Three Essays on Marriage, Housing and Incomplete Markets

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Guangyu Nie, M.A.

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Guangyu Nie, M.A.

Dissertation Advisors: James Albrecht, Ph.D. and Susan Vroman, Ph.D.

Abstract

This dissertation focuses on labor economics and macroeconomics.

Chapter 1 explains the savings rate puzzle in China from the marriage market perspective. Since 2000, China's age profile of the savings rate exhibits a downward-sloping pattern with younger people having higher savings rates than the middle-aged. I explain such a puzzle using the competitive saving motive. That is, single men save in a competitive manner in order to improve their ranking in the marriage market, and this competition gets fiercer with an unbalanced gender ratio. I develop a life-cycle model with a marriage market and calibrate it to the Chinese economy. This model generates a similar downward-sloping age-savings rate profile as observed in the data. Another finding is that the adjustment in marriage age reduces the response of the savings rate to the gender ratio increase by half.

Chapter 2 studies the effect of market incompleteness on business cycles. We find that even without a collateral constraint, market incompleteness by itself plays a quantitatively significant role in the amplified and asymmetric responses in output and housing price to exogenous shocks.

In Chapter 3, I explain why in recent years women with high socioeconomic status in China find it increasingly difficult to find a spouse even with a steadily increasing gender ratio (number of men per woman) in the premarital cohort. In a bilateral search model with positively assortative matching, as the gender ratio increases, women of high quality set up higher requirements for their future spouses while men of high quality become less demanding. I show how this results in the failure of log-concavity

of the stationary distributions among singles which may be the reason why high-quality women have a larger chance of being unmatched even though the marriage market conditions favor women in general.

INDEX WORDS: marriage market, gender ratio, search, matching, savings rate, incomplete market, collateral constraint

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This dissertation is dedicated to my family and friends, for their endless support and love.

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Chapter 1

A Model of Competitive Saving Over the Life-cycle, and its Implications for the Savings Rate Puzzle in China

1.1 Introduction

The permanent income hypothesis predicts that people should save less when their income is growing. However, China's savings rate in recent years exhibits the opposite pattern on both macroeconomic and microeconomic levels. On the one hand, China's urban household savings rate rose by 10 percent of disposable income from 1990 to 2007 against a background of rapid income growth and a constantly low real interest rate. On the other hand, researchers such as Chamon and Prasad (2010) and Song and Yang (2010) document that the age profile of the savings rate exhibits a downward-sloping pattern in early ages with younger households having higher savings rates than middle-aged cohorts even when the lifetime earning profile follows the common hump-shaped path that peaks in middle age. These unusual patterns are denoted as the *savings rate puzzle* in China.

Several recent papers such as Wei and Zhang (2011) and Du and Wei (2013) propose a competitive saving motive as an explanation for the rapid rise of the aggregate savings rate. Single men accumulate savings in a competitive manner in order to improve their relative position in the marriage market, and this competition gets fiercer with an increase in the gender ratio (the number of men per women) in the pre-marital cohort. In Wei and Zhang (2011), the authors estimate that such a saving

motive accounts for about half of the observed increase in the aggregate household savings rate in recent years.

Although the theory about the competitive saving motive is insightful, it leaves many important questions unanswered. For example, why is the age profile of the savings rate downward-sloping in the pre-marital cohorts? Would such a competitive saving motive be mitigated by other responses of single people such as a change in the age gap between spouses - a well documented phenomenon under the marriage squeeze (for example, see Anderson, 2007; Abramitzky et al., 2011)? The goal of this paper is to develop a unified model to clarify these questions. In particular, I will show to what extent the current savings rate puzzle can be explained by a competitive saving motive when single men have the option to postpone their marriage age in addition to increasing their savings.

In this paper I construct an overlapping generations model with two-dimensional matching on the marriage market, in which men are differentiated by wealth and age. Starting from age 1, single men allocate their liquid wealth between consumption and saving with the understanding that when they reach their marital ages, men who accumulate more wealth are strictly preferred[1]. Since their labor income is subject to independently and identically distributed transitory shocks at each age, wealth inequality increases with age which reflects the cumulative differences in the effect of luck on savings. With an unbalanced gender ratio, men from the top of the saving distribution can be matched at their earliest marriageable age, while men from the bottom of the saving distribution who fail to be matched stay single and

[1]In reality, accumulating more saving may not be the only way to improve one's rank. Single men can also choose to invest in their human capital. Since these two channels are complementary, adding human capital into this model framework may dampen the reaction of the savings rate to the gender ratio but should not change the direction of reaction. To limit the dimension of the matching game and keep the model tractable, human capital accumulation is dropped from the model and left for future research.

rejoin the marriage market in the next period. For those men who postpone marriage, although increased age in the next period serves as a disadvantage for them to get a better match, they may compensate for this by accumulating more wealth since they now have additional periods to save and could possibly experience good labor income shocks in those periods. In other words, a young woman may find men with different {saving, age} combinations equally attractive since although marrying an older man provides fewer periods of companionship, she can enjoy more pre-marital saving brought into the family by him. I find that a stable matching always exists in this model but it may fail to be pure due to the trade-off between wealth and age. Some general properties of the matching patterns are derived, and under common functional forms, the stable matching can be characterized.

I show that when the gender ratio in the marriage market is highly unbalanced as is the case in China, the age profile of the savings rate of the pre-marital cohorts is downward-sloping even with an increasing age-income profile due to the different level of uncertainties about marriage prospects single men are subjected to when they approach marriageable age. For a newly born man, no matter what his current income is, his marriage prospect is highly uncertain because he will be subject to other income shocks between his current age and marriageable age, and his position in the marriage market is determined by his wealth at the marriageable age only. As a result, all age 1 men choose to save the most possible part of their income when the chance of improving their marriage prospects is still high. However, as a man approaches the marriageable age, uncertainties in labor income are realized gradually and his position in the marriage market is in large part determined by the wealth he has accumulated. Consequently, there is little chance for him to change his position in the marriage market now. This leads to a reduction in returns to saving and a decrease in the savings rate at this age.

A natural question is how the saving behavior of the women responds to men's saving behaviors when they can share their husbands' wealth after marriage. I assume women's relative position in the marriage market does not depend on their wealth, and thus young women accumulate much less saving than men. After we aggregate the savings of men and women in each age cohort, the overall age profile of the savings rate of single agents is still downward-sloping.

To check whether the model can explain the observed savings rate puzzle, I employ quantitative calibrations and find the following results: (1) The savings rate profile of people younger than 36 exhibits similar patterns to those seen in the data. (2) If the age of marriage is exogenous, as the gender ratio rises from 1 to 1.18, the aggregate household savings rate would rise by 6.50% of GDP. However, if single men can postpone their marriage age, pressure on the savings rate is tempered and the aggregate savings rate rises by 3.35% of GDP instead.

The rest of this paper is organized as follows. Section 1.2 reviews the relevant literature. Section 1.3 presents key facts on household saving in China. Section 1.4 presents the two-period benchmark model to show how a competitive saving motive arises when wealth is a status good. Section 1.5 presents a marriage model with an endogenous age gap between spouses and shows how saving and marital age jointly react to marriage market imbalances. Section 1.6 extends the benchmark model to a multi-period framework. Section 1.7 calibrates the model using data from the Chinese economy and derives numerical results as an explanation of the savings rate puzzle in China. Concluding remarks are offered in Section 1.8.

1.2 Related Literature

This paper is related to several strands of literature. First, there is literature on pre-marital investment. To the best of my knowledge, this is the first paper to propose and solve pre-marital investment and stable matching patterns in a multi-dimensional matching framework. Previous literature on pre-marital investment focuses on exactly one characteristic on which the matching process is based. Various studies have thus investigated the resulting matching patterns and welfare implications (Peters and Siow, 2002; Iyigun and Walsh, 2007; Cole et al., 2001; Chiappori et al., 2009). Several recent papers such as Chiappori et al. (2012), Lindenlaub (2014), Galichon and Salanie (2010) as well as McCann et al., (forthcoming) study the matching model with multidimensional types under transferable utility[2]. One common feature to these papers and mine is that matching is not pure and there are trade-offs between different characteristics. However, since I use a non-transferable utility framework in this paper, my results are not directly comparable to theirs. Besides, all the characteristics are exogenous in these papers and pre-marital investment is not considered.

A second strand of literature is on the marriage squeeze and the age of marriage. The marriage squeeze refers to an imbalance between the numbers of marriageable men and women in the marriage market, potentially due to war, unbalanced gender ratio at birth or shocks to population growth rate since men on average marry younger women. The key potential margin of adjustment to such imbalances is via the age gap at marriage. If the age gap increases in response to excess men in the marriage market, such an imbalance can be reduced, if not eliminated. Existing empirical studies find that the marriage age exhibits considerable flexibility to accommodate the marriage squeeze problem (Bergstrom and Lam, 1989b; Brandt et al., 2008; Abramitzky et al.,

[2]Lindenlaub (2014) provides an excellent survey of the literature on multidimensional matching under transferable utility

2011). For example, Brandt et al. (2008) study the marriage market consequence of the Chinese 1959-1961 famine, and find that the marriage rates of the famine born cohort were not significantly changed. This result is noteworthy given this famine reduced the cohort size by 75%. On the other hand, applied theoretical studies of the age gap between spouses usually either assume age is the only trait that differentiates people, or assume that other socioeconomic characteristics are independent of age (Bergstrom and Lam, 1989a; Anderson, 2007; Bhaskar, 2013). Although such treatment offers a simple and elegant way to study the age gap problems, it comes at the cost of realism. People's socioeconomic status such as income, wealth, profession, education and physical conditions do change with age, and usually these characteristics affect people's choice of marriage age.

This paper is also related to the status good literature, such as Cole et al. (1992), Hopkins and Kornienko (2004) and Bhaskar and Hopkins (2013). One common insight of these papers is that if individuals care about their status defined as their rank in the distribution of spending on one "positional good", consumers' choices will be strategic and depend on the consumption choices of others. In such a situation, spending on the "positional good" generally shows an over-spending pattern. While this literature focuses on welfare analysis of an exogenously changed income distribution, my primary goal is to understand the effect of marriage market imbalance on an individual's choice of saving when each man's ranking in the marriage market is determined by his wealth relative to others.

Finally, there is a sizable literature offering explanations of the savings rate puzzle in China from different perspectives[3]. Among these papers, in Du and Wei (2013) a life-cycle matching model which also highlights the competitive saving motive is

[3]Scholars have offered explanations for the savings rate puzzle from perspectives such as life cycle consideration (Modigliani and Cao, 2004), precautionary saving with borrowing constraints (Wen, 2010; Chamon et al., 2013), demographic change (Wei and Zhang, 2011;

developed. They assume single men are endowed with the same income and save the same amount. In equilibrium, everyone's ranking in the marriage market is determined by a random draw of pizazz. When their model is calibrated to the Chinese economy, they find that the aggregate savings rate will increase by 6% of GDP as the gender ratio rises from 1 to 1.15, which is similar to the result in this paper. What makes this paper different from theirs is that the marriage market here is dynamic in which the age gap between spouses can adapt in response to marriage market imbalances. As illustrated in the calibration, allowing for an endogenous change of the age gap significantly tempers the pressure on the aggregate savings rate from an unbalanced gender ratio. In addition, this paper offers an explanation for the downward-sloping age profile of the savings rate in China.

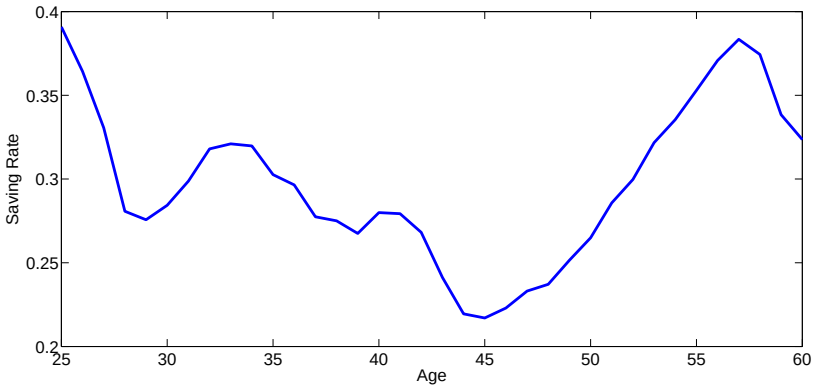
The contributions of this paper are both theoretical and empirical. On the theoretical aspect, to the best of my knowledge this is the first paper which solves the stable matching patterns in a multi-dimensional matching framework with pre-marital investment. On the empirical aspect, this theory can potentially help to understand the savings rate puzzle in China on both macroeconomic and microeconomic levels.

1.3 Stylized Facts

In this section, I show that the concerns for marriage prospects play an important role in forming the downward-sloping age-saving profile in China documented in earlier works (e.g. Chamon and Prasad, 2010; Song and Yang, 2010).

Du and Wei, 2013; Choukhmane et al., 2014), and pension and education reforms (Chamon and Prasad, 2010). See Bussiere et al. (2013) for a review of literature on savings in China.

Figure 1.1: Age-Savings Rate Profile in 2002



I compute the age profile of the savings rate using the 2002 round of the urban sample in the China Household Income Project (CHIP) survey[4]. Following Chamon and Prasad (2010), income and consumption profiles at each age are smoothed by a three-year moving average which includes the age cohorts immediately above and below the given age. The savings rate is defined as 1 minus the ratio of consumption over disposable income. The resulting age profile is plotted in Figure 1.1 which features a U-shaped age profile of the savings rate[5].

In Figure 1.2, I divide the whole sample into two sub-samples with respect to their marital status: single or married, and plot the age-saving profiles for each group. The

[4]The CHIP survey samples are sub-samples of the National Bureau of Statistics annual urban household survey sample. The 2002 CHIP urban samples contains about seven thousand households from 12 provinces which represent major regions in China. Detailed explanation of the data can be found in Li et al. (2008).

[5]Chamon and Prasad (2010) and Song and Yang (2010) document similar results using the annual Urban Household Survey (UHS) conducted by the National Bureau of Statistics. The dataset they use is not publicly available.

age profile of the whole sample is replicated here for the ease of comparison[6]. Two observations stand out. First, at earlier ages, single households save more than married households at each age. Second, the downward-sloping pattern is more pronounced in the single sub-sample. There is actually no such pattern in the married group. The age profile of the whole sample is a weighted average of the age profiles of the two sub-samples and is very close to the profile of married group after age 32 when most people are married.

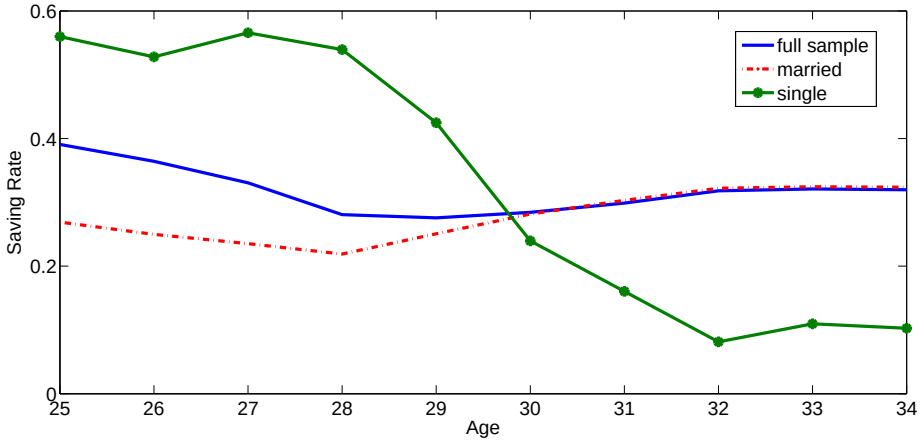
Although these patterns cannot be used directly as evidence for the competitive saving motive as an explanation to the savings rate puzzle, since people with different socioeconomic characteristics may choose their marriage ages differently, they do suggest a link between the savings rates and marital status in the younger cohorts. In the following sections, I model how people's marriage ages and saving behaviors are jointly determined by the marriage market conditions.

1.4 The Benchmark Model

To illustrate how people's saving behavior can be affected by mating considerations, I construct an economy populated by overlapping generations of two-period lived men m and women w whose ages are denoted respectively as young and old. Each generation is characterized by a *gender ratio at birth* R^* defined as the ratio of men to women. Since the focus of this paper is to show how people's savings are affected by an increase in the gender ratio, I assume $R^* \geq 1$. Population in each generation is normalized to one with measures of young men and young women as $M = \frac{R^*}{1+R^*}$ and $W = \frac{1}{1+R^*}$ respectively.

[6] Only the age profile of people younger than 34 is plotted because there are few observations of single households with age above 35 (0.39% of the case), making the estimates of the average savings rates in that range highly inaccurate.

Figure 1.2: Age-Savings Rate Profile with Different Marital Status



Documented empirical patterns on mate preference indicate that males and females have different priorities about traits of potential partners. Men usually choose appearance and personality as their top priority while women on the other hand choose occupation and income[7]. To be consistent with these empirical findings, men and women are treated differently in the model.

Each young man is endowed with a level of wealth $x_{m,1}$ at birth which is an independent draw from an exogenous distribution $H_{m,1}(x)$. $H_{m,1}(x)$ has positive support $[\underline{x}, \overline{x}]$ and density function $h_{m,1}(x) \in (0, +\infty)$, $\forall x_{m,1}$. In contrast, wealth endowments for young women are identical and are normalized to zero. In the meantime, women are distinguished by another trait: a match quality indicator z drawn from a distri-

[7]Using data from a Korean online dating website, Lee (2009) finds around 80% of males choose appearance and personality as their top priority. In contrast, most often (55.6% in the sample) females choose occupation and income. Similar patterns are found in Fisman et al. (2006) and Hitsch et al. (2010). Although we haven't seen much empirical results about mate preference in China, demand for brideprice (usually in the form of housing) is a common practice, while dowry payment is voluntary and not known to the groom's family before marriage.

bution $H_z(z)$ with positive support $[\underline{z}, \overline{z}]$ and density function $h_z(z) \in (0, +\infty), \forall z$. z represents a woman's ability to generate emotional utility in marriage, which may include attractiveness, personality, intelligence, health and ability for domestic work. In addition, everyone earns the same labor income wn at each age where w is the market wage rate and n is a constant labor endowment.

A marriage only happens between a man and a woman at the end of age one if they mutually agree to get married. There are two benefits associated with marriage. First, consumption in a married household may have a public good feature such that individuals consume more in total than aggregate household consumption. For example, couples can share the same couch. The second benefit is happiness from marriage which is determined by the match quality indicator z of the wife.

1.4.1 Utility Function

In the following part of this paper, I label the marital status of an agent by superscript **a** if (s)he is married, or by superscript **s** if (s)he is single.

1.4.1.1 Utility Function of Men

A single man allocates his liquid wealth $x_{m,1} + wn$ in the first period between consumption $c_{m,1}$ and saving $s_{m,1}$, taking other agents' saving choices as given. In the next period, if he remains single, his consumption is the sum of his saving from the first period and labor income in the second period which is

$$c_{m,2}^{\mathbf{s}} = (1 + r)s_{m,1} + wn$$

in which r is the interest rate of saving.

If he is married, he and his wife pool their pre-marital savings and income and thus his second-period consumption is

$$c^{\mathbf{a}}_{m,2} = \kappa[(1+r)(s_{m,1}+s_{w,1})+2wn] \tag{1.1}$$

in which $s_{w,1}$ is the pre-marital saving of his wife, and parameter $\kappa \in [\frac{1}{2}, 1]$ captures the public good feature of consumption in a married household.

The optimization problem for a young man with endowed wealth $x_{m,1}$ is:

$$\begin{aligned} V_{m,1}(x_{m,1}) = \max_{s_{m,1}} & \{\ln c_{m,1} + \beta \mathbb{E}(\ln c_{m,2} + z)\} \\ & s.t. \\ & c_{m,1} + s_{m,1} \leq x_{m,1} + wn;\ s_{m,1} \geq 0 \\ & c_{m,2} = \kappa[(1+r)(s_{m,1}+s_{w,1})+2wn], \quad \text{if he is married} \\ & c_{m,2} = (1+r)s_{m,1} + wn, \quad \text{if he remains single} \end{aligned} \tag{1.2}$$

In Equation (1.2), β is the discount factor which satisfies the following assumption: $\beta(1+r) = 1$. z represents happiness from marriage which equals the match quality of his wife if he is married or 0 if he remains single in the second period. Notice that a man's life-time utility depends on whether and with whom he is matched. The matching pattern will be clear when we come to the description of marriage market.

1.4.1.2 Utility Function of Women

Women's utility function can be described similarly except that their wealth endowment is normalized to 0. I assume each female's match quality z is realized after choice of saving $s_{w,1}$ at age 1. In addition, $s_{w,1}$ is not publicly observable and thus there's no strategic saving consideration on the women's side. Thus all women have the same

optimization problem:

$$V_{m,1} = \max_{s_{w,1}} \{\ln c_{w,1} + \beta \mathbb{E}(\ln c_{w,2} + z)\} \tag{1.3}$$

$$s.t.$$

$$c_{w,1} + s_{w,1} \leq wn;\ s_{w,1} \geq 0$$

$$c_{w,2} = \kappa[(1+r)(s_{m,1} + s_{w,1}) + 2wn], \quad \text{if she is married}$$

$$c_{w,2} = (1+r)s_{w,1} + wn, \quad \text{if she remains single}$$

1.4.2 Marriage Market

Marriage is monogamous and formed under mutual consent. Everyone wants to achieve higher utility after marriage, i.e., $\ln\{\kappa[(1+r)(s_{m,1} + s_{w,1}) + 2wn]\} + z$. Consequently, men are ranked by pre-marital saving $s_{m,1}$ while women are ranked by the realized match quality indicator z since all women's saving $s_{w,1}$ are identical in equilibrium.

Assumption 1.1 *The lower bound of the match quality indicator* $\underline{z} \geq -\log \kappa$.

Lemma 1.1 *Given Assumption 1.1, all matches are acceptable. All agents prefer getting married with any agent of the opposite gender rather than remaining single.*

Proof. A man with saving $s_{m,1}$ compares utility from marriage $\ln\{\kappa[(1+r)(s_{m,1} + s_{w,1}) + 2wn]\} + z$ and remaining single $\log[(1+r)s_{m,1} + wn]$. It is easy to check that the utility from marriage is the larger one. Comparison for women's utilities is similar.

I assume the matching process follows the Gale-Shapley deferred acceptance procedure (Gale and Shapley, 1962; Roth and Sotomayor, 1992), and without search frictions, the stable matching is in principle an assignment game.

Lemma 1.2 *Matching is positively assortative between men's saving $s_{m,1}$ and women's match quality indicator z.*

Proof. Since utility after marriage is increasing in men's pre-marital saving and women's match quality, if we take two men with pre-marital saving levels of $s_{m,1}$ and $s_{m,1}^{'}$ such that $s_{m,1}^{'} > s_{m,1}$, and two women with match quality indicators z and $z^{'}$ such that $z^{'} > z$, the man with $s_{m,1}^{'}$ is preferred by both women and the woman with $z^{'}$ is preferred by both men. As a result, the man with $s_{m,1}^{'}$ matches with the woman with $z^{'}$, and the other two match with each other.

Denote the stationary distribution of men's saving $s_{m,1}$ in the marriage market as $G_s(s)$ with support $[\underline{s}, \overline{s}]$ which is determined endogenously by young men's saving choices. Matching happens along the distributions of $G_s(s)$ and $H_z(z)$ from top to bottom. If $G_s(s)$ has a mass point at s_0, men with saving s_0 match with equal probabilities to women with match quality z in the corresponding interval (See Appendix 1.9.1 for the description of the matching pattern).

If $R^* > 1$, men at the bottom of distribution $G_s(s)$ cannot get matched. Denote the marginal matched saving level

$$s_{m,1}^* = G_s^{-1}(1 - \frac{1}{R^*})$$

[8]Men who choose saving below $s_{m,1}^*$ are expected to remain single at age 2. I assume saving behavior of men is *symmetric* such that men with the same wealth endowment $x_{m,1}$ choose the same level of saving $s_{m,1}$. In other words, there is a saving function $s_{m,1} = s(x_{m,1})$.

Definition 1.1 *An **equilibrium** is a saving function of men $s(x_{m,1})$, a distribution of men's saving in marriage market $G_s(s)$, women's saving choice $s_{w,1}$ as well as*

[8] $G_s^{-1}(p)$ is the quantile function of $G_s(s)$ such that $G_s^{-1}(p) = \inf\{p \leq G_s(s),\ \forall s \in [\underline{s}, \overline{s}]\}$

a matching rule between men's pre-marital saving $s_{m,1}$ and women's realized match quality indicator z such that:

1. *Under the matching rule, matching is positively assortative between $s_{m,1}$ and z. The detailed matching pattern is given in Appendix 1.9.1.*

2. *Saving function $s(x_{m,1})$ is optimal for men such that given $G_s(s)$, $s_{w,1}$ and the assortative matching rule, $s(x_{m,1})$ solves men's optimization problem in Equation (1.2), $\forall x_{m,1}$.*

3. *Given $G_s(s)$ and the matching rule, $s_{w,1}$ solves women's optimization problem in Equation (1.3).*

4. *$G_s(s)$ is consistent with men's saving choice and the distribution of endowment such that $G_s(s^*) = prob\{s(x) \leq s^* | x \sim H_{m,1}(x)\}$.*

For the moment, let's assume there is no mass point in $G_s(s)$. I show this is the case in Lemma 1.4. Based on Part 1 of Definition 1.1, if a woman draws match quality indicator z, she matches with a young man with saving

$$s_{m,1}(z) = G_s^{-1}[1 - \frac{1}{R^*}(1 - H_z(z))] \tag{1.4}$$

On the other hand, a man with saving $s_{m,1}$ matches with a woman with z such that

$$z(s_{m,1}) = H_z^{-1}[1 - R^*(1 - G_s(s_{m,1}))] \tag{1.5}$$

A competitive saving motive arises in this situation and $s_{m,1}$ becomes a "positional good" in the sense that a man not only cares about his absolute value of saving, but also his saving relative to others because whether he can get married and his match quality are determined by his ranking in the saving distribution[9].

[9]I use the term "positional good" in the same way as in Cole et al. (1992) such that agents do not value position itself, but treat it as an instrument for better consumption or marriage opportunities.

Lemma 1.3 *With gender ratio $R^* \geq 1$, women choose first period saving $s_{w,1} = 0$.*

Proof. Based on Lemma 1.1, all women are matched no matter what their realized z is. Thus Equation (1.3) reduces to $\max\{\ln(wn - s_{w,1}) + \beta E_z[\ln \kappa[(1+r)(s_{m,1}(z) + s_{w,1}) + 2wn] + z]\}$ in which $s_{w,1}$ is restricted to be nonnegative. The first-order condition for $s_{w,1}$ shows 0 saving is optimal for women.

No women save since they have constant labor income at each age so there is no need to smooth consumptions by saving. In addition, they can take advantage of the pre-marital savings of future husbands.

1.4.3 Equilibrium Saving Function

Lemma 1.4 *Equilibrium saving function $s(x_{m,1})$ is strictly increasing. In addition, it is continuous and differentiable except for a discrete jump at point $x^*_{m,1} = H^{-1}_{m,1}(1 - \frac{1}{R^*})$.*

Proof. Proof is given in Appendix 1.9.2.

The saving function is increasing since richer people can always achieve the same level of saving with less utility loss in the first period. With an increasing saving function, a man's ranking in the marriage market is the same as his ranking in endowment:

$$G_s[s(x_{m,1})] = H_{m,1}(x_{m,1}) \tag{1.6}$$

Using words from Hopkins and Kornienko (2004), "it takes all the running to keep in the same place." Consequently, men with endowments larger than $x^*_{m,1}$ are matched by the matching rule in Equation (1.5), while men with endowments below $x^*_{m,1}$ remain single for life. In particular, men whose endowment is exactly $x^*_{m,1}$ match with women of match quality $\underline{z}$. For men expected to be single, their optimization problem is solved

by imposing $\delta_m = 0$ in Equation (1.2) and we can get single men's saving function and lifetime utility as

$$s^{\mathbf{s}}(x_{m,1}) = \frac{\beta}{1+\beta} x_{m,1} \tag{1.7}$$
$$V^{\mathbf{s}}_{m,1}(x_{m,1}) = (1+\beta)\log(\frac{x_{m,1}}{1+\beta} + wn)$$

For $x_{m,1} \geq x^*_{m,1}$, the reason $s(x_{m,1})$ is strictly increasing and continuous follows the standard "no atoms, no holes" argument in labor search theory such that any mass point or discontinuity in $G_s(s)$ for $x_{m,1} > x^*_{m,1}$ creates a profitable deviation for some men and thus should be ruled out in equilibrium. Lastly, the saving function is discontinuous at $x^*_{m,1}$ because competition forces a man with wealth $x^*_{m,1}$ to raise his saving high enough such that he is indifferent between matching with women $\underline{z}$ and remaining single.

Based on Lemma 1.4 and Condition 4 in Definition 1.1, $G_s(s)$ is differentiable too. I denote its density function as $g_s(s)$.

Incorporating matching rule $z(s_{m,1})$ in Equation (1.5), young men with wealth $x_{m,1} \geq x^*_{m,1}$ solve the optimization problem

$$\max_{s_{m,1}} \{\log(x_{m,1} + wn - s_{m,1}) + \beta \log[\kappa((1+r)s_{m,1} + 2wn)] + \beta z(s_{m,1})\} \tag{1.8}$$

Differentiating with respect to $s_{m,1}$, we have the following first-order condition:

$$\frac{1}{x_{m,1} + wn - s_{m,1}} = \frac{1}{(1+r)s_{m,1} + 2wn} + \frac{\beta R^* g_s(s_{m,1})}{h_z[1 - R^*(1 - G_s(s_{m,1}))]} \tag{1.9}$$

After plugging in Equation (1.6), we have the following first-order differential equation for the saving function $s(x_{m,1})$:

$$s^{'}(x_{m,1}) = \frac{\beta\varphi(x_{m,1})}{\frac{1}{x_{m,1}+wn-s_{m,1}} - \frac{1}{(1+r)s_{m,1}+2wn}} \tag{1.10}$$

where $\varphi(x_{m,1}) = \frac{R^* h_{m,1}(x_{m,1})}{h_z[1-R^*(1-H_{m,1}(x_{m,1}))]}$ denotes the ratio of the marginal increase in the endowment position to the marginal change in the corresponding match quality.

Proposition 1.1 *The unique equilibrium saving function $s(x_{m,1})$ takes the following form: $\forall x_{m,1} < x^*_{m,1}$, $s(x_{m,1}) = s^s(x_{m,1}) = \frac{\beta}{1+\beta} x_{m,1}$; $\forall x_{m,1} \geq x^*_{m,1}$, $s(x_{m,1})$ is solved by Equation (1.10) with the following boundary condition:*

1. *If $R^* = 1$, everyone gets matched and the boundary condition is $s(\underline{x}) = \max[\frac{1}{2+r}(\underline{x} - wn), 0]$.*

2. *If $R^* > 1$, $s(x^*_{m,1})$ is pinned down by the following equation:*

$$\begin{aligned}&(1+\beta)\log(\frac{x^*_{m,1}}{1+\beta} + wn)\\ =&\log[x^*_{m,1} + wn - s(x^*_{m,1})] + \beta\log[\kappa((1+r)s(x^*_{m,1}) + 2wn)] + \beta\underline{z}\end{aligned}$$

Proof. Proof is given in Appendix 1.9.3.

Proposition 1.1 shows that differential saving equation (1.10) is indeed optimal saving for any single man, given that other people adopt this saving strategy as well. With a balanced gender ratio $R^* = 1$, everyone gets matched. In particular, a man with the lowest endowment $\underline{x}$ is determined to match with a woman of the lowest match quality $\underline{z}$. No matter how much he saves, other men can trump him by saving more as long as it is profitable to do so. As a result, men with $\underline{x}$ simply take their position and matching prospects as given, and choose saving according to Equation (1.12) as described below. Otherwise if $R^* > 1$, only men with endowment higher than $x^*_{m,1}$ get matched. Due to competition, saving of the marginal man with $x^*_{m,1}$ is raised to the level such that he is indifferent between matching with $\underline{z}$ and remaining single. Major findings in this section are summarized by Proposition 1.2.

Proposition 1.2 *The marriage market equilibrium with pre-marital investment **exists** and is **unique**, in which young women do not save while young men's saving*

choices are described by Proposition 1.1, and matching is positively assortative between men's saving and match quality of women.

Proof. The existence of a marriage market equilibrium with pre-marital investment is proved by Lemma 1.2, Lemma 1.3 as well as Proposition 1.1. Since any equilibrium saving function is described by Equation (1.10), uniqueness simply follows from the fundamental theorem of differential equation with boundary condition provided in Proposition 1.1.

For the sake of comparison, I also compute the *cooperative saving function* $s^{\mathbf{c}}(x_{m,1})$ when each young man takes his endowed position $H_{m,1}(x_{m,1})$ as given. In this case, men with endowment below $x^*_{m,1}$ cannot get matched and thus $s^{\mathbf{c}}(x_{m,1}) = s^{\mathbf{s}}(x_{m,1})$, $\forall x_{m,1} < x^*_{m,1}$. For men with wealth higher than $x^*_{m,1}$, his match quality is $\hat{z}(x_{m,1}) = H_z^{-1}[1 - R^*(1 - H_{m,1}(x_{m,1}))]$ and his optimization problem is

$$\max_{s^{\mathbf{c}}_{m,1}} \{\log(x_{m,1} + wn - s^{\mathrm{C}}_{m,1}) + \beta \log[\kappa((1+r)s^{\mathrm{C}}_{m,1} + 2wn)] + \beta \hat{z}(x_{m,1})\}$$

with first order condition

$$\frac{1}{x_{m,1} + wn - s^{\mathbf{c}}_{m,1}} = \frac{1}{(1+r)s^{\mathbf{c}}_{m,1} + 2wn} \tag{1.11}$$

and

$$s^{\mathbf{c}}(x_{m,1}) = \max[\frac{1}{2+r}(x_{m,1} - wn), 0] \tag{1.12}$$

Comparing Equation (1.9) and (1.11), the last term in (1.9) clearly shows how the competitive saving motive arises when a man's status is affected by his saving. However, all those extra savings are "wasted" in the sense that no man's position in the marriage market changes from their positions in wealth endowment. But we cannot claim that cooperative saving is a Pareto improvement relative to the competitive saving case since all the women benefit from men's saving and they strictly prefer men to save more.

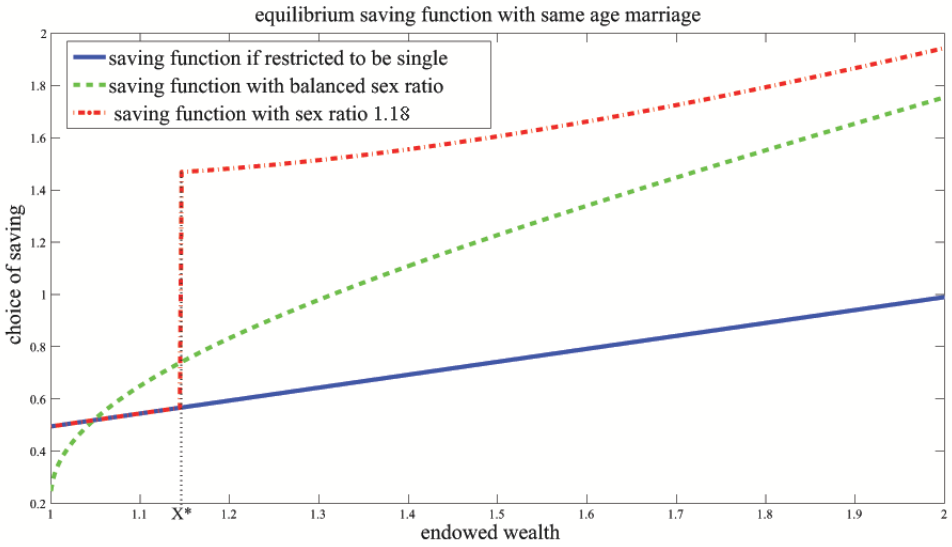


Figure 1.3: equilibrium saving functions with same age marriage

1.4.4 Numerical Example with Same Age Marriage

An illustrative example is given in Figure 1.3. $H_{m,1}(x)$ and $H_z(z)$ are both uniform distributions over the interval $[1, 2]$, labor income wn is set to 0.5 and the discount factor β is chosen to be 0.98. Differential equation (1.10) is solved numerically by the implicit Euler method[10]. The solid line is the saving function for single men $s^{\mathbf{s}}(x)$ and its expression is given by Equation (1.7). With consumption smoothing considerations only and a logarithmic utility function, $s^{\mathbf{s}}(x)$ is proportional to endowed wealth. The dashed line is the equilibrium saving function with $R^* = 1$ in which everyone gets matched. The man with the lowest wealth endowment chooses the cooperative level of saving since he won't attempt to improve his position. His saving level is lower than $s^{\mathbf{s}}(x)$ since he can partly free ride on his wife's future income. However, as

[10] See Page 343 of Judd (1998) for a description of this method.

men's wealth increases, the competitive saving motive arises in interval $(1, 2]$ and the equilibrium savings rate is thus higher than cooperative saving in general.

The dotted line in Figure 1.3 represents equilibrium saving function with $R^* = 1.18$. With an unbalanced gender ratio, there is a jump in the saving function at $x^* = (1 - \frac{1}{R^*})(\overline{x} - \underline{x}) + \underline{x}$. Men at the bottom of the wealth distribution cannot get matched and thus choose $s^{\mathbf{s}}(x)$. At wealth level x^* representing the threshold wealth of getting matched, saving jumps to a point where men with wealth x^* are indifferent between getting matched with match quality $\underline{z}$ and remaining single. The part with $x_{m,1} > x^*$ is characterized by differential equation (1.10).

One key observation here is that a rising gender ratio does not necessarily lead to an increase in aggregate saving, since two opposing effects might arise. As R^* increases from 1 to 1.18[11], on the one hand, men staying in the marriage market raise their saving in a competitive manner; on the other hand, men with the lowest wealth can't get matched in equilibrium and thus quit the saving competition. That's why we see that part of the saving function with an unbalanced gender ratio lies below the saving function with balanced gender ratio. Whether the net effect on aggregate saving is positive or negative depends on the chosen distributions and parameters. I'll address this question in section 1.6 where the model is calibrated to the Chinese economy.

1.5 Marriage Model with Endogenous Age Gap

Section 1.4 illustrates how men's saving choices can be affected if their marriage prospects are determined by their saving relative to others. However, the marriage market considered in the previous section is in essence static and reactions other than changes in savings are not considered. Many studies in economics and demography indicate that the marriage market exhibits great flexibility to the marriage squeeze

[11] China's gender ratio at birth in 2011, data from National Bureau of Statistics

problem and different responses emerge with marriage market imbalances, such as changes in the age gap between spouses, changes in remarriage rates of divorced and widowed people as well as gender biased immigration (Edlund and Lee, 2009; Ebenstein and Sharygin, 2009; Abramitzky et al., 2011). All these responses are likely to reduce the pressure of gender ratio imbalances.

1.5.1 Some Data Patterns on Pre-marital Saving and Age of Marriage

I incorporate the possibility of an age gap between spouses into the model for the following two reasons. First, a change in the age gap is considered to be one of the most important reactions to the marriage squeeze problem. For example, Anderson (2007) studies a similar topic and shows that after age gap responses are considered, an excess of women in the marriage market actually causes dowry deflation instead of inflation. As described in Figure 1.4, the age gap at marriage in China has indeed been increasing since the mid 1990s which is consistent with the changes in gender ratio in the marriage market.

Second, the downward-sloping age profile of the savings rate in the pre-marital cohorts naturally requires consideration about marriage timing. Results from estimating the following regression show there is a negative correlation between wealth and marriage age in China such that men delaying marriage are mostly those with low economic status.[12,13]

[12] Existing literature offers two contrasting hypotheses about the relation between men's earnings (wealth) and their marital age. On the one hand, Keeley (1977) argues the correlation between men's income and their marital age should be negative since men with higher earnings should benefit more from specialization of household production. On the other hand, Bergstrom and Bagnoli (1993) suggest men with high productivity will delay marriage until their productivities are revealed to the public at later ages. Zhang (1995) and Danziger and Neuman (1999) empirically test these two hypothesis and find mixed results.

[13] The regression in Equation 1.13 is similar to the one used in Zhang (1995) except that I use the age gap between spouses as the dependent variable instead of marriage age of the husband.

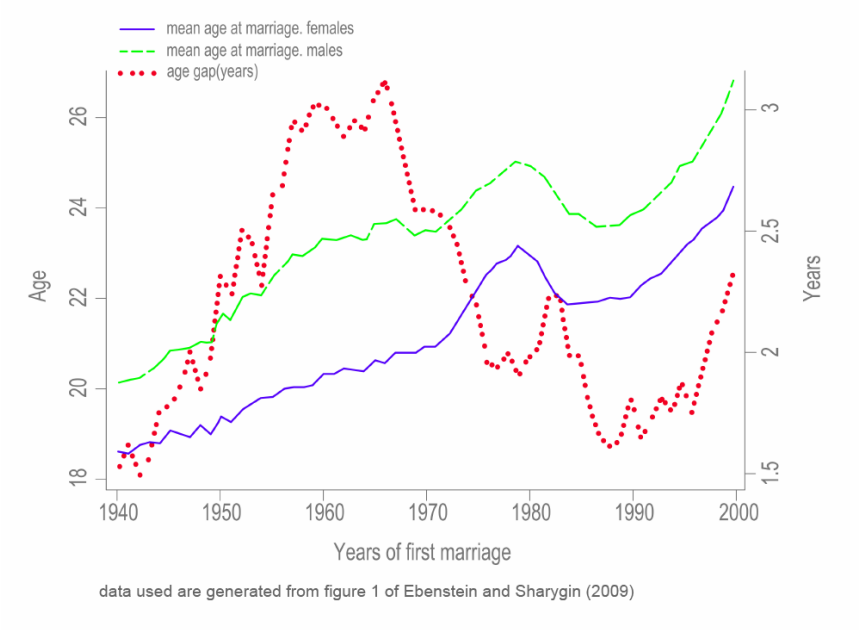


Figure 1.4: marriage age and age gap in China

$$\text{age gap of spouse} = \alpha_0 + \alpha_1 \text{wage} + \alpha_2 \text{education} + \alpha_3 X + \epsilon \tag{1.13}$$

In the regression equation above, X is a list of marriage year and province dummies. Using data from all 9 waves of China Health and Nutrition Survey[14] with observations restricted to married men aged 18-60 years, I find α_1 significantly negative. In other words, men with higher earnings marry earlier and thus have smaller age gap to the ages of their wives. This result is robust with and without an education dummy which takes value 1 if a man has above high-school education and 0 otherwise. Estimations of different specifications are reported in Table 1.1, with numbers in brackets as the absolute values of the corresponding t-statistics. Consequently, a

[14]This dataset is described in detail in Subsection 1.7.1.

Table 1.1: Age Gap between Spouses in China

Variable	**with Education**	**without Education**
Wage (in 2009 dollars)	**-0.101**	**-0.110**
	(5.62)	(5.74)
Marriage Year	**0.556**	**0.550**
	(68.29)	(68.08)
Education	**-1.005**	–
	(5.78)	
R^2	0.29	0.288
F	456.77	498.49

model with a competitive saving motive incorporating the age gap between spouses should be able to generate a similar relationship between the age gap of spouses and the pre-marital wealth (income) of the husband.

1.5.2 Model

I modify the framework in Section 1.4 as follows. Each agent lives for 3 periods: young, middle-aged and old with ages as $j_g = 1, 2, 3 \; \forall g = m, w$. A person may choose to enter the marriage market at either age 1 or age 2 but not both. Each young man allocates his liquid wealth $x_{m,1} + wn$ between consumption $c_{m,1}$ and saving $s_{m,1}$ and then decides his marital age. If he postpones marriage to age 2, next period he becomes middle-aged, allocates liquid wealth $x_{m,2} = (1 + r)s_{m,1} + wn$ between age 2 consumption $c_{m,2}$ and saving $s_{m,2}$, and then joins the marriage market. Given the same wealth endowment, middle-aged single men are potentially wealthier than their young counterparts since they have one more period to save. Single women choose their savings and marital age in the same way as men do except that their level of savings, $s_{w,1}$ and $s_{w,2}$ are not observable before marriage, and their wealth endowment

is normalized to 0. Women choose their marital age before their matching qualities are realized. Unmarried women cannot enjoy match quality z. But after marriage, a woman starts to enjoy z every period even in widowhood. Since the focus here is the marital age of the first marriage, divorce and remarriage are not considered.

1.5.2.1 Utility Function of Married Household

I first describe the utility functions of a married household which single agents use to decide when and with whom to get married. With an endogenous age gap, there are 4 potential age combinations for married couples: $(j_m, j_w) \in \{(2,2), (2,3), (3,2), (3,3)\}$. For notation, I use $A^{\mathbf{a}}_{g,j_m j_w}$ to denote a variable A of husband $(g = m)$ or wife $(g = w)$ in a married household with the age combination (j_m, j_w).

Couples who are both of age 3 simply consume all their family liquid wealth $x^{\mathbf{a}}_{33} = (1+r)s^{\mathbf{a}}_{22} + 2wn$, which is the sum of saving from previous period and current labor incomes. Newlyweds still pool their pre-marital savings such that $s^{\mathbf{a}}_{22} = s^{\mathbf{s}}_{m,2} + s^{\mathbf{s}}_{w,2}$. Thus their utility functions are

$$\begin{aligned} & U^{\mathbf{a}}_{m,33}(s^{\mathbf{a}}_{22}) = U^{\mathbf{a}}_{w,33}(s^{\mathbf{a}}_{22}) = \ln(\kappa x^{\mathbf{a}}_{33}) + z \\ & s.t. \\ & x^{\mathbf{a}}_{33} = (1+r)s^{\mathbf{a}}_{22} + 2wn \end{aligned} \tag{1.14}$$

For couples who are both of age 2, given family wealth $x^{\mathbf{a}}_{22} = (1+r)s^{\mathbf{a}}_{11} + 2wn$, their maximization problem is

$$\begin{aligned} U^{\mathbf{a}}_{m,22}(s^{\mathbf{a}}_{11}) &= U^{\mathbf{a}}_{w,22}(s^{\mathbf{a}}_{11}) \\ &= \max_{s^{\mathbf{a}}_{22}} \{\ln[\kappa(x^{\mathbf{a}}_{22} - s^{\mathbf{a}}_{22})] + z + \beta[\ln(\kappa x^{\mathbf{a}}_{33}) + z]\} \\ s.t &: \\ x^{\mathbf{a}}_{22} &= (1+r)s^{\mathbf{a}}_{11} + 2wn \\ x^{\mathbf{a}}_{33} &= (1+r)s^{\mathbf{a}}_{22} + 2wn;\ s^{\mathbf{a}}_{22} \geq 0 \end{aligned}$$

and from the first-order condition we get

$$\begin{aligned} s^{\mathbf{a}}_{22} &= \frac{1}{1+\beta} s^{\mathbf{a}}_{11} \\ U^{\mathbf{a}}_{m,22}(s^{\mathbf{a}}_{11}) &= U^{\mathbf{a}}_{w,22}(s^{\mathbf{a}}_{11}) \\ &= (1+\beta)\ln[\kappa(\frac{s^{\mathbf{a}}_{11}}{\beta+\beta^2} + 2wn)] + (1+\beta)z \end{aligned} \tag{1.15}$$

The optimization problem for the other 2 age combinations require further assumptions since if one spouse is older than the other, the younger one will be a widow(er) in the next period and thus the optimal level of saving for one spouse is different from the optimal level for the other one. For simplicity I assume that if at least one spouse reaches age 3, the family will not save into the future, and thus the widow(er) can only live on his or her labor income wn next period[15]. Consequently, for a family with an age 3 husband and an age 2 wife, the value functions for each spouse are:

$$U^{\mathbf{a}}_{m,3,2}(s^{\mathbf{a}}_{21}) = \ln[\kappa((1+r)s^{\mathbf{a}}_{21} + 2wn)] + z \tag{1.16}$$

$$U^{\mathbf{a}}_{w,3,2}(s^{\mathbf{a}}_{21}) = \ln[\kappa((1+r)s^{\mathbf{a}}_{21} + 2wn)] + \beta\ln(wn) + (1+\beta)z \tag{1.17}$$

Similarly, for a household with age combination $(2,3)$, the husband is younger and will be a widower next period. The corresponding value functions are:

$$U^{\mathbf{a}}_{m,2,3}(s^{\mathbf{a}}_{12}) = \ln[\kappa((1+r)s^{\mathbf{a}}_{12} + 2wn)] + z + \beta\ln(wn) \tag{1.18}$$

$$U^{\mathbf{a}}_{w,2,3}(s^{\mathbf{a}}_{12}) = \ln[\kappa((1+r)s^{\mathbf{a}}_{12} + 2wn)] + z \tag{1.19}$$

1.5.2.2 Stable Matching in the Marriage Market

At period t, for ages $j = 1, 2$, I denote the measure of single men and women in the marriage market as $M_{j,t}$ and $W_{j,t}$, and the distributions of men's saving as $G_{s,j,t}(\cdot)$, $s_{m,j,t} \in [\underline{s_{j,t}}, \overline{s_{j,t}}]$. Here $M_{1,t} = \frac{R^*}{1+R^*}$ and $W_{1,t} = \frac{1}{1+R^*}$ are exogenous, while $M_{2,t}$, $W_{2,t}$

[15] This assumption is relaxed in Section (1.6) in which couples jointly maximize the household value function which is a weighted sum of value functions of each spouse. Under this setup, the stable matching patterns derived in this section still hold.

and the distributions of saving are endogenous resulting from equilibrium choices of saving and marriage age.

Each agent in the marriage market has 2 characteristics: saving and age $\{s_m, j_m\}$ for men, and match quality indicator and age $\{z, j_w\}$ for women. Notice that unlike what is typically assumed in the matching literature, the two characteristics of men and women are not independent.

Definition 1.2 *Given measures of single agents and distributions of men's saving at each age at period t, a* ***matching*** *at period t is a measure μ on the set $[\underline{z}, \overline{z}] \times \{1,2\} \times [\underline{s_t}, \overline{s_t}] \times \{1,2\}$ such that its marginal distributions equal the distributions of the corresponding groups.*

Intuitively, for set X of women $\subseteq [\underline{z}, \overline{z}] \times \{1,2\}$ and set Y of men $\subseteq [\underline{s_t}, \overline{s_t}] \times \{1,2\}$, $\mu(X, Y)$ denotes the probability that a woman in set X marries a man in set Y. A match is *stable* if no matched agent prefers remaining single to staying with his or her current match, nor do a man and woman both prefer to leave their current situation to form a new match. I first show that if a match is stable, matching is positively assortative between men's saving and women's match quality in each age combination of spouses.

At period t, I denote men's choice of marital age as $\delta_{m,j,t}(s_{m,j,t})$, $\forall j = 1, 2$. $\delta_{m,1,t}(s_{m,1,t}) = 1$ if a young man with saving $s_{m,1,t}$ gets married this period, and 0 otherwise. $\delta_{m,2,t}(s_{m,2,t}) = 1$ if a single middle-aged man with saving $s_{m,2,t}$ gets married this period, and 0 otherwise. Marital indicators for women $\delta_{w,j,t}$, $\forall j = 1, 2$ are defined similarly. Since all women are ex ante identical before the marriage market and make the same choice on marital age, $\delta_{w,1,t}$ and $\delta_{w,2,t}$ are the same for all women in each age cohort. Before we proceed, we can define an equilibrium in the marriage market with pre-marital investment.

Definition 1.3 *An **equilibrium with age gap between spouses** at period t is measures of singles $\{M_{j,t}, W_{j,t}\}$, $\forall j = 1, 2$; distributions of single men's saving $G_{s,j,t}(\cdot)$, $\forall j = 1, 2$ and liquid wealth at age 2 $G_{x,2,t}(\cdot)$; men's saving and marriage timing functions $\{s_{j,t}(x_{m,j,t}), \delta_{m,j,t}(s_{m,j,t})\}$, $\forall j = 1, 2$; women's marital age $\delta_{w,j,t}$, $\forall j = 1, 2$; and a matching rule μ_t such that:*

1. *Saving functions are consistent with individual men's optimal choices. Choices of marital age $\delta_{m,j,t}(s_{m,j,t})$ and $\delta_{w,j,t}$ are also optimal.*

2. *Measures and distributions in the marriage market are consistent with agents' optimal choices and the endogenous distribution of endowment, such that:*

$$\begin{aligned}
G_{s,1,t}(s_{m,1,t}) &= prob\{s_{1,t}(x_{m,1,t}) \leq s_{m,1,t}\}, \forall s_{m,1,t} \\
G_{x,2,t}(x_{m,2,t}) &= prob\{(Rs_{m,1,t-1} + wn \leq x_{m,2,t}) \wedge (\delta_{m,1,t-1}(s_{m,1,t-1}) = 0)\}, \forall x_{m,2,t} \\
G_{s,2,t}(s_{m,2,t}) &= prob\{s_{2,t}(x_{m,2,t}) \leq s_{m,2,t}\}, \forall s_{m,2,t} \\
M_{1,t} &= \frac{R^*}{1+R^*};\ W_{1,t} = \frac{1}{1+R^*} \\
M_{2,t} &= M_{1,t-1} \int [1 - \delta_{m,1,t-1}(s_{m,1,t-1})] dG_{s,1,t-1}(s_{m,1,t-1}) \\
W_{2,t} &= W_{1,t-1}(1 - \delta_{w,1,t-1})
\end{aligned}$$

3. *The matching distribution μ_t is stable.*

I first introduce some qualitative properties of equilibrium which hold true irrespective of the measures and distributions in the marriage market, namely, positive assortative matching in each age combination of couples.

Proposition 1.3 *In each possible age combination among married couples, take two couples with pre-marital savings and match quality indicators as (s_m, z) and (s'_m, z') respectively. Then $s_m \geq s'_m$ if and only if $z \geq z'$.*

Proof. Positively assortative matching results from utilities after marriage (Equations (1.15) to (1.19)) which strictly increase in the other person's trait. As a result, men with higher saving and women with higher match quality would always prefer each other.

Now we come to properties under stationary distributions. Since measures of young agents and distributions of wealth endowment and match quality are constant in each period, we can expect to have a *long-run stationary equilibrium* in which each generation behaves in the same way as previous generations and measures, distributions and saving functions are also the same. Consequently the time subscript t can be dropped. In the remaining part of this Section, I assume the economy is at its long-run stationary equilibrium.

Proposition 1.4 *In long-run stationary equilibrium, all women marry at age 1, and choose pre-marital saving* $s_{w,1} = 0$.

Proof. See Appendix 1.9.4.

Women choose to marry at the earliest marriageable age for two reasons. First, based on utilities after marriage, young men strictly prefer young women while middle-aged men are indifferent about women's ages with the same z. In other words, women can't gain and may fall in ranking if they postpone marriage. Second, remaining single for one additional period causes a utility loss. Thus all women marry at age 1. Since women's income is wn at all ages and $\beta(1+r) = 1$, they would choose 0 saving even if there were no marriage market. When the prospect of marriage is also considered, women have a motivation to borrow since they can take advantage of their future husband's pre-marital saving. Since saving is restricted to be nonnegative, $s_{w,1} = 0$.

If the gender ratio $R^* > 1$, some men cannot marry at age 1, so there are both young and middle-aged men in the marriage market. Since all women in the marriage market are of age 1, we can show women agree in their ranking of men.

Proposition 1.5 *All men in the marriage market are ranked in the same order by women.*

Proof. This can be seen by comparing utilities from marriage in equations (1.15) and (1.17). Since match quality z is additive in the utility functions, all women agree on their ranking of $\{s_m, j_m\}$ combinations in the marriage market. In particular, a young man with saving $s_{m,1}$ and a middle-aged man with saving $s_{m,2}$ are of the same rank if and only if they offer the same level of utilities to women after marriage, i.e.,

$$(1+\beta)\ln(\frac{s_{m,1}}{\beta+\beta^2}+2wn) = \ln(\frac{s_{m,2}}{\beta}+2wn)+\beta\ln(\frac{wn}{\kappa})$$

By Proposition 1.5, we can use women's indirect utility after marriage as a single index to summarize single men's 2-dimensional characteristics of saving and age. In particular, I define men's quality function $q(s_m, j_m)$ as

$$q(s_m, j_m) = \begin{cases} (1+\beta)\ln[\kappa(\frac{s_m}{\beta+\beta^2}+2wn)], \text{if } j_m = 1 \\ \ln[\kappa(\frac{s_m}{\beta}+2wn)]+\beta\ln(wn), \text{if } j_m = 2 \end{cases} \tag{1.20}$$

Men with the same q have the same rank in the marriage market. I denote the equilibrium distribution of q in the marriage market as $H_q(\cdot)$.

In the same spirit as Lemma 1.2, it is easy to show that matching is positively assortative between women's match quality z and men's quality q. With both young and middle-aged single men in the marriage market, let's define the total measure of single men and the gender ratio in marriage market as

$$M = M_1 + M_2;\ \overline{R} = \frac{M}{W_1},$$

I denote the lowest quality of a man who can get matched by $q^* = H_q^{-1}(1 - \frac{1}{\overline{R}})$. Men with quality $q < q^*$ cannot be matched and so remain single this period. Otherwise, a man with quality higher than q^* matches with a young woman of match quality

$$\phi(q) = H_z^{-1}[1 - \overline{R}(1 - H_q(q))] \tag{1.21}$$

For now I take q^* and $H_q(\cdot)$ as given, and explain how they are formed after I describe the utility functions of single men[16].

1.5.2.3 Utility Function of Single Men

Middle-aged Single Men

A middle-aged single man with current liquid wealth $x_{m,2} = (1+r)s_{m,1} + wn$ maximizes the following:

$$\begin{aligned} V_{m,2}(x_{m,2}) &= \max_{s_{m,2},\delta_{m,2}} \{\ln(x_{m,2} - s_{m,2}) + \beta(1-\delta_{m,2})\ln[(1+r)s_{m,2} + wn] \\ &\quad + \beta\delta_{m,2}[\ln[\kappa((1+r)s_{m,2} + 2wn)] + \phi(q(s_{m,2}, 2))]\} \end{aligned} \tag{1.22}$$

subject to quality function (1.20) and matching rule (1.21). Based on Assumption 1, that is, the lowest value of women's match quality $\underline{z} \geq -\ln\kappa$, a middle-aged man prefers marrying any woman to remaining single. Based on matching rule (1.21), the matching threshold saving for middle-aged men is

$$s_2^* = \frac{\beta}{\kappa}\exp[q^* - \beta\ln(wn)] - 2\beta wn$$

If $s_{m,2} < s_2^*$, $\delta_{m,2}(s_{m,2})$ takes the value 0 and this man will remain single for life. His savings rate and value function will be

$$\begin{aligned} s_2^{\mathbf{s}}(x_{m,2}) &= \frac{\beta}{1+\beta}(x_{m,2} - wn) \\ V_{m,2}^{\mathbf{s}}(x_{m,2}) &= (1+\beta)\ln(\frac{1}{1+\beta}x_{m,2} + \frac{\beta}{1+\beta}wn) \end{aligned}$$

[16] Using a similar argument as in Lemma 1.4, we can show that $H_q(q)$ is continuous and strictly increasing $\forall q \geq q^*$.

Otherwise, $s_{m,2}$ is given by the following first-order condition:

$$\begin{aligned} & \frac{1}{(1+r)s_{m,1}+wn-s_{m,2}} - \frac{1}{(1+r)s_{m,2}+2wn} \\ = \quad & \frac{\beta}{s_{m,2}+2\beta wn} \frac{\overline{R}h_q(q)}{h_z[1-\overline{R}(1-H_q(q))]} \end{aligned}$$

with $q = \ln[\kappa(\frac{s_{m,2}}{\beta}+2wn)] + \beta\ln(wn)$.

Young Men

A young single man with endowed wealth $x_{m,1}$ maximizes the following:

$$\begin{aligned} V_{m,1}(x_{m,1}) \quad = \quad & \max_{s_{m,1},\delta_{m,1}} \{\ln(x_{m,1}+wn-s_{m,1}) \\ & +\beta(1-\delta_{m,1})V_{m,2}[(1+r)s_{m,1}+wn] \\ & +(\beta+\beta^2)\delta_{m,1}[\ln(\frac{s_{m,1}}{\beta+\beta^2}+2wn)+\ln\kappa+\phi(q(s_{m,1},1))]\} \end{aligned} \tag{1.23}$$

The threshold level of saving for young men to marry is

$$s_1^* = (\beta+\beta^2)[\frac{1}{\kappa}\exp(\frac{q^*}{1+\beta})-2wn]$$

In equilibrium, the distribution of single men's quality $H_q(\cdot)$ should be consistent with men's choice of marital ages and pre-marital savings who would in return take $H_q(\cdot)$ as given and choose savings and marital ages to maximize the expressions in Equations (1.22) and (1.23). The stationary distribution of $H_q(\cdot)$ and men's pre-marital saving choices cannot be easily characterized, since compared to the model in Section 1.4, now we have mixed matching such that a woman with the same match quality z may match with different {wealth, age} combinations in stable matching. Thus I extend this part into a multi-period model in Section 1.6 to solve the model numerically.

1.6 Multi-period Model

In previous sections, I showed how a competitive saving motive arises when wealth is a status good in the marriage market and how marriage age and saving behavior

can be jointly affected by the marriage market imbalances. In order to see whether this model can offer an explanation for the savings rate puzzle in China, I extend the model into a multi-period framework to show how the age profile of the savings rate and marital ages are affected by changing the gender ratio empirically.

1.6.1 Demographics

Both men and women live for J periods, and people can keep working until retirement age J^{ret}. In contrast to the baseline model, newly born agents don't receive a wealth endowment; that is, the only way to generate wealth is by accumulating saving from labor income each period. In particular, a man receives a labor endowment in each period before retirement depending on his working experience and a transitory shock ε^m :

$$\begin{aligned} \ln n_j^m &= \gamma_0 + \gamma_1 \text{experience} + \gamma_2 \text{experience}^2 + \varepsilon^m \\ \varepsilon^m &\sim N(0, \sigma_{\varepsilon^m}^2) \end{aligned} \tag{1.24}$$

in which γ_0, γ_1 and γ_2 are constants to be calibrated, and experience $= j_m - 1$. The Mincerian equation (1.24) features a typical hump-shaped age profile of income which peaks at middle age.

In order to reduce the dimension of the matching problem, I assume women's labor endowment profile is deterministic over the life cycle:

$$\begin{aligned} \ln n_j^w &= \gamma_0 + \gamma_1 \text{experience} + \gamma_2 \text{experience}^2, \\ \text{experience} &= j_w - 1 \end{aligned} \tag{1.25}$$

Men can choose to get married at age $j_m \in [J_m^{mar}, J_m^{mar} + \Delta]$, in which J_m^{mar} is men's earliest marriageable age, and Δ controls the length of marital stage. Marriageable

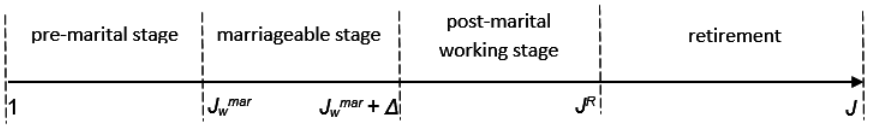


Figure 1.5: Life-cycle of a Representative Woman

ages for women are similarly defined as $[J_w^{mar}, J_w^{mar} + \Delta]$[17]. The life-cycle of a typical woman is summarized in Figure 1.5.

1.6.2 Production

At time t, the production function for aggregate output takes the Cobb-Douglas form:

$$Y_t = AK_t^{\alpha} N_t^{1-\alpha},$$

where Y_t is total output, K_t is aggregate capital, and N_t is the total labor endowment for all working-age people. Factor markets are competitive and capital and labor are paid their marginal products:

$$\begin{aligned} r_{K,t} &= \alpha A K_t^{\alpha-1} N_t^{1-\alpha} \\ w_t &= (1-\alpha) A K_t^{\alpha} N_t^{-\alpha} \end{aligned}$$

There is no capital flow controls in this economy and thus the interest rate r is fixed at the world interest rate. Capital flows freely among countries, and thus by the non-arbitrage condition, the return to capital in production equals the sum of the interest rate r and the depreciation rate δ: $r_K = r + \delta$, and the equilibrium wage and output

[17] I assume women can get married earlier than men. According to China's marriage law, women can be legally married at age 20. The age is 22 for men.

are

$$\begin{aligned} w &= (1-\alpha)A(\frac{r+\delta}{\alpha A})^{\frac{\alpha}{\alpha-1}} \\ Y_t &= A(\frac{r+\delta}{\alpha A})^{\frac{\alpha}{\alpha-1}} N_t \end{aligned}$$

Notice that the interest rate and wage are constant.

National private savings rate is defined as the ratio of aggregate private savings to GDP:

$$\begin{aligned} s_t^P &= \frac{Y_t + rNFA_{t-1} - C_t}{Y_t + rNFA_{t-1}} \\ NFA_{t-1} &= S_{t-1} - K_t \end{aligned} \tag{1.26}$$

where households' disposable income is the sum of output Y_t and interest payments from net foreign assets accumulated in the previous period, NFA_{t-1}.

1.6.3 Period Utility

The choice of period utility is the same as in Section 1.5 with minor modification to the budget constraints. At age j, a single or widowed agent's period utility comes from consumption (c_j):

$$u^{\mathbf{s}}(c_j) = \ln(c_j)$$

subject to the following budget constraint:

$$c_j + s_j \leq wn_j + (1+r)s_{j-1} = x_j$$

where current liquid wealth x_j is the sum of labor income wn_j and wealth accumulated from the previous period. The expression for labor endowment n_j is derived from Equation (1.24) for men or Equation (1.25) for women before retirement age J^{ret}, or 0 otherwise.

Married couples enjoy the same period utility after marriage from consumption and direct utility from marriage z based on the wife's match quality:

$$u^{\mathbf{a}}(C_{j_m j_w}, z) = \ln(\kappa C_{j_m j_w}) + z$$

with household budget constraint

$$C_{j_m j_w} + s_{j_m, j_w} \leq w(n^m_{j_m} + n^w_{j_w}) + (1+r)s_{j_m - 1, j_w - 1}$$

where $s_{j_m-1,j_w-1} = s_{m,j_m-1} + s_{w,j_w-1}$.

1.6.4 Marriage Market

Most of the specification of the marriage market that was developed in Section 1.5 still holds here. Women have to choose a marital age ex ante between J^{mar}_w and $J^{mar}_w + \Delta$, and draw match quality z from distribution $H_z(z)$ after entering the marriage market. Pre-marital saving is not observable and thus there is no competitive saving motive for women. With a deterministic age-labor endowment profile, all women face the same optimization problem before the marital stage, and thus their choice of pre-marital saving and marriage timing are also the same. As in Section 1.5, in equilibrium women choose to marry at their earliest marriageable age J^{mar}_w[18].

Unlike women, single men in the marriage market are of different ages when each generation's gender ratio $R^* > 1$ and not all men can marry at age J^{mar}_m. I denote the measures of single men at age j_m as M_{j_m} and their saving distributions as G_{s,j_m}, $\forall j_m \in [J^{mar}_m, J^{mar}_m + \Delta]$. Consequently men are characterized by combinations $\{s_m, j_m\}$ in the marriage market. As in Proposition 4, all men in the marriage market are ranked in the same order by women and their qualities are summarized by the indicator $q(s_m, j_m)$ based on utilities after marriage. (See Equation (1.31) below.) With a total

[18] In contrast to Proposition 1.4 in Section 1.5, I make this result a conjecture here and then verify that it holds in equilibrium.

measure of single men and a gender ratio in marriage market of

$$M = \sum_{J_m^{mar}}^{J_m^{mar}+\Delta} M_{j_m};\ \overline{R} = \frac{M}{W_1}$$

and an equilibrium distribution of q of $H_q(q)$, matching is positively assortative between q and z.

I denote the lowest value of q among men who can be matched as $q^* = H_q^{-1}(1-\frac{1}{\overline{R}})$. Men with quality $q < q^*$ cannot be matched and remain single this period. Otherwise, a man with quality higher than q^* matches with a young woman of match quality

$$z(q) = H_z^{-1}[1 - \overline{R}(1 - H_q(q))] \tag{1.27}$$

On the other hand, the quality q that a woman of type z matches with is

$$q(z) = H_q^{-1}[1 - \frac{1}{\overline{R}}[1 - H_z(z)]] \tag{1.28}$$

1.6.5 Value Function

1.6.5.1 Single Agents in Post-marital Stage

Single men beyond the age of marriage maximize life-time utility by allocating current liquid wealth between consumption and saving. If the man is still working, he maximizes

$$\begin{aligned} U^{\mathbf{s}}_{m,j_m}(x_{j_m}) &= \max_{s_{j_m}} \left\{\ln c_{j_m} + \beta \mathbb{E} n_{j_m+1} U^{\mathbf{s}}_{m,j_m+1}(x_{j_m+1})\right\} \\ &\quad s.t. \\ c_{j_m} + s_{j_m} &= x_{j_m};\ s_{j_m} \geq 0 \\ x_{j_m+1} &= (1+r)s_{j_m} + wn_{j_m+1} \end{aligned} \tag{1.29}$$

and labor income n_{j_m+1} is subject to income risk according to Equation (1.24). After retirement, his labor income will be 0 and all his liquid wealth is generated by saving from the previous period. At age J, he'll consume all wealth and set $s_J = 0$.

The optimization problem for a single woman after the age of marriage is similar but since her labor income is deterministic, the expectation operator is dropped from the maximization problem above.

1.6.5.2 Married Household

Couples with husband's age j_m and wife's j_w make the consumption-saving decision jointly. If both spouses are working, they maximize:

$$U^{\mathbf{a}}_{j_m j_w}(x_{j_m j_w}, z) = \max_{s_{j_m j_w}} \left\{ \ln(\kappa C_{j_m j_w}) + z + \beta \mathbb{E}_{n_{j_m+1}} U^{\mathbf{a}}_{j_m+1, j_w+1}(x_{j_m+1, j_w+1}, z) \right\} \tag{1.30}$$

with household budget constraint

$$\begin{aligned} C_{j_m j_w} + s_{j_m j_w} &\leq w(n^m_{j_m} + n^w_{j_w}) + (1+r)s_{j_m-1, j_w-1} = x_{j_m j_w} \\ x_{j_m+1, j_w+1} &= w(n^m_{j_m+1} + n^w_{j_w+1}) + (1+r)s_{j_m j_w} \end{aligned}$$

The household's liquid wealth is the sum of saving from the previous period and current labor incomes from both spouses. The expectation operator is included in Equation (1.30) if the husband will still be working in the next period and thus subject to labor income risk. Otherwise, the operator is dropped.

If both spouses are of the same age, when they both reach age J they'll choose to consume all their wealth and save no more: $s_{JJ} = 0$. Otherwise, if only the husband reaches age J while wife's age $j_w < J$, the optimal level of saving for husband is different from optimal saving for wife. Clearly the husband would like to choose to consume all the family wealth today, but by doing so the wife will have no wealth in the next period. In such cases, the optimization problem is

$$\begin{aligned} U^{\mathbf{a}}_{J, j_w}(x_{J, j_w}, z) &= \max_{s_{J, j_w}} \left\{ \ln(\kappa C_{J, j_w}) + z + \mu\beta U^{\mathbf{s}}_{w, j_w+1}(x_{j_w+1}) \right\} \\ &\quad s.t. \\ C_{J, j_w} + s_{J, j_w} &\leq x_{J, j_w} \\ x_{j_w+1} &= (1+r)s_{J, j_w} \end{aligned}$$

where the constant $\mu \in (0,1)$ characterizes the wife's Pareto weight.

Since in equilibrium women marry earlier than men, the general case is that $j_m > j_w$. The husband's life-time utility is:

$$\begin{aligned} U^{\mathbf{a}}_{j_m j_w, m}(x_{j_m j_w}, z) &= \ln(\kappa C_{j_m j_w}) + z + \beta \mathbb{E}_{n_{j_m+1}} U^{\mathbf{a}}_{j_m+1, j_w+1, m}(x_{j_m+1, j_w+1}, z) \\ &= \mathbb{E}_{n_{j_m+1}} \{ \sum_{t=0}^{J-j_m} \beta^t \ln(\kappa C_{j_m+t, j_w+t}) \} + (\sum_{t=0}^{J-j_m} \beta^t) z \end{aligned}$$

and the wife's is

$$\begin{aligned} U^{\mathbf{a}}_{j_m j_w, w}(x_{j_m j_w}, z) &= \ln(\kappa C_{j_m j_w}) + z + \beta \mathbb{E}_{n_{j_m+1}} U^{\mathbf{a}}_{j_m+1, j_w+1, m}(x_{j_m+1, j_w+1}, z) \\ &= \mathbb{E}_{n_{j_m+1}} \{ \sum_{t=0}^{J-j_m} \beta^t \mathbb{E}_{j_m j_w} \ln(\kappa C_{j_m+t, j_w+t}) \} + (\sum_{t=0}^{J-j_w} \beta^t) z \\ &\quad + \mathbb{E}_{n_{j_m+1}} \{ \sum_{t=J-j_m+1}^{J-j_w} \beta^t \mathbb{E}_{j_m j_w} \ln(c^{\mathbf{s}}_{j_w+t}) \} \end{aligned} \tag{1.31}$$

1.6.5.3 Single Agents at Marriageable Stage

A man of marital age $j_m \in [J^{mar}_m, J^{mar}_m + \Delta]$ with liquid wealth $x_{j_m} = (1+r)s_{j_m-1} + wn_{j_m}$ takes the women's saving $s^w_{J^{mar}_w}$, the quality function $q(s_m, j_m)$, its distribution $H_q(q)$ as well as the assortative matching rule (1.27) as given, and allocates his wealth between consumption and saving optimally by solving the following:

$$\begin{aligned} U^{\mathbf{s}}_{m, j_m}(x^m_{j_m}) &= \max_{s^m} \{ \ln c^m + \\ &\quad \max[\beta \mathbb{E}_{n^m_{j_m+1}} U^{\mathbf{a}}_{j_m+1, J^{mar}_w+1, m}(x', z[q(s^m, j_m)]), \\ &\quad \beta \mathbb{E}_{n^m_{j_m+1}} U^{\mathbf{s}}_{j_m+1, m}(x^m_{j_m+1})] \} \end{aligned}$$

subject to:

$$\begin{aligned} c^m + s^m &= x^m_{j_m} \\ x' &= (1+r)(s^m + s^w_{J^{mar}_w}) + w(n^m_{j_m+1} + n^w_{J^{mar}_w+1}) \\ x^m_{j_m+1} &= (1+r)s^m + wn^m_{j_m+1} \end{aligned}$$

Upon meeting with a woman with match quality $z[q(s^m, j_m)]$, a man decides whether to marry by comparing the expected life-time utility from marriage $U^{\mathbf{m}}_{j_m+1,J^{mar}_w+1,m}$ to the utility of remaining single for one more period $U^{\mathbf{s}}_{j_m+1,m}$. A single man of age $J^{mar}_m + \Delta$ who fails to get matched this period remains single for life.

A single woman with wealth x_w at age J^{mar}_w solves

$$U^{\mathbf{s}}_{w,J^{mar}_w}(x_w) = \max_{s^w}\{\ln c_w + \int_z \max[\beta[q(z) + (\sum_{t=0}^{J-j_w} \beta^t)z], \beta U^{\mathbf{s}}_{w,J^{mar}_w+1,}(x'_w)]dG_z(z)\}$$

subject to

$$\begin{aligned} c_w + s_w &\leq x_w;\ s_w \geq 0 \\ x'_w &= (1+r)s_w + wn^w_{j_w+1} \end{aligned}$$

and assortative matching function (1.28) which is the expected utility from marriage.

1.6.5.4 Single Agents at Pre-marital Stage

The optimization problem for single men of age $j_m < J^{mar}_m$ is

$$\begin{aligned} U^{\mathbf{s}}_{m,j_m}(x^m_{j_m}) &= \max_{s^m_{j_m}}\{\ln(x^m_{j_m} - s^m_{j_m}) + \beta \mathbb{E}_{n_{j_m+1}} U^{\mathbf{s}}_{j_m+1,m}(x^m_{j_m+1})]\} \\ &\quad s.t. \\ x^m_{j_m+1} &= (1+r)s^m_{j_m} + wn^m_{j_m+1} \end{aligned}$$

With the competitive saving motive, they keep accumulating wealth to improve their ranking at marital age.

The optimization problem for single women of age $j_w < J^{mar}_w$ is

$$\begin{aligned} U^{\mathbf{s}}_{w,j_w}(x^w_{j_w}) &= \max_{s^w_{j_w}}\{\ln(x^w_{j_w} - s^w_{j_w}) + \beta U^{\mathbf{s}}_{w,j_w+1}(x^w_{j_w+1})]\} \\ &\quad s.t. \\ x^w_{j_w+1} &= (1+r)s^w_{j_w} + wn^w_{j_w+1} \end{aligned}$$

There is no competitive saving motive for women since their wealth is not publicly observable before marriage.

1.6.6 Equilibrium

Definition 1.4 *A **multi-period long-run equilibrium** consists of measures of singles in the marriage market* $\{M_{j_m}, W_{j_w}\}$, $\forall j_m = [J_m^{mar}, J_m^{mar} + \Delta]$, $j_w = [J_w^{mar}, J_w^{mar} + \Delta]$; *single men's saving function* $s_{j_m}(x_{m,j_m})$ *and the associated value functions* $U^{\mathbf{s}}_{m,j_m}(x^m_{j_m})$, *distributions of single men's saving* $G_{s,j_m}(\cdot)$ *and liquid wealth* $G_{x,j_m}(\cdot)$ *at age* j_m $\forall j_m$; *men's and women's conditional marital indicators* $\{\delta_{m,j_m}(s_{m,j_m}), \delta_{w,j_w}\}$, $\forall j_m = [J_m^{mar}, J_m^{mar} + \Delta]$, $j_w = [J_w^{mar}, J_w^{mar} + \Delta]$; *married household's saving functions* $s^{\mathbf{a}}_{j_m j_w}(x)$ *and the associated value functions* $\{U^{\mathbf{a}}_{j_m j_w, m}(x,z), U^{\mathbf{a}}_{j_m j_w, w}(x,z)\}$, *and a matching rule* μ *such that:*

1. *Given the measures and saving distributions in the marriage market, the value functions after marriage, and the value functions for remaining single, then the single men's saving functions* $s_{j_m}(x_{m,j_m})$ *and marital timing choices* $\delta_{m,j_m}(s_{m,j_m})$ *as well as women's marital timing choices* δ_{w,j_w} *are optimal.*

2. *The measures and distributions in the marriage market are consistent with agents' optimal choices and the distribution of endowments, such that:*

$$
\begin{aligned}
G_{s,j_m}(s) &= prob\{s_{j_m}(x_{m,j_m}) \leq s\}, \forall s, j_m \qquad (1.32)\\
G_{x,j_m}(x) &= prob\{((1+r)s_{m,j_{m-1}} + wn^m_{j_m} \leq x) \wedge (\delta_{m,j_m-1}(s_{m,j_{m-1}}) = 0)\}, \forall x, j_m \geq 2\\
M_{j_m} &= M_1, \forall j_m < J_m^{mar}\\
M_{j_m} &= M_{j_m-1} \int [1 - \delta_{m,j_m-1}(s)] dG_{s,j_m-1}(s), \ \forall j_m \geq J_m^{mar}\\
W_{j_w} &= W_1, \forall j_w < J_w^{mar}; \ W_{j_w} = W_{j_w-1}(1 - \delta_{w,j_w-1}), \ \forall j_w \geq J_w^{mar}
\end{aligned}
$$

3. *The saving functions of married households* $s^{\mathbf{a}}_{j_m j_w}(x)$ *solve the optimization problem (1.30) given family wealth* x, $\forall j_m, j_w$.

4. *The matching distribution* μ *is stable.*

1.7 Numerical Results

Does a change in the gender ratio have a significant effect on aggregate saving and on the age profile of saving of the pre-marital cohorts? In this section, I calibrate the multi-period model in Section 1.6 to answer this question quantitatively.

1.7.1 Calibration

The main dataset I use for the calibration is the China Health and Nutrition Survey (hereafter CHNS) which offers detailed information covering economic, demographic and other major social variables on the household and individual levels. This paper uses all nine waves (1989, 1991, 1992, 1997, 2000, 2004, 2006, 2009, 2011) provided by the survey. Most of the waves cover 9 provinces that vary considerably in geography and economic development[19]. Counties in these provinces were stratified by income and 4 counties were selected randomly within each province. In addition, the provincial capital and a lower income county are also included.

The samples used here are limited to male adults in the labor force between the ages of 18 to 60. The CHNS data include both urban and rural samples. I only use the urban part because their income process is more consistent with the purpose of this study. I exclude individuals with annual working hours smaller than 400 or larger than 4000. Individuals reporting as self-employed are also dropped. This leaves us with 8296 observations.

[19]The 9 provinces are Liaoning, Heilongjiang, Jiangsu, Shandong, Henan, Hubei, Hunan, Guangxi, Guizhou. Liaoning participated in all the waves except Wave 1997, and Heilongjiang was not surveyed until Wave 1997. I exclude the observations from Beijing, Shanghai and Chongqing since they are only available in Wave 2011.

Income process: Parameters γ_0, γ_1 and γ_2 in the labor endowment Equation (1.24) are derived by estimating the following equation:

$$y_{i,t} = \gamma_0 + \gamma_1 experience_{i,t} + \gamma_2 experience_{i,t}^2 + d_t + \varepsilon_{i,t}^m$$

where the outcome variable $y_{i,t}$ is the logarithm of real annual labor income of person i in year t. I use annual labor income instead of wage as the dependent variable here because the self-reported data about annual working hours in CHNS are very noisy which reduces the quality of the calculated wage rate. Hence, I only use regressions with wage as the outcome variable in a few specifications to test the robustness of my estimation of $\{\gamma_0, \gamma_1, \gamma_2\}$. Experience is defined as the age of person i minus (7+years of schooling). The set of yearly dummies d_t is normalized to sum to zero: $\sum_t d_t = 0$. The standard deviation of transitory shocks σ_ε in Equation (1.24) matches the standard deviation of the error term $\varepsilon_{i,t}^m$.

Demographics: Demographic parameters are set using a model period of 2 years. An agent lives from real-life age of 18 to real-life age of 80 so that $J = 32$. $J^{ret} = 22$ corresponds to a real-life retirement age of 60. Since I don't have good information about parental savings which constitute a majority part of the brideprice in China, I choose to let single men save for themselves in the model, and set the marital ages relatively late in life. In particular, $J_m^{mar} = 10$ and $J_w^{mar} = 9$ corresponding to real-life age of 36 and 34. I vary the duration of the marital stage Δ from 0 to 3 to empirically evaluate to what extent the pressure on the aggregate savings rate can be tempered by the endogenous age gap between couples. According to the National Bureau of Statistics (NBS) in China, the gender ratio at birth is 1.18 boys per girl in 2011, which is set as the benchmark gender ratio in the model. In cases in which the husband reaches the final age J while the wife is younger, the choice of Pareto weight of the wife μ is taken from Fernandez and Wong (2013) and set as 0.3.

Table 1.2: Parameter Values

Parameters	**Meaning**	**Value**	**Source**
J, J^{ret}	lifespan and retirement age	32, 22	CHNS
J_m^{mar}, J_w^{mar}	marital ages	10, 9	CHNS
R^*	gender ratio at birth	1.18	Population Census 2010
γ_0, γ_1, γ_2	labor endowment parameter	8.9; 0.03; -0.005	CHNS
σ_ε	variance of transitory shock	0.73	CHNS
α, A	production function parameters	0.4; 1	Du and Wei (2013)
δ	depreciation rate	19%	Song et al. (2011)
r, β	interest rate, discount factor	0.035; 0.97	Song et al. (2011)
κ	household consumption factor	0.8	Du and Wei (2013)
μ	wife's Pareto weight	0.3	Fernandez and Wong (2013)
$H_z(\cdot)$	distribution of match quality	$N(1, 0.05)$	Du and Wei (2013)

Interest rate and production: Song et al. (2011) computed the average one-year real deposit rate in China during 1998-2005 and suggest it takes value 0.0175. Since one model period corresponds to 2 years in this paper, I choose $r = 1.0175^2 - 1 = 3.53\%$, and the discount factor β is set as $\frac{1}{1+r} = 0.966$. The production function parameters α and A are chosen to be 0.4 and 1 respectively. Annual depreciation rate of capital in Song et al. (2011) is set to 0.1. Since one model period here corresponds to 2 years, I set $\delta = 1 - (1 - 0.1)^2 = 19\%$.

Other parameters including the distribution of match quality $H_z(\cdot)$ and the parameter for the public good feature of marriage κ are from Du and Wei (2013). $H_z(\cdot)$ is a Normal distribution with mean 1 and standard deviation 0.05, and $\kappa = 0.8$. Parameter values are summarized in Table 1.2.

1.7.2 Results

Using these parameters, I find the following results:

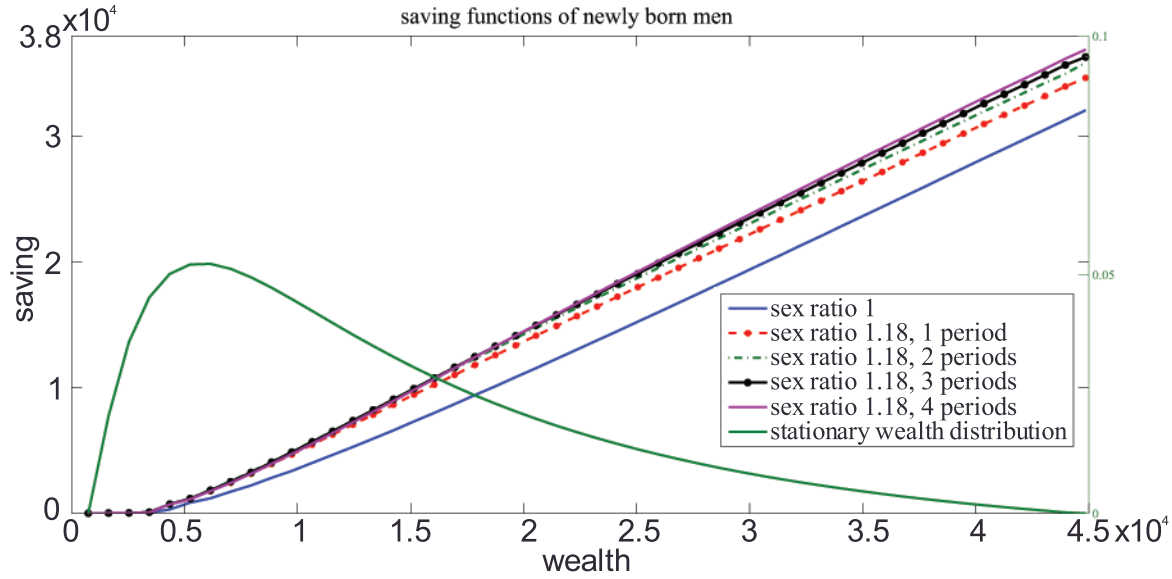


Figure 1.6: Saving Functions for Newly Born Men

1.7.2.1 Competitive Saving Motive in the Stationary Equilibrium

The stationary saving functions of newly born men corresponding to different gender ratios R^* and marriage market durations Δ are plotted in Figure 1.6. It also plots the wealth distribution of newly born men labeled on the right y-axis. I find that the saving functions are strictly increasing with wealth level except for men with little wealth. Since their labor incomes are subject to the lowest transitory shocks this period according to Equation (1.24), and their expected age-income profiles in the future are as good as those of any other men of the same age, zero saving is optimal for them to smooth consumption over their life-cycle. Except for these men, the saving function is strictly increasing with wealth.

One major observation from Figure 1.6 is, with a balanced gender ratio ($R^* = 1$), increasing the duration of the marital stage Δ has no effect on single men's savings. Since everyone prefers getting married to remaining single, in the stationary equilibrium every man gets married at the earliest marital age $J_m^{mar} = 10$ even when $\Delta > 0$ and they have the option to delay marriage. Thus every single man chooses saving as if the marriage market would open once in lifetime ($\Delta = 0$). As R^* increases, men choose to save more at each wealth level because they anticipate more intense mating competition at the marital stage. As a result, they accumulate more premarital saving starting at age 1. This is the reason that the aggregate savings rate is affected by R^*. Furthermore, as the value of Δ increases, men would increase saving even more at each wealth level. This counterituitive result happens for the following reason. when $\Delta = 0$, a single man only competes with other men from his own cohort. When $\Delta > 0$, however, he will compete not only with men from his cohort, but also with single men from older cohorts when he first enters the marriage market, which results in an even fiercer competition. Such competition intensifies the competitive saving motive for men in the premarital cohorts in a stationary equilibrium.

To make this point clearer, let us consider the stationary equilibrium with an unbalanced gender ratio ($R^* = 1.18$) and duration parameter $\Delta = 3$. Thus every man has 4 chances to get married between age J_m^{mar} and $J_m^{mar} + \Delta$. Wealth distributions and saving functions of single men at age $j_m \in [J_m^{mar}, J_m^{mar} + \Delta]$ are plotted in Figure 1.7. Threshold matching wealth at each age j_m, $x_{j_m}^*$ is also labeled in the figure. These threshold values are generated such that an age j_m man with wealth $x_{j_m}^*$ is indifferent between marrying a woman with match quality $\underline{z}$ and remaining single for one more period. When a single man joins the marriage market for the first time at age J_m^{mar}

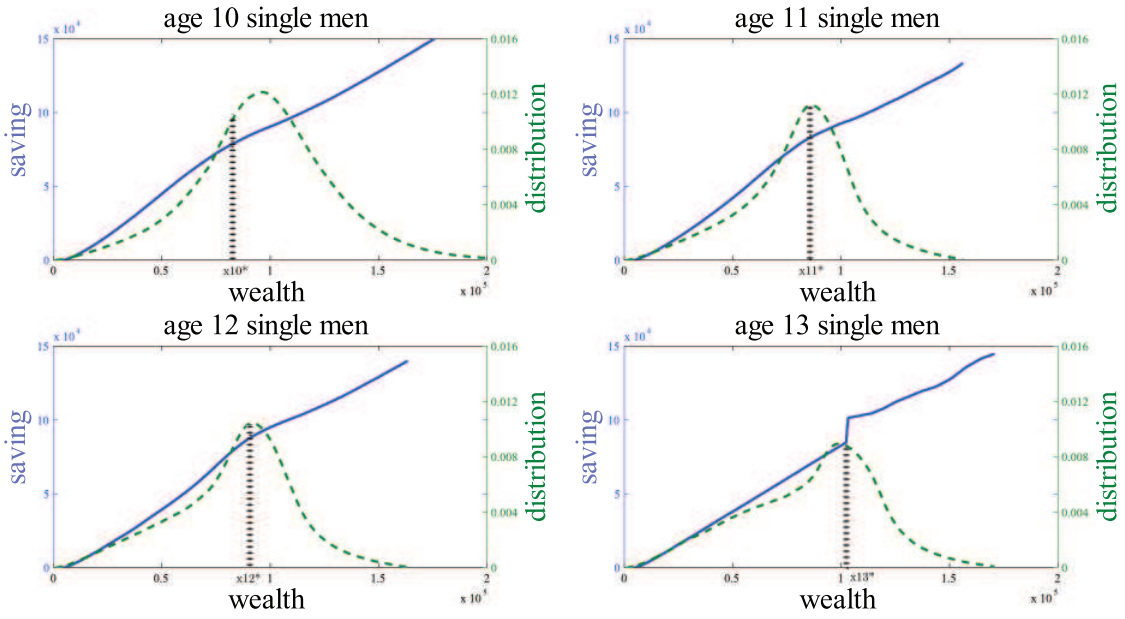


Figure 1.7: Wealth Distributions and Matching Pattern at the Marital Stage

(age 10 in Figure 1.7), he can get married if his wealth is larger than $x^*_{J_{mar}}$ (x^*_{10} in the graph), or he has to remain single this period and try again. In the next period, his age becomes $J_m^{mar}+1$, and his wealth is the sum of his saving from age J_m^{mar} and his labor income at age $J_m^{mar}+1$. If this income draw is high enough and the resulting wealth level is higher than the required matching wealth $x^*_{J_{mar}+1}$ (x^*_{11} in the graph), this man gets married at age $J_m^{mar}+1$, or otherwise he stays single again and rejoin the market at age $J_m^{mar}+2$. This process stops until he gets married or remains single at age $J_m^{mar}+\Delta$. If he still cannot get married at this age, he losses the last chance and will remain single for life.

By increasing Δ, we actually invite extra competitors into the market who might be competitors if they receive good draws of labor income. Fiercer competition forces

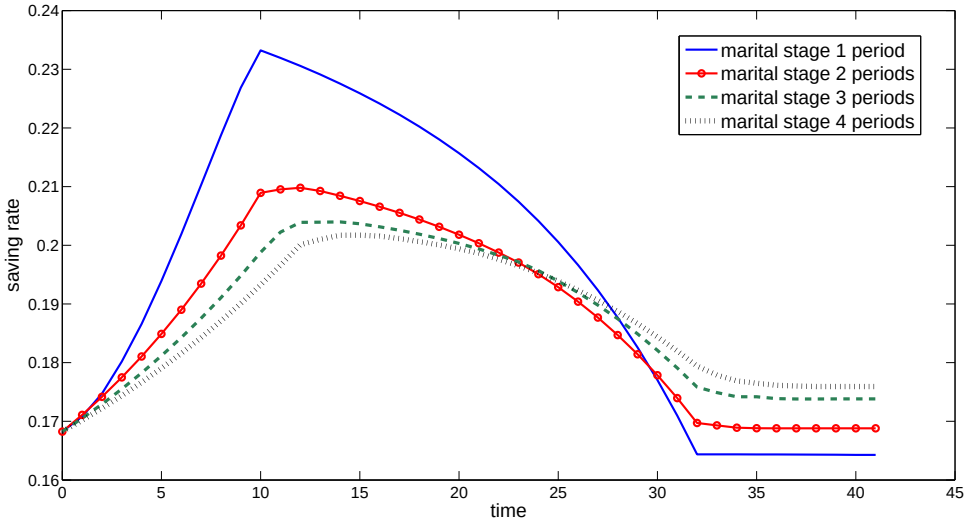


Figure 1.8: Dynamics of Savings Rate after Unexpected Gender Ratio Increase

single men to save more ex ante. In other words, in stationary equilibrium a longer duration of the marital stage actually increases the savings of the pre-marital cohorts.

1.7.2.2 Effect of Gender Ratio on Aggregate Saving

To test the effect of a rising gender ratio on the aggregate savings rate, I next consider the following experiment: At time 0, the gender ratios of all existing generations are 1, and the economy is at its long-run stationary equilibrium. Starting from time 1, the gender ratio of all newly-born generations switches to 1.18 unexpectedly. We'd like to see how the economy-wide savings rate would react to such a permanent shock. An algorithm for such transition dynamics is described in Appendix 1.9.4, and the resultant dynamics of aggregate savings rates corresponding to different marital stage durations are given in Figure 1.8.

In Figure 1.8, I trace the evolution of aggregate savings when R^* changes from 1 to 1.18. The solid line corresponds to the dynamics of the savings rate with $\Delta = 0$ and every single man can only get married at age J_m^{mar}. Over 32 periods (64 years), the savings rate transition path reaches its peak at time 10 when the generation born at time 1 reaches its earliest marriageable age. After that, the increased dis-savings from married households start to offset the increase in the savings rates in the pre-marital generations. At time 32, the economy converges to its new steady state with an unbalanced gender ratio of 1.18 in every living generation. The aggregate savings rate at the peak is 23.3% of GDP which is higher than the savings rate in the original stationary equilibrium by 6.5% of GDP. This result is similar to what Du and Wei (2013) find (6% of GDP).

Then I test how the dynamics of the aggregate savings rate can be affected if we allow an endogenous age gap between spouses. As we increase the duration of marital stage Δ, the dynamics of the savings rates is dampened, generating lower peaks. When the duration parameter Δ is set to 3 corresponding to 8 years as the maximum age gap between couples, the peak of the savings rate dynamics is only 3.35% of GDP higher than the aggregate savings rate in the previous stationary equilibrium. In other words, not considering age gap responses can potentially inflate reactions in savings rate by 94% ((6.5%-3.35%)/3.35%). Such a reduction in the aggregate savings rate arises since men in the affected cohorts have the option to postpone marriage when the duration of marital stage is expanded. Single men in the economy with $\Delta = 3$ unticipate that they will have a maximum of 4 chances to get married, and thus the competitive saving motive is tempered in contrast to men in the $\Delta = 0$ economy where they only have one chance. We can also see that in the long run when the economies converge to their respective steady states with the unbalanced gender ratio 1.18, the economy with $\Delta = 3$ has the highest savings rate while the savings

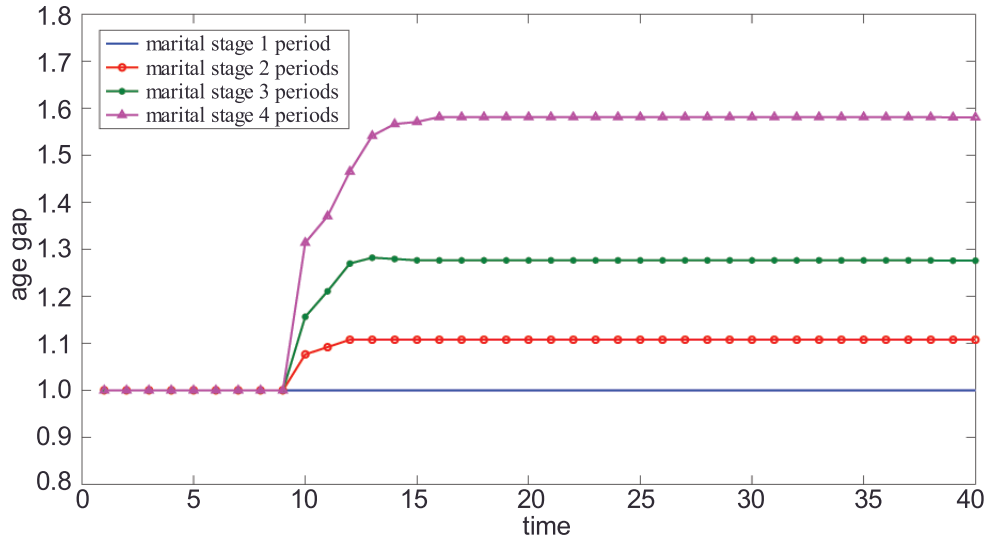


Figure 1.9: Dynamics of Age Gap after Unexpected Gender Ratio Increase

rate in the economy with $\Delta = 0$ is the lowest, for the same reason of their orders exhibited in Figure 1.6.

The corresponding dynamics of the average age gap between newlyweds along the transitional path is reported in Figure 1.9.

Consequently, although a longer duration of the marital stage results in a higher aggregate savings rate in the steady state, it dampens the reaction of the aggregate savings rate to a permanent gender ratio shock along the transitional path. In other words, allowing a longer duration of the marital stage has opposite effects on the aggregate savings rate in stationary equilibrium and during the transition periods. It dampens the reaction of the aggregate savings rate to a gender ratio increase along the transition path, but eventually results in a higher savings rate when the economy reaches its new stationary equilibrium.

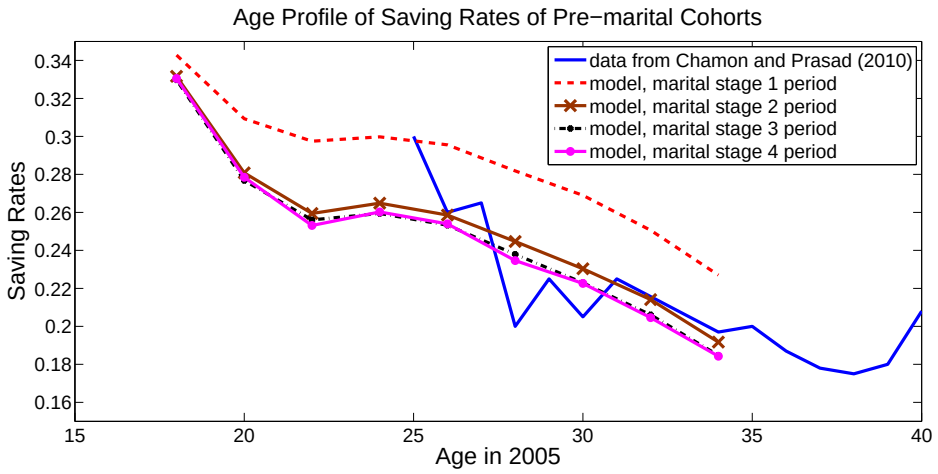


Figure 1.10: Age Profile of the Savings Rate of Pre-marital Cohorts

1.7.2.3 Downward-sloping Age Profile of the Savings Rate

As documented by Chamon and Prasad (2010) and Song and Yang (2010), the age profile of savings in China has had an unusual pattern in recent years with younger households having high savings rates relative to the middle-aged. This is the opposite to what classical life-cycle theory predicts because usually life-cycle saving is hump-shaped with young and old generations saving less than middle-aged generations. As depicted in Figure 1.10, the calibrated age profile of the savings rate in 2005 is generated by matching each age cohort by the gender ratio in births at their birth year. For example, the cohort of age 26 in year 2005 was born in year 1980, and the gender ratio at birth in 1980 was 1.0782. Then I calibrated the model using R^* =1.0782, and recorded the generated savings rate for this cohort at age 26. This process generates the downward-sloping age profile of the savings rate seen in Figure 1.10 which is similar to the empirical data documented in Chamon and Prasad (2010).

The unusual pattern in the age-savings rate profile is consistent with a competitive saving motive among pre-marital cohorts. Without any wealth endowment, men of age 1 (real-life age 18) are quite similar to each other. Although they might earn different income this period according to Equation (1.24), most uncertainties between this period and their earliest marital age (real-life age 36) are yet to be realized. The competitive saving motive is strongest when people are similar to each other because the return to saving is the highest as it is the easiest to overtake one's rivals. This leads to high saving for all newly born men. As men grow older and approach the age of marriage, the competitive saving motive gets weaker. On the one hand, these men become increasingly differentiated by wealth accumulated up to that age. On the other hand, uncertainties of labor income are gradually resolved and thus the chance that a man can significantly improve his position by receiving high income before the marital stage becomes smaller. Eventually as the age of marriage gets nearer, a man's position is mostly determined by his wealth. That is when the competitive saving motive is weakest and the return to saving is low.

Compared to single men, women in equilibrium choose 0 saving in the pre-marital stage, because first, they face a deterministic upward-sloping age profile of income, and second, with gender ratio $R^* \geq 1$, they can get married for sure and take advantage of the savings of their future husbands.

1.7.3 Robustness Checks

I examine the robustness of my findings about the competitive saving motive by once again doing the experiment in Subsection 1.7.2.2 with an unexpected permanent

Table 1.3: Robustness Checks of Maximum Increase in the Aggregate Savings Rate

Checked Parameters	**Meaning**	**Exogenous Age Gap ($\Delta = 0$)**	**Endogenous Age Gap ($\Delta = 3$)**
Baseline Case		6.5%	3.35%
$\mu = 0.2$	Pareto weight of wives	6.5%	3.49%
$\mu = 0.5$	Pareto weight of wives	6.5%	3.24%
$\kappa = 0.5$	private consumption	5.75%	2.28%
$\kappa = 1$	complete public good	6.7%	3.72%
$E(z) = 0.5$	mean of dist. of z	5.68%	2.41%
$E(z) = 2$	mean of dist. of z	7.17%	3.04%
σ_z	z's standard deviation	4.84%	2.71%

change in the gender ratio at birth from 1 to 1.18[20]. The results of robustness checks are reported in Table 1.3.

1.7.3.1 Pareto Weight of Wife μ

μ is the Pareto weight of the wife in cases where couples are of different ages and the older spouse reaches age J, which takes value 0.3 in the benchmark numerical results. Instead of using the value 0.3 as listed in Table 1.2, I now examine a lower value case when $\mu = 0.2$ and a higher value case when $\mu = 0.5$. I find varying μ has very small effects on our results. With $\mu = 0.2$ and exogenous age gap ($\Delta = 0$), the peak of the savings rate path is 6.5% higher than the savings rate with a balanced gender

[20]I also examined the robustness of the feature of the downward-sloping age profile of the savings rate in the pre-marital cohorts under the parameters listed in this subsection, and found that under all these parameters, the simulated age-savings rate profiles are downward-sloping and comparable to the empirical data. Due to limitation of space, this part is not reported here. All the results on robustness checks are available upon request.

ratio. With an endogenous age gap ($\Delta = 3$), the difference is reduced to 3.49%. When $\mu = 0.5$, the maximum changes in the savings rate are 6.5% and 3.24% corresponding to exogenous and endogenous age gaps respectively. With both values of μ, the change in the aggregate savings rate is very close to the results in the benchmark case, which is not surprising since the value of μ only affects the wife's Pareto weight in the last period of marriage if only the husband reaches that last age J. Except for this case, since couples consume the same amount of consumption goods each period, the Pareto weight of wives is implicitly set to 0.5.

1.7.3.2 Public Good Feature of Consumption κ

In our benchmark case, $\kappa = 0.8$. Now I examine two different values. In the first case, $\kappa = 0.5$ corresponding to completely private consumption in a married household. In the second case, $\kappa = 1$ meaning consumption in a married household is a public good without congestion. I find with a larger κ, the response in the aggregate savings rate is also larger with an increase in the gender ratio, because the higher benefit from marriage encourages single men to compete more fiercely by saving more aggressively. When $\kappa = 0.5$, peak increases in the aggregate savings rate are 5.75% and 2.28% corresponding to different age gap setups. Both numbers are smaller than the results in the benchmark case but not by much. Besides, allowing an endogenous age gap dampens the reaction of the aggregate savings rate by a larger fraction $\frac{5.75\%-2.28\%}{5.75\%} \approx 60\%$ than the benchmark case. This is because when marriage becomes less attractive, single men are more willing to postpone marriage. When $\kappa = 1$, the changes in the savings rate are 6.7% and 3.72% corresponding to different age gap setups. Both numbers are higher but close to the results in the benchmark case.

1.7.3.3 Mean of the Distribution of Match Quality z

The distribution of match quality z is important for our results since a large part of the benefit of marriage is from the match quality of wives. The mean of z takes value 1 in the benchmark case. I examine the cases in which the mean can take a higher value ($E(z) = 2$) or a lower value ($E(z) = 0.5$). Similar to the result of changing κ, higher value of $E(z)$ results in a higher benefit from marriage, and thus the competitive saving motive will be stronger. The changes in the savings rate with both values chosen are reported in Table 1.3. I find that changing the value of $E(z)$ to 0.5 or 2 can increase or decrease the responses of the aggregate savings rate in the benchmark model by 15%.

1.7.3.4 Standard Deviation of the Match Quality z

As a robustness check, I increase the standard deviation of z from 0.05 to 0.2 so the distribution of z is more dispersed, and find that the changes in the savings rate are smaller than the results in the benchmark case. This is opposite to what Hopkins and Kornienko (2010) find. By using a one-period tournament model, these authors claim that greater dispersion of rewards forces people into greater effort since the payoff of differentiating oneself from others is larger. Why my result goes in the opposite direction is interesting and left for future research. One candidate explanation is, in my multi-period model, although a more dispersed z increases the return to saving for single men, it also increases the risk of return to saving ex ante since single men's marriage prospects are more uncertain. It seems that the risk part dominates in the numerical results and thus the response in the aggregate savings rate is reduced.

To summarize, changing the parameter values used in the benchmark model doesn't change the responses in the aggregate savings rate by much. With an unex-

pected permanent increase of the gender ratio at birth from 1 to 1.18 and an exogenous age gap between spouses, the peak increase in the aggregate savings rate is around 6% of GDP which is similar to the empirical finding in Wei and Zhang (2011). Allowing single men to postpone their marriage besides increasing their pre-marital savings at given ages can potentially reduce the response in the savings rate to 3% of GDP which is about half the size in the exogenous age gap case.

1.8 Concluding Remarks and Future Research

This paper builds a life-cycle matching model with pre-marital investment. It is a multi-dimensional matching model with non-transferable utilities. I show that under certain conditions, I can summarize the multi-dimensional characteristics of men into a single dimension, thus the stable matching patterns can be characterized. With this model framework, I analyze how a competitive saving motive arises and how people's saving behavior and marital ages will react to marriage market imbalances.

By calibrating this model to China's economy, I analyze whether a competitive saving motive can explain the savings rate puzzle in China, i.e., a rising aggregate savings rate and a downward-sloping age profile of the savings rate among young cohorts. The quantitative result is consistent with empirical findings. After the gender ratio rises, single men increase their saving in a competitive way in order to improve their ranking in the marriage market, and the resultant increase in the aggregate savings rate is economically significant. Although tempered by some men's delaying marriage, the aggregate savings rate still increase by 3.35% of GDP, which is around 30% of the total increase in the savings rate from 1995 to 2007. In addition, the competitive saving motive also generates a downward-sloping age profile of the savings rate of the pre-marital cohorts.

The theory can be extended in a number of directions. Adding some randomness in the matching process in the marriage market should generate more realistic patterns for the ages of marriage. In that case, we may better fit the empirical distribution of marital ages and age gaps between spouses.

1.9 Appendix

1.9.1 Positively Assortative Matching Rule

1. If there is no mass point in $G_s(s)$, a man with saving $s_{m,1}$ and a woman with match quality indicator z match with each other if and only if $R^*[1-G_s(s_{m,1})] = 1 - H_z(z)$ with gender ratio R^*. Men with $s_{m,1} < s^*_{m,1}$ are not matched.

2. If there is a mass point in $G_s(s)$ at s_0, then the matches for men with saving s_0 are independently random draws from the corresponding interval in $H_z(z)$. In particular: (i) if $G_s^-(s_0) \geq 1 - \frac{1}{R^*}$,[21] men with saving s_0 match with women with $z \in [\underline{z}_0, \overline{z}_0]$ with probability $\frac{h_z(z)}{H_z(\overline{z}_0) - H_z(\underline{z}_0)}$, where $\underline{z}_0 = H_z^{-1}[1 - R^*(1 - G_s^-(s_0))]$, and $\overline{z}_0 = H_z^{-1}[1 - R^*(1 - G_s(s_0))]$. (ii) If $G_s(s_0) \geq 1 - \frac{1}{R^*}$ but $G_s^-(s_0) < 1 - \frac{1}{R^*}$, then for any man with s_0, with probability $p_0 = \frac{1 - R^*[1 - G_s(s_0)]}{R^* prob\{s_{m,1} = s_0\}}$ he matches with a woman, and with probability $1 - p_0$ he is not matched. Conditional on matching, the probability of marrying a woman with match quality z is $\frac{h_z(z)}{H_z(\overline{z}_0) - H_z(\underline{z})}$.

1.9.2 Proof of Lemma 1.4

The proof here is adapted from similar proofs in Hopkins and Kornienko (2004).

If s^* is the optimal choice for a man with $x_{m,1}$, then $s^* \geq s^{\mathbf{c}}(x_{m,1})$ the cooperative saving level, or otherwise it is strictly dominated by $s^{\mathbf{c}}(x_{m,1})$ due to consumption smoothing consideration. If the relation holds with equality, then naturally s^* is

[21] $G_s^-(s_0)$ is the left limit of $G_s(s_0)$ at s_0: $G_s^-(s_0) = \lim_{s \to s_0^-} G_s(s_0)$

increasing (maybe weakly) in $x_{m,1}$ since $s^{\mathbf{c}}(x_{m,1}) = \max[\frac{1}{2+r}(x_{m,1} - wn), 0]$. Otherwise if $s^* > s^{\mathbf{c}}(x_{m,1})$, then for the lifetime utility $v(x_{m,1}, s^*) = \ln(x_{m,1} + wn - s^*) + \beta \ln[\kappa((1+r)s^* + 2wn)] + \beta z(s^*)$, $\frac{\partial v^2}{\partial x_{m,1} \partial s^*} = \frac{1}{(x_{m,1} - s^*)^2} > 0$, which implies an increase in wealth leads to an increase in the marginal return on saving, and thus the optimal choice of saving $s(x_{m,1})$ increases.

Next, I show $s(x_{m,1})$ is strictly increasing. Suppose $s(x_{m,1})$ is not strictly increasing in an interval $[x_1, , x_2]$ such that $s(x) = s_0$, $\forall x \in [x_1, x_2]$. Then $G_s(s_0) > G_s^-(s_0)$, i.e., there's a mass point in the distribution of $G_s(s)$. But in this interval, if any man increases his saving by a small amount ε, he gains a discrete increase in his ranking by the matching rule in Equation (1.5) with marginal loss of utility from consumption. The existence of a profitable deviation rules out mass points in the distribution $G_s(s)$. As a result, the optimal saving function $s(x)$ must be strictly increasing.

By a similar argument, $s(x_{m,1})$ must be continuous $\forall x \geq x_{m,1}^*$. Otherwise, there is a jump at wealth level x_0 such that $\lim_{x \to x_0^-} s(x) = \widetilde{s} < s(x_0)$. According to Equation (1.5), all men choosing saving $s \in [s(x), \widetilde{s}]$ have the same ranking. As a result, if a man with wealth x_0 decreases his saving to $\widetilde{s} - \varepsilon$, he gains direct utility from consumption without affecting his ranking; thus reducing saving would be a profitable deviation making a discontinuous saving function not sustainable in equilibrium.

The last step is to show that the saving function $s(x)$ is differentiable. Consider two wealth levels x_1 and x_2 such that $x_1, x_2 \geq x_{m,1}^*$ and $x_2 = x_1 + \Delta$. Denote their saving choice as $s(x_1)$ and $s(x_2)$ respectively. By optimality, we have

$$\begin{aligned} & \ln[x_1 + wn - s(x_1)] + \beta \ln[\kappa((1+r)s(x_1) + 2wn)] + H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))] \\ \geq \; & \ln[x_1 + wn - s(x_2)] + \beta \ln[\kappa((1+r)s(x_2) + 2wn)] + H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] \end{aligned}$$

and

$$\begin{aligned} & \ln[x_2 + wn - s(x_2)] + \beta \ln[\kappa((1+r)s(x_2) + 2wn)] + H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] \\ \geq \; & \ln[x_2 + wn - s(x_1)] + \beta \ln[\kappa((1+r)s(x_1) + 2wn)] + H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))] \end{aligned}$$

By the mean value theorem, $\exists\ s_1, s_2 \in [s(x_1), s(x_2)]$ such that

$$\begin{aligned} & H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))] - H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] \\ \geq \; & (\frac{1}{(1+r)s_1 + 2wn} - \frac{1}{x_1 + wn - s_1})[s(x_2) - s(x_1)] \end{aligned}$$

and

$$\begin{aligned} & H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] - H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))] \\ \geq \; & (\frac{1}{(1+r)s_2 + 2wn} - \frac{1}{x_2 + wn - s_2})[s(x_1) - s(x_2)] \end{aligned}$$

After rearranging the terms and dividing each term by Δ, we have

$$\begin{aligned} & \frac{H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] - H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))]}{\Delta(\frac{1}{x_1 + wn - s_1} - \frac{1}{(1+r)s_1 + 2wn})} \\ \geq \; & \frac{s(x_2) - s(x_1)}{\Delta} \\ \geq \; & \frac{H_z^{-1}[1 - R^*(1 - H_{m,1}(x_2))] - H_z^{-1}[1 - R^*(1 - H_{m,1}(x_1))]}{\Delta(\frac{1}{x_2 + wn - s_2} - \frac{1}{(1+r)s_2 + 2wn})} \end{aligned}$$

As Δ goes to 0, the upper bound and lower bound for $\frac{s(x_2) - s(x_1)}{\Delta}$ converge to the same expression, and we get the derivative of $s(x)$ which converges to Equation (1.10).

1.9.3 Proof of Proposition 1.1

In Lemma 1.4, we've showed that the equilibrium saving function must be strictly increasing and thus men with wealth endowment $x_{m,1} < x_{m,1}^*$ cannot be matched, and their saving function is described by Equation (1.7). For men with wealth $x_{m,1} \geq x_{m,1}^*$, we need to check the differential equation (1.10) resulting from first-order condition

1.9, and matching rule 1.6 actually represents optimal choice of any single man given other people also use the same saving strategy. Following Hopkins and Kornienko (2004), it is sufficient to show that the utility function from Equation (1.8) $U(x_{m,1}, s)$ exhibits pseudo-concavity such that it is decreasing in s for $s > s(x_{m,1})$ and increasing in s for $s < s(x_{m,1})$. Suppose a man with endowment $x > x^*_{m,1}$ chooses saving $s_0 < s(x)$ instead of $s(x)$, then there might be two possible cases.

In the first case $s_0 < s(x^*_{m,1}) = s^*$ such that this man will not be matched. From the optimal saving rule of single men from Equation (1.7), we have $s_0 = \frac{\beta}{1+\beta}x$, and his corresponding life-time utility is $V_0 = (1+\beta)\log(\frac{x}{1+\beta} + wn)$. Notice that we should restrict $x < \frac{1+\beta}{\beta}s^*$ otherwise this man gets matched naturally. But such a choice is strictly dominated by choosing s^* and matching with $\underline{z}$. If he chooses s^*, his life-time utility is $V_1 = \ln(x + wn - s^*) + \beta\ln((1+r)s^* + 2wn) + \beta(\kappa + \underline{z})$. Then we have

$$
\begin{aligned}
&V_1 - V_0 \\
=&\ln(x + wn - s^*) + \beta\ln((1+r)s^* + 2wn) + \beta(\kappa + \underline{z}) \\
&- (1+\beta)\log(\frac{x}{1+\beta} + wn) \\
=&\ln(x + wn - s^*) + (1+\beta)\log(\frac{x^*_{m,1}}{1+\beta} + wn) - \ln(x^*_{m,1} + wn - s^*) - (1+\beta)\log(\frac{x}{1+\beta} + wn) \\
=&[\ln(x + wn - s^*) - (1+\beta)\log(\frac{x}{1+\beta} + wn)] - [\ln(x^*_{m,1} + wn - s^*) - (1+\beta)\log(\frac{x^*_{m,1}}{1+\beta} + wn)] \\
>&0
\end{aligned}
$$

In the calculation above, the second equality comes from the boundary condition stated in Proposition 1.1, and the last inequality is based on

$$
\begin{aligned}
&\frac{\partial}{\partial x}[\ln(x + wn - s^*) - (1+\beta)\log(\frac{x}{1+\beta} + wn)] \\
=&\frac{s^* - \frac{\beta}{1+\beta}x}{(x + wn - s^*)(\frac{x}{1+\beta} + wn)} > 0
\end{aligned}
$$

In the second possible case, $s^* \leq s_0 < s(x)$. Then we can find another $x_0 < x$ such that $s(x_0) = s_0$. From Equation (1.8), conditional on other men adopting the saving

function strategy and $\frac{\partial U}{\partial s} = 0$, we have $\frac{\partial U^2}{\partial x \partial s} > 0 \ \forall x > x^*_{m,1}$. Hence $\frac{\partial}{\partial s}U(x, s_0) > \frac{\partial}{\partial s}U(x_0, s_0) = 0$. As a result, s_0 cannot be optimal saving for men with wealth x. Thus $U(x_{m,1}, s)$ is increasing for $s < s(x_{m,1})$. The proof for $U(x_{m,1}, s)$ decreasing for $s > s(x_{m,1})$ is similar.

What remains is to show that differential equation (1.10) uniquely pins down the equilibrium saving function. This just follows from the fundamental theorem of differential equation with the initial condition given in Proposition 1.1. With the balanced gender ratio $R^* = 1$, every man gets matched. The man with the lowest wealth endowment $\underline{x}$ is determined to match with $\underline{z}$. As a result, this man will not attempt to change his position, and will simply adapt the cooperative saving based on Equation (1.12). Otherwise, if $R^* > 1$, only men with endowment above $x^*_{m,1}$ get matched, and the savings rate of men with $x^*_{m,1}$ is raised to a level such that they are indifferent between getting matched with $\underline{z}$ and remaining single. Thus, there is exactly one solution satisfying the boundary conditions mentioned in Proposition 1.1, and that constitutes the equilibrium saving function.

1.9.4 Proof of Proposition 1.4

A representative woman doesn't save when she is single for the following reasons: Since her income is wn at all ages and $\beta(1 + r) = 1$, she would choose 0 saving even if there were no marriage market due to the consumption smoothing consideration. Now with a marriage market, her motivation to save is further reduced since she can take advantage of her future husband's pre-marital saving. In addition, there is no strategic saving motive since her saving is not observable in marriage market.

Given that women's saving is 0 in equilibrium, for a young man with saving s marrying a young woman with z, based on Equation (1.15), his utility after marriage

is

$$u_{m,22} = (1+\beta)\ln[\kappa(\frac{s}{\beta+\beta^2} + 2wn)] + (1+\beta)z$$

and from Equation (1.18) if he marries a middle-aged woman with same z, he gets

$$u_{m,23} = \ln[\kappa(\frac{s}{\beta} + 2wn)] + \beta\ln(wn) + z$$

Under assumption 1, we can see that $u_{m,22} > u_{m,23}$, and thus young men prefer marrying young women. Similarly, if we compare Equation (1.14) with Equation (1.16), we find middle-aged men consider women of different ages as perfect substitutes. As a result, for all realized z, a woman has no gain and may lose her ranking in the marriage market if she postpones marriage.

On the other hand, for a woman marrying at age 1, based on Equations (1.15) and (1.17), her expected lifetime utilities if she marries a young man or a middle-aged man are respectively

$$\begin{aligned} u_{w,22} &= \ln(wn) + (\beta+\beta^2)\int[\ln(\frac{s_{11}(z)}{\beta+\beta^2} + 2wn) + \ln\kappa + z]dH_z(z) \\ u_{w,32} &= (1+\beta^2)\ln(wn) + \beta\int[\ln(\frac{s_{21}(z)}{\beta} + 2wn) + \ln\kappa + (1+\beta)z]dH_z(z) \end{aligned}$$

in which $s_{11}(z)$ and $s_{21}(z)$ denote the savings of young and middle-aged men she matches with if her realized match quality is z. For a woman marrying at age 2, based on Equations (1.14) and (1.19), her expected lifetime utilities if she marries a young man or a middle-aged man are respectively

$$\begin{aligned} u_{w,23} &= (1+\beta)\ln(wn) + \beta^2\int[\ln(\frac{s_{12}(z)}{\beta} + 2wn) + \ln\kappa + z]dH_z(z) \\ u_{w,33} &= (1+\beta)\ln(wn) + \beta^2\int[\ln(\frac{s_{22}(z)}{\beta} + 2wn) + \ln\kappa + z]dH_z(z) \end{aligned}$$

in which $s_{12}(z)$ and $s_{22}(z)$ denote the savings of young and middle-aged men she matches with if her realized match quality is z. By the argument in the first paragraph

of this proof, we know $s_{11}(z) > s_{12}(z)$, and $s_{21}(z) = s_{22}(z)$. Consequently,

$$\begin{aligned}
& u_{w,22} - u_{w,23} \\
= \; & (\beta + \beta^2) \int [\ln(\frac{s_{11}(z)}{\beta + \beta^2} + 2wn) + \ln \kappa + z] dH_z(z) \\
& -\beta \ln(wn) - \beta^2 \int [\ln(\frac{s_{12}(z)}{\beta} + 2wn) + \ln \kappa + z] dH_z(z) \\
> \; & (\beta + \beta^2) \int [\ln(\frac{s_{11}(z)}{\beta + \beta^2} + 2wn) + \ln \kappa + z] dH_z(z) \\
& -\beta \ln(wn) - \beta^2 \int [\ln(\frac{s_{11}(z)}{\beta} + 2wn) + \ln \kappa + z] dH_z(z)
\end{aligned}$$

$$\begin{aligned}
= \; & \int [(\beta + \beta^2) \ln(\frac{s_{11}(z)}{\beta + \beta^2} + 2wn) - \beta \ln(wn) - \beta^2 \ln(\frac{s_{11}(z)}{\beta} + 2wn)] dH_z(z) \\
& +\beta(z + \ln \kappa) \\
> \; & (\beta + \beta^2) \int [\ln(\frac{s_{11}(z)}{\beta + \beta^2} + 2wn) - \ln(\frac{\beta s_{11}(z) - \beta wn}{\beta + \beta^2} + 2wn)] dH_z(z) \qquad (1.33) \\
& +\beta(z + \ln \kappa) \\
> \; & 0
\end{aligned}$$

Here step (1.33) relaxes the borrowing constraint for the woman and allows her to smooth consumption between age 2 and 3. Similarly,

$$\begin{aligned}
& u_{w,32} - u_{w,33} \\
= \; & (1 + \beta^2) \ln(wn) + \beta \int [\ln(\frac{s_{21}(z)}{\beta} + 2wn) + \ln \kappa + (1 + \beta) z] dH_z(z) \\
& -(1 + \beta) \ln(wn) - \beta^2 \int [\ln(\frac{s_{22}(z)}{\beta} + 2wn) + \ln \kappa + z] dH_z(z) \\
= \; & (\beta - \beta^2) \int [\ln(\frac{s_{21}(z)}{\beta} + 2wn) - \ln(wn) + \ln \kappa + z] dH_z(z) \\
& +\beta z \\
> \; & 0
\end{aligned}$$

As a result, a woman will lose ranking in the marriage market and will suffer a utility loss is she postpones marriage. That is, rational women choose to marry at age 1.

Chapter 2

Amplification and Asymmetric Effects without Collateral Constraint[1]

2.1 Introduction

The seminal contribution of Kiyotaki and Moore (1997) has spurred a vast literature on the importance of collateral constraints in propagating and amplifying shocks to the economy including Iacoviello (2005) and Mendoza (2010). However, several recent papers including Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) obtain these effects without imposing any collateral constraint. The common feature of these papers is that agents borrow through state-incontingent debt, i.e., markets are incomplete. It is possible that a large part of the amplification effect of collateral constraint is actually due to market incompleteness. In this paper, we build and calibrate a model and solve it with and without collateral constraint in order to understand the qualitative and quantitative importance of market incompleteness relative to collateral constraints.

The model is a simplified version of Iacoviello (2005). We have two types of risk-averse agents, entrepreneurs and households, in a one-consumption good economy with fixed supply of an asset: land. There is a land market in which all agents participate. The entrepreneurs combine land with labor supplied by the households to produce consumption good. This production process is subject to aggregate stochastic

[1]This chapter is adapted from my joint work with Professor Dan Cao.

productivity shocks. The households provide labor to earn wages and consume both the consumption good and the housing service. We assume that the entrepreneurs are less patient than the households, and thus they tend to borrow from the households. For the purpose of this study, we consider three alternative financial market structures. In the first one, *the benchmark model* - the model with incomplete markets - the entrepreneurs can borrow from the households using state-incontingent debt only subject to the non-Ponzi condition. In the second model - *the model with a collateral constraint and incomplete markets* - debt is still state-incontingent but the entrepreneurs are subject to the collateral constraint, i.e., borrowing is constrained to be less than a fraction of the expected value (or current value) of the entrepreneurs' land holding. In the last model - *the model with collateral constraints and complete markets* - the entrepreneurs can sell a complete set of state-contingent securities to the households subject to a collateral constraint on each state-contingent security. We solve the Markov equilibria of these economies using the global nonlinear method developed by Kubler and Schmedders (2003) and Cao (2010).

We find that, in the benchmark model with incomplete markets, the equilibrium dynamics exhibit amplification and asymmetric responses to *symmetric* exogenous productivity shocks. For example, land price and output increase after a good shock by less than they decrease after a bad shock of the same size. The amplification and asymmetric effects in this model are due to the *net worth effect* as follows.[2][3] An initial negative shock decreases the net worth of the levered entrepreneurs. Due to risk-aversion, these entrepreneurs try to smooth consumption but their debt is not state-contingent so they have to liquidate some of their land holding to maintain a certain level of consumption. Their selling activities depress the price of land, and

[2]Brunnermeier and Sannikov (2014) call this channel *amplification through prices.*

[3]Geanakoplos (1997) includes a simple 2-period example showing that this effect increases asset price volatility relative to the collateral channel.

further lower their net worth, setting off a vicious circle of falling land price and falling net worth. At the heart of this vicious circle is the pecuniary externality due to market incompleteness a la Geanakoplos and Polemarchakis (1986), i.e., when selling off some of their land holding, each entrepreneur does not take into account the negative effect of falling land price on the net worth of other entrepreneurs.

In the second model with collateral constraint, similar to the findings in the collateral constraint literature, the dynamics of the economy depend crucially on whether the collateral constraint is binding. With a binding collateral constraint, land price and output of the economy are much more sensitive to changes in net worth of the entrepreneurs. The economy also exhibits asymmetric responses to exogenous productivity shocks. We also observe that the probability of a binding constraint decreases rapidly in the size of the shocks due to the precautionary saving motive of the entrepreneurs.

In both benchmark and collateral constraint models, the economy exhibits amplified and asymmetric responses to symmetric exogenous shocks. Quantitatively, the responses are only slightly smaller in the benchmark model compared to the collateral constraint model. These results suggest that market incompleteness alone accounts for a significant part of the responses to shocks in the economy.

To further understand this point, the third financial structure we consider is with collateral constraint and complete markets, in which the entrepreneurs have access to a complete set of state-contingent securities but the sale of these securities has to be collateralized by land. In the long run the economy converges to a single level of wealth distribution, i.e., we have some sort of dynamically complete insurance. At this level of wealth distribution, there is no amplification nor asymmetry effects of exogenous shocks to the economy. These results demonstrate that collateral constraint has to

be coupled with markets incompleteness in order to generate significant amplification and asymmetric responses to shocks.

The main results of the paper can be summarized by Table 2.1. As we will illustrate in Section 2.2, the productivity shock can take on 3 possible values denoted as expansion (high), normal and recession (low), where the high and low shocks are each 3 percent away from normal. Starting from the normal state, we compute the percent changes in land price and output when the aggregate shock in the next period switches to expansion or recession respectively. These changes depend on the wealth distribution between entrepreneurs and households and thus we report the average changes over the stationary distribution of the entrepreneurs' share of wealth[4]. We report our results under different financial market structures - the benchmark model with incomplete markets (row 3), collateral constraint models (rows 4 and 5), collateral constraint with complete markets (row 6), and complete markets (row 7) - with the main parameters calibrated to the U.S. economy from Iacoviello (2005).[5] We compare the responses of land price and output to shocks under different financial markets structures to the responses in the model with complete markets (without collateral constraint).

First, Table 2.1 shows the asymmetric and amplified effects under both the benchmark model and the model with a collateral constraint, i.e. good shocks increase land price and output[6] by less than bad shocks decrease land price and output. For example, in the benchmark model, a 3% fall in productivity generates a larger response

[4]The responses are the averages over the stationary distribution, but if we condition on lower values of the entrepreneurs' wealth, the responses are much larger.

[5]Land price and output stay almost the same as their current values if the economy stays in the normal state in the next period.

[6]Output here is defined as total amount of consumption good produced by the entrepreneurs using land and labor. The definition omits the imputed rental value of land from the housing consumption of the households. The same results hold, however, when we add this imputed rental value of land to the current definition of output.

Table 2.1: Average responses of land price and output to 3% technology shock

	Land price		Output	
Type of constraint	Expansion	Recession	Expansion	Recession
benchmark, incomplete markets	3.21%	-3.45%	3.25%	−3.44%
collateral constraint (IM)	3.30%	-3.76%	3.15%	−3.50%
collateral constraint (alt, IM)	3.32%	-3.83%	3.17%	−3.58%
collateral constraint (CM)	3.01%	-2.99%	2.97%	−2.97%
complete markets	3.00%	-3.00%	2.97%	−2.97%

in output (−3.44%) than the response to a same-size positive shock (3.25%). Second, an amplification effect is present under both incomplete markets and collateral constraint. For example, the responses in output (3.25% to a high shock and −3.44% to a low shock) are larger than the size of the shock (3%) in the benchmark model[7]. In the model with incomplete markets and a collateral constraint, the asymmetric and amplified effects are slightly stronger than in the benchmark model. Lastly, the amplified and asymmetric effects are absent with complete markets, with or without the collateral constraint.

The paper is related to the vast literature on the effects of collateral constraint in addition to the papers cited above. In particular, the benchmark model is a simplified version of Iacoviello (2005). But instead of relying on log-linearization as in Iacoviello (2005), we solve for the global nonlinear dynamics of the equilibrium.[8] To do so, we use the concept of Markov equilibrium and the numerical method developed by Kubler and Schmedders (2003) and Cao (2010) but extend it to a production economy

[7]The magnitude of amplification is similar to the one in Iacoviello (2005).

[8]Guerrieri and Iacoviello (2015) suggests a way to adapt the log-linearization method in a piecewise fashion to handle occasionally binding constraints.

with elastic labor supply, housing consumption, and a natural borrowing limit.[9] This global nonlinear solution also allows us to quantitatively assess the accuracy of the log-linearization solution. Another methodological contribution of this paper is that we extend their numerical methods to allow for a wide range of financial markets structure including incomplete markets with exogenous borrowing constraints, and collateral constraints with complete markets.

In a small open economy framework, Mendoza (2010) also compares the equilibrium under collateral constraint versus the equilibrium under an exogenous borrowing constraint limit and finds that exogenous borrowing limit weakens the amplification effect on Tobin's Q by a factor of 5.75. The difference between our results and his comes from the fact that the supply of the collateral asset (capital) is elastic in Mendoza (2010), while it is completely inelastic in our model. Inelastic supply of the asset implies more volatility in the price of the asset and gives more room for negative shocks to be amplified by the selling activities of the constrained agents, even in the absence of a collateral constraint.[10] Moreover, Appendix 2.6.2 shows that imposing an exogenous borrowing constraint significantly reduces the amplification and asymmetric effects of incomplete markets.

The chapter is organized as follows. Section 2.2 presents the benchmark incomplete markets model and the solution method, as well as reasonable parameters to analyze the solution of this benchmark model. Section 2.3 studies the collateral constraint model and compares it to the benchmark model. Section 2.4 studies the complete markets model with collateral constraint and also compares it to the collateral constraint

[9]Indeed, with housing as durable good, we need to make a change to the timing of production compared to the timing in Iacoviello (2005), without changing key economic forces, in order to apply the solution method in Kubler and Schmedders (2003) and Cao (2010).

[10]Boldrin et al. (2001) show that when capital supply is flexible, it is impossible to match the observed volatility and the equity premium in equity prices in the U.S.

model. Section 2.5 concludes. Additional proofs and constructions are presented as an appendix in Section 2.6.

2.2 Benchmark model

We build a simple model of a production economy in which a durable asset (land) is used as collateral to borrow and as an input in production. Moreover the model is calibrated to the U.S. economy.[11] The solution method, intuition, and results based on this model should carry over to similar models.

2.2.1 Economic environment

Consider an economy inhabited by two types of agents: entrepreneurs and households who are both infinitely lived and of measure one. There is one consumption good. Entrepreneurs produce the consumption good by hiring household labor and combining it with land. Households consume the consumption good and land (housing), and supply labor to the entrepreneurs.

We adopt the standard notation of uncertainty. Time is discrete and runs from 0 to infinity. In each period, an aggregate shock s_t is realized. We assume that s_t follows a finite-state Markov chain. Let $s^t = (s_0, s_1, ..., s_t)$ denote the history of realizations of shocks until date t. To simplify the notation, for each variable x, we use x_t as a shortcut for $x_t(s^t)$.

Households maximize a lifetime utility function given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(c'_t)^{1-\sigma_2} - 1}{1-\sigma_2} + j \frac{(h'_t)^{1-\sigma_h} - 1}{1-\sigma_h} - \frac{1}{\eta}(L'_t)^{\eta} \right\}, \tag{2.1}$$

[11] Cordoba and Ripoll (2004) is another simple extension of Kiyotaki and Moore (1997) to nonlinear production and concave utility functions. In Appendix 2.6.4, we show that the solution method used in this paper applies to their model as well. Despite its simplicity, the model is not calibrated to the U.S. data, and thus is not sufficient to make a quantitative point.

where $\mathbb{E}_0[.]$ is the expectation operator, $\beta \in (0,1)$ is the discount factor, c'_t is consumption at time t, h'_t is the holding of land. L'_t denotes the hours of work. Households can trade in the market for land as well as a state-incontingent bond market.[12] The budget constraint of the households is

$$c'_t + q_t(h'_t - h'_{t-1}) + p_t b'_t \leq b'_{t-1} + w_t L'_t. \tag{2.2}$$

Given land in the utility function of households, we have implicitly $h'_t \geq 0$.

Entrepreneurs use a Cobb-Douglas constant-returns-to-scale technology that uses land and labor as inputs. They produce consumption good Y_t according to

$$Y_t = A_t h_t^{\upsilon} L_t^{1-\upsilon}, \tag{2.3}$$

where A_t is the aggregate productivity which depends on the aggregate state s_t, h_t is real estate input, and L_t is labor input.

In contrast to Iacoviello (2005), the production function uses the contemporaneous land holding of the entrepreneurs instead of the land holding from the previous period. This minor modification turns out to be crucial to apply the concept of Markov equilibrium in Subsection 2.2.2 and the solution method in Subsection 2.2.3.2. However this modification does not affect key economic forces.

We want the entrepreneurs to borrow from the households so we assume that the entrepreneurs discount the future at the rate $\gamma < \beta$. The entrepreneurs maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \frac{(c_t)^{1-\sigma_1} - 1}{1 - \sigma_1} \tag{2.4}$$

subject to the budget constraint

$$c_t + q_t(h_t - h_{t-1}) + p_t b_t \leq b_{t-1} + Y_t - w_t L_t. \tag{2.5}$$

[12] Given their higher discount factor, the households tend to lend to the entrepreneurs so we do not need to impose any borrowing constraint on the households.

Output Y_t is produced by combining land and labor using the production function (2.3). Given the production function of the entrepreneurs, we have implicitly $h_t \geq 0$.

In this benchmark model, we do not impose a collateral constraint as in Iacoviello (2005), so we need to implicitly impose no-Ponzi scheme conditions on the entrepreneurs and households, i.e.,

$$\lim_{t\to\infty} \left(\prod_{t'=0}^{t-1} p_{t'} \right) b_t \geq 0$$

and

$$\lim_{t\to\infty} \left(\prod_{t'=0}^{t-1} p_{t'} \right) b'_t \geq 0.$$

To finish the description of the model, here we assume that the only source of uncertainty is the aggregate productivity A_t. It is straightforward to extend the model to incorporate other sources of uncertainty such as uncertainty in the housing preference parameter j.

2.2.2 Equilibrium

The definition of the sequential competitive equilibrium for this economy is standard.

Definition 2.1 *A **competitive equilibrium** is sequences of prices* $\{p_t, q_t, w_t\}_{t=0}^{\infty}$ *and allocations* $\{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t\}$ *such that (i) the* $\{c'_t, h'_t, b'_t, L'_t\}$ *maximize (2.1) subject to budget constraint (2.2) and the no-Ponzi condition and* $\{c_t, h_t, b_t, L_t\}$ *maximize (2.4) subject to budget constraint (2.5) and the no-Ponzi condition, and production technology (2.3) given* $\{p_t, q_t, w_t\}$ *and initial asset holdings* $\left\{h_{-1}, b_{-1}, h'_{-1}, b'_{-1}\right\}$*; (ii) land, bond, labor, and good markets clear:* $h_t + h'_t = H$, $b_t + b'_t = 0$, $L_t = L'_t$, $c_t + c'_t = Y_t$.

Let ω_t denote the *normalized financial wealth* of the entrepreneurs:

$$\omega_t = \frac{q_t h_{t-1} + b_{t-1}}{q_t H}, \tag{2.6}$$

and ω_t' denote the normalized financial wealth of the households:

$$\omega_t' = \frac{q_t h_{t-1}' + b_{t-1}'}{q_t H}.$$

By the housing and bond market clearing conditions, we have $\omega_t' = 1 - \omega_t$ in any competitive equilibrium. Therefore in order to keep track of the normalized financial wealth distribution between the entrepreneurs and the households, (ω_t, ω_t'), in equilibrium, we only need to keep track of ω_t. To simplify the language, we use the term *wealth distribution* for normalized financial wealth distribution.

Following Kubler and Schmedders (2003) and Cao (2010), we define Markov equilibrium as follows.

Definition 2.2 *A **Markov equilibrium** is a competitive equilibrium in which prices and allocations at time t, as well as the wealth distribution at time $t+1$ under different realizations of the exogenous shocks s_{t+1} depend only on the wealth distribution at time t, ω_t as well as the exogenous state s_t.*

This Markov equilibrium definition features the endogenous state variable ω_t that depends on land price q_t (which by itself depends on the state variable). This equilibrium was first studied in Duffie et al. (1994). Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) use the same type of equilibrium definition in their continuous time models. We are going to use the algorithm developed in Kubler and Schmedders (2003) and Cao (2010) to compute this Markov equilibrium.

2.2.3 Solution

In this Subsection, we first show the equations that characterize a competitive equilibrium. In the absence of borrowing constraints, the model does not have a steady-state in the absence of uncertainty because of the differences in the discount factors

of the households and the entrepreneurs. However, out of steady state, uncertainty prevents the entrepreneurs from borrowing too much because of the precautionary saving motive. Thus, the Markov equilibrium exists with globally bounded amount debt held by the entrepreneurs.

In later sections, when we introduce borrowing constraints, either endogenous collateral constraint or exogenous borrowing limit, a steady-state exists and the Markov equilibrium converges to the steady state when uncertainty vanishes.

2.2.3.1 Equilibrium equations

Given their housing holding at time t, h_t, the entrepreneurs choose labor demand L_t to maximize profit

$$\max_{L_t} \{Y_t - w_t L_t\}$$

subject to the production technology given in (2.3). The first order condition (F.O.C) with respect to L_t implies

$$w_t = (1-\upsilon) A_t h_t^{\upsilon} L_t^{-\upsilon}, \tag{2.7}$$

i.e. $L_t = \left(\frac{(1-\upsilon)A_t}{w_t}\right)^{\frac{1}{\upsilon}} h_t$ and profit

$$Y_t - w_t L_t = \pi_t h_t$$

where $\pi_t = \upsilon A_t \left(\frac{(1-\upsilon)A_t}{w_t}\right)^{\frac{1-\upsilon}{\upsilon}}$ is profit per unit of land.

The first-order conditions with respect to h_t and b_t in the maximization problem of the entrepreneurs imply

$$(\pi_t - q_t) c_t^{-\sigma_1} + \gamma \mathbb{E}_t [q_{t+1} c_{t+1}^{-\sigma_1}] = 0 \tag{2.8}$$

and

$$-p_t c_t^{-\sigma_1} + \gamma \mathbb{E}_t \left[c_{t+1}^{-\sigma_1}\right] = 0. \tag{2.9}$$

Similarly, the F.O.Cs for households are

$$h'_t : -q_t c'^{-\sigma_2}_t + j h'^{-\sigma_h}_t + \beta \mathbb{E}_t \left[q_{t+1} c'^{-\sigma_2}_{t+1} \right] = 0 \tag{2.10}$$

$$b'_t : -p_t c'^{-\sigma_2}_t + \beta \mathbb{E}_t \left[c'^{-\sigma_2}_{t+1} \right] = 0 \tag{2.11}$$

$$L'_t : w_t c'^{-\sigma_2}_t = L'^{\eta-1}_t \tag{2.12}$$

The first order conditions with respect to h_t and h'_t shed light on the determinants of land price. We rewrite (2.8) as

$$\begin{aligned} q_t &= \pi_t + \gamma \mathbb{E}_t [q_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma_1}] \\ &= \mathbb{E}_t [\sum_{s=0}^{\infty} \gamma^s \left\{ \left(\frac{c_{t+s}}{c_t} \right)^{-\sigma_1} \right\} \pi_{t+s}]. \end{aligned}$$

The right hand side of this equation show that, from the entrepreneurs point of view, land price is the net present discounted value of present and future profit from production using land and the discount factor depends on the marginal utility of the entrepreneurs. Similarly, we re-write (2.10) as

$$\begin{aligned} q_t &= \frac{j \left(h'_t \right)^{-\sigma_h}}{c'^{-\sigma_2}_t} + \beta \mathbb{E}_t \left[q_{t+1} \left(\frac{c'_{t+1}}{c'_t} \right)^{-\sigma_2} \right] \\ &= \mathbb{E}_t [\sum_{s=0}^{\infty} \beta^s \left(\frac{c'_{t+s}}{c'_t} \right)^{-\sigma_2} j \left(h'_{t+s} \right)^{-\sigma_h}]. \end{aligned}$$

From the point of view of the households, the house price is the present discounted value of current and future marginal utility from housing.

Despite the fact that we do not impose any constraint on the entrepreneurs' borrowing except for the no-Ponzi condition, the following lemma shows that, in equilibrium, the financial wealth of the entrepreneurs is endogenously bounded from below.

Lemma 2.1 *In any competitive equilibrium, thus any Markov equilibrium, we must have* $\omega_t \geq 0$ *for all* t *and* s^t.

Proof. We prove this result by contradiction. Suppose that in a competitive equilibrium, there is t and s^t such that $\omega_t(s^t) < 0$. Given the formula for the profit maximization of the entrepreneurs above and the definition of financial wealth ω_t, the budget constraint (2.5) can be re-written as

$$c_t + (q_t - \pi_t) h_t + p_t b_t \leq q_t \omega_t.$$

Pick a $\lambda > 1$, and consider an alternative trading and consumption plan $\left\{\tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'}\right\}_{t'=0}^{\infty}$ for the entrepreneurs which is the same as the initial plan for $t' < t$ but for $t' \geq t$:

$$\left\{\tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'}\right\}_{t'=t+1}^{\infty} = \{\lambda c_{t'}, \lambda h_{t'}, \lambda b_{t'}\}_{t'=t+1}^{\infty}$$

and

$$\begin{aligned} \tilde{c}_t &= \lambda c_t - (\lambda - 1) q_t \omega_t > \lambda c_t \\ \tilde{h}_t &= \lambda h_t \\ \tilde{b}_t &= \lambda b_t. \end{aligned}$$

This alternative plan $\left\{\tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'}\right\}_{t'=t+1}^{\infty}$ clearly delivers strictly higher utility to the entrepreneurs while satisfying all the constraints, including the no-Ponzi condition. This contradicts the fact that the initial plan is optimal. Therefore $\omega_t \geq 0$ for all t and s^t.

We interpret this lower bound of the entrepreneurs' wealth as their natural borrowing limit.

2.2.3.2 Global nonlinear method

We also solve the exact nonlinear equilibrium of the model using the algorithm in Kubler and Schmedders (2003) and Cao (2010). In particular, we solve for Markov equilibrium in this economy. The original algorithm in Kubler and Schmedders (2003)

is for an endowment economy. Cao (2010) extends this algorithm to a production economy with capital accumulation. In the current paper, we show that the original algorithm works similarly when we add labor choice as well as housing consumption decision of the households.

Our algorithm looks for a Markov equilibrium mapping from the financial wealth distribution, ω_t - defined in (2.6), and aggregate shock, s_t, to land price, q_t and bond price, p_t, the allocation $\{c_t, h_t, b_t, c'_t, h'_t, b'_t, L'_t\}$ and wage w_t, as well as future financial wealth distribution, ω_{t+1}, depending on the realization of future aggregate shocks. Indeed given the mapping from ω_{t+1} to $\{q_{t+1}, c_{t+1}, c'_{t+1}\}$, for each ω_t and s_t, we can solve for $\{c_t, h_t, b_t, c'_t, h'_t, b'_t, L'_t\}$ and ω_{t+1} using the Equations (2.8), (2.9), (2.10), (2.11), (2.12), the housing and bond market clearing conditions, as well as the future financial wealth distribution for each future state. Here, we follow the procedure in Cao (2010), instead of the one in Kubler and Schmedders (2003) in solving for ω_{t+1} simultaneously with other unknowns. The additional equations needed to solve for ω_{t+1} are Equation (2.6) applied to each of the future state s_{t+1}: $\omega_{t+1} = \frac{q_{t+1}(\omega_{t+1}, s_{t+1})h_t + b_t}{q_{t+1}(\omega_{t+1}, s_{t+1})H}$, in which the mapping from future wealth distribution and exogenous state to land price, $q_{t+1}(\omega_{t+1}, s_{t+1})$, is determined from the previous iteration of the algorithm. It is easy to verify that the number of unknowns are exactly the same as the number of equations.

The algorithm starts by solving for the equilibrium mapping for 1-period economy. Then given the mapping for T-period economy (from period 0 to 1), we can solve the mapping for $(T+1)$-period economy following the procedure described above. The algorithm converges when the mappings for $T-$period economy and $(T+1)$-period economy are sufficiently close to each other.

An important difference relative to Kubler and Schmedders (2003) and Cao (2010) is that there is not any borrowing constraint on the entrepreneurs, therefore when ω_t

is sufficiently low, $c_t = 0$ and the first-order conditions (2.8) and (2.9) are not well-defined. To deal with this issue, we look for the threshold $\underline{\omega}_t$ such that at $\omega_t = \underline{\omega}_t$, $c_t = \underline{c} > 0$, $\underline{c}$ is small and predetermined. At $\underline{\omega}_t$, we know $c_t = \underline{c}$, so we solve for $\{\underline{\omega}_t, h_t, b_t, c'_t, h'_t, b'_t, L'_t, \omega_{t+1}\}$ given the same set of Equations described above. Lemma 2.1 shows that $\underline{\omega}_t > 0$. Numerically, when $\underline{c}$ is close to zero, $\underline{\omega}_t$ is also close to zero.[13,14] The lower bound $\underline{\omega}_t$ should depend on the exogenous shock s_t, as a result, in contrast to the standard algorithm in Kubler and Schmedders (2003) and Cao (2010), the grid for ω_t, $[\underline{\omega}_t, 1]$, depends on the exogenous state s_t, as well as on the horizon of the approximate finite-horizon economy.[15] With the calibrated parameters we use in this paper, the stationary distribution also concentrates around very low levels of ω (around 0.02), so when we discretize $[\underline{\omega}_t, 1]$, we put more points in the range of low values of ω.

2.2.4 Parameter values

We use parameter values from Iacoviello (2005), given in Table 2.2. In particular, the value of υ is chosen to make sure that the value of land holding for entrepreneurs (commercial real estate in the data) in steady state is around 20%.

In order to use the global nonlinear method in Subsection 2.2.3.2, we need to discretize the process for A_t by a finite number of points. For A_t we use a three point process, $A_t \in \{A - \Delta, A, A + \Delta\}$, which corresponds to booms, $s_t = G$, normal times,

[13] When the entrepreneurs have some labor endowment, $\underline{\omega}_t$ can actually be slightly negative.

[14] During iterations, when $\omega_t \leq \underline{\omega}_t$, we extrapolate the functions $x(\omega_t, s_t)$, where $x = q, c, c'$, to obtain the values of x below $\underline{\omega}_t$.

[15] Given households are allowed to borrow from entrepreneurs, the upper bound for ω_t should exceed 1. However, around the steady state, ω_t tends to fall below 1, because of the consumers' tendency to lend given their higher discount factor.

Table 2.2: Baseline Parameter Values

A	1	mean technology
j	0.1	household's weight of housing in utility
H	1	total supply of housing
β	0.99	discount rate of household
γ	0.98	discount rate of entrepreneur
σ_1	1	CRRA coefficient of entrepreneur
σ_2	1	CRRA coefficient of household
σ_h	1	CRRA coefficient of household for housing
η	1	household labor supply elasticity
υ	0.03	share of housing in production

$s_t = N$, and recessions, $s_t = B$.[16] We assume the following form of transition matrix

$$\Pi = \begin{bmatrix} \pi & 1-\pi & 0 \\ \frac{1-\pi_0}{2} & \pi_0 & \frac{1-\pi_0}{2} \\ 0 & 1-\pi & \pi \end{bmatrix}.$$

The exogenous stochastic process of productivity is totally symmetric. However, due to the collateral constraint and incomplete markets, the resulting dynamics of the economy become asymmetric as shown in Subsection 2.2.5 below.

The values of π_0 and π are calibrated using historical US data in the United States. According to the definitions of business cycle expansions and recessions by NBER, there are 12 recessions in total from post-WWII (1945) to December 2013, with an average length of each recession equal to 3.6 quarters. The share of months spent in crisis is 15.7% in total.

Given the transition matrix Π above, we have the average length of each recession is $\frac{1}{1-\pi}$ so $\pi = 72.04\%$. π_0 is chosen such that the probability of a recession in the

[16] A two-state process is enough to illustrate the amplification and asymmetric effects, but in order to match several moments of the productivity process in the U.S. we need at least three states.

stationary distribution for A_t, i.e., $A_t = A - \Delta$ matches the share of months spent in recession which implies $\pi_0 = 87.2\%$.[17]

As we normalize A to 1, in numerical simulations, we vary Δ from 1% to 5%, in order to study the non-linear effects of large shocks. However, in the benchmark set of parameters, we choose, $\Delta = 3\%$ in order to match the standard deviation of productivity in the U.S. economy of about 14%, which is used by Khan and Thomas (2014).[18]

2.2.5 Numerical results

In this Subsection, we present the numerical results for the benchmark incomplete markets economy with the calibrated parameters in Subsection 2.2.4.

The key feature of the solution method presented in Subsection 2.2.3.2 is to solve for the endogenous lower bound $\underline{\omega}_t$ as defined in Subsection 2.2.3.2 in each iteration.[19] When we solve for $T-$period economy, the lower bound is decreasing in T and approaches 0 from above as T goes to infinity. Figure 2.1 shows how the lower bound $\underline{\omega}_t$ changes over time and across states s_t (thick blue lines for good state $s_t = G$, dashed purple lines for normal state $s_t = N$, and dotted red line for bad state $s_t = B$). The lower bounds are lower under the good state than under the bad state, which is intuitive because the good state leads to a high profit for the entrepreneurs so they can borrow more from the households.

[17] From the transition matrix for A_t, the probability of recession in the stationary distribution is $\frac{1-\pi_0}{2(2-\pi-\pi_0)}$.

[18] Given the exogenous process for productivity, the standard deviation of productivity is given by $\sqrt{\frac{1-\pi_0}{1-\pi-\pi_0}}\Delta$. When Δ is close to 3% and π and π_0 are chosen in the text and the standard deviation of productivity is around 15% as in the data.

[19] This method also applies to the case with strictly positive labor endowment for the entrepreneurs. In this case $\underline{\omega}_t$ is negative.

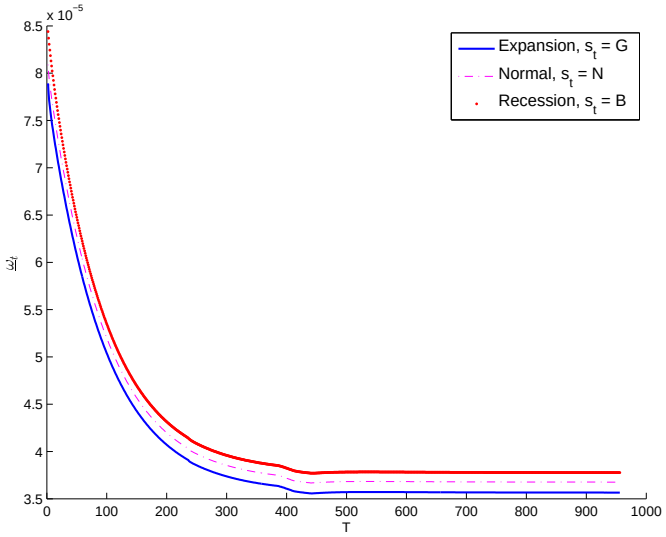


Figure 2.1: Lower bound for financial wealth

Another important ingredient of the solution method is that for each (s_t, ω_t), we have to solve for the future wealth distribution ω_{t+1} for each realization of s_{t+1}. The left panel of Figure 2.2 shows ω_{t+1} as functions of ω_t and s_{t+1} given that $s_t = N$. Given that the entrepreneurs are more exposed to the productivity shock, their wealth increases (relative to the households') as the good shock hits next period (solid blue line), and decreases as the bad shock hits next period (dotted red line). If s_{t+1} stays at the normal state, then the wealth distribution remains almost unchanged as the future wealth function (dashed purple line) stays close to the 45^o line (dashed black line). The transition functions for the wealth distribution combined with the transition matrix of the exogenous states determines the stationary wealth distribution in the right panel (we plot the density of the distribution). Given the small share of land

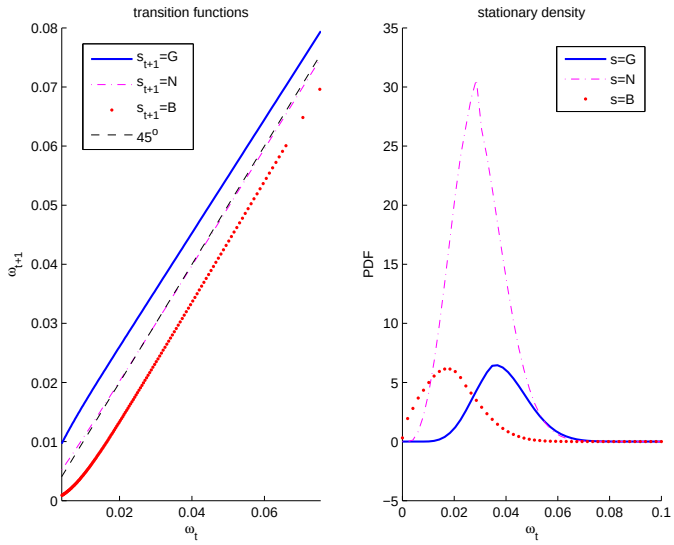


Figure 2.2: Transition and stationary distribution of wealth

in the aggregate production function, the wealth share of the entrepreneurs always stays below 10% in the steady state.

Figure 2.3 shows the policy (consumption of the entrepreneurs and the households and aggregate output) and pricing functions (land price) conditional on the exogenous state s_t and the endogenous state ω_t.[20] Even though the global nonlinear methods solves for the policy and pricing functions for the whole range of ω_t, Figure 2.3 is restricted to the values of ω_t in the support of the stationary distribution of ω_t. We observe that, despite the absence of collateral constraint, land price and output

[20] A more precise measure of output should include imputed rental value of land consumed by the households, i.e. $\tilde{Y}_t = Y_t + \frac{(h'_t)^{-\sigma_h}}{(c'_t)^{-\sigma_2}} h'_t$. But the results are essentially the same with the current output measure Y_t.

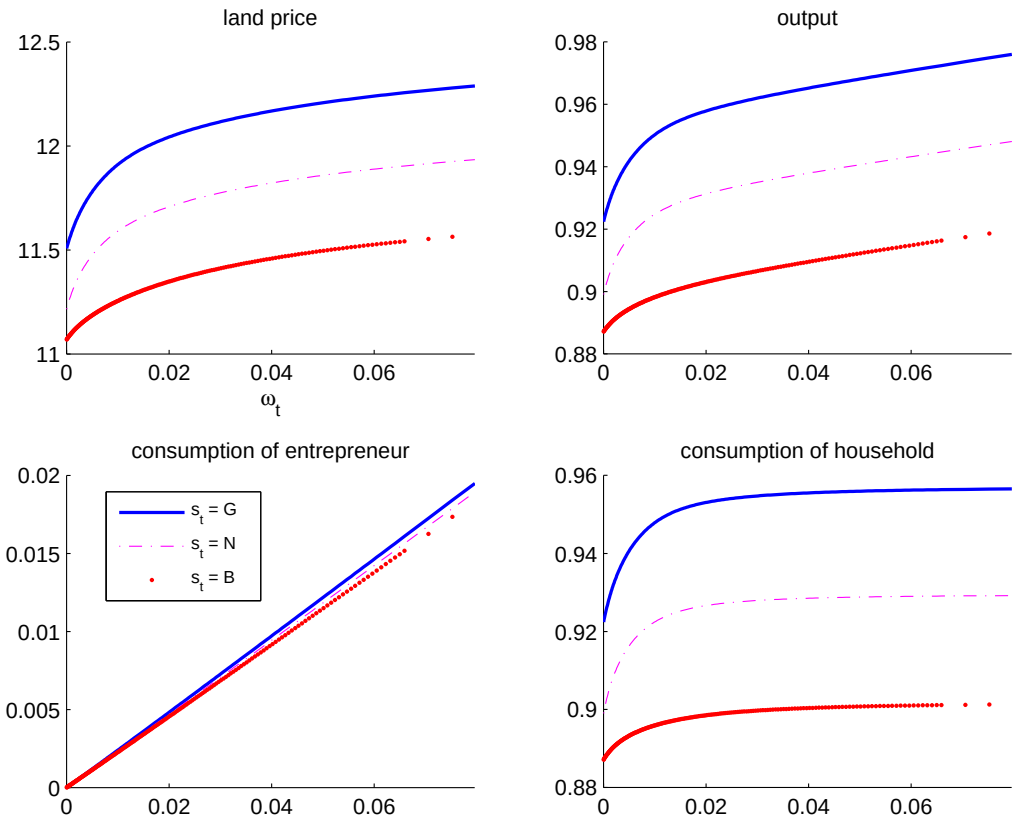


Figure 2.3: Policy and pricing functions for incomplete markets

functions are nonlinear in wealth distribution. In particular they are more sensitive to changes in the wealth distribution when the wealth of the entrepreneurs is low.[21]

Nonlinearity implies asymmetric responses of the equilibrium land price and output with respect to productivity shocks. Starting from $s_t = N$, a good shock,

[21] Another way to see the nonlinearity is to look at $\frac{dx_t}{d\omega_t}$, $x = q$ or Y, as functions of ω_t, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 2.3 suggests that this function is decreasing in ω_t and is much higher at lower ω_t.

i.e., $s_{t+1} = G$ increases the entrepreneurs' wealth and the bad shock $s_{t+1} = B$ decreases the entrepreneurs' wealth as shown in Figure 2.2. But conditional on the same change in entrepreneurs' wealth, land price and output increase after a good shock by less than they decrease after a bad shock due to nonlinearity. This leads to the asymmetric responses of equilibrium land price and output to symmetric shocks, as shown quantitatively in Table 2.1.

The asymmetric responses come from the *net worth effect* as follows. Initially, a negative shock decreases the net worth of the levered entrepreneurs. Due to risk-aversion, these entrepreneurs try to smooth consumption but their debt is not state-contingent so they have to liquidate some of their land holding to maintain a certain level of consumption.[22][23] Their selling activities depress the price of land, further lowering their net worth, setting off the vicious circle of falling land price and falling net worth. At the heart of this circle is the pecuniary externality a la Geanakoplos and Polemarchakis (1986) due to market-incompleteness. i.e., when selling off their land holding, each entrepreneur does not take into account the negative effect of falling land price on the net worth of other entrepreneurs. To illustrate this point, Figure 2.4 plots the portfolio choice (land and bond holdings) of the entrepreneurs as functions of the wealth distribution given the current state is normal (solid blue lines). The figure also plots the portfolio choice next period if the economy stays in the normal state (dashed purple lines) or if the economy enters a recession (dotted red lines).

[22] The lower left panel of Figure 2.3 shows that, unlike land price or output, the consumption of the entrepreneurs only changes linearly with their wealth even when their wealth is very low.

[23] This corresponds to the fire-sale phenomenon described in Shleifer and Vishny (1997) because, the entrepreneurs (the specialists) are the only agents in the economy who can use land to produce output, as the result, they are the only natural buyers. The households can only consume land and their marginal utility from land consumption is decreasing. So when all entrepreneurs sell a part of their land holding, land price falls significantly.

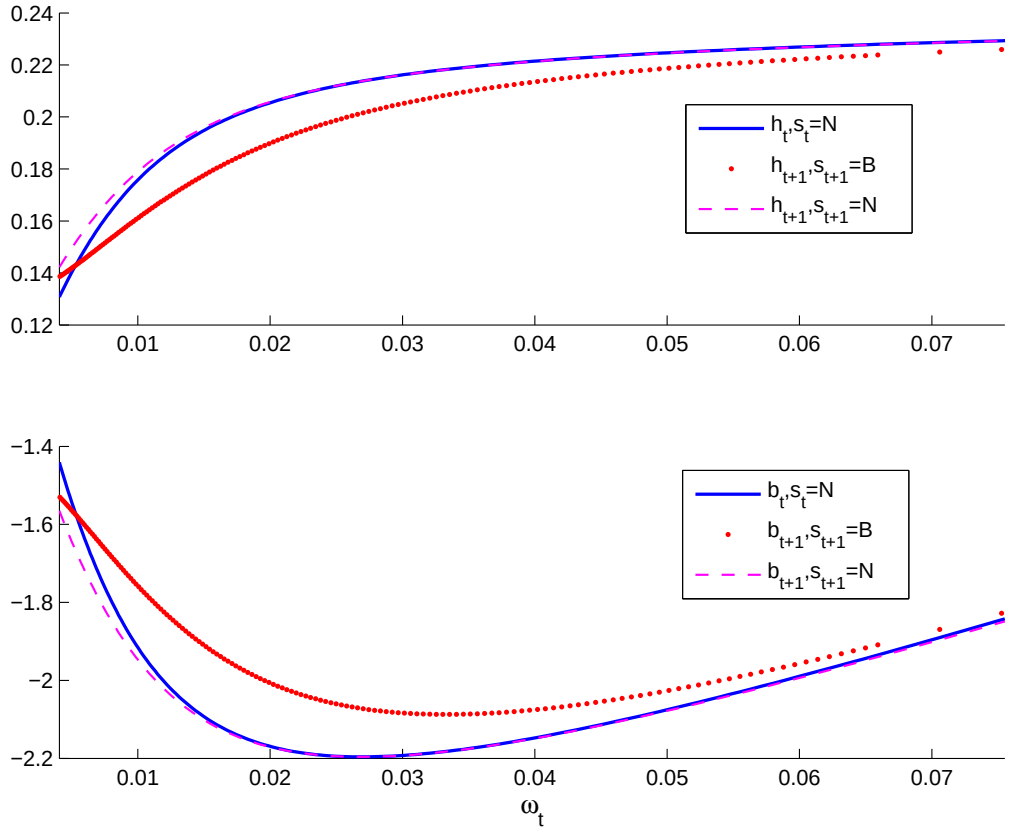


Figure 2.4: Portfolio choice for incomplete markets

The figure shows that after a bad shock, the entrepreneurs reduce their land holding, as well as borrowing.[24]

Quantitatively, Row 2 in Table 2.1 shows the average (over the stationary distribution) of changes in land price and output given the current normal state, $s_t = N$:

[24]The entrepreneurs can also smooth consumption by borrowing more from the households, but similarly they do not take into account that their increased borrowing increases the interest rate for other entrepreneurs. That is, there is also a pecuniary externality in the interest rate as well as in the land price. Indeed, interest rates increase so much that the entrepreneurs actually reduce their borrowing, as shown in Figure 2.4.

$\frac{x_{t+1}-x_t}{x_t}$, where $x = q$ or Y. We observe significant amplification and asymmetric effects under incomplete markets, even though these effects are smaller compared to the responses in the model with collateral constraint below. The last row of Table 2.1 shows the the changes of land prices to shocks in the long run of the complete markets equilibrium, presented in Appendix 2.6.1. Compared to the complete markets outcomes, the model with incomplete markets exhibits both amplification and asymmetric effects. Lastly, Table 2.1 only shows the average responses, due to the nonlinearity of the solution shown in Figure 2.3, the amplification and asymmetric effects are also much larger conditional on the lower values of ω_t.

2.3 Incomplete markets with collateral constraint

While the model with incomplete markets delivers amplification and asymmetric responses of the economy to exogenous shocks, adding a collateral constraint as in Kiyotaki and Moore (1997) will a priori exacerbate these responses. To quantitatively examine the significance of this constraint in addition to incomplete markets channel presented in the last section, we impose a collateral constraint on the borrowing decision of the entrepreneurs. We use the same global solution method presented in Subsection 2.2.3.2 to solve for the dynamic stochastic general equilibrium in this model. In addition, with the collateral constraint, the model has a steady state, so we can log-linearize around the steady state (assuming the collateral constraint is always binding) as in Iacoviello (2005). We can then compare the accuracy of the two solution methods.

As in Kiyotaki and Moore (1997) and Iacoviello (2005), we assume a limit on the obligations of the entrepreneurs. Suppose that, if borrowers repudiate their debt obligations, the lenders can repossess the borrower's assets by paying a proportional

transaction cost $(1-m)\,\mathbb{E}_t\left[q_{t+1}\right]h_t$. In this case the maximum amount that a creditor can borrow is bounded by $m\mathbb{E}_t\left[q_{t+1}\right]h_t$, i.e.

$$b_t + m\mathbb{E}_t\left[q_{t+1}\right]h_t \geq 0, \tag{2.13}$$

where b_t is the saving ($-b_t$ is borrowing) of the entrepreneurs.

Let μ_t denote the Lagrangian multiplier for entrepreneur's collateral constraint. The F.O.C for entrepreneurs with respect to land holding is

$$(\pi_t - q_t)c_t^{-\sigma_1} + \mu_t m\mathbb{E}_t\left[q_{t+1}\right] + \gamma\mathbb{E}_t\left[q_{t+1}c_{t+1}^{-\sigma_1}\right] = 0 \tag{2.14}$$

and the complementary-slackness condition is

$$\mu_t\left(b_t + mh_t\mathbb{E}_t\left[q_{t+1}\right]\right) = 0. \tag{2.15}$$

The F.O.C for the entrepreneurs with respect to bond holding is

$$-p_t c_t^{-\sigma_1} + \mu_t + \gamma\mathbb{E}_t\left[c_{t+1}^{-\sigma_1}\right] = 0. \tag{2.16}$$

We rewrite (2.14) as

$$q_t = \pi_t + \gamma\mathbb{E}_t\left[q_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma_1}\right] + \mu_t m\mathbb{E}_t\left[q_{t+1}\right]c_t^{\sigma_1}.$$

The first two terms on the right hand side of this equation show that, from the entrepreneurs point of view, land price is the net present value of present and future profit from production using land. In addition, the last term on the right hand side shows the collateral value of land in the land valuation of the entrepreneurs. Iterating this equation forward, we obtain the expression for land price as the present discounted value of profit, with the discount factor depending on the marginal utility

of the entrepreneurs as well as on the multiplier on the collateral constraint:[25]

$$q_t = \pi_t + \mathbb{E}_t \left[q_{t+1}\gamma \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma_1} + \mu_t m c_t^{\sigma_1} \right\} \right]$$
$$= \mathbb{E}_t \left[\sum_{s=0}^{\infty} \gamma^s \prod_{r=0}^{s-1} \left\{ \left(\frac{c_{t+r+1}}{c_{t+r}} \right)^{-\sigma_1} + \mu_{t+r} m c_{t+r}^{\sigma_1} \right\} \pi_{t+s} \right].$$

Other conditions are the same as in the benchmark incomplete markets model in Section 2.2. Similar to Lemma 2.1 in Section 2.2, we can easily show that $\omega_t \geq 0$ in any competitive equilibrium under collateral constraint.

2.3.1 Solution

We can apply the nonlinear global solution method as before, but here a steady state exists, so we can also log-linearize around the steady-state and examine the accuracy of the log-linear solution.

2.3.1.1 Steady state

Becker (1980) shows that in a neoclassical growth model with heterogeneous discount factors, long run wealth concentrates on the most patient agents, in this case the households. However, in our model, due to the collateral constraint, the entrepreneurs can only borrow against a fraction of their future wealth to consume in the current period. Therefore, despite their lower discount factor, their wealth does not disappear in the long run. In particular, the model admits a long run steady state in the absence of uncertainty. In this subsection, we solve for the steady state in our model.

Suppose that there is no uncertainty, i.e., $A_t(s_t) \equiv A$. In steady state, all variables are constant, so we can omit the subscript t. For the ease of notation, denote $\gamma_e =$

[25] This formula is similar to the one in Mendoza (2010). In particular, when the entrepreneurs cannot borrow against their land holding, i.e., $m = 0$, there will not be any collateral premium in the pricing of land.

$m\beta + (1-m)\gamma$. The first order condition for b', Equation (2.11), implies that $p = \beta$. Because $\gamma < \beta$, the entrepreneur wants to borrow as much as possible up to the collateral constraint. Indeed, the first order condition for b implies that the collateral constraint is strictly binding and Lagrangian multiplier μ on the constraint is strictly positive:

$$\mu = (\beta - \gamma)\, c^{-\sigma_1} > 0.$$

Given that the collateral constraint is binding, we have $b = -mqh$.

From the first-order condition (2.14), we have

$$q = \frac{1}{1-\gamma_e} \upsilon A h^{\upsilon - 1} L^{1-\upsilon}.$$

The steady state version of Equation (2.7) is

$$w = (1-\upsilon) A h^{\upsilon} L^{-\upsilon}. \tag{2.17}$$

From the budget constraint of the entrepreneurs, we obtain

$$c = \frac{(1-\gamma)(1-m)\upsilon}{1-\gamma_e} A h^{\upsilon} L^{1-\upsilon}.$$

Combining with the market clearing condition in the market for consumption good, we have $c' = Ah^{\upsilon} L^{1-\upsilon} - c$. The market clearing conditions in the housing market and labor market imply, $h' = H - h$ and $L' = L$.

So in the steady-state all variables can be expressed as functions of two unknowns, h and L. The first-order conditions on h' and L' of the households provide two equations that help determine the two unknowns:

$$-q\,(c')^{-\sigma_2} + j\,(h')^{-\sigma_h} + \beta q\,(c')^{-\sigma_2} = 0$$

and

$$w\,(c')^{-\sigma_2} = (L')^{\eta - 1}\,.$$

For example, when $\sigma_2 = 1$ and $\sigma_h = 1$ as in Iacoviello (2005), the second equation, combined with the labor choice equation at the steady state (2.17) implies

$$L = [\frac{1-\upsilon}{1-\frac{(1-\gamma)(1-m)}{1-\gamma_e}\upsilon}]^{\frac{1}{\eta}}$$

From the first equation, h is determined as

$$\frac{h}{H} = \frac{\upsilon\,(1-\beta)}{\upsilon\,(1-\beta) + j[(1-\gamma_e) - (1-\gamma)(1-m)\upsilon]}.$$

Given the determination of the steady state level of h and L, the steady state level of wealth distribution defined in (2.6) is $\omega = \frac{(1-m)h}{H}$.

2.3.1.2 Log-linearization

Following Iacoviello (2005), we assume that the collateral constraint always binds around the steady state. Relative to the standard log-linearization technique, we need to solve for the shadow value of the collateral constraint, i.e., the multiplier μ_t, in addition to prices and allocation. Given a variable x_t, let $\widehat{x}_t$ denote the percentage deviation of x_t from its steady state value, i.e., $\widehat{x}_t = \frac{x_t - x}{x}$.

Given the exogenous processes for the technology shock $\hat{A}_t$, we solve for the endogenous variables $\widehat{c}_t$, $\widehat{c}'_t$, $\widehat{h}_t$, $\widehat{h}'_t$, $\widehat{b}'_t$, $\widehat{q}_t$, $\widehat{w}_t$, $\widehat{p}_t$, $\widehat{L}_t$, $\widehat{\mu}_t$ using the method of undetermined coefficients.[26] The following linear system characterizes the dynamics of the economy around the steady state:

$$(\hat{q}_t - \sigma_2 \hat{c}'_t) = -\sigma_h(1-\beta)\hat{h}'_t + \beta \mathbb{E}_t[(\hat{q}_{t+1} - \sigma_2 \hat{c}'_{t+1})]$$

$$\hat{p}_t = \sigma_2(\hat{c}'_t - \mathbb{E}_t \hat{c}'_{t+1})$$

$$\hat{w}_t - \sigma_2 \hat{c}'_t = (\eta - 1)\hat{L}_t$$

[26] Given the special 3-state structure of the stochastic shocks assumed in Subsection 2.2.4, we cannot directly use Dynare to solve for the log-linearized version of the model.

$$
\begin{aligned}
&(1-\gamma_e)[\hat{A}_t + (\upsilon-1)\hat{h}_t + (1-\upsilon)\hat{L}_t - \sigma_1\hat{c}_t] - (\hat{q}_t - \sigma_1\hat{c}_t) \\
&+ m(\beta-\gamma)(\hat{\mu}_t + \mathbb{E}_t\hat{q}_{t+1}) + \gamma\mathbb{E}_t(\hat{q}_{t+1} - \sigma_1\hat{c}_{t+1}) \\
&= 0
\end{aligned}
$$

$$\beta(\hat{p}_t - \sigma_1\hat{c}_t) = (\beta-\gamma)\hat{\mu}_t - \gamma\sigma_1\mathbb{E}_t(\hat{c}_{t+1})$$

$$\hat{w}_t = \hat{A}_t + \upsilon\hat{h}_t - \upsilon\hat{L}_t$$

$$\hat{b}'_t = \hat{h}_t + \mathbb{E}_t\hat{q}_{t+1}$$

$$c^*\hat{c}_t + c^{*\prime}\hat{c}'_t = Y^*[\hat{A}_t + \upsilon\hat{h}_t + (1-\upsilon)\hat{L}_t]$$

$$h^*\hat{h}_t + h'^*\hat{h}'_t = 0$$

$$
\begin{aligned}
&c'\hat{c}'_t + qh'(\hat{h}'_t - \hat{h}'_{t-1}) + \beta b'^*(\hat{p}_t + \hat{b}'_t) \\
&= b'^*\hat{b}'_{t-1} + wL(\hat{w}_t + \hat{L}_t).
\end{aligned}
$$

2.3.2 Numerical results

In this subsection, we report the properties of the numerical solution of our benchmark model with the parameters given above, in particular the size of the productivity shock is chosen at 3%. Moreover, we set the margin $m = 0.89$ as in Iacoviello (2005). The most important properties are the following.

First of all, the fully nonlinear solution for Markov equilibrium features an occasionally binding collateral constraint. Collateral constraint (2.13) binds when the entrepreneurs' wealth is sufficiently low. In this binding region, endogenous variables including land price and output are more sensitive to changes in wealth distribution. Second, the equilibrium is asymmetric with respect to bad shocks versus good shocks despite the fact that the stochastic structure of the shocks is totally symmetric. For example, on average, good shocks increase the land price less than bad shocks decrease

the land price. Third, the log-linearization solution, by assuming always binding collateral constraint, over-estimates the effect of the shocks. Lastly, the probability of a binding collateral constraint decreases rapidly with the size of the shocks due to precautionary saving motive of the entrepreneurs. Indeed, in the stationary distribution, the binding probability is 83.76% for a 1% standard deviation of the shocks and 5.43% for a 5% standard deviation of the shocks.

Figure 2.5 shows the equilibrium price, output, and consumption of the consumers and entrepreneurs for 3% shocks as function of ω in recession (thick blue lines) compared against the same functions under incomplete markets only in the benchmark model (thin-dashed red lines). Under collateral constraint, the functions exhibit significantly more nonlinearity when ω_t is close to zero. This is because the collateral constraint is binding when ω_t is close to 0. When the collateral constraint binds, the standard feed-back effect in Kiyotaki and Moore (1997) kicks in: after a negative productivity shock, even temporary, in order to smooth consumption the entrepreneurs have to cut back their land holding, h_t. This reduction in land demand depresses the land price, q_t, which through the collateral constraint, forces the entrepreneurs to reduce their debt, partly by reducing consumption, and further cut back their land holding. This vicious circle results in significant decline in land price, as well as, in entrepreneurs' land holding and total output. There is also an intertemporal feed-back process: lower current wealth of the entrepreneurs leads to lower future wealth and lower future land prices. Given that the current land price is the sum of current per unit profit and the discounted future land price, as shown in the asset pricing equations, lower future land price in turns leads to lower current land price. The entrepreneurs use land to produce so a significant decrease in land holding leads to a significant decrease in output.

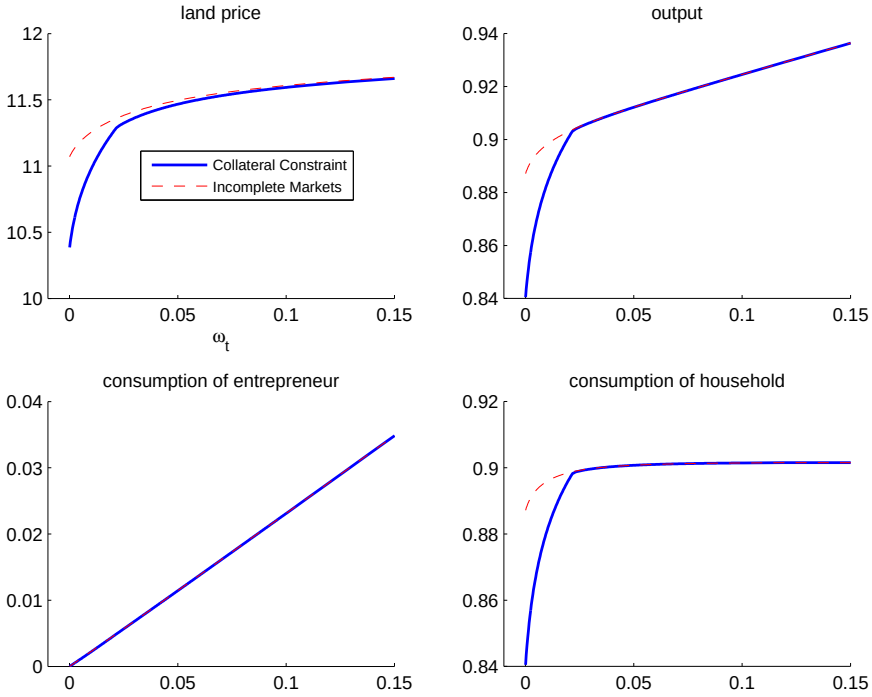


Figure 2.5: Policy functions for collateral constraint versus incomplete markets, $s_t = B$.

The advantage of the nonlinear solution method used in this paper is that, we can see clearly two regions of the state space in which the economy follow different dynamics. In one region, when the collateral constraint is binding (or nearly binding), the feed-back effect is important. In the other region when the collateral constraint is far from binding, asset price and output are less sensitive to changes in wealth distribution.[27]

[27] As noticed in footnote 21, we can see different dynamics between the two regions by looking at $\frac{dx_t}{d\omega_t}$, $x = q, Y$, as functions of ω_t, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 2.5 shows that this function is decreasing in ω_t and is much higher at lower ω_t when the collateral constraint is binding.

Figure 2.6 corresponds to Figure 2.2 for this case with the collateral constraint. Definition 2.2 of Markov equilibrium and the algorithm in Subsection 2.2.3.2 provides the evolution of wealth distribution over time. To illustrate the method, the left panel of Figure 2.6 shows ω_{t+1} as functions of ω_t when the current state $s_t = N$. Next period's wealth distribution depends on the realization of the exogenous state s_{t+1}. By assumption on the stochastic structure of shocks described in Subsection 2.2.4, s_{t+1} can be G, N or B. As shown in the figure, when ω_t close to $\underline{\omega}_t$, we have $\omega_{t+1} > \omega_t$, thus the lowest levels of wealth are never reached. This result is in contrast to the one in the benchmark model (or in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013)). Using these transition functions for wealth distribution, we can compute the stationary distribution for wealth distribution, ω_t, over the business cycles. The right panel of Figure 2.6 shows the stationary distribution (density) for ω_t conditional on the exogenous state s_t. The density becomes zero before the lowest thresholds $\underline{\omega}_t$.

The most importance difference between the model with collateral constraint and the benchmark incomplete markets model is the possibility of a binding collateral constraint. Figure 2.7 illustrates this point. The upper panel shows the portfolio choice of the entrepreneurs as a function of ω_t and $s_t = N$. On the right of the vertical green line (in both panels), the collateral constraint is binding. This yields the (conditional) binding probability in normal times of 23.49%. The binding probabilities for other states are given in Table 2.3. Because of the important nonlinearity when the collateral constraint is binding, dynamically the entrepreneurs try to avoid this region by precautionary saving. Precautionary saving decreases the likelihood of a binding constraint, and significantly so when shocks are large.

Figure 2.8 compares the impulse-response of the log-linearization versus the global nonlinear method. Starting from the long run mean level of wealth, we assume the

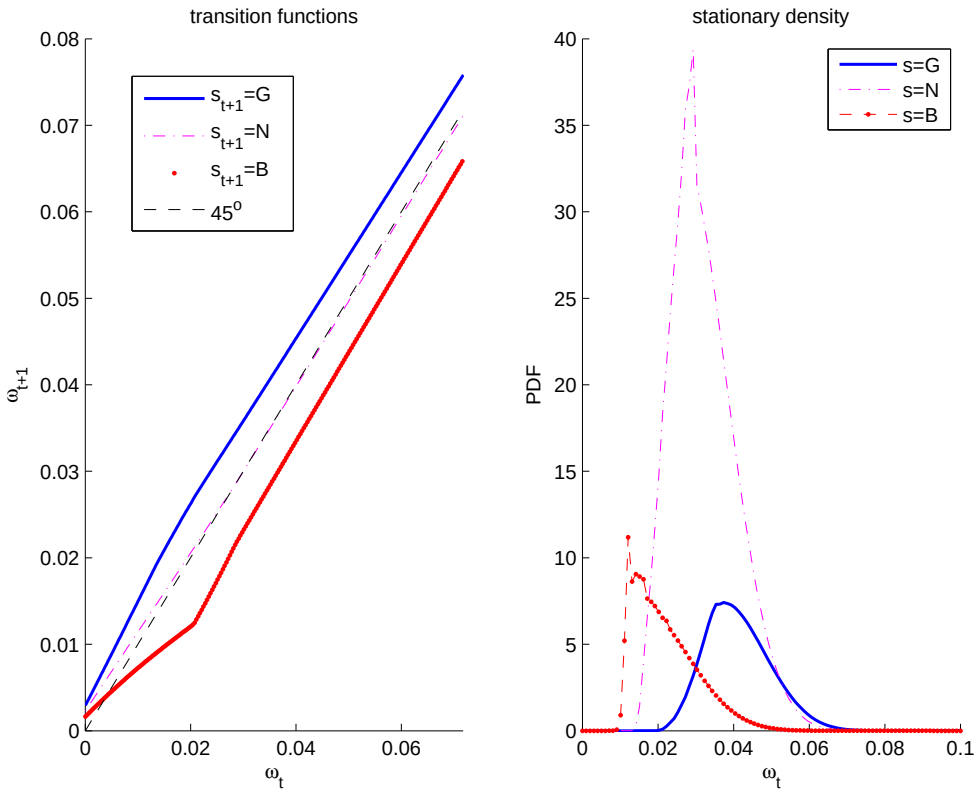


Figure 2.6: Wealth distribution transition in normal times

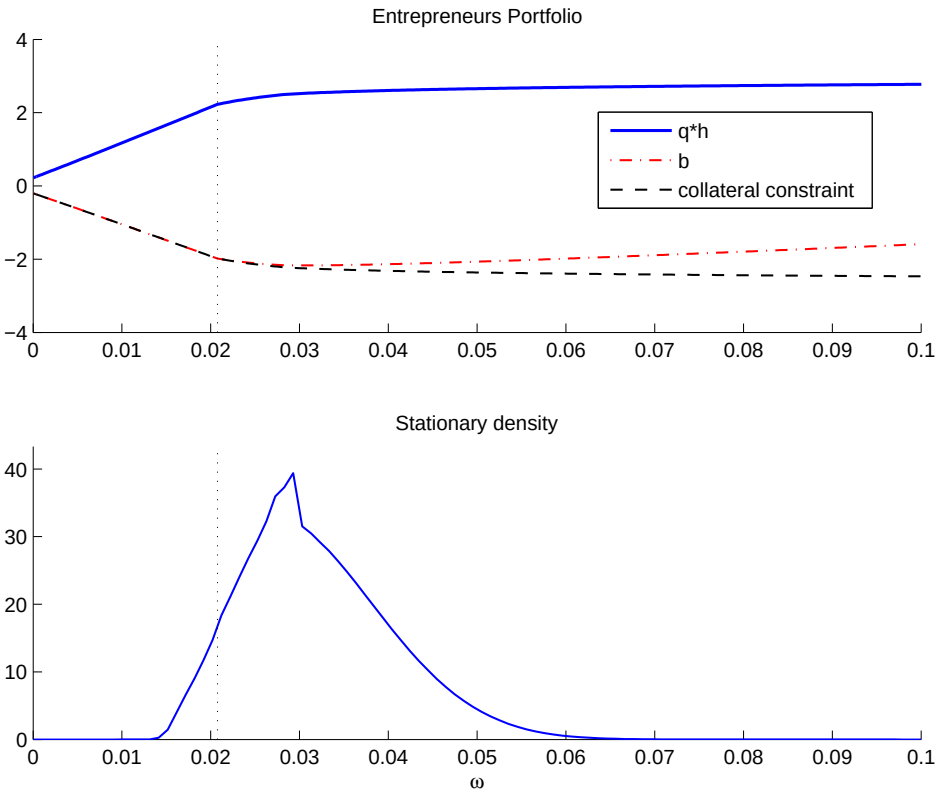


Figure 2.7: The entrepreneurs' portfolio choice and the stationary distribution of wealth in normal times

economy is hit by a sequence of 4 good shocks (dotted red line) and 4 bad shocks (solid blue line) respectively and returns to the normal state afterwards.[28,29] In the case of bad shocks, we plot the minus of relative changes in land price. This figure illustrates the asymmetric effect of collateral constraint. Positive shocks increase the land price by only 3.2% at impact, while negative shocks decrease land price by 3.8% at impact. Negative shocks also have more persistent effect on land price. The black dotted line shows the IRF under log-linearization. Under log-linearization, responses to shocks are perfectly symmetric so we only plot the IRF under positive shocks. As shown in the figure, by assuming that the collateral constraint is always binding, log-linearization overstates the effects of shocks. Land price changes by close to 5% at impact and the changes are also more persistent.

Another way to capture the asymmetric effect of collateral constraint is to calculate the average (weighted by the stationary distribution) percentage change in land price and output in normal times, i.e., $s_t = N$, when the shocks hit the economy. The third row of Table 2.1 shows that good shock changes price by 3.30% and output by 3.15% on average while bad shock changes price by 3.76% and output by -3.5%. Compared to the complete markets outcomes (last row), the collateral constraint exhibit both amplification and asymmetric effects.[30]

The difference between the log-linearization solution and fully nonlinear solution depicted in Figure 2.8 comes from the assumption that the collateral constraint always

[28] As documented in Subsection 2.2.4, the average length of recessions is 3.6 quarters, so we use 4 shocks for the impulse responses.

[29] Another way to plot the impulse-response is to simulate the economy using the transition matrix Π in Subsection 2.2.4 starting from a good shock or a bad shock, and take the average dynamics of the economy across simulations.

[30] Even though the amplification effect here is relatively small - land price declines by 3.76% after bad shock, on average, compared to 3% under complete markets - the effect will be significantly larger if we increase the share of land, ν, in the production function as in Kocherlakota (2000). Moreover, land price also declines much more after bad shocks when the collateral constraint is binding.

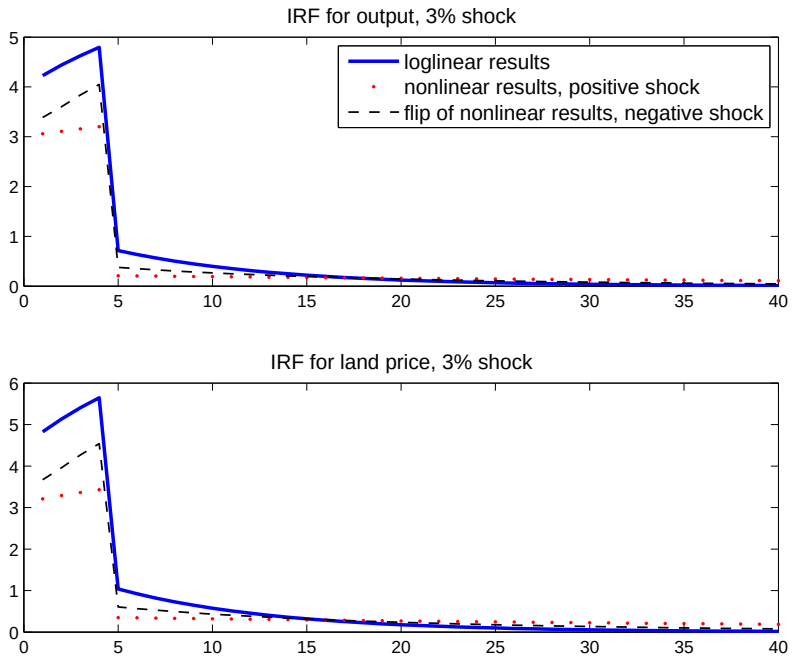


Figure 2.8: IRF for non-linear versus log-linear model

binds around the steady state. This assumption becomes less accurate since as the size of shock increases because agents would engage more in precautionary saving. As a result the collateral constraint will not bind all the time. Table 2.3 shows the binding probability in the long run stationary distribution for wealth distributions as function of the size of the shock. Column 2-4 shows the binding probabilities conditional on the realization of the exogenous shocks, and column 5 shows the unconditional binding probabilities. The binding probabilities decrease very fast in the size of the shock.[31]

[31]In Appendix C of Iacoviello (2005), the author shows that the binding probability is close to 1 when the size of the shocks is calibrated to the U.S. data. The binding probability is much smaller in this paper because of our different stochastic structure of the productivity shocks. For example, our stochastic structure implies more persistent shocks (which leads to more precautionary saving) given the same standard deviation of the shocks.

Table 2.3: probabilities of binding constraint

$\triangle A$	Expansion	Normal	Recession	Overall
1%	5.28%	98.04%	99.83%	83.76%
2%	1.51%	33.37%	86.06%	36.64%
3%	0%	23.49%	70.22%	27.14%
4%	0%	4.89%	40.40%	9.70%
5%	0%	0.53%	32.30%	5.43%

We end this subsection by noting that our fully nonlinear solution does not exhibit the *volatility paradox* presented in Brunnermeier and Sannikov (2014), i.e., lower exogenous risk can lead to higher endogenous risk. As shown in Table 2.3, as we decrease the size of the exogenous shocks, the binding probability goes to 1 and the nonlinear solution becomes closer to the log-linear solution, and both converge to the steady state with no endogenous risk. The difference between our solution and Brunnermeier and Sannikov (2014)'s comes from the fact that we do not allow the households to start producing when the entrepreneurs' wealth goes to zero. If we assume, as in Brunnermeier and Sannikov (2014), that the households can use an alternative, inefficient production technology, $\underline{A}h^{v}L^{1-v}$, where $\min(A_t) < \underline{A} < mean(A_t)$ to produce, then we should recover the volatility paradox. In Appendix 2.6.4, we present a simple model with this feature and show that the solution method in this paper applies for that model as well.

2.3.3 Alternative collateral constraint

In the collateral constraint (2.13), we use the expected future land price. This constraint can be micro-founded under limited commitment and has been used in a large number of papers including Kiyotaki and Moore (1997), Iacoviello (2005), and Cao

(2011).[32] However, in practice collateralized contracts are often written using current asset prices and this is also assumed in a large number of papers, for example Mendoza (2010). Figure 2.6 shows that wealth distribution moves very slowly over the business cycles, as a result the land price also moves slowly. Therefore using current or expected future land price should not imply quantitatively significant differences between the two models. We can show this result rigorously by solving an alternative model in which the collateral constraint (2.13) is replaced by the following alternative collateral constraint:

$$b_t + mq_t h_t \geq 0.$$

Fortunately this case is a special case of the general Markov equilibrium definition and solution method in Kubler and Schmedders (2003) and Cao (2010). We solve for the Markov equilibrium under this alternative collateral constraint for the parameters in Subsection 2.2.4. The solution is quantitatively similar to the one in our benchmark model. For example, Row 4 in Table 2.1, shows that the amplification and asymmetric effects are only slightly higher than the ones in the benchmark model.[33]

2.4 Complete markets with collateral constraint

The comparison between the benchmark incomplete markets model in Section 2.2 and the collateral constraint model Section 2.3 suggests that one of the main ingredients for the amplification and asymmetric effects is market incompleteness beside the

[32] Cao (2011) shows that when the lender can seize a fraction of the asset upon default, the collateral constraint arises endogenously and has the form

$$b_t + mh_t \min_{s^{t+1}|s^t} q_{t+1}\left(s^{t+1}\right) \geq 0.$$

in which $s^{t+1}|s^t$ refers to all the s^{t+1} in the support conditional on s^t.

[33] However, the binding probability is significantly higher at 46.54% compared to 27.14% in the benchmark model.

collateral constraint. To further demonstrate this point, we study a variation of the collateral constraint model, which we call the collateral constraint with complete markets model. In this model, we maintain the collateral constraint, however we allow the agents to trade a complete set of Arrow securities, subject to the collateral constraint. In history s^t, let $p_t(s_{t+1})$ denote the price of the Arrow security that pays off one unit of consumption good if s_{t+1} happens and nothing otherwise. Let $\phi_t(s_{t+1})$ and $\phi'_t(s_{t+1})$ denote the holdings of the entrepreneurs and the households, respectively, of these securities. The definition of competitive equilibrium as well as Markov equilibrium are exactly the same as in the benchmark model, except now we need to impose the condition that the markets for the Arrow securities clear, i.e. $\phi_t(s_{t+1}) + \phi'_t(s_{t+1}) = 0$.

In this model, the budget constraint of the entrepreneurs becomes:

$$c_t + q_t(h_t - h_{t-1}) + \sum_{s_{t+1}|s^t} p_t(s_{t+1})\,\phi_t(s_{t+1}) \leq \phi_{t-1}(s_t) + Y_t - w_t L_t. \tag{2.18}$$

and the collateral constraint (2.13) is now

$$\phi_t(s_{t+1}) + mq_{t+1}h_t \geq 0 \ \forall s_{t+1}|s^t. \tag{2.19}$$

in which $s_{t+1}|s^t$ includes all the possible s_{t+1} conditional on s^t. Let $\widetilde{p}_t(s_{t+1}) = \frac{p_t(s_{t+1})}{\Pr(s_{t+1}|s_t)}$ and $\mu_t(s_{t+1})\Pr(s_{t+1}|s_t)$ denote the multiplier on the constraint (2.19) for each $s_{t+1}|s^t$. The first-order condition on $\phi_t(s_{t+1})$ implies

$$-\widetilde{p}_t(s_{t+1})\,c_t^{-\sigma_1} + \mu_t(s_{t+1}) + \gamma\left(c_{t+1}\left(s^t, s_{t+1}\right)\right)^{-\sigma_1} = 0 \tag{2.20}$$

and the first-order condition on h_t implies

$$(\pi_t - q_t)c_t^{-\sigma_1} + m\mathbb{E}_t\left[\mu_t(s_{t+1})\,q_{t+1}\right] + \gamma\mathbb{E}_t[q_{t+1}c_{t+1}^{-\sigma_1}] = 0. \tag{2.21}$$

Similarly, the budget constraint of the households changes to:

$$c'_t + q_t(h'_t - h'_{t-1}) + \sum_{s_{t+1}|s^t} p_t(s_{t+1})\,\phi'_t(s_{t+1}) \leq \phi'_{t-1}(s_t) + w_t L'_t.$$

From the optimal decision of the households, we have

$$-\widetilde{p}_t\left(s_{t+1}\right)c_t'^{-\sigma_2}+\beta\left(c'_{t+1}\left(s^t,s_{t+1}\right)\right)^{-\sigma_2}=0. \tag{2.22}$$

Other conditions, including the first-order condition with respect to h'_t of the households stay the same as in the collateral constraint with incomplete markets model.

Figure 2.9 shows the differences between the price and policy functions in the collateral constraint with complete markets model and the collateral constraint with incomplete markets model (when $s_t = B$). In contrast to the incomplete markets model, the land price in the collateral constraint with complete markets model differs from the land price in the collateral constraint with incomplete markets model for intermediate levels of wealth of the entrepreneurs.

More importantly, numerical simulations show that, unlike the cases with incomplete markets (with or without collateral constraint), the long run stationary distribution of wealth is degenerate and concentrates on ω^*, regardless of the exogenous state. At $\omega_t = \omega^*$, the collateral constraint is binding for all future states, and land demand of the entrepreneurs is given by h^*. Therefore: $\omega^* = \frac{q_{t+1}h^* - mq_{t+1}h^*}{q_{t+1}H} = (1-m)\frac{h^*}{H}$. In Appendix 2.6.3, we present the other equations that determine the equilibrium at ω^*. The parameters in Subsection 2.2.4 imply ω^* around 0.0235.

At ω^*, Table 2.1 shows that both the amplification and asymmetric effects disappear. In order to understand how the complete set of Arrow securities help the entrepreneurs to insure against negative shocks, Figure 2.10, lower panel, shows the entrepreneurs' optimal choice of $\phi_t\left(s_{t+1}\right)$ as function of the wealth distribution and the future exogenous state s_{t+1}, given the current state $s_t = N$. For clarity we also plot the choice of ϕ_t for $s_{t+1} = G$ and $s_{t+1} = B$. Below ω^*, the collateral constraints are binding for all future states. However, above but close to ω^*, the collateral con-

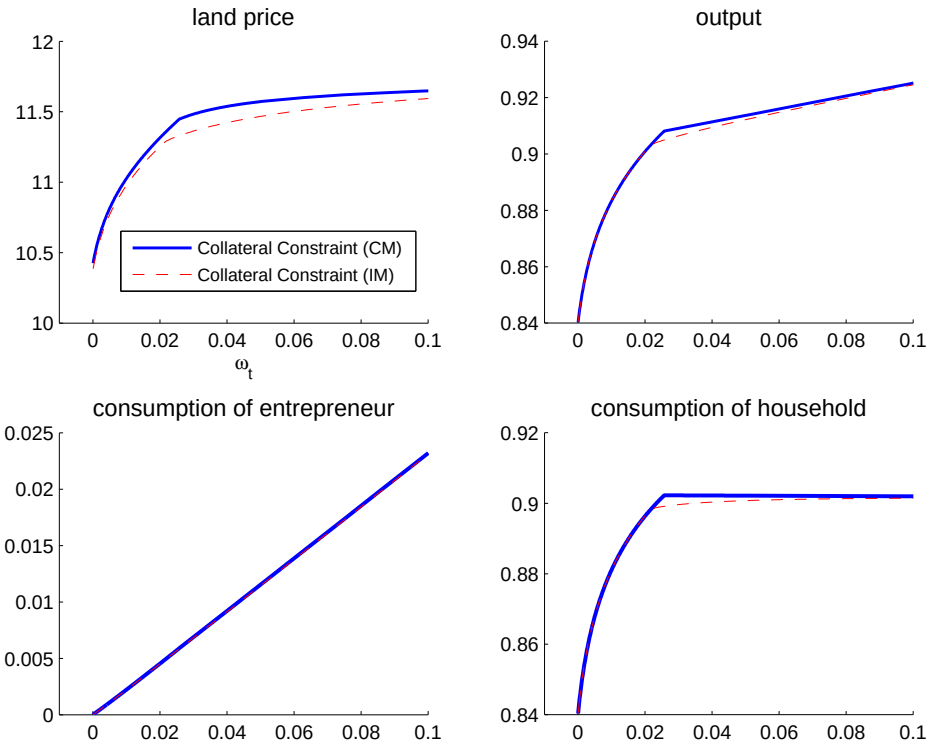


Figure 2.9: Price and policy functions under complete insurance versus collateral constraint markets, $s_t = B$.

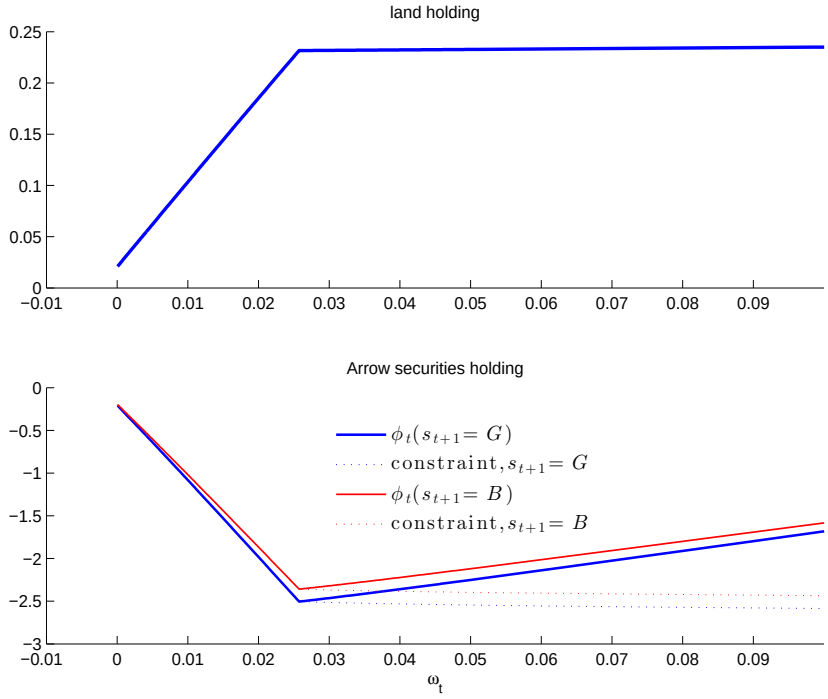


Figure 2.10: Land and arrow securities holdings, $s_t = N$

straints are not binding, and the entrepreneurs borrow relatively more from the future good state compared to the future bad state.

2.5 Conclusion

In this paper, we have shown that market incompleteness, independently of the collateral constraint, plays a quantitatively significant role in the amplified and asymmetric responses of the economy to exogenous shocks. There is only type of shock - productivity shocks. However, it is easy to extend the paper to incorporate other shocks such as housing preference shocks. It would also be interesting to incorporate money into the model to consider the effect of monetary shocks as in Iacoviello (2005).

The current model does not have capital. We can consider adding capital into the model to examine how the amplification and asymmetric effects affect the capital accumulation and the aggregate production processes. Cao (2010) offers a way to introduce capital into this kind of model which enables the use of a similar global nonlinear solution method to the one in Subsection 2.2.3.2. However, in this case we need to keep track of two endogenous state variables: wealth distribution and aggregate capital.

The importance of market incompleteness shown in this paper also suggests that state-contingent debt can be an important macro-prudential policy tool. Using a model with collateral constraint, Geanakoplos (2010) argues that leverage should be restricted in booms to avoid the fire-sale externality and financial crises in subsequent recessions. Given that market incompleteness plays an important role, designing debts with some insurance for downturns should also be effective in reducing the magnitude of the subsequent recessions. Theoretically, Section 2.4 shows that complete state-contingent assets, even being subject to collateral constraints, can nullify the amplification and asymmetric effects. In practice, for example, Mian and Sufi (2014) argue that share-responsible mortgages, i.e., mortgages that reduce principal and mortgage payments upon significant declines in housing prices, can significantly reduce the size of the financial and economic crisis 2007-2008 in the U.S.

2.6 Appendix

2.6.1 Complete Markets

When markets are complete, the entrepreneurs are less patient than the households, so they tend to consume all their future net worth at the present. When they are risk-neutral as in Brunnermeier and Sannikov (2014) they will consume their entire

net worth at time 0, but here they are risk-averse, so they consume their net worth overtime, and their wealth relative to the households' wealth goes to zero as time goes to infinity. The entrepreneurs can do so buy issuing equity to households thanks to frictionless financial markets. We consider the long run limit, when the wealth of the entrepreneurs is close to zero. The economy converges to a representative household economy, with the production technology of the entrepreneurs. Due to the Markovian feature of uncertainty, the endogenous variables depend only on the aggregate state s_t: $x_t = x(s_t)$.

Given that the households own the whole production sector, the marginal utility from the marginal profit per unit of land should be equal to the marginal utility of one unit of land consumption, i.e., $(c'_t)^{-\sigma_2} \pi_t = j (h'_t)^{-\sigma_h}$. To evaluate the marginal utility of consumption, we observe that the households consume the whole output in the long run so

$$c'_t = Y_t = A_t h_t^{\upsilon} L_t^{1-\upsilon}.$$

From the equalization of the marginal utility from the marginal profit and marginal utility from land consumption, and using the expression for wage and profit in Subsection 2.2.3, we have

$$j (H - h_t)^{-\sigma_h} = \left(A_t h_t^{\upsilon} L_t^{1-\upsilon}\right)^{-\sigma_2} \nu A_t h_t^{\nu-1} L_t^{1-\nu}. \tag{2.23}$$

The consumption and labor trade-off equation (2.12) implies

$$(1-\nu) A_t h_t^{\nu} L_t^{-\nu} \left(A_t h_t^{\upsilon} L_t^{1-\upsilon}\right)^{-\sigma_2} = L_t^{\eta-1}. \tag{2.24}$$

The two Equations (2.23) and (2.24) help us solve for the two unknowns (h_t, L_t) for each A_t.

Now, given π_t as function of A_t,h_t, and L_t, land price is determined by the pricing kernel using the marginal utility of the representative households:

$$q\left(s_t\right)-\pi\left(s_t\right)=\sum_{s_{t+1}|s_t}\beta\frac{u'\left(c_{t+1}\left(s_{t+1}\right)\right)}{u'\left(c_t\left(s_t\right)\right)}q\left(s_{t+1}\right)\Pr\left(s_t,s_{t+1}\right).$$

2.6.2 Incomplete Markets with Exogenous Borrowing Constraint

In this Appendix, we examine an alternative model with exogenous borrowing constraint. The model has the same ingredients as the one in Section 2.2 except for the following borrowing constraint instead of the collateral constraint (2.13):

$$b_t\geq-\overline{B}. \tag{2.25}$$

The borrowing constraint $\overline{B}$ is chosen at the steady state level of debt of the original model. Let μ_t denote the Lagrangian multiplier associated to this borrowing constraint. The first-order condition with respect to h_t and b_t in the maximization problem of the entrepreneurs implies

$$(\pi_t-q_t)c_t^{-\sigma_1}+\gamma\mathbb{E}_t[q_{t+1}c_{t+1}^{-\sigma_1}]=0 \tag{2.26}$$

and

$$-p_tc_t^{-\sigma_1}+\mu_t+\gamma\mathbb{E}_t\left[c_{t+1}^{-\sigma_1}\right]=0, \tag{2.27}$$

and the complementary-slackness condition is satisfied:

$$\mu_t\left(b_t+\overline{B}\right)=0. \tag{2.28}$$

Other conditions are the same as in the model with endogenous borrowing constraint in Section 2.2.

We first solve for the steady state of this model. As in Section 2.2, from the first-order condition of the households, we have $p=\beta$. From the first-order condition (2.27), we have

$$\mu=\left(\beta-\gamma\right)c^{-\sigma_1}.$$

From the first-order condition (2.26), we have

$$q = \frac{1}{1-\gamma} \upsilon A h^{\upsilon-1} L^{1-\upsilon}.$$

This expression of price is different from the one in Section 2.2 in the discount factor γ instead of γ^e. Given that $\gamma < \gamma^e$, given the same steady state level of h and L, the land price is lower under exogenous borrowing constraint than under endogenous borrowing constraint. Consequently, this model with exogenous borrowing constraint and the collateral constraint model do not share the same steady state.

Outside the steady state, we can use the global nonlinear solution method presented in Subsection 2.2.3.2 to solve for Markov equilibrium in this economy. Table 2.4 is the counter part of Table 2.1 for this model with exogenous borrowing constraint. In particular, Row 2 of Table 2.4 corresponds to Row 2 in Table 2.1, in which there is no upper bound on the borrowing of the entrepreneurs. When we tighten the exogenous constraint, the amplification and asymmetric effects are actually reduced. At first sight, this result seems counter-intuitive. However, this result is in line with the discussions in Mendoza (2010) and Kocherlakota (2000). The exogenous borrowing constraint reduces the borrowing of the entrepreneurs, and thus reduces the net worth effect in the benchmark incomplete markets model. An important difference here compared to Kocherlakota (2000), is that under uncertainty, it is possible to have infinite exogenous borrowing constraint in the incomplete markets model (the entrepreneurs limits themselves from borrowing too much because of the precautionary saving motive). Infinite exogenous borrowing constraint leads to maximal net worth effect, thus significant amplification and asymmetric effects.

Table 2.4: Average land price and output changes in normal state, 3% shock

	Land price		Output	
Type of constraint	Expansion	Recession	Expansion	Recession
incomplete markets ($\bar{B} = \infty$)	3.21%	-3.45%	3.25%	−3.44%
incomplete markets ($\bar{B} = 2$)	3.21%	−3.23%	3.15%	−3.18%
incomplete markets ($\bar{B} = 1$)	3.14%	-3.13%	3.04%	−3.04%
complete markets	3.00%	-3.00%	2.97%	−2.97%

2.6.3 Stationary State under Collateral Constraint with Complete Markets

We look for an equilibrium in which prices and allocations depend only on the exogenous state s_t. In this case we simplify the notation of state contingent prices and bond holdings by $\widetilde{p}_t(s_{t+1}) = \widetilde{p}(s_t, s_{t+1})$ and $\phi_t(s_{t+1}) = \phi(s_t, s_{t+1})$. Moreover, we look for the equilibrium in which collateral constraints (2.19) are all binding, i.e., $\phi^*(s_t, s_{t+1}) = -mq(s_{t+1})h(s_t)$. This implies $\omega_{t+1} = (1-m)\frac{h_t}{H}$. In order for ω_{t+1} not to depend on s_{t+1}, we must have then $h_t = h^*$ independent of the exogenous state.

From the budget constraint of the entrepreneurs, Equation (2.18), we have

$$\begin{aligned} c(s_t) = & \sum_{s_{t+1}|s_t} p(s_t, s_{t+1}) mq(s_{t+1}) h^* \\ & - mq(s_t) h^* + A(s_t)(h^*)^{\nu}(L_t)^{1-\nu} - w(s_t) L_t \end{aligned}$$

and by the market clearing condition for consumption good, we have

$$c'(s_t) = A(s_t)(h^*)^{\nu}(L(s_t))^{1-\nu} - c(s_t).$$

Given h^* and $L(s_t)$, $w(s_t)$ is determined by the first-order condition from the entrepreneurs' optimal choice of L_t, i.e., Equation (2.7). Therefore, we only need to

solve for

$$\{h^*, q(s_t), \widetilde{p}(s_t, s_{t+1}), L(s_t)\}.$$

The first-order condition with respect to $\phi'_t(s_{t+1})$ implies

$$\widetilde{p}(s_t, s_{t+1}) = \beta \left(\frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2}.$$

We can choose $\mu(s_t, s_{t+1})$ so that Equation (2.20) is satisfied

$$\begin{aligned} \mu(s_t, s_{t+1}) &= \widetilde{p}(s_t, s_{t+1}) c_t^{-\sigma_1} - \gamma (c_{t+1}(s_{t+1}))^{-\sigma_1} \\ &= \beta \left(\frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2} c_t^{-\sigma_1} - \gamma (c_{t+1}(s_{t+1}))^{-\sigma_1} \\ &> 0. \end{aligned}$$

Plugging this expression for $\mu(s_t, s_{t+1})$ into Equation (2.21), we obtain another set of equations that help determine $\{q(s_t)\}$. The equations that determine $\{L(s_t)\}$ comes from the optimal labor-consumption decision of the households, Equation (2.12). And lastly, h^* must be determined so that the first-order conditions on the housing choice of the household are satisfied in all exogenous states:

$$\{q(s_t) - \mathbb{E}_t [q(s_{t+1}) \widetilde{p}(s_t, s_{t+1})]\} (c'(s_t))^{-\sigma_2} = j (H - h^*)^{-\sigma_h},$$

given the expression of $\widetilde{p}(s_t, s_{t+1})$ derived above.

2.6.4 Simpler Model

In this Appendix, we simplify our model in the spirit of Brunnermeier and Sannikov (2014) as well as Cordoba and Ripoll (2004) and Kiyotaki and Moore (1997). We assume that the households does not have a preference for housing but have access to an inefficient production function

$$Y' = \underline{A} h^{\nu'} L^{1-\nu'} \tag{2.29}$$

with $\min(A_t) < \underline{A} < mean(A_t)$. Households maximize a lifetime utility function given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(c'_t)^{1-\sigma_2} - 1}{1-\sigma_2} - \frac{1}{\eta}(\widetilde{L}_t)^{\eta} \right\}, \tag{2.30}$$

where $\widetilde{L}_t$ is the hours of work (instead of L'_t in the benchmark model). The budget constraint of the households is

$$c'_t + q_t(h'_t - h'_{t-1}) + p_t b'_t \leq b'_{t-1} + w_t \widetilde{L}_t + Y'_t - w_t L'_t. \tag{2.31}$$

Housing is no longer in the utility function of households, so we have to impose explicitly

$$h'_t \geq 0.$$

Given their land holding at time t, h_t, the households choose labor demand L'_t to maximize profit

$$\max_{L'_t} \{Y'_t - w_t L'_t\}$$

subject to their production technology (2.29). The first order condition with respect to L'_t implies

$$w_t = (1-\upsilon')\underline{A} h_t^{\upsilon'} L_t^{-\upsilon'},$$

i.e. $L'_t = \left(\frac{(1-\upsilon')A_t}{w_t}\right)^{\frac{1}{\upsilon'}} h'_t$ and profit

$$Y'_t - w_t L'_t = \pi'_t h'_t$$

where $\pi'_t = \upsilon' \underline{A} \left(\frac{(1-\upsilon')\underline{A}}{w_t}\right)^{\frac{1-\upsilon}{\upsilon}}$ is profit per unit of land for the households.

Definition 2.3 *A **competitive equilibrium** is sequences of prices* $\{p_t, q_t, w_t\}_{t=0}^{\infty}$ *and allocations* $\left\{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t, \widetilde{L}_t\right\}$ *such that (i)* $\left\{c'_t, h'_t, b'_t, L'_t, \widetilde{L}_t\right\}$ *maximize (2.30) subject to budget constraint (2.31) and the production technology (2.29), and* $h'_t \geq 0$ *and* $\{c_t, h_t, b_t, L_t\}$ *maximize (2.4) subject to budget constraint (2.5), collateral constraint (2.13), and production technology (2.3) given* $\{p_t, q_t, w_t\}$ *and initial*

asset holdings $\{h_{-1}, b_{-1}, h'_{-1}, b'_{-1}\}$*; (ii) land, bond, labor, and good markets clear:* $h_t + h'_t = H$, $b_t + b'_t = 0$, $L_t + L'_t = \widetilde{L}_t$, $c_t + c'_t = Y_t$.

In the steady state, the entrepreneurs own the whole supply of land. Outside the steady state, we can use the definition of Markov equilibrium and the associated solution method as in the benchmark model in Section 2.2. The main difference between the solution of this model and the benchmark model is that at the natural borrowing limit for the entrepreneurs, i.e. $\omega_t = 0$, the households start producing using their inefficient production function. This puts a lower bound on the total output as well as land price.

Chapter 3

Why the Marital Rate of Successful Women in China Decreases When There Are "Excess Men"

3.1 Introduction

Starting from the mid-1980s, China has witnessed a steadily increasing gender ratio at birth which reached 118 boys for each 100 girls by 2011. Some scholars such as Wei and Zhang (2011) estimate that there are 30 million "excess men" who cannot find a wife. It seems that with a higher gender ratio, single women should find acceptable spouses easier and marry faster. However, from 2005 to 2010, the proportion of women remaining single after age 27 increased dramatically[1]. These women usually live in urban areas, are highly educated and have well-paying jobs. In short, these are successful women based on general standards, and more importantly, they are doing better than their counterparts, single men in the same age cohort. According to the Sixth Population Census in 2010, there are 12 million single men and 6 million single women between age 30 and 39. Although women are significantly outnumbered in this group, they have higher education and income: 48% of them have a master's degree and 36% of them have monthly income of $2500 or higher. On the contrary, the numbers are only 37% and 29% among single men.

[1]First proposed in 2007 by the All-China Women's Federation, a quasi-governmental agency, and then officially introduced by Ministry of Education, the term "leftover women" refers to women who remain single over age 27. Such a derogative name reveals the pressure faced by these single women from family, friends and social media, etc.

Why are these highly educated, well paid women postponing marriage when there are "excess men" available in the marriage market? Some researchers associate women's delaying marriage with their higher social standings such as increased access to education, more job opportunities as well as higher economic independence. For example, Dixon (1978) and Salaff (1976) argue that women who are successful at work have a higher opportunity cost in marriage and thus delay marriage to avoid compromising their career development. Using interviews of Japanese working women, Nemoto (2008) argues from another perspective that women postpone marital age due to a fear of gender inequalities within marriage resulting from social norms. However, the phenomenon observed in China can hardly be explained in this way since women's economic and social status didn't change much over those 5 years. Based on the Gender Inequality Index provided by United Nations, gender inequality in China was greatly reduced from 2000 (index value 0.572) to 2005 (index value 0.219) then remained almost constant until now. In 2010, the index value was 0.209[2]. Other scholars (e.g., Angrist, 2002, Abramitzky et al., 2011) empirically analyze the relationship between the gender ratio and marriage timing. They find that women marry earlier, find better spouses and enjoy more bargaining power within marriage when an unbalanced gender ratio favors women. This explanation is contrary to the phenomenon we observe in China. Goldin and Katz (2002) attribute soaring marriage ages of U.S. women after 1970s to the diffusion of birth control pills which significantly reduced the probability of "shotgun marriage" and thus lowered women's cost in committing to long-duration education. Although this view is quite insightful, it cannot be a good explanation here either since birth control methods have been pub-

[2]Gender Inequality Index is defined as percentage loss in human development due to gender inequalities, reflecting disparities between men and women in 3 aspects: reproductive health, empowerment and the labor market. See technical note 3 at http://hdr.undp.org/en/media/HDR_2013_EN_TechNotes.pdf for detailed information.

licly available since the mid-1980s in China as one method to curb population growth. Finally, using U.S. marriage data since the 1930s, Bronson and Mazzocco (2013) argue that marriage rates are negatively correlated with cohort sizes. Their paper analyzes the group of women as a whole and thus doesn't focus on successful women's marriage timing.

In this paper, I explain this phenomenon by China's hypergamous practice which is reinforced by the unbalanced gender ratio (Qian, 2012). In China, men usually marry women with equal or lower socioeconomic status than themselves. On the other hand, women usually marry up. If A-quality men marry B-quality women, B-quality men marry C-quality women, C-quality men marry D-quality women, eventually there will be A-quality women and D-quality men left in the marriage market. Such a hypergamous marriage pattern is strengthened by the "excess men" problem since all women ex ante set up higher reservation values for the quality of their future husbands, and all men choose lower reservation values with adverse marriage market conditions. Furthermore, intelligent and educated women who are unmarried find it harder to find their "Mr. Right" as they get older, since high quality men are less choosy and leave the market faster. Eventually, we observe a larger proportion of A-quality unmarried women in the population.

This work primarily relates to two strands of literature. The first literature studies the effect of changes in the job offer arrival rate on the duration of unemployment in a classical Diamond-Pissarides-Mortensen job search framework. It is known that a change in the job offer arrival rate has two opposite effects on the hazard rate of unemployment (or the duration of unemployment). On the one hand, a higher offer arrival rate has a positive effect on the hazard rate since job seekers have more opportunities to exit unemployment in each period. On the other hand, there is a negative effect because job seekers become more selective with the higher offer arrival

rate. The net effect depends on the relative strength of these two effects and thus the sign of the net effect is ambiguous. Researchers including Vroman (1985), Burdett (1981), Jensen and Vishwanath (1985), and van den Berg (1994) have shown that for the net effect to be positive, a sufficient condition is that the wage distribution is log-concave[3]. These papers focus on the classical job search model with homogenous workers, exogenous wage distribution and linear utility function, which cannot be directly applied to the question studied in this paper.

The second literature is the bilateral search and matching literature. Existing literature in this field mainly focuses on the matching pattern resulting from bilateral search, i.e., whether matching is positively (or negatively) assortative. Representative works include Burdett and Coles (1997), Shimer and Smith (2000), Smith (2006), Jacquet and Tan (2007) among others. In recent years, several researchers have extended the concept of assortative matching into a multidimensional framework, such as Lindenlaub (2014), McCann et al., (forthcoming), Chiappori et al. (2012), etc. A major result offered by Smith (2006) is that in a non-transferable utility framework, positively assortative matching arises when the payoff function is log-supermodular. This is a quite strong result since once this condition is satisfied, the matching pattern is independent of the distributions of types on both sides of the market. However, none of these papers discusses the effect of changes of the offer arrival rate on the duration of search in a bilateral search model, which is the focus of this paper.

In this paper, I will show that in a bilateral search model with non-transferable utility and positively assortative matching, agents (both men and women) of medium quality have a larger hazard rate from the market than do agents with high or low

[3]If the wage distribution is $F(x)$, the sufficient condition is $\log(1-F(x))$ is concave. A weaker sufficient condition given by van den Berg (1994) is that $\log(1-F(e^x))$ is concave.

quality. Such a feature would turn the stationary distribution of singles in the marriage market towards a V-shaped distribution since those medium-quality agents represent a smaller share in the market. Thus the log-concavity of the stationary distributions fails endogenously, possibly resulting in a negative sign of the net effect of an increasing offer arrival rate on the hazard rate. Or in the current situation, some single women may spend longer time searching for spouses when the gender ratio increases.

The rest of the paper is organized as follows: Section 3.2 formally analyzes this phenomenon using a life-cycle model with bilateral search in the marriage market. Section 3.3 calibrates the model and quantitatively shows how the women with high-quality get married slower when the gender ratio is higher. Section 3.4 concludes. In the Appendix, I use a typical bilateral search model to explain why the log-concavity of the stationary distributions may fail in this type of model.

3.2 Model

3.2.1 Demographics

I develop an overlapping generation model with two genders: men (m) and women (w) who randomly search for each other in a unified marriage market. Both men and women live for J periods with ages denoted as j ($j = 1, 2, 3, \ldots, J$). The measure of *single* men and women of age j are M_j and W_j respectively. The gender ratio for newborn agents is R^* (R^* men to 1 woman). I maintain the measure of age 1 agents each period as 1, so the measures of age 1 men and age 1 women are

$$M_1 = \frac{R^*}{1+R^*};\ W_1 = \frac{1}{1+R^*}$$

Each agent is endowed with labor productivity n with distribution $G_1^n(\cdot)$ based on which they earn labor income each period before retirement age J^R. The discount factor between periods is β.

Single agents can search for spouses before age J^R. For single agents at marriageable ages, each period is further divided into 2 stages. In the first stage, they allocate wealth between consumption c and housing next period h'. In the second stage, single agents go to the marriage market. If a marriage is formed under mutual consent, the couples pool their premarital housing together. Those who fail to marry this period will keep searching in the next period as long as he or she is of a marriageable age. Married agents and single agents with age $j > J^R$ skip the second stage. To keep things simple, divorce and remarriage are not allowed.

3.2.2 Matching Technology

The marriage market is frictional, and characterized by the classical search and matching literature. All single agents of marriageable ages search in the same marriage market, and everyone has at most one date each period. Thus, each single agent can meet with a potential date of any age $j \in \{1, 2, \ldots, J^R\}$, or no date if he or she is unlucky.

The total measures of single agents in the marriage market are:

$$M = \sum_{j=1}^{J^R} M_j;\ W = \sum_{j=1}^{J^R} W_j, \tag{3.1}$$

and the gender ratio in the marriage market is $\overline{R} = \frac{M}{W}$. In the spirit of Pissarides (1990), the total number of matches takes the Cobb-Douglas form:

$$\eta = \lambda W^{\upsilon} M^{1-\upsilon}. \tag{3.2}$$

Each period, the probability of a man meeting a woman is

$$\lambda_M = \frac{\eta}{M} = \lambda \overline{R}^{-\upsilon} \tag{3.3}$$

and with probability $\lambda_M \frac{W_j}{W}$ he meets a woman of age j. With probability $1 - \lambda_M$, he ends up with no date this period.

A woman's probability of meeting a man is

$$\lambda_W = \frac{\eta}{W} = \lambda R^{1-\upsilon} \tag{3.4}$$

and with probability $\lambda_W \frac{M_j}{M}$ she meets a man of age j. With probability $1 - \lambda_W$, she ends up with no date this period.

Distributions of labor productivity among age j single men and women are $G^n_{m,j}(n)$ and $G^n_{w,j}(n)$ respectively. These distributions might be different from the distribution of endowment $G^n_1(\cdot)$ since some people marry faster than the others.

3.2.3 Period Utilities

Before we characterize matching patterns in the marriage market, we need to define people's utilities and value functions. For ease of exposition, I denote agents' marital status using superscript **m** for married and **s** for single.

An age j single (widowed) agent's period utility comes from consumption (c_j) and housing service (h_j)

$$u^{\mathbf{s}}(c^{\mathbf{s}}_j, h^{\mathbf{s}}_j) = \theta \ln c^{\mathbf{s}}_j + (1-\theta) \ln h^{\mathbf{s}}_j$$

subject to the following budget constraint:

$$c^{\mathbf{s}}_j + q h^{\mathbf{s}}_{j+1} \leq \widetilde{n}_j + q h^{\mathbf{s}}_j$$

in which $\widetilde{n}_j$ takes the value of endowed productivity n for working age agent ($j \leq J^R$), or 0 if retired. q is the housing price which is constant in the long-run stationary equilibrium considered here. Housing stock h_j is determined one period before[4].

Based on this setup, housing plays 3 roles in this economy. First, it is a consumption good providing housing service each period. Second, it is a form of a wealth

[4]Age 1 agents have no housing endowment. As a modification, period utility for age 1 agents are defined as $u(c_1) = \theta \ln c_1$

transferring resource between periods. Third, it is also a status good. Given age and productivity endowment n, people with larger pre-marital housing are more welcome in marriage market.

Housing is a public good within a married household, and couples make consumption-housing decisions together. Period utility for household with age j_m husband and age j_w wife (h' is shorthand notation for choice of housing next period) is

$$\begin{aligned} u(c^{\mathbf{m}}_{m,j_m}, c^{\mathbf{m}}_{w,j_w}, h^{\mathbf{m}}_{j_m j_w}, \mu) &= \mu[\theta \ln c^{\mathbf{m}}_{m,j_m} + (1-\theta)\ln h^{\mathbf{m}}_{j_m j_w}] \\ &\quad +(1-\mu)[\theta \ln c^{\mathbf{m}}_{w,j_w} + (1-\theta)\ln h^{\mathbf{m}}_{j_m j_w}] \\ &= \theta[\mu \ln c^{\mathbf{m}}_{w,j_w} + (1-\mu)\ln c^{\mathbf{m}}_{w,j_w}] + (1-\theta)\ln h^{\mathbf{m}}_{j_m j_w} \end{aligned}$$

with family budget constraint

$$c^{\mathbf{m}}_{m,j_m} + c^{\mathbf{m}}_{w,j_w} + q h^{\mathbf{m}}_{j_m+1,j_w+1} \leq \widetilde{n}_{m,j_m} + \widetilde{n}_{w,j_w} + h^{\mathbf{m}}_{j_m j_w}$$

μ is Pareto weight of husband which is determined by Nash bargaining before marriage. The newlyweds pool their pre-marital housings together and in this case $h^{\mathbf{m}}_{j_m j_w} = h^{\mathbf{s}}_{m,j_m} + h^{\mathbf{s}}_{w,j_w}$. Labor incomes $\widetilde{n}_{m,j_m}$ and $\widetilde{n}_{w,j_w}$ equal their endowed productivity for working age couples, or 0 if retired.

3.2.4 Value Functions

3.2.4.1 Post-marital Single Agents

Widowed and single agents with age $j > J^R$ maximize the following:

$$\begin{aligned} V^{\mathbf{s}}_j(h_j, n) &= \max_{h_{j+1}} \{\theta \ln c_j + (1-\theta)\ln h_j + \beta V^{\mathbf{s}}_{j+1}(h_{j+1}, n)\} \qquad (3.5) \\ &\quad s.t. \\ c_j + q h_{j+1} &= \widetilde{n}_j + q h_j \end{aligned}$$

with $\widetilde{n}_j = n$ if $j \leq J^R$; otherwise, $\widetilde{n}_j = 0$. The first-order condition of h_{j+1} is

$$\frac{\theta q}{\widetilde{n}_j + q(h_j - h_{j+1})} = \beta \frac{\partial V^{\mathbf{s}}_{j+1}(h_{j+1}, n)}{\partial h_{j+1}}$$

Age J people simply consume all their liquid wealth.

3.2.4.2 Married Household

For households with age j_m husband and age j_w wife, they jointly maximize (I use shorthand notation h for current housing stock, and h' for choice of housing next period)

$$\begin{aligned}
& \widehat{V}^{\mathbf{m}}_{j_m j_w}(h, n_m, n_w, \mu) \qquad (3.6)\\
= \;& \mu V^{\mathbf{m}}_{j_m j_w, M}(h, n_m, n_w, \mu) + (1-\mu) V^{\mathbf{m}}_{j_m j_w, W}(h, n_m, n_w, \mu)\\
= \;& \max_{c_m, c_w, , \mathbf{s}} \{\mu \left[\theta \ln c_m + (1-\theta)\ln h + \beta V^{\mathbf{m}}_{j_m+1, j_w+1, M}(h', n_m, n_w, \mu)\right]\\
& + (1-\mu)\left[\theta \ln c_w + (1-\theta)\ln h + \beta V^{\mathbf{m}}_{j_m+1, j_w+1, W}(h', n_m, n_w, \mu)\right]\}\\
= \;& \theta\left[\mu \ln c_m + (1-\mu)\ln c_w\right] + (1-\theta)\ln h\\
& + \beta \widehat{V}^{\mathbf{m}}_{j_m+1, j_w+1}(h', n_m, n_w, \mu)
\end{aligned}$$

with constraint

$$c_m + c_w + qh' = \widetilde{n}_m + \widetilde{n}_w + qh$$

I denote total household consumption as $C = c_m + c_w$. Based on first-order conditions, we have

$$\begin{aligned}
c_m &= \mu C\\
c_w &= (1-\mu) C
\end{aligned}$$

If both individuals are at age J, they simply consume all their liquid wealth. Otherwise, if one spouse reaches the last age J while the other one is younger, they will prefer a different level of family housing next period. The age J spouse would

like to consume all the wealth, but doing so is bad for the younger one since (s)he will have nothing to consume next period. In this case, I assume couples choose h' to maximize (assume the husband is of age J)

$$\begin{aligned} V^{\mathbf{m}}_{Jj_w}(h,\mu) &= \max_{h'} \theta\left[\mu \ln c_m + (1-\mu)\ln c_w\right] + (1-\theta)\ln h \\ &\quad + \frac{1}{2}\beta V^{\mathbf{s}}_{w,j_w+1}(h') \\ &\quad s.t.: c_m + c_w + qh' = qh \end{aligned}$$

in which $V^{\mathbf{s}}_{w,j_w+1}(h')$ is the wife's value function next period as she will be widowed. Labor incomes don't appear here since both couples are retired. The first-order condition is

$$\frac{\theta}{h-h'} = \frac{\beta}{2}\frac{\partial V^{\mathbf{s}}_{w,j_w+1}(h')}{\partial h'}$$

Otherwise, if both couples are younger than J, they maximize (3.6) with first-order condition for h' :

$$\frac{\theta q}{q(h-h') + \widetilde{n}_m + \widetilde{n}_w} = \beta \frac{\partial \widehat{V}^{\mathbf{m}}_{j_m+1,j_w+1}(h', n_m, n_w, \mu)}{\partial h'}$$

3.2.4.3 Single Agents of Marriageable Ages

A single man at age $j_m \leq J^R$ first chooses consumption and housing, then goes to the marriage market. He takes his chance of meeting a woman λ_M, measures of single women W_{j_w}, distributions of single women's productivity $G^n_{w,j_w}(n)$ as well as single women's housing function $H^w_{j_w}(n)\ \forall j_w \leq J^R$ as given, and chooses h_{j_m+1} to maximize:

$$\begin{aligned} V^{\mathbf{s}}_{j_m M}(h_{j_m}, n_m) &= \max_{h_{j_m+1}} \{\theta \ln c_{j_m} + (1-\theta)\ln h_{j_m} + \beta(1-\lambda_M) V^{\mathbf{s}}_{j_m+1 M}(h_{j_m+1}, n_m) \\ &+ \beta\lambda_M \sum_{j_w=1}^{J^R} \frac{W_{j_w}}{W} \int \max[V^{\mathbf{s}}_{j_m+1 M}(h_{j_m+1}, n_m), V^{\mathbf{m}}_{j_m+1,j_w+1,M}(h', n_m, n_w, \mu)] dG^n_{j_w W}\} \end{aligned}$$

in which housing after marriage $h' = h_{j_m+1} + H^w_{j_w}(n_w)$. W and λ_M are given in Equations (3.1) and (3.3), and the value function after marriage $V^{\mathbf{m}}_{j_m+1,j_w+1,M}(h', n_m, n_w, \mu)$ is given by Equation (3.6). With probability $\lambda_M \frac{W_{j_w}}{W}$ he meets an age j_w woman and upon meeting her, he decides whether to get married by comparing the value function of marriage and the value function of remaining single.

The case for women is similar. An age j_w woman takes her chance of meeting a man λ_W (given by Equation (3.4)), measures of single men M_{j_m}, distributions of single men's productivity $G^n_{m,j_m}(n)$ as well as equilibrium housing function $H^m_{j_m}(n)$ $\forall j_m \leq J^R$ as given, and chooses h_{j_w+1} to maximize

$$V^{\mathbf{s}}_{j_w W}(h_{j_w}, n_w) = \max_{h_{j_w+1}} \{\theta \ln c_{j_w} + (1-\theta) \ln h_{j_w} + \beta(1-\lambda_W) V^{\mathbf{s}}_{j_w+1,W}(h_{j_w+1}, n_w)$$
$$+\beta\lambda_W \sum_{j_m=1}^{J^R} \frac{M_{j_w}}{M} \int \max[V^{\mathbf{s}}_{j_w+1,W}(h_{j_w+1}, n_w), V^{\mathbf{m}}_{j_m+1,j_w+1,W}(h', n_m, n_w, \mu)] dG^n_{j_m,M}\}$$

with $h' = H^m_{j_m}(n) + h_{j_w+1}$.

3.2.5 Reservation Values

Agents in the marriage market have 3 characteristics: age j, productivity n and their pre-marital housing. However, equilibrium housing will be a function of j and n such that the housing choice of an age j man or woman with productivity n is consistent with the market housing function $H^m_{j_m}(n)$ and $H^w_{j_w}(n)$. As a result, we can reduce the dimension of characteristics of single agents to 2: $\{n, j\}$.

For a man with $\{n_m, j_m\}$, I use $M^M_{j_m j_w}(n_m)$ to denote the set of age j_w women's productivity n_w who will marry him upon contact. Since we use a transferable utility framework and the Pareto weight in marriage μ is determined by Nash bargaining before marriage, a match is formed as long as matching surplus is positive. Similarly, for a woman with $\{n_w, j_w\}$, I use $M^W_{j_m j_w}(n_W)$ to denote the set of age j_m men's productivity n_m who will marry her upon contact.

3.2.6 Marriage Market Aggregates

Measures and distributions of single agents in the marriage market should be consistent with individuals' optimal choices. For an age j_m single man with productivity n_m, his chance of marrying an age j_w woman is:

$$\pi^M_{j_m j_w}(n_m) = \lambda_M \frac{W_{j_w}}{W} \int_{\{n_w \in M^M_{j_m j_w}(n_m)\}} dG^n_{j_w,W}(n_w),$$

and his probability of getting married is

$$\pi^M_{j_m}(n_m) = \sum_{j_w=1}^{J^R} \pi^M_{j_m j_w}(n_m)$$

For all age j_m single men, a fraction $\pi^M_{j_m j_w}$ of them will marry age j_w women, where

$$\pi^M_{j_m j_w} = \int \pi^M_{j_m j_w}(n_m) dG^n_{j_m,M}(n_m)$$

and a fraction $\pi^M_{j_m}$ of them can get married this period:

$$\pi^M_{j_m} = \sum_{j_w=1}^{J^R} \pi^M_{j_m j_w}$$

The total measure of single men with age $2 \leq j_m \leq J^R$ is

$$M_{j_m} = M_{j_m-1}(1 - \pi^M_{j_m-1}),$$

and the distribution of n_m for age j_m men in the marriage market is:

$$G^n_{j_m M}(n_m) = \frac{\int_{\underline{n}}^{n_m} [1 - \pi^M_{j_m-1}(n)] dG^n_{j_m-1,M}(n)}{\int_{\underline{n}}^{\overline{n}} [1 - \pi^M_{j_m-1}(n)] dG^n_{j_m-1,M}(n)}$$

The same formulas apply for single women if we change the subscripts.

3.2.7 Definition of Equilibrium

An *equilibrium* is marriage market aggregates $\{W_{j_w}, M_{j_m}\}$ and distributions $\{G^n_{j_w,W}(n_W), G^n_{j_m,M}(n_M)\}$, men and women's matching sets $\{M^M_{j_m j_w}(n_m), M^W_{j_m j_w}(n_w)\}$ market

housing functions $H^m_{j_m}(n)$ and $H^w_{j_w}(n)\ \forall j_m, j_w = 1, 2, 3, \ldots, J^R$, and housing and value functions for individual single agents $h_{m,j_m+1}(h_{j_m}, n)$, $h_{w,j_w+1}(h_{j_w}, n)$, $V^{\mathbf{s}}_{m,j_m}(h_{j_m}, n)$, $V^{\mathbf{s}}_{w,j_w}(h_{j_w}, n)$ as well as housing and value functions for married households such that:

1. The marriage market aggregates and distributions are consistent with individual agents' choices of housing and marriage.

2. Given marriage market aggregates and distributions, housing and marriage decisions of single agents are optimal.

3. Housing and value functions of married households are optimal.

4. Market housing functions $H^m_{j_m}(n)$ and $H^w_{j_w}(n)$ are consistent with individual single agents' housing functions $h_{m,j_m+1}(h_{j_m}, n)$ and $h_{w,j_w+1}(h_{j_w}, n)$ such that

$$\begin{aligned} H^m_1(n) &= h_{m,2}(0, n) \\ H^m_{j_m}(n) &= h_{m,j_m+1}[H^m_{j_m-1}(n), n], \forall j_m \in [2, J^R] \\ H^w_1(n) &= h_{w,2}(0, n) \\ H^w_{j_w}(n) &= h_{w,j_w+1}[H^w_{j_w-1}(n), n], \forall j_w \in [2, J^R] \end{aligned}$$

5. Housing market clears. Total demand of housing equals a fixed supply of housing $\overline{H}$.

3.3 Numerical Analysis

Can the actual gender ratio increase observed in the data generate significant changes in the marriage rate? I answer this question in this section by calibrating this model.

3.3.1 Calibration

Demographic parameters are set using a model period of 5 years. An agent lives from real-life age of 20 to real-life age of 65 so that $J = 10$. $J^R = 8$ corresponds to real-life

Table 3.1: calibrated parameters

Parameters	**Meaning**	**Value**
J, J^R	lifespan and retirement age	10, 8
R^*	gender ratio at birth	1.18
β	discount factor	0.77
θ	consumption weight in utility	0.76
λ; υ	matching technology	0.7; 0.5
$\overline{H}$	supply of housing	1
$G_1^n(\cdot)$	distribution of match quality	$Uniform$[0.1, 7.1]

retirement age of 55. According to the National Bureau of Statistics (NBS) in China, the gender ratio at birth was 1.18 boys per girl in 2011, which is set as the benchmark gender ratio in the model.

Two parameters λ and υ in Equation (3.2) control matching technology. Based on Giolito (2004), υ is chosen to be 0.5. The search friction parameter λ should be small enough that the probabilities of meeting each period are smaller than 1. I set $\lambda = 0.7$ in this paper. The annual discount factor is 0.95 so I choose $\beta = 0.95^5 = 0.77$. Based on Kiyotaki et al. (2011), the weight of consumption in period utility is $\theta = 0.76$. Total supply of housing $\overline{H}$ is normalized to 1, and distribution of labor endowment is uniform between 0.01 and 0.71. The lower and upper bounds are based on Kiyotaki et al. (2011) such that the most productive worker is 71 times more productive than the worker with the lowest productivity. The parameters are summarized in table (3.1).

3.3.2 Numerical Results

I compute a long-run stationary equilibrium based on the definition in Section 3.2.7, and get the following results.

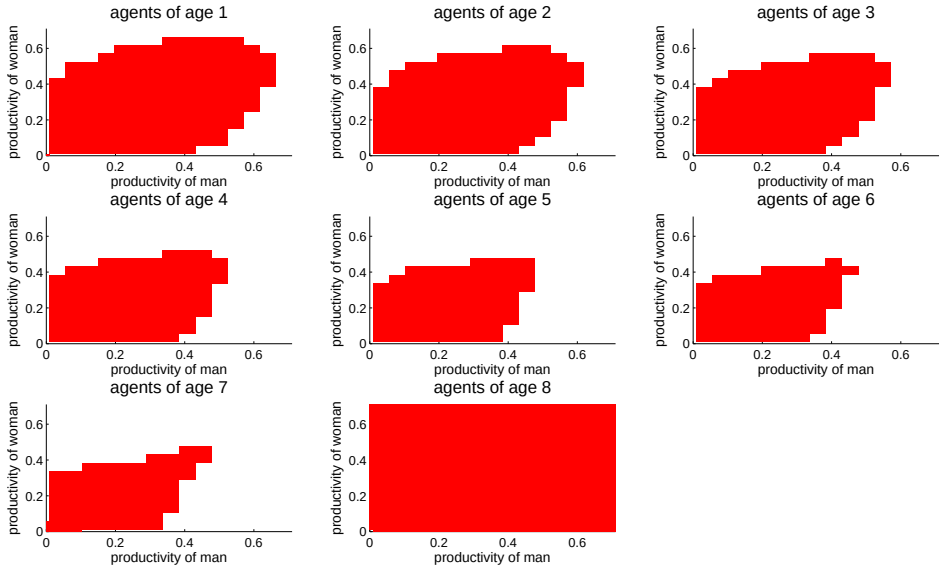


Figure 3.1: matching sets of same-age marriage

3.3.2.1 Assortative Matching

The first observation is that matching is positively assortative in productivity. When one's productivity increases, his or her reservation value for spouse's productivity also increase for each age group $j \in [1, J^R]$. For example, equilibrium matching sets for same-age marriage are plotted in Figure 3.1 as the red patched area. To save computing time, I choose 16 grid points for productivity so there are some zigzags on the edge.

There are 64 age combinations for couples $\{j_m, j_w\} \in \{1, 2, \ldots, J^R\}^2$ because under the unified marriage market assumption, single men and women of any ages can potentially meet with each other. I only plot matching sets of productivity pairs $\{n_m, n_w\}$ when men and women are of the same age. Matching for the other age

combinations also exhibit positively assortative patterns. Figure 3.1 clearly shows that matches are assortative along the productivity grid, but the degree of assortativeness gradually decreases as their ages increase. For aged single agents, if they reject their current dates, they won't have as many opportunities for marriage as they used to have during younger ages. Eventually, at the last marriageable age J^R, single agents are willing to accept any date upon meeting.

3.3.2.2 Comparative Statics

Whether women postpone marriage when the gender ratio R^* increases is unclear since there are two opposite effects with increasing R^*: On the one hand, single women have a better chance of meeting someone; on the other hand, their reservation values may increase in response to favorable market conditions. A similar argument applies to single men's marriage timing as well. In order to see whether an increasing gender ratio could have a significant effect on men and women's marriage age and marital rates and explain why successful women delay marriage in China, I conduct comparative statics by exogenously increasing the gender ratio from the balanced ($R^* = 1$) to the unbalanced case ($R^* = 1.18$).

Men and women's marital rates at each age are given in Figures 3.2 and 3.3. No matter whether the gender ratio is balanced, agents (men and women) with medium productivity have higher marital rates at each age because they are willing to accept the majority of candidates, and there's a majority of people who are also willing to accept them. People on both ends of the productivity distribution marry slower. High-productivity people are choosy and thus have a hard time finding an acceptable spouse. Low-productivity people, on the contrary, can hardly be accepted by most people. This feature is changed at the last marriageable age J^R when high-

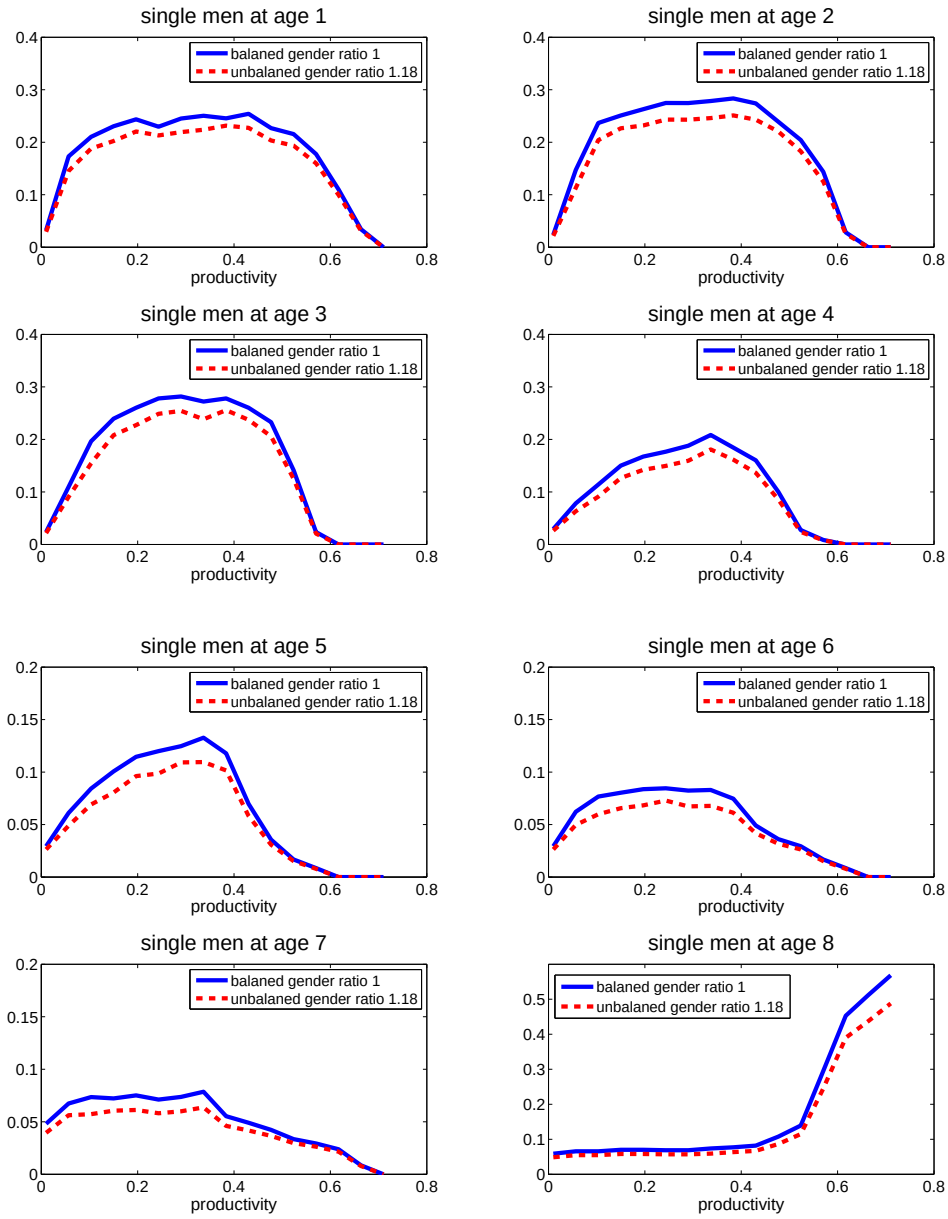


Figure 3.2: single men's hazard rate at each age

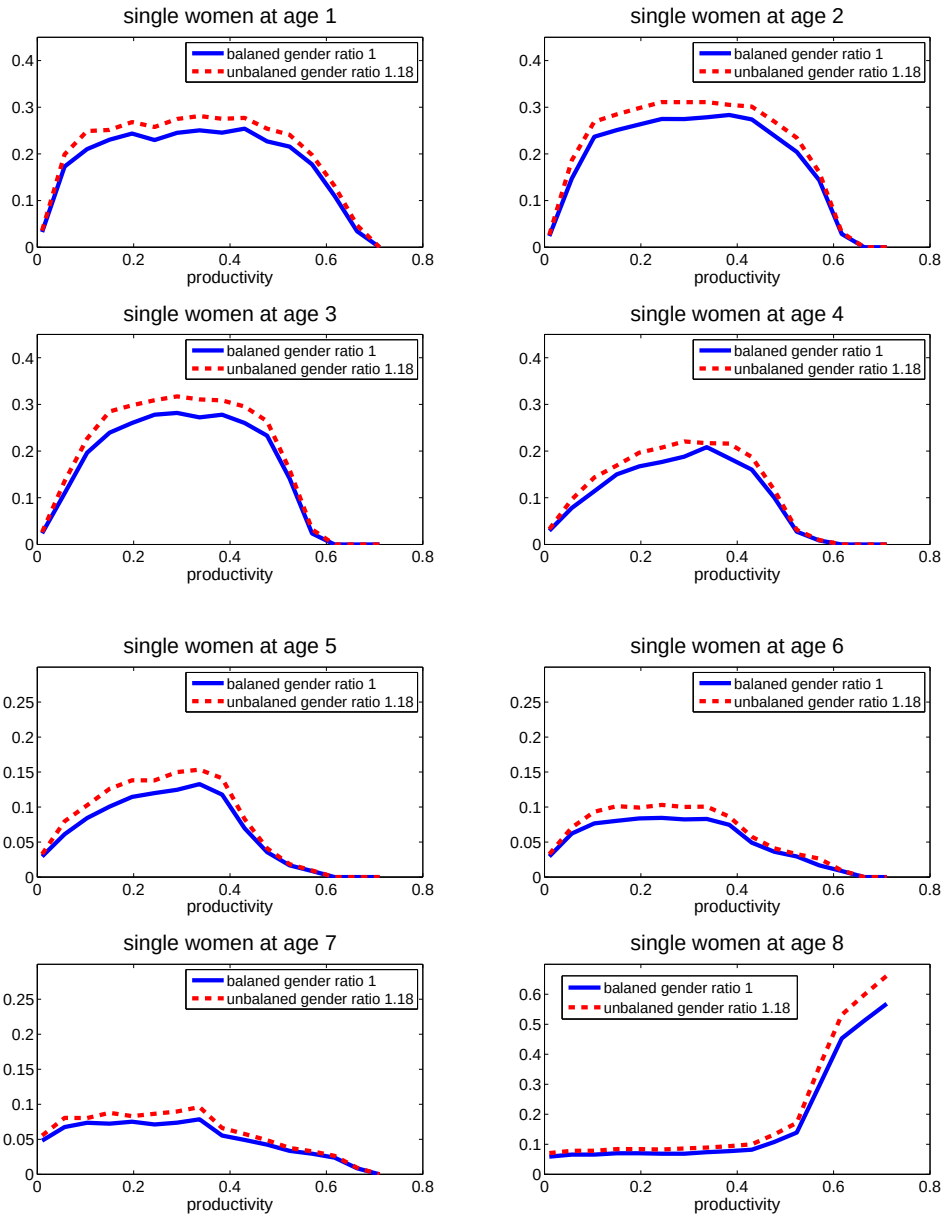


Figure 3.3: single women's hazard rate at each age

productivity people significantly reduce their reservation values and have the highest marital rates in this cohort.

Based on the comparative static analysis, when the gender ratio R^* increases, marital rates of men are reduced in general. Among men in each age group, medium-productivity men are most affected, but marital rates of men with high and low productivity don't change much. For high-productivity men, they reduce their reservation values and accept more in order to marry faster. For low-productivity men, they mostly marry low-productivity women who don't change their reservation values much as R^* increases. Changes in women's marital rates display a similar composition pattern but with opposite direction. As R^* increases, medium-productivity women marry much faster, but marital rates for women with low and high productivity don't change much. In particular, although high-productivity women have more candidates to choose from, they raise their reservation values in response. Consequently, the increase in their marital rates is relatively small.

Figures 3.4 and 3.5 show stationary distribution of productivity among single agents. One clear feature is that for both men and women, agents with medium productivity marry faster while people with low or high productivity marry slower at each marriageable age. Consequently, as age grows, people with low and high productivity have increasing shares in the marriage market, while the share of medium-productivity people decreases with age.

When the gender ratio rises, due to different marital rate changes, the shares of low-productivity and high-productivity men in the marriage market are reduced while the share of medium-productivity men increases. On the contrary, the shares of low-productivity and high-productivity women in marriage market increase, while the share of medium-productivity women decreases. This is consistent with our explanation for the "leftover women" phenomenon.

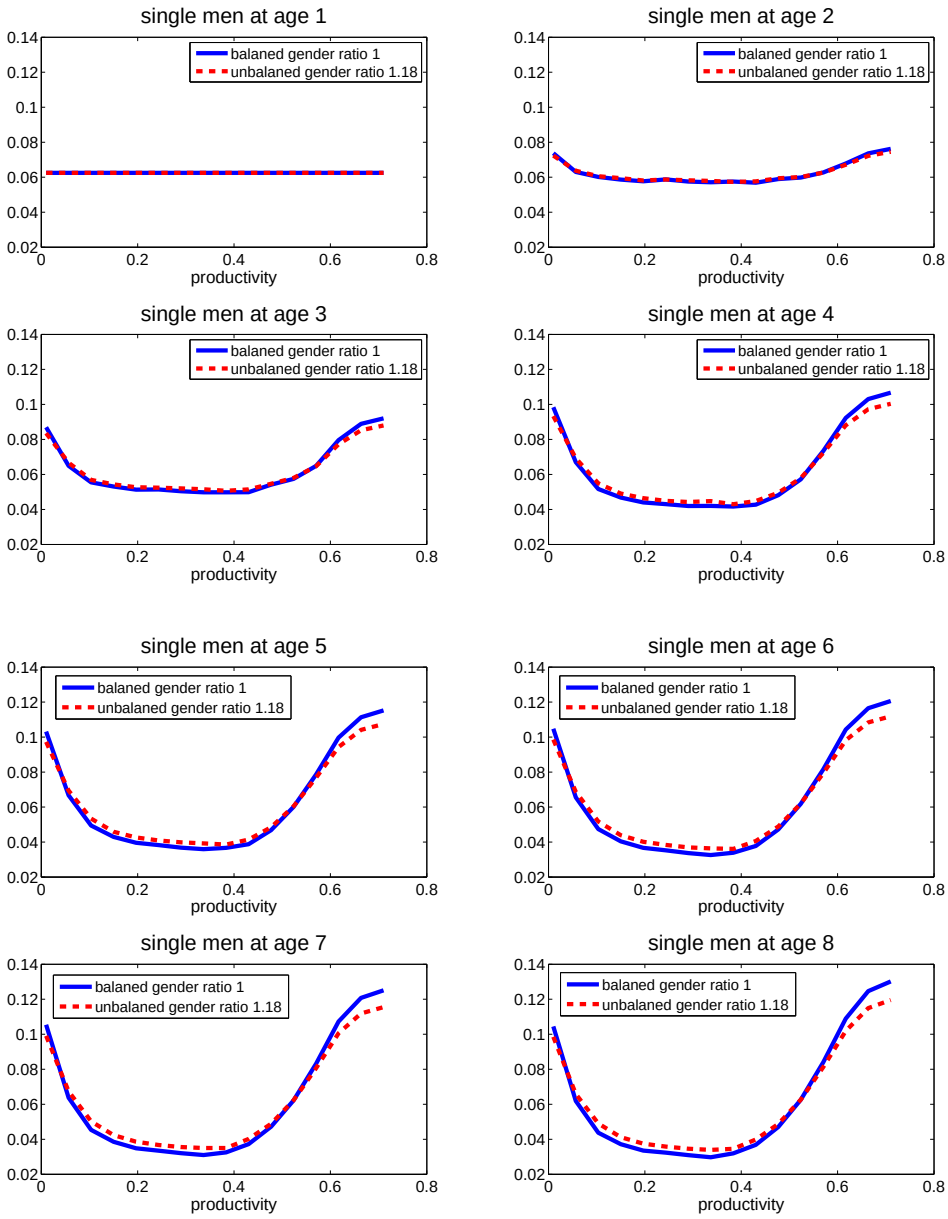


Figure 3.4: stationary distributions of men's productivity

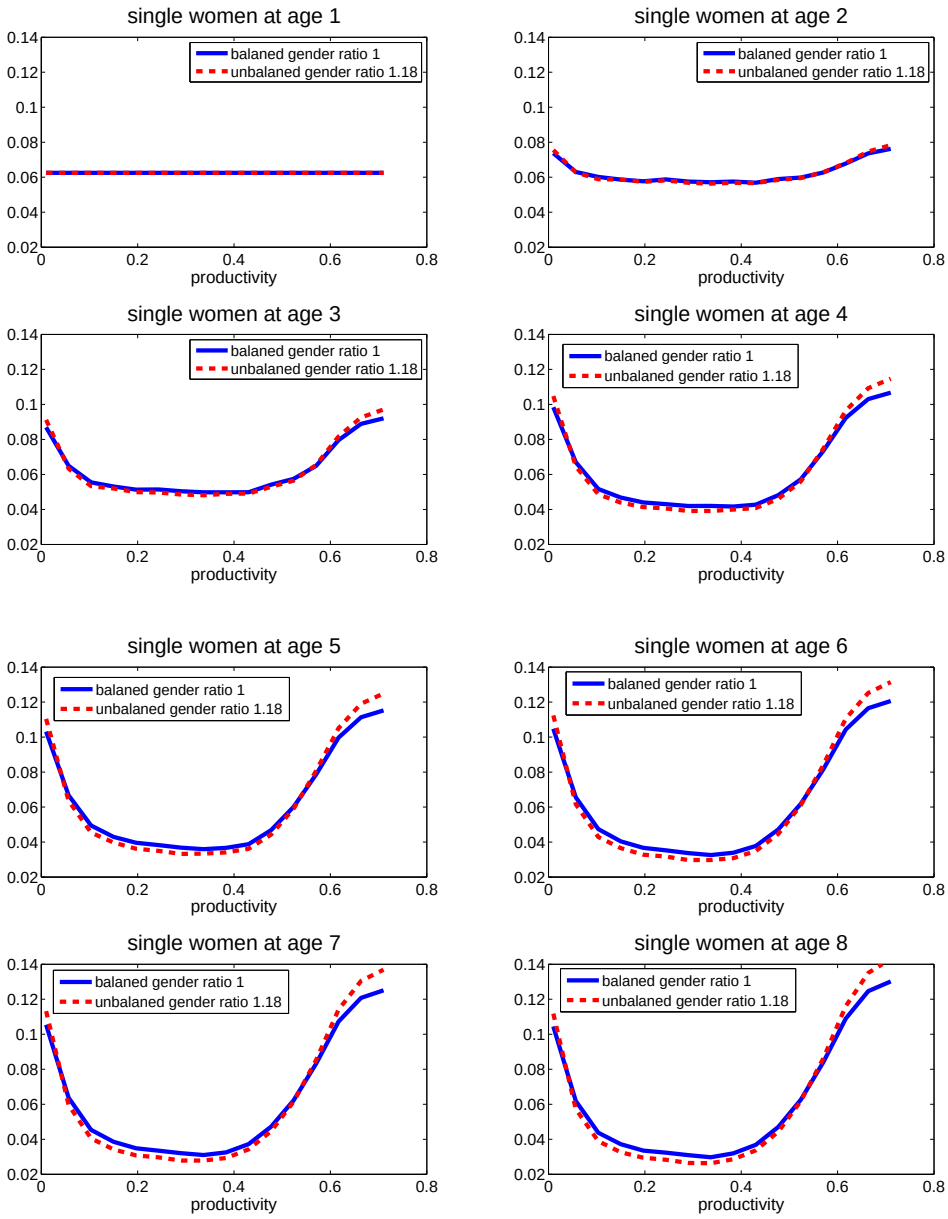


Figure 3.5: stationary distributions of women's productivity

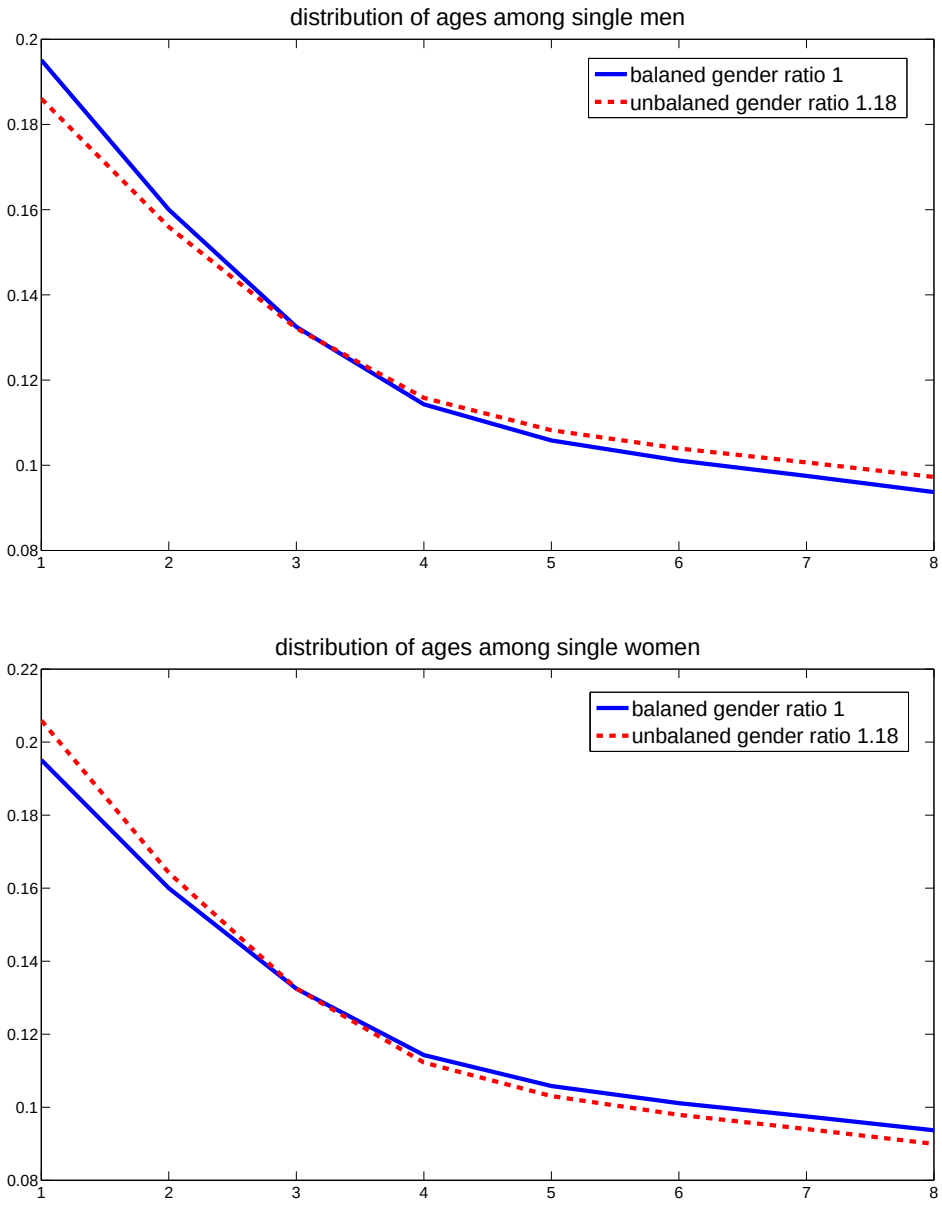


Figure 3.6: distribution of ages among single agents

Lastly, I plot the age distribution among single agents in Figure 3.6. As R^* increases, the average marital rate of men decreases while women's marital rate increases on average. As a result, there are a smaller proportion of young men and a larger proportion of young women in the marriage market. In other words, although there are proportionately more women of high quality in the marriage market, men are still in excess supply and have a harder time getting married with the unbalanced gender ratio.

3.4 Concluding Remarks

This paper focuses on the marriage rates and marital age of successful women in China. In particular, I'd like to address why marriage rates of highly educated, well paid women in China decreased when there is an excess supply of men in the marriage market. This question is of particular interest to policy makers. First, too many single men have been accused of being one major cause for several social problems such as increased crime rates, prevalence of prostitution and sexually transmitted infections in China (Ebenstein and Sharygin (2009); Edlund, Li, Yi, and Zhang (2013)). Unmarried "leftover women" obviously exacerbated this problem. Secondly, it has long been documented that children's education attainment is significantly correlated with the parents' education (see Holmlund et al. (2011) for a survey of different empirical methods for estimating the causal effect of a parents' schooling on a child's schooling). The increasing marital rate of these high-quality women would naturally increase the nation's stock of future human capital.

Using an overlapping generation model with bilateral search in the marriage market, I explain the "leftover women" phenomenon by looking at China's hypergamous marriage practice which is reinforced by the unbalanced gender ratio. Young

women with high socioeconomic status set up higher standards when marriage market tightness favors them, and thus have less probability of being matched in each single period. Furthermore, it's harder for them to find their "Mr. Right" since their counterparts, high-quality men become less choosy with a larger gender ratio and leave the marriage market faster. Consequently, although there are excess men in the marriage market, high-quality women have a smaller chance of getting married.

Random search in the model is a strong assumption which assumes that people with different incomes and ages have the same probabilities of meeting with each other. It will be interesting to see what happens if matching is not purely random. For example, people with similar income can form a submarket and meet with each other with higher probabilities, or people can use different signals in the marriage market to direct the search process. I leave this part for future research.

3.5 Appendix: A Bilateral Search Model

3.5.1 Setup

In this appendix, I use the framework of Smith (2006) to show how log-concavity of the stationary distributions fails in a bilateral search framework. There is no premarital investment and each agent is characterized by a single variable as his/her type. In particular, each man's type x is a random draw from an exogenous continuous distribution $G_m(x)$ with $x \in [\underline{x}, \overline{x}]$ and density function $g_m(x)$. Correspondingly, each woman's type y is drawn from $G_w(y)$ with $y \in [\underline{y}, \overline{y}]$ and density function $g_w(y)$. Both $\underline{x}$ and $\underline{y}$ are positive, and $g_m(x)$,$g_w(y) > 0$ over their respective range. Every man (woman) randomly searches for a woman (man) to match with, and a match is formed under mutual agreement upon contact. Each agent of type x earns flow payoff $f(x, y) > 0$ in a match with type y, and zero if unmatched. I assume that $f(x, y)$

is log-supermodular and thus the matching is positively assortative[5], i.e., people of higher types are more picky.

Assume new agents are born into the economy at each period with total measure 1 and a gender ratio R^* with exogenous distributions of types of men and women as $F_m(x)$ and $F_w(y)$ respectively. Thus the measures of new men and new women are $M_1 = \frac{R^*}{1+R^*}$ and $W_1 = \frac{1}{1+R^*}$. Notice that $F_i(\cdot)$ may be different from $G_i(\cdot)$ $(i = m, w)$ since some agents have higher hazard rates. Assume the stationary measures of single men and women are M and W. The offer arrival rates for men and women are given in Equations (3.3) and (3.4), $\lambda_m = \lambda \overline{R}^{-\upsilon}$, and $\lambda_w = \lambda \overline{R}^{1-\upsilon}$ in which λ is the matching technology and $\overline{R} = \frac{M}{W}$. Besides, each agent faces a constant mortality rate δ, and discount the future at rate r.

3.5.2 Value Functions and Equilibrium

Denote the value function of a type x single man as $V(x)$, and the value function if he matches with a type y woman as $V(x|y)$. In addition, assume that only women with type below $b_m(x)$ accept him upon contact. Then his Bellman equation is

$$V(x) = \frac{\lambda_m}{r+\delta} \int_{\underline{y}}^{b_m(x)} \max\{V(x|y) - V(x), 0\}\, dG_w(y)$$

We can show that $V(x|y) = \frac{f(x,y)}{r+\delta}$, and that this man only accepts women with type higher than a reservation value $a_m(x)$ upon contact. Thus we have

$$V_m(x) = \frac{\int_{a_m(x)}^{b_m(x)} \frac{f(x,y)}{r+\delta} dG_y}{\frac{r+\delta}{\lambda_m} + \int_{a_m(x)}^{b_m(x)} dG_y}$$

with

$$(r+\delta)V_m(x) = f[x, a_m(x)]$$

[5] See Smith (2006) for the proof of this result

Similarly, given that a woman of type y can only be accepted by men whose types are below $b_w(y)$, we can derive her value function and reservation value as

$$V_w(y) = \frac{\int_{a_w(y)}^{b_w(y)} \frac{f(y,x)}{r+\delta} dG_x}{\frac{r+\delta}{\lambda_w} + \int_{a_w(y)}^{b_w(y)} dG_x}$$

with

$$(r+\delta)V_w(y) = f[y, a_w(y)]$$

With positively assortative matching, we know that $a_m(x)$, $b_m(x)$, $a_w(y)$ and $b_w(y)$ are weakly increasing. In addition, $b_m(x) = \sup\{y|a_w(y) \leq x\}$ and $b_w(y) = \sup\{x|a_m(x) \leq y\}$. In a steady state, the inflow (new agents) and outflow of the marriage market (due to marriage or death) should balance in equilibrium, and we have

$$\begin{aligned} M_1 f_x(x) &= [\lambda_m \int_{a_m(x)}^{b_m(x)} dG_y + \delta] M g_x(x) \\ W_1 f_y(y) &= [\lambda_w \int_{a_w(y)}^{b_w(y)} dG_x + \delta] W g_y(y) \end{aligned}$$

3.5.3 An Illustrative Example

I assume $F_m(x)$ and $F_w(y)$ are uniform distributions between 0 and 1, $\delta = 0.1$, $r = 0.3$, $\lambda = 2$, $\nu = 30$ and $f(x,y) = e^{xy}$. Notice that both $F_m(x)$ and $F_w(y)$ are log-concave. $f(x,y)$ is log-supermodular and matching is positively assortative. With a balanced gender ratio (and thus the matching set is symmetric and the distributions of single men and women are the same), the stationary distribution of single women $G_w(y)$ is plotted in Figure 3.7. We can see that even if the distribution of types among inflow agents is log-concave (uniform distribution), the stationary distribution $G_w(y)$ is not. In fact, it is "V-shaped" such that women with medium quality represent a smaller share compared to women on both ends of the distribution. The question is why these medium-quality women leave the market faster than the others. The question can be

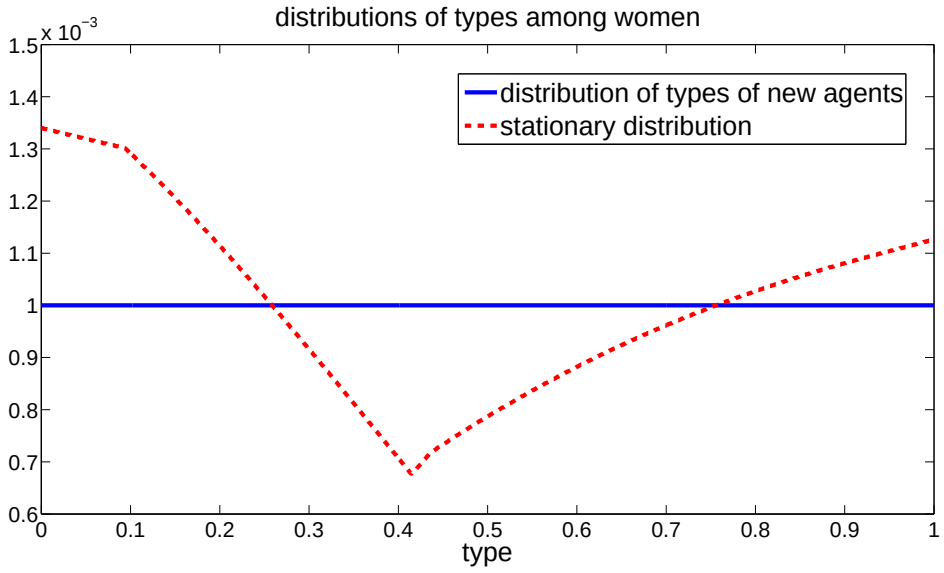


Figure 3.7: stationary distribution of single women's type

answered by the equilibrium matching set in Figure 3.8, in which the lower bound (women's reservation value of men's type) and upper bound (the highest type of men who accept her) of the matching set are plotted. Since the payoff function $f(x, y)$ is log-supermodular, both curves are strictly increasing in women's own type unless they reach the boundaries of the distribution. For women with medium-type, both the lower and upper ends of their matching sets are strictly increasing, and thus given a uniform distribution, they have larger hazard rates than women with low or high quality, whose matching sets have reached the boundaries. As a result, the stationary distribution of singles is V-shaped which is not log-concave.

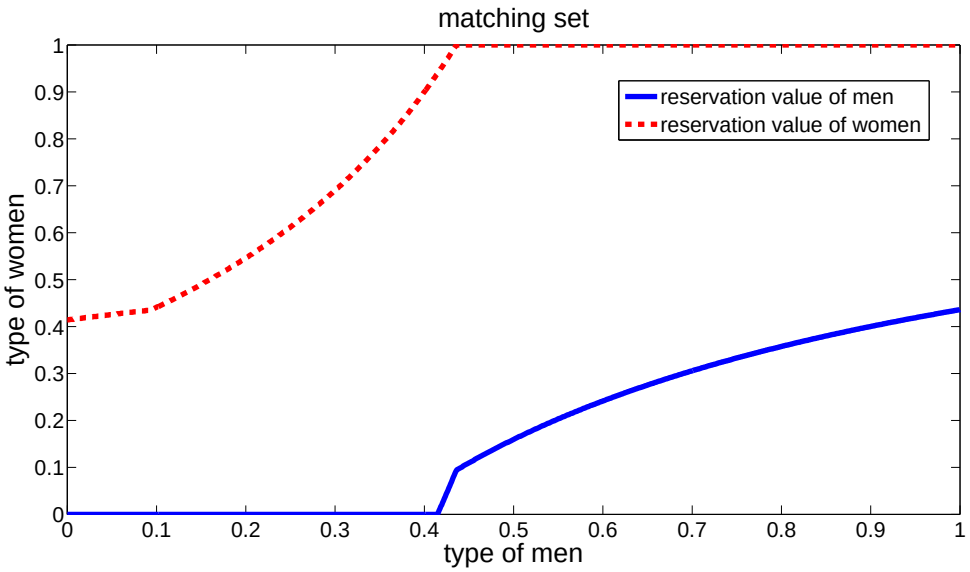


Figure 3.8: matching set

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