ABSTRACT

Title of dissertation: ENTANGLEMENT AND INFORMATION IN ALGEBRAIC QUANTUM THEORIES

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The algebraic approach to physical theories provides a general framework encompassing both classical and quantum mechanics. Accordingly, by looking at the behaviour of the relevant algebras of observables one can investigate structural and conceptual differences between the theories. Interesting foundational questions can be formulated algebraically and their answers are then given in a mathematically compelling way. My dissertation focuses on some philosophical issues concerning entanglement and quantum information as they arise in the algebraic. These two concepts are connected in that one can exploit the non-local character of quantum theory to construct protocols of information theory which are not realized in the classical world.

I first introduce the basic concepts of the algebraic formalism, by reviewing von Neumann's work on the mathematical foundations of quantum theories. After presenting the reasons why von Neumann abandoned the standard Hilbert space formulation in favour of the algebraic approach, I show how his axiomatic program remained a mathematical "utopia" in mathematical physics.

The Bayesian interpretation of quantum mechanics is grounded in information-theoretical considerations. I take on some specific problems concerning the extension of Bayesian statistical inference in infinite dimensional Hilbert space. I demonstrate that the failure of a stability condition, formulated as a rationality constraint for classical Bayesian conditional probabilities, does not undermine the Bayesian interpretation of quantum probabilities. I then provide a solution to the problem of Bayesian noncommutative statistical inference in general probability spaces. Furthermore, I propose a derivation of the a priori probability state in quantum mechanics from symmetry considerations.

Finally, Algebraic Quantum Field Theory offers a rigorous axiomatization of quantum field theory, namely the synthesis of quantum mechanics and special relativity. In such a framework one can raise the question of whether or not quantum correlations are made stronger by adding relativistic constraints. I argue that entanglement is more robust in the relativistic context than in ordinary quantum theory. In particular, I show how to generalize the claim that entanglement across space-like separated regions of Minkowski spacetime would persist, no matter how one acts locally.

ENTANGLEMENT AND INFORMATION

IN ALGEBRAIC QUANTUM THEORIES

by

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Chapter 1

Introduction

> Mathematics and theoretical physics have actually a good deal in common. ... [T]he attitude that theoretical physics does not explain phenomena, but only classifies and correlates, is today accepted by most theoretical physicists. This means that the criterion of success for such a theory is simply whether it can, by a simple and elegant classifying and correlating scheme, cover very many phenomena, which without this scheme would seem complicated and heterogeneous, and whether the scheme even covers phenomena which were not considered or even not known at the time when the scheme was evolved. (These two latter statements express, of course, the unifying and the predicting power of a theory.) Now this criterion, as set forth here, is clearly

> to a great extent of an aesthetical nature. For this reason it is very closely akin to the mathematical criteria of success, which, as you shall see, are almost entirely aesthetical. [von Neumann (1947)]

Mathematics is one of the richest and most outstanding human intellectual activities. Despite its character of abstractness, it is successfully applied to a large number of empirical sciences. The fact that its conceptual foundations are not entirely understood makes this success somewhat amazing. Galileo's metaphor that Nature is a book written in the language of mathematics may not be taken too seriously. That is, one may not believe in the ontological claim that the world has a fundamentally mathematical structure. Yet, it is true that without grasping, and perhaps mastering, the mathematical language that structures our best scientific theories one is doomed to wander "in a dark labyrinth", as Galileo himself put it.

Theoretical physics is the branch of science where mathematics finds its chief applications. Theoretical models describing empirical phenomena are built by using sophisticated mathematical procedures. Mathematical terms are associated with physical entities. The quantitative aspect of the formulas thus obtained enables one to make predictions that can then be tested by experiments. Moreover, the availability of a common language offers a unifying framework for the treatment of different phenomena. In particular, it puts one in a position to merge various physical theories. Differently than pure mathematics, though, theoretical physics ought not to abstract completely from the empirical context. The unifying and predictive stages typically come after experimental physics has raised a problem. Only in this way one may hope to attain any achievement in the discipline.

To be sure, not all the mathematical terms appearing in a theory correspond to physical quantities. A non-trivial part of the formalism has a mere connecting function. However, the more one proceeds in the process of abstraction, the harder it gets to identify the terms which have a genuine physical counterpart. It may be seen as a philosophical problem for the foundations of physics to interpreting the mathematical symbols of a theory. Furthermore, the unification of different theories does not reduce to a purely technical procedure, but it requires merging the different (possibly heterogeneous) concepts underlying each theory as well. Philosophy of physics is thus concerned with the issue of providing a consistent conceptual interpretation of physical theories and their mutual relations.

John von Neumann was one of the greatest mathematicians of the last century. He made substantial contributions to theoretical physics, especially in quantum mechanics. His work on foundations of physics is often referred to as an example for the mathematical treatment of philosophical problems arising in the physical sciences. Far from taking an intransigent attitude toward the demand of mathematical rigor in physics, he actually regarded technical and conceptual difficulties of theoretical physics as an inspirational source for his work as a mathematician. As he emphasized in a paper titled *The Mathematician* which he wrote in 1947, where he compared the method of pure mathematics with that of the empirical sciences, "[s]ome of the best inspirations of modern mathematics (I believe, the best ones) clearly originated in the natural sciences". The construction of von Neumann algebras theory, which is a branch of operator algebras theory, is a paradigmatic case. In fact, he developed the algebraic formalism to provide a general framework for quantum theories.

The algebraic approach to physical theories is nowadays a well-developed area of modern theoretical physics. The algebraic formulation encompasses both classical and quantum mechanics. Thus, it represent a powerful tool to investigate structural and conceptual differences between the theories. It also yields a neat treatment of quantum systems in infinite dimensions. Interesting foundational questions can thus be formulated algebraically and their answers are then given in a compelling mathematical way. This explains the growing interest of philosophers of physics toward algebraic physical theories.

Furthermore, the algebraic formalism offers a clean and rigorous axiomatization of quantum field theory, that is the synthesis of quantum mechanics and special relativity, avoiding the usual approximations of the standard formulation that lead to divergences. The price to pay for formal precision is that one somehow loses a direct connection to the physical scenario, but all the relevant properties of micro-objects, whether these are (free) quantum fields or elementary particles, can actually be reconstructed. As Rudolph Haag, who is the prominent figure in Algebraic Quantum Field Theory, commented in his seminal book *Local Quantum Physics*, the algebraic approach does not provide a theory, but just a language. One of its remarkable virtues, though, is the suggestive harmony between physical questions and mathematical structures that arises in such a framework.

> The theory of von Neumann algebras and C*-algebras offers an adequate language for the concise formulation of the principle of locality in special relativistic quantum physics and tools for further development of the physical theory. But more striking are cases where the discussion of physical questions and the development of a mathematical structure proceed in parallel, ignorant of each other and motivated by completely disjoint objectives, till some time the close ties between them are noticed and a mutually fruitful interaction between physicists and mathematicians sets in. [Haag (1996), p.323]

The purpose of this dissertation is to address some philosophical questions arising in the algebraic approach. In particular, as the title indicates, I focus on some issues concerning entanglement and information. The original part of the work is based on a few papers that I have already published and some others that are in preparation or under review.

The first chapter offers an overview of von Neumann's work on the mathematical foundations of quantum theories. I show that his abandoning the standard Hilbert space formalism of quantum mechanics, which he himself originally worked out, in favor of the algebraic formulation was vindicated by the search for a physically sound and intuitively satisfactory interpretation of the theory. In doing this, I also pinpoint the conceptual motivation behind the technicalities involved in von Neumann algebras theory. As I argued in Valente (2008), von Neumann's work on quantum foundations was inspired by Hilbert's Sixth Problem concerning the geometrical axiomatization of physics: in his view geometry was so tied to logic that he ultimately developed a logical interpretation of quantum probabilities. There I also presented the reasons why his axiomatic program remained an "unsolved problem" in mathematical physics. Specifically, I discussed the consequences of a result by Huzimiro Araki, proving that no algebra with a tracial state defined on it, such as the type II_1 factors, which von Neumann regarded as the proper limit of quantum mechanics in infinite dimensions, can support any (regular) representation of the canonical commutation relations.

A more rigorous introduction of the basic concepts of the algebraic approach is given in the second chapter. There I also discuss the philosophical consequences of entanglement in quantum mechanics. The presence of entangled states across separated physical systems is in fact responsible for the non-local character of the theory. Then, I spell out the close connection between entanglement and quantum information. Specifically, the possibility of long-distance correlations enables one to construct protocols of information theory which are not realized in the classical world. Moreover, since von Neumann algebras theory represents a non-commutative generalization of classical probability theory, one can study the behaviour of quantum probabilities in great detail and contrast it to the classical case. I conclude by taking on some difficulties arising for the Bayesian interpretation of quantum mechanics, which is grounded in information-theoretical considerations. In particular, there are problems concerning the extension of Bayesian statistical inference in infinite dimensional Hilbert space. A stability condition for Bayesian conditional probabilities, which Redei (1992, 1998) formulated as a rationality constraint holding in classical probability theory, is shown to fail in quantum mechanics. In Valente (2007) I demonstrated that Redei's argument does not apply to quantum theory and I provided a solution to the problem of Bayesian noncommutative statistical inference arising from the violation of stability condition in general probability spaces. Moreover, in Valente (2009) I propose a derivation of the *a priori* probability state from symmetry considerations.

Finally, in the last chapter, I address a few philosophical problems of relativistic quantum mechanics. After introducing the axiomatics of Algebraic Quantum Field Theory, I raise the question of whether or not quantum correlations are made stronger by adding relativistic constraints. I first show that, despite the local structure of the relevant algebras, locality is maximally violated by quantum field systems. Then, I demonstrate that entanglement is more robust in the relativistic context than in ordinary quantum theory. In particular, there is a sense in which entangled states across space-like separated regions of Minkowski spacetime would persist, no matter how one acts locally. In fact, Clifton and Halvorson (2001) proved that one cannot destroy entanglement between type III factors by performing any local operation. I offer a generalization of their result and then argue that, contrary to what Clifton and Halvorson claim, the persistence of entanglement is not peculiar only to local algebras. Furthermore, I present a result I obtained with Miklos Redei which shows that, due to the local character of local operations, AQFT fares better than non-relativistic quantum mechanics in a field-theoretic paradigm proposed by Einstein in his 1948 *Dialectica* paper.

Chapter 2

John von Neumann's Mathematical "Utopia" in Quantum Theory

In 1954 John von Neumann was invited to give a lecture on "Unsolved Problems in Mathematics" at the International Congress of Mathematicians held in Amsterdam (September 2-9). From an historical point of view, the typescript recording the talk constitutes a fundamental document of his late conception of quantum mechanics since it offers the last presentation of the ideas he developed after his celebrated 1932 book on the mathematical foundations of the theory. Under explicit request of the Committee of the Congress, von

Neumann's address had the same character as the famous lecture David Hilbert gave to the second International Congress of Mathematicians that took place in Paris in 1900, in which he presented a list of twenty-three unsolved problems in all areas of pure and applied mathematics; solving such open questions was expected to contribute to progress in mathematical science. In particular the Sixth Problem, concerning the geometrical treatment of axioms in physics, had great influence on von Neumann's work in mathematical physics:

> Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. [Hilbert (1900)]

As Hilbert put it, by investigating the foundations of geometry one would get a model for the axiomatization of those physical disciplines in which mathematics plays a prominent role, especially mechanics and the calculus of probability. Von Neumann's aim was just to formulate a geometrical axiomatisation of quantum theories in the spirit of Hilbert.

During his lecture von Neumann discussed operator theory and its connections with quantum mechanics and noncommutative probability theory, pinpointing a number of unsolved problems. In his view geometry was so tied to logic that he ultimately outlined a logical interpretation of quantum probabilities. The core idea of his program is that probability is invariant under the symmetries of the logical structure of the theory. This is tantamount to a formal calculus in which logic and probability arise simultaneously. The problem that exercised von Neumann then was to construct a geometrical characterization of the whole theory of logic, probability and quantum mechanics, which could be derived from a suitable set of axioms.

Importantly, geometry plays a two-fold role in the foundations of the theory. On the one hand, it offers a model for how to carry out the axiomatization of quantum mechanics. On the other hand, projective and continuous geometries provide a framework for the formal structure of the theory. However, as I shall argue, this notion of geometry has nothing to do with, and indeed for von Neumann one should keep it sharply separated from, any spatial considerations. As he himself finally admitted, he never managed to set down the sought-after axiomatic formulation in a way that he felt satisfactory. Furthermore, the conceptual details of the logical interpretation of quantum mechanics are scattered in a number of papers (some unfinished) rather than presented in a systematic picture. A reconstruction of such ideas can be found in Bub (1961a, 1961b), who first quoted the typescript, and, in a more extensive way, in Redei (1996, 1998, 1999, 2001).

The first part of this chapter recalls the main reasons why von Neumann abandoned Hilbert space in favor of the type II_1 factor, a particular sort of von Neumann algebra, as the "proper limit" of quantum theories in infinite dimensions. In particular, section 1.1.3. is also meant to provide a conceptual introduction of the mathematical details of the algebraic approach, which will be recovered in a more rigorous way in the next chapter. The crucial aspects of his unified theory of quantum logic and quantum probability, as well as its inheritance from Hilbert's Sixth problem, are then discussed in detail in section 2. In section 3, by following Valente (2008), I explain in what sense it remained an unsolved problem in mathematical physics. I also supply evidence for the claim that von Neumann intended to complete his project by including information theory and by extending the algebraic treatment to relativistic quantum field theory. These two topics, and their connection to entanglement, will be then developed independently from von Neumann's work in chapter 2 and chapter 3, respectively.

Von Neumann's appealing to the type II_1 factor as the proper mathematical arena for quantum mechanics has been criticized on many grounds. Yet, no argument was supplied that demonstrates that such an algebraic structure cannot, in principle, recover quantum theory. Certainly, in practice, physics never really departed from the Hilbert space formalism. A comment by the mathematical physicist Huzihiro Araki (1990) is particularly evocative:

> ... a specific example provided by type II von Neumann algebras [the type II_1 factor] seemed to be a (mathematical) Utopia for quantum calculus, i.e. calculus of unbounded operators. [Araki (1990), p.119]

Araki has recently produced an unpublished result which appears to challenge the suitability of factors of type II_1 to support any representation of the Canonical Commutation Relations[1]. This would prove that what von Neumann was after is indeed a mathematical "utopia" in quantum theory. I take this up in the last section of the chapter.

[1] I am grateful to Prof. Araki for kindly letting me use the result. I also wish to thank Miklos Redei for showing me the hand-written paper containing the proof that Prof.Araki first passed to him during the Von Neumann Centennial Conference held in Budapest in 2003.

2.1 Rise and Fall of Hilbert Space

2.1.1 The axiomatic method: prolegomena to Hilbert space quantum mechanics

The Hilbert space formalism of quantum mechanics was formulated in his 1932 seminal book on the mathematical foundations of the theory (von Neumann (1932)), that followed a series of three papers that he published in 1927 (von Neumann (1927a, 1927b, 1927c)). Some of the basic ideas developed there are to be traced back to a paper he published in 1926 together with Nordheim and Hilbert, summarizing a series of lectures on quantum theory which Hilbert himself gave in Göttingen in the same year. Two aspects of this joint work are particularly relevant to von Neumann's subsequent development of quantum mechanics, both at a technical and at a conceptual level. On the one hand, the three authors proposed a quite interesting discussion of the axiomatic method in physics in the spirit of the Sixth Problem. On the other one, they emphasized the crucial role of probability in quantum mechanics and connected it with the axioms of the theory.

Hilbert, Nordheim and von Neumann distinguished the axiomatic approach adopted in the physical sciences from the sort of axiomatics that is peculiar of formal systems and languages. The former in fact requires a less strict way of axiomatizing than the latter. Accordingly, there are three components characterizing the structure of any physical theory. One first formulates a set of semi-formal requirements, the physical axioms, which are empirically grounded and capture the observational content of the theory. A mathematical description of the physical quantities involved in the axioms is then given by the formalism, the so-called analytic machinery of the theory. Finally, the physical axioms and the analytic machinery are connected by the physical interpretation. Ideally, one should specify the physical requirements in a such a way that the quantities occurring in the formalism are unambiguously fixed. Geometry would provide the model for such a complete axiomatization.

In particular, as quantum-mechanical observations provide the ground only for probabilistic statements, probability is taken as the primitive concept in the axiomatization of quantum theory. The following passage by Hilbert, Nordheim and von Neumann (1926) makes this point explicit.

> The way leading to this theory is the following: one formulates certain physical requirements concerning these probabilities, requirements that are plausible on the basis of our experiences and developments and which entail certain relations between these probabilities. Then one searches for a simple analytic machinery

> in which quantities appear that satisfy exactly these relations. This analytic machinery and the quantities occurring in it receive a physical interpretation on the basis of the physical requirements. The aim is to formulate the physical requirements in a way that is complete enough to determine the analytic machinery unambiguously. This way is then the way of axiomatising, as this had been carried out in geometry, for instance. The relations between geometric shapes such as point, line, plane are described by axioms, and then it is shown that these relations are satisfied by an analytic machinery, namely, linear equations. Thereby one can deduce geometric theorems from properties of the linear equations. [Hilbert, Nordheim and von Neumann (1926), p.105, translated in Redei and Stöltzer (2006)]

Nevertheless, in the practice of science the analytic machinery is usually conjectured (and uniquely determined) before a complete axiomatics is laid down. In fact, it is by interpreting the formalism that one identifies the basic physical relations of the theory. To establish a set of axioms rich enough is thus the task of the physical interpretation, that is where, quite contrary to formal axiomatics, "a certain freedom and arbitrariness" is actually allowed in mathematical physics. Redei and Stöltzer (2006) defined this attitude as opportunistic soft axiomatization and claim that it characterized von Neumann's work in quantum foundations.

In the case of quantum mechanics, the formal quantities ought to correspond to probabilistic assignments. As I will show later on in the chapter, in von Neumann's view, the axiomatic machinery of quantum mechanics was given in the framework of operator theory, generalizing Hilbert space. Allegedly, the physical interpretation advocated by the axiomatic method was the logical interpretation of quantum probabilities. Such an approach was indeed at the bottom of von Neumann's construction, and eventually of his abandonment as well, of Hilbert space quantum mechanics.

In the Hilbert-Nordheim-von Neumann paper, quantum-mechanical probabilities have the form of probability densities for the distribution of values of physical quantities. That is, the probability that the quantity A takes its value x in the interval $[a, b]$ provided that the quantity B has value y is found by integrating the probability density $w(x, y, A, B)$ with respect to dx, i.e. $\int_a^b w(x, y, A, B)dx$. Importantly, the authors referred to w as the "relative" probability density in the sense that it is not normalized. The problem, which they themselves felt quite uneasy with, is that computing the probability for a physical quantity to have value x provided that the same quantity has value y requires employing the Dirac function (so that w is given by $\delta(x - y)$), which von Neumann regarded as an illegitimate mathematical entity. To avoid appealing to such an "improper" function, he developed what later became the standard Hilbert space formalism of quantum mechanics.

Let us conclude this section by recalling the basic definition of Hilbert space. That is a linear vector space with complex coefficients equipped with a scalar product $\langle\cdot|\cdot\rangle$. The latter associates a complex number to any pair of elements of the space and induces a norm $||\eta||$ defined by the relation $||\eta||^2 = \langle\eta, \eta\rangle$ for any vector η of the space. The dimension $dim(\mathcal{H}) = n$ of a Hilbert space $\mathcal{H}$ is determined by the maximal number of linearly independent[2] vectors in it. If $n < \infty$, then the space is finite dimensional and we denote it by $\mathcal{H}_n$; if, instead, $n = \infty$, then the space is infinite dimensional and we denote it by $\mathcal{H}$. The Hilbert space is also required to be complete with respect to the norm induced by the scalar product, in the sense that every Cauchy sequence of elements of $\mathcal{H}$ converges in norm to a vector belonging to $\mathcal{H}$ itself. Moreover, the Hilbert space is said to be separable just in case there exists a countable dense set of vectors in it (an assumption that von Neumann made throughout his 1932 book).

Any vector ψ of Hilbert space, whether finite or infinite dimensional, can then be associated with the state of a quantum system. The observables in the theory are selfadjoint operators defined on $\mathcal{H}$. This guarantees that the

[2] A finite or infinite sequence of vectors η_n is called linearly independent if $\Sigma_n \lambda_n \eta_n = 0$ implies $\lambda_n = 0$ for all n.

spectrum, namely the set of possible values which an observable is bound to take by the properties of the system, is a subset of the real numbers. Importantly, measured values of observables are independent of the specific state of the system. In quantum mechanics, the preparation of the system in a state ψ does not determine which value a given observable A would take in that state; however, it fixes a unique expectation value $\langle\psi|A\psi\rangle$ by means of the scalar product. A special class of observables are the projections on Hilbert space. A projection P is an idempotent linear self-adjoint operator acting on $\mathcal{H}$. That is, it enjoys the property $P^2 = P$. This means that the projections of the lattice $\mathcal{P}(\mathcal{H})$ are physical quantities which can possess only two values, namely 0 and 1. Hence, they are particularly suitable to represent any experiment that yields a yes/no result.

2.1.2 The derivation of quantum probabilities via statistical inference

Redei and Stöltzer (2006) reconstructed the earlier work that von Neumann pursued alone on the mathematical foundations of the theory on the basis of the three components of the axiomatic scheme proposed by Hilbert, Nordheim and von Neumann. Here, I spell out the relevant details of their analysis to show how probability was, once again, the central notion in the axiomatization of quantum mechanics. Quantum probabilities are actually regarded as conditional probabilities. Illustrating this fact within Hilbert space hints at the structural role of geometry in von Neumann's approach: indeed, it does not only offer a model for the axiomatic treatment of quantum theory, but it supplies an actual framework for its formalism.

The analytic machinery of the theory is identified with the set of all self-adjoint operators acting on a Hilbert space $\mathcal{H}$. Two physical axioms require expectation values of physical quantities to be both linear and positive. The physical interpretation then associates the operators $A, B, ...$ with the physical quantities $a, b, ...$, so that the operator $\alpha A + \beta B + ...$ represents the physical quantity $\alpha a + \beta b + ...$ for any complex number $\alpha, \beta, ...$ and, given any function f, the operator $f(A)$ represents the physical quantity $f(a)$. It follows that the expectation values are of the form $E(a) = Tr(WA)$, with the positive, linear mapping W being the (unnormalized) statistical operator. Accordingly, general probability statements can be written as

$$\phi(P^A_{d'}) = Tr(WP^A_{d'}) \tag{2.1}$$

where the spectral projection $P^A_{d'}$ corresponds to the event that the value of A lies in the Borel set d' of the real line.

At the initial stage of his work on quantum foundations, von Neumann regarded quantum probabilities as relative frequencies. Indeed, the statistical operator W is interpreted as describing the statistical ensemble $\mathcal{E}$ of systems upon which one computes the occurrences of the event $P^A_{d'}$. Then, based on the recognition that quantum-mechanical probabilities are conditional probabilities, he developed the framework of noncommutative statistical inference. If a measurement has shown that the physical quantity B takes its value in the Borel set d, the (unnormalized) probability of the event $P^A_{d'}$ reads $Tr(P^B_d P^A_{d'})$, which derives from formula (2.1) when the statistical ensemble is associated with the spectral projection P^B_d of B.

Yet, there still remained the problem of accounting for the case in which no particular event is specified. For this case, he made the assumption of an *elementary unordered ensemble.* That is, an ensemble of systems $\mathcal{E}$ of which one does not have any specific knowledge, as it reflects the idea that "all possible states are in the highest possible degree of equilibrium, and no measuring action can alter this" (von Neumann (1932), p.346). So, its statistical operator is naturally associated with the identity operator, that is $W = I$; hence the *a priori probability* of the event $P^A_{d'}$ is $Tr(IP^A_{d'}) = Tr(P^A_{d'})$. All the systems of which one knows more are then obtained from the *a priori* ensemble by selection, namely by collecting those elements for which a certain property holds.

In footnote 156 of the *Mathematical Foundations of Quantum Mechan-*

ics von Neumann explicitly advocates von Mises' frequency interpretation of probability. He identifies his notion of quantum mechanical ensembles of systems with von Mises' notion of Kollectiv, which is a mathematical abstraction comprising an infinite sequence of trials. The axiom of convergence assures that, as a sequence of trials is extended, the proportion of favorable outcomes, i.e. number of occurrences of the property, tends toward a definite mathematical limit. The probability of a certain property in a Kollectiv would thus amount to the limiting relative frequency of the occurrence of such a property in the Kollectiv. Yet, to define probabilities as frequencies, one also needs the axiom of randomness which requires that the limiting value is the same for every possible subsequence of trials chosen by a rule of place selection within the sequence. In other words, the outcomes must be "randomly" distributed. The concept of randomness is quite controversial, though: as von Mises (1928) himself admitted, not all the procedures of selecting a subsequence within the sequence are acceptable.

A probability space admits a frequency interpretation if there exists a single, fixed statistical ensemble, say $\mathcal{E} = \{s_1, s_2, ..., s_N\}$, such that one can always check and decide unambiguously without altering the ensemble whether a certain property A belongs or not to any element s_i of $\mathcal{E}$. Accordingly, the probability of A is $p(A) = \lim_{N\to\infty} \frac{\sharp(A)}{N}$, where $\sharp(A)$ denotes the number of times A occurs in the ensemble. In particular, the following relation must holds

$$p(A \cup B) + p(A \cap B) = p(A) + p(B) \tag{2.2}$$

which is called *strong additivity.* Von Neumann insisted that quantum-mechanical probabilities ought to satisfy such a property.

Notice that the frequentist requirement that the operation of checking a property does not alter the ensemble seems to be in patent conflict with quantum mechanics, where performing a measurement always disturbs the measured system, and hence the ensemble itself. Von Neumann tried to get round such conceptual difficulty by arguing that, even if non-commuting quantities cannot be measured simultaneously, their probability distribution in a single ensemble can be computed with arbitrary accuracy when the number N of elements in the ensemble is sufficiently large. His reasoning hinges on the fact that in order to account for the distribution of values of a quantity R one does not need to perform a statistical analysis on the whole ensemble of N elements, but it is sufficient to limit the investigation to any subset of $M < N$ elements, say $\{s_1, s_2, ..., s_M\}$, provided that M is still large (but very small with respect to N). Consequently, only a fraction M/N of the ensemble is actually affected by the measurement and thus, when N is sufficiently large, the effect can be neglected. Analogously, one can conceive the statistics of the measurement of another quantity, say S, in the sub-ensemble $\{s_{M+1}, s_{M+2}, ..., s_{2M}\}$, where $M < N$. So, even if R and S are not compatible, they can be measured simultaneously on different systems without disturbing each other; moreover, given that $2M < N$ is sufficiently small, these measurements change the ensemble $\{s_1, s_2, ..., s_M\}$ in which they are performed only by an arbitrarily small amount. Nonetheless, Redei (2001) stressed that von Neumann's argument implicitly relies on the doubtful notion of randomness, as it makes the assumption that the relative frequency of each quantity is the same when computed in the whole ensemble and in an arbitrary sub-ensemble.

Conceptual difficulties concerning the frequency interpretation of quantum probabilities apart, the derivation of quantum-mechanical probability statements via statistical inference confirms the strict connection of quantum mechanics and probability theory. One better appreciates the fact that quantum measurements can be regarded as conditionalizing probability by focusing on the finite-dimensional case. Let us consider an observable with discrete spectrum. For instance, $B = \sum_i \lambda_i P_i^B = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, with the index i running from 1 to n. The one-dimensional spectral projections P_i^B map onto the closed subspaces of $\mathcal{H}_n$ spanned by the eigenvectors $|\psi_i\rangle$ and the λ_i denote the corresponding eigenvalues.

In particular, one can perform a measurement represented by the single projection P_i^B that is tantamount to asking whether or not the physical quantity B takes on the real value λ_i. The transition to the new quantum state is prescribed by the so-called von Neumann-Lüders rule

$$W \longrightarrow W' = \frac{P_i^B W P_i^B}{Tr(P_i^B W P_i^B)}$$

In case one does not have any specific knowledge before the measurement, the statistical operator W would coincide with the identity I. Then, from the above formula one immediately obtains $W' = P_i^B$.

Suppose the system is in a pure state ϕ. The probability of finding the system in the state ψ_i is thus given by

$$\begin{aligned} Tr(P_i^B|\phi\rangle\langle\phi|) &= Tr(|\psi_i\rangle\langle\psi_i|\phi\rangle\langle\phi|) \\ &= Tr(\langle\psi_i|\phi\rangle\langle\phi|\psi_i\rangle) \\ &= |\langle\psi_i|\phi\rangle|^2 \end{aligned}$$

which is the square of the cosine of the angle, say θ, between the two vectors. In other words, from a purely geometrical point of view, quantum-mechanical transition probabilities correspond to angles.

The Hilbert space formalism of quantum mechanics nicely illustrates the geometrical meaning of probabilities:

> ... geometrically speaking, it would be quite sufficient to say that what one has postulated is that the concept of an angle should

> apply there. The elegant way to state it is that one talks not about the angle but about the cosine and actually not about the cosine but about the inner product of two vectors. So a Hilbert space is defined by the existence of the inner product.
> [von Neumann (1954), p.232]

The inner product is thus the crucial element in the formal machinery of Hilbert space, as it enables the latter to be a suitable mathematical arena for quantum probabilities. That explains von Neumann's emphasis on this concept, which he maintained in the subsequent work on the foundations of quantum theory as well. As we shall see, operator theory deals with linear operations on a Hilbert space. What the space of all operators inherits from $\mathcal{H}$ is, together with the vector state structure, just the existence of an inner product.

2.1.3 The pathological behaviour of Hilbert space

As we pointed out, for von Neumann, operators of physical meaning are supposed to be self-adjoint. Yet, if selfadjoint operators act on Hilbert space, one runs into trouble when dealing with unbounded operators, such as position and momentum, because they cannot be defined everywhere on $\mathcal{H}$. Recall that an operator is unbounded if its norm induced by the inner product exceeds all finite bounds. Conversely, an operator A is bounded if and only if there exists a positive finite number N such that $||A\psi|| \leq N||\psi||$ for all $\psi \in \mathcal{H}$. It is a consequence of the Hellinger-Toeplitz theorem that the domain of definition of self-adjoint unbounded operators cannot comprise the entire Hilbert space, but it must be restricted to a dense subset of the latter.

The impossibility of defining fundamental physical quantities on $\mathcal{H}$ has an even more dramatic consequence, since the Canonical Commutation Relations (CCRs) expressed in terms of position P and momentum Q cannot be defined everywhere either.

> The quite decisive phenomena is that the two operators which play a fundamental role in quantum mechanics, namely those which stand for the basic mechanical concepts for a coordinate and for its conjugate momentum, had to satisfy a certain algebraic condition of which it is quite easy to show that it can never be satisfied by bounded operators. I mean the Heisenberg commutation relation which I will write down
>
> $$PQ - QP = iI \qquad (2.3)$$
>
> P and Q are the two operators in question, i is a number: it is the imaginary unit and I stands for the unit operator... The

> relation, of course, expresses the non-commutative of P and Q but expresses a good deal more. It is not at all difficult to show that bounded operators can never satisfy this relation. [von Neumann (1954), p.233]

While the problem with the domain of definition of unbounded operators was still open in the 1932 work, a solution to this second problem was found in the theorem of uniqueness of the Schrödinger representation of the CCRs, which von Neumann established in 1931 by refining a previous result by Stone. Incidentally, that also established the extent to which the two competing versions of quantum theory appeared in the late 1920s, namely Schrödinger's wave mechanics and Heisenberg's matrix mechanics, are equivalent.

Specifically, one gets rid of the unboundness of the operator $PQ - QP$ (namely the left hand side of (2.3)) by rewriting position and momentum in terms of the corresponding unitary, and hence bounded, Weyl operators. In fact, Stone's theorem assures that, if A is a (possibly unbounded) selfadjoint operator, then, given a real number t, the strongly continuous map $t \longrightarrow e^{itA}$ defines a one-parameter family of unitary operators. Accordingly, P and Q are the infinitesimal generators of the unitaries $U(a) = e^{iaP}$ and $V(a) = e^{ibQ}$, respectively. The Heisenberg's Canonical Commutation Relation thus becomes:

$$U(a)V(b) = e^{iab}V(b)U(a) \tag{2.4}$$

Notice that such a formula proves an irreducible representation of the CCRs if the only subspaces of Hilbert space which are left invariant by $U(a) \bigcup V(b)$ are $\{0\}$ and $\mathcal{H}$ itself[3]. Then, the Canonical Commutation Relations can actually be defined on all $\mathcal{H}$. The price to pay, though, is that now one has to assume that Hilbert space is infinite-dimensional. Appealing to infinite dimensions marked another crucial difference with respect to the Hilbert-Nordheim-von Neumann formulation of quantum theory, where Hilbert space was spanned by an orthonormal basis of only finitely many vectors. This, in turn, gave rise to a deeper physical pathology of Hilbert space quantum mechanics than the use of the Dirac function. Indeed, infinite probabilities appear in the theory.

Recall that von Neumann derived quantum-mechanical probability statements via statistical inference. Within that framework, the finiteness of probability is guaranteed just if one requires the statistical operator W to be normalized. Nonetheless, the identity operator I is not normalized at all. Therefore, the *a priori* probability, being the trace of any projection acting

[3]Von Neumann actually recast the Weyl form of the CCRs by means of the two parameter family $S(a,b) = e^{(-\frac{1}{2}ab)}U(a)V(b)$, but we do not need to go into the mathematical details of his proof here.

on infinite-dimensional Hilbert space, would be infinite. For instance, if the Borel set d' contains parts of the continuous spectrum of the observable A, the *a priori* probability of $P^A_{d'}$ computed by the formula $Tr(P^A_{d'})$ does not yield a finite value.

Von Neumann regarded probability assignments failing to be finite as an unacceptable fact for a physical theory. The pathological behaviour of probabilities in infinite dimensions, as well as the fact that unbounded operators are not defined everywhere, eventually induced von Neumann to abandon the Hilbert space formalism and to look elsewhere to find the proper mathematical arena of quantum theory. In the "Rings of Operators" (1936), von Neumann, together with Murray, discovered algebraic structures of operators that are not tied to Hilbert space, in which the above difficulties are avoided if certain suitable requirements of topological closure are satisfied. Dixmier later on proposed to call such structures *von Neumann algebras*.

2.1.4 The "proper limit" of quantum mechanics in infinite dimensions

The idea behind Murray and von Neumann's theory of rings of operators was the search for subsystems of operators retaining the same algebraic properties as the system of all operators. Hence, they found plausible that any ring of systems ought to be at least closed under addition, multiplication and subtraction, so that it forms an algebra. Moreover, some appropriate topology should be demanded in order to characterize the ring. Accordingly, one defines a von Neumann algebra $\mathcal{M}$ as a subalgebra of the space of all bounded operators on Hilbert space $\mathcal{B}(\mathcal{H})$ which is closed under the strong (and weak) operator topology[4].

An alternative, purely algebraic, definition can be given on the basis of von Neumann's double commutant theorem. The commutant of an algebra amounts to the set of all operators commuting with each element of the algebra. To put it technically, given any set of operators $\mathcal{N}$, its commutant is the set $\mathcal{N}' \equiv \{Q \in \mathcal{B}(\mathcal{H}) | QR = RQ, \forall R \in \mathcal{N}\}$. The double commutant theorem states that $\mathcal{M}$ is a von Neumann algebra if and only if it coincides with its double commutant, that is $\mathcal{M} = \mathcal{M}''$. Von Neumann's own words help one illustrate the meaning of this mathematical result (in the following quote $\mathcal{S}$ plays the same role as $\mathcal{M}$ in our discussion).

> One has a heuristic feeling that $\mathcal{S}$, if it is at all reasonably closed in some reasonable sense, ought to be the same as $\mathcal{S}''$. In other words, any sensible concept of algebraic topological closure should have this property that, if an operator A commutes with every-

[4] $\mathcal{M}$ must be also provided with the unit operator I. It is *-closed in the sense it contains the adjoint of each of its elements.

> thing that commutes with all $\mathcal{S}$, then A should be obtainable from $\mathcal{S}$ algebraically, in other words, if $\mathcal{S}$ is a closed ring, then A should belong to it: $\mathcal{S}$ should be equal to $\mathcal{S}''$. [von Neumann (1954), p.238]

From the double commutant theorem, it follows as a corollary that a von Neumann algebra is generated by its projection lattice $\mathcal{P}(\mathcal{M})$ in the sense that $\mathcal{M} \equiv \mathcal{P}(\mathcal{M})''$. So, one can get insight into the structure of the algebra itself by looking at its lattice structure.

Such a close connection between the projection lattice and the structure of its containing algebra was employed by Murray and von Neumann to construct a concept of dimensionality in terms of classes of equivalence of projections. The relevant notion would arise just as the Cantorean concept of equivalent cardinality emerged in number theory. This intuition was the basis for their classification of von Neumann algebras. Actually, von Neumann's aim was to work out a non-classical analogue of Cantor's set theory. By identifying the *a priori* probability with the dimension function of the projection lattice, he could then develop a noncommutative probability theory arising as the generalization of classical probability spaces.

We now spell out the details of this project. Let us begin with the notion of equivalence. To determine that two projections in the algebra are equivalent is to establish an isomorphism between the corresponding closed linear subspaces of Hilbert space. That can be carried out by means of an operator in the algebra representing a rotation. However, some care must be taken when choosing such an operator. Consider the set of operators in the commutant $\mathcal{M}'$ of $\mathcal{M}$ representing all the rotations of the space which leave the elements of $\mathcal{M}$ fixed. Such unitary operators map each closed linear subspace whose projection is in $\mathcal{M}$ onto, and only onto, itself. To the contrary, as the algebra is non-commutative, the rotations of $\mathcal{M}$ do not commute with all the elements of $\mathcal{M}$; thus, in general, they do not belong to $\mathcal{M}'$. As a consequence, they could well transform a closed linear set in $\mathcal{M}$ into something else. This motivates the following definition: $A, B \in \mathcal{P}(\mathcal{M})$ are *equivalent relative to* $\mathcal{M}$, i.e. $A \sim B$, if there exists an operator $V \in \mathcal{M}$ which maps isometrically from the subspace A onto the subspace B and takes its complement, namely the subspace $A^{\perp}$, into zero. In other words, V is required to be a partial isometry (that is an operator such that $V^*V = A$ and $VV^* = B$).

Provably, the relation $\sim$ is an equivalence relation in $\mathcal{P}(\mathcal{M})$, which Murray and von Neumann called *dimension*. It actually resembles Cantor's notion of cardinality in set theory, although the definition of the latter is more general in that it deals with all sets rather than just with closed linear subsets. Accordingly, one can introduce classes of equivalence, formed by those elements of $\mathcal{P}(\mathcal{M})$ which project to subspaces having equal relative dimension. Let us denote the set of equivalence classes as $\mathcal{P}(\mathcal{M})_{\sim}$. With the aid of $\sim$ one can also define a partial order relation $\preceq$ in $\mathcal{P}(\mathcal{M})$, which plays a crucial role in the classification of von Neumann algebras: $A \preceq B$ if there exists a projection $B' \leq B$ such that $A \sim B'$. This corresponds to the intuition that, relative to $\mathcal{M}$, the dimension of A is not greater than the dimension of B.

At this point a lot of technical complications can be avoided by focusing on the so-called *factors* ("simple rings" in Murray and von Neumann's original terminology). These are the simplest algebraic structures out of which any other von Neumann algebra can be constructed. Accordingly, in order to classify von Neumann algebras, one would just need to specify different types of factors. A factor von Neumann algebra $\mathcal{M}$ is such that its center, namely the intersection of the algebra and its commutant, consists only of complex multiples of the identity, that is $\mathcal{M} \cap \mathcal{M}' = \{\lambda I\}$. Conceptually, this means that factors are so non-commutative that they contain no operator, except for (the complex multiples of) the identity, which commutes with all other elements of the algebra. Then, if $\mathcal{M}$ is a factor, $\mathcal{P}(\mathcal{M})_{\sim}$ turns out to be totally ordered with respect to $\preceq$, that is for any A and B either $A \preceq B$ or $B \preceq A$[5]. It follows that any isomorphism between factors preserves the order

[5]This actually follows from the "Comparison theorem", according to which, given two projections A and B, there is a $Z \in \mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M})''$ such that $ZAZ \preceq ZBZ$ and $(I -$

and the lattice structure, therefore two factors cannot be isomorphic unless the order types of the corresponding $\mathcal{P}(\mathcal{M})_\sim$ are the same.

The classification of von Neumann algebras by factor types is finally completed by defining in a rigorous manner a *dimension function* on the lattice of projections of a factor $\mathcal{M}$. That is a map $d : \mathcal{P}(\mathcal{M}) \longrightarrow [0, \infty]$ (unique up to multiplication by a constant) satisfying the following properties:

(i) $d(A) = 0$ if and only if $A = 0$

(ii) If $A \perp B$, then $d(A + B) = d(A) + d(B)$

(iii) $d(A) < \infty$ if and only if A is a finite projection[6]

(iv) $d(A) = d(B)$ if and only if $A \sim B$

(v) $d(A) \leq d(B)$ if and only if $A \preceq B$

(vi) $d(A \vee B) + d(A \wedge B) = d(A) + d(B)$

$Z)A(I - Z) \preceq (I - Z)B(I - Z)$. The name of the theorem is justified in the sense that any two elements of $\mathcal{P}(\mathcal{M})$ can be cut by a projection commuting with every element of $\mathcal{M}$ into two pieces that one can compare in the ordering $\preceq$. Now, if Z, which belongs to $\mathcal{M}'$, is a complex multiple of the identity operator, $\mathcal{P}(\mathcal{M})_\sim$ is in fact totally ordered with respect to $\preceq$.

[6]A projection is *finite* in the case it is not equivalent to any proper subprojection of itself, i.e. $A \sim B \leq A$ implies $A = B$.

Such a function always exists for a factor. Also, since it is an algebraic invariant, its range proves the same for isomorphic factors. In particular, property (iv) shows that d is constant on the equivalence classes of $\mathcal{P}(\mathcal{M})$, hence it can be regarded as a function on them. The order type of the range of d thus traces the order type of $\mathcal{P}(\mathcal{M})_{\sim}$. Murray and von Neumann then determined all the possible ranges of the dimension functions of factors, identifying five principle types. The following quotation emphasizes the parallelism of Murray and von Neumann's dimension theory and Cantor's theory of alephs.

> One can prove most of the Cantoreal properties of finite and infinite, and, finally, one can prove that given a Hilbert space and a ring in it, a simple ring in it, either all linear sets except the null set are infinite (in which case this concept of alephs gives you nothing new), or else the dimensions, the equivalence classes, behave exactly like numbers and there are two qualitatively different cases. The dimensions either behave like integers, or else they behave like all real numbers. There are two subcases, namely there is either a finite top or there is not. [von Neumann (1954), p.240]

Accordingly, if d takes on integer values, one obtains factors of *type* I: when its range is finite, i.e. $\{1, ..., n\}$, the factors are of type I_n; when it is infinite, i.e. $\{1, ..., \infty\}$, the factors are of type I_∞. Any type I factor von Neumann algebra is isomorphic to Hilbert space quantum mechanics. The algebra $\mathcal{B}(\mathcal{H}_n)$ of bounded operators acting on a finite n-dimensional Hilbert space is an example of the type I_n, whereas the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators acting on infinite dimensional Hilbert space is an example of the type I_∞. However, if d does not take on integer values, there arise von Neumann algebras which do not correspond to any Hilbert space and exhibit quite different properties. If its range is continuous, the factors are of type II: when there is a bounded limit, d maps (up to a suitable renormalization) onto the unit interval $[0, 1]$ of the real line and the factors are of type II_1; otherwise, if the dimensionality is not finite, the factors are of type II_∞. Finally, in the case the dimensionality takes on only zero or infinite values (i.e. d ranges over $\{0, \infty\}$), one obtains configurations of type III.

As the lattice of projections of a von Neumann algebra corresponds to a non-commutative space of events, the dimension function can be associated with the *a priori* probability. The more so because property (vi) of d is tantamount to the strong additivity requirement for a frequentist interpretation of probabilities, which von Neumann insisted on. This offers another way to see how probability assignments in quantum mechanics always yield finite values for $\mathcal{H}_n$, as opposed to the case of infinite-dimensional Hilbert space, where the dimensionality, and thus the *a priori* probability, may well be infinite.

Von Neumann particularly liked the type II_1 case because there is defined a normalized dimension function d despite its lattice containing an infinite number of projections. Hence, probabilities are bound to be finite, as any physical theory would require. Furthermore, the type II_1 factors are well-behaved from the point of view of unbounded operators. Indeed,

> although there are plenty of unbounded operators here, one can show that any finite number of them, in fact any countable number of them, are simultaneously defined on an everywhere dense set; one can prove that one can indulge in operations like adding and multiplying operators and one never gets into any difficulty whatever. The whole symbolic calculus goes through. [von Neumann (1954), p.240]

That is, contrary to the situation in Hilbert space, unbounded operators are not only contained in such an algebraic structure, but they are also defined in a common (everywhere dense) domain, so that they themselves can form an algebra. These reasons motivated von Neumann to regard the II_1 factor as the "proper limit" of quantum mechanics in infinite dimensions.

At the core of the algebraic program there was the study of the lattice of projections of the algebra. Von Neumann in fact realized that focusing on the selfadjoint part of an abstract operator algebra (that is, Jordan algebras theory), in which he was interested at the beginning, was less insightful than the investigation of its lattice-theoretic structure. How this approach was also motivated by logical considerations will be discussed in the next section. It will be shown, furthermore, that the nice features of type II_1 factors, as well as the pathological behaviour of Hilbert space, can be understood from a logico-geometrical perspective too.

2.2 The Logical Interpretation of Quantum Mechanics

2.2.1 The geometrical structure of quantum logic

In 1936 von Neumann, together with Birkhoff, produced a paper titled "The Logic of Quantum Mechanics", which marks the official birth of quantum logic. The purpose of their work was to demonstrate that

> *the propositional calculus of quantum mechanics has the same structure as an abstract projective geometry.* [Birkhoff and von Neumann (1936)]

Accordingly, the basic concepts of quantum logic ought to retain the structural features of a projective geometry, that are defined by means of intersections and unions of the fundamental geometrical elements built up from points, lines and planes. That this is a peculiar aspect of quantum mechanics emerges in contrast to the classical case, where there is a correspondence between logic and set theory.

In classical mechanics the states of a system are represented by points in the phase-space Σ. In particular, for a one point particle, any *pure* state, namely a state providing a maximal piece of information about the system, can be captured by a sequence of six real numbers $\langle r_1, ..., r_6 \rangle$, where the first three numbers correspond to the position-coordinates and last three ones correspond to the momentum-coordinates. Ascribing probabilities to physical events means assigning weights to single points. In fact, the structure of classical events can be described in terms of set theory, so that probabilistic statements are captured by measure-theoretic concepts. A probability function is then defined as a normalized measure over the Boolean algebra of subsets of Σ. However, there exist an infinite number of such classical probability functions that one can introduce over the phase-space. This means, among other things, that in classical mechanics probability is not uniquely determined by logic.

In quantum mechanics the phase-space is replaced by a Hilbert space. The pure states represented by vectors of $\mathcal{H}$ cannot be parametrized by points of Σ. If follows that one is not entitled to predict with certainty the result of an experiment, even when being provided with a complete mathematical description of a physical system. As it turns out, classical set theory is not a suitable framework for quantum logic. The set-theoretical structure is replaced by a geometrical framework. In fact, in a projective geometry quantum-mechanical probabilities are fully defined.

> [A]s soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. This actually is rigorously true and any axiomatic elaboration of the subject bears it out. So in order to have logics you need in this set a projective geometry with a concept of orthogonality in it.
>
> In order to have probability all you need is a concept of all angles, I mean angles other than 90°. Now it is perfectly true that in a geometry, as soon as you can define the right angle, you can define all angles. [von Neumann (1954), p.244]

Once an element of the projective geometry is specified together with its orthogonal complement, that is once a statement of quantum logic is given together with its negation, one can immediately introduce a transition probability. Furthermore, as opposed to the classical case, such a probability function is uniquely determined. Indeed, any automorphism of the logic which leaves orthogonality intact leaves any angle between vectors invariant too.

Yet, classically the probability measures are defined independently of whether one is dealing with a finite or an infinite set. To the contrary, in quantum mechanics there is no obvious way to extend the above procedure to infinite dimensions. As we explain in the rest of the section, von Neumann maintained that the generalization of projective geometries cannot be carried out in infinite-dimensional Hilbert space, but requires a continuous-dimensional structure which proves isomorphic to II_1 factors von Neumann algebra.

Birkhoff and von Neumann's axiomatization began from the recognition that, in any physical model, it makes sense to ask whether a system $\mathcal{S}$ possesses a certain property or not. Pure states allow one to answer precisely, namely by yes or no, to such a question. The logical structure of a physical theory, then, hinges on the notion of experimental propositions about $\mathcal{S}$, which assert that a given physical quantity takes on a certain value. Accordingly, an experimental proposition is associated with the collection of all the pure states for which it holds, that form a subset of the phase-space of the theory.

In classical mechanics the mathematical representatives of experimental propositions amount to the subsets of Σ. The inclusion relation $\subseteq$ between the elements of the power set $\mathcal{P}(\Sigma)$ is the analogue in set theory of the notion of logical implication in classical logic. Moreover, one can define the operations of intersection $\cap$, union $\cup$ and relative complement $-$, which are regarded as the set-theoretical counterparts of the classical logical connectives *and*, *or* and *not*, respectively.

In order to identify the mathematical representatives of the experimental propositions in quantum mechanics, Birkhoff and von Neumann resorted to a characteristic feature of the formalism of the theory which is not shared by the classical case. That is the principle of superposition. According to it, any linear combination of pure states gives rise to a new pure state. Suppose that some pure states ψ_i, where $i = 1, 2, ...$, verify a certain proposition, then their linear combination $\psi = \sum_i c_i \psi_i$ (with complex coefficients $c_i \neq 0$) is a pure state verifying the same proposition. It follows that the mathematical representative of any experimental proposition is required to be closed under finite and infinite linear combinations. Therefore, it corresponds to a closed linear subspace of Hilbert space. In the last analysis, since there is an isomorphism between projections and closed linear subspaces of $\mathcal{H}$, the non-boolean algebra of quantum logic is provided by the projection lattice $\mathcal{P}(\mathcal{H})$.

The basic logical notions and operations between experimental propositions are then defined on this framework. There is a relation of inclusion of closed linear subspaces of Hilbert space that naturally characterizes the logical implication. Also, the intersection of two closed subspaces is again a closed subspace, hence the logical connective *and* is represented by the set theoretical intersection, like in the classical case. Yet, the analogy to classical logic does not hold for the disjunction. In fact, the union of two closed subspaces is not a closed subspace. Instead, one needs to take the supremum of two closed subspaces, that is the smallest closed subspace including both of them, as the representative of the logical connective *or*.

Furthermore, the set theoretical complement of a closed subspace is not a closed subspace of Hilbert space either. Thus, it cannot define the logical negation at all. Birkhoff and von Neumann constructed the negative of a quantum-mechanical experimental proposition as the orthogonal complement of the mathematical representative of the proposition itself. Specifically, the orthogonal complement $X^{\perp}$ of a subspace X of Hilbert space is the set of all vectors which are orthogonal to all the vectors forming X. That is

$$X^{\perp} \equiv \{\psi \in \mathcal{H} | \langle \psi, \phi \rangle = 0, \forall \phi \in X\}$$

So, the concept of orthogonality is defined just in terms of the inner product. As transition probabilities are related to the angles between vectors, this actually illustrates the idea contained in the above quote by von Neumann that probabilities in quantum mechanics are (uniquely) determined by specifying a logical statement and its negation.

The lattice-theoretic characterization of the logic of quantum propositions makes its connection with geometry more rigorous. In fact, one can show that the lattice of all linear subspaces of a (finite dimensional) linear space, say $\mathcal{V}$, constitutes a projective geometry. In particular, the atoms of the lattice, namely the one dimensional subspaces of $\mathcal{V}$, correspond to the points of the projective geometry. Let us recall the definition of a lattice to show how quantum logic ties to noncommutative probability theory.

Lattice theory was first formulated and developed by Birkhoff. The spaces of events of both classical and quantum mechanics can be viewed as two particular cases of the lattice structure. A lattice is a partially ordered set $(\mathcal{L}, \leq)$, namely a set $\mathcal{L}$ in which a partial order $\leq$ is defined. For any two elements A and B in the lattice there exist the "least upper bound" denoted by $A \vee B$ and the "greatest lower bound" denoted by $A \wedge B$. If any subset in $\mathcal{L}$ has a greatest lower bound and a least upper bound, the lattice is said to be complete[7]. An important lattice-theoretic concept is orthocomplementation. That is a map $A \mapsto A^{\perp}$ such that the following conditions hold for every $A \in \mathcal{L}$: (i)$(A^{\perp})^{\perp} = A$, (ii) if $A \leq B$, then $B^{\perp} \leq A^{\perp}$, (iii) $A \vee A^{\perp} = I$, (iv) $A \wedge A^{\perp} = 0$. If $\mathcal{L}$ is equipped with such a map, then the lattice is called

[7]Moreover, a lattice is atomic if for every $B \in \mathcal{L}$ there exists an element A (called atom) such that $A \leq B$ implies $B = A$ or $A = 0$.

orthocomplemented.

A Boolean algebra, namely the lattice of classical propositions, enjoys distributivity in the sense that the equation

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

holds for every element A, B, C. Such a property actually fails for the lattice of projections on Hilbert space, in case the latter is at least two dimensional. Thus, the lattice of quantum logic is not distributive. Instead, it can be proved that $\mathcal{P}(\mathcal{H})$ satisfies only a weaker condition, namely orthomodularity[8], which is expressed by the relation

$$\text{if } A \leq B \text{ and } A^{\perp} \leq C \text{ then } A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

However, if Hilbert space is finite-dimensional, the following property called *modularity*, which is weaker than distributivity but stronger than orthomodularity, holds for any projection of the lattice

$$\text{if } A \leq B \text{ then } A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Although they were fully aware that modularity is violated by the projections of infinite-dimensional Hilbert space, Birkhoff and von Neumann regarded it

[8]According to Kalmbach (1983), it was K. Husimi that first formulated the notion of orthomodularity in 1937; anyway, it is not clear whether such property of the Hilbert space was actually known to von Neumann.

as an essential condition for the logic of quantum propositions. There is a two-fold justification for insisting on such an assumption.

On the one hand, it has a methodological underpinning in the axiomatic method. Indeed, Birkhoff and von Neumann explicitly invoked Hankel's principle of "perseverance of formal laws" that applies to limiting procedures. Accordingly, an appropriate axiomatization of a formal system would require that crucial properties which are present in finite dimensions are preserved when taking the limit. Since modularity is satisfied by any projection acting on $\mathcal{H}_n$, it seems reasonable to demand that the projection lattice ought to be modular in infinite dimensions too. As a result, Hilbert space proves unsuitable for the purpose of constructing quantum logic.

On the other hand, from a structural point of view, modularity is advocated by Birkhoff and von Neumann because its presence "is closely related to the existence of an 'a priori thermo-dynamic weight of states' " [Birkhoff and von Neumann (1936)]. To explain this fact, we need to introduce the dimension function of a lattice. That is a map $d : \mathcal{L} \longrightarrow [0, \infty]$ satisfying the properties:

(i) $d(A) < d(B)$ if $A < B$

(ii) $d(A \vee B) + d(A \wedge B) = d(A) + d(B)$

Provably, the lattice $\mathcal{L}$ is modular only if $d < \infty$. As in the theory of dimensionality of von Neumann algebras, the dimension function is identified with the *a priori* probability. Therefore, the failure of modularity implies that the latter cannot be finite, thus confirming the pathological behaviour of probabilities in infinite-dimensional Hilbert space quantum mechanics.

Footnote 23 of the "Logics of Quantum Mechanics" hints at a continuous-dimensional model as a better candidate than Hilbert space to extend the propositional calculus of the theory in infinite dimensions. The sought-after generalization of projective geometries was achieved by von Neumann (1936) in the framework of continuous geometries, which are (complete) ortho-complemented lattices satisfying modularity[9]. A continuous dimensionality is actually constructed on these structures by means of an equivalence relation, in analogy to the work on the "Rings of Operators".

Given any $A \in \mathcal{L}$, its complement A^c is an element of the lattice such that $A \vee A^c = 0$ and $A \wedge A^c = I$. Then, two elements of $\mathcal{L}$ are called perspective just in case they have the same complement. It was von Neumann's exceptional mathematical achievement to demonstrate that the relation of perspectivity is not only reflexive and symmetric, but also transitive. The geometrical classes of equivalence thus obtained in the lattice behave just like the real

[9]Such lattices are also continuous in the sense that, given a subset $\mathcal{S}$ of $\mathcal{L}$, for every $A \in \mathcal{L}$, (i) $A \wedge (\bigvee_{S \in \mathcal{S}} S) = \bigvee_{S \in \mathcal{S}} (A \wedge S)$, (ii) $A \vee (\bigwedge_{S \in \mathcal{S}} S) = \bigwedge_{S \in \mathcal{S}} (A \vee S)$

numbers. The correspondence with factors is enforced by the requirement of irreducibility. $\mathcal{L}$ is said to be irreducible if its center, namely the set of elements having a unique complement in $\mathcal{L}$, consists only of 0 and 1. Any irreducible continuous geometry is thus a factor and determines a unique dimension function d ranging over the interval $[0, 1]$ of the real line. Finally, continuous geometries can be proven to be all isomorphic to the projection lattices of type II_1 factors, as well as of type I_n factors. Remarkably, no appeal to Hilbert space is made at all. The extension of quantum logic to infinite dimensions is derived from purely geometrical considerations.

2.2.2 Strict- and Probability- Logic

From the point of view of operator theory, the abandonment of the Hilbert space formalism in favor of a theory of finite dimensionality corresponded to a switch from the "vectorial-spatial" approach to the algebraic approach. The following passage, taken from a letter that von Neumann wrote to Birkhoff during the period in which they were working together on the paper on quantum logic, well summaries the development of his attitude in the mathematical foundations of quantum theory.

> I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After

> all Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of "conserving the validity of formal rules". This is very clear, if you consider the axiomatic-geometric definition of Hilbert-space... Thus Hilbert-space is the straightforward generalization of Euclidean space, in one considers the *vectors* as the essential notions.
>
> Now we begin to believe that it is not the *vectors* which matter, but the lattice all linear (closed) subspaces.
>
> ... But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions, then one does not obtain Hilbert space, but that configuration, which Murray and I called "case II_1". [von Neumann (1935), Letter to Birkhoff, in Redei (2005)]

Under this approach, physical states are no more represented by vectors of Hilbert space. Vector states now reduce to derived quantities. The primitives of quantum theory are instead the closed linear subspaces of a linear vector space (not necessarily $\mathcal{H}$). They are the mathematical representatives of what von Neumann called, with somewhat odd terminology, "physical qualities". If the space is n-dimensional, a state is a point in the corresponding $n-$

1-dimensional projective geometry. A physical quality would then be any subset of the latter. As von Neumann put it, the "physically significant" physical qualities are those which can be associated with a linear subspace of the projective geometry; all the others are just "hypothetical", since no experiment could adequately describe them.

Although this suggests how the concept of physical qualities would be formalized, it does not clarify in what sense they should be the primitive, "phenomenological given", notion of quantum theory. His correspondence with Birkhoff supplies some details of what an operational definition of the concept may look like. Accordingly, a physical quality S is described by a two-stage procedure, comprising the measurement of a certain physical quantity, represented by a self-adjoint operator, and a subsequent computation determining either one of a pair of values, say "yes" and "no". An individual physical system Γ possesses the quality S just in case performing the above procedure would yield a positive result with certainty. In addition, Γ should not be affected by the experiment, that is its state should not change. Von Neumann refrained from characterizing the latter requirement in terms of entropy-free measurements "because all matters concerning entropy are somewhat controversial" (November 6, 1935, p.53). It is true, however, that quantum measurements which leave the state of the system unchanged are quite special circumstances. As a consequence, it does not seem that his definition can be carried out in the general case at all. The lack of generality is hardly acceptable if physical qualities ought to be the primitives of quantum theory.

Be that as it may, it should be clear at this point that the emphasis on linear closed subspaces, rather than on the vectors of Hilbert space, as the fundamental concepts of quantum mechanics had a logico-geometrical, as well as an algebraic, underpinning[10]. There remains now to show how the program of deriving the probabilistic structure of quantum theory from continuous geometries was completed. The document that expresses most compellingly von Neumann's logical interpretation of quantum probabilities is an unfinished manuscript titled "Quantum Logics (Strict- and Probability-Logics)" that he wrote about 1937.

The aim of this work was to provide a mathematical description of the physical world by spelling out the general logical structure which underlies various physical theories. In analogy to the logic of quantum mechanics, the logical system $\mathcal{L}$ is restricted to the propositional calculus and has a direct

[10]Actually von Neumann gave a definition of physical qualities in terms of classes of equivalence of experimental propositions too. By means of the partial order $\leq$ one determines an equivalence relation $\sim$ on the lattice $\mathcal{L}$, whereby $a \sim b$ if and only if $a \leq b$ and $b \leq a$. Logically equivalent projections correspond to the same experimental proposition, and hence represent the same physical quality.

appeal to the experimental ground.

> Let $\mathcal{S}$ be a physical system, or rather the mathematical model of a physical system, to which we wish to apply logics (sic). The system $\mathcal{L}$ of logics is then the set of all statements $a, b, c, ...$ which can be made concerning $\mathcal{S}$. Such a statement is always one concerning the outcome of a certain measurement, which is to be performed on $\mathcal{S}$. [von Neumann (1937)]

Two fundamental logical notions are defined on the lattice $\mathcal{L}$: the relation of *implication* $\leq$ and the operation of *negation* $\neg$. Specifically, $a \leq b$ means that, in the case that measuring a on $\mathcal{S}$ has shown a to be true, then measuring b on $\mathcal{S}$ immediately after a will show b to be true with absolute certainty. The experimental proposition $\neg a$, instead, describes the same measurement as a, where the negation obtains if the result of the measurement yields "no" rather than "yes". One can then derive conjunction and disjuction by taking the multiplication ab and the addition $a + b$ as the greatest lower bound and the least upper bound of a and b, respectively. Von Neumann uses the term *strict logic[s]* to refer to a lattice $\mathcal{L}$ with this structure.

On the other hand, physical theories deal with probabilistic statements about the outcomes of measurements. So, applying $\mathcal{L}$ to physical reality demands a further constraint, concerning probability, which enriches the structure of $\mathcal{L}$. In fact, for any reasonable mathematical model of the system $\mathcal{S}$ (or, in von Neumann's words, "for any well defined state of our knowledge concerning the mathematical description of physical reality") a probability function $p(a, b)$ does exist. Given a real number $\theta \in [0, 1]$, the transition probability $p(a, b) = \theta$ has the following meaning: if measuring a on $\mathcal{S}$ has shown a to be true, then measuring b on $\mathcal{S}$ immediately after a will show b to be true with probability equal to θ. Von Neumann called *probability logic[s]* the structure that $\mathcal{L}$ acquires when being equipped with the real number valued function $p(a, b)$.

Interestingly, the impossibility of regarding probability statements of the form $p(a, b) = \theta$ as frequency statements led von Neumann to give up the frequency interpretation of probabilities, as the following quotation shows.

> This view, the so-called "*frequency theory of probability*" has been brilliantly upheld and expounded by R. V. Mises. This view, however, is not acceptable to us, at least not in the present "logical" context. [von Neumann (1937)]

According to the frequentist account, one considers a sequence of identical copies of a physical system $\mathcal{S}$, that is an ensemble $\mathcal{E} = \{s_1, s_2, ..., s_N\}$, with N being a large number. Measuring the quantity a on each copy of the system collects those M elements of $\mathcal{E}$ in which a are true. The procedure of statistical inference is then completed by measuring the quantity b immediately after a, which selects the sub-ensemble of $\mathcal{E}$ composed by the M' copies of $\mathcal{S}$ in which a and b are both found true. As a result, the statement $p(a, b) = \theta$ could be interpreted in terms of relative frequencies only by stipulating that the limit of the ratio of M' and M equals the number $\theta \in [0, 1]$ when N tends to infinity. Yet, von Neumann objected that, from a strict mathematical point of view, the expression $\frac{M'}{M} \longrightarrow \theta$ is not a well-defined convergent-statement. Indeed, requiring the limit of relative frequencies to exist for $N \to \infty$ makes sense only if one is dealing with infinite ensembles. Accordingly, one would extend the physical terminology to the ideal situation involving an *infinite* sequence of systems; however, as he observed, "we are not prepared to carry out such an extension at this stage". In the last analysis, von Neumann gave up the frequency interpretation of quantum probabilities due to its inconsistency with the logical (lattice-theoretic) approach to quantum mechanics[11]. As the previous argument suggests though, this did not imply the abandonment of the ensemble characterization of the theory.

[11]Redei (2003) pointed out another inconsistency between the frequentist and the logical account lying in the fact that it is not clear how von Neumann could interpret the experimental proposition $A \wedge B$ expressing the joint occurrance of two events in terms of relative frequencies in the case A and B are not simultaneously decidable.

One can demonstrate that the system of strict logic is a part of the system of probability logic. In fact, for $0 \leq \theta \leq 1$, the implication and the negation can be defined by means of the transition probability function. If a is true, by taking the relations $p(a, b) = 1$ and $a \leq b$ as equivalent, the fact that b is true with probability one is tantamount to b being true with absolute certainty. On the other hand, one can indirectly define the negation from the equivalence of the relations $p(a, b) = 0$ and $a \leq \neg b$: for any $u \neq 0$, $\neg a$ is the unique element c of $\mathcal{L}$ such that $p(u, c) = 1$ (i.e. $u \leq c$) implies $p(u, \neg a) = 0$ (i.e. $u \leq \neg a$) and vice versa.

The converse, however, is not true. When θ takes on the value 0 or the value 1, the above equivalence relations allow one to describe $p(a, b) = \theta$ by means of strict logic. Nevertheless, for $0 < \theta < 1$, this cannot be done at all. For let us recall that statements of the form $p(a, b) = \theta$ are ill-defined convergence-statements. They would make sense only as approximations of limiting statements involving ensembles of infinite length. Carrying out such an approximation via Bernoulli's law of large numbers employes a probabilistic argument. Hence logical terms cannot suffice to define $p(a, b)$. This led von Neumann to the conclusion that:

> *Probability logics cannot be reduced to strict logics, but constitute an essentially wider system than the latter, and statements of the*

> *form* $p(a,b) = \theta$ $(0 \leq \theta \leq 1)$ *are perfectly new and* sui generis *aspects of physical reality.*
>
> So probability logics appear as an essential extension of strict logic. This view, the so-called "logical theory of probability" is the foundation of J. N. Keynes's work on this subject. [von Neumann (1937)]

Quantum mechanics is a special case of the logic of a physical theory. The remarkable fact about it, which is at the bottom of his logical interpretation of quantum probabilities, is that in the lattice of quantum propositions the probability logic does reduce to the strict logic. This was rigorously demonstrated by von Neumann in the axiomatization of "Continuous geometries with transition probability" (1936). The purpose of this manuscript was to constrain an abstract system $\mathcal{L}$ to describe the structure of a physical model retaining the features of quantum theories.

In the first chapter von Neumann formulated 13 axioms and provided each of them with a phenomenological interpretation. *Axioms I-V* require $\mathcal{L}$ to be an (ortho-)complemented modular lattice, with $\leq$ being the partial ordering and $a^{\perp}$ being the orthogonal complement of $a \in \mathcal{L}$, so that its elements $a, b, c, ...$ are associated with physical events. *Axioms VI-VIII* define a "transition probability from a to b" as a map $p(a,b)$ on $\mathcal{L}$, where $a \neq 0$.

Then, *Axiom IX* and *Axiom X* reflect the conditions of completeness and continuity[12] respectively, whereas *Axiom XI* requires irreducibility. That gives $\mathcal{L}$ the structure of a continuous geometry with a dimension function d. Finally, *Axiom XII* fixes the transition probability function under any $(\leq, \perp)$-automorphism T of $\mathcal{L}$ and *Axiom XIII* guarantees the existence of such $(\leq, \perp)$-automorphisms of the lattice. This means that the transition probability is given by the relation $p(a, b) = p(Ta, Tb)$ for all the transformations T of $\mathcal{L}$ onto itself that leaves the fundamental lattice operations invariant. It follows (*Theorem XIII*) that, whenever $a \geq b$, the transition probability reads

$$p(a, b) = \frac{d(b)}{d(a)}$$

which, since $0 \leq d(b) \leq d(a) \leq 1$, yields a numerical value θ comprised in the interval $[0, 1]$. Any structure fulfilling this axiomatics is provably isomorphic to finite von Neumann algebras. Thus, continuous geometries with transition probability can only be type II_1 factors.

The last two axioms characterizing continuous geometries with transition probability capture the content of von Neumann's logical interpretation of quantum theories. They indeed show how the probabilities of the theory are uniquely determined by the (geometrical) symmetries of the logic of quantum

[12]It was stressed that *Axiom X* is not necessary, since by Kaplansky's theorem one can show that continuity comes from *Axioms I-IX* along with *Axiom XIII*. For a critical review of von Neumann's axiomatization see Halperin (1961).

propositions. This is, of course, a fundamental difference with respect to other physical theories, in particular classical mechanics. In fact, differently from the classical case, there exist isomorphisms of $\mathcal{L}$ onto itself which leave the strict logic of quantum mechanics invariant. Since transition probabilities are also fixed under such mappings, probability logic and strict logic actually arise together.

Von Neumann also proved that the dimension function $d(a) = p(1, a)$, namely the *a priori* probability, on the lattice of a continuous geometry naturally extends to a unique trace τ on the algebra. The derivation of the type II_1 factor as the proper geometrical structure of events with a well-behaved quantum-mechanical probability function was finally achieved. A letter to Stone shows von Neumann's deep excitement at such a discovery.

> Recently, I could prove, that every abstract algebra in which a trace can be defined (...) is isomorphic to a suitable ring $\mathcal{M}$ of bounded operators in a Hilbert space $\mathcal{H}$; and that this trace is unique if and only if $\mathcal{M}$ is a factor of class (I_n) and (II_1) (...). The notion still haunts me, that this may be applicable to quantum theory: After all this means that one can enumerate all abstract algebras which contain a uniquely determined probability-theory - and while the complete matrix-rings of finite-dimensional Eu-

clidean spaces are naturally such cases (I_n), the limiting case for $n \to \infty$ is *not* (I_∞), *but* (II_1)!

What do you think about these things? I am very agreeably surprised, because the pathology of unbounded operators did really not look so, that one could expect such a *dénoument*- if this is a *dénoument*! [Von Neumann, Letter to Stone (1935), published in Redei (2005)]

2.2.3 On the type II_1 factor von Neumann algebra

In this section I review the behaviour of the type II_1 factors as a mathematical arena for quantum mechanics and contrast it to Hilbert space in both finite and infinite dimensions. The synoptic table below resumes some of the analogies and the differences between factors of type I_n, type I_∞ and type II_1 with respect to key features characterizing von Neumann's logical interpretation of quantum probabilities. The analysis is made in light of some latest developments of the theory of von Neumann algebras.

Type of factor von Neumann algebra	I_n	I_∞	II_1
Cardinality	$< \aleph_0$	$\aleph_0$	$\aleph_1$
Range of the dimension function	$\{1, ..., n\}$	$\{1, ..., \infty\}$	$[0, 1]$
Example	$\mathcal{B}(\mathcal{H}_n)$	$\mathcal{B}(\mathcal{H})$	$\bigotimes_n \mathcal{M}_2$
Projection lattice	$\mathcal{P}(\mathcal{H}_n)$	$\mathcal{P}(\mathcal{H})$	$\mathcal{P}(\mathcal{M})$
Modularity	$\surd$	$\times$	$\surd$
Orthocomplementation	$\surd$	$\times$	$\surd$
Atomicity	$\surd$	$\surd$	$\times$
Continuous geometry	$\surd$	$\times$	$\surd$
***A priori* probability**	$\frac{1}{n}Tr$	Tr	τ
Finiteness	$\surd$	$\times$	$\surd$
Uniformity	$\surd$	$\surd$	$\times$
Unitary invariance	$\surd$	$\surd$	$\surd$
Strong additivity	$\surd$	$\times$	$\surd$
Noncommutative conditional expectation	$\surd$	$\times$	$\surd$

The property of unitary invariance for the probability measures and the (non-)existence of non-commutative conditional expectations will be discussed in section 3.3., when addressing the Bayesian interpretation of quantum mechanics. I already surveyed some other points appearing on the table, such as the problem of finiteness of *a priori* probability on a factor and its dependence on the dimension function of its projection lattice. Let us notice here that the *a priori* probability on $\mathcal{P}(\mathcal{H}_n)$ is defined as $d(A) = \frac{1}{n}Tr(A)$ for every projection A. One can normalize the trace by the factor $\frac{1}{n}$ exactly because the dimension function d is finite. The functional Tr is still defined in the case of general Hilbert space $\mathcal{H}$; yet, it cannot be normalized. To the contrary, a unique finite trace, and thus a tracial state τ, exists on a type II_1 factor.

A property von Neumann particularly insisted on is strong additivity of probabilities. In general, it requires that each (normal) state ϕ on $\mathcal{P}(\mathcal{M})$ should satisfy the relation $\phi(A \vee B) + \phi(A \wedge B) = \phi(A) + \phi(B)$ for all $A, B \in \mathcal{M}$. Petz and Zemanek (1988) proved that ϕ enjoys strong additivity if and only if it is a tracial state. Thus, states on factors of type I_n and type II_1 naturally retain strong additivity. An explicit example of its failure in the case a state is defined on the lattice of projections on infinite-dimensional Hilbert space is offered by Szabo (2003).

A lattice theoretical property that the type II_1 does not share with the type I_n (and with the type I_∞) is atomicity. Whereas the lattice of projections on Hilbert space is atomic, there are no atoms in $\mathcal{P}(\mathcal{M})$ if $\mathcal{M}$ is of type different than I. A projection $A \neq 0$ is an atom just in case there is no non-zero projection A' strictly less than A. Now, for any A belonging to the lattice of a type II_1 factor, one can always consider another element B of the lattice such that $d(B) = kd(A)$, where $0 < k < 1$. So, there exists a non-zero projection A' strictly less than A which is equivalent to B, namely such that $d(A') = d(B) = k$. In other words, there cannot be any smallest (non-zero) dimensional projection in the lattice. The loss of atomicity in the type II_1 factor is the consequence of appealing to a continuous dimensionality, that is a finite probability on a lattice comprising infinite projections.

Actually, requiring a finite trace over a non-finite lattice of projections does also prevent the *a priori* probability from being uniform. Indeed, if all elements of an infinite set of disjoint events were assigned the same positive probability value, their sum would not be finite, and hence the probability over the entire set could not be normalized. Hence, von Neumann's requirement of finite physical probabilities is salvaged at the prize of giving up uniformity.

These differences between the type I_n and the type II_1 did not seem to bother von Neumann. In fact, one should expect that not all the properties holding in finite dimensions are maintained by the limiting procedure. Such properties, however, should not be crucial for the foundations of the theory. In particular, von Neumann did not regard atomicity as an indispensable property that the axiomatization of quantum mechanics should preserve under the limit in infinite dimensions. That the lattice of a physical theory does not need to contain atoms follows already from classical mechanics. Specifically, requiring propositions about a classical system to be atomic

> ... is easily seen to be unrealistic; for example, how absurd it would be to call an "experimental proposition", the assertion that the angular momentum (in radians per second) of the earth around the sun was at a particular instant a rational number!
>
> Actually, at least in statistics, it seems best to assume that it is the *Lebesgue-measurable* subsets of a phase-space which correspond to experimental propositions, two subsets being identified, if their difference has *Lebesgue-measure* 0. [Birkhoff and von Neumann (1936)]

This means that experimental propositions are not strictly represented by Borel sets. They are instead represented by equivalence classes of sets having the same Lebesgue measure, that restricts the number of "unrealistic" elements which are contained in the class of all Borel sets. Importantly, the lattice of such classes of equivalent sets are nonatomic. Incidentally, this is the sense in which the *a priori* probability measure on a type II_1 factor really is the noncommutative generalization of the Lebesgue measure on classical spaces of events.

Interestingly, an example of type II_1 factor, which was constructed by Birkhoff von Neumann (1936), is the hyperfinite von Neumann algebra $\bigotimes_n \mathcal{M}_2$, that is the n-fold (infinite) product of the matrix algebra $\mathcal{M}_2$ of 2-by-2 two complex matrices by itself[13]. Such a structure describes the structure of a lattice gas in the state of infinite temperature. Hence, examples of von Neumann algebras of type II_1 have been later on demonstrated to have physical significance. This actually provides a counter-example to Jauch (1968), who maintained, against von Neumann, that the lattice of events of a physical theory needs to contain atoms, and thereby could not be a continuous geometry.

2.3 The Unsolved Problem

2.3.1 Information, geometry and "pseudo-problems"

The reconstruction of von Neumann's work on the mathematical foundations of the theory that I offered so far highlights the connection between his logical interpretation of quantum probabilities and Hilbert's Sixth Problem. On the one hand, type II_1 factors von Neumann algebras are obtained by the axiomatization of those physical systems which are isomorphic to quantum

[13]See section XIV.1 in Takesaki (2003).

mechanics. On the other hand, they retain the structure of continuous geometries with transition probability. Yet, at the end of his 1954 address, von Neumann expressed his discontent toward mathematical models of this sort. Apparently, he regarded the question of what algebraic structure should constitute the proper arena of quantum theories still as an unsolved problem in mathematical physics.

> All the existing axiomatisations of this system are unsatisfactory in this sense, that they bring in quite arbitrarily algebraic laws which are not clearly related to anything that one believes to be true or that one has observed in quantum theory to be true. So, while one has very satisfactorily formalistic foundations of projective geometry of some infinite generalizations of it, of generalizations of it including orthogonality, including angles, none of them are derived from intuitively plausible first principles in the manner in which axiomatizations in other areas are.
>
> Now I think that in this point lies a very important complex of open problems, about which one does not know well how to formulate them now, but which are likely to give logics and the whole dependent system of probability a new slam. [von Neumann (1954), p.245]

Understood in terms of von Neumann's axiomatic method in physics, the passage suggests that the formal aspect of the axiomatization of the quantum system of logic, geometry and probability, although it was fully developed in infinite dimensions, was not satisfactorily connected to empirically grounded physical axioms. The mathematical problem which von Neumann left unsolved then was to derive the algebraic framework in such a way to derive a complete axiomatics from purely quantum-mechanical facts.

During the almost twenty years going from the work on "Continuous Geometries with Transition Probability" and the talk at the Congress of International Mathematicians, Von Neumann did not succeed in constructing the sought-after unified theory of logic and probability. On more than one occasion though, he claimed that the implementation of such continuous dimensional (quasi-)quantum mechanical systems could have been achieved and that only contingent hindrances, such as his appointments in US intelligence in the Second World war, prevented him from accomplishing the project.

In this section we outline possible directions in which von Neumann might have intended to complete the axiomatization of his unified theory of logic, probability and quantum mechanics. A hint is contained in a manuscript displaying the plan of his lecture at the International Congress of Mathematicians. The last three points, which he did not actually manage to address in the talk, reveal that he regarded information as a central concept in the logical interpretation of quantum probabilities.

> *22. 4m*[14] Integration of logics and probability theory. "Quantitative" aspects of logic. Simplest case: Information theory.
>
> *23. 3m* Classical: Measure-volume estimates. Quantum: What is it then?
>
> *24. 5m* Inherent value of this integration. A "thermodynamical" theory of logics-probability. Its value as an heuristic guide in mathematics, etc. What will the role of the "quantum" approach be? [von Neumann (1954), in Redei and Stöltzer (2001), p.248]

By "quantitative" aspect of logic he seems to mean logic with quantifiers. In fact, the axiomatization of continuous geometries with transition probability was based on the propositional calculus. That is, the elements of the projection lattice of type II_1 factors are associated with elementary experimental propositions. A logical interpretation of quantum theory would then be completed by including a transcendental calculus of quantified statements. Arguably, von Neumann had in mind extending the logical characterization of physical theories to both strict- and probability logics with quantifiers[15].

[14] The numbers followed by m indicates the number of minutes he intended to dedicate to each point.

[15] As, von Neumann (1937) put it,

Information theory was the simplest formulation of such quantified logics. Yet, what the meaning of his alleged "thermodynamical" theory integrating logic and probability ought to be, and what role information would then play in it, is less easy to guess.

In his earlier work on the mathematical foundations of quantum mechanics he introduced an entropy function based on a gedanken experiment formulated on the ground of thermodynamics. Given any statistical operator $W = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, the so-called von Neumann entropy is expressed by the formula $S(W) = -k \sum_i \lambda_i \log \lambda_i$, where k is an additive constant. Whether this quantity is really equivalent to thermodynamical entropy is still subject to debate (see Petz (2001) and Shenker and Hemmo (2006)). In any case, although he never made such a connection explicitly, von Neumann entropy is commonly interpreted as a measure of the amount of information encapsulated in a quantum state. In fact, it generalizes the classical Shannon entropy in that it reduces to the latter in the particular case in

> Once the propositional calculus has been well established, the next task will be to extend this *elementary* system of logics (i.e. the propositional calculus) to a complete *transcendental* one (i.e. one with quantifiers)

Unfortunately, it did not expand this passage, nor there is actual evidence that he ever attempted the alleged further step in the axiomatization of quantum logics.

which the vector-states $|\psi_i\rangle$ are orthogonal[16]. $S(W)$ definitely increases if a state change occurs due to a measurement (and remains constant otherwise). Accordingly, measurements in quantum mechanics are typically irreversible processes. This also suggests a patent analogy to the second law of thermodynamics. Be that as it may, the huge development of quantum information theory in the last fifty years certainly demonstrates how keen von Neumann's insight on mathematical physics was.

That information-theoretical considerations were relevant to his interpretation of quantum theory may be inferred from a letter von Neumann wrote to Schrödinger in 1936 too. Schrödinger worried that the instantaneous change affecting the state of a quantum system when operating on another spatially separated quantum system entangled with it would infringe on the relativistic constraint of locality. In his letter, von Neumann argued that similar statistical correlations between spatially separated systems occur in classical mechanics as well, thus no special sort of *action at distance* is really involved in the quantum case. As Redei (2005) pointed out, the underlying assumption is that probabilities are degrees of ignorance. In other words, changes of quantum states are not to be interpreted as physical changes, but are, instead, changes of information. Von Neumann could then conclude that the

[16]Interestingly, Shannon himself acknowledged von Neumann for the suggestion of calling entropy the information measure function which is named after him.

alleged incompatibility between quantum mechanics and special relativity is just a "pseudo-problem".

This terminology, as well as its connections to von Neumann's conception of geometry, is explained in a letter to Ortvay (March 29, 1939 ?). The spatial description of processes connected to life, he wrote, can hold only approximatively. Indeed, the geometrical localization of physical bodies is possible only within certain limits. "Pseudo-problems" arise whenever one tries to extend the spatial approximation beyond such limits. Examples of "pseudo-problems" are the simultaneity of distant events in special relativity or the impossibility of measuring position and momentum at the same time in quantum mechanics. Some non-spatial description is therefore required in order to account for non-localized processes. The case of the observer in quantum mechanics illustrates this idea.

> I have thought a great deal since last year about the nature of "observer" in quantum mechanics. This is a kind of quasi-psychological, auxiliary concept. I think I know how to describe it in an abstract manner divested from its pseudo-psychological complications, and this description gives a quite worthwhile insight regarding how it might be possible to describe intellectual processes (therefore ones essentially connected to life) in a non-

> geometrical manner (without locating them spatially). [von Neumann (1939), in Redei (2005), p.201]

The observer is thought of as a non-geometrical concept, whose existence in space is quite irrelevant to quantum mechanics. To avoid pseudo-problems arising, the intellectual processes ascribed to the observer should not be described in spatial terms. This does not imply that they must be described in psychological terms though. Some other description is in fact available. Von Neumann did not elaborate on this point further, except for suggesting that transformation theory in quantum mechanics would provide the framework for such a non-spatial description. Recall that quantum probabilities are defined by invariance under the transformations of the logical structure onto itself. Arguably, information theory offers a quantitative, non-psychological account of intellectual processes of the observer, such as computation, that integrates logic and probability.

The separation between spatial and non-spatial description bears on the role of geometry in physics, and hence on Hilbert's Sixth Problem. One can in fact distinguish two senses of geometry coexisting in von Neumann's program. On the one hand, he refers to geometry as describing space and the objects located in it. Physical processes taking place in space are in fact in need of a (spatial) geometrical account. On the other hand, he appealed to (non-spatial) geometrical structures serving as a framework for the axiomatization of quantum mechanics. Accordingly, projective and continuous geometries are mere devices for computation, which have nothing to do with spatial considerations.

This remark fits well with Redei's (2005) analysis of von Neumann's work on quantum field theory. There are two distinct problems of infinity arising with the quantum treatment of electromagnetic fields: they diverge in proximity of electric charges and they can be characterized only by infinite number of degrees of freedom. The former is a spatial problem, which von Neumann solved by quantizing space-time: singularities do not appear in a discrete space as one cannot get arbitrarily close to point-like charges (see letter to Dirac, January 27, 1934). The latter is a problem connected with the mathematical description of the physical systems under investigation, which does not depend on their spatial location at all. Its solution therefore is just a matter of refining the computational methods required to treat quantum-mechanical systems having infinite degrees of freedom. A systematic survey of von Neumann's ideas concerning the formalization of quantum electrodynamics has not been done yet. On more than one occasion, he suggested that his algebraic formulation of quantum mechanics could be extended to the relativistic case too. As he wrote to Jordan,

> These continuous- but finite dimensional quantum mechanics can be brought into a quite amusing relation with "second quantization". In view of the recent crunch in quantum mechanics of elementary particles this possibility merits perhaps some more attention. [von Neumann, (December 11, 1949), in Redei (2005), p.151-152].

It remains to be demonstrated, though, whether any of the models for quantum fields that he sketched would ultimately lead one to the factors of type II_1.

We shall see in the last chapter that, contrary to von Neumann's expectations, a complete algebraic axiomatization of quantum field theory was later on given within the framework type III factors. The primitive notion of Algebraic Quantum Field Theory is the net of observable algebras $\mathcal{A}(\mathcal{O})$ associated with any region $\mathcal{O}$ of Minkowski space. In this regard, it is interesting to notice that the philosophical literature on the subject has revived the issue of whether the Hilbert space formulation or the algebraic formulation is more fundamental. The non-uniqueness of representations of the CCRs for infinitely many degrees of freedom has a counterpart in the existence of inequivalent GNS representations $(\pi, \mathcal{H})$ of the relevant algebras in the Hilbert space[17]. Indeed, the net of algebras $\mathcal{A}(\mathcal{O})$ can be embedded in $\mathcal{H}$ in many different ways. Ruetsche (2002) distinguishes two attitudes one may take here, depending on whether any representation of the net of algebras in Hilbert space is supposed to have ontological significance or not. According to Algebraic Imperialism, the physical content of a quantum field theory is fully encoded in the net of algebras. Hence, any representation $(\pi, \mathcal{H})$ may be at most an aid to calculation. Hilbert space Conservatorism, instead, argues that the algebraic structure is not rich enough for the task. In order to capture the physical content of a quantum field theory, one needs to specify a particular representation of $\mathcal{A}(\mathcal{O})$ in $\mathcal{H}$, which thus ought to play a privileged ontological role. Hilbert space would then be restored as a primitive of the theory.

2.3.2 Can the type II_1 factors support any representation of the CCRs?

In the previous sections we discussed the reasons why von Neumann was still unsatisfied with his latest work in quantum foundations. Other criticisms have been raised against him. Just the fact that quantum physics never really departed from Hilbert space would discourage one to take his proposal

[17]See section 2.1.1. for the notion of GNS representations of an algebra in the Hilbert space.

seriously. Yet, the problems with the generalization of quantum mechanics in infinite dimensions that he entertained are genuine conceptual worries. Thus, one may wonder whether factors von Neumann algebras of type II_1 could, at least in principle, represent a mathematical arena for quantum theory.

We now present a result by Huzimiro Araki which establishes that no algebra with a finite trace on it can support any (regular) representation of the Canonical Commutation Relations[18]. This implies, in particular, that type II_1 factors would be unsuitable for quantum-mechanical purposes, thus proving von Neumann wrong. In reporting Araki's proof below we remain as faithful as possible to his original hand-written note, although we add some brief explanation connecting the technical passages.

ARAKI'S PROOF

The proof begins by recasting the CCRs in terms of the Weyl operators $U(a) = e^{iaX}$ and $V(b) = e^{ibP}$, representing X and P respectively. Accordingly, $[X, P] = iI$ becomes

$$U(a)V(b) = e^{-iab}V(b)U(a) \tag{2.5}$$

To recover the CCRs, however, one should require $U(a)$ and $V(b)$

[18]The content of such a result is first published in Valente (2008) with the relevant discussion.

to be differentiable on a dense domain, and hence at least continuous in a and b with respect to the strong operator topology. For unitary operators, this is tantamount to requiring continuity with respect to the weak operator topology.

Suppose that Φ is a tracial vector. Accordingly, one obtains

$$\langle U(a)V(b)\Phi, \Phi\rangle = \langle V(b)U(a)\Phi, \Phi\rangle = e^{iab}\langle U(a)V(b)\Phi, \Phi\rangle \quad (2.6)$$

Comparing the left-hand side and the right-hand side of the previous equality yields

$$\langle U(a)V(b)\Phi, \Phi\rangle = 0 \quad (2.7)$$

for $0 < ab < 2\pi$. Weak continuity then implies

$$\lim_{a\to 0}\lim_{b\to 0}\langle U(a)V(b)\Phi, \Phi\rangle = \langle \Phi, \Phi\rangle = 0 \quad (2.8)$$

which finally shows that the tracial vector Φ must be null!

The upshot of the proof is that representations of the Canonical Commutation Relations expressed by the Weyl operators $U(a)$ and $V(b)$ can be constructed on a given algebra just in case $\Phi = 0$. A tracial vector is an element of Hilbert space generating a tracial state on the algebra via the inner product, in the sense that $\tau(A) = \langle A\Phi, \Phi \rangle$ for any operator A. If Φ is null, then τ is not defined. In other words, the relevant algebra cannot be a finite factor von Neumann algebra, such as the type II_1. Notice that formula (2.6) is crucial in the proof. The first equality is justified by the property that a trace is insensitive to non-commutativity: in fact, the order of $U(a)$ and $V(b)$ does not affect the value of the inner product. The second equality, instead, follows from plugging in the Weyl form of the CCRs; here, of course, the order of the terms does matter. In other words, the formula expresses the two requirements at the bottom of von Neumann's abandonment of Hilbert space, namely the existence of a finite *a priori* probability and the presence of unbounded operators for which one can define the Canonical Commutation Relations. It is almost cruelly ironic that their combination leads to Araki's *no go* result. This would show not only that type II_1 factors may not be appropriate candidates for the mathematical foundations of quantum mechanics, but also that there is a deep inconsistency within von Neumann's work itself.

Let us point out that the proof relies on the assumption of *regularity* of the representations of the CCRs. Indeed, the double limit appearing in the left-hand side of the last formula is what allows one to reduce (2.6) to the inner product of Φ. That is guaranteed by the unitary operators being weakly, equivalently strongly, continuous. So, the content of Araki's result is really that an algebra on which a tracial state is defined cannot support any regular representation of the Canonical Commutation Relations. If one gives up continuity, though, one blocks his conclusion in the general case. Slawny (1972) constructed an abstract C*-algebra (that is a more general structure than a von Neumann algebra) on the basis of the existence and uniqueness of type II_1 factors and then showed that it supports non-regular, i.e. non necessarily continuous, representations of the CCRs. What one may wonder, then, is whether or not regularity can be dropped without losing any physical meaning of the algebra at all.

Von Neumann's 1931 proof of uniqueness of the Schrödinger representation of Heisenberg's commutation relation, that we mentioned in section 2.1., relies on the assumption of regularity. A technical consequence of working with regular representations of the CCRs is that the underlying Hilbert space is separable. Separability is a topological requirement of the space: $\mathcal{H}$ is said to be separable if it contains a countable dense subset. Thus, giving up regularity is tantamount to appealing to non-separable Hilbert space. A generalization of the Stone-von Neumann theorem to non-regular representations of the CCRs was actually given by Cavallaro-Morchio-Strocchi (1999).

Consider the measure space (Γ, Ω, μ), with μ being a finitely additive measure, and let U be an operator-valued function mapping from Γ onto the lattice of projections on an Hilbert space. U is then measurable w.r.t. μ

just in case there exists a sequence U_n converging μ-almost everywhere to it, that is just in case the set $\{x \in \Gamma : |U_n(x) - U(x)| \geq \varepsilon\}$ has measure zero in the limit as $n \to \infty$. In particular, weak measurability for U requires that the complex-valued function $\langle U(x)\psi, \phi\rangle$ defined on Γ is measurable for any Hilbert space vector ψ, ϕ. Strong continuity of unitary operators follows from U being weakly measurable and $\mathcal{H}$ being separable. Weak measurability proves necessary, but not sufficient in order to obtain (irreducible) representations of the CCRs. Hence, if one is willing to drop the separability of Hilbert space, one needs to replace for it with some other topological condition.

A weakly measurable operator-valued function U is also *strongly measurable w.r.t.* μ if $U(x)\psi$ is μ-almost separably-valued for every element ψ of a non-separable Hilbert space. To put it more technically, there must exist $\Gamma_0 \subset \Gamma$ such that the collection of vectors $\{U(x)\psi \in \mathcal{H} | x \in \Gamma/\Gamma_0\}$ is separable and $\mu(\Gamma_0) = 0$. This means that, although there is no countably dense subset of a non-separable $\mathcal{H}$, there exists a set of vectors $U(x)\psi$ which (modulo a subset of Γ of null measure) is a separable subset of $\mathcal{H}$. Accordingly, strong measurability provides a sort of "local separability". Non-regular representations can thus be recovered with the aid of measure-theoretical conditions. Cavallaro, Morchio and Strocchi could then prove that strongly measurable, not necessarily regular, representations of the Canonical Commutation Relations are well defined. Furthermore, the CCR algebras so constructed can be implemented in actual models of quantum systems, hence proving that non-separable Hilbert space quantum mechanics makes physical sense.

The question that remains to be answered in the light of Araki's proof reduces to an historical one: would von Neumann have been willing to drop separability? In the 1932 book on the mathematical foundations of quantum mechanics he explicitly appealed to the Hausdorff's requirement of separability for topological spaces without providing any justification for such an assumption. Later, his attitude became less rigid. In the 1954 talk he concedes that separability is a "plausible but not terribly decisive" property of $\mathcal{H}$ because, although it may constrain the mathematical study of quantum systems in some cases, most of the times it "simply means that parts of the space which could be handled separately anyhow, are singled out" [von Neumann (1954), p.232]. A letter to Kaplansky (March 1, 1950) even hints at some result that apparently he obtained on operator rings in non-separable Hilbert spaces, but no further reference or explanation is provided. Arguably, the theory of dimensionality for the corresponding algebraic structures would contain a classification of continuous dimensions with the cardinality of the continuum. In the last analysis, one could conclude, although von Neumann was likely not aware of the devastating consequences of the assumption of regular representations of the CCRs for the factors of type II_1, he may well have been ready to give it up without affecting the conceptual cornerstones of his program, in particular the quest for an *a priori* finite probability trace and the treatment of unbounded operators.

Chapter 3

Information and The Bayesian Interpretation of Quantum Probabilities

In this chapter I develop some of the ideas stemming from von Neumann's technical and conceptual work on quantum foundations. Section 3.1. is a review of the main concepts of the algebraic approach, as they have been refined during the last few decades. It should be intended as a mathematical appendix for the rest of the dissertation. In particular, I spell out various notions of independence between algebras that are employed in the study of correlations between spatially and space-like separated physical systems.

Long distance correlations acquire a special status in quantum mechanics due to the presence of entanglement. In section 3.2.1, I introduce this concept and explain how it represents the main point of departure of the quantum formalism from the classical world. Entangled states are responsible for the non-local behaviour of quantum systems, which has puzzled physicists and philosophers of physics since the earlier formulation of the theory. The peculiar kind of non-locality arising from entanglement takes the form of the violation of the famous inequality derived by John Bell in 1964. The algebraic formulation of the locality condition from which Bell's inequality follows, as well as its decomposition into two distinct provisions, namely parameter independence and outcome independence, is analyzed in section 3.2.2.

The recent development of quantum information has shed new light on the nature of entanglement. Quantum non-locality is now exploited as a resource rather than regarded as a source of conceptual problems. In the quantum world one can manipulate information in ways that classical protocols do not allow one to do. This also motivates one to look at quantum foundations from an information-theoretical perspective. A theorem by Clifton, Bub and Halvorson (2003), which I discuss in section 3.2.3, is the basis of an axiomatic-type approach that can actually be seen as the realization of von Neumann's project of appealing to information theory to derive the algebraic structure of quantum mechanics.

In the last part of the chapter I survey the Bayesian interpretation of quantum mechanics, which relies on quantum information. Accordingly, quantum states amount to states of information, or, more specifically, states of rational belief of the observer. Quantum measurements provide new information in light of which the observer updates her (subjective) probabilities. The framework of statistical inference is thus particularly suitable to account for such a process. Yet, just as in the case of von Neumann's quest for the "proper limit" of quantum mechanics, the extension of non-commutative statistical inference in infinite dimensional Hilbert space poses non-trivial mathematical difficulties for a Bayesian interpretation of quantum probabilities.

As we have seen, the choice of which properties of quantum theories of finite degrees of freedom are to be preserved under the limit in infinite dimensions is motivated by physical considerations, for example the requirement of finite probabilities. In addition, there are some properties which depend on any specific interpretation of quantum mechanics: in fact, different interpretations of the theory would disagree on the set of properties to be regarded as crucial. Physical constraints are strong, since their failure in infinite dimensions means that the limit has no meaning at all. Conceptual constraints, instead, are less strict. If they are not fulfilled, that is if some conceptual property is not preserved under the limit, one would be just entitled to conclude that such a property is not a crucial property of the theory. Then, any interpretation requiring a non-preserved property would prove inconsistent with quantum mechanics.

To put it in another way: one obtains *no go* results for any interpretation of the theory if the limit cannot be carried out in such a way that any relevant property holds. After showing in section 3.3.1. how quantum statistical inference generalizes the Bayes conditionalization rule in von Neumann algebras theory, which is the proper noncommutative probability theory, I reply to an argument by Miklos Redei (1992, 1998) against the Bayesian interpretation of quantum mechanics (section 3.3.3.). The claim is that a stability condition, holding as a rational constraint for Bayesian noncommutative statistical inference in classical theory as well as in finite dimensions, fails in infinite dimensional Hilbert space. In section 3.2.3, I discuss whether the *a priori* probability state of a quantum observer can be derived by symmetry considerations.

3.1 The Algebraic Approach

3.1.1 Algebras of physical observables

In section 1.1.3. some basic concepts of von Neumann algebras were introduced by spelling out the reasoning underlying von Neumann and Murray's work on "Rings of Operators". Here I provide a more technical and general discussion of the use of algebras of observables in physics. The algebraic formalism supplies a rigorous framework encompassing classical mechanics and quantum mechanics, both relativistic and non-relativistic. This has the advantage that one gets insight into the structure of the theories, and hence one is in a position to pinpoint crucial differences between them. Also, it yields a powerful tool for the treatment of quantum mechanics in infinite dimensions. In fact, one can extend in a natural way the description of observables with discrete spectra, that is finite-dimensional Hilbert space, to the description of observables with continuous spectra, that is infinite-dimensional Hilbert space. Moreover, one can deal with systems with infinitely many degrees of freedom, that is the case of quantum field theory.

The idea behind the algebraic approach is that it is not the observables, but rather the algebras of observables that play a crucial role in the mathematical formulation of physical theories. Segal (1947) proposed that one should take the C*-algebras of all bounded operators as the basic mathematical entities with physical meaning. Recall that an algebra $\mathcal{A}$ is a Banach algebra in case for, any two elements A and B, the norm defined on the algebra is such that $||AB|| \leq ||A||||B||$. It is called involutive if there exists an involution, namely a map * from the algebra onto itself, with the following properties:

1. $(A^*)^* = A$

2. $(AB)^* = A^*B^*$

3. $||A^*|| = ||A||$

An abstract C*-algebra is defined without any reference to Hilbert space as an involutive Banach algebra $\mathcal{A}$, that enjoys the property $||AA^*|| = ||A||||A^*||$ for every element $A \in \mathcal{A}$.

Any abstract C*-algebra thus defined is provably isomorphic to a concrete C*-algebra. The definition of the latter is actually given in terms of Hilbert space. The set $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$ is in fact a *-algebra, that is an algebra where the involution * corresponds to taking the adjoint. Any subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a concrete C*-algebra which is closed under the operator norm topology. The latter requires that, if the sequence of operators $\{A_n\}$ in $\mathcal{A}$ converges in norm to some operator $A \in \mathcal{B}(\mathcal{H})$, i.e. $||A_n - A|| \longrightarrow 0$, then $A \in \mathcal{A}$. The algebra is called unital if it contains the identity transformation I on $\mathcal{H}$. Here I always assume that $\mathcal{A}$ is unital.

A *state* ϕ on the C*-algebra $\mathcal{A}$ is a continuous linear map from the algebra into the complex numbers which is positive, namely $\phi(A^*A) \geq 0$, and normalized, namely $\phi(I) = 1$. Also, ϕ is said to be faithful just in case $\phi(AA^*) = 0$ only if $A = 0$. In particular, a state is normal if it is ultraweakly continuous. This means that ϕ satisfies the property that $\phi(\sum_n P_n) = \sum_n \phi(P_n)$ for any countable family of mutually orthogonal projections P_n in $\mathcal{A}$, and hence it is countably additive. It is well known that any normal state on $\mathcal{B}(\mathcal{H})$ is given by $\phi(A) = Tr(\rho A)$ for some unique density matrix ρ acting on $\mathcal{H}$. Now, every state on a general von Neumann algebra $\mathcal{M}$ is the restriction to $\mathcal{M}$ of a state of $\mathcal{B}(\mathcal{H})$. It is also expressed by a trace Tr, but the density matrix is no longer unique.

Given a state on $\mathcal{A}$, one can construct a Hilbert space representation of the algebra by means of the Gelfand-Naimark-Segal theorem, which establishes an isomorphism between $\mathcal{A}$ and the set of bounded operators over some $\mathcal{H}$. That is, for any state ϕ, the GNS-construction determines a unique (up to unitary equivalence) triple $(\mathcal{H}_\phi, \Omega_\phi, \pi_\phi)$, where Ω_ϕ is a unit vector on the Hilbert space $\mathcal{H}_\phi$ and π_ϕ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}(\mathcal{H}_\phi)$. Then, one obtains

$$\phi(A) = \langle \Omega_\phi, \pi_\phi(A)\Omega_\phi \rangle \tag{3.1}$$

for any operator A in the algebra $\mathcal{A}$ and the set $\pi_\phi(\mathcal{A})\mathcal{H}_\phi$ is dense in $\mathcal{H}_\phi$.

Von Neumann algebras are defined as special cases of C*-algebras. We already saw that there are two equivalent definitions of a von Neumann $\mathcal{M}$. Algebraically, $\mathcal{M}$ coincides with its double commutant, that is $\mathcal{M} = \mathcal{M}''$. Topologically, a von Neumann algebra is required to be closed under the strong operator topology, according to which, given the sequence of operators $\{A_n\}$ in $\mathcal{M}$, if for all states Φ on $\mathcal{H}$ there exists some operator $A \in \mathcal{B}(\mathcal{H})$ such that $A_n\Phi \longrightarrow A\Phi$, then $A \in \mathcal{M}$. The strong operator topology and the norm operator topology coincide in finite-dimensional Hilbert space; however, the former is strictly stronger than the latter if $\mathcal{H}$ is infinite-dimensional. Therefore, any von Neumann algebra is a C*-algebra, whereas the converse is not true in general.

The classification of von Neumann factor algebras into five types, namely I_n, I_∞, II_1, II_∞ and III, which depends on the range of the dimension function of their projection lattice can be extended to general von Neumann algebras. Indeed, any von Neumann algebra can be decomposed into a direct sum of factors. The differences between the various types of factors can be characterized in terms of the nature of the projections they contain. A projection $P \in \mathcal{M}$ is said to be minimal just in case there is no subprojection of P in the algebra. That implies that the projection is abelian too, that is that the algebra $P\mathcal{M}P$ is commutative. Moreover, P is called infinite just in case there exists a projection P_0 which is equivalent to P such that $P_0 < P$; otherwise, P is called finite. Accordingly, von Neumann factor algebras are classified as follows:

1. $\mathcal{M}$ is type I if it contains an abelian projection

2. $\mathcal{M}$ is type II if it contains a (non-zero) finite projection, but no abelian projection

3. $\mathcal{M}$ is type III if it contains no finite projection and no abelian projection

Also, a von Neumann algebra is finite if the identity I is finite in $\mathcal{M}$; otherwise, the algebra is infinite. Examples of the former are the types I_n and II_1, whereas examples of the latter are the types I_∞, II_∞ and III. As it was pointed out in section 2.2.3., the finiteness of a von Neumann algebra coincides with the existence of a tracial state, namely a state τ such that $\tau(AB) = \tau(BA)$ for any $A, B \in \mathcal{M}$. Importantly, the operation of taking the commutant preserves the type. That is, if $\mathcal{M}$ is a von Neumann algebra of type I (respectively, II and III), then $\mathcal{M}'$ is type I (respectively, II and III) too.

Some rules of composition for different types appeal to the tensor product structure, which plays a crucial role in quantum theory. If $\mathcal{M}$ is type I_n and

$\mathcal{N}$ is type I_m, then the tensor product $\mathcal{M} \otimes \mathcal{N}$ is type I_{nm} and acts on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ generated by operators of the form $M \otimes N$, where $M \in \mathcal{M}$ and $N \in \mathcal{N}$. If one of the summands is type III, then, irrespective of the type of the other one, $\mathcal{M} \otimes \mathcal{N}$ is type III. If none of the summands is type III, but at least one is type II, then $\mathcal{M} \otimes \mathcal{N}$ is type II as well. Also, any von Neumann algebra of type I is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$ with $\mathcal{A}$ being commutative. Therefore all abelian algebras are type I, although they cannot be factors, of course, since they coincide with their commutant, and thus their center cannot be a multiple of the identity (unless they are trivial).

Let us conclude by reviewing a few basic facts about the Tomita-Takesaki modular theory, which is presented in Takesaki (2003), as it has remarkable applications in mathematical physics. Specifically, it supplies the technical basis for the sub-classification of von Neumann algebras of type III developed by Connes (1974), and it is therefore relevant to Algebraic Quantum Field Theory. Moreover, the notion of modular automorphism group is employed in Redei's argument against the Bayesian interpretation of non-commutative probabilities that I discuss and criticize in the last section of this chapter.

Some preliminary definitions of a cycling and separating vector of Hilbert space for an algebra should be recalled first. A vector Ω is *cyclic* for a C*-

algebra $\mathcal{A}$ just in case the closed linear span of the set $\{A\Omega | A \in \mathcal{A}\}$ coincides with all Hilbert space $\mathcal{H}$. It is *separating* for $\mathcal{A}$ just in case $A\Omega = 0$ entails that the operator A in the algebra is equal to 0. Under these assumptions for Ω, one can introduce an operator S_0 on $\mathcal{H}$ such that

$$S_0 A\Omega = A^*\Omega \tag{3.2}$$

for any operator A belonging to the von Neumann algebra $\mathcal{M}$. This operator extends to a closed anti-linear operator S, whose polar decomposition reads $S = J\Delta^{\frac{1}{2}}$, where J is called modular conjugation J of the pair $(\mathcal{M}, \Omega)$ and Δ is called modular operator. The modular conjugation is an anti-unitary operator, i.e. J^2 yields the identity I and $J = J^*$, which has the effect of transforming the algebra into its commutant by the relation $J\mathcal{M}J = \mathcal{M}'$. The modular operator, instead, is a positive operator, namely its spectrum takes on only positive real values, such that $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$, where Δ^{it} is unitary for any value of the real parameter t. Accordingly, $J\Omega = \Omega = \Delta\Omega$.

Let ω denote the (faithful) normal state generated by Ω, namely $\omega(A) = \frac{1}{\|\Omega\|^2}\langle\Omega, A\Omega\rangle$. Then, the strongly continuous unitary group $\{\Delta^{it}\}$ induces a one-parameter automorphism group expressed by

$$\sigma_t^{\omega}(A) = \Delta^{it} A \Delta^{-it} \tag{3.3}$$

which is called modular automorphism group of $\mathcal{M}$ (relative to Ω). It follows that ω is invariant under $\{\sigma_t^\omega\}$, in the sense that $\omega \circ \sigma_t^\omega = \omega$. The spectrum of Δ offers a measure of the periodicity of the modular automorphism. In fact, the smaller the spectrum, the closer σ_t^ω is to the identity. The extreme case obtains, for instance, when ω is a tracial state: the modular operator corresponding to the latter takes on only the value 1.

Finally, one defines the modular spectrum of $\mathcal{M}$ as the intersection $S(\mathcal{M}) = \bigcap sp\Delta_\omega$ over all faithful normal states ω of $\mathcal{M}$, with Δ_ω being the corresponding modular operator. $S(\mathcal{M})$ is an algebraic invariant of $\mathcal{M}$. This yields a sub-classification of the type III factors in the following manner[1]:

1. $\mathcal{M}$ is type III_0 if and only if $S(\mathcal{M}) = \{0, 1\}$
2. $\mathcal{M}$ is type III_1 if and only if $S(\mathcal{M}) = (0, \infty)$
3. $\mathcal{M}$ is type III_λ with $\lambda \in (0, 1)$ if and only if $S(\mathcal{M}) = \{\lambda^n\} \bigcup \{0\}$

where the n in the last expression ranges through the integer numbers.

3.1.2 Noncommutative probability theory

In von Neumann's development of the "Rings of Operators", factors were constructed as a non-classical analogue of Cantor's set theory. Von Neumann

[1]Actually, Connes' classification employs the concept of the "period of the flow of weights", where the notion of a weight generalizes that of a state.

algebras were then demonstrated to be a noncommutative generalization of measure theory. In fact, just as set theory informs classical probability theory, states defined on von Neumann algebras are regarded as the proper noncommutative probabilities. This puts one in a position to extend classical concepts of probability theory, such as central theorems of stochastic processes and the problem of statistical inference, in a more general framework. Accordingly, the behaviour of noncommutative probabilities with respect to these concepts determines whether or not any interpretation of probability, like the frequentist or the Bayesian, which is based on them would hold in quantum theory.

Let us now show how classical probability theory is actually a special case of von Neumann algebras theory. The triple (Γ, Ω, μ) is a finite measure space comprising a σ-algebra Ω of subsets of a set Γ and μ is a finite measure on Γ. This becomes a classical probability space if the measure is bounded to take its values on the real unit interval, i.e. $\mu : \Omega \longrightarrow [0, 1]$. To such a space there is associated the algebra $L^{\infty}(\Gamma, \Omega, \mu)$ of all essentially bounded complex-valued functions on Γ. That is indeed a commutative von Neumann algebra acting on the Hilbert space $L^2(\Gamma, \Omega, \mu)$ of all square μ-integrable complex-valued functions on Γ. Conversely, one can also prove that every commutative von Neumann algebra associated to any measure space, which is a direct sum of finite measure spaces, is isomorphic to $L^{\infty}(\Gamma, \Omega, \mu)$.

Physical applications of probability theory appeals to the concept of random variables, which represent the observables of the theory. Classically, these are real-valued functions f belonging to the algebra $L^\infty(\Gamma, \Omega, \mu)$. A particular class of random variables is given by the characteristic functions. Specifically, $\chi_A(x)$ takes on the value 1 if x falls into the subset A and 0 otherwise. Probabilities are thus defined by

$$\mu(A) = \int_\Gamma \chi_A(x) d\mu(x) \tag{3.4}$$

Clearly, the characteristic functions amount to the projections in a commutative von Neumann algebra. Then, the probability measure μ defines a unique state ϕ on $L^\infty(\Gamma, \Omega, \mu)$ by the relation $\phi(f) = \int_\Gamma f(x) d\mu(x)$. If μ is a σ-additive measure, ϕ is a normal state.

A general noncommutative probability space is given by the triple $(\mathcal{M}, \mathcal{P}(\mathcal{M}), \phi)$, where the state ϕ assigns to each projection P in the lattice $\mathcal{P}(\mathcal{M})$ of the von Neumann algebra $\mathcal{M}$ a real number in the interval $[0, 1]$. A normal state thus determines a σ-additive probability measure on the noncommutative space of events. Gleason's theorem (1957) shows the converse of this fact for Hilbert space quantum mechanics, when the dimension of $\mathcal{H}$ is greater than 2. A generalization of such a result was later on obtained for any von Neumann algebra (see Maeda (1990) for a review): if $\mathcal{M}$ has no direct summand of type I_2, then any (finitely and infinitely) additive probability measure $\mu : \mathcal{P}(\mathcal{M}) \longrightarrow [0, 1]$ extends uniquely to a state on $\mathcal{M}$.

In the next few sections I discuss some interesting differences between commutative and noncommutative probability states, which have to do with some notions of independence between algebras, the Bell inequality and the (non-)existence of conditional expectations. These concepts actually prove relevant for the philosophical investigation of algebraic quantum theories.

3.1.3 Types of independence between algebras

Since each physical system is associated with an algebra of observables, correlations between distant systems can be studied by looking at the mutual relations holding between the corresponding algebras. The type of independence between the latter thus reflects the type of independence between two systems. There are various notions of algebraic independence offered in the literature that coincide in the classical case, but are in general quite different for non-commutative algebras. Whether they hold or not is a consequence of structural properties of the algebras as well as the existence of certain states across them.

This section is meant to be an overview of the main types of independence employed in quantum theory. The most basic mathematical formulation of independence is *kinematical independence*. Accordingly, two quantum systems are kinematically independent just in case any element of the algebra associated with one system commutes with all the observables of the algebra associated to the other one. Another definition captures the idea of *statistical independence*. Accordingly, two quantum systems are statistically independent just in case each can be prepared in any state, no matter how the other system has been prepared.

Within such a qualitative distinction, one can proceed to supply the basic definitions of the relevant notions of independence. In what follows, $\mathcal{A}_1$ and $\mathcal{A}_2$ denote two C*-subalgebras of a C*-algebra $\mathcal{A}$, on which there are defined the states ϕ_1 and ϕ_2, respectively. Likewise, $\mathcal{M}_1$ and $\mathcal{M}_2$ denote two von Neumann subalgebras of a von Neumann algebra $\mathcal{M}$, on which there are defined the states ϕ_1 and ϕ_2, respectively. Let us begin with *C*-independence*.

> $\mathcal{A}_1$ and $\mathcal{A}_2$ are C*-independent if, for any ϕ_1 and ϕ_2, there is a state ϕ on $\mathcal{A}$ such that
>
> - $\phi(A) = \phi_1(A)$ for any $A \in \mathcal{A}_1$
> - $\phi(B) = \phi_2(B)$ for any $B \in \mathcal{A}_2$
>
> That is, the state of each system is given by the restriction of ϕ to the corresponding sub-algebra of $\mathcal{A}$.

This means that the states ϕ_1 and ϕ_2 can be prepared in the same preparation process. Hence, no preparation of states of $\mathcal{A}_1$ can exclude any preparation of the system described by $\mathcal{A}_2$.

Analogously, one can introduce the notion of *W*-independence* in the context of von Neumann algebras, although the states are here required to be normal.

> $\mathcal{M}_1$ and $\mathcal{M}_2$ are W*-independent if, for any ϕ_1 and ϕ_2, there is a (normal) state ϕ on $\mathcal{M}$ such that
>
> - $\phi(A) = \phi_1(A)$ for any $A \in \mathcal{M}_1$
> - $\phi(B) = \phi_2(B)$ for any $B \in \mathcal{M}_2$
>
> That is, the state of each system is given by the restriction of ϕ to the corresponding sub-algebra of $\mathcal{M}$.

A stronger notion of independence between von Neumann algebras, namely *W*-independence in the product sense* is defined by placing a further constraint on the states.

> $\mathcal{M}_1$ and $\mathcal{M}_2$ are W*-independent in the product sense if they are W*-independent and there is a (normal) state ϕ on $\mathcal{M}$ that extends both ϕ_1 and ϕ_2, that is

$$\phi(AB) = \phi_1(A)\phi_2(B) \tag{3.5}$$

for all $A \in \mathcal{M}_1$ and all $B \in \mathcal{M}_2$.

Accordingly, the joint state ϕ is a product state[2] across the pair $(\mathcal{M}_1, \mathcal{M}_2)$.

Another notion of statistical independence formulated in the framework of von Neumann algebras theory is *strict locality*. Notice that the following relation is provably symmetric if $\mathcal{M}_1$ and $\mathcal{M}_2$ commute, although it is not known yet whether this is also the case in general.

> $\mathcal{M}_1$ and $\mathcal{M}_2$ are strictly local if, for any $A \in \mathcal{P}(\mathcal{M}_1)$ and for any (normal) state ϕ_2, there exists a (normal) state ϕ on $\mathcal{M}$ such that
>
> - $\phi(A) = 1$
> - $\phi(B) = \phi_2(B)$ for any $B \in \mathcal{M}_2$

The meaning of strict locality is that no preparation of any state of the system described by the sub-algebra $\mathcal{M}_2$ can exclude the occurrence of any probability of whatever event represented by an element of the projection lattice of the other subalgebra $\mathcal{M}_1$. In fact, there exist states ϕ and ψ on $\mathcal{M}$, whose restrictions to $\mathcal{M}_2$ coincide with ϕ_2, such that $\phi(I - A) = 1$ and

[2] The notion of product and its connection with entanglement will be explained in detail in section 3.2.1.

$\psi(A) = 1$. Then, for any number $\lambda \in [0,1]$, one can always construct a probability state φ that yields such a numeric value for A: i.e. $\varphi(A) = \lambda\psi(A) + (1-\lambda)\phi(A) = \lambda$.

If two von Neumann algebras are commuting, that is $\mathcal{M}_1 \subseteq \mathcal{M}'_2$, one can also introduce the *split property*. Accordingly,

> $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to enjoy the split property if there exists a type I factor $\mathcal{B}(\mathcal{H})$ such that

$$\mathcal{M}_1 \subseteq \mathcal{B}(\mathcal{H}) \subseteq \mathcal{M}'_2 \tag{3.6}$$

Such a property will be particularly important in chapter 4, when discussing the correlations between space-like separated systems in Algebraic Quantum Field Theory.

The mutual relations between these notions of independence is not clear in general. However, in case the algebras commute, and thus are statistically independent, it was demonstrated that the split property implies W*-independence, that in turn implies C*-independence. Then, the latter can be shown to be equivalent to the so-called *Schlieder property*.

> $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to satisfy the Schlieder property just in case the non-zero projections $P_1 \in \mathcal{P}(\mathcal{M}_1)$ and $P_2 \in \mathcal{P}(\mathcal{M}_2)$ are such that $P_1 \wedge P_2 \neq 0$.

Here the operation $\wedge$ is defined on the projection lattice $\mathcal{P}(\mathcal{M})$: the projection $P_1 \wedge P_2$ thus formed maps onto the intersection of the closed subspaces of Hilbert space onto which P_1 and P_2 project. Hence, the Schlieder property is the analogue of classical logical independence.

3.2 Entanglement and Quantum Information

3.2.1 The "characteristic trait" of quantum mechanics

The formalism of quantum mechanics allows one to describe the joint state of two (or more) spatially separated systems. A remarkable aspect of the theory, that became evident to the physicists engaged in its foundations already in the early 1930's and has been a source of philosophical debate since then, is the fact that quantum systems which physically interacted in the past maintain some kind of strong correlation, no matter how far apart they are displaced later on. This happens when the systems share a so-called entangled state. Accordingly, the state of each sub-system of the composite system would not be conceived as independent from the state of the other in any obvious manner.

The presence of entanglement can be regarded, by echoing Schrödinger's words, as the "characteristic trait" of quantum mechanics, that distinguishes the latter from the classical case. Indeed, no correlation between classical systems retains the same features as that of an entangled pair of quantum particles. Entanglement is actually at the bottom of various conceptual puzzles, such as those arising from the Einstein-Podolsky-Rosen paradox and the violation of the Bell's inequality, which highlight the peculiar non-locality of quantum theory. I will address the latter issue in the next section. Here I discuss the interesting properties that the notion of entanglement exhibits in itself.

The term *entanglement* was coined by Schrödinger, who was one of the first to point out the radical novelty that this concept brought into physics:

> When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives have become entangled. [Schrödinger (1935), p. 555]

It is the tensor product structure that enables one to account for the entangled state describing the joint system generated by the interaction of the sub-systems within the formalism of quantum mechanics, whereby the mathematical representatives of quantum states are the vectors of Hilbert space.

The tensor product allows one to put vector spaces together to form larger vector spaces. In particular, the tensor product of Hilbert spaces is employed whenever one deals with the union (or separation) of quantum systems. Suppose one considers the set of all mutually commuting observables associated with a system $\mathcal{S}_1$ as well as the set of all mutually commuting observables associated with a system $\mathcal{S}_2$. Then any observable A_1 in the first set ought to commute with all the observables A_2 of the second one, and conversely. Moreover, each element of the two sets is an observable of the composite system $\mathcal{S}_1+\mathcal{S}_2$. Such properties of the union of distinct systems are incorporated in the mathematical description of quantum states.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces associated with systems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then, one can form their tensor product $\mathcal{H}_1 \bigotimes \mathcal{H}_2$ which describes the joint system. Its elements are linear combinations of the vector $\psi \otimes \upsilon$, where $\psi \in \mathcal{H}_1$ and $\upsilon \in \mathcal{H}_2$. One can also define a linear operator $A_1 \otimes A_2$ on the tensor product Hilbert space by the relation:

$$A_1 \otimes A_2(\Sigma_i a_i \psi_i \otimes \upsilon_i) = \Sigma_i a_i A_1 \psi_i \otimes A_2 \upsilon_i \tag{3.7}$$

with a_i being complex coefficients. The operator A_1 on $\mathcal{H}_1$ corresponds to the operator $A_1 \otimes I$ acting on $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. Likewise, A_2 on $\mathcal{H}_2$ corresponds to the operator $I \otimes A_2$ on the joint Hilbert space.

Such a definition has a straightforward generalization. As a matter of fact, given any finite number n of quantum systems, the tensor product can be used to provide a representation of their joint system. The Hilbert space $\mathcal{H}_1 \bigotimes ... \bigotimes \mathcal{H}_n$ would thus yield a mathematical structure describing a multi-particle system $\mathcal{S}_1 + ... + \mathcal{S}_n$. Importantly, the tensor product of Hilbert spaces is uniquely defined. This guarantees that the physical properties of the compound system are entirely determined by the component sub-systems. In particular, any projection on the tensor product Hilbert space corresponds to a yes/no measurement performed on the joint system.

Yet, the tensor product structure does not entail that the state of the compound system is uniquely determined by the states of the composite systems. Entanglement is indeed the culprit. In quantum-mechanics a state $|\psi\rangle$ of the n-particle system does not correspond, in general, to the product of the n states ψ_i of the individual sub-systems. That is,

$$|\psi\rangle \neq |\psi_1\rangle \otimes ... \otimes |\psi_n\rangle \tag{3.8}$$

Indeed, the superposition principle allows such a state to be a linear superposition $|\psi\rangle = \Sigma_{i_n}\lambda_{i_n}|i_n\rangle$, where the complex parameters λ_{i_n} are eigenvalues of the basis $|i_n\rangle = |i_1\rangle \otimes ... \otimes |i_n\rangle$ of the total Hilbert space. In fact, it can be decomposed in different (non-equivalent) ways depending on the orthogonal bases available in the Hilbert space. Hence, just by looking at the general state ψ a single vector state cannot be assigned to each particle.

It is true that one encounters many more linear superpositions than pure states. One may hope, though, that at least in the particular circumstance in which one deals with a pure state, that is a state of maximal knowledge, one could fully determine the states of the subsystems. Remarkably, this is not the case in quantum theory, as the following example illustrates. Let a bipartite system be described by the tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \bigotimes \mathcal{H}_2$, with the Hilbert spaces associated each subsystem having dimension 2. A possible joint state of two entangled particles is the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow_1\rangle|\uparrow\rangle_2) \tag{3.9}$$

where the states $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ may represent the fact that the spin of the particle (labeled by $i = 1, 2$) in a given direction is up and down, respectively. Notice that the symbol of the tensor product is conveniently dropped. This state is pure on the compound system. Nevertheless for each particle there is equal probability $\frac{1}{2}$ of having spin up or spin down. Thus, the state of each subsystem is not fully determined. As Schrödinger put it,

> Another way of expressing the peculiar situation is: the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of all its *parts*, even though they may be entirely separate and therefore virtually capable of being "best possibly known", i.e., of possessing, each of them, a representative of its own. The lack of knowledge is by no means due to the interaction being insufficiently known - at least not in the way that it could possibly be known more completely - it is due to the interaction itself. [Schrödinger (1935), p. 555]

So, the fact that the individual states of spatially separated sub-systems sharing a pure entangled state need not be pure is a consequence of the physical interaction that originally entangled the systems. This is the peculiar aspect of quantum theory, as in classical mechanics maximal knowledge of the physical setting allows one infer the states of the component subsystem from their joints state. One then defines entanglement by contrast to classical correlations.

Classically, the state of a composite system is given by a probability measure μ on the product space $\Gamma_1 \times \Gamma_2$ of the probability spaces of the single subsystems, that is $(\Gamma_1, \Sigma_1, \mu_1)$ and $(\Gamma_2, \Sigma_2, \mu_2)$, respectively. Such a probability measure can be represented as a limit of convex combinations of measures concentrated in one point. As the point measures on a product space are product measures, any classical probability state on a joint (product) space is the limit of convex combinations of product measures. This suggests how to characterize *classically correlated* states.

An uncorrelated state of a composite quantum system described by $\mathcal{H}_1 \bigotimes \mathcal{H}_2$ is given by a density matrix of the form $\rho = \rho_1 \otimes \rho_2$, with ρ_i being the density matrix describing the state of each subsystem on the corresponding Hilbert space $\mathcal{H}_i$. The expectation value of any observable $A_1 \otimes A_2$ always factorizes, that is

$$\begin{aligned} Tr(\rho A_1 \otimes A_2) &= Tr(\rho \cdot A_1 \otimes I) Tr(\rho \cdot I \otimes A_2) \\ &= Tr(\rho_1 A_1) Tr(\rho_2 A_2) \end{aligned}$$

The classical multiplication rule for probabilities thus applies. Suppose that the density matrix is written as $\rho = \sum_i \lambda_i \rho_{1i} \otimes \rho_{2i}$. The corresponding state does not factorize, but it proves to be a convex combination of product states. In fact, $Tr(\rho A_1 \otimes A_2) = \sum_i \lambda_i Tr(\rho_{1i} A_1) Tr(\rho_{2i} A_2)$. Any state whose density matrix can be approximated (under some suitable topology) by density matrices of this form is said to be classically correlated. Accordingly, the statistical properties of a quantum system can be reproduced by classical probabilities. Otherwise, if ρ is not approximated by any convex combination of product states, the corresponding state is defined as an entangled state.

Such a definition can be recast in the general framework of von Neumann algebras theory. Let us consider two commuting von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. The algebra of observables of the composite system is then given by $\mathcal{M} \equiv \mathcal{M}_1 \bigvee \mathcal{M}_2 = (\mathcal{M}_1 \bigcup \mathcal{M}_2)''$. A joint state ϕ is a product state just in case

$$\phi(A_1 A_2) = \phi(A_1)\phi(A_2) \tag{3.10}$$

for any observable $A_1 \in \mathcal{M}_1$ and $A_2 \in \mathcal{M}_2$. That is the closest one can get to the classical notion of mutual independence in a noncommutative setting. Given $\mathcal{M}_1 \equiv L^\infty(\Gamma_1, \Sigma_1, \mu_1)$ and $\mathcal{M}_2 \equiv L^\infty(\Gamma_2, \Sigma_2, \mu_2)$, a state ϕ on the joint algebra is associated with the measure μ on $\Gamma_1 \times \Gamma_2$. If such a state is a product state, then the random variables, i.e. the observables, of the subsystems are mutually independent.

Under certain circumstances[3], which are of physical relevance as they involve many applications of quantum theory, the von Neumann algebra $\mathcal{M}_1 \bigvee \mathcal{M}_2$ proves equivalent to the tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$. The

[3] That is whenever there exists a normal conditional expectation T mapping from the von Neumann algebra $\mathcal{M}_1 \bigvee \mathcal{M}_2$ onto its von Neumann subalgebra $\mathcal{M}_1$. See section 3. for a precise definition of noncommutative conditional expectation.

algebra of observables of the compound system is thus modeled by the von Neumann algebra $\mathcal{M}_1 \otimes \mathcal{M}_2 \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \simeq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, with the subalgebra $\mathcal{M}_1$ being identified with the tensor product $\mathcal{M}_1 \otimes I$ and the other subalgebra $\mathcal{M}_2$ being identified with the tensor product $I \otimes \mathcal{M}_2$. A state ϕ on $\mathcal{M}$ is called *separable*, or classically correlated, if it is a mixture of (normal) product states on $\mathcal{M}_1 \otimes \mathcal{M}_2$. Otherwise, that is if it does not belong to the closure of the convex hull of (normal) product states on $\mathcal{M}$, the state ϕ is entangled across $(\mathcal{M}_1, \mathcal{M}_2)$.

Different notions of entanglement may be defined depending on what topology one requires. According to the metric topology induced by the norm on linear functionals, separable states ought to be the norm limit of convex combinations of product states. According to the weak-* topology, instead, they ought to be only the weak-* limit of convex combinations of product states. Recall that the norm of a state ϕ on $\mathcal{M}$ is defined as $\|\phi\| = \sup\{|\phi(A)| : |A| < 1\}$. Two states ϕ_1 and ϕ_2 that are closed to each other in norm, i.e. $\|\phi_1 - \phi_2\| \longrightarrow 0$, dictate close expectation values uniformly for all observables in the algebra. On the other hand, a sequence of states $\{\phi_n\}$ on $\mathcal{M}$ converges to ϕ just in case $\phi_n(A) \longrightarrow \phi(A)$ for all $A \in \mathcal{M}$. As weak-* convergence need not be uniform on all elements of $\mathcal{M}$, it is weaker than convergence in norm. Therefore, by requiring one to take the weak-* limit of sets of states, rather than the norm limit, one obtains a stronger notion of entanglement.

The above quote also emphasizes that the characteristic features of entangled states occur even if the subsystems are distinct and spatially separated from each other. In fact, quantum correlations can manifest themselves between distant systems, no matter how far apart they are displaced. Schrödinger then pointed out some puzzling consequences of entanglement that are closely related to Einstein's philosophical worries about the foundations of quantum mechanics, in particular those expressed in the famous EPR paper he wrote with Podolsky and Rosen in 1935. There exist matching correlations between both positions and momenta of two spatially separated particles. Performing a measurement of, say, position on one particle allows one to predict with certainty the outcome of a position measurement on the other one, and the same is true if one performs any measurement of the momentum observable. Quantum-mechanical measurements of position and momentum are mutually exclusive, that is establishing position destroys the correlation between the momenta of the two systems. Of course, one can predict either position or momentum of the first system without interfering with it

> ... and since system No.1, like a scholar in an examination, cannot possibly know which of the two questions I am going to ask first:

> it so seems that our scholar is prepared to give the right answer to the *first* question he is asked, *anyhow*. Therefore he must know both answers; which is an amazing knowledge; quite irrespective of the fact that after having given his first answer our scholar is invariably so disconcerted or tired out, that all the following answers are "wrong". [Schrödinger (1935), p.559]

Moreover, since one can reconstruct all the properties of a (classical) physical system by specifying both the canonical conjugate dynamical quantities position and momentum, the first "system-scholar"

> ... does not only know these two answers but a vast number of others, and that with no mnemotechnical help whatsoever, at least with none that we know of. [Schrödinger (1935), p.559]

The core idea of EPR is to exploit the fact that any measurement of position on one particle disturbs its momentum correlations with the other entangled particle and to conclude that the quantum state of the particle pair is incomplete. Indeed, on the basis of the state of the compound system one cannot assign labels to each individual particle that could determine completely the correlated values of the outcomes of position and momentum measurements. Such labels would amount to the common causes explaining the correlations between the two particles in terms of their initial interaction.

A simpler version of the argument involves measurements of spin on a bipartite system $\mathcal{S}_1 + \mathcal{S}_2$ described by the singlet state expressed by formula (3.9), which is in fact also called the EPR state. We know that it induces a mixed state of spin up and spin down on each sub-system. Suppose such a state represents a complete description of the state of the particle pair. Then, in case a measurement of spin in a given direction performed on S_2 reveals that the particle has spin up, by the eigenvalue-eigenvector link one knows with certainty that the state associated with S_1 is the pure state $|\downarrow\rangle$. However, if one assumes a locality principle, according to which no measurement on one particle can cause any real change in the other particle, one must infer that S_1 had already spin down before performing the measurement on S_2. This implies, though, that the EPR state could not have been a complete description of the state of the two sub-systems. Hence, the quantum-mechanical description of reality fails to be complete.

What Schrödinger found unsettling about entangled correlations is that it seems that the state of the sub-system not subjected to any direct measurement depends on what observations one arbitrarily decides to make on the other sub-system. He actually claimed that in general, by suitably acting on one part of an entangled pair, a sophisticated experimenter would be able to drive the other part into any state she chooses. Let us consider the vector Ψ describing the state of the composite system on $\mathcal{H}_1 \otimes \mathcal{H}_2$. By the biorthogonal decomposition it can be uniquely written as

$$|\Psi\rangle = \sum_i a_i |\alpha_i\rangle \otimes |\beta_i\rangle \tag{3.11}$$

where the set of unit vectors $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ represent orthogonal bases in the respective Hilbert spaces. Such vectors can be viewed as eigenvectors of some observables A and B that can be measured on each individual sub-system. The state of $\mathcal{S}_1$ is the mixed state given by the density matrix $\rho_1 = \sum_i |a_i|^2 P_i$, with the one-dimensional projectors P_i mapping onto the subspace of $\mathcal{H}_1$ spanned by the corresponding a_i. Suppose that one then measures a different observable X on $\mathcal{S}_2$. The set of eigenvectors $\{|\chi_k\rangle\}$ which are the eigenvectors associated with the eigenvalues λ_k of X spans the Hilbert space $\mathcal{H}_2$. Accordingly, the vector states of system $\mathcal{S}_1$ are transformed into the normalized, but not orthogonal vectors $|\alpha'_k\rangle = \sum_i a_{ik} |\alpha_i\rangle$. Indeed, any such a vector of $\mathcal{H}_1$ belongs to the range of the density matrix ρ_1 as well. The total state of the bipartite system takes the following form:

$$|\Psi'\rangle = \sum_k \omega_k |\alpha'_k\rangle] \otimes |\chi_k\rangle \tag{3.12}$$

where the constants ω_k depends on the basis selected by measuring X. If the measurement on $\mathcal{S}_2$ yields the outcome λ_k, one ought to assign the vector state α'_k to the other sub-system. This means, according to Schrödinger, that by a suitable choice of χ_k one could "steer" the state of $\mathcal{S}_1$ into any state lying in the range of ρ_1 with a non-vanishing probability $\|\omega_k\|^2$. Furthermore, in case all coefficients λ_i are distinct and non-zero, and hence the range of ρ_1 coincides with $\mathcal{H}_1$, one would obtain any chosen mixture of quantum states.

However, Schrödinger's argument does not show that the statistics of $\mathcal{S}_1$ can be changed at the whim of the experimenter. The term "steering" is indeed a bit misleading. The state of $\mathcal{S}_1$ is a mixed state, which can be expressed non-uniquely as different mixtures of pure states. The only thing that one can do by means of an appropriate choice of observable is to correlate the outcomes of a measurement on $\mathcal{S}_2$ with some specific mixture. Since the outcome of such a measurement is random, and thus the experimenter does not determine it, she cannot actually drive $\mathcal{S}_1$ into any particular state that she chooses. In other words, by operating on a quantum system, the experimenter has just the freedom to constrain the state of another spatially separated entangled system to evolve in any arbitrary set of states. To be sure, this is a remarkable non-local property of the quantum world due to the presence of entanglement, which has no classical analogue.

Schrödinger associated the non-locality arising from the possibility of "steering" with some sort of *action at a distance* between quantum systems, which he did not find acceptable at all. If the systems are separated far enough from each other, an experiment performed on one of them ought not to interfere with the other. This led him to doubt that entangled states are actually instantiated in nature.

> For it seems hard to imagine a complete separation, whilst the system are still so close to each other, that, from a classical point of view, their interaction could still be described as an unretarded *actio in distans*. And ordinary quantum mechanics, on account of its thoroughly unrelativistic character, really only deals with the *actio in distans* case. The whole system (comprising in our case both systems) has to be small enough to be able to neglect the time that light takes across the system, compared with such periods of the system as are essentially involved in the changes that take place. [Schrödinger (1936), p. 451]

So, he conjectured that entanglement does not persist over long distances. The physical systems $\mathcal{S}_1$ and $\mathcal{S}_2$ may share an entangled state only if their distance is such that the amount of time which is required for a light signal to travel from one to the other is comparable to the characteristic time periods of the changes occurring in the compound system. If their spatial separation becomes large enough, the entanglement between the two systems would spontaneously decay. Nevertheless, the persistence of entangled states between spatially separated systems was later on demonstrated experimentally, for instance by Aspect *et al.* (1982).

Incidentally, notice that Schrödinger also predicted that, on the basis of the non-local character of entangled correlations, any attempt to make quantum mechanics relativistic is bound to fail.

> Though in the mean time some progress seemed to have been made in the way of coping with this condition (quantum electrodynamics), there now appears to be strong probability (as P.A.M. Dirac has recently pointed out on a special occasion) that this progress is futile. [Schrödinger (1936), p. 451]

Contrary to his expectations, though, quantum theory has been extended to relativistic physics. As we shall see in the last chapter, quantum field theory provides an empirically successful model of relativistic quantum mechanics. Ironically, it can even be shown that entanglement is more robust in a relativistic context than in the ordinary case.

The peculiar form of non-locality characterizing quantum mechanics which arises from the presence of entanglement is properly expressed in a theorem by Bell (1964). Specifically, he derived an inequality from a locality condition which proves violated by certain quantum systems sharing an entangled state. This is actually what marks the main difference between classical and quantum correlations. I address this issue in the next section, where I cast

Bell's inequality in algebraic terms as well.

3.2.2 Bell-type non-locality: algebraically

The work by Bell (1964) can be regarded as an extension of the EPR argument. The content of the theorem he proved is that any realist physical theory that agrees with all statistical predictions of quantum mechanics is non-local in a specific sense. The underlying assumption of realism consists in postulating that the quantum state is supplemented by some further parameters in order to provide a complete description of reality. Such parameters would amount to the "hidden variables" of the theory. A Bell-type experiment is modeled as follows. Two particles are prepared in an entangled state described by the hidden variable λ and then move apart. Experimental apparatuses are then set to perform measurements on each sub-system. The model is local in the sense that the probabilities of joint outcomes factorize into the probabilities of the outcomes for each individual particle. From such an assumption of locality one derives a certain inequality, which is provably violated by matching correlations between two-valued observables of the two separated systems. What makes Bell's argument more general than the EPR case is that he considered correlations between different observables, not just the same observable.

In the algebraic framework the outcomes of an experiment are represented by projectors of a C*-algebra. In general, the observables of a Bell-type system are the elements of a C*-algebra $\mathcal{A}$ (with unit I) containing a pair of commuting C*-subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$, which are associated with the two subsystems. So, the outcomes of the measurements on each subsystem are given the projectors $E \in \mathcal{A}_1$ and $F \in \mathcal{A}_2$, respectively. Rather than dealing with projections, it is convenient to work with their self-adjoint contractions, that is $A := 2E - I$ and $B := 2F - I$. One thus obtains the observables $A_i \in \mathcal{A}_1$ and $B_j \in \mathcal{A}_2$ with $i, j = 1, 2$. Accordingly, the assumption of locality expressed by *factorizability* of joint probabilities reads

$$pr_{\lambda A_i B_j}(x, y) = pr_{\lambda A_i}(x) \cdot pr_{\lambda B_j}(y) \tag{3.13}$$

where x and y denote the results of A_i and B_j, respectively, and can take on either the value $+1$ or the value -1.

Jarrett (1984) showed that such a condition is the conjunction of two separated independence conditions on a single marginal probability. Specifically, one requires that the probability $pr_{\lambda A_i}$ is independent of the choice of the quantity to be measured in the other wing; the other requires that the probability of the outcomes of a measurement performed on one wing is statistically independent of the probability of any specific outcome obtained by a measurement performed on the other wing. The first provision is expressed by the relation

$$pr_{\lambda A_i}(x) = pr_{\lambda A_i B_j}(x) := pr_{\lambda A_i B_j}(x, y = 1) + pr_{\lambda A_i B_j}(x, y = -1) \qquad (3.14)$$

and, by following Shimony's (1986) terminology, it is called *parameter independence*[4]. The second provision stating

$$pr_{\lambda A_i Bj}(x, y) = pr_{\lambda A_i Bj}(x) \cdot pr_{\lambda A_i Bj}(y) \qquad (3.16)$$

is instead called *outcome independence*.

Whether Jarrett's decomposition of Bell's locality condition gives any real philosophical insight has been much debated. For instance, Maudlin (1994) objects that such a distinction is not just useless, but actually misleading. On the contrary, Howard (1985) argues that, whereas parameter independence embodies the relativistic constraint of no superluminal signaling, one ought to interpret outcome independence as a principle of spatio-temporal separability underlying Einstein's worries concerning quantum mechanics (in particular, the EPR paradox). Accordingly, to avoid a possible conflict with Special

[4]There is of course a similar condition for B_j, that is

$$pr_{\lambda B_j}(y) = pr_{\lambda A_i B_j}(y) := pr_{\lambda A_i B_j}(x = 1, y) + pr_{\lambda A_i B_j}(x = -1, y) \qquad (3.15)$$

Relativity arising from the failure of the no signaling constraint, one ought to maintain parameter independence. This would mean that the culprit in the violation of Bell's inequality is the failure of outcome independence.

Be that as it may, Butterfield (1995) demonstrated that the distinction between parameter independence and outcome independence can be recast in algebraic terms. The conceptual import of this formulation will become explicit in the last chapter. In fact, Bell's inequality does not hold in relativistic quantum mechanics either. Since Algebraic Quantum Field Theory introduces relativistic locality, and thus parameter independence, as an axiom, the failure of Bell-type locality must be entirely traced back to the failure of outcome independence. Below I present the details of the algebraic version of Jarrett's decomposition of Bell's inequality.

The initial state of the pair of systems is captured by a fixed state ϕ on the joint C*-algebra $\mathcal{A}$. The locality condition required by Bell-type models is retained by ϕ in a quite natural way if the sub-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ commute. Schlieder (1968) in fact proved that two bounded self-adjoint operators A and B acting on Hilbert space commute if and only if, for any partition $\{I_i\}$ of the spectrum of B, any state ϕ of A is such that

$$\phi(A) = \sum_i \phi[\int_i dP_\lambda A \int_i dP_\lambda] \tag{3.17}$$

where P_λ denote the projectors in the spectral resolution of B. This means that one observable's statistics is preserved under any measurement of the other observable. That is a version of the so-called no-signaling theorem, according to which instantaneous communication cannot be achieved between space-like separated systems by means of operations enacted by the (non-selective) Lüders rule.

To put it in another way, let us represent the fact that a measurement of the contraction A of $\mathcal{A}_1$ has outcome x by the symbol A_i^x and, similarly, the fact that a measurement of the contraction B of $\mathcal{A}_2$ has outcome y by the symbol B_j^y. The joint probability is thus given by $\phi(A_i^x B_j^y)$. So, factorizability requires

$$\phi(A_i^x B_j^y) = \phi(A_i^x) \cdot \phi(B_j^y) \tag{3.18}$$

Then, the linearity of the state ϕ and the fact that the subalgebras possess a common unit to which the outcomes of the measurements on each wing sum, i.e. $\sum_x A_x = \sum_y B_y = I$, guarantee that single probabilities are independent of what is measured in the other wing. Indeed, parameter independence can be rewritten as

$$\phi(A_i^x) = \phi[A_i^x(\sum_y B_1^y)] = \phi[A_i^x(\sum_y B_2^y)] \tag{3.19}$$

That is, the marginal probability of outcome x for A_i in the context of measuring B_j is obtained by summing out all the possible results of the latter. On the other hand, outcome independence reads

$$\phi(A_i^x B_j^y) = \phi[A_i^x(\sum_y B_j^y)] \cdot \phi[(\sum_x A_i^x)B_j^y] \tag{3.20}$$

hence the joint probabilities factorize into their marginals. That completes the re-formulation of Jarrett's decomposition of Bell's inequality in the algebraic framework.

The algebraic form of Bell's inequality was derived by Summers and Werner (1985) and by Landau (1987). To achieve such a result, one first defines the *Bell correlation*

$$\beta(\phi, \mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2} \sup \phi(A_1(B_1 + B_2) + A_2(B_1 - B_2)) \tag{3.21}$$

where the supremum is taken over the observables A_i and B_j. Accordingly, Bell's inequality is expressed by the relation $\beta(\phi, \mathcal{A}_1, \mathcal{A}_2) \leq 1$. It was demonstrated by Bell that maximal violation of such a bound, namely $\beta(\phi, \mathcal{A}_1, \mathcal{A}_2) = \sqrt{2}$, is attained in quantum mechanics. In particular, it occurs in case the sub-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ contain copies of the two-by-two complex Pauli spin matrices and the joint state ϕ is taken to be the singlet state (3.9). This is actually archetypal in the sense that, for $\phi|_{\mathcal{A}_1}$ and $\phi|_{\mathcal{A}_2}$

faithful states, whenever maximal violation of Bell's inequality is obtained the operators A_1, A_2 and $A_3 \equiv -\frac{i}{2}[A_1, A_2]$ are a realization of the Pauli matrices in $\mathcal{A}_1$, and so are the B_j in $\mathcal{A}_2$.

Bell's inequality is known to hold for classical correlations. Therefore, classical mechanics is a local theory. This is not particularly surprising as Bell's inequality was indeed derived from a hypothesis of local hidden variables corresponding to a classical model for the probabilities expected in an experiment such that considered by EPR. Pitowsky (1989) proved that the satisfaction of Bell's inequality is a necessary and sufficient condition for the existence of probability distributions over a classical probability space reproducing the relevant data. Yet, this is true only within the Hilbert space formalism. In the more general algebraic framework, Bell's inequality can be satisfied by quantum-mechanical states as well. Specifically, the bound $\beta(\phi, \mathcal{A}_1, \mathcal{A}_2) \leq 1$ for the Bell correlation is ensured by ϕ being a separable state, even though the algebras are non-commutative. As Bacciagaluppi (1993) argues,

> ... Pitowsky's results are not vindicating any correspondence between classicality and the Bell's inequalities rather than in the precise form he gave: as a matter of fact, *it is just the correlations that are classical.* Indeed, the catch-phrase "Bell's inequal-

> ities means that everything is classical" is wrong. Theorems ... show that, contrary to the case of Hilbert space quantum mechanics, in algebraic quantum mechanics there are systems that satisfy the Bell inequalities in all possible states, and nevertheless are *not* entirely classical - indeed, one of the subsystems may be completely quantum mechanical. [Bacciagaluppi (1993)]

In particular, Bacciagaluppi refers to a theorem by Raggio (1981), which he himself further elaborates in his paper. Such a result establishes that the following three conditions on two (general) C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are equivalent:

1. each state on the tensor product $\mathcal{A} \bigotimes \mathcal{B}$ is separable
2. $\mathcal{A}$ *or* $\mathcal{B}$ is commutative
3. each state on $\mathcal{A} \bigotimes \mathcal{B}$ satisfies the Bell's inequality

As statement 2 emphasizes, existence of unentangled states, and thus satisfaction of the Bell's inequality, does not require both algebras to be commutative. In fact, if at least one algebra is classical, then all the states across $\mathcal{A}$ and $\mathcal{B}$ are separable.

Furthermore, one should stress that not any non-separable state of a composite quantum system admits a description within the framework of local classical (hidden-variable) models. For Werner (1989) constructed examples of not classically correlated states of a composite quantum system which satisfy Bell-type inequality. Indeed, contrary to the case of entangled pure states described by unit vectors of $\mathcal{H}_1 \otimes \mathcal{H}_2$, it is not the case that all entangled mixed states violates Bell's inequality. In the last analysis, although entanglement is a necessary condition for Bell-type non-locality, it is not actually sufficient.

It should be clear how many conceptual puzzles that entertain physicists and philosophers of quantum mechanics are deeply rooted on the behaviour of entangled states. Since the beginning of the 90's, though, with the growing interest in quantum information, attitudes have started to change. Rather than a source of conceptual difficulties, entanglement has become a resource to exploit. Here is how Popescu and Rohrlich describe the switch to the new perspective in the introduction of their work "The Joy of Entanglement":

> ... today, the EPR paradox is more paradoxical than ever and generations of physicists have broken their heads over it.
>
> Here we explain what makes entanglement so baffling and surprising. But we do not break our heads over it; we take a more positive approach to entanglement. After decades in which everyone talked about entanglement but no one did anything about it,

> physicists have begun to *do* things with entanglement. [Popescu and Rohrlich (1998)]

In the next sections of this chapter I review and discuss some attempts to understand the foundation of quantum mechanics from an information-theoretical point of view.

3.2.3 On the Clifton-Bub-Halvorson's theorem

The attitude of quantum information theorists toward the basic concepts of quantum mechanics is eminently pragmatic. They are in fact interested in developing protocols in which the information encoded in the quantum states can be used in some manner. Specifically, one now treats

> ... entanglement as a resource that allows us to teleport quantum states and construct unbreakable codes, a resource that we can extract, purify, distribute and consume. The applications of entanglement lead us to develop new conceptual tools and to adapt old ones - in particular, the concept of entropy. Like Carnot, we face fundamental questions about how to use this resource most efficiently, and the concept of entanglement helps us exploit entanglement just as it helps us exploit energy efficiently. [Popescu and Rohrlich (1998)]

There is in fact a close connection between entropy and entanglement. For two systems A_1 and A_2 sharing an entangled state represented by the density ρ, the von Neumann entropy $S_{A_1}(\rho) = -Tr\rho_{A_1} \ln \rho_{A_1}$ of system A_1 varies with the degree of its entanglement with A_2. For instance, if ρ is a product state then the relative von Neumann entropy is 0, whereas if the systems are in a singlet state one has $S_{A_1}(\rho) = \ln 2$. Specifically, the more entrenched the entanglement between A_1 and A_2, the more disordered A_1 becomes in the sense that there will be several states available and the system's probabilities of occupying any such state will approach equality. Popescu and Rohrlich (1997) also argued that, just as the second laws of thermodynamics (where entropy in introduced) applied to Carnot's heat cycle implies that it is impossible to construct a *perpetuum mobile*, the impossibility of creating entanglement between two systems by acting locally on one of them implies that the von Neumann entropy of either member of the pair yields the (unique) measure of their entanglement when they share a pure state.

The analogy with thermodynamics can be put forward at the level of the foundations of quantum theory too. Thermodynamics is in fact the paradigm example of a class of theories that Einstein labeled "principle theories" as opposed to "constructive theories". Bub (2000) proposes to understand quantum mechanics as a principle theory. The technical basis for this claim is a theorem which he proved together with Clifton and Halvorson in 2001 (that is referred to as CBH theorem from the initials of the three authors), whereby the structure of quantum mechanics is derived from information-theoretical constraints, describing what one can or cannot do with information in the quantum world.

The distinction between principle and constructive theories was introduced by Einstein in an article written in 1919 for the issue of the London *Times* published on November 28. That article illustrates how he arrived at the formulation of his theories of relativity.

> We can distinguish various kinds of theories in physics. Most of them are constructive. They attempt to build up a picture of the more complex phenomena out of the materials of a relatively simple formal scheme from which they start out. Thus the kinetic theory of gases seeks to reduce mechanical, thermal, and diffusional processes to movements of molecules – i.e., to build them up out of the hypothesis of molecular motion. When we say that we have succeeded in understanding a group of natural processes, we invariably mean that a constructive theory has been found which covers the processes in question.
>
> Along with this most important class of theories there exists a second, which I will call "principle-theories." These employ the

> analytic, not the synthetic, method. The elements which form their basis and starting-point are not hypothetically constructed but empirically discovered ones, general characteristics of natural processes, principles that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy. Thus the science of thermodynamics seeks by analytical means to deduce necessary conditions, which separate events have to satisfy, from the universally experienced fact that perpetual motion is impossible. [Einstein (1919)]

Einstein regarded constructive theories as providing a more fundamental account for physical phenomena than principle theories. As he himself admitted, Einstein first looked for a constructive theory describing the properties of matter and radiation, but he eventually gave up and proposed the special theory of relativity in 1905 as a principle theory. An alternative constructive theory would be Lorentz's mechanical model of the electrodynamics of moving bodies, that derives the Lorentz transformations from some assumptions about the transmission of molecular forces through the ether. Yet, Lorentz theory is not acceptable in its original form.

As is well known, special relativity relies on two postulates. The first is the equivalence of inertial frames for all physical theories and the second is the constancy of the velocity of light in all inertial frames. The structure of spacetime thus arises as Minkowski geometry. Remarkably, Einstein characterizes the requirement of invariance of the laws of mechanics and electromagnetism under the Lorentz transformations from one inertial frame to another as "a restrictive principle for natural laws, comparable to the restricting principle for the non-existence of the *perpetuum mobile* which underlies thermodynamics" (Einstein (1949)). So, in analogy to the first and second law of thermodynamics, one can construct a principle theory out of some empirically justified and mathematically well formulated *no go* constraints.

Clifton, Bub and Halvorson's purpose is to claim that quantum mechanics is not a (constructive) mechanical theory of waves and particles in the first place, but ought to be viewed as a (principle) theory about the impossibility of information transfer. The structure of quantum theory, involving both noncommutativity and nonlocality, is equivalent to three information theoretical principles, that is (1) no signaling, (2) no cloning and (3) no bit commitment. The CBH theorem is entirely derived within the general framework of C*-algebras. Accordingly,,

1. the impossibility of superluminal information transfer between two physical systems implies, and is implied by, the kinematic independence of the algebras corresponding with the latter;

2. the impossibility of perfectly broadcasting the information contained in an unknown physical state implies, and is implied by, the non-commutativity of the algebra associated with each system;

3. the impossibility of unconditionally secure bit commitment implies, and is implied by[5], the fact that spacelike separated physical systems occupy (at least sometimes) entangled states.

The algebraic formalism allows one to appreciate that, if a physical theory is characterized in terms of such constraints, then it cannot be classical. Hence, the CBH theorem sets quantum mechanics apart from classical mechanics. The above principles are in fact features of quantum information.

I explained in the previous section how the impossibility of superluminal transfer of information from one system, say A_1, to another physically distinct system, say A_2, is connected to the fact that the relevant C*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ (which, as usual, are subalgebras of a larger C*-algebra $\mathcal{A} = \mathcal{A}_1 \bigvee \mathcal{A}_2$ describing the compound system $A_1 + A_2$) are kinematically independent. Since this means that each observable in one algebra commutes with all the elements of the other, no (non-selective) measurement performed on A_1 can convey any information to A_2: the statistics of the latter remains invariant

[5]Actually, this implication was left unproven by Clifton, Bub and Halvorson (2003). The derivation was completed by Halvorson (2004).

in the sense that the expectation values for the outcomes of measurements do not change.

Cloning is a special case of an information-theoretical protocol known as broadcasting, according to which a ready state σ of system A_2 and the state ω of system A_1 to be broadcast (that is the *input* pair), are transformed by means of some measurement-operation to a new state ϕ on the joint algebra $\mathcal{A}$ (that is the *output* pair), where the restriction of the latter state to the relevant subalgebras, namely $\phi|_{\mathcal{A}_1}$ and $\phi|_{\mathcal{A}_2}$, are both equivalent to ω. In cloning, one just deals with pure states, and hence σ and ω are both transformed to two copies of ω. Contrary to classical information theory, neither cloning nor broadcasting are possible in elementary quantum mechanics. Indeed, a pair of input pure states can be cloned if and only if they are orthogonal; more generally, a pair of input mixed states can be (perfectly) broadcast if and only if they are represented by mutually commuting density operators.

Therefore, as Clifton, Bub and Halvorson showed, if any two states of a C*-algebra can be broadcast, then all pure states are orthogonal, and hence the algebra must be commutative. This means that no cloning (and, more generally, no broadcasting) is equivalent to $\mathcal{A}$ being non-commutative. As the physical manifestation of non-commutativity is the quantum phenomenon of interference, the information-theoretical counterpart of the latter is just the impossibility of copying the information contained in an unknown quantum state. Equivalently, one may say that quantum states cannot be cloned by means of any measurement-operation because the latter always disturbs the state of a system on which they are performed.

It is interesting to point out that some important procedures of quantum information, such as quantum teleportation, rely on no-cloning. Teleportation, which was first proposed by Bennett *et al.* (1993), is one of most striking phenomena arising from the quantum formalism as entanglement assisted communication. Supposed two experimenters, Alice and Bob, located in spatially separated sites can operate on distinct physical systems sharing the singlet state (3.9). Alice receives another spin-particle system prepared in an unknown quantum state $\chi = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ with complex coefficients α and β. Alice measures an observable with four possible outcomes, corresponding to the four Bell states, on the two particles in her possession. The four outcomes of Alice's measurement are correlated with four states of Bob's system, which turn out to be either the same as the unknown state χ, or unitarily related to χ. If Alice sends the outcome of her measurement, which amounts to two bits of classical information, to Bob, then Bob can reconstruct the state χ by either doing nothing or by performing an appropriate unitary tranformation on his particle.

Notice, however, that the alleged teleportation is just seeming. First of all, it is not matter, namely the particle, which is teleported, but just a quantum state. Furthermore, the trick is made possible by the fact that Alice cannot clone the state χ. If measurements did not always change the state of a quantum system, the particle entangled to Alice's system may well remain in the state χ. In such a case, once the above protocol is completed, one would not be in a position to claim that teleportation was achieved at all.

The last statement of the CBH theorem involves the no bit commitment protocol, which supposedly guarantees entanglement maintenance over distance. In a bit commitment protocol Alice supplies an encoded bit to Bob as a warrant for her commitment to either one of two values, say 0 and 1. The amount of information available does not allow Bob to ascertain the value of the bit at the initial stage. 'At a later opening stage Alice reveals her commitment and communicates additional information to Bob. The additional information is required to be sufficient for Bob to be convinced that Alice was not able to cheat, in the sense that the encoded bit could not have been encoded in such a way that Alice was free to reveal either of the values. Bennett and Brassard (1984) showed that by encoding 0 and 1 in two quantum mechanical mixtures represented by the same density operator ρ Alice can actually cheat by adopting an EPR-type strategy. Suppose she prepares pairs of particles A_1 and A_2 in the same entangled state ϕ such that $\phi|_{\mathcal{A}_2} = \rho$. If she keeps one pair and sends the other to Bob, then she can reveal either bit at will by steering Bob's particles into the desired mixture by performing an appropriate measurement on her particle. Bob is not able to detect the cheating strategy. This means that unconditionally secure bit commitment is not possible in quantum mechanics.

Actually, unconditionally secure bit commitment is impossible for classical systems as well. Yet, the reason is quite different from the quantum case, since the fact that the protocol is not secure depends on issues of computational complexity. Classically, Alice's commitment to either 0 and 1 is equivalent to the truth of an exclusive disjunction. She can send the relevant (encrypted) information to Bob only if such information is biased toward one of the alternative disjuncts, but there is no principle of classical mechanics which prevents him from extracting the information. On the contrary, what makes it possible for Alice to cheat in quantum mechanics is the fact that, *pace* Schrödinger, remote steering is indeed possible.

> What thwarts the possibility of using the ambiguity of mixtures in this way to implement an unconditionally secure bit commitment protocol is the existence of nonlocal entangled states between Alice and Bob...
>
> So what would allow unconditionally secure bit commitment in a nonabelian theory is the absence of physically occupied nonlo-

> cal entangled states. One can therefore that take Schrödinger's remarks as relevant to the question of whether or not secure bit comiitment is possible in our world. In effect, Schrödinger believes that we live in a quantum-like world in which secure bit commitment is possible. [Clifton, Bub and Halvorson (2003)]

The theorem by Raggio and Bacciagaluppi that I spelled out in the previous section guarantees the possibility of entangled states if at least one of the algebras is noncommutative, irrespective of whether or not the corresponding systems are spatially separated. Thus, since noncommutativity follows from the no-cloning principle, the latter is sufficient to entail that non-local entangled states exist. As a consequence, one may argue, appealing to no bit commitment in the context of the CBH theorem is not actually necessary. It could be, though, that in a more general framework than the algebraic setting, namely that of convex sets, the role of no bit commitment in the derivation of quantum mechanics from information theoretical principles could be actually restored[6].

The work by Clifton, Bub and Halvorson makes explicit the structural connections between information theory and the foundations of quantum theory. Allegedly, the philosophical import of their theorem is that quantum

[6] See Barnum *et al.* (2008).

mechanics is a theory about information. This view leads one to regard quantum states as states of information, rather than complete descriptions of states of affairs. A position that heavily relies on this attitude is the Bayesian interpretation of quantum mechanics. I now address some problems arising in the context of such an approach.

3.3 The Problem of Bayesian Statistical Inference

3.3.1 The noncommutative generalization of the Bayes rule

According to the Bayesian interpretation of quantum mechanics (Caves, Fuchs and Schack 2002, Pitowsky 2003), quantum states reflect the degrees of belief of an observer, who is in turn conceived as an ideally rational agent. Furthermore, one relies on the assumption that a quantum measurement is tantamount to performing a Bayesian statistical inference. Supposedly, this puts one in a position to elude the measurement problem, as the so-called wave-function collapse would just reduce to a quantum analogue of the classical Bayes rule, whereby the agent-observer revises her degrees of belief on the basis of the evidence acquired by performing a measurement.

That is, quantum measurements are tantamount to Bayesian statistical inferences. So, a necessary condition for any Bayesian interpretation of quantum mechanics is the existence of a noncommutative conditionalization rule generalizing the Bayes rule. This forces the probability states on $\mathcal{B}(\mathcal{H})$ to retain certain properties. As von Neumann algebras theory represents the proper non-commutative probability theory, one can recast some of these properties in algebraic terms and see whether they hold in the quantum case.

We saw in the first chapter that the problem of statistical inference in quantum mechanics was first posed by von Neumann in his earlier works on the mathematical foundation of quantum mechanics. Statistical inference was required in order to derive the probability statements of the theory. As he emphasized, it characterizes both classical and quantum theories.

> If anterior measurements do not suffice to determine the present state uniquely, then we may still be able to infer from those measurements, under certain circumstances, with what probabilities particular states are present. (Von Neumann 1932, 337)

Since measurements cannot determine quantum-mechanical states uniquely, the problem of how to infer probabilities in quantum mechanics became compelling to complete von Neumann's mathematical formulation of the theory.

Then, noncommutative statistical inference is required in the context of a Bayesian interpretation of quantum probabilities as well. Redei (1989, 1992, 1998) set the problem within the algebraic framework. We now see how the classical Bayes rule is fully generalized in von Neumann algebras theory.

Classically, the problem of Bayesian statistical inference can be formulated as follows: how should a rational agent revise her *a priori probability* μ about any event $A \in \Omega$, on the basis of knowing the probability of some other event? In the simplest case, when she happens to know that an event B has occurred (that is, it is assigned probability 1), the Bayes rule establishes that the *conditional probability* μ', namely her revised degrees of belief, is given by

$$\mu'(A) \equiv \frac{\mu(A \cap B)}{\mu(B)} \tag{3.22}$$

One then considers the more general scenario in which the agent happens to learn the probabilities of a set of events belonging to a subalgebra Ω_0 of Ω (rather than the probability of a single event). Recall that the probability measure μ on Ω corresponds to the *a priori* state w on $L^\infty(\Gamma, \Omega, \mu)$, so the collection of probabilities the agent is presented with amount to the state w_0 on the von Neumann subalgebra $L^\infty(\Gamma, \Omega_0, \mu_0)$ of $L^\infty(\Gamma, \Omega, \mu)$, with μ_0 being the restriction of μ to Ω_0 (i.e. $\mu_0 = \mu|_{\Omega_0}$). Algebraically, the Bayesian recipe gives the conditional probability of any event in Ω/Ω_0 by

$$w'(\chi_A) \equiv w_0(T\chi_A) \tag{3.23}$$

The mapping T from $L^\infty(\Gamma, \Omega, \mu)$ onto $L^\infty(\Gamma, \Omega_0, \mu_0)$ denotes the classical *conditional expectation.*

Importantly, the Radon-Nykodym theorem guarantees that such a map always exists and that it also preserves the *a priori* probability μ. In particular, when Ω_0 is generated by a countable set of disjoint elements B_i, the conditional expectation takes the form

$$T(\chi_A) = \sum_i \frac{\mu(B_i \cap A)}{\mu(B_i)} \chi_{B_i} \tag{3.24}$$

where one immediately recognizes the Bayes rule (Loeve 1962, 340).

The problem of noncommutative statistical inference as formulated by Redei (1989, 1992, 1998) recasts the classical case completely in algebraic terms. The agent's *a priori* probability is a state ϕ on a general von Neumann algebra $\mathcal{M}$. Then, if she learns the probabilities of all the events belonging to the von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ (with common unit), which are encapsulated in the probability state ψ_0, the noncommutative conditionalization rule is expressed by:

$$\phi'(A) = \psi_0(TA) \tag{3.25}$$

for any event $A \in \mathcal{P}(\mathcal{M})$, where T represents the noncommutative conditional expectation mapping from $\mathcal{M}$ onto $\mathcal{M}_0$.

Let us apply the thus defined Bayesian recipe to the case of quantum measurements. The agent-observer's *a priori* probability is defined by a state ϕ on the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Performing a measurement provides the evidence on the basis of which she revises her degrees of belief. Any measured observable B generates the Boolean algebra comprising all the bounded operators commuting with its spectral projections. This corresponds to a subalgebra $\mathcal{B}_0$ of $\mathcal{B}(\mathcal{H})$. Then, the quantum conditional expectation map would depend on the details of the physical quantity which is measured.

For instance, suppose the measured observable has a discrete spectrum, that is $B = \sum_i \lambda_i P_i$ (with $i = 1, 2, ..., n$), where $P_i = |\psi_i\rangle\langle\psi_i|$ denote the spectral projections and λ_i the eigenvalues corresponding to the eigenvectors $|\psi_i\rangle$. That immediately selects the Boolean algebra $\mathcal{B}_0 \equiv \{A_0 \in \mathcal{B}(\mathcal{H}_n) | A_0 P_i = P_i A_0\}$, with $\mathcal{H}_n$ being a finite n-dimensional Hilbert space. The conditional expectation T^B projecting any $A \in \mathcal{B}(\mathcal{H}_n)$ onto an element $T^B(A) = A_0$ of $\mathcal{B}_0$ is expressed by the quantum-mechanical Lüders rule

$$T^B(A) = \sum_i P_i A P_i \tag{3.26}$$

Accordingly, the conditional probability state determined by performing a measurement of B reads $\phi'(A) = \psi_0(T^B A)$. The claim that the Lüders rule generalizes the classical Bayes rule to quantum mechanics in finite dimensions was first put forward by Bub (1977).

To resume, the general problem of Bayesian statistical inference is characterized as follows. Let the state ϕ on an arbitrary von Neumann algebra $\mathcal{M}$ be the *a priori* probability of the agent. The *evidence*, namely the probabilities of a set of events, which she is presented with is given by a state ψ_0 on a von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$. What is the *conditional* probability state ϕ', that is the extension of ψ_0 to $\mathcal{M}$?

The answer to such a question rests on the existence of a proper conditional expectation in a two-fold sense:

1. conditional probabilities $\phi' = \psi_0 \circ T$ exist on $\mathcal{M}$ only if a map T can be defined

2. whether or not conditional probabilities obey rationality constraints depends on the properties of such map T

We shall see in section 3.3.3. that the solution to the problem of Bayesian noncommutative statistical inference is not at all straightforward. Difficult technical and philosophical issues in fact arise both at the level of quantum mechanics and at the level of general probability spaces. Even before determining how the agent should revise her degrees of belief on the basis of certain evidence, though, one is needs to establish the probability she ascribes to any event on the basis of *no* evidence. That is the *a priori* probability state ϕ on $\mathcal{M}$. This is the topic we address in the next section.

3.3.2 Can the *a priori* probability be derived from symmetry considerations?

By definition, the *a priori* probability reflects the agent's most natural probability assignment in those circumstances in which she does not have any specific knowledge. From the point of view of a Bayesian interpretation of quantum mechanics, that is tantamount to establishing what state ϕ on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators acting on Hilbert space captures the agent-observer's degrees of belief before performing any measurement. In particular, one may ask whether an *a priori* quantum state can be determined by the physical symmetries of quantum-mechanical systems and, if so, whether such a state is actually unique. The analysis I offer in this section is contained in Valente (2009).

The problem of determining the *a priori* probability has been debated at length since the eighteenth century, as Laplace provided a definition of the (classical) probability of an event as the ratio of favourable cases to the total number of equipossible cases. Specifically, whenever there is no evidence favoring one possibility over another, equipossible cases are ascribed equal probabilities: this is the principle of indifference. Several arguments have been formulated against such a claim, which can be subsumed under the family of so-called Bertrand's paradoxes (Bertrand 1889). Accordingly, unless some further constraint is specified, one cannot determine unambiguously the *a priori* probability on the basis of the principle of indifference alone.

The class of arguments leading to such a conclusion against the principle of indifference has been investigated, among others, by van Fraassen (1989). He provides the following example that well characterizes the family of Bertrand's paradoxes. Let us suppose that a factory produces iron cubes whose edge length is ≤ 2 units, then one is asked for the probability that a random cube has length ≤ 1. Now, if one assumes the length as parameter the principle of indifference would led one to assign probability $\frac{1}{2}$; however if one considers the area or even the volume, instead, the probability would be $\frac{1}{4}$ or $\frac{1}{8}$ respectively.

E.T. Jaynes (1968, 1973) observed that Bertrand's paradox can be solved by focusing on the geometrical symmetries in the problem. For instance, if one throws a fair dice, one would ascribe equal probability $\frac{1}{6}$ to each of the six faces: intuitively, the "fairness" of the dice implies that it is a perfectly symmetric object, hence there is no evidence to favor one possibility on the others. Thus, one can derive the *a priori* probability by invariance under the group of symmetries characterizing the physical situation. In light of this, Jaynes advocated a Bayesian interpretation of probability in physics, whereby the agent's *a priori* probability is uniquely determined by invariance under the group of symmetries characterizing the physical problem. Specifically, by relying on the basic desideratum that "in two problems where we have the same prior information, we should assign the same prior probabilities" [Jaynes (1968)], any rational agent is compelled to ascribe the same probability by the so-called principle of maximum entropy. Nonetheless, he ended up proving that classical probability depends on the choice of the relevant symmetry transformation, and hence it cannot be uniquely fixed. In fact, when dealing with continuous parameter spaces, the results lack invariance under changes of parameter.

However, one may expect that a physical theory with an intrinsic probability structure, such as quantum mechanics, can be so tightly constrained that a symmetric *a priori* probability is naturally defined on it. Recall that in the axiomatic treatment of "Continuous geometries with transition probability" von Neumann showed that on the factors of type I_n and II_1 a (transition) probability function is uniquely fixed by the symmetries of the projection lattice. In fact, *Axiom XII* requires quantum probability to be invariant under any automorphism of the lattice which leaves the logic of the theory invariant too; on the other hand, *Axiom XIII* establishes the existence of such automorphisms. The uniqueness of the probability function followed by placing the latter constraint, which actually marks a structural difference with respect to classical theories. As von Neumann himself commented,

> It is worth emphasizing, that while these two kinds of logics have been used in classical mechanics too, their connection there was much looser. In particular there is no equivalent in the classical case for our second mentioned connection (expressed by Axiom XIII). [von Neumann (1936)]

Of course, as a special case of transition probabilities, the *a priori* probability is also determined by the symmetries of the lattice of projections, if the von Neumann algebras are of type I_n and type II_1.

Let us see how such insight can be translated into more general algebraic terms. The dimension function on the projection lattice of any finite factor can be uniquely extended to a finite trace. So, the *a priori* probability state on a type II_1 factor $\mathcal{M}$ is a tracial state τ. That is just the unique state which is fixed under all unitaries on the algebra. This implies that the *a* *priori* probability of any event $A \in \mathcal{P}(\mathcal{M})$ is defined by the property

$$\tau(A) = \tau(U^* A U) \tag{3.27}$$

for all unitary operators U. Accordingly, as Redei (1998) pointed out, the notion *a priori* acquires a precise physical meaning: it reflects the symmetry of the system.

> Since the physical symmetries of the system are generally expressed as representations of the symmetry group on the algebra of observables by unitaries, the existence of a unique trace means physically that the probability is determined uniquely as the only (positive, linear) assignment of values $[0, 1]$ to the events that is invariant with respect to any conceivable symmetry. [Redei (1998)]

To put it into the language of noncommutative probability theory, a symmetric *a priori* probability is defined in a general probability space $(\mathcal{M}, \phi)$ if and only if there exists a unique state on $\mathcal{M}$ which is invariant under the group of unitaries $\mathcal{U}$. That constrains ϕ to be a tracial state.

Then, in the framework of Hilbert space quantum mechanics, one must determine whether a (possibly finite) trace which is invariant under the unitary group can actually be defined in any von Neumann algebra factor of type

I. From a Bayesian point of view, the presence of such a functional means that the agent-observer's degrees of belief would be consistently determined by the natural symmetries of quantum-mechanical systems. Furthermore, its uniqueness would compel different agent-observers to agree upon the same *a priori* probability state.

One can easily check that the trace over Hilbert space is invariant under unitaries. By definition any unitary operator satisfies the relation $UU^* = I$. When acting on Hilbert space, U leaves the angle between vectors (namely the transition probability) invariant, in the sense that $\langle \xi, \eta \rangle = \langle \xi, U\eta \rangle$ for any $\xi, \eta \in \mathcal{H}$. Moreover, all symmetry transformations $U^*(\cdot)U$ map any projection in $\mathcal{P}(\mathcal{H})$ onto another projection. As the trace is insensitive to noncommutativity, namely $Tr(AB) = Tr(BA)$ for all elements A and B in the algebra of observables, for any projection P_d^A in the lattice, one can write

$$Tr(IP_d^A) = Tr(UU^*P_d^A) = Tr(U^*P_d^A U)$$

Then to obtain a probability state, Tr needs to be extended to a positive, linear functional taking values in the unit interval $[0, 1]$, which requires a renormalization of the trace. If this can be done, the *a priori* probability of any event is invariant under all unitaries U.

For instance, if Hilbert space is n-dimensional, the *a priori* probability is given by $\tau = \frac{1}{n}Tr$, which retains the sought-after symmetry property. In infinite dimensions the construction of Tr by unitaries can be carried out. However, as we have emphasized a number of times, the trace on $\mathcal{B}(\mathcal{H})$ is not finite, and hence it cannot be normalized. Therefore, the *a priori* probability cannot be a tracial state. Indeed, no tracial state τ is defined on any von Neumann algebra factor of type I_∞. The failure of the symmetry property to be preserved under the limit of Hilbert space quantum mechanics in infinite dimensions would thus imply that the agent-observer's *a priori* probability cannot be determined by the physical symmetries of any quantum-mechanical system.

So, the pathological behaviour of Hilbert space in infinite dimensions that bothered von Neumann, namely the fact that the *a priori* probability can be infinite, seems to pose some difficulty for a Bayesian interpretation of quantum mechanics too. Yet, one may solve the problem by dropping the assumption of normality of states, which is tantamount to the probabilities being countably additive. In fact, if one drops the requirement that, for any family of projection $\{P_i\}$ of $\mathcal{M}$, the relation

$$\phi(\sum_i^n P_i) = \sum_i^n \phi(P_i) \tag{3.28}$$

holds for n, the state ϕ is no longer normal. If the above formula is not satisfied for $n = \infty$, the states on $\mathcal{M}$ are just *singular* states, which entails probabilities being only finitely additive. As a matter of fact, noncommutative probability spaces $(\mathcal{M}, \phi)$ are defined also if ϕ is a singular state. Diximier (1966) actually demonstrated that singular tracial states τ can be defined on $\mathcal{B}(\mathcal{H})$.

The move of relaxing the assumption of countable additivity to circumvent conceptual problems in infinite dimension was also made by Halvorson (2001). He proposes to appeal to singular states to solve some difficulties for the ignorance interpretation of quantum mechanics which arise when measuring continuous observables. His argument, though, hinges on the weak claim that normal states are not necessary. From a Bayesian point of view, instead, dropping normal states can be conceptually justified. In fact, de Finetti (1974) rejected countable additivity as a crucial property for subjective probabilities, as it proves inconsistent with the requirement of equiprobability, which he regarded as a more fundamental constraint. The latter demands that any event is assigned the same non-zero probability. This means that the probability distribution is uniform. However, if one considers a countable set of events, the total probability cannot be normalized, since the sum of the probabilities of all events would diverge. Therefore, insisting on equiprobability forces the probability to be only finitely additive.

Although the symmetry property (3.27) can now be extended to infinite dimensional Hilbert space quantum mechanics, if one assumes only singular states, a new conceptual problem arises. Indeed, one can also show that there exist more than one non-normal trace on $\mathcal{B}(\mathcal{H})$ (see Kadison and Ringrose 1997). Hence, the *a priori* probability, despite being invariant under all unitaries, is not uniquely defined by their symmetry group. From a Bayesian point of view, one may interpret this result in the following way: the *a priori* probability in quantum-mechanics is determined by the physical symmetries of the system, but (unless some further constraint is added) the agent-observer would still maintain a certain arbitrariness in her degrees of belief, which she could associate with any different finite trace.

It should also be notice that the non-existence of unique tracial state on $\mathcal{B}(\mathcal{H})$ undermines only a Bayesian interpretation of quantum mechanics that relies on the existence of a symmetric *a priori* quantum probability. Yet, it does not rule out a Bayesian interpretation of quantum probabilities simpliciter. Indeed, one may advocate a more extreme form of subjectivism, whereby the *a priori* probability is assigned arbitrarily by any individual agent, rather than being uniquely determined for all possible agents by non-subjective constraints. According to such a view, the agent-observer's degrees of belief before performing any measurement may well be represented by a general state ϕ that does not need to be tracial. The only requirement is that the degrees of belief of different agent-observers converge to the same probability assignment in the long run, that is after repeating a measurement for a sufficiently large number of times.

Instead, some more compelling threat for any Bayesian interpretation of quantum probabilities, whether mildly or radically subjectivist, would come from the failure of conditional probability states ϕ' to exist and to satisfy constraints of rationality. As we discuss in detail now, this poses serious problems for the solution of the general problem of noncommutative statistical inference.

3.3.3 Placing constraints on quantum conditional probabilities: against Redei's stability condition

A necessary condition for quantum states to be interpreted as subjective probabilities is the formal adequacy of the relationship between Bayesian probability theory and the structure of quantum mechanics. Bayesianism requires the agent's degrees of belief to obey constraints of rationality. In particular, such requirements are placed upon the noncommutative Bayes rule which is supposed to represent any quantum measurement. Thus, the failure of a rationality constraint in quantum mechanics would prove the Bayesian interpretation of the theory inconsistent.

Redei (1992, 1998) formulated a *stability condition* as a rationality constraint characterizing Bayesian statistical inference. Acoordingly, a noncommutative conditional expectation map is required to retain some specific mathematical property. He then showed that, although it is satisfied by classical probabilities, the stability condition does not hold in quantum theory. Hence, the Bayesian interpretation of quantum mechanics would be inconsistent. More generally, this also questions the extension of Bayesian probability theory to noncommutative spaces of events.

Specifically, Redei's stability condition can be summed up as follows:

> If, after the first statistical inference $\mu \longrightarrow \mu'$, a rational agent revises her new degrees of belief μ' on the basis of the *same* evidence again, her newly revised degrees of belief μ'' will not differ from μ'

Arguably, a stability property really characterizes Bayesian statistical inference. For, if $\mu' \neq \mu''$, then the agent would end up having different degrees of belief in light of the same evidence; as a consequence, without further information, she will be unable to decide rationally between μ' and μ'', so either she "should choose one of them irrationally (for instance by tossing a coin), in which case the chosen new degrees of belief could no longer be considered as degrees of *rational* belief, or the agent's degrees of belief become undefined" (Redei 1992, 130). Moreover, the agent cannot count on a third inference in order to decide between them, as her new degrees of belief μ''' will be in general different from both μ' and μ'', thus increasing her frustration. In the final analysis, the stability condition seems to place a rationality constraint on Bayesian probability theory.

For instance, the Bayes rule (3.22) trivially fulfills such property. In fact, since $\mu'(B) = 1$, one obtains

$$\mu''(A) = \frac{\mu'(A \cap B)}{\mu'(B)} = \mu'(A \cap B) = \frac{\mu(A \cap B)}{\mu(B)} = \mu'(A) \tag{3.29}$$

Then, one is compelled to extend the stability condition to general probability spaces. That is what Redei proposes to do by spelling out the mathematical details.

Redei's first step is to account for commutative statistical inferences, wherein the agent happens to learn the probabilities of all the events in the subalgebra Ω_0 of Ω. Supposedly, this would retain a more general notion of evidence: in fact, the agent is not presented with the probability of a single event as in the previous case (in which the event $B \in \Omega$, as it had occurred, was assigned probability one), but rather with a whole set of probabilities. Algebraically, this is captured by the probability state w_0, namely the restriction of the *a priori* state w to $L^\infty(\Gamma, \Omega_0, \mu_0)$. Conditionalizing on the basis of this evidence yields a conditional state $w' = w_0 \circ T$, where the conditional expectation T is applied to any event in the algebra $L^\infty(\Gamma, \Omega, \mu)$. Then, if the agent performs a second statistical inference, her newly revised degrees of belief amount to the probability state $w'' = w_0 \circ T \circ T$. So, the stability condition claiming $w' = w''$ holds if and only if $T = T^2$. Classically, such a property is always satisfied. In particular, the classical conditional expectation is defined by the relation

$$\int_B \chi_A d\mu = \int_B T\chi_A d\mu_0 \tag{3.30}$$

for any $B \in \Omega_0$. That shows that the map T acts as the identity in $L^\infty(\Gamma, \Omega_0, \mu_0)$; hence, it is a projection from $L^\infty(\Gamma, \Omega, \mu)$ onto $L^\infty(\Gamma, \Omega_0, \mu_0)$.

The second step in Redei's generalization is to extend this result to non-commutative probability theory, where the agent's degrees of belief are probability states on a general von Neumann algebra $\mathcal{M}$. Again the evidence is conceived in the broad sense as a collection of probabilities: accordingly, the agent is presented with a probability state ψ_0 on any von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$. If she is presented with the same evidence again, that is with the probabilities of all the events belonging to $\mathcal{M}_0$, the stability condition requires that the agent's newly revised degrees of belief remain the same, that is $\phi'' = \phi'$.

By analogy with the classical case, Redei argues that the property $T = T^2$ of conditional expectation should be maintained in order for a rational agent to perform stable statistical inferences. This constrains the map T to be a *projection* of norm one from $\mathcal{M}$ onto $\mathcal{M}_0$. The proper conditional expectation for Bayesian statistical inference would then be a linear mapping $T : \mathcal{M} \rightarrow \mathcal{M}_0$ with the following properties

- (i) $T(A) \geq A$ if $A \geq 0$ (positivity)
- (ii) $T(I) = I$ (unit preservation)
- (iii) $\phi(T(A)) = \phi(A)$ for all $A \in \mathcal{M}$ (ϕ-preservation)
- (iv) $T(AT(B)) = T(A)T(B)$ for every $A, B \in \mathcal{M}$

That is also called a *ϕ-preserving conditional expectation*. Remarkably, for $A_0 \in \mathcal{M}_0$, property (iv) can be broken down into (iv') $T(AA_0) = T(A)A_0$ and (iv'') $T(A_0) = A_0$, which implies $T = T^2$.

Nonetheless, as Redei pointed out, norm-one projections rarely exist. In particular, a ϕ-preserving conditional expectation cannot be defined in quantum mechanics if the evidence is associated with any subalgebra of $\mathcal{B}(\mathcal{H})$ generated by a measurement. Therefore, the stability condition is violated in general. The non-existence of norm-one projections in quantum mechanics and, more generally, in any noncommutative space of events patently violates a rationality constraint for Bayesian statistical inference; furthermore, it implies that conditional probability states are not always defined. As a consequence, a *proper* noncommutative conditional expectation map would not exist.

Therefore, we can distinguish two problems stemming from Redei's argument at different levels:

1. On the one hand, it undermines a Bayesian interpretation of quantum mechanics, as quantum measurements cannot be described by a noncommutative Bayes rule. That is a problem of philosophy of physics, which arises when translating the algebraic formalism into the language of the theory.

2. On the other one, it blocks the extension of Bayesian probability theory to general spaces of events. Whether a proper conditional expectation for noncommutative statistical inference can be defined is a genuine question of philosophy of probability.

I first reject the conclusion in the first point and then, toward the end of the section, I answer the second question. Let us now highlight some further details of the argument, on which my subsequent discussion will be based.

In particular, the domain of existence of norm-one projections is established by a theorem due to Takesaki. Specifically, given a von Neumann algebra $\mathcal{M}$, its von Neumann subalgebra $\mathcal{M}_0$ and a state ϕ on $\mathcal{M}$,

> **Takesaki theorem** (1972): a ϕ-conditional expectation T projecting from $\mathcal{M}$ onto $\mathcal{M}_0$ exists (and is known to be unique) if and

only if $\mathcal{M}_0$ is invariant under the one-parameter automorphism group σ_t^ϕ generated by ϕ.

The one-parameter automorphism group σ_t^ϕ, which is uniquely associated with the *a priori* state ϕ, is taken by Redei to represent the dynamics on $\mathcal{M}$ with respect to time t. So, according to the theorem, a necessary and sufficient condition for a norm-one projection to exist is $\mathcal{M}_0$ being stable under the dynamics, i.e. $\sigma_t^\phi(\mathcal{M}_0) = \mathcal{M}_0$. That is a rather rare circumstance.

One should stress that interpreting the modular automorphism group as a time-evolution may be a little inappropriate. Indeed, we know that it is trivial if ϕ is tracial state. Hence, there is no actual evolution in time, as σ_t^ϕ acts as the identity at any instant t. This would actually happen in important physical cases, for instance when dealing with finite-dimensional Hilbert space quantum mechanics. However, the introduction of a dynamics allows one to make some further remarks.

Indeed, Redei is not explicit on how one should apply the conditionalization rule the second time. He does not say at all how the agent is supposed to update the conditional state $\phi' = \psi_0 \circ T$ she previously obtained by revising the original *a priori* state ϕ on the basis of knowing the probabilities of all the events in $\mathcal{M}_0$. Arguably, a probability state ψ_0' on $\mathcal{M}_0$ is extended to a state on $\mathcal{M}$ by the conditional expectation T, so that $\phi'' = \psi_0' \circ T$. As the agent performs the second statistical inference at any time t after the first statistical inference, the new evidence ψ_0' amounts to the probabilities of all the events belonging to $\mathcal{M}_0$ at time t. That is $\psi_0' = \psi_0 \circ \sigma_t^\phi$.

Importantly, the statement of the stability condition contains an implicit requirement, namely that both the first and the second statistical inference are carried out in light of the *same* evidence. So, in order to apply the stability condition, one has to assume $\psi_0 = \psi_0'$. This means that the given probability state on the subalgebra $\mathcal{M}_0$ ought not to change under the modular automorphism group. I will demonstrate that such a requirement does not hold in general in the quantum case, and hence, pace Redei, the stability condition for Bayesian statistical inference cannot really fail there.

In finite-dimensional Hilbert space quantum mechanics one can easily check that, if the *a priori* state ϕ on $\mathcal{B}(\mathcal{H}_n)$ is given by a density matrix ρ commuting with all spectral projections P_i of the discrete measured observable B, the map T^B defined by (3.26) is a ϕ-preserving conditional expectation. Therefore, it fulfills the stability condition. Nonetheless, when the physical qunatity B has a continuous spectrum, the Lüders rule cannot be applied: indeed no eigenvector is associated with the spectral measures in the continuum, hence there is no projection such as $P_i = |\psi_i\rangle\langle\psi_i|$. In the infinite-dimensional case the Boolean algebra onto which T^B projects amounts to the subalgebra $\mathcal{B}_0 = \{A_0 \in \mathcal{B}(\mathcal{H}) | P_d^B A_0 = A_0 P_d^B\}$ of $\mathcal{B}(\mathcal{H})$, where P_d^B represents the possibility that the value of B lies in the Borel set d. A general version of the Lüders rule would then require extending the sum to an integral. Davies (1976, 60) proved that there is no way to define such an integral so that it converges. More dramatically, he showed that a norm-one projection $T^B : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}_0$ does not exist.

As a result, Redei claimed that the stability condition fails to hold in quantum mechanics, and thus quantum states do not obey a rationality constraint. Moreover, as a conditional probability state $\phi' = \psi_0 \circ T^B$ cannot be defined in general, the agent-observer would not even be able to revise her degrees of belief on the basis of knowing the probabilities of all the events of the form 'B has its value in the set d of real numbers' (Redei 1992, 130). Allegedly that proves the violation of a necessary condition for a Bayesian interpretation of quantum mechanics, since quantum measurements could not be regarded as Bayesian statistical inferences. Alternatively, one would be left with the rather unsatisfactory claim that quantum states are degrees of belief of a rational agent-observer only in special cases, namely when the measured observables have a discrete spectrum[7].

At this point, one should stress that Redei's argument relies on a hidden assumption, namely that quantum states are normal states. If one as-

[7]Notice, however, that the stability condition is violated also in the finite dimensional case when one performs Positive Operator-Valued Measurements (POVMs).

sumes singular states, instead, a norm-one projection embedding $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}_0$ is known to exist also in the case the measured observable B does not have a pure-point spectrum (Srinivas 1980). Specifically, by Stone's theorem, any observable $B \in \mathcal{B}(\mathcal{H})$ uniquely generates a (strongly continuous) one-parameter group of unitary operators $\mathcal{U}_B = \{e^{itB} | t \in \mathcal{R}\}$. Then, there exists a conditional expectation map E^B_η such that

$$tr(\rho T^B_\eta(A)) = \eta_x tr(\rho e^{iBx} A e^{-iBx}) \tag{3.31}$$

for any density operator ρ. The symbol η_x denotes the *invariant mean*, which represents the rigorous analogue of the flat probability measure on the real line[8]. Whenever the observable B has a discrete spectrum, $T^B_\eta(A)$ coincides with the Lüders rule. Hence, formula (3.31) generalizes quantum statistical inference in infinite dimensions.

In the last analysis, if probabilities are assumed to be only finitely additive, conditional probability states $\phi' = \phi_0 \circ T^B_\eta$ are always defined in Hilbert space quantum mechanics. As I mentioned in the previous section, Bayesian probabilities do not actually need to be σ-additive. Nevertheless, there still

[8]An invariant mean on the real line is defined over the space of all bounded continuous complex-valued functions f on $\mathcal{R}$ with the norm $||f|| = sup_{x \in \mathcal{R}} |f(x)|$. In particular, it assigns the same value to a function $f(x)$ and its translated function $f(x+a)$ for every real number a.

remains an open problem. There exist infinitely many invariant means η on the real line. Thus, the conditional expectation T_η^B is not uniquely defined. Moreover, as no physical reason seems to indicate which invariant mean one should choose, many conditional probability states are possible. Contrary to the non-uniqueness of the *a priori* probability, this fact raises a serious difficulty for any Bayesian interpretation of quantum mechanics.

Indeed, no matter how the *a priori* probabilities are determined, on the basis of a quantum measurement any single agent-observer would infer more than one conditional probability, without being able to decide rationally which one represent her degrees of belief. That is a very embarassing situation even for an extremely subjectivist approach, as it means that the agent-observer's degrees of belief would become indefinite. Clearly, the non-uniqueness of conditional expectation is even worse than the failure of stability condition, as the agent-observer would end up with several different, but equally justified, degrees of belief already after the first statistical inference. Thus, one would need a compelling reason for selecting a particular map T_η^B. In the absence of something like a rationality constraint for Bayesian probability theory that uniquely fixes a (singular) conditional probability state, one must attack Redei's argument on a different ground.

The non-existence of a ϕ-preserving conditional expectation in quantum mechanics means that the algebra $\mathcal{B}_0$ generated by a continuous observable B

falls outside the conditions dictated by the Takesaki theorem. In fact, it is not stable under the one-parameter automorphism group σ_t^ϕ. As $\sigma_t^\phi(\mathcal{B}_0) \neq \mathcal{B}_0$, there is some event $A_0 \in \mathcal{B}_0$ whose time-evolution $\sigma_t^\phi(A_0) \notin \mathcal{B}_0$. Therefore, the probabilities of all the events belonging to $\mathcal{B}_0$, namely the evidence ψ_0 on the basis of which the agent-observer performs the first statistical inference, are *not the same* as the probabilities of all the events belonging to $\mathcal{B}_0$ at time t, namely the evidence ψ_0' on the basis of which the agent-observer performs the second statistical inference. That is, $\psi_0 \neq \psi_0 \circ \sigma_t^\phi = \psi_0'$.

The fact that one cannot have the same evidence twice implies that the stability condition is not applicable in quantum mechanics. Intuitively, that can be also justified by the following consideration. Performing a yes/no measurement of a continuous observable, say position, yields an interval in the real line, i.e. a Borel set d: if one *repeats* the same measurement again, one does not obtain the same interval, but just an interval overlapping with the previous one. Therefore, one finally gets round Redei's argument: indeed, if a rationality constraint does not apply, one cannot claim its failure. As a consequence, there is no violation of a necessary condition for a Bayesian interpretation of quantum probabilities. Clearly, the non-applicability of the stability condition means that quantum mechanics is characterized by a weaker form of rationality than Bayesian classical probability theory.

Let me conclude by addressing the problem of philosophy of probability which stems from Redei's argument. The non-existence of norm-one projections $T : \mathcal{M} \to \mathcal{M}_0$, and thus the failure of the stability condition in the general case, challenges the extension of Bayesian probability theory to noncommutative spaces of events. To this extent, Redei identifies two alternative conclusions.

- One cannot regard noncommutative probabilities as degrees of belief of a rational agent. In fact, a rationality constraint for Bayesian probability theory is violated in general; moreover, if a conditional expectation cannot be defined at all, then conditional probability states do not exist.
- One can regard noncommutative probabilities as degrees of belief of a rational agent only under special circumstances, that is whenever a norm-one projection exists. Accordingly, the Takesaki theorem would draw the limits of noncommutative Bayesian statistical inference.

As the latter possibility allows for a generalization of Bayesian probability theory to general spaces of events, the special circumstances under which one would able to perform stable statistical inferences define what it takes for noncommutative states to be degrees of belief of a rational agent. The Takesaki theorem requires that the von Neumann subalgebra $\mathcal{M}_0$ is left invariant by the dynamics σ_t^{ϕ} generated by the *a priori* state ϕ on $\mathcal{M}$; otherwise, if

$\mathcal{M}_0$, and thus the collection of probabilities the agent happens to learn, does not remain the same through time, the agent's revised degrees of belief cannot be determined unambiguously. The one-parameter automorphism group σ_t^{ϕ} also determines a dynamics on the space of all states φ on $\mathcal{M}$ by the relation $\mathcal{D}_t(\varphi) = \varphi \circ \sigma_t^{\phi}$.

As Redei suggests, from a Bayesian point of view such dynamics should be interpreted as the time-evolution of the agent's degrees of belief. Accordingly, "the agent can perform statistical inference only on the basis of knowing the probabilities of events whose collection forms a constant, recognizable unit with respect to the natural dynamics of his measures of degrees of rational belief" (Redei 1992, 131). Moreover, the *a priori* state ϕ remains constant through time, as it generates the one-parameter automorphism group σ_t^{ϕ} (in fact $\phi = \phi \circ \sigma_t^{\phi} = \mathcal{D}_t(\phi)$), whereas all the other probability states evolve according to the dynamics $\mathcal{D}_t$. That means that agent's degrees of belief would change in time spontaneously, even without gaining any further evidence.

One may observe, though, that under this interpretation the Takesaki theorem would betray a clash with the requirement of stable statistical inferences (which, instead, it is supposed to guarantee). Indeed, after the first statistical inference, the conditional state ϕ' capturing the agent's revised degrees of belief evolves in time: so, at any time t the agent may perform the second statistical inference, her degrees of belief are given by a state $\mathcal{D}_t(\phi')$

which is in general different than ϕ'. On the contrary, the stability condition supposes that, if *no* new evidence is acquired, the agent's degrees of belief should not differ from ϕ'.

However this may be, Redei concludes his argument by leaving the question of noncommutative statistical inference still open.

> I regard this latter position very reasonable but also highly speculative, since, after all, one does not expect mathematics, especially particular mathematical theorems, to give insight into the psychic processes of human mind. Also, however, categorically rejecting the possibility to interpret non-commutative probabilities as degrees of rational belief seems to be unsatisfactory because it is too strong a claim; furthermore, it is at least as speculative as the other position and for the same reason.
>
> Thus, I think, one ought to look for further answers to the question: 'When can a non-commutative statistical inference be Bayesian?' [Redei (1992), p.131]

In the next section we provide an answer to such a question, which gets round both Redei's alternatives.

The fact that in quantum mechanics the agent cannot be presented with the same evidence twice demonstrates that the stability condition is not always applicable. Interestingly, the conditions dictated by the Takesaki theorem also prove necessary and sufficient for the applicability of the stability condition. Indeed, after the first statistical inference due to knowing ψ_0, at any time t the agent may perform the second statistical inference the probabilities of all the events in $\mathcal{M}_0$ are given by the state $\psi_0' = \psi_0 \circ \sigma_t^\phi$. As both probability states are defined on the subalgebra $\mathcal{M}_0$ of $\mathcal{M}$, the equivalence $\psi_0 = \psi_0'$ holds if and only if $\mathcal{M}_0$ is invariant under the one-parameter automorphism group σ_t^ϕ.

That means that the domain of applicability and the domain of fulfillment of the stability condition coincide. In other words, whenever the agent is actually presented with the same evidence twice, she is able to perform stable statistical inferences and her degrees of belief are determined unambiguously; alternatively, outside this domain the stability condition cannot be applied, and hence one cannot claim the violation of a rationality constraint. Nonetheless, if $\sigma_t^\phi(\mathcal{M}_0) = \mathcal{M}_0$, norm-one projections do not exist, therefore conditional probability states are not defined. As a result, the extension of Bayesian probability theory to noncommutative spaces of events is still blocked.

To solve the problem of Bayesian statistical inference a conditional expectation which always exists and satisfies rationality constraints is required. A conditional expectation in the sense of Redei fulfills stability condition but cannot be defined in general. So, it does not really represent a *proper* conditional expectation. Thus, one should look for another map enacting a noncommutative conditionalization rule in the theory of von Neumann algebras.

In particular, one should relax some of the defining properties of norm-one projections (see section 2.2.). No good candidate for conditional expectation can give up linearity and conditions (i) and (ii). In fact, in order to extend ψ_0 to a state $\phi' = \psi_0 \circ T$, that is a linear, positive, normalized functional on $\mathcal{M}$, T is required to be a linear, positive and unit-preserving mapping. However, the other provisions are mathematically less strict. Redei himself hints at possibility:

> if ... one drops the requirement that $T : \mathcal{M} \longrightarrow \mathcal{M}_0$ preserve a state, then a T projection might exist. [Redei (1998), p.133]

Thus, one could abandon property (iii) of ϕ-preserving conditional expectations.

Accardi and Cecchini (1982) defined a conditional expectation which is completely positive, *always* exists and does not preserve the state ϕ. That is called a *ϕ-conditional expectation* T_ϕ to distinguish it from the ϕ-preserving conditional expectation T and it generalizes the measure theoretic construction of classical conditional expectations in von Neumann algebras (see Petz

(1988) for a review). That means that performing a statistical inference would not leave the agent's *a priori* probability invariant in the sense that $\phi \neq \phi \circ T_\phi$. Yet, such map is not in general a projection mapping from $\mathcal{M}$ onto $\mathcal{M}_0$. Remarkably, the fixed point algebra of T_ϕ, namely the set of elements $A_0 \in \mathcal{M}_0$ such that $T_\phi(A_0) = A_0$, amounts to the largest subalgebra of $\mathcal{M}_0$ which is invariant under the one-parameter automorphism group σ_t^ϕ associated with ϕ. Therefore, within the conditions dictated by the Takesaki theorem, and only within such conditions, a ϕ-conditional expectation coincides with a norm-one projection $T : \mathcal{M} \to \mathcal{M}_0$.

Accordingly, whenever the stability condition is applicable, the mapping T_ϕ guarantees its fulfillment. Furthermore, as a ϕ-conditional expectation exists in general, a (unique) conditional probability state $\phi' = \psi_0 \circ T_\phi$ can be defined even outside such domain. In the final analysis, Accardi and Cecchini's map provides the proper conditional expectation. That assures, pace Redei's argument, that noncommutative statistical inference can always be Bayesian.

Chapter 4

Correlations between Space-like Separated Quantum Systems

Quantum Field Theory (QFT) is the synthesis of quantum mechanics and special relativity. If one may have any doubt concerning the peaceful co-existence of quantum non-locality with Einstein's theory, the experimental success of QFT would thus serve as a strong note of caution toward any claim of incompatibility. Relativistic quantum mechanics is indeed regarded as the *locus* where the two theories are fully developed together. An interesting philosophical question that can be investigated in such a framework is whether or not quantum correlations between spacelike separated physical systems become stronger by adding relativistic constraints. This is the topic I survey in the present chapter.

Section 4.1 begins by offering a brief overview of the foundations of quantum field theory. My main focus is to study the structural features of the models constructed for free quantum field systems, hence I will not deal with the physical account of interactions between the fields. Then, I introduce the basic ideas of Algebraic Quantum Field Theory (AQFT). It is in this context that the formalism developed by von Neumann finds its most outstanding application in physics. Differently from his expectations, though, the proper mathematical arena for AQFT is representated by the von Neumann factor algebras of type III. The algebraic approach is based on an axiomatic treatment of QFT. This makes it possible to cope with some serious mathematical problems that plague the other standard approaches. In fact, AQFT is widely recognized as our most rigorous description of non-interacting quantum field systems. On top of that, the algebraic formulation is conceptually very neat, which is the reason why philosophers of physics interested in quantum field theory have turned their attention to it. The principles of special relativity, namely Einstein's requirement of locality and the invariance under Lorentz transformations, are naturally built in as axioms. So, Algebraic Quantum Field Theory is particularly suitable to discuss the issue of long-distance correlations of relativistic quantum systems.

A crucial aspect of AQFT is its local character. The fundamental objects of the theory are the (nets of) local algebras of observables, whose elements represent operations that can be performed within a finite region of Minkowski spacetime. Yet, as it is recalled in section 4.2.1, Bell's inequality is maximally violated. Moreover, a theorem by Reeh and Schlieder (1961) seems to indicate that vacuum correlations give rise to a quite peculiar type of non-locality. Notwithstanding these facts, I argue in section 4.2.2 that the existence of operations which are local in a well-defined sense entails that Algebraic Quantum Field Theory fares much better than non-relativistic quantum mechanics in a field theoretic paradigm inspired by Einstein (1948).

The last section is entirely devoted to investigate the nature of entangled states across local algebras. Remarkably, entanglement proves more robust in AQFT than in ordinary quantum theory. Section 4.3.1 shows that there are plenty of spacelike separated regions of Minkowski space that are maximally entangled. I then explain that, against what is often maintained in the literature, the overwhelming presence of entangled states is not a consequence of the Reeh-Schlieder theorem. Finally, in section 4.3.2, I critically address and develop an argument by Clifton and Halvorson (2001). In particular, they prove that, contrary to the non-relativistic case, not all entangled states of a global field system can be destroyed by performing local operations. Such a fact depends on a structural difference between type I and type III factors. As I demonstrate at the end of the chapter, though, the persistence of entanglement is not peculiar only to Algebraic Quantum Field Theory.

4.1 Foundations of Algebraic Quantum Field Theory

4.1.1 The mathematical description of relativistic quantum fields

A relativistic extension of quantum theory is required to account for a range of physical events that fall beyond the domain of ordinary quantum mechanics. There exist fundamental particles, such as photons, whose rest mass is null and thus travel in the vacuum at the speed of light. Hence, their behavior cannot be described by a non-relativistic theory. One of the elementary processes in high-energy physics is in fact the emission or annihilation of a photon corresponding to a change of momentum of an electron (or of its anti-particle, that is the positron). Quantum Field Theory provides a mathematical and conceptual framework for the treatment of such phenomena.

As a preliminary step to outline the basic ideas of QFT, one needs to introduce the notion of Fock space, which is a generalization of Hilbert space for multi-particle systems. The statistics of a composite system of an arbitrary number of identical particles depends on the particle species. There are two different cases. On the one hand, particles that are symmetric under permutations obey Bose statistics, for which reason they are called bosons; on the other hand, particles that are anti-symmetric under permutations obey Fermi statistics, and are called fermions. Empirically, the former have integer spin and the latter have half integer spin. Let $\mathcal{H}^{\bigotimes N}$ represent a system of N particles. The Fock space $\mathcal{F}(\mathcal{H})$ is constructed as a direct sum of all tensor product Hilbert spaces corresponding to each (increasing) number of particles, i.e. $\mathcal{F}(\mathcal{H}) = \bigotimes_0^N \mathcal{H}^{\bigotimes N}$. A state $\Psi_{(n)}$ is a normalized vector state of the Fock space associated with the occupation number distribution (n). The latter is defined as an infinite list

$$(n) = n_1, n_2, ...$$

of finite sequences of occupation numbers, as n_k indicates the number of particles in the k-th state. An annihilation operator a_k, as well as a creation operator a_k^*, can also be defined for a particle in state k: Applying such operators to $\Psi_{(n)}$ has the effect of raising, respectively lowering, by one unit the total number of particles. In the bosonic case a_k and a_k^* satisfy mutual commutation relations, whereas in the fermionic case they anti-commute.

The original purpose of QFT was to develop a quantum version of Maxwell's electrodynamics. The quantization methods, which enact the transition from classical to quantum mechanics, were applied to electromagnetic fields satisfying Maxwell equations. Canonical commutation relations can be explicitly formulated for field systems in the absence of charged matter. A free scalar field $\Phi(x)$ defined on Minkowski spacetime does not commute with its corresponding conjugate field at a fixed time. In particular, the canonical quantization of a free scalar field obeying the Klein-Gordon equation leads to the Fock space of a multi-particle system obeying Bose statistics. Accordingly, the states of the field can be interpreted in terms of particle configurations. This procedure is known as “second quantization”.

Quantum Field Theory thus arises as a relativistic extension of quantum mechanics. There is some important difference with ordinary quantum theory, though. In the non-relativistic case the wave-function identifies the quantum-mechanical state of the system. As it corresponds to a vector of Hilbert space, it is acted upon by operators representing the observables. Instead, in QFT one assigns a field value to each space-time point x by means of an operator. That is, a quantum field is an operator-valued quantity and, as such, it itself acts on the space of states. This means, first of all, that the concept of state of a field system loses any direct spatio-temporal significance. Furthermore, since operators represent what one can measure, contrary to classical fields, the operator-valued quantum field $\Phi(x)$ would not be associated with a definite value of a physical quantity any more.

Nevertheless, there emerges a thorny difficulty here, that afflicts the standard formalism of the theory. From a physical point of view, performing a measurement at a single point x would require an infinite amount of energy. Hence, a quantum field at x cannot be an honest observable. In fact, $\Phi(x)$ is not an operator in the Fock space. It is, instead, an operator-valued distribution over Minkowski space. Specifically, finite values for the matrix element $\langle\Psi_1|\Phi(x)|\Psi_2\rangle$ are obtained just in case the vector states Ψ_1 and Ψ_2 lie in some dense subspace $\mathcal{D}$ of $\mathcal{F}(\mathcal{H})$. In particular, this implies that different field operators cannot be directly multiplied at x. One can overcome such a problem by approaching QFT in an axiomatic manner. A set of axioms were proposed by Wightman (1957) that define the mathematical representative of a quantum field as a proper operator on the vectors of $\mathcal{D}$. The solution consists in averaging Φ with a smooth function f with domain in Minkowski space, that is

$$\Phi(f) = \int \Phi(x) f(x) d^4x$$

Then, taking f to be a test function[1] assures that $\Phi(f)$ is an operator on $\mathcal{F}(\mathcal{H})$. Accordingly, the field is not evaluated at x but it is "smeared out" in its neighbourhood. The technical inconvenience with Wightman axiomatics is, however, that the operator thus defined is unbounded, which makes it very problematic to treat it mathematically.

[1]That is an infinitely differentiable function which decreases, together with its derivative, faster than any power as x goes to infinity in any direction.

The algebraic reformulation of QFT has the advantage that one can work with bounded operators. Segal (1947) first suggested to account for relativistic quantum mechanics within the framework of C*-algebras. Yet, the formal aspect of dealing with fields is that one works with systems having infinitely many degrees of freedom. It is known that von Neumann's uniqueness theorem for irreducible representations of the canonical commutation relations fails to hold in this case. The canonical quantities of a quantum field would not suffice to specify all the observables of QFT. This has an algebraic counterpart in the availability of inequivalent irreducible representations of a C*-algebra. Hence, if one insists on the Hilbert space representation, the other representations should be dismissed as non-physically relevant. To avoid such a loss of algebraic structure, Haag proposed that one should not focus on a single algebra: one recovers the physical significance of inequivalent representations by providing a mathematical description of quantum field systems in terms of a net of algebras. The physical information is contained in the way in which these algebras are linked together. Haag's intuition underlies Algebraic Quantum Field Theory, whose reference exposition is offered by Haag (1996). There are two main versions of this approach, depending on whether or not one makes explicit use of operators on Hilbert space: concrete AQFT, developed by Haag and Araki, appeals to von Neumann algebras, whereas abstract AQFT, developed by Haag and Kastler, appeals to C*-algebras.

According to Haag's approach, the primitive object in the mathematical model of a quantum field is the mapping

$$\mathcal{O} \longrightarrow \mathcal{A}(\mathcal{O}) \tag{4.1}$$

from the bounded region $\mathcal{O}$ of Minkowski space M onto the net of algebras $\mathcal{A}(\mathcal{O})$. The elements of the latter represent operations that one can perform in the corresponding spacetime region. Restricting to finite regions rules out those physical quantities, such as total charge and total energy of a field, which are allowed as observables by standard QFT but are actually unobservable, since their measurement would have to take place throughout infinitely extended regions, namely the whole universe. Algebraic Quantum Field Theory is local in the sense that only local observables are considered.

The set $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ is called a *covariant net of strictly local observables.* Its partition into subalgebras yields the relevant information about any physical quantity. In fact, one does not need to specify what observables have physical significance. It is sufficient to investigate the net structure of local algebras. Interpreting the theory in terms of local operations has an empirical justification. In fact, the experimental data that one is presented with in quantum field theory, such as a blackening in a photoemulsion, a track in a bubble chamber or an interference pattern on a screen, are always relative to the space-time localization of micro-systems. From the latter all the other physical properties can be inferred.

Accordingly, neither fields nor particles are the fundamental entities of AQFT. The more so because nets of algebras can be defined without reference to fields at all.

> In quantum physics just as in classical physics the concept of "fields" serves to implement the principle of locality. In particular, a "quantum field" should not be regarged as being more or less synonimous with a "species of particles". While it is true that with each type of particle we may associate an "incoming field" and an "outcoming field", these free fields are just convenient artifacts...
>
> This suggests that the *net of algebras* $\mathcal{A}$ ... constitutes the intrinsic mathematical description of the theory. The mentioned physical interpretation establishes the tie between space-time and events. The role of "fields" is only to provide a coordinatization of this net of algebras. [Haag (1996), p.105]

In fact, local algebras are associated with a quantum field in the sense of Wightman axioms in a well-defined way. The algebraic structure $\{\mathcal{A}(\mathcal{O})\}$ is generated by the field operators smeared out with test functions f whose support lies in the corresponding spacetime regions, i.e. $Supp(f) \subset \mathcal{O}$. However, the relation between nets of local algebras and Wightman fields $\Phi(f)$ is one-to-many. Different fields may describe the same local net. Hence, while specifying $\mathcal{A}$ suffices to determine the corresponding quantum field, the converse is not true.

4.1.2 The axioms of AQFT and their consequences

To complete the algebraic formulation of Quantum Field Theory, one postulates that the net of local algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ satisfies certain mathematical properties, which are introduced by placing physically motivated axioms. In this section I review the axiomatics of AQFT as formulated in Haag's approach and show that it constrains the mathematical description of quantum field systems to be given in the framework of type III_1 factors. Here is below a list of the basic postulates of Algebraic Quantum Field Theory.

Axiom 1

Isotony: $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$

This expresses the fact that any observable which can be measured in a region $\mathcal{O}_1$ is measurable in a larger region $\mathcal{O}_2$ containing $\mathcal{O}_1$ as well. As such a condition holds for any region of M, the net of algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ is an inductive system. So, isotony can be also justified in the sense that there is an inductive limit C*-algebra $\mathcal{A}$ generated by all local algebras. That is a quasilocal algebras whose elements can be uniformly approximated by local observables.

Axiom 2

> *Microcausality:* if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated regions, then the corresponding local algebras commute, i.e. $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$.

This captures the intuition that measurements of observables located in spacelike separated regions of Minkowski spacetime should not disturb each other. They are in fact copossible. We have already seen how microcausality reflects Einstein's relativistic principle of locality. Notice that it is not always required that the operators representing observables associated with spacelike separated regions commute. Indeed, Fermi field operators anti-commute. Hence, one needs superselection rules to distinguish observables, namely the elements of $\mathcal{A}(\mathcal{O})$, from unobservable quantities represented by field operators (I will not enter into the discussion of the Doplicher, Haag and Roberts superselection theory, though). Importantly, for many spacetime regions one can strengthen the locality axiom to obtain a property called *local duality*, which reads

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})' \tag{4.2}$$

where the region $\mathcal{O}'$ is the causal complement of $\mathcal{O}$, comprising all the points in M which are spacelike separated from every element of $\mathcal{O}$.

Axiom 3

Relativistic Covariance: There exists a continuous representation α of the Poincaré group $\mathcal{P}$ by automorphisms α_g on $\mathcal{A}$ such that, given any $g \in \mathcal{P}$,

$$\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O}) \tag{4.3}$$

for all regions $\mathcal{O}$ of Minkowski spacetime M.

This expresses the Lorentz covariance of AQFT in terms of the net of algebras $\{\mathcal{A}(\mathcal{O}) | \mathcal{O} \subset M\}$. Such a condition marks the main difference from ordinary quantum mechanics, whose group of symmetry is just the Galilean group, thus guaranteeing the extension of the theory to the relativistic context. Then, by generalizing the GNS theorem, one can show that a state ϕ on $\mathcal{A}(\mathcal{O})$ which is invariant under the Poincaré group $\mathcal{P}$ gives rise to a Hilbert space $\mathcal{H}_\phi$ carrying a unitary representation U of $\mathcal{P}$, in the sense that for any $A \in \mathcal{A}(\mathcal{O})$ one has

$$U(g)\pi_\phi(A)U^*(g) = \pi(\alpha_g(A)) \tag{4.4}$$

Moreover, the GNS vector Ω_ϕ is invariant under the action of any $\mathcal{P}$ on $\mathcal{H}_\phi$, that is $U(g)\Omega_\phi = \Omega_\phi$ for all g.

One also postulates that there exists at least one physical representation of the algebra $\mathcal{A}$ given by the Poincaré invariant *vacuum* state ϕ_0. That is, the following provision is satisfied by the GNS representation $(\mathcal{H}_{\phi_0}, \Omega_{\phi_0}, \pi_{\phi_0})$:

Axiom 4

> *Spectrum Condition:* $P_0^2 \geq 0$ and $P_0^2 + P_1^2 + P_2^2 + P_3^2 \geq 0$, for the generators P_i (where $i = 0, 1, 2, 3$) of the translation sub-group of the Poincaré group $\mathcal{P}$.

This corresponds to the requirement that energy is positive in every Lorentz frame. In fact, such an axiom mandates that the spectrum of the self-adjoint operators P_i, which has the physical interpretation of the global energy-momentum spectrum of the theory, lies in the closed forward light cone. Clearly, the spectrum condition makes sense only relative to a representation, wherein the infinitesimal generators of the spacetime translation group of M can actually be defined via Stone's theorem.

In the Haag-Araki approach all the local algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ are von Neumann algebras acting on some Hilbert space $\mathcal{H}$, so that $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$.

Accordingly, one supposes that a translationally invariant vacuum state Ω_{ϕ_0} exists. However, the requirement that all abstract nets have a representation satisfying the vacuum does not entail that one must pass to such a representation to compute expectation values. As Haag and Kastler (1964) argued, all concrete representations of a net, whether equipped with a translationally invariant vacuum state or not, are physically equivalent in some precise sense. Here I assume that in the representation π defined by a given state, say the (unique) vacuum, the net consists of local von Neumann algebras $\pi(\mathcal{A}(\mathcal{O}))'' = \mathcal{N}(\mathcal{O})$ for which the above conditions hold.

Axioms 1-4 determine algebraic quantum field theories which are very general. To obtain theoretical models that can describe more concrete properties of quantum fields one needs to supply further constraints depending on the specific physical circumstances. In particular, the following condition is introduced to derive important properties and theorems of AQFT.

Axiom 5

> *Weak Additivity:* $\mathcal{N}(\mathcal{V}) = \{\mathcal{N}(\mathcal{O}) | \mathcal{O} \subset \mathcal{V}\}''$ for any (possibly unbounded) regions $\mathcal{V}$ in M.

This states that there is no minimal distance between regions of Minkowski space. It then follows that spacetime is homogeneous. The physical motivation for such an assumption is that the theory should not allow for any smallest length scale. As a result, all observables localized in $\mathcal{O}$ are generated by means of algebraic operations of observables from arbitrarily small regions $\mathcal{V}$.

Another axiom can be added that appeals to the notion of the *causal hull* $\mathcal{O}^-$ of a spacetime region $\mathcal{O}$. The latter comprises the set of points p in Minkowski space such that all timelike straight lines containing p intersect $\mathcal{O}$.

Axiom 6

Local Primitive Causality: $\mathcal{A}(\mathcal{O}^-) = \mathcal{A}(\mathcal{O})$

This captures the hyperbolic character of the time evolution of the theory. Accordingly, all the quantities which are observable in $\mathcal{O}$, and hence belong to the corresponding local algebra, are fixed by what is observable in a region $\mathcal{O}^-$ of M that causally determines $\mathcal{O}$.

The physical justifications underlying each axiom make the appeal to the algebraic framework less abstract. By constraining the general mathematical structure one obtains a rigorous description of free quantum fields.

> Many concrete models satisfying these conditions have been constructed, though none of them is an interacting quantum field in four spacetime dimensions. Of course, no such model has *ever* been constructed, so one can hardly attribute the source of the

> problem to the set of "axioms" above. On the contrary, we are convinced that the above conditions are operationally natural and express the minimal conditions to be satisfied by any local relativistic quantum field theory in the vacuum Minkowski space. So we view consequences of these assumptions to be generic properties in the stated context. [Redei and Summers (2007)]

An immediate consequence of such assumptions is that specific spacetime configurations are determined, within which well-defined implementations of models for quantum field theory are available.

Typical spacetime regions in AQFT are the double-cones. Let us consider a point x in Minkowski spacetime and another point $y \in M$ lying in x's forward light cone. An open double cone is the intersection of the causal future of x and the causal past of y. The double cones thus obtained for all pairs of such points of the manifold form the set $\mathcal{K}$ of all double cones. Two double cones are tangent if they are spacelike separated and their closure intersects at a single point. If the latter provision does not hold, they are said to be strictly separated.

Another kind of spacetime configuration is given by the wedge regions. Differently than double cones, which are finite, such regions of Minkowski space are actually unbounded. For any fixed point $x_0 \in M$, the right wedge is defined as the set $W_R \equiv \{x \in M | x > |x_0|\}$; likewise, the left wedge amounts to the set $W_L \equiv \{x \in M | x < |x_0|\}$. The collection of wedge regions are obtained as the set of all Poincaré transforms of W_R. Complementary wedge regions are such that $W'_R = W_L$. These are tangent regions, as their closures intersect at one point.

It remains to show that the proper mathematical arena for Algebraic Quantum Field Theory is given by von Neumann factor algebras of type III. That was first argued by Araki (1964). Later on, a uniqueness theorem by Buchholz *et al.* (1987) proved that local algebras for relativistic free fields are type III_1. The basis for such a result is the fact that the physical information in the theory is encoded in the relation between the algebras of different regions, rather than in the individual algebras of observables. This reflects the hyperfinite structure of $\mathcal{A}(\mathcal{O})$. Indeed, a von Neumann algebra is hyperfinite if it is the weak closure, namely the W*-inductive limit, of an ascending sequence of finite dimensional (not necessarily hyperfinite) algebras.

Yet, hyperfiniteness characterizes other types of von Neumann algebras as well. Hence, further constraints are to be put in place. In particular, *non-triviality*, namely the requirement $\mathcal{A}(\mathcal{O}) \neq cI$ for any spacetime region $\mathcal{O}$ (with c ranging in the complex numbers), implies that local algebras are properly infinite, and thus it rules out the case I_n as well as the case II_1.

Then, one can construct $\mathcal{A}(\mathcal{O})$ as a unique type III_1 hyperfinite factor from the underlying Wightman theory by adding the assumption of *scaling limit.* Let us define a monotonic function N mapping from real numbers into real numbers and a scale transformation of the test functions $f \longrightarrow f_\lambda$ such that $f_\lambda = N(\lambda)f(\lambda^{-1}x)$. The fields satisfy asymptotic scale invariance just in case for some field Φ with vanishing vacuum expectation values, $N(\lambda)$ can be suitably chosen in such a way that the scale field operators $\Phi(f_\lambda)$ has the following properties:

- the expectation values $\langle \Omega_0, \Phi(f_\lambda)^* \Phi(f_\lambda) \Omega_0 \rangle$ converge in the limit $\lambda \longrightarrow 0$ for all f, and it is nonzero for some test functions
- the norms $\|\Phi(f_\lambda)^* \Phi(f_\lambda)\Omega_0\|$ and $\|\Phi(f_\lambda)\Phi(f_\lambda)^*\Omega_0\|$ are finite in the limit $\lambda \longrightarrow 0$

where Ω_0 represents the vacuum state. In the last analysis, once the above axioms for AQFT are made, if a model, such as the free Bose field of null (and positive) mass, has a non-trivial scaling limit, then the local algebras are uniquely type III_1.

> The universality of ... $[\mathcal{A}(\mathcal{O})]$ may be seen as analogous to the situation in quantum *mechanics* where we can associate to each system or subsystem an algebra of type I, i.e. an algebra isomorphic to the set of all bounded operators on Hilbert space.

> The change from the materially defined systems in mechanics to "open subsystems" corresponding to sharply defined regions in space-time in a relativistic local theory forces the change in the nature of the algebras from type I to type III_1. [Haag (1996), p.267]

The difference between type III and type I factors and the unsuitability of the latter to account for relativistic quantum mechanics, even for systems requiring infinite dimensions, such as in the I_∞ case, can be illustrated with a though experiment formulated by Fermi (1932) and further discussed by Hegerfeldt (1994) and Ingvason (2005).

Fermi's *gedankenexperiment* envisages two atoms a_1 and a_2 separated by a distance d. The composite system is described by the tensor product $\mathcal{M} = \mathcal{M}_1 \bigotimes \mathcal{M}_2$ of the algebras associated to each atom. At the initial time 0 the first atom is in its ground state ω_1, whereas the other one is in an excited state ω_2. By neglecting the contribution of the radiation field, the joint state is $\omega_0 = \omega_1 \otimes \omega_2$ on $\mathcal{M}$. Due to the decay of a_2, there is a non-zero probability that at time $t > 0$ the atom a_1 absorbing the emitted radiation will be in an excited state. The evolution of the state of the total system

$$\omega_t(\cdot) = \omega_0(e^{itH} \cdot e^{-itH}) \tag{4.5}$$

is governed by the Hamiltonian operator H. The probability of finding a_1 excited at time t is then computed by $p(t) = \omega_t(E)$, where the projection $E = (I - |\psi_1\rangle\langle\psi_1|) \otimes I$ is determined by the Hilbert space vector ψ_1 associated with the ground state of the first system.

Since the effect of the decay cannot propagate faster than the speed of light, the state of a_1 would remain unchanged at least until a time equal to that required by c to cover the distance d has passed. Thus, one might expect that $p(t) = 0$ for $t < \frac{d}{c}$. However, the analyticity following from the stability assumption about H implies that, if $\omega_t = 0$ during some time interval, then it must be zero for all t, but this cannot be the case[2]. As a consequence, for any excitation of a_1 to take place, it should occur immediately after the decay of a_2, hence infringing on the prohibition of superluminal signaling. Actually, by appealing to factors of type III, instead of factors of type I, one can avoid such a counterintuitive conclusion.

Let us denote with ω_0' the state of the total system in case atom a_2 is unexcited; then, ω_t' represents its evolution in time. In order to evaluate the effect of the decay, these states ought to be compared with the state ω_0 and its evolution ω_t associated with a_2 being initially excited. Specifically, the

[2]Since there is a lower bound for energy, as the Hamiltonian has a semi-bounded spectrum, the vector valued function $Ee^{itH}\phi$ can be continuously extended for any ϕ to an analytic function of t, where the imaginary part of the latter is positive.

deviation

$$D(t) = \sup_{A \in \mathcal{M}_1} |\omega_t(A) - \omega'_t(A)| \tag{4.6}$$

yields a proper measure of the effect of the decay with respect to all measurements A that can be performed in the region $\mathcal{O}_1$ where the first atom. At time $t = 0$ the value of the above quantity vanishes for any distance less than d. The axiom of local primitive causality guarantees that $D(t) = 0$ for $t < \frac{d}{c}$ without any failure of the relativistic constraint of locality.

The alleged paradox arises for type I factors because the projector E mapping onto the orthogonal complement of ψ_1 may belong to the algebra associated with $\mathcal{O}_1$, and thus would be a candidate to test the excitation in the latter region. Accordingly, $\omega'_t(E) = 0$ for all t. On the contrary, all projections are properly infinite in a type III factor, and thus the observable E is not necessarily local in the region where the atom a_1 is localized. Therefore, a non-null expectation value of ω_t does not imply $D(t) = 0$. In fact, the deviation $\omega_t(E) - \omega'_t(E)$ can be zero for a positive expectation value $\omega'_t(E)$. This means that $\omega_t(E) > 0$ is not a consequence of the decay of atom a_2. Hence, causality is not violated in Fermi's two-atom system.

4.2 How Local is Local Quantum Field Theory?

4.2.1 Maximal Violation of the Bell's Inequality and the Reeh-Schlieder Theorem

As it should be clear from section 4.1.1., the local structure of Algebraic Quantum Field Theory manifests itself in a two-fold way. On the one hand, locality is introduced via the axiom of microcausality. On the other hand, the observables of the theory are required to be local in the sense that they are connected with finite regions of Minkowski space. As Horuhzy (1990) put it,

> there is a fundamental property which appears already at the early stage and deeply affects the conceptual (and, consequently, the mathematical) structure of algebraic quantum field theory. This property is locality, which is, in its turn, a combination of two properties: localization and causality. The former means that since any physical experiment takes place in a finite space-time region, each physical quantity determined directly from the experiment is also associated with some region (localized in it). As to the latter property, one should keep in mind that no signal

> velocity can exceed the velocity of light, and no processes taking place in spacetime regions separated by spacelike intervals can affect each other (Einstein's causality principle). Consequently, each observable must also be causal, i.e. compatible with any other observable if their localization regions are mutually spacelike. Observables having the described properties of localization and causality are called *local observables.* This concept is the real cornerstone of the algebraic approach in quantum field theory. [Horuhzy (1990), p.3]

The compatibility of any two observables belonging to regions separated by a non-null spacelike distance is indeed the most important feature of relativistic systems, which embodies Einstein's principle of causality. Of course, such a requirement is trivially fulfilled by classical systems too, as no incompatible observables are present at all in commutative algebras. However, in relativistic quantum mechanics, where the relevant algebras are non-commutative, one has to impose locality as a geometrical condition: any observable in the region $\mathcal{O}$ commutes with all observables in its causal complement $\mathcal{O}'$.

As to the localizability property, it retains the idea that quantum field theory is a statistical theory accounting for the outcomes of local measurements. Accordingly, the experimental content of QFT comes from measurements that can be performed in finite regions of spacetime. As a consequence, global observables cannot belong to local algebras, and thus are ruled out from AQFT. It is worthwhile to emphasizing a point that I mentioned in the previous section: that is that no characteristic of observables other than their localization in spacetime is necessary for a description of quantum fields. Empirical data in quantum theory always refer to the localization properties of microscopic objects. Experimental apparatuses, such as detectors and counters, only register that a particle appeared at a certain spacetime region. This is the sense in which the outcomes of experiments in elementary particle physics are just geometrical facts. The structure of the net of algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ is sufficient to reconstruct all the physically relevant information concerning the system under investigation. The set of all local observables belonging to $\mathcal{A}(\mathcal{O})$ is actually fixed by what one can measure by means of experiments performed in the corresponding finite spacetime region $\mathcal{O}$.

> Thus, we have defined observables and states, the basic physical quantities of the algebraic approach, and found that locality is the main specific principle in the observable-state formalism for relativistic quantum theory.

...

> Therefore, fundamentally the algebraic approach was conceived as a general algebraic theory of local structures for relativistic quantum physics or, as it is called more briefly, local quantum theory. [Horuzhy (1990), p.4]

A sharpening of the locality principle that, as Haag (1996) suggested, makes it possible to define "finitely extended subsystems" in AQFT is the split property. It requires that for any two spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ of Minkowski spacetime, there exists a type I von Neumann algebra $\mathcal{M}$ such that

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathcal{A}(\mathcal{O}_2)' \tag{4.7}$$

Accordingly, local algebras associated with spacelike separated regions not only commute, but are statistically independent as well. In fact, the split property is a necessary and sufficient condition for the existence of a (normal) product state ϕ on the total algebra $\mathcal{A}(\mathcal{O}_1) \bigvee \mathcal{A}(\mathcal{O}_2)$ which factorizes into $\phi(AB) = \phi_1(A)\phi_2(B)$ for all $A \in \mathcal{A}(\mathcal{O}_1)$ and $B \in \mathcal{A}(\mathcal{O}_2)$, where ϕ_i is a state on $\mathcal{A}(\mathcal{O}_i)$ with $i = 1, 2$. Therefore, across any pair of local algebras which are split there is some unentangled joint state. The split property has been verified by many models of quantum fields localized in strictly spacelike separated double cones, whereas wedge regions are known to be non-split.

The fact that the primitive concept of Algebraic Quantum Field Theory is the locality of the algebras of observables does not mean that there is not a sense in which the theory is non-local. Indeed, Bell's inequality is provably violated by a pair of relativistic systems in a stronger way than in non-relativistic quantum mechanics. Furthermore, a theorem by Reeh and Schlieder (1961) that can be derived from the axioms of AQFT seems to entail a quite peculiar kind of non-locality. I now discuss these two aspects of relativistic quantum field theory.

The general algebraic form of the maximal Bell correlation that was spelled out in section 3.2.2. can be straightforwardly applied to nets of algebras. Given two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ and a joint state ϕ across the corresponding local algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$, Bell's inequality is expressed by

$$\beta(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq 1 \tag{4.8}$$

In QFT there are many states in which the above bound is violated. Indeed, typical local algebras contain an infinite product of copies of the algebra $\mathcal{M}_2(C)$ associated with Pauli matrices. Two (nets of) algebras are said to be maximally correlated if all possible states across them maximally violate the

Bell's inequality, that is the Bell correlation takes on the value $\sqrt{2}$ for any ϕ. Let us stress that, according to Jarrett's decomposition of Bell's inequality (see section 3.2.2.), Bell-type non-locality in relativistic quantum mechanics must be a consequence of the failure of outcome independence. Indeed, parameter independence naturally holds in AQFT because it is embodied in the axiom of microcausality, whereby one presupposes $\mathcal{O}_1 \subseteq \mathcal{O}_2'$.

For every pair of strictly spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ there is a state ϕ such that $\beta(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = \sqrt{2}$. Actually, for such regions quantum field theory predicts the existence of several states that maximally violate Bell's inequality, no matter how far apart $\mathcal{O}_1$ and $\mathcal{O}_2$ are located. Furthermore, as Summers and Werner (1988) demonstrated for both double cones and wedge regions, if two spacetime regions in AQFT are tangent, then they are maximally correlated. The proof hinges on the fact that the vacuum ϕ_0 being faithful on the von Neumann algebra $\mathcal{A}(\mathcal{O}_1) \bigvee \mathcal{A}(\mathcal{O}_2)$ maximally violates Bell's inequality. It thus follows that no pair of local algebras corresponding to tangent regions can fulfill the split property, otherwise there would be some state satisfying Bell's inequality.

Since tangent regions are maximally correlated, whereas strictly spacelike separated regions are not, it is interesting to study how the Bell correlation varies with the degree of separation between $\mathcal{O}_1$ and $\mathcal{O}_2$ in an irreducible vacuum representation. Summers and Werner (1987) demonstrated a sharp short-distance bound for the Bell correlation in the vacuum. Specifically, in any irreducible vacuum sector with a positive mass gap $m > 0$ the value of the maximal Bell correlation decreases exponentially from $\sqrt{2}$ to 1 according to the formula

$$\beta(\phi_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq 1 + e^{-md(\mathcal{O}_1, \mathcal{O}_2)} \tag{4.9}$$

where $d(\mathcal{O}_1, \mathcal{O}_2)$ indicates the maximal timelike distance that the region $\mathcal{O}_1$ can be translated before it is no longer contained in the causal complement of $\mathcal{O}_2$. Summers and Werner (1995) then improved the above bound to account for spacelike separations smaller than $d(\mathcal{O}_1, \mathcal{O}_2)$. Under the same conditions, one obtains

$$\beta(\phi_0, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq \sqrt{2} - \frac{\sqrt{2}}{7 + 4\sqrt{2}}(1 - e^{-md(\mathcal{O}_1, \mathcal{O}_2)}) \tag{4.10}$$

This means that there exist many pairs of regions of Minkowski space for which the maximal Bell correlation ranges in the semi-open interval $[1, \sqrt{2}[$. In these cases, the Bell's inequality is neither satisfied nor maximally violated. As no specific assumption concerning the nature of the relevant spacetime configurations is made, such a result does not depend on the geometry of $\mathcal{O}_1$ and $\mathcal{O}_2$. Hence, it holds for strictly spacelike separated wedge regions too. As a consequence, the maximal Bell correlation computed for the latter does not need to attain the bound $\sqrt{2}$: in fact, models of quantum fields localized in these type of spacetime regions predict maximal violation of Bell's inequality in all possible states only for the case of massless theories.

A theorem by Halvorson and Clifton (2000) captures the idea that the failure of Bell's inequality is "typical" in Algebraic Quantum Field Theory. Let $(\mathcal{M}_1, \mathcal{M}_2)$ be a pair of von Neumann algebras on a (separable) Hilbert space satisfying the Schlieder property and such that $\mathcal{M}_1 \subseteq \mathcal{M}_2'$. Then, if both algebras are properly infinite, there is a dense set of vectors in $\mathcal{H}$ that induce joint states across $(\mathcal{M}_1, \mathcal{M}_2)$ which violate Bell's inequality. Since the identity operator is equivalent to any of its subprojection in a type III factor, and therefore the latter is properly finite, the theorem applies to nets of local algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$. Accordingly, most states across $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ being *arbitrary* spacelike separated regions of Minkowski space, violate Bell's inequality.

As I just mentioned, the vacuum state contains sufficiently strong correlations between spacelike separated measurements that relativistic quantum field theory predicts maximal violation of the Bell's inequality for ϕ_0. Vacuum correlations have been largely investigated in the literature. Globally, ϕ_0 is the state of lowest energy, which, by definition, has no particle present in it. Accordingly, it yields eigenvalue 0 for both particle and anti-particle operators. Nonetheless, as Redhead (1995) pointed out, locally "it is seething with activity": observable phenomena are produced by the fluctuations exhibited by local observables, such as charge densities, confined in the vacuum. It is then quite remarkable that no local procedure carried out in the spacetime region $\mathcal{O}$ can establish whether one is in the vacuum state. In fact, the projection P_{ϕ_0}, mapping on the closed subspace of Hilbert space spanned by the vector state Ω_0 which generates ϕ_0, does not belong to the local algebra $\mathcal{A}(\mathcal{O})$. In fact, for the sake of *reductio ad absurdum*, suppose that it does. Then, the operator $I - P_{\phi_0}$ would be in $\mathcal{A}(\mathcal{O})$ too (where I is the identity in the algebra), which implies $(I - P_{\phi_0})\Omega_0 = 0$; yet, such a relation would be satisfied only for P_{ϕ_0} coinciding with the identity and this cannot be the case since P_{ϕ_0} is a one-dimensional projection. Consequently, the question whether we actually are in the state Ω_0 can never be answered from a local perspective, but it would require one to survey the entire spacetime manifold.

The correlations associated with the vacuum are actually responsible for some non-local effects of quantum fields which are not predicted by non-relativistic quantum mechanics. That is the content of the Reeh-Schlieder theorem. Before spelling it out, let us first recall some relevant definitions of the algebraic approach. A vector state x on a Hilbert space $\mathcal{H}$ is cyclic for a von Neumann algebra $\mathcal{M}$ if the norm closure of the set $\{Ax|A \in \mathcal{M}\}$ spans the whole of $\mathcal{H}$. In other words, by applying any operator belonging to the algebra to x, one can obtain any state in the underlying Hilbert space.

Furthermore, x is separating for $\mathcal{M}$ just in case $Ax = 0$ implies $A = 0$ for any element A in the algebra. In particular, if $\mathcal{M}$ possesses a separating state, then every state of $\mathcal{M}$ is a vector state. It was established by Kadison and Ringrose (1997) that a vector state is cyclic for a von Neumann algebra $\mathcal{M}$ if and only if it is separating for its commutant $\mathcal{M}'$.

The Reeh-Schlieder theorem connects the property of cyclicity of a global state x of a field to the physical fact that the latter has bounded energy.

> **Reeh-Schlieder Theorem:** If x has bounded energy, then x is cyclic for any local algebra $\mathcal{A}(\mathcal{O})$.

Let E denote the spectral measure of the global Hamiltonian of the field. Energy being bounded in the state x requires the existence of a finite r such that $E([0, r])x = x$. Of course, the vacuum vector state Ω_0 is a state of bounded energy because the corresponding eigenvalue of E is zero. Therefore, the Reeh-Schlieder theorem immediately applies to the vacuum. This means that, by acting on Ω_0 with some operation localized in the spacetime region $\mathcal{A}(\mathcal{O})$, it is possible to approximate an arbitrary state of the field in any other region of Minkowski space, even if it is spacelike separated from $\mathcal{O}$.

Indeed, by operating locally on a finite region of M one can reconstruct any state of the field in its causal complement. In case the expectation values of the vacuum confined to $\mathcal{O}$ agree with the expectations of a global state of the field on the algebra $\mathcal{A}(\mathcal{O}')$, the latter state is said to be localized in $\mathcal{O}$. To put in another way, all localized states are given by "excitations" of the vacuum.

> To achieve this the operator must judiciously exploit the small but nonvanishing long distance correlations which exist in the vacuum as a consequence of the spectral restrictions for energy-momentum in the theory. The theorem shows that the concept of *localized states*, if used in a more than qualitative sense, must be handled with care. [Haag (1996), p.102]

Haag then characterizes the conceptual import of the Reeh-Schlieder theorem as a "superficial paradox". Some more dramatic sentiments toward such a peculiar non-local aspect of relativistic quantum field theory were expressed by Segal (1964), Segal and Goodman (1965) and Fleming (1999), who characterize the possibility of producing space-like distant effects by means of local physical actions as "striking", "bizarre" and "amazing!", respectively.

In order to prove the theorem one just needs to appeal to the spectrum condition and the axioms of weak additivity. If one adds microcausality, one can derive a corollary which has further remarkable implications. Specifically, if the casual complement of $\mathcal{O}$ is non-null, it follows from the Reeh-Schlieder theorem that a state of bounded energy is cyclic in $\mathcal{A}(\mathcal{O}')$, and hence it is separating for its commutant. Microcausality then guarantees that $\mathcal{A}(\mathcal{O}')'$ properly contains $\mathcal{A}(\mathcal{O})$. Therefore, the given state of the field is separating for the latter local algebra too. That is, one can conclude

> **Corollary:** If x has bounded energy, then x is separating for any local algebra $\mathcal{A}(\mathcal{O})$, where $\mathcal{O}' \neq 0$.

This means that the structure of the vacuum Ω_0, as well as of any other vector state x of bounded energy, is rich enough to discriminate the action of distinct elements of any local algebra. In fact, if two elements of $\mathcal{A}(\mathcal{O})$, say operation A and B, have the same effect on Ω_0, it follows that $(A-B)\Omega_0 = 0$. The vacuum being separating for the algebra then assures A and B are the same operator.

A first consequence of such a result is that, just as locally we can never know if the present state is the vacuum state, it is never a local question whether we are in a N-particle state either. Let ψ denote the vector state representing a system composed of a certain number N of particles. Certainly, ψ is orthogonal to vacuum, where there is no particle. Hence, one has $P_\psi \Omega_0 = 0$, but by the above corollary of the Reeh-Schlieder theorem the projection P_ψ, which is an operation in the local algebra, must be zero. In the last analysis, particle states are non-local objects.

Moreover, and quite surprisingly, one can show that every possible outcome of a local measurement in $\mathcal{O}$ has a non-vanishing finite probability of occurring in the vacuum. The probability $p = Prob^{\Omega_0}(P = 1)$ that a local measurement represented by the projection $P \in \mathcal{A}(\mathcal{O})$ yields a positive result in the vacuum is equal to $\|P\Omega_0\|^2$. So, if such a probability is zero, then $P\Omega_0$ would be null too and the corollary of the Reeh-Schlieder theorem would further imply $P = 0$. By contraposition, for any non-zero projection one obtains $p \neq 0$. In other words, in the long run any possible excitation of the field will manifest itself in the vacuum.

According to the Reeh-Schlieder theorem, operations performed in the spacetime region $\mathcal{O}_1$ can effectively produce changes in the spacelike separated regions of Minkowski space $\mathcal{O}_2$. Thus, given two local operations $P_1 \in \mathcal{A}(\mathcal{O}_1)$ and $P_2 \in \mathcal{A}(\mathcal{O}_2)$, the conditional probability $p = Prob^{\Omega_0}(P_2 = 1|P_1 = 1)$ being equal to one means that a positive outcome of the measurement associated with P_1 turns Ω_0 into a state lying in the range of P_2 as well. Actually, the "force" of such non-local effects is not independent of the degree of separation between $\mathcal{O}_1$ and $\mathcal{O}_2$. In fact, a theorem by Fredenhagen (1985), on which the derivation of formula (4.9) by Summers and Werner is based, describes the exponential decreasing of vacuum correlations with the

Lorentz distance $d(\mathcal{O}_1, \mathcal{O}_2)$. Specifically,

$$\langle P_1 P_2 \rangle_{\Omega_0} - \langle P_1 \rangle_{\Omega_0} \cdot \langle P_2 \rangle_{\Omega_0} \leq e^{-md(\mathcal{O}_1,\mathcal{O}_2)} \| P_1 \rangle \Omega_0 \| \cdot \| P_2 \rangle \Omega_0 \|$$

Redhead (1995) exploited such a bound to estimate the probability of a measurement of P_2 to give a positive outcome in terms of the probability relative to a previously performed measurement of P_1, that is $\langle P_1 \rangle_{\Omega_0} \leq \frac{e^{-md(\mathcal{O}_1,\mathcal{O}_2)} \langle P_2 \rangle_{\Omega_0}}{(1 - \langle P_2 \rangle_{\Omega_0})^2}$. He thus concluded that

> ... for a given probability of P_2 happening, the maximally correlated P_1 must have a probability of occurring that falls off exponentially with the distance between $\mathcal{O}_1$ and $\mathcal{O}_2$.
>
> This again shows how difficult it would be in practice to observe the long-range correlations in the vacuum. But, of course, it does not show that they do not exist! [Redhead (1995), p.134]

Such remarks emphasize the significance that vacuum correlations have for the issue of non-locality. Redhead also observed that they pose a threat for the possibility of localizing particles in AQFT. As a reaction, Fleming (2000) proposed a localization scheme that avoids the non-local consequences of the Reeh-Schlieder theorem, although it requires that QFT ought to be formulated in the framework of type I factors. I will not enter into this debate, but one may see Halvorson (2001) for a reply to Fleming.

4.2.2 Einstein's Field Theoretic Paradigm in Relativistic Quantum Mechanics

Einstein gave his argument against the completeness of quantum mechanics (at least) four times after the publication of the EPR paper in 1935. It has been pointed out by several authors, for instance Howard (1985) and Clifton and Halvorson (2001), that Einstein's dissatisfaction with quantum theory hinges on the incompatibility between the standard Hilbert space formulation and what can be called a *field theoretic paradigm*. The latter is spelled out in his 1948 "Dialectica" paper.

> If one asks what is characteristic of the realm of physical ideas independently of the quantum theory, then above all the following attracts our attention: the concepts of physics refer to a real external world, i.e. ideas are posited of things that claim a "real existence" independent of the perceiving subject (bodies, fields, etc.), and these ideas are, on the other hand, brought into as secure a relationship as possible with sense impressions. Moreover, it is characteristic of these physical things that they are conceived of as being arranged in a spacetime continuum. Further, it appears to be essential for this arrangement of the things introduced in physics that, at a specific time, these things claim an existence

> independent of one another, insofar as these things "lie in different parts of space". Without such an assumption of mutually independent existence (the "being-thus") of spatially distant things, an assumption which originates in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such a clean separation. Field theory has carried out this principle to the extreme, in that it localizes within infinitely small (four dimensional) space-elements the elementary things existing independently of one another that it takes as basic as well as the elementary laws it postulates for them.
>
> For the relative independence of spatially distant things (A and B), this idea is characteristic: an external influence on A has no *immediate* effect on B; this is known as the "principle of local action", which is applied consistently only in field theory. The complete suspension of this basic principle would make impossible the idea of the existence of (quasi-)closed systems and, thereby, the establishment of empirically testable laws in the sense familiar to us. [Einstein (1948), translation taken from Howard (1985)]

Then, in another passage he added:

> Matters are different, however, if one seeks to hold on principle II - the autonomous existence of the real states of affairs present in two separated parts of space R_1 and R_2 – simultaneously with the principles of quantum mechanics. In our example the complete measurement on $\mathcal{S}_1$ of course implies a physical interference which only effects the portion of space R_1. But such an interference cannot immediately influence the physically real in the distant portion of space R_2. From that it would follow that every measurement regarding $\mathcal{S}_2$ which we are able to make on the basis of a complete measurement on $\mathcal{S}_1$ must also hold for the system $\mathcal{S}_2$ if, after all, no measurement whatsoever ensued on $\mathcal{S}_1$. That would mean that for $\mathcal{S}_2$ all statements that can be derived from the postulation of ψ_2 or ψ_2', etc. must hold simultaneously. This is naturally impossible, if ψ_2, ψ_2', are supposed to signify mutually distinct real states of affairs of $\mathcal{S}_2$. [Einstein (1948), translation taken from Howard (1985)]

Accordingly, the field theoretic paradigm which Einstein finds incompatible with quantum mechanics relies on three main concepts: the "principle of local action", the "assumption of mutually independent existence (the 'being-thus') of spatially distant things" and the related requirement that spatially distant systems be “quasi-closed”.

It is indeed true that standard Hilbert space quantum mechanics is not field theoretical in the sense that observables in non-relativistic quantum theory are not “conceived of as being arranged in a spacetime continuum”: the observable quantities in quantum theory do not carry labels that would indicate their spatiotemporal location in a four dimensional spacetime continuum; hence, quantum measurements and operations are also not conceived of as possessing spatiotemporal tags explicitly. Nor is Hilbert space quantum mechanics covariant with respect to a relativistic symmetry group. In fact the relevant symmetry group for standard Hilbert space quantum mechanics is the Galilean group: the generators of projective and continuous representations of this group are typically identified with observables. Thus it is not surprising that quantum mechanics does not meet requirements of relativistic locality interpreted in the sense of a field theoretical paradigm.

However, a local and relativistically covariant quantum mechanics exists. Algebraic Quantum Field Theory supplies a particularly well-developed and mathematically exact version of it. Then, it is very natural to ask: To what extent does AQFT comply with the field theoretical paradigm as Einstein formulates it?

Clifton and Halvorson (2001) claim that, in a sense AQFT fares even worse than standard quantum mechanics: in their view the crucial element in Einstein's criticism of standard quantum mechanics is the openness of quantum systems, and in their interpretation "What makes quantum systems open for Einstein is that [...] quantum systems can occupy entangled states in which they sustain non-classical EPR-correlations with other systems outside their light cones". It follows then that the local systems in AQFT are much more radically open systems than standard quantum mechanical systems because, as I shall discuss in section 4.3.2, entanglement is far more dramatic and robust in relativistic quantum field theory than in non-relativistic quantum mechanics. Thus, ironically, relativistic quantum field theory would violate even more dramatically what Einstein takes as necessary conditions for a physical theory to be acceptable from a field theoretical viewpoint.

However, their argument goes through only if one takes the presence of entanglement to be the crucial point in Einstein's criticism against quantum theory. Actually, Einstein does not mention entanglement in his 1948 criticism of quantum mechanics at all - in spite of the fact that entanglement had been well-known to him: he corresponded with Schrödinger in connection with Schrödinger's 1935 papers that analyzed entanglement systematically. Nor does entanglement appear in the two other publications in which Einstein formulated his incompleteness argument after the EPR paper, that is Einstein (1936) and Einstein (1949). So, one can read and interpret Einstein's

1948 criticism of non-relativistic quantum mechanics somewhat differently, in a way that seems very natural and closer to Einstein's idea (and text) than Halvorson and Clifton's interpretation of focusing on openness and identifying it with presence of entanglement. In this section I argue that relativistic quantum field theory fares far better than standard non-relativistic quantum mechanics in that it *does* satisfy Einstein's criteria without which "physical thought in the sense familiar to us would not be possible".

According to Redei and Valente's (2008) interpretation of the 1948 Dialectica paper quoted above, Einstein formulates (informally) the following three requirements for a physical theory to be compatible with a field theoretical paradigm:

1. **Spatio-temporality** "... physical things [...] are conceived of as being arranged in a spacetime continuum..."

2. **Independence** "... essential for this arrangement of the things introduced in physics is that, at a specific time, these things claim an existence independent of one another, insofar as these things 'lie in different parts of space'. "

3. **Local operation** "... an external influence on A has no *immediate* effect on B; this is known as the 'principle of local action' "; "... measurement on $\mathcal{S}_1$ of course implies a physical interference which only

> effects the portion of space R_1. But such an interference cannot immediately influence the physically real in the distant portion of space R_2."

Each of these provisions is indeed met in the framework of Algebraic Quantum Field Theory. The requirement of Spatio-temporality is explicitly incorporated in the axioms of AQFT since the observables of the theory are localized in regions of the spacetime continuum. As to the second condition, we saw in the previous section that there is a rich hierarchy of notions of independence that are satisfied by local algebras of observables. Then, there remains to translate the constraint of local operation in algebraic terms and to see in what sense it would be fulfilled by AQFT.

The notion of *operations* generalizes that of quantum measurements. An operation T is a completely positive map from a C*-algebra $\mathcal{A}$ onto itself, that is σ-weakly continuous and satisfies $0 \leq T(I) \leq I$, where I is the unit in $\mathcal{A}$. An operation is non-selective if it is unit preserving, whereas it is selective if $T(I) < I$. The dual T^* of an operation is an embedding from the state space of $\mathcal{A}$ into itself: $T^*\phi = \phi \circ T$ thus captures the effect of T on the state ϕ of a system. T is normal if T^* maps normal states onto normal states. Operations are the mathematical representatives of physical operations, namely physical processes that take place as a result of physical interactions with the system.

With the aid of such a notion, one can formulate an idea of operational independence that is contained in the two definitions below. The first is operational C*-independence.

Definition 1: A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C*-subalgebras of a C*-algebra $\mathcal{A}$ is *operationally C*-independent* in $\mathcal{A}$ if any two operations on $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, have a joint extension to an operation on $\mathcal{A}$; i.e. if for any two completely positive unit preserving maps

$$T_1 : \mathcal{A}_1 \longrightarrow \mathcal{A}_1$$

$$T_2 : \mathcal{A}_2 \longrightarrow \mathcal{A}_2$$

there exists a completely positive unit preserving map $T : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$T(A_1) = T_1(A_1) \text{ for all } A_1 \in \mathcal{A}_1$$

$$T(A_2) = T_2(A_2) \text{ for all } A_2 \in \mathcal{A}_2$$

The second is operational W*-independence.

Definition 2: A pair $(\mathcal{M}_1, \mathcal{M}_2)$ of von Neumann subalgebras of a von Neumann algebra $\mathcal{M}$ is *operationally W*-independent* in $\mathcal{M}$ if any two normal operations on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, have a joint extension to a normal operation on $\mathcal{M}$.

Operational C*-independence (respectively, W*-independence) expresses that any operation, i.e. procedure, state preparation, etc., on system $\mathcal{S}_1$ is co-possible with any such operation on system $\mathcal{S}_2$, if these systems are represented by C*-algebras (respectively, W*-algebras).

As usual, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spacelike separated spacetime regions, $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ be local observable algebras in AQFT pertaining to spacetime regions $\mathcal{O}_1$ and $\mathcal{O}_2$ and regarded as observables of a larger system $\mathcal{A}(\mathcal{O})$ localized in spacetime region $\mathcal{O}$ with $\mathcal{O} \supseteq \mathcal{O}_1, \mathcal{O}_2$. Let T be an operation on $\mathcal{A}(\mathcal{O})$ that can be regarded also as representing an operation carried out on system $\mathcal{A}(\mathcal{O}_1)$ viewed as a subsystem of $\mathcal{A}(\mathcal{O})$. Let ϕ be a state on $\mathcal{A}(\mathcal{O})$. Redei and Valente (2008) call $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ a *local system.*

The requirement of Local operation means that local systems be such that the operation-conditioned state $T^*\phi = \phi \circ T$ should coincide with ϕ on $\mathcal{A}(\mathcal{O}_2)$. This idea is fixed in the form of the following definitions:

Definition 3: Let $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ be a local system. The operation $T : \mathcal{A}(\mathcal{O}) \longrightarrow \mathcal{A}(\mathcal{O})$ is said to be *localizable* in

$\mathcal{A}(\mathcal{O}_1)$ if the latter is invariant under T; i.e. if $T(A) \in \mathcal{A}(\mathcal{O}_1)$ whenever $A \in \mathcal{A}(\mathcal{O}_1)$.

It is important to realize that localizability of T in $\mathcal{A}(\mathcal{O}_1)$ defined above does not require that the operation does not affect $\mathcal{A}(\mathcal{O}_2)$ in the sense of changing its state. The problem is precisely whether operations localized in $\mathcal{A}(\mathcal{O}_1)$ in the above sense can have this feature or whether operations that are carried out on $\mathcal{A}(\mathcal{O}_1)$ "behave badly causally" in the sense of affecting the state of $\mathcal{A}(\mathcal{O}_2)$. Causal well behaving is spelled out in the next definition.

Definition 4: The local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ with an operation T on $\mathcal{A}(\mathcal{O})$ that is localizable in $\mathcal{A}(\mathcal{O}_1)$ is defined to be *operationally separated* if the operation conditioned state $T^*\phi = \phi \circ T$ coincides with ϕ on $\mathcal{A}(\mathcal{O}_2)$, i.e. if $\phi(T(A)) = \phi(A)$ for all $A \in \mathcal{A}(\mathcal{O}_2)$.

The question now is: are local systems in AQFT operationally separated?

Given a local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T_p)$ with the operation defined by the projection postulate $T_p(X) = \sum_i P_i X P_i$, where the P_i are the spectral projections of discrete observable being measured, the local commutativity requirement of AQFT entails that the operation T_p is the identity map on $\mathcal{A}(\mathcal{O}_2)$. Then, the following propositions hold:

Proposition 1: The local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T_p)$ with T_p describing the projection postulate is operationally separated for every state ϕ.

One can generalize some of the characteristic features of T_p. As we learnt in section 3.3.3, a norm-one projection, that is positive, linear, unit preserving map T from the C*-algebra $\mathcal{A}$ onto a C*-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ whose restriction to $\mathcal{A}_0$ is equal to the identity map is called a conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_0$. Furthermore, for a state φ on $\mathcal{A}$, a norm-one projection also preserves φ. Such a φ-preserving conditional expectation is denoted by T_p^{φ}. Clearly, a local system $(\mathcal{A}, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \varphi, T_p^{\varphi})$ with T_p^{φ} mapping from $\mathcal{A}(\mathcal{O})$ onto $\mathcal{A}(\mathcal{O}_1)$ is operationally separated.

However, the projection postulate has limited applicability. Not every interaction with (operation on) a quantum system can be described by a T_P of the above form. For instance, if the observable to be measured does not have a discrete spectrum then one cannot directly generalize the Lüders rule to obtain a completely positive map. More generally, a norm-one projection mapping from $\mathcal{A}$ onto $\mathcal{A}_0$ exists in very rare circumstances. So, one ought to replace such a map with a φ-preserving conditional expectation in the sense of Accardi and Cecchini (1982): if φ is a faithful normal state on von Neumann algebra $\mathcal{M}$ then there always exists a φ-preserving completely positive map

T^{φ} from $\mathcal{M}$ into any subalgebra $\mathcal{M}_0$.

Consider now the vacuum state ϕ_0 and the local net of algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \subset M\}$ in the vacuum representation. By the Reeh-Schlieder theorem ϕ_0 is faithful on every local von Neumann algebra $\mathcal{A}(\mathcal{O})$. Hence, there exists the Accardi-Cecchini ϕ_0-preserving conditional expectation $T^{\phi_0} : \mathcal{A}(\mathcal{O}) \longrightarrow \mathcal{A}(\mathcal{O}_1)$. Obviously, this T^{ϕ_0} is localizable in $\mathcal{A}(\mathcal{O}_1)$ since its range is in the latter local algebra. Furthermore T^{ϕ_0} cannot be the identity on $\mathcal{A}(\mathcal{O}_2)$ because it takes $\mathcal{A}(\mathcal{O}_2)$ into $\mathcal{A}(\mathcal{O}_1)$, which commutes with $\mathcal{A}(\mathcal{O}_2)$ by local commutativity of the net. Accordingly, if T^{ϕ_0} were the identity on $\mathcal{A}(\mathcal{O}_2)$, then the latter local algebra would be commutative, which of course it is not. It follows that there is an observable $X \in \mathcal{A}(\mathcal{O}_2)$ such that $T^{\phi_0}(X) \neq X$. As a consequence, there exists a normal state ω_2 on $\mathcal{A}(\mathcal{O}_2)$ such that $\omega_2(T^{\phi_0}(X)) \neq \omega_2(X)$, and ω_2 can be extended from $\mathcal{A}(\mathcal{O}_2)$ to a normal state ω on $\mathcal{A}(\mathcal{O})$ by the Hahn-Banach theorem.

In the last analysis, if $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ are local von Neumann algebras in the vacuum representation of a local net in AQFT then there exists a (normal) state ω on $\mathcal{A}(\mathcal{O})$ such that the local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \omega, T^{\phi_0})$ with the Accardi-Cecchini ϕ_0-preserving conditional expectation T^{ϕ_0} is operationally not separated. It is clear that this argument can be repeated with any state in the place of the vacuum state that is faithful. This means that the following proposition has been shown:

Proposition 2: There exist operationally not separated local systems in AQFT.

Thus it would seem that the Local operations requirement is violated in AQFT, contrary to the claim in the introductory part of the section. But this conclusion would be too quick.

One can argue that the mere existence of operationally not separated local systems should not be interpreted as the proper incompatibility of AQFT with the Local operations requirement because one can not expect a theory such as AQFT to exclude causally non-well-behaving local systems necessarily. But it is reasonable to demand that AQFT allow a locally equivalent and causally acceptable description of an operationally not separated local system. In other words, one can say that it may happen that the possible causal bad behavior of the local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ is due to the non-relativistically conforming choice of the operation T on $\mathcal{A}(\mathcal{O})$ representing an operation localized in $\mathcal{A}(\mathcal{O}_1)$, and there may exist another operation T' on $\mathcal{A}(\mathcal{O})$ that is localizable in $(\mathcal{O}_1)$ and which has the same effect on $\mathcal{A}(\mathcal{O}_1)$ as that of T, (i.e. $T'(X) = T(X)$ for all $X \in \mathcal{A}(\mathcal{O}_1)$) and such that the system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T')$ is causally well-behaving.

This idea of reducibility of operational non-separation is explicitly fixed in the form of the following weakening of the definition of operational separation:

> **Definition 5:** The local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ is called *operationally separable* if it is operationally separated, *or*, if it is not operationally separated, then there exists an operation $T' : \mathcal{A}(\mathcal{O}) \longrightarrow \mathcal{A}(\mathcal{O})$ such that $T'(X) = T(X)$ for all $X \in \mathcal{A}(\mathcal{O}_1)$ and such that the system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T')$ is operationally separated.

Algebraic Quantum Field Theory is said to satisfy the **Local operations** requirement if the local systems in AQFT are operationally separable.

If one interprets Einstein's requirement of Local operations as the requirement that local systems in Algebraic Quantum Field Theory should be operationally separable in the above sense, then one is led to ask the question: are local systems in AQFT operationally separable?

Consider the local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$. If the pair $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ is operationally C*-independent, then taking the restriction $T|_{\mathcal{A}(\mathcal{O}_1)}$ of T to $\mathcal{A}(\mathcal{O}_1)$ and the identity map $id_{\mathcal{A}(\mathcal{O}_2)}$ as operation on $\mathcal{A}(\mathcal{O}_2)$, the two operations $T|_{\mathcal{A}(\mathcal{O}_1)}$ and $id_{\mathcal{A}(\mathcal{O}_2)}$ have a joint extension T' to $\mathcal{A}(\mathcal{O})$ and since T' is the identity on $\mathcal{A}(\mathcal{O}_2)$, it is clear that the local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T')$ is operationally separated. In short we have:

> **Proposition 3:** If the pair $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ is operationally C*-

independent, then for every ϕ and every T which is localizable in $\mathcal{A}(\mathcal{O}_1)$ the local system $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi, T)$ is operationally separable.

Since operational C*-independence holds for local algebras $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ associated with strictly spacelike separated double cone regions $\mathcal{O}_1$ and $\mathcal{O}_2$, one can conclude that AQFT typically satisfies the requirement of local operations: "typically" because the double cone regions are the typical spacetime regions with which the local algebras are associated. It still remains to prove that this is also the case for tangent double cones and for wedge regions, whether strictly separated or tangent. To establish such a positive result one ought to show that the relevant spacetime regions are operationally W*-independent. Accordingly, a W*-version of operational separability needs to be formulated. As there is no known *no go* result ruling out such a possibility, one may well conjecture that operational separability, and thus Einstein's paradigm for field theories including the requirement of local operations, can be extended in general to any local algebras of AQFT.

Interestingly, the notion of Operational Separability is quite independent of Bell-type correlations. Indeed, the local operation T does not play any role in the relation $\beta(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2), \phi)$. Moreover, Bell's inequality is maximally violated for the vacuum state ϕ_0, even for strictly spacelike separated double cones, which, as we have just seen, give rise to operationally separable local systems. Therefore, the violation of Bell's inequality does not imply (operational) non-separability.

4.3 Entanglement between space-like separated regions of Minkowski spacetime

4.3.1 Cyclicity and intrisically entangled states

In the previous section I argued that in local quantum field theory one has a variety of notions of algebraic independence which allow one to characterize physical systems being independent from each other. There is a precise sense in which local measurements performed in a certain spacetime region $\mathcal{O}$ would not affect any system located in its causal complement $\mathcal{O}'$, although there may well be entangled states across the two regions. Such a result is quite surprising, especially in light of the fact that entanglement between quantum fields is more robust than in non-relativistic quantum mechanics, as I show in the conclusive part of this chapter.

The definition of entanglement of a field in AQFT follows straightforwardly from the algebraic characterization of non-separable states (see section 3.2.2.). It is convenient here to use the Haag-Kastler concrete version of AQFT. Given the net of local algebras $\{\mathcal{A}(\mathcal{O})|\mathcal{O} \in M\}$ acting on Hilbert space $\mathcal{H}$, a global state ϕ of a field is given on the von Neumann algebra $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$. Such a state is entangled across a pair of spacelike separated regions $(\mathcal{O}_1, \mathcal{O}_2)$ just in case its restriction $\phi|_{\mathcal{A}_{12}}$ to $\mathcal{A}_{12} = [\mathcal{A}(\mathcal{O}_1) \bigvee \mathcal{A}(\mathcal{O}_2)]''$ falls outside the weak*-closure of the convex hull of product states on $\mathcal{A}_{12}$. There is a physical motivation for the choice of weak*-topology. Indeed, one cannot verify the norm convergence of a sequence of states in a laboratory. Thus, experimental reasons motivate one to appeal to a strong notion of entanglement.

Let me spell out two facts concerning entanglement in relativistic quantum field theory which are direct applications of general properties of von Neumann algebras. The first fact is derived by the Raggio-Bacciagaluppi theorem according to which there must exist some non-separable states across two commuting non-abelian algebras. Since the local algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are both non-commutative and, by definition, they fulfill microcausality too, the assumptions of the theorem are satisfied. Therefore, one has the following

Fact 1: There is at least one entangled state ϕ across $(\mathcal{O}_1, \mathcal{O}_2)$

The second fact follows from a well-known result of quantum information. That is that entanglement cannot be created by local operations. To see this, recall that it is a consequence of the Kraus representation theorem that any operation $T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ can be rewritten as

$$T(\cdot) = \sum_i K_i^*(\cdot)K_i \tag{4.11}$$

where $\{K_i\}$ is a countable family of Kraus (bounded) operators such that the weakly*-converging sum $\sum_i K_i K_i^*$ takes its value in the real interval $[0, 1]$. The claim is that, if the state ω across two quantum systems described by the von Neumann algebra $\mathcal{M}_1$ and $\mathcal{M}_2$ is not entangled, then by performing a local operation T on the first system one cannot turn the joint state into an entangled state.

Suppose ω is a product state on $\mathcal{M} = \mathcal{M}_1 \bigvee \mathcal{M}_2$, whose restrictions to $\mathcal{M}_1$ and $\mathcal{M}_2$ are the states ω_1 and ω_2, respectively. For the sake of the proof, but without loss of generality, we can consider pure operations. Accordingly, $T(\cdot) = K^*(\cdot)K$ and the state resulting from applying such a map reads $\omega \circ T = T^*\omega = \frac{\omega(K^* \cdot K)}{\omega(K^*K)}$. So, for any element $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$

$$T^*\omega(AB) = \frac{\omega(K^*(AB)K)}{\omega(K^*K)} = \frac{\omega(K^*AKB)}{\omega(K^*K)}$$

where the last equality holds because the Kraus operator K obviously commutes with B. The state thus factorizes into

$$T^*\omega(AB) = \frac{\omega_1(K^*(A)K)}{\omega_1(K^*K)}\omega_2(B) = T^*\omega_1(A)\omega_2(B)$$

and hence $T^*\omega$ is again a product state across the pair $(\mathcal{M}_1, \mathcal{M}_2)$. The same reasoning can then be extended to the case in which $\omega = \sum_n \lambda_n \omega_n$ to show that T preserves convex combinations of product states on $\mathcal{M}$. In fact, one obtains $T^*\omega = \sum_n \lambda_n^T T^*\omega_n$ by setting the coefficients to be $\lambda_n^T = \frac{\omega_n(K^*K)}{\omega(K^*K)}$. In particular, if one restricts the attention to a global state of a field defined on the von Neumann algebra $\mathcal{A}_{12}$, this conclusion entails the following

> **Fact 2:** If any state ϕ across $(\mathcal{O}_1, \mathcal{O}_2)$ is separable, then the state $\phi \circ T$ resulting from the application of a local operation T on $\mathcal{O}_1$ (or on $\mathcal{O}_2$) will not be entangled either.

Some information about the nature of entangled states in Algebraic Quantum Field Theory can be inferred from the analysis of the maximal Bell correlation relative to the local algebras associated with finite regions of Minkowski space, which I offered in section 4.2.1. Entanglement is in fact a necessary condition for the violation of Bell's inequality. From the fact that tangent regions are maximally correlated it follows that any state across such spacetime configurations is maximally entangled. All joint states across any pair of strictly spacelike separated wedge regions are also entangled, for the Bell correlation is such that $1 \leq \beta(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) \leq \sqrt{2}$, although in this case ϕ is not necessarily maximally entangled. The only circumstance where there exists some separable state is that of strictly spacelike separated double cones, for which the split property holds.

The overwhelming presence of entangled states characterizing local algebras seems to be what prompted Haag's remark below.

> ... it is evidently not obvious how to achieve a division of the world into parts to which we can assign individuality. We have moved far away from Maxwell's ode on atoms: "Though in the course of ages catastrophes have occurred... the foundation stones of the material universe remain unbroken and unworn. They continue this day as they were created - perfect in number and measure and weight." Instead we used a division according to regions in space-time. This leads in general to open systems. Under special circumstances we can come from there to the materially defined systems of quantum mechanics, claiming for instance that in some large region of space-time we have precisely an electron and a photon whose ties to the rest of the world may be neglected. One of the essential elements in singling out such a material system and assigning to it an individual, independent (at least temporary) existence is its *isolation*, i.e; the requirement that in a

> large neighborhood we have a vacuum-like situation. However this is not always enough... There may be persistent correlations of a non-classical character... [Haag (1992), p.298-299]

Accordingly, (bounded) regions of Minkowski space are the basic elements of the description of the world that local quantum physics offers. As Clifton and Halvorson's (2001) commented in relation to this quote, the fundamental aspect of such spacetime "blocks" is that they are *intrisically open* to entanglement. Indeed, even in the case of strictly spacelike separated double cones, no spacetime region $\mathcal{O}$ can be isolated from its environment, where the latter is identified with the causal complement $\mathcal{O}'$. The duality relation assures that the von Neumann algebra $\mathcal{A}(\mathcal{O}')$ associated with the latter region is equal to $\mathcal{A}(\mathcal{O})'$, that is the commutant of the local algebra corresponding to $\mathcal{O}$. Now, it is a general property of von Neumann algebras theory that any state across the pair $(\mathcal{M}, \mathcal{M}')$ maximally violates the Bell's inequality (see Summers (1990)). Thus, all the joint states across $(\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O})')$ are maximally entangled. As a consequence, the region $\mathcal{O}$ is always intrinsically entangled with its environment.

This has some further interesting consequences for AQFT. Connes and Størmer (1978) provided a definition of type III_1 factors in terms of unitary operators: $\mathcal{M}$ acting on a (separable) Hilbert space $\mathcal{H}$ is type III_1 just in case, given any two states ρ and τ on $\mathcal{B}(\mathcal{H})$, the norm $\|\rho - \tau^{UU'}\|$ is bounded by positive real numbers ϵ for some unitaries that $U \in \mathcal{M}$ and $U' \in \mathcal{M}'$. Presumably, two states converging to each other in norm ought to be assigned close degrees of entanglement; on the other hand, Vedral *et al.* (1997) showed that, in order to construct a reasonable measure of entanglement, such as the standard von Neumann entropy measure, invariance under unitary operators on the separated systems is a necessary condition. Clifton and Halvorson (2001) then argued that all measures of entanglement in a type III_1 factor would be trivial.

> The Connes-Størmer characterization immediately implies the impossibility of distinguishing in any reasonable way between the different degrees of entanglement that states might have across ... $[(\mathcal{M}, \mathcal{M}')]$.
>
> [Clifton-Halvorson (2001), p.26, where the notation has been suitably changed]

Nevertheless, such a conclusion is unwarranted. Since all states across a von Neumann algebra and its commutant are maximally entangled, they all possess the same degree of entanglement. That explains away the alleged triviality. It is true, though, that a unitary invariant measure of entanglement in AQFT is yet to be found. A major difficulty is that the von Neumann entropy is not defined at all: for a type III factor contains only infinite projections, and hence no density operator, whose spectral projections are necessarily finite, can exist. In the absence of such a measure, one can hardly give a full characterization of entangled states in relativistic quantum mechanics.

Let us now spell out the connection between the property of cyclicity and the existence of entanglement. This can be done in the general algebraic framework, without any immediate reference to the Reeh-Schlieder theorem. Consider two noncommutative von Neumann subalgebras $\mathcal{M}_1$ and $\mathcal{M}_2$ of a von Neumann algebra $\mathcal{M}$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_2'$. Halvorson and Clifton (2000) proved that, if a vector state x is cyclic for either one of these subalgebras, then the corresponding joint state ω_x on $\mathcal{M}$ is non-separable. The reasoning proceeds by contradiction. Suppose ω_x is actually not entangled. Then, cyclicity of x guarantees that all states on $\mathcal{M}$ can be approximated by means of local operations applied to ω_x. As entanglement cannot be created locally, such states ought to be all separable. However, by the Raggio-Bacciagaluppi theorem, there is at least an entangled state across the pair $(\mathcal{M}_1, \mathcal{M}_2)$. Therefore, ω_x must be entangled in the first place.

Such a result was strengthened by Clifton and Halvorson (2001), by adding the requirement that there is a separating vector for $\mathcal{M}$. The cyclicity of x is actually preserved under the action of invertible operators. Let A be an invertible element of $\mathcal{M}$. Then, since one can act on it by the operator A^{-1} to obtain x again and from there one can of course approximate any other state on $\mathcal{H}$, the vector state Ax is cyclic too. Applying to x all the invertible operators in the algebra gives rise to a dense set of vector states. So, the presence of a cyclic vector for $\mathcal{M}$ entails that the underlying Hilbert space contains a dense set of cyclic vectors for the algebra. Moreover, since $\mathcal{M}$ possesses a separating vector, all its states are vector states. This means that a (norm) dense set of the latter must be cyclic, and thus entangled. The fact that non-separable states are open in the weak-* topology (and hence in the norm topology too) completes the following

> **Generic Result:** If either $\mathcal{M}_1$ or $\mathcal{M}_2$ possesses a cyclic state, and $\mathcal{M}$ possesses a separating vector, then most states on $\mathcal{M}$ are entangled across $(\mathcal{M}_1, \mathcal{M}_2)$.

Such a result applies to local algebras in relativistic quantum field theory. In case the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ satisfies the two antecedent conditions of the above statement, typical states on $\mathcal{A}_{12}$ are entangled across the two spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ of Minkowski space. This shows, once again, that separable states are very rare in AQFT.

Notice that this is an algebraic fact that does not depend on the Reeh-Schlieder theorem. In fact, the latter is just sufficient, under specific circumstances, for the Generic Result to hold. As states of bounded energy are involved, it implies the existence of some cyclic state on any local algebra. Together with the requirement that the complement of the union of the relevant spacetime regions is non-empty, i.e. $(\mathcal{O}_1 \bigcup \mathcal{O}_2)' \neq \emptyset$, it then guarantees that the second antecedent condition is fulfilled as well, as by the corollary of the theorem a separating vector exists in any local algebra. On the other hand, there are cyclic states which have no bounded energy. Specifically, Borchers (1965) proved that for non-trivial operators $A \in \mathcal{A}(\mathcal{O}_1)$ any vector of the form Ax never has bounded energy; yet, provided that x is cyclic, such a vector is also cyclic for A being invertible. In conclusion, the Reeh-Schlieder theorem is not necessary for the application of the Generic Result in Algebraic Quantum Field Theory.

4.3.2 On the persistence of entanglement in relativistic quantum field theory

The overwhelming majority of states being entangled across any pair of regions of Minkowski space associated with local algebras is one way in which the robustness of entanglement manifests itself in AQFT. Actually there is another interesting way in which this happens, that I discuss in detail in this section. We already know by *Fact 2* that entanglement cannot be created by acting locally on either $\mathcal{O}_1$ or $\mathcal{O}_2$. Yet, one may wonder whether an entangled state across such regions can be destroyed by performing any local operation. The main technical result of a paper by Clifton and Halvorson (2001) is that this is never the case in AQFT. My purpose here is to develop and complete their argument.

Allegedly, the persistence of entanglement of field systems marks a fundamental structural difference between relativistic and non-relativistic quantum mechanics. In the latter case, indeed, one can always find a local operation that disentangles any composite quantum system. This can be shown with a simple example. Let x be any vector state on the tensor product of two-dimensional Hilbert space $\mathcal{H}_1 \bigotimes \mathcal{H}_2$ representing a bi-partite non-relativistic quantum system. Subsystem A may be given to Alice and subsystem B to Bob. Suppose that the joint state x is initially entangled. The corresponding density operator ρ_x projects onto the closed subspace spanned by the vector state. If Alice measures a discrete observable with (one-dimensional) eigen-projections P_+ and P_- on subsystem A, then the new state of the system is described by the density matrix

$$\rho_x' = (P_+ \otimes I)\rho_x(P_+ \otimes I) + (P_- \otimes I)\rho_x(P_- \otimes I)$$

Since one gets $(P_\pm \otimes I)x = a_x^\pm \otimes b_x^\pm$ for some non-zero vectors $a_x^\pm \in \mathcal{H}_1$ and

$b_x^{\pm} \in \mathcal{H}_2$, the previous expression takes a more explicit form, namely

$$\rho'_x = Tr[(P_+ \otimes I)\rho_x]P_+ \otimes P_{b_x^+} + Tr[(P_- \otimes I)\rho_x]P_- \otimes P_{b_x^-}$$

That shows that ρ'_x is a convex combination of product states, and hence represents an unentangled state. Therefore, Alice succeeded in isolating A from Bob's system. In terms of the Kraus representation theorem, Alice performed a nonselective operation T, that can be decomposed into the Kraus operators $K_1 = P_+ \otimes I$ and $K_2 = P_- \otimes I$. Similarly, she could apply a selective map, involving just one of these Kraus operators, but the resulting state would still be separable. The disentangling procedure thus described does not at all depend on the details of x, nor on the degrees of entanglement between the two sub-systems. Moreover, as it is shown below in more general terms, it can be extended to any finite and infinite dimensional Hilbert space.

In fact, no matter what the initial joint state is, entanglement between quantum systems described by type I factors can be destroyed by local operations performed on one subsystem. By contrast, it is just such a possibility that fails to hold in relativistic quantum field theory. As Clifton and Halvorson stressed at the beginning of their paper, from an algebraic point of view this difference lies at a deep structural level.

We end section 4. by connecting the type III character of the

> algebra of a local field system with the inability, in principle, to perform local operations on the system that will destroy its entanglement with other spacelike separated systems. We offer this result as one way to make precise the sense in which AQFT requires a radical change in paradigm - a change that, regrettably, has passed virtually unnoticed by philosophers of quantum theory. [Clifton-Halvorson (2001), p.5]

Although the emphasis on the importance of their result is totally warranted, the conclusion of Clifton and Halvorson's argument is not as strong as they claim. I will suggest how to improve it.

To begin with, let us make sense to what the authors mean by the inability *in principle* of destroying entanglement in AQFT. This expression should be understood as opposed to the practical limitations that an experimenter encounters when trying to disentangle a pair of spacetime regions, that indeed explain Streater and Wightman's (1989) comment that "it is difficult to isolate a system described by fields from outside effects" (p.139).

Any element A of a von Neumann algebra $\mathcal{M}$ is the strong limit of a sequence of invertible operators $\{A_n\} \subseteq \mathcal{M}$ (see Dixmier and Maréchal (1971) for the proof). That is, $A_n x \longrightarrow Ax$ for $n \to \infty$. Let us a consider a state x of bounded energy across the pair of regions $(\mathcal{O}_1, \mathcal{O}_2)$ of Minkowski space. Since a local operation A in the algebra $\mathcal{A}(\mathcal{O}_1)$ does not preserve the boundedness of energy, the Reeh-Schlieder theorem does not apply to Ax, and therefore the latter vector does not need to be cyclic. Accordingly, one expects that Alice's measurement could disentangle x. On the contrary, as we just learnt, all the vectors of the form $A_n x$ are cyclic, and hence they certainly represent entangled states. However, insofar as the Kraus operator $\frac{A_n}{|A_n|}$ lies in the strong neighborhood of the Kraus operator $\frac{A}{|A|}$, the experimenter cannot distinguish between them from a purely operational point of view. In particular, instead of performing the (possibly) disentangling local operation represented $\frac{A}{|A|}$, Alice may well have performed any non-disentangling local operation represented $\frac{A_n}{|A_n|}$. So, the practical limitations she is subjected to stem from the lack of experimental means to specify what operation she actually applied. Furthermore, it follows from Dixmier and Maréchal's result that the states generated by the vectors Ax and $A_n x$ are close to each other in norm, i.e. $\|\phi_{Ax} - \phi_{A_n x}\| \longrightarrow 0$. As a consequence, Alice would not even be able to determine whether the global state she ended up with by acting locally on the field is entangled or not.

There are also obstacles to isolating entangled quantum field systems in practice which do not depend on the implications of the Reeh-Schlieder theorem. The Generic Result derived at the end of the previous section poses severe difficulties to the possibility of achieving disentanglement. The fact that most of the states across $(\mathcal{O}_1, \mathcal{O}_2)$ are entangled means that, even if Alice's local operation on $\mathcal{A}(\mathcal{O}_1)$ produced a separable state, she would need an extraordinary ability to distinguish the new global state of the field from the typical states on $\mathcal{A}_{12}$. This could be achieved if further resources, such as a well-defined measure of the degrees of entanglement, were actually available in a type III factor von Neumann algebra.

In the last analysis, the practical limitations that Clifton and Halvorson refer to have to do with the inability of the experimenter to ascertain that the state resulting from the action of local operations is indeed separable. This is the sense in which the above arguments do not establish an impossibility *in principle* to destroy any entangled state of a field. Moreover, and perhaps more importantly, such limits are not characteristic only of AQFT. Indeed, they also arise in non-relativistic quantum mechanics. On the one hand, cyclicity is a property of some von Neumann factor algebras of type I. For instance, whenever $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same dimension, there is a local cyclic state on each of the corresponding subalgebras of the algebra of all bounded operators on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. On the other hand, the Generic Result holds for composite quantum systems with at least three particles described by infinite-dimensional Hilbert spaces too.

The impossibility *in principle* of disentangling any state on local algebras must thus be demonstrated on a different ground. The relevant *no go* result should also be formulated in such a way that it only applies to relativistic quantum systems. To be sure, there are plenty of spacetime configurations in Minkowski space that are intrinsically entangled, in the sense that all states across the local algebras associated with them are entangled. This means that, in general, disentanglement can never be achieved in AQFT, no matter what one does. However, separable states exists on local algebras which are split. So, the hope of isolating a field system from other entangled field systems by performing local operations would make sense just for strictly spacelike separated double cones[3]. Clifton and Halvorson's *no go* result becomes interesting within such a domain.

I reconstruct their proof below in a way that will put me in a position to highlight its conceptual consequences. The crucial definition to recall is that of *abelian* projections: a projection P of a von Neumann algebra $\mathcal{M}$ is abelian just in case the algebra $P\mathcal{M}P$, generated by its application to all the elements of $\mathcal{M}$, is abelian. Clifton and Halvorson consider a local operation performed on the spacetime region $\mathcal{O}_1$ represented by a selective map, where by the Kraus operator K coincides with a projection. That is $T_P(\cdot) = P(\cdot)P$.

[3]The existence of product states between local algebras satisfying the split property was proven by Buchholz (1974), who also constructed a concrete example of such algebras. The split property is in fact verified in a number of physically relevant quantum field models. In particular, Buchholz demonstrated that it holds for free neutral massive scalar fields.

The argument then proceeds by distinguishing the non-relativistic case and the relativistic case.

Type I

Suppose the joint state ϕ is entangled across $(\mathcal{A}, \mathcal{B})$. The projection P can be chosen to be abelian, hence the map T_P has the effect to make the algebra $P\mathcal{A}P$ commutative. As a consequence of the theorem by Raggio and Bacciagaluppi any state across the pair $(P\mathcal{M}P, \mathcal{B})$ is separable. Therefore, $T_P^*\phi$ must be unentangled.

Type III

In this case there is no abelian projection, and hence P cannot be chosen to be abelian. As a result, the algebra $P\mathcal{A}(\mathcal{O}_1)P$ is noncommutative. By *Fact 1* there exists at least one state φ across the pair $(P\mathcal{A}(\mathcal{O}_1)P, \mathcal{A}(\mathcal{O}_2))$ that is entangled. Such a state is the image under T_P^* of a state ϕ on $\mathcal{A}_{12}$, i.e. $\varphi = \phi \circ T_P$. Then, *Fact 1* allows one to conclude that the initial state ϕ must have been entangled as well.

The upshot of the proof is not that entanglement across type III von Neumann algebras associated with strictly spacelike separated regions of Minkowski space can never be destroyed, but there is *some* joint state of a global quantum field that will remain entangled irrespective of how one acts locally by means of T_P. In fact, other states across $(\mathcal{O}_1, \mathcal{O}_2)$ may well get disentangled. Yet, there is no way to determine which ones[4]. Such a result is truly remarkable and demonstrates a feature of the local algebras identified by factors of type III which is not at all shared by the algebras of non-relativistic quantum mechanics. Notwithstanding this, some care needs to be taken.

First of all, Clifton and Halvorson's conclusion is not as general as it may appear. It just holds for selective measurements. Nevertheless, one would like to maintain that there is no disentangling local operation of any kind in Algebraic Quantum Field Theory. Restricting the argument to maps of the form $T_P(\cdot) = P(\cdot)P$ leaves open the possibility that some other $T \in \mathcal{A}(\mathcal{O}_1)$ can disentangle all states on $\mathcal{A}_{12}$. Furthermore, resorting to selective operations for the purpose of isolating a quantum field system from other systems located in spacelike separated spacetime regions does not really seem appropriate in the context of Clifton-Halvorson's work.

In their view the difference between non-selective and selective measure-

[4]Actually, there is dense set of entangled states across the pair of local algebras $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ that will not become separable under the action of T_P (we do not need to go into the details of this part of the proof). However, this does not mean that some joint state is actually disentangled.

ments reflects the distinction between physical and conceptual operations. In fact, a (completely positive) non-selective map describes the physical interaction of a quantum system with the measuring apparatus. That is tantamount to separating an initial ensemble of identical copies of the measured system into sub-ensembles, each one of which is associated with a Kraus operator K_i. Non-selective measurement always yield a positive result. Instead, a selective measurement can elicit different answers. It thus corresponds to throwing away those sub-ensembles that are not associated with the selected outcome. Clifton and Halvorson regarded the latter as a conceptual procedure with no physical counterpart. This is, they claim, the key to dissolve the puzzle arising from the non-local behavior of quantum fields predicted by the Reeh-Schlieder theorem. Accordingly, the "physically quite surprising" fact that one can approximate all states of any local algebra by means of arbitrarily local operations reduces to a purely mathematical fact due to the "innocuous" conceptual component of selective measurements. Irrespective of whether or not one agrees with this interpretation, though, one should find it quite at odds with the choice of T_P as the relevant disentangling operation. For, supposedly, the destruction of entanglement is a physical process. Thus, a selective measurement being a conceptual operation would not be relevant to disentanglement in any interesting way.

In order to fill these gaps in Clifton and Halvorson's argument one needs to extend their result to any local operations T that may be performed on $\mathcal{A}(\mathcal{O}_1)$. My reconstruction of their proof indicates that the crucial elements to prohibit disentanglement to take place is the lack of abelian projections in a type III factor. The other ingredients, namely *Fact 1* and *Fact 2*, are consequences of general properties of the algebraic approach that apply to the type I case as well. More generally, what allows for any entangled state to be transformed into a separable state is the existence of a local operation T making the relevant algebra abelian. This intuition is captured by the following proposition[5].

> **Proposition:** If T is a local operation on the noncommutative von Neumann algebra $\mathcal{M}$ and $T(\mathcal{M})$ is commutative, then T disentangles $\mathcal{M}$ from any algebra, in particular from its commutant $\mathcal{M}'$.

Since T is local, it can be written in terms of isometries $\{V_i\} \subseteq \mathcal{M}$, that is $T(\cdot) = \sum_i V_i^*(\cdot)V_i$. Any $A' \in \mathcal{M}'$ is invariant under T. In fact,

$$T(A') = \sum_i V_i^* A' V_i = \sum_i V_i^* V_i A' = A'$$

where the last two equalities are guaranteed by V_i commuting with all elements of the commutant of $\mathcal{M}$ and by the property $V_i^* V_i = I$ of any isometry,

[5] I am indebted to Hans Halvorson for suggesting this strategy.

respectively. Given an arbitrary state ω on the joint algebra $\mathcal{M} \bigvee \mathcal{M}'$, it follows from Lemma 3.12 in Summers (1990) that $\omega(T(AA')) = \omega(T(A)A')$. Then, by the commutativity of $T(\mathcal{M})$ one can also infer the factorization

$$\omega(T(A)A')) = \rho(T(A))\tau(A')$$

for some state ρ on $T(\mathcal{M})$ and some state τ on $\mathcal{M}'$. Accordingly, $T^*\omega$ is a product state, and thus it is separable. In particular, this is true even if ω is chosen to be entangled.

I now claim that a map with such a disentangling property cannot be defined in Algebraic Quantum Field Theory. That is the content of the conjecture stated below. If one proves that the latter is true, then one derives a general result holding in AQFT.

> **Conjecture:** If $\mathcal{M}$ is a von Neumann algebra of type III, then there is no local operation T such that $T(\mathcal{M})$ is a commutative algebra.

In the language of Algebraic Quantum Field Theory, this would mean that no local operation performed on $\mathcal{A}(\mathcal{O}_1)$ can disentangle any state ϕ across the spacelike separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$ of Minkowski space. In other words, an experimenter would never be able, *in principle*, to isolate a quantum field system from another field system by acting locally on any of them. That completes the generalization of the Clifton-Halvorson result.

By demonstrating a feature of type III factors that is not shared by ordinary quantum mechanics, one may expect to have found an aspect of entanglement that is characteristic of the relativistic setting. As Clifton and Halvorson put it,

> the advantage of the formalism of AQFT is that it allows us to see clearly just how much more deeply entrenched entanglement is in *relativistic* quantum theory. At the very least, this should serve as a strong note of caution to those who would quickly assert that quantum nonlocality cannot peacefully exist with relativity. [Clifton-Halvorson (2001), p.28]

This is certainly true. Nonetheless, the claim they draw from their result is too strong. Indeed, type II von Neumann factor algebras lack abelian projections as well. Therefore, the impossibility of destroying any entangled state by performing local measurements is not peculiar just to local algebras of AQFT. Nor does it seem that one can appeal to structural differences between factors of type II and type III to show that the non-existence of disentangling maps is related uniquely to relativistic quantum theory. The only aspect on which the two algebraic configurations diverge, as far as projections are involved, is that in the type II case not all projections are required to be infinite. Yet, nothing in their argument rests on the fact that P is infinite. So, *pace* Clifton and Halvorson, the persistence of entanglement under local operations is not a direct consequence of associating the (bounded) regions of Minkowski spacetime in which the entangled quantum field systems are located with nets of algebras described by type III_1 factors.

Let us conclude with some remarks on whether quantum field theory is at all testable in such a deeply entangled world. Clifton and Halvorson suggest that, ironically, testing the theory in the relativistic case is easier than in ordinary quantum mechanics. Performing a testing measurement on a field system in the region $\mathcal{O}_1$ requires that the experimenter first prepares a state ρ on $\mathcal{A}(\mathcal{O}_1)$. For split regions such a preparation can be achieved by means of a local operation on a region $\tilde{\mathcal{O}}_1$ which properly contains $\mathcal{O}_1$. Let us call a local operation $\tilde{T}$ in $\mathcal{A}(\tilde{\mathcal{O}}_1)$ *approximatively* local on $\mathcal{O}_1$, as $\tilde{\mathcal{O}}_1$ can be chosen to be arbitrarily larger than $\mathcal{O}_1$. Summers (1990) proved that by acting with $\tilde{T}$ Alice has the freedom to prepare whatever state on $\mathcal{A}(\mathcal{O}_1)$ that she pleases. In particular, $\rho = \phi_1 \circ \tilde{T}$ for any initial state ϕ_1 of the local field on $\mathcal{O}_1$. Furthermore, any mapping $\tilde{T}$ thus constructed would disentangle all states across the pair of local algebras $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$.

So as soon as we allow Alice to perform *approximatively* local op-

> erations on her field system, she *can* isolate it from entanglement with other strictly-separated field systems, while simultaneously preparing its state as she likes and with relative ease. God is subtle, but not malicious. [Clifton-Halvorson (2001), p.29]

Notice that the experimenter's ability of of destroying entanglement is enacted by the existence of a type I von Neumann algebra $\mathcal{M}$ splitting the local algebras associated with $\mathcal{O}_1$ and $\tilde{\mathcal{O}}_1$, in the sense that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathcal{A}(\tilde{\mathcal{O}}_1)$. The mapping $\tilde{T}$ is indeed an extension to $\mathcal{A}(\tilde{\mathcal{O}}_1)$ of a disentangling operation in $\mathcal{M}$. The latter takes the general form $\sum_i P_i(\cdot)P_i$, where $\{P_i\}$ is a family of mutually commuting atomic projections. The type III character of the local algebra associated with $\tilde{\mathcal{O}}_1$ assures that any such projection is equivalent to the identity in $\mathcal{A}(\tilde{\mathcal{O}}_1)$. Moreover, ρ has a counterpart in a state defined on $\mathcal{M}$, whose corresponding density matrix is an infinite convex combination of P_i. Then, one can conclude $\tilde{T}(A) = \rho(A)I$ for any $A \in \mathcal{A}(\mathcal{O}_1)$, thus completing the local preparation of arbitrary states on the region $\mathcal{O}_1$ of Minkowski spacetime. Since P_i can be chosen to be abelian for $\mathcal{M}$, the usual argument for disentangling any state across a pair of algebras goes through. Also, $\tilde{T}$ may just be a non-selective operation.

Disentanglement of the pair $(\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is thus possible if one performs local operations on $\tilde{\mathcal{O}}_1$[6]. From a practical point of view, this requires that Alice is actually allowed to step out of her spacetime region and then operate in a larger laboratory. Although $\tilde{\mathcal{O}}_1$ can be arbitrarily close to $\mathcal{O}_1$, Alice is subjected to some restriction. In fact, she can destroy entanglement only if $\mathcal{O}_2$ is contained in the causal complement of $\tilde{\mathcal{O}}_1$. In such a case, $\tilde{T}(B) = B$ for any $B \in \mathcal{A}(\mathcal{O}_2)$. This implies $\phi(\tilde{T}(AB)) = \phi(\tilde{T}(A)B)$, which can be rewritten as $\phi(\rho(A)B)$. It follows that

$$\tilde{T}^*\phi(AB) = \rho(A)\phi(B)$$

That is, the state resulting from Alice's (approximatively) local action is a product state. Therefore, any entangled state ϕ of the global field can made separable. However, the requirement $\mathcal{O}_2 \subseteq \tilde{\mathcal{O}}_1$ could be quite strong depending on the size of the region $\mathcal{O}_1$ and the spacelike separation of the latter from $\mathcal{O}_2$. Furthermore, the above reasoning rests on the assumption that the relevant regions of Minkowski space enjoy the split property[7]. Hence, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are not strictly spacelike separated double cones, testing quantum field theory in the way described by Clifton and Halvorson is not possible; nor can one achieve disentanglement by performing any approximatively local

[6]To be sure, though, the action of $\tilde{T}$ would not disentangle $\tilde{\mathcal{O}}_1$ from $\mathcal{O}_2$!

[7]Concretely, disentanglement could thus be achieved if the relevant model is that of the free neutral massive scalar field.

operation.

To conclude, an experimenter confined in a certain region of spacetime can never destroy entanglement with another region by acting locally; however, under very restrictive circumstances, if she is allowed to step out of her region she would be able to isolate her field systems from other field systems. *God is indeed benevolent, but quite demanding!*

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