Essays in Mathematical Finance

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To my parents

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Stockholm, June 2009
Agatha Murgoci

Contents

Chapter 1

Introduction and Summary

This thesis consists of three papers. The first two papers deal with the implications of market incompleteness for pricing and managing counterparty risk in the context of over-the-counter (OTC) derivatives. The technique employed for this analysis is good deal bound pricing. In contrast, the last paper develops methods for solving a large class of financial and economic problems which are time inconsistent in the sense that they do not admit a Bellman optimality principle. This is done by viewing them within a game theoretic framework, and looking for Nash subgame perfect equilibrium points. The connecting thread between the papers is the theory of stochastic optimal control and its applications to financial problems: the good deal bounds pricing problems are classical stochastic optimal control problems from a mathematical point of view; the last paper deals with extending the classical stochastical optimal control framework in order to allow the analysis of more complex and interesting financial and economic questions.

As mentioned before, the first two papers deal with counterparty risk in the context of OTC derivatives. Counterparty risk has been brought to the forefront by recent events. The current financial crisis has underlined the importance of good pricing and risk management tools for counterparty risk. The papers approach the issue by developing tools which address the market incompleteness due to the counterparty risk.

In the context of derivatives, the source for counterparty risk is the fact that the products are traded over-the-counter (OTC). According to the Bank of International Settlements, in December 2007, the OTC notional amounts outstanding were 417 trillion US dollars. By comparison, at the end of the same period, the notional amounts outstanding in exchange traded futures were 28 trillion US dollars and the notional amounts outstanding in exchange traded option were 52.5 trillion (see (BIS 2008)). Since the market for OTC derivatives is big, managing counterparty risk for OTC derivatives is essential. If traded on an organized exchange, the counterparty risk associated with the derivatives disappears due to the presence of the market maker. The market incompleteness comes from not having liquidly exchange-traded financial products (credit derivatives) that would help pin down the market price of risk for the counterparty's default. This is a classic case of market incompleteness.

As a way of solving the pricing issues raised by the market incompleteness, I propose the good deal bounds method. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. One has to note that by eliminating unusually good deals, we do not eliminate extreme market outcomes, but extreme attitudes toward risk (i.e. investors asking for extreme compensation for the risks taken).

To put good deal bounds in a general context, we remember that one of the consequences of having an incomplete market setup is the fact that we no longer have a unique stochastic discount factor or a unique equivalent martingale measure, and consequently not a unique price. One could simply calculate the bounds of the prices, generated by the interval of all possible risk-neutral measures (or all possible stochastic discount factors). These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.

Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterium, chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature dealing with this approach see Schweizer (2001), Henderson and Hobson (2004), Barrieu and Karoui (2005). However, there is no clear cut way of choosing between different criteria and some of them are somewhat ad-hoc, in the sense that they do not have a clear economic interpretation.

In contrast to this, Cochrane and Saa-Raquejo (2000) proposed the method of good deal bounds. The good deal approach aims at obtaining an interval of "reasonable" prices in incomplete markets, rather than concentrating at obtaining a unique price. Since the no-arbitrage bounds are too large to be used, Cochrane and Saa-Raquejo (2000) suggested to rule out not only arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered "too good to be true", hence the name of "good deal bounds"(GDB). One possible measure for the "goodness"of a deal is its Sharpe Ratio (SR), and thus, trades/portfolios which have a SR above a certain threshold are eliminated. Since the SR links the return of financial assets to the risk undertaken, it is not extreme events which are eliminated from the set, but extreme compensation for the risk undertaken. The SR is chosen as a measure for the "goodness of the deal"because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Hence, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.

The next step in developing a theory for "good deal bounds"was done by Björk and Slinko (2005). They proposed a new frame for solving the optimization problem defined by Cochrane and Saa-Raquejo (2000) while at the same time allowing for more complex dynamics for the underlying assets, such as jump-diffusion processes, to be taken into account. This formulation of the good deal bounds will be used in the current project.

Previous literature on counterparty risk and good deal bounds involved structural models (e.g. Hung and Liu 2005). The first paper of this thesis also approaches the issue form the same point of view. More exactly, we present a unified framework for the pricing of vulnerable options (options where the counterparty can default) using structural models, i.e. models for credit risk that takes into account the value of the assets of the option writer (counterparty) in order to define default. The main ingredients for such a framework are the dynamics of the stock and the dynamics of the assets of the counterparty. The current paper starts from the traditional framework of pricing vulnerable options in complete markets and extends it to incomplete markets. We start by presenting elements of pricing vulnerable options in complete markets and by streamlining the results of Klein (1996) for a European vulnerable call; to this end, we use the more-tractable technique of change of numeraire. In the structural framework, default happens when the assets of the counterparty fall below the level of the claims against the counterparty. One major assumption for pricing vulnerable options is the fact that we can observe the assets of the counterparty and that they are liquidly traded. By relaxing the later assumption and allowing the assets of the counterparty not to be liquidly traded, we are in an incomplete market setup. As explained before, this implies that we no longer have a unique no-arbitrage price, but an interval of possible prices. We are going to employ the good deal bounds technique in order to compute the highest and the lowest possible price, given a constraint on the Sharpe Ratio of all portfolios that can be formed on this market. The paper also extends well known results for using vanilla non-vulnerable options in the pricing of more complex exchange and barrier options. We show how, in a similar manner, one can use the results for vulnerable options in pricing barrier and exchange options. This is done both for the complete and for the incomplete market setup.

The structural models have the advantage of allowing us to model explicitely the correlation between various firms and it provides a relationship between various securities issued by the same firm (equity, bonds, convertible bonds, etc.). However, structural models assume one can observe accurately the value of the firm analysed. Another major drawback with the classical framework of structural models is the fact that default is predictable. Also, the models become quickly untractable and predict credit spreads which are too low when compared to the observed ones.

The next step in credit risk modelling has been the introduction of intensity based models. They represent default in a jump process framework. Thus, they can fit the market observed credit spreads and incorporate the more realistical feature of unpredicable defaults. Intensity based models are also more tractable than structural models and hence, can be used more effectively for the pricing and hedging of credit derivatives. Duffie and Lando (2001) have linked the structural and intensity based models for credit risk by showing that, when bond holders have incomplete information abou the value of the firm, he structural models yields unpredicable default times. Hence, there is an intensity based formulaion for structural models with incomplete information about the value of the firm.

In the second chapter of this thesis, we allow for counterparty risk to be given by intensity-based models, nowadays a standard tool in credit-risk pricing and management. Previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure -which is not unique (e.g. Brigo and Masetti 2005, Brigo and Pallavicini 2008). We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. Furthermore, we study how the interval of prices induced by the good deal bounds changes with different important parameters in the model: i.e. the current intensity of default, the parameters of the intensity process, the good deal bound constant chosen by the modeler, the recovery rate. Results show that the current intensity of default and the recovery rate impact the GDB price interval more than the chosen GDB constant. After we have studied the effect of counterparty risk on one OTC financial derivative at a time, we

turn our attention to how good deal bounds are computed and "behave" in a portfolio framework. Before this, however, we have to check if good deal bounds meet the the requirements to be a good risk measure. Such requirements were put forward by Artzner and Heath (1999) and the resulting risk management instrument carries the name of coherent risk measures. The first to notice the link between good deal bounds and coherent risk measures were Jaschke and Küchler (2001). However, they dismiss the good deal bounds on the Sharpe Ratio à la Cochrane as not satisfying the monotonicity requirement. Under the new re-formulation of the GDB based on the SR done by Björk-Slinko, one can notice that the lower GDB trivially satisfies the formal properties proposed by Artzner and Heath (1999) and hence, it is a coherent risk measure.
We study the effect of adding more assets traded with the same counterparty to our portfolio and see how the GDB behave in this context. Then, we also check how adding a new counterparty is going to affect the lower good deal bound.

From a mathematical point of view, the good deal bounds pricing belongs to the class of stochastical optimal control problems and, when solving such problems, we employ the powerful tools of dynamic programming. However, in the financial and economic theory, there are problems such as the mean-variance asset allocation problem or macroeconomic problems with hyperbolic discounting which do not allow the use of dynamic programming. Solving these problems leads to time inconsistency - i.e. if for some fixed initial point, we determine the control law which maximizes the objective function, then at some later point the same control law will no longer be optimal.

In the third paper, we undertake a rigorous study of time inconsistent control problems in a reasonably general Markovian framework, and in particular we do not want to tie ourselves down to a particular applied problem. We have therefore chosen a setup of the following form.

- We consider a **general controlled Markov process** X, living on some suitable space (details are given below). It is important to notice that we do not make any structural assumptions whatsoever about X, and we note that the setup obviously includes the case when X is determined by a system of SDEs driven by a Wiener and a point process.

- We consider a functional of the form

$$J(t,x,\mathbf{u}) = E_{t,x}\left[\int_t^T C\left(x, X_s^{\mathbf{u}}, \mathbf{u}(X_s^{\mathbf{u}})\right) ds + F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right).$$

We see that with the choice of functional above, time inconsistency enters at several different points. Firstly we have the appearance of the present state x in the local utility function C, as well as in the functions F and G, and this leads of course to time inconsistency. Secondly, in the term $G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right)$ we have, even forgetting about the appearance of x, a non linear function G acting on the conditional expectation, again leading to time inconsistency.

Note that, for notational simplicity we have not explicitly included dependence on running time t. This can always be done by letting running time be one component of the state process X, so the setup above also allows for expressions like $F(t, x, X_T^{\mathbf{u}})$, thus allowing (among many other things) for hyperbolic discounting.

This setup is studied in some detail in continuous as well as in discrete time. The discrete time results are parallel to those in continuous time, and our main results in continuous time are as follows.

- We provide a precise definition of the Nash equilibrium concept. (This is done along the lines of Ekeland and Lazrak (2006) and Ekeland and Pirvu (2007)).
- We derive an extension of the standard Hamilton-Jacobi-Bellman equation to a non standard system of equations for the determination of the equilibrium value function V.
- We prove a verification theorem, showing that the solution of the extended HJB system is indeed the equilibrium value function, and that the equilibrium strategy is given by the optimizer in the equation system.
- We prove that to every time inconsistent problem of the form above, there exists an associated **standard, time consistent**, control problem with the following properties:
  - The optimal value function for the standard problem coincides with the equilibrium value function for the time inconsistent problem.
  - The optimal control law for the standard problem coincides with the equilibrium startegy for the time inconsistent problem.

- We solve some specific test examples.

Thus, while in the first papers of this thesis one uses the tools of dynamic programming in order to answer questions about the range of possible prices of counterparty risk on OTC markets, the last paper develops tools needed in order to address non-standard problems in economics and finance.

Chapter 2

Vulnerable Options and Good Deal Bounds - Structural Model

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Vulnerable Options and Good Deal Bounds - Structural Model

We price vulnerable options - i.e. options where the counterparty may default. These are basically options traded on the OTC markets. Default is modeled in a structural framework. The technique employed for pricing is Good Deal Bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. We also provide tight pricing bounds for European calls and show how to extend the call formula to pricing other financial products in a consistent way. Specific examples for exchange options and barrier options are computed. Finally, we analyze numerically the behaviour of the good deal pricing bounds interval and analyze the factors that impact its size.

1 Introduction

Vulnerable options are options that bear counterparty risk - in other words, the writer of the option may not deliver the underlying. The main reason for having a counterparty risk is the fact that these options are traded over-the-counter (OTC). If traded on an organized exchange, the counterparty risk associated with the option disappears due to the presence of the market maker. According to BIS, the OTC equity-linked option gross market value in the first half of 2008 USD 863 bln. Also, the volume of equity linked derivatives has increased during the last years. Thus, there is a necessity to have consistent pricing of vulnerable options.

In the previous literature, an important assumption is that vulnerable options are traded in complete markets, i.e. that in addition to the asset underlying the derivative, also the assets of the counterparty are the price of a traded asset. Papers pricing vulnerable options in a complete market setup include Johnson and Stulz (1987), Jarrow and Turnbull (1995), Hull and White (1995), Klein (1996). In real life, vulnerable options are traded mainly over-the-counter, and the assets of the counterparty are not traded assets on the market. Thus, we are in a classical case of incomplete markets. The first to notice this inconsistency were Hung and Liu (2005)

Hung and Liu (2005) have priced the vulnerable options using the structural model set up by Klein (1996) and using "good deal bounds". The good deal bounds were first introduced by Cochrane and Saa-Raquejo (2000) and constitute a "hybrid" between no-arbitrage pricing and utility-based pricing. They narrow the wide bands of possible prices obtained with no arbitrage pricing, while avoiding the model-sensitivity implied by utility-based pricing. The method imposes a new restriction in the arbitrage free model, by setting upper bounds on the Sharpe ratio-s of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Björk and Slinko (2005) translate the stochastic discount problem to an equivalent martingale problem and use martingale methods to solve the optimization problem. Thus, the calculations are more tractable and the price processes can be characterised by point processes, besides the traditional Wiener framework.

In this paper, we present a unified framework for the pricing of vulnerable options using structural models. Traditionally, vulnerable options were analysed in a structural framework, i.e. a model for credit risk that takes into account the value of the assets of the option writer(counterparty) in order to define default. The main ingredients for such a framework are the dynamics

of the stock and the dynamics of the assets of the counterparty. The current paper starts from the traditional framework of pricing vulnerable options in complete markets and extends it to incomplete markets.
Section 2 presents elements of pricing vulnerable options in complete markets. We start by streamlining the results of Klein for a European vulnerable call; to this end, we use the more-tractable technique of change of numeraire. Then, we extend some well-known results for non-vulnerable options to the vulnerable case. More specifically, these results concern pricing of derivatives with linearly homogeneous payoffs and pricing of barrier options.
In section 3, we deal with pricing vulnerable options in incomplete markets. To this purpose, we use the good deal bounds framework proposed by Björk and Slinko (2005) which allows for higher degree of tractability. Besides pricing vulnerable European vulnerable options, we also show how results obtained for the complete markets can be extended in the incomplete market case, i.e. we price exchange options and barrier options. Section 4 presents a few numerical results and section 5 concludes.

2 The Complete Market case - The Klein Model

2.1 Setup

This section takes the setup proposed by Klein (1996) and calculates the price of a vulnerable option by applying a different method, the change of numeraire. This method will alow for a better tractability of the old results, as well as for extending the results for a call option to other vanilla products, such as min or exchange options.
The option is written on the stock S and has maturity T and strike K. For simplicity, we will assume through the entire paper that the stock S is traded on the market. The case of an option written on an untraded stock is a straightforward extension for the incomplete market case analysed in subsequent sections. Since the option to be priced is traded over-the-counter, there is also counterparty risk to be taken into account.
In a first setting, default will depend on the assets of the counterparty - the writer of the option. They are denoted by Y. As stated in Klein (1996), the assets of the counterparty are defined such that they include all assets of the counterparty, marked to market, as well as all derivative positions.
To begin with, we consider Y traded on the market. This assumption implies we are in a complete market setup. The total value of the claims against the counterparty is denoted by D and we are not concerned with modelling D.

We assume a riskless bond B with interest rate r is traded on the market. We proceed by giving the main features of the market:

Assumption 2.1

1. *Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given; $\underline{\mathcal{F}}$ is the internal filtration generated by the Wiener process $\tilde{W}$, which will be defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dY_t &= \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t \\ dS_t &= \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t \\ dB_t &= B_t r dt \end{aligned}$$

   where Y_t denotes the assets of the counterparty underwriting the option, S_t the price of the stock on which the option is contracted and B_t the bank account. The assets of the counterparty are defined such that they include all assets of the counter-party, marked to market, as well all derivative positions.

3. *μ_t and α_t are scalar deterministic functions of time, $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are (1,2) row vector deterministic functions of time, specified as follows:*

$$\begin{aligned} \bar{\gamma}_t &= \begin{pmatrix} \gamma_t, & 0 \end{pmatrix} \\ \bar{\sigma}_t &= \begin{pmatrix} \sigma_t \rho, & \sigma_t \sqrt{1-\rho^2} \end{pmatrix} \end{aligned}$$

4. *Let $\tilde{W}$ be a two dimensional P-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2 \end{pmatrix}$$

   with $\tilde{W}^1$ and $\tilde{W}^2$ independent scalar P-Wiener processes.

5. *Assume that both the assets of the counterparty underwriting the option and the stock are traded on the market.*

6. *The payoff of a vulnerable European call option, $X = \Phi(S_T, Y_T, T)$, is given by*

$$X = \Phi(S_T, Y_T, T) = \max(S_T - K, 0) I(Y_T \geq D) + \mathcal{R} I(Y_T < D)$$

   where D is the value of the total value of the claims against the counter-party.

Before proceeding, we need to clarify a few things. First, note that while $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are defined as row vector coefficients of the diffusion terms, sometimes it is more convienient to work with the scalar values of σ_t and γ_t. However, it is straightforward when one uses scalar notation instead of vector notation and switching to the vector notation does not pose any technical difficulty.
Next, we are going to give an intuition for the payoff function of a vulnerable option. The general payoff function for vulnerable options has two components - the payoff of the claim in case of no default and the recovery payoff, i.e. the payoff in case the counterparty defaults. We denote the general payoff by X and the recovery payoff by $\mathcal{R}$. If there is no default, the payoff of the claim is the standard option payoff, i.e. $\max(S_T - K, 0)$; if the counterparty defaults, the payoff is the recovery payoff $\mathcal{R}$. The default occurs if the value of the assets of the counterparty at time T, Y_T, falls below the value of the claims written against the counterparty, D. All payments are done at time T.

For the complete market setup, the value of the recovery payoff $\mathcal{R}$ is given by:

$$\mathcal{R} = (1 - \beta)\frac{Y_T}{D}\max(S_T - K, 0)$$

The logic behind the above formula is straightforward. One gets a proportional part of the value of the claim, corresponding to how much the assets of the counter-party have fallen below the value of the claim. However, there are some deadweight costs associated to the bankrupcy procedure. These costs are captured by the β parameter. This recovery specification is very close to the specification for recovery of treasury.

Assumption 2.2 *Let the recovery payoff be given by:*

$$\mathcal{R}I(Y_T < D) = (1 - \beta)\frac{Y_T}{D}\max[S_T - K, 0]I\left\{Y_T < D\right\}$$

Having defined the main assumptions of the model, we will now proceed to price the vulnerable option in the complete markets setup, by using the change of numeraire technique.

2.2 Pricing the vulnerable options by change of numeraire

Change of numeraire for the case with zero recovery

First, I will calculate the value of the claim, for the case of zero recovery. The assumption of zero recovery is adopted only for the duration of the current section. However, starting from this simplifying case gives a better clarity in exposition and calculations are more tractable. The payoff function becomes $X = \max[S_T - K, 0]I[Y_T \geq D]$ and we obtain:

$$
\begin{aligned}
X &= \max[S_T - K, 0]I[Y_T \geq D] = (S_T - K)I[S_T \geq K]I[Y_T \geq D] \\
&= S_T I[S_T \geq K]I[Y_T \geq D] - KI[S_T \geq K]I[Y_T \geq D]
\end{aligned}
$$

In order to price the claim with payoff X, we will apply the change of numeraire technique. For details, see Björk (2004). For the first term,$S_T I[S_T \geq K]I[Y_T \geq D]$, we do a change of measure to Q^S, the measure corresponding to S_t as numeraire. For the second term,$KI[S_T \geq K]I[Y_T \geq D]$, the change of measure is to the forward measure, Q^T.

Denoting by Π the price of the vulnerable option, we will start from the following pricing expression:

$$
\Pi = S_0 Q^S[S_T \geq K; Y_T \geq D] - Kp(0,T)Q^T[S_T \geq K; Y_T \geq D] \tag{2.1}
$$

and we will first attack the second term, Q^T, and then the first, Q^S. The calculations are detailed below.

- Under the forward measure, Q^T, we need to calculate:

  $$
  Q^T(Y_T \geq D, S_T \geq K) = Q^T\left(\frac{Y_T}{p(T,T)} \geq D, \frac{S_T}{p(T,T)} \geq K\right)
  $$

  since $p(T,T) = 1$.

  We denote $Z_Y(t) = \frac{Y_t}{p(t,T)}$ and $Z_S(t) = \frac{S_t}{p(t,T)}$ and want to calculate $Q^T(Z_Y(T) \geq D, Z_S(T) \geq K)$. Under the forward measure, $Z_Y(t)$ and $Z_S(t)$ are martingales (as the price of asset with the forward price as numeraire).

  $$
  \begin{aligned}
  dZ_Y(t) &= Z_Y(t)\bar{\sigma}_t dW_t^T \\
  dZ_S(t) &= Z_S(t)\bar{\gamma}_t dW_t^T
  \end{aligned}
  $$

The solutions to the above equations are:

$$\begin{aligned} Z_Y(T) &= Z_Y(0)\exp\left\{-\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T \bar{\sigma}_t dW_t^T\right\} \\ Z_S(T) &= Z_S(0)\exp\left\{-\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^T\right\} \end{aligned}$$

We notice that the equations above have a very similar structure: the exponent is the sum of a time integral and a stochastic integral with a deterministic integrand, which leads to the entire exponent being normally distributed. The variance of the exponent is $\int_0^T \|\bar{\sigma}_t\|^2 dt$ for the first equation, respectively $\int_0^T \|\bar{\gamma}_t\|^2 dt$ for the second equation.
Now, we need to transform the two lognormal variables into standard-normal variables. In order to perform this easy transformation, we use the following string of equivalent inequalities:

$$Y_T \geq D$$

$$Z_Y(T) \geq D$$

$$Z_Y(0)\exp\left\{-\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T (\bar{\sigma}_t) dW_t^T\right\} \geq D$$

$$\frac{\ln Z_Y(T) - \ln Z_Y(0) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \geq \frac{\ln D - \ln Z_Y(0) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$\xi \geq \underbrace{\frac{\ln \frac{Dp(0,T)}{Y(0)} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}}_{-b_2}$$

where ξ is standard-normally distributed.
Following the same steps, we have:

$$S_T \geq K \Leftrightarrow Z_S(T) \geq K$$

and by writing explicitely $Z_S(T)$, taking logs and suitably transforming the resulting normal variable, we obtain the equivalent inequality:

$$\eta \geq \underbrace{\frac{\ln \frac{Kp(0,T)}{S(0)} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}}_{-a_2}$$

where η is standard normally distributed.

Summing up the calculations above, one obtains:

$$Q^T(Y_T \geq D, S_T \geq K) = Q^T(\eta \geq -a_2, \xi \geq -b_2, \rho_1) \tag{2.2}$$

where η and ξ are standard normal, ρ_1 is the correlation coefficient between η and ξ. The constants a_2 and b_2 are given above.
The first step in clarifying the right-handside term in (2.2) is calculating the correlation coefficient ρ_1. We formulate the result as a lemma.

Lemma 2.1 *Given assumptions 3.1 and η and ξ defined as in (2.2), the correlation coefficient between η and ξ is given by:*

$$\rho_1 = \frac{\rho \int_0^T \sigma_t \gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}}.$$

Proof. We know that

$$\xi = \frac{\ln\left(\frac{Z_Y(T)}{Z_Y(0)}\right) + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}}$$

$$\eta = \frac{\ln\left(\frac{Z_S(T)}{Z_S(0)}\right) + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}$$

The only stochasticity in the formulas comes from the expressions A_1 and A_2 defined below:

$$A_1 = \ln\left(\frac{Z_Y(T)}{Z_Y(0)}\right) = -\frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt + \int_0^T \bar{\sigma}_t dW_t^T$$

$$A_2 = \ln\left(\frac{Z_S(T)}{Z_S(0)}\right) = -\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^T$$

Simplifying even further, we obtain that:

$$\rho_1 = Corr[A_1; A_2] = Corr\left[\int_0^T \bar{\sigma}_t dW_t^T; \int_0^T \bar{\gamma}_t dW_t^T\right]$$

where $\bar{\sigma}_t = \left(\sigma_t \rho, \quad \sigma_t \sqrt{1-\rho^2}\right)$ and $\bar{\gamma}_t = \left(\gamma_t, \quad 0\right)$ and $W^T = \left(W^{1;T}, \quad W^{2;T}\right)'$, with $W^{1;T}$ and $W^{2;T}$ independent T-Wiener processes[1]. By direct calculation, it follows that

$$\rho_1 = \frac{Cov\left[\int_0^T \bar{\sigma}_t dW_t^T; \int_0^T \bar{\gamma}_t dW_t^T\right]}{\sqrt{Var\left[\int_0^T \bar{\sigma}_t dW_t^T\right] Var\left[\int_0^T \bar{\gamma}_t dW_t^T\right]}} = \frac{\rho \int_0^T \sigma_t \gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}}.$$

If σ_t and γ_t are constant, we have $\rho_1 = \rho$. ∎

At this point, we need to introduce some new notation.

Definition 2.1 *Let $\mathcal{N}(a, b, \mathbf{r})$ be the probability $P[X \leq a; Y \leq b]$, where X and Y are standard normal variables with correlation coefficient* $\mathbf{r}$.

We are going to use this notation to express the right handside of (2.2) in terms of CDF-s. In order to do this, we need to study the behaviour of the bivariate standard normal distribution, when the two variable are correlated, which will be done by means of characteristic functions. Let $\Phi_1(t_1, t_2)$ be the characteristic function for the bivariate normal distribution with correlation coefficient $\mathbf{r}$, $N[0, 0, 1, 1, \mathbf{r}]$:

$$\Phi_1(t_1, t_2) = \exp\left\{-\frac{1}{2}[t_1^2 + t_2^2 + 2\mathbf{r}t_1 t_2]\right\}$$

Since the characteristic function $\Phi_1(t_1, t_2)$ is a real function, we know the distribution is symmetric (see Gut (2005)), or

$$P[X \geq a; Y \geq b] = P[X \leq -a; Y \leq -b] = \mathcal{N}[-a; -b; \mathbf{r}], \tag{2.3}$$

where X, Y are standard normal distributed and have correlation coefficient $\mathbf{r}$.
Then, we try to transform $P[X \geq a; Y \leq b]$ into a CDF. We know that if (X, Y) $N[0, 0, 1, 1, \mathbf{r}]$, then $(-X, Y)$ $N[0, 0, 1, 1, -\mathbf{r}]$. Hence, we conclude that:

$$P[X \geq a; Y \leq b] = P[-X \leq -a; Y \leq b] = \mathcal{N}[-a; b; -\mathbf{r}] \tag{2.4}$$

[1]The independence of $W^{1;T}$ and $W^{2;T}$ follows from the independence of the P-Wiener processes $\tilde{W}^1$ and $\tilde{W}^2$ (see assumption 3.1). If we denote by φ^T the Girsanov kernel between the objective probability measure P and the equivalent martingale measure Q^T, it is easy to see that $dW_t^{1;T} W_t^{2;T} = (-\varphi^T dt + d\tilde{W}_t^1)(-\varphi^T dt + d\tilde{W}_t^2) = 0$

Note the change of signs in the correlation coefficient.
Thus, we obtain $Q^T(\eta \geq -a_2, \xi \geq -b_2, \rho_1) = \mathcal{N}(a_2, b_2, \rho_1)$.

Going back to (2.2), we conclude that:

$$Q^T(Y_T \geq D, S_T \geq K) = \mathcal{N}(a_2, b_2, \rho_1) \tag{2.5}$$

where ρ_1 is given by lemma (2.1) and:

$$a_2 = -\frac{\ln \frac{Kp(0,T)}{S(0)} + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}}$$

$$b_2 = -\frac{\ln \frac{Dp(0,T)}{Y(0)} + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}$$

- We need to calculate $Q^S(Y_T \geq D, S_T \geq K)$ before being able to give a solution to the the initial pricing equation (2.1).

  To this purpose, we start by identifying the dynamics of the assets under the new probability measure Q^S. The dynamics of Y_t under Q^S are given by standard theory as:

$$dY_t = Y_t(\mu_t + \bar{\sigma}_t \varphi_t^S)dt + Y_t \bar{\sigma}_t dW_t^S$$

  where φ^S is the Girsanov kernel for the transformation $P \rightarrow Q^S$. The Girsanov kernel φ^S is obtained by imposing the martingale condition under Q^S for $\frac{Y_t}{S_t}$:

$$\mu_t - \alpha_t - \sigma_t \gamma_t \rho + 2\gamma_t^2 + (\bar{\sigma}_t - \bar{\gamma}_t)\varphi_t^S = 0$$

  and for $\frac{B_t}{S_t}$:

$$r - \alpha_t + 2\gamma_t^2 - \bar{\gamma}_t \varphi_t^S = 0.$$

  Since we have a system of two equations with two unknowns, φ^S is completely identified and we have:

$$\mu_t + \varphi_t^S \bar{\sigma}_t = r + \gamma_t \sigma_t \rho$$

The solution to SDE describing the dynamics of Y_t is:

$$Y_T = Y_0 \exp\left(\int_0^T (r + \gamma_t \sigma_t \rho)dt - \frac{1}{2}\int_0^T \sigma_t^2 dt + \int_0^T \bar{\sigma}_t dW_t^S\right) \tag{2.6}$$

Since the exponent part from (2.6) is formed by a time integral and a stochastic integral with a deterministic integrand, it is clear that the exponent of Y_t is normally distributed with variance $\int_0^T \|\bar{\sigma}_t\|^2 dt$.

We need to transform the lognormal variables into standard normal variables:

$$
\begin{aligned}
&Y_T \geq D \\
&Y(0)\exp\left(\int_0^T (r + \gamma_t \sigma_t \rho)dt - \frac{1}{2}\int_0^T \sigma_t^2 dt + \int_0^T (\bar{\sigma}_t) dW_t^S\right) \geq D \\
&\xi \geq \underbrace{\frac{\ln \frac{D}{Y(0)} - \int_0^T (r + \gamma_t \sigma_t \rho)dt + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}}_{-b_1}
\end{aligned}
$$

where ξ is standard normally distributed.
For S_T we reformulate:

$$S_T \geq K \Leftrightarrow \frac{1}{S_T} \leq \frac{1}{K} \Leftrightarrow \frac{p(T,T)}{S_T} \leq \frac{1}{K}$$

We know that $\frac{p(t,T)}{S_t}$ is a martingale under Q^S. The dynamics for $\frac{p(t,T)}{S_t}$ can be calculated by the Ito formula. Since, in this model, the interest rate is deterministic and the price of a riskless bond is given by $p(t,T) = \exp\{-r(T-t)\}$, the dynamics are:

$$d\left[\frac{p(t,T)}{S_t}\right] = -\frac{p(t,T)}{S_t}\bar{\gamma}_t dW_t^S$$

The solution to the above equation is:

$$\frac{p(T,T)}{S_T} = \frac{p(0,T)}{S_0}\exp\left(-\frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt - \int_0^T \bar{\gamma}_t dW_t^S\right)$$

The exponent is normally distributed with variance $\int_0^T \|\bar{\gamma}_t\|^2 dt$.
We have:

$$S_T \geq K$$

$$\frac{p(T,T)}{S_T} \leq \frac{1}{K} \Leftrightarrow \log \frac{p(T,T)}{S_T} \leq \log \frac{1}{K}$$

$$\eta \leq \underbrace{\frac{\ln \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}}}_{a_1}$$

Now, we have:

$$Q^S(Y_T \geq D, S_T \geq K) = Q^S(\eta \leq a_1, \xi \geq -b_1, \mathbf{r})$$

where η and ξ are standard normal variables; $\mathbf{r}$ is the correlation coefficient between the two standard normal variables; a_1 and b_1 are defined above. Following the same steps as before, it is easy to show by direct computation that $\mathbf{r} = -\rho_1$.
Also, the properties of the bivariate normal distribution derived before allow us to change the generic probability into a CDF. Thus, we obtain:

$$Q^S(Y_T \leq D, S_T \geq K) = \mathcal{N}(a_1, b_1, \rho_1)$$

where ρ_1 is given by lemma (2.1) and

$$\begin{aligned} a_1 &= \frac{\ln \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\ b_1 &= -\frac{\ln \frac{D}{Y_0} - \int_0^T (r + \gamma_t \sigma_t \rho) dt + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}. \end{aligned}$$

Up to this point, we have performed all calculations necessary to obtain a pricing expression for vulnerable options, in the case of complete markets, for the zero recovery payoff. We needed to calculate the probabilities that appear in the transformation of the general risk neutral pricing formula:

$$\Pi(0,X) = S_0 Q^S[S_T \geq K; Y_T \geq D] - Kp(0,T)Q^T[S_T \geq K; Y_T \geq D]$$

Now we can gather the results from the last calculations and obtain a pricing expression similar to the Black Scholes equation.

Proposition 2.1 *Under the assumptions 3.1, the price for a vulnerable option with maturity T, strike price K, and zero recovery, Π_0^1, is given by:*

$$\Pi_0^1 = S_0 \mathcal{N}(a_1, b_1, \rho_1) - Kp(0,T)\mathcal{N}(a_2, b_2, \rho_1)$$

where

$$\begin{aligned}
a_1 &= \frac{\ln \frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_1 &= \frac{\ln \frac{Y_0}{p(0,T)D} + \int_0^T \gamma_t \sigma_t \rho dt - \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\
a_2 &= \frac{\ln \frac{S_0}{Kp(0,T)} - \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_2 &= \frac{\ln \frac{Y_0}{Dp(0,T)} - \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\
\rho_1 &= \frac{\rho \int_0^T \sigma_t \gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}}
\end{aligned}$$

Change of numeraire for the recovery payoff

Now, we can go back to the original case and price the recovery payoff for complete markets. We recall from the assumption 4.1 that the recovery payoff is given by:

$$\begin{aligned}
\mathcal{R}I(Y_T < D) &= (1-\beta)\frac{Y_T}{D}\max[S_T - K, 0] I\{Y_T < D\} \\
&= (1-\beta)\frac{Y_T}{D}(S_T - K) I\{S_T \geq K\} I\{Y_T < D\} \\
&= \frac{1-\beta}{D} Y_T S_T I\{S_T \geq K\} I\{Y_T < D\} \\
&\quad - \frac{1-\beta}{D} K Y_T I\{S_T \geq K\} I\{Y_T < D\}
\end{aligned}$$

Since the calculations about to follow are cumbersome, it is for the benefit of the reader to split the above formula in 2 parts, to be analysed separately:

$$R_1 = \frac{1-\beta}{D} Y_T S_T I\{S_T \geq K\} I\{Y_T < D\} \tag{2.7}$$

$$R_2 = \frac{1-\beta}{D} K Y_T I\{S_T \geq K\} I\{Y_T < D\}. \tag{2.8}$$

The recovery payoff is given by the equality $\mathcal{R}I(Y_T < D) = R_1 - R_2$. We will use the general risk neutral pricing formula for a general claim $\mathcal{X}(T, Z_T)$:

$$\Pi_t(\mathcal{X}) = e^{\{r(T-t)\}} E^Q\left[\mathcal{X}(T, Z_T) | \mathcal{F}_t\right]$$

where $\Pi_t(\mathcal{X})$ denotes the price of a claim $\mathcal{X}$ at time t.
Then, we apply the technique of change of numeraire. For R_2, I will use the martigale measure which has the Y_t as numeraire. For R_1, the situation is not so straight forward. It seems that the most appropriate numeraire would be $S_t Y_t$. However, this is not the price of a traded asset. Hence, one cannot say that we apply the traditional change of numeraire. Even if $S_t Y_t$ is not a traded asset, one can still perform an appropriate change of measure in order to calculate easier $e^{\{r(T-t)\}} E^Q\left[\mathcal{X}(T, Z_T) | \mathcal{F}_t\right]$. More details upon the exact change of measure will follow in the paper.

- We will start calculations with R_2. We apply the risk-neutral pricing formula and a change of numeraire from the bank account to Y_t. We should calculate $Q^Y(S_T \geq K, Y_T < D)$ and we will use the following chain of inequalities:

  $$Y_T < D \Leftrightarrow \frac{1}{Y_T} > \frac{1}{D} \Leftrightarrow \frac{p(T,T)}{Y_T} > \frac{1}{D}$$

  Under Q^Y, $\frac{p(t,T)}{Y_t}$ is a martingale with dynamics:

  $$d\frac{p(t,T)}{Y_t} = -\frac{p(t,T)}{Y_t} \bar{\sigma} dW_t^Y$$

  Since p(t,T) is deterministic, we obtain:

  $$\frac{p(T,T)}{Y_T} = \frac{p(0,T)}{Y_0} \exp\left(-\frac{1}{2}\int_0^T \sigma_t^2 dt - \int_0^T \bar{\sigma}_t dW_t^Y\right)$$

The exponent is normally distributed with variance $\int_0^T \sigma_t^2 dt$. We have:

$$\begin{aligned}
&Y_T < D \\
&\frac{p(T,T)}{Y_T} > \frac{1}{D} \\
&\ln \frac{p(T,T)}{Y_T} > \ln \frac{1}{D} \\
&\xi > \underbrace{\frac{\ln \frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}}_{-b_4}
\end{aligned}$$

where ξ is standard normally distributed.

Now, we turn to the first part of the probability to compute. The dynamics of S_t under Q^Y are given by:

$$dS_t = (\alpha_t + \bar{\gamma}_t \varphi_t^y) S_t dt + S_t \bar{\gamma}_t dW_t^Y$$

The Girsanov kernel is obtained by imposing the martingale condition to the dynamics of the asset on the market, expressed in the new numeraire, Y_t. The assets in case are $\frac{S_t}{Y_t}$ and $\frac{B_t}{Y_t}$ and the derived conditions are:

$$\begin{aligned}
\alpha_t - \mu_t - \sigma_t \gamma_t \rho + 2\sigma_t^2 + [\bar{\gamma}_t - \bar{\sigma}_t]\varphi_t^y &= 0 \\
r - \mu_t + 2\sigma_t^2 - \bar{\sigma}_t \varphi_t^y &= 0.
\end{aligned}$$

Since we have two equations with two unknowns, the Girsanov kernel is completely identified:

$$\varphi_t^y = \left(\frac{r - \alpha_t + \sigma_t \gamma_t \rho}{\gamma_t}, \quad \frac{\gamma_t (r - \mu_t + 2\sigma_t^2) - \sigma_t \rho (r - \alpha_t + \sigma_t \gamma_t \rho)}{\sigma_t \gamma_t \sqrt{1-\rho^2}} \right)$$

and we obtain:

$$\alpha_t + \bar{\gamma}_t \varphi_t^y = r + \sigma_t \gamma_t \rho.$$

Then, we proceed to solve the stochastic differential equation above, which yields:

$$S_T = S_0 \exp\left(\int_0^T (r + \sigma_t \gamma_t \rho) dt - \frac{1}{2} \int_0^T \|\bar{\gamma}_t\|^2 dt + \int_0^T \bar{\gamma}_t dW_t^Y \right)$$

The exponent of S_t is normally distributed with variance $\int_0^T \|\bar{\gamma}_t\|^2 dt$. As before, we need to transform the lognormal variable into a standard normal variable:

$$\begin{aligned}
&S_T \geq K \\
&S(0)\exp\left(\int_0^T (r+\sigma_t\gamma_t\rho)dt - \frac{1}{2}\int_0^T \gamma_t^2 dt + \int_0^T \bar{\gamma}_t dW_t^Y\right) \geq K \\
&\eta \geq \underbrace{\frac{\ln\frac{K}{S(0)} + \int_0^T [r+\sigma_t\gamma_t\rho]dt + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}}}_{-a_4}
\end{aligned}$$

where η is standard normally distributed. Thus,

$$Q^Y(S_T \geq K, Y_T < D) = Q^Y(\eta \geq -a_4, \xi > -b_4, \rho_2)$$

where η and ξ are standard normal and

$$\begin{aligned}
a_4 &= -\frac{\ln\frac{K}{S_0} + \int_0^T [r+\sigma_t\gamma_t\rho]dt + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_4 &= -\frac{\ln\frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}}
\end{aligned}$$

By ρ_2, we denote the correlation coefficient between the two standard normal variables. As in the previous subsection, by direct calculation and following the same steps, it is straightforward to show

$$\rho_2 = -\frac{\rho\int_0^T \sigma_t\gamma_t dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} = -\rho_1$$

. Using the properties of the bivariate normal distribution derived before, we obtain:

$$Q^Y(S_T \geq K, Y_T < D) = \mathcal{N}(a_4, b_4, -\rho_1)$$

- Calculation for R_1 are detailed below. We take each part of the calculation separetely for a better exposition. As before, the starting point for the calculations is a change of measure. However, since no trivial numeraire leads to esier computations, we will use a change of measure rather than a change of numeraire.

Starting from a general expression for the value of R_1 at time t, which we will denote by $\Pi_t[R_1]$:

$$\begin{aligned}\Pi_t[R_1] &= E^Q\left[e^{-\int_t^T r_s ds} S_T Y_T I[S_T \geq K, Y_T < D] \middle| \mathcal{F}_t\right] \\ &= E^Q\left[e^{-\int_t^T r_s ds} X_T Z \middle| \mathcal{F}_t\right]\end{aligned}$$

where $X_T = S_T Y_T$ and $Z = I[S_T \geq K, Y_T < D]$, we rewrite the above expression as:

$$R_1 = E^Q\left[e^{-\int_t^T r_s ds} m_T R_T Z \middle| \mathcal{F}_t\right] \tag{2.9}$$

where $m_T = E^Q[X_T]$ and $R_T = \frac{X_T}{E^Q[X_T]}$.

We assume that $Y_T \geq 0$. Since S_T is the price of a traded stock, we have $S_T \geq 0$. Thus, we have $X_T \geq 0$ P-a.s. Also, we note that $E^Q[R_T] = 1$. These facts allow us to use R_T as a Radon-Nycodim derivative in a change of measure and define a measure $\hat{Q}$ by:

$$d\hat{Q} = R_T dQ \text{ on } \mathcal{F}_T$$

Using Bayes' Theorem, we can re-write (2.9) as:

$$R_1 = e^{-\int_t^T r_s ds} m_T E^Q\left[R_T | \mathcal{F}_t\right] E^{\hat{Q}}\left[Z | \mathcal{F}_t\right]$$

If we define the likelihood process L_t, $0 \leq t \leq T$, by:

$$d\hat{Q} = L_t dQ \text{ on } \mathcal{F}_t$$

we have by standard theory:

$$L_t = E^Q\left[L_T | \mathcal{F}_t\right] = E^Q\left[R_T | \mathcal{F}_t\right] \tag{2.10}$$

Note that even if $L_T = R_T$, we cannot draw the conclusion $L_t = R_t$ for $t < T$. This is a consequence of the fact that $S_T Y_T$ is not a traded asset.

In order to proceed, we need to calculate the following:

- m_T,
- $E^Q\left[R_T | \mathcal{F}_t\right]$,
- the dynamics for L_t in order to indentify the Girsanov transformation $Q \to \hat{Q}$,
- $E^{\hat{Q}}\left[Z | \mathcal{F}_t\right]$

(a) In order to calculate m_T, we need to obtain the dynamics of S_tY_t under Q:

$$d(S_tY_t) = S_tY_t(2r + \gamma_t\sigma_t\rho)dt + S_tY_t(\bar{\gamma}_t + \bar{\sigma}_t)dW_t \tag{2.11}$$

Hence, $m_T = S_0Y_0 \exp\left\{\int_0^T (2r + \gamma_t\sigma_t\rho)ds\right\}$.

(b) Using (2.11), we obtain $E^Q\left[X_T|\,\mathcal{F}_t\right] = S_tY_t \exp\left\{\int_t^T (2r + \gamma_t\sigma_t\rho)ds\right\}$, so

$$E^Q\left[R_T|\,\mathcal{F}_t\right] = \frac{S_tY_t}{S_0Y_0}\exp\left[-\int_0^t (2r + \gamma_t\sigma_t\rho)ds\right] \tag{2.12}$$

(c) Since L_t is a martingale under Q, we assume the dynamics of L_t are of form:

$$dL_t = L_t\varphi_t dW_t \tag{2.13}$$

From (2.10),(2.12) and (2.13), we obtain:

$$dL_t = L_t(\bar{\gamma}_t + \bar{\sigma}_t)dW_t \tag{2.14}$$

The Girsanov transformation $Q \to \hat{Q}$ is now identified and we can write

$$dW_t = (\bar{\gamma}_t + \bar{\sigma}_t)'dt + d\hat{W}_t$$

where $\hat{W}$ is $\hat{Q}$-Wiener.

(d) By applying this Girsanov transformation to S_t and Y_t, we obtain the following dynamics under $\hat{Q}$:

$$\begin{aligned} dS_t &= S_t[r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\gamma}_t']dt + S_t\bar{\gamma}_t dW_t \\ dY_t &= Y_t[r + (\bar{\gamma}_t + \bar{\sigma}_t)\bar{\sigma}_t']dt + Y_t\bar{\sigma}_t dW_t \end{aligned}$$

Hence, we obtain:

$$\begin{aligned} S_T &= S_0 \exp\left[\int_0^T (r + \sigma_t\gamma_t\rho + \frac{1}{2}\gamma_t^2)dt + \int_0^T \bar{\gamma}_t d\hat{W}_t\right] \\ Y_T &= Y_0 \exp\left[\int_0^T (r + \sigma_t\gamma_t\rho + \frac{1}{2}\sigma_t^2)dt + \int_0^T \bar{\sigma}_t d\hat{W}_t\right], \end{aligned}$$

which yields:

$$
\begin{aligned}
& S_T \geq K \\
& S_0 \exp\left[\int_0^T (r + \sigma_t\gamma_t\rho + \frac{1}{2}\gamma_t^2)dt + \int_0^T \bar{\gamma}_t d\hat{W}_t\right] \geq K \\
& \eta \geq \underbrace{\frac{\ln\frac{K}{S_0} - \int_0^T [r + \sigma_t\gamma_t\rho + \frac{1}{2}\gamma_t^2]dt}{\sqrt{\int_0^T \gamma_t^2 dt}}}_{-a_3} \\
& Y_T < D \\
& Y_0 \exp\left[\int_0^T (r + \sigma_t\gamma_t\rho + \frac{1}{2}\sigma_t^2)dt + \int_0^T \bar{\sigma}_t d\hat{W}_t\right] < D \\
& \xi < \underbrace{\frac{\ln\frac{D}{Y_0} - \int_0^T [r + \sigma_t\gamma_t\rho + \frac{1}{2}\sigma_t^2]dt}{\sqrt{\int_0^T \sigma_t^2 dt}}}_{b_3}
\end{aligned}
$$

Hence, by combining the previous results, we obtain:

$$
\begin{aligned}
\Pi_t(R_1) &= e^{-r(T-t)} m_T E^Q\left[R_T|\,\mathcal{F}_t\right] E^{\hat{Q}}\left[Z|\,\mathcal{F}_t\right] \\
&= e^{-r(T-t)} S_0 Y_0 \exp\left[\int_0^T (2r + \gamma_t\sigma_t\rho)ds\right] \frac{S_t Y_t}{S_0 Y_0} \\
&\quad \exp\left[-\int_0^t (2r + \gamma_t\sigma_t\rho)ds\right] \hat{Q}[S_T \geq K, Y_T < D] \\
&= e^{-r(T-t)} S_t Y_t \exp\left[\int_t^T (2r + \gamma_t\sigma_t\rho)ds\right] \hat{Q}(\eta \geq -a_3, \xi < b_3, \rho_2)
\end{aligned}
$$

where η and ξ are standard normal, ρ_1 is the correlation coefficient between the two standard normal variables and

$$
\begin{aligned}
a_3 &= -\frac{\ln\frac{K}{S_0} - \int_0^T [r + \sigma_t\gamma_t\rho + \frac{1}{2}\gamma_t^2]dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_3 &= \frac{\ln\frac{D}{Y_0} - \int_0^T [r + \sigma_t\gamma_t\rho + \frac{1}{2}\sigma_t^2]dt}{\sqrt{\int_0^T \sigma_t^2 dt}}
\end{aligned}
$$

We follow the same steps as before, and show by direct calculation that

$$
\rho_2 = -\frac{\rho\int_0^T \sigma_t\gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}}.
$$

Using the properties of the bivariate normal distribution, we transform the probability in the above formula for $\Pi_t(R_1)$ in a CDF and have:

$$\Pi_t(R_1) = S_t Y_t \exp\left[\int_t^T (r + \gamma_t \sigma_t \rho) ds\right] \mathcal{N}(a_3, b_3, -\rho_1)$$

At this point, we have obtained all the necessary information in order to price the recovery payoff for the vulnerable option. We are going to collect the last calculations by presenting them in the following proposition:

Proposition 2.2 *Let assumptions 3.1 and 4.1 hold. Then, the fair price for the recovery payoff,* Π_2*, is:*

$$\begin{aligned} \Pi_2 &= \frac{1-\beta}{D} S_0 Y_0 \exp\left[\int_0^T (r + \gamma\sigma\rho) ds\right] \mathcal{N}(a_3, b_3, -\rho_1) \\ &\quad - \frac{1-\beta}{D} K Y_0 \mathcal{N}(a_4, b_4, -\rho_1) \end{aligned}$$

where

$$\begin{aligned} a_3 &= \frac{\ln \frac{S_0}{p(0,T)K} + \int_0^T [\sigma_t \gamma_t \rho + \frac{1}{2}\gamma_t^2] dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\ b_3 &= \frac{\ln \frac{p(0,T)D}{Y_0} - \int_0^T [\sigma_t \gamma_t \rho + \frac{1}{2}\sigma_t^2] dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\ a_4 &= \frac{\ln \frac{S_0}{p(0,T)K} - \int_0^T [\sigma_t \gamma_t \rho] dt - \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\ b_4 &= -\frac{\ln \frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\ \rho_1 &= \frac{\rho \int_0^T \sigma_t \gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}} \end{aligned}$$

Collecting the results

So far, we were concerned with pricing vulnerable options in a complete market setup. In the previous subsections, we have obtained separate expressions for the payoff of a vulnerable option in the case of zero recovery and for the recovery payoff, respectively. For both expressions,we have started from the risk-neutral pricing formula and employed the change of numeraire. Now, we are going to collect the previous results into one formula.

Proposition 2.3 *Let assumptions 3.1 and 4.1 hold. Then, the price for a vulnerable option at time 0,Π, is given by:*

$$\begin{aligned}
\Pi_0 &= S_0\mathcal{N}[a_1, b_1, \rho_1] - Kp(0,T)\mathcal{N}[a_2, b_2, \rho_1] \\
&+\frac{1-\beta}{D}S_0 Y_0 \exp\left\{\int_0^T (r+\gamma_s\sigma_s\rho)ds\right\}\mathcal{N}[a_3, b_3, -\rho_1] \\
&-\frac{1-\beta}{D}KY_0\mathcal{N}[a_4, b_4, -\rho_1]
\end{aligned}$$

with:

$$\begin{aligned}
a_1 &= \frac{\ln\frac{S_0}{p(0,T)K} + \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_1 &= \frac{\ln\frac{Y_0}{p(0,T)D} + \int_0^T \gamma_t\sigma_t\rho dt - \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\
a_2 &= \frac{\ln\frac{S_0}{Kp(0,T)} - \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_2 &= \frac{\ln\frac{Y_0}{Dp(0,T)} - \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\
a_3 &= \frac{\ln\frac{S_0}{p(0,T)K} + \int_0^T [\sigma_t\gamma_t\rho + \frac{1}{2}\gamma_t^2]dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_3 &= \frac{\ln\frac{p(0,T)D}{Y_0} - \int_0^T [\sigma_t\gamma_t\rho + \frac{1}{2}\sigma_t^2]dt}{\sqrt{\int_0^T \sigma_t^2 dt}}
\end{aligned}$$

$$\begin{aligned}
a_4 &= \frac{\ln \frac{S_0}{p(0,T)K} - \int_0^T [\sigma_t \gamma_t \rho] dt - \frac{1}{2}\int_0^T \gamma_t^2 dt}{\sqrt{\int_0^T \gamma_t^2 dt}} \\
b_4 &= -\frac{\ln \frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \sigma_t^2 dt}{\sqrt{\int_0^T \sigma_t^2 dt}} \\
\rho_1 &= \frac{\rho \int_0^T \sigma_t \gamma_t dt}{\sqrt{\int_0^T \sigma_t^2 dt}\sqrt{\int_0^T \gamma_t^2 dt}}
\end{aligned}$$

2.3 Extensions to other products

Vulnerable exchange options

In the current subsection, I will show how one can easily extend the formula for a European call to price other options. The example used will be that of a vulnerable exchange option. An exchange option is a contract that gives the right, but not the obligation to exchange one stock for another. In this section, we will modify a bit the previous assumptions. We need to have a market consisting of two stock price processes. Also, the payoff function is being modified.
An exchange option has the payoff $\max[S_T^1 - S_T^2, 0]$. In its vulnerable form, the payoff of an exchange option becomes:

$$\mathcal{X} = \Phi(S_T^1, S_T^2, Y_T) = \max[S_T^1 - S_T^2, 0] I \{Y_T \geq D\} + \mathcal{R} I(Y_T < D)$$

where the recovery payoff, $\mathcal{R}$ is given by:

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T^1 - S_T^2, 0]$$

The assumptions needed in order to price a vulnerable exchange option are:

Assumption 2.3

1. *Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration given by the 3-dimensional P-Wiener process $\tilde{W}$, which is defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$
\begin{aligned}
dS_t^1 &= \alpha_1 S_t^1 dt + S_t^1 \bar{\gamma}_1 d\tilde{W}_t \\
dS_t^2 &= \alpha_2 S_t^2 dt + S_t^2 \bar{\gamma}_2 d\tilde{W}_t \\
dY_t &= \mu Y_t dt + Y_t \bar{\sigma} d\tilde{W}_t \\
dB_t &= B_t r dt
\end{aligned}
$$

where Y_t is denoting the assets of the counterparty underwriting the option, S_t^1 and S_t^2 the price processes of the stocks on which the option is contracted and B_t the bank account.

3. *In the equations above, μ, α_1 and α_2 be scalars, and $\bar{\sigma}$, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are (1,3) row vectors specified as follows:*

$$
\begin{aligned}
\bar{\gamma}_1 &= \left(\gamma^1, \quad 0,, \quad 0\right) \\
\bar{\gamma}_2 &= \left(\gamma^2 \rho_{12}, \quad \gamma^2 \sqrt{1-\rho_{12}^2}, \quad 0\right) \\
\bar{\sigma} &= \left(\sigma \rho_{13}, \quad \sigma \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad \sigma \sqrt{1-\rho_{13}^2 - \left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}\right]^2}\right)
\end{aligned}
$$

4. *$\tilde{W}$ is a three dimensional P-Wiener process:*

$$
\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2, \\ \tilde{W}^3 \end{pmatrix}
$$

with $\tilde{W}^1$, $\tilde{W}^2$ and $\tilde{W}^3$ being independent scalar P-Wiener processes.

5. *Assume that both the assets of the counterparty underwriting the option and the stock are traded on the market.*

Remark 2.1 *Note that, in this section, the model parameters μ, α_1, α_2, σ, γ_1, γ_2 are constants. This is done for notational convenience. In the case of time varying but deterministic coefficients, the calculations are easily extended, but become very messy.*

Since derivations are very similar to the ones in the previous section, we are going to summarize the results in the following proposition. A proof for the results follows.

Proposition 2.4 *Let Assumptions 4.2 hold. Then, the price for a vulnerable option at time zero,* $\Pi(0;\Phi)$*, is given by:*

$$\begin{aligned}\Pi(0;\Phi) &= S_0^1\mathcal{N}(a_1,b_1,\rho) - S_0^2\mathcal{N}(a_2,b_2,\rho)\\ &+\frac{1-\beta}{D}Y_0S_0^1\exp\left\{T(r+\gamma_1\sigma\rho_{13})ds\right\}\mathcal{N}(a_3,b_3,-\rho)\\ &-\frac{1-\beta}{D}Y_0S_0^2\exp\left\{T(r+\gamma_2\sigma\rho_{23})\right\}\mathcal{N}(a_4,b_4,-\rho).\end{aligned}$$

where

$$\begin{aligned}a_1 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+\frac{1}{2}T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}{\sqrt{T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}}\\ b_1 &= \frac{\ln\left(\frac{Y_0}{D}\right)+T(r+\sigma\gamma_1\rho_{13})-\frac{1}{2}T\sigma^2}{\sqrt{T\sigma^2}}\\ a_2 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)-\frac{1}{2}T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}{\sqrt{T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}}\\ b_2 &= \frac{\ln\left(\frac{Y_0}{D}\right)+T(r+\sigma\gamma_2\rho_{23})-\frac{1}{2}T\sigma^2}{\sqrt{T\sigma^2}}\\ a_3 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)+T[\sigma\gamma_1\rho_{13}-\sigma\gamma_2\rho_{23}+\frac{1}{2}(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)]}{\sqrt{T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}}\\ b_3 &= \frac{\ln\left(\frac{D}{Y_0}\right)-T[r+\gamma_1\sigma\rho_{13}+\frac{1}{2}\sigma^2]}{\sqrt{T\sigma^2}}\\ a_4 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right)-T[\sigma\gamma_1\rho_{13}-\sigma\gamma_2\rho_{23}+\frac{1}{2}(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)]}{\sqrt{T(\gamma_2^2-2\gamma_1\gamma_2\rho_{12}+\gamma_1^2)}}\\ b_4 &= \frac{\ln\left(\frac{D}{Y_0}\right)-T[r+\gamma_2\sigma\rho_{23}+\frac{1}{2}\sigma^2]}{\sqrt{T\sigma^2}}\\ \rho &= -\frac{\gamma_2\rho_{23}-\gamma_1\rho_{13}}{\sqrt{(\gamma_1)^2+(\gamma_2)^2-2\gamma_2\gamma_1\rho_{12}}}\end{aligned}$$

Proof. By a change of numeraire from the risk neutral measure Q to Q^2, the EMM with S_t^2 as numeraire, we obtain

$$\Pi(0;\Phi) = S_0^2 E^2 \left[\max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{Y_T \geq D\right\} + \frac{(1-\beta)Y_T}{D} \max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{Y_T < D\right\}\right]$$

We denote $\frac{S_t^1}{S_t^2}$ by Z_t and from standard theory we know that Z is a Q^2-martingale, i.e. it has a zero rate of return.
Our goal is now to apply the formula for the simple European call writen on Z_t, with strike $K = 1$ and local rate of return 0.
However, we cannot apply the previous result directly. In the formula for a vulnerable option, both the underlying stock and the assets of the counterparty have the same rate of return r. Under Q^2, the underlying stock has zero rate of return, but the assets of the counterparty do not. From the standard Girsanov transformation, we obtain the following dynamics for Y_t:

$$dY_t = (r + \gamma_2 \sigma \rho_{23}) Y_t dt + \bar{\sigma} dW_t^2$$

We denote $r + \gamma_2 \sigma \rho_{23} = c$, and we can re-write Y_t as:

$$Y_t = e^{ct} \tilde{Y}_t$$

where $\tilde{Y}_t$ is defined below.

Definition 2.2 *We define $\tilde{Y}_t$ as the stochastic process given by*

$$\tilde{Y}_t = Y_t \exp\left\{-(r + \gamma_2 \sigma \rho_{23}) T\right\}.$$

The point of this is that $\tilde{Y}_t$ is a Q^2-martingale. Thus the price of the vulnerable exchange option can be written as:

$$\begin{aligned} \Pi(0;\Phi) &= S_0^2 E^2 \left[\max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{\tilde{Y}_T \geq e^{-cT} D\right\}\right] \\ &+ S_0^2 e^{cT} E^2 \left[\frac{(1-\beta)\tilde{Y}_T}{D} \max\left[\frac{S_T^1}{S_T^2} - 1, 0\right] I\left\{\tilde{Y}_T < e^{-cT} D\right\}\right] \end{aligned}$$

Since both Z and $\tilde{Y}$ are Q^2-martingales, we obtain the price of a vulnerable exchange option simply by tranfering the result for the price vulnerable European call option, written on Z_t, with strike 1, local rate of return 0; the assets of the counterparty are given by $\tilde{Y}_t$ and the default barrier becomes $e^{-cT} D$. The only thing we will need to calculate the correlation coefficient

$\tilde{\rho}$ between Z_t and Y_t. A reasoning similar to the one in the proof of lemma (2.1) gives us:

$$\tilde{\rho} = Corr[(\bar{\gamma}_2 - \bar{\gamma}_1)W_T^1, \bar{\sigma}W_T^1]$$

By applying the definition of the correlation coefficient, one obtains:

$$\tilde{\rho} = \frac{\gamma_2\rho_{23} - \gamma_1\rho_{13}}{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - 2\gamma_2\gamma_1\rho_{12}}} \tag{2.15}$$

Thus, by transfering results from Proposition 2.3, we obtain the price for the vulnerable exchange option given in Proposition 3.4. ∎

Linearly Homogeneous Payoffs

Let Assumption 4.2 hold. In this section, we extend results from section 2.3 to a more general class of contracts. More specifically, we now consider a T-claim $X = \Phi(S_T^1, S_T^2)$. In order to do so, we need a homogeneity assumption.

Assumption 2.4 *We assume $\Phi(x, y)$ is a linearly homogenous function, i.e.*

$$\Phi(\lambda x, \lambda y) = \lambda\Phi(x, y), \forall\lambda \geq 0$$

Furthermore, we define the contract function Υ by

$$\Upsilon(z) = \Phi(z, 1)$$

A well known result in mathematical finance relates the non-vulnerable pricing problem of Φ to the simpler problem of pricing Υ. We would like to see if it is possible to find such a relation between vulnerable versions of the contracts defined above. We denote the vulnerable version of the contract function $\Phi(S_t^1, S_t^2)$ by $\Phi^V(S_t^1, S_t^2, Y_t)$ and the vulnerable version of the contract function $\psi(S_t)$ by $\psi^V(S_t, Y_t)$. In general, the vulnerable version of a contract function $F(x)$, denoted by $F^V(x, y)$ is given by:

$$F^V(x, y) = F(x)I\{y \geq D\} + \frac{(1-\beta)y}{D}F(x)I\{y < D\}$$

By applying the risk neutral valuation formula to the claim $X^V = \Phi^V(S_t^1, S_t^2, Y_t)$, we obtain the following expression for the price of the claim, $\Pi(0, X^V)$:

$$\begin{aligned}\Pi(0, X^V) &= e^{(-rT)}E^Q\left[\Phi^V(S_t^1, S_t^2, Y_t)\right] = S_t^2E^2\left[\Phi\left(\frac{S_T^1}{S_T^2}, 1\right)I\{Y_T \geq D\}\right] \\ &+ S_t^2E^2\left[\frac{(1-\beta)Y_T}{D}\Phi\left(\frac{S_T^1}{S_T^2}, 1\right)I\{Y_T < D\}\right]\end{aligned}$$

where $E^2[\bullet]$ is the expectation operator taken under the equivalent martingale measure Q^2 where S^2 is numeraire.
The present argument follows the same lines as the reasoning outlined in the previous section. We denote $\frac{S_t^1}{S_t^2}$ by Z_t. Under Q^2, Z is a martingale, and has a zero rate of return. In order to obtain a similar calculation formula to the one used in the non-vulnerable claims case, we need Y_t also to be a Q^2-martingale. Since this is not the case, we rewrite Y_t as:

$$Y_t = \tilde{Y}_t e^{ct}$$

where $c = r + \gamma_2 \sigma \rho_{23}$ and the process $\tilde{Y}_t$ is defined in Definition 2.2. We remember that $\tilde{Y}_t$ is a martingale under Q^2.
Thus, we can write the price of the claim X^V, as

$$\begin{aligned}
\Pi(0, X^V) &= S_t^2 E^2 \left[\Phi(Z_T, 1) I \left\{ \tilde{Y}_T \geq De^{-cT} \right\} \right] \\
&+ S_t^2 E^2 \left[\frac{(1-\beta)\tilde{Y}_T}{De^{-cT}} \Phi(Z_T, 1) I \left\{ \tilde{Y}_T < De^{-cT} \right\} \right] \\
&= S_t^2 E^2 \left[\Upsilon(Z_T, \tilde{Y}_T) \right]
\end{aligned}$$

where the default barrier for the vulnerable claim $\mathcal{X}^V = \Upsilon(Z_T, \tilde{Y}_T)$ is De^{-cT}. The result is summarized in the following proposition. Again, we reduce the problem of pricing a contract written on two assets S^1 and S^2 to the pricing problem of a contract written for a single asset, Z.

Proposition 2.5 *Let Assumptions 4.2 and 2.4 hold. Then, we have the following equivalence between two pricing problems:*

$$\Pi[0, \Phi^V(S_T^1, S_T^2, Y_T)] = S_t^2 \Pi[0, \Upsilon^V(Z_T, \tilde{Y}_T)]$$

where Z_t and Y_t are defined as above. The claim Υ^V is prices in a world of zero local return and the default barrier for the vulnerable claim $\mathcal{X}^V = \Upsilon(Z_T, \tilde{Y}_T)$ is De^{-cT}.

Barrier Options

In this subsection, we are going to show how one can use the formula for a vulnerable European call to price a vulnerable barrier option. In order to be more concrete, we are going to price a vulnerable down-and-out barrier option. Let assumptions 3.1 and 4.1 hold. The payoff of a down-and-out barrier option is

$$C_{LO} = \begin{cases} \max[S_T - K, 0], & \text{if } S_t > L \text{ for all } 0 < t < T \\ 0, & \text{if } S_t \leq L \text{ for some } 0 < t < T \end{cases}$$

If the down-and-out barrier option is not vulnerable, it can be shown (see Björk (2004)) that the price of the barrier option is given by:

- If the barrier L is lower than the strike price K, $L \leq K$:

$$c_{LO}(t, s, K) = c(t, s, K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma}} c(t, \frac{L^2}{s}, K)$$

  where $c_{LO}(t, s, K)$ is the price of the down-and-out barrier option with strike K, barrier L, evaluated at the initial point $S_t = s$; $c(t, s, K)$ is the price of a European call with strike K, evaluated at the initial point $S_t = s$; $c(t, \frac{L^2}{s}, K)$ is the price of a European call with strike K, evaluated at the initial point $S_t = \frac{L^2}{s}$; $\tilde{r} = r - \frac{1}{2}\gamma_2$

- If the barrier L is higher than the strike price K, $L > K$:

$$c_{LO}(t, s, K) = g(t, s, L, K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma}} g(t, \frac{L^2}{s}, L, K)$$

  where

$$g(t, z, L, K) = c(t, z, L) + (L - K)h(t, z, L)$$

  and $h(t, z, L)$ is the price generated by a digital option that pays 1 if $S_T > L$ and 0 otherwise.

We will un fact prove below that that the same relationship holds for vulnerable down-and-out options, with the only difference that the prices used before are the prices of corresponding vulnerable European calls and vulnerable digital options. In order to do so, we are need to introduce the following elements:

Definition 2.3 *For any process X and real number L, the process X_L denotes the process X with (possible) absorbtion at L*

Definition 2.4 *For any function $\Phi(x, y)$, we define the function $\Phi_L(x, y)$ as*

$$\Phi_L(x, y) = \begin{cases} \Phi(x, y), & x > L \\ 0, & x \leq L. \end{cases}$$

Definition 2.5 *Let X be a simple T-claim with contract function $\Phi(S_T)$. Then, we denote by*

- X^V *the corresponding vulnerable claim, with contract function* $\Phi^V(S_T, Y_T)$
- X_{LO} *the corresponding down and out claim,*
- X_{LO}^V *the corresponding vulnerable down-and-out claim.*

Next, we are going to show how to link the price of a vulnerable down-and-out barrier derivative to the price of the corresponding vulnerable derivative. Let the payoff of the derivative to be priced be $\Psi(S_T)$, then the payoff of the vulnerable version of the same derivative $\Psi^V(S_T, Y_T)$ is:

$$\begin{aligned}\Psi^V(S_T, Y_T) &= \Psi(S_T)I\{Y_T \geq D\} + \frac{(1-\beta)Y_T}{D}\Psi(S_T)I\{Y_T < D\} \\ &= \Psi(S_T)[I\{Y_T \geq D\} + \frac{(1-\beta)Y_T}{D}I\{Y_T < D\}]\end{aligned}$$

Thus, we see that the payoff function for a vulnerable claim can be written as:

$$\Psi^V(S_T, Y_T) = \Psi(S_T)F(Y_T),$$

where

$$F(Y_T) = I\{Y_T \geq D\} + \frac{(1-\beta)Y_T}{D}I\{Y_T < D\}.$$

This, in turn, allows us to derive the following expression for the price of the down-and-out barrier claim at time 0, $\Pi(0, \Psi_{LO}^V)$:

$$\begin{aligned}\Pi(0, \Psi_{LO}^V) &= e^{-rT}E_{0,s}^Q[\Phi_{LO}] = e^{-rT}E_{0,s}^Q\left[\Psi^V(S_T, Y_T)I\left\{\inf_{0\leq t\leq T} S_t > L\right\}\right] \\ &= e^{-rT}E_{0,s}^Q\left[\Psi_L^V(S_L(T), Y_T)I\left\{\inf_{0\leq t\leq T} S_t > L\right\}\right] \\ &= e^{-rT}E_{0,s}^Q\left[\Psi_L^V(S_L(T), Y_T)\right]\end{aligned}$$

where S_L denotes the process S with absorbtion at L. From standard theory, it is easy to derive the following equation for S:

$$S_t = \exp\{X_t\}$$

where the process X is given by:

$$\begin{aligned}dX &= (r - \frac{1}{2}\gamma^2)dt + \bar{\gamma}dW \\ X_0 &= \ln s\end{aligned}$$

We remember that before we have denoted $r - \frac{1}{2}\gamma^2$ by $\tilde{r}$.
This allows us to write $S_L(T)$ as:

$$S_L(T) = \exp\left\{X_{\ln L}(T)\right\}$$

where X_{lnL} denotes the process X with absorbtion at $\ln L$. We can perform the same exercise for Y:

$$Y_t = \exp\left\{Z_t\right\}$$

where the process Z is given by:

$$\begin{aligned} dZ &= \underbrace{(r - \frac{1}{2}\sigma^2)}_{\bar{r}}\, dt + \bar{\sigma} dW \\ Z_0 &= \ln y \end{aligned}$$

We denote the density of the the stochastic variable $\big(X_{lnL}(T),\ \ Y(T)\big)$ by $f(x,z)$. The proposition B.1 tells us that $f(x,z)$ is a combination of bivariate normal densities:

$$\begin{aligned} & f(x,z) = \mathbf{n}_{x,z}[(\ln s + \tilde{r}\sqrt{T}, \ln y + \bar{r}\sqrt{T}), (\gamma\sqrt{T}, \sigma\sqrt{T}), \rho] \\ & - \exp\left\{-\frac{2\tilde{r}(\ln s - \ln L)}{\gamma^2}\right\} \times \\ & \mathbf{n}[(2\ln L - \ln s + \tilde{r}\sqrt{T}, \ln y + 2\rho(\ln L - \ln s) + \bar{r}\sqrt{T}), (\gamma\sqrt{T}, \sigma\sqrt{T}), \rho] \end{aligned}$$

where we keep the same notation as in Appendix B.
We introduce the above result in the pricing formula for a down-and-out vulnerable claim X^V_{LO} with payoff Ψ^V_{LO}. Thus, we obtain:

$$\begin{aligned} \Pi(0, \Psi^V_{LO}) &= e^{-rT} E^Q_{0,s,y}\left[\Psi^V_{LO}\right] \\ &= e^{-rT} E^Q_{0,s,y}\left[\Psi^V_L(S_L(T), Y_T)\right] \\ &= e^{-rT} E^Q_{0,x,z}\left[\Psi^V_L(\exp\left\{X_{lnL}(T)\right\}, \exp\left\{Z_T\right\})\right] \\ &= e^{-rT}\left\{E^Q_{0,s,y}\left[\Psi^V_L(S_T, Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s}, y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi^V_L(S_T, Y_T)\right]\right\}. \end{aligned}$$

Proposition 2.6 *Let Assumptions 3.1 and 4.1 hold. Then, the price of a vulnerable down-and-out claim Ψ^V_{LO} is given by:*

$$\Pi(0, \Psi^V_{LO}) = e^{-rT}\left\{E^Q_{0,s,y}\left[\Psi^V_L(S_T, Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s}, y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi^V_L(S_T, Y_T)\right]\right\}. \tag{2.16}$$

The above proposition links the pricing of a down-and-out barrier claim to the much simpler problem of pricing a certain simple claim. The vulnerable component does not play a role when setting the criteria for the barrier. Hence, if $\Psi^V(S_T, Y_T, K)$ is the payoff for a vulnerable European call with strike K, $\Psi_L^V(S_T, Y_T, K)$ is given by:

- for $L < K$,
$$\Psi_L^V(S_T, Y_T) = \Psi^V(S_T, Y_T, K)$$
- for $L > K$,
$$\Psi_L^V(S_T, Y_T) = \Psi^V(S_T, Y_T, L) + (L - K)H^V(S_T, Y_T, L)$$
where $H^V(S_T, Y_T, L)$ denotes the payoff of a vulnerable digital option that pays 1 if $S_T > L$ and 0 otherwise.

One can show that ,in complete markets, the price of a vulnerable digital option,denoted by $\mathcal{H}$ below, is:

$$\mathcal{H} = p(0,T)\mathcal{N}[-a_1, -b_1, \rho] + \frac{1-\beta}{D} Y_0 \mathcal{N}[-a_2, -b_2, \rho] \tag{2.17}$$

with:

$$\begin{aligned}
a_1 &= \frac{\ln \frac{Kp(0,T)}{S_0} + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_1 &= \frac{\ln \frac{Dp(0,T)}{Y_0} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\
a_2 &= \frac{\ln \frac{K}{S_0} + \int_0^T [\alpha_t + \varphi_t^y \bar{\gamma}_t] dt + \frac{1}{2}\int_0^T \|\bar{\gamma}_t\|^2 dt}{\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}} \\
b_2 &= \frac{\log \frac{Y_0}{p(0,T)D} + \frac{1}{2}\int_0^T \|\bar{\sigma}_t\|^2 (t) dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}} \\
\rho &= \frac{\rho \int_0^T \sigma_t \gamma_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}
\end{aligned}$$

By combining the results above, we obtained the result we looked for.

Proposition 2.7 *Let Assumptions 3.1 and 4.1 hold. Then, the price of a vulnerable down-and-out option $c^V_{LO}(s, y, K, L)$ is given by:*

- *If the barrier L is lower than the strike price K, $L \leq K$:*

$$c^V_{LO}(t, s, K, L) = c^V_y(t, s, K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma}} c^V_{y\left(\frac{L}{s}\right)^{2\rho}}(t, \frac{L^2}{s}, K)$$

  where K is the strike price of the option and L is the barrier; $c(t, s, K)$ is the price of a European call with strike K, evaluated at the initial point $S_t = s$; $c(t, \frac{L^2}{s}, K)$ is the price of a European call with strike K, evaluated at the initial point $S_t = \frac{L^2}{s}$; the subscript denotes the initial value to be taken for the assets of the counterparty when evaluating the vulnerable option.

- *If the barrier L is higher than the strike price K, $L > K$:*

$$c^V_{LO} = g^V_y(t, s, L, K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma}} g^V_{y\left(\frac{L}{s}\right)^{2\rho}}(t, \frac{L^2}{s}, L, K)$$

  where

$$g^V_x(t, z, L, K) = c^V_x(t, z, L) + (L - K)h^V x(t, z, L)$$

  and $h^V_x(t, z, L)$ is the price of a vulnerable digital option that pays 1 if $S_T > L$ and 0 otherwise.

The prices $c^V(t, z, L)$ and $h^V(t, z, L)$ are given by Proposition 2.3 and equation (2.17). Notice that results in Propositions 2.6 and 2.7 are very similar with results obtained for non-vulnerable down-and-out claims.

3 Incomplete Markets and Good Deal Bounds

One of the main limitations of the previous approach is the assumption that the assets of the counterparty, or the default "trigger", are liquidly traded on the market. It is a stong assumption, which allows us to obtain a unique price for the vulnerable option. If both the stock and the assets of the counterparty are traded on the market, we have a complete market model and, hence, a unique price.
However, if the assets of the counterparty are not liquidly traded, we are not in a complete market setup, and hence, we are not entitled to use the formula derived in the previous section. One of the consequences of having an

incomplete market setup is the fact that we no longer have a unique EMM, and consequently not a unique price. One could simply calculate the bounds of the prices, generated by the interval of all possible risk-neutral measures. These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.
Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterium, chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature dealing with this approach see Schweizer (2001), Henderson and Hobson (2004), Barrieu and Karoui (2005) However, there is no clear cut way of choosing between different criteria and some of them are somewhat ad-hoc, in the sense that they do not have a clear economic interpretation.
In contrast to this, Cochrane -Saa-Requejo proposed in Cochrane and Saa-Raquejo (2000), the method of good deal bounds. The good deal approach aims at obtaining an interval of "reasonable" prices in incomplete markets, rather than concentrating at obtaining a unique price. Since the no-arbitrage bounds are too large to be used, Cochrane and Saa-Raquejo (2000) propose to rule out not only arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered "too good to be true", hence the name of "good deal bounds". One possible measure for the "goodness" of a deal is its Sharpe Ratio, and thus, trades/portfolios which have a Sharpe Ratio (SR) above a certain threshold are eliminated. The SR is chosen as a measure for the "goodness of the deal" because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Thus, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.
The next step in developing a theory for "good deal bounds" was done by Björk and Slinko (2005). They proposed a new frame for solving the optimization problem defined by Cochrane and Saa-Raquejo (2000) while at the same time allowing for more complex dynamics for the underlying assets, such as jump-diffusion processes, to be taken into account.

3.1 Setup

First, we will consider the classical structural model, dropping only the market completeness assumption. The model is identical to the one presented in the previous section, except for one feature. The assumption that the assets of the counterparty are traded on the market is dropped. We make the following assumptions:

Assumption 3.1

1. *Let* $(\Omega, \mathcal{F}, P, \mathbf{F})$ *be given.*

2. *The market model is given by the following dynamics under the objective probability measure* P*.*

$$\begin{aligned} dY_t &= \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t, \\ dS_t &= \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t, \\ dB_t &= B_t r dt. \end{aligned}$$

   Here Y_t *is denoting the assets of the counterparty underwriting the option,* S_t *the price of the stock on which the option is contracted and* B_t *the bank account.*

3. μ_t *and* α_t *are scalar deterministic functions of time,* $\bar{\sigma}_t$ *and* $\bar{\gamma}_t$ *are positive deterministic functions of time specified as follows:*

$$\begin{aligned} \bar{\gamma}_t &= \begin{pmatrix} \gamma_t, & 0 \end{pmatrix} \\ \bar{\sigma}_t &= \begin{pmatrix} \sigma_t \rho, & \sigma_t \sqrt{1-\rho^2} \end{pmatrix} \end{aligned}$$

4. $\tilde{W}$ *is a two dimensional* P*-Wiener process:*

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2 \end{pmatrix}$$

   with $\tilde{W}^1$ *and* $\tilde{W}^2$ *being independent scalar* P*-Wiener processes.*

5. *We assume that the assets of the counterparty underwriting the option are not traded on the market and that the stock is traded.*

6. *The payoff of a vulnerable European call option,* $X = \Phi(S_T, Y_T)$*, is given by*

$$X = \Phi(S_T, Y_T) = \max(S_T - K, 0) I(Y_T \geq D) + \mathcal{R} I(Y_T < D)$$

   where D *is the total value of the claims against the counter-party.*

7. *Recovery payoff is given by:*

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T - K, 0]$$

Notice that the above assumptions are identical to assumptions 3.1 and 4.1, with the exception of point 5, which leads to market incompleteness.

Q-dynamics:

Since we are in an incomplete market set-up, we do not have a unique equivalent martingale measure (EMM), but a whole class of EMM. For any potential EMM $Q \sim P$ we define the corresponding likelihood process L by:

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \tag{2.18}$$

Since $\mathcal{F}_t = \mathcal{F}_t^W$, L_t must have dynamics of the form:

$$\begin{aligned} dL_t &= L_t \varphi_t' d\tilde{W}_t && (2.19) \\ L_0 &= 1 && (2.20) \end{aligned}$$

where $\varphi_t = \left(\varphi_t^1, \quad \varphi_t^2\right)'$ is adapted to $\underline{\mathcal{F}}$. From Girsanov's theorem, it follows that:

$$d\tilde{W}_t = \varphi_t dt + dW_t$$

where W_t is a Q-Wienner process.
Thus, the dynamics of the two assets under the potential martingale measure Q are:

$$\begin{aligned} dY_t &= (\mu_t + \bar{\sigma}_t \varphi_t) Y_t dt + Y_t \sigma_t dW_t \\ dS_t &= (\alpha_t + \bar{\gamma}_t \varphi_t) S_t dt + S_t \gamma_t dW_t \\ dB_t &= B_t r dt \end{aligned}$$

Since S_t is a traded asset, its drift must equal the risk free interest rate under an equivalent martingale measure. Thus, in order for Q to be a martingale measure, φ has to satisfy the **martingale condition**:

$$r = \alpha_t + \bar{\gamma}_t \varphi_t \tag{2.21}$$

i.e

$$r = \alpha_t + \gamma_t \varphi_t^1 \tag{2.22}$$

The martingale condition does not determine a unique Girsanov kernel φ_t, but only the first term of the φ_t. Thus we do not have a unique equivalent martingale measure, but we obtain a class of martingale measures. They are defined as the class of measures obtained by (2.18)- (2.20) and satisfying the martingale condition (2.21).

3.2 Optimization Problem

As mentioned before, the "good deal bound" valuation framework rests on the idea of placing constraints on the Sharpe ratio of the claim to be priced. The problem becomes that of finding the highest and the lowest arbitrage free price processes, subject to a constraint on the maximum Sharpe Ratio (SR). However, if we want to be consistent, we should look for a framework which allows us to place an upper bound on the SR not only of the derivative unde rconsideration, but also of all the **portfolios** that can be formed on the market consisting of the underlying assets, the derivative claim and the money account. It then turns out that binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds.
An extended version of the Hansen Jaganathan bounds is derived and proven in Björk and Slinko (2005). This inequality provides the bounds for the Sharpe ratio of the assets on the market, as well as for all derivatives and self financing portfolios formed on the market, and reads as follows:

$$|SR_t|^2 \leq \|\lambda_t\|^2.$$

Here we denote by λ_t the market price of risk and by SR_t the Sharpe ratio on a particular asset derivative or self financing portfolio on the market; $\| \bullet \|$ stands for the Euclidian norm. As we can see, the Sharpe ratio is bounded by the norm of the price of risk on the market. Standard theory gives us the relationship between the Girsanov kernel, φ_t, and the market price of risk:

$$\varphi_t = -\lambda_t.$$

Thus, our pricing problem can be reformulated as follows: we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk or equivalently, a bound on the Girsanov kernel φ_t for every t. Dealing with the market price of risk translates to dealing with the Girsanov kernel of the equivalent martingale measures.

Following the above reasoning, we can now define the good deal bounds.

Definition 3.1 *The **upper good deal bound** price process for a vulnerable option is defined the optimal value process for the following optimal control problem:*

$$\begin{aligned}\max_{\varphi} \quad & E^Q[e^{-r(T-t)}(\max[S_T - K, 0]I\{Y_T \geq D\} + \mathcal{R}I\{Y_T \leq D\})] \\ & dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t \\ & dS_t = rS_t dt + S_t\bar{\gamma}_t dW_t \\ & \alpha_t + \bar{\gamma}_t\varphi_t = r \\ & \|\varphi_t\|^2 \leq C^2\end{aligned}$$

*The **lower good deal bound** process is the optimal value process for a similar optimal control problem, with the only difference that we minimize the expression, subject to the same constraints.*
*We denote the optimal value process by $V(t, S_t, Y_t)$, where V is the **optimal value function**.*

Before proceeding, let us comment on the structure of the optimization problem. The objective function is the arbitrage-free price for the payoff function, where the expectation is computed under the risk neutral measure generated by φ. Since we have to select this measure from a continuum of eligible EMM, we maximize with respect to the Girsanov kernel φ.
The optimization is subject to the dynamics of the assets on the market, under the appropiate probability measure.
The constraints:

$$\begin{aligned} dS_t &= rS_t dt + S_t\bar{\gamma}_t dW_t \\ \alpha_t + \bar{\gamma}_t\varphi_t &= r \end{aligned}$$

are the usual constraints on the drift of the traded assets on the market that establish the probability measure as a risk neutral measure.
If all the elements of φ could be identified from these constraints, we would be in a complete market setup and would be able to find a unique price. Since the number of traded assets is smaller than the number of risk sources, we cannot price all the risk factors and need the last inequality in order to tighten the no arbitrage price bounds. We will refer to this inequality:

$$\|\varphi_t\|^2 \leq C^2, \quad 0 \leq t \leq T$$

as the good deal bounds condition.

3.3 The Hamilton Jacobi Bellman equation

The optimization problem stated above is a standard stochastic optimal control problem and we will solve it with the aid of the Hamilton Jacobi Bellman equation. We restrict ourselves to the case when the market price of risk depends only on the stock and the assets of the counterparty; thus, we have $\varphi_t = \varphi(t, S_t, Y_t)$
According to the general theory of dynamic programing, the optimal value function satisfies the following PDE, also known as the Hamilton Jacobi Bellman equation, where $\mathcal{A}$ denotes the infinitesimal operator for (S, Y).

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,s,y) + \sup_{\varphi} \mathcal{A}V(t,s,y) - rV(t,s,y) = 0 \\ &V(T,s,y) = \Phi(s,y). \end{aligned}$$

Here

$$\Phi(s,y) = \max(s-K,0)I(y \geq D) + \mathcal{R}(s,y)I(y \leq D)$$

and

$$\mathcal{R}(s,y) = (1-\beta)\frac{y}{D}\max[s-K,0]$$

The infinitesimal operator is given by:

$$\begin{aligned} \mathcal{A}V &= \frac{\partial V}{\partial s} rs + \frac{\partial V}{\partial y}(\mu_t + \bar{\sigma}_t \varphi_t) y \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2} s^2 \bar{\gamma}_t \bar{\gamma}_t\prime + \frac{1}{2}\frac{\partial^2 V}{\partial y^2} y^2 \bar{\sigma}_t \bar{\sigma}_t\prime + \frac{\partial^2 V}{\partial s \partial y} sy \bar{\gamma}_t \bar{\sigma}_t\prime \end{aligned}$$

The first step in solving the PDE is to solve for each t, s, y the embedded static maximization problem, corresponding to $sup\mathcal{A}V$ subject to constraints. In our case, for fixed t, s, y, the static problem takes the form:

$$\max_{\varphi} \quad \frac{\partial V}{\partial y}\sigma\varphi y \tag{2.23}$$

$$\alpha + \bar{\gamma}\varphi = r \tag{2.24}$$

$$\|\varphi\|^2 \leq C^2 \tag{2.25}$$

We notice that the above problem is in fact a linear optimization problem and therefore, the solution will be a boundary solution. Thus, both constraints are binding. Since the Girsanov kernel φ is a (2,1) column vector, by solving the system of equations:

$$\begin{aligned} &\alpha + \bar{\gamma}\varphi = r \\ &\|\varphi\|^2 = C^2 \end{aligned}$$

we obtain:

$$\hat{\varphi}(t,s,y)' = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \pm\sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right) \tag{2.26}$$

Thus, we have two candidates for the optimal φ and it remains to determine which is the optimal one. Since our objective function is linear in φ:

$$\frac{\partial V}{\partial y}\sigma\varphi y$$

and σ and y are positive by assumption, we need to investigate the sign of $\frac{\partial V}{\partial y}$ in order to decide which of the possible Girsanov kernels we choose.

Lemma 3.1 *Under assumptions 3.1 and if φ does not depend on s and y, we have*

$$\frac{\partial V}{\partial y} \geq 0 \tag{2.27}$$

Proof. We are going to prove that $\frac{\partial V}{\partial y} \geq 0$, or, equivalently, that the value function is increasing in y. We do this by first showing that the payoff function is increasing in y. Then we prove that this implies that the associated pricing function is increasing in y, and hence, the optimal value function is too.

To see that the payoff function $\Phi(s,y)$ is non-decreasing in y, we note that for $y \geq D$,

$$\Phi(s,y) = \max(s-K,0),$$

which does not depend on the value of y, hence, it is non-decreasing in y. For $y < D$, the payoff function is

$$\Phi(s,y) = \mathcal{R}(s,y) < \max(s-K,0)$$

and thus, $\Phi(s,y)$ is non-decreasing as $y = D_-$. Also, the recovery payoff is a linear function of the assets of the counterparty,

$$\mathcal{R}(s,y) = (1-\beta)\frac{y}{D}\max[s-K,0],$$

with the coefficient of y positive. Hence, $\Phi(s,y)$ is non-decreasing in y.

Let $\Pi^Q(t,s,y)$ be a pricing function, i.e.

$$\Pi^Q(t,s,y) = E^Q[e^{-r(T-t)}\Phi[S_T,Y_T]|S_t = s, Y_t = y]$$

where Q is some admisible EMM.
We now want to prove that if the payoff function $\Phi(s, y)$ is increasing in y and the Girsanov kernel is a deterministic function of time

$$\varphi(t, s, y) = \varphi(t),$$

then also the pricing function $\Pi^Q(t, s, y)$ is increasing in the variable y.
We solve the SDE giving the dynamics of Y_t under Q:

$$dY_t = (\mu_t + \bar{\sigma}_t \varphi_t) Y_t dt + Y_t \bar{\sigma}_t dW_t$$

and obtain the following formula for Y_T, given $Y_t = y$:

$$Y_T = y \exp\left(\int_t^T \left[\mu_t + \bar{\sigma}_t \varphi_t - \frac{1}{2}\sigma_t^2 \right] dt + \int_t^T \bar{\sigma}_t dW_t \right)$$

Thus, for a given φ which does not depend on s and y, we can write $Y_T = yZ$, where Z is a lognormal variable that does not depend on y.
One can easily see that if $\Phi(s, y)$ is increasing in the second variable, than also the pricing function $\Pi^Q(t, s, y)$is increasing in the variable y.
In our case, we know that

$$V = \Pi^Q$$

when Q is generated by $\hat{\varphi}$. Since we see from (2.26) does not depends on s and y, we conclude that $\Pi^Q(t, s, y)$ and thus V is nondecreasing in y. ∎

In conclusion, the optimal Girsanov kernel is:

$$\hat{\varphi}_t = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r - \alpha_t}{\gamma_t})^2} \right)'$$

Proposition 3.1 *Under assumptions 3.1, the Girsanov kernel corresponding to the upper good deal bound EMM is*

$$\hat{\varphi}'_{max} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r - \alpha_t}{\gamma_t})^2} \right)$$

The Girsanov kernel for the lower good deal bound EMM is given by

$$\hat{\varphi}'_{min} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{B^2 - (\tfrac{r - \alpha_t}{\gamma_t})^2} \right)$$

3.4 A closed form solution

Introducing the result above in the HJB equation, we obtain:

$$\begin{aligned} V_t + V_s rs + V_y(\mu_t + \bar{\sigma}_t \hat{\varphi}_t)y + \frac{1}{2}V_{ss}s^2\bar{\gamma}_t\prime\bar{\gamma}_t & \\ +\frac{1}{2}V_{yy}y^2\bar{\sigma}_t\bar{\sigma}_t + +V_{sy}sy\bar{\gamma}_t\prime\bar{\sigma}_t - rV &= 0 \\ V(T,s,y) &= \Phi(T,s,y) \end{aligned}$$

where $\hat{\varphi}_t \in \{\hat{\varphi}_{min}, \hat{\varphi}_{max}\}$.
By applying Feynman Kac to the above equation, we have obtained the following formula:

$$\Pi(t,s,y) = V(t,s,y) = E^{\hat{Q}}\left[e^{-r(T-t)}\Phi(s,y)\middle|\mathcal{F}_t\right] \tag{2.28}$$

where is $\hat{Q}$ is defined by the Radon Nykodim derivative:

$$\begin{aligned} L_t &= \frac{d\hat{Q}}{dP} \\ dL_t &= L_t\hat{\varphi}_t' dW_t \end{aligned}$$

where $\hat{\varphi}_t \in \{\hat{\varphi}_{min}, \hat{\varphi}_{max}\}$. By using the change of numeraire and similar techniques to the ones presented in the section dealing with good-deal bounds, one can obtain a closed form solution for the price of vulnerable options in incomplete markets. We present the result below, followed by the proof.

Proposition 3.2 (Incomplete markets) *Let assumptions 3.1 hold. The upper good deal bound price of a vulnerable option is given by:*

$$\begin{aligned} \Pi(t) &= S_t\mathcal{N}[a_1,b_1,\rho_2] - e^{-r(T-t)}K\mathcal{N}[a_2,b_2,\rho_2] \\ &+ \frac{1-\beta}{D}S_tY_t\exp\left\{\int_t^T[\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \sigma_s\gamma_s\rho]\,ds\right\}\mathcal{N}[-a_3;b_3;-\rho_2] \\ &- e^{-r(T-t)}\frac{K(1-\beta)}{D}Y_t\exp\left\{\int_t^T(\mu_s + \bar{\sigma}_s\hat{\varphi}_s)ds\right\}\mathcal{N}(a_4,b_4,-\rho_2) \end{aligned}$$

where

$$
\begin{aligned}
a_1 &= \frac{\ln\frac{S_t}{K} + \int_t^T \left\{r + \frac{1}{2}\gamma_s^2\right\} ds}{\sqrt{\int_t^T \gamma_s^2 ds}} \\
b_1 &= \frac{\ln\frac{Y_t}{D} + \int_t^T \left[\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \sigma_s\gamma_s\rho - \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}} \\
a_2 &= \frac{\ln\frac{S_t}{K} + r(T-t) - \frac{1}{2}\int_t^T \gamma_s^2 ds}{\sqrt{\int_t^T \gamma_s^2 ds}} \\
b_2 &= \frac{\ln\frac{Y_t}{D} + \int_t^T \left[\mu_s + \bar{\sigma}_s\hat{\varphi}_s - \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}} \\
a_3 &= \frac{\ln\frac{S_t}{K} + \int_t^T \left\{r + \frac{1}{2}\gamma_s^2 + \sigma_s\gamma_s\rho\right\} ds}{\sqrt{\int_t^T \gamma_s^2 ds}} \\
b_3 &= \frac{\log\frac{D}{Y_t} - \int_t^T \left\{\mu_s + \bar{\sigma}\hat{\varphi} + \gamma\sigma\rho + \frac{1}{2}\sigma^2\right\} ds}{\sqrt{\int_t^T \|\bar{\sigma}\|^2 ds}} \\
a_4 &= \frac{\ln\frac{S_t}{K} + \int_t^T \left[r + \gamma_s\sigma_s\rho - \frac{1}{2}\gamma_s^2\right] ds}{\sqrt{\int_t^T \gamma_s^2 ds}} \\
b_4 &= \frac{\ln\frac{D}{Y_t} + \int_t^T \left[\mu_s + \bar{\sigma}\hat{\varphi} + \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\bar{\sigma}\bar{\sigma}'(T-t)}} \\
\rho_2 &= \frac{\rho\int_t^T \sigma_s\gamma_s ds}{\sqrt{\int_t^T \sigma_s^2 ds}\sqrt{\int_t^T \gamma_s^2 ds}} \\
\hat{\varphi}_t &= \left(-\frac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\right)\prime.
\end{aligned}
$$

The lower good deal bound price is given by a similar pricing formula, with the only exception that

$$
\hat{\varphi}_t = \left(-\frac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{B^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\right)\prime.
$$

Proof. After a few easy transformations on (2.28), we obtain the following expression:

$$\begin{aligned}
\Pi(t,s,y) &= E^Q\left[e^{-r(T-t)}S_T I\left\{S_T > K, Y_T > D\right\}\middle|\,\mathcal{F}_t\right] \\
&- e^{-r(T-t)}KQ_t[S_T > K, Y_T > D] \\
&+ \frac{1-\beta}{D}E^Q\left[e^{-r(T-t)}S_T Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\,\mathcal{F}_t\right] \\
&- e^{-r(T-t)}\frac{K(1-\beta)}{D}E^Q\left[Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\,\mathcal{F}_t\right]
\end{aligned}$$

As before, we need to calculate several expectations and we will start calculating the easiest and moving towards the more complicated ones.

- We start with the expression $Q[S_T > K, Y_T > D]$. From the dynamics of S_t and Y_t under Q, we obtain:

$$\begin{aligned}
S_T &= S_t \exp\left(r(T-t) - \frac{1}{2}\int_t^T \gamma_s^2 ds + \int_t^T \bar{\gamma}_s dW_s\right) \\
Y_T &= Y_t \exp\left(\int_t^T \left[\mu_s + \hat{\varphi}_s\bar{\sigma}_s - \frac{1}{2}\sigma_s^2\right]dt + \int_t^T \bar{\sigma}_s dW_s\right)
\end{aligned}$$

  Through a chain of inequalities similar to the ones performed in the section on complete markets, one obtains:

$$S_T > K \Leftrightarrow \eta > \underbrace{\frac{\ln\frac{K}{S_t} - r(T-t) + \frac{1}{2}\int_t^T \gamma_s^2 ds}{\sqrt{\int_t^T \gamma_s^2 ds}}}_{-a_2}$$

$$Y_T > D \Leftrightarrow \xi > \underbrace{\frac{\ln\frac{D}{Y_t} - \int_t^T \left[\mu_s + \bar{\sigma}_s\hat{\varphi}_s - \frac{1}{2}\sigma_s^2\right]ds}{\sqrt{\int_t^T \sigma_s^2 ds}}}_{-b_2}$$

  where η and ξ are standard normal with correlation coefficient ρ_2. Summarizing the last computations, we can say:

$$Q_t[S_T > K, Y_T > D] = \mathcal{N}[a_2, b_2, \rho_2] \tag{2.29}$$

  with a_2 and b_2 as above. We obtain ρ_2 through similar computations to the ones in the previous section and obtain:

$$\rho_2 = \frac{\rho\int_t^T \sigma_s\gamma_s ds}{\sqrt{\int_t^T \sigma_s^2 ds}\sqrt{\int_t^T \gamma_s^2 ds}} \tag{2.30}$$

- The next expectation we are going to calculate is

$$E^Q\left[e^{-r(T-t)}S_T I\left\{S_T > K, Y_T > D\right\}\middle|\mathcal{F}_t\right]$$

. We can rewrite the expectation as $S_0 E^{\tilde{Q}}\left[I\left\{S_T > K, Y_T > D\right\}\middle|\mathcal{F}_t\right]$, where $\tilde{Q}$ is defined by:

$$d\tilde{Q} = L_t dQ$$
$$dL_t = \bar{\gamma} dW_t$$

Through a chain of inequalities similar to the ones before, one obtains:

$$S_T > K \Leftrightarrow \eta > \underbrace{\frac{\ln\frac{K}{S_t} - \int_t^T\left\{r + \frac{1}{2}\gamma_s^2\right\}ds}{\sqrt{\int_t^T \gamma_s^2 ds}}}_{-a_1}$$

$$Y_T > D \Leftrightarrow \xi > \underbrace{\frac{\ln\frac{D}{Y_t} - \int_t^T\left[\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \sigma_s\gamma_s\rho - \frac{1}{2}\sigma_s^2\right]ds}{\sqrt{\int_t^T \sigma_s^2 ds}}}_{-b_1}$$

and

$$E^Q\left[e^{-r(T-t)}S_T I\left\{S_T > K, Y_T > D\right\}\middle|\mathcal{F}_t\right] = S_0\mathcal{N}[a_1, b_1, \rho_2] \qquad (2.31)$$

with a_1 and b_1 as above and ρ_2 the coefficient of correlation between η and ξ, given by the equation (2.30).

- Now we are going to turn to $E^Q\left[Y_T I\left\{S_T > K, Y_T < D\right\}\middle|\mathcal{F}_t\right]$. In order to calculate this expectation, we are going to use a variant of the change of numeraire technique as follows:

$$E^Q\left[Y_T \underbrace{I\left\{S_T > K, Y_T < D\right\}}_{Z}\middle|\mathcal{F}_t\right] = E^Q\left[\underbrace{E^Q[Y_T]}_{m_T}\underbrace{\frac{Y_T}{E^Q[Y_T]}}_{R_T}Z\middle|\mathcal{F}_t\right]$$
$$= m_T E^Q\left[R_T\middle|\mathcal{F}_t\right]E^{\tilde{Q}}\left[Z\middle|\mathcal{F}_t\right]$$

where $\tilde{Q}$ is the equivalent martingale measure defined by:

$$d\tilde{Q} = L_T dQ \qquad (2.32)$$
$$L_T = R_T \qquad (2.33)$$
$$L_t = E^Q\left[R_T\middle|\mathcal{F}_t\right] \qquad (2.34)$$

By calculating each of the parts of the formula separately, we obtain:

– $m_T = Y_0 \exp\left\{\int_0^T (\mu_s + \bar{\sigma}_s \hat{\varphi}_s) ds\right\}$

– $E^Q\left[R_T | \mathcal{F}_t\right] = \frac{Y_t}{Y_0} \exp\left\{-\int_0^t (\mu_s + \bar{\sigma}_s \hat{\varphi}_s) ds\right\}$

– In order to calculate the last expectation, we need the dynamics of L_t, given by $dL_t = L_t \bar{\sigma}_t dW_t$ from (2.34).

– Before obtaining a formula for $E^{\tilde{Q}}\left[Z | \mathcal{F}_t\right]$, we must have the dynamics of S_t and Y_t under $\tilde{Q}$. By applying Girsanov's transformation, we have:

$$\begin{aligned} dS_t &= (r + \gamma_t \sigma_t \rho) S_t dt + S_t \bar{\gamma}_t dW_t \\ dY_t &= (\mu_t + \bar{\sigma}_t \hat{\varphi}_t + \sigma_t^2) Y_t dt + Y_t \bar{\sigma}_t dW_t \end{aligned}$$

and from the formula for the geometric brownian motion and a chain of inequalities similar to the ones in the previous section, we obtain:

$$S_T \geq K \Leftrightarrow \eta \geq \underbrace{\frac{\ln \frac{K}{S_t} - \int_t^T \left[r + \gamma_s \sigma_s \rho - \frac{1}{2}\gamma_s^2\right] ds}{\sqrt{\int_t^T \gamma_s^2 ds}}}_{-a_4}$$

$$Y_T \leq D \Leftrightarrow \xi \leq \underbrace{\frac{\ln \frac{D}{Y_t} - \int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}}}_{b_4}$$

By summing up the last calculations, we obtain the following equality:

$$E^Q\left[Y_T I\left\{S_T > K, Y_T < D\right\} | \mathcal{F}_t\right] = Y_t \exp\left\{(\mu_s + \sigma_s \hat{\varphi}_s)(T-t)\right\} \mathcal{N}(a_4, b_4, -\rho_2) \tag{2.35}$$

where a_4 and b_4 are defined as above and ρ_2 is the correlation coefficient between ξ and η, given by equation (2.30).

- The last expectation to be calculated is

$$E^Q\left[e^{-r(T-t)} \underbrace{S_T Y_T}_{X_T} \underbrace{I\left\{S_T > K, Y_T < D\right\}}_{Z} \middle| \mathcal{F}_t\right]$$

and we will apply the same technique as above. We denote $E^Q[X_T]$ by m_T and $\frac{X_T}{E^Q[X_T]}$ by R_T and rewrite the expectation to be calculated as

$e^{-r(T-t)} m_T E^Q \left[R_T \middle| \mathcal{F}_t \right] E^{\bar{Q}} \left[Z \middle| \mathcal{F}_t \right]$, where $\bar{Q}$ is defined by

$$d\bar{Q} = L_T dQ \quad (2.36)$$

$$L_T = R_T \quad (2.37)$$

$$L_t = E^Q \left[R_T \middle| \mathcal{F}_t \right] \quad (2.38)$$

Before proceeding, we need to calculate the dynamics of $S_t Y_t$ under Q. An easy application of Ito's lemma yields:

$$d(S_t Y_t) = S_t Y_t \left[r + \mu_t + \bar{\sigma}_t \hat{\varphi}_t + \sigma_t \gamma_t \rho \right] dt + S_t Y_t (\bar{\sigma}_t + \bar{\gamma}_t) dW_t \quad (2.39)$$

From (2.39), we obtain:

- $m_T = S_0 Y_0 \exp \left\{ \int_0^T \left[r + \mu_s + \bar{\sigma}_s \hat{\varphi}_s + \sigma_s \gamma_s \rho \right] ds \right\}$
- $E^Q \left[R_T \middle| \mathcal{F}_t \right] = \frac{S_t Y_t}{S_0 Y_0} \exp \left\{ - \int_0^t \left[r + \mu_s + \bar{\sigma}_s \hat{\varphi}_s + \sigma_s \gamma_s \rho \right] ds \right\}$
- the Girsanov kernel corresponding to the dynamics of L_t, given by $\tilde{\varphi}_t = (\bar{\sigma}_t + \bar{\gamma}_t)'$
- $E^{\bar{Q}} \left[Z \middle| \mathcal{F}_t \right]$, from the dynamics of S_t and Y_t under $\bar{Q}$. According to the definition of $\bar{Q}$ and Girsanov's transformation, the dynamics of S_t and Y_t are given by:

$$dS_t = S_t [r + \gamma_t^2 + \sigma_t \gamma_t \rho] dt + S_t \bar{\gamma}_t dW_t$$

$$dY_t = Y_t [\mu_t + \hat{\varphi}_t \bar{\sigma}_t + \gamma_t \sigma_t \rho + \sigma_t^2] dt + Y_t \bar{\sigma}_t dW_t$$

By solving the above stochastic differential equations for S_T and, respectively, for Y_T, we obtain the following inequalities:

$$S_T \geq K \Leftrightarrow \eta \geq \underbrace{\frac{\ln \frac{K}{S_t} - \int_t^T \left\{ r + + \sigma_s \gamma_s \rho + \frac{1}{2} \gamma_s^2 \right\} ds}{\sqrt{\int_t^T \gamma_s^2 ds}}}_{-a_3}$$

$$Y_T < D \Leftrightarrow \xi < \underbrace{\frac{\ln \frac{D}{Y_t} - \int_t^T \left\{ \mu_s + \bar{\sigma}_s \hat{\varphi}_s + \gamma_s \sigma_s \rho + \frac{1}{2} \sigma_s^2 \right\} ds}{\sqrt{\int_t^T \sigma_s^2 ds}}}_{b_3}$$

where η and ξ are standard normal. The correlation coefficient between the two standard variables is denoted by ρ_2 and it is given by (2.30).

We can write the initial expectation

$$A = E^Q \left[e^{-r(T-t)} S_T Y_T I \left\{ S_T > K, Y_T < D \right\} \middle| \mathcal{F}_t \right]$$

as:

$$A = S_t Y_t \exp\left\{\int_t^T [\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \bar{\sigma}_s \bar{\gamma}_s'] \, dt\right\} \mathcal{N}[a_3; b_3; -\rho_2] \quad (2.40)$$

By summing up the calculations from (2.29), (2.31), (2.35) and (2.40), we obtain the closed form solution from proposition (3.2)∎

3.5 Extension to other products

Exchange Options and GDB

One of the advantages of the GDB is that one can transfer results for the European calls to other simple vanilla products on a manner similar to the one used in the complete market case. Below, we exemplify how to do this for the case of an exchange option. The assumptions needed are similar to the ones stated for the complete market case, except that the assets of the counter-party underwriting the option are not traded.

Assumption 3.2

1. *Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration given by the P-Wiener process $\tilde{W}$, which is defined below.*

2. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dS_t^1 &= \alpha_1 S_t^1 dt + S_t^1 \bar{\gamma}_1 d\tilde{W}_t \\ dS_t^2 &= \alpha_2 S_t^2 dt + S_t^2 \bar{\gamma}_2 d\tilde{W}_t \\ dY_t &= \mu Y_t dt + Y_t \bar{\sigma} d\tilde{W}_t \\ dB_t &= B_t r dt \end{aligned}$$

where Y_t is denoting the assets of the counterparty underwriting the option, S_t^1 and S_t^2 the price processes of the stocks on which the option is contracted and B_t the bank account.

3. *Let μ, α_1 and α_2 be scalars, $\bar{\sigma}$, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be (1,3) row vectors specified*

as follows:

$$
\begin{aligned}
\bar{\gamma}_1 &= \left(\gamma_1, \quad 0,, \quad 0\right) \\
\bar{\gamma}_2 &= \left(\gamma_2\rho_{12}, \quad \gamma_2\sqrt{1-\rho_{12}^2}, \quad 0\right) \\
\bar{\sigma} &= \left(\sigma\rho_{13}, \quad \sigma\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad \sigma\sqrt{1-\rho_{13}^2-\left[\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}\right]^2}\right)
\end{aligned}
$$

4. *Let $\tilde{W}$ be a three dimensional P-Wiener process:*

$$
\tilde{W} = \begin{pmatrix} \tilde{W}^1, \\ \tilde{W}^2, \\ \tilde{W}^3 \end{pmatrix}
$$

with $\tilde{W}^1$, $\tilde{W}^2$ and $\tilde{W}^3$ being independent scalar P-Wiener processes.

5. *Assume that the two stocks are traded on the market, but the assets of the counterparty underwriting the option are not liquidly traded.*

Remark 3.1 *Note that, in this section, the model parameters μ, α_1, α_2, σ, γ_1, γ_2 are constants. This is done for notational convenience. In the case of time varying coefficients, calculations are easily extended, but become very messy.*

Before we continue, we remember that an exchange option has the payoff $\max[S_T^1 - S_T^2, 0]$. In its vulnerable form, the payoff of an exchange option becomes:

$$
\mathcal{X} = \Phi(S_T^1, S_T^2, Y_T, T) = \max[S_T^1 - S_T^2, 0] I\left\{Y_T \geq D\right\} + \mathcal{R} I(Y_T < D)
$$

where the recovery payoff, $\mathcal{R}$ is given by:

$$
\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max[S_T^1 - S_T^2, 0]
$$

Definition 3.2 *The **upper good deal bound** price process for a vulnerable exchange option is defined as the optimal value process for the following* *optimal control problem:*

$$\max_{\varphi} \quad E^Q[e^{-r(T-t)}\mathcal{X}]$$
$$dY_t = (\mu + \bar{\sigma}\varphi_t)Y_t dt + Y_t\bar{\sigma}dW_t$$
$$dS_t^1 = rS_t^1 dt + S_t^1\bar{\gamma}_1 dW_t$$
$$dS_t^2 = rS_t^2 dt + S_t^2\bar{\gamma}_2 dW_t$$
$$\alpha_1 + \bar{\gamma}_1\varphi_t = r$$
$$\alpha_2 + \bar{\gamma}_2\varphi_t = r$$
$$\|\varphi_t\|^2 \leq C^2.$$

*The **lower good deal bound** is the optimal value process for a similar optimal control problem, except that we minimize instead of maximizing subject to the same constraints.*

Our aim is to obtain an equivalent good deal bounds problem expressed under Q^2, the measure where S^2 is numeraire. We are going to show how to obtain this equivalent good deal bounds problem which allows a direct transfer from the pricing problem of a vulnerable exchange option to the pricing problem of a simple vulnerable European call, which is more simple.
We will do this by obtaining equivalent expressions to the objective function and the constraints, under the new measure Q^2 and involving Girsanov kernel corresponding to the change of measure $P \to Q^2$, denoted by ψ.
We are going to present how we have obtained the equivalent problem:

- We apply a standard change of measure to the **objective function** of the upper good deal bound problem and we obtain:

  $$E^Q[e^{-rT}\mathcal{X}] = S_0^2 E^2[\mathcal{Z}]$$

  where

  $$\mathcal{Z} = \max\left[Z_T - 1, 0\right] I\left\{Y_T \geq D\right\} + \mathcal{R}(Z_T) I\left\{Y_T < D\right\}$$

  and $Z_T = \frac{S_T^1}{S_T^2}$. We have

  $$\mathcal{R}(Z_T) = (1-\beta)\frac{Y_T}{D}\max[Z_T - 1, 0]$$

  and $E^2(\bullet)$ denotes the expectations operator under Q_2.
  We denote by ψ the Girsanov kernel corresponding to the change of measure $P \to Q^2$.

- Since our objective function is under Q^2, we would like to have also the **dynamics** of the assets under the same measure. From the dynamics of S^1 and S^2 given by equations (2.41) and (2.41) , we can derive the following dynamics under Q^2. We obtain the the dynamics of Y_t from the standard Girsanov transformation. The dynamics for $\frac{S_t^1}{S_t^2}$ is a Q^2-martingale, according to the definition of Q^2.

$$dY_t = (\mu + \bar{\sigma}\psi_t)Y_t dt + Y_t \bar{\sigma} dW_t^2$$
$$d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2)dW_t^2$$

where W_t^2 is Q^2-Wiener.

- The next step is deriving the **martingale conditions** corresponding to Q^2. They are obtained in the following way: we calculate the dynamics of $\frac{S_t^1}{S_t^2}$ and $\frac{B_t}{S_t^2}$ under the P-measure; we perform a Girsanov transformation $P \to Q^2$. We know that both $\frac{S_t^1}{S_t^2}$ and $\frac{B_t}{S_t^2}$ are martingales under Q^2 and hence we impose the drift of the two processes to be zero. We obtain:

$$r - \alpha_2 = \bar{\gamma}_2\psi_t - \gamma_2^2$$
$$\alpha_1 - \alpha_2 = \gamma_1\gamma_2\rho_{12} - \gamma_2^2 - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t$$

- The next step in our equivalence problem is to take the good deal bound condition for the transformation $P \to Q$:

$$\|\varphi_t\|^2 \leq C^2$$

and find an equivalent good deal bound condition for the transformation $P \to Q^2$. We define the following transformations:

  - $P \to Q$, defined by the following relationship:

$$\begin{aligned} L &= \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \\ dL &= L\varphi' d\tilde{W} \end{aligned}$$

  - $P \to Q^2$, defined by the following relationship:

$$\begin{aligned} L^2 &= \frac{dQ^2}{dP} \quad \text{on } \mathcal{F}_T \\ dL^2 &= L^2\psi' d\tilde{W} \end{aligned}$$

– $Q \to Q^2$, defined by the following relationship:

$$\begin{aligned} L^{1,2} &= \frac{dQ^2}{dP} \quad \text{on } \mathcal{F}_T \\ dL^{1,2} &= L^{1,2}\bar{\gamma}_2 dW \end{aligned}$$

We notice that

$$\frac{\frac{dQ^2}{dP}}{\frac{dQ}{dP}} = \frac{dQ^2}{dQ}.$$

The above equation together with the dynamics of the three Radon-Nikodym derivatives yield the following relation between φ and ψ:

$$\varphi = \psi - \bar{\gamma}_2'$$

Thus, the good deal bounds contraint becomes:

$$\|\psi - \bar{\gamma}_2'\|^2 \leq C^2$$

Hence, the problem below is equivalent to the original upper good deal bound problem:

$$\begin{aligned} \max_{\psi} \quad & S_t^2 E^2[\mathcal{Z}] \\ & dY_t = (\mu + \bar{\sigma}\psi)Y_t dt + Y_t \bar{\sigma} dW_t^2 \\ & d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2) dW_t^2 \\ & r - \alpha_2 = \bar{\gamma}_2 \psi_t - \bar{\gamma}_2 \bar{\gamma}_2' \\ & \alpha_1 - \alpha_2 = \bar{\gamma}_1 \bar{\gamma}_2' - \bar{\gamma}_2 \bar{\gamma}_2' - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t \\ & \|\psi - \bar{\gamma}_2'\|^2 \leq C^2 \end{aligned}$$

Thus, we have reduced the problem of pricing a vulnerable claim written on 2 assets to the problem of pricing a vulnerable claim written on one asset.

Proposition 3.3 *The upper good deal bound price process defined in 3.2 is also the optimal value process for the optimal control problem given below:*

$$\begin{aligned} \max_{\psi} \quad & S_t^2 E^2[\mathcal{Z}] \\ & dY_t = (\mu + 2\bar{\sigma}\psi - \bar{\sigma}\bar{\gamma}_2)Y_t dt + Y_t \bar{\sigma} dW_t^2 \\ & d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2) dW_t^2 \\ & r - \alpha_2 = \bar{\gamma}_2 \psi_t - \gamma_2^2 \\ & \alpha_1 - \alpha_2 = \gamma_1 \gamma_2 \rho_{12} - \gamma_2^2 - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t \\ & \|\psi - \bar{\gamma}_2'\|^2 \leq C^2 \end{aligned}$$

The lower good deal bound is the optimal value process for a similar optimal control problem, where we minimize subject to the same constraints as above.

By a reasoning very similar to the one in the previous section, we calculate the upper good deal bound Girsanov kernel, $\psi_u = (\psi_u^1, \psi_u^2, \psi_u^3)'$, as given below:

$$\psi_u = \begin{pmatrix} \frac{r-\alpha_1}{\gamma_1} + \gamma_2\rho_{12}, \\ \frac{1}{\gamma_2\sqrt{1-\rho_{12}}}\left[r - \alpha_2 - \gamma_2\rho_{12}\frac{r-\alpha_1}{\gamma_1}\right] + \gamma_2\sqrt{1-\rho_{12}}, \\ \sqrt{C^2 - (\psi^1)^2 - (\psi^2)2^+(\gamma_2)^2} \end{pmatrix} \tag{2.41}$$

The lower good deal bound Girsanov kernel is given by:

$$\psi_l = \begin{pmatrix} \frac{r-\alpha_1}{\gamma_1} + \gamma_2\rho_{12}, \\ \frac{1}{\gamma_2\sqrt{1-\rho_{12}}}\left[r - \alpha_2 - \gamma_2\rho_{12}\frac{r-\alpha_1}{\gamma_1}\right] + \gamma_2\sqrt{1-\rho_{12}}, \\ -\sqrt{C^2 - (\psi^1)^2 - (\psi^2)2^+(\gamma_2)^2} \end{pmatrix} \tag{2.42}$$

Thus, we can deduce the following proposition concerning the upper and lower GDB prices. This is done by applying the formula for a European call written on $Z_t = \frac{S_t^1}{S_t^2}$ with strike 1 and local rate of return 0 for the process Z_t.

Proposition 3.4 *Let assumptions 3.2 hold. Then, the upper good deal bound price for a vulnerable option at time zero,*$\Pi(S_1, S_2, Y)$*, is given by:*

$$\begin{aligned} \Pi(S_1, S_2, Y) &= S_0^1\mathcal{N}(a_1, b_1, \rho) - S_0^2\mathcal{N}(a_2, b_2, \rho) \\ &+ \frac{1-\beta}{D}Y_0S_0^1\exp\left\{T\left[\mu + \bar{\sigma}\hat{\psi}_u + \sigma\gamma_1\rho_{13}\right]\right\}\mathcal{N}(a_3, b_3, -\rho) \\ &- \frac{1-\beta}{D}Y_0S_0^2\exp\left\{T(\mu + \bar{\sigma}\hat{\psi}_u + \sigma\gamma_2\rho_{23})\right\}\mathcal{N}(a_4, b_4, -\rho). \end{aligned}$$

where

$$
\begin{aligned}
a_1 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + \frac{1}{2}T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}{\sqrt{T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}} \\
b_1 &= \frac{\ln\frac{Y_0}{D} + T\left[\mu + \bar{\sigma}\hat{\psi}_u + \sigma\gamma_1\rho_{13} - \sigma\gamma_2\rho_{23} - \frac{1}{2}\sigma^2\right]}{\sqrt{T\sigma^2}} \\
a_2 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) - \frac{1}{2}T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}{\sqrt{T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}} \\
b_2 &= \frac{\ln\frac{Y_0}{D} + T\left[\mu + \bar{\sigma}\hat{\psi}_u - \frac{1}{2}\sigma^2\right]}{\sqrt{T\sigma^2}} \\
a_3 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + T[\sigma\gamma_1\rho_{13} - \sigma\gamma_2\rho_{23} + \frac{1}{2}(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)]}{\sqrt{T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}} \\
b_3 &= \frac{\ln\left(\frac{D}{Y_0}\right) - T[\mu + \bar{\sigma}\hat{\psi} + \gamma_1\sigma\rho_{13} + \frac{1}{2}\sigma^2]}{\sqrt{T\sigma^2}} \\
a_4 &= \frac{\ln\left(\frac{S_0^1}{S_0^2}\right) + T[\sigma\gamma_1\rho_{13} - \sigma\gamma_2\rho_{23} + \frac{1}{2}(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)]}{\sqrt{T(\gamma_2^2 - 2\gamma_1\gamma_2\rho_{12} + \gamma_1^2)}} \\
b_4 &= \frac{\ln\left(\frac{D}{Y_0}\right) - T\left[\mu + \bar{\sigma}\hat{\psi}_u + \frac{1}{2}\sigma^2 + \gamma_2\sigma\rho_{23}\right]}{\sqrt{T\sigma^2}} \\
\rho &= -\frac{\gamma_2\rho_{23} - \gamma_1\rho_{13}}{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - 2\gamma_2\gamma_1\rho_{12}}}
\end{aligned}
$$

In the above formula $\hat{\psi}_u$ is given by (2.41) The lower good deal bound price is given by the same general formula, but replacing $\hat{\psi}_u$ by the lower bound Girsanov kernel $\hat{\psi}_l$ given by (2.42)

Before proceeding to the next section, we are going to make a short note with respect to claims with payoffs linearly homogenous in the underlying (see section 2.3). The above proposition can be easily generalized to deal with linearly homogenous payoffs. In general, the GDB pricing problem for a linearly homogenous payoff $\Phi^V(S_T^1, S_T^2, Y_T)$ can be reduced to a modified

GDB problem for the associated simple claim $\Upsilon^V(Z_T, Y_T)$:

$$\begin{aligned}
\max_{\psi} \quad & S_t^2 E^2[\Upsilon^V(Z_T, Y_T)] \\
& dY_t = (\mu + 2\bar{\sigma}\psi - \bar{\sigma}\bar{\gamma}_2)Y_t dt + Y_t \bar{\sigma} dW_t^2 \\
& d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2)dW_t^2 \\
& r - \alpha_2 = \bar{\gamma}_2 \psi_t - \gamma_2^2 \\
& \alpha_1 - \alpha_2 = \gamma_1\gamma_2\rho_{12} - \gamma_2^2 - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t \\
& \|\psi - \bar{\gamma}_2'\|^2 \leq C^2
\end{aligned}$$

where we kept the notation used before.

Barrier options and good deal bounds

In this section, we are going to show how to extend the the good deal bounds option pricing setup in order to encompass down-and-out claims as well. We are in the same framework as Section 2.3, except now the assets of the counterparty, denoted by Y_t are not traded. Throughout this section we use the same notation as in section 2.3.

In general, we due to technical reasons, one cannot obtain a closed-form solution the vulnerable barrier option problem by good deal bounds. However, in the case when Y_t and S_t are independent, this is possible.

Remember that, when proving Proposition 2.6, we use the existence of a risk-neutral neasure but not the uniqueness. Thus, the formula:

$$\Pi(0, \Psi_{LO}^V) = e^{-rT}\left\{E_{0,s,y}^Q\left[\Psi_L^V(S_T, Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi_L^V(S_T, Y_T)\right]\right\}.$$

must hold also in incomplete markets as long as we have picked a risk neutral measure Q according to some criteria. When Y_t and S_t are independent, $y\left(\frac{L}{s}\right)^{2\rho} = y$. Then, we can re-write the pricing formula as

$$\begin{aligned}
\Pi(0, \Psi_{LO}^V) &= e^{-rT}\left\{E_{0,s,y}^Q\left[\Psi_L^V(S_T, Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y}\left[\Psi_L^V(Z_T, Y_T)\right]\right\} \\
&= e^{-rT}\left\{E_{0,s,y}^Q\left[\Psi_L(S_T)F(Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y}\left[\Psi_L^V(Z_T)F(Y_T)\right]\right\}
\end{aligned}$$

where $dZ_t = dS_t$ and $Z_0 = \frac{L^2}{s}$ and

$$F(Y_T) = I\left\{Y_T \geq D\right\} + \frac{(1-\beta)Y_T}{D} I\left\{Y_T < D\right\},$$

which in turn yields:

$$\begin{aligned}\Pi(0,\Psi^V_{LO}) &= e^{-rT}\left\{E^Q_{0,s,\frac{L^2}{s},y}\left[\left\{\Psi_L(S_T)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}}\Psi_L(Z_T)\right\}F(Y_T)\right]\right\}\\ &= e^{-rT}\left\{E^Q_{0,s,\frac{L^2}{s},y}[G(S_T,Z_T)F(Y_T)]\right\} = e^{-rT}\left\{E^Q_{0,s,\frac{L^2}{s},y}[G^V(S_T,Z_T,Y_T)]\right\}\end{aligned}$$

where

$$G(S_T,Z_T)=\Psi_L(S_T)-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}}\Psi_L(Z_T)$$

and $G^V(S_T,Z_T,Y_T)$ is the corresponding vulnerable claim.
Then, the upper good deal bound problem for a barrier option can be written as:

$$\begin{aligned}\max_{\varphi}\quad & e^{-rT}\left\{E^Q_{0,s,\frac{L^2}{s},y}[G^V(S_T,Z_T,Y_T)]\right\}\\ & dY_t=(\mu+\bar{\sigma}\varphi_t)Y_tdt+Y_t\bar{\sigma}dW_t\\ & dS_t=rS_tdt+S_t\bar{\gamma}dW_t\\ & dZ_t=rZ_tdt+Z_t\bar{\gamma}dW_t\\ & \alpha+\bar{\gamma}\varphi_t=r\\ & \|\varphi_t\|^2\le C^2.\end{aligned}$$

Note that the above problem is a **standard** GDB problem, but it is formulated for a different vulnerable claim. As usual, the lower good deal bound problem is the corresponding minimization problem subject to the same constraints.
As before, we define the optimal value function V and we set up the HJB equation:

$$\frac{\partial V}{\partial t}(t,s,\frac{L^2}{s},y)+\sup_{\varphi}\mathcal{A}V(t,s,\frac{L^2}{s},y)-rV(t,s,\frac{L^2}{s},y)=0 \quad (2.43)$$

$$V(T,s,\frac{L^2}{s},y)=G^V(s,\frac{L^2}{s},y). \quad (2.44)$$

and solve for each t,s,y the embedded static maximization problem, corresponding to $sup\mathcal{A}V$ subject to constraints. In our case, for fixed t, s, y, the static problem takes the form:

$$\begin{aligned}\max_{\varphi}\quad & \frac{\partial V}{\partial y}\sigma\varphi y\\ & \alpha+\bar{\gamma}\varphi=r\\ & \|\varphi\|^2\le C^2\end{aligned}$$

We notice that the above problem is the same type of problem as analyzed before: a linear optimization problem whose solution φ is obtained by solving the system of equations:

$$\alpha + \bar{\gamma}\varphi = r$$
$$\|\varphi\|^2 = C^2$$

Just as before, we obtain:

$$\hat{\varphi}(t, s, y)'_b = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \pm\sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right)$$

Again, we have two candidates for the optimal φ. Our objective function is linear in φ:

$$\frac{\partial V}{\partial y}\sigma\varphi y$$

and σ and y are positive by assumption. Therefore, we must investigate the sign of $\frac{\partial V}{\partial y}$ before deciding in order to decide which of the possible Girsanov kernels to choose. The proof is basically identical to the previous proof for vulnerable European calls.

Lemma 3.2 *Under assumptions 3.1 and if φ does not depend on s and y, we have*

$$\frac{\partial V}{\partial y} \geq 0$$

Proof. We are going to prove that $\frac{\partial V}{\partial y} \geq 0$, or, equivalently, that the value function is increasing in y. We do this by first showing that the payoff function is increasing in y. Then we prove that this implies that the associated pricing function is increasing in y, and hence, so is the optimal value function. In te following reasoning we denote $\frac{L^2}{s}$ by z.
To see that the payoff function $G(s, z, y)$ is non-decreasing in y, we note that for $y \geq D$, $G^V(s, z, y) = G(s, z)$ which, being the payoff of a barrier option on S_t does not depend on the value of y, hence, it is non-decreasing in y. For $y < D$, the payoff function is

$$G^V(s, z, y) = \mathcal{R}(s, z, y) < G(s, z)$$

and thus, $G^V(s, z, y)$ is non-decreasing as $y = D_-$. Also, the recovery payoff is a linear function of the assets of the counterparty,

$$\mathcal{R}(s, y) = (1 - \beta)\frac{y}{D}G(s, z),$$

with the coefficient of y positive. Hence, $G(s, z, y)$ is non-decreasing in y.
Let $\Pi^Q(t, s, z, y)$ be a pricing function, i.e.

$$\Pi^Q(t, s, z, y) = E^Q[e^{-r(T-t)} G^V[S_T, Z_T, Y_T] | S_t = s, Z_t = z, Y_t = y]$$

where Q is some admisible EMM.
We now want to prove that if the payoff function $G^V(s, z, y)$ is increasing in y and the Girsanov kernel is a deterministic function of time

$$\varphi(t, s, y) = \varphi(t),$$

then also the pricing function $\Pi^Q(t, s, z, y)$ is increasing in the variable y.
We solve the SDE giving the dynamics of Y_t under Q:

$$dY_t = (\mu_t + \bar{\sigma}_t \varphi_t) Y_t dt + Y_t \bar{\sigma}_t dW_t$$

and obtain the following formula for Y_T, given $Y_t = y$:

$$Y_T = y \exp\left(\int_t^T \left[\mu_t + \bar{\sigma}_t \varphi_t - \frac{1}{2}\sigma_t^2 \right] dt + \int_t^T \bar{\sigma}_t dW_t \right)$$

Thus, for a given φ which does not depend on s, z and y, we can write $Y_T = yX$, where X is a lognormal variable that does not depend on y.
One can easily see that if $\Phi(s, z, y)$ is increasing in the third variable, than also the pricing function $\Pi^Q(t, s, z, y)$is increasing in the variable y.
In our case, we know that

$$V = \Pi^Q$$

when Q is generated by $\hat{\varphi}_b$. Since we see that $\hat{\varphi}_b$ does not depends on s and y, we conclude that $\Pi^Q(t, s, y)$ and thus V is nondecreasing in y. ∎

In conclusion, the optimal Girsanov kernel for the upper good deal bound is:

$$\hat{\varphi}_b = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r - \alpha_t}{\gamma_t})^2} \right)'$$

Further more, as we see in section 2.3, the payoff function $\Psi_L^V(s, y, K, L)$ has the following form:

- If the barrier L is lower than the strike price K, $L \leq K$:

$$\Psi_L^V(t, s, K, L) = \Psi^V(t, s, K)$$

- If the barrier L is higher than the strike price K, $L > K$:

$$\Psi_L^V(s,y,L,K) = \Psi^V(s,y,L) + (L-K)H^V(s,y,L)$$

and $H^V(s,y,L)$ is a vulnerable digital option payoff that yields 1 if $S_T > L$ and 0 otherwise.

Hence we can transfer results from Section (3.4).

Remark 3.2 *The upper and the lower good deal bounds of a vulnerable digital option can be easily calculated as:*

$$\begin{aligned}&\mathcal{H}_{u/l}(t) = e^{-r(T-t)}\mathcal{N}[-a_1,-b_1,\rho]\\ &-e^{-r(T-t)}\frac{(1-\beta)}{D}Y_t \exp\left\{\int_t^T (\mu_s + \bar{\sigma}_s\hat{\varphi}_s)ds\right\}\mathcal{N}(-a_2,-b_2,\rho)\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\log\frac{K}{S_t} - r(T-t) + \frac{1}{2}\int_t^T \|\bar{\gamma}\|^2(t)dt}{\sqrt{\int_t^T \|\bar{\gamma}\|^2(s)ds}}\\
b_1 &= \frac{\log\frac{D}{Y_t} - \int_t^T \left[\mu + \hat{\varphi}\bar{\sigma} - \frac{1}{2}\|\bar{\sigma}\|^2(s)\right]ds}{\sqrt{\int_t^T \|\bar{\sigma}\|^2(s)ds}}\\
a_2 &= \frac{\log\frac{K}{S_t} - \int_t^T \left[r + \bar{\gamma}\bar{\sigma} - \frac{1}{2}\|\bar{\gamma}\|^2\right]ds}{\sqrt{\bar{\gamma}\bar{\gamma}'(T-t)}}\\
b_2 &= \frac{\log\frac{D}{Y_t} - \int_t^T \left[\mu + \bar{\sigma}\hat{\varphi} + \bar{\sigma}\bar{\sigma}' - \frac{1}{2}\|\bar{\sigma}\|^2\right]ds}{\sqrt{\bar{\sigma}\bar{\sigma}'(T-t)}}\\
\rho &= \frac{\rho\int_0^T \sigma_t\gamma_t\prime dt}{\sqrt{\int_0^T \|\bar{\sigma}_t\|^2 dt}\sqrt{\int_0^T \|\bar{\gamma}_t\|^2 dt}}\\
\hat{\varphi}_{u/l} &= \left(-\frac{\alpha_t - r}{\gamma_t}, \quad \pm\sqrt{B^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\right)\prime.
\end{aligned}$$

If the underlying stock and the assets of the counterparty are not independent, we can still solve a GDB problem. However, we cannot obtain a closed for solution. We start from:

$$\Pi(0,\Psi_{LO}^V) = e^{-rT}\left\{E^Q_{0,s,y}\left[\Psi_L^V(S_T,Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi_L^V(S_T,Y_T)\right]\right\}.$$

and introduce a stochastic process Z defined by:

$$\begin{aligned} dZ_t &= dS_t, \\ Z_0 &= \frac{L^2}{s}, \end{aligned}$$

and a process X defined by

$$\begin{aligned} dX_t &= dY_t, \\ X_0 &= y\left(\frac{L}{s}\right)^{2\rho}. \end{aligned}$$

We proceed by re-writing the pricing formula as:

$$\begin{aligned} \Pi(0, \Psi_{LO}^V) &= e^{-rT}\left\{E_{0,s,y}^Q\left[\Psi_L^V(S_T, Y_T)\right] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}} E^Q_{0,\frac{L^2}{s},y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi_L^V(Z_T, X_T)\right]\right\} \\ &= e^{-rT} E^Q_{0,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}}\left[\Psi_L(S_T)F(Y_T) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\gamma^2}}\Psi_L(Z_T)F(X_T)\right] \\ &= e^{-rT} E^Q_{0,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}}[G(S_T, Z_T, Y_T, X_T)] \end{aligned}$$

where the function F is defined as previously. We set up the standard GDB problem:

$$\begin{aligned} \max_{\varphi} \quad & e^{-rT}\left\{E^Q_{0,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}}[G(S_T, Z_T, Y_T, X_T)]\right\} \\ & dY_t = (\mu + \bar{\sigma}\varphi_t)Y_t dt + Y_t\bar{\sigma}dW_t \\ & dX_t = (\mu + \bar{\sigma}\varphi_t)X_t dt + X_t\bar{\sigma}dW_t \\ & dS_t = rS_t dt + S_t\bar{\gamma}dW_t \\ & dZ_t = rZ_t dt + Z_t\bar{\gamma}dW_t \\ & \alpha + \bar{\gamma}\varphi_t = r \\ & \|\varphi_t\|^2 \leq C^2. \end{aligned}$$

As usual, the lower good deal bound problem is the corresponding minimization problem subject to the same constraints.
As before, we define the optimal value function V and we set up the HJB

equation:

$$
\begin{aligned}
&\frac{\partial V}{\partial t}\left(t,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}\right)+\sup_{\varphi}\mathcal{A}V\left(t,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}\right)\\
&-rV\left(t,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}\right)=0\\
&V\left(T,s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}\right)=G\left(s,\frac{L^2}{s},y,y\left(\frac{L}{s}\right)^{2\rho}\right).
\end{aligned}
$$

and solve for each t, s, y, x the embedded static maximization problem, corresponding to $sup\mathcal{A}V$ subject to constraints. In our case, for fixed t, s, y,x, the static problem takes the form:

$$
\begin{aligned}
\max_{\varphi} \quad & \left(\frac{\partial V}{\partial y}y+\frac{\partial V}{\partial x}x\right)\sigma\varphi\\
& \alpha+\bar{\gamma}\varphi=r\\
& \|\varphi\|^2\leq C^2
\end{aligned}
$$

Until this point, the two problems seem to be identical and we can already guess that, depending on the sign of $\frac{\partial V}{\partial y}y+\frac{\partial V}{\partial x}x$ the upper GDB Girsanov kernel is going to be one of the following:

$$
\hat{\varphi}(t,s,y,x)_b^* = \left(-\tfrac{\alpha_t-r}{\gamma_t},\quad \pm\sqrt{C^2-(\tfrac{r-\alpha_t}{\gamma_t})^2}\right)
$$

However, we do not have a clear way how to determine the sign of

$$
\frac{\partial V}{\partial y}y+\frac{\partial V}{\partial x}x.
$$

Hence, in the more general case, we cannot come up with a closed form solution for pricing vulnerable barrier options with good deal bounds.

4 Numerical Results

In this section, we are going to analyze a simple numerical example of the good deal bounds applied to vulnerable options. The goal is to explore the sensitivity of the GDB interval to different model parameters. One can separate these parameters in two major groups:

- parameters specific to each transaction;
- parameters which characterize the general market environment.

In the class of relevant parameters related to each specific transaction we mention the distance of the counterparty to default, the volatility of the assets of the counterparty and the correlation between the assets of the counterparty and the underlying asset.
In the second class of parameters, one can include the size of the good deal bound constraint (B) and the dead weight loss parameter (β).
The GDB contraint parameter B is chosen by the modeler as the bound of the Sharpe ratios for the all transactions on the market. We remember that we place an upper bound on the SR of all the portfolios that can be formed on the market consisting of the underlying assets, the derivative claim and the money account; binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds:

$$|SR_t|^2 \leq \|\lambda_t\|^2.$$

where λ_t is the market price of risk and by SR_t the Sharpe ratio on a particular asset derivative or self financing portfolio on the market. This, the choice of the GDB bound B should be dictated in part by the characteristics of the market on which we are performing the transaction. Empirical evidence suggests that, for mature markets, a Sharpe Ratio above 2 is rare. Thus, even if B is chosen by the modeler, its choice should reflect general characteristics about the market on which we deal.
The deadweight loss parameter β represents the deadweight costs associated with bankrupcy reflected in the recovery payoff:

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\max(S_T - K, 0)$$

One gets a proportional part of the value of the claim, corresponding to how much the assets of the counter-party have fallen below the value of the claim, minus deadweight costs associated to the bankrupcy procedure. These costs are captured by the β parameter. Since β reflects inefficiencies of the recovery process embedded in the system, it is also a parameter specific to the general market environment, rather than to each specific transaction.
In order to present numerical results for the size of the GDB interval, we are going to present the following baseline case: the underlying asset price S_t varies from 1 till 60. The volatility of the stock, γ is 0.45 and the drift is $\alpha = 0.1$. The strike price of the option K is 30. The level of the claims against

the counterparty, D, is 30. The volatility of the assets of the counterparty, σ is 0.2 and the drift is $\mu = 0.1$. The interest rate r is 0.04 and the time to maturity is 1. The deadweight loss due to bankrupcy is 0.3. The size of the good deal bound constraint is given by $B = 2.5$.
We start by analysing the influence of the transaction-specific parameters on the good deal bounds. However, since the good deal bound interval is very sensitive to the distance to default of the counterparty, we are going to present results both when the counterparty is near default (modelled as $Y_t = 32$ and $D = 30$) and when the counterparty is far from default ($Y_t = 40$ and $D = 30$). Also we present results for options out-of-the-money, at-the-money and in-the-money. All graphs and tables are presented in the appendix A.

- The distance to default

  The GDB interval is very sensitive to the distance to default. The closer an option is to default, the bigger is the GDB. The intuition behind this is as follows: the closer we are to default, the more important is the risk premium accounting for the possibility of a default from the counterparty. Below, we present a graph comparing the size of the GDB interval for a vulnerable option at-the-money when the distance to default varies. As a measure of the distance to default we use the ratio between the assets of the counterparty and the total claims against the counterparty. Please note that this is not the ratio between equity and debt. According to the basic books of finance, a ratio over 1.3 would be a mark of a stable company. For ratios below 1, the company is bankrupt.

  We notice that the size of the GDB interval decreases rapidly with the distance to default. In the interval between DD = 1 and 1.3, we notice a drop in the GDB interval of more than 1.4. These effects are amplified when calculate them for in-the-money options. This is why we continue to present results for the GDB both when the counterparty is near default (modelled as $Y_t = 32$ and $D = 30$, $\frac{Y_t}{D} = 1.06$) and when the counterparty is far from default ($Y_t = 40$ and $D = 30$, $\frac{Y_t}{D} = 1.33$).

- The volatility of the assets of the counterparty

  We present results for the size of the good deal bounds interval when the volatility of the assets of the counterparty is 0.15, 0.25, 0.4, 05. We notice that the GDB interval increase with the volatility. Due to interaction effects, this increase is more pronounced when the assets of the counterparty are near default. Also, they are more pronounced for options in-the-money. From a mathematical point of view, this is due to the fact that the Girsanov kernel enters the pricing equation only multiplied by the variance of the stochastic process analyzed. From an economic point of view, the explanation envolves the market price of risk. We remember that the market price of risk and the Girsanov kernel are linked by the following inequality:

$$\varphi_t = -\lambda_t.$$

Since the assets of the counterparty are not traded assets, the counterparty risk cannot be hedged. Thus, the more volatile the assets of the counterparty, the bigger the risk that we need to account for and which is not hedge-able and the bigger the good-deal bounds interval.

- The correlation between the assets of the counterparty and the underlying asset

  We present result for the size of the GDB when the correlation of the assets of the counterparty and the underlying asset is 0, 0.3,0.5, 0.9. We notice that the GDB interval is very small for high correlated assets and increases very fast. From a GDB interval of 2.84 for a correlation of 0.9 it grows to 8.5 for asset correlation of 0.5, when the counterparty is near default. When the counterparty is far from default, the interval grows from 0.25 for $\rho = 0.9$ to 4.04 when $\rho = 0.5$.

  The intuition for this variation in the size of the GDB comes as follows: when the assets of the counterparty and the underlying are highly correlated, the big part of the counterparty risk can be hedged directly by traded on the underlying. The size of the risk which cannot be hedged determines the size of the GDB interval.

  Results do not change qualitatively for negative correlation between the stock and the assets of the counterparty.

We now are going to analyse the influence of some general market parameters on the good deal bounds interval:

- The size of the Good Deal Bound constraint

  As one might expect, the size of the good deal bound interval increases with the size of the parameter B. This happens because, by relaxing the good deal bound constraint, we simply increase the set of the admissible equivalent martingale measures and hence the set of possible prices.

- The deadweight cost β

  The good deal bound interval varies with the size of the deadweight cost β. The dead weight cost β acts as a weight on the unhedgeable risk, The higher the deadweight cost, the more important is the counterparty risk in the valuation of the claim, and hence, the higher the GDB interval. For $\beta = 0$ and the assets of the counterparty far from default, we have a GDB interval of 1.35; at the other end of the spectrum, for $\beta = 0.9$ and the assets of the counterparty far from default, the GDB interval observed is 9.42.

When analyzing the GDB interval, we notice another rather striking fact: in general, we have:
Black-Scholes price > Upper GDB > Complete Market price > Lower GDB.
While in general, we would expect the Black-Scholes price to be above the upper good deal bound price, it is striking how close the two are. In most cases, the two of them coincide up to the 4th decimal. Upon closer analysis, we notice this is driven by the following elements in the model:

i) we have a payoff function increasing in the assets of the counterparty;

ii) the assets of the counter-party are driven only by a Wiener process.

Since the payoff is increasing in the assets of the counterparty, when determining the GDB, we will choose the border solution:

$$\hat{\varphi}_t = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{B^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2}\right)'.$$

The assets of the counterparty are driven only by a Wiener process and hence, the risk neutral expectation of Y_T is increasing with the corresponding Girsanov kernel. The probability $P[Y_T > D]$ goes to 1 faster for a higher Girsanov kernel and the probability of default goes to 0 faster. The faster $P[Y_T > D]$ goes to 1, the closer we are to the Black-Scholes case, when there is no counterparty risk.

5 Concluding Remarks

We price vulnerable options - i.e. options where the counterparty may default. These are basically options traded on the OTC markets. Default is modeled in a structural framework. We start by streamlining literature in complete markets and extending results for pricing more complex vulnerable financial derivatives, such as linearly homogenous payoffs in the underlying and barrier options.

Then, we move to the more realistic, incomplete market pricing problem. The technique employed for pricing is Good Deal Bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals, as defined by Cochrane and Saa-Raquejo (2000) and Björk and Slinko (2005). The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tighter pricing bounds. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. We also provide tight pricing bounds for European calls and show how to extend the call formula to pricing other financial products in a consistent way. Specific examples for exchange options and barrier options are computed.

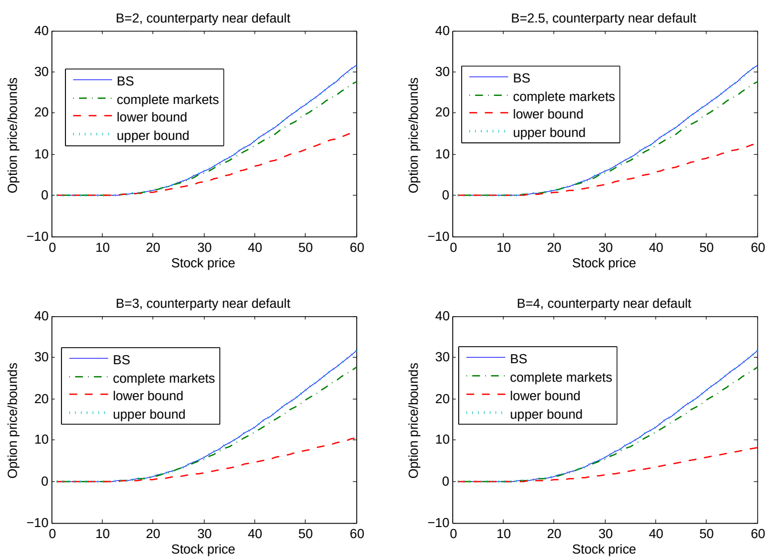
Finally, we analyze numerically the behaviour of the good deal pricing bounds interval and analyze the factors that impact its size. We consider factors specific to each transaction and factors which characterize the general market environment. In the class of relevant parameters related to each specific transaction we mention the distance of the counterparty to default, the volatility of the assets of the counterparty and the correlation between the assets of the counterparty and the underlying asset. In the second class of parameters, we analyze the size of the good deal bound constraint (B) and the dead weight loss parameter (β). We find out that the size of the good deal bound constraint (B) is not the most critical choice in obtaining tight pricing bounds. Rather, factors that impact the importance of the unhedgeable risk (i.e. the counterparty risk) have a bigger influence on the size of the GDB interval. In general, the bigger the probability of default of the counterparty, the bigger the interval of possible prices, calculated according to the GDB method.

A Appendix: Graphs and Tables

- The variable B (or the size of the GDB constraint)

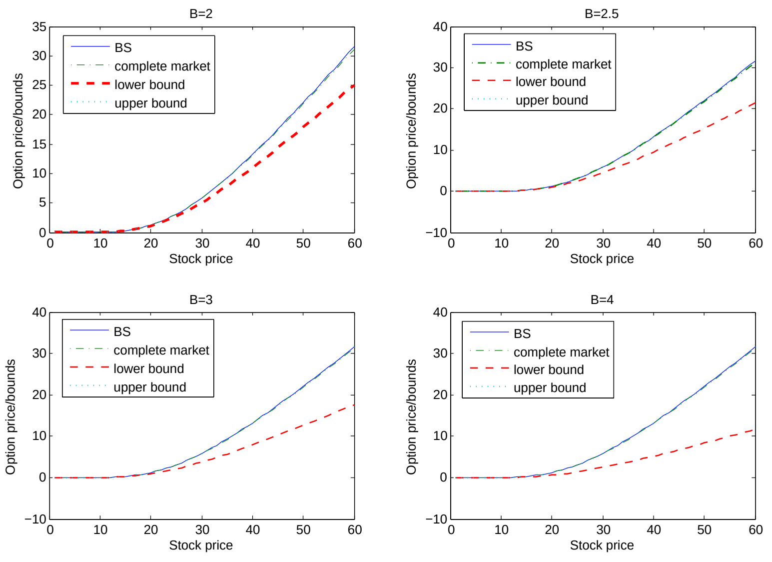
  – when the counterparty is near default

Y_t=32, D=30	B	2	2.5	3	4
	BS	1.2523	1.2523	1.2523	1.2523
S_t=20,K=30	Klein	1.2268	1.2268	1.2268	1.2268
	LB	0.9896	0.8820	0.7818	0.6286
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.2627	0.3703	0.4705	0.6236
	BS	5.8444	5.8444	5.8444	5.8444
S_t=30,K=30	Klein	5.6170	5.6170	5.6170	5.6170
	LB	4.2922	3.8112	3.3909	2.7614
	UB	5.8437	5.8443	5.8444	5.8444
	UB-LB	1.5515	2.0331	2.4535	3.0830
	BS	22.0729	22.0729	22.0729	22.0729
S_t=50,K=30	Klein	20.6565	20.6565	20.6565	20.6565
	LB	15.1785	13.4945	12.0671	9.9116
	UB	22.0635	22.0711	22.0726	22.0729
	UB-LB	6.8850	8.5766	10.0055	12.1612



– when the counterparty is far from default

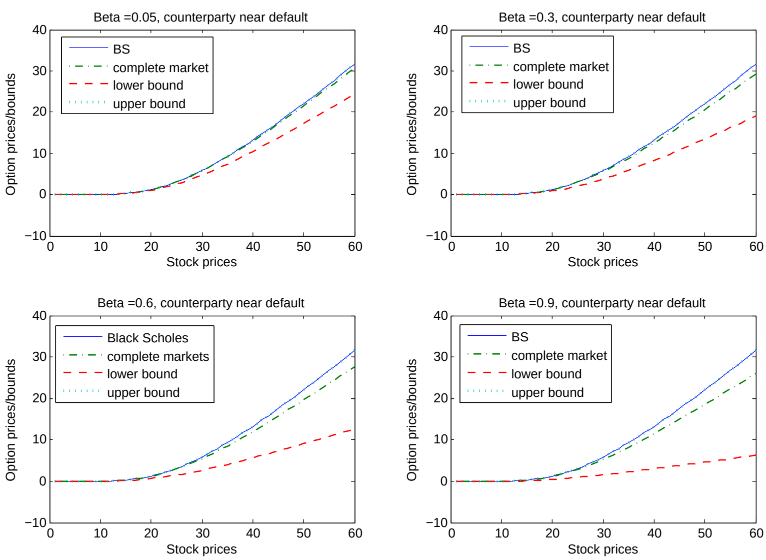
Y_t=40, D=30	B	2	2.5	3	4
	BS	1.2523	1.2523	1.2523	1.2523
S_t=20,K=30	Klein	1.2513	1.2513	1.2513	1.2513
	LB	1.1962	1.1361	1.0486	0.8378
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.0560	0.1162	0.2037	0.4145
	BS	5.8444	5.8444	5.8444	5.8444
S_t=30,K=30	Klein	5.8304	5.8304	5.8304	5.8304
	LB	5.4031	5.0388	4.5763	3.6233
	UB	5.8444	5.8444	5.8444	5.8444
	UB-LB	0.4413	0.8056	1.2681	2.2211
	BS	22.0729	22.0729	22.0729	22.0729
S_t=50,K=30	Klein	21.9416	21.9416	21.9416	21.9416
	LB	19.6099	18.0276	16.2193	12.8539
	UB	22.0728	22.0729	22.0729	22.0729
	UB-LB	2.4629	4.0453	5.8536	9.2190



- the deadweight cost β

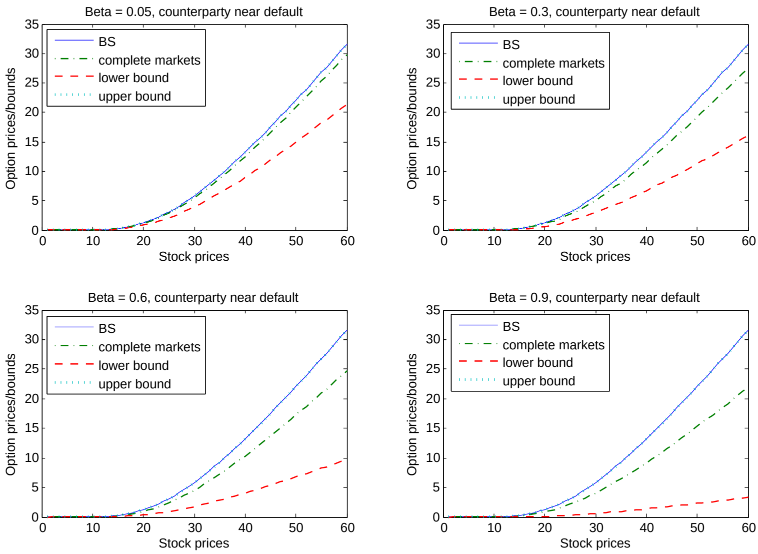
  - near default

Y_t=32, D=30	Beta	0	0.3	0.6	0.9
	BS	1.2523	1.2523	1.2523	1.2523
S_t=20,K=30	Klein	1.2471	1.2268	1.2066	1.1864
	LB	1.1052	0.8820	0.6588	0.4356
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.1471	0.3703	0.5935	0.8167
	BS	5.8444	5.8444	5.8444	5.8444
S_t=30,K=30	Klein	5.7923	5.6170	5.4417	5.2664
	LB	4.9479	3.8112	2.6745	1.5377
	UB	5.8444	5.8443	5.8442	5.8441
	UB-LB	0.8965	2.0331	3.1697	4.3064
	BS	22.0729	22.0729	22.0729	22.0729
S_t=50,K=30	Klein	21.7099	20.6565	19.6032	18.5499
	LB	17.9473	13.4945	9.0418	4.5890
	UB	22.0726	22.0711	22.0696	22.0680
	UB-LB	4.1253	8.5766	13.0278	17.4790



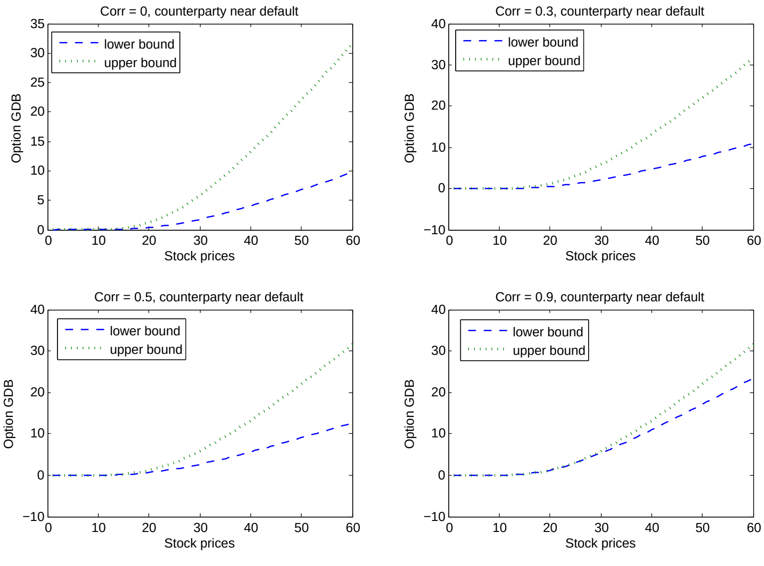
– far away from default

Y_t=40, D=30	Beta	0	0.3	0.6	0.9
S_t=20,K=30	BS	1.2523	1.2523	1.2523	1.2523
	Klein	1.2521	1.2513	1.2504	1.2495
	LB	1.2212	1.1361	1.0509	0.9658
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.0311	0.1162	0.2013	0.2865
S_t=30,K=30	BS	5.8444	5.8444	5.8444	5.8444
	Klein	5.8421	5.8304	5.8187	5.8070
	LB	5.6026	5.0388	4.4749	3.9111
	UB	5.8444	5.8444	5.8444	5.8444
	UB-LB	0.2418	0.8056	1.3695	1.9333
S_t=50,K=30	BS	22.0729	22.0729	22.0729	22.0729
	Klein	22.0487	21.9416	21.8345	21.7274
	LB	20.7189	18.0276	15.3362	12.6448
	UB	22.0729	22.0729	22.0729	22.0729
	UB-LB	1.3540	4.0453	6.7367	9.4280



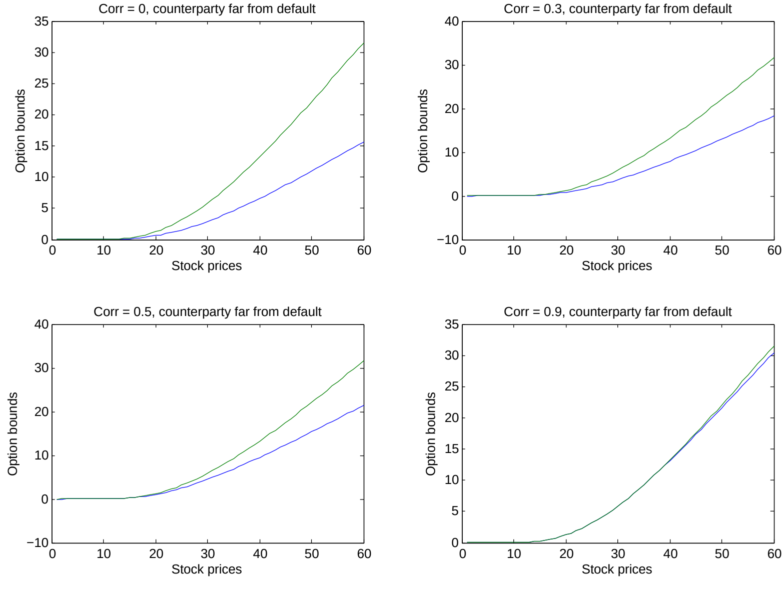
- Correlation
  - when the counterparty is near default

Y_t=32, D=30	ρ	0	0.3	0.5	0.9
S_t=20,K=30	BS	1.2523	1.2523	1.2523	1.2523
	Klein	1.0908	1.1861	1.2268	1.2523
	LB	0.6386	0.7534	0.8820	1.2467
	UB	1.2520	1.2523	1.2523	1.2523
	UB-LB	0.6134	0.4988	0.3703	0.0056
S_t=30,K=30	BS	5.8444	5.8444	5.8444	5.8444
	Klein	5.0907	5.4340	5.6170	5.8366
	LB	2.9806	3.3667	3.8112	5.5746
	UB	5.8431	5.8441	5.8443	5.8444
	UB-LB	2.8625	2.4773	2.0331	0.2698
S_t=50,K=30	BS	22.0729	22.0729	22.0729	22.0729
	Klein	19.2263	20.1190	20.6565	21.6012
	LB	11.2567	12.2579	13.4945	19.2253
	UB	22.0681	22.0705	22.0711	22.0713
	UB-LB	10.8114	9.8125	8.5766	2.8460



– when the counterparty is far from default

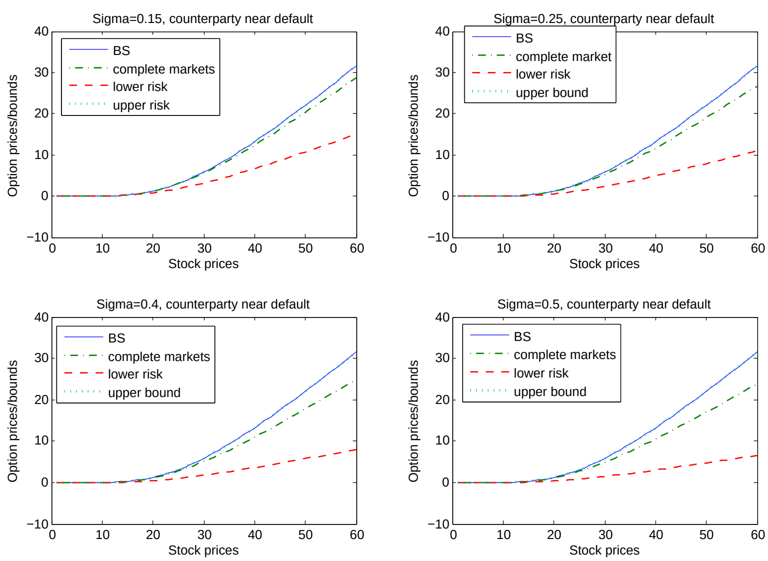
Y_t=40, D=30	ρ	0	0.3	0.5	0.9
	BS	1.2523	1.2523	1.2523	1.2523
S_t=20,K=30	Klein	1.2246	1.2465	1.2513	1.2523
	LB	0.8501	1.0081	1.1361	1.2523
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.4022	0.2442	0.1162	0.0000
	BS	5.8444	5.8444	5.8444	5.8444
S_t=30,K=30	Klein	5.7154	5.8007	5.8304	5.8444
	LB	3.9673	4.5197	5.0388	5.8421
	UB	5.8444	5.8444	5.8444	5.8444
	UB-LB	1.8771	1.3247	0.8056	0.0023
	BS	22.0729	22.0729	22.0729	22.0729
S_t=50,K=30	Klein	21.5855	21.8249	21.9416	22.0687
	LB	14.9836	16.4475	18.0276	21.8186
	UB	22.0728	22.0729	22.0729	22.0729
	UB-LB	7.0893	5.6254	4.0453	0.2543



- Volatility of the assets of the counterparty Y_t

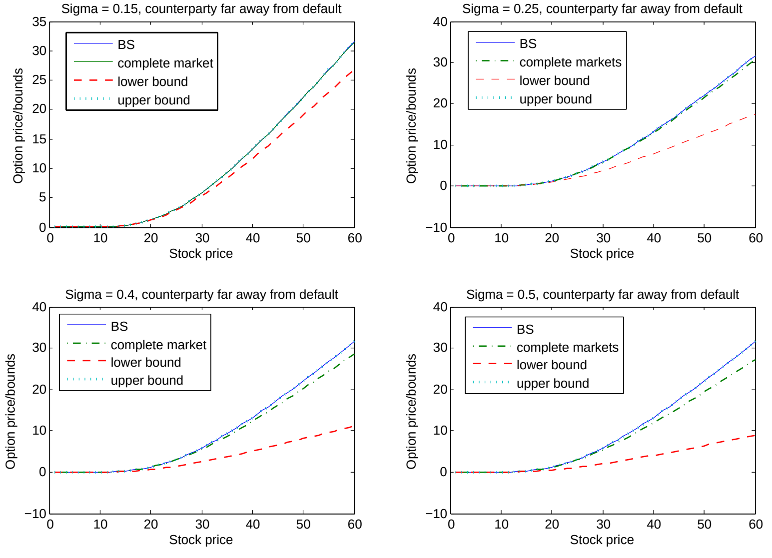
  - when the counterparty is near default

Y_t=32, D=30	σ_v	0.15	0.25	0.4	0.5
	BS	1.2522	1.2522	1.2522	1.2522
S_t=20,K=30	Klein	1.2369	1.2172	1.1905	1.1732
	LB	0.9731	0.8107	0.6480	0.5618
	UB	1.2522	1.2522	1.2522	1.2521
	UB-LB	0.2791	0.4414	0.6041	0.6903
	BS	5.8443	5.8443	5.8443	5.8443
S_t=30,K=30	Klein	5.6979	5.5439	5.34980	5.2279
	LB	4.2462	3.4694	2.6850	2.2763
	UB	5.8443	5.8441	5.8433	5.8426
	UB-LB	1.598	2.3747	3.1583	3.5663
	BS	22.07288	22.0728	22.0728	22.0728
S_t=50,K=30	Klein	21.1043	20.2662	19.2640	18.6531
	LB	15.1410	12.1829	9.1628	7.6150
	UB	22.0723	22.0690	22.0594	22.0508
	UB-LB	6.9313	9.8860	12.8965	14.4358



– when the counterparty is far from default

Y_t=40, D=30	σ_v	0.15	0.25	0.4	0.5
S_t=20,K=30	BS	1.2523	1.2523	1.2523	1.2523
	Klein	1.2522	1.2489	1.2344	1.2209
	LB	1.2187	1.0438	0.8173	0.7024
	UB	1.2523	1.2523	1.2523	1.2523
	UB-LB	0.0336	0.2085	0.4350	0.5499
S_t=30,K=30	BS	5.8444	5.8444	5.8444	5.8444
	Klein	5.8425	5.8043	5.6716	5.5619
	LB	5.5601	4.5346	3.4083	2.8618
	UB	5.8444	5.8444	5.8443	5.8441
	UB-LB	0.2843	1.3098	2.4360	2.9823
S_t=50,K=30	BS	22.0729	22.0729	22.0729	22.0729
	Klein	22.0495	21.7456	20.9129	20.2950
	LB	20.3841	15.9782	11.6411	9.5884
	UB	22.0729	22.0728	22.0713	22.0686
	UB-LB	1.6888	6.0946	10.4301	12.4801



B Appendix: Proposition needed to prove the results in the barrier option case

Denote $W(t) = \big(W^1(t), \quad W^*(t)\big)'$ where $dW^1(t)dW^*(t) = \rho dt$ or

$$W(t) = \big(W^1(t), \quad \rho W^1(t) + \sqrt{1-\rho^2}W^2(t)\big)'$$

with $W^1(t)$ and $W^2(t)$ independent Wiener processes. We assume $W^1(0) = x_0$ and $W^*(0) = y_0$, hence $W^2(0) = \frac{y_0 - \rho x_0}{\sqrt{1-\rho^2}}$.

We denote by $W_A(t) = \big(W_A^1(t), \quad \rho W^1(t) + \sqrt{1-\rho^2}W^2(t)\big)'$, where the notation $W_A^1(t)$ refers to the Wiener process $W^1(t)$ with absorbtion at the barrier 0.We notice that the second component of $W_A(t)$ is not absorbed. We are time $t = 0$ and denote absorbtion time with T. Let $I_1 = (x, x+dx)$ with $x > 0$ and $I^* = (y, y+dy)$ with $y > 0$.
We can write $I = I_1 \times I^*$. If $W^1(t) \in I_1$ and $\rho W^1(t) + \sqrt{1-\rho^2}W^2(t) \in I^*$,then

$$W^2(t) \in \left(\frac{y - \rho(x+dx)}{\sqrt{1-\rho^2}}, \frac{y + dy - \rho x}{\sqrt{1-\rho^2}}\right) = I_2. \tag{2.45}$$

$I_1^r = \{x \in R : -x \in I_1\}$. We want to calculate $P[W_A(t) \in I]$ for $t > 0$. Hence we have

$$\begin{aligned}
P[W_A(t) \in I] &= P[\big\{W_A^1(t) \in I_1\big\} \cap \Big\{\rho W^1(t) + \sqrt{1-\rho^2}W^2(t) \in I^*\Big\}] \\
&= P[\big\{W_A^1(t) \in I_1\big\} \cap \big\{W^2(t) \in I_2\big\}] \\
&= P[W_A^1(t) \in I_1] \times P[W^2(t) \in I_2] \\
&= P[\big\{W^1(t) \in I_1\big\} \cap T > t]P[W^2(t) \in I_2] \\
&= (P[W^1(t) \in I_1] - P[\big\{W^1(t) \in I_1\big\} \cap T < t])P[W^2(t) \in I_2] \\
&= (P[W^1(t) \in I_1] - P[W^1(t) \in I_1^r])P[W^2(t) \in I_2] \\
&= (\phi(x, x_0, \sqrt{t}) - \phi(x, -x_0, \sqrt{t}))\phi(y, \frac{y_0 - \rho x_0}{\sqrt{1-\rho^2}}, \sqrt{t})dxdy
\end{aligned}$$

where $\phi(z, z_0, \sigma)$ denotes the density of the normal distribution of the variable z with mean z_0 and standard deviation σ. We want to translate these results in terms of W^1 and W^* and hence denote by $\mathbf{n}(a, b, \sigma, \gamma, \mathbf{r})$ the joint density of two normal variables with means and standard deviations a and σ, respectively b and γ and correlation $\mathbf{r}$. Then, we can write $P[W_A(t) \in I]$ in terms of the densities of $W^1(t)$ and $W^*(t)$

$$P[W_A(t) \in I] = \Big(\mathbf{n}\left[(x_0, y_0), (\sqrt{t}, \sqrt{t}), \rho\right] - \mathbf{n}\left[(-x_0, y_0 - 2\rho x_0), (\sqrt{t}, \sqrt{t}), \rho\right]\Big)\, dxdy$$

One can easily extend this expression and prove the following proposition:

Proposition B.1 *Let X be a stochastic process defined by:*

$$\begin{aligned} dX(t) &= \alpha dt + dW^1(t), \\ X(0) &= x, \end{aligned}$$

and Y be a stochastic process defined by

$$\begin{aligned} dY(t) &= \mu dt + dW^2(t), \\ Y(0) &= y_0 \end{aligned}$$

where $dW^1(t)dW^2(t) = \rho dt$. We denote by X_β the process with absorbing barrier $\beta \neq 0$, and $x_0 > \beta$.
For a fixed $t > 0$, the density f_β of the stochastic process (X_β, Y) is given by

$$\begin{aligned} &f_\beta = \mathbf{n}[(x_0 + \alpha t, y_0 + \mu t), (\sqrt{t}, \sqrt{t}), \rho] \\ &- \exp\{2\alpha(x_0 - \beta)\}\, \mathbf{n}[(2\beta - x_0 + \alpha t, y_0 + 2\rho(\beta - x_0) + \mu t), (\sqrt{t}, \sqrt{t}), \rho] \end{aligned}$$

Proof. We start by a short digression. Notice that we can re-write the dynamics of X and Y as

$$\begin{aligned} dX(t) &= \alpha dt + \begin{pmatrix} 1, & 0 \end{pmatrix} dW(t), \\ dY(t) &= \mu dt + \begin{pmatrix} \rho, & \sqrt{1-\rho^2} \end{pmatrix} dW(t) \end{aligned}$$

where $W = \begin{pmatrix} W^1, & W^* \end{pmatrix}'$ with W^1 and W^* independent Wiener processes. Let $\underline{\mathcal{F}}$ be the filtration spanned by both W^1 and W^* and P the probability measure under which we have defined the two processes X and Y. We define the probability measure $Q \sim P$ by the process:

$$\begin{aligned} L(T) &= \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \\ dL(t) &= L_t \begin{pmatrix} \alpha, & \frac{\mu - \rho\alpha}{\sqrt{1-\rho^2}} \end{pmatrix} dW \end{aligned}$$

If we denote by V the 2-dimensional Wiener process under Q, the dynamics of X and Y under the new probability measure are:

$$\begin{aligned} dX(t) &= \begin{pmatrix} 1, & 0 \end{pmatrix} dV(t), \\ dY(t) &= [\mu - \rho\alpha - \frac{\mu - \rho\alpha}{\sqrt{1-\rho^2}}\sqrt{1-\rho^2}]dt + \begin{pmatrix} \rho, & \sqrt{1-\rho^2} \end{pmatrix} dV(t) \end{aligned}$$

Also, notice that, by solving the dynamics of L_t and after a bit of algebraic manipulationm we can re-write L_t as:

$$L_t = \exp\left\{\frac{\alpha - \rho\mu}{1-\rho^2}[X(t) - X(0)] + \frac{\mu - \alpha\rho}{1-\rho^2}[Y(t) - Y(0)] - \frac{1}{2}\frac{\mu^2 + \alpha^2 - 2\alpha\mu\rho}{1-\rho^2}t\right\}$$

We notice that both processes have become martingales under Q. We will use Bayes' theorem and the previous proof in order to prove our proposition. Let $I_1 = (x, x + dx)$ with $x > \beta$ and $I_2 = (y, y + dy)$ with $y > 0$.We need to calculate:

$$\begin{aligned}
&P(\{X_\beta(t) \in I_1\} \cap \{Y(t) \in I_2\}) = E^P\left[I\left\{X_\beta(t) \in I_1, Y(t) \in I_2\right\}\right] \\
&= E^Q\left[I\left\{X_\beta(t) \in I_1, Y(t) \in I_2\right\} L(t)\right] \\
&= E^Q\left[I\left\{V^1_\beta(t) \in I_1, V^2(t) \in \left(\frac{y - \rho(x + dx)}{\sqrt{1-\rho^2}}, \frac{y + dy - \rho x}{\sqrt{1-\rho^2}}\right) = I_2*\right\} L(t)\right] \\
&= \mathbf{n}[(x_0 + \alpha t, y_0 + \mu t), (\sqrt{t}, \sqrt{t}), \rho] \\
&- \exp\left\{2\alpha(x_0 - \beta)\right\} \mathbf{n}[(2\beta - x_0 + \alpha t, y_0 + 2\rho(\beta - x_0) + \mu t), (\sqrt{t}, \sqrt{t}), \rho]
\end{aligned}$$

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Chapter 3

Pricing Counterparty Risk Using Good Deal Bounds

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We develop a method for pricing counterparty risk by using good deal bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. Previous literature on counterparty risk and good deal bounds involved structural models. We allow for counterparty risk to be given by intensity-based models. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure - which is not unique. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. Also, we study numerically the tightness of the bounds and underline the use of good deal bounds for risk management. In this context, we also study portfolio effects on the good deal bounds prices.

1 Introduction

Counterparty risk has been brought to the forefront by recent events. The current financial crisis has underlined the importance of good pricing and risk management tools for counterparty risk. This paper approaches the issue by developing tools which address the market incompleteness due to the counterparty risk.

In the context of derivatives, the source for counterparty risk is the fact that the products are traded over-the-counter (OTC). According to the Bank of International Settlements, in December 2007, the OTC notional amounts outstanding were 417 trillion US dollars. By comparison, at the end of the same period, the notional amounts outstanding in exchange traded futures were 28 trillion US dollars and the notional amounts outstanding in exchange traded option were 52.5 trillion. Since the market for OTC derivatives is big, managing counterparty risk for OTC derivatives is essential[1]. If traded on an organized exchange, the counterparty risk associated with the derivatives disappears due to the presence of the market maker. The market incompleteness comes from not having liquidly exchange-traded financial products (credit derivatives) that would help pin down the market price of risk for the counterparty's default. This is a classic case of market incompleteness.

As a way of solving the pricing issues raised by the market incompleteness, I propose the good deal bounds method. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. One has to note that by eliminating unusually goos deals, we do not eliminate extreme market outcomes, but extreme attitudes toward risk (i.e. investors asking for extreme compensation for the risks taken).

To put good deal bounds in a general context, we remember that one of the consequences of having an incomplete market setup is the fact that we no longer have a unique stochastic discount factor or a unique equivalent martingale measure, and consequently not a unique price. One could simply calculate the bounds of the prices, generated by the interval of all possible risk-neutral measures (or all possible stochastic discount factors). These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.

[1]source: BIS report *Statistical Annex* to *Quarterly review Sep 08*- (BIS 2008)

Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterium, chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature dealing with this approach see Schweizer (2001), Henderson and Hobson (2004), Barrieu and Karoui (2005). However, there is no clear cut way of choosing between different criteria and some of them are somewhat ad-hoc, in the sense that they do not have a clear economic interpretation.

In contrast to this, Cochrane and Saa-Raquejo (2000) proposed the method of good deal bounds. The good deal approach aims at obtaining an interval of "reasonable"prices in incomplete markets, rather than concentrating at obtaining a unique price. Since the no-arbitrage bounds are too large to be used, Cochrane and Saa-Raquejo (2000) suggested to rule out not only arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered "too good to be true", hence the name of "good deal bounds"(GDB). One possible measure for the "goodness"of a deal is its Sharpe Ratio (SR), and thus, trades/portfolios which have a SR above a certain threshold are eliminated. Since the SR links the return of financial assets to the risk undertaken, it is not extreme events which are eliminated from the set, but extreme compensation for the risk undertaken. The SR is chosen as a measure for the "goodness of the deal"because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Hence, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.

The next step in developing a theory for "good deal bounds"was done by Björk and Slinko (2005). They proposed a new frame for solving the optimization problem defined by Cochrane and Saa-Raquejo (2000) while at the same time allowing for more complex dynamics for the underlying assets, such as jump-diffusion processes, to be taken into account. This formulation of the good deal bounds will be used in the current project.

Previous literature on counterparty risk and good deal bounds involved structural models (e.g. Hung and Liu 2005). We allow for counterparty risk to be given by intensity-based models, which is a standard tool in credit-risk pricing and management. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure -which is not unique (e.g. Brigo and Masetti 2005, Brigo and Pallavicini 2008). I provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. Furthermore, I study how the interval of prices induced by the good deal bounds changes with different important parameters in the model: i.e. the current intensity of default, the parameters of the intensity process, the good deal bound constant chosen by the modeler, the recovery rate. Results show that the current intensity of default and the recovery rate impact the GDB price interval more than the chosen GDB constant.

Besides the theoretical interest in the link between the risk neutral and objective probability measure, calculating good deal bounds can be useful from a risk management perspective. The good deal bound pricing problem can be reformulated as follows: we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk present on the market. This means that one can use the lower good deal bound as a measure for how low can one expect for the price of a derivative or a portfolio of derivatives to fall, when counterparty risk is taken into account, provided that there are no other changes in the underlying financial product. I prove that this bound is a coherent risk measure according to Artzner and Heath (1999).[2].

I study how stable are the pricing measures induced by good deal bounds in the context of introducing new financial products in a portfolio. Alternatively, keeping the set of financial products traded fixed, I study the quantitative effect of a new counterparty for the pricing measure of the lower good deal bound prices. These investigations are necessary in order to assess how useful is the lower GDB price as a measure of counterparty risk in a more complex setting. The measure is stable with respect to the introduction of new assets traded with existing counterparties. Since we do not have a good model for the correlation of defaults, the lower good deal bounds price is also sensitive to this general drawback of the credit derivatives literature.

The paper is organized as follows. First, I present the GDB methodology. Then, I use vulnerable options as an example for implementing GDB and analyze numerically the results. As a next step, I analyze good deal bounds in the context of portfolio management and introduce the lower good deal bound as a coherent risk measure for portfolio management. Then, I conclude.

[2]The link between GDB and coherent risk measures was first noticed by Jaschke and Küchler (2001). However, he excludes SR based good deal bounds. We show that under the framework of Björk and Slinko (2005), SR based good deal bounds are coherent risk measures as well

2 Good Deal Bounds

One of the main limitations when pricing counterparty risk is the assumption that either the assets of our counterparty or a credit derivative (e.g. a credit default swap, CDS) on our counterparty are liquidly traded on an exchange. In practice, CDS-es are traded OTC and thus bear counterparty risk themselves. This means that it is difficult to pin down whether a change in the CDS spread is due to a change in the risk of default of the CDS name or a change in the risk of default of the CDS counterparty. Figure 1 represents a comparison between a real world measure probability of default like KMV Moody's EDF and the risk-neutral probability of default calibrated from CDS prices (without taking into account CDS counterparty risk). As we see, the risk neutral probability of default varies much more than the objective one and we cannot clearly separate the cause (changing measure effect or additional risk undertaken through the CDS trade). However, since we have one asset (the CDS) and two sources of randomness, we are still in an incomplete market setup.

In order to deal with the market incompleteness, we are going to employ good deal bounds. We eliminate trade opportunities which are considered too favorable to be observed in the real markets. The elimination of the unrealistically good trades is done as follows. From the extended Hansen-Jaganathan bounds, we know that a constraint on the generalized Sharpe Ratio translates into a constraint on the market price of risk (or the Girsanov kernel for the equivalent martingale measure) - for a detailed explanation see Björk and Slinko (2005). We are going to show that these bounds are quite tight and investigate numerically how sensitive to different specific factors the bounds are.

A major difference with the good deal bounds (GDB) approach is the fact that the model is specified under P - the objective probability measure. Most derivative pricing models are specified directly under Q - the risk-neutral probability measure. By doing so, we do not run into the difficulty of separating the probability of default of the name of the CDS from the probability of the counterparty of the CDS, implied by the series of prices. We need, however, a good measure of the real world probability of default. One such measure is KMV Moody's EDF measure. Among the advantages of such a measure is the fact that it is a continuous measure which does not cluster heterogeneous companies together as ratings usually do.

In the next section, we will demonstrate how to price counterparty risk in the context of vulnerable options on equity. Although interest rate derivatives are more widely traded on the OTC markets, they are also more complex

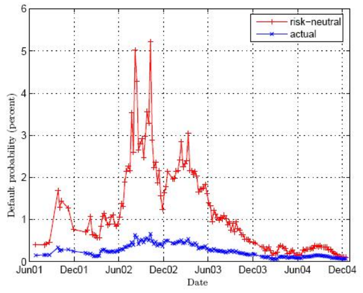


Figure 1: Estimated actual and risk-neutral 1-year default probabilities for Disney.

source:Duffie and Schranz (2005)

products which require much more sophisticated modeling. Also, the most traded fixed income derivatives are swaps, which are two-sided deals. If we take counterparty risk into account, in the case of no recovery, the value of the swap rate for a swap with maturity T_N is given by:

$$\sum_{i=1}^{N} Kp(t,T_i)I[Y_1(T_i)=0] = \sum_{i=1}^{N} L(t,T_{i-1},T,i)p(t,T_i)I[Y_2(T_i)=0]$$

where K is the swap rate, $p(t,T_i)$ is the price of a zero-coupon bond with maturity T_i, $L(t,T_{i-1},T,i)$ is the forward LIBOR rate with maturity T_i, $Y_j(T_i)$ is the probability of survival up to time T_i for the counterparty j, $j=1,2$. Thus, we need to take into account the probability of default of 2 counterparties and potentially the 2 different recovery rates for both participants in the transaction. By comparison, pricing a vulnerable option requires taking into account only the default risk for the writer of the option. Thus, we proceed with computing the good deal bound prices for a vulnerable option.

3 Example on Vulnerable Options

In this section, we are going to show how to implement the good deal bounds for vulnerable options - i.e. options where the counterparty may default. The OTC equity marked options gross market value in December 2007 was 6.2 trillion US dollars. Although a small proportion from the total derivatives transactions in the OTC markets, it is almost a fifth of the exchange traded

futures market[3].
The underlying stock for our chosen derivative is traded and we choose to model the stock as a geometric Brownian motion in order to isolate counterparty risk. Jumps and stochastic volatility extensions are straightforward. However, they would add to the market incompleteness generated by counterparty risk and it would make it harder to separate the impact of counterparty risk on the prices.
Our model is defined under the measure P. The market is formed by the stock and a risk free bank account. We also have a non-traded default indicator Y, which is modeled as a point process with intensity of default λ. Default occurs at the first jump of the process Y. In the main part of the paper, we model λ as an affine process. Appendix A presents computations for the good deal bound problem when λ is constant.
The assumptions we make are summarized as follows:

Assumption 3.1

1. *Let the filtration space $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration generated by the processes W^P, $\tilde{W}^P$ and N, defined below.*

2. *W^P and $\tilde{W}^P$ are P-Wiener processes and $dW^P d\tilde{W}^P = \rho dt$. N is a Cox process with predictable intensity λ_t.*

3. *We assume the intensity of the Cox process λ_t to follow the dynamics*

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t} dW_t^P$$

4. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dS_t &= S_t \alpha_t dt + S_t \gamma_t d\tilde{W}_t^P \\ dB_t &= rB_t dt \end{aligned}$$

   where S_t denotes a traded stock and B_t the money bank account.

5. *α_t and γ_t are scalar deterministic functions of time.*

6. *We assume a European call option is written on the stock. Default of the counterparty/writer of the option is described by the process $Y_t = N_t$. Default occurs at the first jump of the Cox process N_t.*

[3]source: BIS report *Statistical Annex* to *Quarterly review Sep 08*

3.1 The payoff function

The payoff for a vulnerable European derivative is given by

$$X = \begin{cases} \Phi(S_T), & \text{if } Y_T = 0 \\ \mathcal{R}, & \text{if } Y_\tau > 0 \text{ for some } 0 < \tau \leq T \end{cases}$$

where $\mathcal{R}$ denotes the recovery payoff.

We will compute the good deal bound price for the specific example of a vulnerable European call, $\Phi(S_T) = \max[S_T - K, 0]$. However, since the reasoning carries through for more derivatives, we prefer solving for the general case as far as possible and taking the European call as an example only in the last step.

We model the recovery payoff as recovery to market value (RMV). For this type of recovery specification, the payment of the recovery is done immediately after default. Let τ be the time of default. Define the stochastic variable V_t as the market value of the vulnerable option, conditional on no default up to time t:

$$V_t = E^Q \left[e^{-r(T-t)} \Phi(S_T, Y_T) \middle| \mathcal{F}_t, Y_t = 0 \right]$$

where Q is the equivalent martingale measure under which the pricing is done, as defined in the next section. If default occurs at τ, the recovery process is equal to:

$$\mathcal{R} = (1-q)V_{\tau-} \text{ where } 0 < q < 1 \text{ and } V_{t-} = \lim_{s \nearrow t} V_s \tag{3.1}$$

It was proven that, for this recovery specification, the price of a derivative with counterparty risk is:

$$\begin{aligned} \Pi &= E^Q \left[e^{-r(T-t)} (\Phi(S_T) I\{Y_T = 0\} + \mathcal{R} I\{Y_T > 0\}) | \mathcal{F}_t \right] \\ &= E^Q \left[e^{\int_t^T -(r_u + q\lambda_u)du} \Phi(S_T) | \mathcal{F}_t \right] \end{aligned}$$

Besides the mathematical convenience of RMV, the specification is generally preferred for the modeling OTC derivatives counterparty risk, according to Schönbucher (2003).

3.2 Q dynamics

Any intensity-based credit risk model assumes an incomplete market setup, since we have two sources of risk and only one traded asset. Hence, we do not have a unique equivalent martingale measure (EMM), but a whole class of potential EMM. For any potential EMM $Q \sim P$ we define L by:

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \tag{3.2}$$

The fact that $\underline{\mathcal{F}}$ is the internal filtration implies that L_t must have dynamics of the form:

$$\begin{aligned} dL_t &= L_t h_t d\tilde{W}_t^P + L_t g_t \sqrt{\lambda} dW_t^P + L_{t-}\varphi_t(dN_t - \lambda_t dt) && (3.3) \\ L_0 &= 1 && (3.4) \end{aligned}$$

where h_t and g_t are adapted processes and φ_t is a predictable stochastic process. We have chosen to model the Girsanov kernel corresponding to W^P as $g_t\sqrt{\lambda}$ in order to preserve the affine character of λ under the risk neutral measure.
From an economic point of view, $-h_t$ corresponds to the market price of risk for the stock, S_t; φ compensates for the default event itself, while $-g_t\sqrt{\lambda}$ corresponds to the market compensation for the uncertainty over the probability of default.
From Girsanov's theorem, it follows that:

$$\begin{aligned} d\tilde{W}_t^P &= h_t dt + d\tilde{W}_t \\ dW_t^P &= g_t\sqrt{\lambda_t} dt + dW_t \end{aligned}$$

where W_t and $\tilde{W}_t$ are Q-Wiener processes.
Also, the intensity of the Cox process becomes $\lambda_t^Q = (1 + \varphi_t)\lambda_t$. This leads to a **positivity constraint** on φ_t:

$$\varphi_t \geq -1 \tag{3.5}$$

S_t is a traded asset and, from the definition of an EMM, the drift of any traded asset under the EMM must equal the risk free interest rate. Thus, h_t must satisfy the **martingale condition**:

$$r = \alpha_t + \gamma_t h_t \tag{3.6}$$

The class of equivalent martingale measures is defined as the class of measures obtained by (3.2), (3.3) and (3.4) and satisfying the conditions (3.5) and (3.6).

3.3 Optimization Problem

As mentioned in the introduction, we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk, or equivalently, a bound on the Girsanov kernel. Dealing with the market price of risk translates to dealing with the Girsanov kernel of the equivalent martingale measures. Thus, we define the good deal bounds as follows

Definition 3.1 *The **lower good deal bound** price process for a vulnerable option is defined as the optimal value process for the following optimal control problem:*

$$\begin{aligned}
\min_{h,g,\varphi} \quad & E^Q\left[e^{\int_t^T -(r_u+q\lambda_u)du}\Phi(S_T)|\mathcal{F}_t\right] \\
& dS_t = rS_t dt + S_t\gamma_t d\tilde{W}_t \\
& d\lambda_t = [\kappa\,(\theta-\lambda_t) + g_t\sigma\lambda_t]dt + \sigma\sqrt{\lambda_t}dW_t \\
& \lambda_t^Q = \lambda_t(1+\varphi_t) \\
& \alpha_t + \gamma_t h_t = r \qquad (3.7) \\
& \varphi_t \geq -1 \qquad (3.8) \\
& h_t^2 + g_t^2\lambda_t + \varphi_t^2\lambda_t \leq C^2 \qquad (3.9)
\end{aligned}$$

*The **upper good deal bound** process is the optimal value process for a similar optimal control problem, with the only difference that we maximize the expression, subject to the same constraints.*
*We denote the optimal value process by $V(t, S_t, Y_T)$, where V is the **optimal value function**.*

Before proceeding, let us comment on the structure of the optimization problem. The objective function is the arbitrage-free price for the payoff function, where the expectation is computed under the risk neutral measure generated by h_t, g_t, and φ. Since we have to select this measure from a continuum of eligible EMM, we maximize with respect to the Girsanov kernels.
The optimization is subject to the dynamics of the assets on the market, under the appropriate probability measure. The first five constraints are the usual constraints necessary for changing the measure and establishing it as a probability measure, in general (3.8), and a risk-neutral measure (3.7).
If all the Girsanov kernel elements could be identified from these constraints, we would be in a complete market setup and would be able to find a unique

price. Since the number of traded assets is smaller than the number of risk sources, we cannot price all the risk factors and need the last inequality in order to tighten the no arbitrage price bounds. We will refer to this inequality:

$$h_t^2 + g_t^2\lambda_t + \varphi_t^2\lambda_t \leq C^2, \quad 0 \leq t \leq T$$

as the good deal bounds condition.
Notice that φ_t does not appear in equation (3.7), the condition for the drift of the stock under the martingale measure, but in equations (3.8) and (3.9). This separation of the two components of the Girsanov kernel allows us to obtain more elegant closed form solutions.
Classical control theory allows us to solve for the lower good deal bound by solving the **Hamilton Jacobi Bellman equation**, given by the following PDE:

$$\frac{\partial V}{\partial t} + \inf_{h,g,\varphi} \mathcal{A}^{h,g,\varphi}V - rV = 0$$
$$V(T,s,y,\lambda) = \Phi(S_T)$$

where $\mathcal{A}$ is the infinitesimal operator of $(W, \tilde{W}, N)$:

$$\begin{aligned}\mathcal{A}V &= V_s sr + V_\lambda\left[\kappa\left(\theta - \lambda_t\right) + g_t\sigma\lambda_t\right] \\ &\quad + \Delta V\lambda_t^Q + \frac{1}{2}\gamma^2 s^2 V_{ss} \\ &\quad + \frac{1}{2}\sigma^2\lambda V_{\lambda\lambda} + \gamma\sigma s\sqrt{\lambda}V_{s\lambda}\end{aligned}$$

where $\Delta V = V(t,s,1,\lambda) - V(t,s,0,\lambda) = -qV$.
The problem for the upper bound is reduced to an similar PDE, but the inf-problem is replaced by $\sup_{h_t,g_t,\varphi_t} \mathcal{A}V$.
The HJB equation is solved in 2 steps:

- solving for each t, s, λ the embedded static problem, in order to obtain the Girsanov kernel;
- solving the PDE, in order to obtain the price of the vulnerable option

Solving the static problem from the HJB equation reduces to solving the following simple problem:

$$\begin{aligned}\min_{h,g,\varphi} \quad & -qV\lambda\varphi + \sigma\lambda V_\lambda g \\ & \alpha + \gamma h = r \\ & \varphi \geq -1 \\ & h^2 + g^2 + \varphi^2\lambda \leq B^2\end{aligned}$$

This is solved by standard Karoush-Kuhn-Tucker and we find that the *lower bound* Girsanov kernel is given by:

- $\hat{h}_t = \frac{r-\alpha_t}{\gamma_t}$
- $\hat{\varphi}_t = -qV\sqrt{\frac{C^2-h^2}{\lambda[(qV)^2+(\sigma V_\lambda)^2]}}$
- $\hat{g}_t = \sigma V_\lambda\sqrt{\frac{C^2-h^2}{\lambda[(qV)^2+(\sigma V_\lambda)^2]}}$.

We notice the Girsanov kernel depends on the optimal value function V. In a similar way, we can compute the upper GDB Girsanov kernel

Proposition 3.1 *Under assumptions 3.1, the Girsanov kernel for the lower good deal bound EMM as defined in definition 3.1 is given by:*

- $\hat{h}_t = \frac{r-\alpha_t}{\gamma_t}$
- $\hat{\varphi}_t = -qV\sqrt{\frac{C^2-h^2}{\lambda[(qV)^2+(\sigma V_\lambda)^2]}}$
- $\hat{g}_t = \sigma V_\lambda\sqrt{\frac{C^2-h^2}{\lambda[(qV)^2+(\sigma V_\lambda)^2]}}$.

The Girsanov kernel corresponding to the upper good deal bound EMM is given by:

- $\hat{h}_t = \frac{r-\alpha_t}{\gamma_t}$
- $$\hat{\varphi}_t = \max\left[\underbrace{\Delta V\sqrt{\frac{B^2-h^2}{\lambda\left\{(\Delta V)^2+(\sigma V_\lambda)^2\right\}}}}_{L}, \underbrace{-1}_{R}\right]$$
- $$\hat{g}_t = \begin{cases} \sigma V_\lambda\sqrt{\frac{B^2-h^2}{\lambda\left\{(\Delta V)^2+(\sigma V_\lambda)^2\right\}}}, & \textit{if } \hat{\varphi}_t = L \\ -\sqrt{B^2-h^2-\lambda} & \textit{if } \hat{\varphi}_t = R \end{cases}$$

Now, we should plug in the above solution in the HJB equation and solve the PDE. The PDE proves to be unmanageable, and we need to employ some different techniques. We will use a first order Taylor expansion in order to approximate the solution of the HJB equation. We would like to have an approximation that incorporates the tightness of our good deal bounds constraint. Approximating it around C yields explosive solutions. It turns out that the proper variable for this is $y = \sqrt{C^2 - h^2}$, where h was determined by the martingale constraint.

We will do the approximation around the **minimal martingale measure** result given by:

$$\min_{h,g,\varphi} \quad h^2 + g^2 + \varphi^2$$
$$\alpha + \sigma h = r$$

Formally, we define the minimal martingale measure as follows:

Definition 3.2 *Let $Q^{MM} \sim P$, we define L by:*

$$L_t = \frac{dQ^{MM}}{dP} \quad \text{on } \mathcal{F}_T \tag{3.10}$$

with dynamics of the form:

$$dL_t = L_t h_t d\tilde{W}_t^P \tag{3.11}$$
$$L_0 = 1 \tag{3.12}$$

where h_t is given by the equation

$$r = \alpha_t + \gamma_t h_t \tag{3.13}$$

The minimal martingale measure Q^{MM} is defined as the measure obtained by (3.10), (3.11) and (3.12) and satisfying the condition (3.13).

Remark 3.1 *For a thorough analysis on the minimal martingale measure (MMM) and its properties, we refer to Schweizer (1995). Approximations of the good deal bounds solutions around the MMM were first used by Björk and Slinko (2008). In their paper, approximations of GDB are used to price derivatives on underlying with jump-diffusion dynamics and stochastic volatility. The approximations seem to perform well.*
For us, the minimal martingale measure Girsanov kernels are trivially 0. This means that $\lambda^Q = \lambda$ and the dynamics of λ under the minimal martingale measure do not change.

The intuition behind these approximations is as follows: we can re-write the HJB equation by incorporating the Karoush-Kuhn-Tucker constraints:

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,s,y,\lambda)+V_s sr+V_\lambda\left[\kappa\left(\theta-\lambda_t\right)+g_t\sigma\lambda_t\right]-qV\lambda_t(1+\varphi_t)+\frac{1}{2}\gamma^2 s^2 V_{ss}\\&+\frac{1}{2}\sigma^2\lambda V_{\lambda\lambda}+\gamma\sigma s\sqrt{\lambda}V_{s\lambda}-rV(t,s,y,\lambda)\\&+\nu[g_t^2\lambda+\varphi_t^2\lambda+h_t^2-C^2]=0\end{aligned}\tag{3.14}$$

where y is defined as above. If we replace g, h and φ by $\hat{g}$, $\hat{h}$ and $\hat{\varphi}$ as computed above, the solution of (3.14) is the lower good deal bound price. When we compute the lower good deal bound price by approximating around the minimal martingale solution, we obtain:

$$V_{LGB}=V_{MM}+(y_{MM}-y)\frac{\partial V}{\partial y}(y_{MM})\tag{3.15}$$

where V_{LGB} denotes the lower GDB price, V_{MM} the minimal martingale price and $\frac{\partial V}{\partial y}$ represents the sensitivity of V, the solution of PDE (3.14), with respect to the variable y. The variable y_{MM} is taken to be zero. This means that the above equation translates into

$$V_{LGB}=V_{MM}-\sqrt{C^2-h^2}\frac{\partial V}{\partial y}(0)$$

Hence, we need to compute two objects: V_{MM} and $\frac{\partial V}{\partial y}(0)$. The price under the minimal martingale measure is given by:

$$V_{MM}=E_t^{MM}\left[\exp\left\{-\int_t^T(r_u+q\lambda_u)du\right\}\Phi(S_T)\right]$$

where $E_t^{MM}[\bullet]$ denotes the expectations under the minimal martingale measure. If the intensity of default λ and the stock price S are independent, we have a closed-form solution

$$V_{MM}=E_t^{MM}\left[\exp\left\{-\int_t^T(r+q\lambda_u)du\right\}\right]E_t^{MM}\left[\Phi(S_T)\right]$$

where

$$\begin{aligned}dS_t&=rS_tdt+S_t\gamma_t d\tilde{W}_t\\d\lambda_t&=\kappa\left(\theta-\lambda_t\right)dt+\sigma\sqrt{\lambda_t}dW_t\end{aligned}$$

which yields

$$V_{MM} = \exp\left\{-r(T-t) + [A(t,T,q) + B(t,T,q)q\lambda_t]\right\} E_t^{MM}\left[\Phi(S_T)\right]$$

The terms $A(t,T,q)$ and $B(t,T,q)$ can be computed by employing the classical machinery of affine processes. For κ, θ, σ are constants, λ has CIR dynamics and

$$\begin{aligned} B(t,T,q) &= \frac{2\left(e^{\delta(T-t)}-1\right)}{(\delta+q\kappa)\left(e^{\delta(T-t)}-1\right)+2\delta} \\ A(t,T,q) &= \left[\frac{2\delta e^{(q\kappa+\delta)(T-t)/2}}{(\delta+q\kappa)\left(e^{\delta(T-t)}-1\right)+2\delta}\right]^{2q\kappa\theta/\sigma^2} \end{aligned}$$

where $\delta = \sqrt{(q\kappa)^2 + 2q\sigma^2}$. If our claim is a European call, then $\Phi(S_T) = \max[S_T - K]$ and $E_t^{MM}\left[\Phi(S_T)\right]$ is the Black Scholes price.
If the underlying stock for the derivative and the intensity of default are not independent and $\rho \neq 0$, we cannot obtain the V^{MM} in closed form and need to use Monte Carlo simulations. Monte Carlo methods for affine processes and geometric Brownian motion are well known and developed.

Now, we need to compute the sensitivity factor $\frac{\partial V}{\partial y}(0)$. First, we are going to present results for $\rho = 0$.
We denote $\frac{\partial V}{\partial y}(\bullet)$ by $Z(t,\bullet)$. We re-write HJB equation as (3.14) and take the first derivative with respect to y. By applying the envelope theorem and some computations detailed in Appendix B, we obtain:

$$\begin{aligned} &\frac{\partial Z}{\partial t} + Z_s sr + Z_\lambda\left[\kappa\left(\theta-\lambda_t\right)\right] + \frac{1}{2}\gamma^2 s^2 Z_{ss} + \frac{1}{2}\sigma^2\lambda Z_{\lambda\lambda} \\ &+\rho\gamma\sigma s\sqrt{\lambda}Z_{s\lambda} - [q\lambda_t + r]Z + \sqrt{\lambda}\sqrt{(qV_{MM})^2 + \left(\frac{\partial V_{MM}}{\partial\lambda}\sigma\right)^2} = 0 \\ &Z(T,s,\lambda) = 0 \end{aligned}$$

We can solve the above problem by applying Feinman-Kac. The general solution is:

$$Z(t,s,\lambda) = \int_t^T E_{t,s,\lambda}^{MM}\left[\exp\left\{-\int_t^u r(\tau) + q\lambda(\tau)d\tau\right\}\sqrt{\lambda_u}M(u)\right]d\tau$$

where $M(u) = \sqrt{(qV_{MM})^2(u,S_u,\lambda_u) + \left(\frac{\partial V_{MM}}{\partial\lambda}(u,S_u,\lambda_u)\sigma\right)^2}$.
If we have obtained V_{MM} by Monte Carlo simulation (in the general case,

when ρ is different from zero), we will need to compute also the sensitivity of the solution with respect to λ. We can use the likelihood ratio method in order to do so, as explained by Glasserman (2003). Appendix C presents graphs that show the impact of ρ on the lower GDB price.
However, if $\rho = 0$, then we know

$$\frac{\partial V_{MM}}{\partial \lambda}(t) = qB(t,T,q)V_{MM}(t)$$

and, we obtain

$$Z(t,s,\lambda) = \int_t^T E_{t,s,\lambda}^{MM}\left[e^{\left\{-\int_t^u r(\tau)+q\lambda(\tau)d\tau\right\}} qV_{MM}(u)\sqrt{\lambda_u}\sqrt{1+B^2(u,T,q)\sigma^2}\right] du$$

After replacing V_{MM} in the above, we obtain:

$$Z(t,s,\lambda) = qV_{MM}(t)\int_t^T E_t^{\hat{Q}}\left[\sqrt{\lambda_u}\right]\sqrt{1+\sigma^2 B^2(u,T,q)}du$$

where $\hat{Q}$ is as defined in Appendix B. It is a well known fact (see Glasserman 2003,Schönbucher 2003) that λ is non-central chi-square distributed with weighting factor

$$\eta = \frac{q\sigma^2}{4}B(t,T,q), \tag{3.16}$$

degrees of freedom

$$\nu = \frac{\kappa\theta}{\sigma^2}, \tag{3.17}$$

and non-centrality factor[4]

$$\Lambda = \frac{4}{\sigma^2}\frac{\frac{\partial}{\partial T}B(t,T,q)}{B(t,T,q)}\lambda(t). \tag{3.18}$$

This means that $\sqrt{\lambda_t}$ is non-central chi distributed with non-centrality factor Λ and the formula for the mean is given by:

$$E^{\hat{Q}}\left[\sqrt{\lambda}\right] = \sqrt{\frac{\pi}{2}\eta} L_{1/2}^{(\nu/2-1)}\left(-\frac{\Lambda^2}{2}\right) \tag{3.19}$$

where $L_i^{(a)}(x)$ is the generalized Laguerre polynomial and η, ν and Λ are given by (3.27), (3.28) and (3.29). We can summarize our results about Z in the following proposition:

[4]For detailed derivations of the parameters of the non-central chi-square distribution under the $\hat{Q}$ measure, we refer to chapter 7 from Schönbucher (2003)

Proposition 3.2 *Let assumptions 3.1 hold. Let V be the optimal value function that solves the lower good deal bound problem, as defined by Definition 3.1. Let V_{MM} be the price of a vulnerable option computed under the minimal martingale measure defined by Definition 3.2.*

- *The derivative of the optimal value function V with respect to the variable $y = \sqrt{C^2 - h^2}$ and evaluated at $y_{MM} = 0$, $Z(t, s, \lambda)$, is given by the following PDE:*

$$\frac{\partial Z}{\partial t} + Z_s sr + Z_\lambda \left[\kappa\left(\theta - \lambda_t\right)\right] + \frac{1}{2}\gamma^2 s^2 Z_{ss} + \frac{1}{2}\sigma^2 \lambda Z_{\lambda\lambda}$$
$$+\rho\gamma\sigma s\sqrt{\lambda} Z_{s\lambda} - [q\lambda_t + r]Z - \sqrt{\lambda}\sqrt{(qV_{MM})^2 + \left(\frac{\partial V_{MM}}{\partial \lambda}\sigma\right)^2} = 0$$
$$Z(T, s, \lambda) = 0$$

- *The derivative can also be found as:*

$$Z(t, s, \lambda) = \int_t^T E_{t,s,\lambda}^{MM}\left[e^{\left\{-\int_t^u r(\tau) + q\lambda(\tau) d\tau\right\}}\sqrt{\lambda_u} M(u)\right] d\tau$$

where $M(u) = \sqrt{(qV_{MM}(u, S_u, \lambda_u))^2 + \left(\frac{\partial V_{MM}}{\partial \lambda}(u, S_u, \lambda_u)\sigma\right)^2}$.

- *For the case when the underlying stock and the probability of default are uncorrelated ($\rho = 0$), $Z(t, s, \lambda)$ is given by:*

$$Z(t, s, \lambda) = qV_{MM}(t)\int_t^T E_t^{\hat{Q}}\left[\sqrt{\lambda_u}\right]\sqrt{1 + \sigma^2 B^2(u, T, q)} du \qquad (3.20)$$

with $E_t^{\hat{Q}}\left[\sqrt{\lambda_u}\right]$ given by equation (3.30).

Hence, when $\rho = 0$, equations (3.15) and (3.20) imply that the lower bound for the price of a derivative with counterparty risk is

$$V_{LGB} = V_{MM}\left[1 - q\sqrt{C^2 - h^2}\int_t^T E_t^{\hat{Q}}\left[\sqrt{\lambda_u}\right]\sqrt{1 + \sigma^2 B^2(u, T, q)} du\right]$$

We summarize results about the good deal bound prices in the proposition below.

Proposition 3.3 *Let assumptions 3.1 hold. Let V_{MM} be the price of a vulnerable option computed under the minimal martingale measure defined by*

Definition 3.2. Let Z be the derivative of the optimal value function V with respect to the variable $y = \sqrt{C^2 - h^2}$ and evaluated at $y_{MM} = 0$, as in Proposition 3.2. The upper/lower good deal bound price for a vulnerable option is given by:

$$V_{U/LGB} = V_{MM} \pm \sqrt{C^2 - h^2} Z$$

where

$$V_{MM} = E_t^{MM} \left[\exp \left\{ - \int_t^T (r_u + q\lambda_u) du \right\} \Phi(S_T) \right]$$

and

$$Z(t, s, \lambda) = \int_t^T E_{t,s,\lambda}^{MM} \left[\exp \left\{ - \int_t^u r(\tau) + q\lambda(\tau) d\tau \right\} \sqrt{\lambda_u} M(u) \right] d\tau.$$

where $M(u) = \sqrt{(qV_{MM}(u, S_u, \lambda_u))^2 + \left(\frac{\partial V_{MM}}{\partial \lambda}(u, S_u, \lambda_u)\sigma\right)^2}$.
For the special case when $\rho = 0$ and κ, θ, σ are constants, the upper/lower good deal bound price is given by:

$$V_{U/LGB} = V_{MM} \left[1 \pm q\sqrt{C^2 - h^2} \int_t^T E_t^{\hat{Q}} \left[\sqrt{\lambda_u} \right] \sqrt{1 + \sigma^2 B^2(u, T, q)} du \right]$$

with $E_t^{\hat{Q}} \left[\sqrt{\lambda_u} \right]$ given by equation (3.30) and

$$V_{MM} = \exp \left\{ -r(T - t) + [A(t, T, q) + B(t, T, q) q\lambda_t] \right\} E_t^{MM} [\Phi(S_T)]$$

where $E_t^{MM} [\Phi(S_T)]$ is the Black Scholes price and

$$\begin{aligned} B(t, T, q) &= \frac{2\left(e^{\delta(T-t)} - 1\right)}{(\delta + q\kappa)\left(e^{\delta(T-t)} - 1\right) + 2\delta} \\ A(t, T, q) &= \left[\frac{2\delta e^{(q\kappa + \delta)(T-t)/2}}{(\delta + q\kappa)\left(e^{\delta(T-t)} - 1\right) + 2\delta} \right]^{2q\kappa\theta/\sigma^2} \end{aligned}$$

with $\delta = \sqrt{(q\kappa)^2 + 2q\sigma^2}$.

3.4 Variation of the GDB interval for different model specifications

In this section, we are analyzing how the low good deal bound price varies with respect to several model parameters. The upper good deal bound price is either very close or identical with the Black-Scholes price. The intuition

behind this fact is that the higher price we can get for an asset with counterparty risk is the one that we obtain when that particular risk is ignored.
When we analyze the sensitivity of the lower GDB price to different factors, we notice that we can group them in 2 categories: parameters specific to each transaction and parameters specific to the market environment. In the first class, we mention the initial level of the probability of default for our counterparty λ_0, the volatility of the intensity of default σ, the long term level of the intensity of default θ. In the second class, one can include the size of the GDB constraint - the parameter C and, to a certain extent, the loss to default parameter q.
The GDB constraint parameter C is chosen by the modeler as the bound of the Sharpe ratios for the all transactions on the market. We remember that we place an upper bound on the SR of all the portfolios that can be formed on the market consisting of the underlying assets, the derivative claim and the money account; binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds, which state that the SR of all portfolios formed on the market are less or equal to the market price of risk. Thus, the choice of the GDB parameter C should be dictated in part by the characteristics of the market on which we are performing the transaction. Empirical evidence suggests that, for mature markets, a Sharpe Ratio above 2 is rare. Thus, even if C is chosen by the modeler, its choice should reflect general characteristics about the market on which we deal.
The loss to default parameter q reflects both characteristics specific to the counterparty and to the market environment - the dead weight loss due to bankruptcy procedures.
Graphs and tables for this section are presented in the Appendix C. The stock price varies between 1 and 60, the strike price is 30. We present results for the stock being 20, 30 and 50 in order to capture the effect of the moneyness of the option. In the baseline case, λ is 0.03, σ is 0.1 and the long term intensity of default level θ is 0.2. The good deal bound constraint C is 2.5 and the loss to default parameter q is 0.4.
First, we present the impact of changing the size of the parameter C on the lower GDB prices. As one might expect, the size of the good deal bound interval increases with the size of the parameter C. This happens because, by relaxing the good deal bound constraint, we simply increase the set of the admissible equivalent martingale measures and hence the set of possible prices. Result for C equal to 2, 2.5, 3, 4 are tabulated in the appendix. For an option in the money, this translates into a change in price from 18.17 to 15.19. This is a considerable price impact.
However, when we compute the impact of the loss to default parameter, we notice it has a strong influence as well. For a loss to default 0.2, the LGDB

price is 19.73 only to fall at 13.90 for a loss to default of 0.8. The higher is the loss to default, the more significant becomes the impact of the counterparty risk on the price of the derivatives.
The intensity of default when the contract is concluded, λ_0 plays a significant role as well. We compute the price for intensities of 0.01, 0.03, 0.1 and 0.3. The lower good deal price changes from 19.37 to 7.23. Thus, we see the choice of counterparty is crucial for the value of a derivative traded OTC. This is even more striking when we see that the parameters for the dynamics of λ do not have a big impact on the lower good deal bound price. The main explanation for this is the fact that default is a 1-0 phenomenon - we do not care if the intensity of default would revert in the future to a lower probability of default, as much as we care what is the probability of default in the next time interval. Once the default is realized, so are the losses and the notion of intensity of default λ is not meaningful anymore. Thus, the impact of σ and θ on the lower good deal bound price is only of the order of decimals.
From the above exercise, we notice that the choice of GDB constraint is not the most important factor in determining the lower good deal bound price. The "amount of counterparty risk" undertaken, as reflected by the loss to default parameter and the current intensity of default, has a bigger impact on the price then the constraint parameter. The high impact of the price of λ_0 - the current level of the probability of default - also points out toward an "old time wisdom" - i.e. the most important part of risk management is to choose your counterparty and monitor it carefully.

4 Counterparty risk for a portfolio

In the previous sections, we have studied the effect of counterparty risk on one OTC financial derivative at a time. However, for risk management purposes, we are interested more on how good deal bounds are computed and "behave" in a portfolio framework. Before this, however, we have to check if good deal bounds meet the the requirements to be a good risk measure. Such requirements were put forward in Artzner and Heath (1999), and the resulting risk management instrument carries the name of coherent risk measures. They are defined as follows:

Definition 4.1 *A risk measure $\rho : \mathcal{G} \to R$ satisfying the four axioms of*

a) ***translation invariance:***

$$\rho(X + \alpha r) = \rho(X) - \alpha,$$

where X is a risky portfolio, α a real number and r is a reference risk free investment.

b) ***subadditivity**: for all risky portfolios X_1 and $X_2 \in \mathcal{G}$,*

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

c) ***positive homogeneity**: for all $\lambda > 0$ and all $X \in \mathcal{G}$,*

$$\rho(\lambda X) = \lambda \rho(X).$$

d) ***monotonicity**:for all X and $Y \in \mathcal{G}$ with $X \leq Y$; we have $\rho(Y) \leq \rho(X)$.*

is called coherent.

Note that, although abstract, the requirements for coherent risk measures actually have intuitive economic meaning: translation invariance, for example, basically asks that by adding capital to our position we reduce the amount of riskiness of the position; the subadditivity property captures the beneficial effects of diversification.
Artzner and Heath (1999) prove that given the total return R on a reference investment, a risk measure ρ is coherent if and only if there exists a family $\mathcal{P}$ of probability measures on the set of states of nature, such that

$$\rho(X) = \sup \left\{ E^P \left[-\frac{X}{R} \right] | P \in \mathcal{P} \right\} \tag{3.21}$$

The first to notice the link between good deal bounds and coherent risk measures were Jaschke and Küchler (2001). However, they dismiss the good deal bounds on the Sharpe Ration a la Cochrane as not satisfying the monotonicity requirement. Under the new re-formulation of the GDB based on the SR done by Björk-Slinko, one can notice that the lower GDB trivially satisfies (3.21) and hence, it is a coherent risk measure.
In the rest of the paper, we are going to study the effect of adding more assets traded with the same counterparty to our portfolio and see how the GDB behave in this context. Then, we are going to check how adding a new counterparty is going to affect the lower good deal bound.

4.1 Good deal bounds for a portfolio with several assets against one counterparty

As mentioned above, when we deal with counterparty risk, we are more interested in the impact of the risk on the value of a portfolio rather then the impact on the price of each asset. Usually, the two notions coincide since, once the risk-neutral pricing measure is **fixed**, the price of a vulnerable derivative $\Pi^V(X)$ is given by

$$\Pi^V(X) = p_c\Pi(X)$$

where $\Pi(X)$ is the price of the derivative in non-vulnerable form, and p_c is the additional discounting we need to do to account for counterparty risk. However, when using good deal bounds, the choice of a particular set of risk-neutral measures depends on the risk factors on our market, and introducing a new asset traded with the same counterparty changes the choice of EMM used for the pricing of the lower good deal bound.
In order to address this issue, we are going to examine what happens to the prices when we take into account more traded assets in order to fix the lower bound measure. We remember that, when we have included only the underlying of our derivative, the lower good deal bound price is given by:

$$V_{LGB} = V_{MM} - \sqrt{C^2 - h^2}Z(t) \qquad (3.22)$$

where Z_t is the first derivative of the pricing function V with respect to $y = \sqrt{C^2 - h^2}$ and h^2 is the norm of the market price of risk for the traded asset. We notice that neither V_{MM} nor $Z(t)$ do not depend on the value of y. They depend only on the traded underlying and the dynamics of λ under P. Thus, it is straightforward to generalize the above expression to the case when we have n traded assets. Consider a market model with n stocks, a risk-free bank account. We trade n OTC derivatives against the same counterparty. Each derivative is written on one asset.

Assumption 4.1

1. *Let the filtration space $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration generated by the processes W^P, $\tilde{W}_i^P$, with $i = 1, ...n$ and N, defined below.*

2. *W^P and $\tilde{W}_i^P$ are P-Wiener processes and $dW^P d\tilde{W}_i^P = \rho_i dt$. N is a Cox process with predictable intensity λ_t.*

3. *$\tilde{W}_i^P$, $\tilde{W}_j^P$ are independent P-Wiener processes; $(\tilde{W}_i^P)_{i=1,..,n} = \tilde{W}^P$*

4. *We assume the intensity of the Cox process λ_t to follow the dynamics*

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t^P$$

5. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dS_t^i &= S_t^i \alpha_t^i dt + S_t^i \bar{\gamma}_t^i d\tilde{W}_t^P \quad i = 1, ..., n \\ dB_t &= rB_t dt \end{aligned}$$

   where S_t^i denotes the traded stock i, for $i = 1, ..., n$ and B_t the money bank account.

6. *α_t^i are scalar deterministic functions of time and γ_t^i are (1,n) deterministic vector functions of time.*

7. *We assume a European call option is written on each stock. Default of the counterparty/writer of the option is described by the process $Y_t = N_t$. Default occurs at the first jump of the Cox process N_t.*

We denote the payoff function of each European call on stock S_T^i as $\Phi(S_T^i)$ and write the lower good deal problem as

$$\begin{aligned} \min_{h,g,\varphi} \quad & E^Q\left[e^{\int_t^T -(r_u + q\lambda_u)du} \sum_{i=1}^n \Phi(S_T^i) | \mathcal{F}_t \right] \\ & dS_t^i = rS_t^i dt + S_t^i \bar{\gamma}_t^i d\tilde{W}_t \\ & d\lambda_t = [\kappa\left(\theta - \lambda_t\right) + g_t \sigma \lambda_t]dt + \sigma\sqrt{\lambda_t}dW_t \\ & \lambda_t^Q = \lambda_t(1 + \varphi_t) \\ & \bar{\alpha}_t + \bar{\gamma}_t \bar{h}_t = \mathbf{r} \\ & \varphi_t \geq -1 \\ & \sum_{i=1}^n h_i^2 + g^2\lambda + \varphi^2\lambda_t \leq C^2 \end{aligned}$$

where $\bar{\alpha} = (\alpha^i)_{i=1,..n}$ is a (n,1)-vector, $\bar{\gamma} = (\bar{\gamma^i})_{i=1,..n}$ is a (n,n) matrix and $\bar{h} = (h^i)_{i=1,..n}$, g and φ are the Girsanov kernels for the EMM measure change:

$$\begin{aligned} L_t &= \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T \\ dL_t &= L_t \sum_{i=1}^n h_i(t) d\tilde{W}_i^P(t) + L_t g_t \sqrt{\lambda} dW^P(t) + L_{t-}\varphi_t(dN_t - \lambda_t dt) \\ L_0 &= 1 \end{aligned}$$

The Girsanov kernels h_i are computed as the solution to the linear equation system given by:

$$\bar{\alpha}_t + \bar{\gamma}_t \bar{h}_t = \mathbf{r}$$

As before, the minimal martingale measure $\bar{h}$ does not change, but the minimal martingale measure g and φ are 0. We solve the problem through methods similar to the previous sections and the lower good deal bound portfolio value V^n_{LGB} becomes:

$$V^n_{LGB} = V_{MM} - \sqrt{C^2 - \sum_{i=1}^{n} h_i^2 Z(t)}$$

where V_{MM} is the sum of the individual V^i_{MM} computed as in Proposition 3.3 and $Z(t)$ computed as in Proposition 3.3 for the new V_{MM}.

Proposition 4.1 *Let assumptions 4.1 hold. Let V_{MM} be the price of a vulnerable portfolio of options computed under the minimal martingale measure. Let Z be the derivative of the optimal value function V with respect to the variable $y = \sqrt{C^2 - \sum_{i=1}^{n} h_i^2}$ and evaluated at $y_{MM} = 0$. The lower good deal bound price for a vulnerable option is given by:*

$$V_{LGB} = V_{MM} - \sqrt{C^2 - \sum_{i=1}^{n} h_i^2 Z}$$

where

$$V_{MM} = E_t^{MM}\left[\exp\left\{-\int_t^T (r_u + q\lambda_u)du\right\} \sum_{i=1}^{n} \Phi(S_T^i)\right]$$

and

$$Z(t,s,\lambda) = \int_t^T E_{t,s,\lambda}^{MM}\left[\exp\left\{-\int_t^u r(\tau) + q\lambda(\tau)d\tau\right\} \sqrt{\lambda_u} M(u)\right] d\tau.$$

where $M(u) = \sqrt{(qV_{MM})^2(u,(S_u^i)_{i=1}^n,\lambda_u) + \left(\frac{\partial V_{MM}}{\partial \lambda}(u,(S_u^i)_{i=1}^n,\lambda_u)\sigma\right)^2}$.
For the special case when all $\rho_i = 0$, $i = 1,...,n$ and κ,θ, σ are constants, the upper/lower good deal bound price is given by:

$$V_{LGB} = V_{MM}\left[1 - q\sqrt{C^2 - \sum_{i=1}^{n} h_i^2} \int_t^T E_t^{\hat{Q}}[\sqrt{\lambda_u}]\sqrt{1+\sigma^2 B^2(u,T,q)}du\right]$$

with $E_t^{\hat{Q}}[\sqrt{\lambda_u}]$ *given by equation (3.30) and*

$$V_{MM} = \exp\left\{-r(T-t) + [A(t,T,q) + B(t,T,q)q\lambda_t]\right\} \sum_{i=1}^{n} E_t^{MM}\left[\Phi(S_T^i)\right]$$

where $E_t^{MM}[\Phi(S_T^i)]$ *is the Black Scholes price for the option written on the stock* S^i

$$\begin{aligned} B(t,T,q) &= \frac{2\left(e^{\delta(T-t)}-1\right)}{(\delta+q\kappa)\left(e^{\delta(T-t)}-1\right)+2\delta} \\ A(t,T,q) &= \left[\frac{2\delta e^{(q\kappa+\delta)(T-t)/2}}{(\delta+q\kappa)\left(e^{\delta(T-t)}-1\right)+2\delta}\right]^{2q\kappa\theta/\sigma^2} \end{aligned}$$

with $\delta = \sqrt{(q\kappa)^2 + 2q\sigma^2}$.

The individual good deal bound prices for each stock S^j *from the option portfolio are given by:*

$$V_{LGB}^j = V_{MM}^j - \sqrt{C^2 - \sum_{i=1}^{n} h_i^2 Z^j}$$

where $V_{MM}(S^j)$ *and* $Z(S^j)$ *are computed as in Proposition 3.3.*

Proof. It is straightforward, by following exactly the same steps as in Proposition 3.3. ∎

This means it is easy to use the same setup as before in order to price the value of a portfolio of derivatives traded with one counterparty. Appendix C shows that lower good deal bound is fairly stable to the introduction of new assets. For options deep in the money, the prices changes from 17.7698 to 17.8066 when we introduce 9 more assets. There are two main reasons for this phenomenon. First, the value of $Z(t)$ is close to zero. Second, as shown in the last graph, the variable $y = \sqrt{C^2 - \sum_{i=1}^{n} h_i^2}$ changes very little with the introduction of a new asset - by the introduction of 20 new assets, we obtain a change of 0.8.

4.2 Good deal bounds for several counterparties

In this section, we are going to see how the introduction of a new counterparty affects the good deal bound price of an asset. Our vulnerable option will be written on the stock S with dynamics given by:

$$dS_t = \alpha S_t dt + \gamma S_t d\tilde{W}_t$$

We can trade the vulnerable option either with counterparty 1 or with counterparty 2. The default indicator of counterparty i is given by a point process N^i with intensity λ^i, which is modeled as an affine process:

$$d\lambda_t^i = \kappa^i(\theta^i - \lambda_t^i)dt + \sigma^i\sqrt{\lambda_t^i}dW_i^P$$

where W_i^P, $i = 1, 2$ are two Wiener processes, with $dW_1^P dW_2^P = \rho dt$. We denote the correlation between the two counterparties intensities of default as ρ.

Assumption 4.2

1. *Let the filtration space $(\Omega, \mathcal{F}, P, \mathbf{F})$ be given, where $\underline{\mathcal{F}}$ is the internal filtration generated by the processes W^P, $\tilde{W}^P$ and N, defined below.*
2. *W_i^P, $i = 1, 2$ and $\tilde{W}^P$ are P-Wiener processes and $dW_i^P d\tilde{W}^P = \rho_i dt$, $i = 1, 2$ and $dW_1^P dW_2^P = \rho dt$. N^i is a Cox process with predictable intensity λ_t^i.*
3. *We assume the intensity of the Cox process λ_t^i to follow the dynamics*

$$d\lambda_t^i = \kappa^i(\theta^i - \lambda_t^i)dt + \sigma^i\sqrt{\lambda_t^i}dW_i^P$$

4. *κ^i, θ^i, σ^i are scalar deterministic functions of time, $i = 1, 2$.*
5. *The market model under the objective probability measure P is given by the following dynamics:*

$$\begin{aligned} dS_t &= S_t\alpha_t dt + S_t\gamma_t d\tilde{W}_t^P \\ dB_t &= rB_t dt \end{aligned}$$

   where S_t denotes a traded stock and B_t the money bank account.

6. *α_t and γ_t are scalar deterministic functions of time.*
7. *We assume a European call option is written on the stock. Default of the counterparty/writer of the option is described by the process $Y_t^i = N_t^i$. Default of counterparty i, $i = 1, 2$ occurs at the first jump of the Cox process N_t^i.*

I am trying to compute the lower good deal bound price for a derivative on S with counterparty 1. This is formulated as follows:

$$
\begin{aligned}
\max_{h,g_1,g_2,\varphi_1,\varphi_2} \quad & E^Q[\exp\left\{-\int_t^T (r_u + q^1\lambda_u^1)du\right\}\Phi(S_T)|\mathcal{F}_t] \\
& dS_t = rS_t dt + S_t\gamma_t d\tilde{W}_t^Q \\
& \lambda_1^Q = \lambda^1(1+\varphi_1) \\
& \lambda_2^Q = \lambda^2(1+\varphi_2) \\
& d\lambda_t^1 = \left(\kappa^1(\theta^1 - \lambda_t^1) + g_1\sigma^1\sqrt{\lambda_t^1}\right)dt + \sigma^1\sqrt{\lambda_t^1}dW_1^Q \\
& d\lambda_t^2 = \left(\kappa^2(\theta^2 - \lambda_t^2) + g_2\sigma^2\sqrt{\lambda_t^2}\right)dt + \sigma^2\sqrt{\lambda_t^2}dW_2^Q \\
& \alpha_t + \gamma_t h_t = r \\
& \varphi_1 \geq -1 \\
& \varphi_2 \geq -1 \\
& h_t^2 + \varphi_1^2\lambda^1 + \varphi_2^2\lambda^2 + g_1^2\lambda^1 + g_2^2\lambda^2 \leq C^2
\end{aligned}
$$

Please note that λ^2 does not affect the payoff function and, in case of default of the counterparty 2, we do not have a jump in the price process for the asset traded with counterparty 1. However, the coefficients of the process λ_1^Q will change, since $dW_1^P dW_2^P = \rho dt$ - the two Wiener processes are correlated. However, this will have a small impact on the price of our option. If you remember the numerical results from section 3.4, a change in the parameters of the intensity process has only a very small effect on the price of a vulnerable option. Basically, good deal bounds shares the same flaw as the generic credit risk framework of today, in not capturing contagion effects properly.

5 Conclusion

We have developed a method for pricing counterparty risk by using good deal bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, one can obtain tight pricing bounds. Previous literature on counterparty risk and good-deal bounds involved structural models. We allow for counterparty risk to be given by intensity-based models. Also, previous literature on counterparty risk with intensity models uses pricing directly under the risk neutral measure - which is not unique. We provide a link

between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning.
Also, we have studied numerically the tightness of the bounds and the change in the GDB interval due to various factors. Thus, we have noticed that the choice of GDB constraint (which is left at the discretion of the modeler) is not the most important factor in determining the lower good deal bound price. The "amount of counterparty risk" undertaken, as reflected by the loss to default parameter and the current intensity of default, has a bigger impact on the price then the constraint parameter. The high impact of the price of λ_0 - the current level of the probability of default - also points out toward an "old time wisdom"- i.e. the most important part of risk management is to choose your counterparty and monitor it carefully.
Finally, we underline the use of good deal bounds for risk management. We prove the link between the lower good deal bound price and the coherent risk measures. In this context, we also study portfolio effects on the good deal bounds prices. We notice that the GDB are robust to the introduction of new assets in the portfolio traded with our counterparty. Also, the techniques and results used in computing the price of an asset are easily transferable to the risk management framework for a portfolio. However, when it comes to aggregating the effect of different counterparties, good deal bounds are not good at capturing the effect of contagion, a feature transmitted through the generic form of modeling default risk employed today.

A Appendix: The Simple Poisson Process Case

This appendix will deal with the special case when the Cox process has a constant intensity, being reduced to the Poisson process. In this case, we can obtain closed form solutions for the price of the vulnerable options. Results can be easily generalized for the case of Poison processes with piecewise constant intensity.
In the case of deterministic intensity, the upper good deal bound problem becomes:

$$
\begin{aligned}
\max_{h,g,\varphi} \quad & E_t^Q\left[\exp\left\{-\int_t^T (r_s + q\lambda_s^Q)ds\right\}\Phi(S_T)\right] \\
& dS_t = rS_t dt + S_t\gamma_t d\tilde{W}_t \\
& \lambda_t^Q = \lambda_t(1+\varphi_t) \\
& \alpha_t + \gamma_t h_t = r \\
& \varphi_t \geq 1 \\
& h_t^2 + \varphi_t^2\lambda \leq C^2
\end{aligned}
$$

By solving the embedded static problem, we obtain the following optimal Girsanov kernel

- $\hat{h}_t = \frac{r-\alpha_t}{\gamma_t}, \quad \hat{\varphi}_t = \max\left[-\sqrt{\frac{C^2-h^2}{\lambda}}, -1\right].$

The case $\hat{\varphi} = -1$ reduces the jump intensity under Q to zero. Hence forward, we will analyze only the case $\hat{\varphi}_{u/l} = \mp\sqrt{\frac{C^2-(\frac{\alpha-r}{\gamma})^2}{\lambda}}$
We have the following valuation formula: Provided no default occurred until the time of the pricing t, the general pricing formula for a defaultable claim with payoff X and recovery to market value, as defined before is:

$$\Pi_t = E_t^Q\left[\exp\left\{-\int_t^T (r_s + q\lambda_s^Q)ds\right\}X\right]$$

In our case, since both the interest rate and the jump intensity of default are constant, we have:

$$\Pi = \exp\left\{-(r+q\lambda^Q)(T-t)\right\}E_t^Q[\max[S_T-K,0]] \tag{3.23}$$

We obtain the expectation $E_t^Q[\max[S_T-K,0]]$ as $se^{r(T-t)}N[d_1(t,s)]-KN[d_2(t,s)]$. Hence, the upper/lower good deal bound price for a vulnerable option, in the case of constant intensity is given by:

$$\Pi_{u/l} = e^{-(r+q\lambda^Q_{u/l})(T-t)} \left\{ s e^{r(T-t)} N[d_1(t,s)] - KN[d_2(t,s)] \right\} \tag{3.24}$$

where:

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}, \\ \lambda^Q_{u/l} &= \lambda \left(1 \mp \sqrt{\frac{C^2 - (\frac{\alpha - r}{\gamma})^2}{\lambda}} \right) \end{aligned}$$

B Appedix: Computations for $\frac{\partial V}{\partial y}(y_{MM})$ - the sensitivity of V with respect to the GDB constraint

In this section, we detail the computations needed in order to obtain the sensitivity of the lower good deal bounds pricing equation with respect to the variable $y = \sqrt{C^2 - h^2}$ We re-write the HJB as

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,s,y,\lambda) + V_s s r + V_\lambda \left[\kappa(\theta - \lambda_t) + g_t \sigma \lambda_t\right] - qV\lambda_t(1+\varphi_t) + \frac{1}{2}\gamma^2 s^2 V_{ss} \\ &+ \frac{1}{2}\sigma^2 \lambda V_{\lambda\lambda} + \rho\gamma\sigma s\sqrt{\lambda} V_{s\lambda} - rV(t,s,y,\lambda) + \nu[g_t^2\lambda + \varphi_t^2\lambda + h_t^2 - C^2] = 0 \end{aligned}$$

where ν is the Langrange coefficient of the good deal bound constraint. We apply the envelope theorem which allows us to derive the above PDE by $y = \sqrt{C^2 - h^2}$ and denote $\frac{\partial V}{\partial y}$ by Z. As it turns out, we obtain a different PDE for Z:

$$\begin{aligned} &\frac{\partial Z}{\partial t} + Z_s s r + Z_\lambda \left[\kappa(\theta - \lambda_t) + g_t \sigma \lambda_t\right] - qZ\lambda_t(1+\varphi_t) \\ &+ \frac{1}{2}\gamma^2 s^2 Z_{ss} + \frac{1}{2}\sigma^2 \lambda Z_{\lambda\lambda} + \rho\gamma\sigma s\sqrt{\lambda} Z_{s\lambda} - rZ(t,s,y,\lambda) - \nu 2\sqrt{C^2 - h^2} = 0 \end{aligned}$$

From the previous computations, we have obtained the Lagrange coefficient

$$\nu = -\frac{1}{2}\frac{\sqrt{\lambda}\sqrt{(qV)^2 + (\sigma V_\lambda)^2}}{\sqrt{C^2 - h^2}}$$

and replace ν in the PDE for Z. Note that both the values for g and φ and the terms qV and V_λ should be evaluated at the minimal martingale value. The values for g_{MM} and φ_{MM} are 0. Our PDE becomes

$$\begin{aligned}&\frac{\partial Z}{\partial t} + Z_s sr + Z_\lambda \left[\kappa\left(\theta - \lambda_t\right) + g_t \sigma \lambda_t\right] - qZ\lambda_t(1+\varphi_t)\\&+\frac{1}{2}\gamma^2 s^2 Z_{ss} + \frac{1}{2}\sigma^2 \lambda Z_{\lambda\lambda} + \rho\gamma\sigma s\sqrt{\lambda} Z_{s\lambda} - rZ(t,s,y,\lambda) +\\&\sqrt{\lambda}\sqrt{(qV^{MM})^2 + (\sigma V_\lambda^{MM})^2} = 0\end{aligned}$$

We solve this by Feynman-Kac:

$$Z = \int_t^T E^{MM}_{t,s,\lambda}\left[\exp\left\{-\int_t^u (r_\tau + q\lambda_\tau)d\tau\right\}\sqrt{\lambda_u}\sqrt{(qV^{MM})^2 + (\sigma V_\lambda^{MM})^2}\right] du$$

We can distinguish 2 situations:

- for $\rho \neq 0$, we need to insert MC simulation results and solve the PDE numerically. In order to compute V_λ^{MM}, we use the maximum likelihood method as explained in Glasserman (2003)
- for $\rho = 0$, we can go further:
  from $V_\lambda^{MM} = -V_t^{MM} qB(t,T,q)$, we get that:

$$\begin{aligned}Z &= \int_t^T E^{MM}_{t,s,\lambda}\left[e^{\left\{-\int_t^u (r_\tau + q\lambda_\tau)d\tau\right\}}\sqrt{\lambda_u}V^{MM}\sqrt{q^2 + (\sigma qB(u,T,q))^2}\right] du\\&= \int_t^T q\sqrt{1+(\sigma B(u,T,q))^2} E^{MM}_{t,\lambda}\left[\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\}\sqrt{\lambda_u}\right]\\&\quad E^{MM}_{t,s}\left[\Phi(S_T)\right] du\end{aligned}$$

We need to compute

$$\begin{aligned}A &= E^{MM}_{t,\lambda}\left[\underbrace{\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\}}_{M_t}\underbrace{\sqrt{\lambda_u}}_{X}\right] = E^{MM}_{t,\lambda}[M_t X]\\&= E^{MM}_{t,\lambda}[m_T R_T X]\end{aligned} \tag{3.25}$$

where $m_T = \frac{E^{MM}\left[\exp\left\{\int_0^t q\lambda_\tau d\tau\right\}\right]}{E^{MM}\left[\exp\left\{\int_0^T q\lambda_\tau d\tau\right\}\right]}$ and $R_T = \frac{\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\}}{m_T}$.

We have $\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\} \geq 0$ by definition.Also, we note that

$E^{MM}[R_T] = 1$. These facts allow us to use R_T as a Radon-Nycodim derivative in a change of measure and define a measure $\hat{Q}$ by:

$$d\hat{Q} = R_T dQ^{MM} \text{ on } \mathcal{F}_T \tag{3.26}$$

Using Bayes' Theorem, we can re-write (3.25) as:

$$A = m_T E^{MM}\left[R_T | \mathcal{F}_t\right] E^{\hat{Q}}\left[Z | \mathcal{F}_t\right]$$

If we define the likelihood process L_t, $0 \leq t \leq T$, by:

$$d\hat{Q} = L_t dQ^{MM} \text{ on } \mathcal{F}_t$$

by standard theory, we have:

$$L_t = E^{MM}\left[L_T | \mathcal{F}_t\right] = E^{MM}\left[R_T | \mathcal{F}_t\right]$$

Note that even if $L_T = R_T$, we cannot draw the conclusion $L_t = R_t$ for $t < T$. This is a consequence of the fact that $\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\}$ is not a traded asset.
We notice that

$$m_T E^{MM}\left[R_T | \mathcal{F}_t\right] = E^{MM}\left[\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\} \middle| \mathcal{F}_t\right]$$

Thus, in order to proceed, we need to calculate the following:

- $E^{MM}\left[\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\} \middle| \mathcal{F}_t\right]$,
- the dynamics for L_t in order to identify the Girsanov transformation $Q^{MM} \to \hat{Q}$,
- $E^{\hat{Q}}\left[X | \mathcal{F}_t\right]$

We know from the computations linked to V^{MM} that

$$E^{MM}\left[\exp\left\{-\int_t^T (r_\tau + q\lambda_\tau)d\tau\right\} \middle| \mathcal{F}_t\right] = e^{\{-[r\Delta T + A(t,T,q) + B(t,T,q)q\lambda_t]\}}$$

From the definition of L_t, one can easily derive the dynamics of L_t:

$$dL_t = -\sigma q \sqrt{\lambda} B(t,T,q) L_t dW_t$$

and then derive the dynamics of λ under $\hat{Q}$ are given by:

$$d\lambda_t = [\kappa(\theta - \lambda) - \sigma^2 q \lambda B(t,T,q)]dt + \sigma\sqrt{\lambda} d\hat{W}_t$$

Finally, we need to compute $E^{\hat{Q}}\left[\sqrt{\lambda}\middle|\mathcal{F}_t\right]$. For a detailed proof that λ is non-central chi-square distributed with weighting factor

$$\eta = \frac{q\sigma^2}{4}B(t,T,q), \tag{3.27}$$

degrees of freedom

$$\nu = \frac{\kappa\theta}{\sigma^2}, \tag{3.28}$$

and non-centrality factor

$$\Lambda = \frac{4}{\sigma^2}\frac{\frac{\partial}{\partial T}B(t,T,q)}{B(t,T,q)}\lambda(t) \tag{3.29}$$

under the $\hat{Q}$ measure, we refer to chapter 7 from Schönbucher (2003). This means that $\sqrt{\lambda_t}$ is non-central chi distributed with non-centrality factor Λ and the formula for the mean is given by:

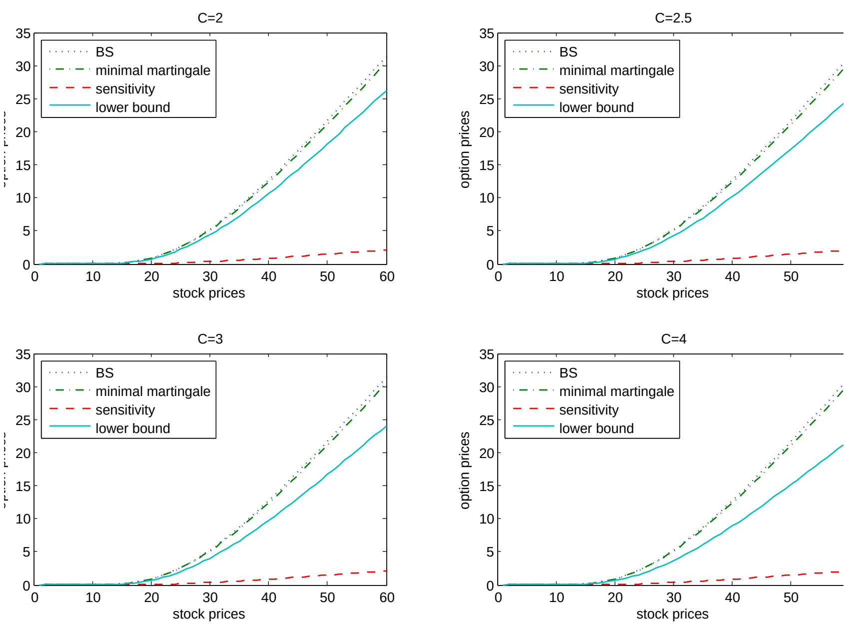
$$E^{\hat{Q}}[\sqrt{\lambda}] = \sqrt{\frac{\pi}{2}\eta L_{1/2}^{(\nu/2-1)}}\left(-\frac{\Lambda^2}{2}\right) \tag{3.30}$$

where $L_i^{(a)}(x)$ is the generalized Laguerre polynomial and η,ν and Λ are given by (3.27), (3.28) and (3.29).

C Appendix: Graphs and Tables

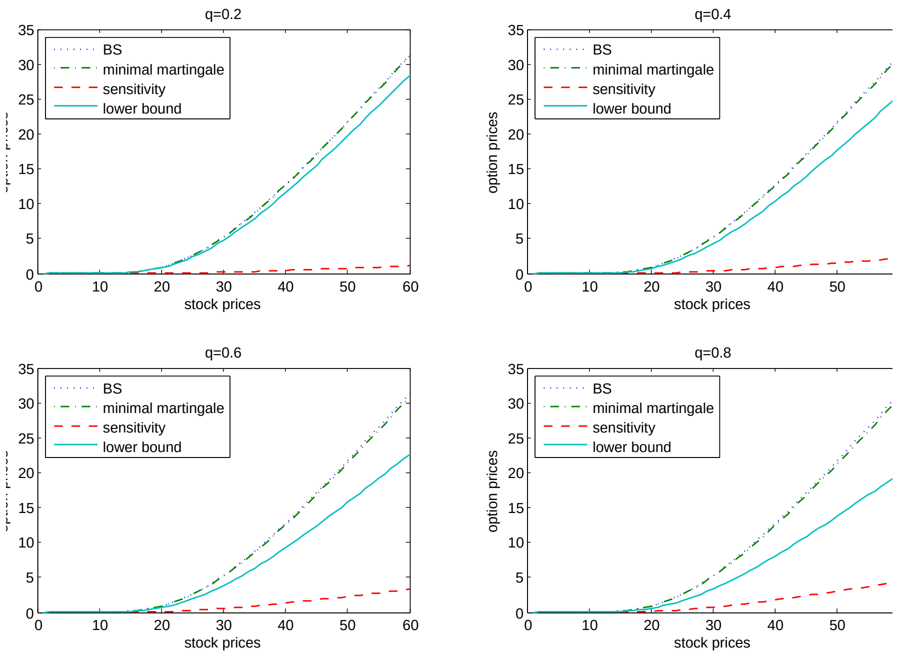
The variable C (or the size of the GDB constraint)

	C	2	2.5	3	4
S_t=20,K=30	BS	0.9291	0.9291	0.9291	0.9291
	MM	0.903	0.903	0.903	0.903
	V_{LGB}	0.7763	0.7445	0.7127	0.6491
S_t=30,K=30	BS	5.2735	5.2735	5.2735	5.2735
	MM	5.1252	5.1252	5.1252	5.1252
	V_{LGB}	4.4063	4.2256	4.0451	3.6843
S_t=50,K=30	BS	21.7537	21.7537	21.7537	21.7537
	MM	21.1418	21.1418	21.1418	21.1418
	V_{LGB}	18.1763	17.4312	16.6866	15.1982



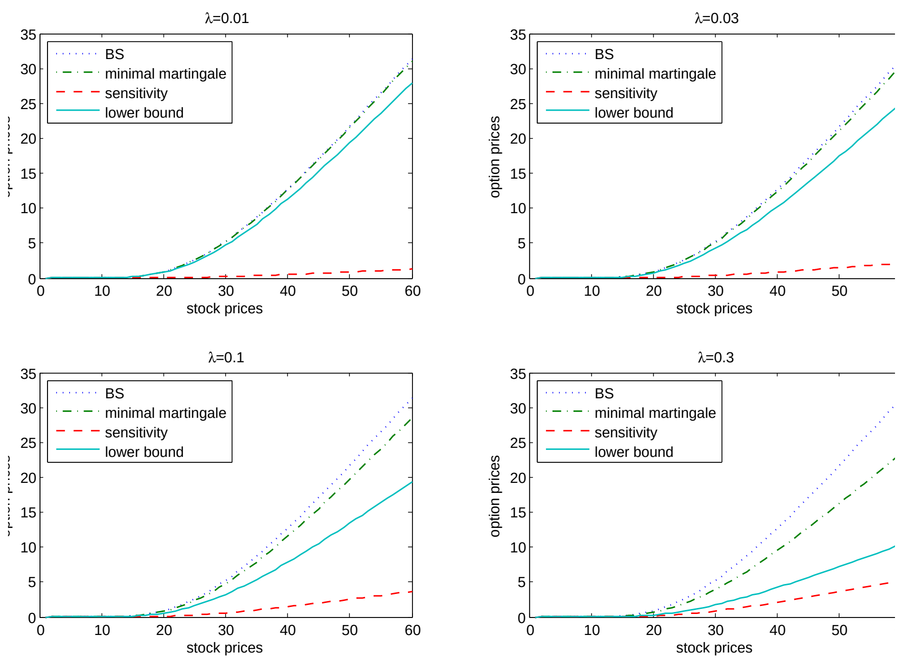
The rate of recovery

	q	0.2	0.4	0.6	0.8
S_t=20,K=30	BS	0.9291	0.9291	0.9291	0.9291
	MM	0.9237	0.9192	0.9161	0.9151
	V_{LGB}	0.8427	0.7579	0.6749	0.5939
S_t=30,K=30	BS	5.2735	5.2735	5.2735	5.2735
	MM	5.243	5.2173	5.1998	5.1938
	V_{LGB}	4.7829	4.3016	3.8309	3.3706
S_t=50,K=30	BS	21.7537	21.7537	21.7537	21.7537
	MM	21.628	21.522	21.4498	21.4248
	V_{LGB}	19.73	17.7446	15.8027	13.9041



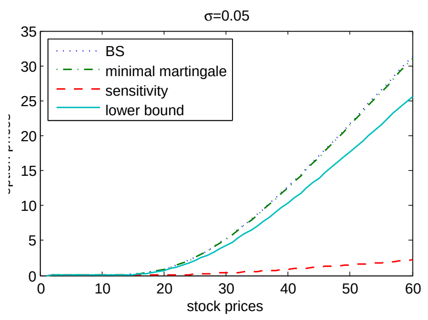
The initial intensity of default

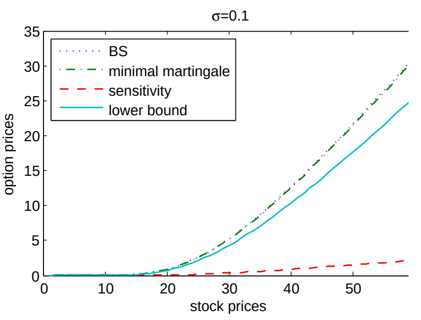
	λ	0.01	0.03	0.1	0.3
S_t=20,K=30	BS	0.9291	0.9291	0.9291	0.9291
	MM	0.9207	0.903	0.8437	0.6948
	V_{LGB}	0.8274	0.7445	0.5733	0.3092
S_t=30,K=30	BS	5.2735	5.2735	5.2735	5.2735
	MM	5.2256	5.1252	4.7885	3.9436
	V_{LGB}	4.6961	4.2256	3.2541	1.7548
S_t=50,K=30	BS	21.7537	21.7537	21.7537	21.7537
	MM	21.5562	21.1418	19.7531	16.2679
	V_{LGB}	19.3719	17.4312	13.4234	7.2389

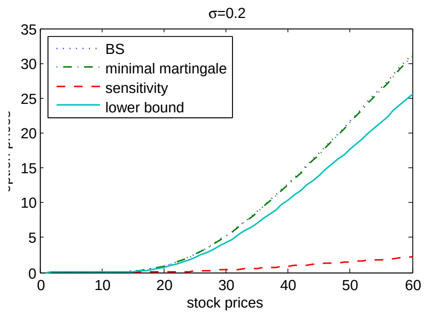


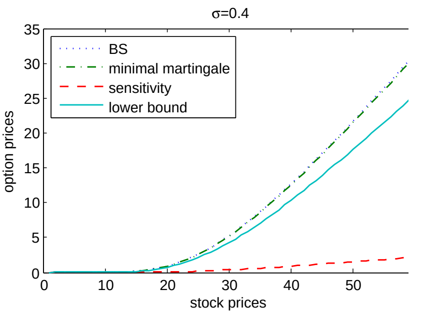
The volatility of the intensity of default process

	σ	0.05	0.1	0.2	0.4
S_t=20,K=30	BS	0.9291	0.9291	0.9291	0.9291
	MM	0.9184	0.9184	0.9185	0.9185
	V_{LGB}	0.7572	0.7571	0.7569	0.7557
S_t=30,K=30	BS	5.2735	5.2735	5.2735	5.2735
	MM	5.2129	5.213	5.2131	5.2135
	V_{LGB}	4.2978	4.2974	4.2958	4.2894
S_t=50,K=30	BS	21.7537	21.7537	21.7537	21.7537
	MM	21.5038	21.504	21.5044	21.5064
	V_{LGB}	17.7291	17.7274	17.7206	17.6943



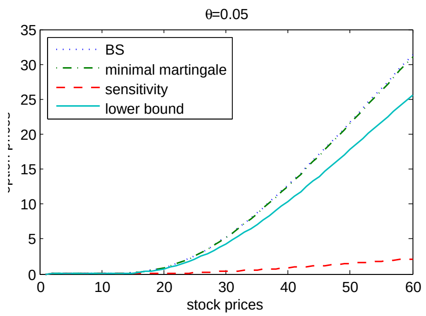


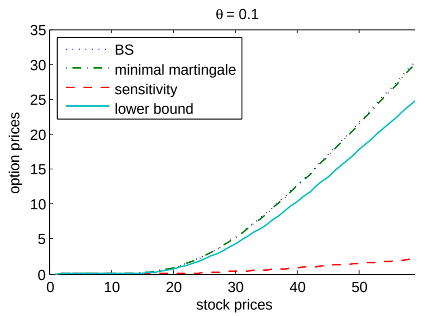


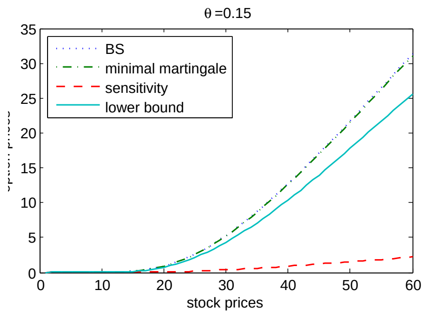


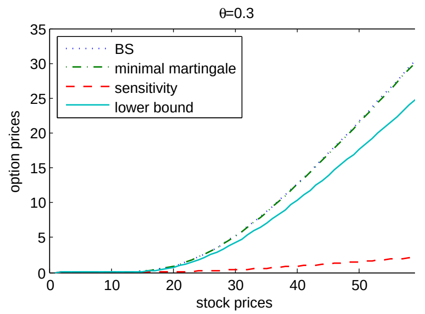
The long term intensity of default

	θ	0.05	0.2	0.4	0.5
S_t=20,K=30	BS	0.9291	0.9291	0.9291	0.9291
	MM	0.9186	0.9192	0.9201	0.9205
	V_{LGB}	0.759	0.7571	0.7545	0.7531
S_t=30,K=30	BS	5.2735	5.2735	5.2735	5.2735
	MM	5.2136	5.2173	5.2222	5.2247
	V_{LGB}	4.3077	4.297	4.2823	4.2747
S_t=50,K=30	BS	21.7537	21.7537	21.7537	21.7537
	MM	21.5068	21.522	21.5423	21.5524
	V_{LGB}	17.7698	17.7256	17.6647	17.6335



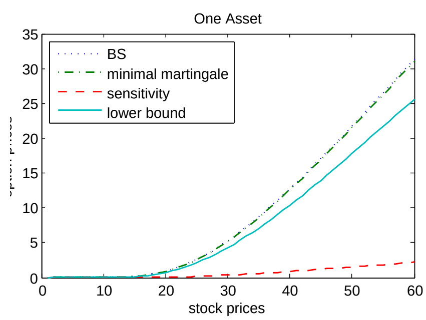


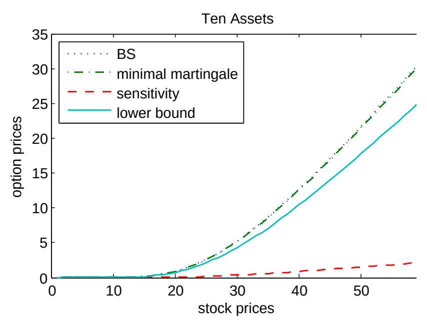


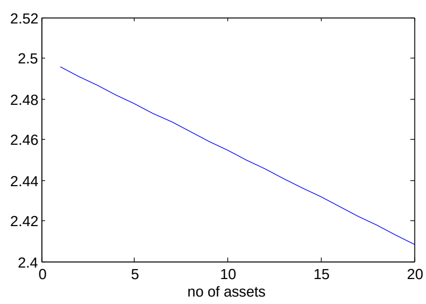


Introducing new assets into the portfolio

	Assets	1	10
S_t=20,K=30	φ	2.4955	2.4546
	Z	0.0647	0.0647
	V_{LGB}	0.759	0.7605
S_t=30,K=30	φ	2.4955	2.4546
	Z	0.3669	0.3669
	V_{LGB}	4.3077	4.3166
S_t=50,K=30	φ	2.4955	2.4546
	Z	1.5137	1.5137
	V_{LGB}	17.7698	17.8066

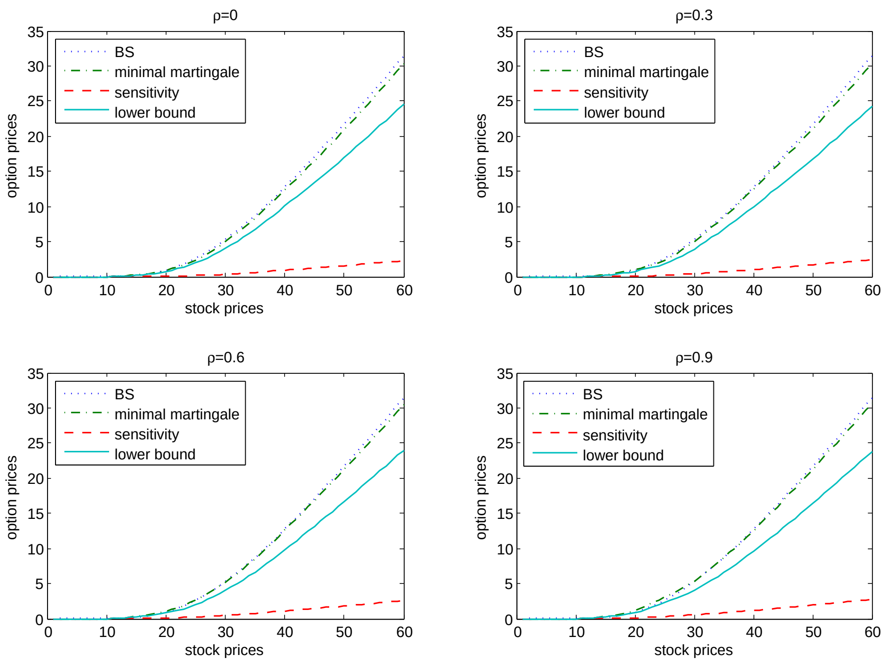






Introducing correlation between S_t and λ_t

	ρ	0	0.3	0.6	0.9
S_t=20,K=30	MM	0.9055	0.9984	1.0672	1.1718
	Z	0.0651	0.08761	0.1073	0.1267
	V_{LGB}	0.743	0.7798	0.7993	0.8556
S_t=30,K=30	MM	4.953	5.012	5.2048	5.4080
	Z	0.3633	0.41833	0.4839	0.5460
	V_{LGB}	4.0462	3.9681	3.9973	4.0455
S_t=50,K=30	MM	20.8957	21.01	21.2336	21.2646
	Z	1.5775	1.6959	1.8234	1.9407
	V_{LGB}	16.9589	16.7778	16.6833	17.1407



Chapter 4

A General Theory of Markovian Time Inconsistent Stochastic Control Problems

joint work with Tomas Bjork

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We develop a theory for stochastic control problems which, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. We attach these problems by viewing them within a game theoretic framework, and we look for Nash subgame perfect equilibrium points. For a general controlled Markov process and a fairly general objective functional we derive an extension of the standard Hamilton-Jacobi-Bellman equation, in the form of a system of non-linear equations, for the determination for the equilibrium strategy as well as the equilibrium value function. All known examples of time inconsistency in the literature are easily seen to be special cases of the present theory. We also prove that for every time inconsistent problem, there exists an associated time consistent problem such that the optimal control and the optimal value function for the consistent problem coincides with the equilibrium control and value function respectively for the time inconsistent problem. We also study some concrete examples.

1 Introduction

In a standard continuous time stochastic optimal control problem the object is that of maximizing (or minimizing) a functional of the form

$$E\left[\int_0^T C\left(s, X_s, u_s\right) ds + F(X_T)\right].$$

where X is some controlled Markov process, u_s is the control applied at time s, and F, C are given functions. A typical example is when X is a controlled scalar SDE of the form

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t,$$

with some initial condition $X_0 = x_0$. Later on in the paper we will allow for more general dynamics than those of an SDE, but in this informal section we restrict ourselves for simplicity to the SDE case.

1.1 Dynamic programming and time consistency

A standard way of attacking a problem like the one above is by using Dynamic Programming (henceforth DynP). We restrict ourselves to **control laws**, i.e., the control at time s, given that $X_s = y$, is of the form $\mathbf{u}(s, y)$ where the control law $\mathbf{u}$ is a deterministic function. We then embed the problem above in a family of problems indexed by the initial point. More precisely we consider, for every (t, x), the problem $\mathcal{P}_{t,x}$ of maximizing

$$E_{t,x}\left[\int_t^T C\left(s, X_s, u_s\right) ds + F(X_T)\right].$$

given the initial condition $X_t = x$. Denoting the optimal control law for $\mathcal{P}_{t,x}$ by $\mathbf{u}_{tx}(s, X_s)$ and the corresponding optimal value function by $V(t, x)$ we see that the original problem corresponds to the problem $\mathcal{P}_{0,x_0}$.

We note that *ex ante* the optimal control law $\mathbf{u}_{tx}(s, X_s)$ for the problem $\mathcal{P}_{t,x}$ must be indexed by the initial point (t, x). However, problems of the kind described above turn out to be **time consistent** in the sense that we have the **Bellman optimality principle**, which roughly says that the optimal control is independent of the initial point. More precisely: if a control law is optimal on the full time interval $[0, T]$, then it is also optimal for any subinterval $[t, T]$. Given the Bellman principle, it is easy to informally derive the Hamilton-Jacobi-Bellman (henceforth HJB) equation

$$\frac{\partial V}{\partial t}(t,x) + \sup_u \left\{ C(t,x,u) + \mu(t,x,u) + \frac{\partial V}{\partial x}(t,x)\frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 V}{\partial x^2}(t,x) \right\} = 0,$$
$$V(T,x) = F(x),$$

for the determination of V. One can (with considerable effort) show rigorously that, given enough regularity, the optimal value function will indeed satisfy the HJB equation. On can also (quite easily) prove a verification theorem which says that if V is a solution of the HJB equation, then V is indeed the optimal value function for the control problem, and the optimal control law is given by the maximizing u in the HJB equation.

We end this section by listing some important points concerning time consistency.

Remark 1.1 *The main reasons for the time consistency of the problem above are as follows.*

- *The integral term $C(s, X_s, u_s)$ in the problem $\mathcal{P}_{t,x}$ is allowed to depend on s, X_s and u_s. It is* **not** *allowed to depend on the initial point (t,x).*
- *The terminal evaluation term is allowed to be of the form $E_{t,x}[F(X_T)]$, i.e the expected value of a non-linear function of the terminal value X_T. Time consistency is then a relatively simple consequence of the law of iterated expectations. We are* **not** *allowed to have a term of the form $G(E_{t,x}[X_T])$, which is a non-linear function of the expected value.*
- *We are* **not** *allowed to let the terminal evaluation function F depend on the initial point (t,x).*

1.2 Three disturbing examples

We will now consider three seemingly simple examples from financial economics, where time consistency fail to hold. In all these cases we consider a standard Black-Scholes model for an underlying stock price S, as well as a bank account B with short rate r.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

We consider a self financing portfolio based on S and B where u_t is the number of dollars invested in the risky asset S, and c is the consumption rate at time t. Denoting the market value process of this portfolio by X, we have

$$dX_t = [rX_t + (\alpha - r)u_t - c_t]dt + \sigma u_t dW_t,$$

and we now consider three indexed families of optimization problems. In all cases the (naive) objective is to maximize the objective functional $J(t, x, \mathbf{u})$, where (t, x) is the initial point and $\mathbf{u}$ is the control law.

1. Hyperbolic discounting

$$J(t, x, \mathbf{u}) = E_{t,x}\left[\int_t^T \varphi(s-t)h(c_s)dt + \varphi(T-t)F(X_T)\right]$$

In this problem h is the local utility of consumption, F is the utility of terminal wealth, and φ is the **discounting function**. This problem differs from a standard problem by the fact that the initial point in time t enters in the integral (see Remark 1.1). Obviously; if φ is exponential so $\varphi(s-t) = e^{-a(s-t}$ then we can factor out e^{at} and convert the problem into a standard problem with objective functional

$$J(t, x, \mathbf{u}) = E_{t,x}\left[\int_t^T e^{-as}h(c_s)dt + e^{-aT}F(X_T)\right]$$

One can show, however, that **every** choice of the discounting function φ, apart from the the exponential, case, will lead to a time inconsistent problem. More precisely, the Bellman optimality principle will not hold.

2. Mean variance utility

$$J(t, x, \mathbf{u}) = E_{t,x}[X_T] - \frac{\gamma}{2}Var_{t,x}(X_T)$$

This case is a continuous time version of a standard Markowitz investment problem where we want to maximize utility of final wealth. The utility of final wealth is basically linear in wealth, as given by the term $E_{t,x}[X_T]$, but we penalize the risk by the conditional variance $\frac{\gamma}{2}Var_{t,x}(X_T)$. This looks innocent enough, but we recall the elementary formula

$$Var[X] = E[X^2] - E^2[X].$$

Now, in a standard time consistent problem we are allowed to have terms like $E_{t,x}[F(X_T)]$ in the objective functional, i.e. we are allowed to have the expected value of a non-linear function of terminal wealth.

In the present case, however we have the term $(E_{t,x}[X])^2$. This is not a non-linear function of terminal wealth, but instead a non-linear function of the expected value of terminal wealth, and we thus have a time inconsistent problem (see Remark 1.1).

3. Endogenous habit formation

$$J(t,x,\mathbf{u}) = E_{t,x}\left[\ln\left(X_T - x + \beta\right)\right], \quad \beta > 0.$$

In this case we basically want to maximize log utility of terminal wealth. In a standard problem we would have the objective $E_{t,x}\left[\ln\left(X_T - d\right)\right]$ where $d > 0$ is the lowest acceptable level of terminal wealth. In our problem, however, the lowest acceptable level of terminal wealth is given by $x - \beta$ and it thus depends on your wealth $X_t = x$ at time t. This again leads to a time inconsistent problem.

1.3 Approaches to handle time inconsistency

In all the three examples of the previous subsection we are faced with a time inconsistent family of problems, in the sense that if for some fixed initial point (t,x) we determine the control law $\hat{\mathbf{u}}$ which maximizes $J(t,x,\mathbf{u})$, then at some later point (s,X_s) the control law $\hat{\mathbf{u}}$ will no longer be optimal for the functional $J(s,X_s,\mathbf{u})$. It is thus conceptually unclear what we mean by "optimality" and even more unclear what we mean by "an optimal control law", so our first task is to specify more precisely exactly which problem we are trying to solve. There are then at least three different ways of handling a family of time inconsistent problems, like the ones above

- We dismiss the entire problem as being silly.
- We fix **one** initial point, like for example $(0,x_0)$, and then try to find the control law $\hat{\mathbf{u}}$ which maximizes $J(0,x_0,\mathbf{u})$. We then simply disregard the fact that at a later points in time such as (s,X_s) the control law $\hat{\mathbf{u}}$ will not be optimal for the functional $J(s,X_s,\mathbf{u})$. In the economics literature, this is known as **pre-commitment**.
- We take the time inconsistency seriously and formulate the problem in game theoretic terms.

All of the three strategies above may in different situations be perfectly reasonable, but in the present paper we choose the last one. The basic idea is then that when we decide on a control action at time t we should explicitly take into account that at future times we will have a different objective functional or, in more loose terms, "our tastes are changing over time". We can then view the entire problem as a non-cooperative game, with one player for each time t, where player t can be viewed as the future incarnation of ourselves (or rather of our preferences) at time t. Given this point of view, it is natural to look for Nash equilibria for the game, and this is exactly our approach.

In continuous time it is far from trivial to formulate this intuitive idea in precise terms. We will do this in the main text below but a rough picture of the game is as follows.

- We consider a game with one player at each point t in time. This player is referred to as "player t". You may think of player t as a future incarnation of yourself, but conceptually it may be easier to think of the game as being played by a continuum of completely different individuals.
- Depending on t and on the position x in space, player t will choose a control action. This action is denoted by $\mathbf{u}(t, x)$, so the strategy chosen by player t is given by the mapping $x \longmapsto \mathbf{u}(t, x)$.
- A control law $\mathbf{u}$ can thus be viewed as a complete description of the chosen strategies of all players in the game.
- The reward to player t is given by the functional $J(t, x, \mathbf{u})$. We note that in the three examples of the previous section it is clear that $J(t, x, \mathbf{u})$ does not depend on the actions taken by any player s for $s < t$, so in fact J does only depend on the restriction of the control law $\mathbf{u}$ to the time interval $[t, T]$. It is also clear that this is really a game, since the reward to player t does not only depend on the strategy chosen by himself, but also by the strategies chosen by all players coming after him in time.

We can now loosely define the concept of a "Nash subgame perfect equilibrium point" of the game. This is a control law $\hat{\mathbf{u}}$ satisfying the following condition:

- Choose an arbitrary point t in time.
- Suppose that every player s, for all $s > t$, will use the strategy $\hat{\mathbf{u}}(s, \cdot)$.

- Then the optimal choice for player t is that he/she also uses the strategy $\hat{\mathbf{u}}(t, \cdot)$.

The problem with this "definition" in continuous time is that the individual player t does not really influence the outcome of the game at all. He/she only chooses the control at the single point t, and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. For a proper definition we need some sort of limiting argument, which is given in the main text below.

1.4 Previous literature

The game theoretic approach to time inconsistency using Nash equilibrium points as above has a long history starting with Strotz (1955) where a deterministic Ramsay problem is studied. Further work along this line in continuous and discrete time is provided in Goldman (1980), Krusell and Smith (2003), Peleg and Menahem (1973), and Pollak (1968).

Recently there has been renewed interest in these problems. In the interesting, and mathematically very advanced, papers Ekeland and Lazrak (2006) and Ekeland and Pirvu (2007), the authors consider optimal consumption and investment under hyperbolic discounting (Problem 1 in our list above) in deterministic and stochastic models from the above game theoretic point of view. To our knowledge, these papers were the first to provide a precise definition of the equilibrium concept in continuous time. The authors derive, among other things, an extension of the HJB equation to a system of PDEs including an integral term, and they also provide a rigorous verification theorem. They also, in a tour de force, derive an explicit solution for the case when the discounting function is a weighted sum of two exponential discount functions. Our present paper is much inspired by these papers, in particular for the definition of the equilibrium law.

In Basak and Chabakauri (2008) the authors undertake a deep study of the mean variance problem within a Wiener driven framework. This is basically Problem 2 in the list above, but the authors also consider the case of multiple assets, as well as the case of a hidden Markov process driving the parameters of the asset price dynamics. The authors derive an extension of the Hamilton Jacobi Bellman equation and manage, by a number of very clever ideas, to solve this equation explicitly for the basic problem, and also for the above mentioned extensions. The paper has two limitations: Firstly, from a mathematical perspective it is somewhat heuristic, the equilibrium concept is never given a precise definition, and no verification theorem is provided. Secondly, and more importantly, the methodology depends heavily on the use of a "total variance formula", which in some sense (partially) replaces the iterated expectations formula in a standard problem. This implies that the basic methodology cannot be extended beyond the mean variance case. The paper is extremely thought provoking, and we have benefited greatly from reading it.

1.5 Contributions of the present paper

The object of the present paper is to undertake a rigorous study of time inconsistent control problems in a reasonably general Markovian framework, and in particular we do not want to tie ourselves down to a particular applied problem. We have therefore chosen a setup of the following form.

- We consider a **general controlled Markov process** X, living on some suitable space (details are given below). It is important to notice that we do not make any structural assumptions whatsoever about X, and we note that the setup obviously includes the case when X is determined by a system of SDEs driven by a Wiener and a point process.
- We consider a functional of the form

$$J(t,x,\mathbf{u}) = E_{t,x}\left[\int_t^T C\left(x, X_s^{\mathbf{u}}, \mathbf{u}(X_s^{\mathbf{u}})\right) ds + F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right).$$

We see that with the choice of functional above, time inconsistency enters at several different points. Firstly we have the appearance of the present state x in the local utility function C, as well as in the functions F and G, and this leads of course to time inconsistency. Secondly, in the term $G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right)$ we have, even forgetting about the appearance of x, a non linear function G acting on the conditional expectation, again leading to time inconsistency.

Note that, for notational simplicity we have not explicitly included dependence on running time t. This can always be done by letting running time be one component of the state process X, so the setup above also allows for expressions like $F(t,x,X_T^{\mathbf{u}})$ etc, thus allowing (among many other things) for hyperbolic discounting.

This setup is studied in some detail in continuous as well as in discrete time. The discrete time results are parallel to those in continuous time, and our main results in continuous time are as follows.

- We provide a precise definition of the Nash equlibrium concept. (This is done along the lines of Ekeland and Lazrak (2006) and Ekeland and Pirvu (2007)).
- We derive an extension of the standard Hamilton-Jacobi-Bellman equation to a non standard system of equations for the determination of the equilibrium value function V.
- We prove a verification theorem, showing that the solution of the extended HJB system is indeed the equilibrium value function, and that the equilibrium strategy is given by the optimizer in the equation system.
- We prove that to every time inconsistent problem of the form above, there exists an associated **standard, time consistent**, control problem with the following properties:
  - The optimal value function for the standard problem coincides with the equilibrium value function for the time inconsistent problem.
  - The optimal control law for the standard problem coincides with the equilibrium startegy for the time inconsistent problem.
- We solve some specific test examples.

Our framework and results extends the existing theory considerably. As we noted above, hyperbolic discounting is included as a special case of the theory. The mean variance example from above is of course also included. More precisely it is easy to see that it corresponds to the case when

$$C = 0, \quad F(x,y) = y - \frac{\gamma}{2}y^2, \quad G(x,y) = \frac{\gamma}{2}y^2$$

We thus extend the existing literature by allowing for a considerably more general utility functional, and a completely general Markovian structure. The existence of the associated equivalent standard control problem is to our knowledge a completely new result.

1.6 Structure of the paper

Since the equilibrium concept in continuous time is a very delicate one, we start by studying a discrete time version of our problem in Section 2. In discrete time there are no conceptual problems with the equilibrium concept, but the arguments are sometimes quite delicate, the expressions are rather complicated, and great care has to be taken. It is in fact in this section that the main work is done. In Section 3 we study the continuous time problem by taking formal limits for a discretized problem, and using the results of the Section 2. This leads to an extension of the standard HJB equation to a system of equations with an embedded static optimization problem. The limiting procedure described above is done in an informal manner and it is largely heuristic, so in order to prove that the derived extension of the HJB equation is indeed the correct one we also provide a rigorous proof of a verification theorem. In Section 4 we prove the existence of the associated standard control problem, and in Section 5 we study some examples.

2 Discrete time

Since the theory is conceptually much easier in discrete time than in continuous time, we start by presenting the discrete time version.

2.1 Setup

We consider a given **controlled Markov process** X, evolving on a measurable state space $\{\mathcal{X}, \mathcal{G}_X\}$, with controls taking values in a measurable control space $\{\mathcal{U}, \mathcal{G}_U\}$. The action is in discrete time, indexed by the set $\mathbf{N}$ of natural numbers. The intuitive idea is that if $X_n = x$, then we can choose a control $u_n \in \mathcal{U}$, and this control will affect the transition probabilities from X_n to X_{n+1}. This idea is formalized by specifying a family of **transition probabilities**,

$$\{p_n^u(dz; x) : n \in \mathbf{N}, x \in \mathcal{X}, u \in \mathcal{U}\}.$$

For every fixed $n \in \mathbf{N}$, $x \in \mathcal{X}$ and $u \in \mathcal{U}$, we assume that $p_n^u(\cdot; x)$ is a probability measure on $\mathcal{X}$, and for each $A \in \mathcal{G}_X$, the probability $p_n^u(A; x)$ is jointly measurable in (x, u). The interpretation of this is that $p_n^u(dz; x)$ is the probability distribution of X_{n+1}, given that $X_n = x$, and that we at time n apply the control u, i.e.,

$$p_n^u(dz; x) = P\left(X_{n+1} \in dz \,|X_n = x, u_n = u\right).$$

To obtain a Markov structure, we restrict the controls to be **feedback control laws**, i.e. at time n, the control u_n is allowed to depend on time n and state X_n. We can thus write

$$u_n = \mathbf{u}_n(X_n),$$

where the mapping $\mathbf{u} : \mathbf{N} \times \mathcal{X} \to \mathcal{U}$ is measurable. Note the boldface notation for the mapping $\mathbf{u}$. In order to distinguish between functions and function values, we will always denote a control law (i.e. a mapping) by using boldface, like $\mathbf{u}$, whereas a possible value of the mapping will be denoted without boldface, like, $u \in \mathcal{U}$.

Given the family of transition probabilities we may define a corresponding family of operators, operating on function sequences.

Definition 2.1 *A* **function sequence** *is a mapping $f : \mathbf{N} \times \mathcal{X} \to R$, where we use the notation $(n, x) \longmapsto f_n(x)$.*

- *For each $u \in \mathcal{U}$, the operator $\mathbf{P}^u$, acting on the set of integrable function sequences, is defined by*

  $$\left(\mathbf{P}^u f\right)_n(x) = \int_{\mathcal{X}} f_{n+1}(z) p_n^u(dz, x).$$

  The corresponding discrete time "infinitesimal" operator $\mathbf{A}^u$ is defined by

  $$\mathbf{A}^u = \mathbf{P}^u - \mathbf{I},$$

  where $\mathbf{I}$ is the identity operator.

- *For each control law $\mathbf{u}$ the operator $\mathbf{P}^{\mathbf{u}}$ is defined by*

  $$\left(\mathbf{P}^{\mathbf{u}} f\right)_n(x) = \int_{\mathcal{X}} f_{n+1}(z) p_n^{\mathbf{u}_n(x)}(dz, x),$$

  and $\mathbf{A}^{\mathbf{u}}$ is defined correspondingly as

  $$\mathbf{A}^{\mathbf{u}} = \mathbf{P}^{\mathbf{u}} - \mathbf{I},$$

In more probabilistic terms we have the interpretation.

$$\left(\mathbf{P}^u f\right)_n(x) = E\left[\left. f_{n+1}(X_{n+1}) \right| X_n = x, u_n = u\right],$$

and $\mathbf{A}^u$ is the discrete time version of the continuous time infinitesimal operator. We immediately have the following result.

Proposition 2.1 *Consider a real valued function sequence $\{f_n(x)\}$, and a control law $\mathbf{u}$. The process $f_n(X_n^{\mathbf{u}})$ is then a martingale under the measure induced by $\mathbf{u}$ if and only if the sequence $\{f_n\}$ satisfies the equation*

$$\left(\mathbf{A}^{\mathbf{u}} f\right)_n (x) = 0, \quad n = 0, 1, \ldots, T-1.$$

Proof. Obvious from the definition of $\mathbf{A}^{\mathbf{u}}$. ∎

It is clear that for a fixed initial point (n, x) and a fixed control law $\mathbf{u}$ we may in the obvious way define a Markov process denoted by $X^{n,x,\mathbf{u}}$, where for notational simplicity we often drop the upper index n, x and use the notation $X^{\mathbf{u}}$. The corresponding expectation operator is denoted by $E_{n,x}^{\mathbf{u}}[\cdot]$, and we often drop the upper index $\mathbf{u}$, and instead use the notation $E_{n,x}[\cdot]$. A typical example of an expectation will thus have the form $E_{n,x}[F(X_k^{\mathbf{u}})]$ for some real valued function F and some point in time k.

2.2 Basic problem formulation

For a fixed $(n, x) \in \mathbf{N} \times \mathcal{X}$, a fixed control law $\mathbf{u}$, and a fixed time horizon T, we consider the functional

$$J_n(x, \mathbf{u}) = E_{n,x}\left[\sum_{k=n}^{T-1} C\left(x, X_k^{\mathbf{u}}, \mathbf{u}_k(X_k^{\mathbf{u}})\right) + F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{n,x}\left[X_T^{\mathbf{u}}\right]\right), \tag{4.1}$$

Obviously, the functional J depends only on the restriction of the control law $\mathbf{u}$ to the time set $k = n, n+1, \ldots, T-1$.

The intuitive idea is that we are standing at (n, x) and that we would like to choose a control law $\mathbf{u}$ which maximizes J. We can thus define an indexed family of problems $\{\mathcal{P}_{n,x}\}$ by

$$\mathcal{P}_{n,x}: \qquad \max_{\mathbf{u}} J_n(x, \mathbf{u}),$$

where *max* is shorthand for the imperative "maximize!". The complicating factor here is that the family $\{\mathcal{P}_{n,x}\}$ is time inconsistent in the sense that if $\hat{\mathbf{u}}$ is optimal for $\mathcal{P}_{n,x}$, then the restriction of $\hat{\mathbf{u}}$ to the time set $k, k+1, \ldots, T$ (for $k > n$) is not necessarily optimal for the problem $\mathcal{P}_{k,X_k^{\mathbf{u}}}$. There are two reasons for this time inconsistency:

- The shape of the utility functional depends explicitly on the initial position x in space, as can be seen in the appearance of x in the expression $F(x, X_T)$ and similarly for the other terms. In other words, as the X process moves around, our utility function changes, so at time t this part of the utility function will have the form $F(X_t, X_T)$.

- For a standard time consistent control problem we are allowed to have expressions like $E_{n,x}[G(X_T)]$ in the utility function, i.e. we are allowed to have the expected value of a non linear function G of the future process value. Time consistency is then a relatively simple consequence of the law of iterated expectations. In our problem above, however, we have an expression of the form $G\left(E_{n,x}\left[X_T^{\mathbf{u}}\right]\right)$, which is not the expectation of a non linear function, but a nonlinear function of the expected value. We thus do not have access to iterated expectations, so the problem becomes time inconsistent. On top of this we also have the appearance of the present state x in the expression $G\left(x, E_{n,x}\left[X_T^{\mathbf{u}}\right]\right)$.

The moral of all this is that we have a family of time inconsistent problems or, alternatively, we have time inconsistent preferences. If we at some point (n, x) decide on a feedback law $\hat{\mathbf{u}}$ which is optimal from the point of view of (n, x) then as time goes by, we will no longer consider $\hat{\mathbf{u}}$ to be optimal. To handle this problem we will, as outlined above, take a game theoretic approach and we now go on the describe this in some detail.

2.3 The game theoretic formulation

The idea, which appears already in Strotz (1955), is to view the setup above in game theoretic terms. More precisely we view it as a non-cooperative game where we have one player at each point n in time. We refer to this player as "player number n" and the rule is that player number n can only choose the control u_n. One interpretation is that these players are different future incarnations of yourself (or rather incarnations of your future preferences), but conceptually it is perhaps easier to think of it as one separate player at each n.

Given the data (n, x), player number n would, in principle, like to maximize $J_n(x, \mathbf{u})$ over the class of feedback controls $\mathbf{u}$, but since he can only choose the control $\mathbf{u}_n$, this is not possible. Instead of looking for "optimal" feedback laws, we take the game theoretic point of view and study so called subgame perfect Nash equilibrium strategies. The formal definition is as follows.

Definition 2.2 *We consider a fixed control law* $\hat{\mathbf{u}}$ *and make the following construction.*

1. *Fix an arbitrary point* (n, x) *where* $n < T$*, and choose an arbitrary control value* $u \in \mathcal{U}$*.*
2. *Now define the control law* $\bar{\mathbf{u}}$ *on the time set* $n, n+1, \ldots, T-1$ *by setting, for any* $y \in \mathcal{X}$*,*

$$\bar{\mathbf{u}}_k(y) = \begin{cases} \hat{\mathbf{u}}_k(y), & \text{for} \quad k = n+1, \ldots, T-1, \\ u, & \text{for} \quad k = n. \end{cases}$$

We say that $\hat{\mathbf{u}}$ *is a subgame perfect Nash* **equilibrium** *strategy if, for every fixed* (n, x)*, the following condition hold*

$$\sup_{u \in \mathcal{U}} J_n(x, \bar{\mathbf{u}}) = J_n(x, \hat{\mathbf{u}}).$$

If an equlibrium control $\hat{\mathbf{u}}$ *exists, we define the* **equilibrium value function** V *by*

$$V_n(x) = J_n(x, \hat{\mathbf{u}}).$$

In more pedestrian terms this means that if player number n knows that all players coming after him will use the control $\hat{\mathbf{u}}$, then it is optimal for player number n also to use $\hat{\mathbf{u}}_n$.

Remark 2.1 *An equivalent, and perhaps more concrete, way of describing an equilibrium strategy is as follows.*

- *The equilibrium control* $\hat{\mathbf{u}}_{T-1}(x)$ *is obtained by letting player* $T-1$ *optimize* $J_{T-1}(x, \mathbf{u})$ *over* u_{T-1} *for all* $x \in \mathcal{X}$*. This is a standard optimization problem without any game theoretic components.*
- *The equilibrium control* $\hat{\mathbf{u}}_{T-2}$ *is obtained by letting player* $T-2$ *choose* u_{T-2} *to optimize* J_{T-2}*, given the knowledge that player number* $T-1$ *will use* $\hat{\mathbf{u}}_{T-1}$*.*
- *Proceed recursively by induction.*

Obviously; for a standard time consistent control problem, the game theoretic aspect becomes trivial and the equilibrium control law coincides with the standard (time consistent) optimal law. The equilibrium value function

V will coincide with the optimal value function and, using dynamic programming arguments, V is seen to satisfy a standard Bellman equation.

The main result of the present paper is that in the time inconsistent case, the equilibrium value function V will satisfy a system of non linear equations. This system of equations extend the standard Bellman equation, and for a time consistent problem they reduce to the Bellman equation.

2.4 The extended Bellman equation

In this section we assume that there exists an equilibrium control law $\hat{\mathbf{u}}$ (which may not be unique) and we consider the corresponding equilibrium value function V defined above. The goal of this section is to derive an system of equations, extending the standard Bellman equation, for the determination of V. This will be done in the following two steps:

- For an arbitrarily chosen control law $\mathbf{u}$, we will derive a recursive equation for $J_n(x, \mathbf{u})$.
- We will then fix (n, x) and consider two control laws. The first one is the equilibrium law $\hat{\mathbf{u}}$, and the other one is the law $\mathbf{u}$ where we choose $u = \mathbf{u}_n(x)$ arbitrarily, but follow the law $\hat{\mathbf{u}}$ for all k with $k = n+1, \dots T-1$. The trivial observation that

$$\sup_{u \in \mathcal{U}} J_n(x, \mathbf{u}) = J_n(x, \hat{\mathbf{u}}) = V_n(x),$$

  will finally give us the extension of the Bellman equation.

The reader with experience from dynamic programming (DynP) will recoginize that the general program above is in fact more or less the same as for standard (time consistent) DynP. However, in the present time inconsistent setting, the derivation of the recursion in the first step is much more tricky than in the corresponding step from DynP, and it also requires some completely new constructions.

The recursion for $J_n(x, \mathbf{u})$

In order to derive the recursion for $J_n(x, \mathbf{u})$ we consider an arbitrary initial point (n, x), and we consider an arbitrarily chosen control law $\mathbf{u}$. The value taken by $\mathbf{u}$ at (n, x) will play a special role in the sequel, and for ease of reading we will use the notation $\mathbf{u}_n(x) = u$.

We now go on to derive a recursion between J_n and J_{n+1}. This is conceptually rather delicate, and sometimes a bit messy. In order to increase readability we therefore carry out the derivation only for the case when the objective functional does not contain the sum $\sum_{k=n}^{T-1} C\left(x, X_k^{\mathbf{u}}, \mathbf{u}_k(X_k^{\mathbf{u}})\right)$ in (4.1), and thus has has the simpler form

$$J_n(x, \mathbf{u}) = E_{n,x}\left[F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{n,x}\left[X_T^{\mathbf{u}}\right]\right). \tag{4.2}$$

We provide the result for the general case in Section 2.4. The derivation of this is completely parallel to that of the simplified case.

We start by making the observation that X_{n+1} will only depend on x and on the control value $\mathbf{u}_n(x) = u$ motivating the notation X_{n+1}^u. The distribution of X_k for $k < n+1$ will, on the other hand depend on the control law $\mathbf{u}$ (restricted to the interval $[n, k]$) so for $k > n+1$ we use the notation $X_k^{\mathbf{u}}$.

We now go on to the recursion arguments. From the definition of J we have

$$J_{n+1}(X_{n+1}^u, \mathbf{u}) = E_{n+1}\left[F(X_{n+1}^u, X_T^{\mathbf{u}})\right] + G\left(X_{n+1}^u, E_{n+1}\left[X_T^{\mathbf{u}}\right]\right), \tag{4.3}$$

where for simplicity of notation we write $E_{n+1}\left[\cdot\right]$ instead of $E_{n+1, X_{n+1}^u}\left[\cdot\right]$. We now make the following definitions which will play a central role in the sequel.

Definition 2.3 *For any control law* $\mathbf{u}$*, we define the function sequences* $\{f_n^{\mathbf{u}}\}$ *and* $\{g_n^{\mathbf{u}}\}$*, where* $f_n^{\mathbf{u}}, g_n^{\mathbf{u}} : \mathcal{X} \times \mathcal{X} \to \mathbf{R}$ *by.*

$$\begin{aligned} f_n^{\mathbf{u}}(x, y) &= E_{n,x}\left[F\left(y, X_T^{\mathbf{u}}\right)\right], \\ g_n^{\mathbf{u}}(x) &= E_{n,x}\left[X_T^{\mathbf{u}}\right]. \end{aligned}$$

We also introduce the notation

$$f_n^{\mathbf{u},y}(x) = f_n^{\mathbf{u}}(x, y).$$

The difference between $f_n^{\mathbf{u},y}$ and $f_n^{\mathbf{u}}$, is that we view $f_n^{\mathbf{u}}$ as a function of the two variables x and y, whereas $f_n^{\mathbf{u},y}$ is, for a fixed y, viewed as a function of the single variable x.

From the definitions above it is obvious that, for any fixed y, the processes $f_n^{\mathbf{u},y}(X_n^{\mathbf{u}})$ and $g_n^{\mathbf{u}}(X_n^{\mathbf{u}})$ are martingales under the measure generated by $\mathbf{u}$. We thus have the following result.

Lemma 2.1 *For every fixed control law* $\mathbf{u}$ *and every fixed choice of* $y \in \mathcal{X}$*, the function sequence* $\{f_n^{\mathbf{u},y}\}$ *satisifes the recursion*

$$\begin{aligned} \left(\mathbf{A}^{\mathbf{u}} f^{\mathbf{u},y}\right)_n(x) &= 0, \quad n = 0, 1, \ldots, T-1. \\ f_T^{\mathbf{u},y}(x) &= F(y, x). \end{aligned}$$

The sequence $\{g_n^{\mathbf{u}}\}$ satisifes the recursion

$$\begin{aligned} \left(\mathbf{A}^{\mathbf{u}} g^{\mathbf{u}}\right)_n (x) &= 0, \quad n = 0, 1, \ldots, T-1. \\ g_T^{\mathbf{u}}(x) &= x. \end{aligned}$$

Going back to (4.3) we note that, from the Markovian structure and the definitions above, we have

$$\begin{aligned} E_{n+1}\left[F(X_{n+1}^u, X_T^{\mathbf{u}})\right] &= f_{n+1}^{\mathbf{u}}(X_{n+1}^u, X_{n+1}^u) \\ E_{n+1}\left[X_T^{\mathbf{u}}\right] &= g_{n+1}^{\mathbf{u}}(X_{n+1}^u). \end{aligned}$$

We can now write (4.3) as

$$J_{n+1}(X_{n+1}^u, \mathbf{u}) = f_{n+1}^{\mathbf{u}}(X_{n+1}^u, X_{n+1}^u) + G\left(X_{n+1}^u, g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right).$$

Taking expectations gives us

$$E_{n,x}\left[J_{n+1}(X_{n+1}^u, \mathbf{u})\right] = E_{n,x}\left[f_{n+1}^{\mathbf{u}}(X_{n+1}^u, X_{n+1}^u)\right] + E_{n,x}\left[G\left(X_{n+1}^u, g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right)\right],$$

and, going back to the definition of $J_n(x, \mathbf{u})$, we can write this as

$$\begin{aligned} E_{n,x}\left[J_{n+1}(X_{n+1}^u, \mathbf{u})\right] &= J_n(x, \mathbf{u}) \\ &+ E_{n,x}\left[f_{n+1}^{\mathbf{u}}(X_{n+1}^u, X_{n+1}^u)\right] - E_{n,x}\left[F(x, X_T^{\mathbf{u}})\right] \\ &+ E_{n,x}\left[G\left(X_{n+1}^u, g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right)\right] - G\left(x, E_{n,x}\left[X_T^{\mathbf{u}}\right]\right). \end{aligned}$$

At this point it may seem natural to use the identities $E_{n,x}\left[F(x, X_T^{\mathbf{u}})\right] = f_n^{\mathbf{u}}(x, x)$ and $E_{n,x}\left[X_T^{\mathbf{u}}\right] = g_n^{\mathbf{u}}(x)$, but for various reasons this is not a good idea. Instead we note that

$$E_{n,x}\left[F(x, X_T^{\mathbf{u}})\right] = E_{n,x}\left[E_{n+1}\left[F(x, X_T^{\mathbf{u}})\right]\right] = E_{n,x}\left[f_{n+1}^{\mathbf{u}}(X_{n+1}^u, x)\right],$$

and that

$$E_{n,x}\left[X_T^{\mathbf{u}}\right] = E_{n,x}\left[E_{n+1}\left[X_T^{\mathbf{u}}\right]\right] = E_{n,x}\left[g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right].$$

Substituting these identities into the recursion above, we can now collect the findings so far.

Lemma 2.2 *The value function J satisfies the following recursion.*

$$\begin{aligned} J_n(x, \mathbf{u}) &= E_{n,x}\left[J_{n+1}(X_{n+1}^u, \mathbf{u})\right] \\ &- \left\{E_{n,x}\left[f_{n+1}^{\mathbf{u}}(X_{n+1}^u, X_{n+1}^u)\right] - E_{n,x}\left[f_{n+1}^{\mathbf{u}}(X_{n+1}^u, x)\right]\right\} \\ &- \left\{E_{n,x}\left[G\left(X_{n+1}^u, g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right)\right] - G\left(x, E_{n,x}\left[g_{n+1}^{\mathbf{u}}(X_{n+1}^u)\right]\right)\right\}. \end{aligned}$$

The recursion for $V_n(x)$

We will now derive the fundamental equation for the determination of the equlibrium function $V_n(x)$. In order to do this we assume that there exists an equilibrium control $\hat{\mathbf{u}}$. We then fix an arbitrarily chosen initial point (n, x) and consider two strategies (control laws).

1. The first control law is simply the equilibrium law $\hat{\mathbf{u}}$.
2. The second control law $\mathbf{u}$ is slightly more complicated. We choose an arbitrary point $u \in \mathcal{U}$ and then defined the control law $\mathbf{u}$ as follows

$$\mathbf{u}_k(y) = \begin{cases} u, & \text{for } k = n, \\ \hat{\mathbf{u}}_k(y), & \text{for } k = n+1, \ldots, T-1. \end{cases}$$

We now compare the objective function J_n for these two control laws. Firstly, and by definition, we have

$$J_n(x, \hat{\mathbf{u}}) = V_n(x),$$

where V is the equilibrium value function defined earlier. Secondly, and also by definition, we have

$$J_n(x, \mathbf{u}) \leq J_n(x, \hat{\mathbf{u}}),$$

for all choices of $u \in \mathcal{U}$. We thus have the inequality

$$J_n(x, \mathbf{u}) \leq V_n(x),$$

for all $u \in \mathcal{U}$, with equality if $u = \hat{\mathbf{u}}_n(x)$. We thus have the basic relation

$$\sup_{u \in \mathcal{U}} J_n(x, \mathbf{u}) = V_n(x). \tag{4.4}$$

We now make a small variation of Definition 4.3.

Definition 2.4 *For arbitrary (k, z, y) we define the function sequences $\{f_k\}_{k=0}^T$ and $\{g_k\}_{k=0}^T$, where $f_k, g_k : \mathcal{X} \times \mathcal{X} \to \mathbf{R}$ by.*

$$\begin{aligned} f_k(z, y) &= E_{n,z}\left[F\left(y, X_T^{\hat{\mathbf{u}}}\right)\right], \\ g_n(z) &= E_{n,z}\left[X_T^{\hat{\mathbf{u}}}\right]. \end{aligned}$$

We also introduce the notation

$$f_k^y(z) = f_k(z, y).$$

Using Lemma 2.2, the basic relation (4.4) now reads

$$\sup_{u\in\mathcal{U}} \Big\{ E_{n,x}\left[J_{n+1}(X^u_{n+1},\mathbf{u})\right] - V_n(x)$$

$$- \left(E_{n,x}\left[f^{\mathbf{u}}_{n+1}(X^u_{n+1},X^u_{n+1})\right] - E_{n,x}\left[f^{\mathbf{u}}_{n+1}(X^u_{n+1},x)\right]\right)$$

$$- \left(E_{n,x}\left[G\left(X^u_{n+1}, g^{\mathbf{u}}_{n+1}(X^u_{n+1})\right)\right] - G\left(x, E_{n,x}\left[g^{\mathbf{u}}_{n+1}(X^u_{n+1})\right]\right)\right)\Big\} = 0.$$

W now observe that, since the control law $\mathbf{u}$ conicides with the equilibrium law $\hat{\mathbf{u}}$ on $[n+1, T-1]$, we have the following equalities

$$\begin{array}{rcl} J_{n+1}(X^u_{n+1},\mathbf{u}) & = & V_{n+1}\left(X^u_{n+1}\right), \\ f^{\mathbf{u}}_{n+1}(X^u_{n+1},x) & = & f_{n+1}(X^u_{n+1},x), \\ g^{\mathbf{u}}_{n+1}(X^u_{n+1}) & = & g_{n+1}(X^u_{n+1}). \end{array}$$

We can thus write the recursion as

$$\sup_{u\in\mathcal{U}} \Big\{ E_{n,x}\left[V_{n+1}\left(X^u_{n+1}\right)\right] - V_n(x)$$

$$- \left(E_{n,x}\left[f_{n+1}(X^u_{n+1},X^u_{n+1})\right] - E_{n,x}\left[f_{n+1}(X^u_{n+1},x)\right]\right)$$

$$- \left(E_{n,x}\left[G\left(X^u_{n+1}, g_{n+1}(X^u_{n+1})\right)\right] - G\left(x, E_{n,x}\left[g_{n+1}(X^u_{n+1})\right]\right)\right)\Big\} = 0.$$

The first line in this equation can be rewritten as

$$E_{n,x}\left[V_{n+1}\left(X^u_{n+1}\right)\right] - V_n(x) = \left(\mathbf{A}^u V\right)_n(x).$$

The second line can be written as

$$\begin{array}{rl} & E_{n,x}\left[f_{n+1}(X^u_{n+1},X^u_{n+1})\right] - E_{n,x}\left[f_{n+1}(X^u_{n+1},x)\right] \\ = & E_{n,x}\left[f_{n+1}(X^u_{n+1},X^u_{n+1})\right] - f_n(x,x) - \left(E_{n,x}\left[f_{n+1}(X^u_{n+1},x) - f_n(x,x)\right]\right) \\ = & \left(\mathbf{A}^u f\right)_n(x,x) - \left(\mathbf{A}^u f^x\right)_n(x). \end{array}$$

To avoid misunderstandings: The first term $\left(\mathbf{A}^u f\right)_n(x,x)$, can be viewed as the operator $\mathbf{A}^u$ operating on the function sequence $\{h\}_n$ defined by $h_n(x) = f_n(x,x)$. In the second term, $\mathbf{A}^u$ is operating on the function sequence $f^x_n(\cdot)$ where the upper index x is viewed as a fixed parameter.

We rewrite the third line of the recursion as

$$\begin{array}{l} E_{n,x}\left[G\left(X^u_{n+1}, g_{n+1}(X^u_{n+1})\right)\right] - G\left(x, E_{n,x}\left[g_{n+1}(X^u_{n+1})\right]\right) \\ = E_{n,x}\left[G\left(X^u_{n+1}, g_{n+1}(X^u_{n+1})\right)\right] - G(x, g_n(x)) \\ - \left\{G\left(x, E_{n,x}\left[g_{n+1}(X^u_{n+1})\right]\right) - G(x, g_n(x))\right\}. \end{array}$$

In order to simplify this we need to introduce some new notation.

Definition 2.5 *The function sequence $\{G \diamond g\}_k$ and, for a fixed $z \in \mathcal{X}$, the mapping $G^z : \mathcal{X} \to \mathbf{R}$ are defined by*

$$
\begin{aligned}
(G \diamond g)_k(y) &= G(y, g_k(y)), \\
G^z(y) &= G(z, y).
\end{aligned}
$$

With this notation we can write

$$
\begin{aligned}
&E_{n,x}\left[G\left(X^u_{n+1}, g_{n+1}(X^u_{n+1})\right)\right] - G\left(x, E_{n,x}\left[g_{n+1}(X^u_{n+1})\right]\right) \\
&= \mathbf{A}^u (G \diamond g)_n(x) - \{G^x(\mathbf{P}^u g_n(x)) - G^x(g_n(x))\}.
\end{aligned}
$$

We now introduce the last piece of new notation.

Definition 2.6 *With notation as above we define the function sequence $\left\{\mathbf{H}^u_g G\right\}_k$ by*

$$
\left\{\mathbf{H}^u_g G\right\}_n(x) = G^x(\mathbf{P}^u g_n(x)) - G^x(g_n(x)).
$$

Finally, we may state the main result for discrete time models.

Theorem 2.1 *Consider a functional of the form (4.2), and assume that an equilibrium control law $\hat{\mathbf{u}}$ exists. Then the the equilibrium value function V satisfies the following equation.*

$$
\begin{aligned}
\sup_{u \in \mathcal{U}} \quad & \left\{(\mathbf{A}^u V)_n(x) - (\mathbf{A}^u f)_n(x,x) + (\mathbf{A}^u f^x)_n(x)\right. && (4.5) \\
& \left. - \mathbf{A}^u (G \diamond g)_n(x) + \mathbf{H}^u_g G_n(x)\right\} = 0, \\
& V_T(x) = F(x,x) + G(x,x), && (4.6)
\end{aligned}
$$

where the supremum above is realized by $u = \hat{\mathbf{u}}_n(x)$.
Furthermore, for every fixed $y \in \mathcal{X}$ the function sequence $f^y_n(x)$ is determined by the recursion

$$
\begin{aligned}
\mathbf{A}^{\hat{\mathbf{u}}} f^y_n(x) &= 0, \quad n = 0, \ldots, T-1, && (4.7) \\
f^y_T(x) &= F(y,x), && (4.8)
\end{aligned}
$$

and $f(x,y)$ is given by

$$
f(x,y) = f^y(x).
$$

The function sequence $g_n(x)$ is determined by the recursion.

$$
\begin{aligned}
\mathbf{A}^{\hat{\mathbf{u}}} g_n(x) &= 0, \quad n = 0, \ldots, T-1, && (4.9) \\
g_T(x) &= x, && (4.10)
\end{aligned}
$$

In these recursions, the $\hat{\mathbf{u}}$ occurring in the expression $\mathbf{A}^{\hat{\mathbf{u}}}$ is the equilibrium control law.

We now have some comments on this result.

- The first point to notice is that we have a **system** of recursion equation (4.5)-(4.10) for the simultaneous determination of V, f and g.
- In the case when $F(x, y)$ does not depend upon x, and there is no G term, the problem trivializes to a standard time consistent problem. The terms $\left(\mathbf{A}^u f\right)_n (x, x) + \left(\mathbf{A}^u f^x\right)_n (x)$ in the V-equation (4.5) cancel, and the system reduces to the standard Bellman equation

$$\begin{aligned} \left(\mathbf{A}^u V\right)_n (x) &= 0, \\ V_T(x) &= F(x). \end{aligned}$$

- In order to solve the V-equation (4.5) we need to know f and g but these are determined by the equilibrium control law $\hat{\mathbf{u}}$, which in turn is determined by the *sup*-part of (4.5).
- We can view the system as a fixed point problem, where the equilibrium control law $\hat{\mathbf{u}}$ solves an equation of the form $M(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$. The mapping M is defined by the following procedure.
  - Start with a control $\mathbf{u}$.
  - Generate the functions f and g by the recursions

    $$\begin{aligned} \mathbf{A}^{\mathbf{u}} f_n^y(x) &= 0, \\ \mathbf{A}^{\mathbf{u}} g_n(x) &= 0, \end{aligned}$$

    and the obvious terminal conditions.
  - Now plug these choices of f and g into the V equation and solve it for V. The control law which realizes the *sup*-part in (4.5) is denoted by $M(\mathbf{u})$. The optimal control law is determined by the fixed point problem $M(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$.

  This fixed point property is rather expected since we are looking for a Nash equilibrium point, and it is well known that such a point is typically determined as fixed points of a mapping. We also note that we can view the system as a fixed point problem for f and g.
- In the present **discrete time** setting, the situation is, however, simpler than the fixed point argument above may lead us to believe. In fact; looking closer at the recursions, it turns out that the system for V, f, and g is a formalization of the recursive strategy outlined in Remark 2.1.

The general case

We now consider the more general functional form given in (4.1).

$$J_n(x,\mathbf{u}) = E_{n,x}\left[\sum_{k=n}^{T-1} C\left(x, X_k^{\mathbf{u}}, \mathbf{u}_k(X_k^{\mathbf{u}})\right) + F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{n,x}\left[X_T^{\mathbf{u}}\right]\right). \tag{4.11}$$

The arguments for the C terms in the sum above are very similar to the arguments for the F term. It is thus natural to introduce an indexed function sequence defined by

$$c_n^k(x,y) = E_{n,x}\left[C(y, X_k^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_k)\right], \quad 0 \le n \le k-1,$$

where, as usual, $\hat{\mathbf{u}}$ denotes the equilibrium law. The result is as follows. We omit the proof, which is a small variation of the previous one.

Theorem 2.2 *Consider a functional of the form (4.11), and assume that an equilibrium control law $\hat{\mathbf{u}}$ exists. Then the the equilibrium value function V satisfies the following equation.*

$$\sup_{u\in\mathcal{U}}\Big\{\left(\mathbf{A}^u V\right)_n(x) + C(x,x,u) - \sum_{k=n+1}^{T-1}(\mathbf{A}^u c^k)_n(x,x) + \sum_{k=n+1}^{T-1}(\mathbf{A}^u c^{k,x})_n(x)$$

$$-\left(\mathbf{A}^u f\right)_n(x,x) + \left(\mathbf{A}^u f^x\right)_n(x) - \mathbf{A}^u\left(G\diamond g\right)_n(x) + \mathbf{H}_g^u G_n(x)\Big\} = 0,$$

$$V_T(x) = F(x,x) + G(x,x),$$

where the supremum above is realized by $u = \hat{\mathbf{u}}_n(x)$.
Furthermore, for every fixed $y \in \mathcal{X}$ the function sequence $f_n^y(x)$ is determined by the recursion

$$\begin{aligned} \mathbf{A}^{\hat{\mathbf{u}}} f_n^y(x) &= 0, \quad n = 0,\dots,T-1, \\ f_T^y(x) &= F(y,x), \end{aligned}$$

and $f(x,y)$ is given by

$$f(x,y) = f^y(x).$$

The function sequence $g_n(x)$ is determined by the recursion.

$$\begin{aligned} \mathbf{A}^{\hat{\mathbf{u}}} g_n(x) &= 0, \quad n = 0,\dots,T-1, \\ g_T(x) &= x \end{aligned}$$

For every $k = 1, \ldots, T$, the function sequence $c_n^k(x,y) = c_n^{k,y}(x)$ is defined by

$$\begin{aligned}(\mathbf{A}^{\hat{\mathbf{u}}} c^{k,y})_n(x) &= 0, \quad 0 \le n \le k-1, \\ c_k^{k,y}(x) &= C(x, y, \hat{\mathbf{u}}_k(x)).\end{aligned}$$

In these recursions, the $\hat{\mathbf{u}}$ occurring in the expression $\mathbf{A}^{\hat{\mathbf{u}}}$ is the equilibrium control law.

3 Continuous time

We now turn to the more delicate case of continuous time models. We start by presenting the basic setup in term of a fairly general controlled Markov process. We then formulate the problem and formally define the continuous time equilibrium concept. In order to derive the relevant extension of the Hamilton-Jacobi-Bellman equation we discretize, use our previously derived results in discrete time, and go to the limit. Since the limiting procedure is somewhat informal we need to prove a formal verification theorem, showing the connection between the extended HJB equation and the previously defined equilibrium concept.

3.1 Setup

We consider, on the time interval $[0, T]$ a **controlled Markov process** in continous time. The process X lives on a measurable state space $\{\mathcal{X}, \mathcal{G}_X\}$, with controls taking values in a measurable control space $\{\mathcal{U}, \mathcal{G}_U\}$. The way that controls are influencing the dynamics of the process is formalized by specifying the controlled infinitesimal generator of X.

Definition 3.1 *For any fixed $u \in \mathcal{U}$ we denote the corresponding infinitesimal generator by $\mathbf{A}^u$. For a control law $\mathbf{u}$, the corresponding generator is denoted by $\mathbf{A}^{\mathbf{u}}$.*

As an example: of X is a controlled SDE of the form

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t,$$

then we have, for any real valued function $f(t,x)$, and for any **fixed** $u \in \mathcal{U}$

$$\mathbf{A}^u f(t,x) = \frac{\partial f}{\partial t}(t,x) + \mu(x,u)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x)$$

For a **control law** $\mathbf{u}(t,x)$ we have

$$\mathbf{A}^{\mathbf{u}} f(t,x) = \frac{\partial f}{\partial t}(t,x) + \mu(x,\mathbf{u}(t,x))\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x,\mathbf{u}(t,x))\frac{\partial^2 f}{\partial x^2}(t,x).$$

By the Kolmogorov backward equation, the infinitesimal generator will, for any control law $\mathbf{u}$, determine the distribution of the process X, and to stress this fact we will use the notation $X_t^{\mathbf{u}}$. In particular we will have, for each $h \in \mathbf{R}$ an operator $\mathbf{P}_h^{\mathbf{u}}$, operating on real valued functions of the form $f(t,x)$, and defined as

$$\mathbf{P}_h^{\mathbf{u}} f(t,x) = E\left[f(t+h, X_{t+h}^{\mathbf{u}}) \middle| X_t = x \right]. \tag{4.12}$$

We also recall that

$$\mathbf{A}^{\mathbf{u}} = \frac{d\mathbf{P}_h^{\mathbf{u}}}{dh}|_{h=0}. \tag{4.13}$$

3.2 Basic problem formulation

For a fixed $(t,x) \in [0,T] \times \mathcal{X}$, a fixed control law $\mathbf{u}$, we consider the functional

$$J(t,x,\mathbf{u}) = E_{t,x}\left[\int_t^T C\left(x, X_s^{\mathbf{u}}, \mathbf{u}(X_s^{\mathbf{u}})\right) ds + F(x, X_T^{\mathbf{u}}) \right] + G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right). \tag{4.14}$$

As in discrete time we have the game theoretic interpretation that, for each point t in time we have a player ("player t") choosing u_t who wants to maximize the functional above. Player t can, however, only affect the dynamics of the process X by choosing the control u_t exactly at time t. At another time, say s, the control u_s will be chosen by player s. We again attack this problem by looking for a Nash subgame perfect equilibrium point. The intuitive picture is exactly like in continuous time: An equilibrium strategy $\hat{\mathbf{u}}$ is characterized by the property that if all players on the half open interval $(t,T]$ uses $\hat{\mathbf{u}}$, then it is optimal for player t to use $\hat{\mathbf{u}}$.

However, in continuous time this is not a bona fide definition. Since player t can only choose the control u_t exactly at time t, he only influences the control on a time set of Lebesgue measure zero, and for most models this will have no effect whatsoever on the dynamics of the process. We thus need another definition of the equilibrium concept, and we follow Ekeland and Lazrak (2006) and Ekeland and Pirvu (2007), who were the first to use the definition below.

Definition 3.2 *Consider a control law* $\hat{\mathbf{u}}$ *(informally viewed as a candidate equlibrium law). Choose a fixed* $u \in \mathcal{U}$*, a fixed real number* $h > 0$*. Also fix an arbitrarily chosen initial point* (t, x)*. Define the control law* $\mathbf{u}_h$ *by*

$$\mathbf{u}_h(s,y) = \begin{cases} u, & \text{for } t \leq s < t+h, \quad y \in \mathcal{X} \\ \hat{\mathbf{u}}(s,y), & \text{for } t+h \leq s \leq T, \quad y \in \mathcal{X} \end{cases}$$

If

$$\liminf_{h \to 0} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_h)}{h} \geq 0,$$

for all $u \in \mathcal{U}$*, we say that* $\hat{\mathbf{u}}$ *is an* **equilibrium** *control law. The equilibrium* **value function** V *is defined by*

$$V(t,x) = J(t,x,\hat{\mathbf{u}}).$$

Remark 3.1 *This is our continuous time formalization of the corresponding discrete time equilibrium concept. Note the necessity of dividing by* h*, since for most models we trivially would have*

$$\lim_{h \to 0} \{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_h)\} = 0.$$

We also note that we do not get a perfect correspondence with the discrete time equilibrium concept, since if the limit above equals zero for all $u \in \mathcal{U}$*, it is not clear that this corresponds to a maximum or just to a stationary point.*

3.3 The extended HJB equation

We now assume that there exists an equilibrium control law $\hat{\mathbf{u}}$ (not necessarily unique) and we go on to derive and extension of the standard Hamilton-Jacobi-Bellman (henceforth HJB) equation for the determination of the corresponding value function V. As in the discrete case we restrict ourselves to the simpler case when the integral term in (4.14) is absent. The general case is very similar and will be treated in Section 3.3.To clarify the logical structure of the derivation we outline our strategy as follows.

- We discretize (to some extent) the continuous time problem. We then use our results from discrete time theory to obtain a discretized recursion for $\hat{\mathbf{u}}$ and we then let the time step tend to zero.
- In the limit we obtain our continuous time extension of the HJB equation. Not surprisingly it will in fact be an equation system.

- In the discretizing and limiting procedure we mainly rely on informal heuristic reasoning. In particular we have do **not** claim that the derivation is a rigorous one. The derivation is, from a logical point of view, only of motivational value.
- We show that our extended HJB equation is in fact the "correct" one, by proving a rigorous verification theorem.

Deriving the equation

In this section we will, in an informal and heuristic way, derive a continuous time extension of the HJB equation. Note again that we have no claims to rigor in the derivation, which is only motivational. To this end we assume that there exists an equilibrium law $\hat{\mathbf{u}}$ and we argue as follows.

- Choose an arbitrary initial point (t, x). Also choose a "small" time increment $h > 0$.
- Define the control law $\mathbf{u}_h$ on the time interval $[t, T]$ by

$$\mathbf{u}_h(s, y) = \begin{cases} u, & \text{for } t \le s < t+h, \quad y \in \mathcal{X} \\ \hat{\mathbf{u}}(s, y), & \text{for } t+h \le s \le T, \quad y \in \mathcal{X} \end{cases}$$

- If now h is "small enough" we expect to have

$$J(t, x, \mathbf{u}_h) \le J(t, x, \hat{\mathbf{u}}),$$

  and in the limit as $h \to 0$ we should have equality if $u = \hat{\mathbf{u}}(t, x)$.

If we now use our discrete time results, with n and $n+1$ replaced by t and $t+h$, we obtain the inequality

$$\left(\mathbf{A}_h^u V\right)(t,x) - \left(\mathbf{A}_h^u f\right)(t,x,x) + \left(\mathbf{A}_h^u f^x\right)(t,x) - \mathbf{A}_h^u \left(G \diamond g\right)(t,x) + \left(\mathbf{H}_h^u g\right)(t,x) \le 0$$

where

$$\left(\mathbf{A}_h^u V\right)(t,x) = E_{t,x}\left[V(t+h, X_{t+h}^u)\right] - V(t,x)$$

and similarly for the other terms. We now divide the inequality by h and let h tend to zero. The the operator $\mathbf{A}_h^u$ will converge to the infinitesimal operator $\mathbf{A}^u$, but the limit of $h^{-1}\left(\mathbf{H}_h^u g\right)(t,x)$ requires closer investigation.

We have in fact

$$\left(\mathbf{H}_h^u g\right)(t,x) = G^x(E_{t,x}\left[g(t+h, X_{t+h}^u)\right]) - G^x(g(t,x)).$$

Furthermore we have the approximation

$$E_{t,x}\left[g(t+h, X_{t+h}^u)\right] = g(t,x) + \mathbf{A}^u g(t,x) + o(h),$$

and using a standard Taylor approximation for G^x we obtain

$$G^x(E_{t,x}\left[g(t+h, X_{t+h}^u\right]) = G^x(g(t,x)) + G_y^x(g(t,x)) \cdot \mathbf{A}^u g(t,x) + o(h),$$

where

$$G_y^x(y) = \frac{\partial G^x}{\partial y}(y).$$

We thus obtain

$$\lim_{h \to 0} \frac{1}{h} \left(\mathbf{H}_h^u g\right)(t,x) = G_y^x(g(t,x)) \cdot \mathbf{A}^u g(t,x).$$

Collecting all results we arrive at our proposed extension of the HJB equation. To stress the fact that the arguments above are largely informal we state the equation as a definition rather than as proposition.

Definition 3.3 *The extended HJB system of equations for the Nash equilibrium problem is defined as follows.*

$$\begin{aligned}
\sup_{u \in \mathcal{U}} \quad & \left\{ \left(\mathbf{A}^u V\right)(t,x) - \left(\mathbf{A}^u f\right)(t,x,x) + \left(\mathbf{A}^u f^x\right)(t,x) \right. \\
& \left. - \mathbf{A}^u \left(G \diamond g\right)(t,x) + \left(\mathbf{H}^u g\right)(t,x) \right\} = 0, \quad 0 \le t \le T \\
& \mathbf{A}^{\hat{\mathbf{u}}} f^y(t,x) = 0, \quad 0 \le t \le T, \\
& \mathbf{A}^{\hat{\mathbf{u}}} g(t,x) = 0, \quad 0 \le t \le T, \\
& V(T,x) = F(x,x) + G(x,x), \\
& f(T,x,y) = F(y,x), \\
& g(T,x) = x.
\end{aligned}$$

Here $\hat{\mathbf{u}}$ *is the control law which realizes the supremum in the first equation, and* f^y, $G \diamond g$, *and* $\mathbf{H}g$ *are defined by*

$$\begin{aligned}
f^y(t,x) &= f(t,x,y) \\
\left(G \diamond g\right)(t,x) &= G(x, g(t,x)), \\
\mathbf{H}^u g(t,x) &= G_y(x, g(t,x)) \cdot \mathbf{A}^u g(t,x), \\
G_y(x,y) &= \frac{\partial G}{\partial y}(x,y).
\end{aligned}$$

We now have some comments on the extended HJB system.

- The first point to notice is that we have a **system** of recursion equation (4.5)-(4.10) for the simultaneous determination of V, f and g.

- In the case when $F(x, y)$ does not depend upon x, and there is no G term, the problem trivializes to a standard time consistent problem. The terms $(\mathbf{A}^u f)(t, x, x) + (\mathbf{A}^u f^x)(t, x)$ in the V-equation cancel, and the system reduces to the standard Bellman equation

$$\begin{aligned} (\mathbf{A}^u V)(t, x) &= 0, \\ V(T, x) &= F(x). \end{aligned}$$

- In order to solve the V-equation we need to know f and g but these are determined by the optimal control law $\hat{\mathbf{u}}$, which in turn is determined by the *sup*-part of the V-equation.

- We can view the system as a fixed point problem, where the optimal control law $\mathbf{u}$ solves an equation of the form $M(\mathbf{u}) = \mathbf{u}$. The mapping M is defined by the following procedure.

  - Start with a control $\mathbf{u}$.
  - Generate the functions f and g by the ODEs

  $$\begin{aligned} \mathbf{A}^{\mathbf{u}} f^y(t, x) &= 0, \\ \mathbf{A}^{\mathbf{u}} g(t, x) &= 0, \end{aligned}$$

    and the obvious terminal conditions.
  - Now plug these choices of f and g into the V equation and solve it for V. The control law which realizes the *sup*-part in the V-equation is denoted by $M(\mathbf{u})$. The optimal control law is determined by the fixed point problem $M(\hat{\mathbf{u}}) = \hat{\mathbf{u}}$.

  This fixed point property is rather expected since we are looking for a Nash equilibrium point, and it is well known that such a point is typically determined as fixed points of a mapping. We also note that we can view the system as a fixed point problem for f and g.

- The equations for g and f^y state that the processes $g(t, X_t^{\hat{\mathbf{u}}})$ and $f^y(t, X_t^{\hat{\mathbf{u}}}))$ are martingales. From the boundary conditions we then have the interpretation

$$\begin{aligned} f(t, x, y) &= E_{t,x}\left[F(y, X_T^{\hat{\mathbf{u}}})\right], \\ g(t, x) &= E_{t,x}\left[X_T^{\hat{\mathbf{u}}}\right]. \end{aligned}$$

  A version of g function above appears, in a more restricted framework, already in Basak and Chabakauri (2008).

The general case

We now turn to the more general functional form given in(4.14) as

$$J(t,x,\mathbf{u}) = E_{t,x}\left[\int_t^T C\left(x, X_s^{\mathbf{u}}, \mathbf{u}(X_s^{\mathbf{u}})\right) ds + F(x, X_T^{\mathbf{u}})\right] + G\left(x, E_{t,x}\left[X_T^{\mathbf{u}}\right]\right). \tag{4.15}$$

Arguing as before we are led to the following definition.

Definition 3.4 *The extended HJB system of equations for the Nash equilibrium problem with the functional (4.15) is defined as follows.*

$$\begin{aligned}
\sup_{u\in\mathcal{U}} \quad & \left\{ \left(\mathbf{A}^u V\right)(t,x) + C(x,x,u) - \int_t^T \left(\mathbf{A}^u c^s\right)_t(x,x)ds + \int_t^T \left(\mathbf{A}^u c^{s,x}\right)_t(x)ds \right. \\
& \left. - \left(\mathbf{A}^u f\right)(t,x,x) + \left(\mathbf{A}^u f^x\right)(t,x) - \mathbf{A}^u\left(G \diamond g\right)(t,x) + \left(\mathbf{H}^u g\right)(t,x) \right\} = 0, \quad 0 \le t \le T \\
& \mathbf{A}^{\hat{\mathbf{u}}} f^y(t,x) = 0, \quad 0 \le t \le T \\
& \mathbf{A}^{\hat{\mathbf{u}}} g(t,x) = 0, \quad 0 \le t \le T \\
& (\mathbf{A}^{\hat{\mathbf{u}}} c^{s,y})_t(x) = 0, \quad 0 \le t \le s \\
& V(T,x) = F(x,x) + G(x,x), \\
& c_s^{s,y}(x) = C(x,y,\hat{\mathbf{u}}_s(x)), \\
& f(T,x,y) = F(y,x), \\
& g(T,x) = x.
\end{aligned}$$

Here $\hat{\mathbf{u}}$ is the control law which realizes the supremum in the first equation, and f^y, $c^{s,y}$, $G \diamond g$, and $\mathbf{H}g$ are defined by

$$\begin{aligned}
f^y(t,x) &= f(t,x,y) \\
c_t^{s,y}(x) &= c_t^s(x,y), \\
\left(G \diamond g\right)(t,x) &= G(x, g(t,x)), \\
\mathbf{H}^u g(t,x) &= G_y(x, g(t,x)) \cdot \mathbf{A}^u g(t,x), \\
G_y(x,y) &= \frac{\partial G}{\partial y}(x,y).
\end{aligned}$$

A simple special case

We see that the general extended HJB equation is quite complicated. In many concrete cases there are, however, cancellations between different terms in the equation. The simplest case occurs when the objective functional has the form

$$J(t,x,\mathbf{u}) = E_{t,x}\left[F(X_T^{\mathbf{u}})\right] + G\left(E_{t,x}\left[X_T^{\mathbf{u}}\right]\right),$$

and X is a scalar diffusion of the form

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t.$$

In this case the extended HJB equation has the form

$$\sup_{u \in \mathcal{U}} \left\{ \mathbf{A}^u V(t,x) - \mathbf{A}^u \left[G\left(g(t,x)\right) \right] + G'(g(t,x)) \mathbf{A}^u g(t,x) \right\} = 0,$$

and a simple calculation shows that

$$-\mathbf{A}^u \left[G\left(g(t,x)\right) \right] + G'(g(t,x)) \mathbf{A}^u g(t,x) = -\frac{1}{2} G''(g(t,x)) \sigma^2(x,u) g_x^2.$$

Thus the extended HJB equation becomes

$$\sup_{u \in \mathcal{U}} \left\{ \mathbf{A}^u V(t,x) - \frac{1}{2} G''(g(t,x)) \sigma^2(x,u) g_x^2 \right\} = 0, \tag{4.16}$$

We will use this result in Section 5 below.

A Verification Theorem

As we have noted above, the derivation of the continuous time extension of the HJB equation is rather informal. It seems reasonable to expect that the system in Definition 3.4 will indeed determine the equilibrium value function V, but so far nothing has been formally proved. However, the following two conjectures are natural.

- Assume that there exists an equilibrium law $\hat{\mathbf{u}}$ and that V is the corresponding value function. Assume furthermore that V is regular enough to allow allow $\mathbf{A}^u$ to operate on it (in the diffusion case this would imply $V \in C^{1,2}$). Define f and g by

$$\begin{aligned} f(t,x,y) &= E_{t,x}\left[F(y, X_T^{\hat{\mathbf{u}}})\right], & (4.17)\\ g(t,x) &= E_{t,x}\left[X_T^{\hat{\mathbf{u}}}\right]. & (4.18)\end{aligned}$$

  Then V satisfies the extended HJB system and $\hat{\mathbf{u}}$ realizes the supremum in the equation.

- Assume that V, f, and g solves the extended HJB system and that the supremum i the V-equation is attained for every (t,x). Then there exists an equlibrium law $\hat{\mathbf{u}}$, and it is given by the optimal u in the in the V-equation. Furthermore, V is the corresponding equilibrium value function, and f and g allow for the interpretations (4.17)-(4.18).

In this paper we do not attempt to prove the first conjecture. Even for a standard time consistent control problem, it is well known that this is technically quite complicated, and it typically requires the theory of viscosity solutions. We will, however, prove the second conjecture. This obviously has the form of a verification result, and from standard theory we would expect that it can be proved with a minimum of technical complexity.

Theorem 3.1 (Verification Theorem) *Assume that V, f, g is a solution of the extended system in Definition 3.4, and that the control law $\hat{\mathbf{u}}$ realizes the supremum in the equation. Then $\hat{\mathbf{u}}$ is an equilibrium law, and V is the corresponding value function. Furthermore, f and g can be interpreted according to (4.17)-(4.18).*

Proof. The proof consists of two steps:

- We start by showing that V is the value function corresponding to $\hat{\mathbf{u}}$, i.e. that $V(t,x) = J(t,x,\hat{\mathbf{u}})$, and that f and g have the interpretations (4.17)-(4.18).
- In the second step we then prove that $\hat{\mathbf{u}}$ is indeed an equilibrium control law.

To show that $V(t,x) = J(t,x,\hat{\mathbf{u}})$, we use the V equation to obtain:

$$\begin{aligned}\left(\mathbf{A}^{\hat{\mathbf{u}}}V\right)(t,x) - \left(\mathbf{A}^{\hat{\mathbf{u}}}f\right)(t,x,x) + \left(\mathbf{A}^{\hat{\mathbf{u}}}f^{x}\right)(t,x) & \\ -\mathbf{A}^{\hat{\mathbf{u}}}\left(G \diamond g\right)(t,x) + \left(\mathbf{H}^{\hat{\mathbf{u}}}g\right)(t,x) &= 0,\end{aligned}$$

where

$$\mathbf{H}^{\hat{\mathbf{u}}}g(t,x) = G_y(x, g(t,x)) \cdot \mathbf{A}^{\hat{\mathbf{u}}}g(t,x).$$

Since V, f, and g satsifies the extended HJB, we also have

$$\left(\mathbf{A}^{\hat{\mathbf{u}}}f^{x}\right)(t,x) = 0, \tag{4.19}$$

$$\mathbf{A}^{\hat{\mathbf{u}}}g(t,x) = 0, \tag{4.20}$$

and we thus have the equation

$$\left(\mathbf{A}^{\hat{\mathbf{u}}}V\right)(t,x) - \left(\mathbf{A}^{\hat{\mathbf{u}}}f\right)(t,x,x) - \mathbf{A}^{\hat{\mathbf{u}}}\left(G \diamond g\right)(t,x) = 0. \tag{4.21}$$

We now use Dynkin's Theorem which says that if X is a process with infinitesimal operator $\mathbf{A}$, and if $h(t,x)$ is a sufficiently integrable real valued function, then the process

$$h(t,X_t) - \int_0^t \mathbf{A}h(s,X_s)ds$$

is a martingale. Using Dynkin's Theorem we thus have

$$E_{t,x}\left[V(T,X_T)\right] = V(t,x) + E_{t,x}\left[\int_t^T \mathbf{A}^{\hat{\mathbf{u}}} V(s,X_s^{\hat{\mathbf{u}}})ds\right],$$

and from (4.21) we obtain

$$\begin{aligned} E_{t,x}\left[V(T,X_T^{\hat{\mathbf{u}}})\right] &= V(t,x) + E_{t,x}\left[\int_t^T \mathbf{A}^{\hat{\mathbf{u}}} f(s,X_s^{\hat{\mathbf{u}}},X_s^{\hat{\mathbf{u}}}ds)\right] \\ &\quad + E_{t,x}\left[\int_t^T \mathbf{A}^{\hat{\mathbf{u}}}\left(G \diamond g\right)(s,X_s^{\hat{\mathbf{u}}})ds\right] \end{aligned}$$

We again refer to Dynkin and obtain

$$\begin{aligned} E_{t,x}\left[\int_t^T \mathbf{A}^{\hat{\mathbf{u}}} f(s,X_s^{\hat{\mathbf{u}}},X_s^{\hat{\mathbf{u}}})ds\right] &= E_{t,x}\left[f(T,X_T^{\hat{\mathbf{u}}},X_T^{\hat{\mathbf{u}}})\right] - f(t,x,x), \\ E_{t,x}\left[\int_t^T \mathbf{A}^{\hat{\mathbf{u}}}\left(G \diamond g\right)(s,X_s^{\hat{\mathbf{u}}})ds\right] &= E_{t,x}\left[G(X_T, g(T,X_T^{\hat{\mathbf{u}}}))\right] - G(x,g(t,x)). \end{aligned}$$

Using this and the boundary conditions for f and g we get

$$\begin{aligned} E_{t,x}\left[F(X_T^{\hat{\mathbf{u}}},X_T^{\hat{\mathbf{u}}}) + G(X_T^{\hat{\mathbf{u}}},X_T^{\hat{\mathbf{u}}})\right] &= V(t,x) + E_{t,x}\left[F(X_T^{\hat{\mathbf{u}}},X_T^{\hat{\mathbf{u}}})\right] - f(t,x,x) \\ &+ E_{t,x}\left[G(X_T^{\hat{\mathbf{u}}},X_T^{\hat{\mathbf{u}}})\right] - G(x,g(t,x)), \end{aligned}$$

i.e.

$$V(t,x) = f(t,x,x) + G(x,g(t,x)). \tag{4.22}$$

Now, from (4.19)-(4.20) it follows that the processes $f^y(s,X_s^{\hat{\mathbf{u}}})$ and $g(s,X_s^{\hat{\mathbf{u}}})$ are martingales, so from the boundary conditions for f and g we obtain

$$\begin{aligned} f(t,x,y) &= E_{t,x}\left[F(y,X_T^{\hat{\mathbf{u}}})\right], \\ g(t,x) &= E_{t,x}\left[X_T^{\hat{\mathbf{u}}}\right]. \end{aligned}$$

This shows that f and g have the correct interpretation and, plugging it into (4.22) we obtain

$$V(t,x) = E_{t,x}\left[F(x,X_T^{\hat{\mathbf{u}}})\right] + G(x,E_{t,x}\left[X_T^{\hat{\mathbf{u}}}\right]) = J(t,x,\hat{\mathbf{u}}).$$

We now go on to show that $\hat{\mathbf{u}}$ is indeed an equilibrium law. To that end we construct, for any $h > 0$ and an arbitrary $u \in \mathcal{U}$, the control law $\mathbf{u}_h$ defined in Definition 3.2. From Lemma 2.2, applied to the points t and $t+h$, we have

$$
\begin{aligned}
J_n(x,\mathbf{u}) &= E_{t,x}\left[J(t+h,X^u_{t+h},\mathbf{u})\right] \\
&- \left\{E_{t,x}\left[f^{\mathbf{u}}(t+h,X^u_{t+h},X^u_{t+h})\right] - E_{t,x}\left[f^{\mathbf{u}}(t+h,X^u_{t+h},x)\right]\right\} \\
&- \left\{E_{t,x}\left[G\left(X^u_{t+h},g^{\mathbf{u}}(t+h,X^u_{t+h})\right)\right] - G\left(x,E_{t,x}\left[g^{\mathbf{u}}(t+h,X^u_{t+h})\right]\right)\right\}.
\end{aligned}
$$

where, for ease of notation, we have suppressed the lower index h of $\mathbf{u}_h$. By the construction of $\mathbf{u}$ we have

$$
\begin{aligned}
J(t+h,X^u_{t+h},\mathbf{u}) &= V(t+h,X^u_{t+h}), \\
f^{\mathbf{u}}(t+h,X^u_{t+h},x) &= f(t+h,X^u_{t+h},x), \\
g^{\mathbf{u}}(t+h,X^u_{t+h}) &= g(t+h,X^u_{t+h}),
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
J_n(x,\mathbf{u}) &= E_{t,x}\left[V(t+h,X^u_{t+h})\right] \\
&- \left\{E_{t,x}\left[f(t+h,X^u_{t+h},X^u_{t+h})\right] - E_{t,x}\left[f(t+h,X^u_{t+h},x)\right]\right\} \\
&- \left\{E_{t,x}\left[G\left(X^u_{t+h},g(t+h,X^u_{t+h})\right)\right] - G\left(x,E_{t,x}\left[g(t+h,X^u_{t+h})\right]\right)\right\}.
\end{aligned}
$$

Furthermore, from the V-equation we have

$$
\begin{aligned}
\left(\mathbf{A}^u V\right)(t,x) - \left(\mathbf{A}^u f\right)(t,x,x) + \left(\mathbf{A}^u f^x\right)(t,x)& \\
-\mathbf{A}^u\left(G\diamond g\right)(t,x) + \left(\mathbf{H}^u g\right)(t,x) &\leq 0,
\end{aligned}
$$

for all $u\in\mathcal{U}$. Discretizing this gives us

$$
\begin{aligned}
E_{t,x}\left[V(t+h,X^u_{t+h})\right] - V(t,x) - \big\{E_{t,x}\left[f(t,X^u_{t+h},X^u_{t+h})\right] - f(t,x,x)\big\}& \\
+E_{t,x}\left[f(t,X^u_{t+h},x)\right] - f(t,x,x)& \\
-E_{t,x}\left[G(t+h,g(t+h,X^u_{t+h})\right] + G(x,g(t,x))& \\
+G(x,E_{t,x}\left[g(t+h,X^u_{t+h})\right] - G(x,g(t,x)) &\leq o(h),
\end{aligned}
$$

or, after simplification,

$$
\begin{aligned}
V(t,x) &\geq E_{t,x}\left[V(t+h,X^u_{t+h})\right] - E_{t,x}\left[f(t,X^u_{t+h},X^u_{t+h})\right] + E_{t,x}\left[f(t,X^u_{t+h},x)\right] \\
&- E_{t,x}\left[G(t+h,g(t+h,X^u_{t+h})\right] + G(x,E_{t,x}\left[g(t+h,X^u_{t+h})\right] + o(h).
\end{aligned}
$$

Using the expression for $J_n(x,\mathbf{u})$ above, and the fact that $V(t,x)=J(t,x,\hat{\mathbf{u}})$, we obtain

$$
J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}) \geq o(h),
$$

so

$$
\liminf_{h\to 0}\frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u})}{h} \geq 0,
$$

and we are done. ∎

4 An equivalent time consistent problem

The object of the present section is to provide a surprising link between time inconsistent and time consistent problems. To this end we go back to the general continuous time extended HJB equation. The first part of this reads as

$$\begin{aligned}\sup_{u\in\mathcal{U}}\{\left(\mathbf{A}^u V\right)(t,x) + C(x,x,u) - \int_t^T (\mathbf{A}^u c^s)_t(x,x)ds + \int_t^T (\mathbf{A}^u c^{s,x})_t(x)ds \\ - \left(\mathbf{A}^u f\right)(t,x,x) + \left(\mathbf{A}^u f^x\right)(t,x) - \mathbf{A}^u\left(G\diamond g\right)(t,x) + \left(\mathbf{H}^u g\right)(t,x)\} &= 0.\end{aligned}$$

Let us now assume that there exists an equilibrium control law $\hat{\mathbf{u}}$. Using $\hat{\mathbf{u}}$ we can then construct c, f and g by solving the associated equations in Definition 3.4. We now define the function h by

$$\begin{aligned}h(t,x,u) &= C(x,x,u) - \int_t^T (\mathbf{A}^u c^s)_t(x,x)ds + \int_t^T (\mathbf{A}^u c^{s,x})_t(x)ds \\ &- \left(\mathbf{A}^u f\right)(t,x,x) + \left(\mathbf{A}^u f^x\right)(t,x) - \mathbf{A}^u\left(G\diamond g\right)(t,x) + \left(\mathbf{H}^u g\right)(t,x).\end{aligned}$$

With this definition of h, the equation for V above and its boundary condition become

$$\begin{aligned}\sup_{u\in\mathcal{U}}\left\{\left(\mathbf{A}^u V\right)(t,x) + h(t,x,u)\right\} &= 0, \\ V(T,x) &= F(x,x) + G(x,x).\end{aligned}$$

We now observe, by inspection, that this is a standard HJB equation for the standard time consistent optimal control problem to maximize

$$E_{t,x}\left[\int_t^T h(s,X_s,u_s)ds + F(X_T,X_T) + G(X_T,X_T)\right]. \tag{4.23}$$

We have thus proved the following result.

Proposition 4.1 *For every time inconsistent problem in the present framework there exists a standard, time consistent optimal control problem with the following properties.*

- *The optimal value function for the standard problem coincides with the equilibrium value function for the time inconsistent problem.*
- *The optimal control for the standard problem coincides with the equilibrium control for the time inconsistent problem.*

- *The objective functional for the standard problem is given by (4.23).*

We immediately remark that the Proposition above is mostly of theoretical interest, and of little "practical" value. The reason is of course that in order to formulate the equivalent standard problem we need to know the equilibrium control $\hat{\mathbf{u}}$. In our opinion it is, however, quite surprising.

Furthermore, Proposition 4.1 has modeling consequences for economics. Suppose that you want to model consumer behavior. You have done this using standard time consistent dynamic utility maximization and now you are contemplating to introduce time inconsistent preferences to obtain a richer class of consumer behavior. Proposition 4.1 then tells us that from the point of view of revealed preferences, nothing is gained by introducing time inconsistent preferences: Every kind of behavior that can be generated by time inconsistency can also be generated by time consistent preferences. We immediately remark, however, that even if a concrete model of time inconsistent preferences is, in some sense, "natural", the corresponding time consistent preferences may look extremely "weird".

5 Example 1: Mean-variance control

In this section we will illustrate the theory developed earlier, and as a first test example we will consider dynamic mean variance optimization. This is a continuous time version of a standard Markowitz investment problem, where we penalize the risk undertaken by the conditional variance. As noted in the introduction, in a Wiener driven framework this example is studied intensively in Basak and Chabakauri (2008), where the authors also consider the case of multiple assets, as well as the case of a hidden Markov process (unobservable factors) driving the parameters of the asset price dynamics. For illustrative purposes we first consider the simplest possible case of a Wiener driven single risky asset and, without any claim of originality, rederivethe corresponding results of Basak and Chabakauri (2008). We then extend the model in Basak and Chabakauri (2008) and study the case when the risky asset is driven by a point process as well as by a Wiener process.

5.1 The simplest case

We consider a market formed by a risky asset with price process S and a risk free money account with price process B. The price dynamics are given by

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt, \end{aligned}$$

where α and σ are known constants, and r is the constant short rate.

Let u_t be the amount of money invested in the risky asset at time t. The value X_t of a self-financing portfolio based on S and B will then evolve according to the SDE

$$dX_t = [rX_t + (\alpha - r)u_t]dt + \sigma u_t dW_t. \tag{4.24}$$

Our value functional is given by

$$J(t,x,\mathbf{u}) = E_{t,x}\left[X_T^{\mathbf{u}}\right] - \frac{\gamma}{2} Var_{t,x}\left(X_T^{\mathbf{u}}\right),$$

so we want to maximize expected return with a penalty term for risk. Remembering the definition for the conditional variance

$$Var_{t,x}[X_T] = E_{t,x}[X_T^2] - E_{t,x}^2[X_T],$$

we can re-write our objective functional as

$$J(t,x,\mathbf{u}) = E_{t,x}\left[F(X_T^{\mathbf{u}})\right] - G(E_{t,x}\left[X_T^{\mathbf{u}}\right])$$

where $F(x) = x - \frac{\gamma}{2}x^2$ and $G(x) = \frac{\gamma}{2}x^2$. As seen in the previous sections, the term $G(E_{t,x}\left[X_T\right])$ leads to a time inconsistent game theoretic problem.

The extended HJB equation is then given by the following PDE system:

$$\begin{aligned} \sup_u \left\{ [rx + (\alpha - r)u]V_x + \frac{1}{2}\sigma^2 u^2 V_{xx} - \mathcal{A}^u (G \circ g) + \mathbf{H}^u g \right\} &= 0, \\ V(T,x) &= x, \\ \mathcal{A}^{\hat{u}} g &= 0, \\ g(T,x) &= x, \end{aligned}$$

where lower case index denotes the corresponding partial derivative. This case is covered in Section 3.3, and from (4.16) we can simplify to

$$\begin{aligned} V_t + \sup_u \left\{ [rx + (\alpha - r)u]V_x + \frac{1}{2}\sigma^2 u^2 V_{xx} - \frac{\gamma}{2}\sigma^2 u^2 g_x^2 \right\} &= 0 \\ V(T,x) &= x \\ \mathcal{A}^{\hat{u}} g &= 0 \\ g(T,x) &= x \end{aligned}$$

Given the linear structure of the dynamics, as well as of the boundary conditions, it is natural to make the *Ansatz*

$$\begin{aligned} V(t,x) &= A(t)x + B(t) \\ g(t,x) &= a(t)x + b(t). \end{aligned}$$

With this trial solution the HJB equation becomes

$$A_t x + B_t + \sup_u \left\{ [rx + (\alpha - r)u]A - \frac{\gamma}{2}\sigma^2 u^2 a^2 \right\} = 0, \tag{4.25}$$

$$a_t x + b_t + [rx + (\alpha - r)\hat{u}]a = 0, \tag{4.26}$$

$$\begin{aligned} A(T) &= 1, \\ B(T) &= 0, \\ a(T) &= 1, \\ b(T) &= 0. \end{aligned}$$

We first solve the static problem embedded in (4.25). From the first order condition, we obtain the optimal control as

$$\hat{u}(t,x) = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2}\frac{A(t)}{a^2(t)},$$

so the optimal control does not depend on x. Substituting this expression for $\hat{u}$ into (4.25) we obtain:

$$A_t x + B_t + Arx + \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}\frac{A^2}{a^2} = 0.$$

By separation of variables we then get the following system of ODEs.

$$\begin{aligned} A_t + Ar &= 0, \\ A(T) &= 1, \\ B_t + \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}\frac{A^2}{a^2} &= 0, \\ B(T) &= 0. \end{aligned}$$

We immediately obtain

$$A(t) = e^{r(T-t)}.$$

Inserting this expression for A into the second ODE yields

$$B_t + \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}\frac{e^{2r(T-t)}}{a^2} = 0, \tag{4.27}$$

$$B(T) = 0.$$

This equation contain the unknown function a, and to determine this we use equation (4.26). Inserting the expression for $\hat{u}$ into (4.26) gives us

$$\begin{aligned} a_t x + b_t + rxa + \frac{1}{\gamma}\frac{(\alpha-r)^2}{\sigma^2}\frac{e^{r(T-t)}}{a} &= 0, \\ a(T) &= 1, \\ b(T) &= 0. \end{aligned}$$

Again we have separation of variables and obtain the system

$$\begin{aligned} a_t + ar &= 0, \\ b_t + \frac{1}{\gamma}\frac{(\alpha-r)^2}{\sigma^2}\frac{e^{r(T-t)}}{a} &= 0. \end{aligned}$$

This yields

$$a(t) = e^{r(T-t)},$$

and the ODE for b then takes the form

$$\begin{aligned} b_t &= \frac{1}{\gamma}\frac{(\alpha-r)^2}{\sigma^2}, \\ b(T) &= 0. \end{aligned}$$

We thus have

$$b(t) = \frac{1}{\gamma}\frac{(\alpha-r)^2}{\sigma^2}(T-t).$$

Introducing the results in the optimal control formula, we get

$$\hat{u}(t,x) = \frac{1}{\gamma}\frac{\alpha-r}{\sigma^2}e^{-r(T-t)}.$$

Using the expression for a above, we can go back to equation (4.27) which now takes the form

$$B_t + \frac{1}{2\gamma}\frac{(\alpha-r)^2}{\sigma^2} = 0,$$

so

$$B(t) = \frac{1}{2\gamma}\frac{(\alpha-r)^2}{\sigma^2}(T-t).$$

Thus, the optimal value function is given by

$$V(t,x) = e^{r(T-t)}x + \frac{1}{2\gamma}\frac{(\alpha-r)^2}{\sigma^2}(T-t)$$

We summarize the results as follows:

Proposition 5.1 *For the model above we have the following results.*

- *The optimal amount of money invested in a stock is given by*

$$\hat{\mathbf{u}}(t,x) = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2}e^{-r(T-t)}.$$

- *The equilibrium value function is given by*

$$V(t,x) = e^{r(T-t)}x + \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}(T-t).$$

- *The expected value of the optimal portfolio is given by*

$$E_{t,x}\left[X_T\right] = e^{r(T-t)}x + \frac{1}{\gamma}\frac{(\alpha - r)^2}{\sigma^2}(T-t).$$

Using Proposition 4.1 we can also construct the equivalent standard time consistent optimization problem. An easy calculation gives us the following result.

Proposition 5.2 *The equivalent (in the sense of Proposition 4.1) time consistent problem is to maximize the functional*

$$\max_u E_{t,x}\left[X_T - \frac{\gamma\sigma^2}{2}\int_t^T e^{2r(T-s)}u_s^2 ds\right]$$

given the dynamics (4.24).

We note in passing that

$$\sigma^2 u_t^2 dt = d\langle X\rangle_t.$$

5.2 A point process extension

We will now present an extension of the mean variance problem when the stock dynamics are driven by a jump diffusion. We consider a single risky asset with price S and a bank account with price process B. The results below can be easily extended to the case of multiple assets, but for ease of exposition, we restrict ourselves to the scalar case. The dynamics are given by

$$\begin{aligned} dS_t &= \alpha(t,S_t)S_t dt + \sigma(t,S_t)S_t dW_t + S_{t-}\int_{\mathcal{Z}}\beta(z)\mu(dz,dt), \\ dB_t &= r dt. \end{aligned}$$

Here W is a scalar Wiener process and μ is a marked point process on the mark space $\mathcal{Z}$ with deterministic intensity measure $\lambda(dz)$. Furthermore, $\alpha(t,s)$, $\sigma(t,s)$ and $\beta(z)$ are known deterministic functions and r is a known constant.

As before u_t denotes the amount of money invested in the stock at time t, and X is the value process for a self financing portfolio based on S and B. The dynamics of X_t are then given by

$$dX_t = [rX_t + (\alpha(t,S_t,Y_t) - r)u]dt + \sigma(t,S_t,Y_t)udW_t + u_{t-}\int_Z \beta(z)\mu(dz,dt).$$

Again we study the case of mean-variance utility, i.e.

$$J(t,x,\mathbf{u}) = E_{t,x}\left[X_T^{\mathbf{u}}\right] - \frac{\gamma}{2}Var_{t,x}\left(X_T^{\mathbf{u}}\right).$$

The extended HJB system now has the form

$$\begin{aligned} \sup_u \left\{\mathcal{A}^u V(t,x,s) - \mathcal{A}^u(G\circ g)(t,x,s) + \left(\mathbf{H}^u g\right)(t,x,s)\right\} &= 0, & (4.28)\\ V(T,x,s) &= x, \\ \mathcal{A}^{\hat{u}} g &= 0, & (4.29)\\ g(T,x,s) &= x. \end{aligned}$$

As before, we make the *Ansatz*

$$\begin{aligned} V(t,x,s) &= A(t)x + B(t,s), \\ g(t,x,s) &= a(t)x + b(t,s), \\ A(T) &= 1, \\ B(T,s) &= 0 \\ a(T) &= 1, \\ b(T,s) &= 0. \end{aligned}$$

After some simple but tedious calculations, equation (4.28) can be re-written as

$$\begin{aligned} &\sup_u\{A_t x + B_t + A\left[rx + (\alpha - r)u\right] + \alpha s B_s + \frac{1}{2}\sigma^2 s^2 B_{ss} + Au\int_Z \beta(z)\lambda(dz) \\ &+ \int_Z [\underbrace{B(t,s(1+\beta(z))) - B(t,s)}_{\Delta_\beta B(t,s,z)}]\lambda(dz) - \frac{1}{2}(\sigma u)^2\gamma[au + b_s s]^2 \\ &-\frac{\gamma}{2}\int_Z [au\beta(z) + \underbrace{b(t,s(1+\beta(z))) - b(t,s)}_{\Delta_\beta b(t,s,z)}]^2\lambda(dz)\} = 0 \qquad (4.30) \end{aligned}$$

First, we solve the embedded static problem in (4.30)

$$\max_u \quad \Big\{(\alpha - r)Au + Au\int_Z \beta(z)\lambda(dz) - \frac{1}{2}\sigma^2 u^2[au + b_s s]^2$$
$$-\frac{\gamma}{2}\int_Z [au\beta(z) + \Delta_\beta b]\lambda(dz)\Big\}$$

and obtain the optimal control

$$\begin{aligned}\hat{\mathbf{u}}(t,x,s) \quad &= \quad \frac{\left[\alpha(t,s) - r + \int_Z \beta(z)\lambda(dz)\right] A(t)}{\gamma a^2(t)[\sigma^2(t,s) + \int_Z \beta^2(z)\lambda(dz)]} \\ &\quad - \frac{\sigma(t,s)b_s(t,s)s + \int_Z \beta(z)\Delta_\beta b(t,s,z)\lambda(dz)}{a(t)[\sigma^2(t,s) + \int_Z \beta^2(z)\lambda(dz)]}\end{aligned}$$

Again we see that the optimal control does not depend on x. We can plug the optimal control into equation (4.30) and as before, we can separate variables to obtain an ODE for $A(t)$ and a PIDE for $B(t,s)$. The ODE for A is

$$\begin{aligned} A_t + rA &= 0 \\ A(T) &= 0 \end{aligned}$$

with solution $A(t) = e^{r(T-t)}$. The PIDE for $B(t,s)$ becomes

$$B_t + (\alpha - r)\hat{u} + \alpha s B_s + \frac{1}{2}\sigma^2 s^2 B_{ss} + A\hat{u}\int_Z \beta(z)\lambda(dz) \tag{4.31}$$
$$+ \int_Z [\Delta_\beta B]\lambda(dz) - \frac{1}{2}\sigma^2\gamma[a(t)\hat{u} + b_s s]^2 \tag{4.32}$$
$$-\frac{\gamma}{2}\int_Z [a(t)\hat{u}\beta(z) + \Delta_\beta b]^2\lambda(dz) = 0 \tag{4.33}$$
$$B(T,s) = 0 \tag{4.34}$$

In order to solve this we need to determine the functions $a(t)$ and $b(t,s)$. To this end we use (4.29). This can be rewritten as

$$a_t x + b_t + [rx + (\alpha - r)\hat{u}]a + \alpha s b_s + \frac{1}{2}\sigma^2 s^2 b_{ss}$$
$$+a\hat{u}\int_Z \beta(z)\lambda(dz) + \int_Z \Delta_\beta b\lambda(dz) = 0. \tag{4.35}$$

with the appropriate boundary conditions for a and b. By separation of variables we obtain the ODE

$$\begin{aligned} a_t + ra &= 0, \\ a(T) &= 1, \end{aligned}$$

and the PIDE

$$\begin{aligned} b_t + (\alpha - r)\hat{u}a + \alpha s b_s + \frac{1}{2}\sigma^2 s^2 b_{ss} + a\hat{u}\int_Z \beta(z)\lambda(dz) + \int_Z \Delta_\beta b \lambda(dz) &= 0, \\ b(T,s) &= 0. \end{aligned}$$

From the ODE we have $a(t) = e^{r(T-t)}$ and, plugging this expression into the previous formula for $\hat{u}$, gives us

$$\hat{u} = \frac{\alpha - r + \int_Z \beta(z)\lambda(dz)}{\gamma[\sigma^2 + \int_Z \beta^2(z)\lambda(dz)]} e^{-r(T-t)} - \frac{\sigma b_s s + \int_Z \beta(z)\Delta_\beta b \lambda(dz)}{[\sigma^2 + \int_Z \beta^2(z)\lambda(dz)]} e^{-r(T-t)}$$

We can now insert this expression, as well as the formula for a, into the PIDE for b above to obtain the PIDE

$$\begin{aligned} & b_t + \left[\alpha - \frac{\left[\alpha - r + \int_Z \beta(y)\lambda(dy)\right]\sigma^2}{[\sigma^2 + \int_Z \beta^2(y)\lambda(dy)]}\right] s b_s + \frac{1}{2}\sigma^2 s^2 b_{ss} + \frac{\left[\alpha - r + \int_Z \beta(y)\lambda(dy)\right]^2}{\gamma[\sigma^2 + \int_Z \beta^2(y)\lambda(dy)]} \\ & \int_Z \Delta_\beta b(t,s,z)\left\{1 - \frac{\alpha - r + \int_Z \beta(y)\lambda(dy)}{[\sigma^2 + \int_Z \beta^2(y)\lambda(dy)]}\beta(z)\right\}\lambda(dz) = 0 \\ & b(T,s) = 0 \end{aligned}$$

This rather forbidding looking equation cannot in general be solved explicitly, but by applying a Feynman-Kac representation theorem we can represent the solution as

$$b(t,s) = E^Q_{t,s}\left[\int_t^T \frac{\left[\alpha(\tau, S_\tau) - r + \int_Z \beta(z)\lambda(dz)\right]^2}{\gamma[\sigma^2(\tau, S_\tau) + \int_Z \beta^2(z)\lambda(dz)]} d\tau\right]. \tag{4.36}$$

Here the measure Q is absolutely continuous w.r.t. P, and the likelihood process

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t,$$

has dynamics given by

$$dL_t = L_t \varphi dW_t + L_{t-}\int_Z \eta(z)\left[\mu(dz,dt) - \lambda(dz)dt\right],$$

with φ and η given by

$$\begin{aligned}\varphi(t,s) &= -\frac{\left[\alpha(t,s) - r + \int_Z \beta(y)\lambda(dy)\right]\sigma(t,s)}{\left[\sigma^2(t,s) + \int_Z \beta^2(y)\lambda(dy)\right]},\\ \eta(t,s,z) &= -\frac{\left[\alpha(t,s) - r + \int_Z \beta(y)\lambda(dy)\right]}{\left[\sigma^2(t,s) + \int_Z \beta^2(y)\lambda(dy)\right]}\beta(z).\end{aligned}$$

From the Girsanov Theorem it follows that the Q intensity λ^Q, of the point process $\mu(dt,dz)$ is given by

$$\lambda^Q(t,s,dz) = \left\{1 - \frac{\left[\alpha(t,s) - r + \int_Z \beta(y)\lambda(dy)\right]}{\left[\sigma^2(t,s) + \int_Z \beta^2(y)\lambda(dy)\right]}\beta(z)\right\}\lambda(dz),$$

and that

$$dW_t = -\frac{\left[\alpha(t,S_t) - r + \int_Z \beta(y)\lambda(dy)\right]\sigma(t,S_t)}{\left[\sigma^2(t,S_t) + \int_Z \beta^2(y)\lambda(dy)\right]}dt + dW_t^Q$$

where W^Q is Q a Wiener process. A simple calculation now shows that the Q dynamics of the stock prices S are given by

$$dS_t = rS_t dt + S_t\sigma(t,S_t)dW_t^Q + S_{t-}\int_{\mathcal{Z}} \beta(z)\left[\mu(dt,dz) - \lambda^Q(t,S_t,dz)\right]$$

so the measure Q is in fact a risk neutral martingale measure, and it is easy to check (see for example **?**)) that Q is in fact the so called "minimal martingale measure" used in the context of local risk minimization and developed in **?**) and related papers. This fact was, in a Wiener process framework, observed already in Basak and Chabakauri (2008).

Performing similar calculations, one can show that the solution of the PIDE (4.33)-(4.34) can be represented as

$$\begin{aligned}B(t,s) &= E_{t,s}^Q\left[\int_t^T \frac{(\alpha - r + \int_Z \beta(z)\lambda(dz))^2}{\gamma[\sigma^2 + \int_Z \beta^2(z)\lambda(dz)]}d\tau\right]\\ &\quad - E_{t,s}^Q\left[\int_t^T \frac{1}{2}\sigma^2\gamma[e^{r(T-\tau)}\hat{u} + b_s s]^2 d\tau\right]\\ &\quad - E_{t,s}^Q\left[\int_t^T \frac{\gamma}{2}\int_Z [e^{r(T-\tau)}\hat{u}\beta(z) + \Delta_\beta b]^2\lambda(dz)d\tau\right], \qquad (4.37)\end{aligned}$$

with Q as above.

We can finally summarize our results.

Proposition 5.3 *With notation as above, the following hold.*

- *The optimal amount of money invested in a stock is given by*

$$\hat{\mathbf{u}} = \frac{(\alpha - r + \int_Z \beta(z)\lambda(dz))}{\gamma[\sigma^2 + \int_Z \beta^2(z)\lambda(dz)]} e^{-r(T-t)} - \frac{(\sigma b_s s + \int_Z \beta(z)\Delta_\beta b(z)\lambda(dz))}{[\sigma^2 + \int_Z \beta^2(z)\lambda(dz)]} e^{-r(T-t)}$$

- *The mean-variance utility of the optimal portfolio is given by*

$$U(t,x) = e^{r(T-t)}x + b(t,s)$$

  where $b(t,s)$ is given by stochastic representation (4.36).

- *The expected terminal value of the optimal portfolio is*

$$E_{t,x}\left[X_T\right] = xe^{r(T-t)} + B(t,s)$$

  where x is the present portfolio value and $B(t,s)$ is given by the stochastic representation (4.37).

6 Conclusions and future research.

In the current chapter, we have derivde an extension of the standard Hamilton-Jacobi-Bellman equation to a non standard system of equations for the determination of the equilibrium value function V which would allow us to solve a larger of non-standard economic and financial problems. We have also proven a veriffication theorem, showing that the solution of the extended HJB system is indeed the equilibrium value function, and that the equilibrium strategy is given by the optimizer in the equation system. To every time inconsistent problem of th form analyzed, there exists an associated standard, time consistent, control problem. While this observation is mostly of theoretical interest, and of little "practical" value, it has interesting consequences for economic modelling. from the point of view of revealed preferences, nothing is gained by introducing time inconsistent preferences: Every kind of behavior that can be generated by time inconsistency can also be generated by time consistent preferences. We immediately remark, however, that even if a concrete model of time inconsistent preferences is, in some sense, "natural", the corresponding time consistent preferences may look extremely "weird".

There is still a lot of scope for research to be done in this area. First we have to investigate several other relevant examples. Then, we would like to investigate the same type of problems with parameter uncertainty. Interesting questions about optimal stopping and time inconsistent problems are another interesting problem to look into.

Bibliography

Artzner, Delbaen, E. and D. Heath (1999). Coherent measures of risk. *Math. Finance 9*, 203–228.

Barrieu, P. and N. E. Karoui (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics 9*, 269–298.

Basak, S. and G. Chabakauri (2008). Dynamic mean-variance asset allocation. Preprint, London Business School.

BIS (2008). *OTC derivatives market activity in the first half of 2008*. Bank for International Settlements.

Björk, T. (2004). *Arbitrage Theory in Continuous Time, 2:nd ed.* Oxford University Press.

Björk, T. and I. Slinko (2005). Towards a general theory of good deal bounds. *Review of Finance 10*, 221–260.

Björk, T. and I. Slinko (2008). Approximation of good deal bound solutions. manuscript, Stockholm School of Economics.

Brigo, D. and M. Masetti (2005). Risk neutral pricing of counterparty risk. In *Pykhtin, M. (Editor), Counterparty Credit Risk Modeling: Risk Management, Pricing and Regulation*. Risk Books.

Brigo, D. and A. Pallavicini (2008). Counterparty risk and contingent cds under correlation. *Risk Magazine February*.

Cochrane, J. and J. Saa-Raquejo (2000). Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy 108*, 79–119.

Duffie, D. and D. Lando (2001). Term structure of credit spreads with incomplete accounting information. *Econometrica 69*, 633–664.

Duffie, Berndt, D. F. and D. Schranz (2005). Measuring default-risk premia from default swap rates and edfs. Preprint, Graduate School of Business, Stanford University.

Ekeland, I. and A. Lazrak (2006). Being serious about non-commitment: subgame perfect equilibrium in continuous time. Preprint, University of British Columbia.

Ekeland, I. and T. Pirvu (2007). Investment and consumption without commitment. Preprint, University of British Columbia.

Glasserman, P. (2003). *Monte Carlo methods in financial engineering.* Springer-Verlag.

Goldman, S. (1980). Consistent plans. *Review of Economic Studies 47*, 533–537.

Gut, A. (2005). *Probability: A Graduate Course.* Springer-Verlag.

Henderson, V. and D. Hobson (2004). Utility indifference pricing - an overview. In *to appear in Utility Indifference Pricing.* Princeton University Press.

Hull, J. and A. White (1995). The impact of default risk on the prices of options and other derivative securities. *Journal of Banking and Finance 19*, 299–322.

Hung, M.-W. and Y.-H. Liu (2005). Pricing vulnerable options in incomplete markets. *Journal of Futures Markets 25*, 135–170.

Jarrow, R. and S. Turnbull (1995). Pricing derivatives on financial securities subject to credit risk. *Journal of Finance 50*, 53–85.

Jaschke, S. and U. Küchler (2001). Coherent risk measures and good deal bounds. *Finance and Stochastics 5*, 181–200.

Johnson, H. and R. Stulz (1987). The pricing of options with default risk. *Journal of Finance 42*, 267–280.

Klein, P. (1996). Pricing black-scholes options with correlated credit risk. *Journal of Banking and Finance 20*, 1211–1229.

Krusell, P. and A. Smith (2003). Consumption and savings decisions with quasi-geometric discounting. *Econometrica 71*, 366–375.

Peleg, B. and E. Menahem (1973). On the existence of a consistent course of action when tastes are changing. *Review of Economic Studies 40*, 391–401.

Pollak, R. (1968). Consistent planning. *Review of Economic Studies 35*, 185–199.

Schönbucher, P. (2003). *Credit derivatives pricing models - models, pricing and implementation.* JWS.

Schweizer, M. (1995). On the minimal martingale measure and the föllmer-schweizer decomposition. *Stochastic Analysis and its Applications 13*, 573 599.

Schweizer, M. (2001). A guided tour through quadratic hedging approaches. In *E. Jouini, J. Cvitanic, M. Musiela (eds.),'Option Pricing, Interest Rates and Risk Management'*, pp. 538–574. Cambridge University Press.

Strotz, R. (1955). Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies 23*, 165–180.

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2007

Books

Andersson, Per, Ulf Essler and Bertil Thorngren (eds). *Beyond mobility.* EFI Yearbook 2007. EFI/Studentlitteratur.
Einarsson, Torbjörn and Filip Wijkström. *Analysmodell för sektorsöverskridande statistik: fallet vård och omsorg.* EFI Civil Society Reports.
Ericsson, Daniel. *Musikmysteriet: organiserade stämningar och motstämningar.*
Samuelson, Lennart (ed). *Bönder och bolsjeviker: den ryska landsbygdens historia 1902–1939.*

Dissertations

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2006

Books

Lundeberg, Mats, Pär Mårtensson and Magnus Mähring (eds) *IT & business performance: a dynamic relationship.* EFI Yearbook 2006. EFI / Studentlitteratur.

Thodenius, Björn. *Organisering av kunskap: en studie av Wallenberg Consortium North.* EFI Civil Society Reports.

Wijkström, Filip and Torbjörn Einarsson. *Från nationalstat till näringsliv?: det civila samhällets organisationsliv i förändring.*

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