TEACHERS' INSTRUCTIONAL PRACTICES WITHIN CONNECTED CLASSROOM TECHNOLOGY ENVIRONMENT TO SUPPORT REPRESENTATIONAL FLUENCY

By

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To Serkan

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LIST OF ABBREVIATIONS

3PL	Three-Parameter Logistics
ARS	Audience Response Systems
CCMS	Classroom Connectivity for Mathematics and Science Achievement
CCSS	Common Core State Standards
CCSSO	Council of Chief State School Officers
CCT	Connected Classroom Technology
CRESST	Center for Research on Evaluation, Standards, and Student Testing
IAEEA	International Association for the Evaluation of Educational Achievement
IRE	Initiate-Respond-Evaluate
IRT	Item Response Theory
NAEP	National Assessment of Educational Progress
NCES	National Center for Educational Statistics
NCTM	National Council of Teachers of Mathematics
NMAP	National Mathematics Advisory Panel
OECD	Organization for Economic Co-operation and Development
PD	Professional Development
PISA	Program for International Student Assessment
TI	Texas Instruments
TI-Nspire	Texas Instruments Nspire
TIMSS	Trends in Mathematics and Science Study

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The purpose of this study was to compare the ways that teachers use Connected Classroom Technology (CCT) to potentially support achievement on translation problems that require moving from one representation to another. Representational fluency is essential for students' mathematical conceptual understanding. Previous research has studied how communication and technologies serve as instructional strategies that may separately support how students develop representational fluency. However, students often leave schools without obtaining these abilities (e.g., Herman, 2007). This dissertation extends prior research by examining how communication in CCT environments may improve students' representational fluency abilities in mathematics classes.

Four mathematics classrooms were chosen based on their gain scores on translation problems in algebra pre- and posttest examinations. The classrooms chosen were the two with the highest and the two with lowest gain scores among the classrooms with pre-test scores that were below 50%. This study used video-recorded

observational data from each classroom to examine the similarities and differences of classrooms where teachers used CCT.

This study analyzes the general classroom description, psychological environments, and general teaching approaches as well as each teacher's representational practices. It then identifies and details representational fluency themes, focusing on how fluency is practiced in effective and less effective classrooms. This study found that both teachers in effective classrooms created environments wherein students could interpret representations by linking them to real-world scenarios; moreover, these students used multiple representations simultaneously and translate between representations through discussion. Additionally, the students and teachers co-constructed different translations. Contrastingly, it shows that teachers in less effective classrooms fostered environments wherein students interpreted representations superficially, used representations independently, and missed opportunities to translate representations through discussion. Students in less effective classrooms generally observed translations. Implications for the results, including this study's limitations and further research are discussed.

CHAPTER 1
INTRODUCTION

The U.S. educational system has undergone reform based on international comparison studies such as Program for International Student Assessment (PISA, 2003). Since its introduction, PISA has influenced the way that high school mathematics education continues to be reformed (Tienken, 2008). Later, President George W. Bush created the National Mathematics Advisory Panel (NMAP) in 2006 to make policy recommendations based on concerns that U.S. students were not competitive globally. The members of the advisory panel emphasized their concern about the increasing number of retiring workers as well as the need for technologically knowledgeable workers as both a safety issue for the nation and an issue affecting the quality of life. As the Final Report of the NMAP (2008) notes, the United States employs many technically talented people from abroad, which affects the nation's autonomy in mathematics, natural sciences, and engineering. The aim of the NMAP report was to make recommendations that would establish the United States as an international leader in the global economy. The U.S. rank in international tests, however, is very low for an international leader.

The U.S. students' international achievement scores come from Trends in International Mathematics and Science Study (TIMSS) and PISA (McGrath, 2011). TIMSS measures fourth and eighth graders' mathematics and science achievement. Forty-eight countries participated in the TIMMS 2007 eighth-grade mathematics assessment. According to the International Association for the Evaluation of Educational Achievement (IAEEA, 2007), the United States ranked 9th, and the average score for U.S. students was 508 in mathematics, whereas the average TIMSS score across all countries was 500. PISA measures 15-year-old students' mathematics, science, and reading literacy achievement. With 65 countries and educational systems participating, PISA 2009 is the most recent international test. Compared to the Organization for Economic Co-operation and Development (OECD) countries' average score of 496, the U.S. average was 487 in mathematics literacy (IAEEA, 2009). In general, United States students' average scores were close to the average score across countries in TIMMS 2007 and PISA 2009. Many feel, however, that since the U.S. is an international leader, its students should be scoring higher than average (NMAP, 2008).

The National Governors Association Center for Best Practices and the Council of Chief State School Officers (CCSSO) organized the Common Core State Standards (CCSS) initiative to augment instruction so that U.S. students may compete successfully in a global economy (CCSSO, 2010). One of the Standards for Mathematical Practice (CCSSO, 2010) states that "mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends" (p. 6), which are necessary abilities for representational fluency.

Representational fluency is a cognitive competence that includes being able to interpret and construct representations as well as translate flexibly between them (Sandoval, Bell, Coleman, Enyedy, & Suthers, 2000). Translating between representations refers to the ability to interchange between forms, for instance, moving from a graph to an equation, or vice versa. Representational fluency is an essential skill for the professional use of mathematics in modeling situations (Lesh & Zawojewski, 2007). The ability to competently model problems requires using a diversity of mathematical representations, such as words, graphs, pictures, and images.

Students need to have opportunities to engage in problem solving to be mathematically proficient (CCSSO, 2010). Mathematical problems generally occur in multi-representational environments (Lesh, Post, & Behr, 1987), which serve as tools for problem solving (NMAP, 2008). Students use a variety of representations to simplify, explain, justify, or predict problem solutions (Lesh et al., 1987). In fact, one recommendation for improving mathematical problem solving is “to teach students how to use visual representations” (National Center for Education Statistics [NCES], 2012, p. 23).

The mathematics education community in the U.S. has paid increasing attention to representational fluency because of new mathematical proficiency standards and the need to develop 21st century skills (e.g., critical thinking). The National Council of Teachers of Mathematics (NCTM, 2000) calls for at least three instructional strategies to increase students’ opportunities to learn representational fluency. First, students should be provided opportunities to use both conventional representations and non-conventional representations (diSessa, Hammer, Sherin, & Kolpakowski, 1991; Greeno & Hall, 1997). Second, students should be able to identify and create connections between representations, which also supports understanding (Brenner et al., 1997). Third, students need multiple opportunities to flexibly work with representations to solve problems and to defend their solutions in order to develop representational fluency (diSessa et al., 1991; Greeno & Hall, 1997; Warner, Schorr, & Davis, 2009).

This study aims to examine the ways in which teachers used connected classroom technology (CCT) to potentially support representational fluency. It investigates the effects of technology and the role of communication on the development of representational fluency.

This chapter explores the relationship between representational fluency, conceptual understanding, and achievement. Then the potential effects of technology and classroom communication on developing representational fluency are examined. The need for representational fluency and the difficulty of developing representational fluency are discussed under the Statement of the Problem section. Finally, this chapter concludes with the purpose of the study and research question.

Representational Fluency, Mathematical Understanding, and Achievement

Hiebert and Lefevre (1986) defined conceptual knowledge as “knowledge that is rich in relationships” (p. 3). These researchers used a metaphor of a “connected web of knowledge” (p. 3) to illustrate this notion. For example, students exhibit conceptual knowledge based on the degree to which they can recognize the relationship between different forms of information such as multi-digit subtraction and the place value position of digits in a number.

Representational fluency and conceptual understanding are two sides of the same coin; they are intertwined constructs. One source of evidence for conceptual understanding is “us[ing] and interrelat[ing] models, diagrams, manipulatives, and varied representations of concepts” (National Assessment of Educational Progress [NAEP], 2003). In other words, conceptual understanding is reflected in the ability to translate between representations, such as between graphs, tables, and words. Representational fluency is therefore a marker or indicator of conceptual understanding.

Conceptual understanding is developed when connections between multiple representations are formed through their active use during instruction (Pape & Tchoshanov, 2001). Models of translations between representations (e.g., Lesh et al., 1987) show that mathematical concepts can be understood in multiple ways. If a learner makes connections between these abstractions, then they will likely develop meaningful understanding (Cramer, 2003). That is, students understand mathematical concepts better when they are able to switch between and understand the connection across representations (Brenner et al., 1997; Cramer, 2003; Even, 1998; Herman, 2007; Lesh et al., 1987; Lesh & Zawojewski, 2007; NCTM, 2000; Pape & Tchoshanov, 2001). Representational fluency not only has an impact on conceptual understanding, it also plays a significant role in helping students solve problems correctly (Ainsworth, Bibby, & Wood, 2002; Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009).

Even though some researchers have suggested that representational fluency by itself is not enough to solve a problem correctly (Bieda & Nathan, 2009), it is an important component of problem solution (Ainsworth et al., 2002; Nistal et al., 2009). Students perform more accurately when they use more and combined representations (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007), use non-symbolic representations (Suh & Moyer, 2007), and have the ability to translate between representations (Brenner et al., 1997). In addition, the nature of particular problem structures may require representational fluency since problem solvers need to be flexible in their use and need to be able to change naturally to more efficient representations during problem solving (Lesh et al., 1987). In other words, if students have the ability to translate between representations flexibly, they can move to representations that are effective (i.e., adaptive use of representations) (Ainsworth et al., 2002; Nistal et al., 2009), or use representations that are more comfortable for them. Thus, they will become more likely to solve problems than students who do not have the ability to translate representations.

Factors that Support Representational Fluency

To develop students' representational fluency abilities, students need opportunities to see the relationship between different representations. This can be done by creating discussion environments that require students to state the relationship between representations and how they are aligned to one another during problem-solving activities. This study therefore argues that two tools, communication and technology, may support the development of representational fluency.

Mathematically proficient students are able to "justify their conclusions, communicate them to others, and respond to the arguments of others" (CCSS, 2010, p. 6-7). Students therefore need an environment in which they can critique their peers' representations and respond to others' critiques about their representations (diSessa et al., 1991; Warner et al., 2009). Teachers' and peers' roles are critical in students' interactions and discussions about representations.

Active engagement in the construction and interpretation of representations can include both creating new representations, such as non-standard representations (e.g., drawings) and reformulating or refining existing, standard representations (e.g., symbolic representations) (diSessa et al., 1991; Greeno & Hall, 1997; Warner et al., 2009). Students use and value others' representations in such engagements, and they also change their representations with the aim of explaining and generalizing (Warner et al., 2009). These interactions create the opportunity for the development of representational fluency for the individual as well as the group (diSessa et al., 1991; Warner et al., 2009). That is, representation is both socially negotiated and individually constructed (Cobb, Yackel, & Wood, 1992).

The use of graphing calculators, such as the Texas Instruments Nspire (TI-Nspire) calculators, enables students to translate between and use multiple representations (Bieda & Nathan, 2009; Bostic & Pape, 2010; Herman, 2007; Knuth, 2000) because they allow multiple representations (i.e., symbolic, tabular, and graphical representations) to be viewed on one calculator screen (Bostic & Pape, 2010). Calculator use can support cognitive links between representations by providing quick access which in turn may increase students' ability to flexibly translate between representations (Bostic & Pape, 2010). In addition, when students are given the opportunity, they are able to use non-symbolic representations such as graphical representations rather than traditional symbolic and computational strategies (Bostic & Pape, 2010). Because evidence suggests that communication and technology may separately support students' developing representational fluency, the present study investigates instruction that is characterized by the use of technology with the aim of examining the relationship between these instructional strategies and increasing such fluency.

Connected Classroom Technology

CCT systems are "wireless communication systems that connect the teacher's computer and students' handheld technology" (Pape, Irving, Owens, et al., 2013, p. 169). These systems are designed to provide greater opportunities to discuss connections among multiple representations. The recent studies about these types of classroom communication systems (e.g., Hegedus & Moreno-Armella, 2009) emphasize a sociocultural perspective that focuses on the relationship between learning opportunities and students' abilities to take advantage of these opportunities during learning (Gee, 2008). CCT provides at least two learning opportunities that support the development of students' representational fluency. These opportunities include "the mobility of multiple representations of mathematical objects" and "the ability to flexibly collect, manipulate and display to the whole-classroom representationally-rich student constructions, and to broadcast mathematical objects to the class" (Hegedus & Moreno-Armella, 2009, p. 403). Jim Kaput once postulated, "wireless connectivity 'inside' the classroom would change the communicative heart of the mathematics classroom" (Hegedus & Penuel, 2008, p. 171). CCT technology's progression has enacted this transformation.

This technology has evolved from Audience Response Systems (ARS) to second-generation CCT systems. Pape et al. (2013) called the Texas Instruments (TI) Navigator a second-generation CCT system to demonstrate the difference of this system from ARS systems. The TI-Navigator system has four components to support learning: Quick Poll, Learn Check, Screen Capture, and Activity Center (Pape et al., 2013). The first two components are similar to ARS systems; however, the last two components, which are not present in ARS systems, show the sophisticated structure of the CCT. A summary of the TI-Navigator system components is given in Appendix A, and the details of each component are explained in Chapter 2.

With Screen Capture, teachers can show each student's response on the projector as a "snapshot" of each calculator. This feature allows both teachers and students to compare the solutions or representations through productive discussion.

With Activity Center, the teacher can show a shared coordinate plane onto which students can submit their points, equations, and graphs. One of the affordances of the second-generation CCT is that the activity center promotes examination and analysis of patterns as well as justification of mathematical generalizations, which may support representational fluency and conceptual understanding. For example, students start to generalize the effect of changing parameters of an equation on the graph's position and shape as their classmates or themselves submit their equations (Hegedus & Moreno-Armella, 2009).

These last two components provide a context for effective classroom discourse because they are designed to publicly display multiple linked representations (Hegedus & Moreno-Armella, 2009; Pape et al., 2013; Roschelle, Vahey, Tatar, Kaput, & Hegedus, 2003). The public display of students' mathematical constructions in conjunction with the communication of ideas and strategies fosters representational expressivity (Hegedus & Moreno-Armella, 2009). That is, projection of students' mathematical thinking by using public display provides a context in which students communicate about representations. In addition, because students' contributions are anonymous, they can see the differences between their work and the group's work comfortably, which provides a suitable context for conjectures and generalization. The public display shows "the juxtaposition of ideas, often literally a debate rather than a resolution or synthesis" (Stroup, Ares, & Hurford, 2005, p. 188). Also, the activities in the activity center with multiple representations may support translation between representations (Bostic & Pape, 2010; Herman, 2007), which are distinguishing characteristics of mathematical proficiency (CCSSO, 2010; Kilpatrick, Swafford, & Findell, 2001).

Although teachers have found CCT to be an efficient means of instruction, there is little evidence that demonstrates or evaluates the effectiveness of its mechanisms (Vahey, Tatar, & Roschelle, 2007). Thus, researchers have suggested qualitative studies that explore learning and teaching in CCT classrooms (e.g., Hegedus & Moreno-Armella, 2009; Hegedus & Penuel, 2008; Pape et al., 2013). Given the current attention called to CCT environments, there is a need to examine this potential for supporting representational fluency in second-generation CCT environments under realistic conditions.

Statement of the Problem

To benefit from representations, students need to be able to comprehend each representation and make connections between them (e.g., Ainsworth, 1999; Brenner et al., 1999; Cramer, 2003; Knuth, 2000; Lesh & Doerr, 2003). As students' representational repertoire expands, they should learn to evaluate the strengths and weaknesses of these representations, know their different purposes, and be able to translate between representations.

Research has shown, however, that students lack these skills at the middle school (Ainsworth et al., 2002), high school (Knuth, 2000), and college (Herman, 2007) levels. Current research suggests two reasons for the paucity of these skills: the overemphasis of symbolic representations within instruction and curriculum (e.g., Knuth, 2000), and the cognitive difficulty of representational fluency (e.g., Ainsworth et al., 2002). Students have difficulty translating between and within representations (Ainsworth et al., 2002; Davis & Maher, 1997; Even, 1998; Lesh et al., 1987). For example, students often identify functions with symbolic representations only (Bostic & Pape, 2010; Herman, 2007; Kaldrimidou & Ikonomou, 1998; Knuth, 2000). This may cause difficulty for researchers in examining representational fluency in typical classrooms. But the CCT environment may provide a context for researchers to examine representational fluency by providing students the opportunity to see interconnected representations and discuss the relationship between representations.

Since representational fluency is important to student learning, instructional contexts that support the development of representational fluency need to be better understood. Few studies have examined environments that help develop these skills (Brenner et al., 1997). The present study argues that implementing CCT technology within a discourse-rich environment supports students' developing representational fluency.

Purpose of the Study

CCT provides a context for possible communication by increasing teachers' knowledge about their students' understanding and projecting students' mathematical constructions. "CCTs encompass a broad range of devices that network a teacher's computer with students' handheld devices employed to increase communication among and between students and teachers" (Pape, Irving, Bell, et al. 2012, p. 178). For example, a teacher may project an image from the Internet superimposed on a coordinate grid, and students may be asked to submit the equation of a curve that fits the arc within the image. Irving et al. (2010) indicate, the "simultaneous display of multiple mathematical representations (e.g., equations, graphs, data tables) creates opportunities for rich mathematical discourse and supports the design and implementation of inquiry lessons related to the coordinate plane" (p. 6-7).

Moreover, classroom communication within CCT-enhanced environments may support the development of representational fluency and conceptual understanding. Students exhibit stronger performance in problem solving when they have conceptual knowledge or representational fluency (Herman, 2007). Therefore, this investigation seeks to examine the effects of CCT on the development of representational fluency, classrooms were ranked by using students' achievement on translation problems. Among the classrooms that evidenced initially low achievement, the two classrooms with the highest gain and the two classrooms with the lowest gain were selected to analyze through classroom observation videos. Thus, the purpose of the present study was to compare the ways in which teachers used CCT to potentially support representational fluency within initially lower achieving classrooms that showed distinct progress on translation problems.

Research Question

How do teachers' uses of CCT differ between classes that were initially low achieving but then showed differential improvement on translation problems?

CHAPTER 2
LITERATURE REVIEW

Chapter 1 revealed the importance of representational fluency for learning mathematics and using mathematics professionally. In Chapter 2, more precise meaning is given to the relevant terms. These terms are not simply defined but are associated with the significant connections to representations in the educational research literature. This section examines the definitions of representations and representational fluency, the impacts of representational fluency on mathematical understanding and achievement, and the factors that influence the development of representational fluency, such as communication and technology. Finally, the effect of CCT on representational fluency is explored.

Representational Fluency

The Concept of Representation

Representations are tools to help record, analyze, solve, reason, understand, justify, explain, and communicate mathematical concepts (Greeno & Hall, 1997; Pape & Tchoshanov, 2001; Preston & Garner, 2003). Some researchers have conceptualized representations as a language of mathematics (Coulombe & Berenson, 2001). Students use representations when they interpret mathematical phenomena, much like an artist interprets the world through mediums such as paint, sculpture, or literature. Greeno and Hall (1997) compare the expressive and inventive properties of mathematical representations to those of painting, sculpting, and literature in that “particular uses of expression and communication are flexibly constructed, and are open to multiple interpretations” (p. 367).

One of the definitions for representation is “a configuration of signs, characters, icons, or objects that can somehow stand for, or ‘represent’ something else” (Goldin, 2003, p. 276). In this definition, interpretations of representations include “correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize” (Goldin, 2003, p. 276). For instance, a graph as a representation can correspond to an algebraic function, depict a set of data, or denote a linear relationship. The problem with this definition is that it only emphasizes concrete properties of representations. For example, a graph is an *object* or its corresponding algebraic function is an *action*.

Referring to something as a representation means that one needs to interpret and give a meaning to notations (Greeno & Hall, 1997). According to this view, a table of values is a representation if a child interprets and gives a meaning to it; otherwise it is just as meaningless notation. Representations are constructed flexibly, open to multiple interpretations, and used for communication and expression of mathematical ideas in the learning and teaching of mathematics. A definition that emphasizes the process and product features of representations is used in the present study. According to NCTM (2000), representation means “the act of capturing a mathematical concept or relationship in some form and to the form itself” (p. 67).

Representations have been categorized as internal and external (Goldin, 2003; Goldin & Kaput, 1996; Pape & Tchoshanov, 2001). Internal representations refer to an individual’s mental constructions. In other words, those “presumed to be encoded in the brain but mainly described at more holistic levels (such as verbal and syntactic configurations, visual imagery, internalized mathematical symbols, rules and algorithms, heuristic plans, schemata, and so forth)" (Goldin, 2002, p. 207). These internal representations, which are an "abstraction of mathematical ideas or cognitive schemata" (Pape & Tchoshanov, 2001, p. 119), are developed through an individual's learning experiences.

External representations refer to physical configurations that can be observed in the present environment such as real-world objects or events, pictures, spoken or written words, formulas and equations, geometric figures, graphs, base ten blocks, or computer-based microworld configurations (Goldin, 2002; Goldin & Kaput, 1996). For example, 5 (numeral) and five (number name) are the external representations that stimulate an image of a set of five objects, which is an internal representation (Pape & Tchoshanov, 2001). The relationship between external and internal representations is called "cognitive representation," which is conceptualized "as a zone of interaction of external and internal representations" (Pape & Tchoshanov, 2001, p. 126). This study focuses only on external representations because they are observable. Also, these are the types of representations that have mostly been studied by researchers (e.g., Herman, 2007).

Types of Representations

There are at least five different types of representations: real scripts, manipulative models, static pictures, written symbols, and spoken language (Lesh et al., 1987). In real or experience-based scripts, knowledge is arranged around real-world situations providing a context to interpret and solve other types of problems. The relationships and operations related to the manipulative models have meaning rather than the elements themselves. Examples include arithmetic blocks, fraction bars, and number lines. Similar to manipulative models, static pictures include static figurative models, and they “can be internalized as ‘images’” (p. 33). Written symbols include “specialized sentences and phrases” (p. 34), such as 3x + 5 = 8, and spoken language contains “specialized sublanguage” (p. 33), which is related to a domain such as logic. A model of these types is shown in the unshaded part of Figure 2-1.

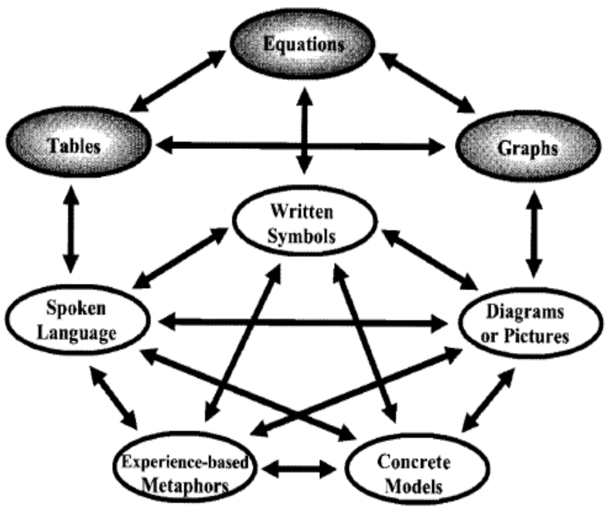


Figure 2-1. Meanings of conceptual systems are distributed across a variety of representational media. Adapted from *Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving* (p. 12), by R. Lesh and H. Doerr, 2003, in R. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Mahwah, NJ: Erlbaum.

Each type of representation has different advantages and disadvantages for cognitive processes. For example, verbal representations are helpful for understanding and communicating about a problem or interpreting its results. However, they can mislead students or cause ambiguity because they depend on personal style (Friedlander & Tabach, 2001). That is, using different representations is powerful because users do not need to limit themselves by the weakness of only one representation (Kaput, 1989). When students have a comprehensive representational repertoire and translation abilities, they can use different representations to illuminate different aspects of a complex concept or relationship. Thus, students need experience working with different representations as well with ways of linking them together (Amit & Fried, 2005).

Symbolic, pictorial, tabular, verbal, and graphical representations are used in the present study. The descriptions of these representations, which are adaped from Bostic (2012) and Preston and Garner (2003), are included in Appendix B.

Translation between and Transformation within Modes of Representations

As students' repertoires of representations expand, they begin to learn to translate between representations (NCTM, 2000). To translate between representations, students need to link the representations with each other. Kaput (1989) has stated that "cognitive linking of representations creates a whole that is more than the sum of its parts…it enables us to see complex ideas in a new way and apply them more effectively" (p. 179-180). In other words, seeing the big picture as a whole with connected parts is more effective than knowing parts of the whole separately. The important premise is that the relationship between these representations should be explicit (Goldin, 2002).

Researchers have defined translation and transformation between representations as an important part of the problem-solving process (Lesh et al., 1987). *Translation* among representations refers to moving between representations, from one representation to a different type of representation, such as translation from a graph to an equation. During translation, one reinterprets an idea from one representation to another, which demonstrates the understanding of concepts in multiple ways. *Transformation* among representations refers to moving within representations, from one representation to another of the same kind, such as transformation from a graph to another graph. It is most common for researchers to use the term translation to discuss both of these constructs. For the purpose of this study, the term “translation” is used to refer to the process of moving from one representation to another representation regardless of representation type.

The Concept of Representational Fluency

Most researchers define representational fluency as an ability to translate between representations (e.g., Bieda & Nathan, 2009). However, representational fluency is much more than the translation between representations. Instead, it is the interaction between each representation inside the individual’s cognition in a meaningful way (Zbiek, Heid, Blume, & Dick, 2007). Representational fluency is “translation across representation, drawing a meaning of a mathematical entity by using different representations, and generalizations across different representations” (Zbiek et al., 2007, p. 1192). Some authors use different words to describe representational fluency. For example, Hong and Thomas (2002) define representational versatility “to include both fluency of translation and the ability to interact procedurally and conceptually with individual representations” (p. 1002).

This definition also aligns with the Zbiek and colleagues’ (2007) definition and adds the interaction with representations procedurally. Sandoval and colleagues (2000) provide an alternative definition:

> …being able to interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link multiple representations in meaningful ways. (p. 6)

This definition is comprehensive and extends prior definitions to include necessary knowledge and abilities for representational fluency. Sandoval and colleagues' (2000) definition is therefore adapted for the present study.

The Characteristics of Representational Fluency

Students benefit from being able to translate between representations. Moving from one representation to another and understanding the connections between representations demonstrate a deep understanding of concepts (Lesh & Zawojewski, 2007; NCTM, 2000; Pape & Tchoshanov, 2001). The cognitive results of learning involve improvement in students' ability to represent functional relations in words, tables, figures, and translation among representations (Brenner et al., 1997). Representational fluency is not only related to understanding, but also plays a large role in helping students solve problems correctly (Ainsworth et al., 2002; Even, 1998; Nistal et al., 2009). The following sections explain the relationship between representational fluency, mathematical understanding, and achievement.

The Relationship between Representational Fluency and Mathematical Understanding

Students translate between and within representations by reinterpreting one representation for another (Cramer, 2003). This reinterpretation process reflects a reconstruction of knowledge and impacts students' deep conceptual understanding. Deep understanding occurs when students appreciate each representation and its connection with concepts, as well as the links between representations (Duncan, 2010). In literature on representational fluency, the focus is generally on relational understanding (diSessa et al., 1991; Duncan, 2010, Suh, Johnston, Jamieson, & Mills, 2008; Suh & Moyer, 2007) as well as abstraction (diSessa et al., 1991; Warner et al.,

2006) and generalization (Bieda & Nathan, 2009; Nathan & Kim, 2007; Warner et al., 2009).

Researchers have explored the impact of representational fluency on relational understanding. Duncan (2010) examined 12 teachers' views from six schools in Scotland to examine whether dynamically linked representations improve students' relational understanding. Teachers used TI-Nspire software and calculators in this investigation and were given six days of professional development (PD). In the first two days of the PD, the teachers were trained to use the software and discussed possible lessons and teaching approaches that focused on multiple representations and relational understanding. The teachers were also introduced to the outline of the study and terminology, and discussed the evidence for relational understanding. On the other days of PD, teachers discussed their experiences and issues.

The researchers examined teachers' responses by looking at the evidence of relational understanding and multiple representations. Student feedback and researcher observations were then used for triangulation. Duncan (2010) concluded that 80% of teachers stated that the use of multiple representations improved the students' relational understanding. Most teachers in this study mentioned that they observed students making specific use of multiple representations, detailing verbal and written responses, and "aha" moments, which demonstrated their understanding. For example, a teacher observed his or her students' discussion about the fact that a square is a special kind of rectangle.

While some studies explicitly state that they are investigating the relationship between linked representations and relational understanding (e.g., Duncan, 2010), others imply this relationship. For example, even though diSessa and colleagues' (1991) study on metarepresentational expertise in children does not explicitly mention linked representations and relational understanding, their research reveals that the students understood the connection between duration, speed, and distance by translating between representations when inventing graphs. Similarly, Suh and Moyer (2007) and Suh and colleagues (2008) discussed this kind of relationship only without using the term relational understanding. Instead, these authors used relational learning and thinking (Suh & Moyer, 2007) or flexibility in thinking (Suh et al., 2008).

Suh and Moyer (2007) explored 36 third graders' algebraic reasoning and relational learning as they used physical and virtual manipulatives to develop their representational fluency in a one-week unit. The aim of the unit was to use different algebraic modes to develop students' relational thinking. There were two groups of students. The first group of students used a virtual manipulative (applet) named Virtual Balance Scale. During this exercise, unit boxes represented numbers, and blue boxes represented unknown values such as x. The goal was to get the remaining x-value alone on one side by removing the same amount of unit boxes from both of the sides. This virtual environment provided students "(a) explicit linking of visual and symbolic modes; (b) guided step-by-step support in algorithmic processes; and (c) immediate feedback and a self-checking system" (p. 165). The second group of students used a physical manipulative tool, Hands-On Equations®, which included a balance scale mat, number cubes, and pawn pieces representing an unknown value such as x. The goal was same as the goal for the students who used the virtual manipulatives. This physical environment provided students "(a) tactile features; (b) more opportunities for invented strategies; and (c) more mental mathematics" (p.164). Overall, the students in both of the environments made gains in algebraic reasoning, and they demonstrated flexibility in their translation with word problems, pictorial and symbolic representations, and manipulatives. The researchers concluded that students' understanding was improved by using different kinds of representations.

Flexibility in the use of representations has an impact on thinking about concepts more comprehensively. Suh and colleagues (2008) conducted a study to support this relationship and examined a collaborative study about teaching addition and subtraction of decimal numbers to improve students' representational fluency and proficiency in mathematics. The representations were helpful for the teachers to assess their students' conceptual understanding. The students made a link between whole numbers and decimal numbers by using representations. The teachers had concerns about potential student confusion of flat manipulatives as a representation for 100 in whole numbers and for representing units in decimal numbers. But the researchers argued that representational flexibility helped students to make a connection between decimal and whole numbers. Suh and colleagues (2008) stated that students who understand decimal numbers just extend the notion of whole numbers by using their representational flexibility. In addition, the researchers noted that the flexible use of representations helps students to make generalizations about place value.

Students are able to abstract and generalize after they gain representational flexibility competence. Warner and colleagues (2006) examined how five middle school teachers interacted with their students with respect to constructing, linking, and modifying their representations and then their movement toward abstraction and generalization. The teachers helped their students through these investigations, and the students imitated their teachers' behaviors such as justifying their solutions, asking hypothetical problems (i.e., what if…), or providing opportunities for generalization and abstract representations. Overall, there was an increase in teachers' encouragement related to students' building, explanation, and justification of their peers' and their own representations as well as discussion about their peers' and their own thinking. This study found that when hypothetical situations were employed during these exercises, students improved their abilities in generalizing and abstracting representations. In addition, the students were encouraged to discover their own errors. Teachers asked students to justify their ideas rather than providing them the answers or telling them that their answers were incorrect.

Students who possess representational fluency competence (in constructing, linking, and modifying representations, for instance) are able to progress toward abstraction and generalization. diSessa and colleagues (1991) also conducted a study where the students moved to abstraction through the development of metarepresentational competence by inventing representational activities. There is a clear connection between metarepresentational competence and representational fluency. Metarepresentational competence includes "generat[ing], critique[ing] and refin[ing] representational forms" (diSessa et al., 1991, p.2) and representational fluency includes constructing, translating and interpreting representations (Sandoval et al., 2000). Thus, the present study uses diSessa and colleagues (1991) to define representational fluency.

The sixth-grade students in diSessa and colleagues' (1991) study were able to understand concepts related to motion as well as move to abstraction. Overall, most of the students already knew the graph, but they reinvented the graph which provided evidence of representational fluency. In addition, diSessa and colleagues noted that students improved their understanding by using other students' representations. Some studies, which will be discussed later, used the terminology of representational fluency and illustrated students' abilities at making generalizations after gaining representational fluency. Bieda and Nathan (2009) asserted that if the students were able to translate from a graph to an equation, they would be more likely to make generalizations. Moreover, students were able to generalize problems with the development of representational fluency (Warner et al., 2009).

In summary, through representations teachers are able to better examine students' thinking and understanding of concepts (Greeno & Hall, 1997; Suh et al., 2008; Zbiek et al., 2007). In addition, when students understand the inter-related nature of mathematical representations, they can improve their understanding of mathematical concepts. Bostic and Pape (2010) argued that "well-formed" (p.140) links between representations aid in students' understanding of mathematical concepts. Thus, representational fluency and conceptual understanding are two sides of the same coin; they are intertwined constructs. Representational fluency is very significant for students' development of conceptual understanding (Bostic & Pape, 2010; Duncan, 2010; Herman, 2007; Lesh & Zawojewski, 2007; NCTM, 2000; Nistal et al., 2009; Pape & Tchoshanov, 2001; Warner et al., 2009), specifically relational understanding (diSessa et al., 1991; Duncan, 2010, Suh & Moyer, 2007; Suh et al., 2008), and generalization and abstraction (Bieda & Nathan, 2009; Nathan & Kim, 2007; Suh et al., 2008; Warner et al., 2006; Warner et al., 2009).

The Relationship between Representational Fluency and Achievement

Cognitive flexibility theory is related to achievement and representational fluency. According to this theory, the ability to construct and move between representations in a domain is necessary for successful learning (Ainsworth et al., 2002). If students are able to reconstruct their knowledge to meet the demands of specific tasks, they will be more successful (Nistal et al., 2009). That is, if students have the ability to translate between representations flexibly, they can move to representations that are effective (adaptive use of representation) or use representations with which they are more comfortable. Thus, these students will more likely become better problem solvers than students who do not have the ability to translate representations.

Students who use two or more representations to solve a problem are more successful than students who use only one representation to solve the problem (Nathan & Kim, 2007). Nathan and Kim explored the development of representational fluency in middle school students by looking at their abilities in pattern generalization. The researchers conducted a cross-sectional study with 372 middle school students (122 sixth-graders, 115 seventh-graders, and 135 eighth-graders) and a longitudinal study with 81 sixth-graders through their seventh- and eighth-grade years.

The students were asked to solve problems with two modes: discrete mode (i.e., point-wise graph) and continuous mode (i.e., line graph). Each problem included three tasks: Near Prediction (i.e., NP; reading the data), Far Prediction (i.e., FP; reading between data), and Abstraction (i.e., AB; reading beyond the data). The students were randomly assigned to solve six problems with one type of representation (i.e., verbal, graphical, or combined). The researchers examined the differences in students' pattern generalization performance between different tasks (i.e., NP, FP, and AB), different presentation modes (i.e., continuous and discrete patterns), and different representations (i.e., verbal, mathematical, and combined representations). For the cross-sectional study, the researchers found an overall advantage of verbal representations when solving problems correctly, especially with continuous patterns. In addition, sixth-graders had a verbal advantage on the discrete patterns and difficulty in solving problems mostly with graphical representations. Overall, the highest performance in pattern generalizations was observed when the representations were combined. The researchers therefore suggested that teachers focus on combined or verbal representations, especially for younger students.

Similar to the cross-sectional study, the longitudinal study indicated that there was a verbal advantage over graphical representations for continuous patterns in the longitudinal study. For discrete patterns, the verbal advantage dropped from the seventh to the eighth grade, though rose during the eighth grade. The importance of Nathan and Kim's (2007) study is that it showed that students performed better with verbal representations than they did with graphical representations. In addition, this study indicated that students who used combined representations performed better than the students who used verbal and graphical representations in pattern generalizations. In other words, students who have representational competence perform better than those without such fluency.

Moreover, students perform better at solving and representing function word problems when they are able to translate between representations. Brenner and colleagues (1997) designed a unit that included 20 days of instruction to improve students' skills in translation and in applying representations. The authors used four mathematics reform principles in this instruction. The first mathematics reform principle focused on the representation skill that included representing a problem situation with a mental representation, such as drawing conclusions from word problems, rather than symbolic manipulation skills such as using arithmetic and algebraic processes. The second mathematics reform principle encourages students to use meaningful thematic contexts rather than isolating problems. The third mathematics reform principle emphasized the process rather than the product during problem solving. The fourth principle centered on a guided discovery approach rather than direct instruction. Their study involved 128 seventh- and eighth-graders from six classes in three schools. One of the three teachers' classes was randomly assigned as a comparison group (n=56) and the other two were randomly assigned as the experimental group (n=72) in which the students were taught with traditional and experimental curricula, respectively. In the traditional curricula, the emphasis was on building symbolic manipulation skills that involved using algebraic and arithmetic procedures through direct instruction. In the experimental curricula, the focus was on developing problem representation skills that involved the translation and application of tables, graphs, pictures, and diagrams through the guided discovery approach. In addition, the problems were open-ended which gave students opportunities to solve problems in multiple ways.

In general, the students who answered correctly used an appropriate representation to solve the problems. The students in the experimental group used correct representations more frequently than the students in the comparison group. In addition, the students in the treatment group showed greater improvement in representing and solving the function word problems as well as in translating word problems into tables, equations, or graphs than the students in the comparison group. The students in the treatment and comparison groups showed similar improvement in solving two-step problems, which required problem representation or symbolic manipulation skills. The students in the comparison group were better at using arithmetic and algebraic procedures, which required symbolic manipulation skills, than were the students in the treatment group.

Thus, if the goal is to perform symbolic manipulation, based on Brenner and colleagues four mathematics reform principles, the conventional method is more beneficial than instruction with multiple representations. However, if the goal is to solve function word problems by creating and constructing representations, instruction with multiple representations is more beneficial. Brenner and her colleagues (1997) noted that to solve an algebra word problem one needs to have both representation skill during the solution planning and monitoring phase and symbolic manipulation skill during the solution execution phase. Representation skill includes being able to use verbal, symbolic, tabular, and graphical representations. Symbolic manipulation skill includes being able to use algebraic and arithmetic processes. In schools, students are generally taught symbolic manipulation skills and do not understand functional relationships.

Furthermore, students are more successful when they use non-symbolic representations. As discussed earlier, Suh and Moyer (2007) explored physical and virtual manipulative effects on algebraic reasoning and relational thinking and also analyzed students' use of eight pictorial, eight symbolic, and two word problems from their posttests. The researchers used a scoring rubric to categorize students' understanding. Seventy-eight percent of the students were more successful with pictorial representations than with symbolic representations or word problems. The students used pictorial representations to solve the problems and translated from symbolic to pictorial representations which indicated that students could fluently translate between representations. The importance of Suh and Moyer's study is that it demonstrates that students gained proficiency in algebraic reasoning in both physical and virtual manipulative environments. The researchers conclude that translation from one representation to another during problem solving reflects achievement in algebraic reasoning.

In summary, even though representational fluency by itself is not sufficient to solve a problem correctly (Bieda & Nathan, 2009), it is an important component for solving problems successfully (Ainsworth et al., 2002; Nistal et al., 2009). Students perform better when they use more and multiple representations (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007), when they have the ability to translate between representations (Brenner et al., 1997), and when they prioritize non-symbolic representations (Suh & Moyer, 2007). Thus, good problem solvers are flexible in their uses of representations and are able to change their representations naturally to more convenient representations during problem solving (Lesh et al., 1987).

Factors that Influence the Development of Representational Fluency

Representational fluency involves students' choice of representations as well as translation between representations while solving problems. In this section, factors that affect students' choice of representation and translation between representations are explored briefly, and two of the major factors that influence and support the development of representational fluency – communication and technology – are examined in depth.

Three factors that the literature suggests affect the ability to make flexible representational choice are task, subject, and context (Nistal et al., 2009). Task factors include compatibility models, which emphasize using different representations for different tasks. These models state that if students determine the demands of a task and select the most suitable representation for these task demands, they perform better than individuals who do not (Nistal et al., 2009). For instance, students who select a graph typically perform better than individuals who select a different representation to solve spatial tasks. Thus, representational fluency is related to the ability to match the representation with the task's demands.

Individual student factors include "prior conceptual and procedural knowledge about representations, abstract conditional knowledge about representations, domain-specific knowledge, representational preference and affective factors" (Nistal et al., 2009, p. 630). Abstract conditional knowledge means knowing when and why a representation is used for solving a task (Nistal et al., 2009). All knowledge types, except domain-specific knowledge, affect subjects' selection of appropriate representations. There is disagreement, however, about whether domain-specific knowledge is a requirement for students to interact with representations (Nistal et al., 2009). Also, students' representational preference is influenced by affective factors, such as curiosity and frustration (Nistal et al., 2009).

Context factors include an “environment, which provides active guidance in representational selection, [and an] environment which encourages active comparison and evaluation of representations” (Nistal et al., 2009, p. 630). Students need to be asked to evaluate their classmates’ use of different representations and their use of inappropriate representations.

For the development of students’ representational fluency, teachers need to be aware of the factors that impact both students’ representational choices and their translations between representations. Ainsworth and colleagues (2002) explored the factors that influence translation between representations in a multiple representational environment. The students were taught computational estimation, which included the process of making problems simple and solving with certain procedures to get a suitable and satisfactory answer by calculating mentally. There were four conditions: one control and three different experimental conditions, including pictorial-pictorial (pictorial condition), in which the students used a pair of pictorial representations; mathematical-mathematical (mathematical condition), where the students used a pair of mathematical representations; and pictorial-mathematical (mixed condition), where the students used a mix of pictorial and mathematical representations.

The researchers examined students’ estimation accuracy and their judgments in these conditions. For each experimental condition, the researchers used one continuous representation that included the magnitude and direction information as well as one categorical representation that included only the information of magnitude. Ainsworth and colleagues used the Computational Estimation Notation-Based Teaching System (CENTS) in two experiments to support students’ understanding while practicing with and reflecting on computational estimations. In addition, students were mean to see how the effect of transforming numbers helps understanding on the accuracy of answers during estimation. The focus of their research was to understand how transforming numbers affect the accuracy of the answers when estimating. In Experiment 1, 48 year-5 pupils participated in the study, and in Experiment 2, 48 year-5 and year-6 pupils participated. In Experiment 1, students spent 80-100 minutes (2 times) with CENTS, and all students in all conditions became more accurate estimators. However, while only the students in the mathematical and pictorial conditions improved their ability to judge the accuracy of their estimation, the students in the mixed condition did not improve this ability. The researchers first thought that this failure might be because of the cognitive load of the tasks.

Thus, in Experiment 2, the researchers allotted extra time to remove a possible time effect for translation for the students in the mixed condition group. The students in Experiment 2 spent 150-200 minutes (4 times) with CENTS. This time, students in all conditions became more accurate estimators and improved their ability to judge the accuracy of their estimations. Although the students in the mixed condition mastered the mathematical (continuous) representation that helped them solve the task successfully, they did not master the pictorial (categorical) representations. Whether students could accurately estimate was based on their use of one representation that included the necessary information to solve the given task. The students in the mixed condition did not master the categorical representation, showing that the reason behind the failure of the translation in Experiment 1 was not due to the cognitive load of the task. Instead, it was based on the difficulty of translating between two different representations.

The researchers provided three reasons why they believed the students in the pictorial and mathematical conditions translated between representations instead of using them separately. The first reason was that if the students used representations individually, they would become experts in judging the accuracy of their estimations before becoming experts in translation. It was reversed for the mathematical condition, however, in which the students were experts in the translation but not experts in their judgment of accuracy of their estimations for Experiment 1. The second reason was that if the students used representations separately without translation, the result would not change based on the representations with which they were paired. However, in this study, the pictorial (categorical) representation was paired with the numerical (continuous) representation, and the students did not solve the problem with the categorical representation. But when the same representation was paired with pictorial (continuous) representations, the students solved the problem with the categorical representation. The third reason was cognitive economy. If a student completed many tasks with one representation, the process would be difficult. To simplify the process, they could remember the outcome and use the information for another representation.

The researchers also suggested factors related to why the students were able to translate in the pictorial and mathematical conditions, while the students in the mixed condition were not. The translation between different pictorial representations was easy for the students because the representations were based on the same metaphor in which formats and operators were similar. Also, the way in which students interacted with the representations for pictorial representations was through direct manipulation.

The students in this age group also had much more experience with these representations.

The students also translated successfully between different mathematical representations because both representations used the same numbers, which made students believe they were equivalent representations. The students in the mixed condition failed to translate between representations because the representations had different numbers and different modalities (i.e., graphical versus textual). Also, these representations are different in terms of interaction method (i.e., direct manipulation versus keyboard use), and in terms of representations (i.e., mathematical versus non-mathematical).

In addition, Ainsworth and colleagues suggested that differences in the appearance of the representations affect students' ability to recognize similarities between them. They pointed out the need for integration between the represented world and representing world to explain the factors that affect the translation between representations. Palmer (1978) describes the represented and representing world in the following way: "representations can differ in two ways, either in the information they express or in the way that information is presented, that is, the represented and representing worlds" (as cited in Ainsworth et al., 2002, p. 31). For the represented world, the factors that affect the translation between representations might be "the amount of available information, the resolution of information, and information redundancy" (Ainsworth et al., 2002, p. 58). For the representing world, the factors might be "the modality of the representations (textual vs. graphical), level of abstraction, type of representations (static vs. dynamic), type of strategies, and interfaces of representations” (Ainsworth et al., 2002, p. 58). Thus, if two representations’ formats and operators are similar in the representing world, students can translate between them. In addition, to solve a problem successfully in the represented world, at least one representation must cover all necessary information in the problem.

Thus, exploring and examining students’ development of representational fluency is challenging. Although students should be able to demonstrate the ability to translate between representations fluently, research has shown that students lack these skills at middle school (Ainsworth et al., 2002), high school (Knuth, 2000), and college (Herman, 2007) levels. That is, students leave school without developing representational fluency (Knuth, 2000). Since representational fluency is important, then instructional contexts that support the development of representational fluency need to be more fully understood. There are at least two tools, however, that may support students in developing representational fluency: communication and technology. In the following sections, these tools are explored separately.

The Role of Communication

Representational flexibility is developed both when students explain the logical aspects of their reasoning in terms of representations to their peers and teachers and when their teachers and peers question their solutions (Warner et al., 2009). In these types of conversations, students are faced with two difficulties. The first difficulty is that students need to solve a task, and the second difficulty is that they need to believe in themselves as well as to prove themselves to others (Warner et al., 2009).

Studies have focused on students’ invention of new representations or the construction and interpretation of representations while they explain, critique, and respond to critiques (diSessa et al., 1991; Greeno & Hall, 1997; Warner et al., 2009). As discussed earlier, diSessa and colleagues (1991) explored eight sixth graders' meta-representational competence, which means "the faculty to generate, critique, and refine representational forms" (p. 118). The students graphically represented various objects in motion. The students collaboratively discussed the activities by creating new representations, which were more applicable and flexible than school-learned representations. The activities allowed students to develop conceptual and interactional skills as well as build their interest and their ownership in inventing representations. The researchers focused on the flow of the students' discourse and moment-by-moment interaction. During the activities, the students discussed the ideas in a roundtable format in a respectful manner.

The researchers examined students' representations by having them invent representations, critique the representations, respond to the critiques, and invent further representations. This study was conducted over five days with 30-40 minutes per day during a full-year elective course. In the initial days of the study, the students worked in pairs to simulate motion for a Logo-like turtle. The teacher asked them to describe the motion with five words. After that, the students started to draw pictures. First, the students invented representations. Second, the teacher asked the students to describe the motion with more words than they did before as well as to clarify and describe their first day's representations. Third, one of the students combined the discrete representation of motion segments with constant speed and created a continuous form of the representation. Another student suggested grids for graphing the motion. On the fourth day, the students played a game that included using many representations for a given motion. They discussed whether a representation looked like a hill. Finally, the students discussed whether a “stop” occurred in the motion when reversing the direction. The teacher’s questions of the students helped them to redesign their representations by learning from others’ ideas. The students praised the other students’ representations and were able to identify the advantages of their representations.

The students also used the following criteria for the quality of the representations: “transparency, homogeneity, compactness, conceptual clarity, objectivity, appropriate abstractness, faithfulness, completeness, economy, quantitative precision, and consistency” (diSessa et al., p. 148). The students fully engaged themselves by giving suggestions, asking questions, and evaluating as well as valuing representations of other students. They used and elaborated on each other’s ideas. The students compared representations based on their advantages and disadvantages and became capable of determining differences between representations. In addition, the teacher had a very important role. She prompted students to explain their ideas to each other and refused to give the correct answer. The teacher made critical decisions about where to lead the discussion in order to make classroom interactions more efficient.

The cycle of inventing representations, critiquing representations, responding to the critiques, and re-inventing may increase students’ meta-representational expertise. In addition, invented representations are more applicable and flexible than school-learned representations. Furthermore, this study reflects how using criteria to determine the quality of representations, and using and praising other students’ representations to redesign their own representations potentially improved their representational fluency.

Similar to diSessa and colleagues, Greeno and Hall (1997) emphasized that the construction and interpretation of representations must include communication and reasoning. They mentioned that non-standard representations such as narratives or drawings served these purposes better than standard representations, such as algebraic expressions or formulas. But Greeno and Hall (1997) also noted that standard forms of representations are essential and “a sizable community shares their [standard forms of representations] conventions of interpretation” (p. 362). So students should learn how to use them. Also, the students need to actively join in discussions to construct and interpret representations and to discuss their advantages and limitations (diSessa et al., 1991).

Every child needs to experience a multiple representational environment and learn how to construct representations (Greeno & Hall, 1997). Representations are very useful for students to monitor their ideas and structure their ongoing work. Greeno and Hall mentioned the aims of building representations during communication and solving problems. Students built representations “to see patterns and perform calculations and taking advantage of the fact that different forms provide different supports for inference and calculation” (p. 365). In addition, students used multiple representations where some were taught and some were invented. The way that students practice representations, such as evaluating the efficiency of representations and reviewing the representations of others is very important for learning and teaching mathematics (diSessa et al., 1991; Greeno & Hall, 1997). In addition, the researchers emphasized that students need to be in a school environment where a variety of representations are used frequently.

Similar to diSessa and colleagues (1991) and Greeno and Hall (1997), Warner and colleagues (2009) explored students’ development of representational fluency as a result of defending their solutions or posing questions to their peers. Unlike diSessa and colleagues (1991), the students in Warner and colleagues' (2009) study did not invent representations. Rather, they reformulated their old representations. Warner's team conducted a case study of one student as well as her peers and a teacher to investigate the effect of peer interaction and student-teacher interaction on the development of representational systems. The study included 10 classroom sessions over 6 months and included the university researcher and the teacher who facilitated the students' group work. The students reformulated their representations during the peer/teacher interaction process, which made their thinking more explicit and conscious. The researchers noted the places and the times when students "modified existing representations, asked questions that seemed to contribute to a modification or change in representation, responded to other students' or the teacher's requests for explanations, and posed or shared extensions to the problem" (Warner et al., 2009, p. 667). The researchers primarily focused on one student named Aiesha, assessing her representational flexibility, in addition to that of two other students in an eight-grade class.

The students were presented with similar handshake problems at different times. The first handshake problem was: "John is having a Halloween party. Every person shakes hands with each person at the party once. Twenty-eight handshakes take place. How many people are at the party? Convince us" (Warner et al., 2009, p.668). In Aiesha's initial solution, she drew a pictorial representation and multiplied seven by eight, but she did not figure out that she was supposed to divide 56 by two. Two weeks later, the students were presented with a similar handshake problem. Aiesha's peer's questioning was critical for her, and using her friend's representation, she reconstructed her own representation. After Aiesha realized why dividing by two was necessary, she started to work with larger numbers of people in the problem. Next, she moved to a symbolic representation to generalize a formula for handshake problems. The other students' questioning helped her to make connections between the representations.

After six months, the students were asked a similar question. Many students in the classroom used Aiesha's formula, which included both symbolic and pictorial representations. In addition, she wrote a verbal explanation of her reasoning. With this result, Warner and colleagues (2009) demonstrated that Aiesha's representation became "a declarative represented tool to explain her reasoning as well as to display a general formula" (p. 675). One of the group members retrieved the representation very easily because she was involved in the discussion when Aiesha built and explained her representation. In addition, a student who asked a question at the beginning about why Aiesha divided the result by two now solved the problem based on reasoning provided in the explanation of dividing by two.

The researcher presented five features for the development of the representations. First, students asked very interesting questions about the other students' reasoning without teacher intervention. Second, the questions were sparks for students to reorganize, rethink, and reconstruct their answers, which were also helpful for using representations flexibly. Third, repairing a representation was shown to be a norm for students. Fourth, whether or not the representations were flexible, they were not of a single type. The fifth feature was that the students used different types of representations when they felt there was a need.

Thus, students have different preferences according to the purpose of using the representations such as for generalization or explanation. For clarifying and defending their reasoning, students need to create representations as well as explain their representations to their peers and teachers. In addition, the new representations that students create are not arbitrary. Instead, they are reconstructions or reorganizations of the representations that have been used in the class previously. In addition, the Warner team's study indicates that Aiesha, her group member, and a student who questioned Aiesha were able to improve their representational fluency within a discussion based environment.

In summary, students need an environment where they can criticize their peers' representations and respond to others' critiques about their representations (diSessa et al., 1991; Warner et al., 2009). Active engagement in the construction and interpretation of representations is essential (Greeno & Hall, 1997). In these students' interactions and discussions about representations, teachers' and peers' roles are very critical (diSessa et al., 1991; Warner et al., 2009). Generally, students are shown to develop representational fluency by both explaining their own representations and by responding to other students' questions about their choices (diSessa et al., 1991; Warner et al., 2009). In these engagements, students also use others' representations (diSessa et al., 1991; Greeno & Hall, 1997; Warner et al., 2009) and learn to value others' representations (diSessa et al., 1991). Finally, students are shown to change their representations with the aim of explaining and generalizing (Warner et al., 2009).

The Role of Technology

Research suggests that one way students may develop representational fluency is by using technology such as graphing calculators and easy-to-use computer software (Bostic & Pape, 2010; Herman, 2007; Knuth, 2000; Nathan & Kim, 2007; NCTM, 2000; Suh & Moyer, 2007). Because the present study focuses on graphing calculators within a connected classroom network, the results of studies using TI-Nspire CAS is reported. One important feature of TI-Nspire CAS is that it can display multiple representations on one screen. This capability exemplifies the type of feature that may change students' overdependence on symbolic representations.

Students rely heavily on symbolic representations in high schools (Bostic & Pape, 2010; Knuth, 2000) and in college (e.g., Herman, 2007). In Herman's (2007) study, 38 college students used TI-83+ graphing calculators within a 10-week advanced algebra course. This study investigated the students' choice of representations, the course's effect on the number of representations used by students, and the correctness of their solutions, as well as the students' beliefs about multiple representations and their effect on understanding. The algebra course was reported to have two aims: (a) to help students realize that different problems can be solved with the same model even if they seem unrelated, and (b) to help students realize that the problems could be solved using multiple representations and that some representations may prove more efficient than others. Thus, the course's focus was on the connection between symbolic, graphical, and tabular representations in polynomial, exponential, and logarithmic functions.

To identify students' choice of representations, the course's influence on the number of representations students learned to use, and the correctness of their solutions, the researcher posed six problems on the pre- and posttest. To explore students' beliefs about multiple representations and their effect on understanding, the researcher designed and implemented two questionnaires for students, a questionnaire for the instructor, and semi-individual interviews with seven students who applied more representations than their classmates or changed their representation on the posttest compared to their pre-test. Even though the students had the opportunity to use graphical and tabular representations, they were more likely to use symbolic representations. Herman concluded that this was due to the students' beliefs that symbolic representations are more mathematical. Students also used the words "algebra" or "math" as synonymous for symbolic representations in their statements.

The students stated that they did not want to depend on calculators and therefore used calculators for checking their answers. They believed that instructors would require them to have a strong knowledge of symbolic representations in future courses. In addition, the students thought that even though their teachers liked graphs, the teachers emphasized symbolic representation, which was confirmed by the teachers' statements on the teacher questionnaire. The same situation applied to using tables. The students did not like to use tables for checking their answers, which was a reflection of their teachers' preferences.

Generally the students' use of the representations increased from the pre-test to posttest except for problems with which they were familiar because they appeared on the pre-test. In addition, the students who were more likely to come up with correct responses used more varied types of representations. Thus, although the students felt comfortable and confident using the calculators and believed that multiple types of representations were beneficial to understanding, they thought that symbolic representations were more mathematically correct.

TI-83+ graphing calculators were used in Herman's (2007) study. Some researchers compared instruction with TI-Nspire CAS to TI-83+ calculators. The difference between TI-83+ and TI-Nspire CAS is that TI-Nspire CAS allows students to see symbolic, tabular, and graphical representations in one screen of the calculator. Bostic and Pape (2010) compared the impact of two instructional methods on students' achievement, perceptions about the use of the technology, problem-solving success, and problem-solving representations. Like Herman (2007), Bostic and Pape also explored the relationship between the number of representations used and students' problem-solving success.

Bostic and Pape selected four Algebra II classrooms for their investigation, two of which were honors classes. The researchers randomly assigned one regular and one honors class to the treatment and comparison groups. The treatment group consisted of 30 students who used TI-Nspire CAS, and the comparison group consisted of 23 students who used TI-83+ in class. Both of the groups experienced lecture-based instruction, and the students were given the opportunity to work individually or in pairs. Instruction in the comparison group was less-calculator based, and the students generally worked individually. Data sources included: (1) students' achievement on a unit test, (2) students' responses to two prompt questions, and (3) students' solutions to one problem. They concluded that there was no statistically significant difference between treatment and comparison groups relating to student achievement, students' perception about the use of technology, students' perceptions about technology's benefits for their understanding, students' problem-solving success, or the number of strategies used by students.

While the students in the comparison group tended to use symbolic representations predominantly, the students in the treatment group tended to use graphical representations. The researchers therefore stated that the students who experienced TI-Nspire CAS used more efficient and effective representations to solve the problem. In addition, as in the Herman (2007) study, Bostic and Pape found a relationship between the number of representations that the students used and their problem-solving success across groups. Therefore, the feature of the TI-Nspire CAS calculator that shows multiple representations in one window may promote the use of more varied representations (Bostic & Pape, 2010). Suggestions for further research include longitudinal studies (Herman, 2007) and larger sample size and expanded instruments (Bostic & Pape, 2010).

The use of graphing calculators has been suggested for students who have difficulty in representational fluency (Bieda & Nathan, 2009; Knuth, 2000). Bieda and Nathan (2009) analyzed 38 middle school students' gestures and speech to understand how students generalize patterns in a Cartesian graph. There were five questions related to the graph of which two far prediction tasks served as the focus of the study. The first question provided an x-value close to the limit of the x-axis on the graph and requested a y-value that was beyond the highest value represented on the y-axis. In the second question, both the x-value given and y-value requested were beyond the graph. In these tasks, the students were faced with a dead-end with the first representation, and therefore, they tried to find another way to solve the tasks. These tasks allowed the researchers to explore translations because the problems were designed to force students to translate between representations.

Three constraints that students demonstrated as they solved the pattern generalization tasks included physically, spatially, and interpretatively grounded. Building on previous literature, the researchers stated,

> grounding is commonly used to describe the mapping that a person makes between an unfamiliar or abstract representation, and a more concrete or familiar referent. … One's perceptions and reasoning processes can be inappropriately bound to the representations they are grounded to, which can impose superfluous or incorrect constraints on the representations themselves and the strategies that draw upon them, thereby negatively affecting problem-solving performance or transfer. (Bieda & Nathan, 2009, p. 638)

The physically-grounded students saw the graph with a bounded view with regard to its numerical and physical limitations. They were not able to go beyond the graph. The spatially-grounded students tried to extend the graph but could not translate to another representation to solve the task. However, interpretatively-grounded students demonstrated representational fluency. These students were able to translate the graph to an equation, which is a more abstract representation. The researchers explained that the physically- and spatially-grounded students demonstrated representational disfluency "where perceived shortcomings in using representations to solve problems motivate students to modify or translate among representations" (p. 637).

Thirty-six percent of the students were physically grounded to the graph. In addition, unbounded gestures had a significant effect on representational fluency, which means if students saw a graph without being bound to its physical and numerical limits, they were able to translate from the graphical to symbolic representation flexibly because they went beyond the graph. In conclusion, the researchers suggested that teachers should consider the negative effects of the grounding and emphasize the unbounded nature of the function. The limitation of the Cartesian graph needs to be discussed with students to help them realize the patterns and unbounded nature of the graphs. The researchers suggested using graphing calculators or larger sized paper to help the spatially-grounded students. The spatially-grounded students have both bounded and unbounded views of evidence, making it seem like these students are making progress from disfluency to fluency, contrary to the authors' coding of these students as representationally disfluent. Since students could also zoom, turn, or lengthen graphs in TI-Nspire calculators to see the graphs as limitless, these calculators may be helpful for students who are between representational disfluency and fluency because it is helpful for students to see an unbounded view of graphs.

In summary, although students rely heavily on symbolic representations (Bostic & Pape, 2010; Herman, 2007; Knuth, 2000) or are bounded with physical and numerical limits of representations (Bieda & Nathan, 2009), these researchers suggest using graphing calculators such as the TI-Nspire CAS to support students' development of representational fluency. For instance, in the treatment group of Bostic and Pape's (2010) study, the students tended to use graphical representations, which means that when students are given the opportunity, they are able to use more effective non-symbolic representations such as graphical representations more than traditional symbolic and computational strategies.

In addition, NCTM (2000) called for the use of technology and stated that it is helpful for obtaining accuracy and immediate feedback as well as for visualizing tables, graphs, and equations and their relationships. In contrast, in Herman's (2007) study, the students' interactions with representations provided evidence of using symbolic representations by hand, and tabular and graphical representations by calculator, which may have affected students' translation between representations. Similar to Herman, Ainsworth and colleagues (2002) mentioned that one of the factors causing difficulty in translation between two different representations is the different way of interaction with representations.

Both Herman's (2007) and Bostic and Pape's (2010) studies showed that students' use of a variety of representations increased after instruction. Both studies concluded that there was a relationship between the number of representations used and the number of problems solved correctly. The students in Knuth's (2000) study mastered translating from equations to graphs but not from graphs to equations. However, TI-Nspire CAS provides links between representations. Bieda and Nathan (2009) suggest the use of graphing calculators for spatially grounded students who have bounded and unbounded views of graphs in order to make progress from representational disfluency to fluency. That is, students can zoom, turn, and lengthen graphs by using graphing calculators (NCTM, 2000), which helps them see representations without a bounded view regarding their numerical and physical limitations. This new version of handheld calculators may also enhance bidirectional translation (i.e., graph-symbolic and symbolic-graph). Ozgun-Koca and Edwards (2009) analyzed 19 pre-service, 26 in-service teachers' and 54 middle school students' views on benefits and weaknesses of TI-Nspire after their first use. The purpose of their study was to explore whether teachers are ready to use this novel technology. Participants manipulated a graph corresponding to its symbolic representation in a quadratic equation. Then the researchers conducted a survey included Likert type and open-ended questions. Results showed that in-service teachers believed in the capability of novel TI-Nspire properties more than pre-service teachers although they had similar views on this technology. Additionally, 97% of the students liked multiple representations and 96% of them liked moving the graph and examining the equation. Thus, changing a graph's structure and seeing changes in the symbolic form may help students who have difficulty translating from graphical to symbolic representation, which was the case for students in Knuth's (2000) study. NCTM (2000) calls for research in graphing technology because calculators and computers alter the use of conventional representations and increase the number of representations.

The Role of CCT on the Development of Representational Fluency

Technology is a very helpful tool when it is used efficiently for both conceptual and procedural understanding. CCSSM (Common Core State Standards Initiative [CCSSI], 2010) emphasized the use of technology to explore students' understanding. For example, according to these standards "mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator" (CCSSI, 2010, p. 7).

Technology (e.g., tools, software, hardware) has evolved very quickly. Graphing calculators are commonly used and increase student achievement by reducing extra cognitive load and allow students to focus more on conceptual understanding (Ellington, 2003). Researchers (e.g., Herman, 2007) have used graphing calculators in their interventional research on ways to help improve teaching and learning. Furthermore, studies have found that technologically enhanced classrooms create more interactive learning environments. For instance, Roschelle et al. (2003) has found that the classroom network and response system technologies with specific questioning pedagogy increases participation. These technologies are being employed to transform classrooms into more learner- and community-centered environments (Bransford, Brown, & Cocking, 2000).

As discussed earlier, classroom communication and technology may impact the development of representational fluency (e.g., Pape & Bostic, 2010; Warner et al., 2009); however, communication and technology have primarily been examined separately in research. The combination of these factors can be seen in networked classrooms such as CCT. This technology has evolved from ARS to TI-Navigator. In this section, the first generation of CCT (ARS systems) is analyzed, followed by the second generation of CCT (TI-Navigator). Then, the effect of second-generation CCT on representational fluency is explored.

First-generation CCT: ARS systems

Networked classrooms started more than one decade ago with what is most well-known as ClassTalk™ (Abrahamson, 1998, 2000). ARS systems such as electronic voting, personal response, and clickers, which are the first basic forms of CCT, provide opportunities to exchange information electronically between teachers and students. In educational environments, these systems enable students to use remote control devices (e.g., clickers) to send their answers (e.g., true/false, yes/no, numeric, multiple-choice) to the teacher's computer. Teachers can determine which students have not responded, and the results are shown to the class anonymously (e.g., histogram graph), which allows the teacher and class to review and discuss the responses.

During the 1960s and 1970s, ARS had a positive effect on student enthusiasm but did not increase their learning (Pape et al., 2013). More recent research on ARS's current effectiveness in classrooms, however, shows that it facilitates student-centered teaching and has increased attendance, participation, collaborative learning, student engagement, student comprehension, student class satisfaction, and conceptual gain (Paschal, 2002; Judson & Sawada; 2002; Pape et al., 2013).

Not only is a first generation CCT, ARS instructional system more beneficial than a non-CCT environment, but research shows that second generation connected classrooms allow for even more instructional opportunities by facilitating platforms wherein students may submit multiple answers and employ a variety of representations (Hegedus & Moreno-Armella, 2009). In a traditional classroom, generally one student is allowed to answer each question; however, with second generation CCT, every student is allowed and even expected to answer the question. If an intervention requires more student responses, it increases learning (e.g., Greenwood, Delquardi, & Hall, 1984). Also second generation CCT provides richer types of formative assessments than clickers. Students can answer with numerical, symbolic, or graphical representations rather than only multiple-choice options, allowing teachers to check students' understanding and modify their instruction if needed.

Second-generation CCT: TI-Navigator

Second-generation CCT (see Figure 2-2) consists of "wireless communication systems that connect the teacher's computer and students' handheld technology" (Pape et al., 2013, p. 169). The TI-Navigator system has four components to support learning: Quick Poll, Learn Check, Screen Capture, and Activity Center. With Quick Poll, the teacher can send an individual question to the students to explore their prior knowledge. With Learn Check, the teacher can send questions (e.g., quizzes) to the students. The questions may vary in response type including open-ended, true-false, Likert-type, or multiple-choice questions. A related component, Class Analysis, accumulates students' responses as a bar graph in the teacher's computer for class examination.

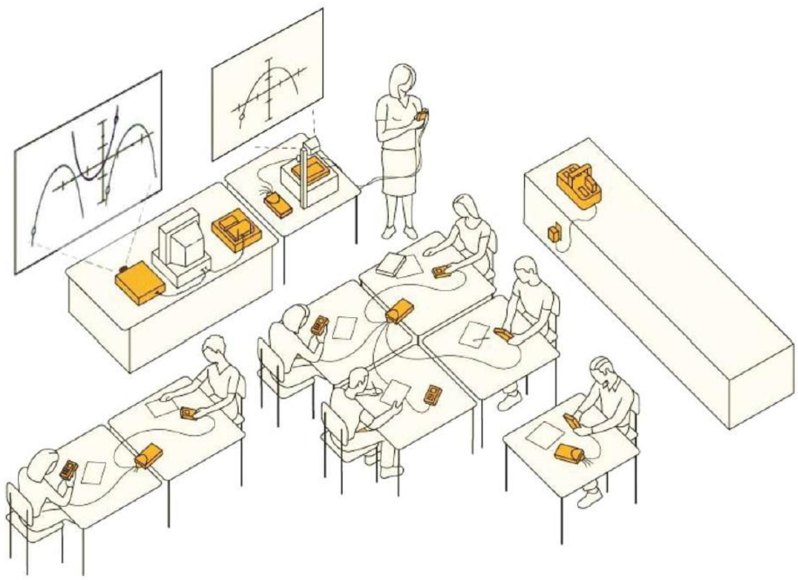


Figure 2-2. Depiction of the TI-Navigator within a classroom. Adapted from *Classroom connectivity in algebra I classrooms: Results of a randomized control trial* (p. 171), by S. J. Pape et al., 2013, Effective Education, 4, 169-189.

The first two components can be seen in ARS systems, while the last two components, Screen Capture and Activity Center, show the sophisticated structure of the CCT. With Screen Capture, teachers can show each student's response on the projector as a "snapshot" of each calculator. This feature allows both teachers and students to compare the solutions in productive discussion. With Activity Center, the teacher can show a shared coordinate plane for students to send their points, equations, and graphs. A summary of TI-Navigator's system components is provided in Appendix A.

By displaying multiple linked representations publicly, the last two components facilitate effective classroom discourse (Hegedus & Moreno-Armella, 2009; Pape et al., 2013; Roschelle et al., 2003). In addition, these components provide an environment that allows students to interact with each other while exploring patterns and constructing knowledge. That is, “instead of constraining the learning experience to be narrowly individualistic, this technology supports socially situated interaction and investigation. Moreover, the group itself owns the learning trajectories and the processes of knowledge construction, rather than outside experts or programmers” (Stroup et al., 2005, p. 183).

Impact of CCT

Second-generation CCT has four affordances (Pape et al., 2012). First, Activity Center promotes examination and analysis of patterns as well as justification of mathematical generalizations, which may support representational fluency and conceptual understanding. For example, students start to generalize the effect of changing parameters of the equations on a graph’s position and shape as their classmates or themselves submit their equations (Hegedus & Moreno-Armella, 2009). In addition, the ability to submit student work anonymously and to project this work for public examination allows students the opportunity to see the differences between their work and the group’s work. The public display shows “the juxtaposition of ideas, often literally a debate rather than a resolution or synthesis” (Stroup et al., 2005, p. 188).

Second, the ability to modify the discourse pattern changes students’ interaction with concepts (Stroup et al., 2005). Teachers may use students’ responses, errors, and misconceptions as tools for productive discourse, which changes the traditional discourse pattern of initiate-respond-evaluate (IRE) (Pape et al., 2010) and provides essential learning gains (Black & William, 1998). Thus, CCT mediates discourse practices (e.g., communication infrastructure) and enhances critical thinking (Hegedus & Moreno-Armella, 2009). Public display with comparison of ideas and strategies also opens up ways for students to communicate through representations (Hegedus & Moreno-Armella, 2009).

Third, the public display of students mathematical reasoning improves students' identities while engaging in mathematical thinking and changing the discourse process (Hegedus & Penuel, 2008). In this kind of environment, students play a larger role in classroom discussions and debates as opposed to the traditional classroom wherein a teacher plays the primary role in discussion (Hegedus & Penuel, 2008). A teacher's role thus changes from an instructor to a facilitator in this kind of environment (Leng, 2011).

Allowing students to see their input publicly, receive critique from others, and immediately make changes will foster "forms of identity and identification of ones contribution to a mathematical argument" (Hegedus & Moreno-Armella, 2009, p. 404). To put it another way, when the IRE discourse pattern is modified, students have more opportunities to engage in classroom discussions. Thus, students' dynamic representations move from private to public display, which positively affects students' identity, rich discourse (Hegedus & Moreno-Armella, 2009), and mathematical and scientific knowledge and reasoning (Stroup et al., 2005).

Fourth, public display supports formative assessment that can be used by students for self-assessment, which improves students' metacognition (Hegedus & Kaput, 2004). In addition, connected classrooms improve teachers' knowledge about their students' present understanding, which can prompt teachers' future decisions about specific feedback and instruction (William & Thompson, 2008).

Some researchers used SimCalc MathWorlds within the TI-Navigator system (e.g., Hegedus & Kaput, 2004). This system uses interactively linked representations, simulations (representational infrastructure) and TI-Navigator (communication infrastructure). The SimCalc representational infrastructure has four key properties: “(1) hot-links between graphs and simulations, (2) visually editable, piecewise-definable graphs of functions, (3) hot-links between rates and total graphs and (4) importing physical data into the computational notation” (Hegedus & Moreno-Armella, 2009, p. 401).

SimCalc MathWorlds within TI-Navigator system influences participation and classroom social structure in two ways: (1) it provides the opportunity to prepare novel activities and allows teachers to instantly collect and analyze class contributions, and (2) students can make better sense of their contributions by generalizing and reasoning their work alongside that of their classmates’ contributions (Hegedus & Moreno-Armella, 2009). Hegedus and Moreno-Armella (2009) explored two “new forms of participatory activity” that are made possible through the use of “representationally-rich software with wireless networks” (p. 403). The first is “mathematical performances” where representations are created by students on their own or as a group. The second is “participatory aggregation to a common display” (p. 403). These activities include “systematic variations” within small groups or across groups to discover patterns, evolve generalizations, show special cases, and help students focus on group work as opposed to individual work.

The second generation CCT increases participation because students can submit their responses anonymously, allowing for more equitable access; “[it] broadens the ‘bandwidth’ of classroom collaboration” (White, 2006, p. 359) by “expand[ing] the range of collaborative ‘frequencies’ through which students participate in a small group”

(p. 380). Reducing the hierarchy increases participation, or in other words, anonymous participation decreases student anxiety and allows students to see all other students' work in a non-threating environment. In this type of environment, even when students' answers are not correct, they do not feel alone. Instead, they receive feedback to improve themselves (Davis, 2003), which enhances the motivational climate of a class (Owens et al., 2004).

Several quasi-experimental designs were implemented in connected classrooms to demonstrate the impact of SimCalc materials (Hegedus & Kaput, 2003). Hegedus and Kaput (2003) analyzed the performance of the middle and high school students who attended an after-school, five-week algebra program that used Simcalc integrated into a classroom network. Pre- and posttests included 20 items consisting of multiple choice, short answer, and open-response questions. The combination of these affordances improved students' learning in high achieving middle schools as well as with at-risk ninth-grade students even though they lacked prior knowledge about core algebraic ideas. Additionally, critically important skills such as graphical interpretation were improved. Recent studies also showed similar increases in students' learning and positive changes in student attitudes toward learning in connected classrooms (Hegedus et al., 2007).

To evaluate the impact of SimCalc MathWorlds on students' conceptual and procedural knowledge, Tatar and colleagues (2008) conducted a pilot experiment including 21 seventh-grade mathematics teachers in Texas. They used a replacement unit that was three weeks long to compare students in SimCalc classrooms to the students in a control group with a regular curriculum. Their most important finding was the growth in students' mathematics knowledge. The students in the treatment condition had greater gains and higher scores than the students in the control group on the posttest. Most of the gains came from function-based items (i.e., conceptual items). That is, the students' scores on function-based items were statistically significant whereas there was no statistical difference in formula-based items (i.e., procedural). Thus, the students in the treatment condition made significant gains in challenging mathematics problems. In addition, their knowledge about formula-based mathematics did not change. Not only did the students in the SimCalc condition learn more complex and conceptually difficult mathematics, but they also maintained the same progress on procedural knowledge as the students in the control group.

Another study focusing on the effect of second generation CCT on students' knowledge was conducted by Leng (2011). Leng's study was comprised of 35 secondary-school students in Singapore to explore how TI-Nspire improves calculus learning and teaching. He first introduced TI-Nspire Navigator by conducting 14, 30-minute training sessions about using TI-Nspire with enriched activities before he integrated its use into the classroom. Qualitative data (i.e., classroom observations, self-reflections, interviews) were then collected. Leng selected eight students randomly to conduct structured interviews to explore students' conceptual understanding of derivatives. He found that with the help of TI-Nspire, students in calculus were able to make better generalizations and visualizations about the relevant properties of mathematics by using appropriate graphical, numerical, and symbolic representations. The students linked the representations, especially symbolic and graphical representations that helped them develop better conceptual understanding and problem-solving abilities. The researcher determined six methods of using TI-Nspire: exploratory, graphing, confirmatory, problem solving, visualization, and the calculation tool. Even though it was not a specific aim of his study, the researcher found that TI-Nspire Navigator provides a learning community that allows students to communicate mathematical concepts by increasing their participation in the learning process.

Finally, Pape and colleagues (2013) examined data from the first year of a four-year control trial in Algebra I classrooms to show the impact of connected classroom implementation on student achievement. The intervention included both PD and second-generation CCT. The PD focused on the effective use of CCT, such as rich mathematical tasks, multiple representations, formative assessment, classroom interaction, discourse, and sustained engagement. The teachers in the treatment group were provided PD and they used CCT in their classrooms. On the other hand, the teachers in the control group used graphing calculators only. Eighty-two teachers (39 treatment, 43 control) and 1224 students were included in the data analysis (53.4% treatment, 54.1% female). HLM analysis showed that there was a treatment effect on the treatment group's posttests. Treatment status was associated with visual, symbolic, and mechanical items after controlling for teachers' years of experience. Teacher knowledge about students' understanding and improvement was positively associated with the posttest and three subscores. Through TI-Nspire Navigator, the teacher was able to immediately see students' difficulties and know students' understanding so he/she can modify the lesson based on that knowledge. Teacher knowledge about student understanding and student achievement is enhanced as a result of TI-Navigator use, because the technology provides more feedback, and PD focuses on responsive student thinking. The treatment (i.e., PD and technology together) had an effect on overall and subtest (i.e., visual, symbolic, mechanical) achievement. The effect size was 0.30, a medium effect size, is rare in randomized experiments in education.

Thus, CCT (1) gives access to patterns, conjectures and generalizations, (2) changes discourse patterns, (3) provides an environment that fosters engagement, and (4) increases achievement. Furthermore, CCT also provides the context for representational expressivity, which is the combination of representational infrastructure and communication infrastructure. These features are detailed in the next section.

Second Generation CCT effect on representational fluency

Two CCT properties may have significant impacts on representational fluency. The first is “the mobility of multiple representations of mathematical objects”, and the second is “the ability to flexibly collect, manipulate and display to the whole-classroom representationally-rich student constructions, and to broadcast mathematical objects to the class” (Hegedus & Moreno-Armella, 2009, p. 403). These researchers explored “the meaning of mathematical representations through enhanced communicative forms” (p. 410), which can be visualized as an intersection of representation infrastructure and communication infrastructure, as seen in Figure 2-3.

The representation infrastructure allows for the creation of a social network and increased communication (Hegedus & Moreno-Armella, 2009). The authors defined communication as “human actions in terms of speech or physical movement (e.g., gesture) or digital inscriptions through modern-day interfaces” (p. 400). That is, students can communicate with their teachers and peers through verbalization, body language, and digital input via their handhelds.

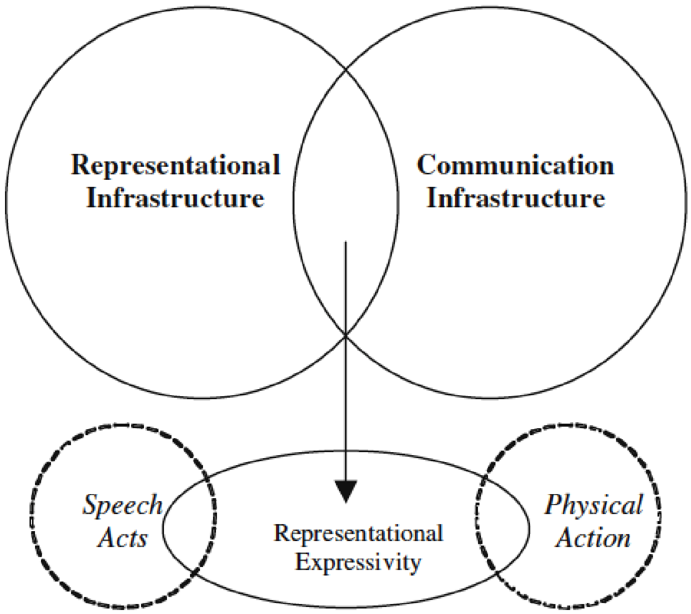


Figure 2-3. Model of RI and CI intersecting. Adapted from *Intersecting representation and communication infrastructures* (p. 400), by S. J. Hegedus and L. Moreno-Armella, 2009, ZDM Mathematics Education, 41, 399-412.

The authors also defined communication infrastructure as “the organizational structure of the various communication inlets and outlets available in society. A digital infrastructure is composed of networks, wires, and servers to create information flow of communication acts and services to various populations” (p. 400). In networked classrooms, the web structure of digital communication infrastructures consists of graphing calculators, networks, wires, and servers to provide the flow of information among the students and their teacher.

In second generation CCT, representations are more visual, interactive, and dynamic than ever before, and new hardware allows a connected wireless system to be “more portable in terms of its hand-heldability” (Hegedus & Moreno-Armella, 2009, p. 399). These properties affect representation and communication infrastructures (Hegedus & Moreno-Armella, 2009). Even though these infrastructures are largely developed separately, the authors postulated that when these two infrastructures develop/co-develop together, they affect each other's development, allowing for new forms of activities to be designed.

The authors defined representational expressivity, which is the intersection of representation and communication infrastructure, as "where learners can express themselves through the representational layers of software and where a participatory structure enables learners to express themselves in natural ways through speech acts (e.g., metaphors, informal registers, and deixis) and physical actions (e.g., gestures or large body movements)" (p. 400). In other words, students express their thinking representationally by using multiple representations, verbalization, and body language, all of which form a learning environment that is interactive.

Although teachers found CCT to be an effective means of instruction, there is limited evidence of the mechanisms for its effectiveness (Vahey et al., 2007). Therefore, researchers have called for more qualitative studies that explore learning and teaching in CCT classrooms (e.g., Hegedus & Moreno-Armella, 2009; Hegedus & Penuel, 2008; Pape et al., 2013). Further studies of CCT implementation could reveal which instructional changes yielded better student achievement, as well as identify which components of the technology and the PD created those instructional changes (Pape et al., 2013).

Currently, research needs to explore representational fluency in second-generation CCT environments under realistic conditions. These types of studies would be feasible for the future since TI-Navigator technology is relatively inexpensive (Hegedus & Moreno-Armella, 2009; Pape et al., 2013) and affordable to use in the education system. Hegedus and Moreno-Armella (2009) also suggest further work based on pedagogical actions that would explore teacher knowledge about effective practices determined by measurable learning gains. Due to verified affordances of CCT environments, future work should examine its potential for supporting representational fluency in second-generation CCT environments under realistic conditions.

Summary

Representational fluency is not only the ability to translate between representations but also the ability to interpret and construct representations, to recognize specific representations for the aim of demonstration or explanation, to use representations for justifications of claims, and to link multiple representations in a meaningful way (Sandoval et al., 2000). Representational fluency is considered both a mechanism for supporting the development of deep conceptual understanding (Bostic & Pape, 2010; Duncan, 2010; Lesh & Zawojewski, 2007; NCTM, 2000; Nistal et al., 2009; Pape & Tchoshanov, 2001; Warner et al., 2009) and a means of assessing conceptual understanding (Suh et al., 2008). That is, representational fluency and conceptual understanding are two sides of the same coin; they are intertwined constructs.

Representational fluency not only supports conceptual understanding but also generalization and abstraction (Bieda & Nathan, 2009; Nathan & Kim, 2007; Suh et al., 2008; Warner et al., 2006; Warner et al., 2009). It is also an essential component for solving a problem correctly (Ainsworth et al., 2002; Nistal et al., 2009). Students are more successful when they possess the ability to translate between representations as well as use multiple and non-symbolic representations (Bostic & Pape, 2010; Brenner et al., 1997; Herman, 2007; Nathan & Kim, 2007; Suh & Moyer, 2007).

Through research and CCSSM document, the education community continues to emphasize the importance and need for developing representational fluency (CCSSI, 2010; NCTM, 2000). However, students in middle, high, and even college levels leave school without attaining proper representational fluency (Ainsworth et al., 2002; Cramer, 2003; Davis & Maher, 1997; Even, 1998; Gerson, 2008; Herman, 2007; Lesh et al., 1987; Kaput, 1989; Knuth, 2000) often because representational fluency is a cognitively difficult ability for students to develop (Ainsworth et al., 2002).

There are at least two factors that potentially support the development of representational fluency: communication and technology. Through communication, representational fluency may be supported by active engagement in the discussions about interpretation, construction, evaluation, comparison, generalizing of representations, justifying the representations in solutions, and criticizing/questioning or explaining/responding to the critiques of their peers or their own representations (diSessa et al., 1991; Warner et al., 2009). The development of representational fluency may be supported by the following interrelations between technology and communication: a calculator's allowance for quick-access to multiple representations (e.g., symbolic, tabular, and graphical), accuracy and immediate feedback, step-by-step support in algorithmic processes, a self-checking system, and flexibility in actions on representations (Bieda & Nathan, 2009; Bostic & Pape, 2010; Herman, 2007; Knuth, 2000). Because evidence suggests that communication and technology may separately support students' developing representational fluency, the present study investigates instruction that is characterized by the use of CCT with the aim of examining the relationship between these instructional strategies and increasing such fluency.

CHAPTER 3
METHOD

Context of the Study

The present study reported on classroom observations conducted during the third year of a four-year project. The *Classroom Connectivity for Mathematics and Science Achievement* (CCMS) project examined the effect of CCT on students' achievement in Algebra I and Physical Science (Pape et al., 2013). In the first year of the project, teachers who volunteered to participate in the study were randomly assigned to treatment or control groups. The treatment group participated in a one-week summer institute where the teachers learned about TI-Navigator. The teachers "engaged in demonstration, practice, and discussion of appropriate teaching with CCT for Algebra I as well as brief lectures on formative assessment, classroom discourse, and self-regulated learning" (Pape et al., 2010, p. 10). Teachers also received follow-up PD at an annual conference, *Teachers Teaching with Technology*, and were supported by a listserv, online training modules, and telephone interviews (Irving et al., 2010; Owens et al., 2008; Pape et al., 2010; Pape et al., 2013).

Teachers in the control group taught the Algebra I class with graphing calculators without CCT in their first year. In the second year, the control teachers were able to use CCT. The research design of the larger study was a randomized control trial where the teacher participants were randomly assigned to treatment or control groups in the first year of the project. Thus, the overall design was a randomized field trial with a wait list control design.

To examine the effects of CCT on the development of representational fluency, this study first identified the classrooms with initial mean pre-test scores below 50%.

Then, the two classrooms with the highest and two classrooms with the lowest gain scores were selected for analysis. This research employed a qualitative study design, with the intent of providing contrasting or illustrative instances in instructional use of representations. Classroom observations conducted on these four classes were examined to determine the similarities and differences that may potentially be associated with these classes' gain-score differences. Thus, the purpose of the present study was to compare the ways in which teachers used CCT to potentially support representational fluency within initially lower achieving classrooms that showed distinct progress on translation problems.

The following research question guided this inquiry: How do teachers' uses of CCT differ between classes that were initially low achieving but then showed differential improvement on translation problems?

Participants

Initially, 127 teachers participated in the study, with 66 of them assigned to the experimental (treatment) group and 61 assigned to the control group (Irving et al., 2010). Some teachers were excluded from the study due to personal reasons such as resignation from their teaching position or health problems. The majority of the teachers were white, female, and held mathematics degrees. Their teaching experience ranged from 1 to 36 years across different years of the study. The complete demographic information is provided in Table 3-1. Depending upon the year of the study, between 271 and 696 students participated. Each student completed pre- and post algebra tests as well as surveys related to their perceptions about instruction. Approximately half of the students were female. Student demographic information for the present study is provided in Table 3-2.

Table 3-1. Teacher Demographic Data

Group *	Year of CCT use	N**	% White	% Female	% Math degree	Years Teaching Experience (Median)	% Free/Reduced Lunch (Median)***
Treatment 2Y3	2	28	89.3	76.7	73.3	13	15
Treatment 1Y3	3	19	100	77.3	59.1	12	9

Note. Adapted from *Algebra Achievement over Four Years in TI-Navigator Connected Classrooms*, by D. T. Owens, S. J. Pape, K. E. Irving, L. Abrahamson, D. Silver, V. A. Sanalan, and B. Morton, 2010, April, Poster presented at the Annual Meeting of National Council of Teachers of Mathematics (NCTM), San Diego, CA.
* In year 1, Cohort 1 is the experimental and Cohort 2 is the control group. The last number shows the year of the study. For instance, Control 2Y3 indicates data from Cohort 2 in the third year of the study. Treatment 1Y3 indicates data from Cohort 1 in the third year of the study. In both cohorts, teachers are the same across the years of the study with different students.
**Number of teachers who reported demographic data.
***This column demonstrates school averages as a proxy for classroom composition and should be interpreted with caution.

Table 3-2. Student Demographic Data.

Group	N	% Female	Group	N	% Female
Treatment 1Y3	577	47.2	Treatment 2Y3	696	52.8

Note. Adapted from *Algebra Achievement over Four Years in TI-Navigator Connected Classrooms*, by D. T. Owens, S. J. Pape, K. E. Irving, L. Abrahamson, D. Silver, V. A. Sanalan, and B. Morton, 2010, April, Poster presented at the Annual Meeting of National Council of Teachers of Mathematics (NCTM), San Diego, CA.

Teachers who participated in classroom observations during the third year of the larger study were chosen for the present study because these teachers had sufficient experience using the TI-Navigator. That is, by year three, the teachers would have had ample opportunity to engage with the technology and were thus likely more proficient in deliberately using the affordances of the technology. In addition, the greatest number of teachers was observed during year three of the original study, which provides a substantial set of data. Forty of the 41 teachers' classroom observations were included because the audio was of sufficient quality for analysis. The demographic information for four teachers who were selected for this analysis is listed in Table 3-3 (see below for participant selection procedures).

Table 3-3. Teacher Participants Demographic Data

Teachers	Year of CCT use	Undergraduate Major	Graduate Major	Years Teaching Experience
Ms. BW	2	Pre-Vet Med	Ph.D., Animal Feeding/Animal	13
Ms. MB	3	Communication	MA, Journalism	3
Ms. MA	2	Mathematics	MA, Educational Computing	21
Ms. JR	2	Mathematics	Curriculum and Instruction	20

Data sources

Classroom Observations

Classroom observations typically consisted of two consecutive class periods. The length of each observation is between 48 and 97 minutes, and these observations were collected between April and June in 2008. The classroom videos and their verbatim transcripts were used in this research.

Algebra Pre- and Posttest

A description of the Algebra pre- and posttests as well as their development within the CCMS project was adapted from Irving et al. (2010). The pre-test and posttests are in Appendices C and D.

Algebra pre-test

The Algebra pretest consists of a total of 32 pre-algebra and algebra items (α = 0.81) containing multiple choice, short-answer, and extended constructed-response formats and was administered at the beginning of the school year to measure baseline achievement of participating students. This pretest, which was previously validated by the National Center for Research on Evaluation, Standards, and Student Testing (CRESST, 2004) was based on released items from the National Assessment of

Educational Progress (NAEP) and the California Standards Test. This measure was validated using a three-parameter logistic (3PL) Item Response Theory (IRT) model.

Algebra posttest

The development of the Algebra I posttest (Abrahamson et al., 2006) began by comparing mathematics content standards of 13 states, with Ohio, Texas, New York, and Virginia representing those states from which a majority of CCMS participants were drawn. Thirty-five questions aligned with these standards were selected from released items from California and Virginia standardized mathematics tests and from the TIMSS assessment and NAEP. IRT analysis was conducted to ensure that the technical quality of the measures resulted in the exclusion of five items. The final instrument included 24 multiple-choice items, 5 extended-response items, and 1 three-part short-answer question ($\alpha = 0.85$).

Translation problems in pre- and posttests

To measure representational fluency, translation problems were extracted from the pre- and posttests. Translation problems are those in which the initial representation (i.e., input) and the answer's representation (i.e., output) are different (Nathan et al., 2002). An example of a verbal-graph translation problem is provided in Figure 3-1. Nine verbal-symbolic (i.e., Problems 5, 6, 12, 14, 15, 21, 23, 25, and 26), one verbal-graphical (i.e., Problem 19), and one symbolic-graphical (i.e., Problem 24) translation problems were identified on the pre-test (n=11). Ten verbal-symbolic (i.e., Problems 2, 4, 6, 7, 9, 13, 15, 21, 23, and 28), one verbal-graphical (i.e., Problem 5), three symbolic-graphical (i.e., Problems 8, 19, and 25), three graphical-symbolic (i.e., Problems 18, 20, and 22) problems were identified on the posttest (n=17).

5. Rice is on sale at the price of 3 pounds for $1.00. Which graph shows the relationship between the number of pounds of rice bought and the total cost?

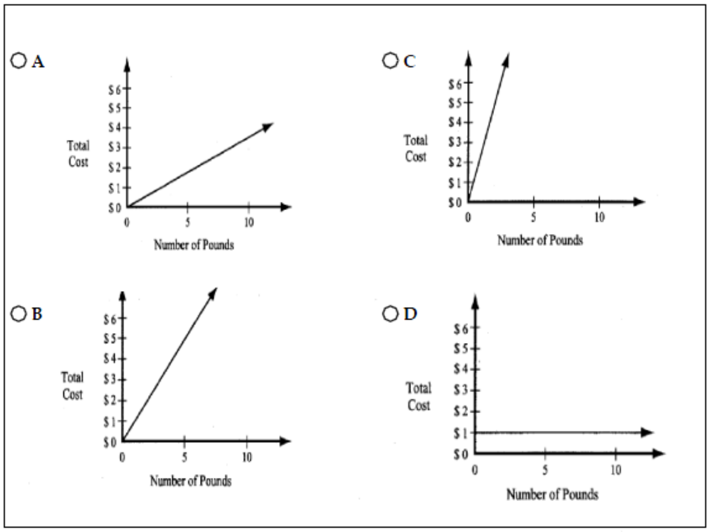


Figure 3-1. An example of verbal-graph translation (Adapted with permission from Stephen Pape).

Procedure

Participant Selection

There were two criteria for participant selection. First, the content of the videos was considered. Only classroom observations that focused on quadratic equations were considered because instruction related to quadratic equations provides more opportunities to observe students' use of representations. Second, classrooms with initial mean pre-test scores below 50% with highest and lowest gain scores were considered. That is, among the classroom observations that focused on quadratic equations, the two teachers' classrooms with the highest gain and the two teachers' classrooms with the lowest gain scores were selected. Gain scores were calculated as the percentage of the maximum possible change (i.e., (Post – Pre)/(Maximum Score (100) – Pre)), which is displayed in Table 3-4. The unit of analysis was the two classrooms' observations per teacher.

Table 3-4. Classroom Gain Scores

Classroom	Gain Score
Ms. BW	0.14773
Ms. MB	0.146815
Ms. MA	-0.06471
Ms. JR	-0.13157

Open-ended item scoring procedure within CCMS project

The algebra pre-test and posttest respectively contained seven and six open-ended items, five of which were common to both measures. Thirteen points were allocated for the open-ended items on each measure. Following initial scoring by the research team, a graduate research associate scored a random selection of 10% of the papers to estimate inter-rater reliability, which ranged from 0.88 and 0.98 reflecting a high degree of consistency.

Translation problem achievement

Students' translation problem achievement on the pre- and posttest were calculated to determine a sub-score by summing their scores on translation problems. The maximum score on the 11 translation pre-test items was 13 points with two problems weighing two points each. The maximum score on the 17 posttest translation items was 20 points with three problems weighing two points each. To create consistent scores on each measure, the students' pre-test scores were divided by 13 and multiplied by 100, and their posttest scores were divided by 20 and multiplied by 100. Students with missing data were deleted from the dataset. The gain score on translation problems, which was the percentage of the maximum possible change, was determined for each student by using the method described above. Finally, the gain score arithmetic mean was calculated for each classroom.

Data Analysis

The main aim of data analysis in a qualitative study is to search for meanings. Hatch (2002) stated the analysis of meaning as “organizing and interrogating data in ways that allow researchers to see patterns, identify themes, discover relationships, develop explanations, make interpretations, mount critiques, or generate theories” (p. 148). Generally constructivist researchers adopt post-positivist procedures to search for these meanings (e.g., Lincoln & Guba, 1985). I have adopted the steps from several analytic methods described by Hatch (2002). The following steps were used for each teacher studied:

First, I began reading the entire transcript of two class periods to become familiar with the context of the class. Then, I reread the transcript to identify the frame of the analysis in order to break the transcript into “analyzable parts” (Hatch, 2002). Tesch (1990) defined these smaller parts of transcript data as “a segment of text that is comprehensible by itself and contains one idea, episode, or piece of information” (p. 116). The frame of analysis for this study was the conversations, which generally started with a teacher’s question to the students and ended when the problem was solved.

Next, I watched the teacher’s classroom observation videos while simultaneously consulting the corresponding transcripts. Throughout the data analysis, I kept a daily research journal in which I included the daily tasks performed along with my overall impressions of each day’s class. The teachers’ behaviors were described without knowledge of whether their students made progress from pre-test to posttest. I read through the descriptions developed while watching the classroom observation video recordings. These descriptions were separated into three sections including classroom context; psychological environment and general teaching strategies; and representational practices. The first section included the total number of students, class involvement, topics covered, setting, and general impressions derived about the class. Then, I described the psychological environment and general teaching strategies and recorded observations such as the way teachers encouraged students to take notes or let them explore. The use of representations, which included translation between representations, construction, and interpretation of representations, were detailed in the last section. Particular descriptions of episodes or instances in certain classes were explained in a couple of words in separate memos, which are “written notes to yourself about the thoughts you have about the data and your understanding of them” (Graue & Walsh, 1998, p. 166). Some examples of when I used memos in my analysis were instances during the classrooms that involved interpreting representations within real-world scenarios and using representations for specific purposes.

Third, I reviewed the teaching strategies potentially related to representational fluency by going back to the related segments of the video and the transcript. During this step, the segments were analyzed in more detail to make sure that the memo provided accurate descriptions and sufficient evidence for the memo. If there was a need, the memos and their descriptions were revised. Next, evidence from transcripts and videos (e.g., screen shots and excerpts) were included to support the memos.

Fifth, I searched for emerging themes across the memos during cross-case analysis. In this step, since the purpose was to look for commonalities and differences across effective and less effective classrooms, the memos were categorized according to effective or less effective classrooms. I then stepped back from individual memos to identify connections between them. The memos were grouped under themes (e.g., formative assessment) based on similarities. Themes were created either by keeping the original form of the memo or by combining more than one memo of similar information. In addition, I also categorized certain memos as belonging to more than one theme. Once the themes were finalized, the memos were re-examined to ensure that they represented their assigned themes. The themes were compared continuously until clear distinctions between them were determined.

While some researchers claim that it is better to take out personal beliefs and characteristics from research reports, other researchers encourage tracking subjectivity systematically during the research to better understand analyses and outcomes (Gall et al., 2007). Including brief notes about the researchers' background and experiences can support trustworthiness and increase readers' understanding of their findings. Thus, I include a brief note about my background and experiences.

Researcher Subjectivity

Before undertaking this study, I will shed some light on my own personal educational experiences, beliefs, perceptions about technology, and expectations for the study. Not until my graduate education did I have access to common technological tools such as graphing calculators or even a laptop computer. Prior to my graduate education, I believed that symbolic representations were very important, and there was no mathematical problem that I could not solve by using symbolic representations.

After I came to the United States, I attended a workshop that discussed the educational benefits of using the TI-Navigator. I was impressed with how teachers and students were using different representations flexibly. A couple of years after that, I had an opportunity to assist on a project in which I prepared lesson plans with a group of instructors and professors. I observed instructors teaching with TI-Navigator. It was very beneficial for me to witness how students and teachers used this technology. In addition, I had a chance to visit both public and private high schools and see how the TI-Navigator was being used to make classrooms more interactive and practical.

Soon afterward, in my doctoral seminar on mathematical processes, I was struck by how representations could be taught and made useful for students. I began to read journal articles and books about the various types of representations and soon realized that being able to make connections between representations increases problem-solving efficiency. Finally, at the University of Florida, I had the opportunity to use the TI-Navigator while teaching and observed how my students' improved their understanding of concepts.

At a personal level, I am particularly interested in how students and teachers in the United States use representations and the TI-Navigator. As an instructor, I hope to witness additional positive aspects of technology and the use of representations among the participants of this study. As a researcher, I acknowledge how my personal and educational experiences may affect my perceptions of the participants and their teaching practices.

Ensuring the Quality and Rigor of Qualitative Research

To increase the credibility and validity of my conclusions, I applied three out of five strategies mentioned by Gall et al. (2007). Two of the strategies could not be applied because one of them requires the participants' involvement in the research and the other includes support from quantitative data.

The first strategy is *usefulness*; that is, making the study "useful to readers" (Gall et al., 2007, p. 474). I achieved this by conducting the study theoretically and practically.

Theoretically, I studied the representational fluency process in a comprehensive way. Practically, I described and recommended contexts that support improved representational fluency abilities.

Second is *the chain of evidence*, which includes “build[ing] clear, meaningful links between research questions, raw data, and the findings” (Gall et al., 2007, p. 474). I achieved this by making an audit trail, which includes documentation of my research process.

The final strategy, *truthfulness and reporting style,* involves “be[ing] honest and straightforward” (p. 474). I achieved this by including direct quotes and describing specific and concrete events from the video recordings.

CHAPTER 4
OVERVIEW OF CLASS SESSIONS

This chapter provides an overview of the Algebra I class sessions including a description of the topics and activities, the class's psychological environment, and the general mathematics teaching approach, as well as discusses the representational practices of each teacher. The psychological environment and general mathematics teaching approaches section describes how the class was set up in terms of fostering or hindering a space conducive to mathematical instruction. The representational practices section explains how individual teachers used representations within instructional strategies. Chapter 5 provides a cross-case analysis of the classroom instruction, which compares the teachers' practices related to each of these characteristics.

Excerpts from transcripts and screenshots from classroom observation video recordings are used as evidence to support the researcher's interpretations and findings. The excerpts are identified with the participant, date, period number (when appropriate), and line number. For example, (MA_05.06.08_Per. 6, 36-45) refers to line numbers between 36 and 45 of a transcript from Ms. MA's class held in period six on May 6th, 2008. The screenshots are identified according to the participant, date, period number (when appropriate), and time. For example, (MB_04.01.08, 16:14) refers to time 16:14 of a classroom observation video recording from Ms. MB's class held on April 1st, 2008. If necessary, pseudonyms are used for students. Otherwise, "S" indicates one student and "SS" refers to two or more students.

Ms. BW

One class was observed on two consecutive days. A detailed description of the lesson activities is discussed in this section to provide an overall understanding of the class sessions.

General Classroom Description

Ms. BW's class consisted of approximately 19 students. The majority of the students consistently participated in classroom discussions. The students worked individually or in pairs. Although the students were quiet, they were energetic and engaged.

The teacher began the first class period using the Learn Check component of TI-Navigator to identify the homework questions on which students had received help from others. The students submitted "yes" if they received help from others on a specific question. Next, the teacher shared a class analysis to show the number of students who did or did not need help on each question. If most of the students solved a homework question by themselves, she quickly went over the problem. If most of the students received help from others (e.g., word problems), the teacher provided detailed explanations before collecting their homework. The teacher asked the students to check and correct each other's homework and then return it to its owners. This activity took 20 minutes of class time.

Then, working individually, the students worked on eight exercises related to quadratic equations while using calculators. In these exercises, the students found the roots of an equation by factoring and also identified x-intercepts. In addition, they sketched its graph and determined the axis of symmetry. Finally, the vertex of a quadratic equation was found algebraically and the axis of symmetry was identified by hand. These exercises took 24 minutes of class time.

In the next activity, the students watched a video clip about a real-world detective scenario. In the clip, wherein a man attempted suicide, a parabolic shape was created while he fell from a bridge to the ground (Figure 4-1). The scenario involved a detective who wanted to determine the horizontal distance between the ground under the bridge and the point where the man fell. The students' goal was to predict where the man should have fallen and the distance he should have traveled.

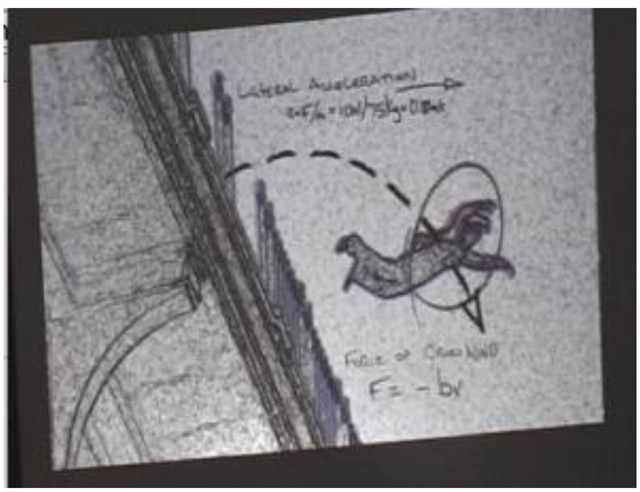


Figure 4-1. Introduction to the detective problem (BW_05.06.08, 49:23).

The students entered the coordinates of the man (values for X and Y that were collected by the detective) into a table (Figure 4-2A) in the graphing calculator to identify its scatter plot (Figure 4-2B). They came up with the corresponding equation (Figure 4-2C) and entered it into the calculator to create its graph (Figure 4-2D). This activity took 17 minutes.

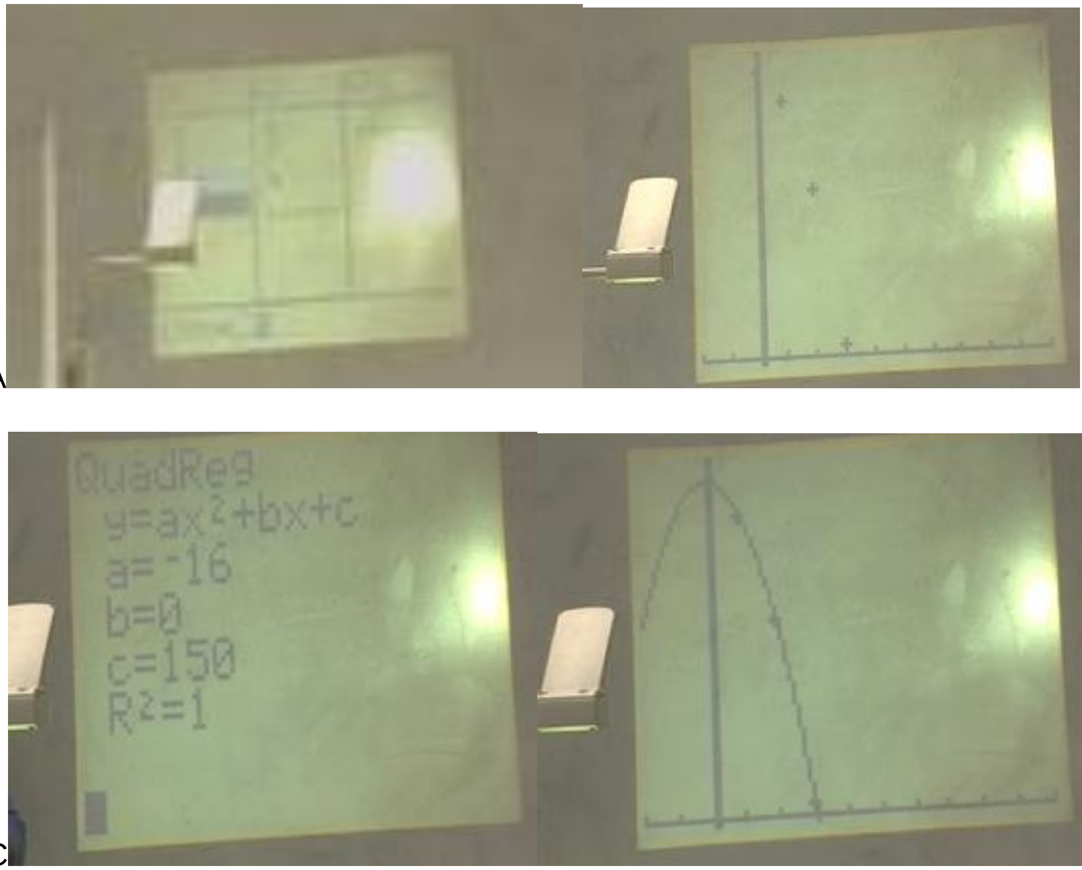


Figure 4-2. Screenshots from the students' work on the detective problem. A) X- and Y-values in tabular representation, B) Scatter plot of the values, C) Quadratic equation, and D) Graph of the equation (BW_05.06.08, 52:52, 55:33, 58:14, & 58:17).

Another real-world problem required drawing the curve for a basketball player's shot to a basket. She asked the students to determine the coordinates identifying a parabola that went through the three points representing the locations of the player, basket, and coach on the X-Y coordinate system. They performed this activity by either using the activity sheet in front of them or by coming up to the board. The students identified and submitted their equations through TI-Navigator to find a parabola that fit between these points (Figure 4-3).

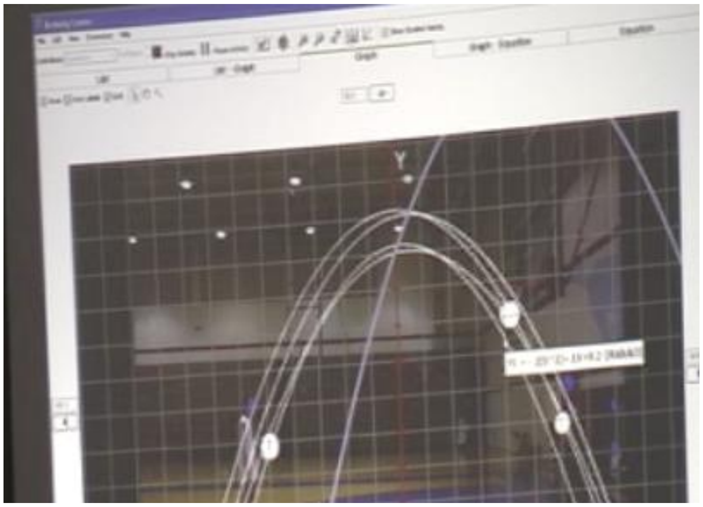


Figure 4-3. Basketball shot problem screenshot (BW_05.06.08, 01:11:23).

The teacher encouraged the students to resubmit their graphs. Yet during the discussion she realized that the students forgot how to modify the location and shape of a parabola by changing the equation's coefficients. She therefore opened a new screen with a coordinate plane in Activity Center and asked the students to determine an equation that matched her parabola on the screen (Figure 4-4). This activity took 14 minutes.

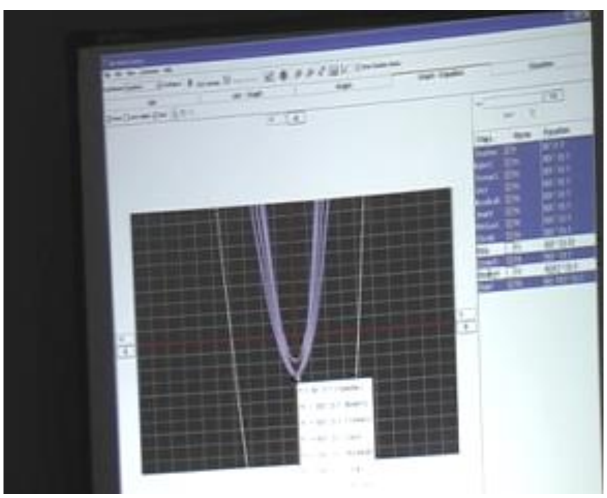


Figure 4-4. Students' submitted parabolas matching with Ms. BW's (BW_05.06.08, 01:21:18).

At the beginning of the second class period, the teacher projected a natural landform picture on the screen and asked the students to submit their quadratic equations individually (Figure 4-5). She assigned a color to each student's submission so they could clearly see their parabolas. The aim was to fit the parabolas between two mountains on the picture.

Figure 4-5. Natural landform problem (BW_05.07.08, 04:45).

Together the students analyzed each equation and discussed how to make them more accurate. The teacher used the Activity Center component for this activity, which took 10 minutes.

Then students exchanged homework and spent six minutes checking each other's work; their homework task had been to find the roots of quadratic equations using the calculator and sketching the graph by hand. The teacher asked one student to share the homework answers with the class, and the other student to correct their classmates' work. Students returned the corrected homework to its owner, and the teacher went over the questions if the students needed detailed explanations.

After the students checked their homework, they worked on a lesson from the book about solving quadratic equations by factoring. They started with warm-up activities including finding the product of a quadratic function and factoring a polynomial. Then the teacher continued the lesson with PowerPoint slides explaining how to solve quadratic equations by graphing. The class spent nine minutes on this exercise.

The students solved two real-world word problems on their worksheet using a graphing calculator. The first problem was to find the length of time a frog remained in the air after jumping straight up. The quadratic function $f(t) = -16t^2 + 12t$ modeled the frog's height above the ground after t seconds. Another real-world problem scenario included a dolphin jumping out of the water. The quadratic equation $y = -16x^2 + 32x$ was used to model the dolphin's height above the water after x seconds. The class found and discussed the length of time the dolphin remained out of the water. After solving these problems, the teacher sent a quiz related to finding roots of quadratic functions to the students using the Learn Check component. The class spent seven minutes each on the real-world problems and the quiz.

Finally, the students performed a group activity to create a quadratic equation to model revenue related to selling flowers. The scenario involved students acting as the owners of a shop. If they sold 100 bouquets, they charged 15 dollars for each bouquet. Every time they lowered the price by 20 cents, the number of bouquets sold increased by four units. First the teacher created a table with the values of X, cost, and number of bouquets by hand (Figure 4-6A). Then the students identified an expression for the price of the bouquets and the number sold so they could use these expressions to create an equation for revenue. Once it was created, they entered it into the calculator to draw a graph for revenue (Figure 4-6B). This activity took 30 minutes.

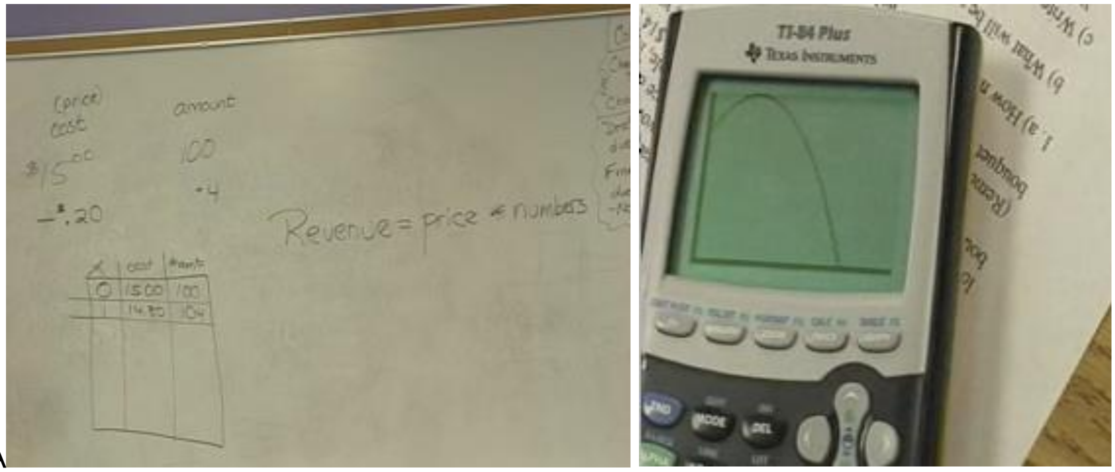
A  B

Figure 4-6. Creating the quadratic equation to model the revenue in the flower problem. A) Table with values and B) Graph of the revenue equation (BW_05.07.08, 54:32 & 01:08:40).

Psychological Environment and General Mathematics Teaching Approach

Four characteristics of this class emerged through the analysis: (a) fostering strong communication skills, (b) promoting student participation and attention, (c) teaching in a fast-paced manner, and (d) encouraging technology-use.

Fostering strong communication skills

Ms. BW engaged the students in classroom activities by fostering effective communication, which was facilitated by her sense of humor. For example, in one instance when her marker dried out, she sarcastically threw the marker in the trash and said it was a good marker. She also joked with a tall student by calling him “little”, and she made fun of her age and said she was getting old and needed a bigger calculator when she misread the numbers on the calculator. She also started a comical exchange with some girls, sarcastically asking, “What’s going on over here, ladies, other than doing hair?” (BW_05.07.08, 2174).

Ms. BW exhibited specific ways of dynamically expressing her feelings. She said, “ah, ah, ah, ah” (BW_05.06.08, 939) when a student answered her question incorrectly and “oh, oh” (BW_05.06.08, 38:37) while teaching an important concept. In addition, she would spend a little time talking with students about non-academic issues such as their birthdays, which tended to support her efforts to build rapport with her students.

Ms. BW explained the correct use of terminology when communicating concepts to her students. In the first class period when identifying the x-intercepts, the students figured out that a parabola that did not intersect with the x-axis had “no solution”. The teacher, however, clarified the distinction between “no solution” and “no real solution” by explaining real and imaginary numbers.

T: At our stage of mathematical knowledge, the answer when it doesn’t cross the x-axis is “no real solution.” It’s not that there is no solution ever; it’s that at our level we’re not cool with imaginary numbers so as far as we know there is no real number solution to it, which is why we say “real solution” because we’re talking about real numbers. Ok? The reason is it does not cross the x-axis, so any time you see one where it doesn’t cross the x-axis, there are no real solutions. (BW_05.06.08, 1082-1087)

The students used this term correctly in the second class period. When the teacher asked a similar question, they responded by saying, “There’s no real number” (BW_05.07.08, 767). As in the previous example, Ms. BW promoted correct use of terminology by clarifying additional misunderstandings when the students expressed confusion over the axis of symmetry and the vertex, which they thought were same. The teacher also corrected a student’s misconception about the difference between a solution and a factor.

Promoting student participation and attention

A high level of class participation was observed in Ms. BW’s classroom. She promoted participation even when reviewing the students’ homework by asking that students provide each of the answers. The teacher initiated discussion by directing a question to one student and would redirect the question only if the student was not able to provide an answer. With this approach, Ms. BW could interact with each student quickly. However, while this approach might be helpful for a teacher to increase students' participation, Ms. BW sometimes used it harshly. In one instance, when a student was not able to provide a correct answer for a question, she repeated the question, and when another student attempted to share his answer without the teacher asking him, Ms. BW angrily replied "You're not Miss _____" (BW_05.07.08, 474).

The teacher sometimes checked whether the students were focused such as following an explanation about a problem, she checked if the students were on track. In one instance, when Ms. BW saw that the students' attention was inadequate, she clapped her hands to reengage them in the lesson. In another instance, she helped one group and left to check other groups; after a few minutes, she came back to check the progress of the first group and still remembered what they had done. She said, "When I left you, you were going to multiply these two together for revenue and nothing happened" (BW_05.07.08, 2178-2179).

Further, the teacher remained in control of the class throughout the period. When the students talked about unrelated topics, she warned them by saying, "There should not be any talking" (BW_05.07.08, 40). Possibly to engage the students better, she mentioned that the quiz available through the Learn Check component of TI-Navigator would be similar to the quiz that would be given the next day.

Ms. BW showed appreciation for students' mathematical work. For example, when drawing the curve for the basketball player's shot, she praised one of her students who submitted an accurate graph, saying, "This is Reddick? That's a good one, Reddick. Just a little low; you're going right through this end so you need to shift a little to the right" (BW_05.06.08, 1876-1877). While working on the natural landform problem, the students' parabolas were all facing the correct direction, and they were all parabolas rather than lines (Figure 4-5). Thus, she said,

T: Everybody did a really good job on this because like I said it's not a perfect parabola. Everybody's got them facing the right direction; that's a start for us. We have all quadratics, which is a really good start because sometimes we end up with a bunch of lines and we don't have any of those this time so we're making good progress on trying to understand what it means to be a quadratic. (BW_05.07.08, 175-179)

The teacher showed her appreciation when the students made good progress in their work such as translating between representations.

The teacher also promoted participation by engaging the students in rich mathematical tasks such as the modeling problems. When performing the last activity about modeling revenue for selling flowers, some students said, "I'm excited" (BW_05.07.08, 1366) and "that's exciting" (BW_05.07.08, 1395). In addition, the teacher increased the students' submission rate by counting down from ten seconds to indicate the end of an activity.

Another way Ms. BW promoted participation was through creating a safe environment for the students to ask questions. Even though the students did not indicate needing help on several homework questions, they wanted to discuss them and asked several questions. In addition, without the teacher's encouragement, the students were comfortable asking questions. For example, a student said, "I have a question. For number six how do you tell there's no real solution here?" (BW_05.07.08, 517). They sometimes asked the teacher to repeat the questions possibly to avoid misunderstanding. At other times, however, Ms. BW sometimes apparently only wanted to hear the correct answers from the students. She said, "Don't send it to me unless you know it's right" (BW_05.06.08, 2047). This may discourage student participation in classroom discussions.

Teaching in a fast-paced manner

The teacher used the entire class period and valued each minute of class time. She assigned tasks to the students even when collecting their homework. For example, she said, "Alright, while I'm collecting homework we're going to be taking notes from the PowerPoint so go ahead and warm yourself up with these" (BW_05.07.08, 610-611). Before the time was up, she would let students know how much remained to finish a given task. But on several occasions, the students needed more time. In one instance, when Ms. BW reminded the students about the remaining time, one said, "I'm still on the first one" (BW_05.07.08, 1160). During another activity, some students did not have enough time to send a graph, so the teacher could not assign a color to them. Sometimes she did not give enough time for the students to think and solve the problems for themselves.

Ms. BW taught in a very fast-paced manner. When a student came up with an incorrect answer, she sometimes explained how to arrive at a solution too quickly. Even if a student answered a question partially, the teacher moved onto the next question without checking if others knew the complete answer. During the last activity of the first class period, the teacher asked the students to match their graphs with hers and said, "Match me" (BW_05.06.08, 2021-2042) several times in a row. The students, however, became stressed by this fast-paced teaching approach and said, "Ok, hold on, Jesus" (BW_05.06.08, 2044).

In several instances Ms. BW limited the students' thinking thorough this fast-paced discourse environment. She occasionally provided answers and gave explanations before letting the students explore an activity. At the beginning of the activity about selling flowers described earlier, the teacher explained that the revenue equation would be quadratic. Even before the students started to determine expressions of the revenue equation, she stated, “Revenue is equal to the price that you are charging times… that’s multiplication… the number of bouquets. And if you do that, or let’s say when you do that, not if… when you do that correctly the revenue equation that you get should be a quadratic, ok?” (BW_05.07.08, 1441-1443). Ms. BW, however, provided thorough explanations and did well to clarify important steps to problems, despite sometimes giving the explanations prematurely.

Encouraging technology-use

Ms. BW provided explicit instruction on how students could use technology to their advantage. She projected instructions on using the technology on the board and distributed printouts to the students. When needed, she also gave instructions verbally and encouraged the students to use the calculator by making statements such as, “The calculator makes life a whole lot easier for us” (BW_05.06.08, 609). She mentioned that finding a solution using the calculator was easier than calculating by hand. When a problem required extensive calculations, the teacher did not overwhelm the students by performing the activity manually. Instead, she encouraged them to use the calculator, saying, “So really we’re going to push a couple of buttons on the calculator if you look at it” (BW_05.06.08, 844-845). Then she went over the steps on the calculator. She reiterated that calculating was “not that much work because most of the work was done in the calculator” (BW_05.06.08, 982-983).

Representational Practices

Five characteristics of Ms. BW's representational practices emerged: (a) fostering the use of representations, (b) using different translations, (c) distributed nature of translations, (d) formative assessment, and (e) promoting translation through teacher questioning.

Fostering the use of representations

Ms. BW fostered the use of representations in two ways: (a) by engaging students with real-world scenarios and (b) by using representations for specific purposes. For example, she introduced the detective problem during the first class period by watching a video clip (Figure 4-2). For this activity, the students were asked to think of themselves as detectives and predict where the man in the picture would fall by determining the quadratic equation for the man's path after he jumped from the bridge, drawing its graph, and identifying the distance that the man fell. The students demonstrated their familiarity with solving these kinds of problems and made connections with another real-world problem they had solved before; at the beginning of this activity when the teacher explained the scenario, the students related it to a previous exercise, stating, "Oh, the basketball thing" (BW_05.06.08, 1355).

During the second class period, the students were asked to determine a frog's jumping height. Ms. BW guided the students in linking their answers to real-world applications through the questions she asked, "What does that [frog's jump] look like on the graph?" (BW_05.07.08, p.19). She also drew a parabola on the board (Figure 4-7), pointed out that the frog started at x = 0 on the coordinate system, and asked where it returned back to the x-axis.

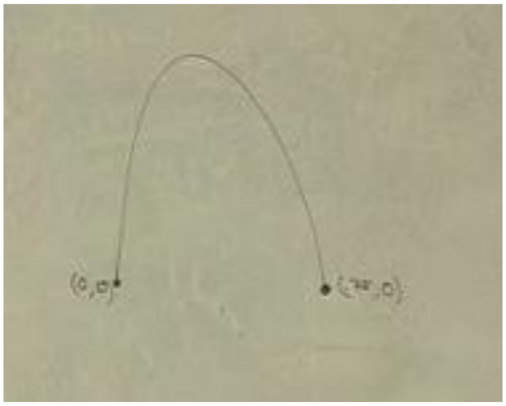


Figure 4-7. Graph for the frog jump calculation (BW_05.07.08, 38:19).

The teacher then asked the students to link the problem to other interpretations by asking the length of time the frog had stayed in the air:

T: I want to know how long he’s in the air.

SS: That’s an intersection. / It would be the X… / Something happens… / …zero on the ground.

T: Right, it will be here when he comes back to zero again.

S: Point seven five.

T: He starts at zero, zero but he comes back at something comma zero.

S: Point seven five, point seven five.

T: Point seven five, so he’s in the air for how long? (BW_05.07.08, 964-976)

Similarly, Ms. BW presented problems (described earlier) involving real-world scenarios including a basketball player, a natural landform, and a dolphin’s jump out of the water.

Another way Ms. BW fostered the use of representations was by using graphical and tabular representations for specific purposes when solving problems. The following excerpt shows how she encouraged the students to use specific representations when solving problems:

T: And I always graph it first because I want to see what I'm looking for, and one of these is going to jump out at you as to why I always look at it first. I mean technically you don't have to look at it first, but I think you should so you have some idea of what they're hunting for. (BW 05.06.08, 1023-1026)

Additionally, Ms. BW emphasized using tabular representations displayed on the calculators to verify answers. After she initially calculated the vertex algebraically, she verified its Y-value by looking at the corresponding X-value in the tabular representation displayed on the calculator.

Using different translations

The use of unidirectional and bidirectional translations were noted during observation of Ms. BW's class. Unidirectional translation refers to translating between different representations within the same activity in the following sequence Symbolic1 → Graphical → Symbolic2. Bidirectional translation refers to translating between the same representations within the same activity, such as Symbolic1 → Graphical → Symbolic1.

Unidirectional translations including two, three, and four translations were observed in Ms. BW's class. For example, when she was using PowerPoint slides to explain how to solve quadratic equations during the second class period, the students initially drew a graph for a quadratic equation using the graphing calculator. They then translated from graphical to symbolic representations to find the X-intercepts of the quadratic equation. This represents a translation from an equation to a graph and then to X-intercepts. At the end of the second class period, Tabular → Symbolic → Graphical translations were used in selling flowers activity. The teacher created a table by hand and the students then found an expression to create an equation for revenue. Finally, students entered the revenue equation into the graphing calculator to draw a graph for the revenue (Figure 4-6B).

Three unidirectional translations observed while solving real-world problems in the second class period are detailed as follows: Symbolic1 → Graphical → Symbolic2 → Verbal. The same type of translation was observed when the students solved the frog and dolphin problems. In the frog example, Ms. BW initially entered the quadratic equation into the calculator to draw its graph. Using calculators, the students then identified the x-intercepts for the parabola. Finally, they discussed the findings (e.g., x-intercepts) by linking them with their real-world meanings as described earlier.

At the beginning of the first class period, four unidirectional translations of the form Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical were observed when the students were solving the quadratic equation exercises. The students initially entered an equation to draw a graph. They used a tabular representation in the calculator to find values of Y for given X-values. Using the data obtained from the calculator, the students finished the activity by sketching its graph.

Bidirectional translation was also observed during of Ms. BW's class observations. During the detective problem, they initially entered the coordinates of the man on the sheet provided and then identified its scatter plot. Then the students came up with an equation by looking at the scatter plot and entering it into the calculator to create its graph. This series of translations can be depicted in the following chain of representations: Pictorial → Tabular → Graphical1 → Symbolic → Graphical1.

Dynamic features of the technology enabled the simultaneous use of representations. This feature allowed students to visualize how each modification they made to their equations would simultaneously change their graphs. For example, when they tried to find the best fit between the basketball player, basket, and coach, they saw the symbolic and graphical representations together and used them jointly (Figure 4-8).

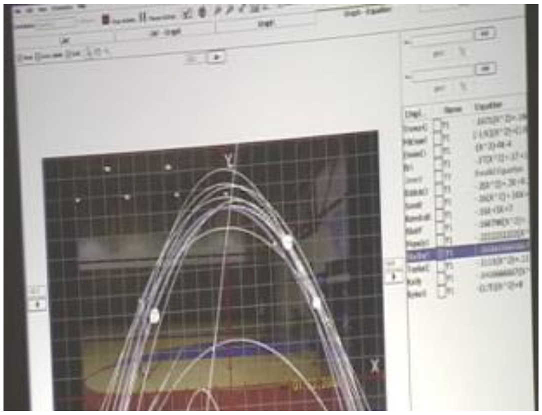


Figure 4-8. Using dynamic representations simultaneously (BW_05.06.08, 01:13:48).

Simultaneous use of representations helped students to translate between representations bidirectionally until they reached the correct solution. Thus, the students translated between two representations iteratively in both directions and multiple times until the answer was determined. This type of bidirectional translation is referred to as cycling between translations and may be depicted by a double arrow (↔). In Ms. BW's class, the students translated between representations in the following sequence Symbolic1 → Graphical ↔ Symbolic2 when the teacher created a parabola and asked the students to find a graph that matched hers (Figure 4-4). While they resubmitted their graphs, they could continue translating bidirectionally until an accurate graph was found.

The students also used cycling translations of the form Pictorial → Symbolic1 → Graphical ↔ Symbolic2. In the natural landform problem, the students were allowed to resubmit their graphs, translating between symbolic and graphical representations until they identified the most accurate graph. The teacher mentioned that the parabola drawn using the calculator and submitted through the TI-Navigator would not be a perfect fit for the natural landform. She, however, encouraged the students to estimate the most accurate graph for this real-world representation and promoted cycling between representations.

Distributed nature of translations

Students were jointly involved in translating between representations in Ms. BW's class. Translations can be collaboratively shared between students and teachers, and they can be mediated through the calculators. In Ms. BW's first class period, the students translated from symbolic to graphical and graphical to symbolic representations with the help of the calculator. The teacher initially entered a quadratic equation into the calculator to draw a graph. Then, using the calculator, the students tried to find the minimum point of the parabola by clicking the "minimum" button. While the point on the parabola descended to the minimum vertex automatically, the X- and Y-values changed at the bottom of the calculator simultaneously (Figure 4-9).

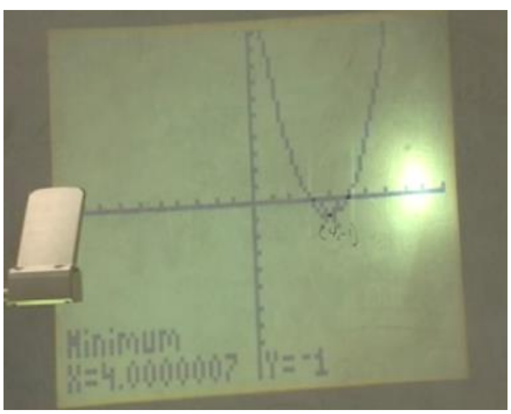


Figure 4-9. Translation from graphical to symbolic representations (BW_05.06.08, 31:11).

Moreover, collaboratively shared translations were also observed, wherein knowledge was distributed across students, the teacher, and the calculator. When solving exercises in Ms. BW’s first class, the students used their calculators to translate from symbolic to graphical and from graphical to tabular representations but translated from tabular to symbolic and from symbolic to graphical representations by hand. In this exercise, the students calculated the x-intercepts, solutions, factors, and vertex, and they then used a calculator to graph the equation. Finally, they used the tabular representation from the calculator to identify the Y-values corresponding to the X-values as shown in Figure 4-10. The following screenshot shows the students finding the Y-value for X=1 with the help of the calculator. Using the data obtained from the calculator, they then sketched a graph that they had already seen.

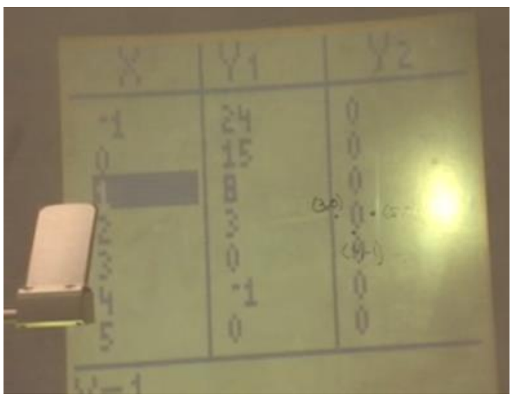


Figure 4-10. Translating from tabular to symbolic representations (BW_05.06.08, time: 33:16).

In the second class period, while working on the frog jump problem, the students translated from symbolic to graphical and graphical to symbolic representations by using their calculator, whereas students translated from symbolic to verbal representations manually. The teacher initially entered the quadratic equation for the frog’s jump into the calculator. Then using the calculator, the students identified the x-intercepts for the parabola. Finally, the students and teacher discussed the findings (e.g., x-intercepts) by associating them to the real-world meanings described earlier.

Another sequential translation was observed when the teacher involved herself in the joint task of creating a equation for the revenue from selling flowers. When creating the equation, the class performed the translation from tabular to symbolic representations. The students tried to find expressions for both the price and number of bouquets after the teacher created a table. The teacher initiated working on this task and the students continued it (Figure 4-6A) leading to the expressions being used to create the revenue equation. The students then entered the equation into the calculator to draw its graph (Figure 4-6B).

Formative Assessment

Ms. BW was aware of her students' occasional discomfort and comfort with the use of representations. It seemed that her students were predisposed to think negatively about solving word problems. For example, when the teacher used the Learn Check component at the beginning of the first class period, most of the students submitted "yes", meaning they had received help from others in solving the word problems included in their homework:

T: 42 and 44 [The numbers of the problems on which the students received help from others]. So these through 44. So you know what they are? I know what they are.

S: The solutions ones?

T: No, they're not the solution ones; they're the…

SS: Word problems.

T: The word problems. Ok, for some reason all [you] have to do is see the words in front of you and all of a sudden, "Ahhh! Can't be done!" (BW_05.06.08, 133-142)

Ms. BW was already aware that her students were having a difficult time solving word problems. With the help of TI-Navigator, she could validate her students' lack of understanding and confidence solving these types of problems.

The teacher effectively monitored her students' in-class work. Ms. BW shared Class Analysis with the students without revealing individual names and scores. She only let the students know about the classroom average and the number of students who scored within each given score range. Other students were curious about their classmates' work, since they could not see who solved the problem correctly or incorrectly. A student—while his classmates were present—asked Ms. BW how well he performed; she didn't share that information, explaining that it was private.

In addition to using these properties of TI-Navigator, Ms. BW also used the Screen Capture component (Figure 4-11) at the end of the first class period to allow the students to examine their mathematical constructions and to make changes. This also allowed the teacher to point out students' technical errors right away instead of having to walk around the classroom.

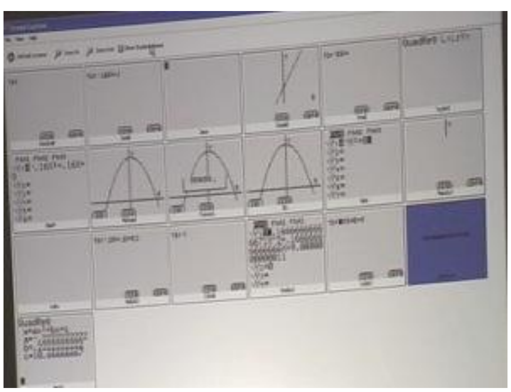


Figure 4-11. Using the Screen Capture component (BW_05.06.08, 01:11:12).

Ms. BW was sometimes flexible and made adjustments in her teaching according to the students' needs. When she realized that some students forgot how to change the shape and location of a parabola, she stopped the activity and opened an empty coordinate plane. Additionally, when some students had difficulty entering the correct boundaries for the graph on their calculators, Ms. BW opened up the discussion to the whole class; she showed them how to change the boundaries on the calculator setting so they could see the graph.

Although the projection of students' mathematical objects provided a vehicle to understand their mathematical thinking, most of the students had a difficult time assessing their work during Ms. BW's first class period. They did not know which graphs on the screen belonged to them unless the teacher dragged the mouse over their parabolas. Thus, in the next class, Ms. BW assigned a color to each of the student's graphs so that they could track their submission. The coloring system made the resubmission process even more effective, because the students' names were kept private. Alternatively, students' names would have to be displayed on the screen and visible to their classmates. This approach, however, did potentially have some negative impact on representational fluency. Some students were afraid of making mistakes, even if there was an opportunity to resubmit. One student said, "I don't want mine if there's a mistake" (BW_05.06.08, 1794). Some students also made fun of their classmates who submitted incorrect graphs. In one instance, when a student submitted a line instead of a parabola, the students laughed sarcastically (BW_05.07.08, 08:08) and asked who submitted the line.

Promoting translation through teacher questioning

In both class periods, Ms. BW asked questions to create an environment where students translated between representations. For instance, while students submitted their equations to fit a parabola to a specific shape, Ms. BW asked questions to promote cycling translations between symbolic and graphical representations. The following exemplifies the types of questions that Ms. BW would ask: “This one is upside down. How do you turn it back around?” (BW_05.06.08, 1811), “What does skinny mean?” (BW_05.06.08, 2189), “What’s wrong with this one?” (BW_05.06.08, 2160), “How would we make him fatter?” (BW_05.07.08, 146), “What do you notice?” (BW_05.07.08, 524). The students usually responded with short answers. In the second class period, while the students were submitting their equations for drawing a curve that would fit between two mountains, the teacher asked, “how do we move him (a low graph)?” (BW_05.07.08, 154). The students responded by saying, “add” (BW_05.07.08, 156). In another instance, the teacher asked, “how would we make him fatter?” BW_05.07.08, 146), and the student replied, “fraction” (BW_05.07.08, 148). The following excerpt highlights the character of these conversations.

T: Stop. Ok, let’s talk about the different ones that we have. What’s wrong with this one?

SS: It’s too long. / Too skinny.

T: Too skinny. How would we make him fatter?

SS: Fraction.

T: Fraction, ok. What’s wrong with this one?

SS: Too low.

T: How do we move him?

SS: Add.

T: Add. Ok, this one’s too low. What’s wrong with number… or black I think it is or dark green?

S: Too wide.

T: Ok, it’s not quite too wide because…

SS: It's too low.

T: …it ends here. It's too far…

S: Over.

T: …right. How would we bring something left?

SS: Positive / Subtract. / Add.

T: Add. Right? Add; if we want to shift it to the left, we add. If we want to shift it to the right, we subtract, ok? (BW_05.07.08, 142-174)

Summary of Ms. BW's Classroom Environment

Ms. BW had a good sense of humor and established a strong rapport with her students. She sometimes talked with them about topics other than those related to the classroom context. The teacher also promoted participation and attention by ensuring that every student had a chance to speak. She sometimes checked whether the students were on track and remained in control of the classroom. If the students began talking about non-class related topics, she usually admonished them to refocus on the class discussion. Ms. BW created an open environment where students could ask questions without hesitation. She used the class time efficiently and did not stop until the end of the period. Her manner of teaching was fast-paced; however, her pace sometimes did not provide adequate time for her students to solve problems. Overall, however, she provided thorough explanations and clarified important steps to problems, even though she sometimes provided an explanation before letting students work out solutions for themselves. The teacher also encouraged students to use calculators and develop competence in using mathematical terminology.

Ms. BW provided opportunities for her students to propose interpretations about representations by linking them to real-world scenarios. She used graphical and tabular representations for specific purposes when solving problems. Ms. BW included many unidirectional and bidirectional translations during classroom activities. In addition, the students used symbolic and graphical representations simultaneously. This encouraged them to translate between these representations until an accurate answer was reached. Students collaboratively translated between representations several times within the same activity. Ms. BW would monitor students' progress often during the class period; at the end, she would share the class analysis with her students. By continuously assessing her students' progress, she identified their discomfort with solving word problems in using representations involving real-world scenarios. And when she realized the students had difficulty in understanding the use of representations, she made adjustments to her teaching to provide better clarification. Finally, Ms. BW created an environment in which she promoted the translation between representations by asking questions throughout the classroom discussion.

Ms. MB

One class was observed on two consecutive days in Ms. MB's Algebra I class. The following is a detailed description of the activities during these lessons to provide an overall understanding of the class sessions.

General Classroom Description

There were approximately 30 students in the class, and most participated in discussions. They generally worked in groups of three, although a few students worked individually. The students had their own calculators during the first class observation, but each group had only one calculator when working on an activity during the second class. There was an additional teacher who helped the students during both periods.

In both classes, the students discussed their graphical representations and corresponding symbolic representations. For example, during the first period, the teacher initially drew two parabolas (Figure 4-12A) and asked the students to compare them including their axes of symmetry. She recorded students' answers by writing them on the board (Figure 4-12B).

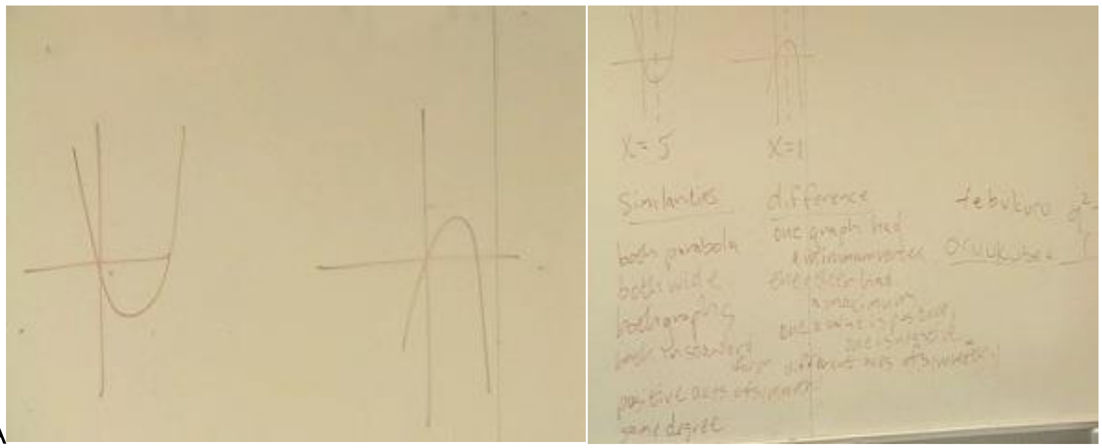
A  B

Figure 4-12. Comparing two parabolas. A) Drawings and B) Similarities and differences (MB_04.01.08, 16:14 & 19:56).

In this exercise, students noticed that both parabolas were wide and their axis of symmetry intersected the x-axis to the right of the origin. While the students mentioned that the graphs had minimum and maximum vertices and positive and negative leading coefficients, the class did not explicitly make the connection between these characteristics. However, the teacher emphasized and the students discussed this connection in the next activities. This activity took 27 minutes.

Next, the teacher modeled how to determine the shape and location of a parabola given pre-determined parameters of the equation (i.e., axis of symmetry between -5 and 5; positive or negative leading coefficient of a quadratic equation or the "a" in the quadratic term) and then asked students to determine an equation given a different set of parameters. The students entered their equations into the calculators to submit to the TI-Navigator (Figure 4-13). Finally, as a class, they discussed why individual graphs did or did not meet the criteria. The teacher used the Activity Center component of TI-Navigator for this activity, which took 20 minutes.

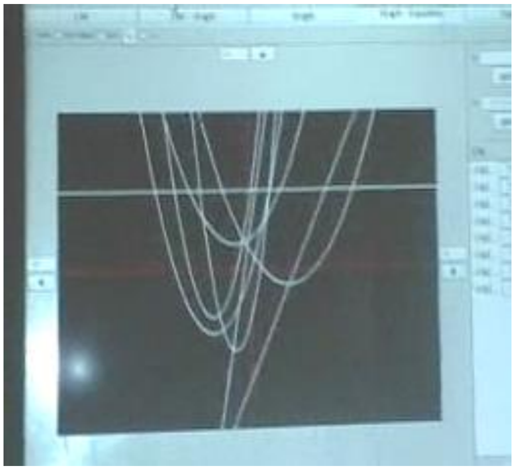


Figure 4-13. Students' submitted equations using pre-determined parameters (MB_04.01.08, 38:30).

During the second class observation, the students initially looked at a picture projected on the screen showing a basketball player shooting a basket. This activity was similar to the last activity during the first period of Ms. BW's class. Unlike her, though, Ms. MB used only two points and excluded the third point representing the coach. Then the students were asked to identify an equation that drew a curve from the location of the player to the basket (Figure 4-14).

Figure 4-14. Basketball shot problem screenshot (MB_04.02.08, 24:21).

The teacher asked the students several questions about the information needed to identify an equation before she allowed them to find it. For example, she questioned students' knowledge about the "a" and "b" values (i.e., the coefficients of the quadratic and linear terms), vertex, axis of symmetry, and y-intercept. Then she let the students discuss the equation within their own groups, which were responsible for submitting one equation. Some students left their groups to look at the picture on the board. When the groups had submitted their equations, they examined the accuracy of each shot, which led to a discussion about changes each group might make. The teacher used the Activity Center component of TI-Navigator for this activity, which took 36 minutes.

At the end of the class, students solved quadratic equations on a worksheet, and the teacher tried to evaluate students' progress through the Learn Check component of TI-Navigator; however, the class period ended before she could check and discuss the students' answers. The class spent 20 minutes on this activity.

Psychological Environment and General Mathematics Teaching Approach

Three characteristics of this class emerged through the analysis: (a) building a class community, (b) promoting a safe environment for exploration, and (c) promoting participation and attention.

Building a class community

The teacher generally used "we" instead of "you" language when communicating with her students. She usually emphasized the word "all" and pressed the students to engage in class activities. In one instance, while she was saying "Now, let's all talk about this together" (MB_04.02.08, 537), she emphasized "all" by moving one of her hands as if she was drawing a circle twice, while elongating the word and increasing the volume of her voice.

Even though the teacher tried to create a sense of community among the students, some students did not respond to this effort. For example, while the students discussed how to identify the most accurate equation for the basketball shot on the second day, one group submitted a very accurate graph. Some students said, “He got the equation already” (MB_04.02.08, 367-368) and attributed all the success to only one of their classmates instead of the whole group. As a consequence of emphasizing students’ exploration, Ms. MB created an environment in which the students engaged in discussions throughout class periods.

Promoting a safe environment for exploration

Ms. MB encouraged the students to share their solutions with their classmates and to share their opinions with others while solving the problems. She said, “listen carefully. One of your colleagues is going to make a recommendation about how he made this equation more accurate” (MB_4.02.08, 560-562), and added, “I want you to try putting in this equation because your colleagues have come up with something that they think will help” (MB_4.02.08, p.587-588).

The teacher also facilitated a learning environment in which the students interacted with her and with one another. The students were not hesitant to share their thoughts about their classmates’ answers. They comfortably commented on their classmates’ answers and indicated whether they agreed with their answers. In one instance, when a student disagreed with one classmate about whether an equation was in standard form, the teacher was excited. She moved both her hands up and down and brought them all the way to her back while exclaiming, “this is fantastic” (MB_04.01.08, 209). Some students, however, reacted negatively to the student’s conflicting comment. But their negativity subsided when this student explained her reasoning. Sometimes Ms.

MB changed the volume of her voice for communicative dynamics such as when she responded back to a student with “ah hah, ah hah” (MB_04.01.08, 143) upon hearing a correct answer.

The class’s safe environment for exploration was reflected through the teacher’s verbal encouragement to use correct terminology. For example, when explaining his answer to a question, one student said, “The first graph has a minimum vertex and the second graph had a maximum” (MB_04.01.08, 64). The teacher responded, “One graph had a minimum vertex…excellent use of language…minimum vertex. The other had a maximum” (MB_04.01.08, 70-71).

Ms. MB facilitated a safe environment wherein students could resubmit their graphs when determining the shape and location of a parabola in the last activity of the first class period. Ms. MB projected the students’ responses anonymously in both class periods (Figure 4-15).

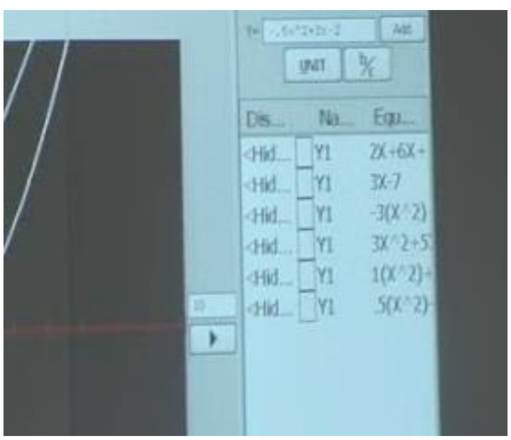


Figure 4-15. Projecting the students’ responses anonymously (MB_04.01.08, 38:06).

Ms. MB also promoted a safe environment for exploration through questioning. In the activity related to the basketball foul shot, Ms. MB asked the students to submit their equation through TI-Navigator to fit their curves on the picture (Figure 4-14). She started the activity by asking an open-ended question, “what kinds of information might help you identify the equation?” (MB_4.02.08, 52). After the students were given a moment to recall prior knowledge, she pressed them to explore by saying, “that’s for you guys to figure out…try to figure out” (MB_4.02.08, 158). By using open-ended questions, the teacher created an environment wherein the students could make connections between concepts. For example, when discussing methods to find an accurate curve in the second class period, she asked for another way to describe the “C” value in the equation to make a connection between this value and the y-intercept.

Promoting student participation and attention

Ms. MB promoted interaction by valuing students’ input. When she explained solutions, she created an interactive environment wherein all students could contribute. For example, when the students were describing the similarities and differences between the two parabolas, she recorded their contribution within a two-column table on the board (Figure 4-12B). If a student was not able to provide a clear explanation, the teacher would ask his/her classmates for help by saying, “who can help us out?” (MB_04.01.08, 155). Finally, when the students answered the questions, the teacher engaged them with encouraging words. For example, “very good; I didn’t even notice that. That’s an excellent observation” (MB_04.01.08, 193-194).

Almost all of the students, however, were engaged and participated in classroom discussions. For example, when the teacher began the activity related to determining the shape and location of a parabola, she encouraged the students to become involved by saying, “we need your help” (MB_04.01.08, 281). They were very excited to engage with the activity and to understand the steps needed to solve the problem. In addition, instead of the same group of students responding, she asked different students to participate in the discussions. Furthermore, when she felt the students' attention starting to wane, the teacher would make sure that they stayed on track. For example, during the second period, one of the students near the window was sitting passively with her head on her desk. After the teacher talked to the student, however, the student involved herself in the class activities.

As a result of the class's high engagement and participation, the classroom was noisy. The teacher sometimes had to give instructions in a very loud setting. She once wrote the instructions on the board rather than providing them verbally. She became angry and irritated because the students could not hear the instructions. Even after the teacher spelled out the activity's instructions on the board, the students still asked for them. Generally, she repeated explanations, warned the students frequently about becoming quieter, and said, "As I said before; did everyone listen? I said this a couple of times. I'll wait till everybody's listening. Ok… Still waiting …. Shhh" (MB_04.02.08, 32-37). Some students warned their disruptive classmates as well by saying "Quiet" (MB_04.02.08, 9) and "Shhh" (MB_04.02.08, 161). In one case, a student raised her hand and tried to get the teacher's attention to answer a question, but, because of the chaotic environment, the teacher did not realize this until two minutes had passed. Despite this, the teacher tried to set up social norms such as encouraging students to raise their hands.

Representational Practices

The representational practices in Ms. MB's classroom were characterized by four aspects: (a) fostering the use of representations, (b) using different translations, (c) distributed nature of translations, and (d) promoting translation through teacher questioning.

Fostering the use of representations

Activities that linked representations to real-world scenarios made some students excited in Ms. MB's class. When the activity involving the basketball player's shot (Figure 4-14) was over, the students made the following comments.

T: Ok? All right, so, you guys have any other comments or questions on this activity?

SS: No./ I like the green picture better.

T: Sorry guys, what was that?

S: With the picture, isn't it more fun that way?

T: It's a little bit more fun that way, right? (MB_04.02.08, 694-703)

When Ms. MB asked her students to determine the best curve between the basketball player and the basket (Figure 4-14), the students initially submitted their equations through TI-Navigator and came up with slightly different graphs for the given picture. Although they all looked correct, the starting and ending points of these graphs were a bit different. So, Ms. MB would help her students make interpretations of their parabolas by encouraging them to link their reasoning to real-world scenarios, as seen here:

T: What kind of shot were they trying to make?

S: Backboard shot.

T: A backboard shot, all right, ok.

S: Yeah.

T: Yes sir.

S: Mine is the orange one; it shows it more clear because he doesn't show from the bottom, just from the top.

T: Ok, everybody listen to K., please. Ladies and gentlemen, everybody listen to K.; he's describing his equation, the orange graph. Go ahead.

K: He shoots it from the top, not the bottom.

T: He shoots it from the top, up here over his head…

SS: Ohhh. / Yeah. (MB_04.02.08, 633-653)

Using different translations

As in Ms. BW's class, different translations including unidirectional and bidirectional were observed in Ms. MB's classroom. The students used unidirectional translation only once. At the beginning of the first period, while comparing two parabolas, Ms. MB promoted a translation from graphical to symbolic representations (Figure 4-12B). In this activity, by looking at parabolas drawn on the board, students identified the axes of symmetry, determined whether the leading coefficient of corresponding quadratic equations was positive or negative, decided whether the parabolas had a minimum or maximum vertex, and so on.

At the end of the first class, Ms. MB provided an opportunity for her students to bidirectionally translate between representations. They initially translated from symbolic to graphical and then back to the same symbolic representations. In this activity, the students created their own equations and entered them into calculators to draw its corresponding graph. Then the students were asked to submit their graphs through TI-Navigator and discuss whether these graphs met the given criteria (Figure 4-13).

Ms. MB also promoted the use of two representations on one screen; thus, a modification made to one of them would change another one simultaneously. At the end of the first period, Ms. MB used the Activity Center component of TI-Navigator so her students could see symbolic and graphical representations simultaneously on the board. She showed the synchronized equations with graphs, and the students made changes to their equations and resubmitted them to TI-Navigator (Figure 4-16).

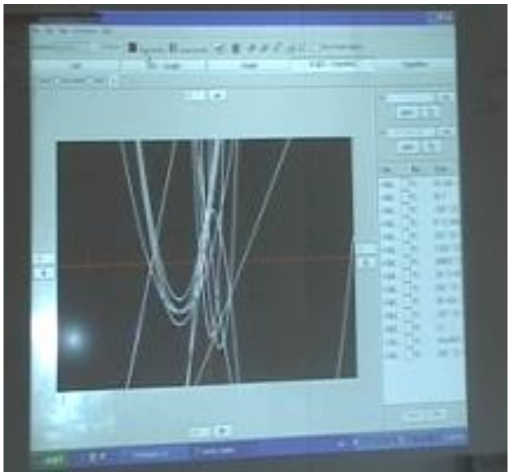


Figure 4-16. Using dynamic representations simultaneously (MB_04.01.08, 39:56).

Cycling translations were fostered by using representations simultaneously in Ms. MB's classroom. When performing the activity to find a curve for the basketball player's shot, a sequence of translations of the form Pictorial → Symbolic → Graphical ↔ Symbolic were followed. The students initially identified the location for the axes of symmetry by analyzing the picture on the screen (Figure 4-14). Then each group of students submitted an equation to draw a curve that would allow the man to shoot a basket. At the end, the entire class discussed the ways to improve the accuracy of the graphs, resulting in cyclical translations of the form Graphical ↔ Symbolic, which were bidirectionally constructed until the correct solution was obtained.

Distributed nature of translations

Co-construction of representations was also observed in Ms. MB's class when the students were asked to create quadratic equations using pre-determined criteria. The students initially determined their own equations and then entered them into their calculators to draw the corresponding graphs, and in doing so they translated from symbolic to graphical representations with the support of their calculator. When all graphs were submitted, the teacher projected them on the board, and the students discussed whether and why the graphs met the given criteria (Figure 4-13). From here, they translated from graphical to symbolic representations and discussed whether the criteria matched with the pre-drawn parabola. Similar to Ms. BW's class, co-construction of translation was observed when students in Ms. MB's class worked on a problem related to a basketball player's shot.

Promoting translation through teacher questioning

Ms. MB promoted translation by asking follow-up, open-ended, and hypothetical questions. She usually asked follow-up questions to advance the students' thinking if she realized that they were unsure of how to respond or if they did not reply promptly and clearly. At the beginning of the first class, the students were asked to compare the concave and convex parabolas drawn on the board by the teacher (Figure 4-12A). They were encouraged to translate from graphical to symbolic representations. For each parabola, they identified the axes of symmetry, determined whether coefficients "a" in quadratic equations were positive or negative, and decided whether the parabolas had a minimum or maximum vertex. If a student could not reach a correct answer, the teacher would rephrase the question. Furthermore, the teacher used constructive language to help students reach the correct solution. Specifically, she directed them from general to specific thinking by asking questions such as "What else did you notice?" (MB_04.01.08, 58) and "How do you know our value is negative?" (MB_04.01.08, 435). Additionally, she said, "Can you be a little more specific with that? I know what you're trying to say, but when you say…I'm not really sure because there's so much to talk about" (MB_04.01.08, 95-96). This may have encouraged students to keep thinking about the problem until they reached a solution and could express it clearly and concisely.

The students also translated between representations on their own. In another instance, when the students noticed similarities between the parabolas that the teacher did not, she replied, “Ok. Both the X- and Y-values of the first vertex are positive. Very good. I didn’t even notice that. That’s an excellent observation, and both the X- and Y-values of the second are negative. Good” (MB_04.01.08, 193-194). There were other moments when students advanced their thinking as the teacher asked questions. For instance, when discussing how to make a parabola wider, one student suggested a smaller value for coefficient “a,” and the teacher asked the follow-up question, “what if we went all the way down to -9? That’s smaller than ½, isn’t it?” (MB_04.01.08, 346). The teacher pointed this out so that students would clearly understand that the absolute value of the leading coefficient of a quadratic equation determines whether the parabola was wider or narrower.

T:	Do you guys remember Ms. ___ decided…I said I wanted a wide parabola and Ms. ___ said oh, then we should be between 0 and 1. Do you guys remember why that…how does that affect the opening of a parabola? Ashley, do you remember?
A:	The smaller numbers the wider…
T:	The smaller the number…
SS:	The wider it gets. / The wider…
T:	The wider the parabola.
S:	[inaudible]
T:	Um hum, but what if we went all the way down to -9, that’s smaller than ½, isn’t it?
SS:	No.
T:	Is -9 going to be wider than…
S:	The negative doesn’t count.

T: Ok, so it's the absolute value of those numbers, ok. (MB_04.01.08, 332-354)

The teacher also asked hypothetical questions that may have helped students translate between graphical and symbolic representations flexibly until they came up with the most accurate representation. She said "what could the group that has the red equation do to make it more accurate? Let's talk about the things that need to change" (MB_04.02.08, 407-409). In another activity, while discussing the location and definition for the axes of symmetry, Ms. MB promoted a translation from graphical to verbal representation. When the students expressed confusion, the teacher clarified what the axes of symmetry were; she also asked about a shot that was the most accurate, and the role of the leading coefficient of a quadratic equation (i.e., the "a" value). The following shows part of the transcript that highlights this conversation.

T: Ok, and what do you think the A value might be?

SS: 5. / The A value's going to be 5. / [inaudible]. / Yeah, like 10. / 10.5.

T: The A value? What's the A value again of the…

S: Isn't it the [inaudible]?

T: What's the A value again?

SS: Ah… / 14.

T: What does the A value determine? Let me ask you that.

S: How wide it's going to be.

T: How wide it's going to be, correct?

S: Yeah.

T: So what do you think the A value might be?

S: It would be a negative.

T: It's going to be negative. Why?

SS: [inaudible] / Because it’s going down.

T: Because it’s going down, good.

SS: Maybe like -1, or 2. / The number is the Y-value.

T: Ok, so what do you think? Do you think it’s going to be close to 1? Do you think it’s going to be between 1 and 0?

SS: Between 1 and 0. / I think the [inaudible] of the vertex [inaudible]. (MB_4.02.08, 209-244)

Even though the teacher created a productive discussion environment, sometimes the discussions lost direction. Some were not very well organized, seemed to be too digressive for students to follow, and (in some instances) Ms. MB would initiate a new conversation without finalizing a previous one. For example, when discussing the accuracy of the students’ graphs during the basketball shot activity, the teacher started to analyze another graph without clarifying the current one.

Summary of Ms. MB’s Classroom Environment

Ms. MB created an environment in which her students could feel a sense of community, even though some of them were reluctant to respond to this effort. She created a safe environment for the students to explore solutions to problems and encouraged them to resubmit their responses when necessary by projecting them anonymously. If other students realized one of their classmate’s mistakes, they would often comment comfortably and share thoughts with the class. Ms. MB was appreciative and verbally encouraged her students when they used terminology correctly. Most of the students participated and appeared engaged during many of Ms. MB’s discussions. She valued her students’ thoughts and promoted an interactive environment wherein her students could contribute. If a student left a question unanswered, the teacher asked other students for their assistance. Ms. MB’s class, however, was noisy at times, and the students sometimes could not hear the instructions well. In these instances, she reiterated explanations several times and frequently warned her disruptive students.

Since the students were excited to solve practical problems, Ms. MB helped them to make interpretations involving real-world scenarios. Unidirectional and bi-directional translations were observed in Ms. MB's classroom. They sometimes used symbolic and graphical representations simultaneously resulting in cycling translations. She ensured that her students did not just observe the translations but collaboratively engaged in translating between representations using the calculator as a tool. She would encourage students to participate in classroom discussions and to translate between representations by asking follow-up, open-ended, and hypothetical questions that helped the students continue thinking about problems. And, to help her students stay on the right path when solving problems, Ms. MB used constructive language and asked questions.

Ms. MA

Two classes were observed on two consecutive days for a total of four class periods in Ms. MA's Algebra I class. The following is a detailed description of the activities during these lessons to provide an overall understanding of the class sessions.

General Classroom Description

There were approximately 18 students in both classes but only a few students participated in discussions. All students had calculators and worked both individually and within groups. Although both classes were generally quiet, the students in period six were noisier than those in period four. In period six, there were a few students talking to each other about non-class related topics. However, in the sixth period, the students were more engaged with the lesson than those in the fourth period.

The activities conducted in both classes were for the most part similar. During the first class period, the students worked individually. The teacher sent five exercises to the students' calculators related to linear equations (4x-4=0; -2x+8=0; 3x-9=0; 1/2x-3=0; 2(x+1)-4=0) by using the Quick Poll component of TI-Navigator. The students solved these equations algebraically on their worksheets and submitted their answers via the TI-Navigator. Then, the teacher shared the class analysis with the students (Figure 4-17). Ms. MA generally identified the students who submitted the correct answers and displayed their worksheets using document camera.

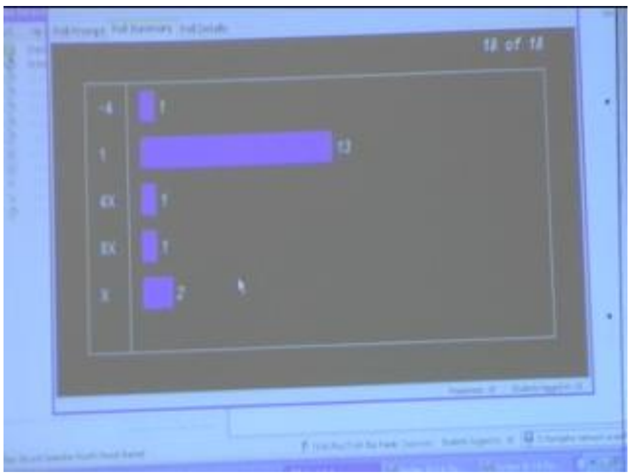


Figure 4-17. Class Analysis showing the number of students who provided each answer (MA_05.06.08_Per. 4, 12:41).

After solving these five linear equations algebraically, Ms. MA used the Activity Center component of TI-Navigator to solve them graphically as follows. The students followed three steps. They initially moved to any point on the line (Figure 4-18A), and then the teacher selected two points and instructed the students to locate these points on their worksheet and to draw a line. Finally, the students moved their points to the x-intercept by using their calculators (Figure 4-18B). This activity took approximately 23 minutes.

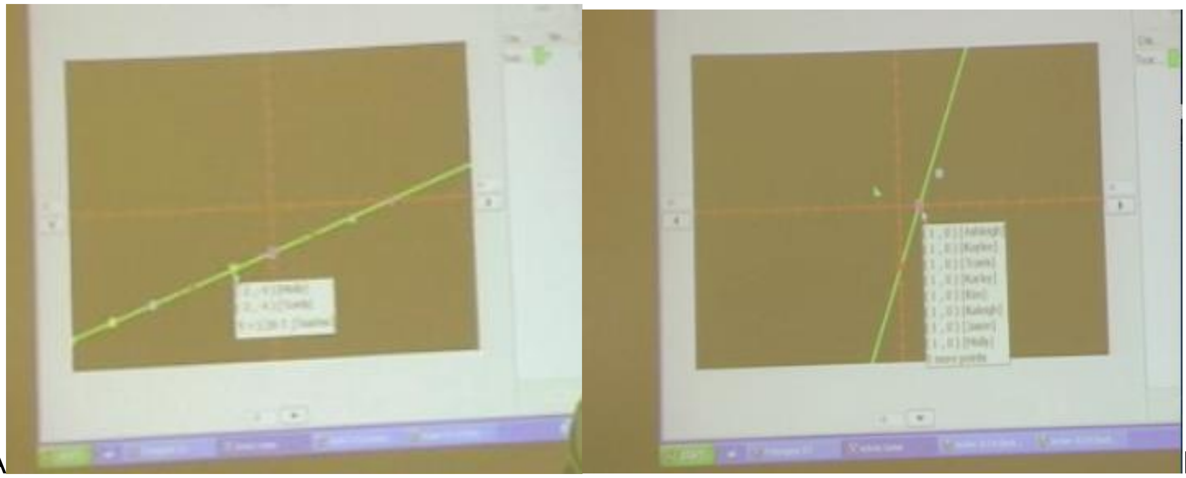
A  B

Figure 4-18. Solving the equations graphically. A) Moving to any point on the line and B) Moving the cursor to the x-intercept (MA_05.06.08_Per. 4, 28:51 & 23:45).

Next, the teacher sent five quadratic equations ($x^2 - 2x - 3 = 0$; $x^2 - 9 = 0$; $x^2 + 5x + 4 = 0$; $x^2 - 9x + 14 = 0$; $2x^2 + 2x - 4 = 0$) to the students' calculators using the Quick Poll component of TI-Navigator. The students found the roots of the equations algebraically by applying several techniques (e.g., guess and test, box ABC, difference of squares, and greatest common factor). The questions were presented as multiple choice questions (Figure 4-19).

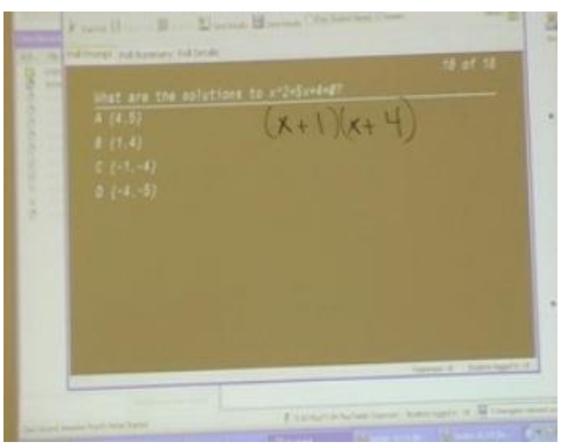


Figure 4-19. An example of a Quick Poll question (MA_05.06.08_Per. 4, 38:03).

Ms. MA solved the first problem, and the students solved the rest. After the students solved each problem, the teacher shared the Class Analysis with the class and solved these problems. Ms. MA then used the Activity Center component of TI-Navigator, and the students solved the quadratic equations graphically (Figure 4-20) by following the same three steps that they had in the previous set of equations. During the second step, however, the teacher selected five points from which the students would draw a parabola on their worksheet. The students spent 19 minutes on this activity.

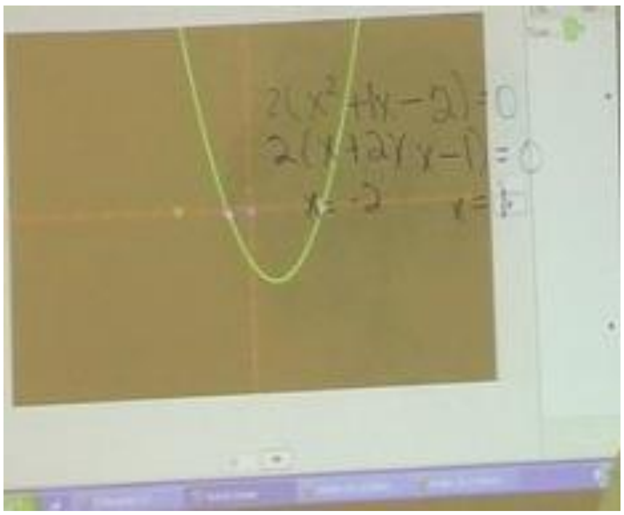
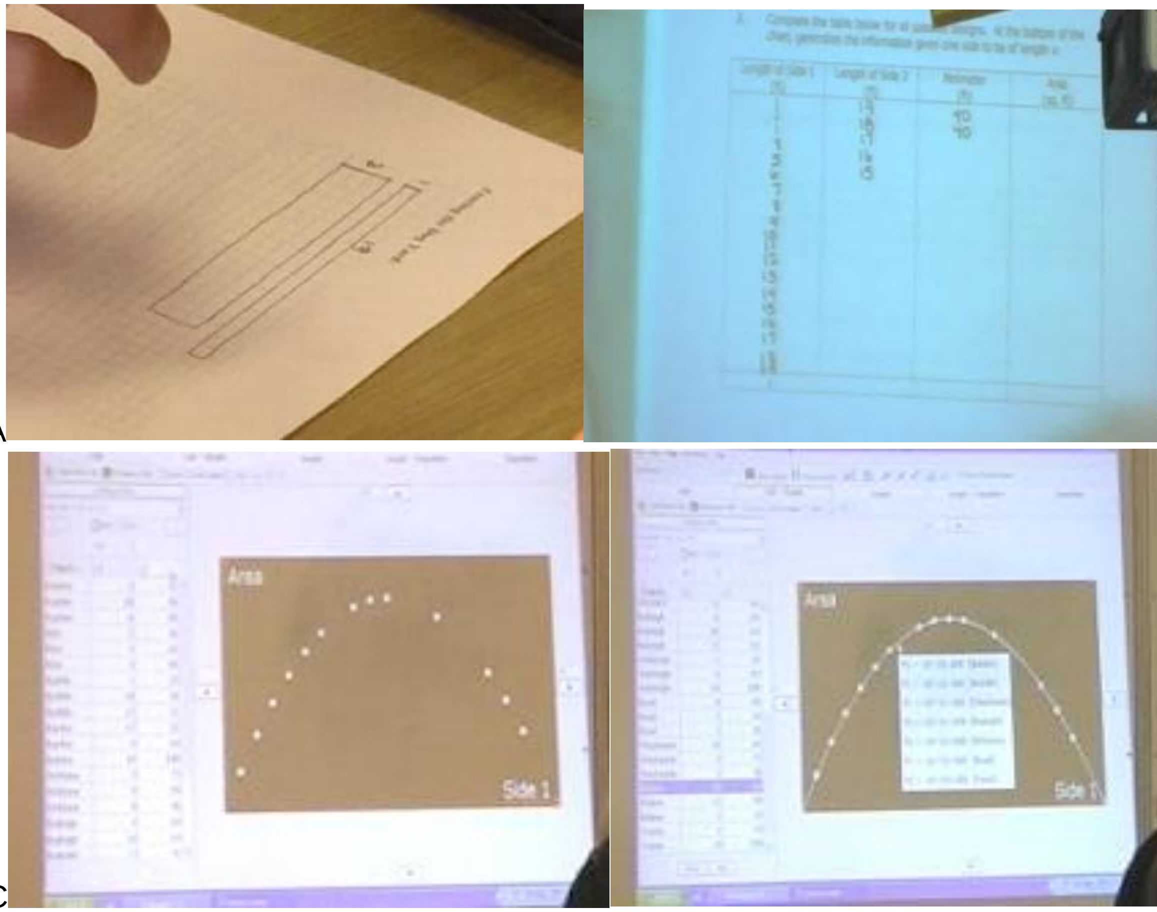


Figure 4-20. Solving the quadratic equations graphically (MA_05.06.08_Per. 6, 46:27).

Finally, Ms. MA demonstrated how to solve a quadratic equation ($x^2 + 4x - 12 = 0$) by projecting her worksheet on the board using a document camera. The students copied down the solution on their worksheets. The X-value for the vertex of the parabola and its corresponding Y-value were calculated initially. After the X-Y table was filled out, the students, with Ms. MA's help, plotted these points on the coordinate grid on their worksheet and joined them to draw a parabola. Finally, they estimated x-intercepts and entered these points into the equation using their graphing calculator. This activity took seven minutes.

During the second day of observations, the students also worked both individually and within groups. They initially graphed the remaining quadratic equations from the previous class. Even though the teacher had stated that the students would have to work on the rest of the questions by themselves, she solved them by herself. They worked on two quadratic equations ($x^2 + 4x + 4 = 0$ and $x^2 - 4x + 6 = 0$). They used the same solution strategies and performed similar analyses such as identifying vertex values, drawing a parabola, and estimating x-intercepts. This activity took 11 minutes.

Next, the students worked in groups of two, three, or four, and each student had a calculator. They explored a scenario wherein they were asked to design a dog run constructed of 40-feet of fencing for a dog named Milk Dud. The design was expected to be a rectangle where the length of each side was to be integer numbers. On the board, the teacher drew a 1 by 19 rectangle as an example of one possible design. Then the students drew all the other possible design dimensions onto a grid by hand (Figure 4-21A). In the table on their worksheets, they filled out each possible length for each of the sides as well as the perimeter and area (Figure 4-21B). The teacher then asked the students to select and send three chart values via the TI-Navigator. Using the values of one side and the area, Ms. MA showed the relationship between these two parameters on one graph with the help of the Activity Center component of TI-Navigator (Figure 4-21C). After showing the relationship between one side and the area, Ms. MA identified the formula (x*(20-x)) and wrote it at the bottom of the area column of the table (Figure 4-21B). The students entered this equation into their calculator to submit its graph to the TI-Navigator (Figure 4-21D).



A B
C D

Figure 4-21. Solving dog run problem. A) Listing possible designs on grid paper, B) Filling out each of the dimension possibilities in a table, C) Showing the relationship between area and one side, and D) Showing graphical representation of the area and one side (MA_05.07.08_Per. 6, 24:11, Per. 4, 33:26, 37:10, & 43:50).

After this activity, the teacher asked the class to determine which rectangular design resulted in the largest area, “So if I want to give Milk Dud the biggest play area, the most area, which Y-value are we going to use?” [MA_05.07.08_Per. 4, 820-821], and to make a generalization using these results. This activity took 10 minutes.

Ms. MA then introduced a word problem involving another real-world scenario and solved it by hand on the worksheet projected on the board. The problem’s scenario involved launching firework rockets from an 80-foot tower with an initial upward velocity of 64 feet per second. The goals of the problem were to determine the length of time it would take for the rocket to reach its highest point, the height of the rocket, and the length of time it would take for the rocket to hit the ground. The equation, $H_T = -16T^2 + 64T + 80$, was given in the problem to show the relationship between the height (H_T) and travel time (T) of the rocket. Ms. MA solved this quadratic equation by factoring, and the students contributed by performing arithmetic operations. This activity took seven minutes.

Finally, with the teacher's guidance, the students drew a graph for an equation that was given in intercept form ((x-6)(x-2)=0) as shown in Figure 4-22. The graph was drawn on the worksheet by determining X-intercepts and the X and Y-values of the vertex. This activity was conducted only in period four and took two minutes. As in the previous activity, the teacher did not use TI-Navigator.

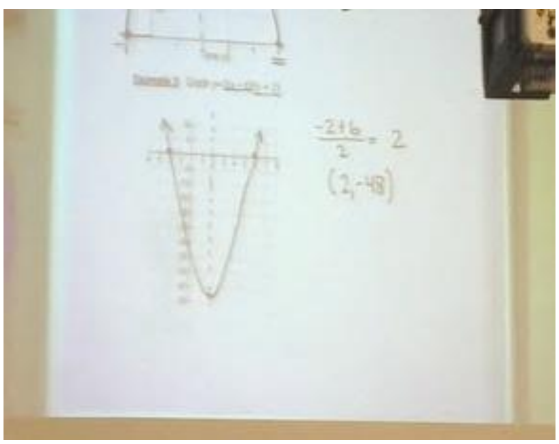


Figure 4-22. Drawing the graph of the equation in intercept form (MA_05.07.08_Per. 4, 56:13).

Psychological Environment and General Mathematics Teaching Approach

Three characteristics of this class emerged through the analysis: (a) showing care for students, (b) fostering politeness in class, and (c) monitoring participation.

Showing care for students

Ms. MA showed care for her students by spending some time talking about their lives before or after class. In one instance, she asked one of her students how his knee was (it appeared that he had been injured). She also inquired whether another student had dental work done and if he was healthy. At the end of the second day of period six, she told one of her students who found a summer job “Have fun, nice outside work in the summer” (MA_05.07.08_Per. 6, 1221). Additionally, she valued what the students thought about the difficulty of the exercises. After solving quadratic equations at the beginning of the second day of period four the students learned how to solve quadratic equations graphically. The teacher asked, “Do you think we’re doing a lot of work here?” (MA_05.07.08_Per. 4, 148) and told the students that she would show them another method that simplified the process.

Fostering politeness in class

Throughout her teaching, Ms. MA cultivated respect by using kind words. She was also a calm and mannerly person. She started her classes by greeting her students and thanking them many times during class when they assisted her with tasks such as distributing materials to classmates or contributed their input. In addition, she honored a researcher who was visiting the class saying, “It really is our great pleasure to have him in our classroom. We don’t normally get that caliber of scientist here in our classes” (MA_05.06.08_Per. 6, 55-56).

The teacher encouraged her students verbally often using the words “good” and “nice” to acknowledge their successes. During the first day of period four, for example, when the students were solving a linear equation algebraically, the teacher asked them how to undo the subtraction, and they responded by saying, “So you would add it?”

(MA_05.06.08_Per. 4, 157). Ms. MA encouraged them by stating, “That’s a good first step” (MA_05.06.08_Per. 4, 159). In another instance while working on the same equation, the teacher was walking around the class and showed appreciation by saying: “Those look good. Very nice. So remember you always undo your additions and subtractions first” (MA_05.06.08_Per. 4, 185-186). When all students arrived at the correct answer, she became excited and said, “Look at you guys go, very nice…and this is what I love to see. All of you were able to get the answer” (MA_05.06.08_Per. 6, 262-266). Additionally, some students corrected their parabolas and the teacher showed her appreciation by saying “good job on fixing that” (MA_05.07.08_Per. 6, 951).

Some students turned the teacher’s words into sarcasm and were a bit sarcastic with one another. Their sarcasm was, at times, close to misbehavior. A few times when students answered questions correctly, other students could be heard saying sarcastically, “Good job, Damon” (MA_05.07.08_Per. 6, 571), “You’re amazing, Sabrina” (MA_05.07.08_Per. 6, 1091), and “What a brain” (MA_05.07.08_Per. 6, 1210).

Monitoring participation

Ms. MA monitored whether the students solved problems throughout both classes. She used the Class Analysis component of TI-Navigator to keep track her students’ progress. She determined the level of difficulty of each question by looking at their submission rates. When solving a problem on the first day during fourth period, the students’ submission rate was 11 out of 18, and she said, “must be a little harder problem” (MA_05.06.08_Per. 4, 247-248). Another time, after the students submitted their points, she realized that there was a gap in the parabola because some of the students had not yet made their submissions. She said, “I think we’re still waiting for a big group of people here" (MA_05.07.08_Per. 6, 776) and asked these students to send their answers through TI-Navigator.

The teacher generally waited for all students to submit their answers to the TI-Navigator before going over the exercises. In one instance, only 12 students submitted their answers within the allotted time. She then gave extra time and waited for about two more minutes for the other students to solve the problem and submit their answers. She often checked if the time provided was sufficient and once asked, "Did I give you enough time to get the line drawn on that graph with those points?" (MA_05.06.08_Per. 4, 320-321). Although some students needed more time, others were usually further ahead in solving the problems. These faster learners waited until all their classmates were finished.

The teacher walked around the class to check whether the students performed the activities. She sometimes made sure that students were on the right track and informed them about their progress. At the end of the first activity, she praised the students' success: "It took us a little bit to get started but you did great in the end" (MA_05.06.08_Per. 6, 393-394).

Representational Practices

The representational practices in Ms. MA's classroom were characterized by the following four aspects: (a) decontextualized representations, (b) using unidirectional translations, (c) observing translations, and (d) a teacher-centered approach.

Decontextualized representations

The problems Ms. MA asked the students to solve were predominantly decontextualized equations, and the solutions were not interpreted within real-world situations, which sometimes resulted in unrealistic answers. For example, in the real-world problem that asked the students to design the dog run, the students initially drew all possible design dimensions on a grid by hand and then identified the dimensions of the most practical running area for the dog. Although she chose a real-world context, Ms. MA's and most of the students' answers did not reflect that they were thinking of the problem in terms of the actual scenario. Ms. MA asked a question that led to an unrealistic answer and most of the students did not realize that the answer did not fit the real-world context. Yet one of the students interpreted the representations by linking it to a real-word scenario without any guiding instruction.

T: So what would be the dimensions that would give Milk Dud the largest running area?

SS: 1 by 19. / 1 by 19…

T: 1 by…

SS: 19. / She's just going on.

T: It may not be a big area but it's a big length so that would be the best.

S: There's no way you and your dog can fit into a 1 by 19 thing.

T: I wouldn't be running with my dog. I wouldn't go with her in there.

SS: I would. / I run with my dog. (MA_05.07.08_Per. 6, 1022-1036)

The student saw that the problem involved an actual location, which practically speaking, needs to be more than 1 foot wide. This shows that he was thinking of the problem in real, rather than hypothetical, terms. The teacher and his classmates did not, however, realize this unrealistic solution.

Additionally, in the rocket problem Ms. MA asked the students to determine the length of time it would take for the rocket to hit the ground. The rocket was, however, broken into pieces after the fire works explosion. Thus, it would never make to the ground.

Using unidirectional translations

Two and three unidirectional translations were observed in Ms. MA's class. In the second period, her students made translations in the form of Symbolic1 → Graphical → Symbolic2 when solving the real-world problem related to launching fireworks. The teacher initially solved the quadratic equation by factoring and then drew its graph on the worksheet while projecting it on the board. By looking at this graph, students identified the answers to such questions as, "How long would it take for the rocket to reach its highest point?" Similar to this activity, the students observed translations of the form Symbolic1 → Graphical → Symbolic2 when the teacher solved the quadratic equation shown in Figure 4-22.

In addition to translations between symbolic and graphical representations, Ms. MA and her students made the following translation Symbolic → Tabular → Graphical. The students entered the equation into their calculator and identified the X-value for the vertex and its corresponding Y-value. They completed the table with the coordinates value that were obtained by adding and subtracting one from the vertex. Since the parabola was symmetric, the values for Y were the same because corresponding X's had same distance (e.g. -2, +2) from vertex. After the table was completed, the students determined these points on the coordinate grid on their worksheet and joined them to draw a parabola.

In the second class, Ms. MA used two unidirectional translations twice within the dog run activity as follows: Pictorial → Tabular → Graphical and Tabular → Symbolic → Graphical. They drew rectangles (Figure 4-21A), determined the possible sides, perimeter, and area, and wrote them in a tabular format (Figure 4-21B). Then the teacher asked them to select and send three chart values via the TI-Navigator. Using the values of one side and the area, the teacher showed the relationship between these two parameters on a graph in the Activity Center component of the TI-Navigator (Figure 4-21C). Then, the students came up with a formula by looking at the pattern in the tabular format (Figure 4-21B), which was then entered into the calculator to represent the graph (Figure 4-21D).

Also, three unidirectional translations were only observed in the first class when the students constructed translations of the form Symbolic1 → Graphical → Symbolic2 → Graphical. The teacher initially entered a linear equation in the TI-Navigator to draw its graph (Figure 4-18A). Then the students moved their cursors to any point on the line, and Ms. MA selected two points for students to draw a line on their worksheet by hand. The same type of translation was performed when solving quadratic equations as well.

While some unidirectional translations were observed, the class's focus was not typically on the dynamic relationships between the representations. These representations were not linked with each other; thus, a modification made on one of them did not change another one simultaneously. That is, the class engaged with the representations independent of one another, which resulted in no cycling translations being observed. In the first class, the students initially solved linear equations algebraically by hand and then solved them graphically. The solutions obtained by these two representations were obviously the same; however, while the teacher pointed this out and called students' attention to it, the students were not provided the opportunity to observe them as dynamically linked because of the way the problems were presented. Ms. MA tried to promote relational learning between solutions and X-intercepts and emphasized the link between symbolic and graphical solution methods. This was done, however, by analyzing different representations on separate screens on the calculator. Thus, the students engaged with the representations independently without translation for both linear (Figure 4-18A and Figure 4-18B) and quadratic functions (Figure 4-19 and 4-20).

Observing translations

Students in Ms. MA's class passively observed translations in the last two activities of the second period. One of these activities was taken from the real-world word problem involving launching firework rockets. The teacher asked several questions related to the given quadratic equation such as the highest point that the rocket could reach. She, however, did not let the students practice this problem. While she was solving the quadratic equation by hand, she projected her worksheet on the screen. So, while the students observed a translation from symbolic to graphical representation, they did not participate in its construction with the teacher. In another activity, Ms. MA drew a graph of an equation to determine the x-intercepts and the X- and Y-values of the vertex. As in the previous activity, the teacher translated from symbolic to graphical representations, and the students observed but did not take part in the activity (Figure 4-22). Ms. MA's class observed other types of translations as well. In the first class period, while Ms. MA was identifying the points of a quadratic equation ($x^2 + 4x - 12 = 0$), the students observed a translation from symbolic to tabular representations performed by the teacher. After the table was filled out, the teacher drew the graph, and similarly the students observed a translation from tabular to graphical representations.

To a lesser degree, students were provided several opportunities to jointly construct translations with the teacher and the calculator. After the students solved linear equations algebraically during Ms. MA's first class, they were asked to solve the same equations graphically. The teacher and the students jointly translated from symbolic to graphical, graphical to symbolic, and symbolic to graphical representations while using the calculator as a tool. When the class found the solution to the linear equation as described earlier, the teacher initially drew the graph, and using Activity Center, the students moved their points to the graph on the screen, and Ms. MA selected some of those points whereby the students could draw the same graph. In this way, the students translated from symbolic to graphical representations (Figure 4-18A). This same solution approach was used to solve quadratic equations (Figure 4-20).

During the second class, other translations between pictorial, tabular, graphical, as well as tabular, symbolic, and graphical representations constructed by multiple parties were observed. While performing the activity involving the design of the dog run, the students were asked to identify the relationship between the sides and area of a rectangle. The first translation observed was from pictorial to tabular representations (Figure 4-21A and 4-21B). The teacher initially showed one possible design, and the students then drew the remaining design options by hand. Using the possible designs drawn on the grid sheet, the students determined the length of each side, the perimeter, and area and then construed a table of values. Following this step, the students translated from tabular to graphical representations by using their calculators (Figures 4-21B and 4-21C). They then sent three pairs of values via TI-Navigator, and the teacher displayed the relationship between the side and the area on one graph with the Activity Center component.

By looking at the table that Ms. MA initiated, the students came up with a formula for the relationship between one side and the area (Figure 4-21B). The teacher and students made this translation from tabular to symbolic representations jointly. Finally, the students entered the equation into the graphing calculator. This time, the students translated from symbolic to graphical representations using their calculators. Even though this activity was rich in terms of the number of translations, Ms. MA primarily facilitated the exercise and did not adequately let the students explore the practices of translating and using representations for themselves.

A teacher-centered approach

Ms. MA generally solved problems and explained them with limited interaction from her students. While solving quadratic equations in the first period, the students were supposed to discuss the answers, but she provided the answers by writing them on the worksheet and projecting them on the screen. In another example, Ms. MA asked the students to solve the rest of the questions, but she primarily kept solving them by herself. The discussions would typically end with her explanations. For instance, when solving a quadratic equation graphically at the beginning of the second class, the teacher asked, “So, I’m saying when are these numbers, when [are] these Y’s going to equal to zero?” and she answered right away, “Y is always zero on the x axis” (MA_05.07.08_Per. 4, 25). Her predominately lecture-based style is easily visible when referring to the transcript; each page of the transcript is mainly composed of the teacher’s lengthy explanations (Appendix E).

Moreover, she rarely asked questions and encouraged students to explore. In one of these rare instances, she asked students to discover, saying, “Just brainstorm…tell me anything you know about this graph before we actually start to graph" (MA_05.07.08_Per. 4, 846-847). The students were also given only a few opportunities to contribute. However, even these contributions were limited to arithmetic operations. Additionally, when Ms. MA reviewed linear equations in the first class period, she explained the solution without first asking for the students' input. While solving one of these equations, she translated between from symbolic to graphical representations and provided the solution by saying: "So our very first equation was 4X minus 4 and because X is to the 1st that makes it a linear equation and its graph is a line" (MA_05.06.08_Per. 4, 306-307). During the first and second day of period six, she involved the students a little more when solving a problem but generally asked them about their steps for calculations.

T: Show me the [inaudible] steps. Show me the algebra. Ok, looks like a lot of people are getting one for this answer so the problem was 2 times the quantity X plus 1 minus 4 is equal to zero. Ok, what did you do as a first step?

S: Got rid of the parentheses.

T: Ok, so she got rid of her parentheses and she distributed. Did anyone do something different as a first step? Ok then let's go with that. Let's multiply and get rid of the parentheses. So two times X is 2X. Two times 1 is 2… Ok, I probably would combine like terms then. And a 2 and a neg. 4 would be a neg. 2. Ok, so then Carly, what would you do as a next step?

C: Add 2.

T: Add 2, so that would give me 2X equals 2 so to finish it up what would our last step be?

SS: Divide by 2.

T: Divide by 2 and we get 1, so 1 is the correct answer. (MA_05.06.08_Per. 6, 278-295)

She also asked whether anyone used a different method for the first step; however, she did not wait and explained that there are different ways that simplify the process of solving this problem right away. Ms. MA mentioned that the intercept form was the easiest way and started solving the problem with this method. Once done, she asked the students, “Wasn’t that less work than the tables?” and the students said “No. / No” (MA_05.07.08_Per. 6, 1190-1191). The students did not find this method easier.

The students would generally follow the solution steps shown by the teacher, which seemed to make them dependent upon Ms. MA for methods and solutions. She asked the students why it was not necessary to calculate the values of two Y points on a parabola whose X points were located at the same distance from the axis of symmetry. But before the students could respond, she provided the answers, not allowing them to identify the solution method. Ms. MA’s teaching method did not seem to allow students to become comfortable with solving difficult problems.

Ms. MA also asked the same questions across class periods to initiate discussion. For example, in the same class period, she asked “What’s the formula that helps me find the X coordinate of my vertex?” (MA_05.07.08_Per. 4, 26) and “How do we find the X-value of the vertex?” (MA_05.07.08_Per. 4, 100-101). Furthermore, when providing explanations, she used the same words and repeated the same information. During these moments, the students tended to begin talking about unrelated topics. In addition, while Ms. MA explained new concepts about quadratic equations, she asked questions calling for short answers. The following excerpt reflects this notion.

T: Ok, so let’s plot those numbers. Negative 5, negative 7; negative 4, negative 12; negative 3, negative 15; negative 2, negative 16; zero, negative 12 and 1, negative 7. Does it look like a parabola?

S: Yes.

T: There’s a nice word. A is 1. Does our parabola open up?

S: Yeah.

T: Does it appear to cross the Y-axis at negative 12? So it fits in with everything that we've talked about this chapter. (MA_05.06.08_Per. 4, 686-696)

When the students had a difficult time reaching a solution, Ms. MA usually corrected them directly instead of allowing them to think through the process of the solution. In the second day of period four, when the students were doing the dog run problem and sent their values for one side and the area through the TI-Navigator, the teacher identified the points that were off the parabola (Figure 4-23) and corrected them herself. The following excerpt shows the dialogue between Ms. MA and her students exemplifying this problem.

T: Do you think our graph is perfectly parabolic?

SS: No.

T: Do you see that point right there? It looks a little bit off. Let's look.

S: Blank.

T: Ok, blank.

S: Ohhh.

T: When you had a side of 16 what was the other side?

SS: 4.

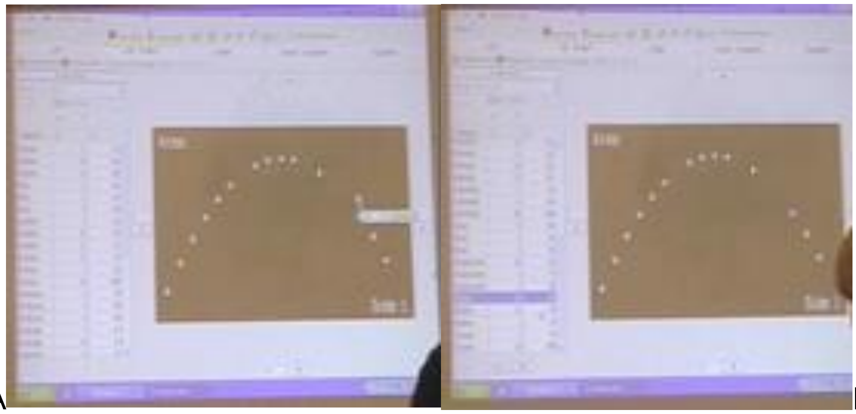
T: Ok, so if you took 16 times 4 would you get 75?

S: No.

T: What would you get?

S: Um, it would be 64.

T: 64, so I'm going to correct that if you don't mind. Oh! That fixed it. (MA_05.07.08_Per. 4, 732-757)



A B

Figure 4-23. Correcting students' work. A) The point that was off the parabola, and B) Corrected point (MA_05.07.08_Per. 4, 42:17 & 42:51).

Ms. MA rarely created a setting where the students could evaluate themselves and discuss their mistakes afterward. She initiated conversation to understand which parts of certain problems students needed more clarification. She sometimes encouraged the students to identify their own mistakes. She once said, “If you wanted to change your answer to one then you tell me how you would get to that answer” (MA_05.06.08_Per. 6, 360-361). She received a number of responses to such statements. One student said that he had made a mistake accidentally. Another student added, “I typed in 4 divided by 4” (MA_05.06.08_Per. 6, 156). Additionally, the teacher asked one of the students why he found an incorrect answer by saying, “Would anyone be willing to say that they got the 1/4th, the .25?” (MA_05.06.08_Per. 6, 243-244) “What did you do?” (MA_05.06.08_Per. 6, 248). He responded “I did negative 2X divided by 8” (MA_05.06.08_Per. 6, 250). Ms. MA would often point out students' mistakes right away, however, instead of letting them figure out the issues.

In the first day of period four, while the students were solving a quadratic equation, Ms. MA provided several hints. She wrote two blank parentheses on the board to direct the students to use the guess-and-test method without considering other options.

T: Ok. As I walk around I see a good start to this problem. I see this, which is good, because it's telling me sort of what technique you're thinking about. What technique are you using to solve this one?

S: Guess and test.

T: Guess and test. Ok. (MA_05.06.08_Per. 4, 481-487)

In the second day of period four, Ms. MA also provided hints while performing the activity related to launching fireworks. She explained that the vertex would be located between the x-intercepts and then asked the students to find the vertex with given x-intercepts by saying, "Now, your vertex is going to be right between those intercepts, so what number do you find right between negative 1 and 5?" (MA_05.07.08_Per. 4, 930-932).

Ms. MA asked the students to make obvious generalizations during class activities. In the first class, they solved linear equations first algebraically and then graphically. Although it was clear that the findings of these different methods were the same, the teacher nevertheless asked, "What was true about the solutions from the top half and the X intercepts from the bottom half?" (MA_05.06.08_Per. 4, 394-395). In the second day of period four, when Ms. MA conducted the dog run activity, she asked if there was a pattern observed between the sides and the perimeter of the rectangular. As seen in Figure 4-21B, the value of one side showed a downward trend when the value of the other side increased. The students replied, "It started going down" (MA_05.07.08_Per. 4, 588). The following excerpt details another example of such questions:

T: On this chart I asked you to generalize. When the first side was X, how was it getting the length of side two? What do you do to one to make 19? What do you do to 2 to make 18? What can you do to 3 to make 17? Is there any pattern between those numbers? Do 1 and 19 do anything the same as 2 and 18, 3 and 17?

S: They add to 20?

T: Oh, they add to 20, so I bet 20 is part of the relationship, so think about 20 and think about 1. What would you do with those numbers to make a 19?

S: Subtract.

T: Oh, subtract, and starting with what number, the 1 or the 20?

S: 20.

T: So if you have a side length of X in this column, it's just found by taking 20, because that seems to be what they add up to be, and subtracting that X. Ok. If this is 40 and this is 40, and this is 40 and this is 40, what's the general pattern for our perimeter answer?

S: It stays the same.

T: Again, that's constantly 40 because you have 40 ft. of fencing. (MA_05.07.08_Per. 4, 688-710)

Summary of Ms. MA's Classroom Environment

Ms. MA exhibited a caring attitude toward her students. She was interested in knowing about her students' lives outside class and was aware of the challenges they faced in their personal lives. She used kind language and verbally encouraged her students when they arrived at a correct solution. Her students would reciprocate with appreciative language, though a few times somewhat sarcastically. Relatedly, they sometimes misbehaved. Ms. MA routinely monitored the students' participation and made sure the students were on the right track. In addition, she usually did not start working on a new problem until all students were finished with the current one. She would walk around the class to check whether her students performed the activity as expected and often informed them about their progress.

Ms. MA sometimes chose real-world problems and most of the students did not interpret the solutions realistically. While two and three unidirectional translations were constructed, they used different representations independently rather than simultaneously. The students either observed or jointly constructed these translations with the teacher and calculator. Some activities were rich in terms of different translations, but Ms. MA primarily completed the exercises and did not adequately let the students explore the practices of translating and using representations for themselves.

The students in Ms. MA's class didn't voice their questions or hold open discussion. She generally solved the problems herself and then provided explanations. The majority of the teacher's questions were related to arithmetic operations. In both of her classes, Ms. MA used similar questions and explanations. Instead of letting students work out the problems by themselves, Ms. MA usually corrected students' answers directly. Additionally, she provided several hints to direct the students to a specific strategy or a correct solution. Finally, Ms. MA asked students to make obvious generalizations regarding solutions during class activities.

Ms. JR

One class was observed on two consecutive periods in Ms. JR's Algebra I class. A detailed description of the activities during these lessons is discussed in this section to provide an overall understanding of the sessions.

General Classroom Description

There were approximately 20 students in the class, but only a few actively participated in discussions. Some students were engaged in activities that did not relate to the lesson. For example, one student was observed reading a book instead of participating. When the teacher used the Class Analysis component of TI-Navigator, the participation increased; however, the class remained quiet during discussions. The students sat in groups of two, but they worked individually and had their own calculators.

During the first day of observations, the teacher reviewed three methods to solve quadratic equations: factoring, square roots, and the quadratic formula. The teacher used the Quick Poll component of TI-Navigator and initially asked the students which method they would use to solve each of six exercises [$x^2 - 6x + 9 = 0$; $x^2 - 49 = 0$; $(x + 4)^2 - 9 = 0$; $3x^2 + 12x = 0$; $x^2 - 2x - 2 = 0$; $3x^2 + 2x + 4 = 0$] without actually solving it. One of them is shown in Figure 4-24.

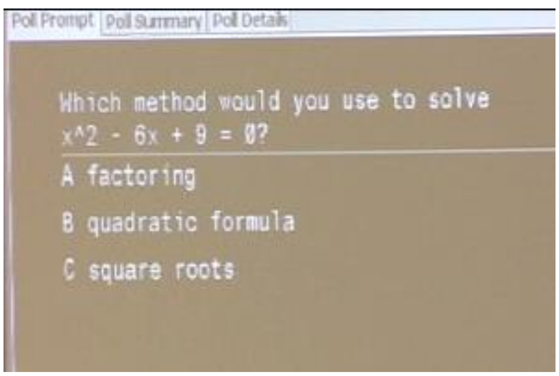


Figure 4-24. An example of a Quick Poll question (JR_05.28.08, 09:17).

The teacher directed the students to find out which method they used in their previous homework solving similar quadratic equations. After the students submitted their answers to the TI-Navigator, Ms. JR showed the students' preferred methods through the Class Analysis component of TI-Navigator (Figure 4-25).

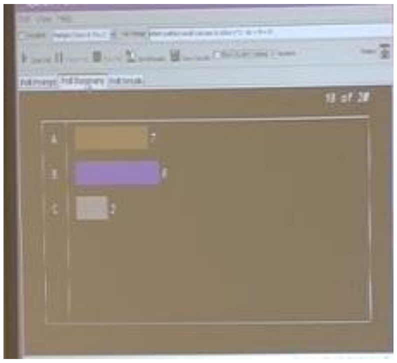


Figure 4-25. An example of Class Analysis (JR_05.28.08, 10:00).

After showing the class analysis, the teacher solved the exercises on the board by hand. She started to solve the equation by using the method most chosen by her students and only interacted with the students when the arithmetic operations were required. When there was a need to perform these operations such as to determine the discriminant, the students used their calculators.

During the second day, the teacher dedicated the first 15 minutes to displaying the answers from the previous assignment on the smart board (Figure 4-26) using five slides [$y = x^2 + 4x - 5$; $y = x^2 - 6x + 5$; $y = x^2 + x + 2$; $y = x^2 - 4$; $y = -2x^2 + 2x + 4$]. The exercises included the standard form of quadratic functions and their graphs. While she was sharing the answers with the students to check their work, she explained each exercise.

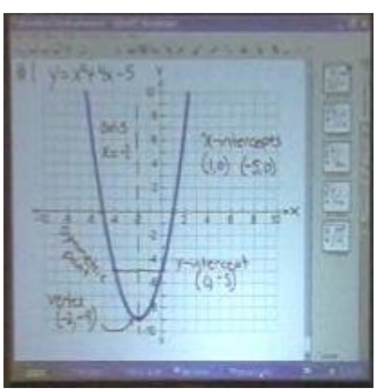


Figure 4-26. An example of reviewing the previous assignment (JR_06.03.08, 12:14).

Ms. JR used the Activity Center component of TI-Navigator to teach students how to move their parabolas on the coordinate plane. She created the function $y = x^2$ on the screen and asked students to move the location (i.e., up, down, right, left, and vertex in third quadrant) and to change the shape of the parabola (i.e., narrower or wider, and upside down). Figure 4-27 shows an example of the students trying to move the teacher's parabola up. The vertex form was explained, as were the coefficients (e.g., "a" in front of x^2) affecting the parabola. This activity took 30 minutes.

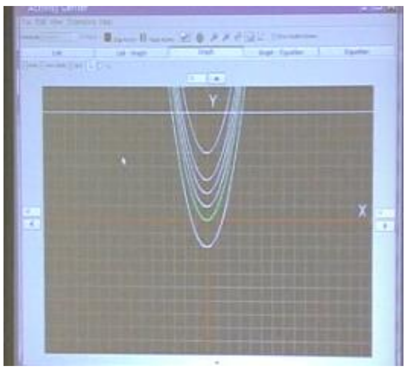


Figure 4-27. Changing the location of a parabola (JR_06.03.08, 24:26).

Finally, Ms. JR went over the first question of the homework. The goal was to draw the parabola for $y = (x-3)^2 - 4$ by determining the vertex, x-intercepts, y-intercept, and axis of symmetry. The students used graphing calculators (Figure 4-28).

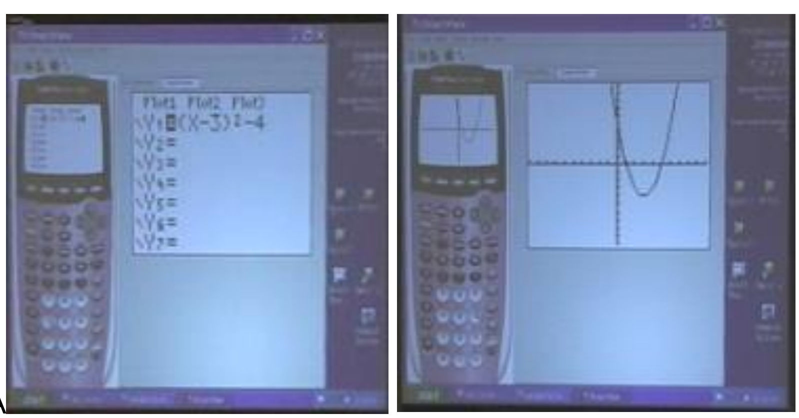
A  B

Figure 4-28. Graphing the quadratic equation using the graphing calculator. A) Equation and B) Graph of the equation (JR_06.03.08, 58:30 &1:00:01).

The teacher initially provided some explanation. She stated that the vertex could be found by looking at the patterns for Y-values, and that the vertex was located in between decreasing and increasing values of Y (Figure 4-29).

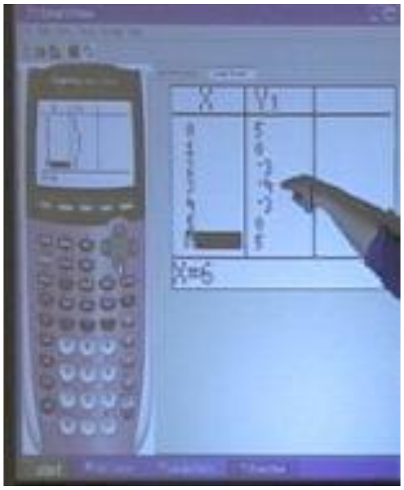


Figure 4-29. Finding the vertex by looking at the patterns for Y-values (JR_06.03.08, 1:00:26).

Additionally, she explained that x-intercepts could be identified by looking at the X-values where Y-values were zero using the calculator. After solving the first homework problem graphically, Ms. JR solved the quadratic equations algebraically to verify the answers. She again asked the students to calculate arithmetic operations. During the last 30 minutes of class, the students started working on the homework questions individually.

Psychological Environment and General Mathematics Teaching Approach

Two characteristics of this class emerged through the analysis: (a) monitoring participation and (b) inefficient use of class time.

Monitoring participation

Ms. JR kept track of her students' presence in class and demonstrating awareness of students who missed the previous class. During the second period, she quickly went over the topics of the previous day with a student who had been absent before covering the next topic. In addition, she used the TI-Navigator to display students' names on the board and checked whether they were logged into the system. When she did not see one of the students' names, she asked, "Did you get this problem here?" (JR_06.03.08, 986). She usually tried to get students' confirmations about their understanding by saying "OK" and provided clarification if needed. In another instance, while the students were working on a problem, she walked around and checked if there was any confusion. Ms. JR inquired when she realized that one of the students erased his work. She wanted to know which part of the problem the student had made mistake on.

Although Ms. JR tried to promote students' input in the class, participation was low, and the same few students contributed to discussions. Although the teacher warned students several times to take notes, some students appeared unmotivated and answered questions with a weak voice. For the most part, the students seemed uninterested and did not listen to Ms. JR's instruction. Instead, many of them were engaged in activities that did not relate to the class content. During the first class, for example, one of the students was listening to music with earphones while others were working on the homework questions. Another student was reading a book instead of working with his calculator or taking notes. During the second period, this same student was again disengaged from the lesson. Another student also did not have the course materials in front of her and was looking into a personal mirror at her desk (JR_06.03.08, 15:45).

Inefficient use of class time

In both periods, the teacher did not use class time efficiently. During the first class, Ms. JR spent the first 10 minutes on instructions for an activity and half the time (40 minutes) on a survey and homework problems at the end. In addition, during the first period, the students spent a good amount of time calculating arithmetic operations. For example, when solving an equation using the quadratic formula, Ms. JR explained the solution and the students helped her with their calculators when needed. In the final step, the teacher asked students to calculate an operation and provide the answer by using their calculators. The students, however, came up with approximately 10 different answers and did not come to an agreement on the right one. The teacher did not have the answer ready, so she could not guide them toward the correct answer.

During the second period, Ms. JR spent about 12 minutes at the beginning on the previous homework, 30 minutes on the next day's homework, and announcements related to senior activities. Thus, the class time was not used efficiently as more than half of the time was dedicated to reviewing old concepts and working on homework.

Representational Practices

The representational practices in Ms. JR's classroom were characterized by the following five aspects: (a) decontextualized representations, (b) using representations explicitly for specific purposes, (c) limited translations, (d) observing translations, (e) a teacher-centered approach.

Decontextualized representations

The problems Ms. JR provided were typically decontextualized, and her students did not interpret representations based on their meaning. Instead, students commented on superficial aspects such as their shape. When the students changed the location and shape of the parabola in the second period, they did not interpret any representations while making connections between the graphical and real-world representations. Instead, they only used metaphors to describe the graphs' shape. For example, the students would liken the shape of the parabolas to a tacos or fountain, as seen in Figures 4-30A and B, respectively.

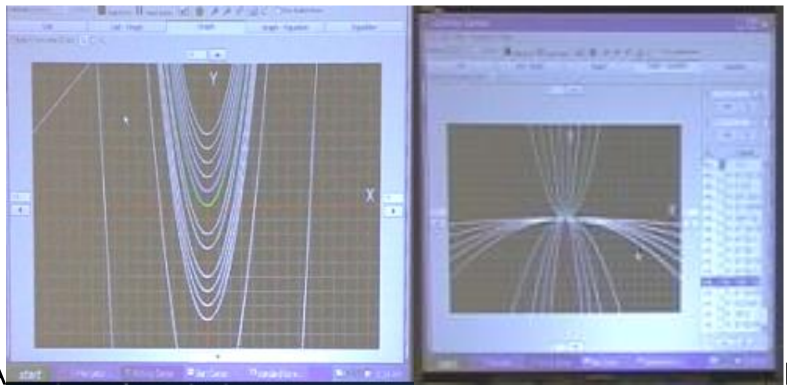
A  B

Figure 4-30. Changing the shape and location of a parabola. A) Taco comparison and B) Fountain comparison (JR_06.03.08, 28:33 & 52:57).

One example of these conversations is as follows:

SS: Oh, that one's mine. / It looks like a…

T: Looks like a…?

S: A taco.

T: A taco?

SS: It does look like a taco. / [inaudible]

T: Like the solar system?

S: Yeah.

T: A taco solar system?

SS: How does it look like a taco? / A taco… (JR 06.03.08, 318-334)

Using representations explicitly for specific purposes

Ms. JR explicitly encouraged the use of specific representations for specific purposes. In her class, there was a poster stating, “Do algebraically, support graphically; do graphically, confirm algebraically” (JR_06.03.08, 3:01), which reflects the different purposes of representations. When solving the homework, a tabular representation was used to find the vertex, x-intercepts, y-intercept, and axis of symmetry for a given equation. The students identified the vertex by looking at the pattern within the Y-values and the intercepts by looking at the values where X=0 or Y=0 (Figure 4-31).

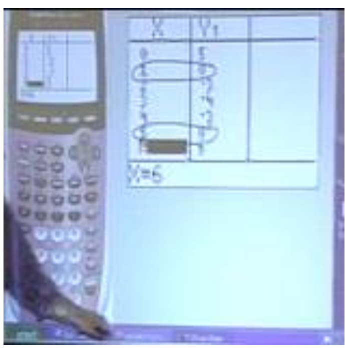


Figure 4-31. Using tabular representation to find X-intercepts (JR_06.03.08, 01:00:59).

At the end of this activity, the students used symbolic representation to verify their answers, as seen here:

T: Oh, so here in this box you’re going to show the work for your X intercepts and in this box you’ll show your work for the Y intercepts. Because the idea was what I wanted you to…to use the tables on the calculator to find the graph. So when you found the answers from the table and the graph first, then you used algebra to verify those answers. (JR 06.03.08, 1208-1211)

Additionally, Ms. JR let the students know about the use of a graphical representation to support a tabular representation. She said, “Looking at the graph helps you decide where to [look into the table]” (JR 06.03.08, 1226-1227). That is, a graphical representation provides an overall idea about what the value is, and students can look at a tabular representation to find the exact numbers.

Limited translations

The students were observed only once performed activities including unidirectional and cycling translations. In the second class, when solving the homework, four unidirectional translations of the form Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical were constructed. Ms. JR initially entered the equation, which was given in vertex form, into the calculator. After seeing its graph, she found the vertex, x-intercept, y-intercept, and axis of symmetry by looking at the tabular format on the calculator. Then she used this information to draw the graph for the parabola (Figure 4-28A and 4-28B).

Ms. JR also used cycling translation once in her second class. She initially created the function $y = x^2$ on the screen and asked the students to change the shape (e.g., narrower) and location of a parabola (e.g., up) (Figure 4-27). The students translated from symbolic to graphical representations bi-directionally until they correctly changed the shape and location of the parabola.

Although cycling translation was observed once, similar to Ms. MA's students, Ms. JR's students generally used multiple representations independently. When the students worked on the five exercises from the previous homework, they saw symbolic and graphical representations in the same PowerPoint slide; however, these representations were static versus dynamic. Changing the equation did not automatically modify its associated graph (Figure 4-26).

Observing translations

Ms. JR constructed representations and translations on her own when performing most of the class activities. Before solving quadratic equations in the first period, Ms. JR used the Quick Poll component of the TI-Navigator to ask students to identify their preferred method of solution. But without students' input, the teacher solved all of them, and the students passively observed the symbolic representations Ms. JR constructed.

At the end of the second period, the teacher used representations while solving the homework questions using the calculator as a tool. Ms. JR initially used the table in the graphing calculator to identify the vertex, x-intercepts, y-intercept, and axis of symmetry (Figure 4-28A and 4-28B). Then she drew a graph by hand based on the information obtained from the calculator. The students only followed the steps, observing the teacher's translations from symbolic to tabular and from symbolic to graphical representations.

Additionally, the students translated between representations when changing the shape and location of a parabola. The students used their calculators to translate from symbolic to graphical and from graphical to symbolic representations. The students initially entered an equation into the graphing calculator to draw its graph, and Ms. JR projected their graphs through the TI-Navigator so that the students could adjust their graphs if needed.

A teacher-centered approach

Ms. JR was predominantly involved in problem solving throughout the observed classes. She sometimes asked good questions; however, she typically provided the answers right away. Without allotting sufficient time for her students to complete current work, she moved to the next question. Generally, Ms. JR asked questions that could be answered very quickly, often completing students' answers or providing detailed explanations herself.

T: I'm not too sure about factoring because I don't know what the equation looks like in that standard form without parentheses. I would have to go through and I'd have to square this, find my terms, and then maybe it can be factored, but it can always use the quadratic formula if I can't factor it. But we're going to stick now to the top choice so we're going to use square roots. How do I solve this using square roots?

S: Square [inaudible] (JR_05.28.08, 269-275)

Ms. JR clarified without interaction if a question was not clearly understood. She usually did not encourage students to discover solutions on their own. Once, when the students were performing an activity related to determining the location of a quadratic equation in the coordinate plane by looking at its vertex form, a student curiously asked Ms. JR to clarify why the location of $y = (x - 3)^2 + 7$ is in the first quadrant of the X-Y coordinate system. Ms. JR provided an explanation without encouraging the student to search for the answer on his own.

T: So if I add that equation, if we take a look at … did I have minus 3? So right three, up seven because we're starting from the very basic X square …. (JR_06.03.08, 669-670)

Ms. JR sometimes responded curtly to students when they gave incorrect solutions or answers. She did not take up and explore students' answers. She also did not allow students time to work through their own mistakes to correct themselves. If she disagreed with a student's answer, she often did so sharply. For instance, when one student arrived at zero for an answer, the teacher told him that he was incorrect by saying "I don't agree with zero" (JR_05.28.08, 397). Another time she said impatiently, "I'm finding wrong answers…I have found four wrong answers" (JR_06.03.08, 69-70).

When the students worked on a difficult problem, Ms. JR let them know about the challenge and preferred to solve the problems by herself without challenging the students to think independently. In the second class, when drawing a graph, she said "So this is that one problem that's really, really tough" (JR_06.03.08, 1231) and continued to explain by saying, "So you need to think about factors of 12 and factors of six. Factors of six give me my first term; factors of 12 give you the second term. You just have to use…guess and check that to see if we can get this…This one you should have a greatest common factor first …" (JR_06.03.08, 1235-1237).

The students repeatedly made the same mistakes on some concepts. During the second period, the students thought that the vertex includes only one coordinate, but it has both x and y coordinates. Additionally, the teacher explained the vertex several times and reviewed several practice problems to show how to find it, but by the end of class one student still admitted, "I still do not get that [vertex]" (JR_06.03.08, 871).

Unlike her usual teaching strategy wherein she would identify the solution method for the students, Ms. JR sometimes foregrounded the students' methodological choices in solving quadratic equations such as during the first period when she asked the students their preference for solving a quadratic equation. She explained the task by saying:

T: Ok, what we're going to work on today is reviewing these three methods; solve by factoring, solve by using square roots, solve by quadratic formula, and we need a couple people to get logged in. So no new methods, just deciding which is the appropriate method; what's going to be the easiest, the fastest or which one you're more comfortable with, so you also need to get out a piece of paper. We're going to go through some quick poll questions and decide and some practice reviewing the solving. So I want your three assignments out so you can look at and say, gee, this problem looks like the first assignment, the second assignment; it will help you decide. (JR_05.28.08, 61-67)

This task may give more freedom to the students in selecting strategies when solving problems, which might enhance representational fluency. The teacher, however, limited the students' choices by directing them to look for similar exercises in their homework.

Similar to Ms. MA, Ms. JR provided hints to her students without allocating time for them to think while solving problems. In the second period, when the students were asked to move a parabola to the left or right and did not initially include parentheses, the teacher told them to add them, but they still could not correctly move the graph as instructed. Thus, Ms. JR provided additional hints saying, “squared outside the parenthesis” (JR_06.03.08, 433). These hints made the problem less challenging and interrupted the students’ independent thinking.

Finally, at the end of the second period, Ms. JR provided hints when going over the first question of the homework, which was related to finding the vertex of a quadratic function. The following excerpt reflects this:

T: I am giving you a hint. Can you tell me what the vertex is before we even look at the graph?

SS: [inaudible]

T: I am giving you a hint. Can you tell me what the vertex is before we even look at the graph?

SS: Negative 3… no / Zero. / No. / 3. / 3 and 4.

T: It’s going to have the 3 and 4 in there.

S: Positive 3 and negative 4. (JR_06.03.08, 850-860)

Ms. JR also asked her students to generalize the findings of the activities, but as in Ms. MA’s class, the students were asked to make obvious generalizations. When changing the location and shape of the parabola, Ms. JR asked her students to move the parabola straight down. After they submitted their parabolas, she asked “x squared minus 6, x squared minus 5, minus 3, minus 2…so what did we have to do to drop the graph?" (JR_06.03.08, 345-346). With this question, the students were asked to generalize how a chosen number changes the location of a parabola. Thus, they related a symbolic representation with a graphical representation, as well as made a translation between them. Ms. JR, however, simplified her question by providing a hint.

Summary of Ms. JR's Classroom Environment

To some degree, Ms. JR attempted to encourage student participation. The participation remained low, however, as the same few students consistently contributed to the discussion. Overall, the students appeared unmotivated and unenthusiastic. Ms. JR also spent time reviewing previous homework and explaining both the assignments and the instructions for that day's activities.

Additionally, Ms. JR's students exhibited difficulties understanding representations. For instance, rather than interpreting the meaning, the students primarily only commented on the shape of graphical representations. She did, however, explicitly direct her students to various representations for different purposes. Tabular, symbolic, and graphical representations were used to find the values of a quadratic equation, to verify answers, and to support tabular representations, respectively. Similar to Ms. MA's students, Ms. JR's class used multiple representations. Four unidirectional and cycling translations were observed. Except in one activity wherein the students translated between representations using their calculator, the students were not involved in constructing representations and translations but only observed the teacher performing them. Ms. JR generally missed opportunities to foster a dynamic, discussion-based environment. She would often provide answers to questions quickly, and if she did not agree with a student's response, she would often respond abruptly or dismissively. Additionally, her students did not exhibit sufficient competence in the course material; they would make the same mistakes repeatedly and did not seem to fully grasp the course's concepts. Similar to Ms. MA, Ms. JR provided hints when students were challenged. However, her hints often interrupted the students' independent thinking. She also asked questions leading to simple generalizations regarding the activities' findings.

CHAPTER 5
REPRESENTATIONAL FLUENCY INSTRUCTIONAL PRACTICES: A CROSS-CASE ANALYSIS

As stated in chapter 3, the cases examined in this study were identified first by examining the gain scores for classes that initially were low performing on the pretest. The effective classes were those that had the greatest gain from pretest to posttest among these initially low-performing classes, and the less effective classrooms were those that experienced the least amount of growth from pre- to posttest (Table 3-4). Once the cases were identified, all identifying information was deleted prior to the initial analysis. The cases were randomly chosen for analysis without knowledge of effectiveness. The aim of Chapter 4 was to explore these teachers' practices in terms of how their students engaged with mathematics and representations. Following the initial coding, the effectiveness category of the classes was identified and the effective and less effective cases are compared through cross-case analysis in chapter 5. Psychological environment and general mathematics teaching approach are discussed briefly, and then teachers' representational practices are analyzed in detail. Five themes that potentially support representational fluency were identified in the present study. In each subsection, a theme and how it is practiced in both effective and less effective classrooms is described.

Psychological Environment and General Mathematics Teaching Approach

Looking across the cases, similarities and differences of the effective and less effective classrooms in terms of general mathematics teaching were identified. Both teachers in the effective classrooms, Ms. BW and Ms. MB, promoted student participation and attention by including rich mathematical tasks. They made sure their students were on track. For example, Ms. BW assisted one group and left them to check the other group's progress at the end of the second period. After a while, she came back and still remembered what the first group had done. The teachers created a safe environment for the students to ask their questions and resubmit their graphs. They appreciated their students' input in classroom discussions and verbally encouraged them when they used correct terminology. Additionally, Ms. BW had a sense of humor, which helped her to build strong rapport and communication with her students. She used the class time efficiently but taught in a fast-paced manner. Although this approach might not give all of the student sufficient time to think, her explanations were clear and concise during problem solving. On the other hand, Ms. MB built a class community by using "we" language and emphasizing the word "all" to encourage students' interaction with each other.

Both teachers in the less effective classrooms, Ms. MA and Ms. JR, monitored participation throughout the class periods. The participation was low, however, and the students seemed uninterested in classroom discussions. In addition, Ms. MA showed care for the students and talked about their lives. She also fostered politeness in her classroom by using kind words and verbally encouraging her students. Ms. JR used the class time inefficiently by allocating approximately half of the time to review previous concepts and homework.

Compared to less effective classrooms, students in effective classrooms engaged in many rich mathematical tasks that required them to use multiple representations. To meet these task demands, the students needed to translate between representations by reconstructing their knowledge (Nistal et al., 2009). As these researchers mentioned, the reconstruction process might make them more successful than the students in less effective classrooms. Participation is increased through classroom response system technologies with specific questioning techniques (Rochelle et al, 2003). Leng (2011) stated that TI-Navigator provides a learning community that allows students to communicate mathematical concepts by increasing their participation in the learning process, which was observed in effective classrooms. Additionally, students are potentially more engaged in using representations in classrooms where the participation is high. The teachers in effective classrooms also increased participation and attention by providing a safe environment for students to ask their questions. Furthermore, CCT creates more student- and community-centered classrooms (Bransford et al., 2000), which was observed in Ms. MB's class. Having a good sense of humor, Ms. BW helped students to focus on classroom discussions. Additionally, Ms. MA talked about students' lives. Although Ms. MB, Ms. BW, and Ms. MA had strong communication with their students, students in Ms. BW and Ms. MB's classrooms communicated more frequently within the context of translating between representations whereas Ms. MA held these conversations outside of mathematics activities, which were either at the beginning or end of the class periods.

Representational Practices

Use of Representations

Representational fluency includes three abilities: constructing representations, interpreting them, and translating between representations (Sandoval et al., 2000). The interpretation of representations is, in a way, a pre-requisite to constructing and translating between representations. If students misinterpret representations, they will not be able to accurately construct representations and correctly translate between them. The interpretation of representations gains more importance in real-world problems that require students to model situations, and the ability to model problems is related to the ability to interpret mathematical representations, such as pictures or images.

Students usually find representations difficult, especially relating to higher-level mathematical concepts. Since mathematics is abstract, it may not initially be enjoyable or interesting to them. But teachers may better capture students' interests by diversifying how they present mathematical representations. One way teachers can do this is by making representations more tangible, hands on, and real-world oriented; by personalizing representations, teachers can better motivate their students to learn them. Each representation provides different advantages and disadvantages during problem solving. For example, verbal representations are helpful when students understand a problem or interpret its results; however, verbal representations can also mislead students if a teacher is not aware of a students' personal style (or language use) (Friedlander & Tabach, 2001). One can use different representations more dynamically if the strengths for their specific use are identifiable and well understood. If students have a strong repertoire related to specific uses of representations, they can use different representations to solve problems (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007).

In the effective classrooms, Ms. BW and Ms. MB included representations from real-world scenarios, encouraged their students to make interpretations, and guided them to link these representations to such scenarios. Additionally, Ms. BW used representations explicitly for specific purposes. She sometimes used graphical representations to provide a global perspective that was used to determine the range of possibilities for an answer, and she sometimes used tabular representations to verify the answers.

In the less effective classrooms, only Ms. MA included some problems from real-world scenarios in her second class period. Both Ms. MA and Ms. JR, however, focused on the characteristics of the representations rather than interpretation within the real-world context. In Ms. JR's class, the students used representations explicitly for specific purposes; her students used tabular representations to find the answers, symbolic representations to verify them, and graphical representations to support tabular ones in determining the range of an answer.

Using Different Translations

Students have been shown to perform better when they can use multiple representations (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007), translate between representations (Brenner et al., 1997), and use non-symbolic representations (Suh & Moyer, 2007). In addition, the new versions of handheld calculators may enhance students' translation capabilities bidirectionally from graphical to symbolic and from symbolic to graphical representations (Ozgun-Koca & Edwards, 2009).

Increasing students' repertoire of representations and translations is useful in sparking their understanding about how concepts relate to one another in various ways. Thus, students need experiences that not only involve working with different representations but that also involve learning how to link them. In this study, unidirectional and bidirectional translations were observed. As described in Chapter 4, unidirectional refers to translating between different representations and bidirectional refers to translating between the same representations.

The students in all four classrooms frequently used different unidirectional translations. In effective classrooms, however, unidirectional translation was observed only once in Ms. MB's classrooms. All unidirectional translation sequences and the longest sequence of translations were observed in Ms. BW's classroom, an effective classroom. Among the less effective classrooms many unidirectional translations were observed in Ms. MA's classroom video recordings. Although Ms. JR did not include any translations in her first class period, the longest unidirectional translation among less effective classrooms was observed in her classroom. Although many unidirectional translations were observed in less effective classrooms, the students did not generally translate between representations instead they observed their teacher's translations. Translations used in the unidirectional category are summarized in the Table 5-1.

Table 5-1. Unidirectional Translations

Classroom	# of Translations	Translation Sequence	Teacher Practiced
Effective	One	Graphical → Symbolic	Ms. MB
	Two	Symbolic1 → Graphical → Symbolic2	Ms. BW
		Tabular → Symbolic → Graphical	Ms. BW
	Three	Symbolic1 → Graphical → Symbolic2 → Verbal	Ms. BW
	Four	Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical	Ms. BW
Less Effective	Two	Symbolic1 → Graphical → Symbolic2	Ms. MA
		Symbolic → Tabular → Graphical	Ms. MA
		Pictorial → Tabular → Graphical	Ms. MA
		Tabular → Symbolic → Graphical	Ms. MA
	Three	Symbolic1 → Graphical → Symbolic2 → Graphical	Ms. MA
	Four	Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical	Ms. JR

In addition to unidirectional translations, one of the main features that differentiated between the effective and less effective classrooms was the presence of bidirectional translations, which were observed only in the effective classrooms. Using four translations including bidirectional translation in Ms. BW's class might have improved her students' translation abilities because it includes many representations and translation processes. Table 5-2 displays the sequences of translations in each of the effective classrooms. Although, there was only one bidirectional translations in each of the effective classes, the activities in which these translations were observed took a substantial amount of class time.

Table 5-2. Bidirectional Translations in Effective Classrooms

# of Translations	Translation Sequence	Teacher Practiced
Two	Symbolic1 → Graphical → Symbolic1	Ms. MB
Four	Pictorial → Tabular → Graphical1 → Symbolic → Graphical1	Ms. BW

Connecting knowledge to another meaningfully and practically can garner deeper, more comprehensive conceptual understanding (Hiebert & Lefevre, 1986). The teacher's role in promoting interactions and discussions about representations is crucial in increasing the quality and quantity of these connections (diSessa et al., 1991). One way for teachers to do this is to show multiple representations on a calculator screen. However, as described earlier, when the students in less effective classrooms were exposed to multiple representations they did so independently rather than dynamically. Thus, viewing multiple representations in one window may not be sufficient unless these representations are dynamically linked and bidirectionally translated. For example, in the dynamically linked situation when students modify the value of a parameter in an equation, they can see the change on the corresponding graph. In addition, CCT allows users to view multiple representations both dynamically and publicly and to change their representations accordingly. Showing more than one representation simultaneously provides a rich environment for mathematical discourse that benefits student learning (Irving et al., 2010). To improve students' representational fluency, they need numerous opportunities to see the relationship between different representations simultaneously. In this study cycling of translations were observed where students iteratively translated between representations bidirectionally until they reached the correct solution.

Both teachers in the effective classrooms provided tasks that allowed their students to view multiple representations simultaneously. These students saw that each modification they made in their equations would simultaneously change their graph. The students in Ms. BW's class could see the symbolic representation of a point on a drawn parabola while changing the location of the point on it. Finally, cycling translations were observed twice in effective classrooms: (a) two translations with one of them cycling and (b) three translations with one of them cycling. On the other hand, in the less effective classrooms, cycling translation between Symbolic → Graphical ↔ Symbolic was observed only once in one of Ms. JR's class during the second period (Table 5-3). This cycling was not, however, used within a real-world context and the students did not interpret the representations. Ms. JR also gave many hints during the cycling process instead of letting the students think for themselves. As stated earlier, an additional difference was the fact that representations were not generally dynamically linked within the less effective classrooms. The students in these classrooms only used representations independent of one another because at least one of the representations was provided on a worksheet.

Table 5-3. Cycling Translations

Classroom	# of Translations	Translation Sequence	Teacher Practiced
Effective	At least two	Symbolic → Graphical ↔ Symbolic	Ms. BW
	At least three	Pictorial → Symbolic → Graphical ↔ Symbolic	Ms. MB and Ms. BW
Less Effective	At least two	Symbolic → Graphical ↔ Symbolic	Ms. JR

Distributed Nature of Translations

In-school teaching and learning places emphasis primarily on an individual's performance; however, work and tasks outside the classroom are usually socially shared. This consideration has prompted researchers to investigate more practical methods for developing instructional strategies. In her research on learning in and out of school, for example, Resnick (1987) examined the commonalities of successful educational programs that focus on thinking skills, learning skills, and higher-order cognitive abilities. She concluded that to be more effective, school programs should include socially-shared intellectual work, jointly accomplished tasks, and cognitive tools (including using computer software and calculators). Not only do technological tools better facilitate learning cognitively-difficult concepts, but they also increase students' learning capacities by encouraging them to think and work more independently (Resnick, 1987). In mathematics classrooms these affordances can be facilitated by teachers using the TI-Navigator, which allows students quick, convenient access to multiple representations in addition to providing ways of translating between them. By using this technology, students can focus more on the meaning of representations and the connections and relationships between them.

Fostering socially-shared thinking and active engagement is potentially important for teachers to prepare their students to be more adaptive learners, especially when using representations. With an instructor's active guidance, students can move to a representation that is more effective in solving a specific type of problem or switch between representations to solve an unfamiliar problem.

In the present study, students passively translated between representations in less effective classrooms, and actively translated between representations in effective classrooms. As described earlier, students in less effective classrooms generally observed while their teacher translated between representations with or without the help of calculators. This is in contrast to the effective classrooms where the translations were distributed between students and teachers and mediated through calculators. Thus, to foster students' ability to learn mathematical concepts through translating between representations, students should be encouraged to create their own representations; by actively creating representations themselves, students also become the main actors in both using representations and translating between them.

The translations were jointly shared between students and the teacher using the calculator as a tool in effective classrooms. The teachers also encouraged students to create their own representations when solving problems. For example, students in Ms. MB's class came up with their own symbolic representations and entered them into the graphing calculator using pre-determined criteria. In addition, Ms. BW usually encouraged the students to use calculators; however, when building a model to identify the revenue gained from selling flowers, she asked the students to fill out a table by hand first. This might help students become less reliant on calculators and therefore help them use representations and translate between them to better understand particular mathematical concepts.

Students should be provided opportunities to share their work during the initial process of translating and using representations. Viewing translations and representations, however, is important for students, especially while learning new concepts. Co-construction of translations was only observed a few times in the less effective classrooms. However, Ms. MA and Ms. JR did not generally provide the students chances to translate between representations on their own. The teachers in the less effective classrooms usually translated for the class rather than allowing the students to translate between representations on their own. They would initiate all translations, and their students would then only follow their steps when solving problems. The students thus imitated translations, but unfortunately the teachers did not encourage them to think of the processes behind this activity.

Formative Assessment

Students who lack understanding about a particular representation may be challenged when solving problems requiring the use of that representation. Since new knowledge is dependent on existing knowledge, a teacher's awareness of students' understanding of and comfort level with specific representations is essential for adapting instruction according to their students' particular needs. When teachers know their students' strengths and weaknesses regarding representations, they use this knowledge to potentially improve their students' competence in solving problems correctly (Nistal et al., 2009).

Assessments provide opportunities for students to learn about their own understanding and help teachers monitor their students' progress and thus make pertinent instructional decisions. The projection of representations through the TI-Navigator, which increases how much teachers know about their students' thinking and reasoning, visualizes students' representational knowledge. That is, by projecting representations, teachers can not only know more about their students' use of representations, but they can also monitor students' understanding during instructional activities. As a result, they can adjust their teaching of mathematical concepts through particular representations or identify misunderstandings that may affect students' learning. Additionally, projecting representations allows students to learn from their own work through self-assessment and recognize where their representations have fallen short or need revision.

In the classrooms that were observed to be effective learning environments, both teachers used the resubmission property of TI-Navigator so that students could compare their graphs with the constraints of a problem and with others' representations. The students initially entered their equations into their graphing calculator and submitted their graphs through the TI-Navigator. The resubmission property allowed students to identify whether their graphs were accurate or inaccurate, and (if inaccurate) how they could modify their graphs by changing the equations. More importantly, these changes were often a subject of the discussion within the classroom. Students were given the opportunity to provide feedback to their classmates related to how they could change the representation. This type of self-assessment helped students to flexibly translate between representations, because they could modify their equations until they obtained the most accurate graph. The following excerpt reflects this benefit:

T: I will let you resubmit so you can look at it up here. So if you send one you can actually see how well it fits and then you can resubmit; edit it. You know how you preview it on your screen? Well on this particular case this is one where you can send it and then replace it. (BW_05.06.08, p. 37)

Students previewed their graphs on their calculator screens before submitting them to compare theirs with others' work. Additionally, the teachers in the effective classrooms projected the students' responses anonymously or with color-coding, thus creating a safe environment for the students to resubmit their graphs. Students might not be afraid of making mistakes while reworking on their resubmissions. In addition, Ms. BW was sometimes flexible and adopted her teaching when needed. Furthermore, she used the Screen Capture component in her class, was aware of her students' representations, and used TI-Navigator to validate their strengths and weaknesses regarding representations.

In the less effective classrooms, there was limited evidence of formative assessment. Ms. JR used the resubmission property only once and in only one of her classrooms. During this activity, she did not create a discussion-rich environment wherein students could also resubmit their graphs. She initially created the function $y = x^2$ on the screen and then asked students to move the location and change the shape of the parabola. The students modified their equations if needed. During this activity, Ms. JR did not initially hide the students' names but later on, she changed the settings of the TI-Navigator to project the students' responses anonymously.

Scaffolding Translation through Teachers' Questioning

To develop students' representational fluency, teachers need to create environments in which students can actively engage in constructing and interpreting representations (diSessa et al., 1991; Greeno & Hall, 1997; Warner et al., 2009). In classrooms with this environment, students learn to value and use others' representations; such an environment also gives them the opportunity to change their own representations with the aim of generalizing and explaining them to others.

Fostering an environment wherein students can talk about their methods and choices increases individual and collective learning; mathematically proficient students should be able to "justify their conclusions, communicate them to others, and respond to the arguments of others" (CCSS, 2010, p. 6-7). Thus, during the learning process students should critique their peers' representations and respond to others' critiques about their own. Teachers' and peers' roles are essential in discussions about representations.

CCT potentially increases communication between the students and their teachers (Pape et al., 2012). Publically displaying work provides a context wherein students can best communicate about representations. Also, teachers' knowledge about the students' representational understanding is better facilitated by projecting students' mathematical thinking. Teachers should therefore use the opportunities provided by CCT to promote translation through discussion.

The teachers in the effective classrooms, Ms. MB and Ms. BW, asked questions to promote students' translation between representations during activities. Ms. BW usually asked questions requiring short answers. When Ms. MB realized that her students seemed lost or confused, or if she needed a student to clarify an answer, she would ask follow-up, open-ended, or hypothetical questions. Ms. MB invited all students to participate. She also encouraged them to share their solutions and opinions while solving the problems.

The teachers in the less effective classrooms, Ms. MA and Ms. JR, missed many opportunities to create discussion-rich environments. Also, by providing hints or asking questions that led to obvious generalizations, both teachers did not adequately challenge their students during problem-solving activities. Thus, the questioning techniques she used did not require the in-depth thinking that would encourage students to make translations between representations. In addition, Ms. MA and Ms. JR did not sufficiently interact with their students when they made mistakes. Instead, they provided explanations right away. When discussing alternative ways of solving a problem, Ms. MA often started using her method without allowing her students adequate time to think for themselves.

Conclusions

The purpose of this study was to compare high school mathematics teachers' use of CCT that potentially supported achievement on translation problems. Research has shown that students at the middle school (Ainsworth et al., 2002), high school (Knuth, 2000), and college (Herman, 2007) levels may lack adequate translation abilities. These studies demonstrate and standards documents highlight the significance of and necessity for the development of representational fluency (CCSS, 2010; NCTM, 2000). Some studies have indicated two solutions to developing the types of representational fluency abilities important for conceptual understanding and achievement (e.g., Lesh & Zawojewski, 2007; Nistal et al., 2009). Communication is suggested as one solution. Research demonstrates that students need an environment where they can actively engage in discussions about interpreting, constructing, evaluating, comparing, and generalizing representations as well as justifying the representations in solutions, criticizing or questioning, and explaining or responding to the critiques of their peers or their own representations (diSessa et al., 1991; Greeno & Hall, 1997; Warner et al., 2009).

Technology is suggested as another potential solution. The development of representational fluency may be supported by the ability to access multiple representations on one screen and use them flexibly (Bieda & Nathan, 2009; Bostic & Pape, 2010; Herman, 2007; Knuth, 2000). Since research shows that communication and technology separately support students' development of representational fluency, the present study investigated instruction that integrated both communication and technology with the goal of examining the relationship between these instructional strategies and increasing representational fluency.

CCT provides a discourse-rich environment by increasing students' use of multiple representations simultaneously (Irving et al., 2010), projecting students' representations, and therefore increasing teachers' knowledge about their students' understanding, as well as students' awareness of their own understanding. Few studies, however, have examined an environment that helps develop representational fluency skills (Brenner et al., 1997). The present study therefore extends prior research by examining the implementation of CCT technology within a discourse-rich environment, specifically high school mathematics classes, to possibly support representational fluency.

To do this, data collected within the CCMS project through classroom observations and algebra pre- and posttests were explored to investigate how teachers' use of CCT in effective and less effective classrooms differs. The two classrooms with the highest and two classrooms with the lowest gain scores from a pool of classes with initial pre-test scores below 50% were selected for the analysis.

TI-Navigator made students' representational thinking and understanding visible for the teachers through projecting students' representations. Among the effective classrooms, only Ms. BW, however, was aware of her students' representational knowledge, even though research shows that knowing students' strengths and weaknesses in using representations is important for supporting their representational fluency abilities (Nistal et al., 2009).

Interpretation of representations is one of the key elements in representational fluency competence (Sandoval et al., 2000). While the students in the effective classrooms interpreted representations within real-world contexts, the students in the less effective classrooms focused on the characteristics of representations without understanding them in a real-world context. Teachers should include activities wherein students can learn mathematical concepts through representations by connecting them to their real-world meanings.

Although knowing the advantages or disadvantages of using representations is important (Friedlander & Tabach, 2001; Kaput, 1989), one teacher from each effective and less effective classroom emphasized the use of representations explicitly for specific purposes. Ms. BW first explained and then let students use representations for specific purposes whereas Ms. JR explained the advantages of using different representations and used these representations on her own in problem-solving activities. Thus, the students in Ms. JR's classroom passively used representations for specific purposes. Students might need an environment in which they explore the advantages and disadvantages of different representations and their uses—through hands-on practice—instead of teachers simply explaining and using the representations for their students.

Teachers in both the effective and less effective classrooms provided an environment where students could use multiple representations on one screen. Only the teachers in the effective classrooms, however, encouraged their students to use multiple representations simultaneously. These students saw that each modification they made in their equations would simultaneously change their graph. On the other hand, the students in the less effective classrooms used representations independently on their activity sheets. Although research indicates that seeing multiple representations on one window may help develop students' flexibility in translation between representations (Bostic & Pape, 2010), this study extends that notion by suggesting that not only is seeing multiple representations on one screen enough, but that students also need an environment where they can use different representations simultaneously.

Students in the less effective classrooms mostly observed representations and translations and were not provided the opportunity to create representations. Conversely, the students in the effective classrooms used representations and translations by jointly sharing them, or creating representations and translations on their own. Resnick (1987) also concluded that school programs should include socially shared intellectual work, jointly-accomplished tasks, and cognitive tools when she examined commonalities in successful programs, which focused on thinking skills, learning skills, and higher-order cognitive abilities. Students should start to create representations and translations on their own, along every stage of the learning path.

Students might learn to become more independent in their learning process regarding translations and representations.

Research shows that students perform better when they use more and multiple representations (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007), when they have the ability to translate between representations (Brenner et al., 1997), and when they use non-symbolic representations (Suh & Moyer, 2007). Although in both the less effective and effective classrooms, students used multiple representations, only the students in the effective classrooms generally used bi-directional and cycling translations. Since the new version of handheld calculators enhances translation bi-directionally (Özgün-Koca & Edwards, 2009) and repairing representations is a norm that the classes should apply (Warner et al., 2009), with bi-directional and cycling representations via TI-Navigator, the teacher creates an environment where students have a chance to make changes to their representations until they arrive at the most accurate one. The students might be more successful if they can use bi-directional and cycling translations along with multiple unidirectional translations.

CCT provides the context that makes increased communication between students and their teacher more possible (Pape, Irving, Bell, et al. 2012). And the public display of student work provides a context in which students can communicate about representations. Although both teachers in the effective classrooms asked questions that prompted students to translate between representations, only Ms. MB created a productive discussion environment to promote translations. Ms. BW, however, accepted partial answers and did not question whether all students understood these translations thoroughly; however, her fast-paced teaching approach might have been a result of her limited time. Although researchers and mathematical educators generally seek ideal teaching environments, it is important for them to be realistic about the challenges high school teachers face given high-stakes testing and the demands of parents and administration.

Implications

This study provides a thick description of four teachers' practices. It may offer examples to mathematics teacher-educators when they prepare pre-service teachers or administer professional development to in-service teachers. While the teachers of the effective classrooms in this study might not serve as perfect examples, they do provide realistic examples of how teachers might construct their classroom to better promote their students' representational fluency abilities.

Teachers should be aware of their students' representational knowledge and seek technological or cognitive tools to visualize their students' thinking. Through the use of classroom connectivity technology such as the TI-Navigator, instructors can monitor and assess their students formatively to adapt their instruction based on their students' needs and misconceptions. Ultimately, teachers should create environments for students to interpret representations by linking them to real-world scenarios. Students should not only be able to see multiple representations on one screen, but they should also see the changes made to one representation simultaneously when another one is modified. Teachers and students should jointly share in the process of translation with the help of cognitive tools because translation is a cognitively difficult process with the goal of enabling students to create representations or translations by themselves. Moreover, they should be provided more opportunities for students to make judgments about the accuracy of their representations and to change them as appropriate during problem-solving activities, which can be accomplished by including activities that require bi-directional and cycling translations. In addition, teachers should frame questions that facilitate students' developing understanding of representations and translations over time. One way to promote such sustained thinking is to foster discussion-based environments.

Limitations and Suggestions for Further Research

As in most research studies, this study has several limitations and opportunities for expansion. Since the data were collected within the CCMS project, which was not conducted for the aim of the present study, all representations that are normally used in these classrooms might not have been captured during the videotaped classes. Furthermore, student work, descriptions of context, and student or teacher interviews related to the purpose of this study were not available. Multiple methods of data collection could not be used to increase the validity of findings (e.g., triangulation). In addition, member checks could not be conducted.

In this study, student achievement was identified based on the gains in their scores on the translations problems only. Since representational fluency includes both the interpretation and construction of representations, complete student understanding cannot be captured with one test; some qualitative data might support, for example, the selection of what are classified as "effective" and "less effective" classrooms in this study. Some supplementary data, for instance, might include interviews with students.

Two classroom observation video recordings were studied for each teacher. Additional video recordings may, however, provide better insights into how these teachers use technology and communication in their typical teaching practices to support representational fluency. In addition to the limitations mentioned, there are potential areas that warrant future research.

Future study may include interviews with teachers or students to include their own words to better understand how students can be supported in developing their representational fluency abilities. On several occasions, it was difficult to understand who was involved in the co-construction of the representations and translations. Interviewing students and their teachers would clear up some of this confusion.

As research has shown, students lack representational fluency or translation abilities in middle school, high school, and college since they do not develop these abilities in their early educational years. Thus, the present study can be extended to focus on students' representational fluency abilities in pre-K or elementary school through long-term observation.

Finally, general mathematics teaching approaches or a classroom's psychological environment might affect the development of representational fluency. Teachers in the effective classrooms were found to possess strong communication skills, promote students' attention and participation, and create discussion-based environments for their students. On the other hand, the students in the less effective classrooms participated infrequently, had short attention spans, and missed opportunities that a more discussion-friendly environment would provide. Studies that deeply examine these factors would help researchers and teacher educators better address the challenges that teachers and students face regarding how representations are currently taught.

APPENDIX A

CONNECTED CLASSROOM (COMPONENTS OF TI-NAVIGATOR)

Components of TI-Navigator	Description
Quick Poll	Sending a single question to the students’ calculators
Learn Check	Sending several questions to the students’ calculators
Class Analysis	Analyzing summarizes students’ responses, which may be displayed as bar graphs
Screen Capture	Taking a “screenshot” of individual student’s calculator screens for display
Activity Center	Allowing teachers to display a coordinate system

APPENDIX B

REPRESENTATION TYPES AND THEIR DESCRIPTIONS

Representation	Description
Symbolic	Expressions that utilize numeric, symbolic or a combination of numeric and symbolic characters
Pictorial	Drawings that represent values, symbols, or real-world objects
Tabular	Stem-and-leaf plots, frequency tables, or charts that categorize and organize data
Verbal	Written statements that use words to represent numbers and mathematical operations
Graphical	Diagrams that exhibits quantitative relationships such as trends, increases, intersections, and minimums

Note. Symbolic, pictorial, tabular, verbal representations' descriptions are adapted from "The Effects of Teaching Mathematics through Problem-Solving Contexts on Sixth-Grade Students' Problem-Solving Performance and Representation Use," by J. D. Bostic, 2012, Doctoral Dissertation, p. 196. Graphical representation description is adapted from "Representation as a vehicle for solving and communicating," by R. Preston and A. Garner, 2003, Mathematics Teaching in the Middle School, 9, p. 42.

APPENDIX C
ALGEBRA PRE-TEST

Draft

National Center for Research on Evaluation, Standards, and Student Testing

UCLA/CRESST
Teaching & Learning

Algebra I Pretest

TODAY'S DATE			GRADE	TEACHER PARTICIPANT CODE	STUDENT PARTICIPANT CODE
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9
0	0	0	0	0	0

ETHNICITY

- ○ American Indian
- ○ Asian
- ○ Black or African-American
- ○ Hispanic or Latino/a
- ○ Native Hawaiian or Pacific Islander
- ○ White, non Hispanic
- ○ Other (please specify) ____________________

TEACHER INITIALS

please write clearly

STUDENT INITIALS

please write clearly

Draft

Draft

SAMPLE QUESTIONS

Read each question carefully. Answer it as well as you can. Do not spend too much time on any one question.

Most of the questions give answer choices. Fill in the circle for the best answer.

● not ☒ ⊘

SAMPLE #1

1. If x equals any number, how many values can $x + 6$ have?

- ○ None
- ○ One
- ○ Six
- ○ Seven
- ● Infinitely many

For some questions, you will write your answer in a box.

SAMPLE #2

2. These numbers are part of a pattern.

2, 4, 6, 8, 10

The next number in the pattern is $\boxed{12}$

Draft

Draft

DIRECTIONS

This test has 30 questions.

Mark your answers in your booklet.

You may use the extra white space for scratch work.

When you reach the end of the test, go back through the test to:

- see if you accidentally skipped a page
- answer some of the harder questions

Draft

Draft

1. Which operation will change the value of any nonzero number?

○ adding zero ○ multiplying by zero ○ multiplying by one ○ dividing by one

2. $\frac{3}{8} + \frac{1}{12} =$ ○ $\frac{1}{5}$ ○ $\frac{1}{6}$ ○ $\frac{11}{24}$ ○ $\frac{11}{48}$

3. Simplify each expression: Answers:

$x^2 \cdot x^4 \cdot y^3 \cdot y^6 =$

$(4x^4)^2 =$

$\frac{3ab^4}{9ab} =$

4. What value of k makes the following equation true?

$k \div 3 = 36$ ○ 108 ○ 98 ○ 39 ○ 12

5. A telephone company charges \$0.05 per minute for local calls and \$0.12 per minute for long-distance calls. Which expression gives the total cost in dollars for x minutes of local calls and y minutes of long-distance calls?

○ $0.05x + 0.12y$

○ $0.05x - 0.12y$

○ $0.17(x + y)$

○ $0.17xy$

Draft

6. A group of hikers climbed from Salt Flats (elevation 55 feet) to Talon Bluff (elevation 620 feet). What is the difference in elevation between Talon Bluff and Salt Flats?

   ○ 565 feet ○ 575 feet ○ 665 feet ○ 675 feet

7. If m represents a positive number, which of these is equivalent to $m + m + m + m$?

   ○ $m + 4$

   ○ $4m$

   ○ m^4

   ○ $4(m + 1)$

8. $(150 \div 3) + (6 \times 2) =$

   ○ 10 ○ 58 ○ 62 ○ 112

9. What value of x makes the equation below true?

   $\frac{x}{8} + 7 = 9$ ○ 2 ○ 16 ○ 54 ○ 128

10. $12 \div -3 =$ ○ 9 ○ 4 ○ $-\frac{1}{4}$ ○ -4

Draft

11. $3^3 + 4(8-5) \div 6 =$

Answer:

12. If x represents the number of newspapers that Sara delivers each day, which of the following represents the total number of newspapers that Sara delivers in 5 days?

- ○ $5 + x$
- ○ $5x$
- ○ $5 \div x$
- ○ $5(x + x)$

13. Solve for x:

$2x + 2 < x + 4$

Your answer:

Show your work here:

14. A bird flew at 17 miles per hour for 2 hours, then at 15 miles per hour for 3 hours. How far did the bird fly in all?

○ 49 miles ○ 62 miles ○ 79 miles ○ 85 miles

Draft

Draft

15. Find the equation for the line that passes through $(3,4)$ and has a slope of $\frac{1}{3}$.

Show your work here:

Your answer:

16. Find x if

$10x - 15 = 5x + 20$

Answer:

17. $\left(\frac{2}{3}\right)^4 =$

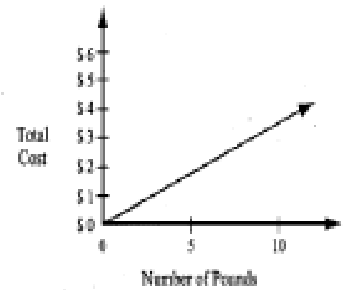
○ $\frac{8}{81}$ ○ $\frac{16}{81}$ ○ $\frac{8}{3}$ ○ $\frac{16}{3}$

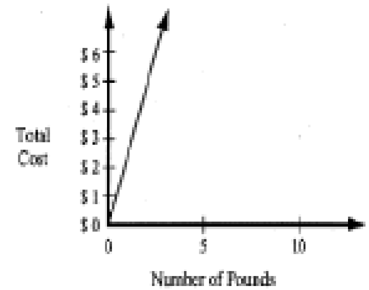
18. What is the least whole number x for which $2x > 11$?

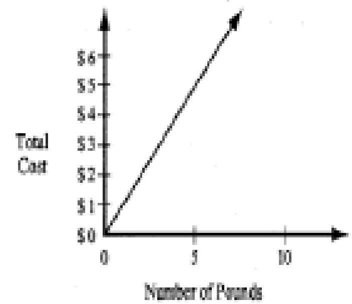
○ 5 ○ 6 ○ 9 ○ 22 ○ 23

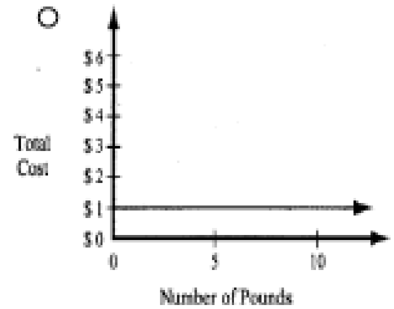
Draft

19. Rice is on sale at the price of 3 pounds for $1.00. Which graph shows the relationship between the number of pounds of rice bought and the total cost?

○ 

○ 

○ 

○ 

20. $\frac{x}{2} < 7$ is equivalent to:

○ $x < \frac{7}{2}$

○ $x < 5$

○ $x < 14$

○ $x > 5$

○ $x > 14$

Draft

21. Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes.

If both cars start at the same time, will Sharon's sedan reach point A, 8 miles away, before, at the same time, or after Victor's van?

○ before ○ same time ○ after

Explain your reasoning:

22. $P = LW$. If $P = 12$ and $L = 3$, then W is equal to:

○ $\frac{3}{4}$

○ 3

○ 4

○ 12

○ 36

23. Juan has 5 fewer hats than Maria, and Clarissa has 3 times as many hats as Juan. If Maria has n hats, which of these represents the number of hats that Clarissa has?

○ $5 - 3n$

○ $3n$

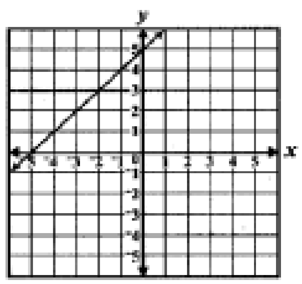
○ $n - 5$

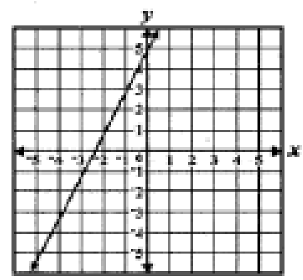
○ $3n - 5$

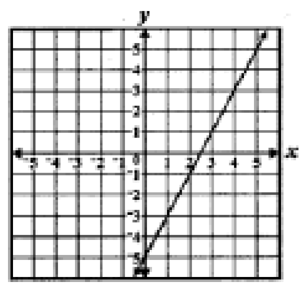
○ $3(n - 5)$

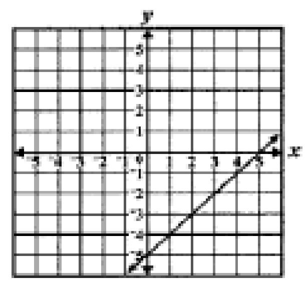
Draft

24. Which best represents the graph of $y = 2x - 5$?

○ 

○ 

○ 

○ 

25. A group of students has a total of 29 pencils and everyone has at least one pencil. Six students have 1 pencil each, 5 students have 3, and the rest have 2. How many students have only 2 pencils?

○ 4 ○ 6 ○ 8 ○ 9

26. Julia earns $42 for 3 hours of work. At that rate, how long would she have to work to earn $840?

○ 12 hours ○ 20 hours ○ 60 hours ○ 140 hours

Draft

27. Which equation is equivalent to $5x - 2(7x + 1) = 14x$?

- ○ $-9x - 2 = 14x$
- ○ $-9x + 1 = 14x$
- ○ $-9x + 2 = 14x$
- ○ $12x - 1 = 14x$

28. The total cost (c) in dollars of renting a truck for n days is given by the equation:

$$c = 120 + 60n$$

If the total cost was $360, for how many days was the truck rented?

○ 2 ○ 4 ○ 6 ○ 8

29. Find the x- and y-intercepts of the graph of $2x + 5y = 30$. Show your work below.

x-intercept:

y-intercept:

Show your work:

30. What is the y-intercept of the graph of $4x + 2y = 12$?

○ -4 ○ -2 ○ 6 ○ 12

Did you skip any pages accidentally? Go back and check your math. Then answer the questions on the following pages.

Draft

APPENDIX D
ALGEBRA POSTTEST

24980

THE OHIO STATE UNIVERSITY

Classroom Connectivity in Mathematics and Science

Algebra I Post Test

TODAY'S DATE			GRADE	TEACHER PARTICIPANT CODE	STUDENT PARTICIPANT CODE
__ __ /	__ __ /	__ __ __ __	__ __	__ __ __ __	__ __ __
1 ○○	1 ○○	1 ○○○○	1 ○○	1 ○○○○	1 ○○○
2 ○○	2 ○○	2 ○○○○	2 ○○	2 ○○○○	2 ○○○
3 ○○	3 ○○	3 ○○○○	3 ○○	3 ○○○○	3 ○○○
4 ○○	4 ○○	4 ○○○○	4 ○○	4 ○○○○	4 ○○○
5 ○○	5 ○○	5 ○○○○	5 ○○	5 ○○○○	5 ○○○
6 ○○	6 ○○	6 ○○○○	6 ○○	6 ○○○○	6 ○○○
7 ○○	7 ○○	7 ○○○○	7 ○○	7 ○○○○	7 ○○○
8 ○○	8 ○○	8 ○○○○	8 ○○	8 ○○○○	8 ○○○
9 ○○	9 ○○	9 ○○○○	9 ○○	9 ○○○○	9 ○○○
0 ○○	0 ○○	0 ○○○○	0 ○○	0 ○○○○	0 ○○○

ETHNICITY

- ○ American Indian
- ○ Asian
- ○ Black or African-American
- ○ Hispanic or Latino/a
- ○ Native Hawaiian or Pacific Islander
- ○ White, non Hispanic
- ○ Other (please specify) ____________________

TEACHER INITIALS

please write clearly

STUDENT INITIALS

please write clearly

GENDER

- ○ Male
- ○ Female

24980

24980

SAMPLE QUESTIONS

Read each question carefully. Answer it as well as you can. Do not spend too much time on any one question.

Most of the questions give answer choices. Fill in the circle for the best answer.

● not ⊠ ⊘

SAMPLE #1

1. If x equals any number, how many values can $x+6$ have?

○ A None

○ B One

○ C Seven

● D Infinitely many

For some questions, you will write your answer in a box.

SAMPLE #2

2. These numbers are part of a pattern.

2, 4, 6, 8, 10

The next number in the pattern is $\boxed{12}$

24980

DIRECTIONS

This test has 30 questions.

Mark your answers in your booklet. For multiple choice questions, select only one answer.

Erase answer changes completely and cleanly.

You may use the extra white space for scratch work.

When you reach the end of the test, go back through the test to:

- see if you accidentally skipped a page.
- answer any questions you may have skipped the first time.
- work any problems again that you had difficulty solving the first time.

<u>There are two parts to this Algebra Post Test:</u>

Part A: Do not use a calculator on Part A (Questions #1-11). Once you have completed Part A and have taken out your calculator for Part B, please do not return to items in Part A unless you have removed your calculator from your desk.

Part B: You may use a graphing calculator for Part B (Questions #12-30).

24980

ALGEBRA POST-TEST PART A

Do not use a calculator on Part A (Questions #1-11).

1. Simplify each expression: Answers:

$x^2 \cdot x^4 \cdot y^3 \cdot y^6 =$

$(4x^4)^2 =$

$\frac{3ab^4}{9ab} =$

2. A telephone company charges $0.05 per minute for local calls and $0.12 per minute for long-distance calls. Which expression gives the total cost in dollars for x minutes of local calls and y minutes of long-distance calls?

○ A $0.17(x+y)$

○ B $0.17xy$

○ C $0.05x+0.12y$

○ D $0.05x-0.12y$

24980

24980

ALGEBRA POST TEST PART A

Do not use a calculator on Part A (Questions #1-11).

3. Solve for *x*:

$$2x+2<x+4$$

Show your work here:

Answer:

4. Find the equation for the line that passes through (3,4) and has a slope of $\frac{1}{3}$.

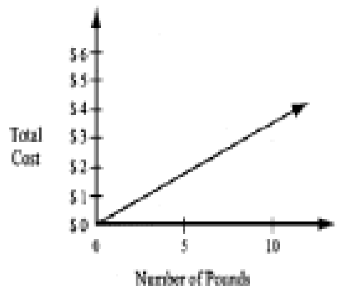
Show your work here:

Answer:

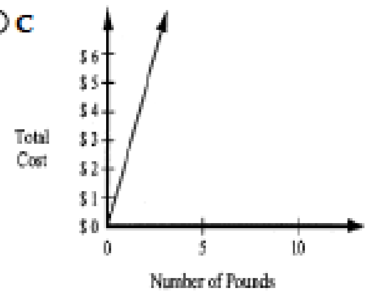
24980

5. Rice is on sale at the price of 3 pounds for $1.00. Which graph shows the relationship between the number of pounds of rice bought and the total cost?

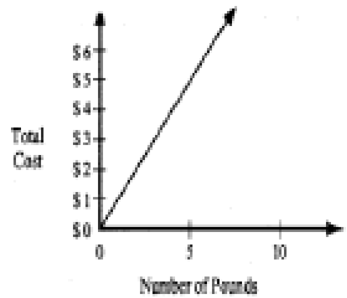
○ A



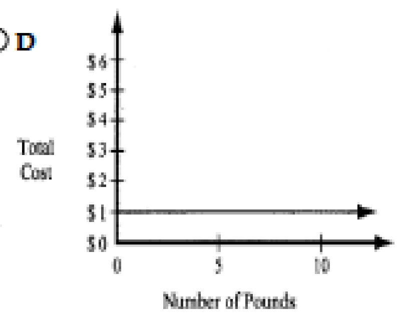
○ C



○ B



○ D



ALGEBRA POST TEST PART A

Do not use a calculator on Part A (Questions #1-11).

6. Victor's van travels at a rate of 8 miles every 10 minutes.
Sharon's sedan travels at a rate of 20 miles every 25 minutes.

If both cars start at the same time, will Sharon's sedan reach point A, 8 miles away, before, at the same time, or after Victor's van?

○ A before ○ B same time ○ C after

Explain your reasoning:

7. Juan has 5 fewer hats than Maria, and Clarissa has 3 times as many hats as Juan. If Maria has n hats, which of these represents the number of hats that Clarissa has?

○ A $5-3n$

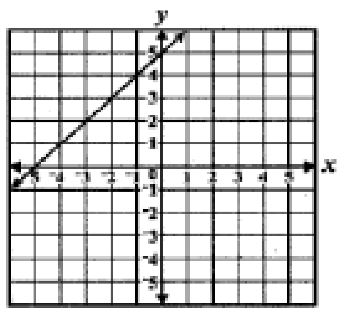
○ B $n-5$

○ C $3n-5$

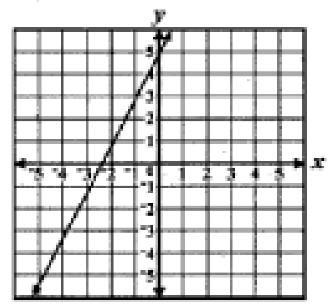
○ D $3(n-5)$

8. Which best represents the graph of $y = 2x - 5$?

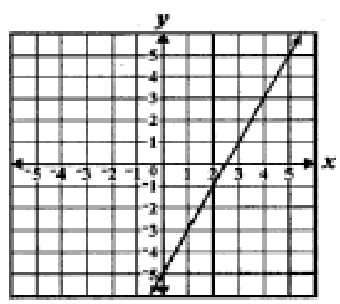
○ A



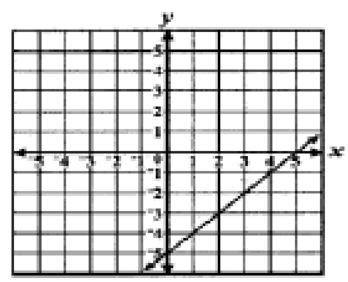
○ C



○ B



○ D



9. A group of students has a total of 29 pencils, and everyone has at least one pencil. Six students have 1 pencil each, 5 students have 3, and the rest have 2. How many students have only 2 pencils?

○ A 4

○ B 6

○ C 8

○ D 9

ALGEBRA POST TEST PART A

Do not use a calculator on Part A (Questions #1-11).

10. Which equation is equivalent to $5x-2(7x+1)=14x$?

- A $-9x-2=14x$
- B $-9x+1=14x$
- C $-9x+2=14x$
- D $12x-1=14x$

11. Find the x- and y-intercepts of the graph of $2x+5y=30$. Show your work below.

x-intercept:

y-intercept:

Show your work:

24980

ALGEBRA POST TEST PART B

You may use a graphing calculator on Part B (Questions #12-30).

12. $\begin{cases} x-y=5 \\ x+y=7 \end{cases}$

What is the solution to the system of equations shown above?

- ○ A $x=6,\ \ y=1$
- ○ B $x=4,\ \ y=3$
- ○ C $x=1,\ \ y=6$
- ○ D $x=-1,\ y=7$

13. The Arcadia Theater charges \$4 for adult tickets and \$3 for student tickets. Mr. Steele purchased 9 tickets (some student and some adult) for \$31. Which system of equations could be used to find a, the number of adult tickets, and s, the number of student tickets Mr. Steele purchased?

- ○ A $\begin{cases} a+s=31 \\ 4a+3s=9 \end{cases}$
- ○ B $\begin{cases} 4a+3s=31 \\ a+s=9 \end{cases}$
- ○ C $\begin{cases} 3a+4s=31 \\ a+s=9 \end{cases}$
- ○ D $\begin{cases} 3a+4s=9 \\ a+s=31 \end{cases}$

24980

ALGEBRA POST TEST PART B

You may use a graphing calculator on Part B (Questions #12-30).

14. Which of the following is a solution to the equation?

$$x^2 - 13x + 40 = 0$$

- A -8
- B 4
- C 5
- D 10

15. Which of the following is an equation of the line through $(-3,4)$ with slope $\frac{1}{2}$?

- A $y-4=\frac{1}{2}(x+3)$
- B $y+3=\frac{1}{2}(x-4)$
- C $y-4=-2(x+3)$
- D $y-4=2(x+3)$

16 If $f(x)=-2x^2+x-5$, what is $f(3)$?

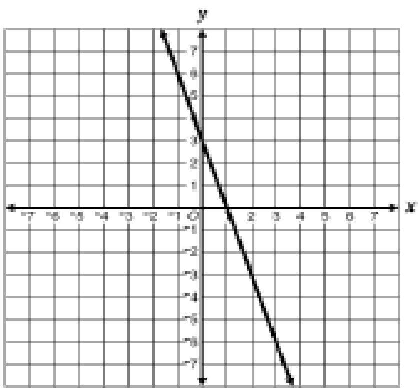
- A -20
- B -14
- C 16
- D 34

17. Which value of x is a zero of the function?

$$f(x) = x^2 - 8x + 7$$

- A 8
- B -7
- C -1
- D 7

18. The graph of the function $f(x) = -3x + 3$ is shown.

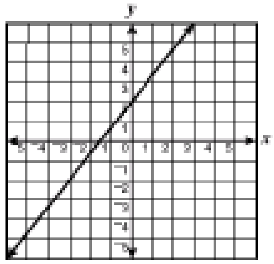


What is the value of $f(3)$?

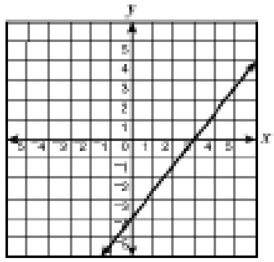
- A 3
- B 0
- C -2
- D -6

19. Which graph best represents the function $y = {}^{-}\frac{4}{3}x + 2$?

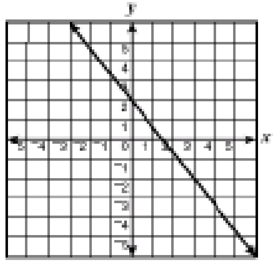
○ **A**



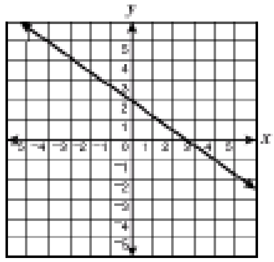
○ **B**



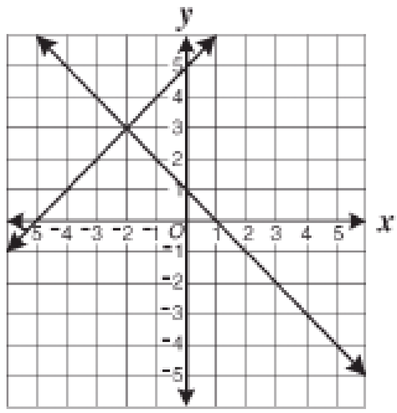
○ **C**



○ **D**



20. This is a graph of a system of equations.



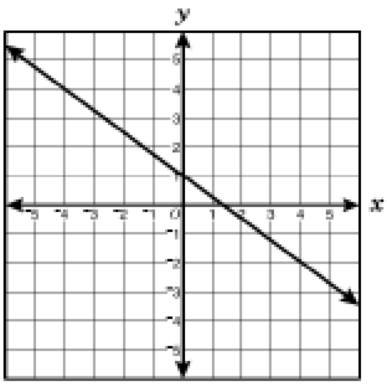
Which is most likely the solution to the system of equations shown?

- A (0,5)
- B (1,0)
- C (3, -2)
- D (-2,3)

21. Jill was looking at a picture of herself and 3 friends. She measured the height of her image as 10 centimeters. If Jill is actually 60 inches tall, which equation can she use to find h, the actual height in inches, of one of her friends who is c centimeters tall in the picture?

- A $h = 10c$
- B $h = 6c$
- C $h = \frac{5}{3}c$
- D $h = \frac{1}{6}c$

22. The graph of $y = {}^{-}\frac{3}{4}x + 1$ is shown.



If the line in the graph is shifted up 2 units, which is the equation of the new line?

- A $y = \frac{3}{4}x + 3$
- B $y = \frac{3}{4}x + 2$
- C $y = {}^{-}\frac{3}{4}x + 2$
- D $y = {}^{-}\frac{3}{4}x + 3$

23. Ben's Bakery charges a fee of $2d + 25$ to deliver d boxes of baked goods while Dan's Bakery charges $3d + 20$. Which expression describes how much more Dan's Bakery charges than Ben's Bakery?

- A $5d + 45$
- B $d - 5$
- C $d + 5$
- D $-d + 5$

24. Solve: $3(x+5)=2x+35$

Step 1: $3x+15=2x+35$

Step 2: $5x+15=35$

Step 3: $5x=20$

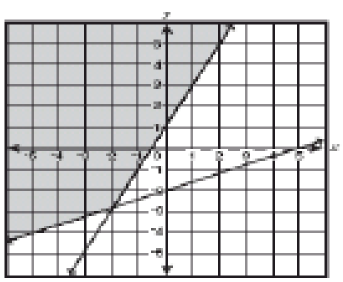
Step 4: $x=4$

Which is the first *incorrect* step in the solution shown above?

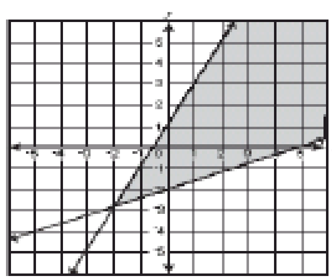
- ○ A Step 1
- ○ B Step 2
- ○ C Step 3
- ○ D Step 4

25. Which graph *best* represents the solution to this system of inequalities?

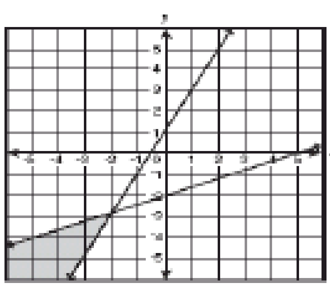
$$\begin{cases} 2x \geq y-1 \\ 2x-5y \leq 10 \end{cases}$$



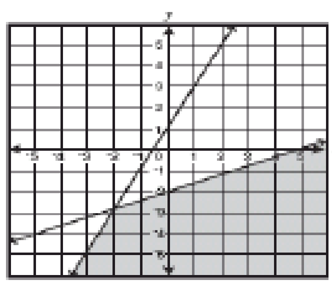
○ A



○ C



○ B



○ D

26. Which of the following is the x-intercept of the line $y = 2x - 5$?

- A $(0, -5)$
- B $(-\frac{5}{2}, 0)$
- C $(0, \frac{5}{2})$
- D $(\frac{5}{2}, 0)$

27. Solve the equation: $\frac{2x}{3} + \frac{1}{2} = \frac{x}{4} - \frac{1}{3}$

- A $-\frac{1}{3}$
- B 2
- C -2
- D -7

In questions 28 and 29, Shamrock Apartments bought a $200,000 building and the value of the building decreases by $8,000 per year over a 25-year period.

28. Find an equation for the value $v(t)$ of the building as a linear function of the time since the building was placed in service.

Show your work here.

Answer:

29. What does the slope of $v(t)$ represent?

- ○ A The initial cost of the house.
- ○ B The amount the house decreases each year.
- ○ C The amount the house decreases after 25 years.
- ○ D The final value of the house.

30. What is the solution to the inequality shown below?

$$-2x+3>7$$

- ○ A $x<-5$
- ○ B $x<-2$
- ○ C $x<2$
- ○ D $x<3$

Did you skip any pages accidentally? Go back and check your math.

APPENDIX E
LONG EXCERPT FROM MS. MA'S TRANSCRIPT

S: It's one.

T: Ok, so if you were a 4X or an 8X, keep that in mind. So as far as numbers go it looks like one person answered neg. 4 and thirteen of you answered 1. Who answered 1? Hailey, would you bring your worksheet up and let's just show how you solved that. Ok, so she said to undo the subtract you add four to both sides so the neg. 4 and the 4 zero out leaving us with 4X on the left side and zero plus 4 gives us 4 on the right side. So then she undid the multiply by 4 by dividing by 4 and what is your answer? If you didn't get one could you please correct it and have a one there in that blank when you're finished? Thank you. Alright.

SS: [work quietly]

T: The second problem was the equation neg. 2X plus 8 equals zero. Tell me what you worked that answer out to be. One more. Ok, I'm glad not to see any X's this time. I saw some solutions and we have some useful answers. If I went with the majority, it looks like most of you said 4, so who in here did give an answer of 4? Sammy, can we borrow your paper and check it? Ok, you always undo your additions and subtractions first, so Sammy chose to undo the 8 by subtracting 8. So that zeros out the 8 and leaves us with neg. 2X on the left and zero plus neg. 8 is neg. 8. Then she undid the multiply by neg. 2 by dividing by neg. 2 and she realized that two neg. became a pos. Ok, those of you who answered neg. 4 could it be just a sign error? Yeah? Ok. Will the person who answered 16 maybe explain what they might have done? Anybody share? I guess my guess would be instead of taking 8 and dividing by 2, what did they probably do?

S: Multiply.

T: Multiplied, so there I would choose to undo with division rather than multiplication. Excellent, Sammy. Do you [inaudible] this?

S: Yes.

T: Problem number 3, tell me what you got for a solution to 3X minus 9 equals zero. 16 of 18 so far; let's see how you're doing. Oh, I love this. Everybody got the same answer of 3. Ok, and now we have 18 of 18. So you add 9 and you divide by 3. Excellent. Ok, now we're getting a little bit harder equation. What did you do with 1/2X minus 3 equals zero? What did you end up solving that to be? Good. What you want to do is be sure you add to both sides so there's your equals. That's what's dividing the sides. So unfortunately you put plus 3 on the same side. It would be nice to move this one over to the other side. Yeah, that would be better. Ok. Ok. Must be a little harder problem 11 of 18 so far. Let's see how it's going. Those are the two answers I would have predicted and what that tells me is I think you worked out the first step correctly. We were working on the equation 1/2X minus 3 equals zero. So Andrea, what would you do as a first step to solving this problem?

A: I'd add 3.

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BIOGRAPHICAL SKETCH

Yasemin Gunpinar earned her undergraduate and graduate degrees from the Department of Elementary Mathematics Education in Marmara University, Istanbul, Turkey. Her master's thesis was titled "The Effect of Project Supported Education in Elementary Number Theory Course on Student Achievement". The Scientific and Technological Research Council of Turkey (TUBITAK) sponsored her while pursuing her master's degree. From 2006 to 2008, she taught mathematics in both private and public schools in Turkey. She co-wrote a publication titled "*Supporting Lesson Book Including Subject Teaching for All Courses in Elementary 6th Grades*". She enjoyed teaching mathematics as much as she enjoyed studying. This feeling influenced her desire to focus on mathematics education as her future career goal. Although she enjoyed her initial work as a teacher, she needed to reach more students and teachers to fulfill her potential as an educator.

While exploring ways to reach her goal, she applied for and won the Scholarship for Graduate Study abroad from the Turkish Ministry of Education. Since 2009, this scholarship enabled her to be enrolled as a doctoral student in the Ph.D. program in Curriculum and Instruction with an emphasis in Mathematics Education at the University of Florida.