PAPPUS REBORN

PAPPUS OF ALEXANDRIA AND THE CHANGING FACE OF ANALYSIS AND SYNTHESIS IN LATE ANTIQUITY

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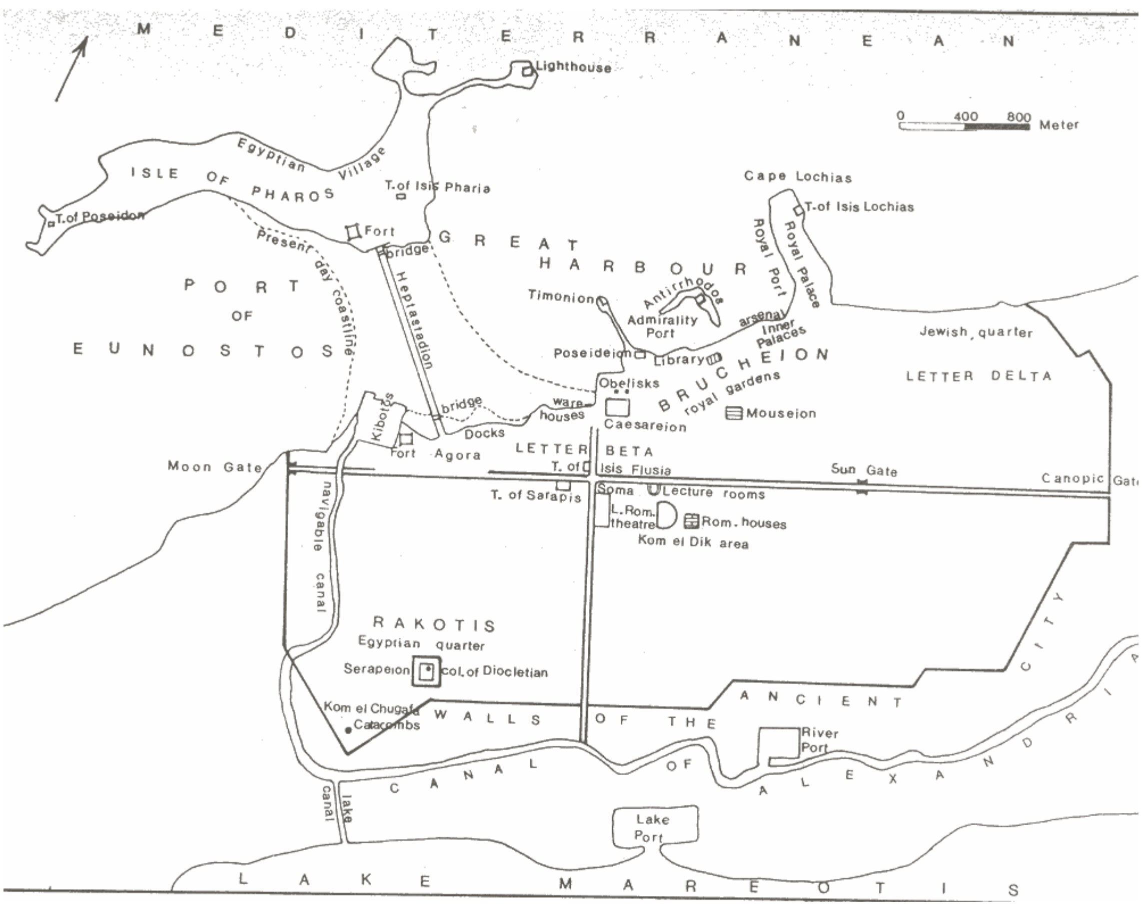

Figure 1: Map of Alexandria from El-Abbadi, Mostafa. The Life and Fate of the Ancient Library of Alexandria. Mayenne, France: Imprimerie Floch, 1990: http://ils.unc.edu/dpr/path/alexandria/geography.htm

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0.1 Acknowledgements

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0.2 Abstract

Despite the recent interest in Pappus of Alexandria,the late antiquarian mathematician remains an enigma in the history of mathematics. Only a handful of chapters and fragments of his seminal work, the **Collection** have been translated into English and there has been little effort to reveal the common mathematical threads therein. To correct this stasis and move Pappian studies into the exploratory stage, I approached the **Collection** with three questions designed to remove the historiographical biases and avoid the missteps that have persisted over the five-hundred years since Commandino's estate published the first Latin translation.

The first question is whether an improved understanding of late antiquity Alexandria and its intellectual environs offer greater insight into Pappus' mathematical style. The second is whether Pappus' infamous exposition on analysis and synthesis from Book 7 of the **Collection** can be reconciled with propositions from the same book. My final question is that after the reconciliation of proposition and exposition in Book 7, what are the consequences for other books in the **Collection** that contain otherwise unacknowledged instances of analysis and synthesis? More specifically, will we find consistency between Books 7, 3, and most surprising of all, 2.

What is revealed through these questions is the problem modern scholars have with Pappus was never actually about his ability but rather the interference of pedagogy in a field where it is least desired: advanced geometry. Pappus demonstrated thoroughly in Books 2 and 3 that his knowledge of analysis and synthesis was more nuanced than he has been given credit for, involving the social practice of mathematics in Alexandria and the application of arithmetic within a supposedly geometrical method.

Part II

Contexts of Analysis

0.3 Introduction

Pappus of Alexandria was, and still is, a popular figure within the history of mathematics. The **Collection**, his signature work, has been translated in its entirety in Latin, French, German, and modern Greek. Translations into English are somewhat more piecemeal: only Books 7 and 8 have been officially published while scholars such as Serafina Cuomo, Alain Bernard, Wilbur Knorr, and Sir. T. L. Heath only share fragments translated from the rest of the **Collection** while their scholarship hints more involved and completed translation. It is said of Heath, one of the most prolific classicists of the 19th and 20th centuries, that he was working on an English translation of Pappus but was dissuaded from the project.[1]

Of the mathematician himself, we know little. Asides from the **Collection** a sizable portion of works attributed to him survive from both the Greek and the Arabic and show a man with interests ranging from geometry to astronomy to alchemy to geography. Infrequently, the plausibility of his actually being from Alexandria is questioned as a means to liven up the discussion. Other than that, he appears to have lived and died a relatively quiet life as his death failed to capture the imaginations of artists and trivia buffs as the deaths of Archimedes, Hypatia, and Socrates have. We do have an idea of when Pappus was active. It is deduced in his commentary on Ptolemy's **Almagest** that the eclipse that he observed in Alexandria occurred on what would be October 18th, 320 CE using the Georgian calendar. This is the date from which Pappus' lifespan of 290 CE to 350 CE is framed despite evidence that states the contrary. In the **Suda Lexicon**, Pappus was believed to be active during the period of Theodosius I (A.D. 379 395). However the Suda omits from posterity the work for which Pappus has become most famous. Another source, a scholium to a Leiden manuscript of chronological tables written by Theon of Alexandria, places Pappus at the time of Diocletian (A.D. 284 305) it notes: "In his time Pappus wrote".[2] It is felt that neither of these dates correspond with the observation of the eclipse but that is a consequence of a lack of discourse in the History of Mathematics about the lifespan of the mathematician versus the

[1]

[2] DSB!!

time-span of a fruitful career.

These are the few facts independent facts about Pappus that one will find in the scholarship and they are treated as little consequence to greater mathematical concerns. However, scholarship has too easily accepted perspectives on the mathematician and his abilities that have no basis in his corpus, the period during which he was active, and the city in which he lived. The influence these three elements had on each other was far from subtle. Pappus had the good foture to live in a multi-cultural city such as Alexandria- all of whom valued mathematics and most supporting the inclusion of women into mathematical education and teaching. The potential of these circumstances is met in the **Collection** where Pappus seems to be a man with an active social life as an Alexandrian mathematician: students and non-mathematicians appear through the work, wither seeking out his services or receiving his advice. An audience of this kind, of course, requires special attention and necessitated from effort on Pappus' part to teach advanced mathematical concepts to people who were obviously inexpert.

In turn, this has left modern appreciation for Pappus' work to be a bit wanting. Instead of accepting Pappus as a writer whose output was incredibly sensitive to the needs of the less mathematically astute in his acquaintance, one is told that he is feeble and his mathematics naive. To accept this portrait of Pappus is to neglect the peculiar harmony that exists in the texts of his that have little in common except that they are grounded in geometry.

This particular harmony I write of is of the drastically different snapshots of analysis and synthesis that are revealed in Books 2, 3, and 7. Traditionally, analysis and synthesis were a powerful tool for problem-solving in geometry. By late antiquity, the needs of the mathematician changed as they devoted more of their study to reflection on their craft. The application of analysis and synthesis changed with it.

What Pappus revealed about analysis and synthesis in late antiquity illustrates the breadth of the divide between the ancients and their commentators. The emphasis on confirmation and what was clearly the receptive inclusion of arithmetic into analytic processes soon after distinguished Pappus from the tradition of his predecessors.

An unexpected result of restaging Pappus' life and practice in this way are the implications in areas that one did not anticipate. Again, analysis and synthesis were traditionally applied to geometrical problems and theorems; even Pappus agreed with this. However, the increased use of arithmetic in the so-called analyses of Heron of Alexandria and Pappus demands that Book 2 of the **Collection** be reevaluated. the original text on which it was based was admitted to by Pappus as being geometrical multiple times in the extant book. As a work that is superficially treated in the scholarship, the possibilities of recasting it in a new light are all too tempting. Only two scholars have acknowledged this truth about Book 2, Frederick Hultsch(who first translated the entire collection into Latin in the 1870s) and Serafina Cuomo(in her 2000 work **Pappus of Alexandria and the Mathematics of Late Antiquity**). Unfortunately, neither thoroughly considered the implications of arithmetic being associated with analysis.

More serious historiographical gaffs can be found throughout modern scholarship. The most common error is evidence in essays and articles such as J.L Berggren and Glen Van Brummelen's *The Role and Development of Geometric Analysis and Synthesis in Ancient Greece and Medieval Islam*. In this essay, Pappus' exposition on analysis and synthesis is evaluated for accuracy against proposition 4 from Archimedes' **Sphere and Cylinder**. Pappus always loses in such comparisons because his description was once an introduction directed towards a student of his. It is the sort of correspondence in which Pappus could be forgiven for being too general initially, but further into Book 7 Pappus' propositions demonstrate that the mathematician was a well-versed in analysis and synthesis, if not more so, than his his predecessors. Most modern scholars tend not to look to the propositions and have created an invisible divide between classical mathematicians and one of their most outspoken supporters. In the few cases where the propositions are included, scholars do not venture to acknowledge Book 3. This text contains some of the more interesting examples of analysis and synthesis in the ancient Greek corpus. Mainly, we witness an analysis being used to show that a problem put forward was impossible, a rarity in a field were incorrect proofs are only published accidently. Central to to proving the problem impossible is, as mentioned previously, the utilization of analysis. Without this perspective, we do not the major role analysis and synthesis played in everyday mathematics as a means for mathematicians to evaluate the validity of their problems and to ensure the fruitfulness of the community.

In order to approach these errors and to portray Pappus' mathematics in a more accurate light, two thematic goals and one practical goals are put forward for this thesis. The first is to prove that there is a significant connection between Pappus' exposition and propositions which is not exclusive to the rest of the **Collection**. Pappus would only be truly feeble if he failed to follow his own practice. The second is to understand how Pappus' practices have been influenced by the cultural environs in which he operated. The practical goal is the translation of Book 2. There are no English translations of the text for now except for the selections Cuomo and Netz provide, particularly of proposition 17, but without too much commentary.[3] Translating a text with little external reference is a challenge few students get to enjoy and offers much by way of reflection as to how Pappus thought and the style in which he wrote.

[3] Cuomo(2000), 181-182; Netz(2008)

Chapter 1

Setting the Scene

Pappus of Alexandria is cast as product and provocateur of late antiquity, where he alternately shines and falters against the backdrop of a period widely acknowledged for its decline. Research on Pappus, be he the star or on the periphery, rarely reflects on the mathematical environs where Pappus conducted himself. This is unfortunate because as a historically international port and an active multi-cultural city, Alexandria allowed for diversity within its mathematical community that was not evidenced elsewhere in first centuries of late antiquity. Bringing Alexandria to the fore depolarizes the situation as moves the audience away from the question of feebleness versus brilliance. Instead, it focuses their attention on the degree to which Pappus saw it necessary to educate his readers, a task that is distinct from simply discussing mathematics with equally skilled peers. Central to this depolarization is understanding what mechanisms and obstacles were in place for one to obtain literacy and numeracy in fourth century CE Alexandria.

1.1 Societal Influences in Literacy

In any city, the demography of its citizens can be divvied-up along socioeconomic, geographic, and political lines but these separations were much more significant in antiquity when the right to citizenship was distributed as a reward to 'outsiders'. By the fourth century CE, the most salient divisions in the city of Alexandria were between the native Egyptian, Greek, Jewish, and Roman communities. These groups did not have equal rights to citizenship, thereby creating an imbalance between the influence each could assert upon the city and vice versa. This tension is relevant as it uncovers which culture maintained the tightest control over the education, and consequently the mathematical practices, of the Alexandrians.

1.1.1 Egyptians

From Herodotus to Proclus, the ancient Egyptians were acknowledged as the key precursors to Hellenistic mathematics, as well as the particular inventors of geometry.[1]

> Any man who was robbed by the river of a part of his land would come to Sesostris and declare what had befallen him; then the king would send men to examine and to find out by measurement how much less the piece of land had become, in order that for the future the man might pay less, in proportion to the rent appointed. From this practice, to my thinking, the Greeks learned the art of geometry. Herodotus **Histories**, 109.

The invention of a numerical system and complex arithmetical processes closely coincided with the emergence of the earliest written text at the end of the fourth millennia BCE.[2] Hieroglyphic writing, also dated to approximately 3200 BCE, preceded similar activity in Greece by almost a millennia. By 1050 BCE, the Egyptians developed a hieroglyphic number system in which numbers up to and including $9,999,999$ were represented respective of quantity and place value through powers of 10.[3] In comparison, the Greek's utilization of large numbers as units of $10,000$ or myriads was a creative method but quickly becomes

[1] Roero(1994), 41-42.
[2] Imhausen(2007), 13.
[3] Roero(1994), 31.

clumsy with use. Knowledge of literacy and numeracy was not accessible to all Egyptians; a person either obtained official religious training, scribal training, or private tutoring by an educated relative.[4] The scribal tradition in Ancient Egypt was nepotic and largely a career for males of the middle-class elite.[5] Scribes-to-be learnt their trade by rewriting classic texts and form documents, bringing into question the true literacy of most members of the scribal schools.[6] The lack of opportunity for originality during their education resulted in their private writing also being derivative and formulaic.[7]

The numeracy of the scribes is not under similar scrutiny as papyrical records show the numerous calculations of bread and barley, compensation for labour, land partition, and building dimensions providing much in the way of numerical and calculative variety.[8] Such accounts show Ancient Egyptians to be practical mathematicians rather than theoretical. There are scholars who will argue that, at least for the Old, Middle, and New Kingdoms, Egyptian mathematics was a pure science.[9] The dearth of surviving materials and the absence of revised translations prevents one from making any further extrapolations at this point.[10]

Nonetheless, it was not necessary to be literate, or male, to make transactions, conduct affairs, or to bring complaints to the courts. Again, papyri reveal that only a small portion of the female population was literate, but most were able to, with the assistance of a scribe, make a record of their taxes, property brought and sold, or dictate a letter.[11] Despite the

[4] Williams(1972), 215.

[5] Manchipwhite(1952), 86. Peasants could also become scribes but they would have to have promise and be brought to the attention of the local nomarch or vizier. Hope would be that the family fortunes would rise with the youngsters training or apprenticeship.

[6] Williams(1972), 215-217.

[7] Ibid., 219.

[8] Imhausen(1997), 8, 37-40, 42-43.

[9] Roero(1994), 43. From the text, Roero quotes A.B. Chance et al. 1927-9 Rhind Mathematical Papyri, Vol. 1, 42-3:"..the Rhind papyri convinced me several years ago that this work is not a mere selection of practical problems especially useful to determine land values, and that the Egyptians were not a nation of shopkeepers...Rather I believe that they studied mathematics and other subjects for their own sakes...Thus we can say that the Rhind papyrus, while very useful to the Egyptian, was also an example of the cultivation of the mathematics as a pure science, even in its first beginnings."

[10] Imhausen(1997), 7; Roero(1994), 43-45

[11] Bagnall(1993), 235-235, 246-247.

fact that many people were unable to write their own names, the existence of the scribal class meant that Ancient Egyptians who were not literate themselves still had access to literacy and were not denied participation from the legal and bureaucratic aspects of Alexandrian life. This has consequences for understanding literacy in late antiquity. Participation in an educational system did not mean that mathematical knowledge or any knowledge for that matter was exclusive to the elite.

1.1.2 Greeks

The Greeks made their presence first known in the port of Naukratis around 630 BCE when merchants permanently established themselves in the port town. Alexander's conquest of Egypt in 332BCE and the subsequent foundation of Alexandria in honour of the Greeks was when the authority of the Hellenes began to rise. The traditional Greek education system shared only the slightest of similarities with the scribal tradition of the Egyptians in their focus on the availability literacy and numeracy to men. Greeks paid more attention to athleticism, general creativity, and civics. Scribes were still in service but not without major changes being made to the usage of the written Egyptian language. Hieroglyphics had given way to hieratic which in turn gave way to demotic.[12] By the first century CE, Greek became the only means of creating written records to the extent that even scribal writings were exclusively in Greek; sometimes, with instructions for the letter to be dictated to the recipient in Egyptian.[13]

1.1.3 Romans

With the inclusion of Alexandria into the Roman Empire in 30 BCE, one would expect Latin to usurp Ancient Greek as the language of choice, but that was not the case. Serafina Cuomo

[12] Bagnall(1993), 235-240; Imhausen(1997), 46-47.

[13] Bagnall(1993), 231; *P.Haun* 14-15; *P. Mich* 679. A translation from the Greek of this papyrus inscription from the Greek is offered by Bülow-Jacobsen and McCarren(1977): "You, whoever you are reading the letter, make a small effort and translate to the women what is written in this letter and tell them".

cites several professions within the Roman Empire that required an application of mathematics in some form or another, but Latin literacy and fluency was not required for employment in Alexandria.[14] By late antiquity, fluency in Greek was still an advantage. Latin did have increased usage but it was limited to governmental correspondence, court proceedings, etc. Most of the Roman elite were bilingual if not literate in Greek and Latin already, thus making it easy to switch between the two when circumstances demanded it. For the Egyptians, the opportunities for their own language decreased. By the 2nd Century CE, there was no written form of Egyptian in everyday use and only a handful of priests could read formal hieroglyphs.[15] Egyptian was still a spoken language in the villages whereas Greek a more urban tongue. Monolinguists of either were able to get by without knowing the other.[16]

In many ways, Rome did not so much as fail to assert its own culture as it assimilated the best practices of those nations and cities which it conquered. The Romans adopted the Hellenic education system of body and mind training and maintained it into late antiquity, with Latin grammar and literature being the only additions into the curriculum. On the political end, Rome's disinterest in the running Alexandria as long as it remained the granary of the empire makes the question of Rome's influence on education and literacy somewhat moot. Without an extensive presence in Alexandria and with the mechanisms within Egyptian culture that allowed people to communicate despite the language barrier, the idea of Rome and Roman education likely never went beyond the inclusion of Latin. Such inclusion of Latin would in any case be a minor contribution the so-called Greco-Roman education system as the Romans merely built upon Greek methods that were already firmly established in the city.

[14] Bagnall(1993), 231-232, 232n10; Cuomo(2000), 9-56; Smith(1974), 155, contends that Latin did start to thrive at the end of the fourth century CE.

[15] Bagnall(1993), 234-235.

[16] Bagnall(1993), 255, 259.

1.1.4 Jewish Inhabitants

The Old Testament and Jewish scriptures chronical the early presence of Jewish inhabitants in Egypt. Excepting the bloody conflicts throughout the four centuries leading to their expulsion from the Holy Roman Empire, they are practically absent from the annals of late antiquity. Ascertaining the standing and influence of the Jewish inhabitants of Alexandria is difficult. The scholars who do include them in their monographs, frequently set the inhabitants apart from the polytheistic majority of the city.[17] The philosopher Philo (20 BCE - 50 CE), wrote that this was not necessarily so. In the third century BCE, Alexandria was divided into three districts and two of these were identified by Philo as Jewish.[18] However, he added that the entire Jewish population was distributed across the city like any other group. In reflecting upon the justification for the discrimination and bloodshed, Josephus (37 CE - 100 CE) understood the justification of the anti-semites in light of the constitutional separations between the Jewish inhabitants and the rest of Alexandria.[19] As an alien population, they enjoyed political freedom and influence not extended to the native Egyptians at all or to the Romans citizens until 200 CE.[20]

Expectations for Jewish students to be integrated into the greco-roman system were not entirely obliterated. Further reading of Philo shows that certain Jewish youth were integrated into the greco-roman education system of Alexandria.

[17]Smith(1974), 21; Taylor(2003), 25n15, 94. Taylor believes that while the Jewish inhabitants would partake in the Greek education system during their childhood and teen years. On the other hand, they would have excluded themselves from the advanced research of the Mouseion because it was the temple of the Muses and overseen by a priest.

[18]Philo, *Flacc.* 8, *Apion I*: 33-35; Josephus, *Contra. Ap.* 55.

[19]Josephus, *Congr.* 74-76.

[20]Bagnall(1993), 55. During the principate of Augustus, the Jews were allowed to organized into a sociopolitical unit known as a *politeuma*. These units were present in Hellenistic cities with significant alien populations and allowed these populations to have a degree of self-governance and political autonomy that was respected throughout the city. The *politeuma* was administered by an ethnarch and he in turn was ruled by a council of elders, a *gerousia*. This was an important and uncharacteristic freedom to have in Alexandria. At no point during the foreign occupation did the native Egyptians have a council to which they could air their grievances and Rome did not grant a *Boule* to its own citizens in the city until 200 CE. Popular assemblies, as seen throughout Greece, were never established.

> They have held them entitled to nurture and late to education of body and soul, so that they may have not only life, but a good life. They have benefitted the body by means of the gymnasium and the training there given, through which it gain muscular vigour and good condition and the power to bear itself and mover with an ease marked by gracefulness and elegance. They have done the same for the soul by means of letters and arithmetic and geometry and music and philosophy as a whole. Philo, *Spec.* 2.229-33. As translated by(add later)

In referencing the gymnasium and exact subjects taught, Philo described not just any method of education. He lauded the Greek education of the Gymnasium and provided one of the few pieces of proof that young Jews at least had the opportunity to receive the same mathematical education as young Greeks or young Romans. That should be more accurately phrased as young Jews from wealthy families who had citizenship, as was the case for Philo himself.[21] It is not known whether this held true for the poor or the devout, or if it even held for the affluent after the pogrom of 38 CE or the riots of 115 - 117CE.[22]

Wealth did much to stave off most of the inequalities that were rampant in Alexandria but it did not ensure that a person would enjoy full citizenship. The Greek way of life was desired and emulated by other groups in a play for official citizenship in place of their affluent alien status. Their labours were in some sense encouraged by Isocrates (436 - 338BCE):

> ...[Athens] has brought it about that the name of the Greeks is no longer denoting a race but an intelligence, and one should call 'Greeks' rather the ones who participate in our culture than those who share our common nature. Isocrates **Panegyricus**, 4.50 (as translated by George Norlin(1980))

[21] Bagnall(1993), 22, 100. Membership into the *gynasmium* was very special. It was typically limited to those certified to the hereditary group entitled to it.

[22] Taylor(2003), 94n34. Boys at least were educated in synagogue related schools.

The massacre of 38 CE shows that this sentiment was not always sustained.

After the Greeks, the Egyptians and the Jews clamoured to share the top spot. The achievements of their ancestors did not give the native Egyptians social leverage. They were seen as too primitive and destitute by their neighbours.[23] However, there is mention in Josephus of an Egyptian priest named Manetho whom Josephus believed was proficient enough in Greek to compose a history of the Egyptians in that language.[24] Priests did receive the same literacy training as the scribal classes but their numbers were in serious decline during the early centuries of the Common Era. When only a single man is able to stand out from under all of the writing completed during that time, all that we have is signs of an overlooked and unacknowledged culture. Hieroglyphs and other forms of strictly Egyptian writing had fallen out of use and without the interest in Egyptian achievements formerly held by the Greeks, the Egyptians all but fade from the record outside of their preserved legal disputes and correspondences. The Jewish inhabitants had a more unstable standing but they argued vigourously for their right to citizenship. They did so either by promoting a recorded history that long preceded the Greek occupation of Egypt or by appealing to their adoption of certain Greek practices (at least, on behalf of the wealthy). This is ironic because it was the Romans who were alternately sympathetic and reactive to the inhabitants.

1.2 Women and Literacy

Such deference to the Greek way of life solidified the dominance of the Greeks in the areas of public education, literacy, and numeracy in Alexandria. Regardless of the inclination towards assimilation over diversity, Alexandria cannot be treated as a typical greco-roman city. It was seen as a city separate from the rest of Egypt, although the Roman themselves characterized it as Greek.[25] Current scholars portray the city as a microcosm of all that

[23] Sly(1996), 42-43, 112-114; Smith(1974), 2, 111; Taylor(2003), 26.

[24] Josephus, *Contra Ap.*, I.73-74.

[25] Haas(1997), 7; Philo, *Legat.* 250; Taylor(2003), 23. According to Haas, papryologists see Alexandria as Mediterranean. It is refereed to as *Alexandria ad Aegyptum*, or Alexandria by Egypt.

happens in antiquity: what occurred in Alexandria is treated as having happened anywhere, anytime.[26] The distinct political and educational autonomy enjoyed by Alexandrian female in the fourth century CE, however, confounds the city's modern treatment and defies some ancient untruths.

In drama and the scant historical evidence, classical Greek women who were able to read and count came from families that were elite or, in the case of Pythagoras' family, able to tutor their own children.[27] Strict moral and societal boundaries prevented women from receiving this education outside of their homes or in the presence of men who were not family or family friends.[28] The textual and archaeological evidence indicates that literate Roman females were a more common occurrence.[29] The attitude was that an educated daughter reflected well upon the family and would be a more desirable partner for her future husband.[30] The preference, for Romans and the Greeks, was for a curriculum of literature, poetry, music, and language. This would be to the greater benefit of the female student. Literacy promised political advantages; married and single women were under the guardianship of their fathers, husbands, or some other appointed male: their *kyrios.* A woman could be exempt from needing a *kyrios* under the roman law of *ius trium liberorum*, the law of three children.

> (Laws have been made), most eminent praefect, which enable women who are honoured with the right of three children to be independent and act without a guardian in all business which they transact, especially those women who know how to write.[31]

[26]Pomeroy(1981), 310.

[27]Harris(1989), 307; Pomeroy(1977), 51-52, 57; In Euripides' *Hippolytus*(856-81), the plot centers around the letter written by Phaedra.

[28]Pomeroy(1977), 51, 52n9, 56; Cole(1981), 223, 227. The archaeological record of vases, terracotta, and funeral monuments suggests that Greek girls were able to avail of education of equal qualities to boys. However, when Women are the focus of similar scenes, holding a stylus or book roll, an immortal or priestess is likely the true subject. Women with the ability to read were scorned if they weren't thought incapable of the task. Still, there were many female poets and philosophers that were active in the fifth and fourth centuries BCE.

[29]Harris(1989), 2,163; Pomeroy(1977), 52n10; Taylor(2003),95-96.

[30]Harris(1989), 263.

[31]*P.Oxy.*, 1467.

While it was the birth of the three children that granted Aurelia Thaisous freedom from the *kyrios*, need for a *kyrios* did not prevent her from writing and submitting her own case; it served to enhance it. As for Jewish inhabitants, there was the monastic and mystical association of the Therapeutae in Philo's **De Vita Contemplativa**. The female members were recognized as being highly educated and embracing a life of contemplation, philosophy, music, and meditation separate from their families.[32] The case has already been made here for an Egyptian woman's access to literacy. During the reign of the Ptolemies, increased rights to their own property and wealth necessitated the increased involvement of women in the legal realm of their villages and cities.[33] Emperor Caracalla's edict in 212 CE, the *Constitutio Antoniniana*, bestowed upon all free men and women under the Roman Empire the same rights as their Roman counterparts.[34] However, it was also Roman law by that time that while women could retain control of their inheritances, they could not act without a *kyrios*. Egyptian women, Alexandrian or otherwise appear to have disobeyed the edict as many women appear in the papyrical records as representing themselves after this date.

Alexandria thus provides an intersection of societal practices and freedoms that did not rescind or fall out of practice in the face of the increased cultural influence of the Greeks or the laws of the Romans.[35] At the very least, a woman's access to literacy, even if for strategic reasons, was valued and rewarded by almost every other culture except the Greeks. But this is to only talk about reading and writing; numeracy seems to be a rarer skill amongst females. Excluding the women of Pythagoras' family, the recorded participation of ancient women in science and mathematics seems to explode in Alexandria by comparison.[36]

[32] Taylor(2003), 54-73, 99-104

[33] Pomeroy(1981), 303-305.

[34] Pomeroy(1981), 312-316.

[35] Pomeroy(1981), 316-317. There are some unusual instances of assimilation. Under Jewish Law, Jewish women did not need a guardian but adopted the practice as they integrated into the Greek ruling class.

[36] Pomeroy(1981), 311. Of these women, Hypatia is the best known. There other females on record: Pandrosion in Book 3 of the **Collection**, Diophilia, who wrote a poem on astronomy, and several female neopythagoreans.

1.3 Affluence and Literacy vs. Mathematics

Affluence in antiquity did not always guarantee literacy and numeracy. Roger Bagnall estimated that families literate in Greek in Roman Egypt were less than half of one percent and even that number may be based on the assumption that affluence always accompanied literacy.[37] Many Alexandrians only needed the technical literacy to sign their own names and there were entire offices in the city that were manned by members of the elite who were illiterate despite their wealth.[38] Still, formal education presented a young person in antiquity with some of the few opportunities to learn mathematics beyond simple counting. It can be assumed from this that many mathematicians in late antiquity were members of families with means or were lucky enough to have been educated by relatives. As the cornerstone of greco-roman education, reading and writing were essential to the training of the mathematician even if mathematics was not similarly valuable to the education of an orator etc. It was possible, though, to have an interest or an capacity for mathematics without being literate.

One should consider the deviations implied by the situation in Alexandria; the possibility to be fluent but not literate in Greek, permitted one to understand the dictation of Pappus' text and simultaneously see the demonstrations drawn out. Those who are neither fluent nor literate still had some form of secondary literacy through the employment of scribes. Surveying Pappus' potential audience so far, those groups that were anticipated to be literate met those expectations. Those that had minimal access to literacy challenge the presumption that they would not have access to mathematics. Such a situation is illustrated in the platonic dialogue **Meno**. In it, Plato related how Socrates guided and illiterate slave boy through the creation of a square double the size of a given square. The intent of the dialogue in this vignette was to promote the innateness of geometrical knowledge because of the immortal soul but the boy's interest in the proceedings and his comprehension on the drawings made offers the

[37] Bagnall(1993), 182-183, 255.

[38] Bagnall(1993) 243, 243n67-68; P.Cain.Isid. p.16, 11.75-76. Aurelias Isidoros held several liturgical offices over the space of two or three decades. Many of these positions were as a tax collector. He left behind a sizable archive but he himself was illiterate. In fact, when he was *sitologos*, the entire college of tax collectors was illiterate. Isidoros' personal land holdings were significantly larger than his literate peers, certainly proving that wealth and literacy do not have to correspond in antiquity.

hope that anyone could learn mathematics.[39]

But Plato harkened back to a much earlier tradition of mathematics. One may call it semi-oral as the public performance of mathematics required the rehearsal and memorization of hundreds of diagrams.[40] The performance of mathematics is at odds with the surviving mathematical accounts of antiquity. No diagrams are extant; the constructions that survive are just over a millennia old.[41] As elegant and simple as drawing on the sand may be, it was actually a highly literate task in which the performer had to understand the significance of what he drew.[42] This was a skill that barely existed in its own right in the greco-roman curriculum.[43] The slave-boy in **Meno** evinced a certain rudimentary, innate potential but mathematics had catapulted into advanced geometry by late antiquity. If literacy was required to draw a diagram and expound on it, then the knowledge required for advanced geometric constructions precluded any possibility of mathematical talent arise from the lower echelons of society as it did in previous centuries.

This is the gap that ultimately exists between the largest possible audience for the **Collection** and those who practice mathematics. An audience can accompany a wide range of skills and abilities but amongst the mathematicians, there were certain standards of education that had to be met. Of course, the education of a mathematician will not be the same as that of a Doctor and within Alexandria, the Museum and the Library are believed to be the very institutions in which the differentiations in higher learning were made. The operation of these places as centres of advanced education is speculative and it is preferable that we look to the efforts of mathematicians in practice to see where the divisions were between careers in mathematics and more applied fields.

[39] Plato *Meno*, 304-320.
[40] Netz(1999), 58-63.
[41] Ibid., 12.
[42] Ibid., 62.
[43] Ibid., 58-59.

1.4 Mathematicians in the Field

Before one can estimate the size of a potential audience for Pappus in Alexandria or elsewhere in the Empire, the number of practicing mathematicians should be calculated first. This is to gauge amongst the upper classes how the size of an audience of Pappus' peers might have compared to an audience whose mathematical interest, on average, lay on the periphery. In his text **Shaping of Deduction in Greek Mathematics**, Reviel Netz estimated that the number was quite small even at the apex of mathematics most productive period. He argued that there was no school of mathematics per se and the death of one mathematician would have been disruptive, shedding some light on Archimedes' particular despondency after the death of Conon.[44] Netz noted that between anonymous and known mathematicians in his text, he listed 144 individuals.[45] He conceded that the absolute number of mathematicians available to anyone, at any time, in late antiquity would have been over twice that, around 300 known mathematicians.[46]

When it comes to just the number of mathematicians that were actually active in late antiquity, Netz believed the number of practitioners changed. During what he considers to be busier times for mathematics, he estimates that a classical mathematician could have about 100 contemporaries, while in Late Antiquity, one could anticipate having 12.5 so-called 'peers'.[47] However, Netz gives over in his later works to seeing late antiquity as a period of widespread decline and in this case, it is a matter of gaps in the mathematical record between important works and/or mathematicians. These numbers are largely based on a combination of presumption and fact. The fact is that many of the mathematicians he listed are not known independently from the mathematicians who mention them, leading Netz to the presumption that mathematicians were not all that numerous in the first place.[48] He goes as far as to guess that the birth rate of geometers, arithmeticians, et al. was low; one birth

[44]Netz(1999), 285-286. Interestingly enough, looking at the tables in section 1.6, there seems to be a few mathematicians active during Archimedes' intellectual lean years.
[45]Ibid., 285.
[46]Ibid., 283.
[47]Ibid.
[48]Ibid.

every two years.[49]

The number 12.5 may be speculative, but it is close to the number of contemporary individuals referred to by Pappus throughout the surviving parts of the **Collection**: Hermodorus was the subject of two dedications, Megethion received a dedication in Book 5, Book 3 is directed to Pandrosion and refers to three of her students, the Philosopher Hierius and 'many others' known to Pappus. Furthermore, unlike Books 2 and 7, Pappus clearly had a group of colleagues who were mathematically inclined even if they were not mathematicians like Pappus. There was also the Greek mathematician Serenus who was active during Pappus' lifetime. Thus, Pappus' contemporaries are at least 8 in number plus an unknown number of 'many others'. And these are just individuals that Pappus directly identified as being interested in what he was producing.

1.5 Mathematical Professions

What of the educated who did not pursue a profession in mathematics but received the same grounding in mathematics? In Cuomo's survey of math-related professions in antiquity, i.e. astrology, architecture, land-surveying, mechanics etc., much of the information regarding the practical uses of mathematics came from the writings of men who were not professional mathematicians.[50] Astrologers, land-surveyors etc., all described their skills in a quantitative language, boasting of the necessity of calculations and measuring in their day to day practice.[51] These claims were dismissed as rhetoric but if it was rhetoric, Cuomo asks, then why would have it been effective? What would their affiliation with the likes of Pappus be and what did they intend to gain?[52]

[49] Ibid.
[50] Cuomo(2000), 9.
[51] Ibid., 9-10.
[52] Ibid., 10. Bernard(2003b), 103. Bernard argues that based fragments from Plutarch (103n51) and Galen (103n52), one cannot assume that a mathematical education sans specialization was widespread.

Sometimes, it was professional necessity that made many proclaim their mathematical prowess. The better their mathematical proficiency, the more an astrologer could charge for a chart; the better that such an astrologer was with using his tools, the better horoscopes he could create. For others, mathematical knowledge gave architects and land-surveyors an edge professional legal disputes. There was also social advantages to having a profession associated with mathematics. In the astrologer's horoscopes, interaction with mathematics was a desirable destiny as those careers offered much in terms of wealth as well as authority through social clout.[53] Diocletian's 301 CE edict of maximum prices, *edictum de pretiis rerum venalium*, illustrated this. There was a maximum set price on what one could charge their students per month and it awarded bigger wages to those with specialized mathematical abilities. Teachers of Arithmetic, [*magistro*] *calculatori*, could charge 50 *denarii* per boy per month whereas a teacher of Architecture could charge 100 *denarii* and a teacher of Greek /or Latin literature and of Geometry could charge the same pupil 200 *denarii* for the same period.[54] One should not understand Diocletian's list as a reverse correlation between the number of practitioners and fees charged; medical doctors and philosophers do not appear on the list whereas lawyers, orators and rhetoricians do and receive high wages for their work.[55] Instead, it seems Diocletian acknowledged that mathematicians and their ilk had more of a presence in the Empire than modern scholars believe. The edict was never enforced but the distinction between the careers listed presents greco-roman education as being more complex than originally believed. The embrace of the applied fields such as Architecture show how mathematical education expands beyond arithmetic and Euclidean geometry at some point in a student's education.

1.6 Historiographical Issues

Late antiquity has always been a troublesome period for Historians of Mathematics. It was traditionally held that late antiquity was a period of overall decline with widespread corrup-

[53]Ibid.,17-20. write about land surveying.
[54]Diocletian, 344.66-346.74.
[55]Cuomo(2000), 30-31.

tion, dying cities, and over-taxed peasantry amongst the many trials that would precipitate the fall of the Roman Empire.[56] This view does not sit well with Roger Bagnall. He accuses modern scholars of holding to an unjustified attitude of perceiving societal and economic struggles as indicators of societal failure.[57] Neither "the absence of complete security and the presence of a high level of risk is necessarily inimical to a vital society" and that advanced and orderly societies might even thrive in those situations.[58] Indeed, a general shift in the perception of late antiquity has occurred and is gradually swaying Historians of Mathematics to reassess the contributions made in the period. They may find little in the way of new mathematical developments but Reviel Netz has extensively outlined how mathematicians and commentators in late antiquity irreversibly transformed the mathematical landscape through practices that are still used today.

1.6.1 Deuteronomic Texts

Using a self-coined term to describe all texts that depended on earlier texts, Netz is able to classify practically all mathematical texts produced in late antiquity as *Deuteronomic*, recalling the book **Deuteronomy** or *second law*.[59] The authors of commentaries, new editions, epitomes, and encyclopedic collections, which form a large body of work dated to this period, borrowed heavily from the works of one or more of their predecessors if they did not outright copy them.[60] Netz's selection of *Deuteronomic* to describe the output of late antiquity is appropriate. The distinction between deuteronomic texts and classical mathematics is deeper than author and audience, teacher and students; it is about the difference in what mathematicians in these periods felt was important. There was a transition in the way the former wrote mathematics, away from original work and into scholia, marginalia, and commentaries.

[56] Bagnall(1993), 3.
[57] Bagnall(1993), 12.
[58] Bagnall(1993), 13.
[59] Netz(1998), 262. **Deuteronomy** will be familiar to many as the fifth book of the Hebrew bible. In it are laws to live by. The title plays on similar words in Greek (*deuteronomion*) and Latin (*deuteronomium*); technically these words literally mean second law but in the Septuagint translation of the Jewish scriptures into Greek, the title Deuteronomy was borrowed from the incorrect rendering of the Hebrew phrase *mishneh ha-torah ha-zot*: "a copy of this law".
[60] Netz(1998), 262-263.

The scholastic and pedantic qualities Netz ascribed to late antiquity and medieval mathematics are of course modern labels. They represent the spirit of the works of these later writers but also share in their tendency to present centuries of classical mathematic development in a singular fashion. That is to say that irrespective of the motivations and aspirations of individual mathematicians and the variations in the quality and quantity of their output, they are written about as if they worked in concert towards a shared goal. This extended to how they wrote about the practice of mathematics as later writers attempted to bring the classical mathematicians into line with newer practices. In his commentary on the first book of Euclid's **Elements**, Proclus stated that all *problems* and theorems contained six parts: *protasis*, *ekthesis*, *diorismos*, *kataskeue*, *apodeixis*, and *sumperasma*.[61] This would be a helpful way to understand propositions in antiquity if not for the absence of such an explicit set up written of elsewhere. It does not occur in the classical or hellenistic periods and it does not appear in the generations immediately preceding Proclus. Normally, this would not be problematic. One can gain from Proclus' paragraph that by his time, mathematicians were immersed in the structural milieu of their practice. It is not uncommon, though, for modern scholars to apply those six parts to propositions that were written without the anticipation of that template.[62] It is anachronistic to presume that the understanding of later mathematicians was shared by their predecessors.

Thus, Netz may have provided a valuable set of categories by which one can identify texts as deuteronomic, but they require more than just a comparison between the original and the inspired text. One must also attend to the integration of scholia and marginalia into the preferred edition of the original. There is a great amount of interference from scholars past and present in this regard for their participation has made them most responsible for the canon-formation of Greek mathematics as it is known today.

Two of Netz's first categories, vertical and horizontal pedantry, highlight the problem of

[61] Netz(1998), 268.

[62] Hintikka(1974), 24-30. Bernard(2003b), 96n17. Despite allusions to Plato in Pappus' work, Bernard cautions about making too strong a link between Pappus and Proclus as Proclus' remarks are neoplatonist.

the inclusion of later scholia into the earlier text. In these situations, the implicit is made explicit. Vertical pedantry is recursive. A figure or relationship that was otherwise presumed by classical mathematicians would be extracted and explicated upon by commentators.[63] That is, the existence of the figure or relationship is unnecessarily discussed and affirmed. Horizontal pedantry is repetitive. Rather than content themselves with the single case of the classical proof, later mathematicians presented all the other cases, which only go to support the first case.[64] As before, all the additional cases were implied by classical mathematicians, with one case believed as sufficient to demonstrate the solution to the problem or the truth of the theorem.

The next set of categories reveal the paradoxical nature of deuteronomic texts. Pedantry encourages there to be more objects, more cases, more steps, and ultimately more triviality to the proof. However, deuteronomic writers actually desired the organization of mathematics into a finite number of categories. The categories were based on characteristics that classical mathematicians were indifferent towards. One such category is standardization.[65] Recalling Proclus, the form of the proposition was one such area of interest. He appended a conclusion to a proposition in his commentary so that Euclid would conform to his template.[66] Geometrical solutions were also classified according to the tools and means used. Geometers strictly adhered to their standardizations and sometimes this led to divisions in the manuscript tradition. The splits are sometimes minor. In the case of Euclid's **Optics**, there is a manuscript in which the lettering of the diagrams within a series of propositions remains consistent instead of the usual practice of the points changing as new objects are added.[67] Two equally important, and related, categories are classification and erudition. For classification, later mathematicians deliberately changed original texts by separating the content into chapters, tying together sections they believed to share one or two main argu-

[63]Netz(1998), 263-265.

[64]Netz(1998) 265-268.

[65]Ibid., 268-271.

[66]Ibid., 270n13. Proclus' analysis is geared towards theorems, but the first proposition of the **Elements** happened to be a problem. Problems tend not to have conclusions but since Proclus was trying to create consistency in his commentary, the problem was treated as a theorem.

[67]Ibid., 277. The division occurs in the one manuscript that is the only Greek mathematical work in which a letter is consistently inscribed on an object: K denoted the center of a circle.

ments.[68] This in turn contributed to the practice of erudition. Able to reference the chapters and propositions of the original work, a late antiquity mathematician could both show off his learning and demonstrate that his work was deserving of a place in history.[69]

The transformation in the outlook on productivity and contributions affirms that there was much development in late antiquity mathematics. Mathematicians and commentators became increasingly engaged in questions of how their predecessors wrote about the mathematics they performed. Netz's picture of mathematics in Late Antiquity is truly intriguing as the search for unification and standardization show a growing appreciation for the differences in methods and their intent along side the growth of practices that capture the evolution of meta-mathematical thought.

1.6.2 The Collection

The style and culture of late antiquity mathematics establishes certain expectations of both the **Collection** and its author. Netz would say that Pappus, as a mathematician of this period, was preoccupied with the form of classical problems and would make alterations to their presentation that would have been foreign to their authors. His corpus would consist largely of surveys, epitomes, and commentaries and thus the **Collection** would be a sketch of what is present in the corpus, with frequent reference to the original text.

By and large, these expectations are fulfilled. Outside of the Collection, much of Pappus' work comprised of commentaries or encyclopedic collections.[70] Four other works survive. With respect to commentaries, there is one on Ptolemy's **Almagest** and another on Book 10 of Euclid's **Elements**. As for an encyclopedic collection, there are seventh century Armenian fragments of a work mentioned in the **Souda** called **Chorography of the Inhabited World**; it followed the arrangement of Ptolemy's **Geography** but included amusing but fan-

[68]Ibid., 270.
[69]Ibid. 277, 279. Classical mathematicians did not do this.
[70]Jones(1986), 293-294.

ciful descriptions of the lands. The final surviving work is known from a ninth century Arabic translation as **Introduction to Mechanics**. It appears to be a stand-alone text but while it essentially agrees with the text of Book 8 of the **Collection**, it includes additional material including the conclusion that is missing from the Greek translation and constructions from straight-edge and compass.[71] Jones thought that it was possible that this was a later revision of Book 8 that somehow did not replace the earlier version in the **Collection**.[72] There were three rather encyclopedic works that are now lost: **The Rivers of Libya**, **Interpretation of Dreams**, and an unnamed astrological almanac. There were also three commentaries, another on the **Elements**, on Ptolemy's **Planispherium**, and Diodorus' **Analemma**. Finally, there was an alchemical oath and formula that appeared under the name of Pappus, but its authorship is questionable.

As for the **Mathematical Collection** itself, one is always balancing between understanding the text as it survives today and how it once was in its complete form. It is accepted that the text consisted only of eight books containing a variety of topics from large numbers to mechanics. The entirety of Book 1, the first half of Book 2 and the final half of Book 8 are lost from the Greek manuscripts.

Contents of the **Collection**	
Book	*Details*
[2]	Multiplication of large numbers
[3]	Geometrical problems: Finding two mean proportionals between two given magnitudes, the theory of means, the paradoxes of Erycinus, and non-Euclidean methods to inscribe five regular solids into a sphere.
[4]	Miscellaneous theorems and special curves
[5]	Isoperimetry after Zenodorus, theory of solids, and the comparison of the five platonic bodies
[6]	Resolution of difficulties in the little domain of Astronomy
[7]	Lemmas to the domain of analysis
[8]	Mechanical problems, scope and divisions of mechanics

However, it is possible that the **Collection** is more complete than first thought. As men-

[71] Ibid., 9.
[72] Ibid.

tioned previously, there is an Arabic translation of a work by Pappus that is a complete and expanded update of Book 8. Jones also mentioned the papal inventories from about 1311 CE refer to a text that is a "...Commentary of Pappus on the difficult things of Euclid and on the rest of geometry".[73] Unless there was a another lost Pappian treatise in Europe at this time, Jones cannot not see this commentary as being anything other than the **Collection** which included Pappus' commentary on the **Elements** Neither of these lost/found texts give any sign that they were excerpts of a larger work.[74] However, because the commentary on Book 10 of the **Elements** is not widely accepted as the missing Book 1 of the **Collection**, there are still some who believe that Book 1, by dint of preceding Book 2, was an arithmetical text.[75] This is not an unreasonable hypothesis as a mathematician with the breadth of interests clearly seen in Pappus could hardly have excluded arithmetic from his oeuvre. It does not take into account that, in a broader sense, the **Collection** is predominantly geometrical. Supposing the topic of Book 1 on the basis of its proximity to a text that is superficially understood is *spurious* at best. Yet, Books 1 and 2 may benefit from a realignment with the overall theme of the **Collection**: Geometry and its application. Little more can be presumed about the content of Book 1 here but recasting Book 2 in this light may improve one's understanding of the text.

[73] Jones(1986), see p.46 for a convincing argument for the Commentary on the **Elements** being the lost Book 1 of the **Collection**

[74] Bulmer-Thomas(1997), 291, 294; Jones(1986) 16, 21. Jones evaluates Bookes 3, 5, 8 as self-standing; 2, 6, 7 as accompaniments to older texts, and Book 4 covering mostly introductory topics. It is also digressive and Archimedean.

[75] Bulmer-Thomas(1997), 294.

Demography of the Collection	
Descriptors	*Names*
[Heath's list]	Apollonius of Perga, Archimedes, Aritaeus the Elder, Aristarchus of Samos, Autolycus, Carpus of Antioch, Charmandus, Conon of Samos, Demetrius of Alexandria, Dinostratus, Diodorus, Eratsthenes, Erycinus, Euclid, Geminus, Heraclitus, Hermodorus, Heron of Alexandria, Hierius, Hipparchus, Megethion, Menelaus of Alexandria, Nicomachus, Nicomedes, Pandrosion, Pericles, Philon of Byzantium, Philon of Tyana, Plato, Ptolemy, Theodosius
[Dedicatees]	Hermodorus, Megethion, Pandrosion
[Unknown mathematical occupation]	Hierius, Megethion, Hermodorus
[Not mentioned in Heath's list]	Sporus, Pandrosion's three students, Pappus' and Hierius' friends, Hippias, Plato, Zenodorus[76]

Thomas L. Heath listed 32 individuals that Pappus' mentioned throughout the **Collection**.[77] Most of them can be confirmed as mathematicians either by Pappus' heavy reliance on their works within the **Collection**, through their own independent works, or more frequent reference elsewhere in Greek mathematics. Heath's list is not exhaustive: Sporus, Hippias, Zenodorus, Plato are not on it. A handful of names are also unknown outside of Pappus. Sometimes, there is a lack clarity in Heath's identification of them. Often it can be difficult to figure out where they stand in the chronology of Greek mathematics; Heath distinguished Heraclitus of Ephesus the Philosopher from a supposed mathematician named Heraclitus in his work, **Greek Mathematics**.[78] If the Heraclitus mentioned was the former, then the chronology of the works in the **Collection** could easily go back as far as 535 BCE, long before Hippias, the earliest mathematician mentioned. The mathematician closest to Pappus in terms of chronology is the commentator Sporus, easily an older contemporary of Pappus, who is followed by the astronomer Ptolemy, who died over 150 years before the **Collection**. On the other hand, while one will not call the topics in Book 3 new even for late antiquity, Pappus does mention a rather large group of people who were active in the mathematical scene in Alexandria either as Mathematicians or interested parties. There are indeed three unnamed students plus Pandrosion, possibly Hierius, and an uncounted number of lovers of mathematics, whose constructions, errors, and solutions are the subjects of Book 3. Because

[77]Heath(1921)ii, 358-360.
[78]Heath(1921),ii, 577.

of the historical value of the problems therein take priority in the history of mathematics, the currency of Book 3 is an aspect of the **Collection** that is always overlooked.

Chronology of The Collection's Demography	
Timeline	*Mathematician*
[460-370 BCE]	Hippias
[428-347 BCE]	Plato
[390-320 BCE]	Dinostratus
[360-300 BCE]	Aristaeus
[360-290 BCE]	Autolycus
[325-265 BCE]	Euclid
[310-230 BCE]	Aristarchus
[287-212 BCE]	Archimedes
[280-210 BCE]	Nicomedes
[280-220 BCE]	Conon, Philon of Byzantium
[276-197 BCE]	Eratosthenes
[262-190 BCE]	Apollonius of Perga
[200-140 BCE]	Zenodorus
[190-120 BCE]	Hipparchus
[160-90 BCE]	Theodosius
[First century BCE]	Diodous
[10 BCE-69 CE]	Geminus
[10 CE-70CE]	Heron of Alexandria
[70-130 CE]	Menelaus
[85-165 CE]	Ptolemy
[240-300 CE]	Sporus
[300 CE onward]	Pappus, Pandrosion, Hierius, Megethion, Hermodorus
[Ambiguous]	Demetrius, Heraclitus
[Unknown date/pre-Collection CE]	Philon of Tyana, Erycinus

Occurrence of Names throughout the Collection	
Book	*Authors*
[2]	Apollonius of Perga, possibly Euclid
[3]	Pandrosion, 3 unnamed students, Hierius, Eratosthenes, Philon*, Heron, Nicomedes, Erycinus, Plato
[4]	Archimedes, Nicomedes, Sporus, Dinostratus, Hippias, Demetrius of Alexandria, Philon of Tyana, Menelaus of Alexandria, Apollonius
[5]	Megethion, Zenodorus, Archimedes
[6]	Ptolemy, Theodosius, Euclid, Menelaus, Autolycus, Aristarchus
[7]	Hermodorus, Euclid, Apollonius, Aristaeus, Eratosthenes
[8]	Hermodorus, Archimedes, Philon, Heron, Carpus of Antioch, Ptolemy

Therefore Books 3, 4 (mentioning Sporus and Menelaus), 6, and 8 (surveying Ptolemy's works) may indicate which topics that had popular interest, as in the case of Book 3, or were venues of recent mathematical activity besides Pappus' own contributions (books 4, 6, and 8). There are about 150 years unaccounted for according to the final list of mathematicians and Pappus does not cover every topic in mathematics. In some ways, this chronology of mathematics in the **Collection** confirms the older interpretation of late antiquity as a period of decline or, more specifically, lack of mathematical progress. One cannot say that there was no progress whatsoever. There were still developments and gaps in the chronology were not as long nor frequent: Diophantus (200 - 284 CE) and Porphyry (233 - 309 CE) do fill in a large portion of that 150+ year gap between Ptolemy and Pappus, with Diophantus producing essentially original material in the field of arithmetic and Porphyry writing ubiquitous late antiquity commentaries. Erycinus may also be a name that can be placed in the gap.

The final question that remains in setting the scene concerns the audience Pappus was writing for and how one should view the structure of the **Collection**. The **Collection** is a text of commentaries, histories, and assembled constructions. It has been established previously in this chapter that Pappus had a significant selection of the Alexandrian population from which he could draw his audience, but what is not known is whether the **Collection** was initially a collection.

The purported manuscripts of Book 1 and Book 8 give no sign that they were part of a larger corpus. If the books of the surviving Greek text were likewise removed from the **Collection**, these too would give no indication as to their placement in a larger text. The books do not contain any of the self-references that are expected from a text that was written with the intent of being part of a Collection or as a deuteronomic text. Late antiquity commentators often titled and subdivided works of older texts in way they felt were appropriate. One could look at the Collection and see that maybe Pappus' own commentators, should they have existed, took what was once a single text and separated the material into appropriate books. However, the existence of dedications in at least four of the surviving books, implies that they were initially separate treatises. The disparity and isolation of their topics certainly suggest this. Whether it was Pappus who did this or some later unnamed mathematicians is not unimportant but it is irrelevant to this discussion. What has priority is that the **Collection** clearly had two types of audiences regardless of the final authorial decision: Eight individual audiences for each book and one audience for a collection.

Heath once suggested that the **Collection** could be a handbook, albeit one intended to revive the classical Greek geometry. It was to be read with the original works (which were still extant) rather than replace them.[79] This is supported in Book 4 when Pappus ends with an analysis of the neusis construction assumed by Archimedes in **On Spirals**, "in order that you will not be nonplussed when you work through the book".[80]

Heath admitted that this did not hold up when the history of a mathematical subject was given. The originals would be too numerous to collect and thus the treatise would serve as its stand in.[81] It seems rather ironic that in many cases Pappus' text has survived whereas the indispensable texts have been lost.

Heath's suggestion also implied authorial uniformity and involvement in the production of

[79]Heath(1921) ii, 358.
[80]Jones(1986), 7.
[81]Heath(1921) ii, 358.

the **Collection** as a collection. As stated, it is not so much who collated a collection as it is for whom it was collated. Support for the books originally being separate treatises presents itself in Book 3. In setting out the historical constructions of the problem of two mean proportionals, none of the solutions presented use conics even though there were several ancient solutions that did. Jones believes that Pappus perhaps thought it too advanced for the intended readers of that particular book.[82] However, conic sections are mentioned in Book 4, a part of which is devoted to special curves. Pappus was clearly knowledgable in these matters but the discrepancy between books is inexplicable unless one believes that each book had a separate audience.

The audience of an individual treatise would easily have larger cumulative audience than a collection. The ability to own any sort of writing was limited to the affluent in antiquity but in an educational setting, an enterprizing student or teacher might make their own copies or disseminate the teachings amongst a larger group. It may thus seem that the identification of Pappus' audience may remain hopelessly vague and the question ultimately fruitless. Instead, it is demonstrated that while Pappus' audience was potentially quite large, he wrote very specifically depending on who he was writing to, taking into account skill level and knowledge and interests. When his audience changed, so did the level of discourse.

To briefly conclude, as both a product of late antiquity as well as the work of a man from Alexandria, the **Collection** and its many influences are often not as well understood as once believed. Standing before it was over a millennium of Egyptian advances in literacy and numeracy as well as a scribal culture that enabled a form of secondary literacy to arise, allowing illiterate citizens such as women to become heavily involved in the government, economics, and legal system of their village. The not-so-sudden influx of Greek and Roman culture was certainly a death knell of written Egyptian, but not so for spoken language, thus necessitating the maintenance of the scribal class that expanded a possible, literate audience for Pappus' work to include people other than the affluent and educated. But, the intellectual demand of mathematics by late antiquity restricted that potential. In its midst, the **Collection** was

[82] Jones(1986), 5.

part of a shift in writing Greek mathematics which gave priority and prominence to the language and format of classical Greek mathematics instead of the mathematics proper. This change influenced the opinion of modern scholars to view the mathematics of late antiquity as vulnerable to the same decline as everything else in antiquity. Nevertheless, Pappus had a wide range of interests, old and new, that allowed the **Collection** to be accessible to audiences with differing levels of mathematical literacy. It has also become one of the more valuable resources in the history of ancient mathematics for the record that Pappus keeps of important mathematical discoveries that are not found elsewhere.

Chapter 2

The Domain of Analysis

The **Collection**, Books 5 and 7 in particular, was acknowledged by Descartes, Newton, and others, as having inspired their own achievements. Unfortunately, the same people were suspicious about the ancients recognition of similar accomplishments in their own works.[1] Of course, Pappus was the most emblematic of this even though it is he who provided the strongest case for the ancients' capabilities with analysis and synthesis.

Apollonius, Aristaeus, Eratosthenes, and Euclid are named by Pappus as authoring the foundational texts of analysis and synthesis. By reputation alone one would not say that these men were inept by modern standards. But one can appreciate the difficulty that Early Modern scholars faced when it came to obtaining multiple the manuscripts needed to verify Pappus' claims. Most of them are lost but Pappus' epitomes motivated many mathematicians and scholars to try and restore the original texts. Still, it is bad historical practice to assume that a practitioner or practitioners for not know what they are doing, especially where the extant text revels otherwise.

The *Domain of Analysis* thereby provides a starting point from which analysis and synthesis, as Pappus knew it can be examined.

[1]Hintikka(1974), 7n1.

2.1 The Domain of Analysis

Critical to Pappus' exposition of analysis and synthesis are the following two passages from the introduction of the **Collection**. The text opens:

> That which is called the *Domain of Analysis*, my son Hermodorus, is, taken as a whole, a special resource that was prepared, after the composition of the *Common Elements*, for those who want to acquire a power in geometry that is capable of solving problems set to them; and it is useful for this alone. It was written by three men: Euclid the Elementarist, Apollonius of Perge, and Aristaeus the elder, and its approach is by analysis and synthesis. Pappus, **Collection** 7.1: 1-10 (as translated by Jones(1986)).

And the end of the introduction closes with a catalogue of works as followed:

> The order of the books of the *Domain of Analysis* alluded to above is this: Euclid, *Data*, one book; Apollonius, *Cutting off of a Ratio*, two; *Cutting off of an Area*, two; Determinate Section, two; *Tangencies*, two; Euclid, *Porisms*, three; Apollonius, *Neuses*, two, by the same, *Plane Loci*, two; *Conics*, eight; Aristaeus, *Solid Loci*, five; Euclid, *Loci on Surfaces*, two; Eratosthenes, *On Means* [two]. These make up 33 books. I have set out epitomes of them, as far as the *Conics* of Apollonius, for you to study, with the number of dispositions and diorites and cases in each book, as well as the lemmas that are wanted in them, and there is nothing wanting for the working through of the books, I believe, that have I left out. Pappus, **Collection** 7.3: 13-25 (as translated by Jones(1986)).

The reference, in the first quote, to Hermodorus is telling as Hermodorus was not Pappus' son. *My son* was used as an endearment that can also mean *my student*. This often occurs in

Egyptian papyri where the 'brothers' and 'uncles' addressed in letter were not related to the sender by any means. Somewhat more profound than this cultural nuance is the consequence of this work being addressed to a student rather than a fellow full-fledged mathematician. Hermodorus may have been a promising student who was deserving of a mathematical treatise on several mathematical texts written just for him. The truth is likely to be less extreme. It could have been that Hermodorus impressed Pappus enough that the older mathematician believed that his student would understand and follow his lengthy lemmas but the youth was early-on in his training. With his student unfamiliar with *Domain of Analysis* and what it entailed, Pappus was pressed to write this text in a manner that departs from other mathematical commentaries and treatises. That is, Pappus needed to provide lots of technical details about each text from the *Domain* that would otherwise have been omitted even by other late antiquity mathematicians. Conversely, one must be wary of Pappus' explanation of analysis and synthesis. If he must tell his student what was in Apollonius' **Conics**, then one cannot anticipate Pappus to get too complicated with the lemmas he provided from that and other *Domain* texts. One can only expect a summary explanation that places the essence of analysis and synthesis ahead of its actual practice.

Pappus does not specify what form the *Domain of Analysis* took; whether it was a collection of complete treatises or a selection of propositions from which Pappus only included lemmas from *Cutting off of a Ratio*, *Cutting off of an Area*, *Determinate Section*, *Neuses*, *Tangencies*, *Plane Loci*, *Porisms*, *Conics*: Books $1-3$, $5-8$, and *Loci on Surfaces*. Propositions from Euclid's *Data*, Aristaeus' *Solid Loci*, and Eratosthenes' *On Means* are not included.[2] Pappus does specify when the *Domain of Analysis* was compiled, on that it was after the composition of the **Common Elements** that Pappus refers to in the first excerpt. According to Jones, such a title would refer to such things that are common to all branches of mathematics.[3] He also commented that **Common Elements** did not refer to Euclid's **Elements**. This is based on the frequency which Pappus used other words to denote Euclid's work, i.e. **First**

[2] Jones(1986), 69. Pappus is the only substantial source of the Domain, but it was known by Marius around 500 CE and Eutocius quoted from it (or Pappus) a century after that.

[3] Jones(1986), 380.

Elements (prota stoicheia) or just **Elements** (Stoicheia).[4] However, looking to both Jones and the Lidell-Scott Greek-English Dictionary prove that *Elements* itself can refer to other works as well. Outside of Book 7, it is used to title at least two non-Euclid texts in the **Collection**.Within Book 7 itself, it was used to describe works by Aristaeus (**Solid Loci**), Apollonius (**Plane Loci**), and Euclid (**Porisms**). It was a term used in a adjectival manner and referential to whichever text was the subject of discussion. It did not necessarily refer to a text that would have been most familiar to Greek readers.

The date when the *Domain* was compiled is not clear either. If one were to assume that the *Domain of Analysis* was compiled by one of the mathematicians whose works were included, the latest possible date for the *Domain of Analysis* would fall within the purported year of the death of its latest contributor. This would be Apollonius, who supposedly died in 190 BCE. One could easily add 5 years to account for students or friends taking up the compilation and preparation the text post-humously. If *Domain of Analysis* was not written by Apollonius, than the earliest it could have been produced is 242 BCE, 20 years after the purported birth year of Apollonius, adjusting for a prodigious youth.

2.2 The Domain and Book 7

It seems that analysis and synthesis was a familiar tool long utilized by classical mathematicians and their descendants. In the few places where one could compare what Pappus wrote to the original text itself, the differences in manuscripts complicate the proper coordination between the texts.

Pappus divided Book 7 into two sections. The first are the epitomes, which offer historical and technical details about each text in the domain. The second section is the lemmas, which make up the majority of Book 7. For Pappus, a lemma is a statement that is used as a

[4]Ibid.

stepping stone toward the proof or result of another statement. They less important than the original proposition but are necessary for the validation of the more mundane details of the proof or for results that are used repeatedly in the work.[5] Keen eyes and minds are clearly needed to match a lemma to its appropriate proposition or theorem. For Heath, Pappus' lemmas were particularly problematic:

> It is difficult to give any summary or any general idea of these lemmas, because they are very numerous, extremely various, and often quite difficult, requiring first-rate ability and full command of all the resources of pure geometry. Their number is also greatly increased by the addition of alternative proofs, often requiring lemmas of their own, and by the separate formulation of particular cases where by the use of algebra and conventions with regard to sign we can make one proposition cover all cases. The style is admirably terse, often so condensed as to make the argument difficult to follow without some filling-out; the hand is that of a master throughout. The only misfortune is that, the books elucidated being lost... it is difficult, often impossible, to see the connexion of the lemmas with one another and the problems of the book to which they relate. Heath,**The Handbook of Greek Mathematics** ii, 404.

This is exactly the case in Jones' commentary on Pappus' lemmas to Apollonius' **Conics**. Excluding the Arabic texts (as the text from which they are based is of unknown provenance), it is believed that Pappus likely worked from an earlier manuscript than Eutocius (early 6th century) consulted for his commentary of **Conics**' Books 1-4.[6] Jones identified numerous discrepancies between the text transmitted to us today and the text that Pappus was responding to. At the end of the epitome, Pappus stated that:

> The eight books of Apollonius' **Conics** contain 487 theorems or diagrams, and

[5] Jones(1986), 71. Jones appraisal of Pappus' lemmas is not promising: Dreary reading, not the most advanced or innovative geometry.

[6] Jones(1986), 474-475, 474n17.

there are 70 lemmas, or things assumed in it. Pappus, **Collection** 7.42:10-12 (as translated by Jones(1986)).

Since there are only 410 propositions in the 7 extant books of the **Conics**, it has been propounded that there may have been an 8th book with 77 proposition. Jones disagrees because that was simply too many propositions.[7] Jones did suggest that Pappus possibly employed a different method of counting or merely over-counted but Pappus was also known to combine or reduce the original propositions in his other texts.[8] However, given the condition of Book 7, Jones' view is more than justified. Lemmas are either bungled (lemmas 244, 276, 279, 298), unexplained(lemmas 242), trivial (lemmas 247, 279, 281 282, 283), or, in rare cases, useful but not otherwise present in the modern text of the **Conics** (lemmas 233, 234, 235, 236, 270, 280). More frequently are identifications, made by Jones and others, of lemmas that could be attached to specific propositions but were of uncertain relevance (lemmas 240, 242, 247, 255, 257, 259, 260, 261, 277, 278, 279). Jones and others were trying to take a text that was written in response to the **Conics** and expected that Pappus would conform to the **Conics**. However, if Pappus is publishing lemmas he found wanting in the originals, than matching **Conics** to the **Collection** will undoubtedly be difficult. Pappus tends to select a single line or a simple geometrical diagram from what might have originally been a complicated curve.

Just as the coordination and fact-checking of the differences between Pappus' lemmas and the **Conics** was out of the scope of Jones' work of Book 7, it is also out of the scope of this thesis, but it is important to keep some things in mind. One cannot be sure if Pappus wrote a commentary based on a secondary source or if he followed its format and provided Hermodorus with lemmas derived from the original material. The **Collection** provides more than ample evidence that Pappus likely had the best access to the original texts than most later commentators and thereby would be capable of achieving the latter. For Book 7 to reflect to the former would be somewhat unheard of and if Book 3 is any proof, Pappus would

[7]Ibid., 498.
[8]Ibid., 498.

make it well-known if he thought there were any mistakes. He is high-handed in addressing mathematicians who were not classical authors.

2.3 Provenance and Originality of Book 7

While the source or sources of Book 7 cannot be completely clarified, the descriptions of analysis and synthesis therein are fully attributable to Pappus. First, he made a clear distinction between what he provided to Hermodorus and what the *Domain of Analysis* offered: "I have set out three epitomes of them,... as well as the lemmas that are wanted in them, and there is nothing wanting..".[9] The *Domain of Analysis*, regardless of its format, clearly motivated Pappus' own work. In stating that the epitomes of Book 7 were set out by him for the benefit of Hermodorus, Pappus thereby claimed to have extensive knowledge of most of the works in the Domain. However, the bulk of Book 7 consists of lemmas that, as mentioned above, were added wherever Pappus thought they were wanting.

Additionally, his reverence towards classical mathematicians is maintained as long as they were reverent to the successes of their predecessors.[10] Otherwise, Pappus becomes increasingly critical of their efforts.

> Our immediate predecessors have allowed themselves to admit meaning to such things, though the express nothing at all coherent when the say 'the (thing contained) by these', referring to the square of this (line) or the (rectangle contained) by these...They who look at these things are hardly exalted, as were the ancients and all who wrote the finer things. When I see everyone occupied with the rudiments of mathematics and of the material for inquiries that nature sets before us, I am ashamed; I for one have proved things that are much more valu-

[9] Ibid., 7.3:20-21.

[10] Bernard(2003b), 100. This reverence may have been drilled into Pappus. The works of ancient philosophers and mathematicians was standard fair for greco-roman education.

able and offer much application. Pappus, **Collection** 7.41:24-29 (as translated by Jones(1986)).

.

If the *Domain of Analysis* was anything other than a genre, Pappus would not have hesitated to comment. Instead, we are told by Jones that most of the mistakes in the **Collection** are due to Pappus himself, with the exception of the final set of propositions appended at the end. And so we continue through Book 7 having learnt two things about Pappus' take on analysis and synthesis. First, is that he has learnt the methods from the best of the classical mathematicians. Second, his transmission of that knowledge is hampered by his interlocutor; a student whom Pappus presumed did not know what the *Domain* contained and implied. Through the comparison between exposition and the propositions the impairment of writing an introduction to someone how is less mathematically astute has also impaired modern scholars ability to truly understand Pappus as they are supposed to. It is in the next couple of chapters that I hope I can begin to push Pappian scholarship into productive period once more.

Chapter 3

Everyone Love Archimedes: A Brief Prelude to Pappian Analysis and Synthesis

Having set the scene for an examination of Pappian mathematics, we will actually take a step back and take a brief look at the analysis and synthesis of Pappus' acknowledged predecessors. The surviving texts of the *Domain of Analysis* are unsuitable for this task: Euclid's **Data** does not include synthesis; Apollonius' **Cutting off of a Ratio**,**Cutting off of an Area**, **Conics**, and **Tangencies** are too involved, or still unknown quantities, for the general example we seek here.

It does not help that the numbers of analysis and synthesis pairs in classical mathematics that are extant are so few. Archimedean corpus has 6 pairs; Apollonius has 6 pairs; Euclid has 5 pairs which are actually post-Euclid insertions.[1]. The five pairs are also theorematical analyses, a rare but naive type of analysis as Jones tells us; obviously not a desirable set to work from. I have selected a problem out of Archimedes' **Sphere and Cylinder**. Although he is not named in the *Domain*, the immediate influence of Euclid would still have been resonant when Archimedes was a young student and he was an older contemporary to

[1] Netz(2000), 140-141

Apollonius and Eratosthenes. Archimedes is referred to elsewhere in the **Collection**, so this inclusion is in keeping with what is known about Pappus' mathematical interests. Ultimately, it is the familiarity and accessibility of Archimedes that make him an ideal representative of analysis and synthesis. We know that he is not a feeble mathematician and can be trusted to sufficiently represent classical mathematical practice of analysis and synthesis.

3.1 Sphere and Cylinder, Book 2, Proposition 3

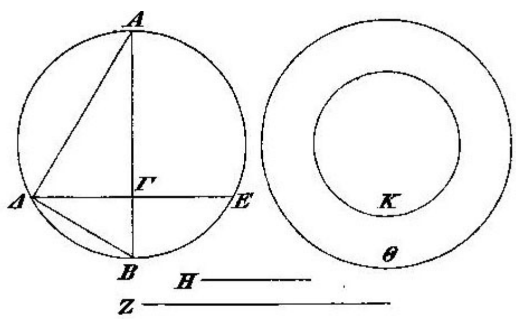

Figure 3.1: Proposition 3, Book 2, **Sphere and Cylinder** from Heiberg

The enunciation of the analysis declares that on is to cut a given sphere with a plane so that the surfaces of the segments may be in a given ratio to one another. The figure is constructed first, with $AB\Delta\Gamma$ representing the great circle of the sphere. AB is the diameter and DE is the plane that cuts it perpendicularly, but the position of the intersection is unknown. A and B are connected at point Δ.[2]

> The (given) ratio is the surface of segment ΔAE to the surface of segment ΔBE[1] but the surface of segment ΔBE is equal to a circle that has a radius equal to $A\Delta$[2]; the surface of segment ABE is equal to a circle with a radius equal to ΔB[3]. Thus the pronounced circles are to each other[4]. The square of AD is to the square of ΔB[5]; this is as $A\Gamma$ is to ΓB[6]. Therefore the ratio of $A\Gamma$ to ΓB is given[7]. Thus point Γ is given[8]. And DE is perpendicular to AB[9].

[2]Archimedes *Sphere and Cylinder*i, 206.5-206.9

Therefore ΔE is given in position[10].

Archimedes, **Sphere and Cylinder** 2.3, 184.10-184.20.

[1]-ΔAE:ΔBE.

[2]-ΔBE= circle with rad. of $A\Delta$.

[3]-ΔAE= circle with rad. of ΔB.

[4]-$A\Delta$:ΔB.

[5]-$A\Delta^2$:ΔB^2.

[6]-$A\Delta^2$:ΔB^2=$A\Gamma$:ΓB.

[7]-$A\Gamma$:ΓB is given.

[8]-Γ is given.

[9]-ΔE is perpendicular to AB.

[10]-ΔE is given in position.

An analysis, as completed by Archimedes, appears to proceed like so. Once the figure is established as having been established, the reader is given a ratio that is by no means supportable from the information they have been given about the diagram. However, since ΔAE:ΔBE is in a given ratio. If this is the goal determined by the enunciation, then the Archimedes will have completed a problem before it even started. Instead, he continues on. Using a proposition that was proven in Propositions 42 and 43 of the first book of **Sphere and Cylinder**, Archimedes knows that he can manipulate ratios and the segments to create an equivalent ratio between the circles that are formed from the hypotenuses of the right-angled triangles $B\Gamma\Delta$ and $\Delta\Gamma A$, which then become similar triangles.

He performs one final manipulation in step [6]. He uses a variation of Book 5, definition 10 from Euclid's Elements: when four magnitudes are in continuous proportion, the first is said to be the cubed ratio of the fourth i.e. if $\alpha\beta$=$\beta\gamma$=$\gamma\delta$ then α:δ=a^3:δ^3. Using only three magnitudes, a duplicated result can be achieved i.e. $\alpha\beta$=$\beta\gamma$ then $\alpha\gamma$=$\alpha^2\beta^2$. Archimedes uses the latter. Since the $B\Gamma\Delta$ and $\Delta\Gamma A$ are similar, the ratio $B\Gamma$:$\Gamma\Delta$=$\Gamma\Delta$:ΓA as per the duplicate result shown above becomes $B\Gamma$:ΓA=$B\Gamma^2$:$\Gamma\Delta^2$. Again, due to similar triangles $B\Gamma^2$:$\Gamma\Delta^2$ can be substituted with $A\Delta^2$:ΔB^2.

Archimedes then switches focus. No particular reason is provided as to why Archimedes preferred $A\Gamma$:ΓB as the given ratio over other possible ratios but it is implicitly obvious by the sequence of the objects given in steps [7], [8], [9], and [10] that Archimedes desired to have the existence of the intersecting plane of ΔE verified. This is unusual since the figure was provided, but all that is given is the given ratio. It seems that in a general sense, Archimedes is using analysis in this case to validate that every point in the diagram can be justified from the given ratio.

Lets see how the analysis compares to the synthesis:

> The synthesis is thus. Let there be a sphere, of a great circle $AB\Delta E$ and diameter AB. The given ratio is Z to H[1] and let AB be cut at Γ. Thus $A\Gamma$ is to $B\Gamma$, as is Z is to H[2], and let the sphere be cut through plane Γ perpendicularly and at right angles to line AB and let ΔE be the cut. And let $A\Delta$, ΔB be joined at the top and let the two circles Θ and K be taken: Θ contains a radius equal to $A\Delta$[3]; K has a radius equal to ΔB[4]. Therefore circle Θ is equal to the surface of segment ΔAE[5] and K to segment ΔBE[6]. This was shown by the example in the first book. Since $A\Delta B$ makes a right-angle triangle and $\Gamma\Delta$ is perpendicular to it, $A\Gamma$ is to ΓB[7], this is Z to H[8] and $A\Delta$ squared to ΔB squared[9], this is also the square of the radius of circle Θ to the square of the radius of circle K[10], this is the surface of segment ΔAE to the surface of segment ΔBE[11]. Archimedes, **Sphere and Cylinder** 2.3, 184.21-185.13.

[1]-Z:H.

[2]-$A\Gamma$:$B\Gamma$=Z:H.

[3]-Θ rad.=$A\Delta$.

[4]-K rad.=ΔB.

[5]-Θ=ΔAE.

[6]-K=ΔBE.

[7]-$A\Gamma$:ΓB.

[8]-$A\Gamma$:ΓB=Z:H.

[9]-[8]=$A\Delta^2$:ΔB^2.

[10]-Θ rad.^{2}=K rad.^{2}.

[11]-ΔAE:ΔBE

Archimedes does quite a few things differently. Instead of revealing the given ratio immediately, it is symbolized by Z:H. This is no less of a given ratio than the ratios given in the analysis but because Z and H are not part of the figure, what they represent is speculative until Archimedes allows there to be an equivalency between Z:H and $A\Gamma$:$B\Gamma$. Recall that in the analysis that final ratio was carefully derived from the given ratio between the surfaces of the two segments at the end of the analysis; here, it is easily provided. This is due to the intersection of planes AB and ΔE, point Γ being provided unlike the analysis. It is easier to presume the existence of a ratio of $A\Gamma$:$B\Gamma$ when the existence of each point can be accounted for.

Once $A\Delta$ and ΔB are joined at Δ, Archimedes again invokes the equivalency between the surfaces of sphere and the surfaces of segments by introducing circles Θ and K to represent circles with radii equal to the hypotenuses of the similar triangles in the segments. As one can see, in moving through the synthesis, we retrace the steps of analysis, albeit in less than perfect order. The introduction of Θ and K may seem redundant insofar as that in the analysis, the reference to the circle allowed for the lines $A\Delta$ and ΔB to fall into the given ratio of the segments. However, since the ratio of the segments is not determined circles Θ and K provide something of a mental bookmark until they are no longer needed to be placeholders for the given ratio. In the conclusion of the synthesis, Pappus works back through the equivalencies that were worked out in the analysis first and is finally able to manipulate Θ and K until ΔAE:ΔBE.

3.2 Thoughts on Archimedean Analysis

Archimedes understood the connection between analysis and synthesis as one in which the start of analysis and the conclusion of the synthesis were one and the same regardless of how they are achieved or what one derives from it. By the same token, the end of analysis and the beginning of synthesis may also be one and the same but they, and the steps that preceded or are derived from it, are expressed in different ways. The use of Z, H, Θ, and K show how in synthesis one can be provided with a figure but if a desired ratio is not evident, auxiliary figures must be introduced so that the proposition can move from step to step i.e. without Z:H, we would not have the ratio $A\Gamma$:ΓB, without Θ and K, $A\Delta^2$:ΔB^2 could not be equivalent to ΔAE:ΔBE.

With regards to analysis, the selection of the objects that are given seems to be arbitrarily chosen by the mathematicians. If a particular point or line is not fully determined in the enunciation but shown in the figure, it is one way Archimedes let the reader know what they would have to find in the analysis. When the reader switches over to the synthesis, the relevancy of the given objects diminishes to a degree. They may be important for a step or two but quickly become a mundane part of the proposition.

It should also be noted that Pappus is approaching analysis and synthesis as a tool for discovery. He may have relied on previous propositions but they were often propositions that he could claim as his own. We know as much because later mathematicians, such as Pappus, would turn to his work as source text. In the previous chapter, we see that Pappus does include his own lemmas but they are based on the original works of earlier mathematicians. Analysis and synthesis clearly had transformed by Pappus' time into something mundane but still essential for mathematical practice.

Thus concludes this brief section on classical analysis and synthesis. At the end of the next chapter, both Pappus' method and practice will be measured against Archimedes to see where Pappus departs from tradition, if he does, and where he falls in line with it, if he does.

Chapter 4

Hermodorus' New Power: Analysis in Book 7

The introduction of Book 7 has been viewed as the most detailed discussion of analysis and synthesis from antiquity.[1] However, in its evaluation, modern scholars are quite disappointed by what they see as a very poor and inaccurate description of those methods. It is forgotten that Book 7 was addressed to an individual probably unfamiliar with analysis and synthesis so that Pappus had to write about it and summarize the key texts on the subject. In the first instance, one sees Pappus engaged in correspondence with someone he viewed as a pupil rather than a peer or colleague. The extent of the rudimentary information about a mathematical method, and one that was associated with classical mathematicians by late antiquity, should serve as an indication of the pedagogical nature of this particular book in the **Collection**. The language that Pappus brought into play did not have the exactness expected of a dialogue between scholarly equals. Instead, it is vague and confusing. This is not because Pappus was incompetent but it is necessarily simplified for the benefit of the supposedly less mathematically adept Hermodorus.

[1] Mäenpää(1997), 201.

4.1 Pappus in Theory

Although I am building up to the new discoveries that are possible when propositions are included, we cannot leave the exposition can be left untouched. Our knowledge of the introduction to Book 7 is imperfect and navigation between what Pappus simplified and what he left under-determined is key to making the most of the propositions that will appear in the following sections. By refining the interpretations of the exposition, more can be made about Pappus' true connection to past efforts by mathematicians such as Archimedes and, of course, his understanding of the method when it is finally challenged by new scenarios in Books 3 and 2.

4.1.1 Analysis

> ἀνάλυσις τοίνυν ἐστὶν ὁδὸς ἀπὸ τοῦ ζητουμένου ὡς ὁμολογούμενου διὰ τῶν ἑξῆς ἀκολούθων ἐπί τι ὁμολογούμενον συνθέσει.
>
> Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis.
> Pappus, **Collection** 7.1:11-13 (as translated by Jones(1986), 82).

.

Pappus began by giving analysis the priority of place in his introduction. However, his brief sketch left it under-specified. The term Pappus invoked to refer to what one is seeking is *ζητούμενον* and this is the beginning of the analysis. It is not sought, but treated as if it were already established. The profile of the *ζητούμενον* as something that was once sought was clearly provided from somewhere other than by the analysis. Pappus may have anticipated the change of the nature of the established *ζητούμενον* into just the *ζητούμενον* but if analysis did truly have the priority of place, than the first step in the beginning of the path should have been introduced in a different manner. Indeed, all Pappus needed to write was that analysis was the movement from what was given to what was sought.

However, Pappus chose to terminate the path with something that is established or given by synthesis, thereby bracketing analysis with two objects that have their nature, existence, and identity provided for them prior to analysis. The first step is from an unknown source as of yet and the final step is from synthesis. However, Pappus did not yet state anything explicit about the relation between these steps prior to analysis. All that is revealed about synthesis is that what is given in that method is the true *ζητούμενον* of analysis insofar that it is what is sought in that technique.[2] There remains no indication that the relationship is reversed during synthesis, with the established *ζητούμενον* of analysis serving as the final step of synthesis.

The third object that Pappus introduced between the first and final step is a series of yet undiscernible *consequences* or ἀκολουθος . All that is known of them is that during analysis, they must be arranged in analysis to follow from the given *ζητούμενον* to something that is given by synthesis.

Without clearly defining what he wanted when he asked for both the established *ζητούμενον* and for what was established in synthesis, Pappus left much available for interpretation. First, with no expansion on what synthesis is at this stage, the connection to analysis is tenuous as the reciprocal appearance of the brief sketch is not elaborated upon as being necessary. There is only the implication that such a relationship exists. At this part Pappus could have otherwise discussed analysis without referring to outside sources. Second, it is unknown if analysis and synthesis are intended to be representative of an entire proof or is only applied to just a part of it. Analysis and synthesis, as it appears in Book 7, is applied to lemmas which are only concerned with parts of a proof but that in no way precludes an analysis or synthesis from being conducted on an entire proof. Finally, there are many parts of a problem or theorem that can be understood to be established. Pappus provided no advice to his protege as to when or what it is in the synthesis that is acceptable as established.

[2] Hintikka(1974), 11-12. This would be analysis in a downward movement.

Granted, this is more of a question of how the established object is identified. Nonetheless, problems and theorems can proceed any number of ways and, in the setting up of the proof, can contain one or many objects which could pass for established.

Thankfully, Pappus was quick in offering more insight:

> ἐν μὲν γὰρ τῇἀναλύσει τὸ ζητούμενον ὡς γεγονὸς ὑποθέμενοι τὸ ἐξ οὗ τοῦτο συμβαίνει σκοπούμεθα καὶ πάλιν ἐκείου τὸ προηγούμενον, ἕως ἂν οὕτως ἀναποδίοντες καταντήσωμεν εἴς τι τῶν ἤδη γνωριζομένων ἢ τάξιν ἀρχῆς ἐχόντων · καὶ τὴν τοιαύτην ἔφοδον ἀνάλυσιν καλοῦμεν, οἷον ἀνάπαλιν λύσιν.

> That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method ‘analysis’, as if to say *anapalin lysis* (reduction backward).[3]
> Pappus, **Collection** 7.1:13-18 (as translated by Jones(1986), 82).

The path of analysis remained unchanged nor was the definitive source of the *ζητούμενον* identified but Pappus conceded that the path from the *ζητούμενον* to the final step was slightly more complicated than he initially let on. Hintikka and Jones offer differing translations for the line *ψνωριζομένων ἢ τάξιν ἀρχῆς ἐχόντων*. The former wrote it as "...already known **and** being first in order" (emphasis mine) whereas the latter "...already known, **or** occupy the rank of first principle".[4] Both attempt to convey that the final step in analysis is important early on in the synthesis. Objects and relations can become known at any time in a proof. Therefore, the position of first rank was significant to, as well as undermined

[3]Mäenpää(1997), 205-206. Heath translated this line in his translation of Book 1 of Euclid's **Element** and translated it as "backward solution". Hintikka follows Heath's precedent but Mäenpää(1997) prefers Jones' translation because Pappus included theorems in his writing, not just problems.

[4]Hintikka(1974), 8.14.

by, Pappus; he still had not admitted if there was any cohesiveness between analysis and synthesis insofar that the prior source of ζητούμενον and other elements of analysis were accessible through synthesis too. Jones used an unfortunate translation of *first principles* which confuses Pappus' intent to indicate rank with elements that are more in the philosophical realm of mathematics. The annoyance with both translations of the line is that neither fully represent what is in the text. Hintikka is perhaps too liberal by translating the η as an AND. It is actually an OR, as Jones translated, but that OR unintentionally gives the idea that something other than what was established in synthesis could be the final step in analysis. An accurate translation in spirit, if not the word, would replace OR with WHICH, as in "...things that are already known WHICH occupy the rank of first principle". Alternatively, the OR can be removed, leaving THAT to demonstrate the same intention. The blame is firmly on Pappus here as he previously stated how the final step in analysis related to synthesis but only now firmly declared that there is some reciprocal nature between analysis and synthesis; what is sought in analysis serves as one of the first steps in synthesis.

One is still set up to see analysis as the first process now that any reference to synthesis has been removed. Pappus lost no time in placing the responsibility for the progress of analysis on the ability of the reader to determine the correct order of steps that best move one from the ζητούμενον to the τάξιν ἀρχῆς, (or things of first order). By recommending that one systematically look for the steps that preceded from the established ζητούμενον, Pappus left little room for misstep. The particular relationship of the ζητούμενον to the τάξιν ἀρχῆς called upon a specific series of step that connected these objects. In this sense, analysis is comparable to many games of strategy in which the mathematicians must look for the steps that follow from the established ζητούμενον with an eye a few steps ahead to the so-called *first principles*. Such an analogy fails because analysis is more than just a 'reduction backwards'; it is an infallible immediacy that must exist between steps. If the first step of analysis sufficiently determined the sequence of the subsequent steps, room for mathematical discovery is eliminated as there is little need for the mathematician to look in the first place.

4.1.2 Synthesis

> ἐν δὲ τῆσυνθέσει ἐξ ὑποστροφῆς τό ἐν τῆἀναλύσει καταληφθὲν ὕστατον ὑποστησάμενοι γεγονὸς ἤδη, καὶ ἑπόμενα τὰ ἐκεῖ προηγούμενα κατὰ φύσιν τάξαντες καὶ ἀλλήοις ἐπισυνθέντες, εἰς τέλος ἀφικνούμεθα τῆς τοῦ ζητουμένου κατασκευῆς · καὶ τοῦτο καλοῦμεν σύνθεσιν. .
>
> In synthesis, by reversal, we assume what was obtained last in analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call ‘synthesis’.
> Pappus, **Collection** 7.1: 11-23 (as translated by Jones(1986), 82).

Synthesis, Pappus claimed, was the reversal of analysis. This is certainly a fair assumption as Pappus had just established a reciprocal link between the final step of analysis and the first step of the synthesis. This line is at the core of the prevalent expectations that Pappus intended for his readers to accept the convertibility of analysis and synthesis.[5] One moves through the consequences like before but the roles and directional relationship of the objects to one another have changed: In analysis the *ἀκολουθος* were consequences of the established *ζητούμενον* whereas in synthesis, one is to approach them as precedents of the synthetic *ζητούμενον* . The description of the precedents is muddled but it is plausible to interpret it as a reference to his earlier phrasing about the *ἀκολουθος* in relation to the established *ζητούμενον* than as an adherence to any order. For this reason, Hintikka and Remes wanted to translate *ἀκολουθος* as *concommitants* instead of *consequences* with respect to the relationship that these secondary objects have with one another.[6] The focus Pappus placed in synthesis was not on the connection of the precedents to each other but rather how the direction of synthesis differed from that of analysis. The only two objects

[5]Hintikka(1974), 18-39.

[6]Hintikka(1974), 15; Mäenpää(1997), 206. Hintikka's argument for this translation is based on their position that the direction of analysis moves from the desired result to the premisses from which it can be deduced. Mäenpää thinks concommitants are implausible because in the deductive tradition of analysis, one must talk of consequences.

that maintain a stable existence despite the change in roles are the first principles and the synthetic *ζητούμενον*. The former *ἀκολουθος* appear to be re-conceptualized to reflect the position of the synthetic *ζητούμενον*.

The lack of grounding in Pappus' first attempt at describing analysis has already been noted as the *ζητούμενον* first being sought was left undetermined. Then, in the final step of synthesis, another *ζητούμενον* is mentioned. While this chapter has tried to absolve Pappus for writing pedagogically, to have the first step of analysis and the last step of synthesis understood as two different objects would be the move of a truly feeble mathematician if no specifications are offered. It is not impossible but, as it will be demonstrated later, uncharacteristic in terms of Pappus' mathematical rhetoric; when Pappus introduces a word, he sticks to its first usage. Conversely, the *ἀκολουθος* /precedents are juxtaposed in analysis and synthesis in relation to the established and synthetic *ζητούμενον* respectively. Should these objects refer to a single object, than Pappus merely adjusted his analytical description around the changed status of the synthetic *ζητούμενον*. For all intents and purposes, Pappus arranged synthesis thusly: The synthetic *ζητούμενον* is the established *ζητούμενον* of analysis and the final step of analysis holds the first rank in synthesis.[7]

Pappus gradually established analysis and synthesis as cohesive methods but he did not commit to them being convertible. It is so far only implied through Pappus' manipulation of objects and vocabulary. But while moving through the now-precedents, three things occur: First, the precedents are set into their natural order; second, fitting them into each other; finally, attaining the construction of what is sought.

What remains to be established is the order of the *ἀκολουθος* during the transition. Pappus allowed lots of flexibility - natural order from first rank only guaranteed natural order, not a reverse order. The instruction to fit the precedents to each other only makes this purpose more prominent for Pappus never explicitly stated that convertibility was mandatory. The requirements for synthesis being the reverse of analysis were never all that strong. All that

[7] Panza(1997), 384.

is admitted is that the beginning and end points of analysis and synthesis were common to both, just reversed. If strict convertibility was desired, would it be necessary to fit them together if all that was needed was a quick flip of the order? Clearly not. There seems to be a recognition that the movement in synthesis was not quite the same as the movement in analysis, requiring different relations to be made explicit at different times in contrast to analysis.[8]

The sudden appearance of the construction or *κατασκευή* over the course of synthesis is intriguing as no mention is made of it in analysis. Here, it is attached to the synthetic *ζητούμενον*. With these two objects entwined as Pappus shows, the construction may potentially be something that is presumed alongside the established *ζητούμενον* in analysis. Maybe Hermodorus should be given more credit. Just as he was to understand the source of the first and final steps of analysis before synthesis was even approached based on Pappus' choice of words, he might have been expected to understand the presence of the construction throughout the analysis. Ironically, this is a sound suggestion. A criticism frequently leveled at synthesis as a problem solving method is that the movement between steps appears arbitrary whereas analysis, as previously mentioned, the movement is deemed more sensible.[9] It is easier to find to find a line on a rhombus if the construction is already provided for in full than it is to find that same line when the synthesis only gradually leads one to create that same form. With one of the goals of synthesis being the *κατασκευή* , it is reasonable to claim that it is upon the completed construction that the analysis can be followed and upon which the analytic connections are examined.

4.1.3 Loose Threads

By the end of the introduction, there are some unresolved issues. While the circular nature of analysis and synthesis can now be accepted, Pappus remained aloof about whether one

[8]Hintikka(1974), 18. Ideally, Hintikka believed that analysis should be reversible but convertibility was only hoped for at that point. It was in synthesis that the problem or theorem would prove convertible.

[9]Hintikka(1974), 12; Knorr(1986), 9. In Knorr's words, analysis had advantages for exegesis because the analysis of a problem exposed in a natural and well motivated way the rational behind each step in a construction. This does not occur in synthesis.

or both methods fully encompassed a problem or theorem and its ultimate direction. One is told that synthesis is the natural order but it is unknown what that order is relative to. Nor does one know how anything that is given or sought can be decided from a proof. This is the setback of relying on literal reading of the text when all Pappus attempted to do was give a brief overview before proceeding immediately into numerous propositions where his definition would be fleshed out in much more detail. This is most relevant to the concerns for convertibility. The reversal that Pappus referred to could be very simple and free, all that was required was the reversal of the first and last steps while everything in between ordered itself as needed. It could also be simple but stringent in which all steps maintain their order between the established *ζητούμενον* and the first principles when the direction reversed. By examining some sample problems later in the chapter, one will see why Pappus chose not to endorse one or over the other.

It is obvious from the outset that analysis would be the focus, but since synthesis is given equal representation early in the introduction, Pappus at least acknowledged that analysis was connected to a method of equal utility if not equal activity. Still, there is the question of whether analysis and synthesis were part of a single method or two separate methods that are reversals of one another in Pappus' mind. Given the predominance of analysis, the answer may actually be just a matter of preference. The description of analysis was written in such a way that implies that the process of synthesis had to occur first in order for the analysis to be completed. Pappus did this through the identification of the *ζητούμενον* . This *ζητούμενον* , however, cannot be obtained elsewhere as the sequence of synthesis, according to Pappus, was due to natural order. If, then, one is to believe that synthesis is the very process that organically occurs in the normal course of the proof then the necessity of there to be something that was once sought but now admitted for the purposes of analysis proves that although synthesis is essential for the operation of analysis, a reciprocal connection cannot be made. There is little room for the *ζητούμενον* to be mistaken for an intermediary step as the intermediary steps are not the things that were originally sought but merely the links that join the premisses to conclusions and vice-versa.

It should be no surprise that Pappus was not nearly as verbose about synthesis as he already labeled its the main objects in his method of analysis. There are two processes that occur: The arrangement of the *ζητούμενον* and the *προηγούμενον* into their natural order, *κατὰ φύσιν*, and the production of a construction, *κατασκευή*. It is expected that a construction towards what is sought is produced to coincide with the reading of the proof. Netz has already written much about the role played by diagrams in Greek mathematics and without restating too much of what he already accomplished, the investigation into Pappus' use will be to the point. It is intriguing that Pappus did not declare the role of the *κατασκευή* in analysis. The text and the diagram, Netz stated, are interdependent.[10] While the diagram cannot be considered as having truth in and of itself, it borrows on what truth the text gives it.[11] In return, the diagram organizes the text in a ways that can clarify underspending point or instructions.[12] Despite being interdependent, the partnership between these two elements is not equal. The creation of a diagram almost relies more on what mathematical sense demands than on the text and in turn, one will find that the text is often not recoverable from the diagram.[13] The text, on the other hand, can provide both the diagram and sometimes offer much needed elucidation.

The *κατασκευή* inadvertently straddles both analysis and synthesis. Borrowing its truth from the text, the construction (presumed with the *ζητούμενον*) is established in analysis with all of its truth inherent. Without the directions recoverable from the synthesis, the construction would not truly be understood until the final step of analysis. Moreover, the strength of analysis is based on the sensible and demonstrable relationships between steps and to this end, the *κατασκευή* does not add to the legitimacy of the analysis for its movement is so narrowly guided that it appears Pappus recognized that the analytical connection could stand for themselves.[14] In contrast, the goal of synthesis is the discovery of a thing sought, which is heavily reliant on the corresponding *κατασκευή* to uphold the relations between steps.[15]

[10]Netz(1999), 19.
[11]Ibid.
[12]Ibid., 30.
[13]Ibid., 25.
[14]Panza(1997), 385.
[15]Panza(1997), 384-385.

Analysis is based on a complete, but unseen, diagram being provided. Even if, logically, one is suppose to assume that the synthesis was completed first, this is nonetheless inconvenient for modern readers. Our command of all the resources of pure geometry, to use Heath's words, is not nearly as implicit or as masterful as that of ancient mathematicians'.

4.1.4 Division of Analysis

Although *κατασκευή* was the last thing that Pappus said about the method of synthesis in exposition, there was still much that was said indirectly about synthesis through the two kinds of analysis:

> There are two kinds of analysis: one of them seeks after truth, and is called 'theorematic'; while the other tries to find what was demanded, and is called 'problematic'. In the case of the theorematic kind, we assume what is sought as a fact and true, then, advancing through its consequences, as if they are true facts according to the hypothesis, to something established, if this thing that has been established is a truth, then that which was sought will also be true, and its proof the reverse of the analysis; but if we should meet with something established to be false, the thing that was sought too will be false. In the case of the problematic kind, we assume the proposition as something we know, then, proceeding through its consequences, as if true, to be something established, if the established thing is possible and obtainable, which is what mathematicians call 'given', the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible, then the problem too will be impossible.
>
> Pappus, **Collection** 7.2: 24-12 (as translated by Jones(1986), 82-83).

In the end, the actual differentiation of analysis is somewhat irrelevant. In order for there to be 'problematic' and 'theorematic' analysis, the source would be a problem or a theorem respectively.[16] The forms of the validation by analysis would already be determined without further specification. The distinction between problems and theorems was hardly relevant to Pappus' lemmas either. Book 7 mostly contained theorems and passed no comments other than when the theorem was proven. The word 'given' is frequently used in these cases as well.

This final description of analysis is unusual because it informs one of the scenarios in which analysis can be used and the proper terminology of its results. It does not, on the other hand, tell one anything new about the technique of analysis. For the most part, theorematic and problematic are indistinguishable from each other for both take what was sought as true, given, known, or fact and proceed through the consequences until it reaches something established. Differences lie in how we describe a successful analysis. If we go through a successful theorematic analysis, one can say that the *ζητούμενον* is true and if the *ζητούμενον* is true, then is follows that the reverse of the analysis is true; as of yet, Pappus did not inform the reader of any other reversal of analysis other than synthesis so one would be safe in accepting that the synthesis is likewise true. For the problematic, a successful case could be called possible.[17] An unsuccessful analysis can be seen as false or impossible, by *reductio ad absurdum*, respectively. The attribution of possibility/impossibility and truth/falsehood to the method of analysis brings the problem of the convertibility between it and synthesis to the fore as Pappus finally provided a defining difference between analysis and synthesis, thereby possibly offering an explanation as to why analysis was a preferred method. For analysis to be valuable it had to provide something, it must perform a function that cannot be promised nor accomplished by synthesis. There has to be some way for analysis to be utilized beyond being the reverse of synthesis. If one was intended accept a correct synthesis intrinsically, there would be little point for analysis. In assigning analysis the role of final validation, there is an attraction to skip the preliminary work that synthesis requires and to jump into

[16] Jones, 67. Theorematically analysis is rare in ancient texts: The first five propositions in Book 13 of **Elements** and in lemmas 225, 226, 231, and 321 in Book 7 of the **Collection**. It shared the same steps as synthesis and did not guarantee correctness or possibility.

[17] Mäenpää(1997), 206. Mäenpää understood Pappus here as saying that an impossible analysis precluded synthesis. My reasons for disagreeing are printed early in this chapter. Pappus did give analysis a priority of place but that does not means that the synthesis was unnecessary in the first instance. The whole point of analysis by late antiquity was to serve as a confirmation tool; it was possible to have a possible synthesis and an impossible analysis.

verifying propositions. Synthesis can easily be constructed afterwards from the information from the analysis, even if it does not accurately reflect the actual practice of mathematics. However, there is a catch to having a built-in proof-check. By allowing analysis to determine ultimate possibility or truth, Pappus did not outline what would transform something that was possible in synthesis to impossible in analysis. Should one look to simply having the *ζητούμενον* in the expected positions or the exact reversal of the steps in analysis in synthesis and reverse? It also entails that convertibility is merely another possible result, a desired one definitely but the failure to obtain would never prevent analysis from occurring. Finding false or impossible propositions is the other point of analysis, it just does not seem that way because the examples proffered are frequently convertible and always seen by Pappus to be correct. There is little evidence from Pappus' efforts of what a failed analysis looks like until later in the **Collection** when there is finally an opportunity to examine one in Book 3.[18]

> Diorite is the preliminary distinction of when, how, and how many ways the problem will be possible. So much, then, concerning analysis and synthesis. Pappus, **Collection** 7.2:10-12 (as translated by Jones).

Diorite was in the right place logically for Pappus' description but the line still looks like it was tacked on in comparison to the thoughtful discussion of analysis and synthesis that preceded it. Diorites are important because, as Pappus informs use, there may be more than one solution and the mathematician is required to find and know all such possibilities. Knorr found their significance in what they can tell us about the outcome of the analysis.[19] He claimed that the reversal of the logical order is possible only when additional conditions are satisfied and diorites determine what the appropriate conditions are. Within Book 7, their quantity is noted in the epitomes of each *Domain* text when needed, but Pappus rarely mentions diorites directly. The diorites of the original problems are sometimes turned into

[18]Knorr(1986), 9. An analysis can still be fruitfully applied, even if the problem/theorem is impossible/false.
[19]Knorr(1986), 110.

propositions in their own right. Diorites are otherwise unimportant to the search for the solution but they are necessary for the presentation of that solution.

4.2 A Modern Opinion

The difficulty with bringing modern scholars into this particular discussion and interpretation of Pappus is due to the want of a scholarly standard about pappier analysis and synthesis. Differences in translations aside, the introduction of Book 7 is hardly studied in its own context. Instead, it is forced to conform to the so-called traditional practices of antiquity (Knorr and Panza), to anticipate Descartes and Vente (Panza and many others), or to be amenable to modern logic (Hintikka and Mäenpää). These themes are not wrong in and of themselves but they often bring beliefs and expectations that are not in simpatico with Pappus circumstances.

The preferential theme of the three listed above endeavoured to place Pappus within the ancient tradition of analysis and synthesis. One would be foolish to think that Pappus' method was uninfluenced by the method of the very mathematicians he mentioned: Euclid, Apollonius, etc. However, Pappus did not completely conform to those traditions; indeed, he was educated about them, but as demonstrated previously, mathematics in his time had a new direction and form. Pappus' method was more likely to embody the new practice than old ways. Pappus' pedagogy also inhibits scholars from making the mathematician fit with the history they created, but Pappus is subsequently characterized as inaccurate and naive rather than necessarily oblique.[20] Again, the theme itself is legitimate, but the direction of the censure is not.

The second and third themes sometimes meld together because their assessments of Pappus bring with them the same consequences. It is easy to fall into anachronistic judgements about Pappus. The requirement for convertibility was essential to 18th century philosophers

[20] Knorr(1986), 354-360; Panza(1997), 385.

and mathematicians; that it was not as important to the ancients should not held against them nor should one read convertibility into Pappus simply to make a stronger connection to later writers.[21] The initiation of modern symbolic logic is much more bothersome because it makes huge presumptions about Pappus' approach to analysis and synthesis that are more compatible with Aristotle.

Sometimes, modern scholarship will introduce problems that have no bearing on the heuristic bearing on the mechanism of analysis and synthesis. Sometimes these problems become so integrated into the academic tradition of the subject that it is arduous work to separate Pappus actual problems from those of scholars. An example of this sort of textual interference is the outstanding debate about the direction of analysis. It began innocuously enough in 1958 when Norman Gulley suggested that rather than the usual downward interpretation of analysis (deduction of consequences from the desired conclusion/initial assumption), Pappus' analysis was actually upwards (movement from desired result to premises from which it could be deduced).[22]

Analysis as a downward movement is the deduction of consequences from the desired conclusion. If A is the *ζητούμενον* and B is the first principle the movement of analysis can be represented like this: $A \longleftarrow A_1 \longleftarrow ... \longleftarrow A_n \longleftarrow B$.[23] Therefore the synthesis written in the following way: $B \longrightarrow A_n \longrightarrow ... \longrightarrow A_1 \longrightarrow A$.[24] Proponents of analysis as an upwards movement say that analysis is the movements from the achieved *ζητούμενον* to the premisses from which it could be deduced. It is said to be a more intuitive method.[25]. With A and B still representing the same objects, analysis would look something like this: $A \longrightarrow A_1 \longrightarrow ... \longrightarrow A_n \longrightarrow B$. Synthesis is the same as the downward movement.

Support for the upward-movement interpretation comes from the second description of analysis where Pappus writes that me begin with what is sought as if it has been achieved and

[21]Mäenpää(1997), 201.
[22]Gulley(1958), 1-14.
[23]Behboud(1992), 56; Hintikka(1974), 13-15.
[24]Ibid.
[25]Behboud(1992), 56

then look for the thing from which it follows, and so on until we reach the first principles. However, it is argued the synthesis is made redundant because the perspective of analysis now is to recreate the proposition by moving backwards through the proof. Hintikka and Remes wrote an entire book based on analysis being an upward movement.[26] Both they and Gulley agreed at first that the two movements are compatible, if all steps are convertible, but they do not let Pappus have it both ways because Pappus would then be contradicting himself, thus they side with an upward movement.[27] Hintikka and Remes disclose that the question is not relevant to the operation of analysis and synthesis but that has not stopped many scholars from roundly refuting it.[28] Knorr posits that ancient writers (especially those in the later editorial tradition) understood analysis in this was: the desired figure is assumed to have already been effected and one is then to deduce properties of this figure until an element of it emerged which is known from prior results to be constructible.[29] This is the downward movement and one that is very much clear from Pappus introduction. Petri Maenpaa found the Hintikka and Remes description, out of sync with the deductive examples typical of Greek mathematics; reductive examples were emblematic of the work of commentators rather than mathematicians with original contributions.[30]

Neither the upwards or downwards-movements do justice to analysis and synthesis but the downwards movements is better reconciled to Pappus' text than the upwards movement. Indeed, Pappus does write about the direction of analysis in the second description as though one was to find premisses that led to the first principles. What is missing from the upwards movement is recognition that the movement from A to B does not involve intuition but rather an ability to do so with geometrical assumptions that avoid the auxiliary figures that are necessary for synthesis. However, Hitikka himself admits often that the direction of analysis doesn't matter in the long run.[31]Thus, the crux of the problem of dealing with modern scholarship. The direction may be irrelevant but the problem is still treated as a critical one.

[26]Hintikka(1974), J; Remes, U. **The Method of Analysis**. D.Reidel Publishing: Dodrecht, 1974.
[27]Hintikka(1974), 14-19
[28]Ibid., 19.
[29]Knorr(1986), 9.
[30]Mäenpää(1997), 201, 206.
[31]Hintikka(1974), 38.

Of the few things that can be taken away from modern scholarship, knowledge that analysis and synthesis in antiquity lack its modern connotations and usages is key. Without it, one could run headlong into yet another historiographical tar pit.

4.3 Pappus in Action

In the final section of this chapter, the promise to survey Book 7 will be fulfilled. Only 14 propositions out of a total of 253 have both an analysis and synthesis. Here, two such propositions will be examined.

Proposition 21 is the first proposition in Book 7 that includes analysis and synthesis together. The original proposition from Apollonius' *Cutting off of a Ratio*, but it provides little assistance in understanding the proposition as Pappus presents it. [32]

4.3.1 Proposition 21

The square brackets refer to the order of the statement in Jones' commentary.

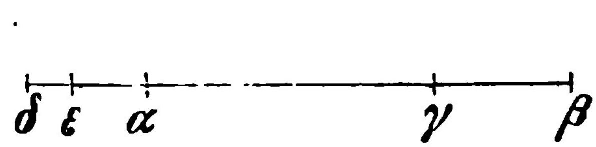

Figure 4.1: Proposition 21, Book 7 of the **Collection** from F. Hultsch.

Given two straight lines AB, $B\Gamma$, and producing line $A\Delta$, to find a point Δ that makes the ratio $B\Delta$ to ΔA the same as that of $\Gamma\Delta$ to the excess by which

[32] Jones(1986), 412, 511-512, 616-619. These pages will provide all the necessary background about the original and it's transmission into Pappus

$AB\Gamma$ together exceeds the line that is equal in square to four times the rectangle contained by AB, $B\Gamma$.

The combination cannot be made any other way unless ΔE, $A\Gamma$ together are equal to the excess EA, and all ΔA to all AB, and furthermore (it is not possible otherwise?) that EA, AB, ΓB have the ratio to one another of a square number to a square number, and that ΓB is twice ΔE.

Let it be accomplished, and let the excess be AE[1]; for we have found this in the forgoing lemma. Then as is $B\Delta$ to ΔA, so if $\Gamma\Delta$ to AE[2]. And alternando[3] and separando[4] and area to area, it follows that the rectangle contained by $B\Gamma$, EA equals the rectangle contained by $\Gamma\Delta$, ΔE[5]. But the rectangle contained by $B\Gamma$, EA is given[6]; hence the rectangle contained by $\Gamma\Delta$, ΔE too is given[7]. And it lies along ΓE, given,[8] exceeding by a square. Hence Δ is given[9].
The synthesis will be made thus. Let the excess be EA, and along ΓE let there be applied the rectangle contained by ΓE, ΔE, exceeding by a square, and equal to the rectangle contained by $B\Gamma$, EA. I say that Δ is the point sought. For since the rectangle contained by $B\Gamma$, EA equals the rectangle contained by $\Gamma\Delta$, ΔE,[10] therefore putting in ratio[11] and componendo[12] and alternando as is BΔ to ΔA, so is $\Gamma\Delta$ to EA,[13] which is the excess. The same also if we try to take a point making, as $B\Delta$ to ΔA, so $\Gamma\Delta$ to the line comprising $AB\Gamma$ together and the line equal in square to four times the rectangle contained by AB, $B\Gamma$. Q.E.D. Pappus **Collection** 7.64:12-36 (as translated by Jones(1986), 138).

Jones' Commentary:

Analysis [1]-$AE=AB+B\Gamma-x$, $x^2=4(AB\cdot B\Gamma)$

[2]-$B\Delta:\Delta A=\Gamma\Delta : AE$

[3]-Thus $B\Delta:\Delta\Gamma=\Delta A : AE$

[4]-Thus $B\Gamma:\Gamma\Delta=\Delta A : AE$

[5]-Thus $B\Gamma\cdot EA=\Gamma\Delta\cdot\Delta E$

[6]-$B\Gamma{\cdot}EA$ given

[7]-Thus $\Gamma\Delta{\cdot}\Delta E$ given

[8]-ΓE given

[9]-Thus Δ given

Synthesis [10]-$\Gamma\Delta{\cdot}\Delta E{=}B\Gamma{\cdot}EA$

[11]-Thus $B\Gamma{:}\Gamma\Delta{=}\Delta E{:}EA$

[12]-Thus $B\Delta{:}\Delta\Gamma{=}\Delta A{:}AE$

[13]-Thus $B\Delta{:}\Delta A{=}\Gamma\Delta{:}EA$

Jones(1986), **Collection** 412:64.

In the analysis, we are given $B\Delta$:ΔA =$\Gamma\Delta$: AE and told to find Δ so that the ratio may be possible. The synthesis, on the other hand starts with the last ratio listed in the analysis and concludes with the ratio listed above. The proposition began with the enunciation of the existence of a line AB, $B\Gamma$. Then we extend the line to a point Δ and the goal is to find the Δ that makes the ratio $B\Delta{:}\Delta A{=}\Gamma\Delta$: the excess by which $AB\Gamma$ together exceeds the line that is equal in square to four time the rectangle contained by $AB{\cdot}B\Gamma$.

The establishment of AE as the excess is determined in a previous proposition, proposition 19 but the carry-over of that particular result between two propositions that have only share minimal similarities in terms of their ratios and construction requires explanation. The relationship between propositions 19 and 21 in terms of the original text has already been discussed, however, their cohesion within the **Collection** is suspect until Jones switched the connections between the original text and Pappus'.

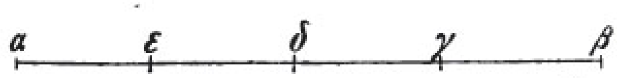

Figure 4.2: Proposition 19, Book 7 of the **Collection** from F. Hultsch

In order for the minimum given ratio to be solvable in the fourteenth disposition to for it to be a ratio of a determined line segment to another determined magnitude. Heath and Jones mentioned this relation in some form of $c{=}a{+}b{-}2\sqrt{(a \cdot b)}$ where, in proposition 19, a and b are AB, $B\Gamma$ respectively.[33] In that lemma, AE comes to represent c insofar that Pappus provided the relationships such as $B\Delta^2 = AB{\cdot}B\Gamma$ and $\Delta E{=}\Gamma\Delta$ that in the context of that particular lemma allow for the logical calculative progression to $AE = (AB + B\Gamma) - x$. Proposition 21, on the other hand, was constructed in a way that does not allow those steps and the application of AE is further complicated when Jones declared that proposition 21 actually furnishes a second point of intersection for the first case of the fourteenth disposition and is unnecessary unless one wished to draw a figure for the Apollonius' summation of that disposition.

The validity of AE is unsettled but if it is expected that AE or similar segment is essential to maintaining the relationship between $AB\Gamma$ and the line that, squared, is equal to four time the rectangle contained by $AB\Gamma$ then it must be treated as every much a given as the ratio of $B\Delta{:}\Delta A{=}\Gamma\Delta{:}$ to the excess of $AB\Gamma$ to what have you. The identification of the value or identity of x is not essential to the resolution of point Δ in 21; in the previous propositions the declaration of a possible x is important to determining AE but nothing else happens.

There are differences in the language Pappus utilized in his lemmas compared to his outline of analysis/synthesis in the introduction. The reader is not provided with the acknowledgement of the *ζητούμενον* or whatever would occupy the rank of first order but if one were to use Pappus' definitions, those objects are possible to identify. It is not until the conclusion of the synthesis that the ratio of $B\Delta{:}\Delta A{=}\Gamma\Delta{:}EA$ is understood as the *ζητούμενον* of synthesis as well as the established *ζητούμενον* of analysis. But, the application of this ratio in these roles is quite different. In synthesis, the *ζητούμενον* plays a passive role; the mathematician may know that the construction built and the proof conceived must bear the ratio out but the ratio itself must be the product of the preceding steps where as in analysis, the established *ζητούμενον* comes into play as soon as it is introduced and all the consequences are drawn

[33]Heath(1921), ii, 405;Jones, 412. Both are likely to be inspired by Hultsch's Latin translation.

from it. In this case, all subsequent ratios are drawn from the original ratio alone without the need for stating additional products and equalities. The identification of the *ζητούμενον* in synthesis is also accompanied by its construction. The existence of the lines AB, $B\Gamma$ is made clear before the beginning of analysis and nowhere else. This is a departure from the introduction where constructions received no mention during analysis and come only at the end of synthesis. There is no indication here that the line ΔB is being constructed, although all points on the line were named and given in synthesis after the first ratio. Since 21 is an exercise in geometrical algebra, disinterest in the creation of a straight line on Pappus' behalf might be forgiven but it heightens the expectation of the integration of complex constructions into analysis/synthesis later in Book 7

The next object needed is something that is established in synthesis. Jones' commentary for the synthesis begins with $\Gamma\Delta{\cdot}\Delta E{=}B\Gamma{\cdot}AE$ but omitted two things that are given: AE as the excess and ΓE as the line upon which the rectangles lie. Also, Pappus says that Δ is the point sought in synthesis. If one were to follow Pappus' definitions strictly, on the basis of it being the last step in analysis, thereby being the true zetoumenon for the analysis, one would expect Δ to be the given in the first instance in synthesis before ΓE. Instead, Δ is said to be sought just after the relationship of the rectangles contained by $\Gamma\Delta E$ and $B\Gamma$,EA is stated. Pappus may have wanted Δ assumed given after the conclusion of the analysis for how else could the rectangle contained by $\Gamma\Delta E$ be possible to construct if Δ did not have the same giveness as ΓE. Δ is nonetheless not named by Pappus as the object that is both established in synthesis and sought in analysis; that title might be best claimed by ΓE.

Were one to evaluate the analysis and synthesis in 21 in terms of its strict adherence to the guidelines Pappus laid out in the introduction of Book 7, it is possible to argue that Pappus was exceptionally consistent; the analysis was followed by its complete and exact reversal by synthesis while a broad interpretation would only require the reversal first and last steps of analysis in synthesis. In Jones' commentary one can see how precise Pappus is in the reversal of the *ἀκολουθος* into their expectant order in synthesis. The only obstacle to the method being a complete reversal is the difference between the final steps of analysis and the first step of synthesis. If, as one sees in the analysis that the given-ness of Δ and ΓE is based on the giveness of the rectangles that contain them, then $\Gamma\Delta{\cdot}\Delta E{=}B\Gamma{\cdot}AE$ can sufficiently satisfy the condition of what is sought in analysis is considered given in synthesis. Still, Δ was specifically invoked by Pappus as that which analysis sought, not AE nor ΓE nor the rectangles. Thus one must be warned against anticipating a strict adherence to the introduction on Pappus' behalf on the basis of what is otherwise an excellent example of Pappian analysis/synthesis. It is clear Pappus may have considered analysis and synthesis to have begun with different steps, after certain pieces of information was provided. To complete this survey of Book 7, one can look to proposition 72 of Pappus' selections from the first book of Apollonius' *Neuses* for an example of analysis/synthesis that does not proceed as expected.

4.3.2 Proposition 72

A *neusis* is the fitting of a line of given length into a diagram, usually between two other lines, such that it passes through a given point. In deference to his predecessors, Pappus cannot but help to mention their discoveries in the epitome he provides for *Neuses*. Some of the applications of the neusis by the ancients were linear, solid, and curvilinear. From the planar uses, the ancients demonstrated many problems but the one relevant to proposition 72 involves a rhombus. Given a rhombus with one side extended, the ancients sought to fit into the outside angle a straight line given in magnitude to make a neusis on the opposite angle.

> Problem, as Heraclitus.
>
> $A\Delta$ being a square (given) in position, to place a given (line) EZ, making a neusis on B Let it be accomplished, and from point E let EH be drawn at right angles to BE; for (BZE) is a straight line. Then since the squares of $\Gamma\Delta$ and ZE equal

the square of $H\Delta$ [1], while the squares of $\Gamma\Delta$ and ZE are given[3], because both are given in magnitude[2], therefore also the square of ΔH is given[4]. Therefore ΔH is given in magnitude[5]. And therefore all BH is given in magnitude[6]. But is it also (given) in position[7]. Therefore the semicircle on BH is given in position[8]. And it passes through E[9], and hence E is on an arc (given) in position. But (it is) also (on) AE (which is given) in position[10]. Hence it is given[11]. but B too is given[12]. Therefore BE is (given) in position[13].

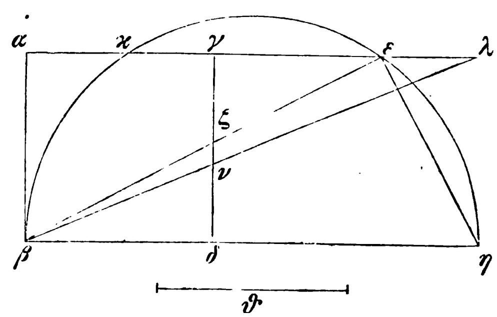

Figure 4.3: Proposition 72, Book 7 of the **Collection** from F. Hultsch

The synthesis of the problem will be made thus. let the square be $A\Delta$, the given straight line Θ, and let the square of ΔH be equal to the squares of $\Gamma\Delta$ and Θ.

Then $H\Delta$ is greater than $\Delta\Gamma$[14]. Hence the rectangle contained by $H\Delta$, ΔB is greater than the square of $\Delta\Gamma$[15]. Therefore the semicircle on BH when drawn will fall beyond point Γ[16]. Let it be drawn, and let it be $BKEH$, and let $A\Gamma$ be produced to E, and let BE, EH be joined. Then the squares of $\Gamma\Delta$ and EZ equal the square of $H\Delta$[17]. But the squares of $\Gamma\Delta$ and Θ were set equal to the square of ΔH[18]. Therefore the squares of $\Gamma\Delta$ and Θ equal the squares of $\Gamma\Delta$ and EZ[19]. Hence the square of Θ equals the square of EZ[20]. Therefore Θ equals EZ[21]. And EZ is given. This EZ solves the problem.
Pappus, **Collection** 7.128-129:1-24 (as translated by Jones(1986)).

Jones' Commentary:

Analysis [1] - $\Gamma\Delta^2+ZE^2=\Delta H^2$

[2] - $\Gamma\Delta$,ZE given mag.

[3] - Thus $\Gamma\Delta^2$,ZE^2 given mag.

[4] - Thus ΔH^2 given mag.

[5] - Thus ΔH given mag.

[6] - Thus BH given mag.

[7] - BH given pos.

[8] - Thus semicircle BH given pos.

[9] - E on semicircle BH

[10] - E on AE, AE given pos.

[11] - Thus E given

[12] - B given

Synthesis [13] - Thus BE given pos.

[14] - $H\Delta>\Delta\Gamma$

[15] - $H\Delta \cdot \Delta B > \Delta\Gamma^2$

[16] - E is not between A, Γ

[17] - Thus $\Gamma\Delta^2+EZ^2=H\Delta^2$

[18] - $\Delta H^2=\Gamma\Delta^2+\Theta^2$

[19] - Thus $\Gamma\Delta^2+\Theta=\Gamma\Delta^2+EZ^2$

[20] - Thus $\Theta^2=EZ^2$

[21] - Thus $\Theta=EZ$

Jones(1986), **Collection** 437.

Initially, the achieved *ζητούμενον* is the relationship $\Gamma\Delta^2+ZE^2=\Delta H^2$ while B is the point that is sought. In the synthesis, a different story comes out in which the first principle is BE but the synthetic *ζητούμενον* is EZ. As with many of the propositions in Book 7, proposition 72 is closely connected to the proposition before it. Both propositions demonstrate the construction of a neusis into a rhombus that is a square. 71 is Apollonius' method and 72 is attributed to the otherwise unknown geometer Heraclitus, who accomplished the same with a semicircle. However, the analysis of 72 takes for granted the pythagorean relationship between line segments it requires which is only explained in 71, not unlike the issue with AE in propositions 19 and 21. In the Apollonian version, the rhombus is gradually constructed as if it was in three parts: Square, rectangle, and a right angle triangle are divided from each other by a determined line segment.

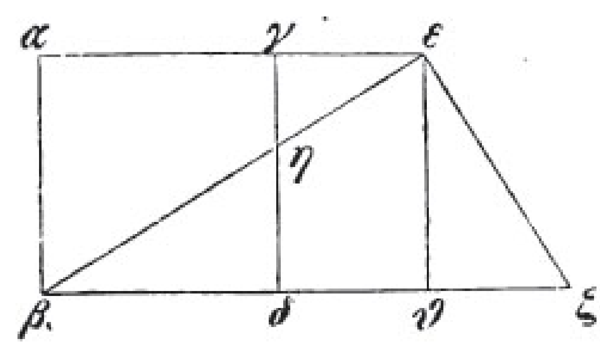

Figure 4.4: Proposition 71, Book 7 of the (Collection) from F. Hultsch

The neusis cuts through the bottom left-hand corner of the square, intersects the line that separates the square from the rectangle, and terminates at the upper right-hand corner of the rectangle. Because each section is clearly defined by the line segments between them, similar angles and triangles are easy to designate. In the end, one is left with the equation of $Z\Delta^2$=$\Gamma\Delta^2$+HE^2. In Heraclitus' version, the partition between the rectangle and the right-angled triangle is removed and additional neusis for another case is added, necessitating the reexpression of the equation to reflect the named points of the new proposition. It is thus transformed into $H\Delta^2$=$\Gamma\Delta^2$+ZE^2. Also similar to proposition 21 is how Pappus made the conduction of the analysis and synthesis possible without the validity of the equation coming into play; all that mattered was that whatever was proven in proposition 71 could be applied as a desired case in 72.

Thus, the analysis with several objects from the construction being given: The square of $A\Delta$ means that the lines $A\Gamma$, $\Gamma\Delta$, ΔB, BA are also given in magnitude while B is given in position when another given line, EZ, makes a neusis on that point. This is followed by the construction of a right angle between EH and BE. There is little reason within the proposition itself as to why they should believe Pappus when he wrote that $H\Delta^2=\Gamma\Delta^2+ZE^2$. None of these segments actually form a triangle within the construction and without the stated equality between EH and BZ, like EZ and BH in 71, it is difficult to see how a similar equation would just as easily be found. Regardless, given that this is an analysis, such relations can be presumed as fully given until Pappus neglects to provide an explanation.

From the equation, Pappus proceeded through some of the possible positions and magnitudes that are given because of it, with the position of BE the last thing given. If one were to follow a broad reading of the introduction so far, the synthetic *ζητούμενον* could be any of the seven objects given and the analytic *ζητούμενον* (and possible object to occupy first rank), is the straight line BE. The best choice for the synthetic *ζητούμενον* is the given line EZ. Recall in proposition 21 that in the analysis, the *ζητούμενον* that was treated as established had an immediate impact on the proposition and EZ accomplishes this by bringing a second given quantity to the equation that permits $H\Delta^2$ to be given.

In this particular demonstration, any expectations for a strict reading of analysis/synthesis must be dashed when starting the synthesis as the order of the objects referenced has changed. All the reader is told upon beginning their construction of the rhombus is that there is a square, a given line Θ, as well as the equation with EZ replaced for Θ. None of these objects is present in the final steps of the analysis or were identified as the analysis *ζητούμενον*, with the exception of $H\Delta$ in the equation. Moreover, the inclusion of the equation so early in the proof may appear to be cheating by Pappus but it was a legitimate move in a problem that is centered around the fitting in a the neusis; it confirms that EZ will be the synthetic *ζητούμενον* as much as one can tell that Θ is standing-in for EZ as the given line in anticipation for the substitution that will be conducted later. Furthermore, Pappus included some unusual steps about the equivalency between $H\Delta$ and $\Delta\Gamma$ that are not present in the analysis: If the former is larger than the latter, it should not be surprising that the square of the latter would be smaller than a rectangle that is contained by a segment equal to the

latter and $H\Delta$. Continuing onward, Pappus also focuses on the semicircle which had little influence on the analysis outside of making the intersection of AE on it a given. In the synthesis, the semicircle is important for establishing $BKEH$, thereby extending $A\Gamma$ and H to E. The joining of BE, EH into a right angle allows for the reintroduction of the equation as the original equation in proposition 71 began with the pythagorean relationship of $Z\Delta^2{=}\Gamma\Delta^2{+}HE^2$; this is work saving measure for Pappus and from here it is confirmed that EZ is the synthetic *ζητούμενον*
σελεςτλανγυαγεενγλιση.

Τηερε ις νοτ μυςη γρουνδ φορ αν ινιτιαλ βροαδ ρεαδινγ ηερε ειτηερ. Τηε αναλψτις *ζητούμενον* and that which holds first rank in synthesis appear to be two or more different things. It was argued with the last proposition that $\Gamma\Delta{\cdot}\Delta E{=}B\Gamma{\cdot}AE$ should sufficiently embody all of the given objects and relations derived from it to compensate for Pappus not incorporating into the synthesis the individual objects that were acknowledged. In this case, that explanation does not work so well. Some of the individual objects actually are used but their relationships sometimes exceed what is established in the analysis. Indeed, with so many objects that could be perceived as given in the context of synthesis, how is one best to approach such a list. Hinktikka and Remes argued that the *ἀκολουθος* should be called concomitants rather than consequences as many of these steps occur simultaneously and sometimes independently of one another. This may not be the best term to use as there is some logical order to the progression through the givens, it does encapsulate the idea that whatever is given in analysis might not necessarily be order in terms of priority but rather as a single package that is assumed at the beginning of analysis. For proposition 72, such a hypothesis works as much of what is discovered to be given through analysis is given with the initial construction of the synthesis in the place of first principles.

4.4 Analysis and Synthesis: An Evaluation Thus Far

Two lemmas are hardly enough to let one firmly pin down Pappus' transition from analysis to synthesis but they do allow for a sufficient demonstration of some of the discrepancies between his theory and practice. One is slightly better off in practice than in theory. The best way find the synthetic *ζητούμενον* in an analysis is to find that given object from which most, if not all, the consequents are in some way derived. The difficulty of correlating the final steps of analysis to the object(s) that are considered given in synthesis, makes a broad reading of Pappian analysis/synthesis as difficult as a strict one if the first and final steps of analysis are unable to be reversed.

One of the first things that Pappus did write about synthesis was that it was organized through natural order and the fitting of objects into one another; the arrangement of the precedents cannot be much looser than that, although proposition 21 does show that the arrangement can be incredibly strict if the proposition lends itself to such treatment.

Through the proposition, Pappus introduced the concept of an object or relationship being *given*. It is first introduced in Euclid's **Data**; a collection of propositions from which show how it one object or relationship is given, so are the objects and relationships associated with it. The propositions often take a form like "If any number of given magnitudes are added together, the magnitude composed of them will also be given".[34] Their form is analytic; Euclid does not provide syntheses for these propositions.

This is the rare occasion where Pappus forgot his usual attachment to his lexicon and used a word for more than one context.[35] *Given* has more pull in analysis, where one must prove the existence of objects or relationships that are presumed at the start , but not yet

[34]Euclid *Data*, 3.

[35]Knorr(1986), 110. Pappus use of *given* and its derivatives would have been another ahistorical strike against the mathematician from Knorr. Knorr stated that *givens* separated terms with distinct logical purposes. What was *given* in analysis was only known conditionally if the construction was assumed. The true *givens* are known by virtue of the definition of the construction which one is to produce, i.e. the synthetic *givens*

verified. Therefore every step after the established *ζητούμενον* is deemed *given* on the basis of the established *ζητούμενον* being established. In proposition 72, from the *ζητούμενον* $\Gamma\Delta^2+ZE^2=\Delta H^2$, any object, position, or magnitude that could be pulled from it would be considered given on the basis that in analysis, such relationships are presumed given at the beginning of the proposition. Everything is given by dint of being possible to derive it from the achieved *ζητούμενον*. This is not so in synthesis where only the *ζητούμενον* is considered *given*.[36] Depending on the proposition, only select objects and relationship are automatically *given* because of the synthetic *ζητούμενον* . Such a scenario is seen in proposition 72, where Θ is given because it served as a placeholder for a desired ratio that was no yet found.

What is truly missing from Book 7 are examples of propositions where the analysis find an false or impossible proof. The distinction between theorematic and problematic analysis may be fairly irrelevant but for analysis to have a purpose akin to fact-checking hints that the paths through the consequences and given objects might not have been so clear. Hintikka and Remes believe that convertibility was a process that could only be hoped for despite what Pappus said. This is a careless statement to make as Pappus, however trivially, did attempt to capture the methods of analysis and synthesis as two interconnected methods which shared common elements opposite to each other. Complete convertibility might not have been Pappus' goal but he did want it understood that certain objects and relations appeared at certain times in both methods, regardless of some exceptions in his own practice.

One finds Pappus ultimately clumsy in his efforts but he did try to stay true to the placement of the synthetic *ζητούμενον* and to a lesser extent, the established *ζητούμενον* . Expectations are mixed; simple linear problems allow for near-complete convertibility while more complex problems made it difficult for Pappus to adhere to his own theory. Recalling the example analysis/synthesis from Archimedes, I agree with scholars such as Berggren when they say that there is a dissonance between Pappus and earlier analyses.[37] But only if one is talking about the exposition only. Pappus omits mention of *givens* but he does attempt to include

[36] Jones, 67. Disagreed with Knorr. The *given* is essential but has a lot of connotations.
[37] Berggren(2000), 5.

the reality that the order of objects between the achieved *ζητούμενον* and the first principles changes depending on the demands of the proposition. When the propositions are included, Pappus is in harmony with the earlier mathematician. Moreso, Pappus seems more conscientious by comparison. He is clear about where the analysis starts and ends as well as and for many propositions he does out of his way to make the *ζητούμενον* , and the first principle obvious.

Pappus and Archimedes are in sync about analysis and synthesis, but what about within Book 7? The difference between the proposition and the exposition is as such that the propositions do not benefit from the exposition. The exposition on the other hand improves with the practical knowledge of what happens in the propositions. It is transformed from being a vague piece of prose into as meaningful a summation of analysis and synthesis one could write. However, we already know why there difference are so. Pappus composed Book 7 for the benefit of a student with less mathematical skills than he. His is exposition is quite simple but the number of propositions would easily show the student how analysis and synthesis is completed full on.

In Books 2 and 3 of the **Collection**, a somewhat different picture of Pappus' ability to wield the techniques of analysis appears for not only will one see Pappus conduct a failed analysis and accurate analysis/synthesis on complex figures, Pappus presented a non-geometrical way of analysis that may change the way we look at Book 2.

Chapter 5

Dear Pandrosion: Mathematical Etiquette and Analysis

The preceding chapters present many reasons as to why the characterization of Pappus as a feeble mathematician are groundless. The supposed Deuteronomic writing was becoming common practice and the second-order mathematics that developed from it was compatible with Pappus' pedagogical style. This make Pappus increasingly vulnerable to being misunderstood and his presentation of analysis and synthesis in Book 7 does little dissuade modern criticism.

In Book 3 of the **Collection** Pappus redeemed himself both in his theory and in his practice. He offered insight into the social practice of mathematics. Analysis and synthesis is applied to an impossible problem. In accomplishing this, valuable analytical techniques are introduced in Book 3 that were unmentioned in Book 7.

In the second part of this chapter, Pappus' mathematical capabilities are revisited and reevaluated in light of Pappus' insight about the value of analysis and synthesis to the mathematical community.

Book 3 can be divided into approximately six sections, five of them geometrical. Pappus opened the text was an admonishment towards a named Pandrosion and her students, but while he absolved the students for their mistakes due to the student's own errors, a much stronger rebuke was indirectly made towards Pandrosion for not instilling the value of analysis into her students.

The second section follows the problem posed by one of the students.[1] The student claimed to have discovered a way to find two mean proportionals in continued proportion between two given straight lines by linear means. The mistakes present both in the student's solution and Pappus' correction are the subject of discussion later in the chapter. Pappus also provided a history of the problem with solutions proffered Eratosthenes, Heron, and Nicomedes which, in reference to the classification of geometric problems that he provided, are essentially 'solid' or mechanical. Also included is a fourth solution that is described as being "discovered by us all", but was specifically attributed to Sporus by Eutocius much later.

The third section is spurred by the failings of a second geometer to present the three means in a semicircle.[2] The definitions of arithmetic, geometric, and harmonic means are first, followed by seven more means (three are ancient, four are recent) and conclude with a series of problems in which these means are calculated through varying relations between given extremes.

The fourth section contains propositions from a lost work, **Paradoxes**, by the otherwise unknown Erycinus.[3] The fifth section includes problems of inscribing five solids within a sphere and does so in analysis/synthesis pairs[4] and the 6th section is a variant on Pappus's solution for the mean proportionals examined in section two.

This particular chapter is itself divided into two parts. The first covers the implication of Pappus' demands for analysis on the social practice of mathematicians from the first sec-

[1]Heath(1921) ii,363. Pappus *Collection* 3, 68.
[2]Heath(1921) ii,365. Pappus *Collection* 3, 103.
[3]Heath(1921) ii,365. Pappus *Collection* 3, 103.
[4]Heath(1921) ii,368. Pappus *Collection* 3, 131.

tion of Book 3. The second and third parts focus on Pappus' use of analysis and synthesis outside of Book 7, but with departures in style. The second part takes on the impossible problem from the second section; a rare opportunity as Pappus only examined possible or true propositions elsewhere in the **Collection**.

5.1 Rules for being a Mathematician

Book 3 begins with a lengthy exposition on how a mathematician must conduct his or herself, with respect to peers and superiors, when sharing problems and theorems. In this social practice of mathematics, Pappus found himself pressed to respond to the efforts of a young man who had been rather careless with the problem he distributed. This carelessness had consequences that Pappus deemed important enough to reiterate not just to the youth, but to his teacher.

> Those wishing to discriminate more precisely between the things sought in geometry, my dear Pandrosion, think it fit to call a problem that which one casts forward something to do or to construct, and a theorem that in which, when certain things are being supposed, one observes what follows from these presupposed things and in general everything that accompanies them; among the ancients some said all things are problems, while others said that all things are theorems.
>
> Pappus, **Collection** 3.1:1-8 (as translated by Bernard(2003b), 118-119)

Pappus informed his female interlocutor, Pandrosion, of the past and present turmoil over the distinction between problems and theorems. The division sounds familiar. Indeed, it is reminiscent of the two forms of analysis, problematical and theorematical, described in the introduction of Book 7. The differences between the passages, however, are acute. In

Book 3, the description casts the mathematician as a key participant in the observation and construction of whatever it is that he or she has put forward. The origin of the objects of analysis is vague in Book 7. The ancients did not share Pappus' division and although they themselves are invoked later, the arguments alluded to here were not followed-up. Reliance on the ancients in this text usually serves to provide historical support for Pappus' arguments. Here, they are meant to illuminate for Pandrosion that the differences between problems and theorems were even acknowledged by earlier mathematicians regardless of their inclination.[5]

The presence of Pandrosion herself is not insignificant. Her visibility in the manuscript tradition is rather shaky. It is accepted that Pandrosion was a female but previous translations and transcriptions have either masculinized her name or eliminated her altogether.[6] As in **Setting the Scene**, Alexandrian women had a better chance of receiving not only a basic education, but one that included mathematics. Thus, it is not impossible for an individual such as Pandrosion to be the recipient of the mathematically-tinged epistle. Still, he likely had enough regard for her abilities to have expected her to be capable of reading the entirety of Book 3.[7]

> Now the person who puts forward the theorem, supervising in one way or another what follows, thinks it fit to search in this way and would not sanely put it forward in another way. On the other hand, he who puts forward a problem, in the case he is ignorant and totally inexpert, even if he prescribed something which was somehow impossible to construct, it is understandable and he should not be blamed. Indeed, the task of the person who is searching is to determine also this: the possible and the impossible, and, if possible, when and how and in how many ways possible. But should he, at the same time, pretend to know something in mathematics and so to speak cast forward something in an inexperienced way, he is not without blame. Pappus, **Collection** 3.1:9-17.

[5]Knorr(1986), vii.
[6]Cuomo(2000), 127n2; Jones(1986), 4n8.
[7]Jones(1986), 16. It makes no sense to only dedicate part of a treatise to a recipient with no explanation.

It is now revealed that Pappus only wished to discuss problems in their present construct, but he does so in a peculiar way. The activity of the theorem is portrayed as restricted due to its disposition towards general cases in encompassing all possibilities. Theorems are not likened to analysis but what Pappus wrote here, again, sounds quote similar in their mutual adherence to sequence of objects. Problems, on the other hand, can lead one astray. Moral judgement is introduced here. The onus is placed on the possibility that a proposed problem is constructible, and if it is possible, the conditions and number of those possibilities (much like a diorism).

However this was evaluated, an individual who is "ignorant and totally inexpert" would not know how he was to go about ensuring that his problem was possible nor how he was to present his solution. Such a case could easily describe anyone who was ever a student of mathematics and maybe this is why Pappus allowed them leniency; if sharing problem and theorems were the foundation of an active mathematics community, there must be room for newcomers to make mistakes. Contrast this with where Pappus does place the blame. It could just be a word of caution towards humility but Pappus is harshest towards the pretenders. One will soon see why; it is not so much about boasting as it is about how one cannot completely disavow the traditions and practices of mathematics simply because they think they are clever. One may call Pappus conservative because of it, but Pappus was referring to more than just the history of problem-solving. It is about using tools and proofs already proven ineffective over ones that do work; it is about not checking one's own work for fallacies and circular arguments. If one truly knew mathematics as they claimed, there is no excuse for sharing impossible problems.

One who has read Book 7 will see in allusions to it in Pappus's prescription to prevent impossible problems. There is no explicit reference in either books to the other but it does seem that Pappus recommended the use of problematical analysis. The best that can be presumed at this point is that the use of analysis was implied between Pappus and Pandrosion and will be demonstrated, or that Pappus was referring to something else and a demonstration of that method will be made.

> Well, recently some who professed to know something in mathematics thanks to you determined for us in an inexperienced manner the enunciations of some problems. About these and some other close subjects, it was necessary to expound some demonstrations in order to help you as well as those who love learning in this third book of the Collection.
>
> Pappus, **Collection** 3.1:17-22 (as translated by Bernard(2003b) 119).

Pappus, presented with several students claiming to know mathematics, resolved to deal with these inexperienced students by dealing with the source: Pandrosion. The revelation that she was their teacher instead of a student changes the perspective of the previous paragraphs. She was a peer of Pappus' but now, he addressed her in a manner different from Hermodorus, as someone who should be grateful for Pappus' corrective measures. His demonstrations are directed to her first of all, but his admonishment of her will benefit lovers of mathematics as well, so that they too avoid copying her methods or those of her students.

> That man gave this to us in writing without including the proof of the proposed problem. And since Heirus the Philosopher and many others of those companions known to me deemed it worthy that I determine this through the proposed diagram, I have waited up to this point for the promised proof to be put forward.
>
> Pappus, **Collection** 3.3:1-6 (as translated by Bernard(2003b), 120-121).

Pappus' attentions were first directed towards an intriguing problem of one student of Pandrosion's. The problem itself will be thoroughly examined in the next section, but for the time being one can skip past it to the exposition that follows it. Besides all the mathematical crimes Pappus accused the student of committing early, Pappus might not have necessarily been motivated to respond as he did had the student not committed what was a final faux pas. As a novel solution to a well-known problem, the claims of the student couldn't help but stir interest amongst Pappus' acquaintances. Furthermore, the student distributed his construction, sans proof, to a number of individuals besides Pappus. Enough additional information was provided so that Pappus was able to understand what the student attempted. Taking Pappus at his word, he was pushed to finally respond to the problem despite its incomplete state. Granted, the sharing of constructions without the accompanying proof was a common practice but given the nature of the problem and the student's promise of a forthcoming proof, the student was being disrespectful towards the social mores of mathematics.

5.2 Pappus' Practice in Book 3

The problem posed by the geometer is to find the two mean proportions between two lines in continuous proportion using only a ruler and compass. This is a reduction of the famous Delian problem of doubling the volume of a cube. Geometrically, one wants to find a cube whose ratio to a given cube equals the ratio of two given lines. The reduction, stated above, can be presented algebraically like so by Archytas, Menaechmus, or, originally, by Hippocrates of Chios:[8]

$$\text{Given line } a,\ b \text{ find } x,\ y \text{ such that } a : x = x : y = y : b$$

This equation can be applied to the cube just so

$$a^3 : x^3 = (a : x)^3 = (a : x)(x : y)(y : b) = a : b$$

This is the same equation Archimedes uses in **Sphere and Cylinder**. It refers to definition 10 from Book 5 of Euclid's Elements but with the duplication of the cube, this solution clearly predates Euclid. Nevertheless, in this format the problem makes a bit more sense. Since we

[8] Heath(1921), i, 245-252.

are looking for a ratio between two cubes, $a^3 : x^3$ instantly produces two cubes that can be manipulated into the desired ratio. The Delian problem was notorious in antiquity for being solvable by many methods except ruler and compass as it was possible to apply curves and mechanical means to meet the ends of

$$a : x = x : y = y : b$$

The problem Pappus writes about is the only recorded attempt at a ruler and compass solution.

To appreciate Pappus' critique, one must do two things: To accept that Pappus' representation of the original construction is faithful and that it is not about whether Pappus is just or fair in his criticism but rather how his techniques are representative of the rest of the **Collection** and of late antiquity in general. One can also appreciate the interesting position Pappus is in. Without the proposition, he proved himself able to complete an analysis by taking sections of the diagram he knew he could derive a desired relationship or object from.

5.3 The Student's Problem

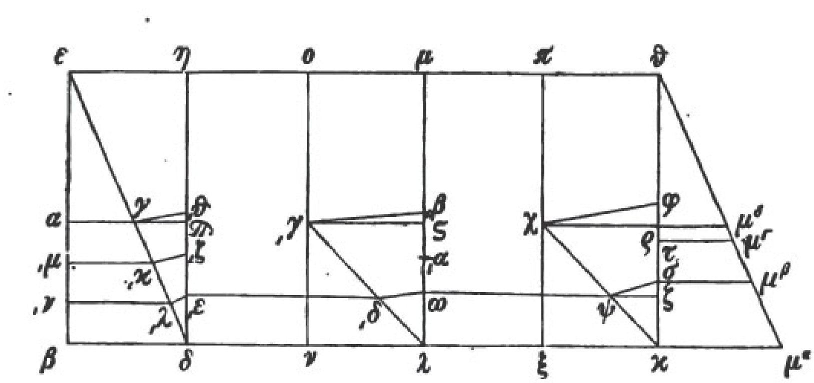

Figure 5.1: A Planar way to solve the Delian Problem, Book 3 of the **Collection** from F. Hultsch

Here is summary of the construction set out by the student. Beginning with making the lines $B\Delta$ and AB perpendicular, the first ideal case is made by making $A\Gamma \parallel B\Delta$. A is then extended vertically to E and $E\Delta$ are connected. $BE \parallel \Delta H$.

$$AB = B\Delta = \Delta N = N\Lambda = \Lambda\Xi = \Xi K = K\mu^{\alpha}$$

Through points N, Λ, Ξ, K

$$BE \parallel NO \parallel \Lambda M \parallel \Xi\Pi \parallel K\Theta$$

$$KP = BA = \Xi X = \Gamma' N$$

KP is cut in two at Σ, therefore $K\Sigma = \Sigma P$.
Then

$$K\Theta : \Theta\Sigma = \Theta\Sigma : \Theta T = T\Phi$$

There are what seems to be four separate cases original to the student's problem: $EB\Delta$, $EH\Delta$, $OMN\Lambda$, and $\Pi\Theta\Xi K$. Starting from point Φ, a process of approximation is expected to occur, repeating with increasing accuracy with B' and finally Θ', thereby proving $M'K'$, $N\prime A'$ are the mean proportionals desired in the problem described above. Exactly what was being approximated and how the contributed to the determination of mean proportions by planar means is unknown and consequently the logic of the rest of the construction, while simple to explain in terms of parallels, is a mystery without more detail. All that can be said is that the case held within $\Theta K\mu^{\alpha}$ are an addition of Pappus' which apparently takes the approximation to its faulty conclusion. Luckily enough, Pappus' objections are encapsulated by the comparison between cases $EB\Delta$ and $\Pi\Theta\Xi K$ and this chapter will focus on those two cases.[9]

The design of the construction is reminiscent of Eratosthenes solution to the Delian problem.[10]

Three rectangles are inscribed between two parallel lines, and each rectangle is further inscribed with a diagonal line, creating several triangles. A point, D, is placed on the outside

[9]Bernard(2003), 122; Cuomo(2000), 130-131; Heath(1921), 269. In the few translations and discussions of Book 3, the case included by Pappus, and the subsequent argument are excluded.
[10]Heath(1921), i, 259.

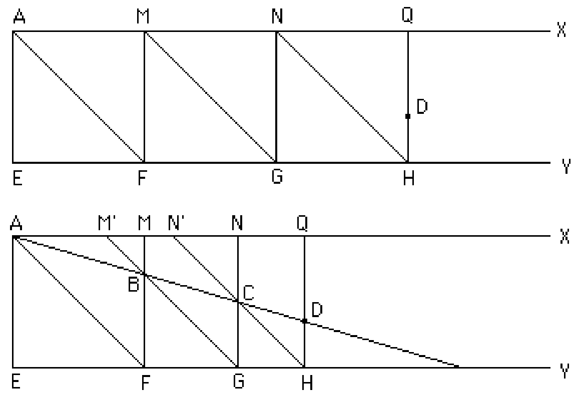

Figure 5.2: Eratosthenes Ruler, **Handbook of Greek Mathematics** from Heath

of the triangle on the extreme right, NQH, creating the two parallel lines between which the means can be found. The two triangles NQH and MNG are shifted to the left as their hypotenuses overlap the sides of the triangles in front of them. The shifting ceases when a line can be construct from point A to the point on the extreme right, through the intersections of both hypotenuses.[11]. The validity of the desired case $EB\Delta$ in the student's construction can thus be proven by mechanical means but the student claimed his solution was not mechanical. Still, the visual similarity cannot go unnoticed by modern scholars regardless of Pappus' silence about whether this similarity was significant or superficial.[12]

Pappus does have much to say about the mistakes made by his main interlocutor, but the fairness of his criticism is questionable. Providing constructions or problems without proofs was a common practice in antiquity but it is important to remain cognizant that all of Pappus' criticisms were on the basis of what he has reconstructed for his own readers. There is the possibility that the student and Hierius provided Pappus with more information about the problem in their correspondence but the value of that eventually becomes moot because what is being examined here are the analytical techniques, not the ultimate accuracy of Pappus' critique. The critique can be without merit but that does not necessarily mean that the results of the analysis are incorrect in and of itself. When it comes to this problem, Pappus is usually accused by modern scholars of being overly vitriolic and unfairly using analysis

[11]Heath(1921) i, 259.

[12]Knorr(1989), 63-72. Both of Knorr's volumes cover the Delian problem and analysis and synthesis extensive but he never brings them together under Pappus.

against someone whose work he may have misrepresented; the use of analysis itself has yet to be deemed incorrect or inappropriate to the problem.

The first objection of Pappus is the movement of point Φ along line ΘK depending on the ratio $K\Theta : \Theta P$.

> Necessarily neither that man nor us find such point of intersection of the third ratio, being Φ. Since confusion of following this kind (of problem) is the responsibility of the man himself, without comprehending these consequences. Being unable to determine the point of intersection of Φ of the third ratio, having not previously establishing the ratio of $K\Theta : \Theta P$, that is the ratio in which $BE : EA$[1],
>
> Not only does that man try to investigate the impossible, but he deemed it worthy that we investigate the impossible. What with that ratio having been established, namely that one which $K\Theta : \Theta P = BE : EA$, and $K\Theta$ having been given, the shortest line of the third ratio ($\Theta\Phi$) is also given and the point Θ is given. Therefore the other limit of the smallest line is given. It is clear that (the point) falls between either ΘP or PT. For we will prove that T falls between $P\Sigma$ and then that Φ sometimes falls between ΘP and other times between P and T according to hypothesis of the ratio given for $K\Theta : \Theta P$. Pappus, **Collection** 3.3:34.9-36.3.

$$[1] K\Theta : \Theta P = BE : EA$$

Before examining the analysis it is essential to understand the ratio Pappus selected. Taking into account that the geometer was applying a method of approximation to his problem, it would be desired that the ratio of the first case, $K\Theta : \Theta P$, would be equal to the goal ratio of $BE : EA$. However, that ratio was not declared and for some reason it has lead Pappus to curiously see point Φ as being undetermined and the problem consequently impossible.

Pappus did not explain how he found Φ undetermined without making the ratio of the first case equal to the desired ratio but it seems that the point would first draw Pappus' interest as a majority of the points in the first case are established in magnitude and position except for T and Φ, which are established by ratios; the T is ignored for now because it is irrelevant to the goal of the desired ratio $BE : EA$. Secondly, Φ would necessarily need to be undetermined in a problem that used approximation. Its relationship to $BE : AE$ is equivalent to the other two cases but yet remain adjustable. For Φ to be fixed would undermine the geometer's attempt to find the mean proportionals.

Thus, the grounds for an analysis: Accepting that the relationship $K\Theta : \Theta P = BE : EA$ is established, find Φ. Why analysis when it seems that synthesis is as legitimate a format? Recall proposition 21 in the previous chapter in which the setup of the problem as almost exact to this. Analysis began with a ratio and proceeded to prove that a point, presumed established in synthesis, was given. In the setting up of a proof, a point such as Φ should be properly established but the similarities should be somewhat less superficial or at least better supported. The set-up of proposition 72 also supports this ordering of mathematical facts; the analysis began with the ratio established and proceeded to line BE being given but the synthesis proceeded in an unexpected way. The only other evidence that can be provided for this being an analysis rather than a synthesis is because Pappus set it up that way. The construction is already provided and if the proof is directed towards finding the two mean proportions, the synthesis could very well be also oriented to the *ζητούμενον* being the desired ratio of the approximate cases.

So moving on,

If

$$K\Theta : \Theta P = BE : EA$$

And using $a : x = x : y = y : b$,

Then

$$BE : EA = BE : EN' = N'E : EM' = M'E : EA$$

Therefore

$$K\Theta : \Theta\Sigma = \Theta\Sigma : \Theta T = T\Theta : \Theta P$$

In order to find Φ,Pappus tried to resolve its position arithmetically using ratios larger than $1 : 1$. Since the method of approximation is being used, P replaces Φ in the ratio given by the student because Φ and its equivalents in the other cases aim to have the ratio shared between $K\Theta : \Theta P = BE : EA$.

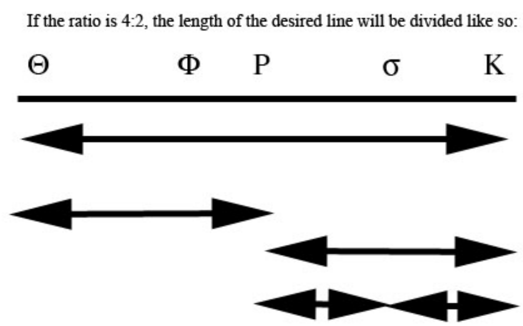

Figure 5.3: A representation of what is going on in Pappus' criticism of the Delian problem. At a ratio of $4 : 2$, PK is found by subtraction of ΘP from $K\Theta$. The remainder is split evenly between $P\Sigma$ and $\Sigma\kappa$. From here, Pappus can calculate the values of his first set of equivalent ratios.

The first ratio Pappus tested was $2 : 1$ or $4 : 2 = K\Theta : \Theta P$. If the ratios are applied to the length of $K\Theta$, then

$$K\Theta = 4, \Theta P = 2;\ K\Theta - \Theta P = PK \text{ or } 2$$

ΘP also equals 2'. Thus

$$KP = K\Sigma + \Sigma P \text{ or } 2 = 1 + 1$$

So, in the ratio to find the two means between two lines,

$$K\Theta : \Theta\Sigma = 4 : 2 + 1,\, 4 : 3$$

For $\Theta\Sigma : \Theta T$,
$4 : 3 = 3 : x$, $x = \frac{9}{4} = 2\frac{1}{4}$
Finally, for $T\Theta : \Theta P$,
$2\frac{1}{4} : 1\frac{33}{48}$.

The desired ratio was $4 : 2$ and the expectation is that $\Theta P = 2$. Instead, it is less than that, approximately 1.7. Lengthwise, if $P = \Phi$, Φ falls between ΘP.

The next ratio Pappus used was $8 : 2$. Using the same sequence, the ratio will be as follows: $8 : 5 = 5 : 3\frac{1}{8} = 3\frac{a}{8} : 1[\frac{61}{64}]$. Again, ΘP is less than the projected 2, but it is very close: 1.95. The final ratio is $10 : 2$, as above:
$10 : 6 = 6 : 3[\frac{3}{5}] = 3[\frac{3}{5}] : 2[\frac{4}{25}]$. This time, the fraction exceeds 2. Two-fold and four-fold, Φ will fall between Θ and P and for a ratio of five-fold, Φ falls between P and T.

This is extraordinary not only because Pappus foregoes the geometrical analysis but that the error of finding two mean proportionals via approximation is displayed so simply and elegantly. Nowhere in Book 7 does Pappus imply that arithmetic could be used in analysis and here it is used to great pomp elsewhere in the second section as well: "Everyone who wants to be convinced of this can do it by following the analysis through the numbers themselves..."[13] This application of arithmetic has serious implications for Book 2, which will be discussed in the next chapter, but for now the use of numerical ratios efficiently outlines what would be lengthy and difficult to demonstrate geometrically: $K\Theta : \Theta P = BE : AE$ will remain that relationship regardless of the numerals inserted and multiple constructions will be of little use because all such cases are meant to be held within it.[14] Furthermore, by not stating what

[13]Pappus, *Collection*3, 48:12-18.
[14]Knorr(1986), 340. There was a shift from problem-solving towards spherics, trigonometry, and numerical

the ratio was, the student failed to uphold the diagram he proposed.

Pappus fumbled here. He did not use any examples that allow Φ to fall as it does in the diagram. There are ratios that let Φ intersect $K\Theta$ at P; the four-fold ratio came vet close. The warning from Pappus here is, however, that one must be clear and correct in specifying their constructions. A ratio alone is insufficient for determining a point.

Cuomo is slightly confused about what Pappus is doing in this objection.[15] At one point, Pappus placed full blame upon the geometer for not taking the responsibility to fix his error in the ambiguity of Φ but in the next moment he allows $BE : EA$ to be given. Cuomo is surprised at the concession but understand that it allows Pappus to prove that the position of Φ is problematic. As one of the few writers on Pappus to overlook the issue of analysis and synthesis in the Collection, Cuomo missed the significance of such an action as it was outlined above.

The second objection of Pappus' is that the process of approximation that the geometer appears to have utilized relied upon a circular demonstration.Pappus tells Pandrosion why the location of Φ was so important. Undetermined, the ratio of $K\Theta : \Theta\Sigma = \Theta\Sigma : \Theta T = T\Theta : \Theta P$ will not exist. $K\Theta : \Theta P$ is already established as the desired ratio through its equality with $BE : EA$. Pappus then assumed, for arguments sake, that Φ was where the student wanted and the ratio therefore existed. A new case, represented by triangle $\Theta K\mu^{\alpha}$, shows what would happen should the approximation process actually achieve its goal. The argument on Pappus' side is that the student takes for granted that he already knows the desired ratio to find the two means and is therefore manipulating the construction to coincide with that knowledge. The student's argument is circular because the ratio of $K\Theta : \Theta\Sigma = \Theta\Sigma : \Theta T = T\Theta : \Theta P$ is what he is trying to find as well. The last of the three objections raised by Pappus has to do with the geometer's ignorance of his mathematical forbears.[16] The Delian problem was a well

methods which dramatically altered geometric and computational astronomy. Regardless of the shift away, the innovations could have also influenced simple analysis.

[15]Cuomo(2000), 129-130.

[16]Cuomo(2000), 132.

known, traditional problem rooted in classical mathematics and it was long recognized that the problem could only be solved using solid methods, not planar as the student proposed. This prejudice, along with Pappus' general reverence for the ancients, might have predisposed Pappus to disapprove of the problem. The classification of geometrical problems in this was an invention of late antiquity, likely beginning with Heron of Alexandria, but Pappus claims that these divisions were discovered by the ancients.[17] Problems can be one of three types: Planar, linear, and solid. The first are problems that can be solved by straight lines and the arc of a circle; the second have a much more complex origin, such as spirals, quadratrices, conchoids, and cissoids; the third employs the surfaces of solid figures, namely conics. The ancients freely used those methods and discovered by planar methods did not work:

> What with the difference of the problems being of this kind, the ancient geometers were unable to construct the aforementioned figure on the problem of two straight lines from nature in the following geometrical method. Since the intersection of the cone is not easy to draw planarly [what with having given two unequal lines it is necessary to take the means of two proportional lines together in proportion] cleverly substituting for instruments they drew in the execution and the diagram fittingly, as it is shown from the arrangement being set out by them... these men agreeing that the problem was solid they made the construction of it only instrumentally...
>
> Pappus *Collection* 3, 54.22-30.

This is followed by a survey of historical solutions proffered for this problem by Eratosthenes, Nicomedes, Heron, and finally, 'the discovery by us'. The fourth solution, as mentioned in the previous chapter, was attributed to Sporus elsewhere but Pappus may have been using the royal *We* in the sense of the contribution of contemporary mathematics rather than classical mathematicians, but that is a stretch.

[17] Knorr(1986), 342; Panza(1997), 385-387.

Pappus' complaint can be summarized as being less about the math and more about the history of mathematics. The student committed two wrongs by proposing to solve the Delian problem in a way that was already determined to be impossible, thereby completely disregarding the methods of his more expertly predecessors. One finds, however, that Pappus does a less than convincing job in upholding the 'solid method' only argument. Two of the solutions provided are mechanical (Eratosthenes and Heron) while the other two are linear (Nicomedes and Pappus/Sporus).[18] There is another solution alluded to, a truly solid one that uses conic sections, but it does not appear in Book 3.[19] It does appear in Books 4 and 5 and there is a lemma in Book 7 that refers to the appropriate proposition in the Conics (5.52).[20] For Wilbur Knorr, Pappus stumbled here by neither acknowledging that there was a planar aspect to the problem of the means of two straight lines.[21] Nor did Pappus satisfactorily demonstrate that the discovery of a solid construction justified the strict classification of the Delian problem as 'solid' when there were solutions that were not solid. Pappus forgot that the only judgement he directly attributed to the ancients is that they could only solve the problem instrumentally, otherwise, they were at odds about what constituted planar, solid, and linear problems.

Alain Bernard agreed with Knorr but for different reasons. It was not so much to teach the student why solid methods work but rather to chastise the student for, as noted, not learning from his mathematical forebears.[22] However, Bernard's view of Pappus' criticism is that the mathematician was working blind; without the proof, Pappus' criticism is baseless, unsupported and is largely used to serve Pappus' rhetorical purposes. Reference to classical mathematicians allows him to link his criticism to the classification of problems but doing so requires him to presume the intentions of the student that, according to Bernard, are not expressed anywhere. This is a rather fuzzy argument because, if we take Pappus' word for it, the student provided the construction with the known goal of solving the problem of two mean proportionals by ruler-and-compass. This statement of purpose was sent not

[18]Knorr(1986), 347.

[19]Jones(1986), 5. Perhaps Pappus thought it too advanced for his audience. A fair decision in light of all the basic mistakes Pandrosion and her students made.

[20]Pappus, 7.276-278.

[21]Knorr(1986), 347.

[22]Bernard(2003), 126. Bernard's concerns about Knorr go a bit deeper. He accuses Knorr of borrowing heavily from Pappus' philosophical cache while denying that there were any such influences in Pappus' work.

only to Pappus, but shared with many others. Even if one excludes the external influence of Hierius and company, those are still very explicit intentions that eliminate any expectations of mechanical, solid, or linear constructions. Without further clarification as to why he feels Pappus was too eager to speculate, Bernard's interpretation can only be half-right: Pappus' deference to the ancients remains the cornerstone of his works but how he went about his mathematics was in no way aberrant to mainstream practice.[23]

5.4 Book 3: A Renewed Look At Analysis and Synthesis

The contributions of Book 3 can be slotted into one of two types: The enhancement of analytical and synthetic techniques and the clarification of the role played by analysis/synthesis in regular problem solving. The incorporation of arithmetic into analysis as a substitute for the geometrical method elegantly resolved the ambiguity of the ratio for the Delian problem that would have been clumsily done by geometry. It is not the only time in Book 3 that numbers are used to elucidate a demonstration of Pappus; section 3 uses numbers to allow readers to work out and confirm the arithmetic, geometric, harmonic means described.

As for the role in regular problem solving, Pappus finally cemented the confirmation element of analysis. Failure to check for all impossibilities can bring censure upon an individual who claimed to have mathematical knowledge. One indeed had a responsibility to pursue mathematics and to treat it and its practitioners with a respect that goes unuttered in Book 7, save for the hero worship of the classical mathematicians. As far as the Delian problem is relevant, the confirmation aspect is the only use of analysis that it pertinent to Pappus.

[23] Bernard(2003), 130; Jones(1986), 24-26. Bernard is willing to cut Pappus a break. He finds himself inline with Jones in seeing much of the **Collection** as foul papers. That is, works that were not meant to be distributed. Jones notes examples of this in the **Conics** section in Book 7. Bernard sees Pappus being a bit more mathematically cunning than that in Book 3. The incompletion here is deliberate; breaks in the demonstration are there for the audience to fill in. Audience participation allowed Pappus to enhance his argument.

There is more than just a successful analysis and synthesis needed to make a problem completely 'right'. The classification of geometrical problems into linear, planar and solid, when necessary, is preeminent in this decision for. In Pappus' mind using the wrong geometric tools could only come back to haunt you. Dismissing the work that came before your efforts meant one missed out on the nuances of why certain methods do not work as well as others. The geometer created a fuss about finally finding a way to solve the Delian problem using only ruler and compass. By failing his only reward was an invalid proof and a reputation for being unfamiliar with his classical mathematics

In the final chapter, this new vision of analysis will be used with Book 2. This text is often maligned but it has yet to be extensively studied. It is seemingly dismissed for its simplistic subject matter of multiplying large numbers. It is hoped that by slightly re-imagining this book as a geometrical text, the arithmetic analysis introduced here can be fully evaluated to the benefit of gaining more information about both Book 2 and Pappus' practice of analysis/synthesis.

Chapter 6

Sing the Wrath of Demeter: Analysis and the Underworld of Arithmetic

In this final chapter, Book 2 will be the grounds for an experiment on Pappian analysis and synthesis. This is an unlikely book to use because its traditional subject matter has been judged as strictly arithmetical, and therefore incompatible with geometrical analysis and synthesis. The unveiling of numbers in Book 3 encourages a re-evaluation of Book 2 that was first suggested in **Setting the Scene**. The outstanding historiographical issues must be considered first.

6.1 The Style of Book 2

The standard description of Book 2 of **Collection** remains unchallenged since T. L. Heath published the following in **The History of Greek Mathematics**:

> The first 13 propositions of Book II evidently, like the rest of the Book, dealt with Apollonius' method of working with very large numbers expressed in successive powers of the myriads, $10,000$. Heath,**ii**, 361.

Heath's succinct summation captured both the essence of Book 2 as well as propagated major misconceptions about the text that continue today. Only the second half of Book 2 has survived, including propositions 14 to 25. The leitmotif of this book is a method of multiplying several numbers that are either divisible by a hundred, divisible by ten, and less than ten. However, the product must be expressed in units and powers of $10,000$ or *myriads*. This is not an arbitrary demand. Despite the expansion in vocabulary evident in some Greek texts, $10,000$ was the single highest individual number that the Greeks named. Therefore, large numbers are represented either as values between $10,000$ and $1,000 \times 10,000$ or *myriads* squared, cubed etc. $100,000$, $1,000,000$ nor $10,000,000$ are ever directly applied. Representation of such whole numbers without necessitating measures by means of myriads has yet to be revealed in contemporaneous language.

The first few propositions required the units Pappus provided to be multiplied indirectly. In order to accomplish this method, Pappus took each unit and separated their parts collectively into two categories; the *πυθμήν* and the *multitudes*. Take the numbers 50 and 400, five and four would be the *πυθμήν* while ten and a hundred are the *multitudes*. Pappus then multiplied the *πυθμήν* together separately from his multiplication of the *multitudes* and then united the two products to create a new product, which in this case is $20,000$. Sometimes, Pappus would include single digits numbers that are less than ten into the series of numbers he wished to see indirectly multiplied. These numbers are not *πυθμήν* and they appear in propositions where there is no restriction of indirect multiplying. Complex whole numbers like 99 or 73 were not used by Pappus. He avoided numbers that prevented him from getting products that were perfectly representable by myriads or, for smaller numbers, divisible by a thousand without remainders.

Interestingly enough, by challenging the reader not to multiply the numbers directly, Pappus hinted that the linguistic limitation might not entirely impeded some expression of numbers larger than $10,000$. Akin to the use of powers of 10 in the Chinese large number system, the Greeks preferred myriads as their place holders. While this is just as evident in the earlier propositions as in the later propositions, it is not until the second half of proposition 25, when the numerical products derived from the so-called 'word game' repeatedly pushes the sets of *myriads* to their thousandths limit, that the myriads most clearly take on this additional role.

Almost every proposition from 14 to the first half of 25 begins with a series of unlabeled and undetermined numbers, of a singular or combination of quantities that are less than a thousand and are divisible by ten, with the exception of those numbers that are less than ten in the beginning. The goal of finding the product without multiplying the numbers is then stated. Proposition 14 is the only proposition where the numbers are immediately introduced. In the proceeding proposition, letters are increasingly used in alphabetical order to represent the units Pappus desired to have multiplied. For example, the units that Pappus wanted multiplied in proposition 15 are never stated but are symbolized as a whole by series B and Γ. The identity of these units is not completely unknown because Pappus does give their product as well their quantity and the degree of the *multitudes*. Still, it is different from a proposition such as 20 in which every letter is assigned a number before being multiplied together.

Pappus repeated the process of multiplying long after the desired method of multiplying was sufficiently demonstrated. Propositions 17 contains an unusual addition to the proposition that is repeated word for word in proposition 21:

> But the multitude of *(the numbers)* from which *(we get)* series A, multiplied with the square of the multitude of *(the numbers)* from which *(we get)* series B and divisible by four, let it leave, as previously, one, and Apollonius inferred that together with the product from the numbers of series A *(and)* B is just as many myriads to the power of Z, this is ten times as many as E, thus as said beforehand as many of the *multitudes* divisible by four leaves behind two, the product of *(the numbers)* from which *(we get)* series A and B are just as many myriads to the power of Z, which is one hundred times the numbers of E, and whenever it might have left three, the product from the numbers is just as many myriads to the

power of Z, which is one thousand times the numbers of E. Pappus, **Collection** 2.5:16-28.

The units given by Pappus do not allow for such an expansion but, for the sake of the example, this passage further demonstrated Pappus' tendency for repetition. If each of the smaller units were gradually increased to being divisible by a hundred, the final product would also increase respect to the increase in the quantity of the *multitudes*. The product will only go as high as a thousand times a myriad to the nth power because, obviously, the next step is a myriad and would thereby bring the product up to a myriad to the $nth + 1$ power.

So, when Pappus says:

$$\frac{A \cdot B}{4} = 1,$$

he means

$$\tfrac{10^5}{10^4} = 10^1 \text{ or } \tfrac{100{,}000}{10{,}000} = 10$$

Therefore,

$$\frac{A \cdot B}{4} = 2,$$

he means

$$\tfrac{10^6}{10^4} = 10^2 \text{ or } \tfrac{1{,}000{,}000}{10{,}000} = 100$$

and

$$\frac{A \cdot B}{4} = 3,$$

he means

$$\frac{10^7}{10^4} = 10^3 \text{ or } \frac{10{,}000{,}000}{10{,}000} = 1,000$$

Thus,

$$\frac{A \cdot B}{4} = 0,$$

he means

$$\frac{10^8}{10^4} = 10^4 \text{ or } \frac{100{,}000{,}000}{10{,}000} = 10,000 \text{ or } 10,000^2$$

In the preceding propositions, letters did not take on the numerical representations that they do in arithmetic: A did not represent 1 and so on. The second half of proposition 25 does just this in a lengthy fashion by calculating the product of the numerical equivalent of all the letters in two sentence. Thus an A does equal 1, K does equal 20, etc. What this proposition borrows mathematically from the previous propositions is the separation and multiplication of the *πυθμήν* and the *multitudes* before multiplying the products together again into multiple steps of myriads.

6.2 Concerns about Heath

Book 2 has not been the subject of as much scrutiny as the rest of the books in the Collection. Its arithmetical content is seemingly out of place amidst the advanced geometry of the other texts and even appears somewhat out of sync with the rest of Pappus' works. If it receives any attention at all, it is fairly brief and takes on a form similar to Heath's. Sometimes a brief translation is added, proposition 17 usually.[1] The translation however, is only used to show the quirkiness of the text and little else as the contribution of Book 2 to the thesis of these current works is inconsequential. With Book 2 only receiving a perfunctory treatment by scholars past and present. Heath's succinct summation has since become fact while the

[1]Cuomo(2000), 181-182; Netz(2008), unpublished. It is unclear why proposition 17 is an appealing example of Book 2 over other propositions..

text offers strong suggestions that one must admit that the so-called facts of Book 2 are not as definite as they appear.

There are three common presumptions when it comes to Book 2. The first is that the first thirteen propositions were taken from the same text as the other twelve. The second is that of the twelve propositions that do survive, that Apollonius of Perga was necessarily the author of all of them; finally, that Book 2 is an arithmetical text. Each of these can be resolved but at the cost of acknowledging that the facts of Book 2 have as of yet to be properly examined.

The first is a fairly minor concern, but it is important in light of the texts previously discussed. Pappus' penchant for jumping from text to text, problem to problem, and mathematician to mathematician should make one less willing to believe that the missing propositions have the same provenance as the surviving ones, not more. Nor can one assume that the previous propositions were of the same nature. If all that survived were propositions 14 to 24, one could not predict that proposition 25 would include three multiplication problems of two different types under Heath's suggestion. In Heath's favour is Pappus' dedication to erudition in this text. The original theorems upon which Book 2 is based in a numerical order that are one higher than the propositions he assigned them i.e. theorem 26 is proposition 25. Pappus intermittently neglects to refer to the main text but even in their sequencing, it is acceptable to assume that much of that scene setting occurred early in the missing propositions.

On the other hand, it is not entirely set in stone that the propositions in Book 2 were written by a single author. Apollonius of Perga is often awarded complete authorship of the original text without explanation.[2] Seven of the eleven propositions reference a previous source and two of those propositions list a source that does not name Apollonius. Propositions 21 and 22 state their source as from the *stoicheiou*, or the **Elements**. Jones suggested that **Elements** is the title of the lost arithmetical text by Apollonius, as it is the title of other works by lesser known Greek mathematicians. But, this reasoning does not acknowledge that any reference

[2] Jones(1986), 3.

to **Elements** is predominantly accepted to refer to Euclid's work nor that the other uses are adjectival in nature. What makes this suggestion valid is that the term has a self-referential use, in that within the discussion of the text, the author might refer to a particular text as the **Elements**, as if to say the, "the elementary text of which we are discussing". The self-referential usage cannot be turned on Pappus because the phrasing would be too awkward even for Pappus; it would be too easy to confuse it with the Apollonian text. So, it refers either to Apollonius or to some other work.

Attributing the original text to Apollonius is appealing for many reasons. Book 7 alone shows how much Pappus engaged Apollonius when it came to advanced geometry. The intent and verifiable source metrical may have long disappeared, but the major advantage of Book 2 is its provision of an Apollonian work in a subject matter he is not associated with. Unified authorship is a desirable quality in an otherwise incomplete text.

If the reference is to Euclid's **Elements**, then there is recourse to a text that survives in its entirety to provide assistance as to what the original text might have looked like. However, identifying the exact propositions in **Elements** that Pappus might have alluded to is difficult. It requires the propositions to be stripped of the arithmetic and reinterpreted in geometric terms with serious consideration as to the original proof. Looking at the straight forward multiplication for the *πυθμήν* and the *multitudes*, it is possible to recast the arithmetic of Book 2 into the comparison of ratios, which alone make room for comparisons between Books 2 and 5 of the **Elements**, all for only two of the surviving propositions.

By following Heath's statement, this effort is unnecessary as Apollonius was suppose to have written an arithmetical text, making the task of turning the arithmetical into the geometrical avoidable. However, while Pappus' calculations are unarguably numerical and his methods are noted to be through arithmetical processes, Pappus always stated that several of the arithmetical propositions of Book 2 were previously proven by Apollonius. This should not be overlooked and brings one to the final issue brought up by Heath. For all of the errors Pappus makes in his criticisms to Pandrosion's student in Book 3, one does witness how strict Pappus was against what he saw as a violation of essential mathematical truths. In that case, it was the use of planar means to solve a solid problem; in Book 2, the same care can be seen when Pappus distinguished between what he does, by *αριθμος* and what happened in the original text, *γραμμικον*.

Γραμμικον alternately can translate into linear or geometrical, although the form from this dual translation is taken from sways more to it being understood as linear. Nonetheless, Pappus was clear that what he did was much different than the original. This differentiation justifies the existence of Book 2: The *γραμμικον* source of the propositions is an afterthought to the *αριθμος* of Pappus. If the source written with *αριθμος* rather than *γραμμικον*, one must ask what the point of Book 2 would be. The purpose of the **Collection** was to collect and give context to works that were forgotten or difficult to get together in one place. The **Collection** was not intended to be a replacement, only time made it that way. Both Books 3 and 7 are significantly derivative of the original texts but Pappus never failed to make it clear that his contributions could be set apart from the rest. In Book 3, it was his criticism and in Book 7, most of the lemmas were his. If Apollonius really did deal with large numbers arithmetically, then what was Pappus doing that would make his multiplication different than Apollonius'? This is the wrong question as Pappus wrote that Apollonius originally conducted his missing work in metrical analysis.

As it appears in Pappus' text,*γραμμικον* ad its derivative *δια των γραμμων* are very important in the area of spherical astronomy, or, more specifically, ancient spherical trigonometry- how the arcs of great circles form a concave quadrilateral. Nathan Sidoli believes that Hiapparchus, in **Commentary** and Ptolemy, in the **Almagest** and **Analemma** use the word to similar effect.[3] In the **Almagest** it is interpreted as metrical analysis; in the **Analemma** it is metrical analysis applied to problems of plane trigonometry. In Hipparchus' **Commentary**, which is the earliest occurrence of *δια των γραμμων* , simply refers to a calculation completed by geometrical analysis i.e. tracking the co-ordinates of principal points with relation to a

[3]Sidoli(2004), 72-73.

setting star.[4] Hipparchus and other who use *γραμμικον* include data tables so that the reader is spared the effort of conducting the same analyses. Pappus would not have been ignorant to such application of and the precedent set by Ptolemy and others when using *γραμμικον* for analytical purposes. Pappus referred to Hipparchus in Book 3 and one of his extant texts was a commentary on Ptolemy's **Almagest**. One can only assume that such influences were behind Pappus' repeated reminders that his *αριθμος* was distinct from Apollonius' *γραμμικον*. None of this is to say that Apollonius' text was related to trigonometry but with the evidence inclining to that direction, it is open to pursuit. The real question now is what motivated Pappus to take a linear text and represent it arithmetically.

The marriage of arithmetic and geometry is an uneasy one. When he speculated as to the original Apollonian text, Netz believed the theorems were initially vague and entirely geometrical, even though any vagueness in the text would have been clarified by a separate diagram.[5] Cuomo called this process *arithmetization*: The application of numbers to a general proposition that has been established and then reformulating the proposition again.[6] This reformulation would be valuable as a method to convince the reader that a proof does or does not work; Pappus used this particular piece of rhetoric to great effect twice in Book 3. First, when he tried to prove that the position of Φ was under-determined and second when, after repeatedly demonstrating to the reader how the proof presented by the student is circular, he did not conclude his argument but leaves it to the readers to complete the calculations using analysis through numbers and see for themselves that he was correct. When present elsewhere, 'arithmetization' is not so abstract. For instance, in Hero of Alexandria's **Metrica**

> First, let there be an equal-sided triangle, each of its sides equals 10[1]. Let it be $AB\Gamma$[2]. Let $A\Delta$ be perpendicular to $B\Gamma$[3]. Since $B\Gamma$, which is equal to AB, is equal to twice $B\Delta$[4], therefore AB^2 is equal to four times $B\Delta^2$[5]. Thus, $A\Delta^2$ equals thrice times ΔB^2[6]. Thus ΔB^2 equals $\frac{1}{4}B\Gamma^2$[7]; and $B\Gamma^2 = \frac{3}{4}A\Delta^2$[8]. The

[4]Sidoli(2004), 80.
[5]Netz(2008)(unpublished)
[6]Cuomo(2000), 180-183.

ratio of $B\Gamma^2$ to $A\Delta^2$ is 4 : 3[9]. All is multiplied by $B\Gamma^2$; $B\Gamma^2$ is multiplied to itself and multiplied to $A\Delta^2$[10]. Therefore the ratio of $B\Gamma^4$ to $A\Delta^2$ times $B\Gamma^2$ is 4 : 3, which is 16 : 12[11]. However, $B\Gamma^2$ times $A\Delta^2$ is like ($B\Gamma$ times $A\Delta$)^2[12]: This is two triangles multiplied to one another. Therefore the ratio between $B\Gamma^4$ to the double triangles squared is 16 : 12[13]. The double triangles squared are four times larger than a triangle squared. Therefore the ratio of $B\Gamma^4$ to one triangle squared is 16 : 3[14]. Since $B\Gamma^4$ is given, $B\Gamma$ is given[15]. Therefore the area of the squared triangle is given and the triangle is given[16]. This has been calculating according to the following analysis. Square 10, this is 100[17]. Square 100, this is 10,000[18]. Multiply that by $\frac{3}{16}$, that is $1,875$[19]. Since this does not have a root, it should be estimated with the difference. The area is $43\frac{1}{3}$[20]. Heron, **Metrica A** 46.23 – 48.29.

[1]	$AB = B\Gamma = \Gamma A = 10$
[2]	$AB\Gamma$
[3]	$A\Delta$ bisect $B\Gamma$
[4]	$B\Gamma = AB = 2B\Delta$
[5]	$AB^2 = 4B\Delta^2$
[6]	$A\Delta^2 = 3\Delta B^2$
[7]	$\Delta B^2 = \frac{1}{4}B\Gamma^2$
[8]	$B\Gamma^2 = \frac{3}{4}A\Delta^2$
[9]	$B\Gamma^2 : A\Delta^2 = 4 : 3$
[10]	$B\Gamma^4 : A\Delta^2 \cdot B\Gamma^2$
[11]	$4 : 3 = 16 : 12$
[12]	$B\Gamma^2 \cdot A\Delta^2 = (B\Gamma \cdot A\Delta)^2$
[13]	$B\Gamma^4 : (B\Gamma \cdot A\Delta)^2 = 16 : 12$
[14]	$B\Gamma^4 : A\Delta \cdot A\Gamma = 16 : 3$
[15]	$B\Gamma^4$ is given; $B\Gamma$ is given
[16]	Area of the squared triangle is given and the triangle is given
[17]	$10^2 = 100$ ($B\Gamma^2$)
[18]	$100^2 = 10,000$ ($B\Gamma^4$)
[19]	$10,000 \cdot \frac{3}{16} = 1875$ ($B\Gamma^4 : A\Delta \cdot A\Gamma$)
[20]	$\sqrt{1875} = 43\frac{1}{3}$

The geometrical problem is not opaque at all. Heron uses a technique to calculate the area of a triangle without using the length from height to base in arithmetical or geometrical ways. Its use above is purely symbolic, a way to represent a single triangle in a visually familiar form. Heron assigned a numerical value to the segments he was interested in calculating and, in conjunction with the proof and the diagram, completed arithmetical products of relations that were easy to express geometrically anyways. In the following set of problems, the geometry is second to the arithmetical method that Heron elucidates here for the reader: It is a technique that one is expected to use repeatedly as one calculates the area of ever-growing polygon with the triangle above inscribed in the center.

The presence of analysis will be discussed later, but what is relevant at the moment in the confirmation of Cuomo's assumption. Arithmetic can even enhance proofs that are the easiest to comprehend. Heron's method of calculating the area was a bit unusual an likely needed numerical assistance to confirm his formula. Now that it is confirmed, Heron did not need to be explicit about it throughout the next set of problems. After a couple of iterations, Heron quickly assumed that one no longer needs the triangle formula offered because it was memorized. The numbers, on the other hand, remained.

6.3 Arithmetic and Geometric

If one accepts arithmetization as the technique Pappus utilized in Book 2, it does make sense of some of the quirks in the text. Propositions 17, 21, and 25 contain additional information about how the magnitude of the product will increase when the degree of the remainder increases. Where this increase will come from is not stated but by understanding arithmetization as being in action, the increases are then given a rather complicated origin. It might possibly be a consequence of the arbitrary nature of the numbers Pappus assigned. Should the reader want to try out Pappus' method on Apollonius' theorems using different combinations of large numbers, the increases would occur as acknowledged even when Pappus' numbers do not allow such products. This is supported in propositions 21 and 23, wherein the numbers are included for the sake of example or *λογου χαϱιν*.[7]

Arithmetization does not reveal anything about the original text. Its presence means only

[7]Cuomo(2000), 18on27. It is significant that this phrase is not seen alongside numbers in **Metrica**

that it is a method, typically available to geometers and commentators, to enhance some demonstrations and to disprove others. Nonetheless, its use elsewhere in the **Collection** and by other mathematicians does offer reason that the original text of Book 2, and by extension Book 2 itself, should be treated as a geometrical text. There are risks to arguing that Book 2 is a geometrical text. Pappus and other mathematicians made the difference between geometry and arithmetic clear. If the original propositions were arithmetical, Pappus would have written that they were arithmetical. However, he did not. All but one of the propositions linked to Apollonius, the original theorems were noted to have been linear. The linearity of the original problems is in no way a barrier for analysis and synthesis to be conducted on the proof. Many of the lemmas in Book 7 were linear propositions, especially *Cutting off of a Ratio*. The only consequence is that the analyses and syntheses are slightly different than those of more complicated lemmas.

The reconceptualizing of Book 2 as a geometric text is another matter as the text necessarily lacks much of the information that would guide a construction. The numerical values are assigned to individual letters whereas in a linear problem the value could be assigned to a line segment or to an area, as Heron does in the **Metrica**. This weakens the argument for a geometrical text of course but in this book, it is the arithmetic that is on for show. The reader was expected to have the Apollonian text on them, just as it was hoped that they would have some of the texts from the Domain of Analysis in Book 7, thereby freeing Pappus from the responsibility of rewriting everything that was in the original proposition.

A.P. Treweek does offer a diagrammatic representation of Book 2's propositions.[8] The construction are taken from Book 2's fragmentary manuscript tradition. Their provenance can only be assumed from the small collection of manuscripts that do have diagrams. The diagram for proposition 14 takes the *grammikon* to the extremes.

The focus on arithmetician is great for Pappus's proposition but it does not seem to be

[8]Treweek(1950),1-6 trans. For more about Treweek and the manuscript tradition of the **Collection** see the Commentary section of this work as well as Treweek(1958).

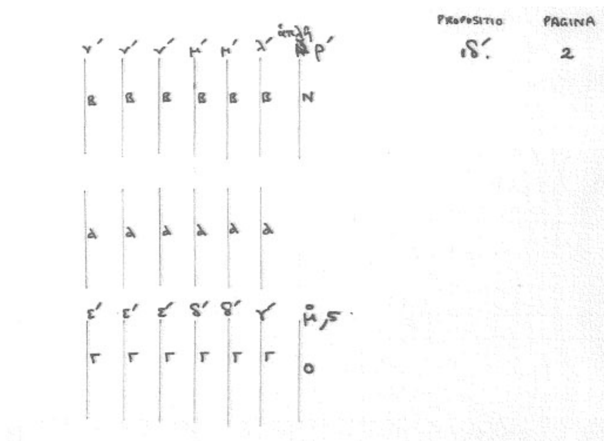

Figure 6.1: Proposition 14, Book 2 of the **Collection** from A. P. Treweek

working here. In the diagram, the proposition is explained like so. B represents all number being hypothesized. N represents the product of the *multitudes*. ($\pi\upsilon\vartheta\mu\acute{\eta}\nu$ are represented by Γ and their own product is represented by an O. The *grammikon* and *arithmos* have been combined to create an abacus-like diagram. This is not sensitive to the nuances of the text. Proposition 17 shows just how convoluted this method can be. In the later series of

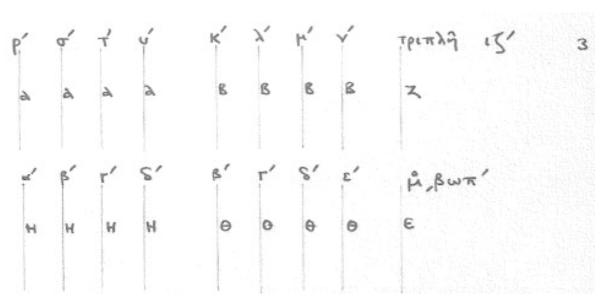

Figure 6.2: Proposition 17, Book 2 of the **Collection** from A. P. Treweek

diagrams, the *multitudes* are no longer represented; the task of calculating their products is left for mental mathematics. As the groupings become larger, numbers that are not assigned specific a letter in the text happen to be assigned them anyways, complicating the ones ability to understand the construction as the symbolism of the place-holders is inconsistent.

It must be conceded that the best proof one can provide of Book 2 as a geometrical text in spirit is the discovery of the constructions that inspired Pappus. As much as the previous diagrams best represent what Pappus intended. Constructions must be sensitive towards the text. In this case, only a few corrections are needed to bring Treweek's drawing in line with the text.

A minor setback to Book 2 being a geometrical text is the second half of proposition 25. In it, both Apollonius and Pappus play a lengthy word game where each letter of a hexameter verse is converted into their numerical values and the product of all the numbers are then multiplied. The Apollonius completed this game arithmetically as per the lost notice at the beginning of Book 2, initially challenges the presumption that the lost text was entirely geometrical. But, it does not change the fact that Pappus continually separated his arithmetical work from the linear efforts of his predecessor. If the Collection has proven anything, it is that a text does not have to be limited to one subject matter and the place most like for that change to occur will be at the end of a text.

6.4 Analysis and Synthesis and Book 2

The appearance of numbers in Book 3 promised to reinvigorate discussion about a text that is widely misunderstood. If the arithmetic of Book 2 was a means to demystify one or more geometric proofs, was it there as a technique of analysis or something else?

One needs more than numbers to conduct an analysis. They are used in Book 3 to emphasize a point about the nature of ratios. It is possible, as the example in Book 3 illustrates, to isolate the arithmetic from the diagram and just calculate the ratios. On its own, this would not be an analysis. As per Book 7, analysis requires an established *ζητούμενον* , *ἀκολουθος* , and things that would hold the rank of first principle in synthesis. In the context of the proposition, these things are provided in the form of the desired ratio, the ratios in between, and the position of Φ respectively. The numbers are set to these objects, not the opposite.

Book 2 was not laid out in that way. Due to its arithmetical nature, the text has to be understood synthetically first and foremost. This means that the proposition must move from first principles to objects that are sought. Of course, there is nothing preventing Book 2 from being read analytically. Many of the proposition display the individual numbers immediately, and they are easy enough to calculate without Pappus' restrictions. In ideal analytical-type of proposition in this case would begin with the product first and seeks the existence of a desired multiplicand. The closest proposition to this ideal is proposition 15. The product and multitudinous value of the multiplicands are the only quantities given. Then it is possible to identify the sets of four or five multiplicands by working backwards from the possible products of the (*πυθμήν* and the *multitudes*. As for the other propositions, too much information is usually given and the order favours the calculation of the (*πυθμήν* first.

The case is stronger for the analytic use of numbers in the **Metrica**. There are a couple of instances where Heron used the phrase "συντηήσεται δὲ ἀκολούθως τ,ῆ ἀναλύσει οὖτως. ".[9] The problem itself is not set up as a typical analysis. Heron took the reader through the creation of the construction and proceeded through the steps in a synthetic way. Heron's goal, to find the area of the triangle, is not the sort of goal used in analysis. As in propositions 21 and 72, are usually a single point or a line. Like many analysis, Heron declared the *givens* at the end of his proof whereas in synthesis these are identified first. Another thing that happens in **Metrica** that also happened in Book 3 was the order in which arithmetic was used. In both propositions, the numbers are applied after the construction and the relationships were completed. Then, the arithmetic would proceed through the proof in the sequence that it was established. For example, **Metrica**'s arithmetic follows the order of the geometrical identities introduced. The analysis is completed not once, but twice.

Of course, **Metrica**'s analysis does have some very synthetic qualities.[10] The main setback of synthesis has been the arbitrariness of the sequence of steps. **Metrica** demonstrated how sequences become comprehensible when numbers are applied to a construction. It make

[9]Heron, 38.26-27, 48.24.

[10]Cuomo(2000), 180. Indeed, **Metrica**'s analysis is deceptive this way. Cuomo is right in that Heron's use of numbers is not just an example but it seems she mistranslated..... and interpreted Heron's method as pseudo-synthetic.

the comparison, addition, ratios et al. of objects sensible. Still, there is more support for the use of numbers to be analytical. Heron's propositions all have figures provided and the first step in an Heronian proposition is often not to start constructing a figure but to assign numbers to what is already there. In the same vein, the arithmetic in the **Metrica** is formed upon the geometry but there is some confusion between the analysis and synthesis proper when he begins his calculations. The technique may have been nascent in Heron's time and by Pappus' mathematicians had become more confident in using it. This is the same for Pappus' first objection in Book 3. Book 2 is the exception. Pappus has stripped his propositions of their geometry, leaving behind only the arithmetic. This could be a point for Book 2 being synthetic at least, but only if one had not seen Pappus do this before. Book 7 also minimally referenced the original propositions because a reader of Book 7 was expected to have his own copy of the original texts accompanying him. From what can be gleaned from this text, the expectation were not likely to be that different.

For what it is worth, Book 2 benefits from this change from an arithmetical text to one more inline with the **Collection** as a whole. Heron and Pappus may coincide on many points but the evaluation must always return to Pappus' adherence to his own rules. One can imagine how a construct would be compatible to any of the propositions in Book 2, but it would be little more than conjecture. Analysis and synthesis may not have had the same impact as in Books 3 and 7 but many of Pappus' decisions and behaviours in those books are reflected in Book 2. Again, the numbers themselves are not the synthesis, but merely are a technique, just as they were for the analyses in Book 3. At the heart of analysis and synthesis is the order in which the geometrical objects fall when the *ζητούμενον* and the first principle alternate their positions. These are objects that Book 2 does not explicitly have but at the same time, it does not need them. In maintaining that Book 2 is based on the lost Apollonian text, the arithmetic of Book 2 is merely a technique of analysis, not the analysis itself.

Chapter 7

Epilogue

Recently a mathematician, glad to meet another who was familiar with Pappus, commented to me that he through Pappus was the intellectually loneliest individual in Greek mathematics. The mathematicians agree that Pappus certainly was active but that his correspondents such as Hermodorus were engaged in mathematics at a much lower level than Pappus. Pappus, on the other hand, was burdened with the monumental task of revitalizing Greek mathematics during a period of intellectual and social decline.

As I have previously argued, the mathematics of late antiquity was not subject to the same decline witnessed elsewhere. Instead it entered an epoch of intense reflection about how their predecessors transmitted mathematics and how they did the same by text. The mathematician did highlight what I see as the mode in which Pappus wrote, which was at odd with the vast knowledge he clearly possessed. Scholars, however, approach Pappus as if he was in his true mathematical element rather than in his more frequent role of pedagogue. As a consequence, Pappus has been accused of many mathematical crimes that do not hold up after a careful reading of the text.

Chief amongst these has been Pappus' clumsiness with the methods of analysis and synthesis. In chapters 3, 4, and 5 Pappus proved that in the structure of analysis and synthesis, he was a conformist to the practice of the classical mathematicians. His motivation was framed by the need to instruct rather than share an interesting theorem, thereby necessitating language and exposition that leave more advanced mathematicians put out.

In chapter 6, we see that Pappus' practice, in comparison to the expositions and propositions of Book 7, is much more involved that previously recognized. Analysis and synthesis is here treated by Pappus as central to the social practice of mathematics. They serve as a public check against impossible problems and theorems accidently becoming accepted practice. The inclusion of arithmetic as a technique within that application of analysis and synthesis is stunning and unexpected. It does not, however, set a precedent: Heron of Alexandria used a nascent method centuries earlier.

Indeed, how can one understand Book 2 know that we have seen the application of arithmetic to geometrical problems? It is a significant discovery, supported by Nathan Sidoli and Serafina Cuomo, that Apollonius' original text was originally geometrical and not arithmetical as presumed by scholarly tradition. It is to this end that Book 2 becomes deserving of more serious treatment.

Pappus has been reborn several times over in this thesis: as a mathematician, in his methods of analysis and synthesis, and in the reevaluation of Book 2.

Part III

Translation and Commentary

Chapter 8

Book 2 Text, Translation, and Commentary

8.1 Greek Numbers

Provided below is the listing of numerals and their alphabetical equivalent

1 - α'; 2 - β'; 3 - γ'; 4 - δ'; 5 - ϵ'; 6 - ς'; 7 - ζ'; 8 - η'; 9 - θ'; 10 - ι'; 11 - $\iota\alpha'$; 20 - κ'; 30 - λ'; 40 - μ'; 50 - ν'; 60 - ξ'; 70 - o'; 80 - π'; 90 -[ ][1]; 100 - ρ'; 200 - σ'; 300 - τ'; 400 - υ'; 500 - ϕ'; 600 - χ'; 700 - ψ'; 800 - ω'; 900 -[ ]; 1000 - α;

Ϝ

[1] Due to the particulars of this language package used, 90 and 900 are manually inserted where necessary as is the subprime for 1000. There are further difficulties with the number 6000. It typesets as a diagramma when it should be an outside sigma with a subprime. The correction has been made were possible.

8.2 Greek Text, with translation and commentary of Book 2 from Pappus of Alexandria's *Mathematical Collection*

** γὰρ αὐτοὺς ἐλάσσονας μὲν εἶναι ἑκατοντάδος μετρεῖσθαι δὲ ὑπὸ δεκάδος, καὶ δέον ἔστω τὸν ἐξ αὐτῶν στερεὸν εἰπεῖν μὴ πολλαπλασιάσαντα αὐτούς.

Ἔστωσαν οὖν οἱ ἀριθμοὶ ν′ ν′ ν′ μ′ μ′ λ′· ἔσονται ἄρα οἱ πυθμένες ε′ ε′ ε′ δ′ δ′ γ′· ὁ ἄρα ἐξ αὐτῶν στερεὸς γίνεται μονάδων ς. καὶ ἐπεὶ τὸ πλῆθος τῶν δεκάδων ἐστὶν ς′ καὶ μετρούμενον ὑπὸ τετράδος λείπει δύο, ἔσται ὁ ἐξ αὐτῶν στερεὸς [τῶν δεκάδων] μυριάδων ἁπλῶν ἑκατόν. καὶ ἐπεὶ ὁ ἐκ τῶν δεκάδων στερεὸς ἐπὶ τὸν ἐκ τῶν πυθμένων στερεὸν ποιεῖ τὸν ἐκ τῶν ἐξ ἀρχῆς στερεόν, αἱ ἄρα μυριάδες ρ′ ἐπὶ τὰς μονάδας ς γενόμεναι ποιοῦσιν μυριάδας ξ′ διπλᾶς, ὥστε ὁ ἐκ τῶν ν′ ν′ ν′ μ′ μ′ λ′ στερεός ἐστιν μυριάδων ξ′ διπλῶν.

[Proposition 14] *(Let there be howeversomany numbers and let the numbers be)* less than one hundred and divisible by ten, and let it be necessary to give the product from them without multiplying them *(directly)*.

Let the numbers be 50, 50, 50, 40, 40, 30[1]. Therefore the puthmenes *(of that series)* will be 5, 5, 5, 4, 4, 3[2]. Consequently the product of these same *(numbers)* will be 6,000 units[3]. And since the multitude of tens is 6[4] and dividing *(the number of zeroes)* by four leaves *(a remainder)* two[5], the product from these same *(numbers)* is 100 straight myriads[6]. And since the product of the tens *(multiplied with)* the product of the puthmenes makes the product from *(the numbers)* from the beginning, consequently the 100 myriads *(multiplied with)* the 6,000 units make 60 myriads squared[7], so that the product of 50, 50, 50, 40, 40, 30 is 60 myriads squared.

ιε′. Ἔστωσαν δὴ πάλιν ὁσοιδηποτοῦν ἀριθμοὶ ἐφ᾽ ὧν τὰ Β, ὧν ἕκαστος ἐλάσσων μὲν χιλιάδος μετρείσθω δὲ ὑπὸ ἑκατοντάδος, καὶ δέον ἔστω τὸν ἐξ αὐτῶν στερεὸν εἰπεῖν μὴ πολλαπλασιάσαντα τοὺς ἀριθμούς.

Γεγονέτω, καὶ ὁ διπλάσιος τοῦ πλήθους αὐτῶν μετρείσθω πρότερον ὑπὸ τετράδος, καὶ ὑποκείσθω ὑπὸ ἕκαστον τῶν Β ἑκατοντὰς ἡ α′, καὶ καθὸ μετρεῖται ἕκαστος τῶν Β ὑπὸ τῆς ἑκατοντάδος ἔστωσαν οἱ ἐφ᾽ ὧν τὰ Γ· πυθμένες ἄρα εἰσὶν οἱ ἐφ᾽ ὧν τὰ Γ τῶν ἐφ᾽ ὧν τὰ Β. ὁ δὲ διὰ τῶν πυθμένων στερεὸς ἔστω ὁ Ε [τουτέστιν μονάδες ρκ′]. δείκνυται οὖν διὰ τῶν γραμμῶν ὁ διὰ τῶν ἐφ᾽ ὧν τὰ Β στερεὸς μυριάδων διπλῶν ρκ, ἐπειδὴ καὶ ὁ διὰ τῶν ἐφ᾽ ὧν τὰ Β στερεὸς ἴσος ἐστὶν τῷ. ὁ διὰ τῶν ἑκατοντάδων στερεῷ ἐπὶ τὸν ἐκ τῶν πυθμένων στερεόν, τουτέστιν διπλῆ μυριὰς α′ ἐπὶ τὰς ρκ′ μονάδας.

Ἀλλ᾽ ὁ διπλάσιος τοῦ πλήθους τῶν ἐφ᾽ ὧν τὰ Β μὴ μετρείσθω ὑπὸ τετράδος· μετρούμενος ἄρα λείψει δυάδα ἐξ ἀνάγκης (τοῦτο γὰρ προδέδεικται), ὥστε καὶ ὁ διπλάσιος τοῦ πλήθους τῶν ἑκατοντάδων μετρούμενος ὑπὸ τετράδος· τὸ ἄρα πλῆθος τῶν ἑκατοντάδων μετρούμενον ὑπὸ δυάδος λείψει μίαν ἑκατοντάδα. ὁ τοίνυν διὰ τῶν ἑκατοντάδων στερεὸς ἔσται μυριάδων ρ′ ὁμωνύμων τῷ Ζ, τουτέστι διπλῶν, ὥστε δῆλον ὅτι ὁ διὰ τῶν ἐφ᾽ ὧν τὰ Β μυριάδες εἰσὶν ρ′ ὁμώνυμοι τῷ Ζ γενόμεναι ἐπὶ τὸν Ε [τὰς ρκ′ μονάδας]. γίνονται μυριὰς μία δισχίλιαι διπλῶν μυριάδων.

[Proposition 15] Again, let there be however so many numbers from which *we get* series B of which each are less than one thousand and let them be divisible by a hundred[1], let it be necessary to give the product from out of these without multiplying the numbers *(directly)*.

Let this be done: Let the square of the multitude *(of those numbers)* be divided by 4 [2]as before and let a hundred be placed below each *(number)* of B, and insofar as each number of B is divided by a hundred[3], Let them be *(the numbers)* from which we get *series* Γ[4]. Consequently the puthmenes which we get from series B are *(the numbers)* for which we get series Γ[5]. Let the product of the puthmenes be E[6]. The product of *(the numbers)* from which we get series B is 120 myriads squared proven through workings[7], and since the product of *(the numbers)* from which we get series B is equal to the product obtained through the hundreds *(multiplied with)* the product of the puthmenes, this is 1 myriad squared *(multiplied with)* 120 units[8].

But let the square of the multitude *(of the numbers)* from which we get series B not be divisible by four *(without remainder)*[9]. Consequently, (the multitude) having divided *(by four)* will leave two necessarily (for this have been brought forth)[10], so that the square of the multitude from the hundreds has been divided by four. Thus the multitude of the hundreds having been divided by two will leave one of the hundreds[11]. Then the product of the hundreds will be 100 myriads to the power of Z, that is squared[12], consequently it is clear that *(the product of the multitude of the numbers)* from which we get series B gives 100 myriads to the power of Z *(multiplied with)* E becomes one myriad and two thousand *(multiplied with)* a myriad squared[13].

ιϛ΄. Ἔστωσαν δύο ἀριθμοὶ οἱ Α Β, καὶ ὁ μὲν Α ὑποκείσθω ἐλάσσων μὲν χιλιάδος μετρούμενος δὲ ὑπὸ ἑκατοντάδος, οἷον μονάδες φ΄, ὁ δὲ Β ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, οἷον μονάδες μ΄, καὶ δέον ἔστω τὸν ἐξ αὐτῶν ἀριθμὸν εἰπεῖν μὴ πολλαπλασιάσαντα αὐτούς.

Ἔστι δὲ φανερὸν διὰ τῶν ἀριθμῶν· οἱ γὰρ ε΄ δ΄ πυθμένες αὐτῶν ὄντες [μο. ε΄ καὶ μο. δ΄] πολλαπλασιασθέντες ποιοῦσι μονάδας κ΄, χιλιάκις δὲ ὁ κ΄ ἀριθμὸς ποιεῖ μυριάδας δύο ποιούσας τὸν ὑπὸ τῶν Α Β γινόμενον. τὸ δὲ γραμμικὸν δῆλον ἐξ ὧν ἔδειξεν Ἀπολλώνιος.

[Proposition 16] Let there be two numbers A *and* B and let A be established as less than one thousand and divisible by a hundred, for example 500[1], let B be established as less than a hundred and divisible by ten, for example 40[2], and let it be necessary to give the number from these without multiplying them *(directly)*

It is clear through arithmetical procedures. For the puthmenes 3 and 4[3] from those numbers having been multiplied makes 20[4] and the number 20 a thousand times makes two myriads[5], making the result from *(multiplying)* A *(and)*B[6]. Apollonius proved the metrical analysis clearly from *(his works)*.

ιζʹ. Ἐπὶ δὲ τοῦ ιηʹ θεωρήματος. Ἔστω πλῆθος ἀριθμῶν τὸ ἐφ᾽ ὧν τὰ Α, ὧν ἕκαστος ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, καὶ ἄλλο πλῆθος ἀριθμῶν τὸ ἐφ᾽ ὧν τὰ Β, ὧν ἕκαστος ἐλάσσων μὲν χιλιάδος μετρούμενος δὲ ὑπὸ ἑκατοντάδος, καὶ δέον ἔστω τὸν ἐκ τῶν ἐφ᾽ ὧν τὰ Α Β στερεὸν εἰπεῖν μὴ πολλαπλασιάσαντα αὐτούς.

Έστωσαν γὰρ πυθμένες τῶν μὲν ἐφ᾽ ὧν τὰ Α οἱ ἐφ᾽ ὧν τὰ Η, μονάδες αʹ καὶ βʹ καὶ γʹ καὶ δʹ, τῶν δὲ ἐφ᾽ ὧν τὰ Β οἱ ἐφ᾽ ὧν τὰ Θ, μονάδες βʹ καὶ γʹ καὶ δʹ καὶ εʹ, καὶ ληφθέντος τοῦ ἐκ τῶν πυθμένων στεροῦ [τῶν βʹ γʹ δʹ βʹ γʹ δʹ εʹ], τουτέστιν τοῦ Ε, μονάδων ὄντος ͵βωπʹ, τὸ πλῆθος τῶν ἐφ᾽ ὧν τὰ Α προσλαβὸν τὸν διπλασίονα τοῦ πλήθους τῶν ἐφ᾽ ὧν τὰ Β μετρείσθω πρότερον ὑπὸ τετράδος [κατὰ τὸν Ζ, μετρεῖ δὲ αὐτούς]. καὶ δείκνυσιν ὁ Ἀπολλώνιος τὸν ἐκ πάντων τῶν ἐφ᾽ ὧν τὰ Α Β στερεὸν μυριάδων τοσούτων, ὅσαι εἰσὶν ἐν τῷ Ε μονάδες, ὁμωνύμων τῷ Ζ ἀριθμῷ, τουτέστιν τριπλῶν μυριάδων ͵βωπʹ.

Ἀλλὰ δὴ τὸ πλῆθος τῶν ἐφ᾽ ὧν τὰ Α, προσλαβὸν τὸν διπλασίονα τοῦ πλήθους τῶν ἐφ᾽ ὧν τὰ Β, μετρούμενον ὑπὸ τετράδος καταλειπέτω πρότερον ἕνα. καὶ συνάγει ὁ Ἀπολλώνιος ὅτι ὁ ἐκ τῶν ἀριθμῶν ἐφ᾽ ὧν τὰ Α Β στερεὸς μυριάδες εἰσὶν τοσαῦται ὁμώνυμοι τῷ Ζ, ὅσος ἐστὶν ὁ δεκαπλασίων τοῦ Ε, ἐὰν δὲ τὸ προειρημένον πλῆθος μετρούμενον ὑπὸ τετράδος καταλείπῃ δύο, ὁ ἐκ τῶν ἀριθμῶν στερεὸς τῶν ἐφ᾽ ὧν τὰ Α Β μυριάδες εἰσὶν τοσαῦται ὁμώνυμοι τῷ Ζ, ὅσος ἐστὶν ὁ ἑκατονταπλάσιος τοῦ Ε ἀριθμοῦ, ὅταν δὲ τρεῖς καταλειφθῶσιν, ἴσος ἐστὶν ὁ ἐξ αὐτῶν στερεὸς μυριάσιν τοσαύταις ὁμωνύμοις τῷ Ζ, ὅσος ἐστὶν ὁ χιλιαπλάσιος τοῦ Ε ἀριθμοῦ.

[Proposition 17] In the case of Theorem 18. Let there be a multitude of numbers from which *(we get)* series A, each being less than a hundred and divisible by ten[1], and the another multitude of numbers from which *(we get)* series B, each being less than one thousand and divisible by a hundred[2], and let it be necessary to give the product from A *(and)* B without multiplying them *(directly)*.

Let the puthmenes of the numbers from which *(we get)* series A be the numbers for which *(we get)* series H, the units 1, 2, 3, 4[3], and *(let the puthmenes)* of the numbers from which *(we get)* series B be the numbers for which we get Θ, the units 2, 3, 4, 5[4], and the product of the puthmenes having been taken, this is **E**[5], being 2880 monades[6], let the multitude *(of the numbers)* from which *(we get)* series A take the square of the multitude *(of the numbers)* from which *(we get)* series B [7] be divided first by four[8]. And Apollonius shows that the product from all of *(the numbers)* from A *(and)* B will be just so many myriads, as many as are equal in units to E, with to the power of Z[9], which is 2880 myriads cubed[10].

But the multitude of *(the numbers)* from which *(we get)* series A, multiplied with the square of the multitude of *(the numbers)* from which *(we get)* series B and divisible by four[11], let it leave, as previously, one. And Apollonius inferred that together with the product from the numbers of series A *(and)* B is just as many myriads to the power of Z, this is ten times as many as E[12], thus as said beforehand as many of the multitudes divisible by four leaves behind two, the product of *(the numbers)* from which *(we get)* series A and B are just as many myriads to the power of Z, which is one hundred times the numbers of E[13], and whenever it might have left three, the product of the numbers is just as many myriads to the power of Z, which is one thousand times the numbers of E[14].

ιη′. Ἐπὶ δὲ τοῦ ιθ′ θεωρήματος. Ἔστω τις ἀριθμὸς ὁ Α ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, καὶ ἄλλοι ὁσοιδηποτοῦν ἀριθμοὶ ἐλάσσονες δεκάδος οἷον οἱ Β Γ Δ Ε, καὶ δέον ἔστω τὸν ἐκ τῶν Α Β Γ Δ Ε στερεὸν εἰπεῖν.

Ἔστω γὰρ καθ᾽ ὃν μετρεῖται ὁ Α ὑπὸ τῆς δεκάδος ὁ Ζ, τουτέστιν ὁ πυθμὴν τοῦ Α, καὶ εἰλήφθω ὁ ἐκ τῶν Ζ Β Γ Δ Ε στερεὸς καὶ ἔστω ὁ Η· λέγω ὅτι ὁ διὰ τῶν Α Β Γ Δ Ε στερεὸς δεκάκις εἰσὶν οἱ Η.

Καὶ ἔστι φανερὸν διὰ τῶν ἀριθμῶν· τοῦ γὰρ Α ὑποκειμένου, φέρ᾽ εἰπεῖν, μονάδων κ′ καὶ τοῦ Β μονάδων γ′ καὶ τοῦ Γ μονάδων δ′ καὶ τοῦ Δ μονάδων ε′ καὶ τοῦ Ε μονάδων ς′, ὁ ἐξ αὐτῶν στερεὸς γίνεται μονάδες ͵ζσ′. ἀλλὰ καὶ τοῦ Ζ ὄντος μονάδων β′, ὅς ἐστι πυθμὴν τοῦ Α, ὁ ἐκ τούτου καὶ τῶν Β Γ Δ Ε στερεὸς δεκάκις γενόμενος ἔσται μονάδες ͵ζσ′, ἴσος τῷ ἐκ τῶν Α Β Γ Δ Ε στερεῷ . τὸ δὲ γραμμικὸν ὑπὸ τοῦ Ἀπολλωνίου δέδεικται.

[Proposition 18] In the case of Theorem 19. Let A be some number less than a hundred and divisible by ten[1] and let however so many other number less than ten, for example B Γ Δ E and let it be necessary to give the product from A B Γ Δ E.

For let there be Z, which is A divided by 10, that is the puthmenes of A[3], and let the product of Z B Γ Δ E be taken and let it be H[4]. I say that the product of A B Γ Δ E is ten times H[5].

And it is clear through the arithmetical procedures: with A lying bunder 20, easy to say, and B under 3 and Γ under 4 and Δ under 5 and E under 6[6], the product from these will be 7200 monades[7]. But with Z being 2 monades[8], which is the puthmenes of A, the product of that and $B\Gamma\Delta E$ ten times will be 7200[9], equal to the product of A B Γ Δ E. The metrical analysis has been shown by Apollonius.

ιϑʹ. Ἀλλὰ δὴ ἔστωσαν δύο ἀριθμοὶ οἱ Α Β, ὧν ἑκάτερος ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, τῶν δὲ Γ Δ Ε ἕκαστος ἐλάσσων δεκάδος ἔστω, καὶ δέον ἔστω τὸν ἐκ τῶν Α Β Γ Δ Ε στερεὸν εἰπεῖν.

Ἔστωσαν γὰρ τῶν Α Β πυθμένες οἱ Ζ Η· λέγω ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ε στερεὸς τοῦ ἐκ τῶν Ζ Η Γ Δ Ε στερεοῦ ἑκατονταπλάσιός ἐστιν.

Φανερὸν δὲ καὶ τοῦτο διὰ τῶν ἀριθμῶν, τοῦ Α ὄντος μονάδων κʹ καὶ τοῦ Β μονάδων λʹ καὶ τοῦ Γ μονάδων βʹ καὶ τοῦ Δ μονάδων γʹ καὶ τοῦ Ε μονάδων δʹ καὶ τοῦ Ζ μονάδων βʹ καὶ τοῦ Η μονάδων γʹ.

ὁ γὰρ ὑπὸ τῶν Α Β Γ Δ Ε στερεός ἐστιν μ^{α} ͵δυʹ, ὁ δὲ ὑπὸ Ζ Η Γ Δ Ε μονάδες ρμδʹ, οὗτος δὲ γενόμενος ἑκατοντάκις ποιεῖ μ^{α} ͵δυʹ. τὸ δὲ γραμμικὸν ἐκ τῶν Ἀπολλωνίου.

[**Proposition 19**] But let there be two numbers A B, each being less than a hundred and divisible by ten[1], let each of Γ Δ E be less than ten[2], and let it be necessary to give the product from out of A B Γ Δ E.

Let Z H be the puthmenes of A B[3]; Let it be said that the product of A B Γ Δ E is a hundred times of the product of Z H Γ Δ E[4].

This is clear through the arithmetical procedures, of A under 20 monades, of B under 30 monades, of Γ under 2 monades, of Δ under 3 monades, of E under 4 monades[5] and of Z under 2 monades, of H under 3 monades[6].

The product of A B Γ Δ E is 14,400[7], the *(product)* of Z H Γ Δ E *(is)* 144[8]. This being a hundred-fold makes 14,400[9. This has been shown from the metrical analysis from *(the works)* of Apollonius.

κʹ. Ἀλλὰ δὴ ἔστωσαν τρεῖς ἀριθμοὶ οἱ Α Β Γ, καὶ ἔστω ἕκαστος αὐτῶν ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, ἕκαστος δὲ τῶν Δ Ε Ζ ἔστω ἐλάσσων δεκάδος, καὶ ἔστωσαν τῶν Α Β Γ πυθμένες οἱ Η Θ Κ, καὶ εἰλήφθω ὁ ἐκ τῶν Η Θ Κ Δ Ε Ζ στερεὸς καὶ ἔστω ὁ Ξ· ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ε Ζ στερεὸς ἴσος ἐστὶν χιλίοις τοῖς Ξ.

Ἔστι φανερὸν διὰ τῶν ἀριθμῶν, τοῦ Α ὄντος λόγου χάριν μονάδων κʹ καὶ τοῦ Β μονάδων λʹ καὶ τοῦ Γ μονάδων μʹ καὶ τοῦ Δ μονάδων βʹ καὶ τοῦ Ε μονάδων γʹ καὶ τοῦ Ζ μονάδων δʹ, τοῦ δὲ Η μονάδων βʹ καὶ τοῦ Θ μονάδων γʹ καὶ τοῦ Κ μονάδων δʹ· ὁ γὰρ ὑπὸ Α Β Γ Δ Ε Ζ στερεός ἐστιν μυριάδων νζʹ ἁπλῶν καὶ μονάδων ς, ὁ δὲ ὑπὸ τῶν Η Θ Κ πυθμένων καὶ τῶν Δ Ε Ζ ἔσται μονάδων φοςʹ, αὗται δὲ χιλιάκις γενόμεναι, τουτέστιν ὁ ἐκ πάντων στερεός, γίνεται μυριάδων ἁπλῶν νζʹ καὶ μονάδων ς.

[Proposition 20] But let there be three numbers A B Γ, and let each of these be less than a hundred and divisible by ten[1], Let each of Δ E Z be less than ten[2] and Let H Θ K be the puthmenes of A B Γ[3] and let the product of H Θ K Δ E Z be taken and let it be Ξ[4]. I say that the product of A B Γ Δ E Z is equal to a thousand-fold Ξ[5].

It is clear through the arithmetical procedures, for the sake of example, of A under 20 monades, of B under 30 monades, of Γ under 40 monades, of Δ under 2 monades, of E under 3 monades, of Z under 4 monades[6], of H under 2 monades, of Θ under 3 monades, of K under 4 monades[7]. The product of A B Γ Δ E Z is 57 straight myriads and 6000 monades[8] and the *(product)* of the puthmenes H Θ K and of Δ E Z is 576 units[9], these being a thousand-fold, this is the product of all *(numbers)* becomes 57 straight myriads and 6000 units[10].

κα΄. Ἀλλὰ δὴ ἔστωσαν πλείους τριῶν οἱ Α Β Γ Δ Ε, καὶ ἕκαστος ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, τῶν δὲ Ζ Η Θ ἕκαστος ἔστω ἐλάσσων δεκάδος.

Τὸ πλῆθος τῶν Α Β Γ Δ Ε πρότερον μετρείσθω ὑπὸ τετράδος κατὰ τὸν Ο. καὶ ἔστωσαν τῶν Α Β Γ Δ Ε πυθμένες οἱ Κ Λ Μ Ν Ξ· ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ζ Η Θ στερεὸς ἴσος ἐστὶν μυριάσιν ὁμωνύμοις τῷ Ο ὅσαι μονάδες εἰσὶν ἐν τῷ στερεῷ τῷ ἐκ τῶν Κ Λ Μ Ν ἐπὶ τὸν ἐκ τῶν Ζ Η Θ.

Ἔστι δὲ φανερὸν διὰ τῶν ἀριθμῶν, τοῦ Α ὑποκειμένου λόγου χάριν μονάδων ι΄ καὶ τοῦ Β μονάδων κ΄ καὶ τοῦ Γ μονάδων λ΄ καὶ τοῦ Δ μονάδων μ΄, καὶ τῶν Κ Λ Μ Ν πυθμένων ὄντων μονάδων α΄ καὶ β΄ καὶ γ΄ καὶ δ΄· ὁ ἄρα ἐκ τῶν Α Β Γ Δ στερεός ἐστιν ἁπλῶν μυριάδων κδ΄, ὁ δὲ ἐκ τῶν Α Β Γ Δ Ζ Η Θ μυριάδων ἁπλῶν ρμδ΄, ὁ δὲ ἐκ τῶν Κ Λ Μ Ν πυθμένων μονάδων κδ΄· οὗτος δὲ γενόμενος ἐπὶ τὸν ἐκ τῶν Ζ Η Θ, ὄντα μονάδων ϛ΄, ποιεῖ μονάδας ρμδ΄, ὅσαι μυριάδες ἁπλαῖ εἰσιν τοῦ ἐκ τῶν Α Β Γ Δ Ζ Η Θ στερεοῦ, διὰ τὸ καὶ τετράδα ἅπαξ μετρεῖν τὸ πλῆθος τῶν Α Β Γ Δ.

Ἀλλὰ δὴ τὸ πλ˜ήθος τῶν Α Β Γ Δ Ε μὴ μετρείσθω ὑπὸ τετράδος· μετρούμενον δὴ ἤτοι α΄ ἢ β΄ ἢ γ΄ λείψει. εἰ μὲν οὖν ἕνα λείψει, ἔσται ὁ ἐκ τῶν Α Β Γ Δ Ε Ζ Η Θ στερεὸς μυριάδων ὁμωνύμων τῷ Ο, ὅσος ἐστὶν ὁ ἐκ τῶν Κ Λ Μ Ν Ξ στερεὸς ἐπὶ τὸν ἐκ τῶν Ζ Η Θ γενόμενος δεκάκις, εἰ δὲ δύο λείψει, ἑκατοντάκις [γενόμενος ὁ εἰρημένος στερεός]. εἰ δὲ τρεὶς λείψει, ὅσων ὁ ἐκ τῶν Κ Λ Μ Ν Ξ ἐπὶ τὸν ἐκ τῶν Ζ Η Θ χιλιάκις γενόμενος ἔσται μονάδων, τοσούτων μυριάδων ὁμωνύμων τῷ Ο. τὸ δὲ γραμμικὸν ἐκ τοῦ στοιχείου δῆλον.

[**Proposition 21**] But let there be more than three numbers A B Γ Δ E and each being less than a hundred and divisible by ten[1], let each of Z H Θ be less than ten[2].

Let the multitude of A B Γ Δ E be first divided by four measured by O[3] and let K Λ M N Ξ be the puthmenes of A B Γ Δ E[4]. I say that the product of A B Γ Δ Z H Θ, *(which)* is myriads to the power of O[5], is equal to the units from out of the product of K Λ M N multiplied with the product of Z H Θ[6].

It is clear from the arithmetical processes, what with assuming, for the sake of example, A under 10 monades, B under 20 monades, Γ under 30 monades, Δ under 40 monades[7], what with the puthmenes K Λ M N being 1, 2, 3, 4[8]. Consequently, the product of A B Γ Δ is 24 straight myriads[9], the (product) of A B Γ Δ Z H Θ is 144 straight myriads[10], the product of the puthmenes K Λ M N is 24[11]. The result being multiplied with the product of Z H Θ, being 6[12], it makes 144 monades[13], which is as many as the straight myriads of the product of A B Γ Δ Z H Θ[14], because the multitude of A B Γ Δ E measures by four once[15].

But let the multitude of A B Γ Δ E not be divided by four, when dividing it shall leave a certain 1, 2, or 3[16]. If it leaves 1, the product of A B Γ Δ E Z H Θ of myriads to the power of O, it is equal to the product of K Λ M N Ξ multiplied with *(the product)* of Z H Θ ten-times the result[17], if it leaves 2, a hundred-times [the product created above]. If it leaves 3, of so many myriads the product of K Λ M N Ξ multiplied with the product of Z H Θ is a thousand-times the monades, the result is as many as many myriads to the power of O. The metrical analysis from the elements is clear.

κβʹ. Ἔστω ὁ μὲν Α ἐλάσσων μὲν χιλιάδος μετρούμενος δὲ ὑπὸ ἑκατοντάδος, ἕκαστος δὲ τῶν Β Γ Δ ἐλάσσων δεκάδος, καὶ δέον ἔστω τὸν ἐκ τῶν Α Β Γ Δ στερεὸν εἰπεῖν.

Κείσθω γὰρ τοῦ μὲν Α πυθμὴν ὁ Ε, ὁ δὲ ἐκ τῶν Ε Β Γ Δ ὁ Ζ· ὅτι ὁ ἐκ τῶν Α Β Γ Δ στερεὸς ἑκατοντάκις ἐστὶν ὁ Ζ.

Φανερὸν δὲ καὶ τοῦτο διὰ τῶν ἀριθμῶν, τοῦ Α ὑποκειμένου, φέρ᾽ εἰπεῖν, μονάδων τʹ καὶ τοῦ Β μονάδων γʹ καὶ τοῦ Γ μονάδων δʹ καὶ τοῦ Δ μονάδων εʹ· ὁ μὲν γὰρ ὑπὸ τῶν Α Β Γ Δ ἐστὶν μ^{α} ͵η, ὁ δὲ ὑπὸ τῶν Ε Β Γ Δ ἐστὶν μονάδων ρπʹ· οὗτος δὲ γενόμενος ἑκατοντάκις ἔσται μ^{α} ͵η. τὸ δὲ γραμμικὸν ἐκ τοῦ στοιχείου δῆλον.

[Proposition 22] Let A be less than a thousand and divisible by a hundred[1], each of B Γ Δ is less than ten[2] and let it be necessary to give the product of A B Γ Δ.

Let E be made the puthmenes of A[3], Z is the product from out of E B Γ Δ[4]. *(I say)* that the product of A B Γ Δ is Z times a hundred[5].

It is clear from the arithmetical procedures, what with A being assumed to be, for the sake of example, under 300 monades and of B under 3 monades, of Γ under 4 monades, of Δ under 5 monades[6]. The *(product)* of A B Γ Δ is 18,000[7]. Thus the *(product)* of E B Γ Δ is 180[8]. This result a hundred times is 18,000[9]. The metrical analysis from the elements is clear.

κγʹ. Ἐπὶ δὲ τοῦ κδʹ Θεωρήματος. Τοῦ Α ὑποκειμένου λόγου χάριν μονάδων σʹ καὶ τοῦ Β μονάδων τʹ καὶ τοῦ Γ μονάδων βʹ τοῦ δὲ Δ μονάδων γʹ καὶ τοῦ Ε μονάδων δʹ, ὁ στερεὸς ἐξ αὐτῶν ἔσται μυριάδων ἁπλῶν ρμδʹ, ἐπεὶ τὸ διπλάσιον τοῦ πλήθους τῶν Α Β μετρεῖται ὑπὸ τετράδος ἅπαξ [κατὰ τὸν Κ],

ὁ δὲ ὑπο τῶν Ζ Η πυθμένων καὶ τῶν Γ Δ Ε ἐστιν μονάδων ρμδʹ.

Ἐὰν δὲ τὸ διπλάσιον τοῦ πλήθους τῶν Α Β μὴ μετρῆται ὑπὸ τετράδος, δῆλον ὅτι μετρούμενον κατὰ τὸν Κ λείψει δύο· τοῦτο γὰρ ἀνώτερον ἐδείχθη. διὰ δὴ τοῦτο [ἐκ τοῦ λείπεσθαι δύο] μυριάδες εἰσὶν ἑκατὸν ὁμώνυμοι τῷ Κ, καὶ ἔστιν ὁ ἐκ τῶν Α Β Γ Δ Ε στερεὸς ὁ Θ ἴσος τῷ ἐκ τῶν Ζ Η Γ Δ Ε στερεῷ ἐπὶ τὰς ἑκατὸν μυριάδας ὁμωνύμους τῷ Κ. τὸ γραμμικὸν ὡς Ἀπολλώνιος.

[**Proposition 23**] In the case of theorem 24. What with A being assumed, for the sake of example, to be under 200 monades, of B being under 300 monades, of Γ being under 2 monades, of Δ being under 3 monades, of E being under 4 monades[1], the product of these *(numbers)* will be 144 straight myriads[2]. Since the square of the multitude of A B is divisible [exactly] by four [measured by K][3].

The *(product)* of the puthmenes Z H and of Γ Δ E is 144[4].

If the square of the myriads of A B is not divisible by four, if it is clear that division *(by)* K will leave 2, for this was proven above. Because of this [from the two remaining], a hundred myriads is to the power of K[5], and Θ is the product of A B Γ Δ E[6], which is equal to the product of Z H Γ Δ E multiplied by a hundred myriads to the power of K[7]. The metrical analysis *(as per)* Apollonius.

κδ′. Ἐπὶ δὲ τοῦ κε′ θεωρήματος. Ἔστω τῶν μὲν Α Β ἑκάτερος ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, ἕκαστος δὲ τῶν Γ Δ Ε [ἔστω] ἐλάσσων δεκάδος, καί δέον ἔστω τὸν ἐξ αὐτῶν στερεὸν εἰπεῖν.

Ἔστωσαν γὰρ τῶν Α Β πυθμένες οἱ Θ Κ, καὶ τῷ ἐκ τῶν Θ Κ Γ Δ Ε στερεῷ ἴσος ἔστω ὁ Λ· ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ε στερεὸς ἴσος ἐστὶν ἑκατὸν τοῖς Λ.

Ἔστι δὲ φανερὸν διὰ τὼν ἀριθμῶν, τοῦ Α ὄντος μονάδων κ′ καὶ τοῦ Β μονὰδων κ′, καὶ τοῦ Γ μονάδων ε′ καὶ τοῦ Δ μονάδων ς′ καὶ τοῦ Ε μονάδων ζ′, καὶ τῶν Θ Κ πυθμένων ὄντων μονάδων Β′·

ὁ γὰρ ὑπὸ τῶν Θ Κ Γ Δ Ε γίνεται στερεὸς μονάδων ωμ′, οὗτος δὲ ἑκατοντάκις γενόμενος ἔσται μυριάδων η′ μονάδων ͵δ, ἴσος τῷ ἐκ τῶν Α Β Γ Δ Ε στερεῷ ἀριθμῷ.

[Proposition 24] In the case of theorem 25. Let each of A B be less than a hundred and divisible by ten[1], Let each of Γ Δ E be less than ten[2]. Let it be necessary to give the product from these *(numbers)*.

Let Θ K be the puthmenes of A B[3], and let Λ be equal to the product of Θ K Γ Δ E[4]; *(I say)* that the product of A B Γ Δ E is equal to a hundred Λs[5].

It is clear from the arithmetical procedures, of A being under 20 monades, of B being under 20 monades, of Γ being under 5monades, of Δ being under 6 monades, of E being under 7 monades[6] and what with the puthmenes Θ K being under 2 monades[7].

The product of Θ K Γ Δ E is going to be 840[8]. This result a hundred times is 8 myriads and 4000 monades[9], which is equal to the product of A B Γ Δ E in number[10].

κε΄.α. Τὸ δ᾽ ἐπὶ πᾶσι θεώρημα κς΄ πρότασιν ἔχει καὶ ἀπόδειξιν τοιαύτην. Ἔστωσαν δύο ἀριθμοὶ ἢ πλείους οἱ Α Β, ὧν ἕκαστος ἐλάσσων μὲν χιλιάδος μετρούμενος δὲ ὑπὸ ἑκατοντάδος, καὶ ἄλλοι ἀριθμοὶ ὁσοιδήποτε οἱ Γ Δ Ε, ὧν ἕκαστος ἐλάσσων μὲν ἑκατοντάδος μετρούμενος δὲ ὑπὸ δεκάδος, καὶ ἄλλοι πάλιν ὁσοιδηποτοῦν ἀριθμοὶ οἱ Ζ Η Θ, ὧν ἕκαστος ἐλάσσων δεκάδος, καὶ δέον ἔστω τὸν ἐκ τῶν Α Β Γ Δ Ε Ζ Η Θ στερεὸν εἰπεῖν.

Ἔστωσαν γὰρ τῶν Α Β Γ Δ Ε πυθμένες οἱ Λ Μ Ν Ξ Ο. ὁ δὴ διπλάσιος τοῦ πλήθους τῶν Α Β μετὰ τοῦ τῶν Γ Δ Ε ἁπλοῦ ἀριθμοῦ ἤτοι μετρεῖται ὑπὸ τετράδος ἢ οὔ.

Μετρείσθω πρότερον ὑπὸ τετράδος κατὰ τὸν Κ, καὶ ὑποτετάχθωσαν τοῖς μὲν Α Β ἑκατοντάδες αἱ Π Ρ, τοῖς δὲ Γ Δ Ε δεκάδες αἱ Σ Τ Υ .

καὶ φανερὸν ὅτι ὁ ἐκ τῶν Π Ρ Σ Τ Υ ἐπὶ τὸν ἐκ τῶν Λ Μ Ν Ξ Ο ἴσος ἐστὶ τῷ ἐκ τῶν Α Β Γ Δ Ε στερεῷ . εἰλήφθω δὴ ὁ ἐκ τῶν Λ Μ Ν Ξ Ο Ζ Η Θ στερεὸς καὶ ἔστω ὁ Φ· ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ε Ζ Η Θ στερεὸς μυριάδες εἰσὶν τοσαῦται ὁμώνυμοι τῷ Κ ὅσαι μονάδες εἰσὶν ἐν τῷ Φ. τοῦτο δὲ γραμμικῶς Ἀπολλώνιος ἀπέδειξεν.

Ἐὰν δὲ ὁ διπλάσιος τοῦ πλήθους τῶν Α Β μετὰ τοῦ πλήθους τῶν Γ Δ Ε μὴ μετρῆται ὑπὸ τετράδος, μετρούμενος ἄρα κατὰ τὸν Κ λείψει ἢ ἕνα ἢ δύο ἢ τρεῖς. εἰ μὲν οὖν ἕνα λείψει, ὁ ἐκ τῶν Π Ρ Σ Τ Υ στερεὸς μυριάδες εἰσὶν δέκα ὁμώνυμοι τῷ Κ, εἰ δὲ δύο, μυριάδες ἑκατὸν ὁμώνυμοι τῷ Κ, εἰ δὲ τρεῖς, μυριάδες χίλιαι ὁμώνυμοι τῷ Κ.

καὶ δῆλον ἐκ τῶν γεγραμμένων ὅτι ὁ ἐκ τῶν Α Β Γ Δ Ε Ζ Η Θ στερεὸς μυριάδες εἰσὶν τοσαῦται, ὅσος ὁ δεκαπλάσιος τοῦ Φ, ὁμώνυμοι τῷ Κ ἀριθμῷ , ἢ ὅσος ὁ ἑκατονταπλάσιος τοῦ Φ, ὁμώνυμοι τῷ Κ, ἢ ὅσος ὁ χιλιαπλάσιος τοῦ Φ, ὁμώνυμοι τῷ Κ.

[**Proposition 25. Part 1**] The 26th Theorem contains this theorem and the following proof. Let A B be two or more numbers, each being less then a thousand and divisible by a hundred[1], and let however so many of the other numbers Γ Δ E, each being less than a hundred and divisible by ten and again[2], however so many number Z H Θ, each being less than ten[3]. And let is be necessary to give the product of A B Γ Δ E Z H Θ.

Let Λ M N Ξ O be the puthmenes of A B Γ Δ E[4]. The square of the multitude of A B plus the straight numbers of Γ Δ E will either be divisible by four or not[5].

Let them be divided first by four, measured by K[6], and let Π P substitute the hundreds of A B[7], Let Σ T Y *(substitute)* the tens of Γ Δ E[8].

It is clear that the *(product)* of Π P Σ T Y multiplied with *(the product)* of Λ M N Ξ O[9], which is equal to the product of A B Γ Δ E[10]. Let the product of Λ M N Ξ O Z H Θ be taken and let that be Φ[11]. *(I say)* that the product of A B Γ Δ E Z H Θ is so many myriads to the power of K is as many as the monades that are in Φ[12]. Apollonius proved this metrically.

If the square of the multitudes of A B is added with the multitude of Γ Δ E, the sum should not be divisible by four, thus dividing as measured by K will leave either 1, 2 or 3[13]. If one is left, the product of Π P Σ T Y is ten myriads to the power of K[14], if two is left, a hundred myriads to the power of K[15], if three is left, 1000 myriads to the power of K[16].

It is clear from the metrical analysis that the product from A B Γ Δ E Z H Θ is so many myriads equal to ten-fold Φ to the power of the number of K or equal to a hundred-fold Φ to the power of K or equal to a thousand-fold Φ to the power of K[17].

κε΄.β. Τούτου δὴ [τοῦ θεωρήματος]προτεθεωρημένου πρόδηλον, πῶς ἔστιν τὸν δοθέντα στίχον πολλαπλασιάσαι καὶ εἰπεῖν τὸν γενόμενον ἀριθμὸν ἐκ τοῦ τὸν πρῶτον ἀριθμὸν ὃν εἴληφε τὸ πρῶτον τῶν γραμμάτων ἐπὶ τὸν δεύτερον ἀριθμὸν ὃν εἴληφε τὸ δεύτερον τῶν γραμμάτων πολλαπλασιασθῆναι καὶ τὸν γενόμενον ἐπὶ τὸν τρίτον ἀριθμὸν ὃν εἴληφε τὸ τρίτον γράμμα καὶ κατὰ τὸ ἑξῆς περαίνεσθαι μέχρι τοῦ διεξοδεύεσθαι τὸν στίχον, ὃν εἶπεν Ἀπολλώνιος ἐν ἀρχῇ [κατὰ τὸν στίκον] οὕτως.

Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι
(τὸ δὲ κλεῖτέ φησιν ἀντὶ τοῦ ὑπομνήσατε).

Ἐπεὶ οὖν γράμματα ἐστιν λη΄ τοῦ στίχου, ταῦτα δὲ περιέχει ἀριθμοὺς δέκα τοὺς ρ΄ τ΄ σ΄ τ΄ ρ΄ τ΄ σ΄ χ΄ υ΄ ρ΄, ὧν ἕκαστος ἐλάσσων μέν ἐστιν χιλιάδος μετρεῖται δὲ ὑπὸ ἑκατοντάδος, καὶ ἀριθμοὺς ιζ΄ τοὺς μ΄ ι΄ ο΄ κ΄ λ΄ ι΄ κ΄ ο΄ ξ΄ ο΄ ο΄ ν΄ ν΄ ν΄ κ΄ ο΄ ι΄, ὧν ἕκαστος ἐλάσσων μέν ἐστιν ἑκατοντάδος μετρεῖται δὲ ὑπὸ δεκάδος, καὶ τοὺς λοιποὺς [σὺν ταῖς μονάσιν] ια΄ τοὺς α΄ ε΄ δ΄ ε΄ ε΄ α΄ ε΄ ε΄ ε΄ α΄ α΄, ὧν ἕκαστος ἐλάσσων δεκάδος, ἐὰν ἄρα [τοὺς δέκα ἀριθμοὺς διπλασιάσωμεν καὶ τοὺς γενμένους κ΄ προσθῶμεν τοῖς εἰρημένοις ἁπλῶς ἀριθμοῖς ἑπτακαίδεκα, τὰ γενόμενα ὁμοῦ λζ΄ ἕξομεν τῶν ὑπ᾽ αὐτοῦ γενομένων ἀναλόγων,

κἄν] τοῖς μὲν δέκα ἀριθμοῖς ὑποτάξωμεν ἰσαρίθμους δέκα κατὰ τάξιν ἑκατοντάδος, τοῖς δὲ ιζ΄ ὁμοίως ὑποτάξωμεν δεκάδας ιζ΄, φανερὸν ἐκ τοῦ ἀνώτερον λογιστικοῦ θεωρήματος ιβ΄ ὅτι δέκα ἑκατοντάδες μετὰ τῶν ιζ΄ δεκάδων ποιοῦσι μυριάδας ἐνναπλᾶς δέκα. [αἱ γὰρ δέκα ἑκατοντάδες δὶς γενόμεναι, τουτέστιν κ΄, καὶ προσλαβοῦσαι τὰς ιζ΄ δεκάδας γίνονται λζ΄ ἀναλόγων ὄντα· μερισθέντα δὲ τὰ λζ΄ εἰς τὸν δ΄ ποιεῖ τὸν᾽ εκ τοῦ μερισμοῦ θ΄ καὶ καταλείπεται α΄, ὡς εἶναι μυριάδας ἐνναπλᾶς δέκα τὰ ἐκ τῶν ἑκατοντάδων δέκα καὶ δεκάδων ιζ΄.] Ἐπεὶ δὲ καὶ πυθμένες ὁμοῦ τῶν μετρουμένων ἀριθμῶν ὑπὸ ἑκατοντάδος καὶ τῶν μετρουμένων ὑπὸ δεκάδος εἰσὶν οἱ ὑποκείμενοι κζ΄: α΄ γ΄ β΄ γ΄ α΄ γ΄ β΄ ς΄ δ΄ α΄ δ΄ α΄ ζ΄ β΄ γ΄ α΄ β΄ ζ΄ς΄ ζ΄ ζ΄ ε΄ ε΄ ε΄ β΄ ζ΄ α΄, ἀλλὰ καὶ τῶν ἐλασσόνων δεκάδος εἰσὶν ια΄, τουτέστιν ἀριθμοὶ οἱ α΄ ε΄ δ΄ ε΄ ε΄ α΄ ε΄ ε΄ ε΄ α΄ α, [**Proposition 25. Part 2**] What with this previous contemplation of how the given line is multiplied and to give the desired number from the first number which is taken from the first letter and multiplied with the second number which is taken from the second letter and the result is multiplied with the third number which is taken from the third letter and so forth to continue until the verse ends, which Apollonius gives in the beginning.

Celebrate the excellent power of Artemis 9 maidens (Celebrate, they say, for the sake of memory).

Since there are 38 letters in the line *(in Greek)*, this contains about 10 letters; $\rho, \tau, \sigma, \tau, \rho, \tau, \sigma$,-$\chi, \upsilon, \rho$, each being less than a thousand and divisible by a hundred and 17 numbers; $\mu, \iota, \emptyset, \kappa, \lambda$,-$\iota, \kappa, \emptyset, \xi, \emptyset, \emptyset, \nu, \nu, \nu, \kappa, \emptyset, \iota$, each being less than a hundred and divisible by ten, and the remaining 11; $\alpha, \epsilon, \delta, \epsilon, \epsilon, \alpha, \epsilon, \epsilon, \epsilon, \alpha, \alpha$, are each less than ten. [If we should double the tens *(of the hundreds)* and if we added *(the quantity of)* the resulting 20 *(multitudes)* to the aforementioned 17 straight numbers, together the result we have is 37 of the desired proportion by those *(numbers)*]

We substitute for ten *(numbers)* ten equivalent numbers to the order of a hundred, we likewise substitute the 17 *(numbers)* with 17 tens. It is clear from the 12 previous theorems that the ten hundreds added with the 17 tens makes nine and ten myriads [for ten hundreds multiplied by two, this is 20, and taking the 17 tens becomes 37 equivalently; dividing 37 by four makes 9 from the division and leaves behind 1, thus nine and ten myriads to be from ten hundreds and 17 tens] *(numbers)* divided by ten, there are 27 digits laid down; $\alpha, \gamma, \beta, \gamma, \alpha, \beta, \varsigma, \delta, \alpha, \delta, \delta, \zeta, \beta, \gamma, \alpha, \beta, \zeta, \varsigma, \zeta, \zeta, \epsilon, \epsilon, \epsilon, \beta, \zeta, \alpha$, but there are 11 *(numbers)* less than ten, there numbers are $\alpha, \epsilon, \delta, \epsilon, \epsilon, \alpha, \epsilon, \epsilon, \epsilon, \alpha, \alpha$.

ἐὰν τὸν ἐκ τούτων τῶν ια′ καὶ τὸν ἐκ τῶν κζ′ πυθμένων στερεὸν δι᾿ ἀλλήλων πολλαπλασιάσωμεν, ἔσται ὁ στερεὸς μυριάδων τετραπλῶν ιθ′ καὶ τριπλῶν ϛλϛ′ καὶ διπλῶν ,ηυπ′

[Ἴσος δὲ τούτῳ συνάγεται καὶ ὁ διὰ τῶν τοῦ στίχου πυθμένων ἅμα ταῖς μονάσιν

Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι,

οἵ εἰσιν α′ α′ γ′ ε′ δ′ α′ δ′ ζ′ β′ β′ γ′ ε′ α′ γ′ ε′ β′ α′ α′ γ′ ζ′ β′ ε′ ϛ′ ζ′ ϛ′ ζ′ ε′ ε′ ε′ ε′ ε′ α′ β′ ζ′ δ′ α′ α′ α′.

ἕν γὰρ ἐπὶ α′ γίνεται α′
ἐπὶ γ′ γίνεται γ′
ἐπὶ ε′ γίνεται ιε′
ἐπὶ δ′ γίνεται ξ′
ἐπὶ α′ γίνεται ξ′
ἐπὶ δ′ γίνεται σμ′
ἐπὶ ζ′ γίνεται ,αχπ′
ἐπὶ β′ γίνεται ,γτξ′
ἐπὶ β′ γίνεται ϛψκ′
ἐπὶ γ′ γίνεται $μ^{α}$ β′ καὶ $μ^{ο}$ ρξ′
ἐπὶ ε′ γίνεται $μ^{α}$ ι′ καὶ $μ^{ο}$ ω′
ἐπὶ α′ γίνεται $μ^{α}$ ι′ καὶ $μ^{ο}$ ω′
ἐπὶ γ′ γίνεται $μ^{α}$ λ′ καὶ $μ^{ο}$,βυ′
ἐπὶ ε′ γίνεται $μ^{α}$ ρνα′ καὶ $μ^{ο}$,β
ἐπὶ β′ γίνεται $μ^{α}$ τβ′ καὶ $μ^{ο}$,δ
ἐπὶ α′ γίνεται $μ^{α}$ τβ′ καὶ $μ^{ο}$,δ
ἐπὶ α′ γίνεται $μ^{α}$ τβ′ καὶ $μ^{ο}$,δ

If we might multiply the product from out of the 11 with the product of the 27 puthmenes, the product will be $19 \times 10{,}000^4$, $6,036 \times 10{,}000^3$ and $8,480 \times 10{,}000^2$ When the puthmenes from the numbers divided by a hundred are added with those from the.

[From those taken together and the product through the puthmenes of the verse and the units Celebrate the excellent power of Artemis nine maidens

The *(numbers are)* $\alpha, \alpha, \gamma, \epsilon, \delta, \alpha, \delta, \zeta, \beta, \beta, \gamma, \epsilon, \alpha, \gamma, \epsilon, \beta, \alpha, \alpha, \gamma, \zeta, \beta, \epsilon, \varsigma, \zeta, \varsigma, \zeta, \epsilon, \epsilon, \epsilon, \epsilon, \varsigma, \alpha$,-$\beta, \zeta, \delta, \alpha, \alpha, \alpha$.

for one multiplied by 1 produces 1
multiplied by 3 produces 3
multiplied by 5 produces 15
multiplied by 4 produces 60
multiplied by 1 produces 60
multiplied by 4 produces 240
multiplied by 7 produces 1680
multiplied by 2 produces 3360
multiplied by 2 produces 6720
multiplied by 3 produces $2 \times 10,000$ and 160
multiplied by 5 produces $10 \times 10,000$ and 800
multiplied by 1 produces $10 \times 10,000$ and $8,000$
multiplied by 3 produces $30 \times 10,000$ and $2,400$
multiplied by 5 produces $151 \times 10,000$ and $2,000$
multiplied by 2 produces $302 \times 10,000$ and $4,000$
multiplied by 1 produces $302 \times 10,000$ and $4,000$
multiplied by 1 produces $302 \times 10,000$ and $4,000$

ἐπὶ γʹ γίνεται μ^{α} ϡλζʹκαὶ μ^{ο} ͵β

ἐπὶ ζʹ γίνεται μ^{α} ϛτνʹκαὶ μ^{ο} ͵δ

ἐπὶ βʹ γίνεται μ^{β} αʹ καὶ μ^{α} ͵βψʹκαὶ μ^{ο} ͵η

ἐπὶ εʹ γίνεται μ^{β} ϛʹ καὶ μ^{α} ͵γφδʹ

ἐπὶ ϛʹ γίνεται μ^{β} ληʹ καὶ μ^{α} ͵ακδʹ

ἐπὶ ζʹ γίνεται μ^{β} ϛξʹϛʹ καὶ μ^{α} ͵ζρξηʹ

ἐπὶ ϛʹ γίνεται μ^{β} ͵αχʹκαὶ μ^{β} ͵γηʹ

ἐπὶ ζʹ γίνεται μ^{γ} αʹ καὶ μ^{β} ͵ασβʹκαὶ μ^{α} ρʹνϛʹ

ἐπὶ εʹ γίνεται μ^{γ} εʹ καὶ μ^{β} ϛιʹ καὶ μ^{α} ͵εσπʹ

ἐπὶ εʹ γίνεται μ^{γ} κηʹ καὶ μ^{β} νβʹ καὶ μ^{α} ϛυʹ

ἐπὶ εʹ γίνεται μ^{γ} ρμʹ καὶ μ^{β} σξγʹ καὶ μ^{α} ͵β

ἐπὶ εʹ γίνεται μ^{γ} ψʹ καὶ μ^{β} ͵ατιʹϛʹ

ἐπὶ εʹ γίνεται μ^{γ} ͵γφʹκαὶ μ^{β} ϛφπʹ

ἐπὶ αʹ γίνεται μ^{γ} ͵γφʹκαὶ μ^{β} ϛφπʹ

ἐπὶ βʹ γίνεται μ^{γ} ͵ζαʹκαὶ μ^{β} ͵γρξʹ

ἐπὶ ζʹ γίνεται μ^{δ} δʹ καὶ μ^{γ} ͵θθʹκαὶ μ^{β} ͵βρκʹ

ἐπὶ δʹ γίνεται μ^{δ} ιθʹ καὶ μ^{γ} ϛλϛʹ καὶ μ^{β} ͵ηυπʹ]

Αὗται δὴ συμπολλαπλασιαζόμεναι ἐπὶ τὸν ἐκ τῶν ἑκατοντάδων καὶ δεκάδων στερεόν, τουτέστι τὰς προκειμένας μυριάδας ἐνναπλὰς δέκα, ποιοῦσιν μυριάδας τρισκαιδεκαπλᾶς π;ϛʹ, δωδεκαπλᾶς τξηʹ, ἑνδεκαπλᾶς ͵δωʹ. [ἐνναπλαῖ γὰρ μυριάδες ἐπὶ μὲν τετραπλᾶς ποιοῦσι τρισκαιδεκαπλᾶς, ἐπὶ δὲ τριπλὰς γενόμεναι ποιοῦσιν δωδεκαπλᾶς, καὶ ὁμοίως ἐπὶ διπλᾶς πολλαπλασιασθεῖσαι γίνονται ἑνδεκαπλαῖ] ταῦτα γὰρ πάντα προδέδεικται.

Φατέον οὖν τὸν ἐξ ἀρχῆς στίχον

multiplied by 3 produces $907 \times 10,000$ and $2,000$

multiplied by 7 produces $6,350 \times 10,000$ and $4,000$

multiplied by 2 produces $1 \times 10{,}000^2$ and $2,700 \times 10,000$ and $8,000$

multiplied by 5 produces $6 \times 10{,}000^2$ and $3,504 \times 10,000$

multiplied by 6 produces $38 \times 10{,}000^2$ and $1,024 \times 10,000$

multiplied by 7 produces $266 \times 10{,}000^2$ and $7,168 \times 10,000$

multiplied by 6 produces $1,600 \times 10{,}000^2$ and $3,008 \times 10,000$

multiplied by 7 produces $1 \times 10{,}000^3$ and $1,202 \times 10{,}000^2$ and $1,056 \times 10,000$

multiplied by 5 produces $5 \times 10{,}000^3$ and $6,010 \times 10{,}000^2$ and $5,280 \times 10,000$

multiplied by 5 produces $28 \times 10{,}000^3$ and $52 \times 10{,}000^2$ and $6,400 \times 10,000$

multiplied by 5 produces $140 \times 10{,}000^3$ and $263 \times 10{,}000^2$ and $2,000 \times 10,000$

multiplied by 5 produces $700 \times 10{,}000^3$ and $1,316 \times 10{,}000^2$

multiplied by 5 produces $3,500 \times 10{,}000^3$ and $6,580 \times 10{,}000^2$

multiplied by 1 produces $3,500 \times 10{,}000^3$ and $6,580 \times 10{,}000^2$

multiplied by 2 produces $7,001 \times 10{,}000^3$ and $3,160 \times 10{,}000^2$

multiplied by 7 produces $4 \times 10{,}000^4$ and $9,009 \times 10{,}000^3$ and $2120 \times 10{,}000^2$

multiplied by 4 produces $19 \times 10{,}000^4$ and $6,036 \times 10{,}000^3$ and $8,480 \times 10{,}000^2$

]

These things being multiplied together with the product of the hundreds and the tens, this is to say nine and ten myriads makes $196 \times 10{,}000^{13}$, $368 \times 10{,}000^{12}$, $4,800 \times 10{,}000^{11}$ [for nine straight myriads multiplied by four makes 13 straight myriads, multiplied by three makes 12 and when multiplying proportionals by two, it makes 11 straight myriads.

It is necessary to explain the verse from the beginning;

Ἀρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι

πολλαπλασιασθέντα δι᾽ ἀλλήλων δύνασθαι μυριάδων πλῆθος τρισκαιδεκαπλῶν ρ;ς′, δωδεκαπλῶν τξη′, ἑνδεκαπλῶν ͵δω′, συμφώνως τοῖς ὑπὸ Ἀπολλωνίου κατὰ τὴν μέθοδον ἐν ἀρχῇ τοῦ βιβλίου προγεγραμμένοις. Πάλιν δεδόσθω στίχος ὁ ὑποκείμενος

Μῆνιν ἄειδε θεὰ Δημήτερος ἀγλαοκάρπου, καὶ εἰλήφθω τά τε ἀνάλογα καὶ οἱ πυθμένες ἅμα ταῖς μονάσιν ὥσπερ ὑπόκεινται

δ′ η′ ε′ α′ ε′ α′ ε′ α′ δ′ ε′ ϑ′ ε′ α′ δ′ η′ δ′ η′ γ′ ε′ α′ ζ′ β′ α′ γ′ γ′ α′ ζ′ β′ α′ α′ η′ ζ′ δ′,
καὶ πεπολλαπλασιάσθωσαν δι᾽ ἀλλήλων οἱ ἀριθμοί· γίνονται τετραπλαῖ μυριάδες δύο, τριπλαῖ ͵αωμϑ′, διπλαῖ ͵δυβ′, ἁπλαῖ ͵εχ′.

Τέσσαρες γὰρ μ^{o} ἐπὶ η′ γίνεται λβ′

ἐπὶ ε′ γίνεται ρξ′

ἐπὶ μίαν γίνεται ρξ′

ἐπὶ ε′ γίνεται ω′

ἐπὶ α′ γίνεται ω′

ἐπὶ ε′ γίνεται ͵δ

ἐπὶ μίαν γίνεται ͵δ

ἐπὶ ε′ γίνεται μ^{α} α′ καὶ μ^{o} ς

ἐπὶ ε′ γίνεται μ^{α} η′

ἐπὶ ϑ′ γίνεται μ^{α} οβ′

ἐπὶ ε′ γίνεται μ^{α} τξ′

ἐπὶ α′ γίνεται μ^{α} τξ′

ἐπὶ δ′ γίνεται μ^{α} ͵αυμ′

ἐπὶ η′ γίνεται μ^{β} α′ καὶ μ^{α} ͵αφκ′

ἐπὶ δ′ γίνεται μ^{β} δ′ καὶ μ^{α} ςπ′

ἐπὶ η′ γίνεται μ^{β} λς′ καὶ μ^{α} ͵ηχμ′

ἐπὶ γ′ γίνεται μ^{β} ρι′ καὶ μ^{α} ͵ε϶κ′

Celebrate the excellent power of Artemis nine maidens

multiplying *(each letter)* respectively *(to one another)* produces a multitude of myriads $196 \times 10{,}000^{13}$, $368 \times 10{,}000^{12}$, $4,800 \times 10{,}000^{11}$, in harmony with the works of Apollonius according to the method in the beginning of the book. Again, let an established line be given

Sing, Goddess, the wrath of Demeter, bearer of beautiful fruit
Let both the proportion be taken and the puthmenes be made like the units simultaneously;

$\delta, \eta, \epsilon, \alpha, \epsilon, \alpha, \epsilon, \alpha, \delta, \epsilon, \theta, \epsilon, \alpha, \delta, \eta, \delta, \eta, \gamma, \epsilon, \alpha, \zeta, \beta, \alpha, \gamma, \gamma, \alpha, \zeta, \beta, \alpha, \alpha, \eta, \zeta, \delta$
and the numbers having been multiplied through each other make $2 \times 10{,}000^{4}$, $1,849 \times 10{,}000^{3}$, $4,402 \times 10{,}000^{2}$, $5,600 \times 10,000$.

four multiplied by 8 makes 32
multiplied by 5 produces 150
multiplied by 1 produces 160
multiplied by 5 produces 800
multiplied by 1 produces 800
multiplied by 5 produces 4,000
multiplied by 1 produces 4,000
multiplied by 4 produces $1 \times 10,000$ and $6,000$
multiplied by 5 produces $8 \times 10,000$
multiplied by 9 produces $72 \times 10,000$
multiplied by 5 produces $360 \times 10,000$
multiplied by 4 produces $1,440 \times 10,000$
multiplied by 8 produces $1 \times 10{,}000^{2}$ and $1,520 \times 10,000$
multiplied by 4 produces $4 \times 10{,}000^{2}$ and $6,080 \times 10,000$
multiplied by 8 produces $36 \times 10{,}000^{2}$ and $8,640 \times 10,000$
multiplied by 3 produces $110 \times 10{,}000^{2}$ and $5,920 \times 10,000$

ἐπὶ ε΄ γίνεται μ^{β} φνβ΄ καὶ μ^{α} ,ϑχ΄

ἐπὶ α΄ γίνεται μ^{β} φνβ΄ καὶ μ^{α} ,ϑχ΄

ἐπὶ ζ΄ γίνεται μ^{β} ,γωο΄καὶ μ^{α} ,ζσ΄

ἐπὶ β΄ γίνεται μ^{β} ,ζψμα΄καὶ μ^{α} ,δυ΄

ἐπὶ α΄ γίνεται μ^{β} ,ζψμα΄καὶ μ^{o} ,δυ΄

ἐπὶ γ΄ γίνεται μ^{γ} β΄ καὶ μ^{β} ,γσκδ΄ καὶ μ^{α} ,γσ΄

ἐπὶ γ΄ γίνεται μ^{γ} ς΄ καὶ μ^{β} ,ϑχοβ΄καὶ μ^{α} ,ϑχ΄

ἐπὶ α΄ γίνεται μ^{γ} ς΄ καὶ μ^{β} ,ϑχοβ΄καὶ μ^{α} ,ϑχ΄

ἐπὶ ζ΄ γίνεται μ^{γ} μη΄ καὶ μ^{β} ,ζψι΄καὶ μ^{α} ,ζσ΄

ἐπὶ β΄ γίνεται μ^{γ} ϙζ΄καὶ μ^{β} ,ευκα΄καὶ μ^{α} ,δυ΄

ἐπὶ α΄ καὶ πάλιν ἐπὶ ά γίνεται μ^{γ} ϙζ΄καὶ μ^{β} ,ευκα΄καὶ μ^{α} ,δυ΄

ἐπὶ η΄ γίνεται μ^{γ} ψπ΄ καὶ μ^{β} ,γτοα΄καὶ μ^{α} ,εσ΄

ἐπὶ ζ΄ γίνεται μ^{γ} ,ευξβ΄καὶ μ^{β} ,γχ΄καὶ μ^{β} ςυ΄

ἐπὶ δ΄ γίνεται μ^{δ} β΄, τριπλαῖ ,αωμϑ΄, διπλαῖ ,δυβ΄, ἁπλαῖ ,εχ΄.

τῶν δὴ ἀναλόγων κβ΄ καὶ μετρουμένων ὑπὸ τετράδος [καὶ δυάδος ὑπολειπομένης] ὅσαι μονάδες γεγόνασι [μέτρωεὶς ε΄], τοσαυτάκις αὐξήσομεν τὸν ἐκβάντα διά τε τῶν μονάδων καὶ [διὰ τῶν πεπολλαπλασιασμένων] πυθμένων ἀριθμόν (λέγω δὲ τοσαυτάκις κατὰ μυριάδων αὔξησιν), ὥστε γίνεσθαι τὸν πρότερον ὑπάρχοντα μυριάδων τετραπλῶν δύο, τριπλῶν ,αωμϑ΄, διπλῶν ,δυβ΄καὶ ἁπλῶν ,εχ΄, νῦν ἐνναπλῶν β΄, ὀκταπλῶν, ,αωμϑ΄, ἑπταπλῶν ,δυβ΄, ἑξαπλῶν ,εχ΄.

Ὅτι δὲ περιλέλειπται τῶν ἀναλόγων δύο, ἅπερ ἐστὶ τῆς ἑκατοντάδος, τοςαυτάκις αὐξὴσομεν τὸν εἰρημένον ἀριθμόν, ὥστε εἶναι μυριάδων ἐνναπλῶν σιη΄, ὀκταπλῶν ,δϡμδ΄, ἑπταπλῶν σνς΄.

Ῥητέον οὖν τὸν ἐξ ἀρχῆς στίχον

Μῆνιν ἄειδε ϑεά Δημέτερος ἀγλαοκάρπου

πολλαπλασιασϑέντα δύνασϑαι μυριάδων πλῆϑος ἐνναπλῶν σιη΄, ὀκταπλῶν ,δϡμδ΄, ἑπταπλῶν σνς΄.

multiplied by 5 produces $552 \times 10{,}000^2$ and $9,600 \times 10,000$

multiplied by 1 produces $552 \times 10{,}000^2$ and $9,600 \times 10,000$

multiplied by 7 produces $3,870 \times 10{,}000^2$ and $7,200 \times 10,000$

multiplied by 2 produces $7,741 \times 10{,}000^2$ and $4,400 \times 10,000$

multiplied by 1 produces $7,741 \times 10{,}000^2$ and $4,400 \times 10,000$

multiplied by 3 produces $2 \times 10{,}000^3$ and $3,224 \times 10{,}000^2$ and $3,200 \times 10,000$

multiplied by 3 produces $6 \times 10{,}000^3$ and $9,672 \times 10{,}000^2$ and $9,600 \times 10,000$

multiplied by 1 produces $6 \times 10{,}000^3$ and $9,672 \times 10{,}000^2$ and $9,600 \times 10,000$

multiplied by 7 produces $48 \times 10{,}000^3$ and $7,710 \times 10{,}000^2$ and $7,200 \times 10,000$

multiplied by 2 produces $97 \times 10{,}000^3$ and $5,421 \times 10{,}000^2$ and $4,400 \times 10,000$

multiplied by 1 produces $97 \times 10{,}000^3$ and $5,421 \times 10{,}000^2$ and $4,400 \times 10,000$

multiplied by 8 produces $780 \times 10{,}000^3$ and $3,374 \times 10{,}000^2$ and $5,200 \times 10,000$

multiplied by 7 produces $5,462 \times 10{,}000^3$ and $3,600 \times 10{,}000^2$ and $6,400 \times 10,000$

multiplied by 4 produces $2 \times 10{,}000^4$, $1,849 \times 10{,}000^3$, $4,402 \times 10{,}000^2$, $5,600 \times 10,000$

What with 22 proportionals being divided by four [and leaving a remainder of two] as many units result [divisible by 5], so many we increase that which comes from the number of units [and through multiplication] the number of puthmenes. (I call so may time the increases according to the myriads) so that the previous result was $2 \times 10{,}000^4$, $1,849 \times 10{,}000^3$, $4,402 \times 10{,}000^2$, $5,600 \times 10,000$ is now $2 \times 10{,}000^9$, $1,849 \times 10{,}000^8$, $4,402 \times 10{,}000^7$, $5,600 \times 10{,}000^6$

I say that two remainders from the proportion which indeed is 100, increasing so many time the aforementioned number. Let it be $218 \times 10{,}000^9$, $4,944 \times 10{,}000^8$, $256 \times 10{,}000^7$

It is necessary to say the verse from the beginning
Sing the wrath of the Goddess Demeter, bearer of the beautiful fruit.
Having been multiplied, a multitude of myriads is produced; $218 \times 10{,}000^9$, $4,944 \times 10{,}000^8$, $256 \times 10{,}000^7$

8.3 General Commentary

[Notes on the Translation] The translation provided is based on the ancient Greek transcribed by Frederick Hultsch in Volume 1 of **Pappus: Collection**. Some of the interpolations which he noted have been removed.

The manuscript tradition of the **Collection** in Europe is well documents in Treweek's Ph.D Thesis **A Critical Edition of the Text of the Collection of Pappus of Alexandria**, his article *Pappus of Alexandria: The Manuscript Tradition of the Collectio Mathematica*, Sabetai Unguru's *Pappus in the Thirteenth Century in the Latin West*, and Alexander Jones' **Book 7 of the Collection**.

There are approximately 40 manuscripts in Europe and America that include a portion of the **Collection**[2] The Vaticanus Graecus 218 is the earliest of these, dating between the 9th and 10th centuries.[3] It consists of 202 folia of parchment but five folia of paper from the 16th-18th centuries. The manuscript begins with an incomplete 6th century work on trick mirrors by Anthemius of Tralles. The rest of the surviving manuscript has been discussed in **Setting the Scene** but there are some addition details to add. When the Vaticanus was first produced, Book 8 is believed to have lost.[4] However, the impression left on the first page of the extant Pappus manuscript supports the hypothesis that the rest of Book 2 was originally copied into the Vaticanus and was lost sometime later due to water damage.

An an exemplar, the Vaticanus is very useful for following the transmission and translation of Book 2. Book 2 is included in 18 of the 44 manuscripts listed by Jones and Treweek. These are listed below. The sigla follow the standard of Treweek, who followed Hultsch. The listing of the date, exemplars, and presence of diagrams follows Jones' style, although it is based on Treweek.

[2] Jones(1986), 30.
[3] Jones(1986), 206.
[4] Jones(1986), 35.

A-Vaticanus: 9th-10th century with inset diagrams.

R-Argentoratensis: < 1544. Exemplar was A but was lost in the bombing of Strasbourg.

P-Oxon Savile: 6th century. Exemplar was R and does not contain diagrams.

E-Escorial T ill, y 17: 1547-1548. Exemplar is A but has no diagrams.

B-Paris gr. 2440: < 1554. Exemplars are R and E. Contains marginal diagrams

Y-Vind Sup. gr. 40: > 1574. Exemplar is B and contains no diagrams.

x: ¡1554. Exemplar was A but is lost. We know it exists because John Wallis referred to in in his 1680s translation of Book 2.

V-Leifen Voss. F. 18: 16th century. The exemplar appears to have been x. Contains inset diagrams.

C-Paris gr. 2368: 1562. Also has x as an exemplar and contains inset diagrams.

S-Leiden Scal.3: > 1562. C is the exemplar and contains inset diagrams.

K-Neap.III c 14: 16th century. E is the exemplar. Contains inset diagrams.

M-Amber.C 266 inf.: 16th century. R is one of many exemplars. No diagrams.

Z-Bn. Burney 105: < 1588. M one of two exemplars. Contains a set of diagrams that does not include a set for Book 2.

I-Laur.Plut. 28,17: 16th century. M is an exemplar. No diagrams.

-**Oxon. Chch 86**: 1688-1710. P is one of two exemplars. Contains inset diagrams.

-Oxon. Savile 60A: 1772. C is the exemplar. No diagrams.

Wa-Oxon Savile 60B: 1680s. P is one of two examplars. No diagrams. This is the manuscript Wallis created for his translation.

Hultsch used manuscripts ABVS for his translation of Books 2 to 7 but it was not until late in his translating that he came across manuscript A. He also included the emendations from John Wallis' translation. In the 1680s, John Wallis was the first person to translate Book 2 in its extant form. Treweek's translation is also based on Hultsch's but increases the manuscripts for Book 2 to included PMCK as well as ABVS. Hultsch and Wallis' emendations.

For all of his extensive research into the Vaticanus and its descendants, Treweek was vague about the content and accuracy of the diagrams, both in his thesis and in his later monographs. With regards to the relevant manuscripts, there is no detail beyond inset, marginal, or no comment on the matter. There is no insight as to whether all Books in the manuscript contained diagrams or merely a few propositions. This is a relevant concern because Hultsch made the editorial decision to omit the diagrams from his translation because of the confusion between the diagram and the text, as introduced in **Sing the Wrath of Demeter**. Treweek makes some criticisms as to the confusion between the diagrams but never about their connection to the text. It would be valuable to know if these diagrams originated in the Vaticanus or from a later manuscript. For the reasons listed in the chapter named above, these diagrams are not included here.

As seen through numerous mathematical examples in this text, the style of the translation and mathematical commentary emulates that of Jones. Sections of the Greek are followed by the English translation. Inset within the English are numbers that refer the reader to the relevant mathematical information in the commentary, as determined by the proposition. There are also some comments about editorial and translation choices.

Order and Content of Book 2 Propositions	
Proposition	*Details*
[14]	Multiplication of complete numerical series in which each of the six numbers is divisible by ten.
[15]	Multiplication of several within a series, in which each of the numbers is divisible by a hundred. First instance where myriads are used to organized the multitudes. First instance in which the most numbers are represented alphabetically.
[16]	Multiplication of two numbers, one divisible by a hundred and another divisible by ten. First reference to Apollonius.
[17]	Multiplication of two separate series of numbers, one series divisible by a hundred and the other divisible by ten. Refers to Apollonius. Unlike proposition 15, demonstrates how the magnitude of the result increases with each possible remainder. Labeled as Theorem 18.
[18]	Theorem 19. Multiplication of five numbers, one divisible by ten and four less than ten.
[19]	Multiplication of five numbers, two divisible by ten and three less than ten. Refers to Apollonius.
[20]	Multiplication of six numbers, three divisible by ten and three less than ten.
[21]	Multiplication of eight numbers, five divisible by ten and three less than ten. First instance of reference to Euclid's *Elements*.
[22]	Multiplication of four numbers, one divisible by a hundred and three less than ten. Reference made to *Elements*.
[23]	Theorem 24. Multiplication of five numbers, two divisible by a hundred and three less than ten. Reference made to Apollonius.
[24]	Theorem 25. Multiplication of five numbers, two divisible by ten and three less than ten.
[25]	Theorem 26. Part 1) Multiplication of eight numbers, two divisible by a hundred, three divisible by ten and three less than ten. Reference made to Apollonius. Part 2) Each letter contained within a hexameter line, attributed to Apollonius, is multiplied in a series by its corresponding number. This is repeated a second time with a verse selected by Pappus.

8.4 Mathematical Commentary

[Proposition 14]

[1]Five numbers: 50, 50, 50, 40, 40, 30

[2]*Puthmenes*: 5, 5, 5, 4, 4, 3

[3]Product of puthmenes: $5 \times 5 \times 5 \times 4 \times 4 \times 3 = 6000$

[4]Number of multitudes: 6 $(10, 10, 10, 10, 10, 10)$

[5]$6 \div 4 = 2$ as in $10^6 \div 104 = 2$ [6]Product of multitudes: $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1,000,000$ or $100 \times 10,000$

[7]Multitudes $\times$ puthmenes: $(100 \times 10,000 \times 6000) = 60 \times 10,000^2$

[1.1] Divisible ($\mu\varepsilon\tau\varrho\acute{\varepsilon}\omega$) - In mathematics, the more precise definition is 'to measure'. However, Aristotle, Euclid and Nicomachus apply it was a term for dividing or specifically measuring magnitudes. In Nicomachus especially, it comes to mean the number of times a number is expressed and produced.[5]

[1.2] Product ($\acute{o}$ $\sigma\tau\varepsilon\varrho\varepsilon\acute{o}\varsigma$) - In his translation of proposition 17, Netz translates this as a 'solid number'; Cuomo, instead, also translates this as product.[6] In Lidell-Scott, the precedent is for the translation to be 'solid' but can also refer to sums and quantities.

[1.3] Give ($\lambda\acute{\varepsilon}\gamma\omega$) - Cuomo and Netz keep to the more familiar definition of 'to say'. The selection of 'to give' is for aesthetic reasons.[7]

Multiply ($\pi o\lambda\lambda\alpha\pi\lambda\alpha\sigma\iota\acute{\alpha}\zeta\omega$) -

[1.4] Numbers ($\acute{o}$ $\alpha\varrho\iota\vartheta\mu\acute{o}\varsigma$) - A natural,positive numbar consisting of one or more digits

[1.5] Puthmenes ($\acute{o}$ $\pi\upsilon\vartheta\mu\acute{\eta}\nu$) - The puthmenes are quotients, without remainder, of the numbers that are divisible by ten or hundred. The exact translation of puthmenes is unclear. Thomas Heath calls such numbers 'bases', as they are factors of smaller numbers which are

[5]Nicomachus(1938), 301

[6]Netz(2008): Cuomo(2000), 181-182

[7]Cuomo(2000), 181-182

less than ten.[8] Netz also calls them 'base numbers', while Cuomo translates the word as 'basic numbers'.[9] However, I feel the word base rather understates the specificity of the numbers Pappus discusses. It is sufficient until one must deal with numbers less than ten that can be broken down into smaller while numbers, ie. $6 = 3 \times 2$, would 3 or 2 be also described as 'base' numbers? Puthmenes is elegant and specifically labels what Pappus intended for the reader to organize in all numerical cases.

[**1.6**] Units (*μοναδόν*) - Used when letters are assigned numerical values and parts of products that are less than 10,000

Multitude (*πλθος*) - The quantity of 10s and 100s that remain when all the puthmenes in a series or a number is removed.

[**1.7**] By four (*Ὑπὸ τετράδος*) - In this case, since there are six numbers that are divisible by ten, the product of the six tens will be 1,000,000. If this is divided by four, 10^4 or $10,000$, it will leave two, 10^2 or 100. The product of the multitudes can be reexpressed as $100 \times 10,000$.

[**1.8**] Straight Myriad (*μυριάδων ἁπλῶν*) - A single $10,000$ unit. Straight, later, is used in proposition 25 to denote a series of three numbers that were divisible by ten. The variation in usage might be due to the 'straight' set of zeroes present.

[**1.9**] Multiply (*ἐπί*) - A shorthand used outside the preamble of most propositions. The context of this preposition in the sentence supports this interpretation. Netz also applied this usage to this brief translation of proposition 25.[10]

[**1.10**] Make (*ποιέω*) - While Pappus does not abandon the use of 'to make' to not the result in later chapters, he does transition to terms that better express equality in the sense of $1 + 2 = 3$. The connotation here is more of a construction.

From the start(*εξ ἀχῆς*) - Despite the lengthy explanation that proceeds from the limitations Pappus places on multiplication, this line establishes that direct method of multiplication did occur. It is also possible that in the previous, but lost, propositions, Pappus had built up to this combination of number, thereby the new product would be easily deduced if a three was substituted for a thirty or an additional fifty was added. The former scenario is most probable.

[8] Cuomo(2000), 181-182; Heath(1921)i, 54-57
[9] Netz(2008): Cuomo(200), 181-182
[10] Netz(2008)

[Proposition 15]

[1]-$SeriesB = B_1,...\ , B_n; < 1000$, divisible by 100

[2]-When n=number of multitudes, Multitudes = 100^n, $(100^n)^2 \div 10^4 = 10^0$.

[3]-$SeriesB\ \div 100 = B_1 \div 100,...\ , B_n \div 100$

[4]-$Series\Gamma = SeriesB\ \div 100$

[5]-Puthmenes $= SeriesB\ \div 100$ therefore Puthmenes$= Series\Gamma$

[6]-$\Gamma_1,...\ ,\Gamma_n = E$

[7]-$B_1 \times ...\ \times B_n = 120 \times 10,000^2$

[8]-$B_1 \times ...\ \times B_n = 100^n \times E = 10,000^2 120$ therefore $100^n = 10,000^2$ and $n = 4$.Thus $SeriesB = B_1 B_2 B_3 B_4$

[9]-Suppose $(100^n)^2$ is not divisible by 4

[10]-$(100^n)^2 \div 10^4 = 10^2$

[11]-$(100^n)^2 \div (10^4)^2 = 10^2$

[12]-$100^n = 100 \times 10,000^z$, z=2

[13]- $B_1,...\ , B_n = 100 \times 10,000^z \times E = 12,000 \times 10,000^2$

In the first case, if $E = 120$, then $G_1 = 2$, $G_2 = 3$, $G_3 = 4$, $G_4 = 5$ and $B_1 = 200$,$B_2 = 300$, $B_3 = 400$, $B_4 = 500$.

However, in the second case there is a remainder of 2 or 10^2,

Therefore Series B contains 5 quantities, $B_5 = 100$

Thus, $B_1 \times\ ...\ \times B_5 = 1,200 \times 10,000^2$.

[2.14] Again(*πάλιν*) - This is not a reference to proposition 14 as the arrangement of numbers in 15 is different from 14. Thus, this is likely a reference to a now lost proposition that was similar in its treatment of numbers within a series as 15.

From which we get series B(*ἐφ᾽ ὧν τὰ B*) - A read-through reveals that B does not represent a single number, as subsequent propositions, but a series of numbers. Cuomo translates this as 'series labelled by the As'; Netz, 'That on which are the As'.[11] Again, I feel my translation, from which we get series..., both stays true to the mathematical integrity of the proposition

[11] Cuomo(2000), 181-182: Netz(2008)

and clarifies what was written in the Greek.

[2.15] Series B - 100, 200, 300, 400, 500

[2.19] Let it be put below(*ὑποκεισθω*) - It seems Pappus expected the reader to write out the equation as they were solving it. This is just a place-holding instruction

[**2.1**] Series G - 1, 2, 3, 4, 5. Since 100 contributed to the magnitude of the result only, it is temporarily removed until the end and expands the result a hundredfold.

[**2.4**] Workings(*ἡ γραμμή*)

[3.15] (Power(*ὁμωνυμος*) - In this section, Pappus establishes a pattern for writing out large results, as demonstrated later; $E \times 10{,}000^{\mathrm{Z}}$. The use of Z in particular is random and is replaced by other letter as the quantity of numbers multiplied increases.

[Proposition 16]

[1]-A is < 1000, $\div 10$; $A = 500$

[2]-B is < 100, $\div 10$. B $= 40$

[3]-Puthmenes: A_p=5; B_p=4. Therefore multitudes: A_m=100; B_m=10

[4]-$A_p \times B_p$=20

[5]-20×1000=$20,000$; 1000=$A_m \times B_m$

[6]-$A \times B$=500×40=$20,000$

[4.1-4.5] It is clear through the arithmetic...linear description(*τῶν ἀριθμῶν ... δὲ γραμμικὸν*) - The difference between arithmetical and linear demonstrations depends on how one chooses to interpret how Pappus utilizes those differences. In Netz's opinion, Greeks only used numbers when the presentation was opaque.[12] When he speculated as to the original Apollonian text, he believes the proofs were initially vague and entirely geometrical. The vagueness in the text would have been clarified by a collection of lines in a separate diagram. The arithmetic also served another purpose, as a climax to the 'word game' that concludes proposition 25. That much is fairly clear from the text itself. Cuomo interprets Book 2 as containing several examples of arithmetizations, the assignation of specific numbers to a general proposition, which already had been established.[13] This reformulation is useful as a method to convince the reader that a proof does or does not work (ie. Book 3). Moreover, it is the exercise of the arithmetic that was as important to this persuasion as the proof itself. Cuomo does not attempt to recreate Apollonius' possible diagrams.

[Proposition 17]

[1]-Series $A = A_1, ..., A_n$; < 100, $\div 10$

[2]-Series $B = B_1, ..., B_n$; < 1000, $\div 100$

[12]Netz(2008)

[13]Cuomo(2000), 180-184

[3]-Series $A\div10$=Series$H$; Series H=1, 2, 3,4 therefore Series A=10, 20, 30, 40

[4]Series $B\div100$=SeriesΘ; Series Θ=2, 3, 4,5 therefore Series B=200, 300, 400, 500

[5]-Series$H\times$ Series$\Theta=E$

[6]-E=2880

[7]-(Multitude of Series A)$\times$(Multitude of Series B)=$10^4\times100^4=10^4\times10^8=10^12$ or $10,000^3$

[8]-Multitude of Series A)$\times$(Multitude of Series B) is evenly divisible by 4

[9]-$A\times B=E\times10,000^z$

[10]-$2800\times10,000^3$

[11]-Multitude of Series A)$\times$(Multitude of Series B)$\div4=1$

[12]-$A\times B=10,000^z\times(E\times10)$

[13]-If remainder is 2, $A\times B=10,000^z\times(E\times100)$

[14]-If remainder is 3, $A\times B=10,000^z\times(E\times1000)$

[5.6] Behold, Theorem 18(*Ἐπὶ δὲ τοῦ ιη′ θεωρήματος*) - Book 2 is largely seen as a commentary of a lost text by Apollonius. However, there are two propositions that are drawn from Euclid's *Elements* as well: Propositions 21 and 22. Proving that Pappus based his work on more than one text. Still, since the propositions that refer to Euclid do so explicitly as do all propositions that refer to Apollonius, it is more than reasonable to say that the theorem more the likely was derived from Apollonius.

[Proposition 18]

[1]-$A < 100, \div 10$

[2]-$B\Gamma\Delta E < 10$

[3]-$A \div 10 = Z$

[4]-$Z \times B \times \Gamma \times \Delta \times E = H$

[5]-$10 \times H = AB\Gamma\Delta E$

[6]-$A = 20$; $B = 3$; $\Gamma = 4$; $\Delta = 5$; $E = 6$

[7]-$AB\Gamma\Delta E = 7200$

[8]-$Z = 2$

[9]-$ZB\Gamma\Delta E \times 10 = 7200$

[10]-$AB\Gamma\Delta E = 7200$

[Proposition 19]

[1]-$A, B < 100$, $\div 10$

[2]-$\Gamma\Delta E < 10$

[3]-$A \div 10 = Z$; $B \div 10 = H$

[4]-$AB\Gamma\Delta E = 100 \times ZH\Gamma\Delta E$

[5]-$A = 20$; $B = 30$; $\Gamma = 2$; $\Delta = 3$; $E = 4$;

[6]-$Z = 2$; $H = 3$

[7]-$AB\Gamma\Delta E = 14,400$

[8]-$ZH\Gamma\Delta E = 144$

[9]-$ZH\Gamma\Delta E \times = 14,400$

[Proposition 20]

[1]-$AB\Gamma < 100$, $div 10$

[2]-$\Delta EZ < 10$

[3]-$A \div 10 = H$; $B \div 10 = \Theta$; $\Gamma \div 10 = K$

[4]-$H\Theta K\Delta EZ = \Xi$

[5]-$AB\Gamma\Delta EZ = 1000 \times \Xi$

[6]-$A = 20$; $B = 30$; $\Gamma = 40$; $\Delta = 2$; $E = 3$; $Z = 4$

[7]-$H = 2$; $\Theta = 3$; $K = 4$ [8]-$AB\Gamma\Delta EZ = (57 \times 10,000) + 6000$

[9]-$H\Theta K\Delta EZ = 576$

[10]-$H\Theta K\Delta EZ \times 1000 = 576 \times 1000 = (57 \times 10,000) + 6000$

[Proposition 21]

[1]-$AB\Gamma\Delta E < 100$, $\div 10$

[2]-$ZH\Theta < 10$

[3]-Multitude of $AB\Gamma\Delta E$ $\div 4 = $o

[4]-$A \div 10 = K$; $B \div 10 = \Lambda$; $\Gamma \div 10 = M$; $\Delta \div 10 = N$; $E \div 10 = \Xi$

[5]-$AB\Gamma\Delta EZH\Theta = n \times 10,000^o$

[6]-$n = K\Lambda MN \times ZH\Theta$

[7]-$A = 10$; $B = 20$; $\Gamma = 30$; $\Delta = 40$

[8]-$K = 1$; $\Lambda = 2$; $M = 3$; $N = 4$

[9]-$AB\Gamma\Delta = 24 \times 10,000$

[10]-$AB\Gamma\Delta ZH\Theta = 144 \times 10,000$

[11]-$K\Lambda MN = 24$

[12]-$ZH\Theta = 6$

[13]-$K\Lambda MNZH\Theta = 144$

[14]-$K\Lambda MNZH\Theta \cong AB\Gamma\Delta ZH\Theta$

[15]-Multitude of $AB\Gamma\Delta E$ $\div 4 = 10^0$

[16]-But 4 does not evenly divide into the multitudes of $AB\Gamma\Delta E$. Remainders are 1,2, or 3

[17]-If 1, $AB\Gamma\Delta EZH\Theta = n \times 10,000^o \cong K\Lambda MN\Xi ZH\Theta \times 10,000^o \times 10$

[18]-If 2, $K\Lambda MN\Xi ZH\Theta \times 10,000^o \times 100$

[19]-If 3, $K\Lambda MN\Xi ZH\Theta{\times}10,000^{o}{\times}1000$

[11.3] Propositions 21 and 22 both refer to Euclid's *Elements* as the source of the proof in the proposition.

[Proposition 22]

[1]-$A < 1000$, $\div 10$

[2]-$B\Gamma\Delta < 10$

[3]-$A{\div}100{=}E$

[4]-$EB\Gamma\Delta{=}Z$

[5]-$AB\Gamma\Delta{=}Z{\times}100$

[6]-$A = 300$; $B = 3$; $\Gamma = 4$; $\Delta = 5$; $E = 3$

[7]-$AB\Gamma\Delta{=}18,000$

[8]-$EB\Gamma\Delta{=}180$

[9]-$EB\Gamma\Delta{\times}100{=}18,000$

[Proposition 23]

[1]-$A = 200$; $B = 300$; $\Gamma = 2$; $\Delta = 3$; $E = 4$

[2]-$AB\Gamma\Delta{=}144{\times}10,000$

[3]-Multitude of $AB{\div}4{=}10,000^{K}{=}10,000$

[4]-$ZH\Gamma\Delta E{=}144$

[5]-If multitude of AB is not divisible by 4, if it leaves 2,$AB\Gamma\Delta E{=}n{\times}100{\times}10,000^{K}$

[6]-$\Theta{=}AB\Gamma\Delta E$

[7]-$\Theta{=}ZH\Gamma\Delta E{\times}100{\times}10,000^{K}$

[Proposition 24]

[1]-A, $B < 100$, $\div 10$

[2]-Γ, Δ, $E < 10$

[3]-$A \div 10 = \Theta$; $B \div 10 = K$

[4]- [4]-$\Theta K \Gamma \Delta E = \Lambda$

[5]-$AB\Gamma\Delta E = 100 \times \Lambda$

[6]-$A = 20$; $B = 20$; $\Gamma = 5$; $\Delta = 6$; $E = 7$

[7]-$\Theta = 2$; $K = 2$

[8]-$\Theta K \Gamma \Delta E = 840$

[9]-$840 \times 100 = 84,000$

[10]-$84,000 = AB\Gamma\Delta E$

[Proposition 25]

Part 1

[1]-A, $B < 1000$, $\div 100$; could be more than two quantities

[2]-Γ, Δ, $E < 100$, $\div 10$

[3]-$ZH\Theta < 10$

[4]-$A \div 100 = \Lambda$; $B \div 100 = M$; $\Gamma \div 10 = N$; $\Delta \div 10 = \Xi$; $E \div 10 = O$

[5]-Multitudes of $AB\Gamma\Delta E$ may be divisible by 4

[6]-$AB\Gamma\Delta E \div = 10,000^K$

[7]-Multitude of $A = \Pi$; Multitude of $B = P$; Π, $P = 100$

[8]-Multitude of $\Gamma = \Sigma$; $\Delta = T$; $E = U$; Σ, T, $U = 10$

[9]-$\Pi P \Sigma T U \Lambda M \Xi O$

[10]-$\Pi P \Sigma T U \Lambda M \Xi O = AB\Gamma\Delta E$

[11]-$\Lambda M \Xi O Z H \Theta = \Phi$

[12]-$AB\Gamma\Delta E Z H \Theta = \Phi \times 10,000^K$:$\Phi$

[13]-(Multitudes of AB)+(Multitudes of $\Gamma\Delta E$) is not divisible by 4; $10^3 \times 10^4 = 10^7$. Then $10^7 \div 10^4 = 10^3$

[14]-If 1 remainder, $\Pi P \Sigma T U = 10 \times 10,000^K$

[15]-If 2 remainder, $100 \times 10,000^K$

[16]-If 3 remainder, $1000 \times 10,000^K$

[17-$AB\Gamma\Delta EZH\Theta = \Phi \times 10,000^K$ and $\times$ either 10, 100, or 1000

Part 2 This part of the proposition is straightforward.

[16.12] Addition($\mu\epsilon\tau\alpha$)

[23.3] Proportional(*ἀνάλογα*) - Alternate way of describing multitudes.

Celebrate the excellent power of Artemis 9 maidens. Calculations of the line organized by myriads

Multiplied by	$10,000^4$	$10,000^3$	$10,000^2$	$10,000$	*units*
5					15
4					60
4					240
7					1680
2					3360
2					6720
3				2	0160
5				10	0800
3				30	2400
5				151	2000
2				302	4000
3				907	2000
7				6350	4000
2			1	2700	8000
5			6	3504	0000
6			38	1024	0000
7			266	7168	0000
6			1600	3008	0000
7		1	1202	1056	0000
5		5	6010	5280	0000
5		28	0052	6400	0000
5		140	0263	2000	0000
5		700	1316	0000	0000
5		3500	6580	0000	0000
2		7001	3160	0000	0000
7	4	9009	2120	0000	0000
4	19	6036	8480	0000	0000

Sing the wrath of the Goddess Demeter, Bearer of the beautiful fruit.

Multiplied by	$10,000^4$	$10,000^3$	$10,000^2$	$10,000$	*units*
8					32
5					160
5					800
5					4000
4				1	6000
5				8	0000
9				72	0000
5				360	0000
4				1440	0000
8			1	1520	0000
4			4	6080	0000
8			36	8640	0000
3			110	5920	0000
5			552	9600	0000
7			3870	7200	0000
2			7741	4400	0000
3		2	3224	3200	0000
3		6	9672	9600	0000
7		48	7710	7200	0000
2		97	5421	4400	0000
8		780	3371	5200	0000
7		5462	3600	6400	0000
4	2	1849	4402	5600	0000

Part IV

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Chapter 9

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