Derivatives Pricing and Term Structure Modeling

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STOCKHOLM SCHOOL
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Stockholm, August 2007
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Introduction and Summery

The thesis consists of three papers and covers two topics: derivatives pricing and term structure modeling. The first paper is concerned with the pricing of equity swaps, in particular so called quanto equity swaps. The second paper deals with the pricing of index-linked instruments such as inflation indexed swaps and swaptions. The third paper investigates the term structure of futures prices.

Paper 1: Pricing Equity Swaps in an Economy with Jumps

Market data reveals that asset prices have discontinuities and that the distribution of asset returns exhibits skewness and excess kurtosis over the normal distribution. This has been empirically tested both for stock prices, and foreign exchange rates by for example Ball and Torous (1985), Campa et al. (1998) and Jorion (1989). One way of obtaining distributions that are consistent with market data and allowing the price processes to have discontinuities is to model the asset price processes by jump-diffusion models. In this paper every asset price processes in the economy is modeled as a jump-diffusion. The processes are generated by a standard multidimensional Wiener process and a general marked point process.

A swap is an agreement between two parties to exchange cash flows. In an equity swap, one of the parties agrees to pay a cash flow equal to the return on a stock or an equity index and in exchange she receives from the other a cash flow that can be either a fixed or floating interest rate or the return on another stock or equity index. Equity swaps are widely used in the international financial markets. They can be used to hedge an existing portfolio, or as a cost effective alternative to a direct investment in the underlying asset.

We provide analytical pricing formulas for a wide range of equity swaps including quanto equity swaps which are settled in one currency but payed out in another. Our results are an extension of the results of Liao and Wang (2003). The pricing formulas are derived by using martingale methods and the technique of convexity corrections. The martingale method is the key that enables the extension to jump-diffusions.

Paper 2: Inflation Indexed Swaps and Swaptions[1]

The modern market for inflation linked products started around the 1980s when several countries including the United Kingdom, Canada and Sweden started to issue inflation indexed bonds. The derivatives market begun approximately at the same time as the United states issued the first inflation indexed bonds. The most common inflation linked derivatives are inflation indexed swaps, caps and floors. Inflation is defined as the percentage change of a particular reference index. Usually the reference index is a consumer price index. Inflation linked products can be used to hedge future cash flow against inflation. That is particularly attractive to investors that seek asset-liability matching such as for example insurance companies. Inflation linked derivatives are sometimes viewed as an asset class of its own and used for risk diversification purposes.

In this paper we provide analytical pricing formulas for inflation indexed swaps and options on such swaps, i.e. swaptions. We model real and nominal interest rates and the reference index by the Heath-Jarrow-Morton (HJM) approach extended to the point process case. For the simplified case when the price processes are only driven by a multidimensional Wiener process, we also show how to hedge inflation indexed swaps. Furthermore we price options on inflation indexed bonds.

Previously Jarrow and Yildirim (2003) has applied the HJM approach without jumps to inflation markets. They assumed a priori that the foreign-currency analogy can be applied. This foreign-currency analogy was introduced by Hughston (1998) and the idea is to regard nominal assets as domestic assets, real assets as foreign assets, and the consumer price index as the exchange rate between the nominal and the real market. It is not clear a priori that this analogy is true, since there are differences between the markets. We show that the HJM approach, allowing for both jumps and stochastic volatility, can be used for the pricing of index linked instruments without assuming the foreign-currency analogy a priori. In fact, this proves the foreign-currency analogy.

Our pricing formulas of inflation indexed swaps are an extension to the point process case of the results by Mercurio (2005). Other studies that assumes non-Gaussian interest rates in an inflation market context include Slinko (2006) and Mercurio and Moreni (2006). To our knowledge no previous studies has considered the pricing of inflation indexed swaptions and options on inflation indexed bonds.

Paper 3: Shifts in the Term Structure of Futures Prices

A standard concept in interest rate theory is the yield curve, which essentially is the curve of bond prices plotted against time of maturity. This curve is also referred to as the term structure of interest rates. Modeling the term structure thus means modeling interest rates of all possible maturities. Like the bond price, the futures

[1]This paper is forthcoming in Journal of Banking and Finance.

price depends on the time of maturity. Thus modeling the term structure of futures prices is equivalent to modeling futures prices of all possible maturities.

We consider an arbitrage free futures price model of HJM type which is driven by both a Wiener process and a marked point process. We thus take the futures prices as a priori given instruments. Assuming the arbitrage free model of HJM type; we investigate if, when and how the natural logarithm of this curve can be represented by a curve that only changes by parallel shifts. We find necessary and sufficient conditions on the stochastic differential equation for the futures price, and the shift function. The analysis is repeated for the case when the log futures price model can only change by proportional shifts.

Affine structures are often desired for tractability reasons. Parallel and proportional shifts are special cases of a single factor affine term structure. Finally we consider all other single factor affine term structures, besides parallel and proportional structures. We find necessary and sufficient conditions for the purely Wiener driven log futures model to admit such single factor affine shifting curve and we characterize the shift functions. In addition we conclude that every model for a futures price with a tradable non-dividend paying underlying asset that admits a parallel shifting log futures price curve implies constant short rates. We also find that every purely Wiener driven log futures model that admits a proportional shifting curve will eventually be absorbed at zero.

In interest rate theory the questions of consistent parallel and proportional shifts of the yield curve have been examined in Armerin et al. (2005). Our work parallels that by Armerin et al. and rests on the approach of invariant manifolds and consistent forward rate curves introduced by Björk and Cristensen (1999) and extended by Filipovic and Teichmann (2003) and Filipovic (2001). Other studies on the term structure of futures prices that takes the HJM approach includes papers by Reisman (1991), Cortazar and Schwartz (1994), Björk and Landén (2002), Björk et al. (2006), Miltersen and Schwartz (1998), Hilliard and Reis (1998) and Gaspar (2004).

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Paper 1

Paper 1

Pricing Equity Swaps in an Economy with Jumps

Empirical evidence confirms that asset price processes exhibits jumps and that asset returns are not Gaussian. We provide a pricing model for equity swaps, when all the asset price processes in the economy are allowed to jump. The market is driven by a general marked point process as well as by a standard multidimensional Wiener process. In order to obtain closed-form solutions of the swap values, we assume that all parameters in the asset price processes are deterministic, but possibly functions of time. We derive swap values using martingale methods and the technique of convexity corrections rather than using replicating portfolios. Our results are an extension of the results of Liao & Wang (2003). The martingale method is the key that enables the extension.

1 Introduction

Market data reveals that asset prices have discontinuities and that the distribution of asset returns exhibits skewness and excess kurtosis over the normal distribution. This has been empirically tested both for stock prices, and foreign exchange rates by for example Ball and Torous (1985), Campa et al. (1998) and Jorion (1989). One way of obtaining distributions that are consistent with market data and allow the price processes to have discontinuities is to model the asset price processes by jump-diffusion models. In this paper we will allow all asset price processes in the economy to be jump-diffusions.

An equity swap is an agreement between parties to exchange cash flows. In an equity swap, one of the parties agrees to pay a cash flow equal to the return on a stock or an equity index and in exchange she receives from the other a cash flow that can be either a fixed or floating interest rate or the return on another stock or equity index. Equity swaps are widely used in the international financial markets. They can serve as a hedging instrument to hedge an existing portfolio. They can also be used as an alternative to invest directly into the stock market, in order to lower transaction costs. This is especially attractive if the stock market has low liquidity or if there is tax or regulatory restrictions in the market.

One of the early studies on the pricing of equity swaps was made by Chance and Rich (1998). They used arbitrage-free replicating portfolios to derive pricing formulas for a number of equity swaps such as plain vanilla equity swaps, variable notional swaps, and cross currency equity swaps. Their result on swaps with a variable notional relies on an assumption of deterministic interest rates. However, their results on swaps with a constant notional are model independent.

Kijima and Muromachi (2001) have studied equity swaps in a stochastic interest rate economy. In their model, the market is driven by Wiener processes and the volatilities of the interest rates and of the equity prices are assumed to be deterministic functions of time, implying a Gaussian economy. They used martingale methods to derive their results. They showed that if the notional principal is variable and not constant, the equity price process affects the swap rate.

Recently, Liao and Wang (2003) have provided a generalized formula for pricing equity swaps with a constant notional principal. They value swaps covering the international capital markets, allowing the underlying equity to be foreign and the notional principal to be specified in an arbitrary currency. Liao and Wang assumes a Gaussian economy where the market is driven by a multidimensional Wiener process and the volatilities of bond prices, equity prices and exchange rates are assumed to be deterministic. Their study shows that for swaps on foreign equity markets, the swap value is dependent of both the dynamics of the equity price process and the exchange rate process. Liao and Wang use the method of arbitrage-free replicating portfolios. Their results rely on the result of Musiela and

Rutkowski (1997) on the pricing of equity-linked foreign exchange options.

The purpose of this paper is to provide a model for the valuation of equity swaps when all asset price processes in the economy are allowed to jump. Every asset price process will be modelled by a standard multidimensional Wiener process and a general marked point process. To ensure closed-form solutions we assume that the intensity of the point process as well as the volatilities of all asset prices and exchange rates, with respect to both the Wiener process and the point process, are deterministic. Our results are an extension of the results of Liao and Wang (2003). We use martingale methods and the technique of convexity corrections developed by Pelsser (2000). By using martingale methods, rather than the method of replicating portfolios, there is in no need for pricing formulas on equity-linked foreign exchange options in order to price equity swaps. Hence the martingale method is the key that enables the extension to jump-diffusions.

This article is organized as follows:

- In the first Section we extend the convexity corrections method of Pelsser (2000) to the case where we have random processes driven by both a Wiener process and a marked point process. This is one of the main tools used to derive our results.
- In Section 3 we present our model of an international capital market and specify our assumptions.
- In Section 4 a general definition of equity swaps are given.
- Section 5 is devoted to pricing. In Section 5.1 we derive the price of the most simple equity swaps, the so called vanilla swaps. The real contribution is in Section 5.2, where we price quanto payoffs. This leads to a generalized pricing formula for equity swaps, presented in Section 5.3.
- Section 6 concludes.
- Appendix A contains two useful formulas on Ito calculus for jump diffusion driven processes that will be used repeatedly in this article. In Appendix B we have stated some of the well known, and for this article crucial, Theorems in measure theory.

2 Convexity corrections

In order to price equity swaps we will assume that the market is free of arbitrage and we will use martingale methods. Thus, pricing equity swaps is essentially equivalent to finding an appropriate numeraire under which we can calculate the expected value of the payoffs. With our choice of numeraire, it turns out that we have to be able to calculate the expected value of the product of two martingales. In this Section, we show how this can be done by extending the convexity corrections method of Pelsser (2000) to the case where we have random processes driven by both a Wiener process and a marked point process. This result will be used repeatedly in the preceeding Sections.

Since the Wiener process that we will use is allowed to be n-dimensional, the volatility with respect to this process is an n-dimensional row vector. We will use the symbol $(\cdot)^*$ to denote the transpose of $(\cdot)$.

Proposition 2.1 *Consider a filtered probability space $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ that carries both an n-dimensional Wiener process W and a general marked point process $\mu(dt, dv)$ on $\mathbb{R}_+ \times V$ with intensity measure $\lambda_t(dv)dt$. The filtration $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by both W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$. Assume that $\sigma^M(t)$, $\sigma^N(t)$, $\delta^M(t,v)$, $\delta^N(t,v)$ and $\lambda_t(dv)dt$ are deterministic. Let $M(t), N(t)$ be two stochastic processes with dynamics:*

$$dM(t) = M(t-)\left[\sigma^M(t)dW(t) + \int_V \delta^M(t,v)\tilde{\mu}(dt,dv)\right] \tag{1}$$

$$dN(t) = N(t-)\left[\sigma^N(t)dW(t) + \int_V \delta^N(t,v)\tilde{\mu}(dt,dv)\right] \tag{2}$$

Note that M and N are P-martingales. Then

$$E_t^P[M(T)N(T)] = M(t)N(t)e^{\int_t^T \left(\sigma^M(u)\sigma^{N*}(u) + \int_V \delta^M(u,v)\delta^N(u,v)\lambda_u(dv)\right)du} \tag{3}$$

Remark 2.1 *The exponential term in equation (3) is sometimes called the convexity correction. See Pelsser (2000)*

Proof. Define $Y(t) = M(t)N(t)$. To find the P-dynamics of $Y(t)$ we use Ito calculus and apply Lemma A.1 stated in Appendix A:

$$\begin{aligned} dY(t) &= Y(t)\left\{\sigma^M(t)\sigma^{N*}(t) + \int_V \delta^M(t,v)\delta^N(t,v)\lambda_t(dv)\right\}dt \\ &+ Y(t)\left\{\sigma^M(t) + \sigma^N(t)\right\}dW(t) \\ &+ Y(t-)\int_V \left\{\delta^M(t,v) + \delta^N(t,v) + \delta^M(t,v)\delta^N(t,v)\right\}\tilde{\mu}(dt,dv) \end{aligned}$$

Since dW and $\tilde{\mu}(dt,dv)$ are P-martingale increments, it follows that

$$E_t^P[Y(T)] = Y(t) + E_t^P\left[\int_t^T Y(u)A(u)du\right]$$

where

$$A(u) = \sigma^M(u)\sigma^{N^*}(u) + \int_V \delta^M(u,v)\delta^N(u,v)\lambda_u(dv)$$

Moving the expectation within the integral sign in the du-integral, noting that A is non stochastic, and defining $m(s) = E_t^P[Y(s)]$ we obtain:

$$m(s) = Y(t) + \int_t^s m(u)A(u)du$$

Taking the s-derivative of this equation, we obtain the ordinary differential equation:

$$\begin{aligned} \dot{m}(s) &= m(s)A(s) \\ m(t) &= M(t)N(t) \end{aligned}$$

Hence

$$m(T) = M(t)N(t)e^{\int_t^T A(u)du}$$

∎

3 The Economy

In this Section we specify the dynamics of the processes governing the stock prices, bond prices, money market accounts and the exchange rates for one domestic market d, and three foreign markets f, g and h.

We consider an international financial market living on a filtered probability space $(\Omega, \mathcal{F}, Q^d, \underline{\mathcal{F}})$ where Q^d is the domestic risk neutral probability measure. The probability space carries both an n-dimensional Wiener process W and a general marked point process $\mu(dt, dv)$ on $\mathbb{R}_+ \times V$ with predictable intensity measure $\lambda_t(dv)dt$. The filtration $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t \geq 0}$ is generated by both W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.

We assume that there are no arbitrage possibilities, i.e. the market is arbitrage free.

Let $r(t), r_f(t), r_g(t)$ and $r_h(t)$ denote the domestic, f-foreign, g-foreign and h-foreign risk free interest rates. We allow all the asset prices to jump and assume that the Q^d-dynamics of the domestic stock price, $S(t)$, and the domestic bond price, $p(t,T)$, of a bond with maturity date T are given by:

$$dS(t) = S(t-)\left[r(t)dt + \sigma(t)dW(t) + \int_V \delta(t,v)\tilde{\mu}(dt,dv)\right] \quad (4)$$

$$dp(t,T) = p(t-,T)\left[r(t)dt + b(t,T)dW(t) + \int_V \beta(t,T,v)\tilde{\mu}(dt,dv)\right] \quad (5)$$

That the drift term is equal to the domestic short rate under the risk neutral measure is standard and follows from the assumption of arbitrage free markets. (see for example Björk (2004) and Björk et al. (1997)).

Let X^i be the exchange rate of the i-foreign currency for $i = f, g, h$. Hence X^f is the price in domestic units of one unit of the f-foreign currency. Similarly, X^g is the price in domestic units of the g-foreign currency and X^h is the price in domestic units of the h-foreign currency. The Q^d-dynamics of the exchange rates are assumed to be given by

$$dX^i(t) = X^i(t-)\left[\{r(t) - r_i(t)\}\, dt + \gamma_i(t)dW(t) + \int_V \xi_i(t,v)\tilde{\mu}(dt,dv)\right] \quad \text{for } i = f, g, h \tag{6}$$

Again general arbitrage free pricing theory explains the specification of the drift term of the exchange rate under the domestic risk neutral measure.

Let B denote the domestic money market account and B^i denote the i-foreign money market account for $i = f, g, h$. The dynamics of the domestic and foreign money market accounts are given by:

$$dB(t) = r(t)B(t)dt \tag{7}$$

$$dB^i(t) = r_i(t)B^i(t)dt \quad \text{for } i = f, g, h \tag{8}$$

To denote the i-foreign risk neutral measures, Q^i are used for $i = f, g, h$. When we mark variables, we will suppress Q, so for example W^f is a Wiener process under Q^f. For the forward measures we use $Q^{T,d}$, $Q^{T,f}$, $Q^{T,g}$ and $Q^{T,h}$ for the domestic, f-foreign, g-foreign and h-foreign measures respectively. Again, we suppress Q when we mark variables. So for example $W^{T,d}$ is a Wiener process under $Q^{T,d}$ and $E_{T_1}^{T_2,d}[\cdot]$ denotes the conditional expectation given $\mathcal{F}_{T_1}$ of $(\cdot)$ under the domestic T_2-forward measure $Q^{T_2,d}$.

In the next Proposition, we derive the dynamics of the foreign stock prices and bond prices under the domestic risk neutral measure. We let $S^f(t)$, $S^g(t)$ and $S^h(t)$ denote the f-foreign, g-foreign and h-foreign stock prices at time t. Similarly let $p(t,T)^f$, $p(t,T)^g$ and $p(t,T)^h$ denote the f-foreign, g-foreign and h-foreign bond prices at time t for bonds with maturity date T.

Proposition 3.1 *The Q^d-dynamics of the foreign stock prices, $S^f(t), S^g(t), S^h(t)$, and the foreign bond prices, $p^f(t,T), p^g(t,T), p^h(t,T)$, of bonds with maturity date*

T are given by:

$$\begin{aligned}
\frac{dS^i(t)}{S^i(t-)} &= \left\{r_i(t) - \gamma_i(t)\sigma_i^*(t) - \int_V \delta_i(t,v)\xi_i(t,v)\lambda_t(dv)\right\} dt \\
&+ \sigma_i(t)dW(t) + \int_V \delta_i(t,v)\tilde{\mu}(dt,dv) \quad \text{for } i = f,g,h \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\frac{dp^i(t,T)}{p^i(t-,T)} &= \left\{r_i(t) - \gamma_i(t)b_i^*(t,T) - \int_V \beta_i(t,T,v)\xi_i(t,v)\lambda_t(dv)\right\} dt \\
&+ b_i(t,T)dW(t) + \int_V \beta_i(t,T,v)\tilde{\mu}(dt,dv) \quad \text{for } i = f,g,h \qquad (10)
\end{aligned}$$

Proof. By standard results, $S^f(t)/B^f(t)$ is a Q^f-martingale. Hence under the f-foreign risk neutral measure, Q^f, the dynamics of $S^f(t)$ are given by

$$dS^f(t) = S^f(t-)\left[r_f(t)dt + \sigma_f(t)dW^f + \int_V \delta_f(t,v)\tilde{\mu}^f(dt,dv)\right]$$

where $W^f(t)$ is a Wiener process under Q^f and $\tilde{\mu}^f(dt,dv) = \mu(dt,dv) - \lambda_t^f(dv)dt$ and $\lambda_t^f(dv)dt$ is the Q^f-intensity.

Define the likelihood process $L_t^{Q,d/Q,f}$ by

$$L_t^{Q,d/Q,f} = \frac{dQ^d}{dQ^f} \quad \text{on } \mathcal{F}_t$$

where $0 \le t \le T$, for some finite T.

Then by the Multi-Currency Change of Numeraire Theorem (Theorem B.3 in Appendix B), it follows that

$$L^{Q,d/Q,f}(t) = \frac{B(t)}{B^f(t)X^f(t)}X^f(0)$$

To get the dynamics of $L^{Q,d/Q,f}(t)$, we use Ito and apply Lemma A.1 to $B^f(t)X^f(t)$, and thereafter Lemma A.2 to $B(t)/(B^f(t)X^f(t))$ and bear in mind that $L^{Q,d/Q,f}$ is a Q^f-martingale. We get

$$dL^{Q,d/Q,f}(t) = L^{Q,d/Q,f}(t-)\left[-\gamma_f(t)dW^f - \int_V \frac{\xi_f(t,v)}{1+\xi_f(t,v)}\tilde{\mu}^f(dt,dv)\right]$$

Thus by the Girsanov Theorem, Theorem B.1 in Appendix B

$$\begin{aligned}
dW^f &= -\gamma_f^*(t)dt + dW \\
\lambda_t(dv) &= \frac{1}{1+\xi_f(t,v)}\lambda_t^f(dv)
\end{aligned}$$

Hence, the dynamics of $S^f(t)$ under the domestic risk neutral measure, Q^d, is

$$\begin{aligned}\frac{dS^f(t)}{S^f(t-)} &= r_f(t)dt - \gamma_f(t)\sigma_f^*(t)dt + \sigma_f(t)dW - \int_V \delta_f(t,v)\lambda^f(dv)dt \\ &+ \int_V \delta_f(t,v)\tilde{\mu}(dt,dv) + \int_V \delta_f(t,v)\lambda_t(dv)dt \\ &= \left\{r_f(t) - \gamma_f(t)\sigma_f^*(t) - \int_V \delta_f(t,v)\xi_f(t)\lambda_t(dv)\right\}dt \\ &+ \sigma_f(t)dW + \int_V \delta_f(t,v)\tilde{\mu}(dt,dv)\end{aligned}$$

Hereby equation (9) is proved for $i = f$. The Proof is completed by proceeding in a similar way for the other five assets, using the standard results that $p^f(t,T)/B^f(t)$ is a Q^f-martingale, $S^g(t)/B^g(t)$, $p^g(t,T)/B^g(t)$ are Q^g-martingales and $S^h(t)/B^h(t)$, $p^h(t,T)/B^h(t)$ are Q^h-martingales. ∎

In subsequent Sections, we will use Proposition 2.1 on convexity corrections. In order to do that we need to assume that the intensity of the point process as well as the volatilities of all asset prices and exchange rates, with respect to both the Wiener process and the point process, are deterministic.

Assumption 3.1 *We assume that* $\lambda_t(dv)$, $\sigma(t)$, $\sigma_f(t)$, $\sigma_g(t)$, $\sigma_h(t)$, $b(t,T)$, $b_f(t,T)$, $b_g(t,T)$, $b_h(t,T)$, $\gamma_f(t)$, $\gamma_g(t)$, $\gamma_h(t)$, $\delta(t,v)$, $\delta_f(t,v)$, $\delta_g(t,v)$, $\delta_h(t,v)$, $\beta(t,T,v)$, $\beta_f(t,T,v)$, $\beta_g(t,T,v)$, $\beta_h(t,T,v)$, $\xi_f(t,v)$, $\xi_g(t,v)$ *and* $\xi_h(t,v)$ *are deterministic.*

Note that the volatilities are independent of the measure under which the dynamics of the assets are specified.

4 Defining Equity Swaps

A swap is an agreement between two counter parties to exchange cash flows. The agreement specifies the cash flows and the dates when the cash flows are to be paid. In an equity swap, one party agrees to pay cash flows equal to the return on a stock or an equity index in a particular currency on a specific notional principal over a pre specified time period. In exchange she receives a cash flow from the other party that can be either a fixed or floating interest rate or the return on another stock or equity index in the same currency and based on the same notional principal.

More specifically, let A and B be to counter parties in an equity swap and consider two assets Z_1 and Z_2. Define the return over period T_{j-1} to T_j on asset Z_i by

$$R_i(T_{j-1}, T_j) = \frac{Z_i(T_j)}{Z_i(T_{j-1})} - 1 \quad \text{for } i = 1, 2 \tag{11}$$

Let N denote the notional principal. Then an equity swap written on Z_1 and Z_2 with constant notional principal can be defined as follows:

- The contract starts at time T_0.
- The payment dates are $T_1, T_2, \ldots, T_m$ and $t < T_1 < T_2, < \cdots, T_m$.
- At time T_j party A pays $NR_1(T_{j-1}, T_j)$ to counterpart B and receives $NR_2(T_{j-1}, T_j)$ from B.

Since the cash flows between party A and B are of the same size, but of opposite signs, it suffice to consider the cash flows of party A. Suppose that A enters into a two year long swap contract today, where the first payment date is in one year and the second and last payment date is in two years. Furthermore, assume that the US market is the domestic market for party A. For example let Z_1 be the IBM stock and Z_2 be the Intel stock. Let the notional principal be \$100. Then, in one year from now, party A will receive the 1-year return on the IBM stock times 100 USD and pay the 1-year return on the Intel stock times 100 USD. In two years from now, party A will receive the next 1-year return on the IBM stock times 100 USD and pay the next 1-year return on the Intel stock times 100 USD. This is a domestic two-way equity swap.

Another type of equity swap is the plain vanilla equity swap. In this swap one party receives a cash flow equal to the return on a domestic stock on a notional principal, expressed in the domestic currency, and pays a cash flow equal to interest at a predetermined fixed rate on the same notional principal. Hence if we exchange R_2, in the example above, from the return on the Intel share to a fixed swap rate, then we have an example of a plain vanilla equity swap where party A pays fixed and receives equity return.

So far we have only considered domestic swaps, but there also exist cross currency swaps. Cross currency swaps belong to the type of derivatives that are called quanto derivatives. It is because the cash flows are defined in terms of an asset that is measured in one currency but payed out in another currency. This currency mismatch may seem unnatural but nevertheless, these contracts exist and are widely used. To illustrate a cross currency equity swap, again let Z_1 be the IBM stock and the notional principal be \$100. In contrast to previous examples, let Z_2 be a foreign asset, for example the Ericsson share on the Swedish stock market. Then, in one year from now, party A will receive the 1-year return on the American IBM stock times 100 USD and pay the 1-year return on the Swedish Ericsson stock times 100 USD. In two years from now, party A will receive the next 1-year return on the

IBM stock times 100 USD and pay the next 1-year return on the Ericsson stock times 100 USD. Note that in this example it is the cash flow that party A will pay that has the mismatch feature. The underlying asset of this cash flow is measured in SEK while it is paid in USD.

Let us consider yet another example of a cross currency equity swap. As in the previous example, let Z_1 be the IBM stock on the US market and Z_2 the Ericsson stock on the Swedish market. However, let the notional principal be specified in British pounds, say 100 GBP. Then, in one year from now, party A will receive the 1-year return on the American IBM stock times 100 GBP and pay the 1-year return on the Swedish Ericsson stock times 100 GBP. In two years from now, party A will receive the next 1-year return on the IBM stock times 100 GBP and pay the next 1-year return on the Ericsson stock times 100 GBP. Since the notional principal is specified in British pounds but the domestic market for party A is the US market, we need the foreign exchange rate between USD and GBP in order to get the domestic cash flow for party A. Let $X(T_j)$ denote the exchange rate between USD and GBP at time T_j. Then in terms of the domestic currency the net cash flow that party A will receive in one years time is:

$$X(T_1) \cdot 100 \cdot (R_{IBM}(0,T_1) - R_{ERI}(0,T_1))$$

where $R_{IBM}(s,t)$ and $R_{ERI}(s,t)$ denotes the return over period $[s,t]$ on the IBM share and Ericsson share respectively. The net cash flow, expressed in the domestic currency, that party A will receive in two years from now is:

$$X(T_2) \cdot 100 \cdot (R_{IBM}(T_1,T_2) - R_{ERI}(T_1,T_2))$$

The last example illustrates that if the notional principal is specified in a currency different from the domestic one, the exchange rate enters into the payoff measured in domestic units. Furthermore it is clear that, without loss of generality, the return R can be exchanged with $1+R$. Taking this into account, we rewrite the definition of an equity swap and we state it from the perspective of party A:

Let N^h denote the notional principal specified in currency h and let $X^h(T_j)$ denote the exchange rate between the units of the domestic currency to the units of the currency h. Then an equity swap written on the assets Z_1 and Z_2, with constant notional principal can be defined as follows:

- The contract starts at time T_0.
- The payment dates are $T_1, T_2, \ldots, T_m$ and $t < T_1 < T_2, < \cdots, T_m$.
- At time T_j party A pays $X^h(T_j)N^h Z_1(T_j)/Z_1(T_{j-1})$ units of the domestic currency to the counterpart and receives $X^h(T_j)N^h Z_2(T_j)/Z_2(T_{j-1})$ units of the domestic currency.

Note that if the notional principal is specified in the domestic currency, i.e. if h is the domestic currency, then $X^h = 1$. Since the notional principal is just a size factor, it can be scaled to 1. Therefore without loss of generality we let $N^h = 1$. Hence the cash flows that party A will pay or receive at time T_j can be written as:

$$X^h(T_j)\frac{Z(T_j)}{Z(T_{j-1})} \tag{12}$$

where Z is a domestic or foreign asset and X^h is equal to the h foreign exchange rate if the notional principal is specified in the h-foreign currency and equal to 1 if the principal is specified in the domestic currency.

Now let us consider the economy specified in Section 3. Then the asset Z is the domestic or foreign stock. In the simplest case, Z is the domestic stock and the notional principal is one domestic unit. Hence equation (12) becomes:

$$\Upsilon = \frac{S(T_j)}{S(T_{j-1})} \tag{13}$$

If instead Z is the f-foreign asset, equation (12) becomes:

$$\Phi = \frac{S^f(T_j)}{S^f(T_{j-1})} \tag{14}$$

If Z is the f-foreign asset and the notional principal is specified in the h-foreign currency, equation (12) becomes:

$$\Psi = X^h(T_j)\frac{S^f(T_j)}{S^f(T_{j-1})} \tag{15}$$

In the next Section we will price equity swaps. Pricing an equity swap is equivalent to pricing the cash flows of the equity swap. If we can price the three different types of cash flows Υ, Φ and Ψ, we can price any equity swap with constant notional principal. In fact, it is enough to be able to price cash flow Ψ. This is because Ψ is the most general case and we can get Φ by letting h denote the domestic currency so that X^h is equal to 1. Similarly we can get Υ by letting both h and f denote the domestic currency so that X^h is equal to 1 and S^f is equal to S. It is merely of illustrative purpose that we study all three cash flows Υ, Φ and Ψ, rather than Ψ only.

5 Pricing Equity Swaps

In this Section we will derive a general pricing formula for equity swaps. We will take as given the economy specified in Section 3 and price the cash flows Υ, Φ

and Ψ defined in the previous Section. When we can price these cash flows, we can calculate the difference between any two of them and sum over the number of payment dates. Hence we can price any swap involving these payments. In Section 5.1 we will focus on the relatively simple case of vanilla swaps that involves only the payments of the type Υ. In Section 5.2 we will price the payments Φ and Ψ and this requires a somewhat deeper analysis. Finally, in Section 5.3 we will provide a generalized pricing formula that can be used to price any type of equity swap with constant notional principal.

We will use martingale methods and the technique of convexity corrections described in Section 2. We will let $\Pi[t,\cdot]$ denote the price, in the domestic currency, of the contract $(\cdot)$.

5.1 Vanilla Swaps

In this Section we will price contracts that has a payoff, Υ, given by equation (13) in Section 4. Even though this can be done by using a simple replicating argument we will, for the sake of completeness, provide a proof by martingale methods.

Proposition 5.1 *Let $t \le T_1 \le T_2$. The value at time t, in the domestic currency, of a contract that pays out $\Upsilon = S(T_2)/S(T_1)$ units of the domestic currency at time T_2 is given by:*

$$\Pi[t,\Upsilon] = p(t,T_1)$$

Proof. By using the domestic stock itself as numeraire, we get that

$$\begin{aligned} \Pi[t,\Upsilon] &= S(t)E_t^{S,d}\left[\frac{\Upsilon}{S(T_2)}\right] \\ &= S(t)E_t^{S,d}\left[\frac{1}{S(T_1)}\right] \end{aligned} \tag{16}$$

Noting that $p(T_1,T_1) = 1$, we have:

$$E_t^{S,d}\left[\frac{1}{S(T_1)}\right] = E_t^{S,d}\left[\frac{p(T_1,T_1)}{S(T_1)}\right]$$

Since $p(t,T_1)/S(t)$ is a $Q^{S,d}$-martingale equation (16) simplifies to

$$S(t)E_t^{S,d}\left[\frac{p(T_1,T_1)}{S(T_1)}\right] = S(t)\frac{p(t,T_1)}{S(t)} = p(t,T_1)$$

∎

Remark 5.1 *This result could also have been achieved by a simple replicating argument: If you at time T_1 use one unit of the domestic currency to buy $1/S(T_1)$ units of the domestic stock $S(T_1)$ you will at time $S(T_2)$ have a payoff of $S(T_2)/S(T_1)$. The value at time t of having one unit of the domestic currency at time T_1 must be equal to $p(t, T_1)$.*

Remark 5.2 *Regardless of if one uses the martingale method or the replicating argument to prove proposition 5.1 it should be noted that no assumptions on the dynamics of the assets are needed. The only assumption needed is that the stock is traded assets. Hence this result is model independent.*

Recall from Section 4 that in a domestic two-way equity swap one party receives the stock index return of a domestic stock and pays the return on another domestic stock. Hence both payments are of the type Υ, so the value of the difference of the payments are zero. Consequently, we have the following Corollary.

Corollary 5.1 *The value, at any time up until maturity, of a domestic two-way equity swap is zero.*

So far we have completely neglected the so called two-way equity swap with exchange rate risk. This is because, from a mathematical point of view, it is equivalent to a domestic two-way equity swap. In the two-way equity swap with exchange rate risk one party pays a cash flow equal to the currency-adjusted return on a foreign stock on a notional principal, expressed in the domestic currency, and receives a cash flow equal to the currency-adjusted return on another foreign stock on the same notional principal. Hence both cash flows are of the type:

$$\frac{X^f(T_2)S^f(T_2)}{X^f(T_1)S^f(T_1)} \tag{17}$$

Note that $S^f(t)$ units of the f-foreign currency $X^f(t)$ are worth $S^f(t)X^f(t)$ in the domestic currency. So, buying the f-foreign currency and investing it in the f-foreign stock is equivalent to investing it in a domestic asset with price process $\hat{S}^f(t)$, where

$$\hat{S}^f(t) = X^f(t)S^f(t)$$

Hence equation (17) can be written in terms of the domestic asset $\hat{S}^f$:

$$\frac{\hat{S}^f(T_2)}{\hat{S}^f(T_1)}$$

This is the same type of cash flow as in a domestic two-way equity swap. Consequently, the value of a two-way equity swap with exchange rate risk must equal the value of a domestic two-way equity swap which is zero.

5.2 Quanto Swaps

In this Section we will price contracts with payoffs equal to $\mathcal{Z}$ and Ψ given by equation (14) and (15) in Section 4. These contracts are payed out in the domestic currency even though they are written on foreign equity. This currency mismatch is the reason to why they are called quanto derivatives.

Both contacts have a payoff that is:

- payed out at time T_2
- determined by foreign assets
- payed out in the domestic currency

The first contract that we consider has the payoff function

$$\Phi = \frac{S^f(T_2)}{S^f(T_1)}$$

This contract pays out an amount in the domestic currency that is based on the foreign stock return over the period T_1 to T_2. So, even though investing in a foreign market, there is no currency risk. For example if the US-market is the domestic market, the notional principal is one USD and S^f is the price of the Swedish Ericsson stock, then this contract gives you, at time T_2, the gross return on the Ericsson stock over period T_1 to T_2 times one dollar. Hence if the price of Ericsson at time T_1 is 20 SEK and at time T_2 is 25 SEK, the contract will pay out 1.25 USD at time T_2.

The second contract that we will price has the payoff function

$$\Psi = X^h(T_2)\frac{S^f(T_2)}{S^f(T_1)}$$

Hence the contract pays out, in the domestic currency, the f-foreign stock return over the period T_1 to T_2 scaled by the exchange rate between the domestic currency and the h-foreign currency. For example if the US-market is the domestic market, the notional principal is one GBP, X^h is the exchange rate between USD and GBP and S^f is the price of the Swedish Ericsson stock, then this contract gives you, at time T_2, the gross return on the Ericsson stock over period T_1 to T_2 times the exchange rate times one dollar. Hence if the price of Ericsson at time T_1 is 20 SEK and at time T_2 is 25 SEK and the exchange rate at time T_2 is 2 USD/GBP, then the contract will pay out 2.50 USD at time T_2.

Before we price the first contract we will state five lemmas. The first two give the dynamics of some processes that will be needed in the subsequent. In the next two, we calculate some expected values that are needed. These two lemmas rely on

Proposition 2.1 on convexity corrections. The last lemma is also about expected values and relies on the two previous ones. The benefit of having these lemmas, is that the Proofs of the Propositions later on follows quite easily.

Lemma 5.1 *Let $0 \leq t \leq T_2$. Let $Q^{T_2,d}$, $Q^{T_2,f}$ and $Q^{T_2,h}$ denote the T_2-forward measure for the domestic, the f-foreign and the h-foreign market respectively. Define the likelihood processes $L_t^{T_2,d/T_2,f}$ and $L_t^{T_2,h/T_2,f}$ as:*

$$L_t^{T_2,d/T_2,f} = \frac{dQ^{T_2,d}}{dQ^{T_2,f}} \quad \text{on } \mathcal{F}_t \tag{18}$$

$$L_t^{T_2,h/T_2,f} = \frac{dQ^{T_2,h}}{dQ^{T_2,f}} \quad \text{on } \mathcal{F}_t \tag{19}$$

Then the dynamics of $L_t^{T_2,d/T_2,f}$ and $L_t^{T_2,h/T_2,f}$ under $Q^{T_2,f}$ are given by:

$$\frac{dL^{T_2,d/T_2,f}(t)}{L^{T_2,d/T_2,f}(t-)} = A(t,T_2)dW^{T_2,f}(t) + \int_V B(t,T_2,v)\tilde{\mu}^{T_2,f}(dt,dv) \tag{20}$$

$$\frac{dL^{T_2,h/T_2,f}(t)}{L^{T_2,h/T_2,f}(t-)} = A_f^h(t,T_2)dW^{T_2,f}(t) + \int_V B_f^h(t,T_2,v)\tilde{\mu}^{T_2,f}(dt,dv) \tag{21}$$

where

$$A(t,T_2) = b(t,T_2) - b_f(t,T_2) - \gamma_f(t) \tag{22}$$

$$B(t,T_2,v) = \frac{\beta(t,T_2,v) - \beta_f(t,T_2,v) - \xi_f(t,v)(1+\beta_f(t,T_2,v))}{1+\beta_f(t,T_2,v)+\xi_f(t,v)(1+\beta_f(t,T_2,v))} \tag{23}$$

$$A_f^h(t,T_2) = b_h(t,T_2) - b_f(t,T_2) - (\gamma_f(t) - \gamma_h(t)) \tag{24}$$

$$\begin{aligned} B_f^h(t,T_2,v) &= \frac{\beta_h(t,T_2,v)+\xi_h(t,v)(1+\beta_h(t,T_2,v))}{1+\beta_f(t,T_2,v)+\xi_f(t,v)(1+\beta_f(t,T_2,v))} \\ &\quad - \frac{\beta_f(t,T_2,v)+\xi_f(t,v)(1+\beta_f(t,T_2,v))}{1+\beta_f(t,T_2,v)+\xi_f(t,v)(1+\beta_f(t,T_2,v))} \end{aligned} \tag{25}$$

$$\tilde{\mu}^{T_2,f}(dt,dv) = \mu(t,dv) - \lambda^{T_2,f}(dv)dt \tag{26}$$

$$\lambda^{T_2,f}(dv) = (1+\beta_f(t,T_2,v)+\xi_f(t,v)(1+\beta_f(t,T_2,v)))\,\lambda(dv) \tag{27}$$

Proof. By the Multi-Currency Change of Numeraire Theorem it follows that

$$L^{T_2,d/T_2,f}(t) = \frac{p(t,T_2)}{p^f(t,T_2)X^f(t)} \frac{p^f(0,T_2)X^f(0)}{p(0,T_2)}, \qquad 0 \leq t \leq T_2$$

$$L^{T_2,h/T_2,f}(t) = \frac{p^h(t,T_2)X^h(t)}{p^f(t,T_2)X^f(t)} \frac{p^f(0,T_2)X^f(0)}{p^h(0,T_2)X^h(0)}, \qquad 0 \leq t \leq T_2$$

To prove equation (20), first apply Lemma A.1 to $p^f(t,T_2)X^f(t)$, thereafter apply Lemma A.2 to $p(t,T_2)/(p^f(t,T_2)X^f(t))$. Since $L^{T_2,d/T_2,f}$ is a martingale under $Q^{T_2,f}$ the drift term is zero. Proceed similarly to prove equation (21). ∎

Lemma 5.2 *Define $Y_1(t) = S^f(t)/p^f(t,T_2)$ for $0 \le t \le T_2$ and $Y_2(t) = p^f(t,T_1)/p^f(t,T_2)$ for $0 \le t \le T_1$. Then the dynamics of $Y_1(t)$ and $Y_2(t)$ under $Q^{T_2,f}$ are given by:*

$$\frac{dY_1(t)}{Y_1(t-)} = \left[C^f(t,T_2)dW^{T_2,f}(t) + \int_V D^f(t,T_2,v)\tilde{\mu}^{T_2,f}(dt,dv)\right] \tag{28}$$

$$\frac{dY_2(t)}{Y_2(t-)} = \left[M^f(t,T_2)dW^{T_2,f}(t) + \int_V N^f(t,T_2,v)\tilde{\mu}^{T_2,f}(dt,dv)\right] \tag{29}$$

where

$$C^f(t,T_2) = \sigma_f(t) - b_f(t,T_2) \tag{30}$$

$$D^f(t,T_2,v) = \frac{\delta_f(t,v) - \beta_f(t,T_2,v)}{1+\beta_f(t,T_2,v)} \tag{31}$$

$$M^f(t,T_2) = b_f(t,T_1) - b_f(t,T_2) \tag{32}$$

$$N^f(t,T_2,v) = \frac{\beta_f(t,T_1,v) - \beta_f(t,T_2,v)}{1+\beta_f(t,T_2,v)} \tag{33}$$

$$\tilde{\mu}^{T_2,f}(dt,dv) = \mu(t,dv) - \lambda^{T_2,f}(dv)dt \tag{34}$$

$$\lambda^{T_2,f}(dv) = (1+\beta_f(t,T_2,v) + \xi_f(t,v)(1+\beta_f(t,T_2,v)))\,\lambda(dv) \tag{35}$$

Proof. Immediate application of Lemma A.2, remembering that $S^f(t)/p^f(t,T_2)$ and $p^f(t,T_1)/p^f(t,T_2)$ are martingales under $Q^{T_2,f}$. ∎

Lemma 5.3 *Let $t \le T_2$. Then*

$$E_t^{T_2,d}\left[S^f(T_2)\right] = \frac{S^f(t)g(t,T_2)}{p^f(t,T_2)} \tag{36}$$

$$E_t^{T_2,h}\left[S^f(T_2)\right] = \frac{S^f(t)g^h(t,T_2)}{p^f(t,T_2)} \tag{37}$$

where

$$g(t,T_2) = e^{\int_t^{T_2}\left\{C^f(u,T_2)A(u,T_2)+\int_V D^f(u,T_2,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \tag{38}$$

$$g^h(t,T_2) = e^{\int_t^{T_2}\left\{C^f(u,T_2)A_f^h(u,T_2)+\int_V D^f(u,T_2,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \tag{39}$$

and $A(u,T_2), B(u,T_2,v), A_f^h(u,T_2), B_f^h(u,T_2,v), C^f(u,T_2)$ and $D^f(u,T_2,v)$ are given by equations (22) to (25), (30) and (31).

Proof. Defining the likelihood process $L_t^{T_2,d/T_2,f}$ as in equation (18), it follows from Bayes Theorem that

$$E_t^{T_2,d}[S^f(T_2)] = \frac{E_t^{T_2,f}[S^f(T_2)L^{T_2,d/T_2,f}(T_2)]}{L^{T_2,d/T_2,f}(t)} \tag{40}$$

Noting that $p^f(T_2,T_2)=1$, we have:

$$E_t^{T_2,f}\left[S^f(T_2)L^{T_2,d/T_2,f}(T_2)\right] = E_t^{T_2,f}\left[\frac{S^f(T_2)}{p^f(T_2,T_2)}L^{T_2,d/T_2,f}(T_2)\right]$$

Since $L^{T_2,d/T_2,f}(t)$ and $S^f(t)/p^f(t,T_2)$ are $Q^{T_2,f}$-martingales with dynamics given by equation (20) and (28) respectively, Proposition 2.1 gives that :

$$E_t^{T_2,f}\left[\frac{S^f(T_2)}{p^f(T_2,T_2)}L^{T_2,d/T_2,f}(T_2)\right] = \frac{S^f(t)}{p^f(t,T_2)}L^{T_2,d/T_2,f}(t)g(t,T_2) \tag{41}$$

where $g(t,T_2)$ is given by equation (38). Finally by inserting equation (41) into equation (40), equation (36) is proved.

Equation (37) is proved by defining the likelihood process $L_t^{T_2,h/T_2,f}$ as in equation (19) and thereafter continuing as above. ∎

Lemma 5.4 *Let $t \le T_1 \le T_2$. Then*

$$E_t^{T_2,d}\left[\frac{1}{p^f(T_1,T_2)}\right] = \frac{p^f(t,T_1)}{p^f(t,T_2)}K(t,T_1,T_2) \tag{42}$$

$$E_t^{T_2,h}\left[\frac{1}{p^f(T_1,T_2)}\right] = \frac{p^f(t,T_1)}{p^f(t,T_2)}K^h(t,T_1,T_2) \tag{43}$$

where

$$K(t,T_1,T_2) = e^{\int_t^{T_1}\left\{M^f(u,T_2)A(u,T_2)+\int_V N^f(u,T_2,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \tag{44}$$

$$K^h(t,T_1,T_2) = e^{\int_t^{T_1}\left\{M^f(u,T_2)A_f^h(u,T_2)+\int_V N^f(u,T_2,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \tag{45}$$

and $A(u,T_2), B(u,T_2,v), A_f^h(u,T_2), B_f^h(u,T_2,v), M^f(u,T_2)$ and $N^f(u,T_2,v)$ are given by equations (22) to (25), (32) and (33).

Proof. Define the likelihood process $L_t^{T_2,d/T_2,f}$ as in equation (18). Then, from Bayes Theorem, it follows that

$$E_t^{T_2,d}\left[\frac{1}{p^f(T_1,T_2)}\right] = \frac{E_t^{T_2,f}\left[\frac{1}{p^f(T_1,T_2)}L^{T_2,d/T_2,f}(T_1)\right]}{L^{T_2,d/T_2,f}(t)} \tag{46}$$

Noting that $p^f(T_1,T_1)=1$ yields

$$E_t^{T_2,f}\left[\frac{1}{p^f(T_1,T_2)}L^{T_2,d/T_2,f}(T_1)\right] = E_t^{T_2,f}\left[\frac{p^f(T_1,T_1)}{p^f(T_1,T_2)}L^{T_2,d/T_2,f}(T_1)\right]$$

Since $L^{T_2,d/T_2,f}(t)$ and $\frac{p^f(t,T_1)}{p^f(t,T_2)}$ are $Q^{T_2,f}$-martingales with dynamics given by equation (20) and (29) respectively, Proposition 2.1 gives that:

$$E_t^{T_2,f}\left[\frac{p^f(T_1,T_1)}{p^f(T_1,T_2)}L^{T_2,d/T_2,f}(T_1)\right] = \frac{p^f(t,T_1)}{p^f(t,T_2)}L^{T_2,d/T_2,f}(t)K(t,T_1,T_2) \tag{47}$$

where $K(t,T_1,T_2)$ is as in equation (44). Finally by inserting equation (47) into equation (46), equation (42) is proved.

Equation (43) is proved by defining the likelihood process $L_t^{T_2,h/T_2,f}$ as in equation (19) and thereafter continuing as above. ▮

Lemma 5.5 *Let $t \le T_1 \le T_2$. Then*

$$E_t^{T_2,d}\left[\frac{S^f(T_2)}{S^f(T_1)}\right] = \frac{p^f(t,T_1)G_f^d(t,T_1,T_2)}{p^f(t,T_2)} \tag{48}$$

$$E_t^{T_2,h}\left[\frac{S^f(T_2)}{S^f(T_1)}\right] = \frac{p^f(t,T_1)G_f^h(t,T_1,T_2)}{p^f(t,T_2)} \tag{49}$$

where

$$G_f^d(t,T_1,T_2) = \frac{e^{\int_t^{T_2}\left\{C^f(u,T_2)A(u,T_2)+\int_V D^f(u,T_2,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}}{e^{\int_t^{T_1}\left\{C^f(u,T_1)A(u,T_2)+\int_V D^f(u,T_1,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}} \tag{50}$$

$$G_f^h(t,T_1,T_2) = \frac{e^{\int_t^{T_2}\left\{C^f(u,T_2)A_f^h(u,T_2)+\int_V D^f(u,T_2,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}}{e^{\int_t^{T_1}\left\{C^f(u,T_1)A_f^h(u,T_2)+\int_V D^f(u,T_1,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}} \tag{51}$$

and $A(u,T_2)$, $B(u,T_2,v)$, $A_f^h(u,T_2)$, $B_f^h(u,T_2,v)$, $C^f(u,T_2)$ and $D^f(u,T_2,v)$ are given by equations (22) to (25), (30) and (31).

Proof. From Lemma 5.3 and Lemma 5.5, we get that

$$\begin{aligned} E_t^{T_2,d}\left[\frac{S^f(T_2)}{S^f(T_1)}\right] &= E_t^{T_2,d}\left[\frac{1}{S^f(T_1)}E_{T_1}^{T_2,d}[S^f(T_2)]\right] \\ &= E_t^{T_2,d}\left[\frac{g(T_1,T_2)}{p^f(T_1,T_2)}\right] \\ &= g(T_1,T_2)E_t^{T_2,d}\left[\frac{1}{p^f(T_1,T_2)}\right] \\ &= g(T_1,T_2)\frac{p^f(t,T_1)}{p^f(t,T_2)}K(t,T_1,T_2) \end{aligned}$$

where

$$\begin{aligned} g(T_1,T_2) &= e^{\int_{T_1}^{T_2}\left\{C^f(u,T_2)A(u,T_2)+\int_V D^f(u,T_2,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \\ K(t,T_1,T_2) &= e^{\int_t^{T_1}\left\{M^f(u,T_2)A(u,T_2)+\int_V N^f(u,T_2,v)B(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du} \end{aligned}$$

Finally, noting that $g(T_1,T_2)K(t,T_1,T_2) = G_f^d(t,T_1,T_2)$, equation (48) is proved. Equation (49) is proved using the same arguments. ∎

Now we are finally ready to price the quanto derivatives. In the next Proposition we price the contract with payoff function $\Phi = S^f(T_2)/S^f(T_1)$. Note that even though the payoff function is determined by the foreign asset S^f, which is measured in the f-foreign currency, the amount that the contract pays out is Φ units of the domestic currency.

Proposition 5.2 *Let $t \le T_1 \le T_2$ and $\Phi = S^f(T_2)/S^f(T_1)$. The value at time t, in the domestic currency, of a contract that pays out Φ units of the domestic currency at time T_2 is given by:*

$$\Pi[t,\Phi] = \frac{p(t,T_2)p^f(t,T_1)G_f^d(t,T_1,T_2)}{p^f(t,T_2)}$$

where $G_f^d(t,T_1,T_2)$ is given by equation (50).

Proof. By using the domestic bond with maturity T_2 as numeraire and applying Lemma 5.5, we get that

$$\begin{aligned} \Pi[t,\Phi] &= p(t,T_2)E_t^{T_2,d}\left[\frac{S^f(T_2)}{S^f(T_1)}\right] \\ &= \frac{p(t,T_2)p^f(t,T_1)G_f^d(t,T_1,T_2)}{p^f(t,T_2)} \end{aligned} \tag{52}$$

with $G_f^d(t, T_1, T_2)$ as above. ∎

Recall from Section 4 that in a cross currency two-way equity swap with domestic notional principal, one party pays the return of a domestic stock and receives the return on a foreign stock. Hence Proposition 5.1 and 5.2 immediately give the following Corollary.

Corollary 5.2 *If one counterpart in the equity swap pays the domestic equity index return R and receives the f-foreign equity index R_f with the notional principal value scaled to one and denominated in the domestic currency, then the swap value at time $t \leq T_0$ is equal to*

$$\sum_{j=1}^{m} p(t, T_j) \left[\frac{p^f(t, T_{j-1}) G_f^d(t, T_{j-1}, T_j)}{p^f(t, T_j)} - \frac{p(t, T_{j-1})}{p(t, T_j)} \right]$$

where $G_f^d(t, T_1, T_2)$ is given by equation (50).

Proposition 5.2 also gives the next Corollary.

Corollary 5.3 *If one counterpart in the equity swap pays a fixed swap rate K and receives the f-foreign equity index R_f with the notional principal value scaled to one and denominated in the domestic currency, then the swap value at time t is equal to*

$$\sum_{j=1}^{m} p(t, T_j) \left[\frac{p^f(t, T_{j-1}) G_f^d(t, T_{j-1}, T_j)}{p^f(t, T_j)} - (1 + K) \right]$$

where $G_f^d(t, T_1, T_2)$ is given by equation (50).

For the swap in Corollary 5.3, we define the par swap rate to be the value of K for which the price of the swap is zero. We denote the par swap rate by $R(t)$, thus:

$$R(t) = \frac{\sum_{j=1}^{m} \frac{p(t, T_j) p^f(t, T_{j-1}) G_f^d(t, T_{j-1}, T_j)}{p^f(t, T_j)} - \sum_{j=1}^{m} p(t, T_j)}{\sum_{j=1}^{m} p(t, T_j)} \tag{53}$$

In the next Proposition we price the contract with payoff function $\Psi = X^h(T_j) S^f(T_j) / S^f(T_{j-1})$. This contract has a payoff function that is determined by assets measured in the f-foreign and h-foreign currency but payed out in the domestic currency.

Proposition 5.3 *Let $t \le T_1 \le T_2$ and $\Psi = X^h(T_2)S^f(T_2)/S^f(T_1)$. The value at time t, in the domestic currency, of a contract that pays out Ψ units of the domestic currency at time T_2 is given by:*

$$\Pi[t, \Psi] = \frac{X^h(t)p^h(t, T_2)p^f(t, T_1)G_f^h(t, T_1, T_2)}{p^f(t, T_2)}$$

where $G_f^h(t, T_1, T_2)$ is given by equation (51).

Proof. Define the likelihood process $L_t^{T_2,d/T_2,h}$ as:

$$L_t^{T_2,d/T_2,h} = \frac{dQ^{T_2,d}}{dQ^{T_2,h}} \quad \text{on } \mathcal{F}_t$$

then use the domestic bond with maturity date T_2 as numeraire and apply Bayes Theorem, to find that

$$\begin{aligned} \Pi[t, \Psi] &= p(t, T_2)E_t^{T_2,d}\left[X^h(T_2)\frac{S^f(T_2)}{S^f(T_1)}\right] \\ &= p(t, T_2)\frac{E_t^{T_2,h}\left[X^h(T_2)\frac{S^f(T_2)}{S^f(T_1)}L^{T_2,d/T_2,h}(T_2)\right]}{L^{T_2,d/T_2,h}(t)} \end{aligned} \tag{54}$$

By the Multi-Currency Change of Numeraire Theorem

$$L^{T_2,d/T_2,h}(t) = \frac{p(t, T_2)}{p^h(t, T_2)X^h(t)}\frac{p^h(0, T_2)X^h(0)}{p(0, T_2)}, \qquad 0 \le t \le T_2$$

Insert the expression for $L^{T_2,d/T_2,h}(T_2)$ into (54). Then

$$\Pi[t, \Psi] = X^h(t)p^h(t, T_2)E_t^{T_2,h}\left[\frac{S^f(T_2)}{S^f(T_1)}\right]$$

Finally, apply Lemma 5.5. ∎

5.3 Generalized Pricing formula

In this Section we will present a generalized pricing formula for equity swaps with constant notional principal. This pricing formula was presented by Liao & Wang (2003) for the case when all assets are Wiener driven, but is here extended to include also the point process case.

In the next Proposition we will specify the pricing formula for a cross currency two-way equity swap with a foreign denominated constant notional principal. Then we will illustrate how this formula can be used to price any type of equity swap with constant notional principal. Hence the pricing formula in the next proposition is the generalized pricing formula for equity swaps with constant notional principal.

Proposition 5.4 *If one counterpart in the equity swap pays the equity index return R_g and receives another equity index return R_f with the notional principal value, denominated in currency h, then the swap value at time $t \leq T_0$ is equal to*

$$\sum_{j=1}^{m} X^h(t)p^h(t,T_j)\left[\frac{p^f(t,T_{j-1})G_f^h(t,T_{j-1},T_j)}{p^f(t,T_j)} - \frac{p^g(t,T_{j-1})G_g^h(t,T_{j-1},T_j)}{p^g(t,T_j)}\right] \quad (55)$$

where

$$G_f^h(t,T_1,T_2) = \frac{e^{\int_t^{T_2}\left\{C^f(u,T_2)A_f^h(u,T_2)+\int_V D^f(u,T_2,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}}{e^{\int_t^{T_1}\left\{C^f(u,T_1)A_f^h(u,T_2)+\int_V D^f(u,T_1,v)B_f^h(u,T_2,v)\lambda^{T_2,f}(dv)\right\}du}}$$

$$G_g^h(t,T_1,T_2) = \frac{e^{\int_t^{T_2}\left\{C^g(u,T_2)A_g^h(u,T_2)+\int_V D^g(u,T_2,v)B_g^h(u,T_2,v)\lambda^{T_2,g}(dv)\right\}du}}{e^{\int_t^{T_1}\left\{C^g(u,T_1)A_g^h(u,T_2)+\int_V D^g(u,T_1,v)B_g^h(u,T_2,v)\lambda^{T_2,g}(dv)\right\}du}}$$

and

$$A_f^h(t,T_2) = b_h(t,T_2) - b_f(t,T_2) - (\gamma_f(t) - \gamma_h(t))$$

$$B_f^h(t,T_2,v) = \frac{\beta_h(t,T_2,u) + \xi_h(t,u)(1+\beta_h(t,T_2,u))}{1+\beta_f(t,T_2,u)+\xi_f(t,u)(1+\beta_f(t,T_2,u))} - \frac{\beta_f(t,T_2,u) + \xi_f(t,u)(1+\beta_f(t,T_2,u))}{1+\beta_f(t,T_2,u)+\xi_f(t,u)(1+\beta_f(t,T_2,u))}$$

$$A_g^h(t,T_2) = b_h(t,T_2) - b_g(t,T_2) - (\gamma_g(t) - \gamma_h(t))$$

$$B_g^h(t,T_2,v) = \frac{\beta_h(t,T_2,u) + \xi_h(t,u)(1+\beta_h(t,T_2,u))}{1+\beta_g(t,T_2,u)+\xi_g(t,u)(1+\beta_g(t,T_2,u))} - \frac{\beta_g(t,T_2,u) + \xi_g(t,u)(1+\beta_g(t,T_2,u))}{1+\beta_g(t,T_2,u)+\xi_g(t,u)(1+\beta_g(t,T_2,u))}$$

$$C^f(t,T_2) = \sigma_f(t,T_2) - b_f(t,T_2)$$

$$D^f(t,T_2,v) = \frac{\delta_f(t,v) - \beta_f(t,T_2,v)}{1+\beta_f(t,T_2,v)}$$

$$
\begin{aligned}
C^g(t,T_2) &= \sigma_g(t,T_2) - b_g(t,T_2) \\
D^g(t,T_2,v) &= \frac{\delta_g(t,v) - \beta_g(t,T_2,v)}{1+\beta_g(t,T_2,v)} \\
\tilde{\mu}^{T_2,f}(dt,dv) &= \mu(t,dv) - \lambda^{T_2,f}(dv)dt \\
\tilde{\mu}^{T_2,g}(t,dv) &= \mu(t,dv) - \lambda^{T_2,g}(dv)dt \\
\lambda^{T_2,f}(dv) &= \left(1+\beta_f(t,T_2,v) + \xi_f(t,v)(1+\beta_f(t,T_2,v))\right)\lambda(dv) \\
\lambda^{T_2,g}(dv) &= \left(1+\beta_g(t,T_2,v) + \xi_g(t,v)(1+\beta_g(t,T_2,v))\right)\lambda(dv)
\end{aligned}
$$

Proof. The swap payments are both of type Ψ, hence the result follows from Proposition 5.3. ∎

The pricing formula in Proposition 5.4 is general in the sense that it can be used to price different types of equity swaps with constant notional principal. We will illustrate by two examples.

First consider a cross currency two-way swap with domestic constant notional principal where one party pays the return on a domestic equity and receives the return on an f-foreign equity. We can price this swap using the general pricing formula above if we substitute both h and g with d. Since $X^d(t) = 1$ for all t, $\gamma_d = 0$ and $\xi_d = 0$, equation 5.4 becomes:

$$
\sum_{j=1}^{m} p(t,T_j)\left[\frac{p^f(t,T_{j-1})G_f^d(t,T_{j-1},T_j)}{p^f(t,T_j)} - \frac{p(t,T_{j-1})}{p(t,T_j)}\right] \tag{56}
$$

Hence equation (55) is reduced to the pricing formula of a cross currency swap with domestic notional principal given by equation (50) in Corollary 5.2.

As another example we will show that equation (55) also can be reduced to the case of a domestic two-way equity swap with domestic notional principal. Here we substitute f with $d1$ and g with $d2$ so that one party will receive the domestic equity index return R_{d1} and pay another domestic equity index return R_{d1}. To make the notional principal be denoted in domestic units we substitute h with d. We have that $X^d(t) = 1$ for all t so $\gamma_d = 0$ and $\xi_d = 0$. Furthermore $p^{d1}(t,T) = p^{d2}(t,T) = p(t,T)$ for all t and for all T so $b_{d1} = b_{d2} = b$, and $\beta_{d1} = \beta_{d2} = \beta$. Hence equation 5.4 reduces to:

$$
\sum_{j=1}^{m} p(t,T_j)\left[\frac{p(t,T_{j-1})}{p(t,T_j)} - \frac{p(t,T_{j-1})}{p(t,T_j)}\right] = 0 \tag{57}
$$

which is in accordance with Corollary 5.1.

Similarly equation 5.4 can be used to price any kind of equity swap with constant notional principal by choosing f, g and h suitable.

6 Conclusions

By using martingale methods and the technique of convexity corrections we have extended the generalized pricing model for equity swaps first presented by Liao & Wang (2003). The extension allows for the international market to be driven by both a standard multidimensional Wiener process and a general marked point process. The martingale method and the convexity correlation result that we achieved is the key that enabled the extension.

A Two Useful Formulas

Let $W(t)$ be an n-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. Let $\mu(dt, dv)$ be an adapted marked point process over $\mathbb{R}_+ \times V$ with predictable intensity $\lambda_t(dv)dt$ and denote $\tilde{\mu}(dt, dv) = \mu(dt, dv) - \lambda_t(dv)dt$. Assume that $\sigma_x(t), \sigma_y(t)$ are adapted, and that $\delta_x(t, v), \delta_y(t, v)$ are predictable processes. Let $X(t), Y(t)$ be two stochastic processes with dynamics:

$$dX_t = X_{t-}\left[\alpha_x(t)dt + \sigma_x(t)dW(t) + \int_V \delta_x(t, v)\tilde{\mu}(dt, dv)\right] \tag{58}$$

$$dY_t = Y_{t-}\left[\alpha_y(t)dt + \sigma_y(t)dW(t) + \int_V \delta_y(t, v)\tilde{\mu}(dt, dv)\right] \tag{59}$$

Lemma A.1 *Let $X(t), Y(t)$ be two stochastic processes with dynamics given by equation (58) and (59). Then the dynamic of the product, $(XY)(t)$ is given by:*

$$\begin{aligned} d(XY)_t &= X_tY_t\left\{\alpha_x(t) + \alpha_y(t) + \sigma_x(t)\sigma_y(t) + \int_V \delta_x(t, v)\delta_y(t, v)\lambda_t(dv)\right\}dt \\ &+ X_tY_t\left\{\sigma_x(t) + \sigma_y(t)\right\}dW(t) \\ &+ X_{t-}Y_{t-}\int_V \left\{\delta_x(t, v) + \delta_y(t, v) + \delta_x(t, v)\delta_y(t, v)\right\}\tilde{\mu}(dt, dv) \end{aligned} \tag{60}$$

Proof.

$$\begin{aligned} d(XY)_t &= X_tY_t\left\{\alpha_x(t) + \alpha_y(t) + \sigma_x(t)\sigma_y(t)\right\}dt \\ &- X_tY_t\left\{\int_V \delta_x(t, v)\lambda_t(dv) + \int_V \delta_y(t, v)\lambda_t(dv)\right\}dt \\ &+ X_tY_t\left\{\sigma_x(t) + \sigma_y(t)\right\}dW(t) \\ &+ X_{t-}Y_{t-}\int_V \left\{\delta_x(t, v) + \delta_y(t, v) + \delta_x(t, v)\delta_y(t, v)\right\}\mu(dt, dv) \end{aligned}$$

$$
\begin{aligned}
&= X_tY_t\left\{\alpha_x(t)+\alpha_y(t)+\sigma_x(t)\sigma_y(t)\right\}dt \\
&- X_tY_t\left\{\int_V\left\{\delta_x(t,v)+\delta_y(t,v)\right\}\lambda_t(dv)\right\}dt \\
&+ X_tY_t\left\{\sigma_x(t)+\sigma_y(t)\right\}dW(t) \\
&+ X_{t-}Y_{t-}\int_V\left\{\delta_x(t,v)+\delta_y(t,v)+\delta_x(t,v)\delta_y(t,v)\right\}\tilde{\mu}(dt,dv) \\
&+ X_{t-}Y_{t-}\int_V\left\{\delta_x(t,v)+\delta_y(t,v)+\delta_x(t,v)\delta_y(t,v)\right\}\lambda_t(dv)dt
\end{aligned}
$$

Simplifying the last expression gives equation (60). ∎

Lemma A.2 *Let $X(t), Y(t)$ be two stochastic processes with dynamics given by equation (58) and (59). Then the dynamic of the ratio, $(X/Y)(t)$ is given by:*

$$
\begin{aligned}
d\left(\frac{X}{Y}\right)_t &= \frac{X_t}{Y_t}\left\{\alpha_x(t)-\alpha_y(t)-\sigma_x(t)\sigma_y(t)+\sigma_y^2(t)\right\}dt \\
&- \frac{X_t}{Y_t}\int_V\frac{\delta_y(t,v)\left(\delta_x(t,v)-\delta_y(t,v)\right)}{1+\delta_y(t,v)}\lambda_t(dv)dt \\
&+ \frac{X_t}{Y_t}\left\{\sigma_x(t)-\sigma_y(t)\right\}dW(t) \\
&+ \frac{X_{t-}}{Y_{t-}}\int_V\frac{\left(\delta_x(t,v)-\delta_y(t,v)\right)}{1+\delta_y(t,v)}\tilde{\mu}(dt,dv)
\end{aligned}
\tag{61}
$$

Proof.

$$\begin{aligned}
d\left(\frac{X}{Y}\right)_t &= \frac{X_t}{Y_t}\left\{\alpha_x(t)-\alpha_y(t)-\sigma_x(t)\sigma_y(t)+\sigma_y^2(t)\right\}dt \\
&+ \frac{X_t}{Y_t}\left\{-\int_V \delta_x(t,v)\lambda_t(dv)+\int_V \delta_y(t,v)\lambda_t(dv)\right\}dt \\
&+ \frac{X_t}{Y_t}\left\{\sigma_x(t)-\sigma_y(t)\right\}dW(t) \\
&+ \frac{X_t}{Y_t}\int_V \frac{(\delta_x(t,v)-\delta_y(t,v))}{1+\delta_y(t,v)}\mu(dt,dv) \\
&= \frac{X_t}{Y_t}\left\{\alpha_x(t)-\alpha_y(t)-\sigma_x(t)\sigma_y(t)+\sigma_y^2(t)\right\}dt \\
&- \frac{X_t}{Y_t}\int_V \left(\delta_x(t,v)-\delta_y(t,v)\right)\lambda_t(dv)dt \\
&+ \frac{X_t}{Y_t}\left\{\sigma_x(t)-\sigma_y(t)\right\}dW(t) \\
&+ \frac{X_{t-}}{Y_{t-}}\int_V \frac{(\delta_x(t,v)-\delta_y(t,v))}{1+\delta_y(t,v)}\tilde{\mu}(dt,dv) \\
&+ \frac{X_{t-}}{Y_{t-}}\int_V \frac{(\delta_x(t,v)-\delta_y(t,v))}{1+\delta_y(t,v)}\lambda_t(dv)dt
\end{aligned}$$

Simplifying the last expression gives equation (61). ∎

B Measure Theory

The Theorems stated below are standard and can be found in Björk (2004), Björk et al. (1997) and Pelsser (2000).

Theorem B.1 *(Girsanov Theorem) Let $W(t)$ be an n-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$. Let $\mu(dt, dv)$ be an adapted marked point process over $\mathbb{R}_+ \times V$ with predictable intensity $\lambda_t(dv)dt$ and denote $\tilde{\mu}(dt, dv) = \mu(dt, dv) - \lambda_t(dv)dt$. Assume that$\varphi(t, v) \geq -1 \forall v \in V$, that φ is predictable and that $h(t)$ is adapted. Choose a fixed T and define L on $[0, T]$ by*

$$
\begin{aligned}
dL(t) &= L(t)h(t)dW(t) + L(t-) \int_V \varphi(t, v)\tilde{\mu}(dt, dv) \\
L(0) &= 1
\end{aligned}
$$

Assume $E^P[L_T] = 1$ Define a new probability measure Q on $\mathcal{F}_T$ via

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

Then

$$dW(t) = h^*(t)dt + dW^Q(t)$$

where W^Q is a Wiener process under the Q-measure. Furthermore the marked point process μ has a predictable Q-intensity λ^Q given by

$$\lambda^Q(dv) = (1 + \varphi(t, v))\lambda_t(dv)$$

Theorem B.2 *(Bayes' Theorem) Let $X(t)$ be a random process on $(\Omega, \mathcal{F}, P)$. Let Q be another probability measure on $(\Omega, \mathcal{F})$, absolutely continuous with respect to P and with Radon-Nikodym derivative*

$$L = \frac{dQ}{dP} \quad \text{on} \quad \mathcal{F}$$

Let $\mathcal{G}$ be a sigma-algebra with $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E^Q[X|\mathcal{G}] = \frac{E^P[L \cdot X|\mathcal{G}]}{E^P[L|\mathcal{G}]} \quad Q - a.s.$$

Theorem B.3 *(Change of Numeraire) Let Q^N be a martingale measure for the numeraire N_t on $\mathcal{F}_T$. Let Q^M be an equivalent martingale measure for the numeraire M_t on $\mathcal{F}_T$. The Radon-Nykodym derivative that changes the equivalent martingale measure Q^M to Q^N is given by:*

$$\frac{dQ^N}{dQ^M} = \frac{N(T)}{M(T)} \frac{M(0)}{N(0)} \quad \text{on} \quad \mathcal{F}_T$$

Theorem B.4 *(Multi-Currency Change of Numeraire) Let* $t \in [0, T]$. *Given an arbitrage free system of economies (d,f), an exchange rate,* $X_t^{(f/d)}$ *and two numeraires* N_t^d *and* M_t^f *within the economies, with associated martingale measures* $Q^{N,d}$ *and* $Q^{M,f}$ *on* $\mathcal{F}_T$ *we have the Radon Nykodym-Derivative:*

$$\frac{dQ^{N,d}}{dQ^{M,f}} = \frac{N(T)}{X^{d/f}(T)M(T)} \frac{M(0)X^{d/f}(0)}{N(0)} \quad \text{on} \quad \mathcal{F}_T$$

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Paper 2

Paper 2

Inflation Indexed Swaps and Swaptions

This article considers the pricing and hedging of inflation indexed swaps, and the pricing of inflation indexed swaptions, and options on inflation indexed bonds. To price the inflation indexed swaps, we suggest an extended HJM model. The model allows both the forward rates and the consumer price index to be driven, not only by a standard multidimensional Wiener process but also, by a general marked point process. Our model is an extension of the HJM approach proposed by Jarrow and Yildirim [16] and later also used by Mercurio [18] to price inflation indexed swaps. Furthermore we price options on so called TIPS-bonds assuming the model is purely Wiener driven. We then introduce an inflation swap market model to price inflation indexed swaptions. All prices derived have explicit closed form solutions. Furthermore, we formally prove the validity of the so called foreign-currency analogy.

1 Introduction

Even though inflation linked products can be traced back to the middle of the 18th century, the modern inflation indexed market did not start until the beginning of the 1980s. Then the United Kingdom issued inflation indexed bonds and shortly thereafter Austria, Sweden and Canada followed. In 1997 the United States began to issue Treasury Inflation Protected Bonds (TIPS) and since then several countries have entered the inflation indexed bond market. About ten years ago, the inflation indexed swap market began to develop in the United Kingdom. Today inflation indexed swaps and other inflation indexed derivatives are traded in for example the United Kingdom, the United States, France, Japan and in the Euro Market. The derivatives market is still young though and most certainly we will experience more products develop.

Inflation is defined as the percentage change of a particular reference index. The choice of reference index varies from country to country but usually it is a consumer price index (CPI). The consumer price index measure the price of a representative basket of goods and services. Thus an increase in the consumer price index over a period of time implies that there has been inflation over that period.

Inflation indexed products are tied to inflation. The main idea of inflation indexed bonds is that investing in the bond and keeping it until maturity will generate a certain real return. Thus, even though the nominal value of the coupons and principal may change, the real value of these remains the same. A zero coupon inflation indexed bond with principal equal to unity will pay out just enough in dollars to buy one unit of the consumer price index basket.

A swap is an agreement between two counter parties to exchange cash flows. The agreement specifies the cash flows and the dates when the cash flows are to be paid. The most common types of swaps are interest rate swaps and currency swaps, but inflation indexed swaps are also traded and have gained more and more interest lately. In an inflation indexed swap at least one of the cash flows is tied to inflation.

A swaption is an option to enter into a swap at a pre specified date for a pre specified swap rate. An inflation indexed swaption is a swaption where the underlying swap is an inflation indexed swap.

Inflation linked products can be used to hedge future cash flow against inflation. That is particularly attractive to investors that seek asset-liability matching such as for example insurance companies. Inflation linked products may also be used for risk diversification and of course for speculation. From the issuers perspective, inflation linked bonds may prove as means of establishing a trustworthy inflation policy.

One of the early studies on inflation derivatives was made by Hughston [14] and introduces a methodology based on the foreign-currency analogy. Here nominal assets are thought of as domestic assets, real assets as foreign assets, and the consumer price index is treated as the exchange rate between the nominal and the real markets. Developing a theory where the dynamics of the consumer price index and the real and nominal discount bonds have the same structure as in a HJM model, Hughston shows how index linked derivatives can be treated in much the same way as foreign-exchange derivatives. However no derivative pricing formulas are calculated explicitly.

Also relying on the foreign-currency analogy, Jarrow and Yildirim [16] have developed a three factor HJM model in order to price TIPS and options written directly on the inflation index. They assume that the volatilities of all asset prices and the consumer price index are deterministic. They use a parameterization of the forward volatility that corresponds to the Hull-White short rate model in order to obtain an explicit formula for the option. Hence they obtain pricing formulas for the case when bond prices are Gaussian.

Mercurio [18] has studied the pricing of zero coupon inflation indexed swaps, year-on-year inflation indexed swaps, as well as inflation indexed caplets and floorlets. The swaps are priced first using the Jarrow & Yildirim model with the Hull-White parameterization and then using two different market model approaches. In the market model, the forward CPI is modelled and is, along with the asset price processes, assumed to be Gaussian. A similar approach, also influenced by market models, have been independently suggested by Kazziha [17] who uses teh model to price options on inflation.

In a more recent paper, Mercurio & Moreni [19] price inflation indexed caplets and floorlets under the assumption of stochastic volatility and are able to derive closed form solutions.

All the above articles, except the last, are based on Gaussian interest rates and none admits jumps in interest rates. In this paper we will allow interest rates to jump. There is strong empirical evidence supporting that interest rates have embedded jumps. Several studies argue that jump-diffusion models more accurately describe the observed term structure of interest rates than pure diffusion models do. General explanations for jumps include surprises in the information flow. For instance jumps can reflect macroeconomic announcements concerning GDP growth, unemployment and inflation, policy shocks and monetary actions by the Federal Reserve. See Das [9], Johannes [12] and Piazzesi [22].

The literature on term structures of interest rates has evolved from pure diffusion models to jump-diffusions models and many of the popularly used interest rate models that originally was pure diffusion models have been extended to include also jumps. There is a rather large literature on interest rate term structures that considers jump-diffusion models. See for example Ahn & Thompson [1], Chacko & Das [11], Björk et al. [6] and Duffie et al. [10].

Information surprises of the type mentioned above could cause jumps, not only in nominal interest rates but also in real interest rates and in the inflation rate. Hence it is natural to extend the literature on inflation derivatives from pure diffusion models to jump-diffusion models. Slinko [23] who considers nominal and real interest rates in a two-country setting, has independently proposed a jump-diffusion model for inflation. From a practical point of view, all bonds including indexed linked bonds may jump and thus both the real and nominal interest rate should be allowed to jump. The need for including jumps in the inflation process is less articulate since today CPIs are usually monitored discretely[1]. For the sake of completeness we will allow also the inflation process to exhibit jumps. Furthermore, Mercurio & Moreni [19] stress the importance of the smile effect in inflation derivatives pricing and allows the forward CPI to have stochastic volatility. Since jumps, as stochastic volatility, can be used to handle smile effects (see Cont & Tankov [8]), it is natural to allow for jumps in the inflation process.

The purpose of this paper is to:

- Specify an extended HJM framework, allowing for both jumps and stochastic volatility for a market consisting of a money market account, zero coupon bonds and indexed zero coupon bonds that are based on a non-traded index.
- Show how this framework can be used for the pricing of indexed derivatives without assuming a priori the foreign-currency analogy.
- In fact, we prove that the foreign-currency analogy holds for a completely arbitrary process, including the case where the process does not even have an economic interpretation. The proof is done with fewer assumptions than what we have seen in the previous literature.
- Once the foreign-currency analogy is established, the additional assumption of deterministic volatilities are introduced. The framework is then used to explicitly calculate prices of inflation indexed derivatives. Compared to the model suggested by Jarrow and Yildirim [16] we allow for more than three factors and also allow for the possibility of jumps in the economy. Thus, in our model the random processes describing the real and nominal market as well as the consumer price index are allowed to be driven by both a standard multidimensional Wiener process and a general marked point process. We assume that the intensity of the point process as well as the volatilities of all asset prices and the consumer price index, with respect to both the Wiener process and the point process, is deterministic. This assumption will assure us of closed-form solutions. The derivatives we price are:
  - Zero coupon inflation indexed swaps[2]
  - Year-on-year inflation indexed swaps

[1] For instance, the CPI is published monthly both in the United States and in Sweden.
[2] This has been proved earlier by Mercurio [18].

  - Zero coupon inflation indexed swaptions
  - Options on TIPS [3]

- Show how to hedge inflation indexed swaps when there is no jump risk.

- Specify two different inflation indexed swap market models. Under these models we price and zero coupon inflation indexed swaptions and year-on-year inflation indexed swaptions.

The article is organized as follows: In the next Section we will present the extended HJM model. We will also prove the validity of the foreign-currency analogy. Section 3 is devoted to inflation indexed swaps. For preparatory purposes we will in Section 3.1 show how to find the model independent price of the zero coupon inflation indexed swap. Then we will price the year-on-year inflation indexed swap, given the model specified in Section 3.2. In the last part of Section 3.1 we study hedging issues of inflation indexed swaps. In Section 4 we will study inflation indexed swaptions. We will introduce two different inflation indexed swap market models and use these to price the year-on-year inflation indexed swaption. This is done in Section 4.1. In Section 4.2 we show that the zero coupon inflation indexed swaption can be priced by using the inflation indexed swap market models as well as by the multi factor HJM model without jumps. In Section 5, options on TIPS are priced using the model specified in Section 3 with zero jump-volatilities. Section 6 concludes.

2 The extended HJM model

In this section we specify an extended HJM model under the objective probability measure. From this specification we derive the dynamics of the nominal bonds, the inflation protected bonds, the inflation process and of what we call the (fictive) real bonds. Furthermore we will be able to define a real martingale measure. Everything will be done without using any prior assumption about the foreign-currency analogy. We will instead show that the foreign-currency analogy does indeed hold.

Assumption 2.1 *We consider a financial market where all objects are defined on a filtered probability space* $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ *where* P *is the objective probability measure. The probability space carries both an n-dimensional Wiener process* W^P *and a general marked point process* $\mu(dt, dv)$ *on* $\mathbb{R}_+ \times V$ *with compensator* $\lambda^P(t, dv)dt$ *. The filtration* $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t \geq 0}$ *is generated by both* W^P *and* μ*, i.e.* $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.

[3] Under the additional assumption of no jumps.

Let $p_n(t,T)$ denote the price in dollars at time t of a nominal zero-coupon bond that pays out one dollar at the maturity date, T. We will refer to these bonds as T-bonds. Let $I(t)$ be any stochastic process at time t. By $p_{IP}(t,T)$ we denote the price in dollars at time t of a contract that pays out $I(T)$ dollars at time T. We will refer to these contracts as T-IP-bonds. If $I(T)$ denotes CPI at time T, then $p_{IP}(t,T)$ is the price at time t of a contract that at maturity will pay out the dollar value of one CPI-unit at time T. Hence, in this case $p_{IP}(t,T)$ is the price of an inflation protected zero-coupon bond. The inflation protected bonds are often called TIPS-bond, where TIPS stands for treasury inflation protected security. Since $I(t)$ can be any process, we can let $I(t) = \tilde{I}(t) \cdot 1$ where $\tilde{I}(t)$ is the temperature at the top of the Eiffel tower at time t and 1 has the unit of number of dollars over the squared temperature. That is I is measured as number of dollars per temperature degree. Then, $p_{IP}(t,T)$ is the price at time t of a contract that will pay out an amount in dollars that is equal to the number of degrees on the scale at the top of the Eiffel tower at time T.

Define $p_r(t,T)$ as

$$p_r(t,T) = \frac{p_{IP}(t,T)}{I(t)}$$

This implies that $p_r(t,t) = 1$. Note that the unit of $p_r(t,T)$ is equal to the number of dollars per one unit of I. Since CPI is expressed as dollars per CPI-basket, then if $I(t)$ is the CPI-index at time t, the unit of $p_r(t,T)$ will be CPI-baskets. Hence $p_r(t,T)$ is the price in CPI-baskets of a (fictive) real bond that pays out one CPI-basket at time T. Suppose for simplicity that the CPI-basket consists of carrots only, then $p_r(t,T)$ will be the price at time t, expressed in carrots, of a contract that pays out one carrot at time T. In the case where I denotes the temperature the interpretation of $p_r(t,T)$ is a bit awkward, but of course this did not prevent us from defining it. If $I(t)$ is the temperature of the top of the Eiffel tower at time t then $p_r(t,T)$ is the price, expressed as degrees, of a contract that pays out one degree at time T.

Assumption 2.2 *We assume that there exists a (dollar) market for T-bonds and T-IP-bonds for all maturities $T > 0$. Furthermore we assume that for every fixed t, $p_n(t,T)$ and $p_{IP}(t,T)$ are differentiable with respect to the maturity T.*

Define, for each fixed T, two types of instantaneous forward rates, contracted at time t by

$$f_i(t,T) = -\frac{\partial \ln p_i(t,T)}{\partial T} \quad \text{for } i = r, n$$

Using these forward rates, we now define two types of interest rates by $r^i(t) = f_i(t,t)$ for $i = r, n$. If I is the CPI, then the forward rates can be interpreted as the nominal and the real forward rates respectively. Analogously the interest rates can be interpreted as the nominal and real short rates. Finally we define B_n and B_r by,

$$B_i(t) = e^{\int_0^t r^i(s)ds} \quad \text{for } i = r, n$$

If I is the CPI then $B_n(t)$ is the nominal money market account at time t, measured in dollars while $B_r(t)$ is a (fictive) real money market account at time t, measured in CPI-baskets.

Assumption 2.3 *We assume that there exists a (dollar) market for the money market account $B_n(t)$*[4]

Assumption 2.4 *Assume that under the objective probability measure P, the dynamics of f_r and f_n for every fixed $T > 0$ and the dynamics of I are given by:*

$$\begin{aligned} df_i(t,T) &= \alpha^i(t,T)dt + \sigma^i(t,T)dW_t^P + \int_V \xi^i(t,v,T)\mu(dt,dv) \quad \textit{for } i = r, n \\ dI(t) &= I(t)\mu^I(t)dt + I(t)\sigma^I(t)dW_t^P + I(t-)\int_V \gamma^I(t,v)\mu(dt,dv) \end{aligned}$$

where σ^r, σ^n, σ^i are $\underline{\mathcal{F}}$-adapted and ξ^r, ξ^n, γ^I and λ^P are $\underline{\mathcal{F}}$-predictable.

Even though the notation only states explicitly that the parameters are functions of time; the parameters are also allowed to be stochastic, as long as the adaptability and predictability requirements are fulfilled. In addition to Assumption 2.4 we need to know that the above processes possess some boundedness and regularity properties, see Björk et al. [6].

Assumption 2.5 *We assume that there are no arbitrage possibilities, i.e. the market is arbitrage free.*

From general no arbitrage theory[5] it follows by Assumption (2.5) and (2.2) that there exists an equivalent martingale measure Q^n such that, for each fixed T,

$$\frac{p_n(t,T)}{B_n(t)} \quad \frac{p_{IP}(t,T)}{B_n(t)} \quad \text{are } Q^n\text{-martingales}$$

This result is the main tool in the proof of the next proposition.

It is the next Proposition, together with Corollary 2.2, that constitutes the proof of the foreign-currency analogy. It is an extension of the analogy proposed by

[4]Björk et al. [5] have shown that this actually follows from the part of Assumption 2.2 concerning the nominal T-bonds.

[5]See for example Björk [4]

Jarrow & Yildirim [16] from a complete market, driven by Gaussian processes, to an incomplete market with processes driven by a multidimensional Wiener process, a marked point process and possibly also stochastic volatility. We would like to emphasize that the proof of Proposition 2.1 is different from what is presented by Jarrow & Yildirim [16], since they make the additional and unneccesary assumption that:

$$\frac{I(t)B_r(t)}{B_n(t)} \quad \text{is a } Q^n\text{-martingale}$$

We believe that this assumption cannot be made a priori. Since the real market is fictive, we cannot consider the real money market account as a traded asset. Neither do we have an inflation protected money market account in the nominal market. Hence we cannot find suitable economic arguments for making this assumption. Instead we prove Proposition 2.1 without this assumption. Then, as a result of Proposition 2.1, we find that $I(t)B_r(t)/B_n(t)$ is indeed a Q^n-martingale. This is done in Corollary 2.2.

Proposition 2.1 *If $f_n(t,T)$, $f_r(t,T)$ and $I(t)$ satisfies Assumption (2.4), then I, p_n, p_{IP} and p_r will under the nominal martingale measure Q^n satisfy:*

$$\frac{dI(t)}{I(t-)} = \{r^n(t) - r^r(t)\}\,dt + \sigma^I(t)dW_t + \int_V \gamma^I(t,v)\tilde{\mu}(dt,dv) \tag{1}$$

$$\frac{dp_n(t,T)}{p_n(t-,T)} = r^n(t)dt + \beta^n(t,T)dW_t + \int_V \delta^n(t,v,T)\tilde{\mu}(dt,dv) \tag{2}$$

$$\frac{dp_{IP}(t,T)}{p_{IP}(t-,T)} = r^n(t)dt + \beta^{IP}(t,T)dW_t + \int_V \delta^{IP}(t,v,T)\tilde{\mu}(dt,dv) \tag{3}$$

$$\frac{dp_r(t,T)}{p_r(t-,T)} = a(t,T)dt + \beta^r(t,T)dW_t + \int_V \delta^r(t,v,T)\tilde{\mu}(dt,dv) \tag{4}$$

where

$$\beta^i(t,T) = -\int_t^T \sigma^i(t,s)ds \quad \text{for } i = r,n$$

$$\beta^{IP}(t,T) = \sigma^i(t) + \beta^r(t,T)$$

$$\delta^i(t,v,T) = e^{-\int_t^T \xi^i(t,v,s)ds} - 1 \quad \text{for } i = r,n$$

$$\delta^{IP}(t,v,T) = \delta^r(t,v,T) + \gamma^I(t,v) + \delta^r(t,v,T)\gamma^I(t,v)$$

$$a(t,T) = r^r(t) - \sigma^i(t)\cdot\beta^r(t,T) - \int_V \delta^r(t,v,T)\gamma^I(t,v)\lambda_t^P(dv)$$

$$\begin{aligned}\tilde{\mu}(dt, dv) &= \mu(dt, dv) - \lambda_t(dv)dt \\ \lambda_t(dv) &= \lambda_t^P(dv)(1 + \rho(t, v)) \\ dW_t^P &= h_t dt + dW_t\end{aligned}$$

We have suppressed Q^n to shorten notation so W is the Q^n-Wiener process, and λ is the intensity of the marked point process under the Q^n-measure. Furthermore h_t and $\rho_t(v)$ are the Girsanov kernels for the transition from P to Q^n with respect to the Wiener process and the marked point process respectively. That is $-h_t$ and $-\rho_t(v)\lambda_t(dv)^P$ are the market price of diffusion risk and jump risk respectively.

Proof. Given the P-dynamics of $f_n(t, T)$ and $f_r(t, T)$, then Björk et al. [6] show that

$$\begin{aligned}\frac{dp_i(t,T)}{p_i(t-,T)} &= \left\{r_i(t) + A^i(t,T) + \frac{1}{2}\|\beta^i(t,T)\|^2\right\} dt + \beta^i(t,T)dW_t^P \\ &+ \int_V \delta^i(t,v,T)\mu(dt,dv) \quad \text{for } i = r, n\end{aligned} \tag{5}$$

where

$$A^i(t,T) = -\int_t^T \alpha^i(t,s)ds \quad \text{for } i = r, n$$

By using the P-dynamics of $p_r(t, T)$ that we just obtained, together with the P-dynamics of $I(t)$ that is given by Assumption 2.4, Ito's lemma gives that

$$\begin{aligned}\frac{dp_{IP}(t,T)}{p_{IP}(t-,T)} &= \left\{r^r(t) + A^r(t,T) + \frac{1}{2}\|\beta^r(t,T)\|^2\right\} dt \\ &+ \left\{\mu^I(t) + \sigma^I(t) \cdot \beta^r(t,T)\right\} dt \\ &+ \left\{\beta^r(t,T) + \sigma^I(t)\right\} dW_t^P + \int_V \delta^{IP}(t,v,T)\mu(dt,dv)\end{aligned}$$

$$\text{where} \quad \delta^{IP}(t,v,T) = \delta^r(t,v,T) + \gamma^I(t,v) + \gamma^I(t,v)\delta^r(t,v,T)$$

Next, we would like to change measure from P to the equivalent (nominal) martingale measure Q^n. By the Girsanov theorem, we know there exist a P-adapted process h_t and a P-predictable process $\rho_t(v) \geq -1 \forall t, v$ such that $dL_t = h_t L_t dW_t^P + L_{t-} \int_V \rho_t(v) \left\{\mu(dt, dv) - \lambda_t^P(dv)dt\right\}$ where $L_T = dQ/dP$ on $\mathcal{F}_T$ so that $dW_t^P = h_t dt + dW_t$ and $\lambda_t(dv) = \lambda_t^P(dv)(1 + \rho_t(v))$. Here W denotes a Q^n-Wiener process and $\lambda_t(dv)dt$ is the intensity measure of the marked point process under the Q^n-measure. We will use $\tilde{\mu}(dt,dv)$ to denote the compensated marked point process under Q^n that is $\tilde{\mu}(dt,dv) = \mu(dt,dv) - \lambda_t(dv)dt$. Hence the dynamics of $I(t)$, $p_n(t,T)$ and $p_{IP}(t,T)$ under Q^n are given by:

$$\begin{aligned} \frac{dI(t)}{I(t-)} &= \left\{\mu^I(t) + h(t)\cdot\sigma^I(t)\right\}dt \\ &+ \int_V \gamma^I(t,v)\,(1+\rho(t,v))\,\lambda^P(t,dv)dt \\ &+ \sigma^i(t)dW_t + \int_V \gamma^I(t,v,T)\tilde{\mu}(dt,dv) \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{dp_n(t,T)}{p_n(t-,T)} &= \mu^n(t,T)dt + \beta^n(t,T)dW_t \\ &+ \int_V \delta^n(t,v,T)\tilde{\mu}(dt,dv) \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{dp_{IP}(t,T)}{p_{IP}(t-,T)} &= \mu^{IP}(t,T)dt + \left\{\beta^r(t,T) + \sigma^I(t)\right\}dW_t \\ &+ \int_V \delta^{IP}(t,v,T)\tilde{\mu}(dt,dv) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu^n(t,T) &= r^n(t) + A^n(t,T) + \frac{1}{2}\|\beta^n(t,T)\|^2 \\ &+ h(t)\cdot\beta^n(t,T) + \int_V \delta^n(t,v,T)\,(1+\rho(t,v))\,\lambda^P(t,dv) \end{aligned} \tag{9}$$

$$\begin{aligned} \mu^{IP}(t,T) &= r^r(t) + A^r(t,T) + \frac{1}{2}\|\beta^r(t,T)\|^2 + \mu^I(t) \\ &+ \sigma^I(t)\cdot\beta^r(t,T) + h(t)\cdot\beta^r(t,T) + h(t)\cdot\sigma^I(t) \\ &+ \int_V \delta^{IP}(t,v,T)\,(1+\rho(t,v))\,\lambda^P(t,dv) \end{aligned} \tag{10}$$

As stated above, the assumption of arbitrage free markets implies that

$$\frac{p_n(t,T)}{B_n(t)} \quad \frac{p_{IP}(t,T)}{B_n(t)} \quad \text{are } Q^n\text{-martingales}$$

hence the drift of $p_n(t,T)$ and $p_{IP}(t,T)$ must equal the nominal short rate, that is $\mu^n(t,T) = \mu^{IP}(t,T) = r^n(t)$. This condition together with equation (7) and (8) immediately gives that the Q^n-dynamics of $p_n(t,T)$ and $p_{IP}(t,T)$ satisfies equation (2) and (3) respectively.

Next we insert the conditions $\mu^n(t,T) = \mu^{IP}(t,T) = r^n(t)$ into the drift equations (9) and (10). By noting that this must hold for all T and using that $\delta^{IP}(t,v,T) = \delta^r(t,v,T) + \gamma^I(t,v) + \gamma^I(t,v)\delta^r(t,v,T)$ we get three drift conditions:

$$\begin{aligned} A^n(t,T) &= -\frac{1}{2}\|\beta^n(t,T)\|^2 - h(t)\cdot\beta^n(t,T) \\ &- \int_V \delta^n(t,v,T)\,(1+\rho(t,v))\,\lambda_t^P(dv) \qquad (11) \\ A^r(t,T) &= -\frac{1}{2}\|\beta^r(t,T)\|^2 - \sigma^I(t)\cdot\beta^r(t,T) - h(t)\cdot\beta^r(t,T) \\ &- \int_V \delta^r(t,v,T)\left(1+\gamma^I(t,v)\right)(1+\rho(t,v))\,\lambda_t^P(dv) \qquad (12) \\ \mu^I(t) &= r^n(t) - r^r(t) - h(t)\cdot\sigma^I(t) \\ &- \int_V \gamma^I(t,v)\,(1+\rho(t,v))\,\lambda_t^P(dv) \qquad (13) \end{aligned}$$

From condition (13) and equation (6) we now see that under Q^n the dynamics of I satisfy equation (1).

By definition $p_r(t,T) = p_{IP}(t,T)/I(t)$, so by Ito's lemma and the equations (1) and (3) we finally see that the Q^n-dynamics of $p_r(t,T)$ satisfies equation (4). ∎

Corollary 2.1 *The drift conditions that have to be satisfied under the objective probability measure in order for the market to be free of arbitrage are:*

$$\begin{aligned} \alpha^n(t,T) &= \sigma^n(t,T)\cdot\left(\int_t^T \sigma^r(t,s)ds - h(t)\right) \\ &- \int_V \{\delta^n(t,v,T)+1\}\,\xi^n(t,v,T)\,(1+\rho(t,v))\,\lambda_t^P(dv) \end{aligned}$$

$$
\begin{aligned}
\alpha^r(t,T) &= \sigma^r(t,T)\cdot\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t) - h(t)\right)\\
&- \int_V \left(1+\gamma^I(t,v)\right)\left(1+\rho(t,v)\right)\left(1+\delta^r(t,v,T)\right)\xi^r(t,v,T)\lambda_t^P(dv)
\end{aligned}
$$

$$
\mu^I(t) = r^n(t) - r^r(t) - h(t)\cdot\sigma^I(t) - \int_V \gamma^I(t,v)\left(1+\rho(t,v)\right)\lambda_t^P(dv)
$$

The corollary follows from the three drift equation (11), (12) and (13) and by taking the T-derivative of the first two.

Corollary 2.2 *Define* $B_{IP}(t)$ *by*

$$B_{IP}(t) = I(t)B_r(t)$$

Then

$$\frac{B_{IP}(t)}{B_n(t)} \quad \textit{is a } Q^n\textit{-martingale}$$

Proof. By the definition of $B_r(t)$ and the Q^n-dynamics of $I(t)$ given in equation (1), Ito's lemma gives that

$$\frac{dB_{IP}(t,T)}{B_{IP}(t-,T)} = r^n(t)dt + \sigma^I(t)dW_t + \int_V \gamma^I(t,v)\tilde{\mu}(dt,dv)$$

∎

The following results follows from the general result of Geman et al. [13].

Proposition 2.2 (Geman et al.) *Define* Q^{IP} *and* Q^{T-IP} *respectively by*

$$
\begin{aligned}
dQ^{IP} &= L_T dQ^n \text{ on } \mathcal{F}_T\\
dQ^{T-IP} &= \tilde{L}_T dQ^n \text{ on } \mathcal{F}_T
\end{aligned}
$$

where

$$
\begin{aligned}
L_t &= \frac{B_{IP}(t)}{B_n(t)}\frac{B_n(0)}{B_{IP}(0)}\\
\tilde{L}_t &= \frac{P_{IP}(t,T)}{B_n(t)}\frac{B_n(0)}{P_{IP}(0,T)}
\end{aligned}
$$

then Q^{IP} *is a martingale measure for the numeraire* B_{IP} *and* Q^{T-IP} *is a martingale measure for the numeraire* $P_{IP}(t,T)$

Thus far we have only considered nominal measures but now we shall also consider real martingale measures. That the next Proposition holds can be realized directly from the analogy to the foreign-currency. Alternatively it can easily be proved from Proposition 2.2.

Proposition 2.3 *Let Π_n denote an arbitrage free price process in the nominal economy. Define the process Π_r by $\Pi_r(t) = \Pi_n(t)/I(t)$. Define Q^r and $Q^{T,r}$ respectively by*

$$\begin{aligned} dQ^r &= L_T dQ^n \text{ on } \mathcal{F}_T \\ dQ^{T,r} &= dQ^n \tilde{L}_T \text{ on } \mathcal{F}_T \end{aligned}$$

where

$$\begin{aligned} L_t &= \frac{B_r(t)I(t)}{B_n(t)} \frac{B_n(0)}{B_r(0)I(0)} \\ \tilde{L}_t &= \frac{P_r(t,T)I(t)}{B_n(t)} \frac{B_n(0)}{P_r(0,T)I(0)} \end{aligned}$$

Then Q^r is a martingale measure for the real numeraire $B_r(t)$ and $Q^{T,r}$ is a martingale measure for the real numeraire $p_r(t,T)$. Furthermore

$$\frac{\Pi_r(t)}{B_r(t)} \quad \text{is a } Q^r\text{-martingale}$$

$$\frac{\Pi_r(t)}{p_r(t,T)} \quad \text{is a } Q^{T,r}\text{-martingale}$$

Corollary 2.3

$$\frac{p_r(t,T)}{B_r(t)} \quad \text{is a } Q^r\text{-martingale}$$

$$\frac{p_r(t,S)}{p_r(t,T)} \quad \text{is a } Q^{T,r}\text{-martingale}$$

From standard no arbitrage theory we can now conclude that the price at time t of a simple contingent claim on I, $\Phi(I_T)$, that is payed out at time T is given by

$$E^{Q,n}\left[\frac{B_n(t)}{B_n(T)} \Phi(I_T) \middle| \mathcal{F}_t \right]$$

where the dynamics of I under Q^n are given by equation (1). Should we find it more convenient we can for example also write the price as

$$E^{Q,r}\left[\frac{I(t)B_r(t)}{I(T)B_r(T)}\Phi(I_T)\middle|\mathcal{F}_t\right]$$

We finish this section by making one additional assumption that will be needed for some of the calculations in the coming sections.

Assumption 2.6 *We assume that σ^r, σ^n, σ^i, ξ^r, ξ^n, γ^I and λ^P are deterministic.*

3 Inflation indexed swaps

In this section we will study inflation indexed swaps. In particular, we will price the zero coupon inflation indexed swap and the year-on-year inflation indexed swap. In what follows $\Pi[t, T, \cdot]$ is used to denote the price at time t, in dollars, of the payoff $(\cdot)$ that is payed out at time T. Furthermore $E_t^{T,r}[\cdot]$ denotes the conditional expectation of $(\cdot)$ given $\mathcal{F}_t$ under the T-forward measure $Q^{T,i}$ where $i = n$ for a nominal measure and $i = r$ for a real measure.

A swap is an agreement between two counter parties to exchange cash flows. The agreement specifies the cash flows and the dates when the cash flows are to be paid. In an inflation-indexed swap at least one of the cash flows is dependent on an inflation index or an inflation protected security. If we let T_0, T_1, $\cdots$, T_M be a fixed set of increasing times and define α_i by

$$\alpha_i = T_i - T_{i-1} \quad \text{for} \quad i = 1, \cdots, M$$

Then, typically the swap starts at time T_0 and the payments occur at the dates T_1, T_2, $\cdots$, T_M. By a receiver swap we refer to a swap where the holder at each payment date receives a fixed amount and pays a floating amount. The fixed payment is known at the start date of the swap while the floating payment is not. By a payer swap we mean a swap where the payments go in the opposite direction to a receiver swap.

3.1 Pricing Zero Coupon Inflation Indexed Swap

In this Section we will price Zero Coupon Inflation Indexed Swap (ZCIIS). Mercurio [18] priced ZCIIS by using martingale methods and showed that the price is model independent. As a preparation for the next Section on Year-on-Year Inflation Swaps we will also show this result. Furthermore we will provide an alternative proof by a simple replicating argument.

In a ZCIIS one party pays a fixed interest rate and receives the inflation rate over the specified time period. The inflation rate is calculated as the percentage return of the consumer price index. The other party of the swap receives the same flows but of opposite signs. As the name indicates, a ZCIIS has only one time interval $[T_0, T]$ with payments at time T and no intermediary payments. That is, cash flows are exchanged only once. Let $Z_0(T, K)$ denote a payer ZCIIS that starts at time T_0, has payment date at time T and has a swap rate equal to K. Then a fixed amount of

$$(1+K)^{T-T_0} - 1$$

is payed out at time T and floating amount of

$$\frac{I(T)}{I(T_0)} - 1$$

is received at time T.

Let $Z_0(t, T, K)$ denote the price of a $Z_0(T, K)$ at time t. Then the payoff to the holder of a $Z_0(T, K)$ is

$$Z_0(T, T, K) = \frac{I(T)}{I(T_0)} - (1+K)^{T-T_0}$$

and

$$\begin{aligned} Z_0(T_0, T, K) &= \Pi\left[T_0, T, \frac{I(T)}{I(T_0)} - (1+K)^{T-T_0}\right] \\ &= \Pi\left[T_0, T, \frac{I(T)}{I(T_0)}\right] - \Pi\left[T_0, T, (1+K)^{T-T_0}\right] \qquad (14) \end{aligned}$$

where the first part is

$$\begin{aligned} \Pi\left[T_0, T, \frac{I(T)}{I(T_0)}\right] &= \frac{p_n(T_0, T)}{I(T_0)} E_{T_0}^{T,n}\left[I(T)\right] \\ &= \frac{p_n(T_0, T)}{I(T_0)} E_{T_0}^{T,n}\left[\frac{I(T)P_r(T, T)}{P_n(T, T)}\right] = P_r(T_0, T) \end{aligned}$$

since $I(t)p_r(t, T)/p_n(t, T) = p_{IP}(t, T)/p_n(t, T)$ is a $Q^{T,n}$-martingale.

The second part is

$$\Pi\left[T_0, T, (1+K)^{T-T_0}\right] = p_n(T_0, T)(1+K)^{T-T_0}$$

Hence equation (14) becomes equal to

$$p_r(T_0, T) - p_n(T_0, T)(1+K)^{T-T_0}$$

This result was first stated in the article by Mercurio [18]. We note that there is also a simple replicating argument that proves the result.

Remark 3.1 *To replicate the floating leg of the swap: At time* T_0 *buy* $1/I(T_0)$ *TIPS-bonds with maturity date* T. *Then at time* T *we will receive the dollar value of* $1/I(T_0)$ *CPI units, that is* $I(T)/I(T_0)$. *The price at time* T_0 *of* $1/I(T_0)$ *TIPS-bonds is* $1/I(T_0)$ *times* $I(T_0)p_r(T_0,T)$, *ie* $p_{IP}(T_0,T)/I(T_0) = p_r(T_0,T)$.

Remark 3.2 *Regardless of whether one uses the martingale method or the replicating argument to price the ZCIIS, it should be noted that no assumptions on the dynamics of the assets are needed. Hence this result is model independent.*

3.2 Pricing Year-on-Year Inflation Indexed Swaps

In this Section we will price Year-on-Year Inflation Indexed Swaps (YYIIS) using the jump-diffusion model specified in Section 2.

Let $Y_m^M(K)$ denote a payer Year-on-Year Inflation Indexed Swap that starts at time T_m with payment dates at $T_{m+1}, T_{m+2}, \cdots, T_M$. For every period $[T_i, T_{i+1}]$ for $i = m, \cdots, M-1$ a fixed amount of

$$\alpha_{i+1}K$$

is payed out at time T_{i+1}. For the same period a floating amount of

$$\alpha_{i+1}\left[X_{i+1} - 1\right]$$

where

$$X_{i+1} = \frac{I(T_{i+1})}{I(T_i)}$$

is received at time T_{i+1}.

If we let $Y_m^M(t,K)$ denote the price of a $Y_m^M(K)$ at time t where $t \leq T_m$, then

$$\begin{aligned} Y_m^M(t,K) &= \sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}\left(X_{i+1} - 1\right)\right] - \sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}K\right] \\ &= \sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right] - (K+1)\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1}) \quad (15) \end{aligned}$$

where we have used standard no-arbitrage pricing theory. We are thus left with the exercise of calculating $\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right]$. However, before we do that, we will define the forward swap rate. We define the forward swap rate of a Year-on-Year Inflation indexed swap to be the value of K for which the price of the swap is zero. We denote the forward swap rate for the swap $Y_m^M(K)$ by $R_m^M(t)$. Hence $Y_m^M(t, R_m^M(t)) = 0$ and so $R_m^M(t)$ is given by:

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right] - \sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1})}{\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1})} \quad (16)$$

Next we want to calculate $\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1} X_{i+1}\right]$. As we will see this expression is model-dependent and in order to calculate it we will use the model that was setup in Section 2. This will enable us to obtain explicit formulas for both the swap price and the forward swap rate.

To calculate $\Pi\left[t, T_{i+1}, X_{i+1}\right]$ when $t < T_i$ we simplify and look at the case when $i = 1$. That is we calculate $\Pi\left[t, T_2, X_2\right]$ when $t < T_1$. The general case $\Pi\left[t, T_{i+1}, X_{i+1}\right]$ is obtained by the obvious extention. $\Pi\left[t, T_2, X_2\right]$ is the value at time t of the payoff $X_2 = \alpha_2 I(T_2)/I(T_1)$ that is payed out at time T_2 and in order to calculate it we will use the T_2-forward measure and iterated expectation along with the fact that $I(t)p_r(t,T_2)/p_n(t,T_2) = p_{IP}(t,T_2)/p_n(t,T_2)$ is a $Q^{T_2,n}$-martingale. We find that

$$
\begin{aligned}
\Pi\left[t, T_2, X_2\right] &= p_n(t,T_2) E_t^{T_2,n}\left[\alpha_2 \frac{I(T_2)}{I(T_1)}\right] \\
&= p_n(t,T_2)\alpha_2 E_t^{T_2,n}\left[\frac{1}{I(T_1)} E_{T_1}^{T_2,n}\left[I(T_2)\right]\right] \\
&= p_n(t,T_2)\alpha_2 E_t^{T_2,n}\left[\frac{1}{I(T_1)} E_{T_1}^{T_2,n}\left[\frac{I(T_2)p_r(T_2,T_2)}{p_n(T_2,T_2)}\right]\right] \\
&= p_n(t,T_2)\alpha_2 E_t^{T_2,n}\left[\frac{p_r(T_1,T_2)}{p_n(T_1,T_2)}\right] \qquad (17)
\end{aligned}
$$

To calculate the expected value in equation (17) we change the numeraire to the $Q^{T_1,n}$-forward measure by using Bayes formula and the likelihood ratio $L_t^{T_2,n/T_1,n}$ where

$$
L_t^{T_2,n/T_1,n} = \left.\frac{dQ^{T_2,n}}{dQ^{T_1,n}}\right|_t = \frac{p_n(t,T_2)}{p_n(t,T_1)} \frac{p_n(0,T_1)}{p_n(0,T_2)}
$$

Hence

$$
\begin{aligned}
E_t^{T_2,n}\left[\frac{p_r(T_1,T_2)}{p_n(T_1,T_2)}\right] &= \frac{E_t^{T_1,n}\left[\frac{p_r(T_1,T_2)}{p_n(T_1,T_2)} L_{T_1}^{T_2,n/T_1,n}\right]}{L_t^{T_2,n/T_1,n}} \\
&= E_t^{T_1,n}\left[\frac{p_r(T_1,T_2)}{p_n(T_1,T_2)} \frac{p_n(T_1,T_2)}{p_n(T_1,T_1)}\right] \frac{p_n(t,T_1)}{p_n(t,T_2)} \\
&= \frac{p_n(t,T_1)}{p_n(t,T_2)} E_t^{T_1,n}\left[p_r(T_1,T_2)\right] \qquad (18)
\end{aligned}
$$

Combining equation (17) and (18) we find that

$$
\Pi\left[t, T_2, X_2\right] = \alpha_2 p_n(t,T_1) E_t^{T_1,n}\left[p_r(T_1,T_2)\right] \qquad (19)
$$

This result is also stated in the article by Mercurio [18]. Note that we have not yet made use of any model assumption. However, the expected value in equation (19) is model dependent. Mercurio calculates it using a diffusion model. We will calculate it using the jump-diffusion model specified in Section 2.

We change measure from the nominal $Q^{T_1,n}$-forward measure to the real $Q^{T_1,r}$-forward measure. Again we use Bayes formula and the expected value in equation (19) can thus be rewritten as

$$E_t^{T_1,n}\left[p_r(T_1,T_2)\right] = \frac{E_t^{T_1,r}\left[\frac{p_r(T_1,T_2)}{p_r(T_1,T_1)} L_{T_1}^{T_1,n/T_1,r}\right]}{L_t^{T_1,n/T_1,r}}$$

Since

$$L_t^{T_1,n/T_1,r} = \left.\frac{dQ^{T_1,n}}{dQ^{T_1,r}}\right|_t = \frac{p_n(t,T_1)}{p_r(t,T_1)I(t)} \frac{p_r(0,T_1)I(0)}{p_n(0,T_1)}$$

the dynamics of $L_t^{T_1,n/T_1,r}$ under $Q^{T_1,r}$ is given by:

$$\begin{aligned}\frac{dL_t^{T_1,n/T_1,r}}{L_{t-}^{T_1,n/T_1,r}} &= \left\{\beta_t^{n,1} - \beta_t^{r,1} - \sigma_t^i\right\} dW_t^{T_1,r} \\ &+ \int_V \frac{\delta_t^{n,1} - \delta_t^{r,1} + \gamma_t^I + \delta_t^{r,1}\gamma_t^I}{1 + \delta_t^{r,1} + \gamma_t^I + \delta_t^{r,1}\gamma_t^I} \tilde{\mu}^{T_1,r}(dt,dv)\end{aligned}$$

where we have used the simplifying notation $\beta_t^{k,j} = \beta^k(t,T_j)$, $\delta_t^{k,j} = \delta^k(t,v,T_j)$ and $\gamma_t^{I,j} = \gamma^I(t,v,T_j)$. Since both $p_r(t,T_2)/p_r(t,T_1)$ and $L_t^{T_1,n/T_1,r}$ are $Q^{T_1,r}$-martingales Ito gives that

$$E_t^{T_1,n}\left[p_r(T_1,T_2)\right] = \frac{p_r(t,T_2)C(t,T_1,T_2)}{p_r(t,T_1)}$$

where

$$C(t,T_1,T_2) = e^{\int_t^{T_1}\left\{(\beta_s^{n,1} - \beta_s^{r,1} - \sigma_s^I)\cdot(\beta_s^{r,2} - \beta_s^{r,1}) + \int_V \Delta_s^{1,2}\lambda_s(dv)\right\}ds} \tag{20}$$

and

$$\Delta_t^{1,2} = \left(\delta_t^{r,2} - \delta_t^{r,1}\right)\frac{\delta_t^{n,1} - \delta_t^{r,1} + \gamma_t^I + \delta_t^{r,1}\gamma_t^I}{1 + \delta_t^{r,1} + \gamma_t^I + \delta_t^{r,1}\gamma_t^I}$$

where we have used that $\lambda^{T_1,r} = (1 + \delta^{r,1})\lambda$ which follows from Radon-Nikodym derivative between the $Q^{T_1,r}$-measure and the Q^n-measure. Equation (19) thus gives that

$$\Pi\left[t,T_2,X_2\right] = \alpha_2 \frac{p_n(t,T_1)p_r(t,T_2)C(t,T_1,T_2)}{p_r(t,T_1)}$$

Changing back to the general case with $1 = i$ and $2 = i+1$ we have that

$$\Pi\left[t,T_{i+1},X_{i+1}\right] = \alpha_{i+1}\frac{p_n(t,T_i)p_r(t,T_{i+1})C(t,T_i,T_{i+1})}{p_r(t,T_i)} \tag{21}$$

Hence the pricing equation (15) is found to be

$$\begin{aligned} Y_m^M(t,K) &= \sum_{i=m}^{M-1} \alpha_{i+1} \frac{p_n(t,T_i)p_r(t,T_{i+1})C(t,T_i,T_{i+1})}{p_r(t,T_i)} \\ &- (K+1)\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t,T_{i+1}) \end{aligned} \tag{22}$$

and the forward swap rate (16) is found to be

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1} \frac{\alpha_{i+1}p_n(t,T_i)p_r(t,T_{i+1})C(t,T_i,T_{i+1})}{p_r(t,T_i)} - \sum_{i=m}^{M-1} \alpha_{i+1}p_n(t,T_{i+1})}{\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t,T_{i+1})} \tag{23}$$

If we choose all volatilities to be zero and the volatilities of the real and nominal forward rates to be $\xi^n(t,T) = ae^{-b(T-t)}$ and $\xi^r(t,T) = ce^{-d(T-t)}$ for some positive constants a, b, c, d as in the model by Jarrow and Yildirim [16] then the pricing formula (22) reduces to that in the article by Mercurio [18].

As the valuation formula appears in equation (22), it looks as if one has to estimate real parameters in order to price a YYIIS. However, the pricing formulas can be rewritten so that they do not depend on any real bond prices or any real volatilities. Instead they will be functions of the inflation protected TIPS-bonds and the volatilities of these inflation protected bonds. This is good news since it means that we do not need to estimate any real parameters in order to price a YYIIS. The trick is just to use that $p_{IP}(t,T_k) = I(t)p_r(t,T_k)$. Hence if we extend the right hand side of equation (22) by $I(t)/I(t)$. The time t price of a $Y_m^M(K)$ can be rewritten as

$$\begin{aligned} Y_m^M(t,K) &= \sum_{i=m}^{M-1} \alpha_{i+1} \frac{p_n(t,T_i)p_{IP}(t,T_{i+1})C(t,T_i,T_{i+1})}{p_{IP}(t,T_i)} \\ &- (K+1)\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t,T_{i+1}) \end{aligned}$$

Since $p_{IP}(t,T_k) = I(t)p_r(t,T_k)$, the Ito formula gives that $\beta^{IP} = \beta^r + \sigma^I$ and $\delta^{IP} = \delta^r + \gamma^I + \gamma^I\delta^r$. This is what we will use to rewrite the correction term $C(t,T_i,T_{i+1})$ to a function of the volatilities of the TIPS-bonds rather than of the volatilities of real bonds. Recall that

$$C(t,T_i,T_{i+1}) = \frac{e^{\int_t^{T_i}(\beta_s^{n,i}-\beta_s^{r,i}-\sigma_s^I)\cdot(\beta_s^{r,i+1}-\beta_s^{r,i})ds}}{e^{-\int_t^{T_i}\int_V \Delta_s^{i,i+1}\lambda_s(dv)ds}}$$

and

$$\Delta_t^{i,i+1} = \left(\delta_t^{r,i+1} - \delta_t^{r,i}\right)\frac{\delta_t^{n,i} - \delta_t^{r,i} + \gamma_t^I + \delta_t^{r,i}\gamma_t^I}{1 + \delta_t^{r,i} + \gamma_t^I + \delta_t^{r,i}\gamma_t^I}$$

In the nominator of $C(t,T_i,T_{i+1})$, we extend the second parenthesis in the exponent by $+\sigma^I-\sigma^I$ and use the relation $\beta^{IP}=\beta^r+\sigma^I$. In the denominator of $C(t,T_i,T_{i+1})$ we extend $\Delta_t^{i,i+1}$ by $(1+\gamma^I)/(1+\gamma^I)$ and use the relation $\delta^{IP}=\delta^r+\gamma^I+\gamma^I\delta^r$. We find that

$$C(t,T_i,T_{i+1})=\frac{e^{\int_t^{T_i}(\beta_s^{n,i}-\beta_s^{IP,i})\cdot(\beta_s^{IP,i+1}-\beta_s^{IP,i})ds}}{e^{-\int_t^{T_i}\int_V \Delta_s^{i,i+1}\lambda_s(dv)ds}}$$

where

$$\Delta_t^{i,i+1}\frac{(\delta_t^{IP,i+1}-\delta_t^{IP,i})(\delta_t^{n,i}-\delta_t^{IP,i})}{1+\delta_t^{IP,i}}$$

Similarly the forward swap rate in equation (23) can be rewritten as

$$R_m^M(t)=\frac{\sum_{i=m}^{M-1}\frac{\alpha_{i+1}p_n(t,T_i)p_{IP}(t,T_{i+1})C(t,T_i,T_{i+1})}{p_{IP}(t,T_i)}-\sum_{i=m}^{M-1}\alpha_{i+1}p_n(t,T_{i+1})}{\sum_{i=m}^{M-1}\alpha_{i+1}p_n(t,T_{i+1})}$$

3.3 Hedging Inflation Indexed Swaps

In this section we will show how to hedge inflation indexed swaps. Given Assumption 2.2, the fixed cash flows of a swap is trivial to hedge. Thus we will only consider the floating cash flows, i.e. the inflation leg.

The hedge of a ZSIIS is given by Remark 3.1. For the YYIIS we will only consider the pure diffusion case, i.e. we will find the hedge for the special case when all jump parameters are zero.

Since the floating leg of a YYIIS is a sum of cash flows we can hedge each cash flow separately. Thus it suffice to consider the cash flow $X(T_{i+1})=I(T_{i+1})/I(T_i)$ that is paid out at time T_{i+1}. By Remark 3.1 we know that at time T_i we can replicate this cash flow with a buy and hold portfolio consisting of $1/I(T_i)$ inflation protected bonds with maturity date T_{i+1}. The price at time T_i of this portfolio is $p_{IP}(T_i,T_{i+1})/I(T_i)$. Thus we are left with the exercise of finding a replicating portfolio for the claim that pays out $p_{IP}(T_i,T_{i+1})/I(T_i)$ at time T_i. This is done in the next Proposition, which shows that replicating portfolio consists of T_i-bonds, inflation protected T_i-bond and inflation protected T_{i+1}-bonds.

Proposition 3.1 *Let $t\le T_i$, and*

$$\begin{aligned}S_1(t)&=p_n(t,T_i)\\S_2(t)&=p_{IP}(t,T_{i+1})\\S_3(t)&=p_{IP}(t,T_i)\end{aligned}$$

and

$$\begin{aligned} h_1(t) &= \frac{\Pi[t, T_{i+1}, X_{i+1}]}{p_n(t, T_i)} \\ h_2(t) &= \frac{\Pi[t, T_{i+1}, X_{i+1}]}{p_{IP}(t, T_{i+1})} \\ h_3(t) &= -\frac{\Pi[t, T_{i+1}, X_{i+1}]}{p_{IP}(t, T_i)} \end{aligned}$$

where

$$\Pi[t, T_{i+1}, X_{i+1}] = \frac{p_n(t, T_i) p_{IP}(t, T_{i+1}) e^{\int_t^{T_i} (\beta_s^{n,i} - \beta_s^{IP,i}) \cdot (\beta_s^{IP,i+1} - \beta_s^{IP,i}) ds}}{p_{IP}(t, T_i)}$$

Finally let $S = [S_1, S_2, S_3]$ and $h = [h_1, h_2, h_3]$. Then h is a self-financing portfolio with value process $V^h(t)$ and

$$V^h(T_i) = \frac{p_{IP}(T_i, T_{i+1})}{I(T_i)}$$

Note that $\Pi[t, T_{i+1}, X_{i+1}]$ is the price process[6] that we calculated in section 3.2.

Proof. The value process of the portfolio h is given by:

$$V^h(t) = \sum_{i=1}^{3} h_i(t) S_i(t) = \Pi[t, T_{i+1}, X_{i+1}]$$

and so

$$V^h(T_i) = \frac{p_{IP}(T_i, T_{i+1})}{I(T_i)}$$

To see that h is self-financing, we apply Ito's lemma to $V^h(t)$

$$\begin{aligned} dV^h(t) &= -V^h(t)(\beta_t^{n,i} - \beta_t^{IP,i}) \cdot (\beta_t^{IP,i+1} - \beta_t^{IP,i}) dt \\ &+ h_1(t) dp_n(t, T_i) + h_2(t) dp_{IP}(t, T_{i+1}) + h_3(t) dp_{IP}(t, T_i) \\ &+ V^h(t) \beta_t^{n,i} \cdot \beta_t^{IP,i+1} dt - V^h(t) \beta_t^{n,i} \cdot \beta_t^{IP,i} dt \\ &- V^h(t) \beta_t^{IP,i} \cdot \beta_t^{IP,i+1} dt + V^h(t) \beta_t^{IP,i} \cdot \beta_t^{IP,i} dt \\ &= \sum_{i=1}^{3} h_i(t) dS_i(t) \end{aligned}$$

∎

[6] here without jumps

4 Inflation Indexed Swaptions

A Swaption is an option to enter into a swap at a pre specified date for a pre specified swap rate. An inflation indexed swaption is an option to enter into an inflation indexed swap. In this Section we will price two types of swaptions, the zero coupon inflation indexed swaption (ZCIISO) and the year-on year inflation indexed swaption (YYIISO). We will start with the latter one.

4.1 YYIISwaption

A YYIISwaption is an option to enter into a YYIIS at a pre specified date for a pre specified swap rate. More specifically, let $YO_m^M(K)$ denote an option to enter into a payer $Y_m^M(K)$ at time T_m with the fixed swap rate K and let $YO_m^M(t, K)$ denote the price of this option at time t. Then the payoff of $YO_m^M(K)$ is:

$$YO_m^M(T_m, K) = max[Y_m^M(T_m, K), 0] \tag{24}$$

where according to equation (15) in Section 3.2

$$Y_m^M(t, K) = \sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right] - (K+1)\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1}) \tag{25}$$

We will state an alternative formulation of the payoff of a YYIISwaption which involves the forward swap rate. To find this formulation, recall from equation (16) the forward swap rate is given by

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right] - \sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1})}{\sum_{i=m}^{M-1} \alpha_{i+1}p_n(t, T_{i+1})} \tag{26}$$

For each pair m, k such that $m < k$, define $S_m^k(t)$ by

$$S_m^k(t) = \sum_{i=m}^{k-1} \alpha_{i+1}p_n(t, T_{i+1})$$

Since S_m^k is a weighted sum of tradable assets, it is a self-financing portfolio. Hence there exist a martingale measure for the numeraire asset S_m^k. We will denote this measure by Q_m^k.

Using $S_m^k(t)$, the forward swap rate can be rewritten as:

$$R_m^M(t) = \frac{\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1}X_{i+1}\right] - S_m^M(t)}{S_m^M(t)} \tag{27}$$

Proposition 4.1 *The forward swap rate R^k_m, is a Q^k_m-martingale.*

Proof. Since both $\sum_{i=m}^{M-1} \Pi\left[t, T_{i+1}, \alpha_{i+1} X_{i+1}\right]$ and S^M_m are self financing-portfolios, the numerator of R^M_m is a self financing portfolio. Hence R^M_m is the value of a self financing portfolio divided by the numeraire S^M_m. ∎

Using the expression for the forward swap rate given in equation (27), the price of the swap $Y^M_m(K)$ can be expressed as:

$$Y^M_m(t, K) = \left(R^M_m(t) - K\right) S^M_m(t) \tag{28}$$

and the payoff of the YYIISwaption in equation (24) can be rewritten as:

$$YO^M_m(T_m, K) = S^M_m(T_m) max[R^M_m(T_m) - K, 0] \tag{29}$$

From which we see that the $YO^M_m(K)$ can be regarded as a call option on the forward swap rate, expressed in units of S^M_m.

The natural choice of measure to use for pricing the YYIISwaption is the Q^M_m-measure. However even if we choose the jump volatilities to be zero in the model specified in Section 2, so that the model is purely Wiener driven, the swap rate will involve a sum of lognormal variables and so the swap rate has a nasty distribution in this model. It seems plausible that there does not exist any explicit formula for the YYIISwaption in this case. The problem is similar to that of pricing standard interest rate swaptions assuming a HJM model for the forward rates.

For standard interest rate swaptions, the Swap Market Model by Jamshidian [15] is a practical solution which has proved to be very useful. This model, which is connected to the Libor Market Model due to Miltersen et al. [20] and Brace et al. [7] is based on the assumption that the forward swap rate is lognormally distributed. One main advantage with this assumption is that it justifies the use of Black's model[7] for the pricing interest rate swaptions.

In the next section we introduce an Inflation Indexed Swap Market Model. We too, make the assumption that the forward swap rate is lognormally distributed. It should be noted that this assumption does not spring from empirical evidence but is made for technical convenience, so as to fit Black's model also in the case of inflation indexed swaptions. Since inflation rates can become negative, the swap rate can as well. Hence this model is less suited for pricing horizons that covers periods of negative inflation. Being aware of this shortcoming, the model provides a simple solution to the pricing problem of inflation indexed swaptions.

In section 4.1 we provide a second solution to the pricing problem of inflation indexed swaptions. In this model the forward swap rates are assumed to be normally distributed instead of lognormally distributed. Thus this model allows for negative swap rates.

[7] See Black [3]

A Lognormal Inflation Indexed Swap Market Model

In this Section we will define an inflation indexed swap market model. Given this model, we will price the YYIISwaption.

Definition 4.1 *Given a set of increasing resettlement times* $T_0, T_1, \ldots, T_M$ *we define* B *to be the set consisting of pairs* (m,k) *of positive integers* m *and* k *such that* $0 \leq m < k < M$. *For any given pair* (m,k) *in* B *we assume that the forward swap rate* R_m^k *has dynamics given by*

$$dR_m^k(t) = R_m^k(t)\sigma_m^k(t)dW_m^k(t) \tag{30}$$

where W_m^k *is a multidimensional Wiener process under the* Q_m^k*-measure and* $\sigma_m^k(t)$ *is a vector of non-stochastic functions of time.*

Proposition 4.2 *For the swap market model (30), the price* $YO_m^M(t,K)$ *at time* t *where* $t \leq T_m$ *of a payer* Y_m^M *is given by*

$$YO_m^M(t,K) = S_m^M(t)\left(R_m^M(t)N(d_1) - KN(d_2)\right)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\Sigma_{m,M}}\left(\ln\left(\frac{R_m^M(t)}{K}\right) + \frac{1}{2}\Sigma_{m,M}^2\right) \\ d_2 &= d_1 - \Sigma_{m,M} \end{aligned}$$

and

$$\Sigma_{m,M}^2 = \int_t^{T_m} \|\sigma_m^M(s)\|^2 ds$$

Proof. By assumption, under the Q_m^M-measure we have

$$R_m^M(T_M) = R_m^M(t)e^{\int_t^{T_m}\sigma_m^M(s)dW_m^M(s) - \frac{1}{2}\int_t^{T_m}\|\sigma_m^M(s)\|^2 ds}$$

Hence conditional on time t,

$$\ln R_m^M(T_M) \sim N\left(\ln R_m^M(t) - \frac{\Sigma_{m,M}^2}{2}, \Sigma_{m,M}^2\right)$$

So by letting $f_X(x)$ denote the density function for a standard normal random variable and using the Q_m^M-measure, no-arbitrage pricing gives that

$$\begin{aligned} YO_m^M(t,K) &= S_m^M(t)E_t^{Q_m^M}\left[max\{R_m^M(T_M)-K,0\}\right] \\ &= S_m^M(t)\int_{-\infty}^{\infty} max\left\{R_m^M(t)e^{-\frac{\Sigma_{m,M}^2}{2}+x\Sigma_{m,M}}-K,0\right\}f_X(x)dx \\ &= S_m^M(t)\left(\int_{x_0}^{\infty} R_m^M(t)e^{-\frac{\Sigma_{m,M}^2}{2}+x\Sigma_{m,M}}f_X(x)dx - KN[-x_0]\right) \\ &= S_m^M(t)\left(R_m^M(t)N[-(x_o-\Sigma_{m,M})]-KN[-x_0]\right) \end{aligned}$$

where

$$x_0 = \frac{\ln\left(\frac{K}{R_m^M(t)}\right)+\frac{\Sigma_{m,M}^2}{2}}{\Sigma_{m,M}}$$

which proves the proposition. ∎

A Normal Inflation Indexed Swap Market Model

In this Section we will define an alternative inflation indexed swap market model under which we will price the YYIISwaption.

Definition 4.2 *Given a set of increasing resettlement times $T_0, T_1, \ldots, T_M$ we define B to be the set consisting of pairs (m,k) of positive integers m and k such that $0 \le m < k < M$. For any given pair (m,k) in B we assume that the forward swap rate R_m^k has dynamics given by*

$$dR_m^k(t) = \sigma_m^k(t)dW_m^k(t) \tag{31}$$

where W_m^k is a multidimensional Wiener process under the Q_m^k-measure and $\sigma_m^k(t)$ is a vector of non-stochastic functions of time.

Proposition 4.3 *For the swap market model (31), the price $YO_m^M(t,K)$ at time t where $t \le T_m$ of a payer Y_m^M is given by*

$$YO_m^M(t,K) = S_m^M(t)\left(R_m^M(t)N(d) - KN(d) + \frac{\Sigma_{m,M}e^{\frac{d^2}{2}}}{\sqrt{2\pi}}\right)$$

where

$$d = \frac{K - R_m^M(t)}{\Sigma_{m,M}}$$

$$\Sigma_{m,M}^2 = \int_t^{T_m} \|\sigma_m^M(s)\|^2 ds$$

Proof. Conditional on time t, under the Q_m^M-measure

$$R_m^M(T_M) \sim N\left(R_m^M(t), \Sigma_{m,M}^2\right)$$

So by letting $f_X(x)$ denote the density function for a standard normal random variable and using the Q_m^M-measure, no-arbitrage pricing gives that

$$\begin{aligned} YO_m^M(t,K) &= S_m^M(t)E_t^{Q_m^M}\left[max\{R_m^M(T_M) - K, 0\}\right] \\ &= S_m^M(t)\int_{-\infty}^{\infty} max\left\{R_m^M(t) + x\ \Sigma_{m,M} - K, 0\right\} f_X(x)dx \\ &= S_m^M(t)\left(\left\{R_m^M(t) - K\right\} N[-x_0] + \int_{x_0}^{\infty} x\ \Sigma_{m,M} f_X(x)dx\right) \\ &= S_m^M(t)\left(\left\{R_m^M(t) - K\right\} N[-x_0] + \frac{\Sigma_{m,M} e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}}\right) \end{aligned}$$

where

$$x_0 = \frac{R_m^M(t) - K}{\Sigma_{m,M}}$$

which proves the proposition. ∎

4.2 ZCIISwaption

In this Section we will price ZCIISwaptions, given a standard HJM model without jumps. A payer ZCIISwaption is an option to enter into a ZCIIS for a given pre specified swap rate at a pre specified time. Let $ZO_0(T,K)$ denote an option with maturity date T_0 to enter into a payer ZCIIS that starts at time T_0, has payment date at time T and a swap rate equal to K. Let $ZO_0(t,T,K)$ denote the price at time t of a $ZO_0(T,K)$. Then the payoff of this option is given by:

$$ZO_0(T_0,T,K) = max\{Z_0(T_0,T,K), 0\} \tag{32}$$

where as in previous sections $Z_0(t,T,K)$ is the price at time t of a payer ZCIIS that starts at time T_0, has payment date at time T and a swap rate equal to K.

Proposition 4.4 *The price, at time t, of the payer ZCIISwaption, $ZO_0(T,K)$, is given by*

$$ZO_0(t,T,K) = p_r(t,T)e^M N[d_1] - p_n(t,T)GN[d_2] \tag{33}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{p_r(t,T)}{Gp_n(t,T)}) + M + \frac{\Sigma^2}{2}}{\Sigma} \\ d_2 &= d_1 - \Sigma \\ M &= \int_t^{T_0} \{a(s,T) - r^n(s)\}\, ds \\ \Sigma^2 &= \int_t^{T_0} \|\beta^r(s,T) - \beta^n(s,T)\|^2 ds \end{aligned}$$

Proof. From Section 3.1 we know that

$$Z_0(T_0,T,K) = p_r(T_0,T) - p_n(T_0,T)(1+K)^{T-T_0}$$

Hence equation (32) is equal to

$$max\left\{p_r(T_0,T) - p_n(T_0,T)(1+K)^{T-T_0}, 0\right\}$$

By letting $(1+K)^{T-T_0} = G$ and defining

$$\Psi(t,T) = \frac{p_r(t,T)}{p_n(t,T)}$$

we can rewrite the payoff again so that equation (32) is equal to

$$p_n(T_0,T)\,(max\,\{\Psi(T_0,T) - G, 0\})$$

Using $p_n(t,T)$ as the numeraire, the price of the swaption at time t is:

$$ZO_0(t,T,K) = p_n(t,T)E_t^{T,n}[max\,\{\Psi(T_0,T) - G, 0\}] \tag{34}$$

Since we assume a standard HJM model without jumps, $\Psi(T_0,T)$ is lognormally distributed. The Q^n-dynamics of $p_n(t,T)$ and $p_r(t,T)$ are given by equation (2) and (4) with the assumption that $\delta^n = 0$ and $\delta^r = 0$. Hence by Ito's lemma the dynamics of $\Psi(t,T)$ under the nominal risk neutral measure Q^n is:

$$\begin{aligned} \frac{d\Psi(t,T)}{\Psi(t,T)} &= \{a(t,T) - r^n(t) + \beta^n(t,T)\cdot(\beta^n(t,T) - \beta(t,T)^r)\}\, dt \\ &+ \{\beta^r(t,T) - \beta^n(t,T)\}\, dW_t \end{aligned}$$

To change measure to the nominal T-forward measure $Q^{T,n}$ we use that

$$L_t^{T,n/n} = \left.\frac{dQ^{T,n}}{dQ^n}\right|_t = \frac{p_n(t,T)}{B(t)}\frac{1}{p_n(0,T)}$$

hence the dynamics of Radon-Nykodym is given by

$$dL_t^{T,n/n} = L_t^{T,n/n}\beta^n(t,T)dW_t^{T,n}$$

So by the Girsanov Theorem

$$dW_t = \beta^{n*}(t,T)dt + dW_t^{T,n}$$

where * denotes transpose. Hence the dynamics of $\Psi(t,T)$ under the nominal T-forward measure $Q^{T,n}$ is

$$\frac{d\Psi(t,T)}{\Psi(t,T)} = \{a(t,T) - r^n(t)\}\,dt + \{\beta^r(t,T) - \beta^n(t,T)\}\,dW_t^{T,n}$$

By Ito's lemma we find that

$$\begin{aligned} d\ln\Psi(t,T) &= \left\{a(t,T) - r^n(t) - \frac{1}{2}\|\beta^r(t,T) - \beta^n(t,T)\|^2\right\}dt \\ &+ \{\beta^r(t,T) - \beta^n(t,T)\}\,dW_t^{T,n} \end{aligned}$$

Hence

$$\ln\Psi(T_0,T) \sim N\left(\ln\Psi(t,T) + M - \frac{\Sigma^2}{2}, \Sigma^2\right)$$

So by letting $f_X(x)$ denote the density function for a standard normal random variable, we can write the expected value in equation (34) as

$$\int_{-\infty}^{\infty} max\left\{\Psi(t,T)e^{M-\frac{\Sigma^2}{2}+x\Sigma} - G, 0\right\} f_X(x)dx$$

The remaining part of the proof contains straight forward calculations similar to those in the proof of Proposition 4.2 and is therefore omitted. ▮

Since a ZCIISwaption is a special case of a YYIISwaption, the ZCIISwpation can also be priced using the inflation swap market models. More precisely $ZO_m(t,T_{m+1},K) = YO_m^{m+1}(t,G)$.

5 TIPStions

In this section we will price a TIPStion, assuming a standard HJM model without jumps. A TIPStion is an option on a TIPS-bond. A call TIPStion gives the buyer the right to purchase a TIPS-bond for a given pre specified price at maturity date. Let t denote the maturity date of a call option on a TIPS-bond that pays out the dollar value of one CPI unit at time T. Let K be the strike price of the option. Then the payoff of the option at maturity is

$$\mathcal{X} = max[p_{IP}(t,T) - K, 0]$$

Proposition 5.1 *The price, at time* 0*, of a TIPStion with payoff* $\mathcal{X}$ *and maturity date* t*, is given by*

$$\Pi[0,t,\mathcal{X}] = p_{IP}(0,T)N(d_1) - p_n(0,t)KN(d_2) \tag{35}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{p_{IP}(0,T)}{p_n(0,t)K}\right) + \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}} \\ d_2 &= d_1 - \Sigma \\ \Sigma^2 &= \int_0^t \left\|\beta^n(s,t) - \beta^{IP}(s,T)\right\|^2 ds \end{aligned}$$

Proof. The price of the option at time 0 is

$$\Pi[0,t,\mathcal{X}] = \Pi[0,t,p_{IP}(t,T)\mathbb{I}_A] - \Pi[0,t,K\mathbb{I}_A] \tag{36}$$

where

$$A = \{p_{IP}(t,T) > K\}$$

Define

$$\Upsilon(s,t,T) = \frac{p_n(s,t)}{p_{IP}(s,T)}$$

Since Υ is the quotient of two traded assets where the inflation protected T-bond is the numeriare, it is a Q^{T-IP}-martingale. The Q^n-dynamics of p_n and p_{IP} are given by equations (2) and (3) with $\delta^n = 0$ and $\delta^{IP} = 0$. By Ito's lemma we find that the volatility of $\Upsilon(s,t,T)$ under Q^n is $\beta^n(s,t) - \beta^{IP}(s,T)$. Since the volatility is preserved under measure changes we have that the Q^{IP}-dynamic of Υ is

$$\frac{d\Upsilon(s,t,T)}{\Upsilon(s,t,T)} = \left\{\beta^n(s,t) - \beta^{IP}(s,T)\right\} dW_s^{T-IP}$$

so under Q^{T-IP}

$$\ln \Upsilon(t,t,T) \sim N\left(\ln \Upsilon(0,t,T) - \frac{1}{2}\Sigma^2, \Sigma^2\right)$$

Define

$$\Gamma(s,t,T) = \frac{p_{IP}(s,T)}{p_n(s,t)} = \frac{1}{\Upsilon(s,t,T)}$$

Since Γ is a Q^t-martingale, the dynamics under Q^t is

$$\frac{d\Gamma(s,t,T)}{\Gamma(s,t,T)} = -\left\{\beta^n(s,t) - \beta^{IP}(s,T)\right\} dW_s^{T-IP}$$

hence

$$\ln \Gamma(t,t,T) \sim N\left(\ln \Gamma(0,t,T) - \frac{1}{2}\Sigma^2, \Sigma^2\right)$$

Using the nominal T-IP forward measure, we calculate the first part of equation (36) to

$$\begin{aligned}
\Pi[0,t,p_{IP}(t,T)\mathbb{I}_A] &= p_{IP}(0,T)E_0^{T-IP}\left[\mathbb{I}_A\right] = p_{IP}(0,T)Q^{T-IP}\left(p_{IP}(t,T) \geq K\right) \\
&= p_{IP}(0,T)Q^{T-IP}\left(\Upsilon(t,t,T) \leq \frac{1}{K}\right) = p_{IP}(0,T)N[d_1]
\end{aligned}$$

Using the nominal t-forward measure we calculate the second part of equation (36) to

$$\begin{aligned}
\Pi[0,t,K\mathbb{I}_A] &= p_n(0,t)KE_0^t\left[\mathbb{I}_A\right] = p_n(0,t)KQ^t\left(p_{IP}(t,T) \geq K\right) \\
&= p_n(0,t)KQ^t\left(\Gamma(t,t,T) \geq K\right) = p_n(0,t)KN[d_2]
\end{aligned}$$

∎

6 Conclusion

We have priced options on TIPS-bonds and zero coupon inflation indexed swaptions given a multidimensional HJM model for the real and nominal forward rates. Furthermore we have priced year-on-year inflation indexed swaps given this finite dimensional HJM model but extended to also allow the bond prices and the consumer price index to jump. We have shown that the price of inflation indexed swaps can be expressed as a funtion of zero coupon bonds, inflation protected zero coupon bonds and volatility parameters of these bonds. Hence there is no need for estimating any real parameters in order to price these swaps. In addition we have shown, for the case when there are no jumps that an inflation indexed swap can be hedged by using zero coupon bonds and inflation protected zero coupon bonds.

We have proposed two inflation swap market models and used these to price year-on-year inflation indexed swaptions. We have priced the zero coupon inflation indexed swaptions also under these models. Furthermore, we have extended and formally proved the validity of the so called foreign-currency analogy.

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Paper 3

Paper 3

Shifts in the Term Structure of Futures Prices

We consider an arbitrage free futures price model of Heath-Jarrow-Morton type which is driven by both a multidimensional Wiener process and a marked point process. We find necessary and sufficient conditions for this model to produce a log futures curve that changes only trough parallel shifts. The same analysis is done for the case when the log futures curve changes only trough proportional shifts. We prove that there exists nontrivial parallel and proportional shifting log futures curves and we show how to specify the futures price model in order to obtain them. Additionally the shift functions are characterized. Finally we consider the case of all other single factor affine models which are neither parallel nor proportional shifting curves. We find necessary and sufficient conditions for the purely Wiener driven log futures model to admit such a strictly affine shifting curve and we characterize the shift functions.

1 Introduction

In this paper we study the term structure of futures prices. We divide the class of single factor affine term structures into parallel shift, proportional shift and strictly affine term structures. We consider an arbitrage free futures price model of Heath-Jarrow-Morton (henceforth HJM) type which is driven by both a Wiener process and a marked point process. We investigate if, when and how this model can produce a log futures curve that changes only trough parallel (proportional) shifts. Using the terminology from Björk and Cristensen [7] we investigate when the family of parallel (proportional) shifting log futures curves is consistent with the dynamics of the futures price model.

We find necessary and sufficient conditions for the log futures price to have a parallel shift term structure and a proportional shift term structure. For the purely Wiener driven case we also find necessary and sufficient conditions for the log futures price to have a strictly affine term structure. Additionally we find the dynamics for the induced spot price that are necessary for admitting parallel shifts. We also find the dynamics for the induced spot price that are necessary for admitting proportional shifts, in the purely Wiener driven case.

Futures prices have been considered by Amin and Pirrong [1], Schwartz [19], Hilliard and Reis [15] and Gibson and Schwartz [14]. These articles all use the so called state space approach. In the state space approach, there is an a priori given state vector from which the futures prices are then derived. An alternative approach is the HJM approach which originate from interest rate theory. In this approach, there is no a priori given state space vector. Instead the entire futures price curve is modeled. Examples of this approach, applied to the term structure of futures prices, can be found in the papers by Reisman [18], Cortazar and Schwartz [10], Björk and Landén [6], Björk et al. [4], Miltersen and Schwartz [16] and Gaspar [13]. In this paper we take the HJM approach. We model the futures curve in a very general setting, allowing the curve to be driven by both a multidimensional Wiener process and a marked point process.

In interest rate theory the questions of consistent parallel and proportional shifts of the yield curve have been examined by Armerin et al. [2]. The questions and arguments in this study essentially parallels those by Armerin et al. [2]. Thus, as [2], this study rests on the approach of invariant manifolds and consistent forward rate curves introduced by Björk and Cristensen [7] and later and extended by Filipovic and Teichmann [12] and Filipovic [11].

This paper is organized as follows: In section 2 we set up the model for the dynamics of the futures prices and state our assumptions. Section 3 is divided into three subsections. In the first subsection we consider parallel shifts. In the second subsection we consider proportional shifts. In the third and last subsection we consider all other (single factor) affine term structures. That is all term structures which are affine but neither a parallel shifting model nor a proportional shifting model.

2 The Model

We begin this section by reproducing the definition of a futures contract, (see Björk [3]).

Definition 2.1 *(The futures contract) Let $\{X_t\}_{t\geq 0}$ be an adapted process. A futures contract on X_T with time of delivery T, is a financial asset with the following properties:*

- *At every point in time t, with $0 \leq t \leq T$, there exists in the market a quoted object $\hat{F}(t,T)$, known as the futures price for X_T at time t for delivery at time T.*
- *At time T of delivery, the holder of the contract pays $\hat{F}(T,T)$ and receives the claim X_T.*
- *During an arbitrary time interval, $(s,t]$, the holder of the contract receives $\hat{F}(t,T) - \hat{F}(s,T)$.*
- *The spot price, at any time t prior to delivery, of obtaining the futures contract, is by definition equal to zero.*

The stochastic process $\{X_t\}$ is commonly referred to as the underlying process of the futures contract.

Assumption 2.1 *All objects on the financial market are defined on a filtered probability space $(\Omega, \mathcal{F}, Q, \underline{\mathcal{F}})$ where Q is the risk neutral probability measure. The probability space carries both an n-dimensional Wiener process W and a marked point process $\mu(dt, dv)$ on $\mathbb{R}_+ \times V$ where the mark space is a finite point set $V = \{1, \ldots, N\}$. The compensator is of the form $\nu(dt, dv)$ and admits a predictable intensity measure $\lambda(t, dv)dt$. The filtration $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by both W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.*

Note that since V is a point set, λ is determined by its point masses i.e. $\lambda_i(t) = \lambda(t, \{i\})$ for $i = 1, \ldots, N$. Thus integrals of the type $\int_V f(v)\lambda(dv)$ are equivalent to sums i.e. to $\sum_{i=1}^N f_i \lambda_i$. We stick to the more general notation, since this allows us to discuss extensions to the general marked point process.

Assumption 2.2 *There are no arbitrage possibilities, i.e. the market is arbitrage free.*

A standard result states that

$$\hat{F}(t,T) = E_t[X_T] \tag{1}$$

where $E_t[\cdot]$ denotes the conditional expectation of $(\cdot)$ given $\mathcal{F}_t$ under the risk neutral measure Q.

One way to look at the relation between the underlying process and the futures price is to consider the underlying process as the primary object and the futures price as the secondary object that can be derived from the underlying process. If for instance $\{X_t\}$ denotes the price process of a stock and we start by assuming that $\{X_t\}$ evolves according to some stochastic differential equation (henceforth SDE), we can calculate the price of $\hat{F}(t,T)$ by using equation (1). This is an example of the so called state space approach.

Another way to look at the relation between the underlying process and the futures price is to consider the futures price as the primary object. This is the case in the Heath-Jarrow-Morton (henceforth HJM) approach, where the whole curve $T \longmapsto \hat{F}(t,T)$ is modeled. Since this curve connects the futures prices with the time of maturity, these models are referred to as term structure models. Here X_t is the secondary object given by the relation

$$X(t) = \hat{F}(t,t). \tag{2}$$

This relation follows directly from the definition of a futures contract and is in agreement with equation (1). The process $\{X_t\}$ is in this approach referred to as the induced spot process. Note that X is not necessarily the price of a traded asset.

In this paper we use the HJM approach. Hence we will not make any assumptions about the underlying claim. Instead we will consider the entire term structure of futures contracts as the primary objects.

Assumption 2.3 *We assume that there exists a market for futures contract of all maturities $T \geq 0$.*

It is clear from equation (1) that for every fixed T, $\hat{F}(t,T)$ is a Q-martingale, which leads us to make the following assumption.

Assumption 2.4 *Assume that under the risk neutral probability measure Q, the dynamics of $F(t,T)$ for every fixed T is given by:*

$$\begin{cases} d\hat{F}(t,T) = \hat{F}(t,T)\tilde{\sigma}(t,T)dW_t + \hat{F}(t-,T)\int_V \tilde{\xi}(t,v,T)\tilde{\mu}(dt,dv), & 0 \leq t \leq T, \\ \hat{F}(0,T) = \hat{F}_0(T), \quad T \geq 0 \end{cases} \tag{3}$$

where σ, is $\underline{\mathcal{F}}$-adapted, ξ is $\underline{\mathcal{F}}$-predictable and $\tilde{\mu}(dt,dv) = \mu(dv) - \lambda_t(dv)dt$ and $T \longmapsto \hat{F}_0(T)$ is a given curve.

Above we have used time of maturity, T, to parameterize the futures price. Even though this probably is the most common parameterization, it is not the only one. For our purposes it is more convenient to parameterize the forward prices by time to maturity, rather than by time of maturity. This is analogous to the Musiela parameterization of forward rates, (see Björk [3] and Björk et al. [4]). Let x denote time to maturity so that $T = t + x$. Next define F as

$$F(t,x) = \hat{F}(t, t+x) \tag{4}$$

then it follow from Ito's formula and equation (3) that

$$\begin{aligned} dF(t,x) &= \frac{\partial F(t,x)}{\partial x} dt + F(t,T)\sigma(t,x) dW_t \\ &+ F(t-,T) \int_V \xi(t,v,x) \tilde{\mu}(dt,dv), \qquad (5) \\ F(0,x) &= \hat{F}_0(x) \qquad (6) \end{aligned}$$

where

$$\begin{aligned} \sigma(t,x) &= \tilde{\sigma}(t, t+x), \\ \xi(t,v,x) &= \tilde{\xi}(t,v,t+x). \end{aligned}$$

It turns out that for computational purposes it is more convenient to study the logarithm of $F(t,x)$ rather $F(t,x)$ itself. For this reason we define

$$q(t,x) = \log F(t,x). \tag{7}$$

We will refer to the curve $x \longmapsto q(t,x)$ at time t as the log futures curve or the q_t-curve.

With $q_0(x) = \ln \hat{F}_0(x)$ equation (5) and (6) gives, by Itos formula, that

$$\begin{cases} dq(t,x) &= \left(\frac{\partial q(t,x)}{\partial x} - \frac{1}{2} \sum_{i=1}^m \sigma_i^2(t,x) - \int_V \xi(t,v,x) \lambda_t(dv) \right) dt \\ &+ \sigma(t,x) dW_t + \int_V \ln\left(1 + \xi(t,v,x)\right) \mu(dt,dv), \\ q(0,x) &= q_0(x). \end{cases} \tag{8}$$

If we consider each fixed time to maturity x at a time, equation (8) can be regarded as a scalar SDE for each fixed time to maturity x.

However if we instead consider x as a (infinite dimensional) vector, equation (8) can be regarded as an infinite dimensional SDE, describing the dynamics of the entire q_t-curve at once. This is the interpretation we will use. The q_t-curve, which is a vector of infinite dimension, can be interpreted as a point in an infinite dimensional space $\mathcal{H}$. The space $\mathcal{H}$, which is defined as the space of all log futures curves, needs to be specified. This was first done by Björk and Svensson [8], where $\mathcal{H}$ is the space of all infinitely differentiable functions satisfying a norm under which $\mathcal{H}$ is complete so that $\mathcal{H}$ is a Hilbert space. Later, the space was extended by Filipovic DF3 and Filipovic and Teichmann [12] to include square root processes, like the CIR process, which was not included with the original specification. We refer to [8], [11], [12] for a detailed description of the space $\mathcal{H}$.

To ensure that the log futures curve process is Markovian, we need to impose some assumptions on the structure of the volatilities and the intensity

Assumption 2.5 *We assume that:*

- *The Wiener volatility structure is of the form*

$$\sigma(t,x) = \sigma(q_t,x)$$

where each component σ_i of the vector $\sigma(q,x) = [\sigma_1(q,x),\cdots,\sigma_m(q,x)]$ is a mapping from $\mathcal{H} \times R_+$ to R.

- *The jump volatility structure is of the form*

$$\xi(t,v,x) = \xi(q_t,v,x) \quad \text{for all } v \in V$$

where ξ is a mapping from $\mathcal{H} \times V \times R_+$ to R.

- *The intensity structure is of the form*

$$\lambda_t(dv) = \lambda(q_t,dv)$$

where for each fixed q, $\lambda(q,.)$ is a nonnegative measure on V.

Each Wiener volatility, σ_i is a functional of q and a function of x. However we can also view σ_i as a mapping that takes q as input and leaves the curve $x \longmapsto \sigma_i(q,x)$ as output. Thus σ_i can be viewed as vector field on $\mathcal{H}$, mapping $\sigma_i : \mathcal{H}$ to $\mathcal{H}$. Similarly, for each fixed v in V, the jump volatility ξ_v can be viewed as a vector field on $\mathcal{H}$.

Assumption 2.6 *We assume that the curves:*

$$x \longmapsto \sigma_i(q,x) \quad \forall i = 1,\ldots,m$$
$$x \longmapsto \xi_v(q,v,x) \quad \forall v \in V$$

belongs to $\mathcal{H}$.

In addition, some regularity conditions are needed. For instance the vector fields σ_i and ξ_v are required to be smooth on $\mathcal{H}$. See Björk and Svensson [8] for further details.

Given our additional assumptions we can now rewrite the dynamics of q_t from equation (8) as

$$\begin{aligned} dq_t &= \left(\mathbf{L}q_t - \frac{1}{2}\mathbf{D}\sigma(q_t) - \int_V \xi_v(q_t)\lambda(q_t, dv)\right) dt \\ &+ \sigma(q_t)dW_t + \int_V \ln\left(1 + \xi_v(q_t)\right)\mu(dt, dv) \end{aligned}$$

where the operators $\mathbf{L}$ and $\mathbf{D}$ are defined by

$$\begin{aligned} \mathbf{L} &= \frac{\partial}{\partial x} \\ \mathbf{D} &= \| \ \|^2_{R^m} \end{aligned}$$

Note that by Assumption 2.5, the volatilities σ and ξ does not depend directly on t, i.e. we have restricted our study to time homogenous systems. Non-homogenous models have been considered by Björk et al. [4] and the results turn out to be straightforward generalizations of homogeneous results.

In the next section we will use results concerning consistency from Björk and Cristensen [7]. For this purpose we need the dynamics of q_t to be written on Stratonovich form. (See Protter [17] for the definition of Stratonovich integrals.) Hence we rewrite the dynamics of q_t once again.

$$\begin{aligned} dq_t &= \left(\mathbf{L}q_t - \frac{1}{2}D\sigma(q_t) - \frac{1}{2}\sigma'(q_t)\sigma_q(q_t) - \int_V \xi_v(q_t)\lambda(q_t, dv)\right) dt \\ &+ \sigma(q_t) \circ dW_t + \int_V \ln\left(1 + \xi_v(q_t)\right)\mu(dt, dv) \end{aligned}$$

where $\circ$ denotes Stratonovich integral and $\sigma_q'(q_t)$ denotes the Fréchet derivative with respect to the variable q.

3 Parallel and Proportional shifts

Definition 3.1 *The log futures curve q_t is said to have a (homogenous) parallel shift term structure if*

$$q_t(x) = h(x) + Z_t \quad \forall (t, x) \in R_+ \times R$$

where h is a time homogenous deterministic function from R_+ to R and Z_t is an adapted process.

Thus as time changes, say from s to t, the entire curve q_t will shift vertically by the size $Z_t - Z_s$. Using vector notation a parallel shift term structure can be written as $q_t = h + Z_t e$ where $e(x) = 1$ for all x. However will suppress the e-vector and just write $q_t = h + Z_t$.

Definition 3.2 *The log futures curve q_t is said to have a (homogenous) proportional shift term structure if*

$$q_t(x) = Z_t g(x) \quad \forall (t,x) \in R_+ \times R$$

where g is a time homogenous deterministic function from R_+ to R and Z_t is an adapted process.

With vector notation this can be written as $q_t = Z_t g$.

Definition 3.3 *The log futures curve q_t is said to have a degenerate shift term structure if*

$$q_t(x) = Z_t \quad \forall (t,x) \in R_+ \times R$$

where Z_t is an adapted process.

Remark 3.1 *The degenerate shift term structures is both a parallel and a proportional shift term structure.*

Definition 3.4 *The log futures curve q_t is said to have a one dimensional (homogenous) affine term structure if*

$$q(t,x) = h(x) + Z_t g(x) \quad \forall (t,x) \in R_+ \times R \tag{9}$$

where h and g are time homogenous deterministic functions, and Z_t is an adapted process.

With vector notation we write $q_t = h + Z_t g$.

Remark 3.2 *Both the parallel shift and the proportional shift term structures are special cases of the affine term structure.*

We normalize by setting $h(0) = 0$ and $g(0) = 1$ thus implying that $q_t(0) = Z_t$. This is without loss of generality since if we start with another equation $q(t,x) = H(x) + G(x)Y_t$ where $H(x) \neq 0$ and $G(x) \neq 1$ we can define $Z = H(0) + G(0)Y$ and g and h as

$$\begin{aligned} g(x) &= \frac{G(x)}{G(0)}, \\ h(x) &= H(x) - \frac{H(0)G(x)}{G(0)}, \end{aligned}$$

which will give us equation (9) with $h(0) = 0$ and $g(0) = 1$.

Unless there is a need to emphasize the word homogenous, it will hereafter be suppressed.

In the next sections we will analyze the parallel shift, the proportional shift term and other affine term structures. However, we will first state some results that applies to all cases and that will be used in each of the coming sections.

Next, the task is to find under what conditions the model for the log futures prices in equation (8) produces affine log futures curves according to definition above. If we start with an initial curve $q_0(x)$ that can be written on the form (9), we would thus like to find out what conditions that has to be satisfied in order to ensure that also $q_t(x)$ will be of the form (9) for arbitrary t. To answer this question we will use results from Armerin et al. [2] which are based on the theory on invariant manifolds and consistent forward rate curves developed by Björk and Cristensen [7].

If we write the affine log futures curve with vector notation so that $q_t = h + Z_t g$ where q, h and g are vectors in $\mathcal{H}$ it is clear that an affine log futures curve will always lie on the line that passes through the vector h and has direction g. It is thus natural to believe that a consistent log futures model will produce q_t-curves that lies on this line. The next Proposition, which follows from Proposition 4.1 by Armerin et al. [2] formalizes this idea.

Proposition 3.1 *The futures model will have a one dimensional affine term structure if and only if the manifold defined by*

$$\mathcal{G} = \{q \in \mathcal{H}; \quad q = h + zg, \quad z \in R\}$$

is invariant under the action of the log futures price equation and $q_0 \in \mathcal{G}$.

The question is now when $\mathcal{G}$ is invariant. The next Proposition, based on Proposition 6.2 by Björk and Cristensen [7], provides the answer.

Proposition 3.2 *Let* $\mathcal{T}_{\mathcal{G}}(q)$ *be the tangent space of* $\mathcal{G}$ *at point* q. *Then* $\mathcal{G}$ *is locally invariant if and only if the following conditions hold for all* q *in* $\mathcal{G}$:

$$\mathbf{L}q - \frac{1}{2}\mathbf{D}\sigma(q) - \frac{1}{2}\sigma'(q)\sigma_q(q) - \int_V \xi_v(q)\lambda(q,dv) \in \mathcal{T}_{\mathcal{G}}(q), \tag{10}$$

$$\sigma_i(q) \in \mathcal{T}_{\mathcal{G}}(q) \quad \text{for } i= 1, \ldots, m, \tag{11}$$

$$\ln(1 + \xi_v(q)) \in \mathcal{T}_{\mathcal{G}}(q) \quad \text{for all } v \in V. \tag{12}$$

Proof. The conditions in equations (10) and (11) follows directly from Proposition 6.2 by Björk and Cristensen [7]. The condition in equation (12) follows from the third invariance condition in Proposition 6.2 in Björk and Cristensen [7] which is:

$$q + \ln(1 + \xi_v(q)) \in \mathcal{G} \quad \forall q \in \mathcal{G}, \quad \forall v \in V$$

This is true if and only if $\ln(1 + \xi_v(q)) \in \mathcal{T}_{\mathcal{G}}(q)$ since the manifold in our setting is linear. ▮

Since the manifold is the line that passes through h with direction g, every point will have the same direction g. Thus g spans the tangential manifold and so $\mathcal{T}_{\mathcal{G}} = \{zg, \quad z \in R\}$. Hence the invariance conditions in Proposition 3.2 can be written as

$$\mathbf{L}q - \frac{1}{2}\mathbf{D}\sigma(q) - \frac{1}{2}\sigma'(q)\sigma_q(q) - \int_V \xi_v(q)\lambda(q, dv) = \psi(q)g, \tag{13}$$

$$\sigma_i(q) = \gamma_i(q)g \quad \text{for i= 1, \ldots, m,} \tag{14}$$

$$\ln(1 + \xi_v(q)) = \delta_v(q)g \quad \text{for all } v \in V. \tag{15}$$

where ψ, γ_i and δ_v are smooth scalar fields on $\mathcal{H}$.

By equation (14) it follows that $\sigma_i'(q)\sigma_i(q)$ lies in $\mathcal{T}_{\mathcal{G}}$ since the Fréchet derivative w.r.t q, $\sigma_i'(q)$, operating on $\sigma_i(q)$ yields

$$\sigma_i'(q)[\sigma_i(q)] = \gamma_i'(q)g[\gamma_i(q)g] = \gamma_i'(q)[\gamma_i(q)g]g = \gamma_i'(q)[g]\gamma_i(q)g$$

where $\gamma_i'(q)[g]$ and $\gamma_i(q)$ are real numbers and g is in $\mathcal{H}$. Thus the first condition given in equation (13) is reduced to

$$\mathbf{L}q - \frac{1}{2}\mathbf{D}\sigma(q) - \int_V \xi_v(q)\lambda(q, dv) = \psi(q)g \tag{16}$$

for some scalar field Ψ.

Plugging the expressions for σ and ξ from the equations (14) and (15) into equation (16) and using that

$$\mathbf{L}q = \frac{\partial}{\partial x}(h + zg) = h' + zg',$$

yields

$$h'(x) + zg'(x) - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q)g^2(x) - \int_V \left(e^{\delta_v(q)g(x)} - 1\right)\lambda(q, dv) = \psi(q)g(x). \tag{17}$$

Note that this is a drift condition, specifying that the drift term in equation (8) is proportional to g. This equation of course simplifies if either $g' \equiv 0$ or $h' \equiv 0$.

We start by analyzing the case when $g' \equiv 0$, which is the case of parallel shifts. Thereafter we continue with the case when $h' \equiv 0$, which is the case of proportional shifts. Lastly, in section 3.3, we will look at the case when both $g' \not\equiv 0$ and $h' \not\equiv 0$.

Before we continue to the next section we state a Lemma, which will be of use in the subsequent sections. The proof follows directly from the definition of linear independence.

Lemma 3.1 *Assume that Y is a linear vector space and S is an arbitrary set. Furthermore, assume that $y_1, y_2, \ldots, y_M$ are linearly independent in Y and that $\alpha_1(s), \alpha_2(s), \ldots, \alpha_M(s)$ are real valued functions $\alpha_i : S \to R$. If $\sum_{j=1}^{M} \alpha_i(s) y_i$ is a constant function of the variable $s \in S$, then for $i = 1, 2 \ldots, M$ each α_i, must be a constant function of $s \in S$.*

3.1 Parallel shifts

In this section we will find conditions under which the model for the log futures prices in equation (8) produces a parallel shift term structure, i.e. takes the form $q_t(x) = h(x) + Z_t$. Furthermore we will find the spot price which is induced by a parallel shift term structure for the futures price. The results in this section also holds for the case of a general marked point process, i.e. it is not necessary for V to be a finite point set.

Proposition 3.3 *If a log futures price model admits parallel shifts, such that $q_t(x) = h(x) + Z_t$, then $h(x)$ must have the form*

$$h(x) = Ax \quad A \in R \tag{18}$$

Proof. If a log futures model admits parallel shifts, then by Propositions 3.1 and 3.2, the three invariance conditions in equations (14), (15) and (17) are satisfied with $g \equiv 1$. Since $g \equiv 1$ implies that $g' \equiv 0$ equation (17) gives that

$$h'(x) = \frac{1}{2} \sum_{i=1}^{m} \gamma_i^2(q) + \int_V \left(e^{\delta_v(q)} - 1\right) \lambda(q, dv) + \psi(q).$$

Integrating over $[0, x]$ and using that $h(0) = 0$ gives that

$$h(x) = \left(\frac{1}{2} \sum_{i=1}^{m} \gamma_i^2(q) + \int_V \left(e^{\delta_v(q)e} - 1\right) \lambda(q, dv) + \psi(q) \right) x.$$

Since the left hand side does not depend on q, the right hand side cannot depend on q. Hence $\frac{1}{2} \sum_{i=1}^{m} \gamma_i^2(q) \int_V \left(e^{\delta_v(q)e} - 1\right) \lambda(q, dv) + \psi(q) = A$ for some $A \in R$. ∎

Proposition 3.4 *The log futures price model in equation (8) admits a parallel shift term structure if and only if*

$$\sigma_i(q,x) = \gamma_i(q) \quad \text{for } i = 1,\dots,m, \tag{19}$$

$$\xi_v(q,x) = e^{\delta_v(q)} - 1 \quad \text{for all } v \in V, \tag{20}$$

$$q_0(x) = a + Ax \quad \text{for some } a, A \in R, \tag{21}$$

where γ and δ_v are scalar fields, $\gamma : \mathcal{H} \to R$ and $\delta_v : \mathcal{H} \to R$ for all v in V.

Thus the necessary and sufficient conditions for existence of parallel shifts in the log futures curve are that the volatilities in the log futures model are independent of time to maturity and that the initial log futures curve is linear in time to maturity.

Proof. That the conditions (19) and (20) are necessary for existence of parallel shifts follows by condition (14) and (15) since $g \equiv 1$. By Proposition 3.3, $q_t(x)$ must be on the form $q_t(x) = Z_t + Ax$, for some A in R. Especially this must hold for $q_0(x)$ which implies condition (21).

Next we prove that the conditions are sufficient for admitting a parallel shift model. We define a function $h(x)$ to be of the type given in equation (18) and choose A to be same real number as in condition (21). We define $\mathcal{G} = \{q \in \mathcal{H}; q(x) = z + h(x), z \in R\}$, thus $q_0(x)$ is in $\mathcal{G}$. We need to show that the invariance conditions in equations (14), (15) and (17) are satisfied for every q such that $q(x) = z + h(x)$ where $z \in R$. Remembering that $g \equiv 1$ equations (19) and (20) immediately gives the invariance conditions (14) and (15). To show that condition (17) is satisfied, we need to check that the expression

$$h'(x) - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q) - \int_V \left(e^{\delta_v(q)} - 1\right)\lambda(q,dv) \tag{22}$$

is independent of x, i.e that it is equal to some scalar field $\psi(q)$. But this follows directly, since $h'(x) = A$.

∎

Corollary 3.1 *The log futures price model in equation (8) admits a non-degenerate parallel shift term structure if and only if the conditions in Proposition 3.4 are satisfied and $A \neq 0$.*

Proposition 3.5 *For every h that satisfies $h(x) = Ax$ with $A \in R$, there exists log futures price models that admits parallel shifts with that particular h as shift function.*

Proof. The model can be obtained by first choosing any real numbers γ_i and δ_v and then defining $\sigma_i(q,x) = \gamma_i$ for $i = 1,\ldots,m$ and $\xi_v(q,x) = e^{\delta_v} - 1$ for all $v \in V$. The intensity $\lambda(q,dv)$ can be freely specified. We can choose any real number b to be the spot value of the log futures curve, i.e. $q(0) = b$ for any b in R. However to ensure that the shift function is exactly $h(x) = Ax$, the initial curve $q_0(x)$ must be specified by $q_0(x) = b + Ax$. ∎

Note that to create a model that admits parallel shifts with a particular shift function $h(x) = Ax$, we can choose **any** volatilities that are time independent. It is the initial curve alone, that determines the shift function. For instance both $\sigma = 5\%$ and $\sigma = 40\%$ can generate a parallel shifting curve with shift function $h(x) = 3x$, as log as in the initial curve has $h(x) = 3$.

Proposition 3.6 *If the dynamics of the log futures price $q_t(x)$ are given by*

$$\begin{aligned} dq(t,x) &= \left(\frac{\partial q(t,x)}{\partial x} - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q) - \int_V \left(e^{\delta_v(q)} - 1\right)\lambda_t(q,dv) \right) dt \\ &+ \sum_{i=1}^{m}\gamma_i(q)dW_t^i + \int_V \delta_v(q)\mu(dt,dv), \qquad (23) \\ q(0,x) &= b + Ax \quad \text{for some } A, b \in R, \qquad (24) \end{aligned}$$

then $q_t(x)$ can be represented as $q_t(x) = Ax + Z_t$ and Z_t has the following dynamics

$$\begin{aligned} dZ_t &= \left\{ A - \frac{1}{2}\sum_{i=1}^{m}\alpha_i^2(Z_t) - \int_V \left(e^{\beta_v(Z_t)} - 1\right)\lambda(Z_t,dv) \right\} dt \\ &+ \sum_{i=1}^{m}\alpha_i(Z_t)dW^i + \int_V \beta_v(Z_t)\mu(dt,dv), \\ Z_0 &= b, \end{aligned}$$

where

$$\begin{aligned} \alpha_i(z) &= \gamma_i(h+z) \quad \text{for } i = 1,\ldots,m, \\ \beta_v(z) &= \delta_v(h+z) \quad \text{for all } v \in V, \\ \lambda(z,dv) &= \lambda(h+z,dv) \quad \text{for all } v \in V. \end{aligned}$$

Proof. Since the futures price dynamics satisfies the conditions in Proposition 3.4, it is clear that it admits a parallel shift term structure model such that $q_t(x) = Ax + Z_t$. Thus

$$\begin{aligned} dq(t,x) &= \left\{ A - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q) - \int_V \left(e^{\delta_v(q)} - 1\right)\lambda_t(dv) \right\} dt \\ &+ \sum_{i=1}^{m}\gamma_i(q)dW_t^i + \int_V \delta_v(q)\mu(dt,dv) \end{aligned} \tag{25}$$

Defining α, β and λ as above and applying Ito to $q_t(x) = h(x) + Z_t$ gives the Z-dynamics. ▮

Since $Z_t = \ln X_t$, the next Corollary follows from Ito's formula.

Corollary 3.2 *The spot price, X_t, induced by a futures price that admits a parallel shift term structure has the following dynamics:*

$$dX_t = X_t A dt + X_t \alpha(\ln X_t) dW_t + X_{t-}\int_V \left(e^{\beta_v(\ln X_t)} - 1\right)\tilde{\mu}(dt,dv) \tag{26}$$

Consider a tradable asset without dividends X. It follows by no arbitrage and Corollary 3.2, that the drift term A has to equal the short rate. Since A is constant over time, the short rate must also be constant. Thus we have that:

Corollary 3.3 *If the futures price of a tradable asset without dividends admits a homogenous parallel shift term structure, then the interest rate is constant.*

It is well known that if the interest rate is constant, then $\hat{F}(t,T) = X_t e^{r(T-t)}$ thus implying that the log futures price has a parallel shift term structure with shift function $h(x) = rx$. However our results states that if the log futures curve has a parallel shift term structure, then the shift function must be h(x)=rx, which implies constant interest rates.

3.2 Proportional shifts

In this section we will search for the conditions under which the model for the log futures prices in equation (8) produces a proportional shift term structure, $q_t(x) = Z_t g(x)$. Since the Propositions 3.2 and 3.1 in the beginning of this section are specified for $q_t(x) = h(x) + Z_t g(x)$, we get the results for proportional shifts by letting $h \equiv 0$.

It turns out to be somewhat easier to find the sufficient conditions for a log futures model to admit a proportional shift term structure, hence we begin with this.

Proposition 3.7 *The following conditions are sufficient for existence of a proportional shift term structure, such that $q_t(x) = g(x)Z_t$*

- *The log futures volatilities are proportional to the shift function for all q:*

$$\sigma_i(q, x) = \gamma_i(q)g(x) \quad \text{for } i = 1, \ldots, m, \tag{27}$$

$$\xi_v(q, x) = e^{\delta_v(q)g(x)} - 1 \quad \text{for all } v \in V. \tag{28}$$

- *There exist a real number A such that*

$$\sum_{i=1}^{m} \gamma_i^2(q) = Aq(0) \quad \forall q \tag{29}$$

- *The integral expression R given by*

$$R(q, y) = \int_V \left(e^{\delta_v(q)y} - 1\right) \lambda(q, dv) \tag{30}$$

satisfies

$$R(q, y) = f(y)q(0) \quad \forall q \tag{31}$$

for some mapping $f : R \to R$.

- *The shift function g satisfies*

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + f(g(x)) \tag{32}$$

where A is the real number in (29), B is any real number and f is the mapping in (31).

- *The initial log futures curve is proportional to the shift function:*

$$q_0(x) = cg(x) \tag{33}$$

where c is any real number except 0.

As will become apparent in the proof, we do not make use of the earlier assumption that the mark space V is a point set. Thus Proposition 3.7 is true also for a general marked point process.

Proof. Define a function g such that $g(0) = 1$, and

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + f(g(x)) \tag{34}$$

where we choose A to be the real number given in equation (29), $f(g(x))$ is the function in equation (31) and B is any real number. Using this g, we define $\mathcal{G} = \{q \in \mathcal{H}; \quad q = zg, \quad z \in R\}$. Note that by condition (33), the initial curve, $q_0(x)$, belongs to $\mathcal{G}$. Equations (27) and (28) immediately gives that conditions (14) and (15) are satisfied for all q in $\mathcal{G}$. To show that also condition (17) is satisfied, we need to check that the expression

$$q(0)g'(x) - \frac{1}{2}\sum_{i=1}^{m} \gamma_i^2(q) - \int_V \left(e^{\delta_v(q)g(x)} - 1\right) \lambda(q, dv)$$

is proportional to the shift function, i.e that there exist a functional $\psi(q)$ such that the above expression is equal to $\psi(q)g(x)$. Plugging in our g from equation (34) yields

$$\frac{Aq(0)}{2} g^2(x) + Bq(0)g(x) + f(g(x))q(0)$$

$$-\frac{1}{2}\sum_{i=1}^{m} \gamma_i^2(q) - \int_V \left(e^{\delta_v(q)g(x)} - 1\right) \lambda(q, dv),$$

which by using condition (31) is equal to

$$\frac{Aq(0)}{2} g^2(x) + Bq(0)g(x) - \frac{1}{2}\sum_{i=1}^{m} \gamma_i^2(q). \tag{35}$$

For all q in $\mathcal{G}$ such that $q(0) \neq 0$, we can substitute A with the expression from equation (29) and thus get:

$$Bq(0)g(x)$$

For all q in $\mathcal{G}$ such that $q(0) = 0$, the expression (35) reduces to 0. Thus there exist a ψ, given by $\psi = Bq(0)$, such that the expression is proportional to g for all q in $\mathcal{G}$. Hence by Proposition 3.2, $\mathcal{G}$ is invariant and hence by Proposition 3.1 admits a proportional term structure. ∎

Remark 3.3 *An obvious way to guarantee condition (31) is by choosing δ and λ such that $\delta_v(q) = \delta_v$ and $\lambda(q, dv) = q(0)\lambda(dv)$.*

It turns out to be somewhat more complicated to prove that these conditions are also necessary. We need to make one additional assumption in order to achieve the proof. We assume that the jump volatility ξ_v is independent of q for all $v \in V$ and thus δ_v is independent for all $v \in V$. Thus the jump volatility is deterministic. This assumption is needed in order to achieve conditions (29), (31) and (32).

Proposition 3.8 *If the log futures price model has deterministic jump volatility, the conditions in Proposition 3.2 are necessary for existence of a proportional shift term structure, such that* $q_t(x) = g(x)Z_t$,

Proof. Assume that there exists a log futures model that admits a proportional shift term structure. It then follows from Propositions 3.2 and 3.1 that there exists a nonempty set $\tilde{R} \in R$ such that the manifold

$$\mathcal{G} = \left\{ q \in \mathcal{H}; \quad q = zg, \quad g(0) = 1, \quad z \in \tilde{R} \right\}$$

is invariant. Thus the three invariance conditions (14), (15) and (17) are satisfied for all $q \in \mathcal{G}$. The conditions (14) and (15) immediately gives condition (27) and (28).

Condition (17), applied to proportional shifts is

$$q(0)g'(x) - \frac{1}{2}\sum_{i=1}^{m} \gamma_i^2(q)g^2(x) - \int_V \left(e^{\delta_v g(x)} - 1\right)\lambda(q, dv) = \psi(q)g(x).$$

Note that the extra assumption gives $\delta_v(q) = \delta_v$ independent of q. Since $g(0) = 1$ for all q in $\mathcal{G}$, we have that $q(0) = z$. By assumption there exists a proportional shift curve, different from the trivial shift $q_t \equiv 0$ for all t. Hence $\tilde{R}$ cannot be the singleton set $\{0\}$. That is, there exists q in $\mathcal{G}$ such that $q(0) \neq 0$. For every q in $\mathcal{G}$ such that $q(0) \neq 0$ we can divide the invariance condition by $q(0)$. Furthermore we use that V is a point set and thus

$$g'(x) = \frac{1}{2q(0)}\sum_{i=1}^{m} \gamma_i^2(q)g^2(x) + \frac{1}{q(0)}\psi(q)g(x) + \frac{1}{q(0)}\sum_{i=1}^{M} \left(e^{\delta_i g(x)} - 1\right)\lambda_i(q). \quad (36)$$

For simplicity we now assume that the real numbers $\delta_1, \delta_2, \ldots, \delta_M$ are distinct. Since $g(0) = 1$ it follows that $g' \not\equiv 0$, thus $g(x)$, $g^2(x)$, $e^{\delta_1 g(x)} - 1$, $e^{\delta_2 g(x)} - 1, \ldots e^{\delta_M g(x)} - 1$ are linearly independent. By Lemma 3.1 it follows that $\sum_{i=1}^{m} \gamma_i^2(q)/q(0) = A$ for some $A \in R$, $\psi(q)/q(0) = B$ for some $B \in R$ and that the last term in equation (36) does not depend on q, i.e. $\frac{1}{q(0)}\sum_{i=1}^{M}\left(e^{\delta_i y} - 1\right)\lambda_i(q) = f(y)$ for some real valued function f. With a slight modification of the arguments, it can be shown that this is true also when not all real numbers $\delta_1, \delta_2, \ldots, \delta_M$ are distinct. Thus we have found that

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + f(g(x)). \quad (37)$$

If $0 \in \tilde{R}$ then there exist $q(0)$ in $\mathcal{G}$ such that $q(0) = 0$. The condition (17) is here reduced to

$$-\frac{1}{2}\sum_{i=1}^{m} \gamma_i^2(q)g^2(x) - \int_V \left(e^{\delta_v g(x)} - 1\right)\lambda(q, dv) = \psi(q)g(x)$$

Similar arguments as used for the case $q(0) \neq 0$ yields that for every q in $\mathcal{G}$ such that $q(0) = 0$ it must hold that $\sum_{i=1}^{m} \gamma_i^2(q) = \psi(q) = \int_V \left(e^{\delta_v(q)g(x)} - 1\right) \lambda(q, dv) = 0$.

Finally $q_0(x)$ must be of the form $q_0(x) = cg(x)$ for some real number c since $q_0(x)$ belongs to $\mathcal{G}$. For non-trivial shifts $c \neq 0$. ∎

Remark 3.4

- *If the jump volatility is deterministic, then the conditions (27)-(33) implies that the global intensity is proportional to $q(0)$, i.e.*

$$\int_V \lambda(q, dv) = q(0)C \quad \textit{for some real number } C \in R.$$

- *If the jump volatility is deterministic and $\delta_1, \delta_2, \ldots, \delta_M$ are distinct real numbers, then the conditions in (27)-(33) implies that every intensity λ_i is proportional to $q(0)$ for all $i \in V$:*

$$\lambda_i(q) = q(0)C_i \quad \textit{for some real number } C_i \in R, \quad \forall i \in V.$$

Corollary 3.4 *The log futures price model in equation (8) admits a non-degenerate proportional shift term structure if and only if the conditions in Proposition 3.7 are satisfied and $B \not\equiv -A/2 - f(1)$.*

Proof. To see that $B \not\equiv -A/2 - f(1)$ is a sufficient extra condition, we just note that if $g \equiv 1$ in equation (32), then $B = -A/2 - f(1)$.

To see that $B \not\equiv -A/2 - f(1)$ is a necessary extra condition, we need to show that $B = -A/2 - f(1)$ implies that $g \equiv 1$. Inserting $B = -A/2 - f(1)$ into equation (32) yields

$$g'(x) = Ag^2(x) - \left(\frac{1}{2}A + f(1)\right) g(x) + f(g(x)).$$

We know that $g(x) = 1$ is one solution to this equation. By equations (30) and (31) it follows that f is differentiable with respect to y. It then follows by Picard's theorem $g(x) \equiv 1$ is the unique solution. ∎

We have seen that the conditions in Proposition 3.7 are both sufficient and necessary[1] for proportional shifts. However, we can by analyzing the conditions in closer detail make some further conclusions.

[1]When the jump volatility is independent of q

Corollary 3.5 *Assume that a log futures price model has deterministic jump volatility and admits proportional shifts i.e.* $q_t(x) = g(x)Z_t$. *Then* Z_t *can take only nonnegative values or only nonpositive values for all* t.

Proof. If the futures price model has a diffusion term then condition (29) must hold. Since $\sum_{i=1}^{m}\gamma_i^2(q)$ is always positive, condition (29) implies that sgn A = sgn $q(0)$. If the model has a jump term, then by Corollary 3.4, $\int_V \lambda(q, dv) = q(0)C$ for some real number C. Since $\int_V \lambda(q, dv)$ always is positive, sgn C = sgn $q(0)$. Since $q(0) = z$, the Corollary is proved.
▮

Proposition 3.9 *Assume that a log futures price model has deterministic jump volatility and admits proportional shifts i.e.* $q_t(x) = g(x)Z_t$. *Then* $g'(x)$ *must satisfy an ODE of the following form:*

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + D\int_V \left(e^{\delta_v g(x)} - 1\right)\lambda(dv) \tag{38}$$

where $\lambda(dv)$ *is a positive finite measure,* A, B *are real numbers and* $D \in \{-1, +1\}$. A *and* D *are nonnegative (nonpositive) when* Z_t *is positive (negative).*

Proof. By assumption, it follows from Proposition 3.8 that

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + f(g(x))$$

where

$$f(y) = \frac{1}{q(0)}\int_V \left(e^{\delta_v y} - 1\right)\lambda(q, dv)$$

Choose an arbitrary $\hat{q} \in \mathcal{G}$. Then in particular it holds that

$$f(y) = \frac{1}{\hat{q}(0)}\int_V \left(e^{\delta_v y} - 1\right)\lambda(\hat{q}, dv)$$

we define $\lambda(dv)$ via

$$\lambda(dv) = \begin{cases} +\frac{\lambda(\hat{q},dv)}{\hat{q}(0)} & \forall v \in V \quad \text{f } \hat{q}(0) \text{ is nonnegative} \\ -\frac{\lambda(\hat{q},dv)}{\hat{q}(0)} & \forall v \in V \quad \text{f } \hat{q}(0) \text{ is nonpositive} \end{cases}$$

Inserting this into the expression for $g'(x)$ we thus get equation (3.9). It is clear that the intensity $\lambda(dv)$ is a positive measure and that D is positive (negative) for nonnegative (nonpositive) $q(0)$. By corollary 3.5, A is positive (negative) for nonnegative (nonpositive) $q(0)$. Finally we remember that $q(0) = z$.
▮

Remark 3.5 *If we in addition to the assumptions in Proposition 3.9 assume that $\delta_1, \delta_2, \ldots, \delta_M$ are distinct real numbers. Then equation (38) can be replaced by:*

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) + D\sum_{i=1}^{M}\left(e^{\delta_i g(x)} - 1\right)\lambda_i \tag{39}$$

where $\lambda_i = \lambda_i(q)/q(0)$ for all $q \in \mathcal{G}$ such that $q(0)$ is nonnegative and $\lambda_i = -\lambda_i(q)/q(0)$ for all $q \in \mathcal{G}$ such that $q(0)$ is nonpositive and with A, B and D as in Proposition 3.9.

Proposition 3.10

1. *For every g such that g' satisfies equation (38) and every positive value of Z_t, there exist a log futures price model that admits proportional shifts, admitting the particular g as the shift function.*

2. *For every g such that g' satisfies equation (38) and every negative value of Z_t, there exist a log futures price model that admits proportional shifts, admitting the particular g as the shift function.*

Proof. To attain the positive log futures model, we first define

$$\mathcal{G} = \{q \in \mathcal{H}; \quad q = zg, \quad z \in R_+\}$$

where g satisfies equation (38). Next we choose any real numbers $\gamma_1, \ldots, \gamma_m$ such that $\sum_{i=1}^m \gamma_i^2 = A$ and define $\sigma_i(q,x) = \gamma_i\sqrt{q(0)}g(x)$. We define $\xi(q,v,x) = \beta(v)g(x)$ for some real valued function $\delta(v)$ on V and $\lambda(q,dv) = q(0)\lambda(dv)$ for some positive finite measure $\lambda(dv)$ on V. It is clear that the conditions in Proposition 3.7 are satisfied for every q in $\mathcal{G}$ an thus we are done.

The negative log futures model is obtained as follows. Define

$$\mathcal{G} = \{q \in \mathcal{H}; \quad q = zg, \quad z \in R_-\}$$

where g satisfies equation (38). Next choose any real numbers $\gamma_1, \ldots, \gamma_m$ such that $\sum_{i=1}^m \gamma_i^2 = A$ and define $\sigma_i(q,x) = \gamma_i\sqrt{-q(0)}g(x)$. Define $\xi(q,v,x) = \beta(v)g(x)$ and $\lambda(q,dv) = -q(0)\lambda(dv)$ for some real valued function $\delta(v)$ on V and some positive finite measure $\lambda(dv)$ on V. ∎

Examples

In Proposition 3.9, equation (38) gives an expression for g' rather than for g since it is not possible to solve for g in the general case. However there are some simple special cases in which equation (38) can be solved analytically. We consider the case where Z_t takes only positive values. First we consider the case of a purely Wiener driven log futures model that admits a proportional shift term structure.

Example 1 (The purely Wiener driven log futures model)
By Proposition 3.9 equation (38) reduces to

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x),$$

which is a Riccati equation that can be solved by separation of variables: By letting $y = g(x)$ we have that

$$\int \left(\frac{\frac{1}{B}}{y} - \frac{\frac{A}{2B}}{\frac{Ay}{2} + B} \right) dy \quad = \quad \int dx + c$$

for some constant c. Thus

$$\frac{1}{B}\ln y - \frac{1}{B}\ln(\frac{Ay}{2} + B) = x + c.$$

Replacing $g(x) = y$, rearranging terms and using the initial condition $g(0) = 1$, finally yields that

$$g(x) = \frac{e^{Bx}}{1 + \frac{A}{2B}\left(1 - e^{Bx}\right)}.$$

It then follows by Proposition 3.8 that the dynamics of q reduces to

$$dq(t,x) = Bq(0)g(x)dt + g(x)\sum_{i=1}^{m}\gamma_i(q)dW_t^i$$

where $\sum_{i=1}^{m}\gamma_i^2(q) = Aq(0)$. Thus we can define a scalar Wiener process $\hat{W}$ by

$$d\hat{W} = \frac{1}{\sqrt{Aq(0)}}\sum_{i=1}^{m}\gamma_i(q)dW^i \tag{40}$$

and rewrite the dynamics of q as:

$$dq(t,x) = Bq(0)g(x)dt + g(x)\sqrt{Aq(0)}d\hat{W}_t.$$

Since $q_t(x) = Z_t g(x)$, Ito gives that

$$dZ_t = BZ_t dt + \sqrt{AZ_t}d\hat{W}_t.$$

This is a CIR process with mean reversion level zero.

We have thus proved the following Corollary.

Corollary 3.6 *If a purely Wiener driven log futures model admits proportional shifts such that $q_t = Z_t g(x)$ then it must hold that*

$$g(x) = \frac{e^{Bx}}{1 + \frac{A}{2B}(1 - e^{Bx})}$$

and

$$dZ_t = BZ_t dt + \sqrt{AZ_t} d\hat{W}_t$$

where A and B are real numbers and $\hat{W}_t$ is a scalar Wiener process.

Remark 3.6 *A CIR process with zero mean reversion level will in finite time be absorbed at 0 a.s. (See Cairns [9].) Thus the only proportional shifts that can be admitted by a purely Wiener driven model are the degenerate shift $q_t(x) = Z_t$ and a shift model which eventually becomes the trivial $q_t(x)$.*

Since $Z_t = \ln X_t$, the next Corollary follows from Ito's formula.

Corollary 3.7 *The spot price, X_t, induced by a futures price that admits a proportional shift term structure has the following dynamics:*

$$dX_t = X_t \ln X_t \left\{ B + \frac{A}{2} \right\} dt + X_t \sqrt{A \ln X_t} dW_t. \tag{41}$$

Example 2 (The pure jump driven log futures model)
Here we consider the special case when the log futures model is a pure jump process. We assume that N_t is a counting process with intensity $\lambda q(0)$ for some positive real number λ and that the log futures curve satisfies the SDE

$$dq(t,x) = \beta g(x) dN_t. \tag{42}$$

Thus the Wiener volatility σ is zero. For tractability reasons we have also assumed that the drift term is zero. If this model admits a proportional shift term structure, then by equation (38),

$$g'(x) = \left(1 - e^{\beta g(x)}\right) \lambda.$$

By letting $f(x) = e^{\beta g(x)}$ this can be rewritten as

$$f'(x) = -\lambda \beta f^2(x) + \beta \lambda f(x),$$

which can be solved by separation of variables. After some calculations and using the initial condition $g(0) = 1$ we find that

$$g(x) = \lambda x - \frac{1}{\beta} \ln(e^{-\beta} - 1 + e^{\lambda \beta x}).$$

Since $Z_t = q_t(0)$ and $g(0) = 1$ we see from equation (42) that

$$dZ_t = \beta dN_t.$$

3.3 Other affine shifts

Recall that the log futures model is said to have an affine term structure if

$$q(t,x) = h(x) + Z_t g(x) \quad \forall (t,x) \in R_+ \times R \tag{43}$$

We divide affine term structure models into four categories:

1. Parallel shifts:
$$g'(x) \equiv 0$$
2. Proportional shifts:
$$h'(x) \equiv 0$$
3. Strictly affine shifts:
$$h'(x) \not\equiv 0 \quad g'(x) \not\equiv 0$$

Thus far we have studied the two first cases of an affine term structure, the parallel structure and the proportional structure. In this section we will analyze the affine structures which are neither parallel nor proportional. We call these other shifts *strictly affine shifts* and define them via item 3 above.

Proposition 3.11 *A log futures model with deterministic jump volatility that admits strictly affine shifts such that $q(t,x) = h(x) + Z_t g(x)$, must have $h'(x)$ linearly dependent on $g(x)$, $g^2(x)$ and $\left(e^{\delta_i g(x)} - 1\right)$ for all $i \in V$.*

Proof. Assume $h'(x)$ is linearly independent of $g(x)$, $g^2(x)$ and $\left(e^{\delta_i g(x)} - 1\right)$ for all $i \in V$. If the model admits a strictly affine term structure model then by Propositions 3.2 there exist a nonempty set $\tilde{R} \subseteq R$ where $\tilde{R} \neq \{0\}$ such that the manifold

$$\mathcal{G} = \left\{ q \in \mathcal{H}; \quad q = h + zg, \quad z \in \tilde{R} \right\}$$

is invariant for all q in $\mathcal{G}$. Thus by Proposition 3.1 the three invariance conditions (14), (15) and (17) are satisfied.

For every q in $\mathcal{G}$ such that $q(0) \neq 0$, equation (17) gives that

$$\begin{aligned} g'(x) &= \frac{1}{2q(0)} \sum_{i=1}^{m} \gamma_i^2(q) g^2(x) + \frac{1}{q(0)} \psi(q) g(x) \\ &+ \frac{1}{q(0)} \sum \left(e^{\delta_i g(x)} - 1\right) \lambda_i(q) - \frac{h'(x)}{q(0)}. \end{aligned}$$

For simplicity we now assume that the real numbers $\delta_1, \delta_2, \ldots, \delta_M$ are distinct. Since $g(0) = 1$ it follows that $g' \not\equiv 0$, thus $g(x)$, $g^2(x)$, $e^{\delta_1 g(x)} - 1$, $e^{\delta_2 g(x)} -$

$1, \ldots e^{\delta_M g(x)} - 1$ are linearly independent. Thus by Lemma 3.1 it follows that $1/q(0)$ is independent of q. This is impossible and thus $h'(x)$ cannot be linearly independent of $g(x)$, $g^2(x)$ and $\left(e^{\delta_i g(x)} - 1\right)$ for all $i \in V$ if it admits a strictly affine term structure. With a slight modification of the arguments, this contradiction can be shown also for the case when not all real numbers $\delta_1, \delta_2, \ldots, \delta_M$ are distinct. ∎

Proposition 3.12 *There exist a purely Wiener driven log futures price model that admits a strictly affine shift term structure, such that $q_t(x) = h(x) + g(x)Z_t$ if and only if*

- *The function h satisfy*
$$h'(x) = ag(x)^2 + bg(x) \tag{44}$$
where a and b are real numbers, not both equal to zero.

- *The log futures volatilities are proportional to the shift function for all q:*
$$\sigma_i(q, x) = \gamma_i(q)g(x) \quad \textit{for } i = 1, \ldots, m, \tag{45}$$

- *There exist a real number A such that*
$$\sum_{i=1}^{m} \gamma_i^2(q) = 2a + Aq(0). \tag{46}$$

- *The shift function g satisfies*
$$g'(x) = \frac{A}{2} g^2(x) + Bg(x) \tag{47}$$
where A is the real number in (46) and B is any real number.

- *The initial log futures curve is strictly affine:*
$$q_0(x) = h(x) + cg(x) \tag{48}$$
where c is any real number except 0.

Proof. It follows by Proposition 3.11 that h must be of the form (44). Thus by Propositions 3.2 and 3.1 there exist a manifold

$$\mathcal{G} = \left\{ q \in \mathcal{H}; \quad q = h + zg, \quad h = ag^2 + bg, \quad z \in \tilde{R}, \quad \tilde{R} \subseteq R \right\}$$

such that for all q in $\mathcal{G}$, the three invariance conditions (14), (15) and (17) are satisfied.

For every q in $\mathcal{G}$ such that $q(0) \neq 0$, equation (17) gives that

$$\begin{aligned} g'(x) &= \frac{1}{2q(0)}\left\{\sum_{i=1}^{m}\gamma_i^2(q) - 2a\right\}g^2(x) + \frac{1}{q(0)}\left\{\psi(q)g(x) - b\right\} \\ &+ \frac{1}{q(0)}\int_V \left(e^{\delta_v(q)g(x)} - 1\right)\lambda(q, dv). \end{aligned}$$

Since the left hand side does not depend on q, the right hand side cannot depend on q either. Since g and g^2 are linearly independent condition (46) follows by Lemma 3.1. The rest of the necessary part follows similarly as in Proposition 3.8.

Next we show that the conditions are sufficient. For some function g, define h as in equation (44). Define the function g such that $g(0) = 1$ and

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) \tag{49}$$

where we choose A to be the real number given in equation (46) and B is any real number. Using the g and h just defined, we define $\mathcal{G} = \{q \in \mathcal{H}; \quad q = h + zg, \quad z \in R\}$. Note that by condition (48) the initial curve is in $\mathcal{G}$. Equation (45) immediately gives that the condition (14) is satisfied for all q in $\mathcal{G}$. To show that also condition (17) is satisfied, we need to check that the expression

$$h'(x) + q(0)g'(x) - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q)g^2(x)$$

is proportional to the shift function, i.e that there exist a functional $\psi(q)$ such that the above expression it is equal to $\psi(q)g(x)$. Plugging in our g and h from equation (49) and (44) respectively yields

$$\frac{(Aq(0) + 2a)}{2}g^2(x) + (Bq(0) + b)\,g(x) - \frac{1}{2}\sum_{i=1}^{m}\gamma_i^2(q). \tag{50}$$

For all q in $\mathcal{G}$ such that $q(0) \neq 0$, we can substitute A with the expression from equation (46) and thus get:

$$(B + b)q(0)g(x).$$

For all q in $\mathcal{G}$ such that $q(0) = 0$, the expression (50) reduces to 0. Thus there exist a ψ, given by $\psi = (B + b)q(0)$, such that the expression is proportional to g for all q in $\mathcal{G}$. Hence by Proposition 3.2, $\mathcal{G}$ is invariant and by Proposition 3.1 it admits a strictly affine term structure. ∎

Proposition 3.13 *If there exists a log futures price model that admits strictly affine shifts, such that* $q_t(x) = h(x) + g(z)$ *and* $h'(x) = ag^2(x) + bg(x)$ *then*

- *$g'(x)$ must satisfy an ODE of the following form:*

$$g'(x) = \frac{A}{2}g^2(x) + Bg(x) \tag{51}$$

where $\beta(v)$ is a real valued function on V, $\lambda(dv)$ is a positive finite measure, A and B are real numbers and $C \in \{-1, +1\}$.

- *When $a \leq 0$ and Z_t is positive (negative), then A must be positive (negative).*

Proof. Suppose that $a \leq 0$. By restriction (46) it follows that if $q(0) > 0$, then A must be positive since $\sum_{i=1}^{m} \gamma_i^2(q)$ is positive. If instead $q(0) < 0$, then A must be negative. ∎

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