gave the researcher insights into how the problem-solving heuristic instructional method could be incorporated into mainstream classroom practice and the possible challenges that might arise in its implementation.

The summary is based on the researcher's field notes and data from researcher's and HODs' observation schedule during the pre-intervention observation. Data gathered was analysed through the following lens

- Type of activities used by mathematics educators in the classroom.
- Medium used by educator to explain mathematical concepts.
- Level of interactivity between the mathematics educators and the learners.
- How the success of the lesson was evaluated.

During the observation, the two educators in the two control schools were identified as educator CE1 and educator CE2 and the two educators in the two experimental schools were identified as educator EE1 and educator EE2. The next section gives a summary of the teaching methods used by each of the Grade 6 mathematics educators who took part in this study.

4.1.1 Summary of teaching methods adopted by educators in all four schools

This section gives a brief description of the teaching methods adopted by the educators as observed and documented by the researcher and HODs of schools used for this study. The summary of the description was based on the class observations by the researcher which was enriched with the class observation of the HODs of the participating schools (also see Appendix D and Appendix E; Table 4.1).

Table 4.1: *S*ummary of teaching method used by educator CE1

<table>
<tr><th colspan="2">Lesson preparation</th></tr>
<tr><td colspan="2">The educator always prepared a lesson plan for the activity for the day. The lesson plan contained the topic to be taught for the day, the learning outcome on which the topic was based and the type of exercises (i.e. homework, class work) to give learners at the end of the lesson.</td></tr>
<tr><th colspan="2">Teaching aids</th></tr>
<tr><td colspan="2">Educator makes use of learners' textbooks</td></tr>
<tr><th colspan="2">Educator/Learner activities</th></tr>
<tr><th>Educator</th><th>Learners</th></tr>
<tr><td>The educator starts the lesson by going straight to implement his lesson preparation. He uses a step-by-step textbook explanation to explain concepts to learners.
The educator periodically asks learners questions which most of the time went unanswered or were answered incorrectly.
Educator refers learners to exercises from the workbook and asks them to write answers.</td><td>Learners write the step-by-step explanation as written by the educator on the chalkboard and try to answer questions posed by the educator.
The only role the learners played in the learning activity was copying what the educator wrote on the chalkboard and writing solutions to the assessment task the educator referred them to in their workbooks.
The learners worked independently and were passive rather than active receivers of knowledge during the entire learning process.</td></tr>
<tr><th colspan="2">Lesson evaluation</th></tr>
<tr><td colspan="2">To evaluate the lesson, the educator refers to exercises in the learners' workbook and asked learners to write the exercise and submit their books.</td></tr>
</table>

In educator CE1's lesson the learners were not active in the creation of knowledge as the educator did close to ninety percent of the talking in trying to impart mathematical concepts to learners. All the learners wrote activities on their own with no form of discussion among themselves or between learners and educator. Also, evidence in the educators' lessons showed that the educator did not give clear specifics on what activities learners should perform in order to show their understanding of the concepts. Table 4.2 describes the teaching methods as observed for educator CE2.

Table 4.2: Summary of teaching method used by educator CE2

<table>
<tr><th colspan="2">Lesson preparation</th></tr>
<tr><td colspan="2">The educator always prepared his lesson plans which was in line with the CAPs’ curriculum and linked up with the previous lessons.</td></tr>
<tr><th colspan="2">Teaching aids</th></tr>
<tr><td colspan="2">No available teaching aids were used in this lesson.</td></tr>
<tr><th colspan="2">Educator/Learner activities</th></tr>
<tr><th>Educator</th><th>Learners</th></tr>
<tr><td>The educator gives a thorough explanation of the concept being taught with his own prepared notes without engaging learners in the process.
After the educator’s explanation of the topic, the educator writes questions on the chalkboard and nominates learners to write the solutions on the chalkboard.
Based on learners’ answers, the educator further explains and gives clarity to the misconceptions identified and where necessary offers corrective measures.</td><td>Learners write notes on the chalkboard, as directed by the educator.
Nominated learners write answers on the chalkboard as directed by the educator.
All learners in the class write the educator’s corrected answers.</td></tr>
<tr><th colspan="2">Lesson evaluation</th></tr>
<tr><td colspan="2">The educator gave learners hardly any activity at the end of the lesson. This was largely because of the time factor as most of the time was used to explain the procedures in the concepts being taught.</td></tr>
</table>

Educator CE2’s lessons were generally educator-centred, as the educator did all the talking but engaged a few learners during the latter part of the learning process. Generally, most learners remained inactive in the learning process and wrote the notes the educator wrote on the chalkboard as directed by the educator. The lesson plan by this educator also did not outline a clear sequence of activity by learners to develop the concept being taught and the lesson did not incorporate any real-life context in the teaching and learning process. Table 4.3 describes a sample of the teaching method adopted by educator EE1.

Table 4.3: Summary of teaching method used by educator EE1

<table>
<tr><th colspan="2">Lesson preparation</th></tr>
<tr><td colspan="2">The educator always prepares her lesson plan which was in line with the CAPS curriculum.</td></tr>
<tr><th colspan="2">Teaching aids</th></tr>
<tr><td colspan="2">The educator makes use of educators’ teaching guides and learners’ workbooks.</td></tr>
<tr><th colspan="2">Educator/Learner activities</th></tr>
<tr><th>Educator</th><th>Learners</th></tr>
<tr><td>• The educator implements her activities set out in the lesson plan.
• The educator uses textbook worked examples to explain the concepts to learners.
• She then uses the remainder of the time to give learners activities that were taken directly from the learners’ textbook which learners answered individually.</td><td>• Learners write the notes as directed by the educator.
• Learners write their solutions to activities given by educator.</td></tr>
<tr><th colspan="2">Lesson evaluation</th></tr>
<tr><td colspan="2">The educator uses the activities given to the learners to evaluate the success of the lesson.</td></tr>
</table>

Apart from the lesson being largely educator-centred, there was no incorporation of authentic real-life problems into the learning process. The only activity learners were observed doing were copying the notes on the board and writing activities as directed by the educator which they answered individually. As in previous lessons the educator did not demonstrate a clear strategy on how she intended to impart these concepts to the learners. Table 4.4 describes a sample model of the teaching method adopted by educator EE2.

Table 4.4: Summary of teaching method used by educator EE2

<table>
<tr><th colspan="2">Lesson preparation</th></tr>
<tr><td colspan="2">All lesson plans for lessons prepared were in line with the lesson and the CAPs' policy document.</td></tr>
<tr><td colspan="2">Teaching aids
Educator made use of learners' workbooks</td></tr>
<tr><th colspan="2">Educator/Learner activities</th></tr>
<tr><th>Educator</th><th>Learners</th></tr>
<tr><td>• Educator starts his lessons by giving a brief description of the content being taught.
• Educator gives learners the opportunity to ask questions on his description of the content.
• Educator answer learners' questions without brainstorming the concepts in these questions with them.
• The educator then refers learners to an activity in their workbook where learners are supposed to answer individually and submit before the class comes to an end.</td><td>• Learners listened to educator's explanation.
• Learners asked educator questions to gain clarity on some of the questions.
• Learners worked on the activities individually as directed by the educator.</td></tr>
<tr><th colspan="2">Lesson evaluation</th></tr>
<tr><td colspan="2">The educator in this class also used the activities given to the learners to evaluate the success of the lesson.</td></tr>
</table>

In Table 4.4 the educator explained concepts to the learners without actively engaging them in the learning process. The educator's explanations generated some questions from the learners but in the process of answering the learners' questions the educator did not access their individual thinking and mathematical development processes regarding the concepts but gave learners direct answers. The educator did not incorporate any real-life context into his teaching and there was no evidence of a clear sequence of activities learners should undergo to develop their conceptual understanding of the concept being taught.

4.1.2 Concluding remarks on pre-intervention classroom observation

All four educators in the participating schools followed the plan as set out in their lesson plan. The lesson plan was not explicit enough as it did not give a clear plan of what activities learners should undertake to develop their conceptual understanding of the mathematical concept being taught. In some cases, educators provided a step-by-step explanation of each procedure; learners then practised the procedure through class exercises. In other cases, the learners were required to listen to the educator explain some completed examples in their textbooks, with solutions, and then learners were directed to some exercises in their textbooks to practise a few of these examples. The mode and structure of the lesson delivered by all four educators was such that most learners were not actively involved in the learning process. Learners in these classrooms worked and learnt independently and became passive rather than active receivers of these procedures. Learners were not given the opportunity to be active constructors of information and the educator remained the only director of what and how learners must learn. The teaching method adopted by the educators did not give them the opportunity to monitor learners' reasoning on the topic being taught in order to enable them to develop it further. None of the educators integrated a real-life context to explain any of the concepts being taught.

Despite these similarities, there were a few differences with regards to the use of teaching aids. Three educators, one in the control group and two in the experimental group, were found to make use of textbooks and workbooks in their lessons. Despite the noted discrepancies in the Grade 6 teaching and learning methods in all four schools, the study concluded that the processes and procedures used in all the classes with regards to teaching and learning were similar. This placed all four schools at a comparable level with regard to resources used in teaching and learning and in effect controlled some of the extraneous variables that could contaminate the effects of the treatment on the experimental group. Through this the true effects of the problem-solving heuristic instructional method could be measured. The next section gives a brief description of the development and implementation of the problem-solving heuristic instructional method and highlights its influence in the learning of algebra by Grade 6 learners.

4.2 DEVELOPMENT AND IMPLEMENTATION OF A PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL MODEL

This section describes how the problem-solving heuristic instructional method was developed for teaching and learning. The description is organized into two main sections, section 4.2.1 describes how a problem-solving heuristic instructional model lesson can be developed for algebra lessons and section 4.2.2 gives a brief description of how the problem-solving heuristic instructional model was implemented in the learning of algebra. Burnard (2004) explains that details of the actual intervention programme diminish if the research only focuses on reporting on the statistical interpretation of the intervention. This section accounts for the descriptive and statistical findings reported in section 4.3.

4.2.1 A problem-solving heuristic instruction model lesson

Topic: Number sentences and introduction to algebraic expressions

Objectives: To develop learners' conceptual understanding in number sentences and algebraic expressions

Resource used: The MEAs developed through the guidance of the modelling and modelling perspective are used as the main medium of interaction between the learners and the educator to develop learners' conceptual understanding in algebra (see Appendix A).

Classroom setting

Learners in class are arranged in groups of 3 to 5. This arrangement creates an atmosphere where each learner is given the opportunity to actively participate in the learning process being undertaken in the classroom. The problem-solving heuristic instructional method by its very design is to activate active participation by each learner in the class in the learning process as the lesson starts with authentic MEAs familiar to each learner in the class.

Activities

Researcher

- Introduces MEA in a whole-class format with emphasis on learners' personal experience with the activity;
- Builds on learners' understanding of the MEA to explain and develop important concepts in algebra to Grade 6 learners by guiding learners to model a solution of the MEA at hand in the lens of the preliminary genetic decomposition (see section 3.5.3.1);
- Offers further explanations and slightly lowers the cognitive demands of some aspects of the MEA at different stages of the preliminary genetic decomposition and uses learners' experience with the MEA as the main vehicle to develop concepts in algebra to learners in groups and at the same time does not compromise learners' individual thinking and knowledge creation.

Learners

- Discuss MEA on their own level of understanding based on their own experience of the problem;
- Engage actively in the solution process of the MEA and start doing the MEA at their own level of understanding;
- Work collaboratively in their respective groups to mathematize the MEA through integrating the learning of algebra with their basic understanding of the MEA. They share ideas, comment on each other's ideas and write their perceived answers on a common group sheet with the guidance of the researcher through the preliminary genetic decomposition;
- Integrate their existing knowledge in algebra and their understanding of the MEA to develop new conceptual understanding and knowledge in algebra.

Reflection

- Activities are given to learners in the form of homework which is written by every individual at home.
- These activities are algebra questions randomly selected from learners' workbooks and textbooks.

- Activities are aimed at enabling learners to reflect on the class activities in relation to the problem-solving heuristic instructional method.
- In the next class session learners discuss activities in their groups and some groups are selected to present their solution in a whole-class format under the guidance of the researcher.

4.2.2 Brief description of implementation of the problem-solving heuristic instructional method and how it influences the learning of algebra

In this section, the researcher gives a brief narrative description of how the problem-solving heuristic instructional method was implemented. It explains the mental constructions and mechanisms the learners exhibited during the implementation of the preliminary genetic decomposition, and the influence the two theoretical frameworks, namely the modelling and modelling perspective and the APOS that guided the teaching had on the learners' achievement in algebra. This narrative description is in the form of lengthy transcribed data from the video and audio recordings (including the quotes documented in the transcripts), vignettes of answers learners wrote on common group sheets and the field notes taken by the researcher and the observers during the teaching treatment.

The excerpts used in this presentation are based on one of the subgroups in the experimental groups that was purposively sampled for further study during the intervention. This enabled the researcher to understand how the problem-solving instructional method influenced the Grade 6 learners' conceptual understanding in algebra. In this way, the study was able to explain how Grade 6 learners' knowledge evolved when learning algebra through the problem-solving heuristic instructional method. Purposive sampling studies a few cases in depth (McMillan & Schumacher, 2001). Meriam (1998) emphasises that purposive sampling is based on the assumption that the researcher wants to discover, understand, and gain insight into the phenomenon under study. Therefore, a sample must be chosen from which the most can be learned. This group was chosen on grounds of the lowest average marks scored by the learners in the pre-test. It must be stressed that all groups in the experimental group were subjected to the same treatment, and the researcher's attention was not concentrated on the group purposively selected for the analysis.

This brief description is based on MEA 1 (see Appendix A) and is organized into three main sections. Section 4.2.2.1 describes the the action conception, section 4.2.2.2 describes the interiorization of the action conception into a process conception, and 4.2.2.3 describes the encapsulation of the process conception into an object conception in the light of the preliminary genetic decomposition as explained in section 3.5.3.1. The purposively sampled group will be referred to as group 1 in this description.

4.2.2.1 Action conception

After distributing the MEA to the various groups the researcher introduced the MEA to the learners in a whole-class format. The learners were asked to discuss their understanding of the problem situation. The following discussion ensued in the episode below.

L_{12}: [L_{12} rereads the problem statement again to the whole group] *We need to find the number of taxis that can take the learners.*

L_{11}: *Is it the same taxi we come to school with?*

L_{14}: *I think so.*

L_{11}: *How many learners can that taxi take?*

L_{14}: *I'm not so sure, but from the question* [He reads the question again] *I think we must also include the educators.*

L_{11}: *I don't know*

L_{13}: *In this question we need to find the number of taxis that are here at school every day.*

L_{11}: *Do you know the number?*

L_{13}: *I have not counted them before.*

L_{12}: *We must find the number of learners going to school.*

L_{11}: *[L_{11} loudly repeats what he said initially]. How many learners can a taxi take at a time? No, we must find the number of taxis we are going to use.*

L_{13}: I think we must count all the taxis that can take learners at the time.

L_{15}: But do you know the number of learners in the school?

L_{11}: I don't know, but we can ask the educator.

L_{12}: Many taxis come to school every day, but we don't know the number.

Learners in the group recognized the problem situation as one that could happen in real life. This leads to all members of the group actively taking part in the ensuing discussion. The groups were able to start a discussion of the problem situation on their own without any prompting from the researcher. The learners' background knowledge of the problem situation coupled with their reflection on the social context placed them on a concrete level of thinking as they could share their ideas on the problem situation. Their knowledge of the situation assisted them in making a start towards the solution. It appears that the learners were able to express their views and share their thinking freely without any impediments because of their knowledge of the situation and the environment which was familiar to them. Their initial discussion was centred on a non-mathematical discussion of the situation. They discussed the problem as they understood it in their daily lives. The intensity of their discussions translated into their understanding and recognition, and increased their reasoning of the solution process. The authentic nature of the problem created a situation of communication among the learners as they verbalised their thoughts towards other members of the groups with the same experience of the problem situation playing a key role. As the discussion continued the learners gradually began to coordinate arithmetic in their quest to find the solution to the problem

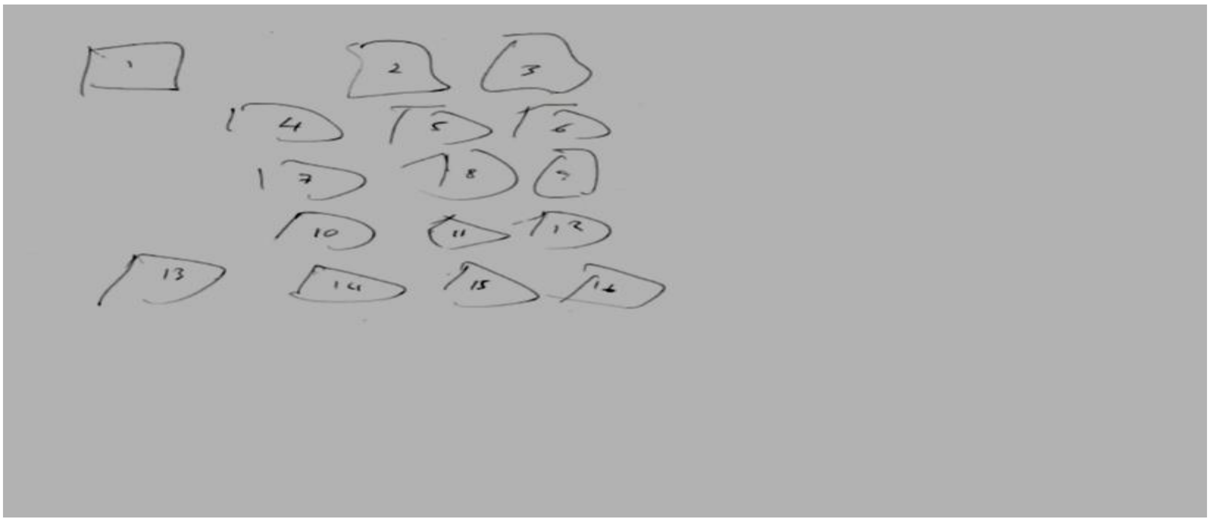
L_{12}: [Rereads the question again and comments.] *The question is not looking for the number of learners or taxis, but rather it is asking for a rule.*

L_{13}: We need to find the number of taxis that is needed to transport all the learners and educators to the school.

L_{11}: But how can we know that since we don't know how many learners and educators come to school every day.

L_{12}: How many learners can sit in a taxi? You people should read the question. The question is asking for a rule.

L_{14}: *We need to find out how many people a taxi can take first* [After a brief visual reflection and flipping of his fingers, he makes a drawing (external representation) on paper, as shown in Vignette 4.1] *sixteen people can sit in a taxi including the driver.*



Vignette 4.1: Mental image by L_{14} of number of people a taxi can transport

L_{12}: *Don't add the driver. The driver is neither a learner nor an educator. So we have to say fifteen people can travel in one taxi.*

L_{15}: *If one taxi carries fifteen passengers, how many taxis are needed to transport all the learners to school?*

L_{11}: *Not only learners but educators too.*

L_{14}: *So, two taxis will carry thirty people. We don't have to add the driver.*

L_{11}: *It will take a lot of work to find the number of taxis needed to transport all the educators and the learners.*

L_{11}: [L_{11} stresses as he rereads the question again.] *The question wants us to find a rule of the number of taxis that can take the learners and the educators.*

L_{12}: *But we still need to find out how many people can sit in the taxi [she reads the question several times].*

L_{11}: *So, can both learners and educators sit in the same taxi?*

L_{13}: *The question did not say so* [he reads the MEA to be sure].

L_{15}: [L_{15} reads the MEA again]*. But the question says the rule for the number of taxis. What is a rule?*

L_{12}: I keep on telling you that.

L_{11}: How can we find the rule then?

L_{11}: I don't know.

L_{13}: We need to ask the educator.

L_{13}: What is a rule?

L_{14}: (Shouts to the researcher)*: We don't know what a rule is.*

The learners gradually integrated their mathematics unconsciously developed from their social interactions to the formal mathematics they study in the classroom. L_{13}'s initial analysis of the problem statement played a role in making learners share their opinions about the problem situation and through their discussion they started promoting mathematics among themselves in the groups.

Even though they had still not grasped what they really had to be doing first, at least they could hypothesize the objective of the MEA. In addition, L_{14}'s mental image of the maximum number of people a taxi can carry at a time is demonstrated by his external physical representation as he sketches the seating arrangement of the taxi he uses to come to school every day. Most of the members in the group not only agreed with him but also started to venture their own opinions about the problem situation.

They had not yet started to introduce the concept of variables and constants. The learners were rather expected to generalize the arithmetic by coming up with a rule that represents a relationship between the number of taxis and the number of educators and learners. They were able to initiate a particular direction to the solution process even though they did not know what was meant by a 'rule'. Nevertheless, their ideas and understanding of the problem guided the direction of the problem-solving heuristic instructional method. The researcher used the learners' experience with the problem situation to guide them in the solution process by asking guiding questions. In the next episode, the researcher guided the learners to identify the key elements in the problem situation and used their knowledge of the problem situation to explain to them what is meant by a variable and a constant.

Researcher: To find our rule, you need to first find all the key elements in the problem situation. Can you identify all the key elements?

L_{13}: What are key elements?

Researcher: I mean anything you can recognize or anything familiar that is mentioned in the problem.

L_{11}: [reads the questions carefully as the other learners listen and a discussion ensues.]

L_{13}: There are taxis.

L_{15}: Learners and educators.

Researcher: Is that all?

L_{15}: Yes.

Researcher: Can you reread the question again?

L_{11}: [reads the question silently]*That is all: learners, educators and taxis.*

Researcher: What about L_{14}'s drawing?

L_{11}: I don't know.

Researcher: L_{14} has made a drawing on the answer sheet; what does that represent?

L_{11}: It is for the number of people who sit in a taxi.

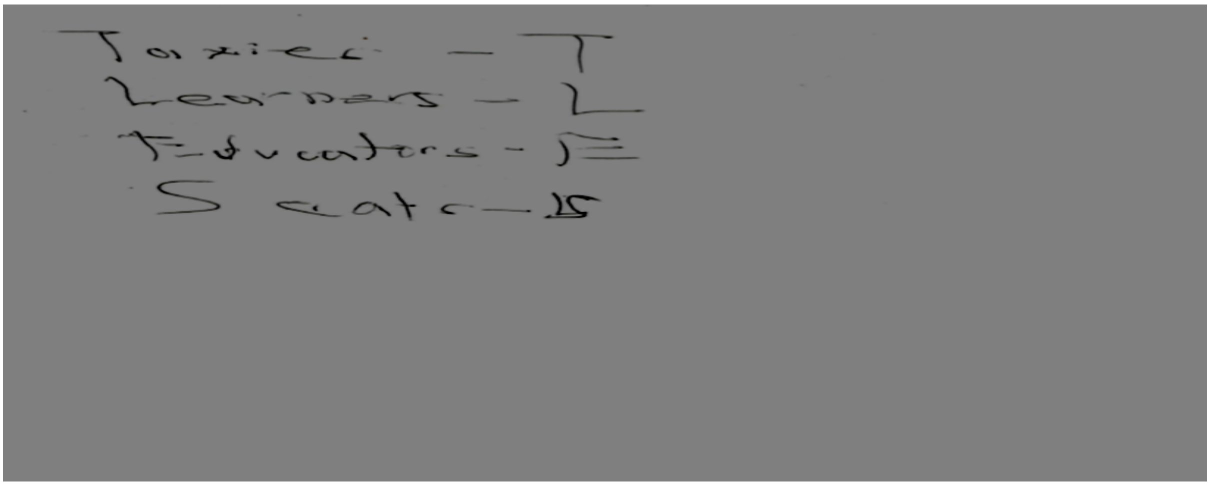
L_{12}: Oh, and the number of people the taxi can take.

Researcher: Is that all you can find?

L_{12}: Yes.

*Researcher**:** Ok. Can you represent the identified elements with any letter of your choice?*

[In Vignette 4.2 the learners in Group 1 decided to represent their identified elements with its initials]



Vignette 4.2: Learners in group 1 present their elements identified with its initials on their answer sheet

As evident in the next episode, the learners steadily, with the guidance of the researcher, identified all the key elements in the MEA. They not only identified all the key elements but they knew and understood what each of the elements represented in the MEA. On their answer sheets the learners represented each element identified in the MEA with its initials. Also with the guidance of the researcher, they categorized their elements into variables and constants. The explanation of a variable and a constant would be guided by the learners' own interpretation of the items they have identified in the MEA.

Researcher: Can you distinguish the items you have identified into variables and constants?

L_{12}: What is a variable?

L_{11}: I have not heard about it before.

$L_{12:}$ And a constant?

L_{11}: I don't know what that is.

L_{13}: We need to ask the educator. [He prompts the researcher to explain to them what a variable and a constant is.]

Researcher: Look at the items you have identified. A variable is an element that is likely to change at any given time whereas constants are items that can never change.

L_{11}: Reads the question repeatedly. [Develop a rule….]

L_{12}: The educator says that variables change but constants don't change.

L_{14}: Educators and ….

Researcher: Educators and what?

L_{14}: Learners.

Researcher: Why do you think the educators and the learners are variables? L_{14}: Sir, they are changing.

Researcher: Why do you think the learners and educators are changing?

L_{11}: They are changing because not everybody comes to school every day, and the educators are absent, sometimes, so both the learners and the educators are variables.

Researcher: OK, looking at the four elements identified; are these the only variables in the problem?

L_{12}: Taxis are also a variable.

Researcher: Why do you say so?

L_{12}: The taxis too can change.

Researcher: Why do you think the number of taxis can change?

L_{11}: The more the learners the more the taxis.

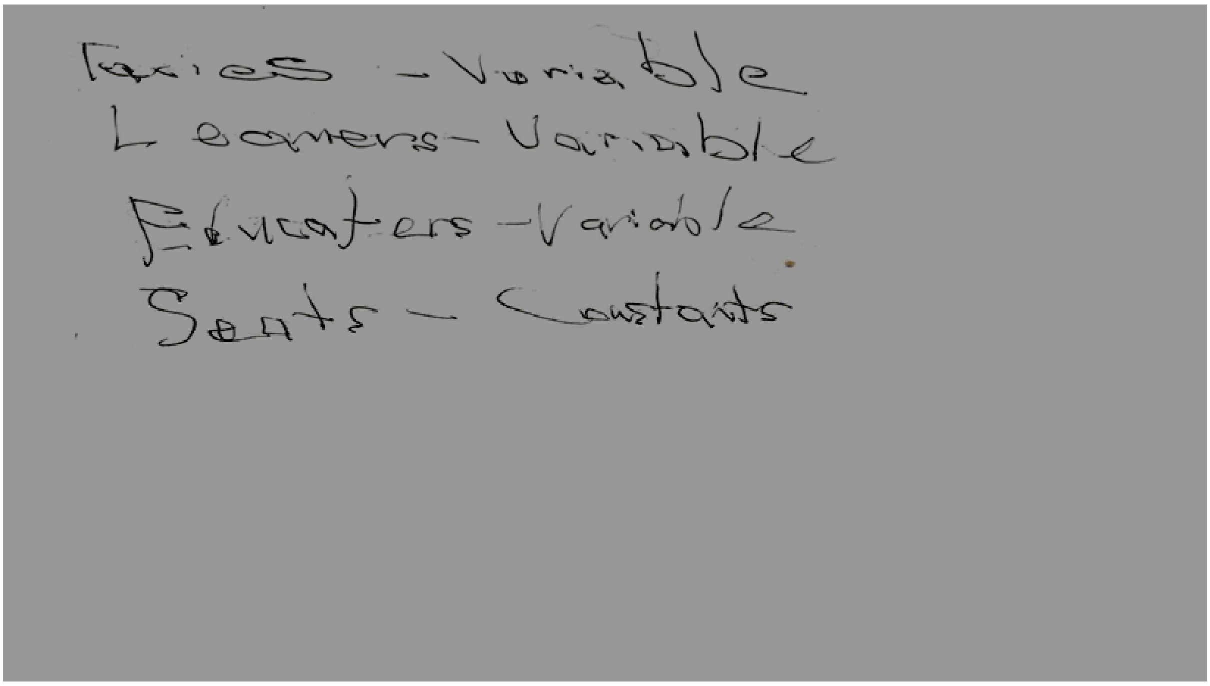
L_{12}: Because if the number of learners reduces, the number of the taxis will reduce, and if the learners increase we will need more taxis.

Researcher: OK, what about our last item, the seats in the taxi? What happens to them? [The learners looked confused at this stage. The researcher draws their attention to the drawing one of them had made and asks them whether the number of seats in the taxi can change. All of them realise they have missed a very simple thing by saying, *"OK".*]

Researcher: Can the number of seats in the taxi change?

L_{11}: No, that means it will be a constant.

Researcher: Assign constants or variables to your elements and write them down. [In Vignette 4.3 learners in Group 1 assign their identified elements into variables and constants.]



Vignette 4.3: Learners in Group 1 categorize the elements in the problem statement into a constant and variable.

The next episode explains the learners' actions on the problem situation as they are introduced to the concept of algebraic expressions. At this stage of the learning process the learners had developed a conception of variables and constants. They were gradually moving from the concrete way of analysing the problem to a more abstract way.

Researcher: Can we use our identified variables and constants to formulate an expression for the number of possible people, both learners and educators, who come *to school in a taxi on a particular day? Let us assume all the taxis used must be filled to capacity.*

L_{12}: Which people should be in the taxi?

L_{14}: The educators and learners who come to school that day.

L_{11}: We are not adding the driver.

L_{14}: We cannot add the driver to this. Remember, we have left the driver out.

L_{12}: The educator said we should use the letters we have already used.

L_{14}: So, it will be L and E.

Researcher: What do you mean by L and E? Represent it mathematically. [The learners were still finding it difficult to understand me, so I will ask more probing questions].

Researcher: You said the number of people on a day will be L and E, so how can I know the total number of people in the taxi?

L_{12}: Do you mean when they are together?

Researcher: How do you put them together mathematically?

L_{13}: Hmm, I am not sure.

Researcher: How can you put L and E together to have the total number of people that board the taxis on a particular school-day? I mean, are you going to subtract, divide, add or multiply them together?

L_{11}: We can times them together.

L_{13}: No! No! No!

L_{11}: What can we do then?

L_{13}: Multiplying the learners and the educators will not work. The number will be too big for the school. I don't think it will give us the number of people that come to school each day.

L_{11}: I said so what do we do then?

L_{13}: I think we simply need to add.

L_{13}: Sir, it should be L plus E.

Researcher: Good, now we are left with two more elements. How can we formulate an expression for the number of people in the taxi using the remaining two elements?

L_{14}: We are left with S and T.

L_{14}: We can add S and T.

Researcher: Why do you add?

L_{14}: To get the number of people that will take the taxi.

Researcher: So, you mean if we add the number of taxis and the number of seats in a taxi we are going to get the number of people who come to school?

L_{14}: Yes.

L_{12}: No! No!

Researcher: If No, then what should we do to S and T to get the number of people who can come to school by taxi? Remember we are assuming that all the seats in all the taxis used are occupied.

L_{12}: We need to know the number of taxis first.

L_{13}: But the number of taxis is now T.

L_{12}: How can we know the number of people in the taxi if we don't know the actual number of the taxis?

L_{13}: I told you, the number of people in the taxi is T, and it is a variable.

L_{11}: We need to know the number of taxis, so we can add the people in the taxi to get the number.

Researcher: Is that the best way to find the number of people that come to school in a taxi if you know the number of taxis? Look back at what L_{14} said: If one taxi can take *fifteen people two taxis can take thirty people. Remember two multiplied by fifteen gives thirty. So how can we link addition and multiplication in terms of the variable?*

$L_{13:}$ We need to add all the taxis that come to school every day.

Researcher: That will be a lot of work. Is that the best way?

L_{13}: Yes, Sir. If we can add the seats of all the taxis that are here every day, it will be easy to determine the number of people who use the taxi.

Researcher: But do you know the number of taxis?

L_{13}: No.

Researcher: So $L_{13,}$ what if the taxis are a thousand? Can you count to there?

L_{13}: I don't know.

Researcher: Remember what I said earlier: if one taxi can take fifteen people, two taxis can take thirty people, and remember the fifteen mentioned here is the number of people the taxi can take.

L_{12}: We can also multiply T and S.

Researcher: Why multiply?

L_{12}: The number of seats in a taxi multiplied by the number of taxis can give us the number of people that use a taxi to school every day.

Researcher: So, what will the expression look like? Write it down.

L_{12}: S times T as he writes it down*).*

The researcher built on the learners' ideas but corrected them where necessary, and ensured that the learners came up with the answer through guiding questions and their understanding of the problem situation. Through probing questions in a meaningful problem situation, the learners were now able to conceptualize how the variables and the constants they had identified influence and relate to each other. Through the MEA the learners are able to construct, manipulate and explain algebraic expressions in terms of the goal of the MEA. The real-life component embedded in the task enabled the learners to make and justify their created algebraic expressions and helped them to build a more powerful conception in operations and computational fluency.

In the next episode, the learners relate their obtained algebraic expressions to form an algebraic rule.

Researcher: Now we are going to use our algebraic expressions to develop an algebraic rule.

L_{14}: What does that mean?

Researcher: To relate the algebraic expressions identified using the equal sign.

L_{11}: Can we use the equal sign here where there are no numbers?

L_{12}: The equal sign can only be used if we are looking for an answer in a problem.

Researcher: The equal sign means anything on either side of it is the same in value.

L_{13}: But it only gives us answers when we are calculating.

Researcher: What did L plus E represent?

L_{13}: L plus E gives us the number of people in the taxi.

Researcher: What does that mean in terms of the number of people taking the taxi to school?

L_{12}: It indicates the number of people that can be in a taxi on a particular day [other members nod their heads in approval to what he said].

Researcher: What about when we multiply T by S? [The learners in the group tried to refer back to what they had already written].

L_{13}: What did we say was S?

L_{11}: S is our constant; the number of people the taxi can take at a time.

L_{12}: We said it was fifteen, T is also the number of the taxis.

L_{13}: So, what can T multiplied by S give us?

L_{11}: Each taxi takes fifteen people, so the more the taxis the more the people.

L_{13}: It can also give us the number of people in the taxi, which is the same as L plus E.

Researcher: The equal sign also means the same. So, can you now link up your two expressions with the equal sign?

L_{13}: That means E plus L is equal to S times T.

Researcher: Can you explain further why you think E plus L is equal to S times T?

L_{13}: Sir, you said anything on either side of the equal sign is the same, and E plus S is giving the number of people the taxi can take, and S times T also gives the number of people the taxi can take. So they are equal.

Researcher: Ok. Write your algebraic rule on your worksheet. [$L_{14,}$ with the support of other members of the group, is able to write down the link between the two expressions; see Vignette 4.4.]

Vignette 4.4: Learners develop a relationship between their algebraic expressions to form an algebraic rule

Researcher: Now you need to simplify your rule and make the number of taxis the subject.

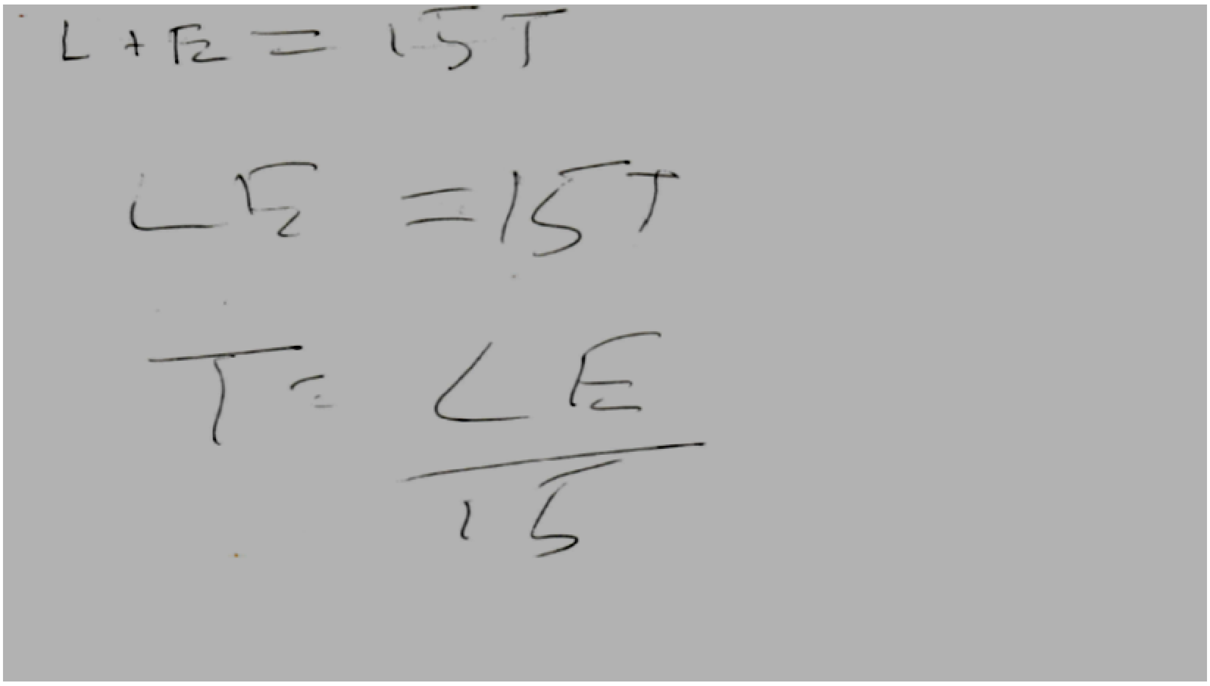
L_{13}: What do you mean when you say 'subject'?

Researcher: The question said we should find the number of taxis, which is T. So, T should be on the one side, and every other variable or constant on the other side.

L_{13}: We will add the educators and learners.

Researcher: Can you write your simplification down? Remember that if an element is multiplying and it crosses the equal sign it divides, and if an element is dividing and it crosses the equal sign, it multiplies. In the same way, if an element is positive and it crosses the equal sign it becomes negative and if an element is negative and it crosses the equal sign it becomes positive.

As discussed in section 2.2.3 the learners thought they had to put the two variables together. As shown in Vignette 4.5, L_{13} simplifies the algebraic expression as the others look on.



Vignette 4.5: Learners' misconception as they simplify an algebraic rule

Vignette 4.5 indicates a common misconception learners make in algebra, namely by multiplying the two variables together. I probed the learners with more questions with the aim of correcting this misconception.

Researcher: Why do you multiply the learners and the educators?

L_{12}: No, but it will give us the total number of people.

Researcher: Yes, is true. They do give us the number of people, but the way you have written it is as if you are multiplying them now.

L_{13}: So how are we going to put them together?

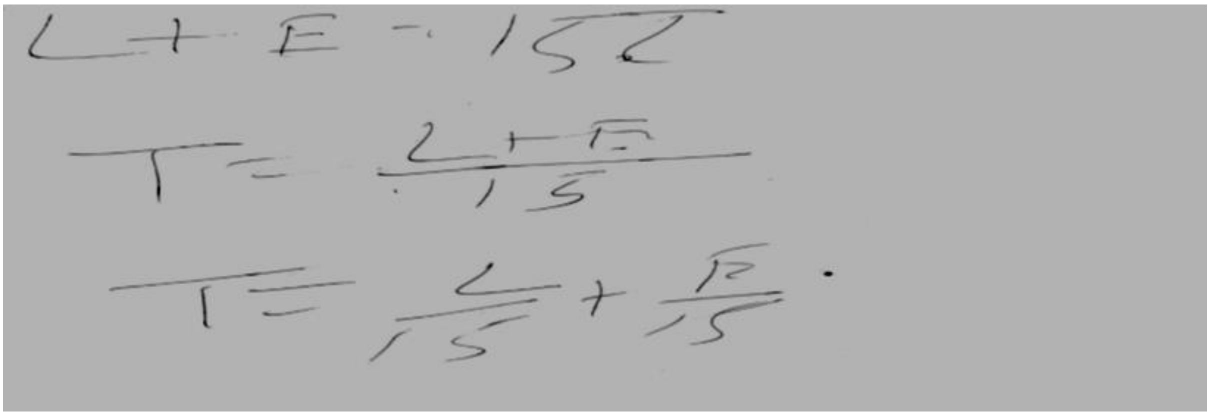
Researcher: You can only write it this way if the two variables are multiplying each other. Here you have educators and learners, if you put the two together what are you going to get? L and E are two different variables.

L_{14}: The two cannot be added together.

L_{12}: Ok. That means we can also not subtract the number of learners from the educators, but we can take fifteen to the other side of the rule.

Researcher: Remember that when a variable or a number is multiplied and is taken to the other side of the equal sign, it divides. You need to also note that when there is an addition or subtraction, the number dividing must divide each element separately.

In Vignette 4.6 the learners are able to correct themselves in simplifying the algebraic expression.



Vignette 4.6: Learners in group write a correct simplification of their rule on their answer sheet

Researcher: So, what can we say about our rule?

L_{14}: We don't understand the question.

Researcher: Can we say what is on the right-hand side is the same as what is on the left-hand side?

L_{14}: Yes.

Researcher: Why?

L_{14}: Because of the equal sign.

The learners were able to conceptually expand their knowledge on the equal sign as not only a symbol that precedes an answer to an arithmetic expression but that the equal sign is a symbol that signifies the equality of two very different expressions. The learners realised that the expressions on either side of the equal sign have the same numerical value and are equivalent. Even though they were totally different, they were able to build an explicit understanding of why the two expressions on both sides of the equal sign are equivalent, and in the process, build a clearer conceptual understanding of algebraic procedures. Developing a relational view of the equal sign in elementary school learners is critical for learning algebra, and a lack of understanding it is a major stumbling block for them as they shift from arithmetic to algebra (Kieran, 1981; Matz, 1982).

The reality principle in the MEA enabled the researcher to explain a common misconception ‘conjoining’ or ‘closure’ as explained in section 2.2.2, to the learners. The learners thought the easiest way of simplifying an algebraic expression was to multiply the two variables together even though there was an addition sign between them. But the learners recognized that those two elements in the problem statement are two distinct entities and cannot be multiplied in that manner. The algebraic rule formulated by learners was not just a mere rule; it conveyed meaning which makes the learners conceptualize an algebraic rule as a rule that can explain the phenomena in an MEA.

In the next episode of the teaching treatment the learners were asked to use their algebraic rules to transform one set of elements in the MEA into another.

Researcher: How many taxis will be needed to transport all the educators in the school if all the educators in your school come to school on that day?

L_{14}*: How many educators are in the school?*

L_{12}*: I am not sure - let me count* [After briefly counting the educators in his mind]. *The number of educators is 19. So, we will need two taxis.*

Researcher: Does your rule explain that?

L_{12}*: I am not sure.*

Researcher: What was the rule supposed to find?

L_{12}: It was supposed to find the number of taxis needed to transport the learners and the educators.

Researcher: Use your rule to verify your answer.

L_{14}: But the rule has L, which represents the learners.

Researchers: Yes, but you can still use the rule to determine the number of taxis that can transport only the educators.

L_{14}: How can we do that?

Researcher: How many educators will be transported?

L_{12}: Nineteen educators.

Researcher: And how many learners will be transported?

L_{12}: You asked only about educators.

Researcher: Yes, I know. But I want you to give me a number.

L_{12}: No learners.

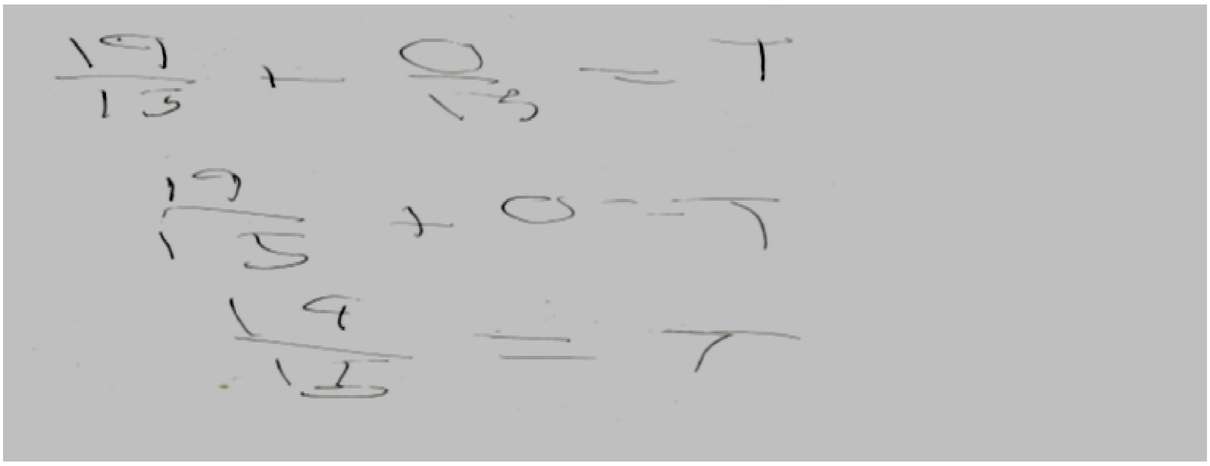
Researcher: How can you represent no learners mathematically?

L_{12}: I'm not sure.

Researcher: I'm asking for a number if there is no learner to be transported.

L_{12}: Zero.

Researcher: Good. So, you can substitute the number of learners with zero. Now you can replace your numbers in the rule you have found to determine the number of taxis needed to transport the educators. [The learners discussed the question among themselves and with the researcher came up with an answer as shown in Vignette 4.7].



Vignette 4.7: Learners' responses to finding the number of taxis that will transport the educators in their classes to school using their model

L_{14}: We will need 19 over 15 taxis to transport all the educators to school.

Researcher: Can we have 19 over 15 taxis? Convert your answer into a mixed fraction.

L_{12}: We are going to have one whole number 4 over 15 taxis.

Researcher: Can we have 4 over 15 taxis?

L_{11}: We don't know.

Researcher: What does the 4 over 15 represent?

L_{11}: It is a fraction.

Researcher: A fraction of what?

L_{11}: The number of educators in a taxi.

Researcher: But what we found was the number of taxis that can accommodate the educators? All the same, you are correct if you say it is a fraction of the number of educators. But I want you to explain it in terms of the number of taxis that will transport the educators to school.

L_{13}: We don't know.

Researcher: What answer did you get?

L_{12}: One whole number, 4 over15.

Researcher: What does the one represent?

L_{12}: The taxi.

Researcher: And what does the 4 over 15 represent in terms of the number of taxis?

L_{14}: That means there is going to be one more taxi to make two taxis.

Researcher: Will the second taxi be full?

L_{14}: No.

Researcher: Why not?

L_{14}: Only 4 people will enter, since the first taxi has taken 4 educators and the educators are 19.

Researcher: Good. In other words, we can say 4 out of the 15 seats will be filled by educators in the second taxi. That is the meaning of the 4 over15?

L_{14}: Yes.

Researcher: Now use your developed rule to find the number of taxis needed to transport the learners in your class.

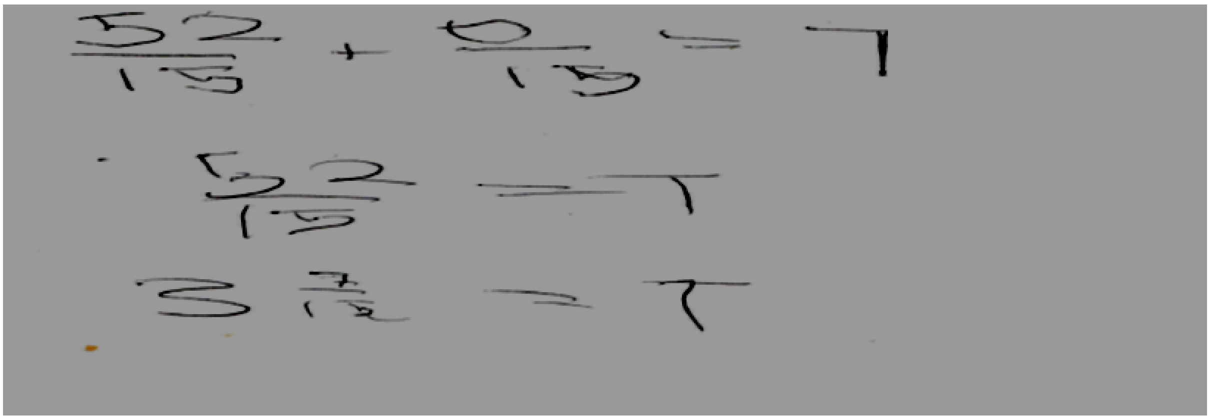
L_{12}: There are 52 learners in this class, and there will be no educator.

L_{11}: That means the number of educators will be zero.

L_{12}: It means E will now be zero.

Researcher: Good. So, you can compute your answer now.

After a brief discussion, the learners in the group came up with the following answer, as indicated in Vignette 4.8.



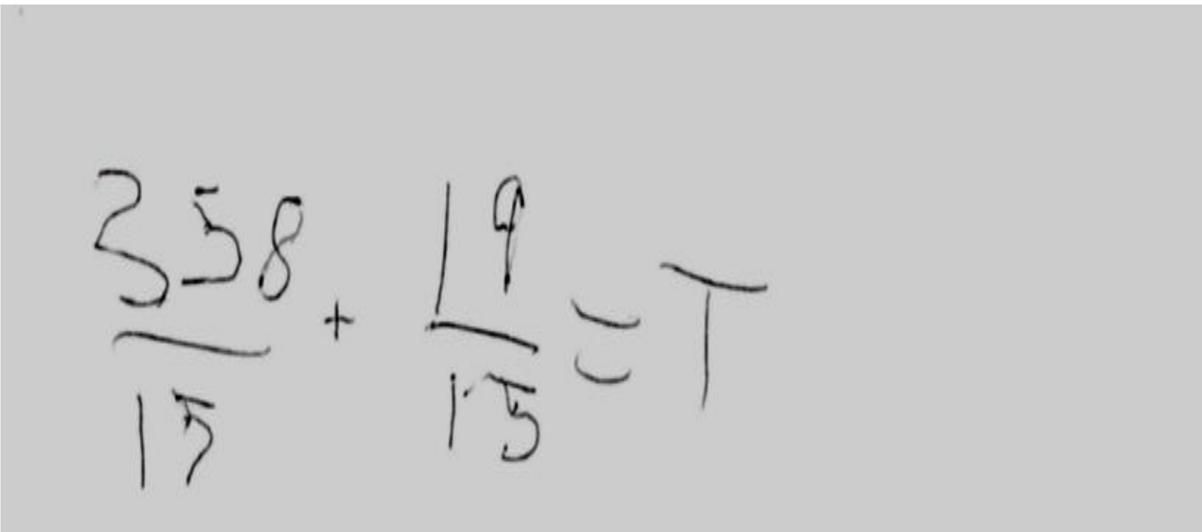
Vignette 4.8: Learners' responses to finding the number of taxis that will transport the learners in their classes to school using their model

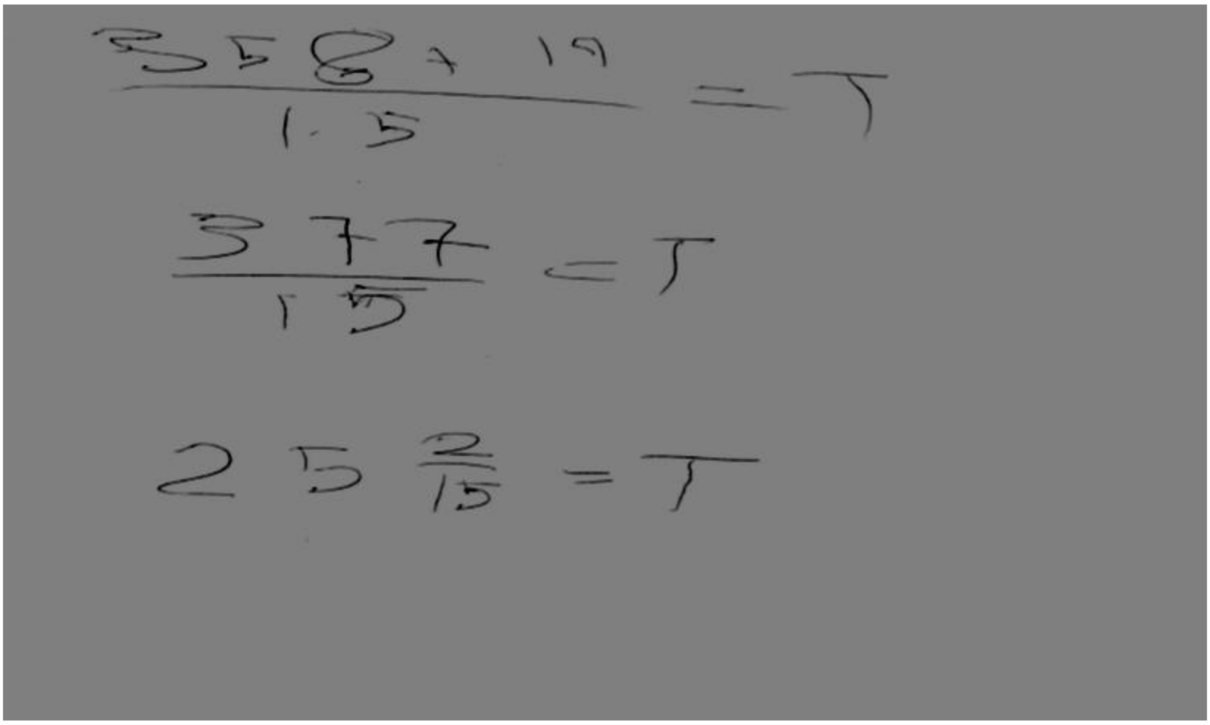
Researcher: What does that mean? Can you explain it in terms of the number of taxis that will transport the learners in your class?

L_{14}: That means we are going to need 4 taxis to transport the learners in our class. But the last taxi will have only 7 seats occupied.

Researcher: Ok, now we can do the last question. How many taxis will be needed to transport all the learners and the educators to the school, assuming that all the learners and educators come to school that day?

The learners were very quick with their answer, as shown in Vignette 4.9.





Vignette 4.9: Learners' responses to finding the number of taxis that will transport the learners and educators in their classes to school using their model

L_{11}: Many taxis will be needed, 25 taxis.

L_{13}: No, there will be 26 taxis. Remember, two over 15.

Researcher: And how many people will be in the last taxi?

L_{13}: Only two people.

In the above episode, the researcher started introducing the learners to the dynamic nature of an algebraic rule to transform one set of elements into another. The numbers displayed were not just numbers to them. They concretized and visualized those numbers to real elements as they linked all the letters, numbers and operations with reality. The learners could now conceptualize an algebraic rule as a rule that transforms one number into an entirely new number. All this was happening in the context of a situation the learners were able to make sense of. Learners' actions were gradually being interiorised into a mental process where learners can picture a rule as an expression that can be used to transform one set of elements (input values) into another set of elements (output values) in a problem situation. At the next section the researcher built on this conception by engaging learners to use the rule to predict input values using output values.

4.2.2.2 Interiorizing action conceptions into a process conception

Researcher: I have twenty taxis. Can you tell me the number of learners I can take to and from school?

L_{13}: Are we going to use the rule?

Researcher: Yes?

L_{11}: Only the learners?

Researcher: Yes. That means we are looking for L now and we know the value of T.

L_{11}: But there are no educators.

Researcher: Remember, we have discussed that.

L_{12}: Yes.

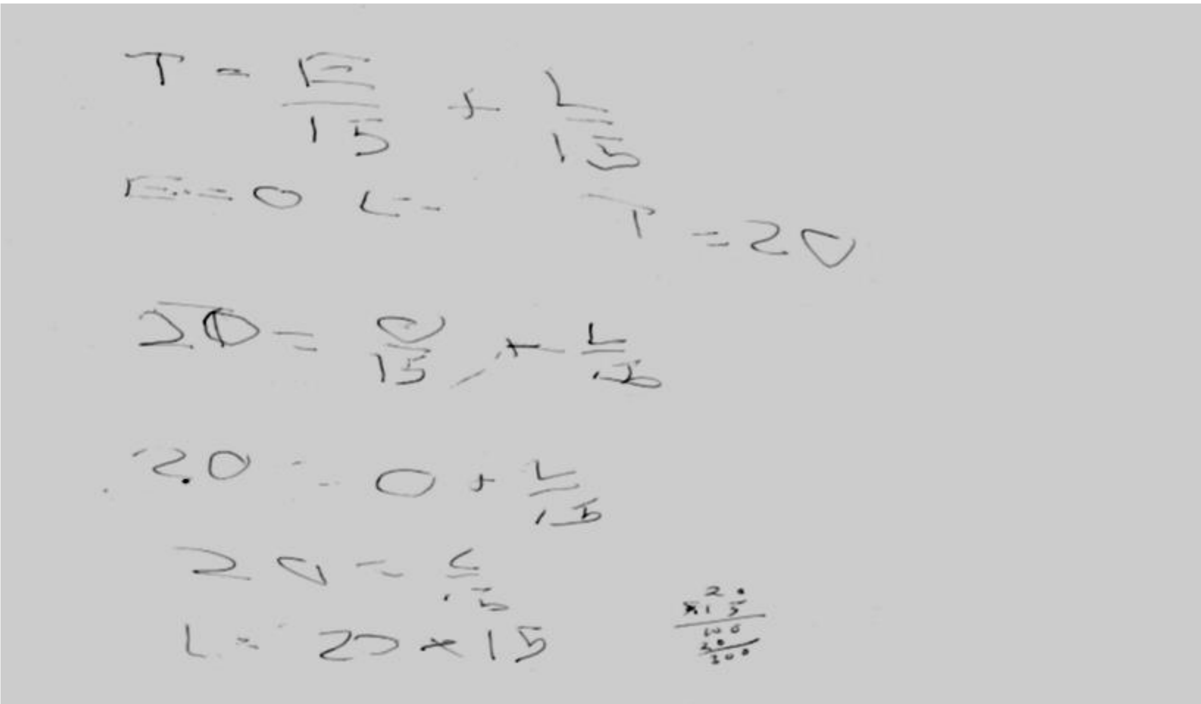
Researcher: So, if there are no educators what is going to be the number of the educators?

L_{11}: I think it should be… it should be zero.

Researcher: Ok. And what is going to be T?

L_{13}: T is the number of taxis which you said is 20.

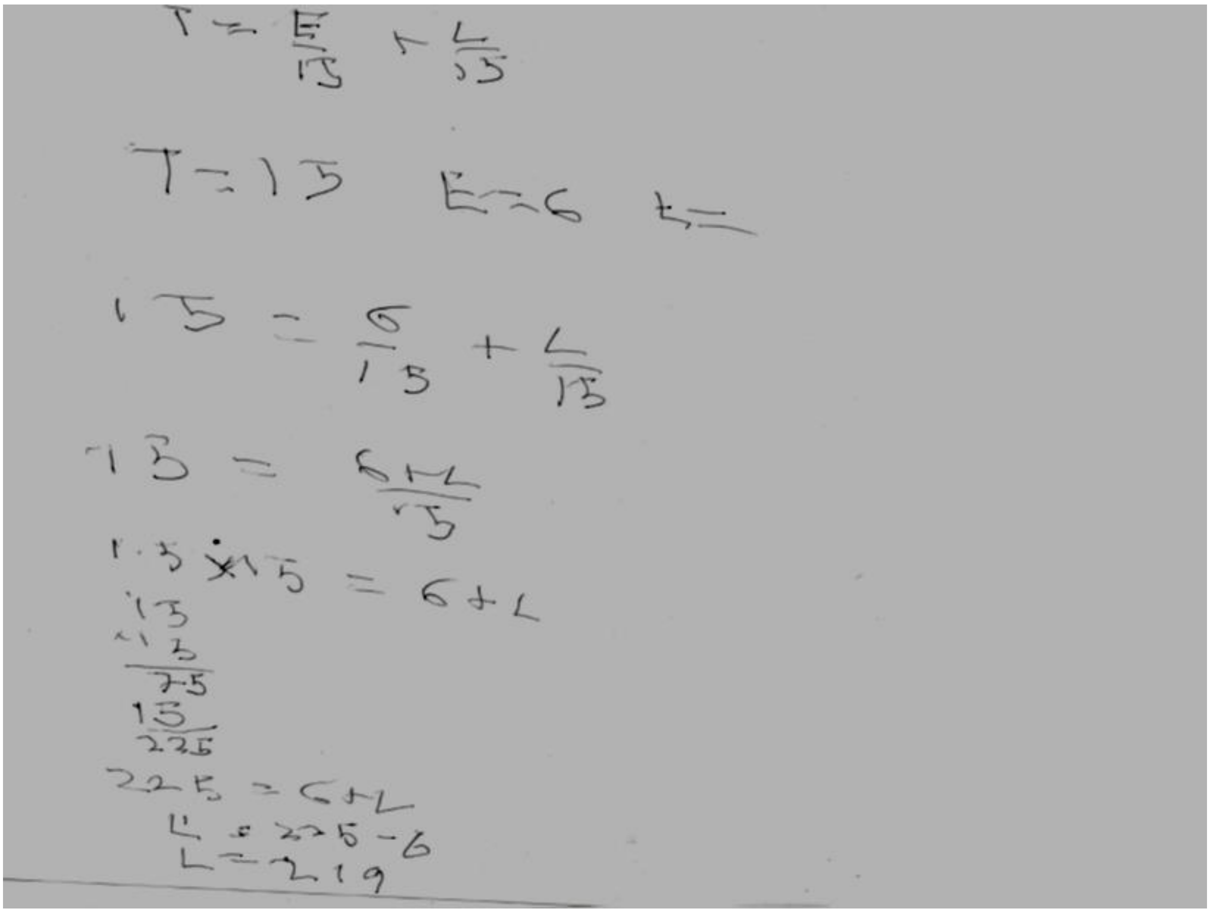
Researcher: Ok. Can you calculate the number of learners that can occupy 20 taxis? Remember when a number or letter is divided and it crosses the equal sign it is multiplied, and when it is multiplied and it crosses the equal sign it is divided. [One of the learners writes their answers as shown in Vignette 4.10.]



Vignette 4.10: Learners in group 1 use the inverse of the algebraic rule to predict input values using output values

L_{14}: For 20 taxis, there will be 300 learners.

Researcher: Correct., Can you use the algebraic rule to predict the number of learners that will be needed to board the 15 taxis, assuming there are 6 educators who will board the taxis. [See learners' response in Vignette 4.11.]



Vignette 4.11: Learners in Group 1 use their algebraic rule to predict the input values using one input value and one output value

L_{15}: Sir, we are going to have 219 learners.

Researcher: That is correct. So, what can you say about the two previous solutions?

L_{11}: Do you mean…?

Researcher: What I mean is, what can you say about the answers you obtained for these last two solutions?

L_{11}: I think it is like the first one - the more the taxis the more the learners.

Researcher: Ok.

In the episode above the learners used the inverse of the algebraic rule to predict the input elements using the output elements which gradually moved them from the action of in-putting a value to get an output to the process of interiorizing an algebraic expression as a rule that transforms one set of elements in one situation to another set of elements in the same situation.

In the next section the learners encapsulate their process conceptions into an object conception by substituting other real-life situations that can be explained by their algebraic rule and perform algebraic operations on the algebraic expression to fit the new situation.

4.2.2.3 *Encapsulation of a process conception into an object conception*

Researcher: The algebraic rule we have just found has been used to determine how many taxis can be used to transport learners and educators to school, given that you know the number of learners and educators, or the other way around, can you think of other situations where the algebraic rule can be applied?

L_{11}: Sir, we don't understand the question.

Researcher: I mean the algebraic rule we have just found helps us to find, at any other time, the number of taxis we may need to pick up the learners, the educators or both. I want you to describe a similar situation that we can use our algebraic rule to predict.

L_{11}: If we can use buses, we can use our rule to determine the number of buses that will be used by the learners and the educators.

L_{13}: The number of classrooms required in a school.

Researcher: But buses can transport more people than taxis.

L_{13}: Yes.

Researcher: So, how can we modify your algebraic rule to cater for the number of people the bus can take?

L_{12}: We don't understand.

Researcher: What did 15 represent in your algebraic expression?

L_{12}: The number of people the taxi can take.

Researcher: So, now we are using buses, we need to know if, when the situation changes, the limits of our rule will also change. Let as assume the bus will take 76 people at a time. Can you modify your algebraic expression to explain this new situation?

L_{13}: We can change the 15.

L_{11}: The bus takes 76 people.

L_{14}: So, what can we do now to the rule?

L_{11}: The taxi was taking 15 people and now the bus is taking 76 people.

L_{13}: Yes.

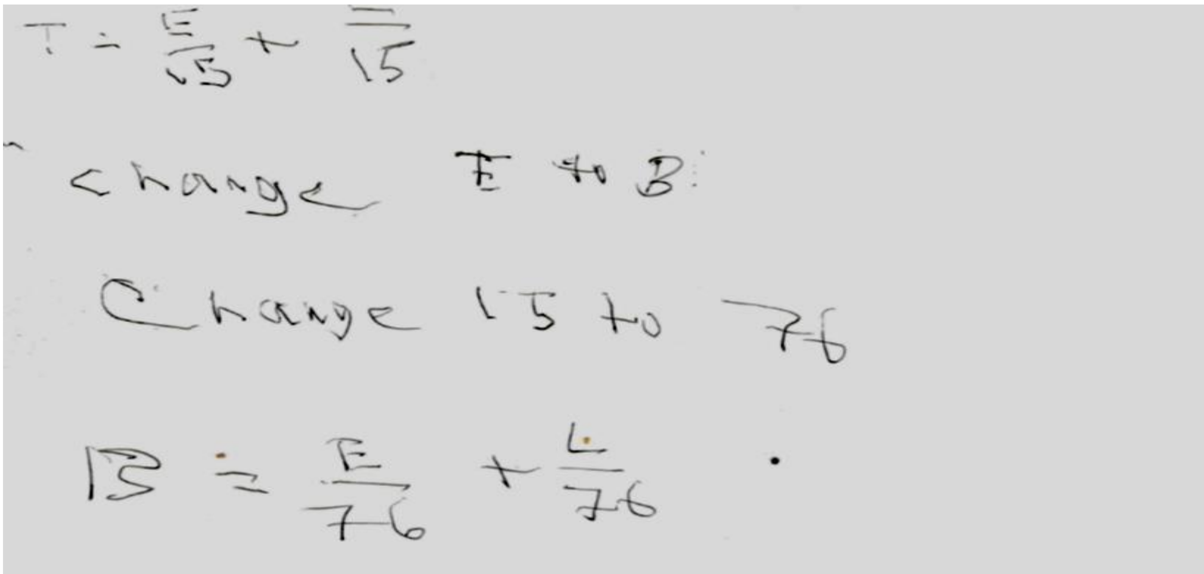
L_{14}: We just have to change the 15 to 76 to get our new rule.

L_{11}: Yes, I can see that.

Researcher: Ok, let's represent the number of buses by the letter B. Will B be a constant or a variable?

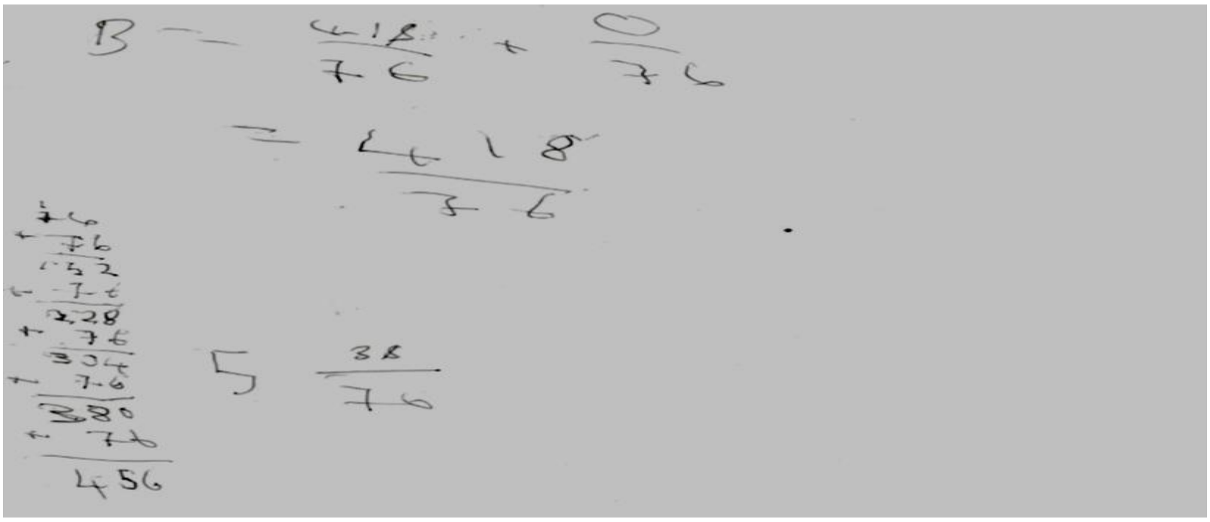
L_{13}: Sir, a variable, because it can change.

Researcher: Ok, good. Can you write down your new rule? [Based on the previous discussion the learners were able to modify their rule without much difficulty, as shown in Vignette 4.12.]



Vignette 4.12: Learners in Group 1 use their modified algebraic rule to determine the number of the buses that take the learners to school

Researcher: Can you write down how many buses will transport only learners to school? [As can be seen in Vignette 4.13, the learners in Group 1 were able to give their answer without much difficulty because they were able to interiorize an action conception into a process conception, according to the APOS theory.]



Vignette 4.13: Learners use their modified algebraic rule to determine the number of buses that can take the learners to school

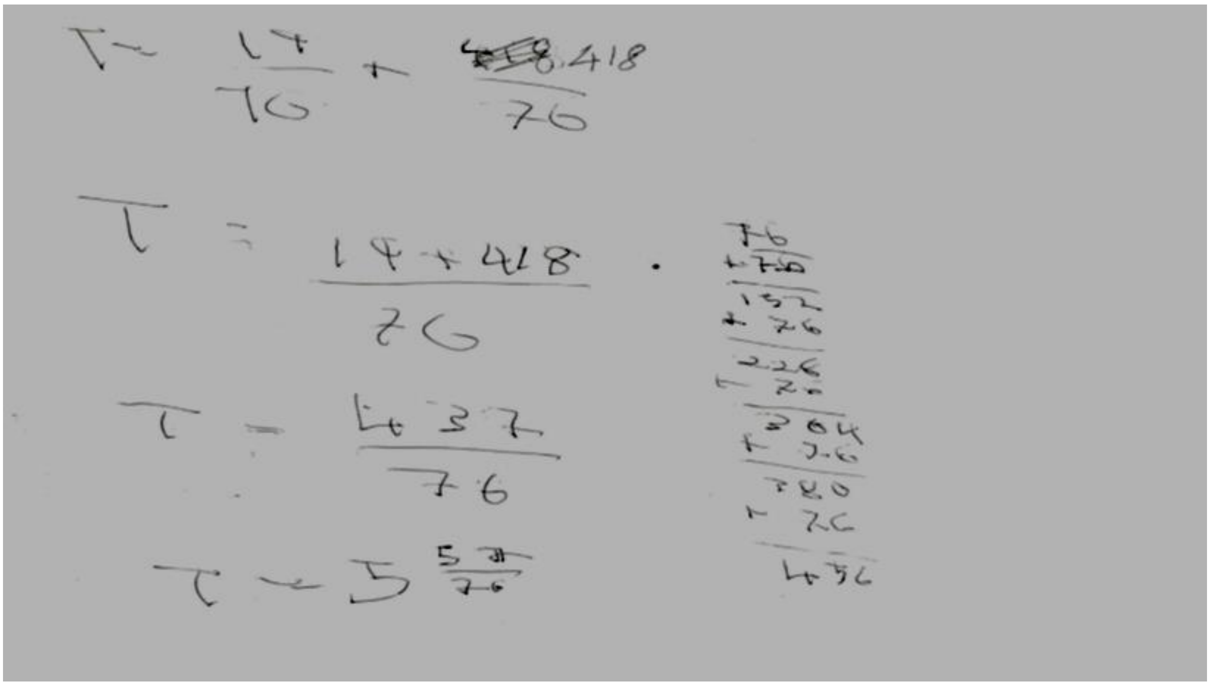
Researcher: So, in effect, how many buses will be required to transport the learners to school?

L_{11}: Six.

Researcher: And how many learners will be in the 6th bus?

L_{11}: 38 learners.

Researcher: Good. Can you also calculate the number of buses required to transport both the learners and the educators to school? [See the learners' solution in Vignette 4.14.]



Vignette 4.14: Learners use modified algebraic expressions to determine the number of buses that can carry all the learners and the educators to school

Researcher: So, how many buses will be required here too?

L_{14}: The same six buses; the last one will take 57 learners.

L_{11}: And also educators.

Researcher: So, can you tell the difference between the two solutions?

L_{14}: I don't understand.

Researcher: You are getting six buses for the last two solutions, but the answers are not the same, so can you tell the difference between the two?

L_{14}: The second one will be more than the first one because there were also educators.

Researcher: Can you explain it in terms of the two answers?

L_{11}: I don't know

Researcher: Looking at the two answers, why will you say the second one is bigger?

L_{15}: The first answer has 38 and the second answer has 57.

Researcher: L_{11} also mentioned the number of classrooms required at school. Can you modify your model for that one in the case where every classroom has space for 43 learners?

L_{12}: Yes, we change the 15 to 43.

Researcher: Can you write it down? But remember the educators don't sit in the classroom, so you must not add the educator.

L_{11}: So, what happens?

Researcher: If there is no educator, what does it mean?

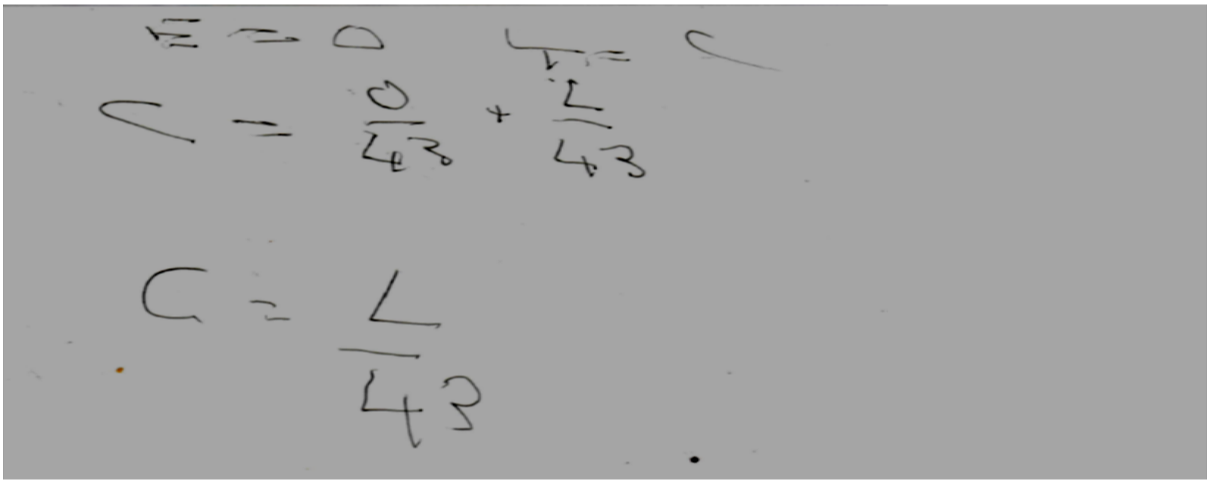
L_{11}: It means there are zero educators.

Researcher: So how can you use this information to modify your rule?

L_{11}: I am not sure.

L_{15}: Yes, we have to replace the E with zero.

Researcher: Can you write what the rule will look like on paper? [In Vignette 4.15 the learners modified their algebraic rules to determine the number of classrooms that can accommodate the learners.]



Vignette 4.15: Learners' modified rule to determine the number of classrooms required to accommodate a particular number of learners

Researcher: From your rule, how many learners can your school accommodate if there are 15 classrooms?

L_{14}: Our classroom is only one.

Researcher: Yes, I know. But I want to know what your algebraic rule says.

L_{14}: How many learners?

Researcher: Yes.

L_{14}: We are going to replace 15 with L so we are going to have 15 over 43.

Researcher: So you are saying, if we you have 15 classrooms in the school, it is going to take 15 over 43 learners, Can that be true?

Whole group: No.

Researcher: So what are you supposed to do?

L_{11}: Not sure.

Researcher: What does C represent?

L_{11}: C represents the number of classes.

Researcher: From the question I asked you, what is the number of classes?

L_{11}: The number of classes is 15.

Researcher: Drawing from the previous example, what will you do to the letter C in your rule?

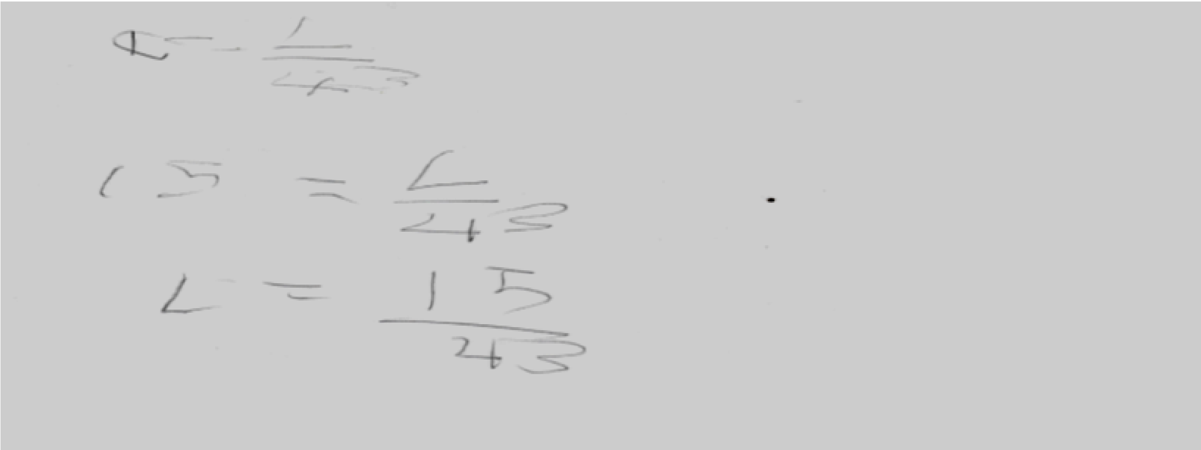
L_{15}: We are going to change it to 15.

Researcher: Good, so can you determine the number of learners your school can accommodate in 15 classrooms?

L_{13}: So, we replace C with 15.

L_{11}: Yes.

Researcher: Can you write your answer on your sheet? [In Vignette 4.16 the learners in Group 1 ascertain the number of learners required if there are 15 classrooms, but could still not recognize that a dividing number or element multiplies when it crosses the equal sign.]



Vignette 4.16: Learners produce a wrong solution for the number of learners required for 15 classrooms

Researcher: What was your answer?

L_{14}: *Fifteen over 43.*

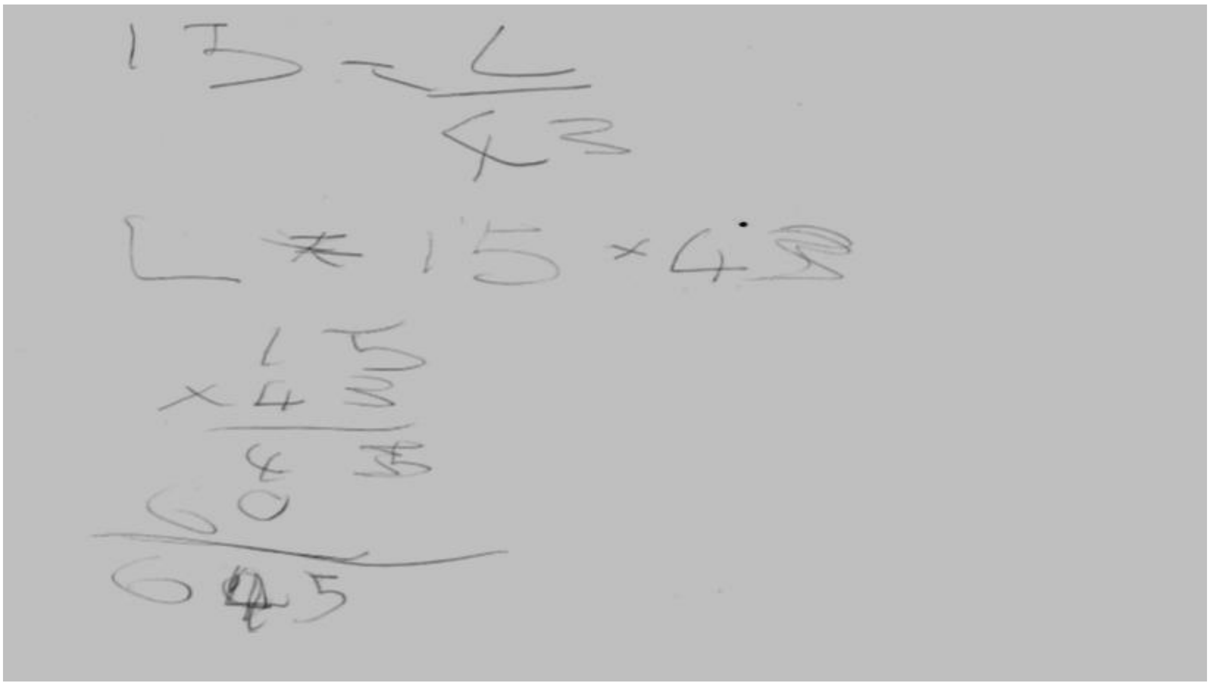
Researcher: Does this make sense to you?

L_{14}: *I'm not sure.*

Researcher: What you have written means you are going to have less than one learner if there are 15 classes in your school, which is not realistic. Remember what I told you at the beginning of the lesson, that a dividing variable or constant will multiply if it crosses the equal sign, and vice versa.

L_{11}: *Yes.*

Researcher: So, can you indicate the correct answer now? [In Vignette 4.17 the learners in Group 1, realising their error, are able to rectify their solution for the number of learners required for a classroom of 15 learners.]



Vignette 4.17: Learners in Group 1 determine the correct solution for the number of learners required for a school with 15 classrooms after guiding questions from the researcher

Through guiding questions the problem-solving heuristic instructional method guided the learners to substitute other real-life situations that could fit their model. The substitution skill is a highly powerful mathematical thinking tool as it involves thinking about new possibilities in the new situation, whilst at the same time considering its limitations. The learners were guided by means of questions to perform arithmetic operations on their already developed algebraic expressions to fit the new situation.

4.2.3 Concluding remarks on the implementation of the problem-solving heuristic instructional method.

i. Integrating the two theories, namely the modelling and modelling perspective and the APOS theory, proved useful in designing algebraic concept construction activities that helped the learners in Grade 6 to develop a sound conceptual understanding of algebra. It also activated learners' interest in the lesson, leading to active participation by most of the learners in the class.

ii. Guiding questions used by the researcher were crucial in helping the learners to reflect on their personal experience with the MEA and their already acquired arithmetic schemas to focus on a strategy towards the solution processes of the MEA.

iii. It was found that the learners were able to make sense of an MEA developed on the basis of the modelling and modelling perspective as it presents the learners with concrete objects which they can perceive. This made it cognitively less demanding for the learners to reflect on what they know in terms of the new problem situation presented to them. This showed the MEA was consistent with the reality principle of the modelling and modelling perspective.

iv. The learners' actions on the MEA led to the discovery and extension of their knowledge on a constant, a variable, the equal sign and an algebraic expression, which led to the formulation of an algebraic rule. The algebraic rules were not merely abstract symbols to the learners, but could be explained in accordance with the goals of the MEA. This was consistent with the modelling construction principle.

v. The learners performed actions on the developed algebraic rules. They used input values to predict output values in line with the goals of the MEA. Based on their understanding and experience with the MEA, the learners could refer their answers to their understanding of the MEA to check whether their answers were correct and made sense by comparing their answers with the reality of the MEA. This is consistent with the self-evaluation principle.

vi. The teaching treatment moved the learners away from an action conception to a process conception by guiding them to reverse the developed algebraic rule to predict input values using output values. Through this, the learners gradually began to interiorize an algebraic rule as a rule that dynamically transforms one set of elements into another in a given situation (MEA).

vii. The learners' discussion of these MEAs enabled the researcher to monitor their progress, conceptions and misconceptions in their quest to develop a conceptual understanding of algebra. It also enabled the researcher to explain to the learners what a variable and what a constant is and how they can be used to formulate algebraic expressions and rules. This was consistent with the model-documentation principle.

viii. The learners encapsulated the process conception into an object conception by using their developed algebraic rules to describe parallel real-life situations and to modify their algebraic rules to fit that situation by performing arithmetic operations on their algebraic rule. An example was the learners manipulating an algebraic rule to fit a classroom problem, which was consistent with the model generalization principle.

ix. The learners encapsulated the process conception into an object conception by using their developed algebraic rules to predict similar situations that fit their model and perform arithmetic operations with them; this was consistent with the simple prototype principle.

x. Grade 6 learners have the capacity to reason functionally in describing the relationships between variables and constants in a given situation. They are also able to keep track of how one variable in a given situation changes with respect to another variable in the same situation.

4.3 EFFECTS OF THE PROBLEM-SOLVING HEURISTIC INSTRUCTIONAL METHOD ON LEARNERS ACHIEVEMENTS IN ALGEBRA

This section is organized into three main parts, namely descriptive statistics, graphical representation, and inferential statistics. These statistical tools are used to explain the effects on learners' achievement in algebra after being taught algebra using the problem-solving heuristic instructional method as explained in section 4.2 above.

4.3.1 Descriptive statistics

The descriptive statistics describe the learners' scores in the control and the experimental groups in terms of the mean scores, standard deviation, variability, and gain scores in the pre-test and post-test.

4.3.1.1 *Mean, standard deviation and range of scores*

Table 4.5 provides the mean, standard deviation, and the maximum and minimum scores in the tests before and after the intervention.

Table 4.5: Mean, standard deviation and range of scores

		Experimental	Control
Pre-test			
	Mean	15.60%	13.54%
	Standard deviation	9.52	9.98
	Minimum mark	0	0
	Maximum mark	45	35
	n	92	106
Post-test			
	Mean	43.73%	13.92%
	Standard deviation	16.8	9.36
	Minimum mark	10	0
	Maximum mark	80	40
	n	92	106

From Table 4.5 the mean score of the control group was 13.6%, whereas that of the experimental group was 15.6%. This score indicates the learners' academic strength in algebra before the intervention. Even though the pre-test mean score for the experimental group was higher than that of the control group, the researcher could not draw any definite conclusions regarding the pre-intervention equivalence of the control and experimental groups, as there were a myriad of factors, such as the natural variation in their respective schools that could have contributed to this. All the learners in both the experimental and the control groups achieved 35% or less in the pre-test.

The difference between the post-test and the pre-test mean scores of the control group indicate a 0.38% (approximately 0%) improvement, whereas those of the experimental group show an improvement of 28.18%. The minimum and maximum marks of the experimental group also improved significantly to 5% and 35% points respectively whereas those of the control group changed by 0% and 5% points respectively. The significant improvement in the experimental group's scores seems to suggest the positive effect of the heuristic teaching method and supported the hypothesis that the heuristic teaching method improved the learners' achievements in algebra. The variation of the pre-test marks around the pre-test mean of the experimental and the control group was approximately the same.

In the post-test, the standard deviation of the experimental group increased by 7.28 whereas that of the control group increased by 0.62, implying that the data variation of the post-test marks around the post-test mean for the experimental group was bigger than that of the control group. A graphical representation of the data is given to indicate more evidence in support of the hypothesis.

4.3.2 Graphical representation pre-test and post-test scores of control and experimental group

The graphical representation in this analysis illustrates the frequency of the pre-test scores of the control and the experimental groups before the intervention, and the frequency of the post-test scores of both the control and the experimental groups after the intervention with the experimental group, highlighting the effects of the treatment on the experimental group.

4.3.2.1 Comparison of the pre-test scores of experimental and control groups

Figure 4.1 illustrates the frequency of the pre-test scores of the experimental and control groups before the intervention with the experimental group. It highlights their knowledge level in algebra before the intervention. The control group is represented by the blue circles and the experimental group by the green circles.

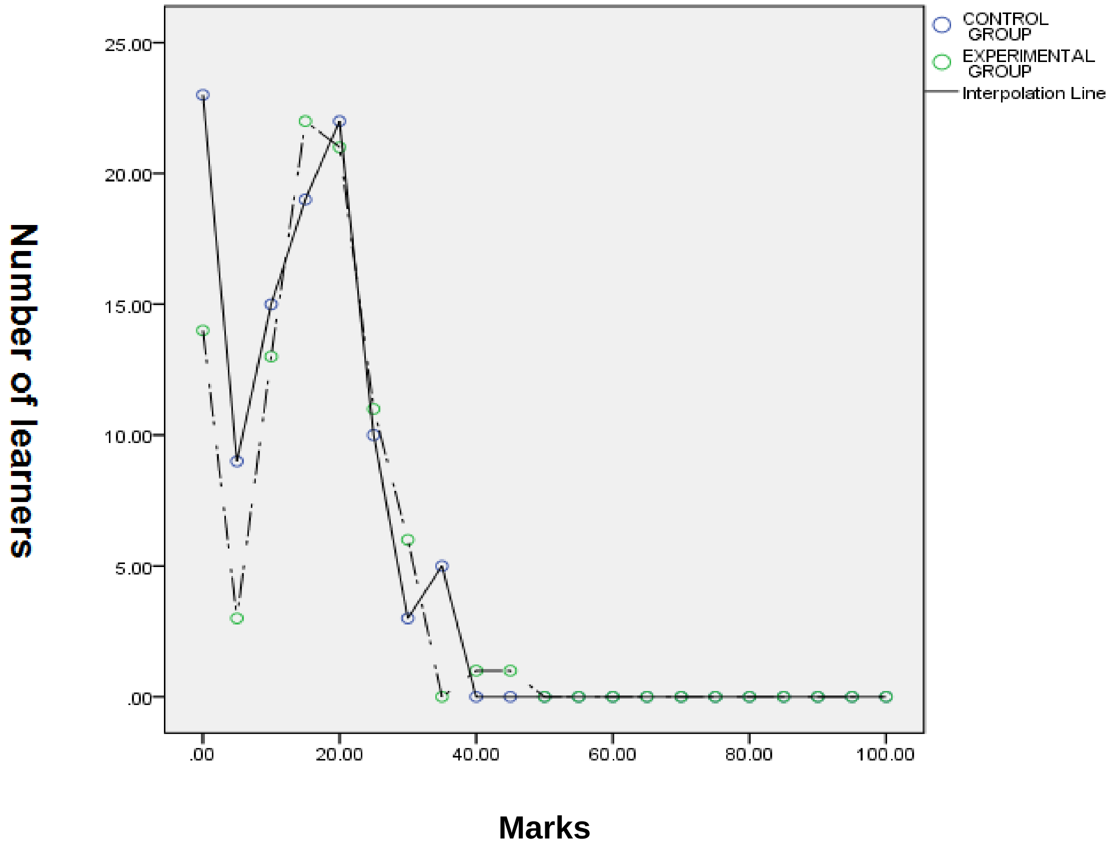


Figure 4.1: Comparison of pre-test marks for the control and the experimental groups

Both graphs in Figure 4.1 are skewed to the right and inconsistent, even though all the groups scored low marks in the algebra pre-test. The marks of the learners in the experimental group were slightly higher compared to those of the learners in the control group.

4.3.2.2 Comparison of post-test marks of control and experimental groups

In Figure 4.2 the frequency of the post-test scores of the experimental group (after participation in the intervention) and the control group (non-participation in intervention) is compared, with the control group represented by the blue lines and the experimental group represented by the green circles

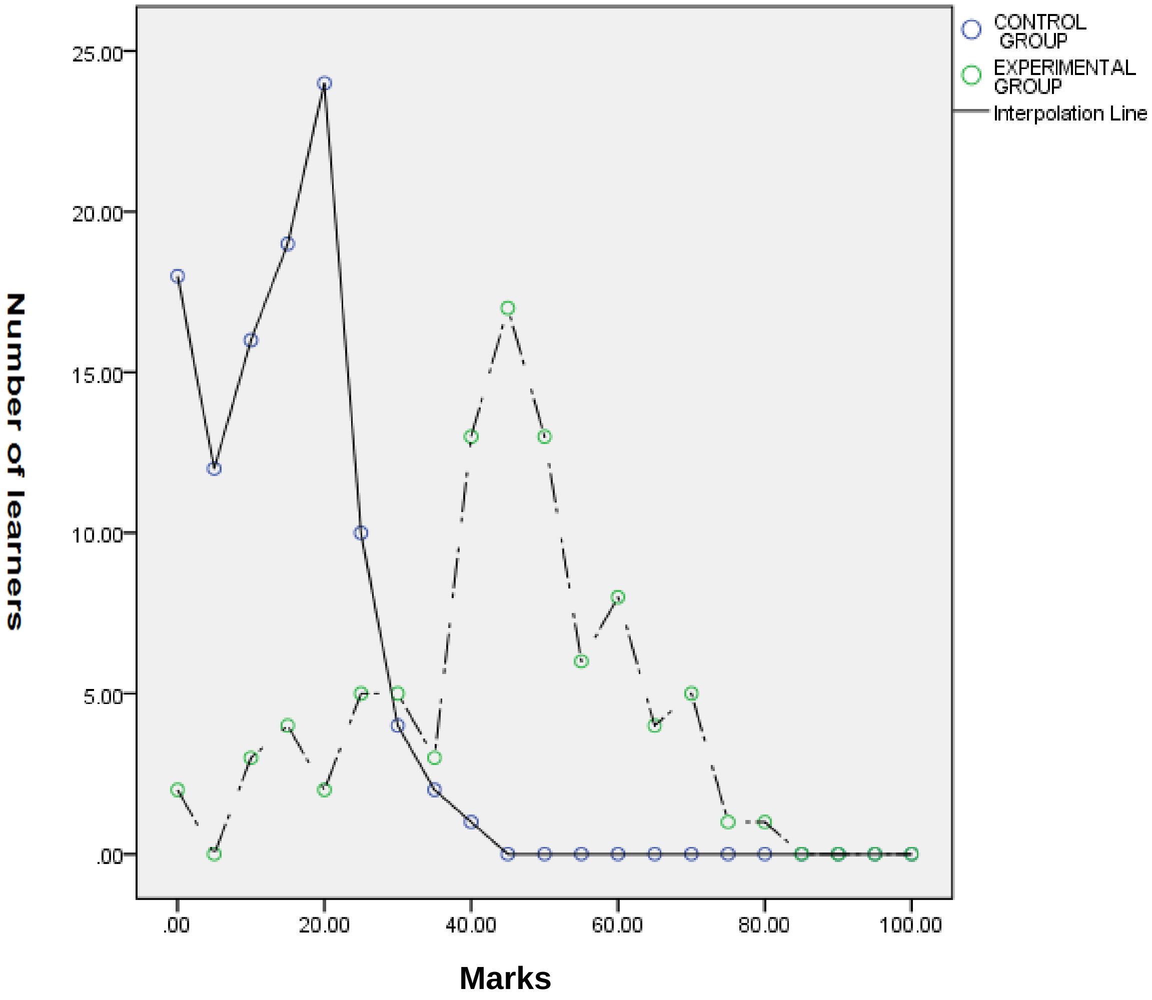


Figure 4.2: Comparison of post-test scores for experimental and control groups

The graph for the control group is further skewed to the left whereas that of the experimental group symmetric around the mean, indicating that the learners in the experimental group scored higher marks after the intervention in contrast to the control group.

4.3.2.3 Concluding remarks on graphical representation

Figures 4.1 and 4.2 indicate a comparison of the frequencies of the pre-test and post-test scores of the learners in the experimental and the control groups, and the changes that occurred from the administration of the pre-test to the participation (the experimental group) and non-participation (the control group) in the heuristic teaching method through to the administration of the post-test.

According to Figures 4.1 and 4.2, the hypothesis is supported as it was observed, namely that the algebraic scores of the learners in the control group did not change much from the pre-test to the post-test, whereas a large number of learners in the experimental group were able to improve their scores in algebra from the pre-test and the post-test. For example, in the pre-test only seven learners in the experimental group were able to score above 30% (elementary achievement, according to the DBE) against nine learners in the control group, whereas in the post-test 75 learners in the experimental group were able to score more than 30% against seven learners in the control group. This remarkable improvement by the learners in the experimental group highlights the effects of the problem-solving heuristic instructional method in the learners' achievements in algebra. The lack of improvement in the scores of the learners in the control group reinforces the hypothesis that the learner participation in the heuristic teaching method improves the learners' achievement of algebra. The observed difference between the pre-test and the post-test scores of the control and the experimental groups provides enough evidence that the learners' improvement in the post-test was due to the positive effects of the heuristic teaching method, and was not as a result of natural variation. To support this claim, further testing was carried out with inferential statistical tools namely the t-test, ANCOVA, HOS and the J-N technique to verify the hypothesis.

4.3.3 Findings from the inferential statistics

Inferential statistics, namely the t-test, ANCOVA, HOS and the J-N technique, were used to further substantiate the claims made in both the descriptive statistics and the graphical representation in sections 4.3.1 and 4.3.2 respectively.

4.3.3.1 Analysis of pre-test scores between control and experimental group

The pre-test for the control and experimental group was further analysed to establish whether the difference in their means were statistically significant. A null hypothesis (H_O) was formulated stating, there is no statistically significant difference between the pre-test scores in the algebra achievement test for the control and experimental group. A corresponding alternative hypothesis (H_A) stated that there is a statistically significant difference between the pre-test scores for learners in the control and experimental group.

H_o: There is no statistically significant difference between the pre-test scores for learners in the control and experimental group
H_A: There is a statistically significant difference between the pre-test scores for learners in the control and experimental group

At 0.05 level of significance level the null hypothesis was accepted because the p value was found to be greater than 0.05($p>0.05$) and the t-statistic (1.57) was found to be less than the t-critical (1.97) which implied the pre-test scores between the control and experimental group was not statistically significant (see Appendix F). This result confirmed that both the knowledge level in algebra for Grade 6 learners in the control group and Grade 6 learners in the experimental group was not statistically significant before the intervention was administered to Grade 6 learners in the experimental group.

4.3.3.2 Analysis of post-test scores between control and experimental group

The post-test scores of the control and experimental group were also analysed to determine whether the difference recorded after the problem-solving heuristic instructional method was administered to the experimental group was statistically significance. A null hypothesis (H_O) was formulated stating there is no significant difference between the post-test in the algebra achievement test for the control and experimental group. A corresponding alternative hypothesis (H_A) stated, there is a statistically significant difference between the post-test scores for learners in the control and experimental group.

H_0: There is no statistically significant difference between the post-test scores for learners in the control and experimental group
H_A: There is a statistically significant difference between the post-test scores for learners in the control and experimental group

At 0.05 significance level the null hypothesis was rejected because the p value was found to be less than 0.05 ($p<0.05$) and the t-statistic (14,98) was found to be greater than the t-critical (1,98) (see Appendix G) hence confirming the statistical significance of the difference in post-test scores between the control and experimental group after the experimental group was exposed to the problem-solving heuristic instructional method.

4.3.3.3 Analysis of pre-test and post-test scores for learners in control and experimental group

Since ANCOVA incorporates covariates, the researcher deemed it appropriate to use ANCOVA and the J-N technique to account for the statistical non-equivalence of the two groups resulting from their pre-knowledge and background in basic algebra. ANCOVA calculates the dependent variable means (adjusted mean) when the covariates are deemed to be equal. By doing so, ANCOVA does not take into account the differences in the covariate mean between the control group and the experimental group, and factors out group effects such as the groups' initial ability and the non-randomization of the sample. This made it possible for the researcher to assess the impact of the intervention with the experimental group using the dependent variable (post-test) group mean. A null and alternative hypothesis was formulated accordingly as follows:

Null hypothesis (H_0): There is no statistically significant improvement in the algebra test scores of the learners who participated in the teaching treatment.

Hypothesis (H_A): There is a statistically significant improvement in the algebra test scores of the learners who participated in the problem-solving heuristic instructional method.

At 95% confidence interval, the post-test scores were compared with the pre-test scores where the pre-test scores were the covariate. The F value obtained was then compared with the critical F value with 1 and 195 degrees of freedom; $F_{(0.05,\ 1,195)}$= 3.89. At 95% confidence interval the research rejected the null hypothesis since the F value (F= 298.85) was greater than the critical F value ($F_{(0.05,\ 1,195)}$= 3.88958864). Furthermore, the p-value (0.001) was far less than α (0.05) (see Appendix H). This implies that the heuristic treatment influenced the post-test scores of the experimental group and that the improved post-test scores obtained by the learners in the experimental group were statistically significant.

None of the assumptions made before the ANCOVA test were met. This included the HOS and the randomization of the sample due the nature of the study. The researcher performed a test for the HOS since it was a condition for using ANCOVA.

Figure 4.3 illustrates a scatter plot for the pre-test scores against the post-test scores for the control and the experimental groups. The control group is represented by the blue lines and the experimental group is represented by the green lines.

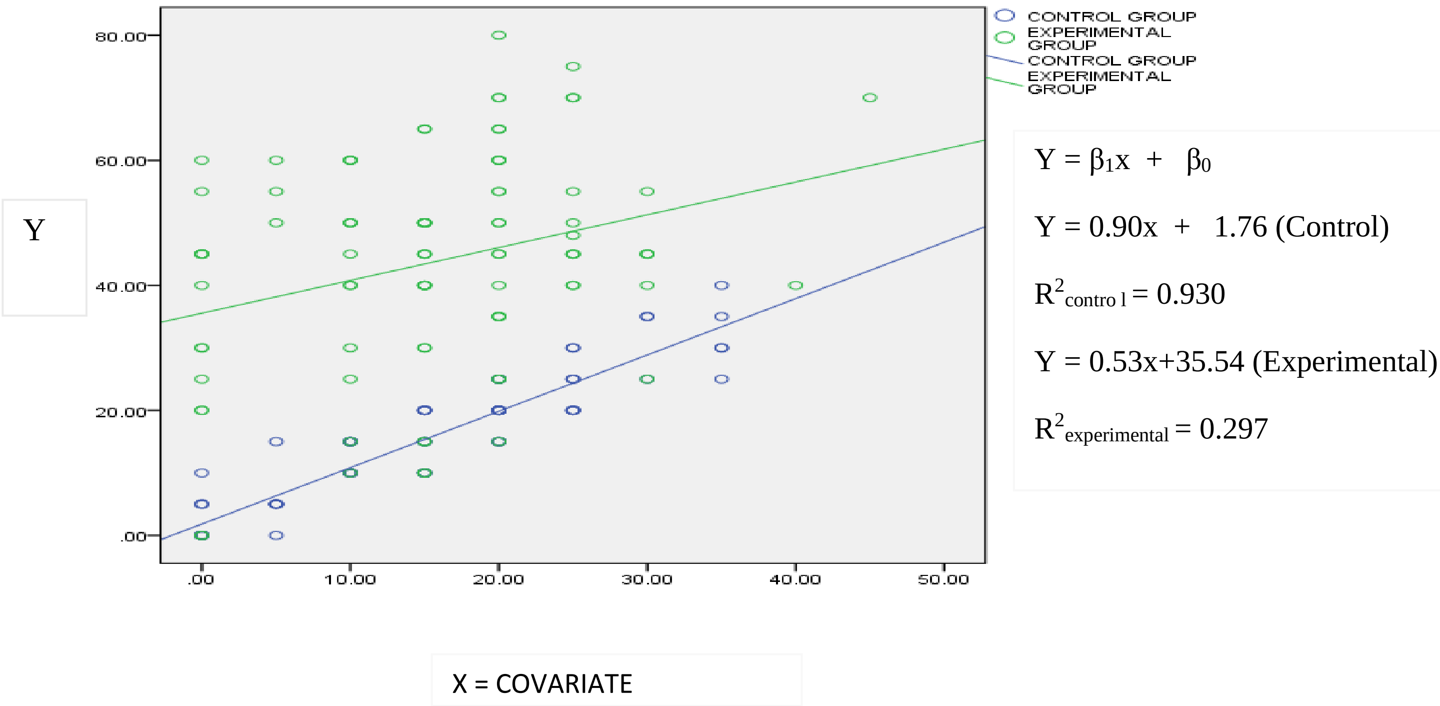


Figure 4.3: Scatter plot for pre-test against post-test and their corresponding regression rule for the experimental and control groups

Figure 4.3 shows the heterogeneous nature or convergence of the two regression lines of the respective control and experimental groups. At α=0.05(95% confidence interval) the p-value (0.026) was less than α=0.05 and the F value was 5.01 which was larger than the critical F value at 1 and 194 degrees of freedom ($F_{0.05,\ 1,194} = 3.88983904$) (see Appendix I). This provides sufficient evidence to reject the null hypothesis (no difference between the two regression slopes). This implies there was a significant difference between the regression slopes. There were many factors that could have contributed to the slopes not being homogenous. This included the fact that the four schools were not randomly selected. The J-N technique was used to correct this defect. The technique determined that the intervention had differential effects on the experimental group.

4.3.3.4 Findings from Johnson-Neyman technique

Testing for homogeneity of the regression slopes is a routine aspect of evaluating the adequacy of the ANCOVA model. The study discovered the heterogeneity of the regression slopes (see Appendix I) of the two groups. When there is heterogeneity between the regression slopes, it implies that the magnitude of the treatment effect was not the same at all levels of the covariate (pre-test) which meant an alternative to ANCOVA should be considered. The J-N technique was used to analyse the intervals of the pre-test where the treatment was effective and the intervals where the treatment was not effective (see Table 4.6).

Table 4.6: Summary of Johnson-Neyman technique

Summary of statistics	**Control**	**Experimental**
Sample size	106	98
Sample mean	13.54	15.60
Sum of squares	29875	30625
Intercept	1.76	35.54
Slope	0.90	0.53
$F_{0.05,1,194}$	3.89	3.89
Within residual (SS_{RES})	**25458.48**	
A	0.1031	
B	-12.99	
C	**1123.50**	
X_{L1}	55.44	**Bounded above by 100**
X_{L2}	196.54	

Where X_{L1} and X_{L2} are limits of the interval where the problem-solving heuristic instructional method was not effective

The results in Table 4.6 indicate that the intervention was effective for the learners in the experimental group who obtained marks of 55.44 (X<55.44) or less in the pre-test and inconclusive for learners that obtain pre-test marks between 55.44 and 100 percent.

4.3.3.5 Calculating the effect size

Table 4.7 shows the calculation of the effective size to quantify the difference between the control and experimental groups.

Table 4.7: Effect size of the intervention

	Sample size	**Mean**	**Standard deviation**
Experimental Group	92	43.73%	16.80
Control Group	106	13.92%	9.36
Cohen's d = $\frac{\bar{x}_E - \bar{x}_c}{SD_{Pooled}}$		**SD** **pooled** = $\sqrt{}\ \frac{(N_E - 1)SD_E^2 + (N_c - 1)SD_C^2}{N_E + N_C - 2}$	
Cohen's *d* = (13.92 - 43.73) ⁄ 13.598706 = 2.19			

The high value of Cohen's *d* confirmed the difference between the post-tests' means of the experimental and control groups and that this difference was not only statistically significant but was also educationally significant.

4.3.3.6 Concluding remarks on inferential statistics

The t-test, ANCOVA, the J-N technique and the effect size were used to check whether the changes observed in the experimental group were due to natural variation or to the learners participating in the heuristic teaching method and to check whether the improved scores were statistically significant. The results obtained from the t-test, ANCOVA, J-N technique and the effect size indicate that most of the changes that occurred in the experimental group could be accounted for by the intervention, which supported rejection of the null hypothesis (see section 3.6.1.1)

At a 95% confidence interval (0.05 significance level), which is deemed to be an acceptable level of significance, the null hypothesis (H_0) of equal means in the post-test scores between the experimental and the control groups after the intervention could be rejected.

This suggests that the improved algebraic scores the learners in the experimental group obtained are statistically significant and the hypothesis of the improved algebraic scores through participation in the problem-solving heuristic instructional method is confirmed. The large value of Cohen's d indicated a large degree of effectiveness of the problem-solving heuristic instructional method.

4.3.4 Concluding remarks on learners' achievements in algebra after participating in the problem-solving heuristic instructional method

The findings from the descriptive and inferential statistics were in line with each other and addressed the main research questions of this study. The findings from the descriptive statistics show that the learners in the experimental group improved their scores in the algebra test after they had participated in the heuristic teaching method whereas the learners in the control group who received traditional classroom teaching showed little or no improvement in the algebra test. This was further corroborated by the inferential statistics. Findings from the t-test showed that there was no statistically significant difference in the pre-test scores between learners in the control and the experimental group before the problem-solving heuristic instructional method was administered with learners in the experimental group, indicating the knowledge level of algebra for the two groups was comparable. The t-test also confirmed a statistically significant difference in the post-test scores between the learners in the control and experimental group after the problem-solving heuristic instructional method was administered to learners in the experimental group. Further findings from ANCOVA support the initial hypothesis of the positive effects of the intervention on the experimental group learners' post-test scores. ANCOVA also confirmed that the post-test means between the experimental and control groups were statistically significantly different.

Due to the heterogeneous nature of the regression lines of the control and the experimental groups, the J-N technique was used to investigate the intervals in the marks from 0 to 100 where the intervention was effective.

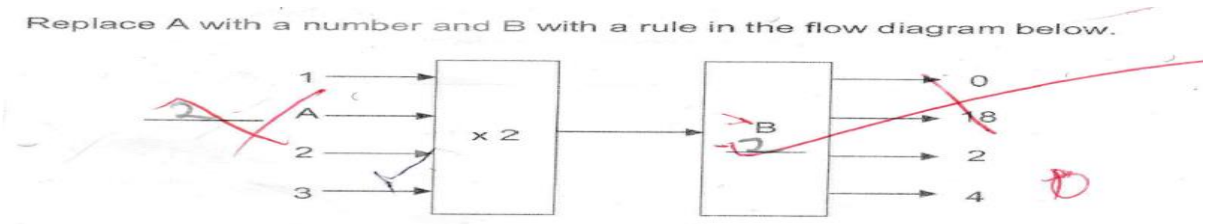
It revealed that all the learners in the experimental group who scored less than 55.44% in the pre-test improved their post-test scores after participation in the intervention. This pointed to the effectiveness of the teaching treatment for low-achieving learners. It is concluded from the study that the quantitative findings support the initial hypothesis of the heuristic teaching method causing the effect of the improved algebraic scores of the learners in the experimental group. The calculated effect size confirms the strength of the difference in post-test means between the control and the experimental group.

4.4 ANALYSIS OF SAMPLED LEARNERS PRE-TEST AND POST-TEST ANSWERS

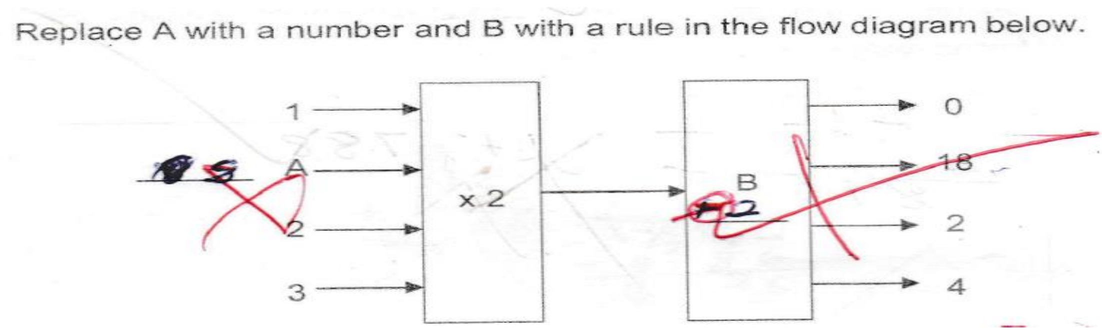
To further corroborate the findings obtained in the descriptive and inferential statistics, pre-test and post-test scripts of two learners each in the control and experimental group were sampled using simple random sampling technique and analysed to further explain the effects of the problem-solving heuristic instructional method on learners' achievement in algebra. Written questions, namely questions 12 and 16, were sampled from the achievement test and analysed (see Appendix B). A simple random technique was used to sample the learners and the questions that were selected. The learners whose scripts were sampled from the control group were identified as CL1 and CL2 and the learners whose scripts were sampled from the experimental group were identified as EL1 and EL2.

4.4.1 Analysis of samples of pre-test written work of control group

Vignettes 4.18 and 4.19 show sample answers for question 12 given by learners in the control group.

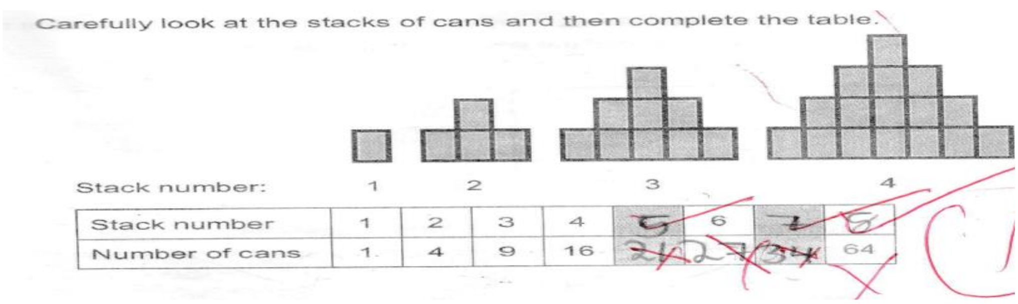


Vignette 4.18: Learner CL1's pre-test answer for Question 12

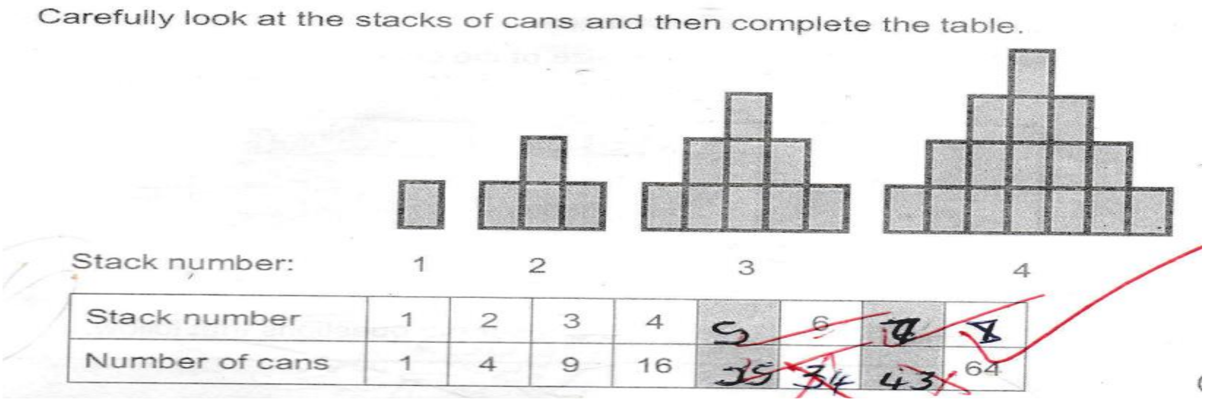


Vignette 4.19: Learner CL2's pre-test answer for Question 12

According to the answers in the vignettes, the two control group learners recognised that there must be the number 2 to replace the letter B. They did not prefix the 2 with an arithmetic operation or used the wrong arithmetic operation which signifying their lack of understanding of the algebraic rule. This translated to learners not being able to identify the rule that can map output values to an input value given in a table, which leads to a wrong answer for letter A. Vignettes 4.20 and 4.21 present a sample of the answers learner CL1 and learner CL2 gave for question 16 in the pre-test.



Vignette 4.20: Learner CL1's pre-test answer for question 16

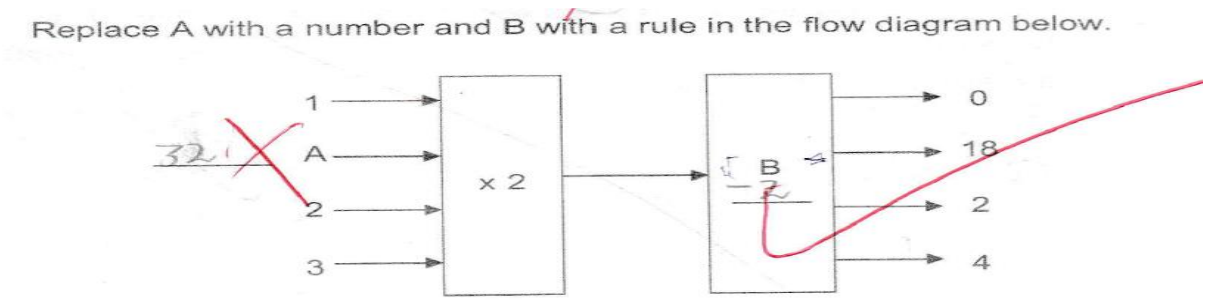


Vignette 4.21: Learner CL2's pre-test answer for question 16

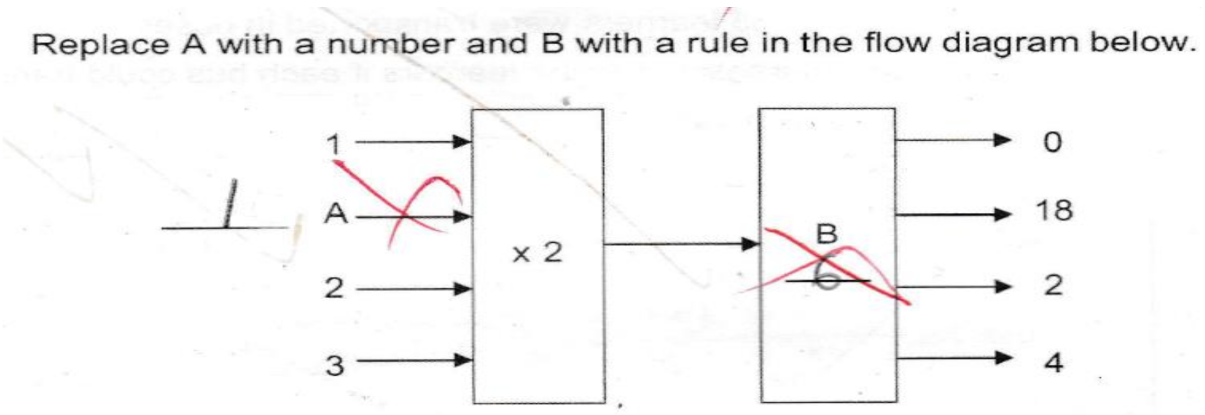
From the answers given by learners in Vignettes 4.20 and 4.21, both learners could recognize the consecutive numbers but were not able to identify the rule that maps input values to output values. This also signifies their lack understanding of an algebraic expression as a rule that maps input elements to output elements given in a table. The next section presents sampled answers given by experimental group learners in the pre-test.

4.4.2 Analysis of samples of pre-test written work of experimental group learners before the intervention

Vignettes 4.22 and 4.23 present samples of answers by learners EL1 and EL2 to question 12 in the pre-test.

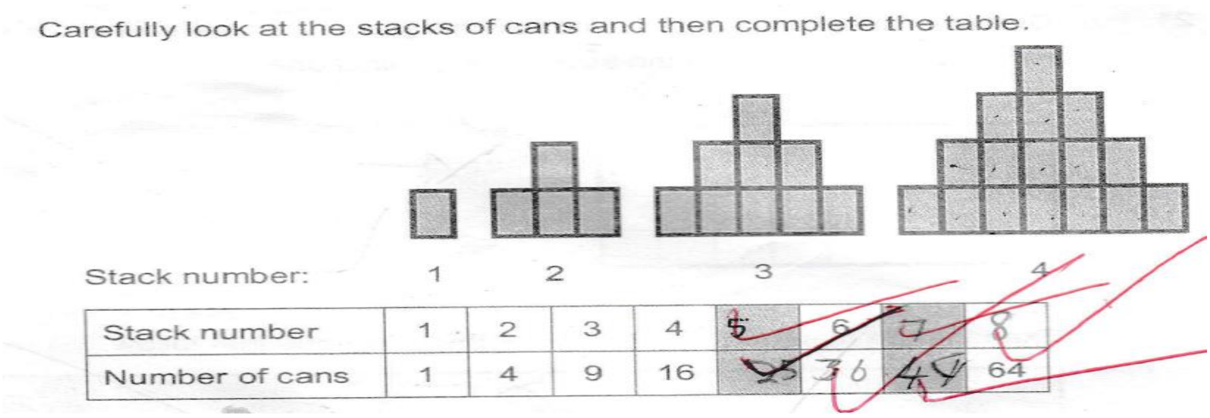


Vignette 4.22: Learner EL1's pre-test answer for question 12

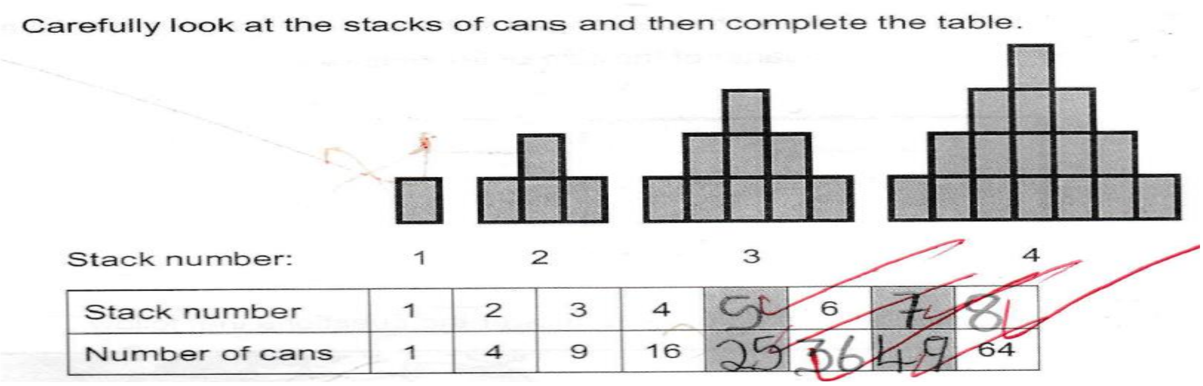


Vignette 4.23: Learner EL2's pre-test answer for question 12

From the vignettes, learner EL1 could identify the algebraic rule as an expression that maps input values to output values which signifies his understanding of an algebraic rule as an expression that maps inputs values to get output values. However, the learner could not use the inverse function of the rule to map output values to get input values which signifies his lack of understanding of the inverse property of an algebraic rule. Learner EL2 identified an incorrect rule that maps input values to output values. It was not so clear how her rule was developed but gave evidence of a misconception of an algebraic rule as an expression that translates input values to get output values and vice versa. This led to the learner's inability to find the correct answer to the input value A. Vignettes 4.24 and 4.25 present answers to question 16 that learners EL1 and EL2 gave in the pre-test.



Vignette 4.24: Learner EL1's pre-test answer for question 16


Carefully look at the stacks of cans and then complete the table.

Stack number: 1 2 3 4

Stack number	1	2	3	4	5	6	7	8
Number of cans	1	4	9	16	25	36	49	64

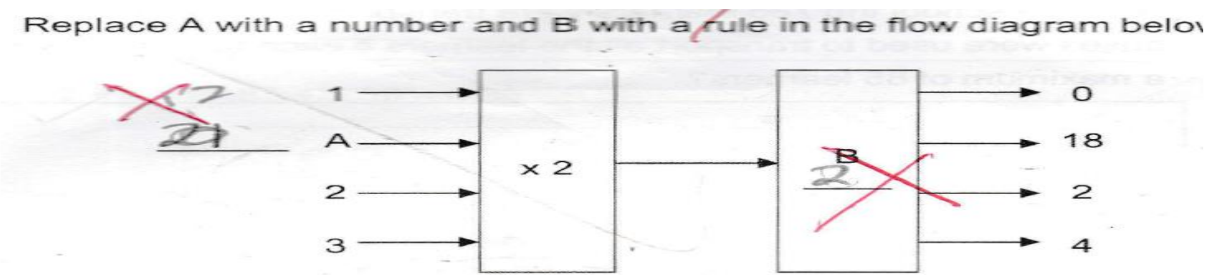
Vignette 4.25: Learner EL2's pre-test answer for question 16

In Vignettes 4.24 and 4.25, the two learners in the experimental group identified the consecutive nature of the input values correctly and could clearly formulate a rule that maps input values to output values signifying their understanding of an algebraic rule as an expression that maps input values to get output values.

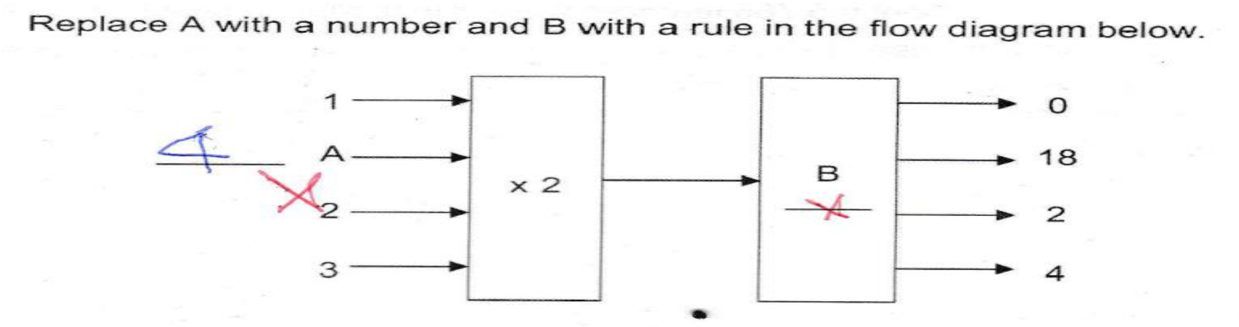
4.4.3 Analysis of samples of post-test written work of control group learners after being taught algebra using the traditional approach

Sampled post-test scripts of learners in the control group were analysed to assess whether some knowledge was gained after learners had been taught using the traditional teaching method only.

Vignettes 4.26 and 4.27 present samples of control group learners' answers by learner CL1 and learner CL2 respectively.



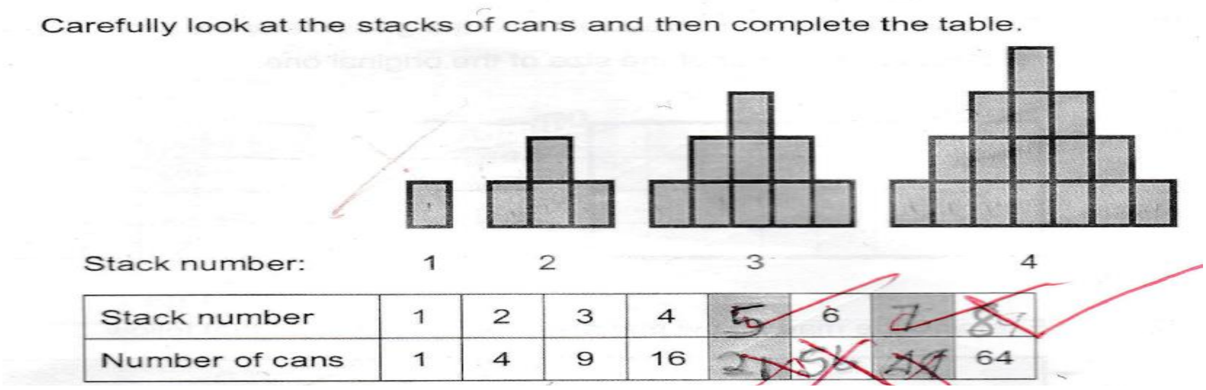
Vignette 4.26: Learner CL1's post -test answer for question 12



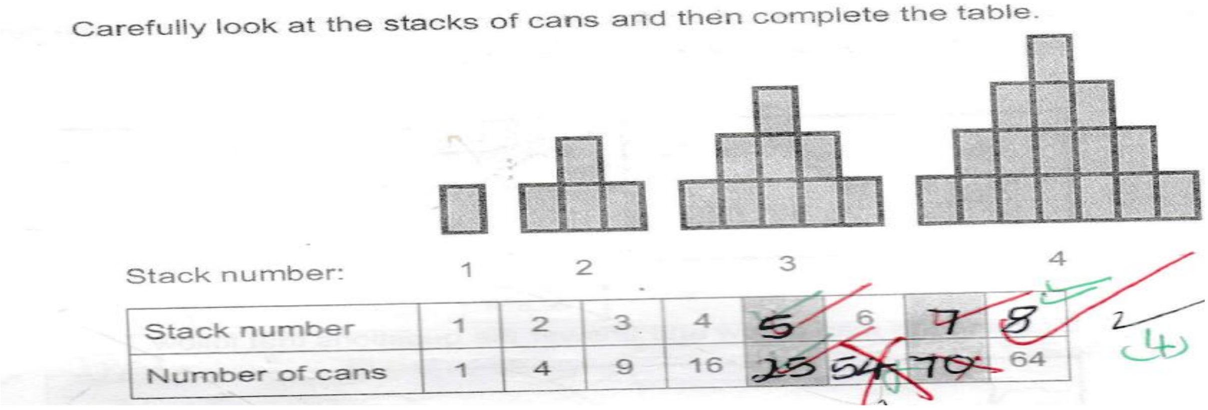
Vignette 4.27: Learner CL2's post-test answer for question 12

Answers provided by learners in Vignettes 4.26 and 4.27 were incorrect answers which differed from the incorrect answers the learners provided in the pre-test. This suggests that the learners had not still grasped the concept of the use of an algebraic rule after being taught using the traditional teaching method. From the vignettes, the learners did not have a clear conceptual understanding on how to map input elements into output elements and vice versa.

Vignettes 4.28 and 4.29 present the answers of control group learners CL1 and CL2 to question 16.



Vignette 4.28: Learner CL1's post-test answer for question 16

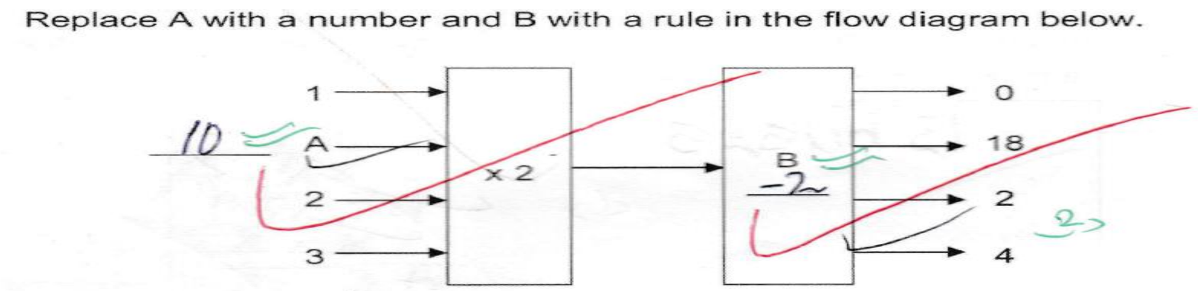


Vignette 4.29: Learner CL2's post-test answer for question 16

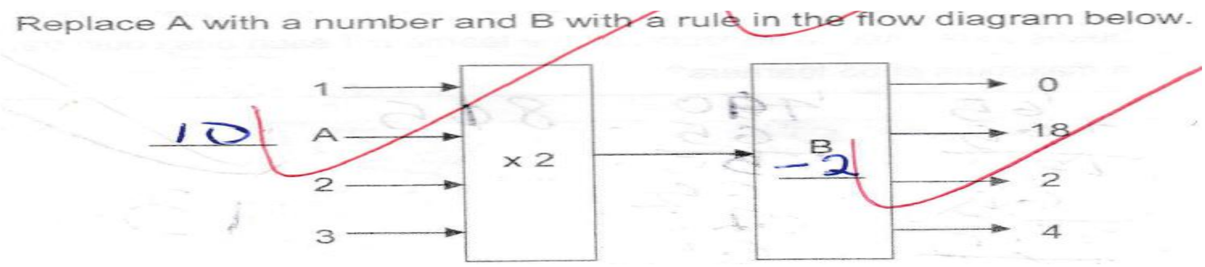
Learners CL1 and CL2 gave the same correct and incorrect answers in the post-test as the answers they had given in the pre-test. This suggests that the knowledge they used to find the correct answers was maintained after being taught through traditional methods but no new knowledge had been added to the knowledge already acquired.

4.4.4 Analysis of samples of post-test written work of the experimental group learners after being taught using the problem-solving heuristic instructional method

Vignettes 4.30 and 4.31 present answers given in the post-test for question 12.

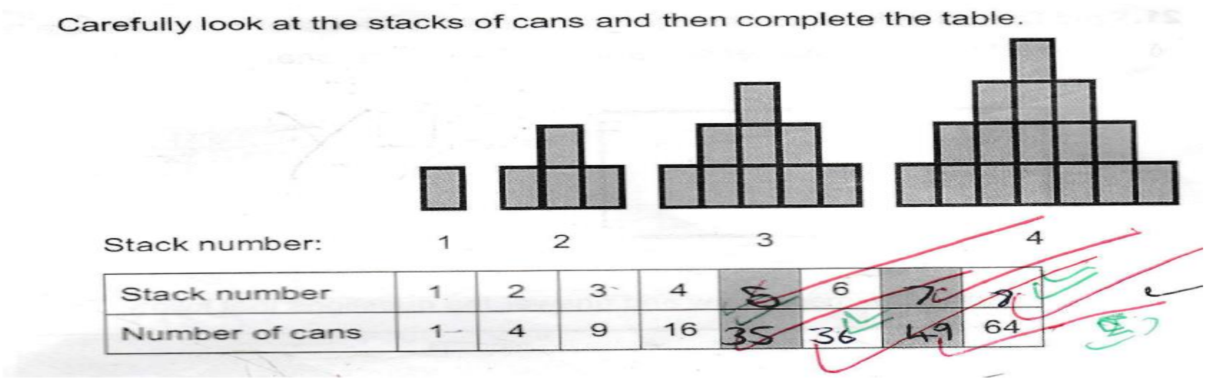


Vignette 4.30: Learner EL1's post-test answer for question 12

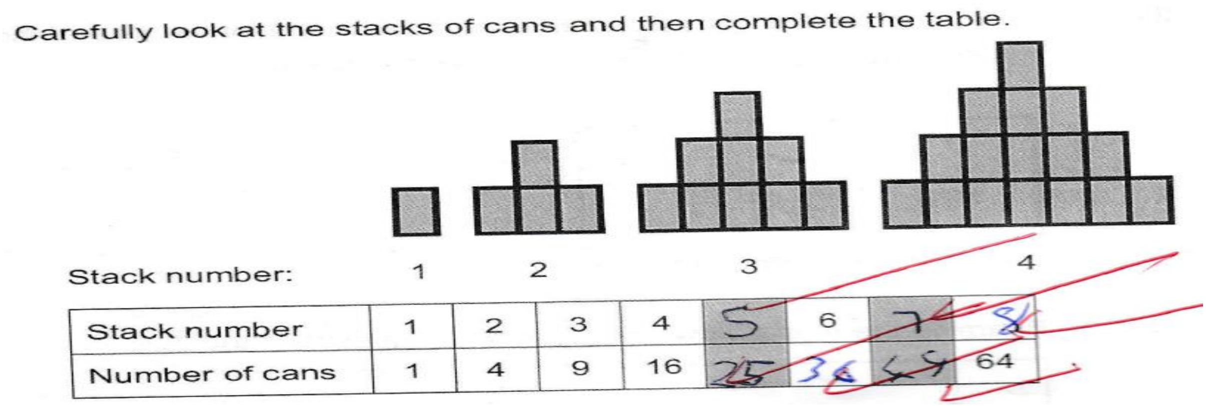


Vignette 4.31: Learner EL2's post-test answer for question 12

Answers from Vignettes 4.30 and 4.31 showed that learners had improved in their answers as compared to the answers learners produced in Vignettes 4.22 and 4.23 after participation in the problem-solving heuristic instructional method. The learners were able to conceptualize an algebraic rule correctly as an expression that maps an input element into an output element and vice versa. These improved answers further validate the positive effects of the problem-solving heuristic instructional method.



Vignette 4.32: Learner EL1's post-test answer for question 16



Vignette 4.33: Learner EL2's post-test answer for question 16

Learners' answers from the above two vignettes indicate they gave the same answers in the post-test as in the pre-test, presented as Vignettes 4.25 and 4.26. This confirmed that learners who participated in the problem-solving heuristic instructional method maintained or gained more knowledge in algebra after participating in the problem-solving heuristic instructional method.

4.4.5 Concluding remarks from analysed pre-test and post-test scripts

Analysis from learners' scripts also supported the initial hypothesis of learners' improved scores after participation in the problem-solving heuristic instructional method. The analysis of sampled scripts of the experimental group learners revealed they gained knowledge after participating in the teaching treatment compared to the control group learners that maintained their knowledge level in the pre-test after being taught using the traditional approach.

4.5 CONCLUDING REMARKS ON FINDINGS OF THE STUDY

Chapter four has presented the results of the empirical investigation on the impact of the problem-solving heuristic instructional method that combines the effects of the modelling and modelling perspective and the APOS theory to develop learners' conceptual understanding of algebra. Firstly, the study established the type of teaching method adopted by all the four schools used in this study compared to the problem solving heuristic instructional method and highlighting its similarities and differences. Although there were a few slight differences, most of the factors examined that influence effective teaching and learning were similar. This included educator-centred teaching, lack of collaborative learning, lack of incorporation of real-life problems into the teaching process, and educators not formulating a clear strategy on how they intended to develop and teach the mathematical concept under consideration for learners. The study then concluded that the quality of teaching and learning in these four schools was comparable. In this way, the true effects of the problem-solving heuristic instructional method could be measured.

Secondly, the study described how the problem-solving heuristic instruction was implemented by highlighting the impact of the two theoretical frameworks that guided and informed the study. Thirdly, the study compared and measured the effects of the traditional teaching method and problem-solving heuristics instructional method on learners' achievements in algebra in the respective groups to check whether there was any statistically significant difference in learners' achievement in algebra after being taught algebra using the respective teaching methods.

Lastly, the statistical findings were validated with answers from learners' scripts to verify how the participation in the respective teaching methods impacted on answers' that learners produced in the post-test achievement test.

The next chapter discusses the summary, the findings in the light of the research questions, the conclusion, the recommendations and limitations of the study

CHAPTER FIVE
SUMMARY OF THE STUDY, DISCUSSION, CONCLUSIONS AND RECOMMENDATIONS

Chapter five presents the summary of the study; discussion of the results; conclusions drawn; recommendations made and limitations of the study.

5.1 SUMMARY OF THE STUDY

At the beginning of the study the researcher indicated the need for a different method of teaching mathematics to learners in the intermediate phase of schools in South Africa. This was informed by the poor performance in mathematics as was corroborated by the deplorable numeracy results obtained by the learners, in the 2012–2014 ANA examinations. Against this background, it was considered that ways and means should be sought to improve learner performance in mathematics, in view of the fact that the traditional teaching methods being used by educators seem to be ineffective.

Literature on this topic supported the use of an instruction that focused on the learners' conceptual understanding of mathematics. Literature also supported the use of the problem-solving instruction approach as a form of teaching with the potential to develop conceptual understanding. This study draws on the modelling and modelling perspective and the APOS theory where the former guided the design of appropriate MEAs, whereas the APOS theory was used to develop particular mental constructions the learners should go though in order to achieve conceptions in algebra through genetic decomposition.

Algebra was identified in the literature as a suitable topic to explore this teaching method because of the impact the conceptual understanding of algebra may have on the overall learning of mathematics.

The empirical investigation was done in four Grade 6 classes in four primary schools. Two of the schools represented the control group and the other two the experimental group. The four schools were conveniently sampled from the same district based on their quintile levels.

A mixed-method research approach was used to determine how the problem-solving heuristic instructional method was used in the teaching of Grade 6 algebra, and whether the heuristic teaching had any effect on learners' achievements in algebra. Classroom observations were made by the researcher and the HOD in each of the four schools while a non-equivalent control group quasi-experimental design was used to contrast the use of the problem-solving heuristic instructional method with the traditional teaching method. The results show that the learners in the control group could not improve on their pre-test scores above their counterparts in the experimental group. The difference in post-test scores of the two groups was found to be statistically significant. Analyses of sample answers of learners' scripts in the control and experimental group corroborated the findings from the non-equivalent control group design. Learners in the experimental group who participated in the problem-solving heuristic instructional method were found to improve on their answers compared to learners who were taught using the traditional teaching method.

5.2 DISCUSSION OF THE RESULTS IN TERMS OF THE RESEARCH QUESTIONS

The section discusses the results of the study by focusing on the effects of the problem-solving heuristic instructional method in learners' achievement in algebra and the development and use of a problem-solving heuristic instructional method in teaching algebra to Grade 6 learners.

5.2.1 How can a problem-solving heuristic instructional method be developed and used in the teaching of algebra to Grade 6 learners?

Based on the data gathered during the design and implementation of the problem-solving heuristic instructional method in section 4.2, the study proposes pathways on how this teaching method could be developed and used in mainstream classroom practice. This teaching method can be developed on the basis of the APOS theory and the modelling and modelling perspective (see an example of the problem-solving heuristic teaching model in section 4.2.1). The six principles of the modelling and modelling perspective must guide the design of the MEAs and must solely be based on learners' daily routine activities and learners' interests. Most importantly, mathematical content that needs to be learnt in class by learners must be embedded in these MEAs, which learners must elicit through engagement in trying to find the solution to the particular MEA (see Appendix A). Learners are likely to participate in class activities if they are at ease with the medium in which the activities are being presented. The content of these activities must be used as the main vehicle in the teaching process to drive learners' development of algebra (see teaching episodes in section 4.2.2.1). Exploring learners' pre-existing knowledge with an MEA to develop their algebraic reasoning improves efficiency in learning as they are able to continuously reflect on a context that remains unchanged, a view supported as follows: "Recurring features of the environment may afford recurring sequences of learning actions" (Brown, Collins & Duguid, 1989, p. 37). Through this, learners can create their own understanding that can be validated through social negotiations in the form of debates and discussions with their peers and their educator (see section 4.2.2.1).

Learners' retention of knowledge is not a context-independent process. The constructivists explain that learning and information transfer can be facilitated by involving a learner in an authentic task through a meaningful context. In other words, transfer of mathematical knowledge is unlikely to take place if learning a particular mathematical concept is de-contextualized (Ertmer & Newby, 1993). Understanding a particular mathematical concept depends on the experience; the authenticity of the experience is critical to the learners' ability to use and develop the particular mathematical concept (Brown et al., 1989). Through MEAs learners can create their own understanding and meanings in algebra from their basic experience of the problem context in the MEA. Learners can build knowledge based on their own interpretation and experience with an MEA. Learners' mathematical knowledge can emerge from a real-life context which is relevant to the learner (Ertmer & Newby, 1993). Brown et al. (1989) explain that learning through authentic real-life activities co-produce knowledge along with cognition.

Learners should be arranged in groups when they engage in MEAs because group discussion of an MEA provides an effective platform for cooperative learning, which requires learners to develop and share powerful mathematical ideas to be able to solve a problem at hand. It also provides a platform for learners to document their own thinking and learning in their development in algebraic concepts (see teaching and learning episodes in sections 4.2.2.1, 4.2.2.2 and 4.2.2.3). Meaningful learning through an MEA with higher cognitive demands is likely to occur when learners learn through cooperative learning in groups rather than in an individual learning context (Kirschner, Paas & Kirschner, 2008, 2009a, 2009b).

MEAs cognitively engage learners and set the scene for learners to begin developing sound conceptions in algebra through the APOS theory framework comprised of the action stage, process stage and object stage, which are coordinated to form a schema. Transfer of knowledge is effective if algebraic concepts are developed together with learners' problem-solving skills using the MEA. This can be done by linking abstract algebraic concepts to the experiences learners bring to an MEA.

The APOS framework aids the educator to hypothesize what actions learners should take in order to develop a conceptual understanding in algebra, which can be outlined through the preliminary genetic decomposition (see section 2.3.2.1 and section 3.5.3.1).

The action stage enabled learners to be able to identify relationships between various elements in the MEAs through their basic understanding of the problem situation in the MEA (see section 4.2.2.1). Learners could then use these relationships to develop an algebraic rule in line with the goals of the MEA. As learners performed these actions they gradually interiorised them into mental processes and began to conceptualize an algebraic rule as an expression that transforms one set of elements in a problem situation into another (see section 4.2.2.2).

As learners reflect on these mental processes, they are able to encapsulate these process conceptions into a mental object where they begin to conceptualize algebraic rules as objects that can themselves be transformed and manipulated from one form to another (see section 4.2.2.3). As learners iterate action, process and object conceptions over a period, they then begin to coordinate the action, process and object conceptions into a schema and begin to have a general conceptual understanding of the concept of algebra. Teaching through MEAs gives the educator the tools to understand how the learners are thinking about a particular mathematical procedure. This can guide the educator to manage a multiplicity of ideas that can be used to support multiple developments of the learners' ideas. The APOS theory is a viable learning framework that can guide the educator through the process of developing the learners' conceptual understanding in algebra as it occurs though the learners' mathematizing an MEA. Educators should place emphasis on the context of the MEA in which learners are capable of manipulating the information it presents and in which most algebraic skills can be developed by learners and applied to other unfamiliar problems.

The APOS framework can guide educators to develop learners' problem-solving skills and aid learners to mathematize a problem in an MEA that enables them to apply algebraic concepts beyond the information given in the problem, to unfamiliar questions.Hypothesizing mental structures and mechanisms that the learners may make in developing conceptions in algebra through the preliminary genetic decomposition before it is presented to them, serves a dual purpose: firstly, it enables the educator to monitor the learners' progress in the development of algebraic concepts; and secondly, it gives the educator the opportunity to document and correct the learners' misconceptions when learning algebra. Improvements in the post-test scores of the experimental group's learners appear to support the preliminary genetic decomposition developed for this study as reasonable mental constructions that can be used to develop learners' conceptual understanding in algebra.

5.2.2 What is the impact of a problem-solving heuristic instructional method on learners' achievements in algebra at the level of Grade 6?

Pre-intervention class observation indicated that most of the four schools used for this study employed similar traditional methods of instruction which varied from the problem solving heuristic instructional method in a number of aspects such as cooperative learning, integrating authentic real life problems in into the learning process and developing mental structures learners need to go through in order to learn a mathematical concept (See sections 4.1 and 4.2). This placed all four schools at a comparable level with regards to teaching and learning resources, this made measuring the effects on the learners' who were taught algebra through the problem-solving heuristics instructional method clearer.

A comparison of pre-test scores between learners in the experimental group and the control group was made and analysed before the intervention to check whether there were any statistically significant results in learners' achievements in algebra between the control and the experimental group (see section 4.3). The descriptive statistics revealed pre-test means between the control and the experimental group were comparable, with the learners in the experimental group having a slightly higher mean score, namely a difference of 2.06 marks. Further analysis with the t-test revealed that the difference in the pre-test mean scores between control and experimental group was not statistically significant. This seemed to be in line with the findings of the pre-intervention class observation that found that the quality of teaching and learning by the mathematics educators in all four schools was comparable.

A post-test was administered to both the control and the experimental group after the experimental group had been taught algebra using the problem-solving heuristics instructional method while the control group had continued learning through the traditional methods of teaching. The post-test revealed an improvement in the mean scores for learners in the experimental group compared to the control group with a difference of 29.81%. A t-test analysis confirmed a statistically significant result for the difference between post-test mean scores for the control group learners and the experimental group learners.

The descriptive analysis between the pre-test and post-test mean scores of the experimental group indicated that the post-test mean scores improved by 28.13% compared to the pre-test mean scores. ANCOVA analysis confirmed that the difference between the pre-test and post-test mean scores of learners in the experimental group was statistically significant.

ANCOVA requires that there should be no difference in the regression slopes between the control group and the experimental group. The test for HOS revealed a heterogeneous relationship between the regression slopes of the control and experimental group, thereby indicating that the effects of the intervention were not equal for all levels of the pre-test. Further analysis with the J-N technique revealed that learners in the experimental group who obtained 55.44 or less in the pre-test improved their scores after being taught algebra through the problem-solving heuristics instructional method.

The effect size was also calculated and the value obtained confirmed the statistical and educational significance of the difference in post-test scores between the control group and the experimental group.

Analysis of sampled answers from the pre-test and post-test scripts of learners in both the control and experimental groups indicated that learners in the experimental group were able to improve on their knowledge level based on their post-test answers compared to their pre-test answers whereas learners in the control group could not record any improvement in their post-test answers compared to their pre- test answers (see section 4.4). This was in line with findings from the descriptive and inferential statistics.

These improvements suggest that there are positive effects from the problem-solving heuristic instructional method on the experimental group of learners who were taught algebra using this teaching method. The problem-solving heuristic instructional method incorporated learner-centred learning approaches where the learners worked collaboratively in groups in a meaningful medium with minimal instructions from the educator. Data gathered from scripts of the experimental group of learners indicated that the learners were able to accommodate the algebraic conceptions during the teaching treatment, and assimilated these conceptions into new, unfamiliar algebraic questions. The teaching treatment grounded the learners' conceptual understanding in algebra and gave them the capability to deal with unfamiliar algebraic questions. A learner "who demonstrates a deep understanding of a concept is capable of dealing with unfamiliar and even new situations using the concept or concepts in question" (Arnon et al., 2013, p. 181). This is in sharp contrast to the traditional teaching methods as observed, where the lessons were educator-centred and learners learnt abstract mathematical concepts with no evidence of incorporation of group learning or the integration of meaningful authentic real-life problems. The educators in the traditional classroom did not demonstrate a clear plan on how they intended to impart mathematical concepts to the learners. This seems to suggest that the lack of these factors contributed to the inability of learners in the control group to improve on their post-test scores.

5.3 CONCLUSIONS

In this study, there was evidence of improvements in achievement in algebra with Grade 6 learners from quintile one schools after being taught algebra through the problem-solving heuristic instructional method. Schools from quintile one were used because they were found to have the lowest-performing learners in the ANA examinations. The ANA examination is the only examination used to diagnose Grade

6 learners' performance in mathematics nationally. Schools from the same quintile usually have comparable teaching and learning resources which translates to a comparable quality of teaching and learning. To further test this theory, a pre-intervention class observation schedule was conducted by the researcher. The researcher also used previous data documented by mandatory HOD class observation to collaborate findings from his data which also enriched and filled the gaps in some aspects of the researcher's data that he might have missed during his own observation.

Based on this data, the study concluded that educators in the four schools used for this study employed comparable teaching strategies in the teaching and learning of mathematics. Highlights of these approaches included: educator - centred teaching; learners inactive in the learning process; learners working individually with no evidence of group work; no integration of authentic real-life problems into the teaching method; and educators' lesson plans lacked a comprehensive plan of the activities to be undertaken by learners in the classroom for educators to impart these concepts to learners. Hence the study was able to measure the true effects of the problem-solving heuristic instructional method, as observed in earlier research: "In an experiment, every effort is made to control for confounding of extraneous variables in order to be more confident of the cause-effect relationship" (Tierney, 2008, p. 2).

The design of the problem-solving heuristics instructional method that combined the effects of the modelling and modelling perspective and the APOS theory activated learners' independent learning in groups, with minimal guidance by the researcher. The modelling and modelling perspective enabled the study to design activities that accommodate group work and gave learners in the group an opportunity to discuss real-life components of the MEAs familiar to their environment collaboratively. Collaborative learning in small groups through a familiar problem-solving environment proved to be very effective in developing the learners' algebraic thinking. The learners could share both their conceptions and misconceptions and assist one another in the common goal of finding a solution to the stated problems.

As a starting point, educators could improve their teaching of algebra and mathematics in general by using innovative authentic real-life problems that learners experience daily. Most importantly, these activities should be designed in line with the mathematical concept the educator wants to impart to learners. The educator can then use guiding questions aimed at fostering learners' independent mathematical thinking and creation of mathematical knowledge to help them engage in the MEA. In the process, this would develop learners' conceptual understanding of a particular mathematical concept using the APOS theory as a framework that hypothesizes particular mental constructions learners should undergo in order to develop a conception of a particular mathematical concept.

This innovative teaching method is in sharp contrast to the traditional teaching methods educators in the participating schools used where mathematics has been taught without relating it to learners' real-life experience and where no clear framework has been adopted to develop learners' conceptual understanding of the subject. Results from the quantitative findings supported those of the qualitative findings as learners in experimental group could improve their scores in algebra in contrast to learners in the control group. In the light of these findings, this study has made a contribution and thrown more light on how instruction through problem-solving can be developed and used in standard classroom practice to improve the effectiveness of everyday mathematics teaching in Grade 6.

5.4 RECOMMENDATIONS

5.4.1 Recommendations for the improved teaching and learning of mathematics in primary schools

Based on the findings of the study the following points may improve the teaching of mathematics in the primary school:

i. Educators should take into consideration the physical objects present in the classroom, or the physical objects that can be perceived by the learners, and embed them in the mathematics concepts they present to learners when designing their instruction. Starting a lesson with the help of physical objects brings intuitive knowledge into the learning process on which educators can capitalize in order to embed the mathematical concepts to be studied.

ii. Concrete objects are not always available in the classroom. One way of introducing learners to physical objects which they can easily perceive is by developing an MEA based on the modelling and modelling perspective.

iii. Educators should hypothesize the particular mental structures and mechanisms learners need to engage in to learn a specific mathematical concept in the form of a genetic decomposition before it is presented to them. This will enable educators to monitor the learners' progress and misconceptions in respect of the mathematical concept being studied.

iv. Research into the development of a genetic decomposition of mathematical concepts in primary schools is still in its infancy. A few genetic decompositions can be found in the literature (Arnon, 1998; Arnon et al., 1999; 2001). Although non-existent in Grade 6 mathematics textbooks, educators should take their cue from the genetic decompositions developed in this study and from those by Arnon (1998) and Arnon et al. (1999; 2001).

v. Another practical way in which primary school educators can incorporate the APOS theory into their everyday teaching is by exploring the strategies used to explain mathematical concepts in a textbook through the lens of the APOS theory. Educators could use them to predict the learners' mathematical constructions on a topic, and assess its influence on the learners' development (Arnon et al., 2013, p. 103).

5.4.2 Recommendations for further research

Many studies have reported on the positive effects of incorporating mathematical modelling into teaching and learning of mathematics in primary schools, among them are Chan (2008), English & Watters (2005), Mousoulides et al. (2008) and Seto, Thomas, Ng, Chan and Widjaja (2012) to mention a few. Chan (2008) explained that incorporation of the mathematical modelling into mathematics lessons serves as a catalyst to promote reasonable and meaningful learning. English and Watters (2005) in their work concluded that the integration mathematical modelling into the teaching and learning of mathematics in primary schools enables learners to encounter important mathematical ideas and processes that they would naturally not encounter through traditional instruction. Mousoulides et al. (2008) emphasized the need to teach mathematical modelling in schools as it has the capacity to develop diverse mathematical strategies and thinking in learners. Seto et al. (2012) in their study also concluded that, the introduction of MEAs into teaching and learning of mathematics in primary schools improves the quality of small group mathematical discussion during a mathematics lesson.

The APOS theory has been used successfully as a developmental and analytical tool in developing a teaching sequence in post-secondary mathematics in areas such as limits of functions (Cottrill et al., 1996; Cottrill, 1999; Dubinsky, 2000; ÇETĐN, 2009); parametric functions (Stalvey, 2014); linear algebra (Possani et al., 2010); and the chain-rule project (Clark, Cordero, Cottrill, Czarnocha, De Vries, John & Vidakovic, 1997), among others. In the South African context, studies have been reported of the use of the APOS theory to explore undergraduate learners' understanding of the derivatives of functions (Maharaj, 2013); continuity of functions (Brijlall& Maharaj, 2013); pre-service educators' mental construction when solving problems involving infinite sets (Brijlall, & Maharaj, 2015); pre-service educators' mental constructions of concepts in matrix algebra (Ndlovu & Brijlall, 2015), to mention but a few. Studies have also been reported on high school learners' understanding of mathematical concepts, for example exploration of learners' mental construction when solving optimization problems (Brijlall & Ndlovu, 2013).

As mentioned earlier in section 2.3.3, the only studies identified in the literature on the development of learners' understanding of mathematical concepts in primary schools on the basis of the APOS theory, are those reported by Arnon (1998) and Arnon et al. (1999; 2001). Arnon (1998) compared the standard instructional sequence to an instructional sequence based on the APOS theory in Grade 4 learners' understanding of part-whole fractions. In those studies, the APOS instruction started with the learners using concrete materials, namely pieces of cardboard, known as partitioning rings, whereas the standard instruction used ready-made circle cut-outs representing various fractions. Both the standard instruction and the APOS instruction were based on Piaget's idea that constructing new mathematical concepts begins with actions applied on physical objects or perceived physical objects, the difference being in the instructional process. Arnon (1998) found that learners who were taught by means of the APOS instruction method performed better in developing a process conception of the APOS theory (Arnon et al., 2013, pp. 151-161).

Arnon et al. (1999, 2001) also developed a teaching sequence on small-group activity based on the APOS theory to teach equivalence fractions in Grade 5 where a software program was developed to act as a concrete (graphical) representation in dealing with the concept of equivalence fractions. Arnon et al. (1999, 2001) came to similar conclusions as Arnon (1998).

To the best of the researcher's knowledge, research into the learning of mathematical concepts in South African primary schools on the basis of the modelling and modelling perspective and the APOS theory is non-existent.

In the light of the above discussion the following suggestions are made:

i. Further research should be carried out to explore how the genetic decomposition proposed in this study can be used to improve the learners' understanding of algebra in other grades in the primary school, which may lead to a revision of the proposed genetic decomposition to improve the instruction of algebra in the Grade 6 classroom. A genetic decomposition may go through a cycle of data analysis and revision to closely reflect a true cognition of a concept before it can be used in instruction that positively affects the learners' learning (Mulqueeny, 2012).
ii. Research on problem-solving heuristic instruction on other mathematical concepts, other than algebra, should be initiated in South African primary schools on the basis of the APOS theory and the modelling and modelling perspective with the goal of improving and finding alternative instructional methods in the teaching of mathematics in South African primary schools by developing a genetic decomposition based on the two theories.
iii. Further research should be carried out on how effective mainstream educators can be in the implementation of genetic decomposition in mainstream classroom practice.
iv. Longitudinal research on a larger scale should be carried out to assess the effects of integrating the APOS theory and the modelling and modelling perspective in a teaching experience for the various quintile levels in the South African school system on a specific mathematical topic in primary schools.
v. Similar research should be carried out to compare gender performance in a chosen mathematical concept.

5.5 LIMITATIONS OF THE STUDY

i. This study was conducted in four schools which all fall under one quintile out of the five quintiles in the Zululand district of Kwazulu-Natal. The scope of the study could not cover the other quintile levels, districts and provinces, due to time and financial constraints. Hence, even though there was strong evidence of the learners' improvement through the use of the heuristic teaching method, the findings of the study cannot be generalized to all quintiles and all primary school learners in South Africa, but could be limited only to learners in quintile 1 schools in the Zululand district of Kwazulu-Natal.

ii. The nature of the study did not allow the researcher to randomize the selection of the four schools used in this study. Firstly, all four schools were required to have comparable characteristics with regards to teaching and learning resources and socio-economic conditions. Secondly, it was required that there should be a minimum distance between any two of the four schools chosen.

iii. The study could not explore the effects of the problem-solving heuristic instructional method in respect of gender.

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LIST OF APPENDICES

APPENDIX A: MODELLING ELICITING ACTIVITIES

MEA1

Context and parameters

Your school gate is always overcrowded every morning with taxis transporting learners and educators before school begins. This makes it difficult for other taxis to get to the school gate and as a result the learners are always late for school. Your principal wants to develop a piece of land near the school where the taxis can park and offload learners, so that other taxis can come through. Your principal wants you to help him determine the number of taxis that come to school daily, based on the number of learners and educators.

Mathematical problem

Develop a rule that predicts the number of taxis that come to your school every day, depending on the number of learners and educators that are in school.

a. How many taxis will arrive at school if only the educators came to school that day?

b. How many taxis will arrive at school if only the learners in your class came to school that day?

c. How many taxis will arrive at school if all the educators and the learners come to school that day?

d. Determine alternative situations in which your developed equation can be applied.

MEA 2

Context and Parameters

Due to water shortages in your school, your school principal has decided to buy water tanks that can store water in the school for at least a week. Your school principal wants you to advise him on the size of water tank he needs to buy.

Mathematical problem

Develop a rule that explains how many litres of water your school needs for a week if learners who are girls consume three times more water than learners who are boys and female educators consume twice as much water as male educators every day.

a. Interpret your model in the real world by answering the following questions.
   i. How many litres of water will be required for all educators?
   ii. How many litters of water will be required for learners in your class?
   iii. How many litres of water will be required for all learners and educators in your school?
b. Write a letter to your principal, in the letter explain to your principal what size of water tank he should buy for your school and why.

MEA3

Context and Parameters

Your school is organizing a sports competition that will include learners from 3 other schools in your community. Due to the large number of learners, your school principal wants to know how many educators will be required to supervise learners during the competition.

Mathematical problem

You are requested to develop a rule that explains the number of educators that will be required to supervise learners if one educator will be required for every 50 foundation phase learners, one educator for every 75 intermediate phase learners and one educator every 100 senior phase learners.

a. Interpret your model in the real world by responding to the following questions
   i. How many educators will be required to supervise learners in your school?
   ii. How many educators will be required to supervise learners from all the four schools in the event including your school?

APPENDIX B: ACHIEVEMENT TEST USED FOR PRE-TEST AND POST-TEST

Answer all questions

Duration: 1 hour

Marks: 40

PRE-TEST AND POST-TEST QUESTIONS

Section A

1. Which of the following will always have the same value as 9 x k?

   A. k + 9

   B. k – 9

   C. k x 9

   D. k + 9

2. 25 x 27 is not equivalent to which of the following?

   A. 25 x (20 x 7)

   B. (20 + 5) x (20 + 7)

   C. 25(20 + 7)

   D. 20(20 + 7) + 5(20 + 7)

3. Which of the following below is equivalent to: 15 x (4 x 9) = ?

   A. (15 x 4) x 9

   B. 15 x 2 x 2 x 3 x 3

   C. (15 x 4) + (15 x 9)

   D. (10 – 1)(15 x 4)

4. Find the variable *c* which satisfies the equation

   8c = 32, c =

   A. 40

   B. 4

   C. 24

   D. 512

5. On the first day of soccer camp, all the players were divided into nine teams. There are ten players in each team. Which equation, when solved, will give the total number of players?

A. $9 \times p = 10$

B. $p \times 10 = 9$

C. $p \times 9 = 10$

D. $p \div 9 = 10$

6. Select a number sentence to match the following statement; seven less than a certain number *m* is equal to twelve

A. $7 - m = 12$

B. $12 - m = 7$

C. $m + 7 = 12$

D. $m - 7 = 12$

7. 734 293,999 x k = 734 293,999, k =

A. 1

B. 0

C. 73293,999

D. 999

8. x + 5 000 000 = 100 000 + 5 000 000

x =

A. 6 000 000

B. 5 100 000

C. 11 000 000

D. 100 000

9. 1 000 000 – y = 0

y =

A. 0

B. 1 000 000

C. 1

D. 1 00 000

10. 400 x 500 = 500 x P

A. P = 500

B. p = 20 000

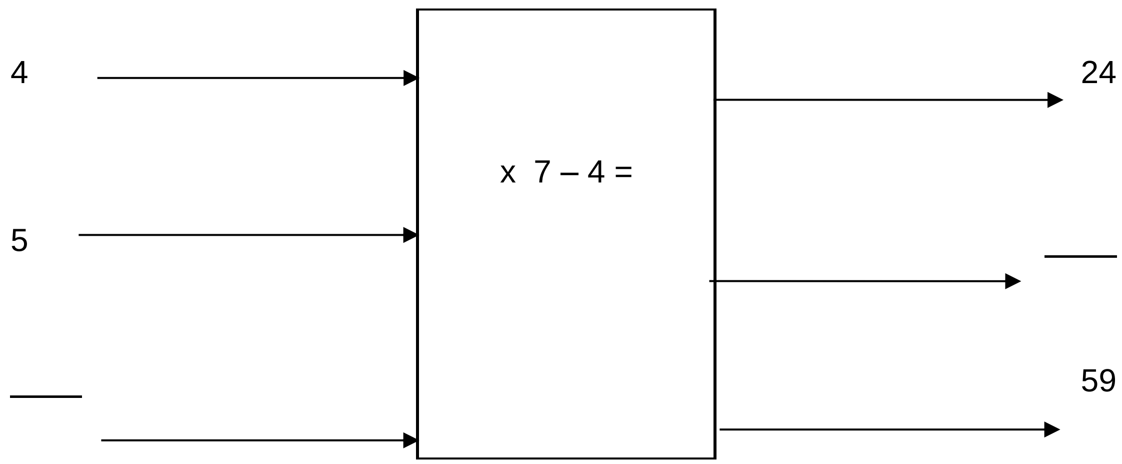
C. p = 400

D. p = 19 500

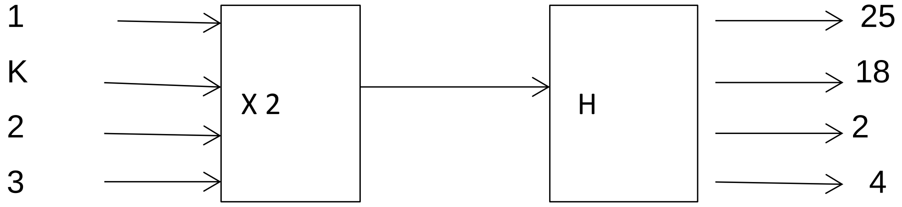
Section B

11.

Complete the following flow diagram.



12. Find the values of k and h



k=________

h=__________

13. Look at this pattern and complete the table

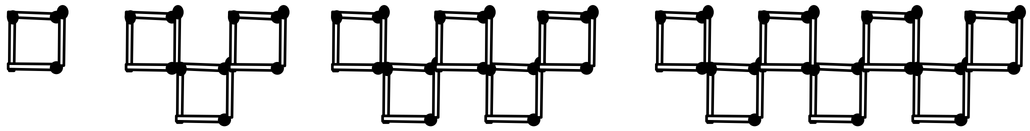
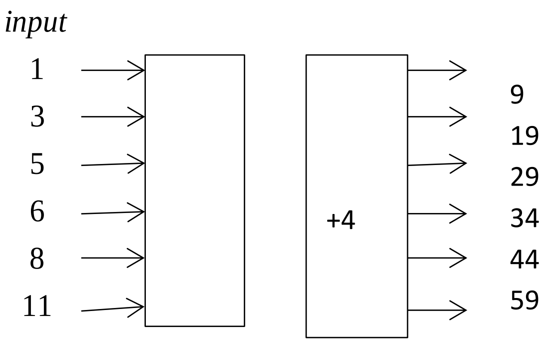


Figure	1	2	3	4	10	25	
Number of squares	1	3	5	7	19		199
Number of matches	4	12	20	28		196	796

14. Look at the input and output numbers and complete the table.

Input Numbers	2	3	4	5	10	
Output Numbers	5	8	11	14		44

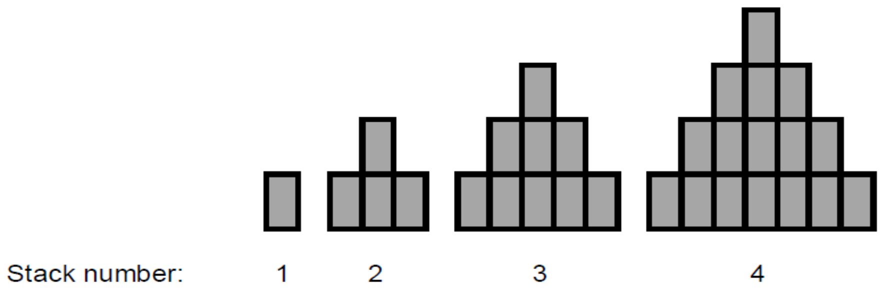
15.



i. What would you write in the empty box? ____________________________

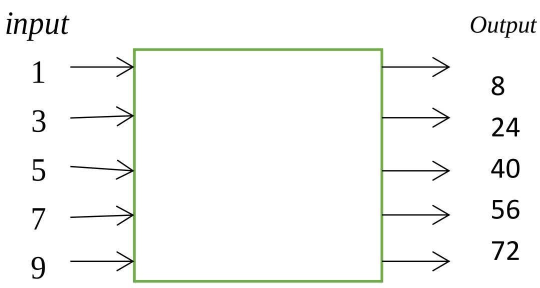
ii. What would you call it? ____

16. Carefully look at the stacks of cans and then complete the table



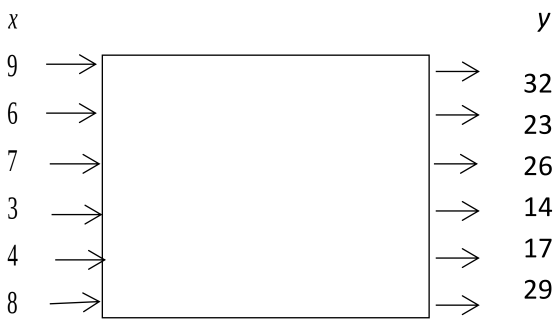
Stack number	1	2	3	4		6		
Number of cans	1	4	9	16				64

17.



i. What will you write in the empty box?___________________________

ii. What will you call it?______________________________

18. 

i. Determine the rule ________________

ii. Write the number sentence______________________

19. The length of a rectangular hall is 10 meters less than 2 time of width. If the width is b meters what is the length?__________________________________

20. Complete the table

X	y
1	3
2	7
3	11
8	

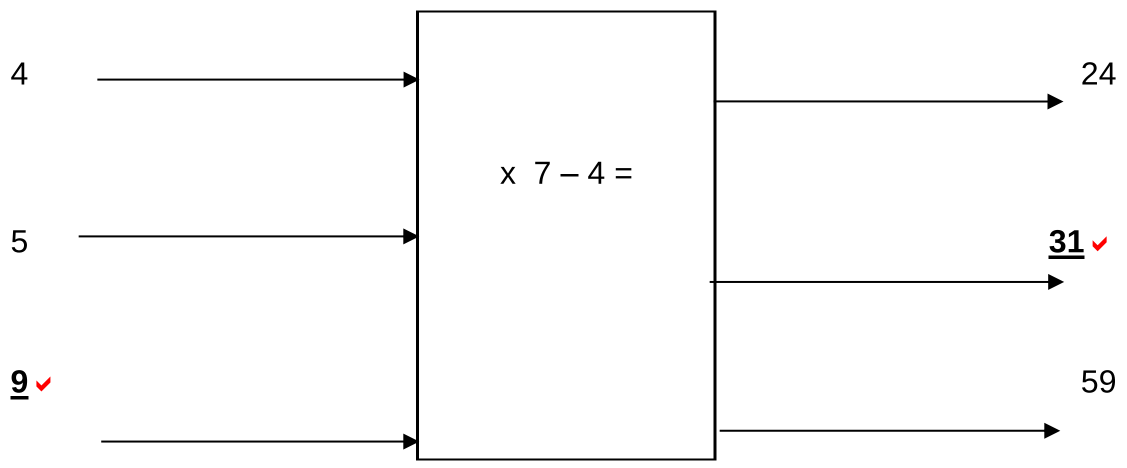
APPENDIX C: SUGGESTED ANSWERS FOR ACHIEVEMENT TEST

Section A

1. C ✓
2. A ✓
3. A ✓
4. B ✓
5. D ✓
6. D ✓
7. A ✓
8. D ✓
9. B ✓
10. C ✓

Section B

11. Complete the following input-output diagram.



12. k= **10** ✓

h=**-2** ✓

13.

Figure	1	2	3	4	10	25	**50** ✓
Number of squares	1	3	5	7	19	**49** ✓	199
Number of matches	4	12	20	28	**76** ✓	196	796

14.

inputs Numbers	2	3	4	5	10	**15** ✓
Output Numbers	5	8	11	14	**29** ✓	44

15.

i. **x5** ✓

ii. algebraic rule ✓

16.

Stack number	1	2	3	4	**5** ✓	6	**7** ✓	**8** ✓
Number of Cans	1	4	9	16	**25** ✓	**36** ✓	**49** ✓	64

17.

i. **<u>x8</u>** ✓

ii. **<u>algebraic rule</u>** ✓

18.

i. **<u>x 4</u>** ✓ **<u>+5</u>** ✓

ii. **<u>9 x 3 +5 = 32</u>** ✓

<u>6 x 3 + 5 = 23</u> ✓

<u>7 x 3 + 5 = 27</u> ✓

<u>3 x 3 + 5 = 14</u> ✓

<u>4 x 3 + 5 = 17</u> ✓

<u>8 x 3 + 5 = 29</u> ✓

19. **<u>2*b*</u>** ✓ **<u>-10</u>** ✓

20. **<u>31</u>** ✓

APPENDIX D: RESEARCHER'S CLASSROOM OBSERVATION TOOL

Name of Educator_______________ **Name of Observer**_________________

Class_______________________ **Subject**_________________________

Topic_______________________ **Period**_____________________

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners' previous knowledge				
2	Evidence of educator's preparation prior to lesson				
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class				
4	Educator appropriately uses teaching aids to facilitate teaching and learning				
5	Educator is able to simplify difficult concepts to learners				
6	Educator encourages learners to answer other learners Questions				
7	Educator provides relevant examples on concepts that relate to their everyday experience				
8	Educator presents well-planned lessons according to the curriculum and the work schedule				
9	Educator general class management				
10	Educator evaluates lessons to check achievement of lesson objectives				

Adapted from Kotoka (2012)

Appendix D₁: Summary of first classroom observation for control group 1 by researcher

Name of Educator: Educator CE1

Name of Observer: Researcher **Class:** Grade 6 **Subject:** Mathematics

Topic: Mental Mathematics

Learning Outcome: Numbers Operation and Relationship

	Educator's teaching style	**Excellent**	**Good**	**Fair**	**Poor**
1	Educator's review of learners' previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning			X	
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives			X	

Appendix D$_2$: Summary of second classroom observation for control group1 by researcher

Name of Educator: Educator CE1

Name of Observer: Researcher **Class:** Grade 6

Subject: Mathematics

Topic: Number sentences (Introduction to Algebraic expression)

Learning Outcome: Patterns, Functions and Algebra

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning			X	
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience			X	
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievements of lesson objectives			X	

Appendix D3: Summary of first classroom observation for control group2 by researcher

Name of Educator: Educator CE2

Name of Observer: Researcher

Class: Grade 6 **Subject:**Mathematics

Topic: Mental Mathematics

Learning Outcome: Numbers Operation and Relationship

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class)				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning				X
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners Questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives			X	

Appendix D4: Summary of second classroom observation for control group1 by researcher

Name of Educator: Educator CE2

Name of Observer: Researcher **Class:** Grade 6

Subject: Mathematics

Topic: Number Sentences(Introduction to Algebraic Expression)

Learning Outcome: Patterns, Functions and Algebra

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning				X
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners' questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives			X	

Appendix D5: Summary of first classroom observation for experimental group1 by researcher

Name of Educator: Educator EE1

Name of Observer: Researcher

Class: Grade 6 **Subject** : Mathematics

Topic: Number sentences (Introduction to Algebraic Expression)

Learning Outcome: Patterns, Functions and Algebra

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class)				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning			X	
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience			X	
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives			X	

Appendix D_6: Summary of second classroom observation for experimental group1 by researcher

Name of Educator: Educator EE1

Name of Observer: Researcher **Class:** Grade 6

Subject: Mathematics

Topic: Mental Mathematics

Learning Outcome: Numbers Operation and Relationship

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners previous knowledge	X			
2	Evidence of educator's preparation prior to lesson	X			
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class)				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning		X		
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners' questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule		X		
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives			X	

Appendix D_7: Summary of first classroom observation for experimental group2 by researcher

Name of Educator: Educator EE2
Name of Observer: Researcher
Class: Grade 6 **Subject :** Mathematics
Topic: Number sentences (Introduction to Algebraic Expressions)
Learning Outcome: Patterns, Functions and Algebra

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners' previous knowledge		X		
2	Evidence of educator's preparation prior to lesson		X		
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class)				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning				X
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule	X			
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives	X			

Appendix D_8: Summary of second classroom observation for experimental group 1 by researcher

Name of Educator: Educator EE2
Name of Observer: Researcher
Class: Grade 6 **Subject:** Mathematics
Topic: Number sentences (Introduction to Algebraic Expressions)

Learning Outcome: Patterns, Functions and Algebra

	Educator's teaching style	Excellent	Good	Fair	Poor
1	Educator's review of learners' previous knowledge			X	
2	Evidence of educator's preparation prior to lesson		X		
3	Educator involves learners in the lesson (e.g. puts learners in groups to perform activities or to have discussion and reports to class)				X
4	Educator appropriately uses teaching aids to facilitate teaching and learning	X			
5	Educator is able to simplify difficult concepts to learners		X		
6	Educator encourages learners to answer other learners questions				X
7	Educator provides relevant examples on concepts that relate to their everyday experience				X
8	Educator presents well-planned lessons according to the curriculum and the work schedule	X			
9	Educator general class management	X			
10	Educator evaluates lessons to check achievement of lesson objectives	X			

APPENDIX E: CLASS OBSERVATION TOOL USED BY HODS
Class visits

Name of Educator: ……………………………………………….

Grade: ………………………………………………………………..

Subject: ……………………………………………………………….

1.

	Lesson Preparation	**Yes/No**	**Comments**
a	Is lesson preparation available?		
b	Is the content clearly outlined?		
c	Is the lesson preparation in line with CAPS policy?		

2.

	Communication Skills	**Yes/No**	**Comments**
a	Is there effective communication between learners and educator?		
b	Is the language used clear, appropriate and understandable?		
c	Are questioning tactics relevant and effective?		

3.

	Presentation	**Yes/No**	**Comments**
a	Was the prior knowledge considered when introducing the lesson?		
b	Were activities varied according to learners' levels?		
c	Were learners actively involved throughout the lesson		
d	Did the lesson accommodate diversity in the classroom?		

4.

	Classroom Management	Yes/No	Comments
a	Was the classroom environment conducive to learning and teaching?		
b	Did the educator maintain discipline throughout the lesson?		
c	Were the learner teacher support materials (LTSM) effectively used and relevant to the classroom?		

5.

	Assessment	Yes/No	Comments
a	Did the educator use different types of assessment?		
b	Were questions in accordance with the demand of the activities?		

Recommendation

………………………………………………………………………………………………………
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…………………………………………………………………………

Signature: ……………………………………….
Date:……………………………………………..

Signature of HOD: ……………………………..

Date:……………………………………………..

Appendix E_1: Summary of classroom observation for control group 1 by HOD

Name of Educator: Educator CE1
Name of Observer: HOD1
Class: Grade 6 **Subject:** Mathematics
Topic: Numeric patterns
Learning Outcome: Patterns, Functions and Algebra

1.

	Lesson Preparation	Yes/No	Comments
a	Is lesson preparation available?	Yes	
b	Is the content clearly outlined?	Yes	
c	Is the lesson preparation in line with CAPS policy?	Yes	

2.

	Communication Skills	Yes/No	Comments
a	Is there effective communication between learners and educator?	Yes	
b	Is the language used clear, appropriate and understandable?	Yes	
c	Are questioning tactics relevant and effective?	No	

3.

	Presentation	Yes/No	Comments
a	Was the prior knowledge considered when introducing the lesson?	No	No reference made to previous lesson
b	Were activities varied according to learners' levels?	Yes	
c	Were learners actively involved throughout the lesson?	No	Educator did all the talking
d	Did the lesson accommodate diversity in the classroom?	Yes	

4.

	Classroom Management	Yes/No	Comments
a	Was the classroom environment conducive to learning and teaching?	Yes	
b	Did the educator maintain discipline throughout the lesson?	Yes	
c	Were the learner teacher support materials (LTSM) effectively used and relevant to the classroom?	Yes	

5.

	Assessment	Yes/No	Comments
a	Did educator use different types of assessment?	Yes	Evidence of Different Assessment Strategies
b	Were questions in accordance with the demand of the activities?	Yes	

Recommendation

Educator must encourage active learner participation and use more effective teaching tactics.

Signature of Educator:.. Date: 15/09/2014

Signature of HOD: .... Date: 15/09/2014

Appendix E2: Summary of classroom observation for control group 2 by HOD

Name of Educator: Educator CE2
Name of Observer: HOD2
Class: Grade 6 **Subject:** Mathematics
Topic: Data handling
Learning Outcome: Data handling

1.

	Lesson Preparation	**Yes/No**	**Comments**
a	Is lesson preparation available?	Yes	
b	Is the content clearly outlined?	No	Write explanation and examples on the Lesson Plan
c	Is the lesson preparation in line with CAPS policy?	Yes	

2.

	Communication Skills	**Yes/No**	**Comments**
a	Is there effective communication between learners and educator?	Yes	Good
b	Is the language used clear, appropriate and understandable?	Yes	
c	Are questioning tactics relevant and effective?	Yes	More questioning should be done

3.

	Presentation	**Yes/No**	**Comments**
a	Was the prior knowledge considered when introducing the lesson?	Yes	
b	Were activities varied according to learners' levels?	Yes	
c	Were learners actively involved throughout the lesson?	No	Less questioning was done
d	Did the lesson accommodate diversity in the classroom?	Yes	

4.

	Classroom Management	Yes/No	Comments
a	Was the classroom environment conducive to learning and teaching?	Yes	
b	Did the educator maintain discipline throughout the lesson?	Yes	
c	Were the learner teacher support materials (LTSM) effectively used and relevant to the classroom?	No	No text-book, only notes are used

5.

	Assessment	Yes/No	Comments
a	Did the educator use different types of assessment?	Yes	
b	Were questions in accordance with the demand of the activities?	No	No activities were indicated on the lesson plan

Recommendation

The duration of the lesson plan, content and activities to be outlined clearly

Signature of Educator: Date: 9/09/2014

Signature of HOD: Date: 9/09/2014

Appendix E3: Summary of classroom observation for experimental group 1 by HOD

Name of Educator: Educator EE1
Name of Observer: HOD3
Class: Grade 6
Subject : Mathematics
Topic: Properties of 2-D objects
Learning Outcome: Space and Shape (Geometry)

1.

	Lesson Preparation	**Yes/No**	**Comments**
a	Is lesson preparation available?	Yes	
b	Is the content clearly outlined?	Yes	
c	Is the lesson preparation in line with CAPS policy?	Yes	

2.

	Communication Skills	**Yes/No**	**Comments**
a	Is there effective communication between learners and educator?	No	
b	Is the language used clear, appropriate and understandable?	Yes	
c	Are questioning tactics relevant and effective?	No	

3.

	Presentation	Yes/No	Comments
a	Was the prior knowledge considered when introducing the lesson?	Yes	
b	Were activities varied according to learners' levels?	No	No evidence of accommodating poor performing learners
c	Were learners actively involved throughout the lesson	No	
d	Did the lesson accommodate diversity in the classroom?	Yes	

4.

	Classroom Management	Yes/No	Comments
a	Was the classroom environment conducive to learning and teaching?	Yes	
b	Did the educator maintain discipline throughout the lesson?	Yes	
c	Were the learner teacher support materials (LTSM) effectively used and relevant to the classroom?	Yes	Evidence of textbook usage

5.

	Assessment	Yes/No	Comments
a	Did the educator use different types of assessment?	Yes	
b	Were questions in accordance with the demand of the activities?	Yes	

Recommendation

Educator must vary learner activities to accommodate learners at different cognitive levels.

Signature of Educator:  .... Date: 12/08/2014

Signature of HOD:  Date: 12/08/2014

Appendix E4: Summary of classroom observation for experimental group 2 by HOD

Name of Educator: Educator EE2
Name of Observer: HOD4
Class: Grade 6 **Subject :** Mathematics
Topic: Number sentences (Introduction to Algebraic Expressions)
Learning Outcome: Patterns, Functions and Algebra

1.

	Lesson Preparation	**Yes/No**	**Comments**
a	Is lesson preparation available?	Yes	
b	Is the content clearly outlined?	Yes	
c	Is the lesson preparation in line with CAPS policy?	Yes	

2.

	Communication Skills	**Yes/No**	**Comments**
a	Is there effective communication between learners and educator?	Yes	
b	Is the language used clear, appropriate and understandable?	Yes	
c	Are questioning tactics relevant and effective?	Yes	

3.

	Presentation	Yes/No	Comments
a	Was the prior knowledge considered when introducing the lesson?	No	Educator did not refer to previous lesson
b	Were activities varied according to learners' levels?	No	Activities were not varied
c	Were learners actively involved throughout the lesson?	No	
d	Did the lesson accommodate diversity in the classroom?	Yes	

4.

	Classroom Management	Yes/No	Comments
a	Was the classroom environment conducive to learning and teaching?	Yes	
b	Did the educator maintain discipline throughout the lesson?	Yes	
c	Were the learner teacher support materials (LTSM) effectively used and relevant to the classroom?	Yes	

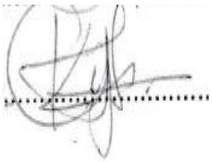
5.

	Assessment	Yes/No	Comments
a	Did the educator use different types of assessment?	No	
b	Were questions in accordance with the demand of the activities?	No	

Recommendation

Educator must make effort to link previous lessons to current lessons and also vary activities to accommodate weak learners.

Signature of Educator:  Date: 18/08/2014

Signature of HOD:  Date: 18/08/2014

APPENDIX F: A T – TEST ANALYSIS OF THE PRE – TEST SCORES BETWEEN THE CONTROL AND EXPERIMENTAL GROUP

	Control Group	*Experimental Group*
Mean	13,54	15,60
Variance	96,36	90,57
Observations	106	92
Hypothesized mean difference	0	
df	194	
t Stat	1,57	
P(T<=t) one-tail	0,060	
t Critical one-tail	1,65	
P(T<=t) two-tail	0,12	
t Critical two-tail	1,98	

APPENDIX G: A T – TEST ANALYSIS OF THE POST – TEST SCORES BETWEEN THE CONTROL AND EXPERIMENTAL GROUP

	Control Group	*Experimental Group*
Mean	13,92	43,73
Variance	92,86	282,90
Observations	106	92
Hypothesized mean difference	0	
df	140	
t Stat	14,98	
P(T<=t) one-tail	3,49E-31	
t Critical one-tail	1,66	
P(T<=t) two-tail	6,98E-31	
t Critical two-tail	1,98	

APPENDIX H: ANCOVA SUMMARY TABLE

Source	SS	df	MS	F	P - Value
Adjusted Treatment	**39016.16**	**1**	**39016.16**	**298.85**	**0.001**
Error (Res $_{within}$)	**25458.48**	**195**	**130.56**		
Total Residuals (Res_{total})	**64474.64**	**196**			
	CRITICAL VALUES		$F_{0.05, 1, 195)}$	3.88958864	
	CRITICAL VALUES		$F_{0.01, 1, 195)}$	6.76663859	
Adjusted means					
Pooled Regression Coefficient	**0.676**				
Control	**14.62**				
Experimental	**42.92**				

APPENDIX I: HOMOGENEITY OF REGRESSION SLOPE TEST

Source	SS	DF	MS	F	p-value
Heterogeneity of slopes	641.69	1	641.69	5.01	0.026
Individual Residuals (res_i)	24816.79	194	127.92		
Within Residual (res_w)	25458.48	195			
H_o: $\beta_1^{Control} = \beta_1^{Experimental}$	CRITICAL VALUES		$F_{0.05, 1, 194)}$	3.88983904	
H_1: $\beta_1^{Control} \neq \beta_1^{Experimental}$					
Compare p – value (0.026) with α = 0.05	If $p < \alpha$ then we reject the null hypothesis				
p- value is much lower than α	Null hypothesis is rejected				
Compare critical value with F	$F_{0.05}$=3.89 with F= 5.01			Reject null hypothesis if the F value is greater than the critical F value	
Null hypothesis is rejected since $F_{0.05}$=3.89 < F= 5.01					

R Squared = 0.687 (Adjusted R Squared = 0.682)
(See Huitema, 2011, p.123-157)

Appendix J: SUMMARY JOHNSON-NEYMAN TECHNIQUE

SUMMARY OF STATISTICS	CONTROL	EXPERIMENTAL	
SAMPLE SIZE	106	98	
SAMPLE MEAN	**13.54**	**15.60**	
SUM OF SQUARES	29875	30625	
INTERCEPT	1.76	35.54	
SLOPE	0.90	0.53	
$F_{0.05,1,194}$	3.8898	3.8898	
WITHIN RESIDUAL(SS_{RES})	25458.48		
A	0.1031		
B	-12.9900		
C	1123.5		
X_{L1}	55.4439		**Bounded above by 100**
X_{L2}	196.545		

(See Huitema, 2011, p.247-256)

Where X_{L1} and X_{L2} are limits of Non-Significance region

$$A = \frac{-F_{(\alpha,1,N\text{-}4)}}{n-4}\left(SS_{res_i}\right)\left(\frac{1}{\sum x_1^2} + \frac{1}{\sum x_2^2}\right) + \left(b_1^{control} - b_1^{experimental}\right)^2$$

A = 0.1031

$$B = \frac{-F_{(\alpha,1,N\text{-}4)}}{N-4}\left(SS_{res_i}\right)\left(\frac{\overline{X}_1}{\sum X_1^2} + \frac{\overline{X}_2}{\sum X_2^2}\right) + \left(b_0^{control} - b_0^{experimental}\right)\left(b_1^{control} - b_1^{experimental}\right)$$

B= -12.9900

$$C = \frac{-F_{(\alpha,1,N\text{-}4)}}{N-4}\left(SS_{res_i}\right)\left(\frac{N}{n_1 n_2} + \frac{\overline{X}_1^2}{\sum X_1^2} + \frac{\overline{X}_2^2}{\sum X_2^2}\right) + \left(b_0^{control} - b_0^{experimental}\right)^2$$

C= 1123.5

Limits of Region of Non – Significance on X is computed using

$X_{L1} = \frac{-B - \sqrt{B^2 - AC}}{A}$ and $X_{L2} = \frac{-B + \sqrt{B^2 - AC}}{A}$

APPENDIX K: ETHICAL CLEARANCE

UNISA university of south africa

22 February, 2013

Mr. Ofori-Kusi Daniel
(49603604)

Dear Mr. Daniel

REQUEST FOR ETICAL CLEARANCE: Investigating the use of Problem solving Heuristics to improve the teaching and learning of Mathematics

Your application for ethical clearance of the above study was received and considered by the ISTE sub-committee in the College of Graduate Studies on behalf of the Unisa Research Ethics Review Committee on 21 February, 2013.

The Committee is pleased to inform you that ethical clearance has been granted for this as set out in your application.

Congratulations on this interesting and relevant study. We would like to wish you well in this research undertaking.

Kind regards.

C. E. OCHONOGOR, *Ph.D; FCAI*
CHAIR: ISTE SUB-COMMITTEE

CC. PROF L. LABUSHAGNE
EXECUTIVE DIRECTOR: RESEARCH

PROF M N SLABBERT
CHAIR- URERC

University of South Africa
Preller Street, Muckleneuk Ridge, City of Tshwane
PO Box 392 UNISA 0003 South Africa
Telephone: +27 12 429 3111 Facsimile: +27 12 429 4150
www.unisa.ac.za

APPENDIX L : PERMISSION TO CONDUCT RESEARCH IN KZN DOE INSTITUTIONS

kzn education
Department:
Education
KWAZULU-NATAL

Enquiries: Sibusiso Alwar | Tel: 033 341 8610 | Ref.:2/4/8/364

Daniel Ofori – Kusi
P. O. Box 3003
KWALINDIZWE
3954

Dear Daniel

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **Investigating the Use of Problem Solving Heuristics to Improve the Teaching and Learning of Mathematics,** in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 01 February 2013 to 31 January 2015.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Mr. Alwar at the contact numbers below.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report / dissertation / thesis must be submitted to the research office of the Department. Please address it to The Director-Resources Planning, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to the Schools and Institutions in the Zululand District.

Nkosinathi S.P. Sishi, PhD
Head of Department: Education

20-02-2013
Date

...dedicated to service a[...]
beyond the call of d[...]

KWAZULU-NATAL DEPARTMENT OF EDUCATION

POSTAL : Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa

APPENDIX M: CONSENT LETTERS
Cover Letter: Request to conduct research at the school.

Armstrong Street

P.O. Box 3003

Kwalindizwe, 3954

Date: 3/05/2014

Dear Parent/Guardian

My name is Daniel Ofori-Kusi, a doctoral student at University of South Africa. I'm conducting research in your ward's school on the use of a heuristic teaching method to improve the learning of algebra. The study demands engaging Grade 6 learners in a teaching experiment for a period of six months. The learners will also write a test before and after the problem-solving heuristic instructional method.

Data taking from your ward will be used for the purposes of research only and your ward is free to withdraw from the exercise at any moment during the course of data collection. The name of your ward will not be used anywhere in the study and will remain anonymous.

I will request you to kindly grant your ward permission to participate in this study by signing the attached informed consent form and the recording consent form. Hoping to get a positive response from you. Thank you

Yours faithfully

....

Daniel Ofori-Kusi

Cc SGB chairperson

1. PARENT INFORMED CONSENT FORM

Topic: An investigation into the use of problem-solving heuristics to improve the teaching and learning of mathematics

Researchers: Name: Daniel Ofori-Kusi **Institution:** University of South Africa

<u>Purpose of Research</u>

The purpose of this study is to explore an alternative teaching strategy in the learning of algebra in grade 6. A pre-test and a post-test will be conducted before and after the teaching treatment to give me information about the effectiveness of the teaching treatment as a learning pedagogy

<u>Specific Procedures to be Used</u>

A heuristic teaching method will be explored in which learners will be presented with an authentic real life problem known as modelling eliciting activities where they will be expected to think about a new mathematical ways of exploring the problem with the guide of the researcher.

<u>Duration of Participation</u>

Data collection will take place over a period of 6 months.

Benefits to the Ward (Learners)

This research will be highly beneficial to your ward, It will offer your ward the opportunity to understand the importance of mathematics and enable child to learn how to apply mathematics to solve real world problems they encounter daily in their community and at school.

<u>Confidentiality</u>

In this study I will comply with ethical issues of confidentiality, anonymity and privacy. To ensure confidentiality, names of the participants will not be disclosed; the name of the schools, teachers, learners, principals (heads) will remain anonymous and will not appear in the thesis. Instead, fictitious names will be used throughout the study. The data gathered in this exercise will be solely and strictly used for the purpose of this research project. During and after completion of the study the raw research data will be appropriately stored by the researcher under lock and key.

Voluntary Nature of Participation

My ward does not have to participate in this research project. If I agree for my ward to participate I can withdraw my wards participation at any time without penalty.

Human Subject Statement:

If I have any questions about this research project, I can contact ***Professor L. D. Mogari, University of South Africa, Institute of Science and Technology Education***. The phone number is 0124293904. The email address is mogarld@unisa.ac.za

I HAVE HAD THE OPPORTUNITY TO READ THIS **CONSENT FORM**, ASK QUESTIONS ABOUT THE RESEARCH PROJECT AND AM PREPARED TO MAKE MY WARD PARTICIPATE IN THIS PROJECT.

____ __________________

Parent's Signature Date

____

Parent's Name

....

3-05-2014

Researcher's Signature Date

NB: This document will be translated into the zulu language for easy understanding of parent.

2. CONSENT TO AUDIO- OR VIDEO-RECORDING & TRANSCRIPTION

Topic: An investigation into the use of problem-solving heuristics to improve the teaching and learning of mathematics

DANIEL OFORI-KUSI, INSTITUTE OF SCIENCE AND TECHNOLOGY EDUCATION, UNIVERSITY OF SOUTH AFRICA

This study involves the audio or video recording of your wards interview with the researcher. Neither your ward's name nor any other identifying information will be associated with the audio or audio-recording or the transcript. Only the research team will be able to listen to (view) the recordings.

The tapes will be transcribed by the researcher and erased once the transcriptions are checked for accuracy. Transcripts of your ward's interview may be reproduced in whole or in part for use in presentations or written products that result from this study. Neither your ward's name nor any other identifying information (such as your voice or picture) will be used in presentations or in written products resulting from the study.

By signing this form, I am allowing the researcher to audio- or video-tape my ward as part of this research. I also understand that this consent for recording is effective until the following date: December, 2016. On or before that date, the tapes will be destroyed.

Parent's signature: ________________________ Date: ________

Parent's name: ____

.... 3-05-2014

Researcher's Signature Date

NB: Information will be translated into the Isizulu language for easy understanding by learner's parent.

APPENDIX N: TURNITIN ORIGINALITY REPORT

Turnitin Originality Report

Final Submission for Examination by Daniel Ofori-Kusi

From Final submission for examination (M&D Student submissions 2017)

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AN INVESTIGATION INTO THE USE OF PROBLEM-SOLVING HEURISTICS TO IMPROVE THE TEACHING AND LEARNING OF MATHEMATICS by DANIEL OFORI-KUSI Submitted in accordance with the requirements for the degree of DOCTOR OF PHILOSOPHY in the subject MATHEMATICS EDUCATION at the UNIVERSITY OF SOUTH AFRICA Supervisor: Prof. L. D. Mogari March 2017 i Declaration I declare that the thesis “An investigation into the use of problem-solving heuristics to improve the teaching and learning of mathematics” is my own work and has not previously been submitted to any other institution of higher education. All sources cited or quoted are acknowledged by means of a comprehensive list of references. ---------------------------------- Daniel Ofori-Kusi ---------------------------------- Date ii ABSTRACT The aim of this study was to explore the effects of a problem-solving heuristic instructional approach on Grade 6 learners’ achievements in algebra. Two main theories inspired the design of this teaching approach, namely the modelling and modelling perspective, and action, process, object, schema (APOS theory). Modelling and modelling perspectives guided the development of modelling-eliciting activities used in the teaching

APPENDIX O: LANGUAGE EDITING CERTIFICATE

This Certificate serves to confirm that

Dr E. M. Murray, PhD (English) University of Johannesburg

Certificate in Editing Principles and Practices, University of Pretoria

813-26th Avenue, Rietfontein, Pretoria 0084

Freelance editor

Judiciously edited a University of South Africa dissertation titled:

AN INVESTIGATION INTO THE USE OF PROBLEM-SOLVING HEURISTICS TO IMPROVE THE TEACHING AND LEARNING OF MATHEMATICS

Issued without prejudice to

DANIEL OFORI-KUSI

On this day

25 January 2017