ADVANCES IN ADAPTIVE CONTROL THEORY: GRADIENT– AND DERIVATIVE–FREE APPROACHES

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ADVANCES IN ADAPTIVE CONTROL THEORY: GRADIENT– AND DERIVATIVE–FREE APPROACHES

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To my wife Gulfem Ipek Yucelen,

the origin of my love and passion.

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SUMMARY

In this dissertation, we present new approaches to improve standard designs in adaptive control theory, and novel adaptive control architectures.

We first present a novel Kalman filter based approach for approximately enforcing a linear constraint in standard adaptive control design. One application is that this leads to alternative forms for well known modification terms such as e–modification. In addition, it leads to smaller tracking errors without incurring significant oscillations in the system response and without requiring high modification gain. We derive alternative forms of e– and adaptive loop recovery (ALR–) modifications.

Next, we show how to use Kalman filter optimization to derive a novel adaptation law. This results in an optimization-based time-varying adaptation gain that reduces the need for adaptation gain tuning.

A second major contribution of this dissertation is the development of a novel *derivative-free*, delayed weight update law for adaptive control. The assumption of constant unknown ideal weights is relaxed to the existence of time-varying weights, such that fast and possibly discontinuous variation in weights are allowed. This approach is particulary advantageous for applications to systems that can undergo a sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, docking, or structural damage, and for rejection of external disturbance processes.

As a third and final contribution, we develop a novel approach for extending all the methods developed in this dissertation to the case of output feedback. The approach is developed only for the case of derivative-free adaptive control, and the extension of the other approaches developed previously for the state feedback case to output

feedback is left as a future research topic.

The proposed approaches of this dissertation are illustrated in both simulation and flight test.

CHAPTER I

Introduction

1.1 Model Reference Adaptive Control

Research in adaptive control is motivated by the fact that models employed in control system design may not adequately represent the actual system dynamics due to idealized assumptions, linearization, model order reduction, external disturbances, and degraded modes of operation. Although robust control design approaches can deal with these sources of uncertainty, they may fail to satisfy performance requirements in the face of high levels of uncertainty. On the other hand, adaptive controllers are able to deal with uncertainty without necessarily sacrificing performance. Furthermore, they require less modeling information than do robust controllers. These facts make adaptive control theory important for engineering applications.

Adaptive controllers can be classified as either direct or indirect. Direct adaptive controllers adapt feedback gains in response to system variations without requiring a parameter estimation algorithm. This property distinguishes them from indirect adaptive controllers which employ an estimation algorithm to estimate the unknown system parameters and adapt the controller gains. This dissertation presents developments for the well known and the most important class of direct adaptive controllers, namely model reference adaptive controllers (MRAC) [4, 29, 33, 36, 38, 44, 49, 61, 68, 71–73].

MRAC has three major components: Reference model, weight (gain) update law, and controller. The reference model specifies the desired behavior of the closed-loop system. The output (resp., state) of an uncertain system is compared to the output (resp., state) of the reference model. This comparison results in an error signal used in the weight update law. The controller employs the weight information from the weight update law to form the adaptive control signal.

MRAC was developed by Whitaker and his colleagues around 1958 [4]. Ref. 64 presented this novel idea based on a gradient method. In this reference, the weight update law is constructed as the negative gradient of a cost function chosen as the square of the norm of the error signal. This procedure drives the trajectories of the uncertain system to the trajectories of the reference model as time goes to infinity. Butchart and Shackcloth [9] and Parks [65] analyzed the stability of this gradient method for the first time using Lyapunov stability theory [54, 55]. Other notable early achievements on MRAC can be found in Refs. 25, 26, 28, 57, 58, 62 and references therein.

In this dissertation, we present not only new approaches to improve standard MRAC designs but also novel adaptation laws and MRAC architectures.

1.2 Linear Constraints and Modification Terms

In the literature, adaptation laws that impose constraints on the weights to improve an existing adaptive law are commonly referred as composite adaptation [71]. In general, the modification terms are found by taking the gradient of a norm of the constraint violation. However, using a gradient method can result in slow parameter convergence towards a local minimum [8]. In addition, modification terms that are gradient based have a fixed adaptation gain, that often have to be chosen large to obtain satisfactory results. The use of a high modification gain can interact negatively with unmodeled dynamics, and amplify the effect of sensor noise.

Many modification terms are reported in the literature [10, 20, 37, 47, 60, 63, 75–77, 84, 90, 91]. Included among these, σ–modification [37] adds pure damping to the adaptive weight update law and turns it into a lag filter. The negative aspect about σ–modification term is that it inhibits the adaptation process. By contrast, e–modification adds variable damping that is proportional to the norm of the error signal. When the error signal is large, it also inhibits the adaptation process. The positive aspect of both σ– and e–modification terms is that they ensure a bounded weight history.

Concurrent learning [20] uses current and past data concurrently in the adaptation process. It allows the adaptation law to continually train in the background based on past data while still being responsive to dynamic changes based on the current data. In this way, concurrent learning incorporates long term learning. Q modification [75–77,90] is similar in spirit to concurrent learning in its intent to improve adaptation performance by using a moving window of the integrated system uncertainty. There is an optimal control theory based modification that improves adaptation in the presence of large adaptive gain [63]. An adaptive loop recovery (ALR) approach [10] has been introduced as a modification with the objective of recovering the loop transfer properties of the reference model. More recently, $\mathcal{K}$ modification [47] has been proposed to add stiffness to the weight update law. When it is used with σ– or e–modification terms, the transient response of the weight history can be controlled by selection of a natural frequency and damping ratio.

In this dissertation all of these modification terms are viewed as having been introduced to reduce the violation of a linear constraint on the weights in an adaptive control algorithm. With this perspective in mind, a Kalman filter (KF) optimization method is developed to arrive at alternative forms for all previously mentioned methods of modification. We develop this approach in the context of approximately enforcing a linear constraint in a standard MRAC design. The approach is shown to result in smaller tracking errors without requiring a high modification gain. The approach is illustrated using a simplified model of aircraft roll dynamics at high angles of attack.

1.3 Linear Constraints and Weight Update Laws

In the previous section we stated that the KF approach can be used for approximately enforcing a linear constraint in a standard MRAC design which leads to alternative forms for well known modification terms. An extension of this approach results in a novel Kalman filter based weight update law for adaptive control. We first show that the standard MRAC weight update law can be viewed as enforcing a linear constraint using gradient optimization. Then, we use KF optimization to replace the standard weight update law with its KF form. This approach results in an optimal time-varying adaptation gain that allows one to achieve a given performance criteria without the need for excessive adaptation gain tuning. The simulation results show that it leads to smaller tracking errors without incurring high frequency oscillations in the system response.

1.4 Derivative-Free Adaptive Control

Standard MRAC approaches are based on Lyapunov stability theory and either assume or derive a weight update law in the form of an ordinary differential equation for the weight estimates. All these methods have in common the underlying assumption that there exists a constant, but unknown, ideal set of weights. Although this assumption seems reasonable and these MRAC architectures work well on many systems, in some failure modes they may require the use of unrealistically high adaptation gain, or may fail to achieve the desired level of performance in terms of failure recovery. MRAC laws that require high gain can excite unmodeled dynamics, typically exhibit an excessive amount of control activity [81,82], amplify the effect of sensor noise, and are not sufficiently robust to time delay [63].

In this dissertation, we develop a derivative-free model reference adaptive control (DF-MRAC) law, which uses the information of delayed weight estimates and the information of current system states and errors [16, 86–89]. The proposed method is an extension of the iterative learning approach adopted in Ref. 17 for purposes of adaptive observer design. We relax the assumption of constant unknown ideal weights to the existence of time-varying weights, such that fast and possibly discontinuous variation in weights are allowed. The DF-MRAC law is advantageous for applications to systems that can undergo a sudden change in dynamics. We prove that the error dynamics are uniformly ultimately bounded using a Lyapunov-Krasovskii functional [31], without employing modification terms in the adaptive law. We consider constant unknown ideal weights as a special case and show that the state tracking error dynamics are asymptotically stable. Finally, we discuss employing various modification terms for further improving the performance and robustness of the adaptively controlled system.

DF-MRAC differs from MRAC in that it does not make use of integration in its weight update law. In fact, DF-MRAC challenges the conventional wisdom of expecting an adaptive law to cancel the effect of matched constant disturbances and uncertainties, since one does not need adaptation to correct for biases. Biases can more reliably be handled using a non-adaptive controller with integral action. Most conventional flight control systems employ integral action for this purpose. Therefore it is advantageous when augmenting an existing flight control system with an adaptive element, that the adaptive law not employ an integral function of the tracking error. Otherwise, there is the potential for a conflict to arise between the existing controller and the augmenting adaptive element. The nature of this conflict is often seen as a slowly varying tracking error that is difficult to remove by adjusting the parameters in the adaptive law. On the other hand, if for some reason it is desirable to treat the effect of biases with a standard MRAC weight update law, then in this case the underlying existing (nominal) controller being augmented with an adaptive element should employ only proportional control. However, it is highly advantageous that the nominal part of the control contain integral action since this ensures that the errors in the regulated output variables go to zero for constant disturbances and constant model errors even if they are unmatched, so long as the closed loop system remains stable. Therefore, since many existing guidance and flight controllers do employ integral action, and if the goal is to augment an existing controller, then it is not desirable that the weight update law have the effect of integral action. In these circumstances, the role of adaptation should be confined to things like maintaining stability and error transient performance, fast upset recover in the event there is a destabilizing failure, and not degrading the time delay margins of the nominal design. DF-MRAC is particularly well suited for these circumstances.

Quantification of transient dynamics, guaranteed transient and steady-state performance bounds, and gain/time-delay sensitivity at the system inputs and outputs are three open problems in adaptive control theory. An analysis based on Singular Perturbation Theory is used to quantify the transient dynamics of an adaptively controlled system in Ref. 52. A similar analysis can be applied to derivative-free adaptive control which should provide a similar result. With regard to the second problem, we show that DF-MRAC does have guaranteed transient and steady-state performance bounds. In order to reduce gain/time-delay sensitivity, we combined derivative-free adaptive control with ALR-modification as proposed in Ref. 10. It is shown in both simulation using NASA's generic transport model (GTM) [1] and in flight test on NASA Airborne Subscale Transport Aircraft Research (AirSTAR) flight test vehicle [41, 42] that DF-MRAC with ALR-modification is successful in dealing with uncertainties and latencies with the loop broken at the plant input.

1.5 Output Feedback Adaptive Control

The assumption of full state feedback leads to computationally simpler adaptive controller algorithms in comparison to output feedback algorithms. Output feedback adaptive controllers, however, are required for applications in which it is impractical or impossible to sense the entire state of the process, such as active noise suppression, active control of flexible structures, fluid flow control systems, combustion control processes, and low cost or expendable unmanned aerial vehicle applications. Models for these applications vary from reasonably accurate low-frequency models, in the case of structural control problems, to less accurate low-order models in the case of active control of noise, vibrations, flows, and combustion processes.

There has been a number of results in the recent decades focused on output feedback adaptive controllers. For example, Esfandiari and Khalil [27] introduced a high gain observer for the reconstruction of the unavailable states. Krstic, Kanellakopoulos, and Kokotovic [49], and Marino and Tomei [56] proposed backstepping approaches for output feedback adaptive control, which are affine with respect to unknown parameters. In Ref. 48, Kim and Lewis suggested a neural network architecture for the observer. Output feedback adaptive control using a high gain observer and neural networks has also been proposed in Ref. 69 by Seshagiri and Khalil for nonlinear systems represented by input output models. Another method involving the design of adaptive observer using function approximators and backsteppting was given by Choi and Farrel [18] for a limited class of systems that can be transformed to output feedback form in which nonlinearities depend on measurements only.

Calise, Hovakimyan, and Idan [11] developed an output feedback adaptive control architecture for nonlinear uncertain dynamical systems without using a state observer but with a stable low-pass filter to satisfy a strictly positive real conditions. It requires that the input vector to the neural networks be composed of current and past input output data. In Ref. 34, Hovakimyan, Nardi, Calise, and Kim considered an output feedback adaptive controller with an error observer instead of a state observer. In this approach, only the relative degree of the regulated system needs to be known and the basis functions employed depend only the available input output history. Volyanskyy, Haddad, and Calise [74] introduced $\mathcal{Q}$-modification to output feedback adaptive control. This approach assumes a restrictive matching assumption.

Applying an adaptive control technique for uncertain dynamical systems in many cases implies replacement of an existing control system. On the other hand, it is highly desirable to consider an adaptive approach that can be implemented in a form of augmenting an existing controller. This rationale has been one of the viewpoints taken in applying adaptive control in Refs. 2, 3, 6, 21, 80.

Hoagg, Santillo, and Bernstein [32] estimate Markov parameters that leads to output feedback adaptive control algorithms with varying set of assumptions. Recently, Lavretsky [51] introduced an adaptive output feedback design using asymptotic properties of LQG/LTR controllers that asymptotically satisfies a strictly positive real (SPR) condition. In this approach, it is assumed that the determinant of the product of the system output and input matrices is not zero, although it is possible to relax this assumption.

In this dissertation, we extend the derivative-free adaptive control architecture mentioned in the previous section to output feedback for adaptively controlling continuous time uncertain dynamical systems using a state observer. The observer is employed in the adaptive part of the design in place of a reference model. A derivative-free weight update law in output feedback form ensures that the estimated states follow both the reference model states and the true states so that both the state estimation error and the state tracking error are bounded. The proposed approach is particularly advantageous for applications to systems that can undergo a sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, docking, or structural damage.

The stability analysis uses a Lyapunov-Krasovskii functional that entails the solution of a parameter dependent Riccati equation [46], rather than a Lyapunov equation, to show that all the error signals are uniformly ultimately bounded (UUB). The complexity of the proposed approach is significantly simpler than many other approaches to output feedback adaptive control, and it can be implemented in a form that augments an existing observer based linear control architecture. The design procedure is illustrated using an aeroelastic generic transport model.

1.6 Organization

The organization of this dissertation is as follows. Chapter II presents the theory of KF based modification in adaptive control. The results of this chapter are extended in Chapter III to obtain a KF based adaptation law. Chapter IV presents the theory of DF-MRAC. The results of this chapter are extended in Chapter V to output feedback adaptive control. Finally, Chapter VI concludes this dissertation.

1.7 Notation

The notation used in this dissertation is fairly standard. $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $a_{i,j} \in A$ denotes the row i, column j element of matrix A, and $S \subset R$ denotes S being a subset of R. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, $\mathrm{vec}(\cdot)$ for the column stacking operator, $A \otimes B$ for the Kronecker product, $\mathrm{diag}[A, B]$ for a block diagonal matrix formed with matrices A and B on the diagonal, $|\cdot|$ for the Euclidean vector norm, $||\cdot||$ for the Euclidean matrix norm, $\mathrm{tr}(\cdot)$ for trace operator, $\mathcal{R}\{\cdot\}$ denotes the range space of a matrix, (a, b) for the open interval in $\mathbb{R}$ from a to $b > a$, and $[a, b]$ for the closed interval in $\mathbb{R}$ from a to $b > a$.

CHAPTER II

Kalman Filter Modification in Adaptive Control

2.1 Introduction

This chapter presents a novel KF based approach for approximately enforcing a linear constraint in adaptive control which leads to an alternative forms for many commonly used modification terms. Simulation results are provided that illustrate that this approach provides smaller tracking errors without incurring significant oscillations in the system response and without requiring a high modification gain.

For illustrative purposes, we develop a KF version of the well known e–modification. It is shown how the standard e–modification term [37] can be interpreted as the gradient of a norm measure of a linear constraint violation, and then this linear constraint is used to develop a Kalman filter based e–modification. As a second example, the linear constraint imposed by ALR–modification [10] is treated using the KF method. The resulting KF-ALR–modification is shown to improve upon standard ALR–modification. Finally, a solution is proposed to the problem of adaptation in the presence of input constraints that does not require modification of the reference model.

2.2 Preliminaries

We begin by presenting a formulation of the model reference adaptive control problem. For this purpose, consider the following uncertain system

$$\dot{x}(t) \quad = \quad Ax(t) + B\big[u(t) + \Delta(x(t))\big] \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices with (A, B) being a controllable pair, and $\Delta(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is the *unknown* matched uncertainty. Furthermore, it is assumed that $x(t)$ is available for feedback and $u(t)$ is restricted to the class of admissible controls consisting of measurable functions.

When controlling a nonlinear system, the matrices $A,\ B$ in (1) are usually obtained by linearizing the nonlinear dynamics at selected equilibrium conditions, and the resulting set of linear models are used to design a gain scheduled controller. In this setting a part of the uncertainty in (1) can be thought of as arising from the modeling error associated with linearization. It is assumed that a nominal (baseline) controller for the system in (1) exists for a neighborhood of each equilibrium point, and can be written in the form

$$u_{\mathrm{n}}(t) \quad = \quad -K_1 x(t) + K_2 r(t) \tag{2}$$

where $r(t) \in \mathbb{R}^r$, $r \leq m$, is a bounded reference command that defines the desired output response, $K_1 \in \mathbb{R}^{m\times n}$ is the state feedback gain matrix, and $K_2 \in \mathbb{R}^{m\times r}$ is the command input gain matrix. It should be noted that many controller designs commonly contain dynamics, in which case one can augment the controller dynamics with the dynamics in (1), and consider an expanded state made up of the system states and the controller states, and rewrite the dynamics and controller in the form of (1) and (2). So there is no loss in generality with respect to dynamic controllers in assuming these forms. However, to further simplify the discussion we introduce the following assumption.

Assumption 2.1. The matched uncertainty in (1) can be linearly parameterized as

$$\Delta(x) \quad = \quad W^{\mathrm{T}}\beta(x) \tag{3}$$

where $W \in \mathbb{R}^{s\times m}$ is the unknown *constant* weight matrix and $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a known vector of basis functions of the form $\beta(x) = [\beta_1(x),\ \beta_2(x), \ldots,\ \beta_s(x)]^{\mathrm{T}} \in \mathrm{I\!R}^s$.

One can construct a reference model for the desired response of the closed loop system as

$$\dot{x}_{\mathrm{m}}(t) \;=\; A_{\mathrm{m}} x_{\mathrm{m}}(t) + B_{\mathrm{m}} r(t) \tag{4}$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$ is the reference state vector, $A_{\mathrm{m}} \in \mathrm{I\!R}^{n\times n}$ is Hurwitz, and $B_{\mathrm{m}} \in \mathrm{I\!R}^{n\times r}$.

Assumption 2.2. A_{m} and B_{m} in (4) are chosen so that:

$$\begin{aligned} A_{\mathrm{m}} &= A - BK_1 \\ B_{\mathrm{m}} &= BK_2 \end{aligned}$$

Consider the augmenting adaptive controller given by

$$u(t) \;=\; u_{\mathrm{n}}(t) - u_{\mathrm{ad}}(t) \tag{5}$$

$$u_{\mathrm{ad}}(t) \;=\; \hat{W}(t)^{\mathrm{T}} \beta(x(t)) \tag{6}$$

Defining

$$e(t) \;\equiv\; x(t) - x_{\mathrm{m}}(t)$$

it is well known [71] that the following adaptive law

$$\dot{\hat{W}}(t) = \gamma\big[\beta(x(t)) e^{\mathrm{T}}(t) P B\big] \tag{7}$$

where γ is a positive adaptation gain (adaptation learning rate), and $P \in \mathbb{R}^{n\times n}$ is a positive definite solution of the Lyapunov equation

$$0 \;=\; A_{\mathrm{m}}^{\mathrm{T}} P + P A_{\mathrm{m}} + Q_{\mathrm{m}} \tag{8}$$

for any $Q_{\mathrm{m}} > 0$, ensures that $\hat{W}(t)$ remains bounded, and that $e(t) \to 0$ as $t \to \infty$. The resulting adaptive control system is illustrated in Figure 1.

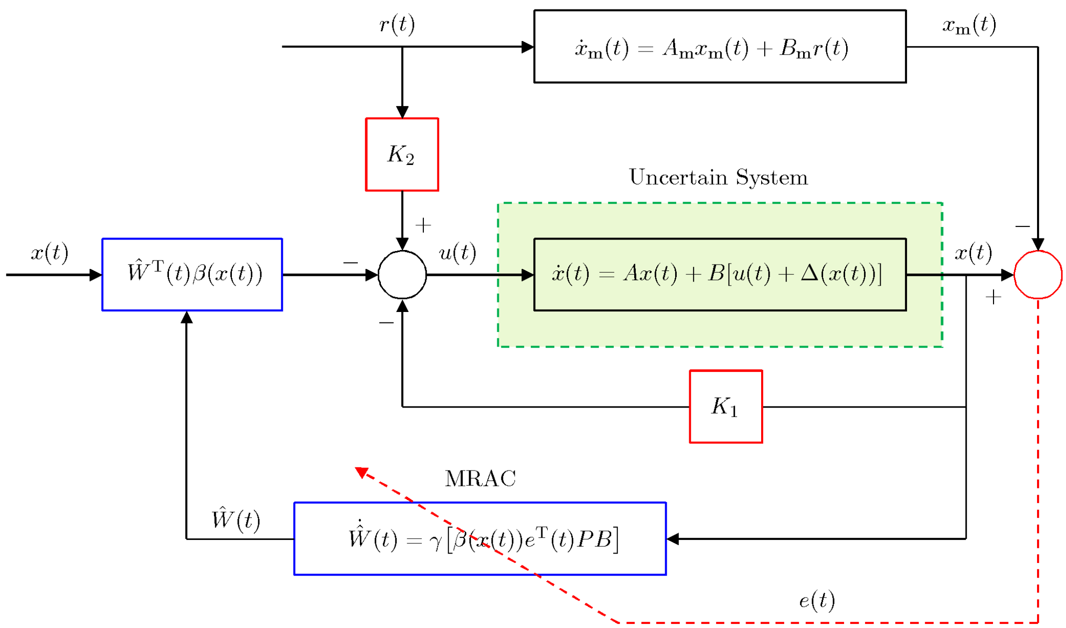

Figure 1: Augmenting adaptive control of a baseline control system.

Definition 2.1 ([30,45])**.** Consider the nonautonomous system $\dot{x} = f(t,x)$, where $f : [0,\infty) \times \mathcal{D} \to \mathbb{R}^n$ is piecewise continuous in t, locally Lipschitz in x on $[0,\infty) \times \mathcal{D}$, and $\mathcal{D} \subset \mathbb{R}^n$ is a domain that contains the origin, $x = 0$. If there exists positive constant $\underline{r}$ and $\bar{r}$, independent of $t_0 \geq 0$, and for every $\eta \in (0, \bar{r})$, there is $T = T(\eta, \underline{r})$, independent of $t_0 \geq 0$, such that $||x(t_0)|| \leq \eta \Rightarrow ||x(t)|| \leq \underline{r}$ for all $t \geq t_0 + T$, then the solution of $\dot{x} = f(t,x)$ is *uniformly ultimately bounded* (UUB) with ultimate bound $\underline{r}$.

In the case where there are bounded external disturbances or when Assumption 2.1 is relaxed to the following assumption, it can be shown that $e(t)$ is UUB.

Assumption 2.3. The matched uncertainty in (1) can be linearly parameterized as

$$\Delta(x) \quad = \quad W^{\rm T}\beta(x) + \varepsilon(x), \quad |\varepsilon(x)| \leq \epsilon, \quad x \in \mathcal{D}_x, \quad \mathcal{D}_x \subset \mathbb{R}^n \tag{9}$$

where $W \in \mathbb{R}^{s\times m}$ is the unknown *constant* weight matrix, $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a known vector of basis functions of the form $\beta(x) = [\beta_1(x),\ \beta_2(x), \ldots,\ \beta_s(x)]^{\mathrm{T}} \in \mathrm{I\!R}^s$, $\varepsilon(x) : \mathrm{I\!R}^n \to \mathrm{I\!R}^m$ is the residual error, and $\mathcal{D}_x$ is a sufficiently large compact set.

In this case the adaptive law in (7) can be modified to guarantee that $\hat{W}(t)$ remains bounded, using either σ–modification [37], e–modification [60], and/or parameter projection [66]. For example with e–modification the adaptation law becomes

$$\dot{\hat{W}}(t) = \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t)\big] \tag{10}$$

where σ is a positive modification gain (modification learning rate).

The state error dynamics are obtained using (1) and (4) with (2), (5), (6), and (10)

$$\begin{aligned}
\dot{e}(t) &= Ax(t) + B\big[u(t) + W^{\mathrm{T}}\beta(x) + \varepsilon(x)\big] - A_{\mathrm{m}}x_{\mathrm{m}}(t) - B_{\mathrm{m}}r(t) \\
&= Ax(t) + B\big[-K_1x(t) + K_2r(t) - \hat{W}(t)^{\mathrm{T}}\beta(x(t)) + W^{\mathrm{T}}\beta(x(t)) + \varepsilon(x)\big] \\
&\quad - A_{\mathrm{m}}x_{\mathrm{m}}(t) - B_{\mathrm{m}}r(t) \\
&= A_{\mathrm{m}}x(t) + B_{\mathrm{m}}r(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + B\varepsilon(x(t)) - A_{\mathrm{m}}x_{\mathrm{m}}(t) - B_{\mathrm{m}}r(t) \\
&= A_{\mathrm{m}}e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + B\varepsilon(x(t))
\end{aligned} \tag{11}$$

where $\tilde{W}(t) \equiv W - \hat{W}(t)$. Since W is a constant matrix, it follows from (10) that

$$\dot{\tilde{W}}(t) = -\gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t)\big] \tag{12}$$

Define $\zeta(t) = [e(t)^{\mathrm{T}}\ \mathrm{vec}(\tilde{W}(t))^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathcal{B}_r = \{\zeta(t) : |\zeta(t)| \le r_\alpha\}$, such that $\mathcal{B}_r \subset \mathcal{D}_x$ for a sufficiently large compact set $\mathcal{D}_x$. Denote $\Omega_\alpha = \{\zeta(t) \in \mathcal{B}_r : \zeta(t)^{\mathrm{T}}\tilde{P}\zeta(t) \le \alpha\}$, where $\tilde{P} = \mathrm{diag}[P,\ \gamma^{-1}I]$ and $\alpha = \min_{|\zeta(t)|=r_\alpha}(\zeta(t)^{\mathrm{T}}\tilde{P}\zeta(t)) = r_\alpha^2\lambda_{\min}(\tilde{P})$. A depiction of these sets is given in Figure 2. Further define $\Theta_e = (\frac{\sigma}{2}||W||^2 + 2||PB||\varepsilon)/\lambda_{\min}(Q_{\mathrm{m}})$, $\Theta_{\tilde{W}} = \frac{1}{2}||W||^2 + \frac{1}{\sqrt{\sigma}}\sqrt{||PB||\varepsilon + \frac{\sigma}{4}||W||^2}$, and $\underline{r}_e^2 = \frac{\lambda_{\max}(P)\Theta_e^2 + \frac{1}{\gamma}\Theta_{\tilde{W}}^2}{\lambda_{\min}(\tilde{P})}$. We can then state the following theorem.

Theorem 2.1. Consider the uncertain system given by (1) subject to Assumption 2.3. Consider, in addition, the feedback control law given by (5), with the nominal

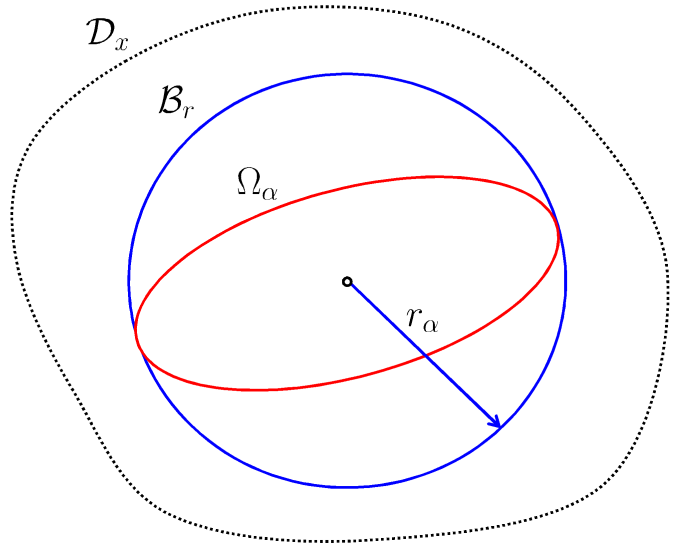

Figure 2: Geometric representation of sets.

feedback control component given by (2), and with the adaptive feedback control component given by (6) that has a weight update law in the form (10). If $\zeta(t_0) \in \Omega_\alpha$ and $r_\alpha > \underline{r}_e$, then either $\zeta(t)$ is UUB with ultimate bound $\underline{r}_e$, or $e(t) = 0$ for all $t \geq T \geq t_0$ and $\tilde{W}(t)$ is bounded.

Proof. Consider the Lyapunov-like function candidate:

$$\mathcal{V}(e(t), \tilde{W}(t)) = \frac{1}{2}e^{\mathrm{T}}(t)Pe(t) + \frac{1}{2\gamma}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] = \frac{1}{2}\zeta^{\mathrm{T}}(t)\tilde{P}\zeta(t) \tag{13}$$

The time derivative of (13) along the trajectories of (11) and (12) can be expressed as

$$\begin{aligned}
\dot{\mathcal{V}}(e(t), \tilde{W}(t)) &= e^{\mathrm{T}}(t)P[A_m e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + B\varepsilon(x(t))] - \frac{1}{\gamma}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\dot{\hat{W}}(t)] \\
&= -\frac{1}{2}e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) + e^{\mathrm{T}}(t)PB\varepsilon(x(t)) + \sigma\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)|e(t)|(W - \tilde{W}(t))] \\
&\leq -|e(t)|\big[c_1|e(t)| + c_2||\tilde{W}(t)|| - c_3 - c_4||\tilde{W}(t)||\big] \\
&= -|e(t)|\big[c_1|e(t)| + \big(\sqrt{c_2}||\tilde{W}(t)|| - \frac{c_4}{2\sqrt{c_2}}\big)^2 - c_3 - \frac{c_4^2}{4c_2}\big]
\end{aligned} \tag{14}$$

where $c_1 = \lambda_{\min}(Q_{\mathrm{m}})/2$, $c_2 = \sigma$, $c_3 = ||PB||\varepsilon$, and $c_4 = \sigma||W||$. Note that $\dot{\mathcal{V}}(e(t), \tilde{W}(t))$ is negative as long as the term in braces is positive and $e(t) \neq 0$.

If $e(t) = 0$ for all $t \geq T$, then $\tilde{W}(t)$ is bounded since $\dot{\tilde{W}}(t) = 0$ from (12). For the case when $e(t) \neq 0$, either $|e(t)| > \Theta_e$, or $||\tilde{W}(t)|| > \Theta_{\tilde{W}}$ renders $\dot{V}(e(t), \tilde{W}(t)) < 0$, where $\Theta_e = \frac{1}{c_1}\big(c_3 + \frac{c_4^2}{4c_2}\big)$, and $\Theta_{\tilde{W}} = \frac{1}{\sqrt{c_2}}\big(\frac{c_4}{2\sqrt{c_2}} + \sqrt{c_3 + \frac{c_4^2}{4c_2}}\big)$. Therefore, $e(t)$ and $\tilde{W}(t)$ are uniformly bounded. Similar to [12], using $e(t)^{\mathrm{T}}Pe(t) \leq \lambda_{\max}(P)|e(t)|^2$, (14) can be written as

$$\dot{\mathcal{V}}(e(t), \tilde{W}(t)) \leq -\frac{\sqrt{e(t)^{\mathrm{T}}Pe(t)}}{\sqrt{\lambda_{\max}(P)}}\Big(c_1\frac{\sqrt{e(t)^{\mathrm{T}}Pe(t)}}{\sqrt{\lambda_{\max}(P)}} + \big(\sqrt{c_2}||\tilde{W}(t)|| - \frac{c_4}{2\sqrt{c_2}}\big)^2 - c_3 - \frac{c_4^2}{4c_2}\Big)$$

which is strictly negative when $\sqrt{e(t)^{\mathrm{T}}Pe(t)} > \sqrt{\lambda_{\max}(P)}\Theta_e$ or $||\tilde{W}(t)|| > \Theta_{\tilde{W}}$. Define $\Omega_\beta = \{\zeta(t) \in \mathcal{B}_r : \zeta(t)^{\mathrm{T}}\tilde{P}\zeta(t) \leq \lambda_{\max}(P)\Theta_e^2 + \frac{1}{\gamma}\Theta_{\tilde{W}}^2\}$. Then, Ω_α is a positively invariant set if $\Omega_\beta \subset \Omega_\alpha$. This requires that $\lambda_{\max}(P)\Theta_e^2 + \frac{1}{\gamma}\Theta_{\tilde{W}}^2 < \alpha$. The minimum size of $\mathcal{B}_r$ that ensures this condition is $r_\alpha > \underline{r}_e$. Therefore, if $\zeta(t_0) \in \Omega_\alpha$, then error dynamics (11) and (12) are UUB with ultimate bound $\underline{r}_e$. □

Remark 2.1. In Theorem 2.1 it should be noted that $\zeta(t)$ is UUB with probability one in the sense that given a joint probability density function for the set of initial conditions $\{e(t_0), \tilde{W}(t_0)\}$, the probability measure of all initial conditions from that set that will result in $e(t) = 0$ for $t \geq T$ is zero, because the space of initial conditions that produces this outcome has dimension less than $n + m \times s$.

Remark 2.2. When the uncertainty is exactly parameterized, e-modification is not needed to prove the weight error remains bounded. In this case $c_2 = 0$ and $c_3 = 0$, and $\dot{\mathcal{V}}(e(t), \tilde{W}(t)) = -e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) \leq -c_1|e(t)|^2$, which is sufficient to prove that $e(t)$ and $\tilde{W}(t)$ are bounded. Since $\mathcal{V}(e(t), \tilde{W}(t))$ is lower bounded and $\dot{\mathcal{V}}(e(t), \tilde{W}(t)) \leq 0$, $\mathcal{V}(e(t), \tilde{W}(t))$ approaches a finite limit as $t \to \infty$. Moreover, $\dot{\mathcal{V}}(e(t), \tilde{W}(t))$ is a uniformly continuous function of time, since its time derivative $\big(\ddot{\mathcal{V}}(e(t), \tilde{W}(t)) = -e^{\mathrm{T}}(t)Q_{\mathrm{m}}\big[A_{\mathrm{m}}e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t))\big]\big)$ is bounded, which is a consequence of the fact that $e(t)$ and $\tilde{W}(t)$ are bounded. Then by Barbalat's lemma [30] it follows that $\lim_{t\to\infty}\dot{\mathcal{V}}(e(t), \tilde{W}(t)) = 0$, and therefore $e(t)$ asymptotically goes to zero.

Remark 2.3. The condition $r_\alpha > \underline{r}_e$ is needed to ensure that $x(t_0)$ lies inside the ball $\mathcal{D}_x$ as defined in Assumption 2.3. Invariancy in the error dynamics together with the fact that $x_{\mathrm{m}}(t)$ is bounded ensures that $x(t)$ remains in $\mathcal{D}_x$. It is apparent from this condition that this implies a lower bound on γ whenever $\sigma > 0$. However, it also implies an upper bound. When γ is sufficiently large, then $\lambda_{\min}(\tilde{P}) = 1/\gamma$ and the limiting from of this condition $\underline{r}_e^2 > \gamma\lambda_{\max}(P)\Theta_e^2$.

In the next section, we introduce an optimization based approach for approximately imposing a linear constraint.

2.3 *Approximate Enforcement of a Linear Constraint by KF Optimization*

We first restrict the form of the constraint of interest. For this purpose, the constraint on the weight estimate in an adaptive control design has the linear form

$$\hat{W}^{\mathrm{T}}(t)\phi(t, x(t), e(t)) = 0 \tag{15}$$

where $\phi(\cdot) : [0, \infty) \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}^{s\times l}$ is known and bounded pointwise in time, with $l \leq s$.

The problem of estimating W while approximately enforcing the linear constraint in (15) can be formulated as an optimization problem. Eq. (15) can be expressed in the following equivalent vector form

$$\mathrm{vec}\big(\phi^{\mathrm{T}}(t, x(t), e(t))\hat{W}(t)\big) \;=\; \Phi(t, x(t), e(t))^{\mathrm{T}}\omega = 0 \tag{16}$$

where $\Phi(\cdot) = I_{m\times m} \otimes \phi(\cdot) : [0, \infty) \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}^{ms\times ml}$ and $\omega = \mathrm{vec}(W) \in \mathrm{I\!R}^{ms}$. Define the stochastic process

$$\dot{\omega} \;=\; q(t) \tag{17}$$

$$z(t) \;=\; \Phi^{\mathrm{T}}(\cdot)\omega + r(t) \tag{18}$$

where $q(t)$ and $r(t)$ are zero-mean, Gaussian, white noise processes with covariances

$$\mathbb{E}\big\{q(t)q^{\mathrm{T}}(\tau)\big\} \;=\; \bar{Q}\delta(t-\tau), \quad \bar{Q} \in \mathrm{I\!R}^{ms\times ms} > 0$$

$$\mathbb{E}\big\{r(t)r^{\mathrm{T}}(\tau)\big\} \;=\; \bar{R}\delta(t-\tau), \quad \bar{R} \in \mathrm{I\!R}^{ml\times ml} > 0$$

and $z(t)$ is regarded as a measurement. The estimate of $z(t)$ is

$$\hat{z}(t) \;=\; \Phi^{\mathrm{T}}(\cdot)\hat{\omega}(t)$$

where $\hat{\omega}(t)$ is an estimate of ω. The Kalman filter associated with this problem formulation is given by [19]:

$$\dot{\hat{\omega}}(t) \;=\; \bar{S}(t)\Phi(\cdot)\bar{R}^{-1}\big[z(t) - \hat{z}(t)\big], \quad \hat{\omega}(0) = 0 \tag{19}$$

$$\dot{\bar{S}}(t) \;=\; -\bar{S}(t)\Phi(\cdot)\bar{R}^{-1}\Phi^{\mathrm{T}}(\cdot)\bar{S}(t) + \bar{Q}, \quad \bar{S}(0) = \bar{S}_0 > 0 \tag{20}$$

where $\bar{S}(t) \in \mathbb{R}^{ms\times ms}$. Since the objective is to approximately satisfy the constraint in (15), the logical choice for $z(t)$ when employing this estimator is $z(t) = 0$.

Lemma 2.1. Let $\bar{R} = I_{m\times m} \otimes R$, with $R \in \mathbb{R}^{l\times l} > 0$, and $\bar{Q} = I_{m\times m} \otimes Q$, with $Q \in \mathbb{R}^{s\times s} > 0$. Then, Eqs. (19) and (20), with $z(t) = 0$, reduces to:

$$\dot{\hat{W}}(t) \;=\; -S(t)\phi(\cdot)R^{-1}\phi^{\mathrm{T}}(\cdot)\hat{W}(t), \quad \hat{W}(0) = 0 \tag{21}$$

$$\dot{S}(t) \;=\; -S(t)\phi(\cdot)R^{-1}\phi^{\mathrm{T}}(\cdot)S(t) + Q, \quad S(0) = S_0 > 0 \tag{22}$$

where $S(t) \in \mathbb{R}^{s\times s}$.

Proof. Substituting $z(t) = 0$, (19) becomes

$$\dot{\hat{\omega}}(t) \;=\; -\bar{S}(t)\Phi(\cdot)\bar{R}^{-1}\Phi^{\mathrm{T}}(\cdot)\hat{\omega}$$

Choosing $\bar{R} = I_{m\times m} \otimes R$ and $\bar{Q} = I_{m\times m} \otimes Q$, $\bar{S}(t)$ in (20) will have the form of $\bar{S}(t) = I_{m\times m} \otimes S(t)$, with $S(t) \in \mathbb{R}^{s\times s}$. Now, it follows from the above equation with

$\hat{\omega} = \text{vec}(\hat{W}(t)) \in \mathrm{IR}^{ms}$ and $\Phi(\cdot) = I_{m\times m} \otimes \phi(\cdot) \in \mathrm{IR}^{ms\times ml}$ that

$$\begin{aligned}
\text{vec}(\dot{\hat{W}}) &= -[I_{m\times m} \otimes S(t)][I_{m\times m} \otimes \phi(\cdot)][I_{m\times m} \otimes R^{-1}] \\
&\quad \times[I_{m\times m} \otimes \phi^{\mathrm{T}}(\cdot)]\text{vec}(\hat{W}(t)) \\
&= -[I_{m\times m} \otimes S(t)][I_{m\times m} \otimes \phi(\cdot)][I_{m\times m} \otimes R^{-1}] \\
&\quad \times\text{vec}(\phi^{\mathrm{T}}(\cdot)\hat{W}(t)) \\
&= -[I_{m\times m} \otimes S(t)][I_{m\times m} \otimes \phi(\cdot)]\text{vec}(R^{-1}\phi^{\mathrm{T}}(\cdot)\hat{W}(t)) \\
&= -[I_{m\times m} \otimes S(t)]\text{vec}(\phi(\cdot)R^{-1}\phi^{\mathrm{T}}(\cdot)\hat{W}(t)) \\
&= -\text{vec}(S(t)\phi(\cdot)R^{-1}\phi^{\mathrm{T}}(\cdot)\hat{W}(t))
\end{aligned}$$

where the fact $(C^{\mathrm{T}} \otimes A)\text{vec}(B) = \text{vec}(ABC)$ is used [5], with $C = I_{m\times m}$. It follows from vec^{-1} operator that the above equation becomes (21). Similarly, it follows from(20) that

$$\begin{aligned}
[I_{m\times m} \otimes \dot{S}(t)] &= -[I_{m\times m} \otimes S(t)][I_{m\times m} \otimes \phi(\cdot)][I_{m\times m} \otimes R^{-1}] \\
&\quad \times[I_{m\times m} \otimes \phi^{\mathrm{T}}(\cdot)][I_{m\times m} \otimes S(t)] + [I_{m\times m} \otimes Q] \\
&= -[I_{m\times m} \otimes S(t)\phi(\cdot)R^{-1}\phi^{\mathrm{T}}(\cdot)S(t)] \\
&\quad +[I_{m\times m} \otimes Q]
\end{aligned}$$

where the fact $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ is used [5], with $A = C = I_{m\times m}$. Since, $[I \otimes A] = [I \otimes B] + [I \otimes C]$ implies $A = B + C$, then the above equation becomes (22). □

Remark 2.4. The Riccati equation in (22) can be associated with the following system of equations:

$$\begin{aligned}
\dot{w}_i &= q_1(t) \\
z_i(t) &= \phi^{\mathrm{T}}(\cdot)w_i + r_1(t)
\end{aligned}$$

in which w_i are the process dynamics associated with the i-th column of W, z_i is the corresponding measurement vector, $\mathbb{E}\{q_1(t)q_1^{\mathrm{T}}(\tau)\} = Q\delta(t-\tau)$, and $\mathbb{E}\{r_1(t)r_1^{\mathrm{T}}(\tau)\} = R\delta(t-\tau)$.

Assumption 2.4. The observability gramian

$$W_0(t_0,t)=\int_{t_0}^{t}\phi(t,x(t),e(t))\phi^{\mathrm{T}}(t,x(t),e(t))\mathrm{d}t \tag{23}$$

of the system defined in Remark 2.4 satisfies the following condition for uniform complete observability [43]:

$$0<\alpha_0(\sigma)I_l\leq W_0(t_0,t_0+\sigma)\leq\alpha_1(\sigma)I_l \tag{24}$$

for all t_0, and for some fixed constant $\sigma>0$.

Remark 2.5. The condition in Assumption 2.4 is sufficient to show that $S(t)$ is positive-definite, and lower and upper bounded, independent of t_0, if $\phi(t,x(t),e(t))$ is bounded pointwise in time. That is, there exists constants $\underline{s}>0$ and $\bar{s}>0$ such that $\underline{s}I_s\leq S(t)\leq\bar{s}I_s$ holds for all $t\geq t_0$.

Using Lemma 2.1 and incorporating (21) as a modification term in the adaptive law given in (10), we arrive at

$$\dot{\hat{W}}(t) \;=\; \gamma(\beta(x(t))e^{\mathrm{T}}(t)PB-\sigma|e(t)|\hat{W}(t)-kS(t)\phi(\cdot)R^{-1}\phi(\cdot)^{\mathrm{T}}\hat{W}(t)) \tag{25}$$

where k has been introduced a positive modification gain, and $kS(t)$ can be interpreted as its variable gain. In this case the weight update error dynamics become

$$\dot{\tilde{W}}(t) \;=\; -\gamma(\beta(x(t))e^{\mathrm{T}}(t)PB-\sigma|e(t)|\hat{W}(t)-kS(t)\phi(\cdot)R^{-1}\phi(\cdot)^{\mathrm{T}}\hat{W}(t))$$

A stability analysis based on the above weight update error dynamics depends on the form of the constraint in (15), and is therefore problem dependent. For illustration purposes, we state and prove the KF alternative to e–modification in the next section.

2.4 Examples on a Model of Wing Rock Dynamics

In this section we first show how the KF approach for enforcing a linear constraint can be used to derive an alternative for the standard e–modification term in adaptive control. Next an alternative to ALR–modification is illustrated [10]. As a third example, we introduce a novel solution to the problem of input constraints imposed by an actuator, and compare it with the method of hedging [39, 40]. All of the examples are treated using an adaptive control approach to the problem of stabilizing uncertain wing rock dynamics [70]. In all cases we chose $Q_{\mathrm{m}} = I_2$, and for the KF based designs we chose $Q = I_3,\ R = 10^{-3} I_3$, and $S_0 = I_3$. The form in Assumption 2.3 was used to approximate the uncertainty with $\beta(x(t)) = [1,\ \beta_1(x(t)),\ \beta_2(x(t))]^{\mathrm{T}}$, where $\beta_i(x) = 1/(1+e^{-x_i})$, $i = 1, 2$, are sigmoidal type basis functions.

2.4.1 A KF Alternative to e–Modification

In this subsection we replace the standard e–modification with KF e–modification. The standard e–modification term in (10) can be viewed as having been introduced to approximately enforce the following linear constraint

$$|e(t)|^{\frac{1}{2}} \hat{W}(t) = 0 \tag{26}$$

by adding a component to $\dot{\hat{W}}(t)$ along the gradient of $||\ |e(t)|^{\frac{1}{2}} \hat{W}(t)\ ||_F^2$ with respect to $\hat{W}(t)$. In this case $\phi(\cdot) = |e(t)|^{\frac{1}{2}} I_s$ in (15). In this case it will be necessary to prove that $\phi(|e|)$ remains bounded pointwise in time as an intermediate step in the proof. Applying the form of constraint in (26) to (21) and (22) leads to the following KF based e–modification term:

$$\begin{aligned} \dot{\hat{W}}(t) &= -|e(t)| S(t) R^{-1} \hat{W}(t) \\ \dot{S}(t) &= -|e(t)| S(t) R^{-1} S(t) + Q \end{aligned}$$

Remark 2.6. If Q, R, and S_0 are chosen in the form $\bar{q}I$, $\bar{r}I$, and $\bar{s}_o I$, respectively, then the above equations reduce to:

$$\dot{\hat{W}}(t) = -\frac{|e(t)| s(t)}{\bar{r}} \hat{W}(t) \tag{27}$$

$$\dot{s}(t) = -|e(t)| \frac{s^2(t)}{\bar{r}} + \bar{q} \tag{28}$$

Replacing the conventional e–modification term in Eq. (10) with the KF based term in Eq. (27) we have

$$\dot{\hat{W}}(t) = \gamma\Big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma_{kf}\frac{|e(t)|s(t)}{\bar{r}}\hat{W}(t)\Big] \tag{29}$$

where σ_{kf} is a positive learning rate and $s(t)$ is defined in (28). In this case, the weight update error dynamics can be given using (29) as:

$$\dot{\tilde{W}}(t) = -\gamma\Big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma_{kf}\frac{|e(t)|s(t)}{\bar{r}}\hat{W}(t)\Big] \tag{30}$$

Furthermore, note that since $\phi(\cdot) = |e(t)|^{\frac{1}{2}}I_s$, it follows from (23) that $W_0(t_0,t) = \int_{t_0}^{t}|e(\tau)|\mathrm{d}\tau I_s$ is positive-definite for all $t \geq t_0$ as long as $|e(\tau)| \neq 0$ over the interval $[t_0,t]$.

Define $\underline{r}^2_{e_{kf}} = \frac{\lambda_{\max}(P)\bar{\Theta}_e^2 + \frac{1}{\gamma}\bar{\Theta}^2_{\tilde{W}}}{\lambda_{\min}(\tilde{P})}$, where $\bar{\Theta}_e = (\frac{\sigma_{kf}\bar{s}}{2\bar{r}}||W||^2 + 2||PB||\epsilon)/\lambda_{\min}(Q_{\mathrm{m}})$, $\bar{\Theta}_{\tilde{W}} = \frac{1}{2}||W||^2 + \frac{\sqrt{\bar{r}}}{\sqrt{\sigma_{kf}\underline{s}}}\sqrt{||PB||\epsilon + \frac{\sigma_{kf}\bar{s}}{4\bar{r}}||W||^2}$, where $\underline{s}$ and $\bar{s}$ are defined in Remark 2.5.

Theorem 2.2. Consider the uncertain system given by (1) subject to Assumption 2.3. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2), and with the adaptive feedback control component given by (6) that has a weight update law in the form (29). If $\zeta(t_0) \in \Omega_\alpha$ and $r_\alpha > \underline{r}_{e_{kf}}$, then either $\zeta(t)$ is UUB with ultimate bound $\underline{r}_{e_{kf}}$, or $e(t) = 0$ for all $t \geq T \geq t_0$ and $\tilde{W}(t)$ is bounded.

Proof. Consider the Lyapunov-like function candidate given by (13). The time derivative of (13) along the trajectories of (11) and (30) can be expressed as:

$$\begin{aligned}
\dot{\mathcal{V}}(e(t),\tilde{W}(t)) &= e^{\mathrm{T}}(t)P[A_m e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + B\varepsilon(x(t))] - \frac{1}{\gamma}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\dot{\hat{W}}(t)] \\
&= -\frac{1}{2}e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) + e^{\mathrm{T}}(t)PB\varepsilon(x(t)) + \sigma_{kf}s(t)\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t) \\
&\quad \cdot|e(t)|(W - \tilde{W}(t))]/\bar{r} \\
&\leq -|e(t)|\big[d_1|e(t)| + d_2(t)||\tilde{W}(t)|| - d_3 - d_4(t)||\tilde{W}(t)||\big] \\
&= -|e(t)|\big[d_1|e(t)| + \big(\sqrt{d_2(t)}||\tilde{W}(t)|| - \frac{d_4(t)}{2\sqrt{d_2(t)}}\big)^2 - d_3 - \frac{d_4^2(t)}{4d_2(t)}\big]
\end{aligned} \tag{31}$$

where $d_1 = \lambda_{\min}(Q_{\rm m})/2$, $d_2(t) = \sigma_{kf}s(t)/\bar{r}$, $d_3 = ||PB||\varepsilon$, and $d_4(t) = \sigma_{kf}s(t)||W||/\bar{r}$. Note that $\dot{\mathcal{V}}(e(t), \tilde{W}(t))$ is negative as long as the term in braces is positive and $e(t) \neq 0$. If $e(t) = 0$ for all $t \geq T$, then $\tilde{W}(t)$ is bounded since $\dot{\tilde{W}}(t) = 0$ from (30).

For the case when $e(t) \neq 0$, either $|e(t)| > \frac{1}{d_1}\big(d_3 + \frac{d_4^2(t)}{4d_2(t)}\big)$ or $||\tilde{W}(t)|| > \frac{1}{\sqrt{d_2(t)}}\big(\frac{d_4(t)}{2\sqrt{d_2(t)}} + \sqrt{d_3 + \frac{d_4^2(t)}{4d_2(t)}}\big)$ renders $\dot{V}(e(t), \tilde{W}(t)) < 0$. At $t = t_0$, $s(t_0) > 0$. Since both $e(t)$ and $s(t)$ are continuous functions of time, there exists a finite time $t_1 > t_0$ such that both $s(t)$ and $e(t)$ are bounded. During this time interval $s(t) > 0$, since $s(t)$ is continuous and must pass through 0 in order to become negative, at which point $\dot{s}(t) = \bar{q} > 0$. In addition, during this time interval $\zeta(t)$ is confined to the largest invariant set, Ω_α, contained in $\mathcal{B}_r$, such that $r > \underline{r}_{e_{kf}}$, since we are assured that $\dot{V}(e(t), \tilde{W}(t)) < 0$ outside the ball $\mathcal{B}_{\underline{r}_{e_{kf}}}$. By repeating this argument for some $t_i > t_{i-1}$, $i = 2, 3, \ldots$, one can extend the time interval indefinitely beyond t_1, to show that $\phi(|e|)$ remains bounded pointwise in time. Therefore, by Remark 2.5 $s(t)$ satisfies $\underline{s} \leq s(t) \leq \bar{s}$ for all $t \geq t_0$.

Next, let $\underline{d}_2 = \sigma_{kf}\underline{s}/\bar{r}$, $\bar{d}_4 = \sigma_{kf}\bar{s}||W||/\bar{r}$, and note from the previous definitions $\bar{\Theta}_e$ and $\bar{\Theta}_{\tilde{W}}$, that $\bar{\Theta}_e = \frac{1}{d_1}\big(d_3 + \frac{\bar{d}_4^2}{4\underline{d}_2}\big)$, and $\bar{\Theta}_{\tilde{W}} = \frac{1}{\sqrt{\underline{d}_2}}\big(\frac{\bar{d}_4}{2\sqrt{\underline{d}_2}} + \sqrt{d_3 + \frac{\bar{d}_4^2}{4\underline{d}_2}}\big)$. Since $\bar{\Theta}_e > \frac{1}{d_1}\big(d_3 + \frac{d_4^2(t)}{4d_2(t)}\big)$ and $\bar{\Theta}_{\tilde{W}} > \frac{1}{\sqrt{d_2(t)}}\big(\frac{d_4(t)}{2\sqrt{d_2(t)}} + \sqrt{d_3 + \frac{d_4^2(t)}{4d_2(t)}}\big)$, $\dot{V}(e(t), \tilde{W}(t))$ is negative for either $|e(t)| > \bar{\Theta}_e$ or $||\tilde{W}(t)|| > \bar{\Theta}_{\tilde{W}}$, independent of t. Similar to the proof of Theorem 2.1, define $\Omega_\beta = \{\zeta(t) \in \mathcal{B}_r : \zeta(t)^{\rm T}\tilde{P}\zeta(t) \leq \lambda_{\max}(P)\bar{\Theta}_e^2 + \frac{1}{\gamma}\bar{\Theta}_{\tilde{W}}^2\}$. Then, Ω_α is a positively invariant set if $\Omega_\beta \subset \Omega_\alpha$. This requires that $\lambda_{\max}(P)\bar{\Theta}_e^2 + \frac{1}{\gamma}\bar{\Theta}_{\tilde{W}}^2 < \alpha$. The minimum size of $\mathcal{B}_r$ that ensures this condition is $r_\alpha > \underline{r}_{e_{kf}}$. Therefore, if $\zeta(t_0) \in \Omega_\alpha$, then the error dynamics (11) and (12) are UUB with ultimate bound $\underline{r}_{e_{kf}}$. □

Next, we compare the standard e–modification based adaptation law in Eq. (10) with the KF version of e–modification in Eq. (29) using a model of wing rock dynamics to illustrate the advantages of the proposed KF approach to modification.

For this purpose, consider the following dynamics

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[u(t) + \Delta(x(t)) \right] \tag{32}$$

where $\Delta(x(t)) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 |x_1(t)| x_2(t) + \alpha_4 |x_2(t)| x_2(t) + \alpha_5 x_1^3(t)$ with $\alpha_1 = 0.2314$, $\alpha_2 = 0.6918$, $\alpha_3 = -0.6245$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0214$. In (32), $x_1(t)$ represents the roll angle, and $x_2(t)$ represents the roll rate. We selected the initial state values as $x(0) = [6^o, 3^o/s]^{\mathrm{T}}$. The reference system is selected to be second order with a natural frequency of 0.5 rad/s, and a damping of 0.707, and to have unity low frequency gain from $r(t)$ to $x_1(t)$. This corresponds to choosing $K_1 = [0.25,\ 0.707]$ and $K_2 = 0.25$. Figures 3-10 present the results.

Figure 3 shows that the baseline control response with uncertainty significantly degrades system performance. Figures 4-8 compare the standard e–modification based adaptive control law with the KF based e–modification adaptive control law for different gains. It is obvious from these figures that performance is good only when we employ KF based e–modification. The standard e–modification based adaptive control law does not produce a reasonable result for any of these cases. In addition, KF based e–modification requires significantly less and smoother control effort. The gain in the KF based approach to e–modification is now time varying due to the presence of $S(t)$, and that its performance is nearly independent of value of gains employed. Since most actuators are rate limited, in Figure 8 we compare $\mathrm{d}u(t)/\mathrm{d}t$ responses. Finally, Figure 9 and 10 show the comparison results under sensor noise, where it is evident that the KF based e–modification achieved the best performance while requiring much less control effort.

At this point the reader may wonder if this comparison of e–modification is biased by the fact that no attempt was made to tune the adaptive law other than to vary the adaptation gains. However this is not the case since all our attempts to tune the adaptive law with conventional e–modification did not significantly change the

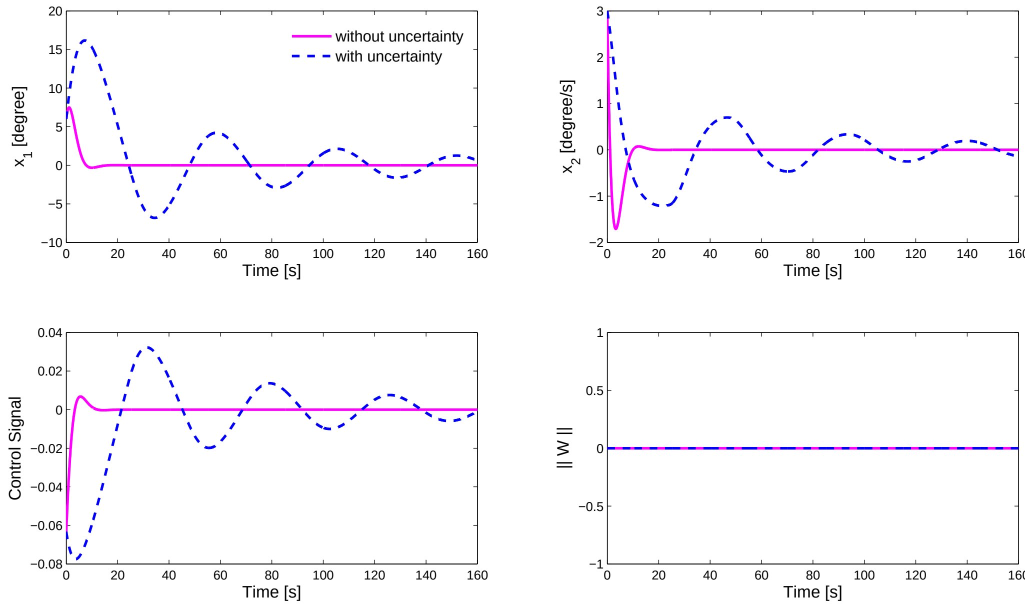

Figure 3: Baseline control response without and with uncertainty.

outcome [13].

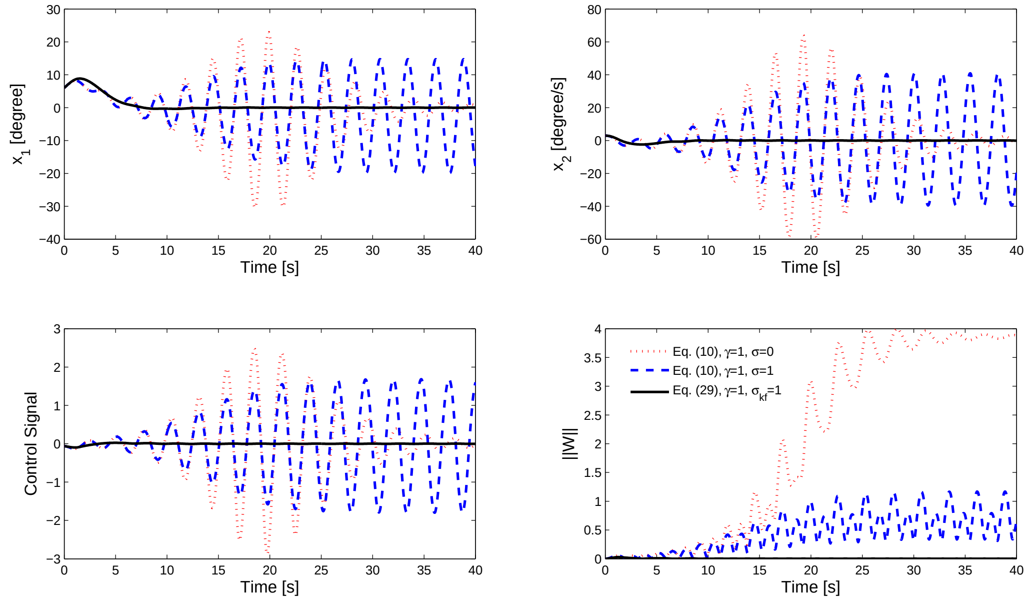

Figure 4: Comparison of adaptive controller responses without modification, with e-modification, and with KF based *e*- modification with all gains set to 1.

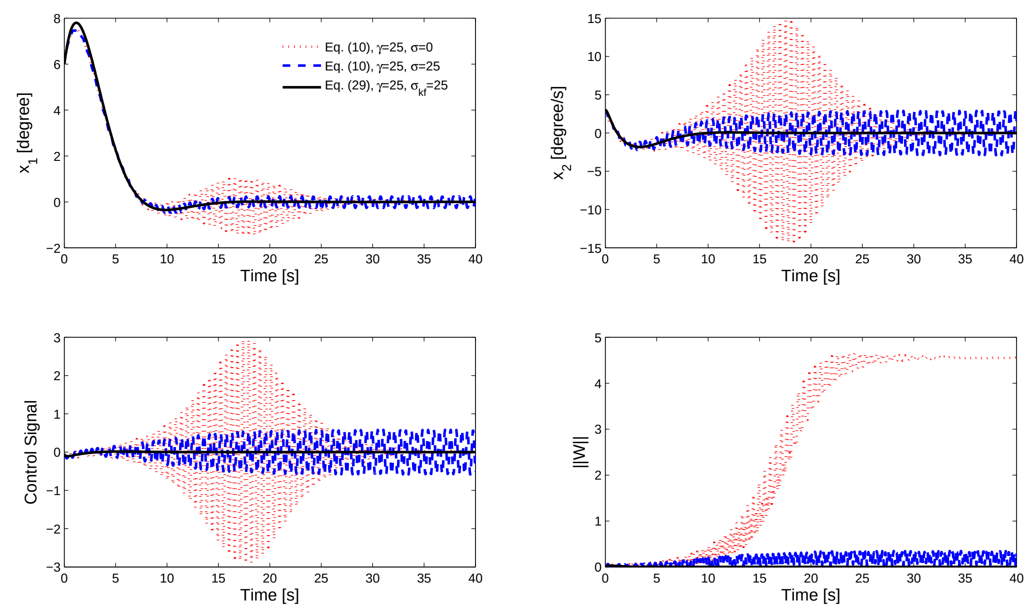

Figure 5: Comparison of adaptive controller responses without modification, with e-modification, and with KF based *e*- modification with all gains set to 25.

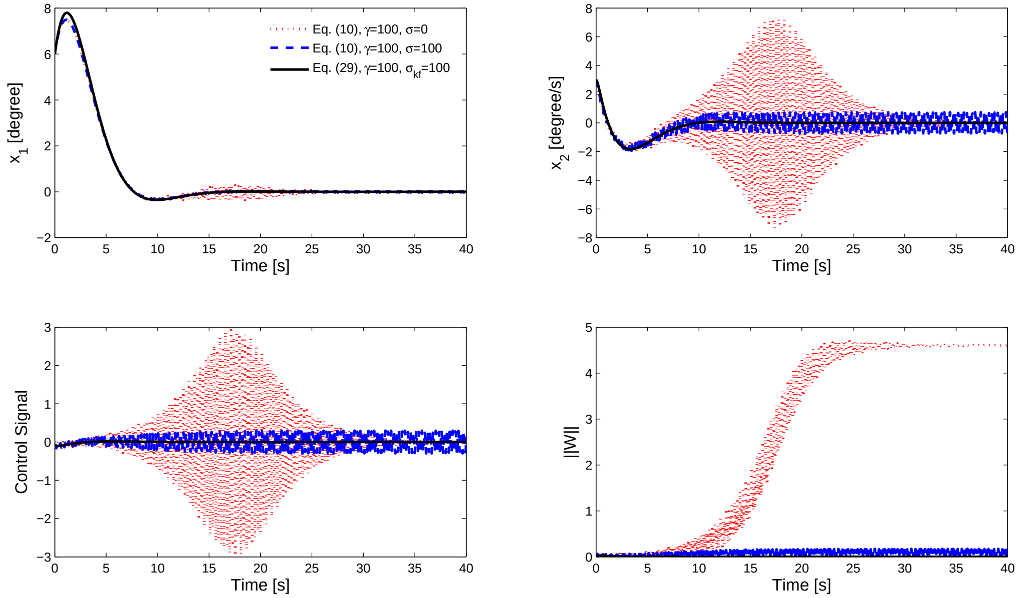

Figure 6: Comparison of adaptive controller responses without modification, with e-modification, and with KF based *e*- modification with all gains set to 100.

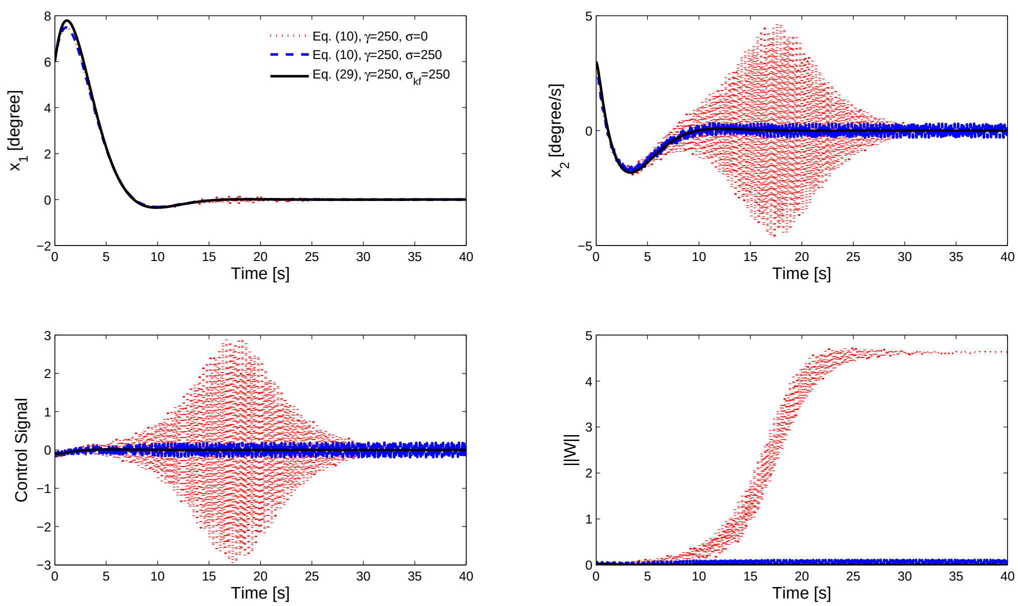

Figure 7: Comparison of adaptive controller responses without modification, with e-modification, and with KF based *e*- modification with all gains set to 250.

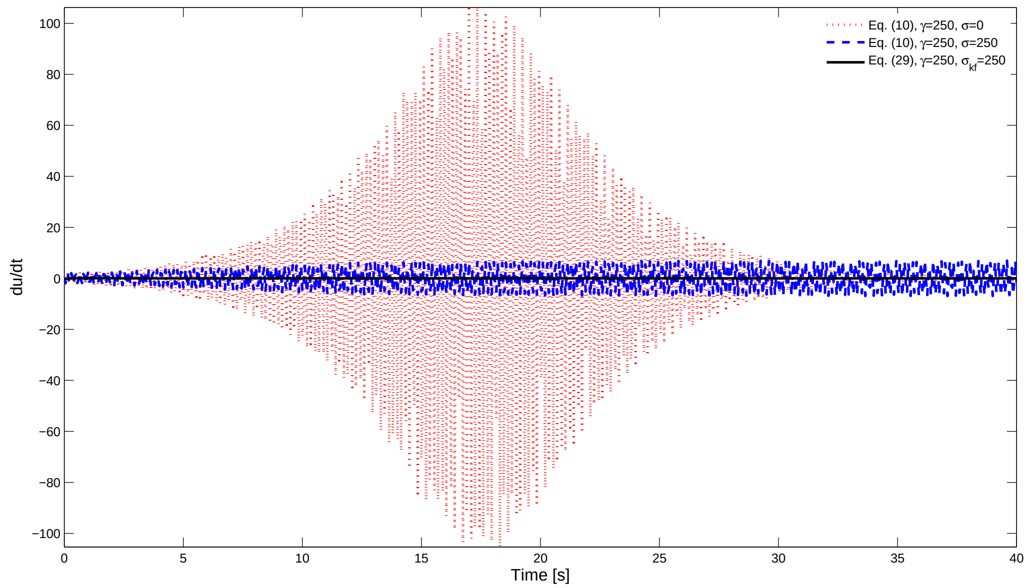

Figure 8: Comparison of $\mathrm{d}u(t)/\mathrm{d}t$ responses without modification, with e-modification, and with KF based e- modification with all gains set to 250.

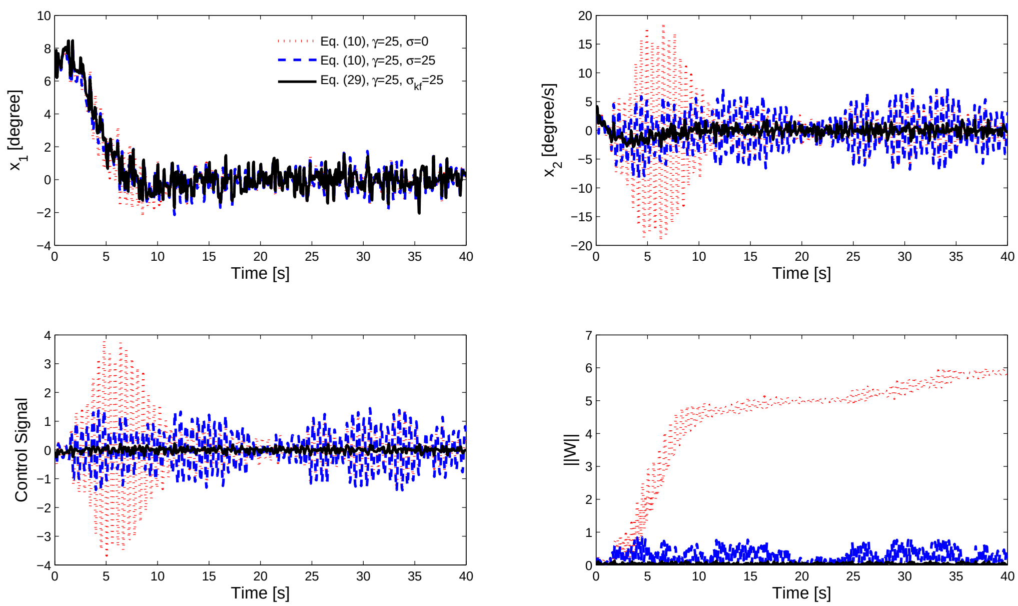

Figure 9: Comparison of adaptive controller responses under sensor noise without modification, with e-modification, and with KF based e- modification with all gains set to 25.

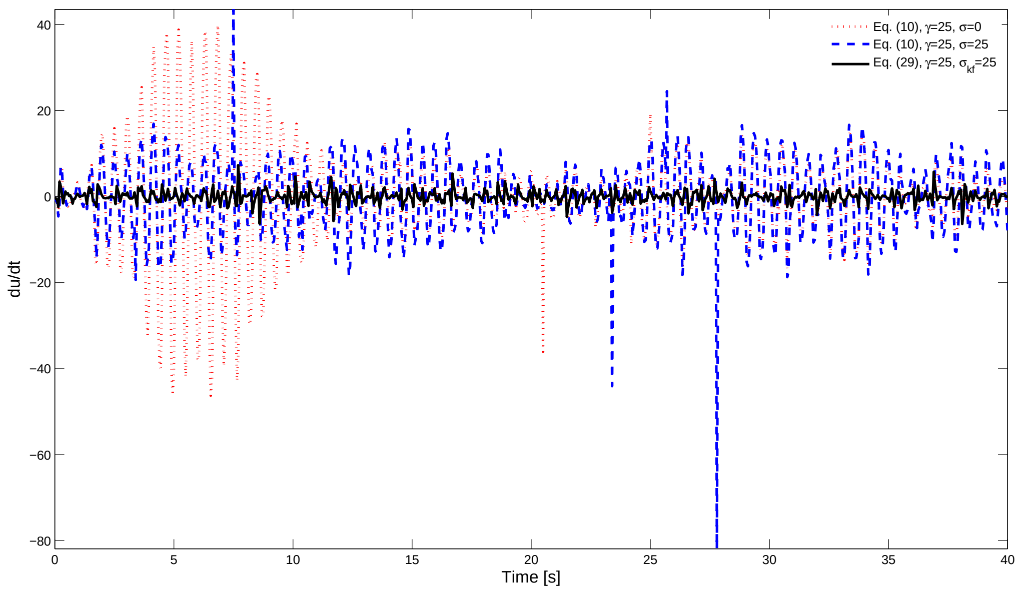

Figure 10: Comparison of $\mathrm{d}u(t)/\mathrm{d}t$ responses under sensor noise without modification, with e-modification, and with KF based e- modification with all gains set to 25.

2.4.2 A KF Alternative to ALR–Modification

In Ref. [10], an ALR–modification term is proposed for use in adaptive control to preserve the stability margins of the reference model defined in (4). This is done by approximately imposing a linear constraint on the weights so that the loop properties of the reference model are asymptotically recovered as the gain on the modification term is increased. Here, the proposed KF optimization approach is applied to the ALR constraint $\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)) = 0$ as a second example, where $\beta_x(x(t)) \equiv \frac{d\beta(x(t))}{dx(t)} \in \mathbb{R}^{s\times n}$ is the derivative of the basis function with respect to $x(t) \in \mathbb{R}^n$. The standard ALR–modification term is found by taking the negative gradient of the following quadratic function in $\hat{W}(t)$:

$$\begin{aligned}\mathcal{J}_{\mathcal{A}}(\hat{W}(t)) &= \frac{1}{2}\mathrm{tr}[(\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)))^{\mathrm{T}}(\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)))] \\ \frac{\partial \mathcal{J}_{\mathcal{A}}(\hat{W}(t))}{\partial \hat{W}(t)} &= \beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\end{aligned}$$

Introducing the above gradient as a modification to adaptive law in (10) results in

$$\dot{\hat{W}}(t) = \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t) - k_{grad}\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\big] \tag{33}$$

where k_{grad} is the gradient learning rate. Since $\phi(\cdot) = \beta_x(x(t))$ for the ALR–modification, using this expression in (21) and (22), and applying the result as a modification to the adaptive law in (10) leads to the following adaptive law

$$\begin{aligned}\dot{\hat{W}}(t) &= \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t) \\ &\quad -k_{kf}S(t)\beta_x(x(t))R^{-1}\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\big] \\ \dot{S}(t) &= -S(t)\beta_x(x(t))R^{-1}\beta_x(x(t))^{\mathrm{T}}S(t) + Q\end{aligned} \tag{34}$$

where k_{kf} is the positive learning rate of this KF based version of ALR.

We compare the performance of the adaptive law in Eq. (33) with the adaptive law in Eq. (34), using the same wing rock dynamics as presented in the previous subsection, with $\sigma = 0.001$. Figures 11-13 show the standard ALR adaptive control responses using Eq. (33) with several different gains. It is clear from these figures that the system response is reasonable only for the case $\gamma = 100$, $k_{grad} = 100$ as depicted in Figure 13. Figures 14-16 show the KF based ALR (KF-ALR) adaptive control responses using Eq. (34) for the same gain settings. It is obvious from these figures that for all cases in which $k_{kf} > 0$, KF-ALR is able to suppress the uncertainty successfully, even with the lowest setting for γ.

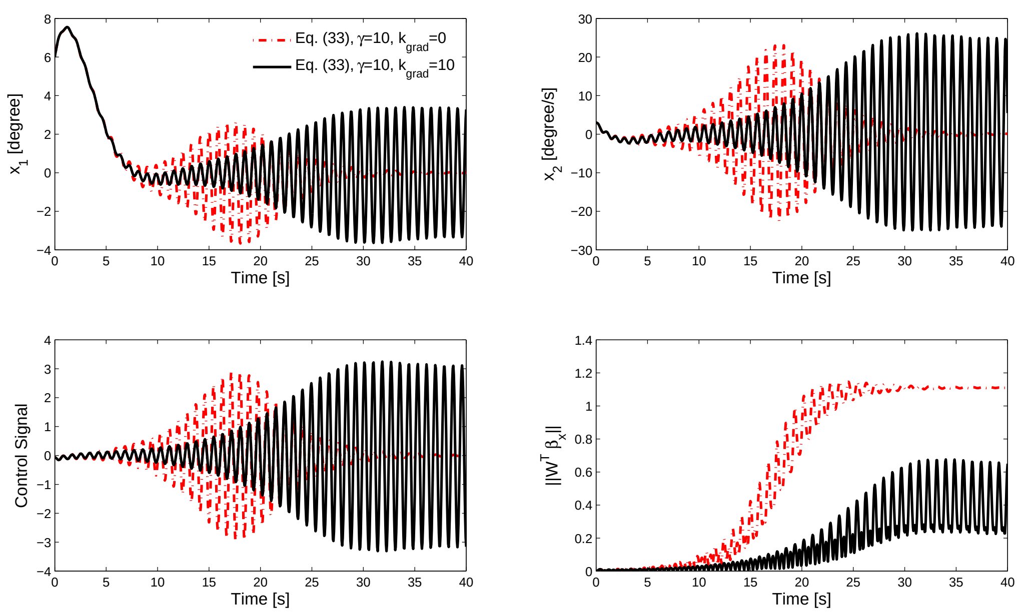

Figure 11: Standard ALR based adaptive control response with uncertainty using Eq. (33) with $\gamma = 10$ and $k_{grad} = 10$.

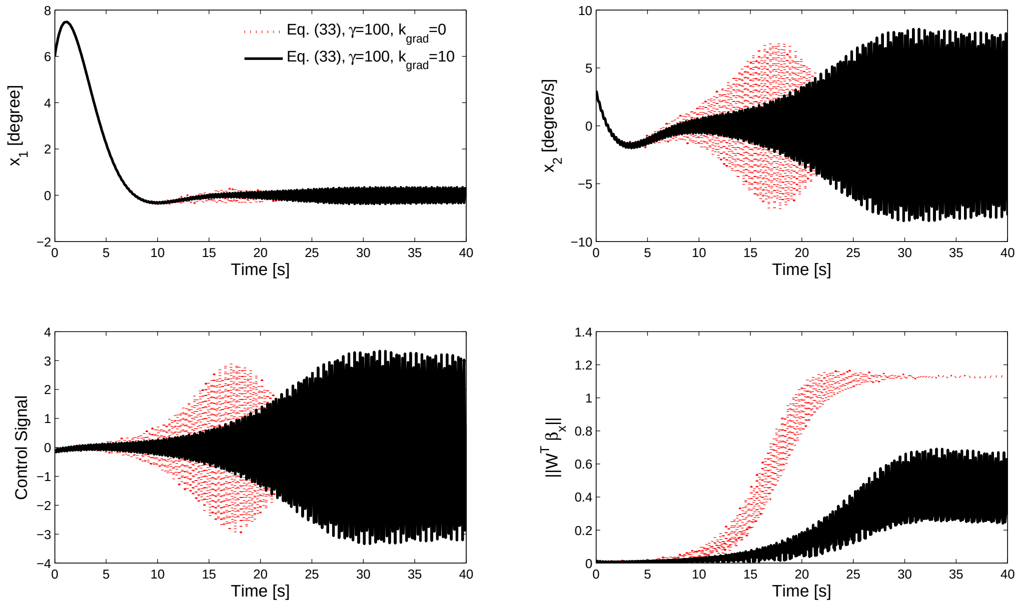

Figure 12: Standard ALR based adaptive control response with uncertainty using Eq. (33) with $\gamma = 100$ and $k_{grad} = 10$.

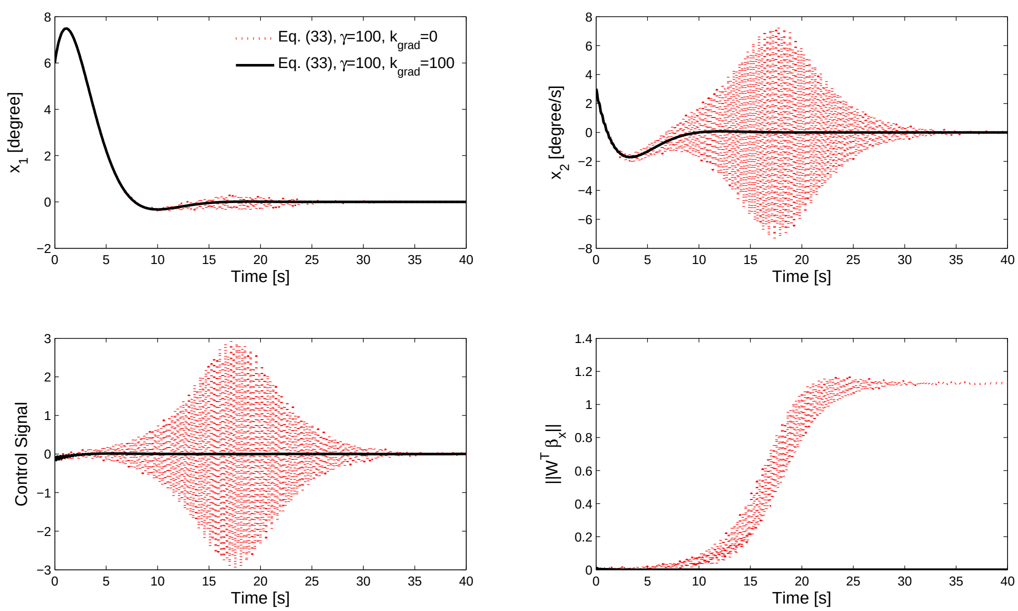

Figure 13: Standard ALR based adaptive control response with uncertainty using Eq. (33) with $\gamma = 100$ and $k_{grad} = 100$.

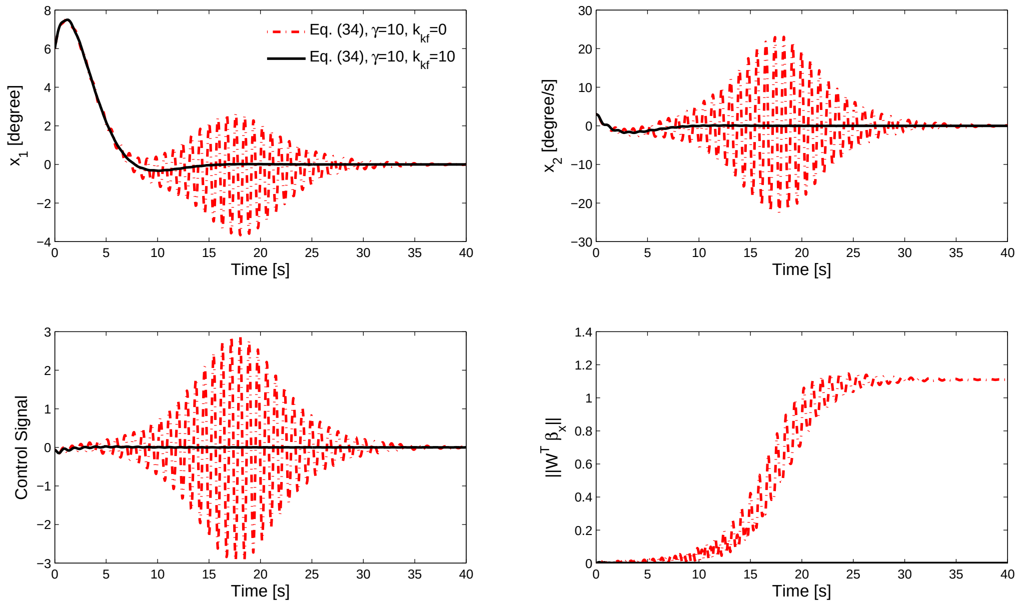

Figure 14: KF-ALR based adaptive control response with uncertainty using Eq. (34) with $\gamma = 10$ and $k_{grad} = 10$.

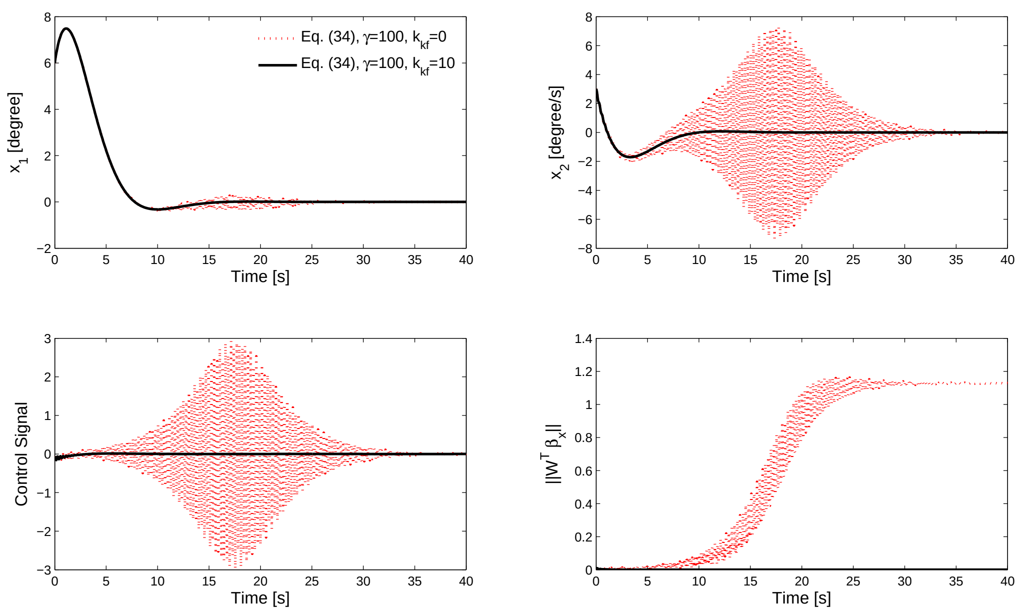

Figure 15: KF-ALR based adaptive control response with uncertainty using Eq. (34) with $\gamma = 100$ and $k_{grad} = 10$.

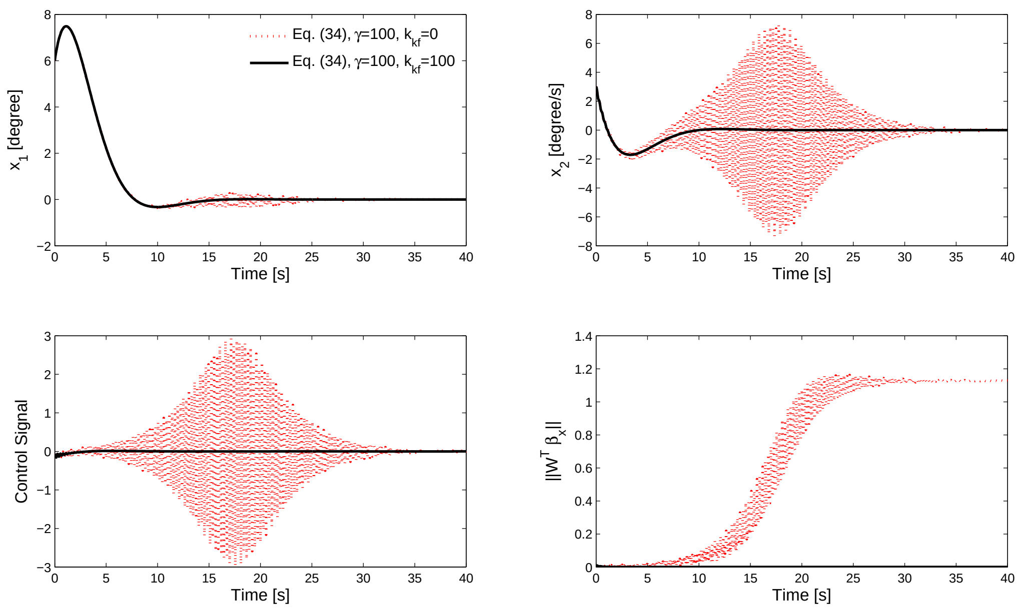

Figure 16: KF-ALR based adaptive control response with uncertainty using Eq. (34) with $\gamma = 100$ and $k_{grad} = 100$.

2.4.3 A New u_e–Modification for Input Constraints

A new treatment of actuator dynamics and limits is proposed using the KF based optimization procedure. We present the following results for illustration purposes without proof of stability. For this purpose, let $u(t)$ be the control commanded to the actuator of a nonlinear uncertain system, and let $u_s(t)$ be the output of the actuator. Defining $u_e(t) = u_s(t) - u(t)$, our aim is to approximately apply the following linear constraint

$$|u_e(t)|^{\frac{1}{2}}\hat{W}(t) = 0 \tag{35}$$

to the adaptive law with σ–modification, given by

$$\dot{\hat{W}}(t) = \gamma[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)], \tag{36}$$

Applying the form in (35) to (21) and (22), results in:

$$\begin{aligned}\dot{\hat{W}}(t) &= -|u_e(t)|S(t)R^{-1}\hat{W}(t)\\ \dot{S}(t) &= -|u_e(t)|S(t)R^{-1}S(t) + Q\end{aligned}$$

Remark 2.7. Since Q, R, and S_0 are chosen in the form $\bar{q}I$, $\bar{r}I$, and $\bar{s}_oI$, respectively, then the above equations reduce to:

$$\dot{\hat{W}}(t) = -\frac{|u_e(t)|s(t)}{\bar{r}}\hat{W}(t) \tag{37}$$

$$\dot{s}(t) = -|u_e(t)|\frac{s^2(t)}{\bar{r}} + \bar{q} \tag{38}$$

The adaptive law in (36) augmented with the u_e–modification term in (37) then becomes

$$\dot{\hat{W}}(t) = \gamma\Big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t) - \sigma_{u_e}\frac{|u_e(t)|s(t)}{\bar{r}}\hat{W}(t)\Big] \tag{39}$$

where σ_{u_e} is a positive learning rate. The main assumption is that the actuator output is known or can be estimated to sufficient accuracy using a model for the actuator.

We compare u_e–modification with the method of hedging [39, 40], that uses the adaptive law in (36) and modifies the reference model in (4) with a so-called hedge signal. The same wing rock adaptive control problem is used, with the design parameters $\gamma = 1$, $\sigma = 3$, and $\sigma_{u_e} = 50$. Furthermore, the following actuator dynamics are included

$$\dot{u}_s(t) \quad = \quad -\tau u_s(t) + \tau u(t) \tag{40}$$

where $\tau = 1/3$.

Figure 17 shows that the presence of actuator dynamics significantly degrades performance when using the adaptive law in Eq. (36). In Figures 18 and 19, hedging was used to modify the reference model dynamics. Figure 13 shows that the response with hedging becomes less oscillatory. However, Figure 18 shows that hedging modifies the reference model into a response that is more oscillatory. Figure 20 shows the response when u_e–modification is employed. The response is improved without modifying the reference model. Figure 21 compares the results of hedging and u_e–modification for the case where, in addition, an amplitude limit of ± 0.25 is applied. Both methods improve the system performance significantly when subjected to a hard limit, however u_e–modification achieves this without changing the reference model. Finally, Figure 22 shows the performance of the adaptive controller with u_e–modification, for the case where the actuator dynamics include an input time delay of 0.1 seconds. The system response is unstable without u_e–modification, and this response is significantly improved when u_e–modification term is employed.

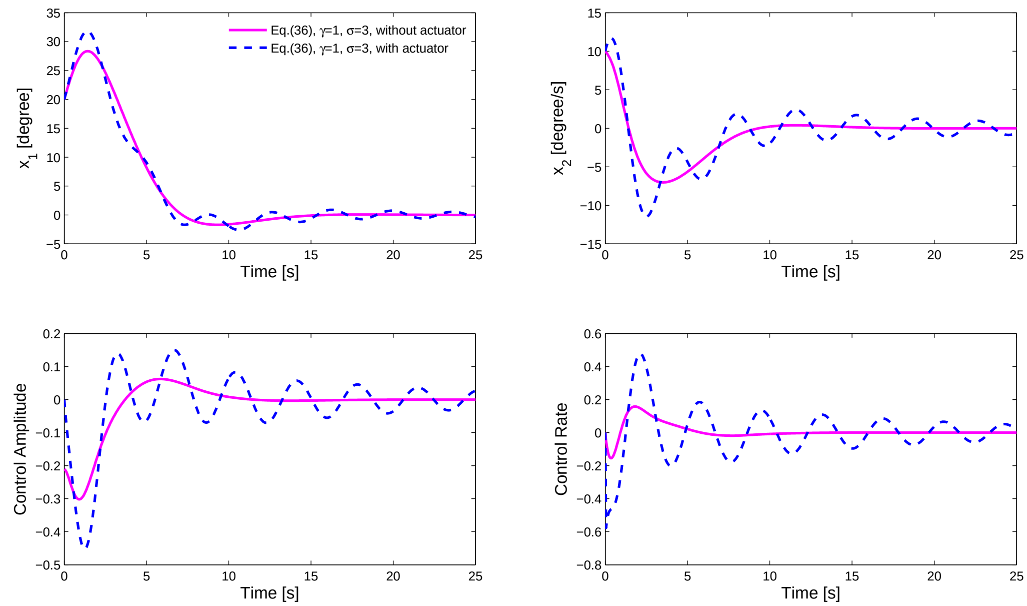

Figure 17: Adaptive control response without and with actuator dynamics.

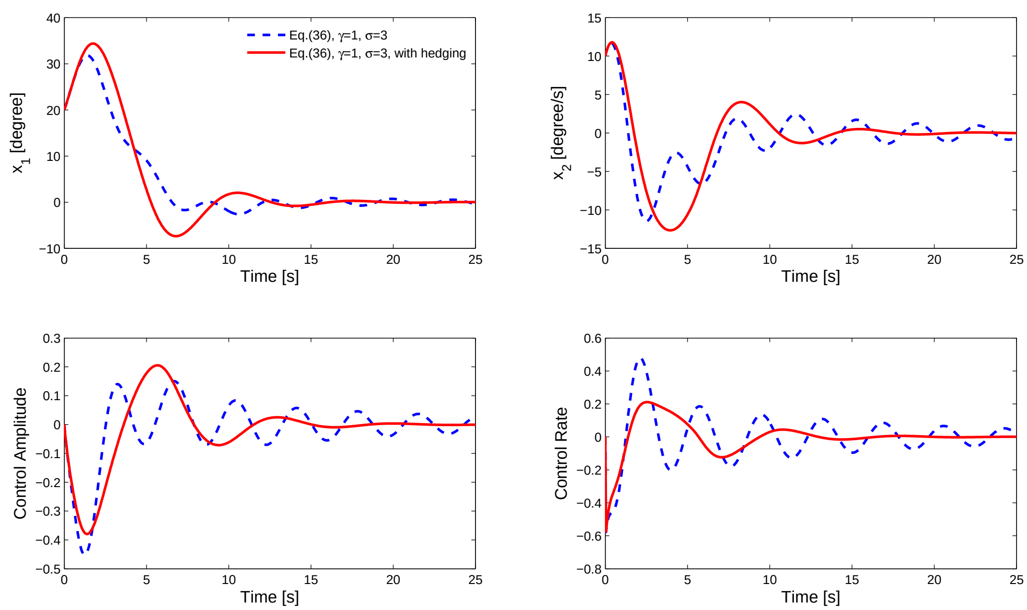

Figure 18: Performance of adaptive controller with hedging, with actuator dynamics.

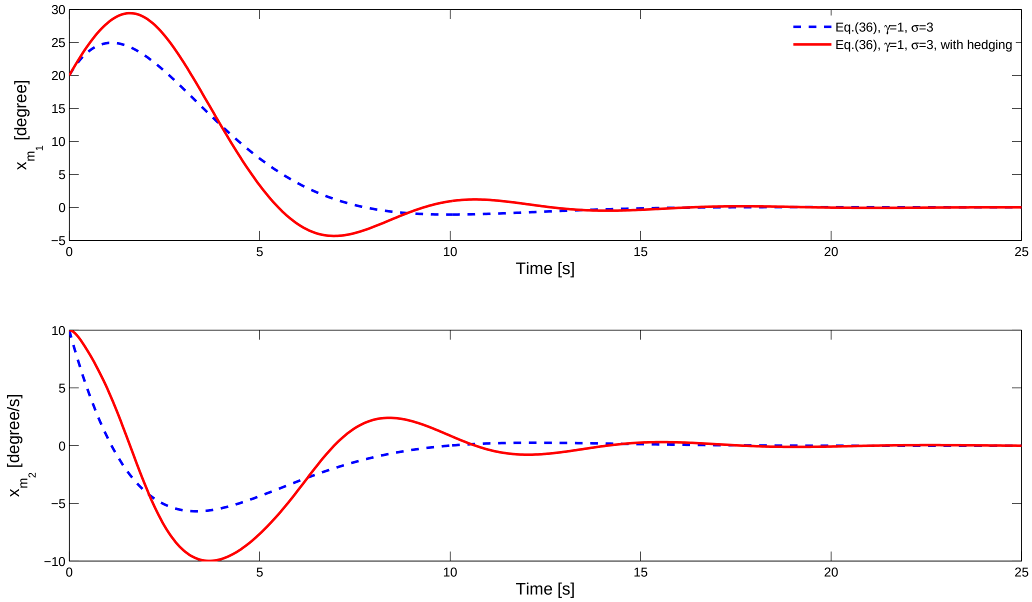

Figure 19: Response of reference model without and with hedging, with actuator dynamics.

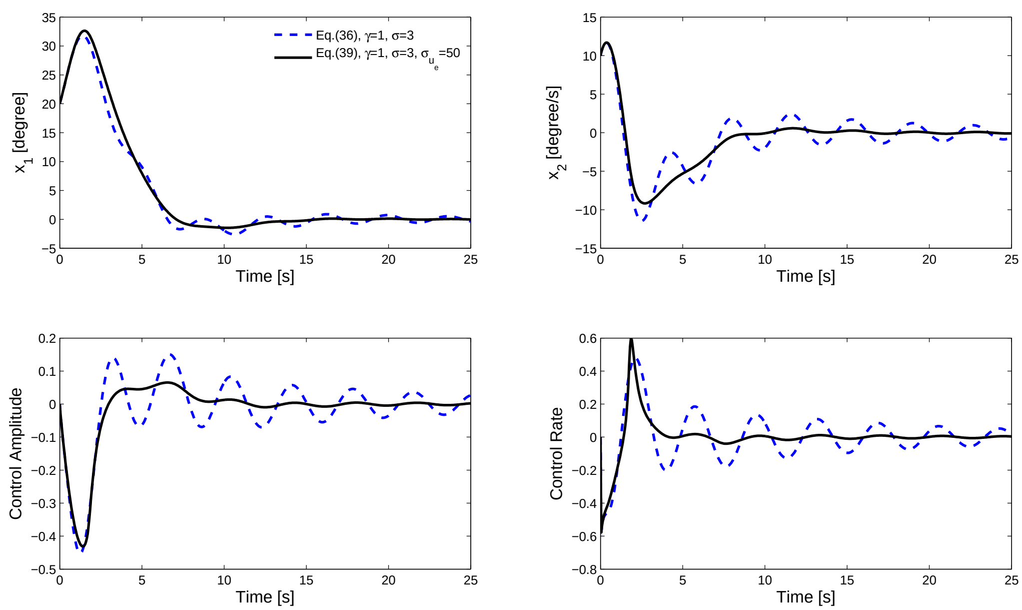

Figure 20: Performance of adaptive controller with u_e-modification, with actuator dynamics.

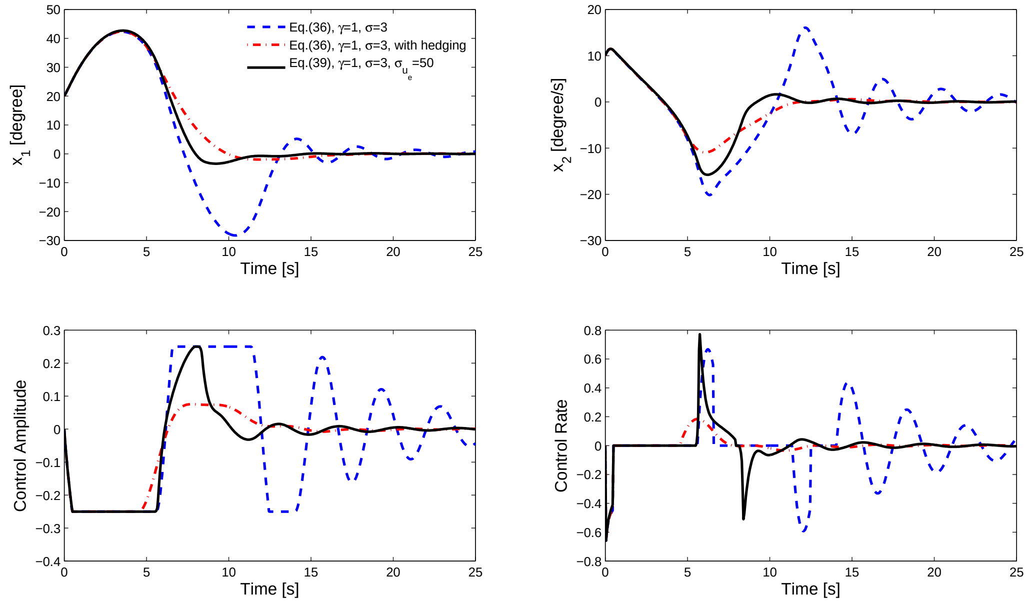

Figure 21: Comparison of adaptive control responses with hedging versus with u_e-modification, with actuator dynamics and limits.

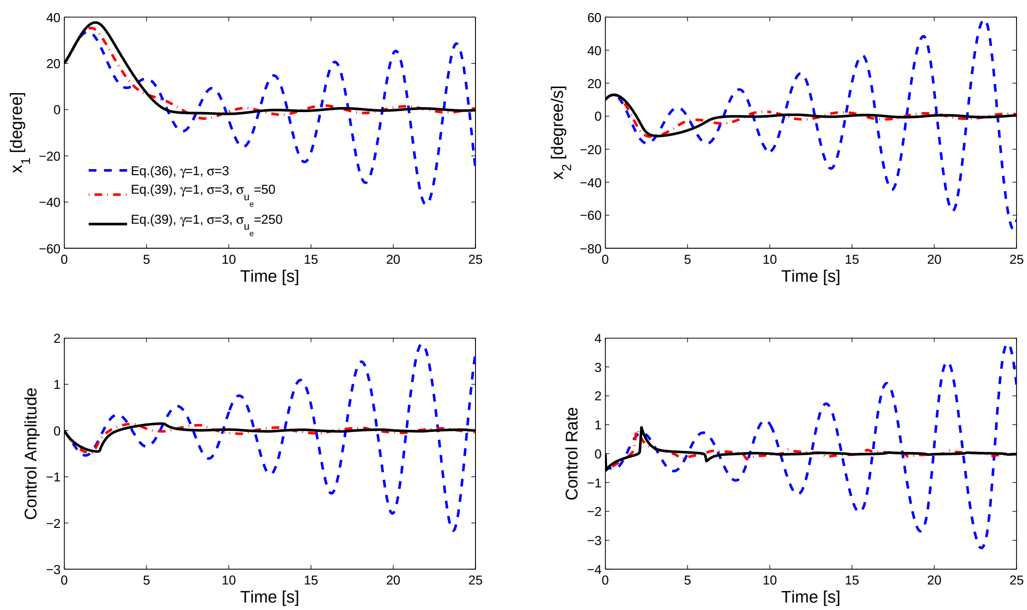

Figure 22: Performance of adaptive controller with u_e-modification, with actuator dynamics and time delay.

Figure 23: TwinSTAR flight test vehicle.

2.5 Conclusion

The intent of this chapter has been to present a procedure that approximately enforces linear constraints to modify an existing adaptive control design by using a Kalman filter optimization approach. The form of the constraint is chosen so that Kalman filter versions of existing modification terms are obtained. The simulation results on a model of wing rock dynamics illustrate the presented theory and show significant improvement over other gradient based modification approaches. One key difference in the Kalman filter based approach to modification is that the resulting gain on the modification term is time varying. The proposed approach to deriving a modification term for an adaptive controller can be used in place of all modification terms that can be equivalently viewed as the gradient of a norm measure of a linear constraint on the adaptation gains. Finally, the proposed approach has been flight tested on TwinSTAR, a conventional twin engine RC aircraft shown in Figure 23, and the results are reported in Ref. 14.

CHAPTER III

Kalman Filter Based Adaptive Control

3.1 Introduction

This chapter presents an approach to construct a novel weight adaptation law using Kalman filter optimization. This approach is an extension of the Kalman filter modification approach presented in the previous chapter. Instead of optimizing the modification gain, the adaptation gain is optimized. This approach results in an optimization based, time-varying adaptation gain that achieves better performance than adaptive laws that employ a fixed gain.

3.2 Preliminaries

In this section, we consider the problem of characterizing adaptive control laws for a class of uncertain systems given by (1), where the full state is available for feedback, the control input $u(t)$ is restricted to the class of admissible controls consisting of measurable functions, and Assumption 2.3 holds. In order to achieve trajectory tracking, we construct a reference system given by (4), where Assumption 2.2 holds. We consider the control law in (5) with the nominal control law given by (2) and the adaptive control law given by (6) where an estimate of W obtained from the adaptive weight update law

$$\dot{\hat{W}}(t) \quad = \quad \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB + \dot{\hat{W}}_{\mathrm{m}}(t)\big], \tag{41}$$

where γ is a positive fixed gain, $e(t) \equiv x(t) - x_{\mathrm{m}}(t)$ is the error, $P \in \mathbb{R}^{n\times n}$ is a positive-definite solution of the Lyapunov equation given by (8), and $\dot{\hat{W}}_{\mathrm{m}}(t) \in \mathbb{R}^{s\times m}$ is a modification term, for example $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma\hat{W}(t)$ for σ–modification term [37] and $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma|e(t)|\hat{W}(t)$ for e–modification term [60], where σ is a positive fixed gain.

We stated the UUB property of the closed loop system error defined by $e(t) \equiv x(t) - x_{\mathrm{m}}(t)$ and $\tilde{W}(t) \equiv W - \hat{W}(t)$ in Theorem 2.1 for (41) when e–modification is employed. For completeness, we now state and prove the UUB property for (41) when σ–modification is employed. For this purpose, the weight update error dynamics can be expressed as:

$$\dot{\tilde{W}}(t) = -\gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big] \tag{42}$$

Let $\zeta(t)$, $\mathcal{B}_r$, Ω_α, and $\mathcal{D}_x$ as in Theorem 2.1 (see Figure 2). Define $\tilde{P} = \mathrm{diag}[P,\ \gamma^{-1}I]$, $\Theta_{e_1} = \sqrt{||W||^2 + ||PB||^2\epsilon^2/(2\xi\sigma)}$, $\Theta_{\tilde{W}_1} = \sqrt{(2\xi\sigma||W||^2 + ||PB||^2\epsilon^2)/(\lambda_{\min}(Q_{\mathrm{m}}) - 2\xi)}$, and $\underline{r}_\sigma^2 = \frac{\lambda_{\max}(P)\Theta_{e_1}^2 + \frac{1}{\gamma}\Theta_{\tilde{W}_1}^2}{\lambda_{\min}(\tilde{P})}$, where $\lambda_{\min}(Q_{\mathrm{m}}) > 2\xi$, $\xi > 0$.

Theorem 3.1. Consider the uncertain system given by (1) subject to Assumption 2.3. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has a weight update law in the form (41) with $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma\hat{W}(t)$. If $\zeta(t_0) \in \Omega_\alpha$ and $r_\alpha > \underline{r}_\sigma$, then $\zeta(t)$ is UUB with ultimate bound $\underline{r}_\sigma$.

Proof. Consider the Lyapunov-like function candidate given by (13) where $P > 0$ satisfies (8). The time derivative of (13) along the trajectories of (11) and (42) can be expressed as

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) &= e^{\mathrm{T}}(t)P[A_{\mathrm{m}}e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + B\varepsilon(x(t))] - \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\beta(x(t))e^{\mathrm{T}}(t)PB] \\
&\quad +\sigma\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\hat{W}(t)] \\
&= \frac{1}{2}e^{\mathrm{T}}(t)[A_{\mathrm{m}}^{\mathrm{T}}P + PA_{\mathrm{m}}]e(t) + \sigma\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\hat{W}(t)] + e^{\mathrm{T}}(t)PB\varepsilon(x(t)) \\
&= -\frac{1}{2}e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) + \sigma\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)(W - \tilde{W}(t))] + e^{\mathrm{T}}(t)PB\varepsilon(x(t)) \\
&\leq -c_1|e(t)|^2 + c_2|e(t)| - c_3||\tilde{W}(t)||^2 + c_4 \\
&\leq -(c_1 - \xi)|e(t)|^2 - c_3||\tilde{W}(t)||^2 + c_4 + \frac{c_2^2}{4\xi},
\end{aligned} \tag{43}$$

where $c_1 = \frac{\lambda_{\min}(Q_{\mathrm{m}})}{2} > \xi > 0$, $c_2 = ||PB||\epsilon > 0$, $c_3 = \frac{\sigma}{2} > 0$, and $c_4 = \frac{\sigma||W||^2}{2} > 0$. Consequently, either $|e(t)| \geq \Theta_{e_1}$ or $||\tilde{W}(t)|| \geq \Theta_{\tilde{W}_1}$ renders $\dot{\mathcal{V}}(e(t), \tilde{W}(t)) < 0$, where $\Theta_{e_1} = \sqrt{\frac{c_4 + \frac{c_2^2}{4}}{c_1 - 1}}$ and $\Theta_{\tilde{W}_1} = \sqrt{\frac{c_4 + \frac{c_2^2}{4}}{c_3}}$. It follows now identical to the proof of Theorem 2.1 that $e(t)$ and $\tilde{W}(t)$ are UUB with ultimate bound $\underline{r}_\sigma$. □

3.3 Motivation

For motivational purposes, we show that σ– and e–modification based *gradient* weight adaptation laws can be obtained by enforcing the constraint

$$\hat{W}^{\mathrm{T}}(t)\Phi_1(t, x(t), e(t)) = \Phi_2(t, x(t), e(t)) \tag{44}$$

on the weight estimates, where $\Phi_1(\cdot) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s\times l}$ and $\Phi_2(\cdot) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m\times l}$.

Remark 3.1. The form of the constraint in (44) differs than the form of the constraint in (15). If $\Phi_2(\cdot) = 0$ in (44), then (44) reduces to (15).

Lemma 3.1. The σ–modification based gradient adaptive control law given by (41) with $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma\hat{W}(t)$ can be obtained by using a gradient approach to enforcing the constraint given by (44) on the weight estimates, where:

$$\Phi_1(\cdot) \equiv \sqrt{\sigma} I \tag{45}$$

$$\Phi_2(\cdot) \equiv \beta(x(t)) e^{\mathrm{T}}(t) P B / \sqrt{\sigma}, \quad \sigma > 0 \tag{46}$$

Proof. Consider the cost

$$\mathcal{J}(\hat{W}(t)) = \frac{1}{2} ||\hat{W}^{\mathrm{T}}(t)\Phi_1(\cdot) - \Phi_2(\cdot)(x(t))||^2 \tag{47}$$

where the negative gradient of $\mathcal{J}(\hat{W}(t))$ with respect to $\hat{W}(t)$ gives

$$-\frac{\partial \mathcal{J}(\hat{W}(t))}{\partial \hat{W}(t)} = \beta(x(t)) e^{\mathrm{T}}(t) P B - \sigma \hat{W}(t) \tag{48}$$

where (45) and (46) are used. Multiplying both sides of (48) by γ results in σ–modification based adaptive control law in (41) with $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma \hat{W}(t)$. □

Lemma 3.2. The e–modification based gradient adaptive control law given by (41) with $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma |e(t)| \hat{W}(t)$ can be obtained by using a gradient approach to enforcing the constraint given by (44) on the weight estimates, where:

$$\Phi_1(\cdot) \equiv \sqrt{\sigma |e(t)|} I \tag{49}$$

$$\Phi_2(\cdot) \equiv \beta(x(t)) e^{\mathrm{T}}(t) P B / \sqrt{\sigma |e(t)|}, \quad \sigma > 0 \tag{50}$$

Proof. Consider the cost given by (47) where the negative gradient of $\mathcal{J}(\hat{W}(t))$ with respect to $\hat{W}(t)$ gives

$$-\frac{\partial \mathcal{J}(\hat{W}(t))}{\partial \hat{W}(t)} = \beta(x(t)) e^{\mathrm{T}}(t) P B - \sigma |e(t)| \hat{W}(t) \tag{51}$$

where (49) and (50) are used. Multiplying both sides of (51) by γ results in e–modification based adaptive control law in (41) with $\dot{\hat{W}}_{\mathrm{m}}(t) = -\sigma |e(t)| \hat{W}(t)$. □

It is shown in Lemmas 3.1 and 3.2 that σ– and e–modification based gradient adaptive control laws can be obtained by imposing the constraint given by (44) on the weight estimates subject to (45) and (46), and (49) and (50), respectively. In the next section, we introduce a different treatment for a general class of linear constraints to obtain Kalman filter based adaptive control (KF-AC) laws.

3.4 Kalman Filter Based Adaptive Control Formulation

The KF-AC architecture is applicable to most direct adaptive control schemes available in the literature in full-state feedback control or output feedback control forms. For simplicity and without loss of generality, we consider the full-state feedback formulation. As highlighted in Lemmas 3.1 and 3.2, (41) can be viewed as a gradient approach for enforcing the linear constraint (44) on the weight estimates.

Next, the problem of estimating W, so that it satisfies the the linear constraint in (44), will be formulated as an optimization problem. For this purpose, (44) can be expressed in the following equivalent vector form

$$\tilde{\Phi}_1^{\mathrm{T}}(t, x(t), e(t))\omega(t) = \tilde{\Phi}_2^{\mathrm{T}}(t, x(t), e(t)) \tag{52}$$

where $\tilde{\Phi}_1(\cdot) \in \mathbb{R}^{ms \times ml}$, $\tilde{\Phi}_2(\cdot) \in \mathbb{R}^{1 \times ml}$, and $\omega(t) = \mathrm{vec}(\hat{W}(t)) \in \mathbb{R}^{ms}$.

Define the stochastic process

$$\dot{\omega}(t) = q(t), \tag{53}$$

$$z(t) = \tilde{\Phi}_1^{\mathrm{T}}(\cdot)\omega(t) + r(t), \tag{54}$$

where $q(t)$ and $r(t)$ are zero-mean, Gaussian, white noise processes with covariances

$$\mathbb{E}\{q(t)q^{\mathrm{T}}(\tau)\} = \bar{Q}\delta(t-\tau),\ \bar{Q} \in \mathbb{R}^{ms \times ms} > 0 \tag{55}$$

$$\mathbb{E}\{r(t)r^{\mathrm{T}}(\tau)\} = \bar{R}\delta(t-\tau),\ \bar{R} \in \mathbb{R}^{ml \times ml} > 0 \tag{56}$$

and $z(t)$ is regarded as a measurement. The estimate of $z(t)$ is

$$\hat{z}(t) = \tilde{\Phi}_1^{\mathrm{T}}(\cdot)\hat{\omega}(t) \tag{57}$$

where $\hat{\omega}(t)$ is an estimate of $\omega(t)$.

The Kalman filter associated with this problem formulation is given by [19]:

$$\dot{\hat{\omega}}(t) = \bar{S}(t)\tilde{\Phi}_1(\cdot)\bar{R}^{-1}\big[z(t) - \hat{z}(t)\big], \quad \hat{\omega}(0) = 0 \tag{58}$$

$$\dot{\bar{S}}(t) = -\bar{S}(t)\tilde{\Phi}_1(\cdot)\bar{R}^{-1}\tilde{\Phi}_1^{\mathrm{T}}(\cdot)\bar{S}(t) + \bar{Q}, \quad \bar{S}(0) = \bar{S}_0 > 0 \tag{59}$$

where $\bar{S}(t) \in \mathrm{IR}^{ms \times ms}$. Since the objective is to approximately satisfy the constraint (44), the logical choice for $z(t)$ when employing this estimator is $z(t) = \tilde{\Phi}_2^{\mathrm{T}}(\cdot)$. Furthermore, choosing $\bar{R} = I_{m\times m} \otimes R$, with $R \in \mathrm{IR}^{l\times l} > 0$, and $\bar{Q} = I_{m\times m} \otimes Q$, with $Q \in \mathrm{IR}^{s\times s} > 0$, Lemma 2.1 of Section 2.3 shows that (58) and (59), with $z(t) = \tilde{\Phi}_2^{\mathrm{T}}(\cdot)$, can equivalently be expressed as:

$$\dot{\hat{W}}(t) = -S(t)\Phi_1(\cdot)R^{-1}\left[\Phi_1^{\mathrm{T}}(\cdot)\hat{W}(t) - \Phi_2^{\mathrm{T}}(\cdot)\right],\ \hat{W}(0) = 0 \tag{60}$$

$$\dot{S}(t) = -S(t)\Phi_1(\cdot)R^{-1}\Phi_1^{\mathrm{T}}(\cdot)S(t) + Q,\ S(0) = S_0 > 0 \tag{61}$$

where $S(t) \in \mathrm{IR}^{s\times s}$.

Remark 3.2. The Riccati equation in (61) can be associated with the following system of equations:

$$\begin{aligned} \dot{w}_i &= q_1(t) \\ z_i(t) &= \Phi_1^{\mathrm{T}}(\cdot)w_i + r_1(t) \end{aligned}$$

in which w_i are the process dynamics associated with the i-th column of W, z_i is the corresponding measurement vector, $\mathbb{E}\{q_1(t)q_1^{\mathrm{T}}(\tau)\} = Q\delta(t-\tau)$, and $\mathbb{E}\{r_1(t)r_1^{\mathrm{T}}(\tau)\} = R\delta(t-\tau)$.

Assumption 3.1. The observability gramian

$$W_0(t_0, t) = \int_{t_0}^{t} \Phi_1(t, x(t), e(t))\Phi_1^{\mathrm{T}}(t, x(t), e(t))\mathrm{d}t \tag{62}$$

of the system defined in Remark 3.2 satisfies the following condition for uniform complete observability [43]:

$$0 < \alpha_0(\sigma)I_l \le W_0(t_0, t_0 + \sigma) \le \alpha_1(\sigma)I_l \tag{63}$$

for all t_0, and for some fixed constant $\sigma > 0$.

Remark 3.3. The condition in Assumption 3.1 is sufficient to show that $S(t)$ is positive-definite, and lower and upper bounded, independent of t_0, if $\Phi_1(t, x(t), e(t))$ is bounded pointwise in time.

3.5 Stability Analysis

The stability analysis of the KF-AC law given by (60) and (61) depends on the constraint given by (44), and hence, it is problem dependent. Therefore, we need to limit our presentation, without loss of generality, by giving the KF-AC forms of σ– and e–modification based adaptive control laws.

From the constraint (44) with (45) and (46), and the equations (60) and (61), σ–modification based KF-AC (KF$_\sigma$–AC) law can be given by:

$$\dot{\hat{W}}(t) = S(t)R^{-1}\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big] \quad (64)$$

$$\dot{S}(t) = -\sigma S(t)R^{-1}S(t) + Q \quad (65)$$

Similarly, from the constraint (44) with (49) and (50), and the equations (60) and (61), e–modification based KF-AC (KF$_e$–AC) law can be given by:

$$\dot{\hat{W}}(t) = S(t)R^{-1}\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t)\big] \quad (66)$$

$$\dot{S}(t) = -\sigma|e(t)|S(t)R^{-1}S(t) + Q \quad (67)$$

Remark 3.4. For the KF$_\sigma$–AC law, the observability gramian given by (62) becomes $W_0(t_0,t) = \int_{t_0}^{t}\sigma \mathrm{d}\tau I_s$, and hence, Assumption 3.1 is satisfied. Furthermore, for the KF$_e$–AC law, the observability gramian given by (62) becomes $W_0(t_0,t) = \int_{t_0}^{t}\sigma|e(\tau)|\mathrm{d}\tau I_s$, and it is positive-definite for all $t \geq t_0$ as long as $|e(\tau)| \neq 0$ over the interval $[t_0,t]$. It will be necessary to prove that $|e(t)|$ remains bounded pointwise in time using arguments similar to those given in the proof of Theorem 2.2.

Remark 3.5. $S(t)R^{-1}$ is a time-varying and bounded Kalman filter gain in the KF$_\sigma$–AC and KF$_e$–AC laws. However, for the following results, we require it to be positive-definite and symmetric as well. For this purpose, we set $R = rI$, $r > 0$.

The weight update error dynamics for KF$_\sigma$–AC law can be given using (64) as:

$$\dot{\tilde{W}}(t) = -S(t)R^{-1}\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big] \quad (68)$$

Define $\zeta(t)$, $\mathcal{B}_r$, Ω_α, and $\mathcal{D}_x$ as in Theorem 2.1 (see Figure 2). Further define $\tilde{P}_{\rm KF}(t) = {\rm diag}[P,\ \Gamma^{-1}(t)]$, $\Gamma(t) = S(t)R^{-1} = S(t)/r$, $\Theta_{e_2} = \sqrt{v_3/v_1}$, $\Theta_{\tilde{W}_2} = \sqrt{v_3/v_2}$, and $\underline{r}^2_{\sigma KF} = \frac{\lambda_{\max}(P)\Theta^2_{e_2} + \bar{\Gamma}^{-1}\Theta^2_{\tilde{W}_2}}{\lambda_{\min}(\underline{\tilde{P}}_{\rm KF})}$, where $\underline{\tilde{P}}_{\rm KF} = {\rm diag}[P,\ \bar{\Gamma}^{-1}]$, $\bar{\Gamma} = \max_{S(t)}\{\Gamma(t)\}$, $v_1 = \frac{\lambda_{\min}(Q_{\rm m})}{2} - \xi > 0$, $\xi > 0$, $v_2 = \frac{1}{2}\lambda_{\min}(Q\bar{\Gamma}^{-1})$, and $v_3 = \frac{\sigma}{2}||W||^2 + \frac{1}{4\xi}||PB||^2\epsilon^2$. The next theorem states the UUB property of KF_σ–AC law.

Theorem 3.2. Consider the uncertain system given by (1) subject to Assumption 2.3. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) together with KF_σ–AC law given by (64). If $\zeta(t_0) \in \Omega_\alpha$ and $r_\alpha > \underline{r}_{\sigma KF}$, then $\zeta(t)$ is UUB with ultimate bound $\underline{r}_{\sigma KF}$.

Proof. Consider the Lyapunov function candidate

$$\mathcal{V}(e(t), \tilde{W}(t)) = \frac{1}{2}e^{\rm T}Pe + \frac{1}{2}{\rm tr}[\tilde{W}(t)^{\rm T}\Gamma^{-1}(t)\tilde{W}(t)] \tag{69}$$

where $P > 0$ satisfies (8). The time derivative of (69) along the trajectories of (11) and (68) can be expressed as

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) &= -\frac{1}{2}e^{\rm T}(t)Q_{\rm m}e(t) + e^{\rm T}(t)PB\varepsilon(x(t)) + \sigma{\rm tr}[\tilde{W}^{\rm T}(t)(W - \tilde{W}(t))] \\
&\quad -\frac{1}{2}{\rm tr}[\tilde{W}^{\rm T}(t)\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t)\tilde{W}(t)] \\
&\leq -\Big(\frac{1}{2}\lambda_{\min}(Q_{\rm m}) - \xi\Big)|e(t)|^2 - \frac{\sigma}{2}||\tilde{W}(t)||^2 + \frac{\sigma}{2}||W||^2 + \frac{1}{4\xi}||PB||^2\epsilon^2 \\
&\quad -\frac{1}{2}{\rm tr}[\tilde{W}^{\rm T}(t)\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t)\tilde{W}(t)]
\end{aligned} \tag{70}$$

where $\xi > 0$. Using $\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t) = rQS^{-1}(t) - \sigma I_s$ in (70) gives:

$$\dot{\mathcal{V}}(\cdot) \leq -v_1|e(t)|^2 - v_2||\tilde{W}(t)||^2 + v_3 \tag{71}$$

Consequently, either $|e(t)| \geq \Theta_{e_2}$ or $||\tilde{W}(t)|| \geq \Theta_{\tilde{W}_2}$ renders $\dot{\mathcal{V}}(\cdot) < 0$. Therefore, $e(t)$ and $\tilde{W}(t)$ are uniformly bounded. Define $\Omega_\beta = \{\zeta(t) \in \mathcal{B}_r : \zeta(t)^{\rm T}\tilde{P}_{\rm KF}(t)\zeta(t) \leq \lambda_{\max}(P)\Theta^2_{e_2} + \Gamma^{-1}(t)\Theta^2_{\tilde{W}_2}(t)\}$. Then, Ω_α is a positively invariant set if $\Omega_\beta \subset \Omega_\alpha$. This requires that $\lambda_{\max}(P)\Theta^2_{e_2} + \Gamma^{-1}(t)\Theta^2_{\tilde{W}} < \alpha$. The minimum size of $\mathcal{B}_r$ that ensures this condition is $r_\alpha > \underline{r}_{\sigma_{KF}}$. Therefore, if $\zeta(t_0) \in \Omega_\alpha$, then $\zeta(t)$ is UUB with ultimate bound $\underline{r}_{\sigma_{KF}}$. □

We next state the UUB property of the KF_e–AC law. For this purpose, the weight update error dynamics for KF_e–AC law can be given using (66) as:

$$\dot{\tilde{W}}(t) = -S(t)R^{-1}\left[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma|e(t)|\hat{W}(t)\right] \tag{72}$$

Define $\Theta_{e_3} = \bar{v}_3/\bar{v}_1$, $\Theta_{\tilde{W}_3} = \sqrt{\bar{v}_3/\bar{v}_2}$, and $\underline{r}^2_{e_{KF}} = \frac{\lambda_{\max}(P)\Theta^2_{e_3} + \bar{\Gamma}^{-1}\Theta^2_{\tilde{W}_3}}{\lambda_{\min}(\underline{\tilde{P}}_{\mathrm{KF}})}$, where $\bar{v}_1 = \frac{\lambda_{\min}(Q_{\mathrm{m}})}{2} > 0$, $\bar{v}_2(t) = \frac{1}{2}\lambda_{\min}(Q\bar{\Gamma}^{-1})$, and $\bar{v}_3 = \frac{1}{2}\sigma||W||^2 + ||PB||\epsilon$.

Theorem 3.3. Consider the uncertain system given by (1) subject to Assumption 2.3. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) together with KF_e–AC law given by (66). If $\zeta(t_0) \in \Omega_\alpha$ and $r_\alpha > \underline{r}_{e_{KF}}$, then either $\zeta(t)$ is UUB with ultimate bound $\underline{r}_{e_{KF}}$, or $e(t) = 0$ for all $t \geq T$ and $\tilde{W}(t)$ is bounded.

Proof. Consider the Lyapunov function candidate (69) where $P > 0$ satisfies (8). Applying similar arguments as in the proofs of Theorems 2.1 and 3.2, the time derivative of (69) along the trajectories of (11) and (72) can be expressed as:

$$\dot{\mathcal{V}}(\cdot) \;\; \leq \;\; -|e(t)|(\bar{v}_1|e(t)| + \bar{v}_2||\tilde{W}(t)||^2 - \bar{v}_3) \tag{73}$$

Note that $\dot{\mathcal{V}}(\cdot)$ is negative as long as the term in braces is positive and $e(t) \neq 0$. If $e(t) = 0$ for all $t \geq T$, then $\tilde{W}(t)$ is bounded since $\dot{\tilde{W}}(t) = 0$ from (72). It now follows identical to the proofs of Theorems 2.2 and 3.2 that if $\zeta(t_0) \in \Omega_\alpha$, then $\zeta(t)$ is UUB with ultimate bound $\underline{r}_{e_{KF}}$. □

3.6 Examples on a Model of Wing Rock Dynamics

In this section we apply our results to a model of wing rock dynamics given by (32) with $\alpha_1 = 0.9814$, $\alpha_2 = 1.5848$, $\alpha_3 = -0.6245$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0215$.

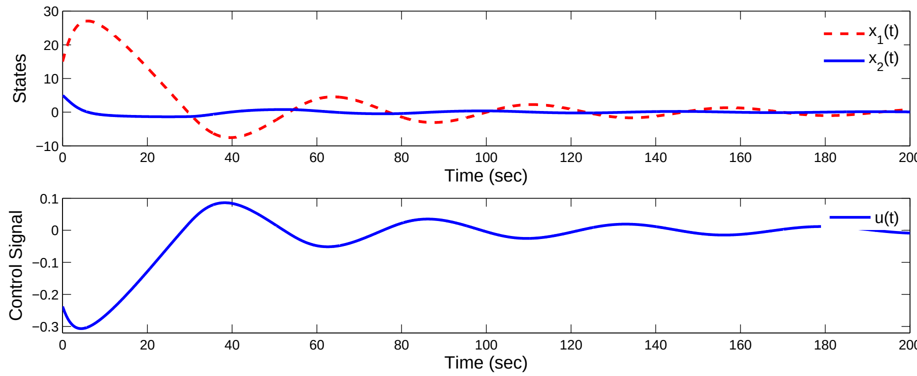

Figure 24: Nominal control response.

The control objective is to minimize the oscillations of the wing rock dynamics in order to stabilize the system at the zero trim condition. We selected the initial state values as $x(0) = [15^o, 5^o/s]^{\mathrm{T}}$. The reference system is selected to be second order with a natural frequency of 1.0 rad/s, and a damping of 0.8, and to have unity low frequency gain from $r(t)$ to $x_1(t)$. This corresponds to choosing $K_1 = [1,\ 1.6]$ and $K_2 = 1$. The form in Assumption 2.3 was used to approximate the uncertainty with $\beta(x(t)) = [\beta_1(x(t)),\ \beta_2(x(t))]^{\mathrm{T}}$, where $\beta_i(x) = 1/(1+e^{-x_i})$, $i = 1, 2$, are sigmoidal type basis functions. Furthermore, we chose $Q_{\mathrm{m}} = I_2$ for (8) and $Q = I_2$, $R = 0.01 I_2$, $S_0 = I_2$ for the KF$_e$–AC law given by (66) and (67).

Fig. 24 shows the nominal control response with the matched uncertainty $\Pi(x(t))$, where $u_{\mathrm{ad}}(t) = 0$. Figs. 25–28 presents the standard, gradient, e–modification based adaptive controller results on this system for different gains. It is obvious from all these figures that the achieved system performance is not desirable due to high frequency oscillations in the system responses. Fig. 29 shows the performance of KF$_e$–AC law on this system, where its time-varying Kalman filter gain is also given in Fig. 30. The performance is significantly improved relative to the results obtained using the standard, gradient, e–modification term. Hence, this approach leads to smaller tracking errors without incurring high-frequency oscillations in the system response.

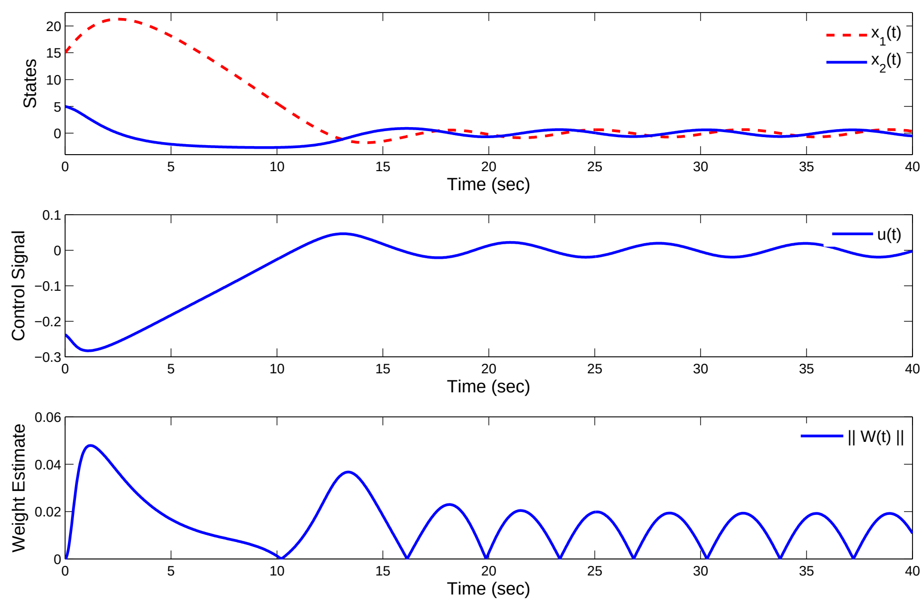

Figure 25: Standard e- modification based adaptive controller, $\gamma = 1$ and $\gamma \times \sigma = 40$.

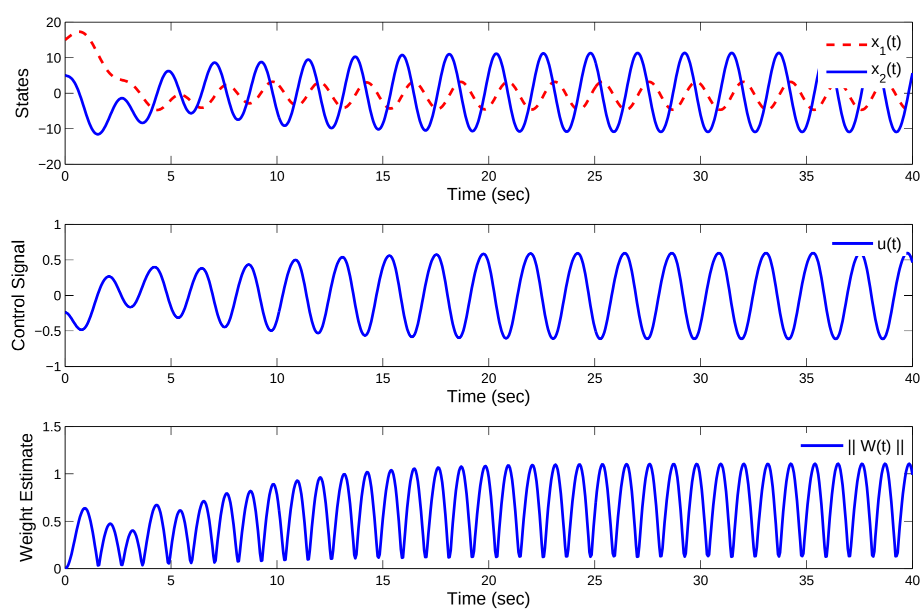

Figure 26: Standard e- modification based adaptive controller, $\gamma = 10$ and $\gamma \times \sigma = 40$.

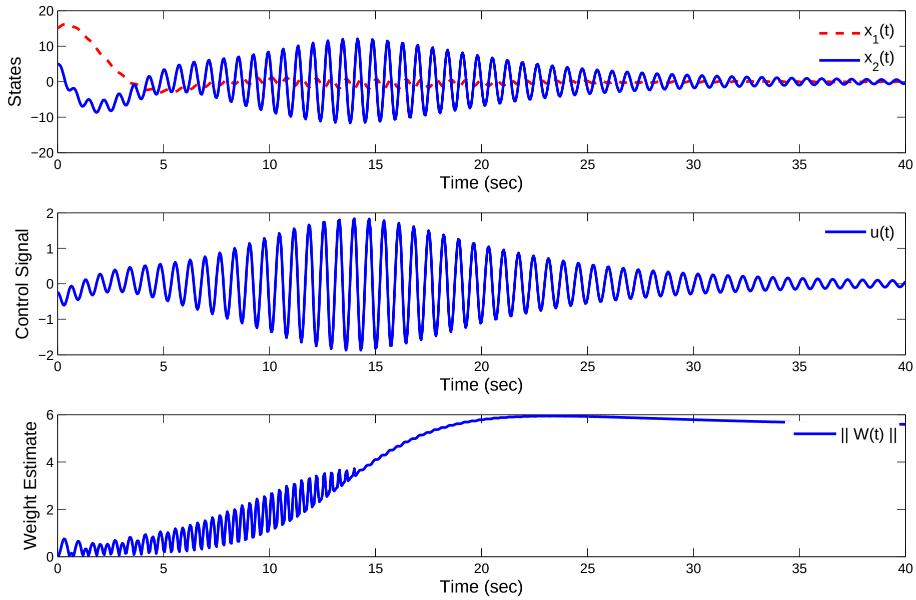

Figure 27: Standard e- modification based adaptive controller, $\gamma = 100$ and $\gamma \times \sigma =$ 40.

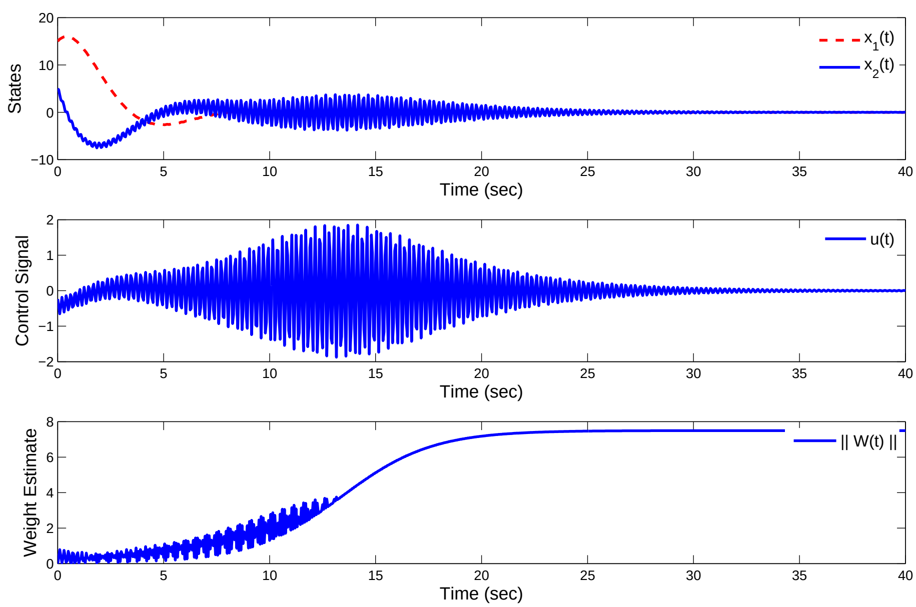

Figure 28: Standard e- modification based adaptive controller, $\gamma = 1000$ and $\gamma \times \sigma =$ 40.

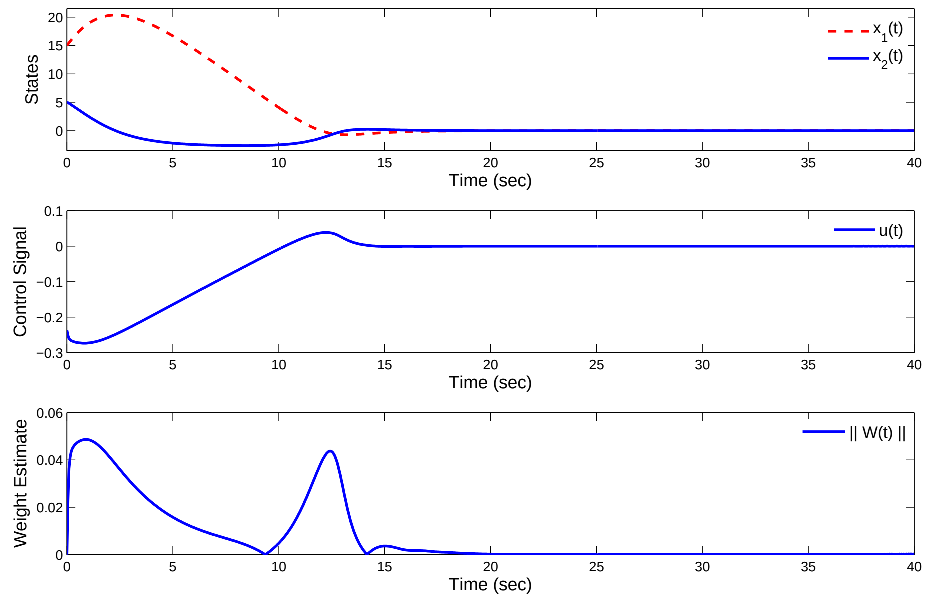

Figure 29: KF_e–AC law, $\sigma = 40$.

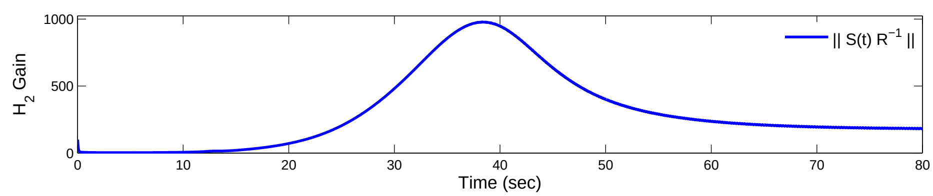

Figure 30: Adaptation gain $\Gamma(t)$ for the KF_e–AC law in Figure 29.

3.7 Conclusion

The intent of this chapter has been to present a Kalman filter based approach to deriving a novel weight adaptation law. A model of wing rock dynamics illustrates the presented theory. The new controllers showed significant improvement over their gradient based counterparts. One key difference in the Kalman filter based approach is that the resulting adaptation gain is time varying, and requires minimal tuning of the design parameters. The proposed approach can be used in place of all adaptive control laws that can be equivalently viewed as the gradient of a norm of the error in the linear constraint. Finally, the proposed approach has been flight tested on TwinSTAR and the results are reported in Ref. 15.

CHAPTER IV

Derivative-Free Adaptive Control: The Full-State Feedback Case

4.1 Introduction

A derivative-free, delayed weight update law is developed for model reference adaptive control of continuous-time uncertain systems, without assuming the existence of constant ideal weights and without requiring discretization. Using a Lyapunov-Krasovskii functional it is proven that the error dynamics are uniformly ultimately bounded, without the need for modification terms in the adaptive law. Estimates for the ultimate bound and the exponential rate of convergence to the ultimate bound are provided. We also discuss employing various modification terms for further improving performance and robustness of the adaptively controlled system. Examples illustrate that the proposed derivative-free model reference adaptive control (DF-MRAC) law is advantageous for applications to systems that can undergo a sudden change in dynamics.

4.2 Preliminaries

In this chapter, we consider the uncertain system given by (1), in which $\Delta(x(t))$ is replaced by $\Delta(t, x(t))$, the full state is available for feedback, and the control input $u(t)$ is restricted to the class of admissible controls consisting of measurable functions. In order to achieve trajectory tracking, we construct a reference system given by (4), where Assumption 2.2 holds. We consider the control law in (5) with the nominal control law given by (2) and the adaptive control law given by (6).

4.3 Derivative-Free Adaptive Control

The following assumption strengthens Assumption 2.3 by setting $\varepsilon(x) = 0$, which can be justified under the assumption that time-variation is allowed in the unknown ideal weight matrix.

Assumption 4.1. The matched uncertainty in (1) can be linearly parameterized as

$$\Delta(t, x) \;\; = \;\; W^{\mathrm{T}}(t)\beta(x), \qquad x \in \mathcal{D}_x \tag{74}$$

where $W(t) \in \mathrm{IR}^{s\times m}$ is an unknown *time-varying* weight matrix that satisfies $\|W(t)\| \leq w^*$ and $\beta : \mathrm{IR}^n \to \mathrm{IR}^s$ is a vector of known functions of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^{\mathrm{T}} \in \mathrm{IR}^s$.

Remark 4.1. Assumption 4.1 expands the class of uncertainties that can be represented by a given set of basis functions. That is, an adaptive law designed subject to Assumption 4.1 can be more effective than an adaptive law designed subject to Assumption 2.3 in dominating a wider class of uncertainties, due to the fact that time-variation is allowed in the unknown ideal weight matrix.

Remark 4.2. Assumption 4.1 does not place any restriction on the time derivative of the weight matrix. However the degree of time dependence will depend on how $\beta(x)$ is chosen.

The state error dynamics in (11) with $\varepsilon(x) = 0$ is now given by

$$\dot{e}(t) \;\; = \;\; A_{\mathrm{m}}e(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) \tag{75}$$

where $\tilde{W}(t) \equiv W(t) - \hat{W}(t)$ is the weight update error. The following theorem presents the main result of this section.

Theorem 4.1. Consider the uncertain system given by (1) subject to Assumption 4.1. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has a derivative-free weight update law in the form

$$\hat{W}(t) \;=\; \Omega_1 \hat{W}(t-\tau) + \hat{\Omega}_2(t) \tag{76}$$

where $\tau > 0$, and $\Omega_1 \in \mathrm{I\!R}^{s\times s}$ and $\hat{\Omega}_2 : \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}^{s\times m}$ satisfy:

$$0 \le \Omega_1^{\mathrm{T}} \Omega_1 < I \tag{77}$$

$$\hat{\Omega}_2(t) \;=\; \kappa_2 \beta(x(t)) e^{\mathrm{T}}(t) P B, \quad \kappa_2 > 0 \tag{78}$$

with $P \in \mathrm{I\!R}^{n\times n}$ satisfying (8) for any symmetric $Q_{\mathrm{m}} > 0$. Then, $e(t)$ and $\tilde{W}(t)$ are UUB.

Proof. Using (76) and defining

$$\Omega_2(t) \;\equiv\; W(t) - \Omega_1 W(t-\tau) \tag{79}$$

where $||\Omega_2(t)|| \le \delta^*$, $\delta^* = w^*(1 + ||\Omega_1(t)||)$, the weight update error can be rewritten as:

$$\tilde{W}(t) \;=\; \Omega_1 \tilde{W}(t-\tau) + \Omega_2(t) - \hat{\Omega}_2(t) \tag{80}$$

The state tracking error dynamics in (75) can be rewritten using (80) as:

$$\dot{e}(t) \;=\; A_{\mathrm{m}} e(t) + B\big[\Omega_1 \tilde{W}(t-\tau) - \hat{\Omega}_2(t) + \Omega_2(t)\big]^{\mathrm{T}} \beta(x(t)) \tag{81}$$

To show that the closed-loop system given by (80) and (81) is UUB, consider the Lyapunov-Krasovskii functional [31]

$$\mathcal{V}(e(t), \tilde{W}_t) = e^{\mathrm{T}}(t) P e(t) + \rho \, \mathrm{tr}\big[\int_{t-\tau}^{t} \tilde{W}^{\mathrm{T}}(s) \tilde{W}(s) \mathrm{d}s\big] \tag{82}$$

where $\rho > 0$ and $\tilde{W}_t$ represents $\tilde{W}(t)$ over the time interval $t-\tau$ to t. The directional derivative of (82) along the closed-loop system trajectories of (80) and (81) is given by

$$\begin{aligned}\dot{\mathcal{V}}(e(t),\tilde{W}_t) \;=\; & -e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) \;+\; 2e^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(x(t)) \\ & -2e^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(x(t)) \;+\; 2e^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}(t)\beta(x(t)) \\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)+\eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) \\ & -\tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau)\big] \end{aligned} \tag{83}$$

where $\eta = 1+\xi$. In what follows we impose the restriction $\xi \geq 0$.

Using (80) to expand the term $\mathrm{tr}[\eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)]$ in (83) produces

$$\begin{aligned}\dot{\mathcal{V}}(e(t),\tilde{W}_t) \;=\; & -e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t)+2e^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(x(t)) \\ & -2e^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(x(t))+2e^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}(t)\beta(x(t)) \\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)-\tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau) \\ & +\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)+\eta\hat{\Omega}_2^{\mathrm{T}}(t)\hat{\Omega}_2(t)+\eta\Omega_2^{\mathrm{T}}(t)\Omega_2(t) \\ & -2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_1\tilde{W}(t-\tau)+2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2(t) \\ & -2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_2(t)\big] \end{aligned} \tag{84}$$

Next, consider the fact $a^{\mathrm{T}}b \leq \gamma a^{\mathrm{T}}a + b^{\mathrm{T}}b/4\gamma$, $\gamma > 0$, that follows from Young's inequality [5, 17] for any vectors a and b. This can be generalized to matrices as $\mathrm{tr}\big[A^{\mathrm{T}}B\big] = \mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(B) \leq \gamma\mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(A) + \mathrm{vec}(B)^{\mathrm{T}}\mathrm{vec}(B)/4\gamma = \gamma\mathrm{tr}\big[A^{\mathrm{T}}A\big] + \mathrm{tr}\big[B^{\mathrm{T}}B\big]/4\gamma$, $\gamma > 0$, for any matrices A and B with appropriate dimensions. Using this, we can write

$$\begin{aligned}\mathrm{tr}\big[2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2(t)\big] \;\leq\; & \mathrm{tr}\big[\gamma\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\big] \\ & +\mathrm{tr}\big[\eta^2\Omega_2^{\mathrm{T}}(t)\Omega_2(t)/\gamma\big], \quad \gamma > 0 \end{aligned} \tag{85}$$

Using (78) with $\kappa_2 \triangleq 1/\rho\eta > 0$ for $\hat{\Omega}_2(t)$, and substituting (85) in (84), it follows that

$$\begin{aligned}\dot{\mathcal{V}}(e(t), \tilde{W}_t) \;\; \leq \;\; & -e^{\mathrm{T}}(t)Q_{\mathrm{m}}e(t) - \kappa_2 e^{\mathrm{T}}(t)PBB^{\mathrm{T}}Pe(t)\beta(x(t))^{\mathrm{T}}\beta(x(t)) \\ & -\rho\xi \ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] - \rho \ \mathrm{tr}\Big[\tilde{W}^{\mathrm{T}}(t-\tau)\big[I - (\eta+\gamma)\Omega_1^{\mathrm{T}}\Omega_1\big]\tilde{W}(t-\tau)\Big] \\ & +\rho(\eta + \eta^2/\gamma)\mathrm{tr}\big[\Omega_2^{\mathrm{T}}(t)\Omega_2(t)\big] \qquad (86)\end{aligned}$$

Using (77) with $\kappa_1 \triangleq 1/(\eta+\gamma) < 1$ for Ω_1 yields

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \;\; \leq \;\; -c_1\|e(t)\|^2 - c_2\|\tilde{W}(t)\|^2 - c_3\|\tilde{W}(t-\tau)\|^2 + d \qquad (87)$$

where the constants c_1, c_2, c_3, and d are given by :

$$c_1 \;=\; \lambda_{\min}(Q_{\mathrm{m}}) > 0 \qquad (88)$$

$$c_2 \;=\; \rho\xi \geq 0 \qquad (89)$$

$$c_3 \;=\; \rho\lambda_{\min}(I - \kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1) > 0 \qquad (90)$$

$$d \;=\; \rho(\eta + \eta^2/\gamma)\delta^{*^2} \geq 0 \qquad (91)$$

where $\rho = 1/\kappa_2\eta$. If $\xi > 0$, then since $\eta = \kappa_1^{-1} - \gamma = 1 + \xi$, $0 < \kappa_1 < 1$, and $\gamma > 0$, it follows that η must lie in the open interval $(1, 1/\kappa_1)$. Either $|e(t)| > \Psi_1$ or $\|\tilde{W}(t)\| > \Psi_2$ or $\|\tilde{W}(t-\tau)\| > \Psi_3$ renders $\dot{\mathcal{V}}(e(t), \tilde{W}_t) < 0$, where $\Psi_1 = \sqrt{d/c_1}$, $\Psi_2 = \sqrt{d/c_2}$, and $\Psi_3 = \sqrt{d/c_3}$, or equivalently:

$$\begin{aligned}\Psi_1 \;&=\; \delta^*\sqrt{\rho\big(\eta + \eta^2/\gamma\big)/\lambda_{\min}(Q_{\mathrm{m}})} \\ &=\; \delta^*/\sqrt{\kappa_2\lambda_{\min}(Q_{\mathrm{m}})\big[1 - \eta\kappa_1\big]} \qquad (92)\end{aligned}$$

$$\begin{aligned}\Psi_2 \;&=\; \Psi_1\sqrt{c_1/c_2} \\ &=\; \Psi_1\sqrt{\kappa_2\lambda_{\min}(Q_{\mathrm{m}})\eta/[\eta - 1]} \qquad (93)\end{aligned}$$

$$\begin{aligned}\Psi_3 \;&=\; \Psi_1\sqrt{c_1/c_3} \\ &=\; \Psi_1\sqrt{\kappa_1\kappa_2\lambda_{\min}(Q_{\mathrm{m}})\eta/\lambda_{\min}(\kappa_1 I - \Omega_1^{\mathrm{T}}\Omega_1)} \qquad (94)\end{aligned}$$

Hence, it follows that $e(t)$ and $\tilde{W}(t)$ are UUB. □

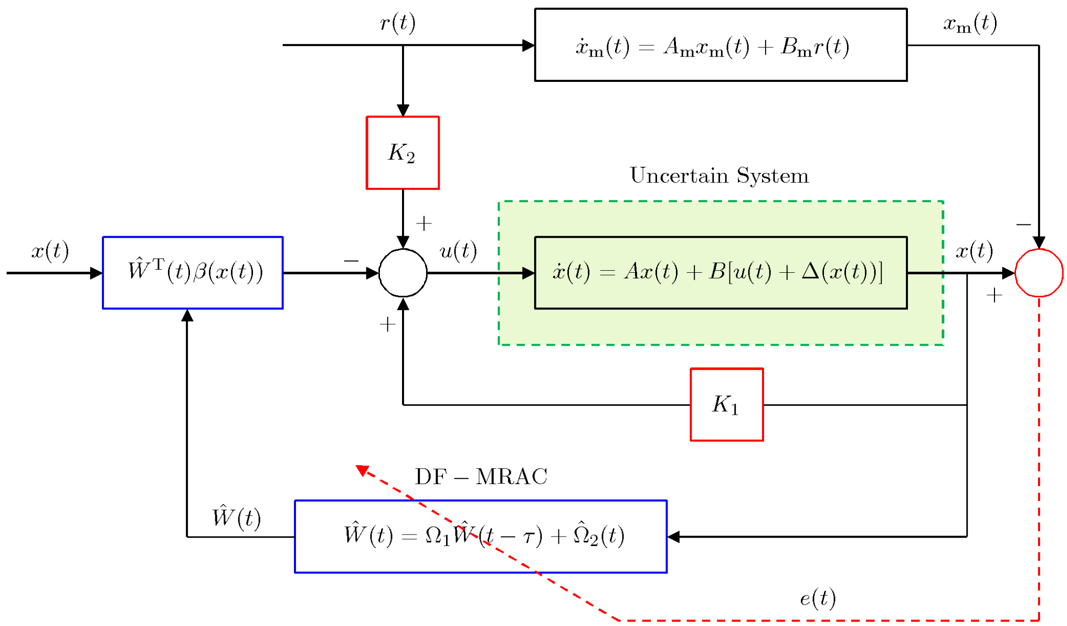

Figure 31: Visualization of the proposed DF-MRAC architecture.

The proposed adaptive control architecture is shown in Figure 31.

Remark 4.3. Using a 1st order Euler method for integration in (7), with $\tau_{\rm s}$ being the step size results in:

$$\hat{W}(t) = \gamma\tau_{\rm s}\big[\beta(x(t))e^{\rm T}(t)PB\big]+\hat{W}(t-\tau_{\rm s}) \tag{95}$$

This form of weight update law is identical to the DF-MRAC law in (76), if $\Omega_1 = I$, $\kappa_2 = \gamma\tau_{\rm s}$, and $\tau = \tau_{\rm s}$, with the exception that the choice $\Omega_1 = I$ is not permitted in DF-MRAC. In DF-MRAC, Ω_1 can be chosen, for example, as ςI where $0 < |\varsigma| < 1$, and τ is not necessarily equal to $\tau_{\rm s}$. This added dimension in the tuning process provides memory to the adaptive law, and can be employed to improve transient behavior without increasing the effective adaptation gain. As an another example, using a 1st order Euler method for integration in (41) with the σ-modification term $\dot{\hat{W}}_{\rm m}(t) = -\sigma\hat{W}(t)$ results in:

$$\hat{W}(t) = \frac{\gamma\tau_{\rm s}}{1+\sigma\gamma\tau_{\rm s}}\big[\beta(x(t))e^{\rm T}(t)PB\big]+\frac{1}{1+\sigma\gamma\tau_{\rm s}}\hat{W}(t-\tau_{\rm s}) \tag{96}$$

This form of weight update law is identical to the DF-MRAC law in (76), if $\Omega_1 = (1+\sigma\gamma\tau_{\rm s})^{-1}I$, $\kappa_2 = (1+\sigma\gamma\tau_{\rm s})^{-1}\gamma\tau_{\rm s}$, and $\tau = \tau_{\rm s}$. However, τ is not necessarily equal to $\tau_{\rm s}$. The value of τ influences the expressions for the ultimate bound and the guaranteed exponential rate of convergence to the ultimate bound, as stated below in Corollaries 4.1 and 4.2.

Remark 4.4. The derivative-free weight update law given by (76) subject to (77) and (78) does not require a modification term to prove the error dynamics, including the weight errors, are UUB.

Define $q(t) \equiv [e^{\rm T}(t), \tilde{v}(t,\tau)]^{\rm T}$, where $\tilde{v}^2(t,\tau) \equiv {\rm tr}\big[\int_{-\tau}^{0} \tilde{W}^{\rm T}(s)\tilde{W}(s){\rm d}s\big]$, and let $\mathcal{B}_r = \{q(t) : ||q(t)|| < r_\alpha\}$, such that $\mathcal{B}_r \subset \mathcal{D}_q$ for a sufficiently large compact set $\mathcal{D}_q$. Then, we have the following corollary.

Corollary 4.1. Under the conditions of Theorem 4.1, an estimate for the ultimate bound, for the case $\xi > 0$, is given by

$$\underline{r} = \sqrt{\frac{\lambda_{\rm max}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\lambda_{\rm min}(\tilde{P})}} \tag{97}$$

where $\tilde{P} = {\rm diag}[P, \rho]$.

Proof. Denote $\Omega_\alpha = \{q(t) \in \mathcal{B}_r : q^{\rm T}(t)\tilde{P}q(t) \le \alpha\}$, $\alpha = \min\limits_{||q(t)||=r} q^{\rm T}(t)\tilde{P}q(t) = r_\alpha^2\lambda_{\rm min}(\tilde{P})$. Since

$$\begin{aligned} \mathcal{V}(e(t),\tilde{W}_t) &= q^{\rm T}(t)\tilde{P}q(t) \\ &= e^{\rm T}(t)Pe(t) + \rho\ {\rm tr}\Big[\int_{t-\tau}^{t} \tilde{W}^{\rm T}(s)\tilde{W}(s){\rm d}s\Big] \end{aligned} \tag{98}$$

it follows that Ω_α is an invariant set as long as:

$$\alpha \ \ge \ \lambda_{\rm max}(P)\Psi_1^2 + \rho\tau\Psi_2^2 \tag{99}$$

Thus, the minimum size of $\mathcal{B}_r$ that ensures this condition has radius given by (97).

□

Remark 4.5. The proofs of Theorem 4.1 and Corollary 4.1 assume that the sets $\mathcal{D}_x$ and $\mathcal{D}_q$ are sufficiently large. If we define $\mathcal{B}_{r^*}$ as the largest ball contained in $\mathcal{D}_q$, and assume that the initial conditions are such that $q(0) \subset \mathcal{B}_{r^*}$, then from Figure 1 we have the added condition that $\underline{r} < r^*$, which implies a lower bound on ρ. It can be shown that in this case the lower bound must be such that $\lambda_{\min}(\tilde{P}) = \rho$. With $\underline{r}$ defined by (97) and $\lambda_{\min}(\tilde{P}) = \rho$, the condition $\underline{r} < r^*$ implies:

$$\rho > \frac{\overline{\lambda}(P)\Psi_1^2}{r^{*2} - \tau\Psi_2^2} \tag{100}$$

Since $\kappa_2 = 1/\rho\eta$, $\eta > 1$, it follows from (100) that r^* should ensure that:

$$\kappa_2 < \frac{r^{*2} - \tau\Psi_2^2}{\overline{\lambda}(P)\Psi_1^2} \tag{101}$$

Therefore, the meaning of $\mathcal{D}_q$ *sufficiently large* in Corollary 4.1 is that

$$r^* > \sqrt{\kappa_2 \lambda_{\max}(P)\Psi_1^2 + \tau\Psi_2^2} \tag{102}$$

and $q(0) \subset \mathcal{D}_{r^*}$. The meaning of $\mathcal{D}_x$ *sufficiently large* is difficult to characterize precisely since $x(t)$ depends on both $r(t)$ and $x(0)$. Nevertheless it can be seen that increasing κ_2 implies increasing the require size of the set $\mathcal{D}_x$.

Corollary 4.2. Under the conditions of Theorem 4.1, the error trajectory approaches the ultimate bound exponentially in time according to

$$|q(t)| \leq k|q(0)|e^{-c_r t}, \quad t < T \tag{103}$$

with a convergence rate given by:

$$c_r = \frac{\tau/(1+\tau)\Psi_1^2}{2\lambda_{\max}(\tilde{P})}, \quad \tau = \frac{\rho(\eta - 1)}{\lambda_{\min}(Q_m)} \tag{104}$$

Proof. Choosing ξ so that $c_2 = c_1\tau$, and using the expressions for c_1 and c_2 in (88) and (89) results in the expression for τ in (104). Then, from (87), we can write:

$$\dot{\mathcal{V}}(e(t), \tilde{W}_t) \leq -c_1|q(t)|^2 + d \tag{105}$$

Define $\hat{c} \equiv d/\mu^2$, where $\mu \equiv \sqrt{\Psi_1^2 + \tau\Psi_2^2}$. Then, when $|q(t)| > \mu$, we have that

$$\begin{aligned} \dot{\mathcal{V}}(e(t), \tilde{W}_t) &\leq -(c_1 - \hat{c})|q(t)|^2 \\ &\leq -k_3|q(t)|^2 \end{aligned} \tag{106}$$

where $k_3 \equiv c_1 - \hat{c}$. Now, $c_1 = d/\Psi_1^2$ and $c_2 = d/\Psi_2^2$, and since $c_2/c_1 = \tau$, it follows that $\Psi_1^2 = \tau\Psi_2^2$, and therefore $\hat{c} = d/(1+\tau)\Psi_1^2$. Therefore $k_3 = \tau/(1+\tau)\Psi_1^2$. Finally, since $k_1|q(t)|^2 \leq \mathcal{V}(e(t), \tilde{W}_t) \leq k_2|q(t)|^2$, where $k_1 = \lambda_{\min}(\tilde{P}) > 0$, $k_2 = \lambda_{\max}(\tilde{P}) > 0$, and $\dot{\mathcal{V}}(e(t), \tilde{W}_t) \leq -k_3|q(t)|^2$, then (103) follows directly from Corollary 5.3 of Ref. 45 where $k \equiv \sqrt{k_2/k_1} = \sqrt{\lambda_{\max}(\tilde{P})/\lambda_{\min}(\tilde{P})}$, and $c_{\rm r} = (k_3/2k_2) = \tau/2(1+\tau)\lambda_{\max}(\tilde{P})\Psi_1^2$.

□

Remark 4.6. For the most common case in which $\lambda_{\max}(\tilde{P}) > \rho$, there exists:

$$\eta^* = (-\varphi_1 + \sqrt{\varphi_1^2 - 4\varphi_2})/2 \tag{107}$$

$$\varphi_1 \triangleq -\frac{2}{1+\kappa_2\lambda_{\min}(Q_{\rm m})} \tag{108}$$

$$\varphi_2 \triangleq \frac{\kappa_1 - \kappa_2\lambda_{\min}(Q_{\rm m})}{\kappa_1(1+\kappa_2\lambda_{\min}(Q_{\rm m}))} \tag{109}$$

where η^* is the the value of η such that the convergence rate $c_{\rm r}$ in (104) attains a maximum on the open interval $(1, 1/\kappa_1)$.

Figures 32 and 33 show plots of the normalized ultimate bound (r/δ^*) in (97) versus convergence rate $c_{\rm r}$ in (104) for $\lambda_{\max}(\tilde{P}) = 2$, and $\lambda_{\min}(\tilde{P}) = \lambda_{\min}(Q_{\rm m}) = 1$,

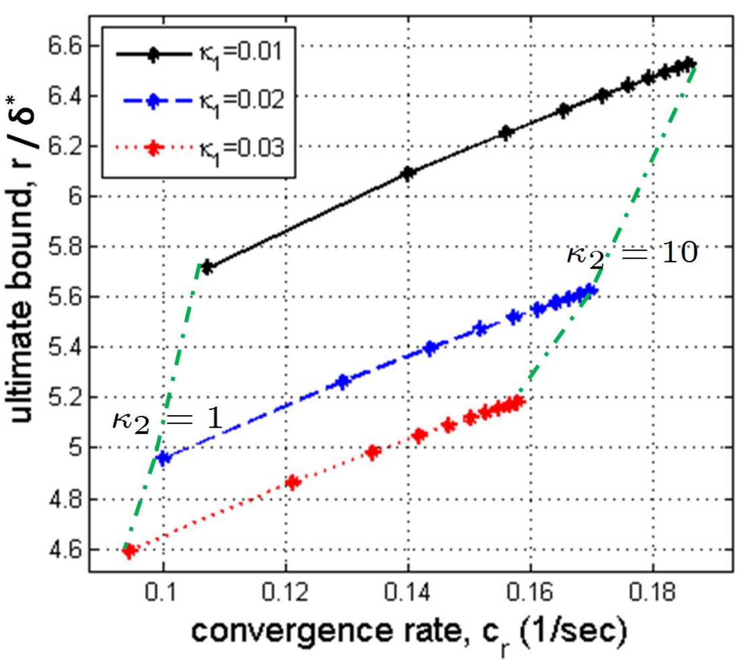

Figure 32: r/δ^* versus c_r for $\kappa_1 = \{0.01, 0.02, 0.03\}$ and $\kappa_2 \in [1, 10]$.

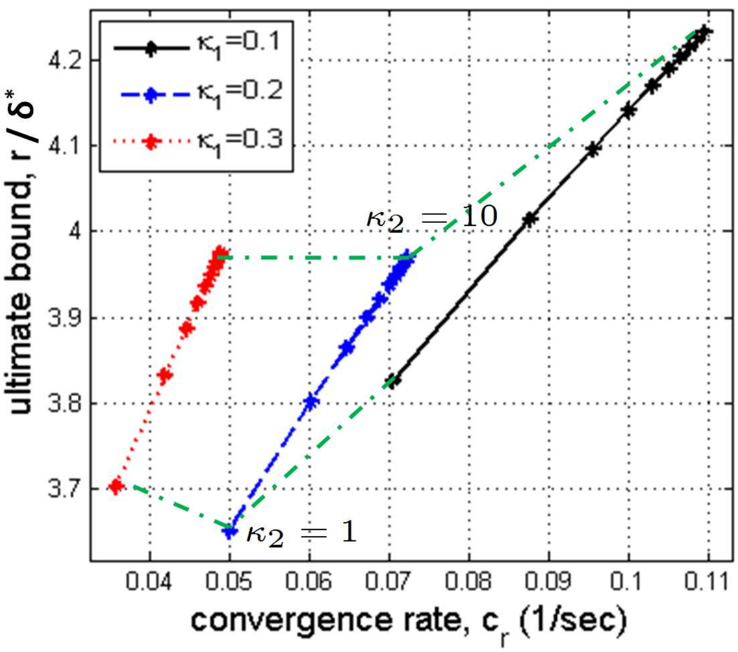

Figure 33: r/δ^* versus c_r for $\kappa_1 = \{0.1, 0.2, 0.3\}$ and $\kappa_2 \in [1, 10]$.

and $\eta = \eta^*$. Figure 32 shows that increasing κ_2 increases the ultimate bound, but it also increases the convergence rate. The fact that the estimate for the ultimate bound increases with the adaptation gain has been previously noted in Refs. 12 and 83. Figure 32 also shows that increasing κ_1 from 0.01 to 0.03 reduces the ultimate bound for any given value of $\kappa_2 \in [1, 10]$. However, if we increase κ_1 from 0.1 to 0.3, then Figure 33 shows that the trend in Figure 32 reverses for values $\kappa_2 \in [1, 10]$. This calculation can be repeated for any given set of parameters in (97) and (103) as a guide to choosing the design parameters of DF-MRAC.

For the case of constant ideal weights in Assumption 4.1, Theorem 4.1 specializes to the following theorem. In this case, we assume that the uncertainty is structured. That is, the vector of known functions in (74) represent the vector of *known* basis functions.

Theorem 4.2. Consider the uncertain system given by (1) subject to Assumption 4.1, where $W \in \mathbb{R}^{s \times m}$ is an unknown *constant* weight matrix. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has the derivative-free weight update law in the form (76) and (78), where $\Omega_1 = I$. Then, $e(t)$ and $\tilde{W}(t)$ approach a subspace in these error variables in which $e(t) = 0$ and $B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) = 0$.

Proof. The result follows directly from the proof of Theorem 4.1 by choosing $\Omega_1 = I$. In this case, $\delta^* = 0$ due to $\Omega_2(t) = 0$ in (79), which follows from the fact that the ideal weights are constant, and $\Omega_1 = I$. Then, the inquality in (85) is not needed since the left hand side vanishes. In this case, $\kappa_1 = \frac{1}{\eta} = 1$, $\xi = 0$, $c_2 = 0$ in (89), $c_3 = 0$ in (90), and $d = 0$ in (91). Therefore, it follows from (87) that the entire error space is invariant. Let $\mathcal{E}$ denote the set of points in this space where $\dot{\mathcal{V}}(e(t), \tilde{W}_t) = 0$. From (87) all points in lie in a subspace, where $e(t) = 0$ and it follows from LaSalle's Theorem [45] that all solutions in the error space approach the largest invariant set $\mathcal{M}$ in $\mathcal{E}$. Now $e(t) = 0$ implies that $\hat{\Omega}_2(t) = 0$ and $\dot{e}(t) = 0$. Then (81), with all of the above taken together, implies that $\mathcal{M}$ is comprised of all points in the error space in which $e(t) = 0$ and $B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) = 0$. □

Remark 4.7. The system is said to be sufficiently excited if $r(t)$ is such that the conditions $e(t) = 0$, $B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) = 0$ admit only the solution $\tilde{W}(t) = 0$ in the limit $t \to \infty$. It is straightforward to show that this amounts to the standard MRAC condition for persistency of excitation [71].

Remark 4.8. As noted in the introduction, DF-MRAC does not employ an integrator in its weight update law. This is advantageous from the perspective of augmenting a nominal controller that employs integral action to ensure that the regulated output variables track $r(t)$ for constant disturbances, regardless of how these disturbances may enter the system. An example that illustrates this advantage is provided in Section 4.6.2.

Remark 4.9. The expressions for Ψ_1 and Ψ_2 represent ultimate bounds for $|e(t)|$ and $||\tilde{W}(t)||$, respectively. These expressions depend on $\eta > 1$. In Remark 4.6 it is shown that there exists an optimal value η^* such that the convergence rate c_r in (104) attains a maximum on the open interval $\{1,\ 1/\kappa_1\}$, and curves are provided that show the trade-off between the UUB and c_r for a range of values for κ_1 and κ_2.

4.4 Modifications to Derivative-Free Adaptive Control

Although the derivative-free weight update law does not require a modification term to prove the error dynamics are UUB, one may wish to employ a modification term in order to improve performance or robustness of the system. The following theorem extends Theorem 4.1 to a general form of modified DF-MRAC.

Theorem 4.3. Consider the uncertain system given by (1) subject to Assumption 4.1. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has a derivative-free weight update law in the form given by (76), where $\tau > 0$ is a time delay design value, and $\Omega_1 \in \mathbb{R}^{s\times s}$ and $\hat{\Omega}_2(t) \equiv \hat{\Omega}_2(x(t), e(t))$, $\hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s\times m}$, satisfy:

$$0 < \Omega_1^{\mathrm{T}}\Omega_1 < \kappa_1 I, \quad 0 < \kappa_1 < 1 \tag{110}$$

$$\hat{\Omega}_2(t) = \kappa_2[\beta(x(t))e^{\mathrm{T}}(t)PB - \kappa_{\mathrm{m}}S(t)\hat{W}(t)], \quad \kappa_{\mathrm{m}} > 0 \tag{111}$$

where κ_{m} is the modification gain, $P \in \mathbb{R}^{n\times n}$ satisfies

$$0 \;=\; A_{\mathrm{m}}^{\mathrm{T}}P + PA_{\mathrm{m}} - {\kappa_{\mathrm{m}}}^2 PBB^{\mathrm{T}}P + Q_{\mathrm{m}} \tag{112}$$

for any symmetric $Q_{\mathrm{m}} > 0$, and $S(t) \in \mathbb{R}^{s\times s}$ satisfies $||S(t)|| < s^*$. Then, $e(t)$ and $\tilde{W}(t)$ are UUB.

Proof. The result follows directly from the proof of Theorem 4.1, with (88) - (91)

changed to:

$$
\begin{aligned}
c_1 &= \lambda_{\min}(Q_{\mathrm{m}}) > 0 && (113)\\
c_2 &= \rho\xi + {s^*}^2\kappa^* > 0 && (114)\\
c_3 &= \rho\lambda_{\min}(I - [1+\eta+\gamma]\Omega_1^{\mathrm{T}}\Omega_1) > 0 && (115)\\
d &= \rho(1+\eta+\eta^2/\gamma){\delta^*}^2 + {\omega^*}^2{s^*}^2\kappa^* > 0 && (116)
\end{aligned}
$$

where $\kappa^* = 1 + 2\kappa_{\mathrm{m}} + (1+\xi)^{-1}{\kappa_{\mathrm{m}}}^2 > 0$. □

Remark 4.10. If $S(t)$ is not assumed to be bounded by s^*, e.g. $S(t) = ||e(t)||$ in the case of e- modification [37], then one can show that $e(t)$ and $\tilde{W}(t)$ are UUB by applying a projection operator [66] to the weight estimates given by (76).

Remark 4.11. In the case of σ- modification [60], we have $S(t) = I$ in (111). ALR modification [10] can be introduced by letting $S(t) = \beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))$, where $\beta_x(x(t)) \triangleq \mathrm{d}\beta(x(t))/\mathrm{d}x(t) \in \mathrm{I\!R}^{s\times n}$ and in this case $s^* = 1$. Table 1 summarizes DF-MRAC laws for these two and other modification terms as well. Note that since $\hat{W}(t)$ appears in both left and right hand sides of some DF-MRAC laws with modification terms, then one needs to solve these expressions for $\hat{W}(t)$ for implementation.

Table 1: DF-MRAC laws for various modification terms

Modification	DF-MRAC Law for $-1 < \varphi_1 < 1$, $\varphi_2 > 0$, $\hat{\varphi}_2 > 0$
Original (76)	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB\big]$.
σ [60]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \hat{W}(t)\big]$.
e [37]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \\|e(t)\\| \hat{W}(t)\big]$.
ALR [10]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \beta_x(x(t)) \beta_x^{\mathrm{T}}(x(t)) \hat{W}(t)\big]$.
$\mathcal{Q}$ [75, 77]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \big(\hat{W}^{\mathrm{T}} q(t) - c(t)\big) q(t)\big]$; where $q(t) \triangleq \int_{t_{\mathrm{d}}}^{t} \beta(x(s)) \mathrm{d}s, t_{\mathrm{d}} > 0$, $c(t) \triangleq e_n(t) - e_n(t_{\mathrm{d}})$ $- \int_{t_{\mathrm{d}}}^{t} B^{\mathrm{T}} A_{\mathrm{m}} \times A_{\mathrm{m}} e(s) \mathrm{d}s + \int_{t_{\mathrm{d}}}^{t} u_{\mathrm{ad}}(s) \mathrm{d}s$, and $e_n(t)$ is the $n-$th component of the vector $e(t)$, and $t_{\mathrm{d}} > 0$.
Optimal [63]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB + \hat{\varphi}_2 \beta(x(t)) \beta(x(t))^{\mathrm{T}}$ $\times \hat{W}(t) B^{\mathrm{T}} P A_{\mathrm{m}}^{-1} B\big]$.
CMRAC [50]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \beta_{\mathrm{f}}(x(t)) e_{\mathrm{Y}}^{\mathrm{T}}(t)\big]$; where $\beta_{\mathrm{f}}(x(t)) = \mathcal{L}^{-1}\{\frac{\lambda_{\mathrm{f}}}{s+\lambda_{\mathrm{f}}}\} \beta(x(t))$, where $\mathcal{L}^{-1}$ denotes the Laplace operator, and $e_{\mathrm{Y}}(t)$ is the predictor estimation error defined in Ref. 50.
$\mathcal{K}$ [47]	$\hat{W}(t) = \varphi_1 \hat{W}(t-\tau) + \varphi_2 \big[\beta(x(t)) e^{\mathrm{T}} PB - \hat{\varphi}_2 \int_{t-t_{\mathrm{d}}}^{t} \hat{W}(s) \mathrm{d}s\big]$; where $t_{\mathrm{d}} > 0$.

4.5 Extensions to the Input Uncertainty Case

Consider the system with matched uncertainty and input uncertainty of the form

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) + B\Delta(t, x(t)) \tag{117}$$

where $\Lambda \in \mathrm{I\!R}^{m\times m}$ is an uncertain diagonal matrix with diagonal elements $\lambda_i \geq 0$.

Assumption 4.2. $\mathcal{R}\{B\} = \mathcal{R}\{B\Lambda\}$.

Remark 4.12. Assumption 4.2 is automatically satisfied when $\lambda_i > 0$, therefore it is only of importance in situations wherein one or more of the $\lambda_i = 0$ corresponds to actuator failure. In this case Assumption 4.2 implies that there exists actuator redundancy. An actuator is considered redundant if its failure does not change the range space of B. This is approximately true in aircraft applications where the means of control is primarily through moment generation. Therefore, Assumption 4.2 is approximately satisfied when there is sufficient actuator redundancy. For situations in which there is no actuator redundancy, and one or more $\lambda_i = 0$, it may still be possible to control the aircraft through a hierarchal adaptive approach [35], which does not require Assumption 4.2. In Section 4.7 we illustrate the application of Assumption 4.2 for the case of redundant control. The manner in which redundant controls can be implemented is also illustrated in Section 4.7.

Under Assumption 4.2, the dynamics (117) can equivalently be written in the form

$$\dot{x}(t) = Ax(t) + B\Lambda\big[u(t) + \Delta_{\mathrm{e}}(t, x(t))\big] \tag{118}$$

where $\Delta_e(x(t))$ also satisfies Assumption 4.1. The state tracking error dynamics in (75) becomes

$$\begin{aligned} \dot{e}(t) &= A_{\mathrm{m}}e(t) + B\big[\Lambda - I_m\big]K_1x(t) + B\big[\Lambda - I_m\big]K_2r(t) \\ &\quad + B\Lambda\big[-u_{\mathrm{ad}}(t) + W^{\mathrm{T}}(t)\beta(x(t))\big] \end{aligned} \tag{119}$$

Assumption 4.2 allows us to write (119) in the equivalent form

$$\dot{e}(t) = A_{\mathrm{m}}e(t) + B\Lambda\big[-u_{\mathrm{ad}}(t) + W^{\mathrm{T}}(t)\beta(x(t)) + K_{1e}^{\mathrm{T}}x(t) + K_{2e}^{\mathrm{T}}r(t)\big] \quad (120)$$

The adaptive law is now chosen in the form

$$u_{\mathrm{ad}}(t) = \hat{W}^{\mathrm{T}}(t)\beta(x(t)) + \hat{K}_{1e}^{\mathrm{T}}(t)x(t) + \hat{K}_{2e}^{\mathrm{T}}(t)r(t) \quad (121)$$

Using (121) in (120) yields

$$\dot{e}(t) = A_{\mathrm{m}}e(t) + B\Lambda\big[\tilde{W}^{\mathrm{T}}(t)\beta(x(t)) + \tilde{K}_{1e}^{\mathrm{T}}(t)x(t) + \tilde{K}_{2e}^{\mathrm{T}}(t)r(t)\big] \quad (122)$$

where

$$\tilde{K}_{1e}(t) \equiv K_{1e} - \hat{K}_{1e}(t) \quad (123)$$

$$\tilde{K}_{2e}(t) \equiv K_{2e} - \hat{K}_{2e}(t) \quad (124)$$

The following theorem states an extension to Theorem 4.1 for the case of input uncertainty.

Theorem 4.4. Consider the uncertain system given by (117) subject to Assumptions 4.1 and 4.2. Consider, in addition, the feedback control law given by (5), with the nominal feedback control component given by (2) subject to Assumption 2.2, and with the adaptive feedback control component given by (121), that has derivative-free weight update laws in the form given by (76), (77), (78), and

$$\hat{K}_{1e}(t) = \Xi_{11}\hat{K}_{1e}(t-\tau) + \hat{\Xi}_{12}(t) \quad (125)$$

$$\hat{K}_{2e}(t) = \Xi_{21}\hat{K}_{2e}(t-\tau) + \hat{\Xi}_{22}(t) \quad (126)$$

where $\tau > 0$, and $\Xi_{11} \in \mathbb{R}^{n\times n}$, $\Xi_{21} \in \mathbb{R}^{r\times r}$, $\hat{\Xi}_{12} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\hat{\Xi}_{22} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r\times m}$ satisfy:

$$0 \le \Xi_{i1}^{\mathrm{T}}\Xi_{i1} < I, \quad i = 1, 2 \quad (127)$$

$$\hat{\Xi}_{12}(t) = \kappa_2 x(t)e^{\mathrm{T}}(t)PB, \quad \kappa_2 > 0 \quad (128)$$

$$\hat{\Xi}_{22}(t) = \kappa_2 r(t)e^{\mathrm{T}}(t)PB \quad (129)$$

with $P \in \mathrm{IR}^{n\times n}$ satisfying (8) for any symmetric $Q_{\mathrm{m}} > 0$. Then, $e(t)$, $\tilde{W}(t)$, $\tilde{K}_{1e}(t)$, and $\tilde{K}_{2e}(t)$ are UUB.

Proof. Let $W_e(t) \equiv \left[W^{\mathrm{T}}(t), K_{1e}^{\mathrm{T}}, K_{2e}^{\mathrm{T}}\right]^{\mathrm{T}}$, $\hat{W}_e(t) \equiv \left[\hat{W}^{\mathrm{T}}(t), \hat{K}_{1e}^{\mathrm{T}}(t), \hat{K}_{2e}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, $\tilde{W}_e(t) \equiv W_e(t) - \hat{W}_e(t)$, and $\beta_e(\beta(x(t)), x(t), r(t)) \equiv \left[\beta^{\mathrm{T}}(x(t)), x^{\mathrm{T}}(t), r^{\mathrm{T}}(t)\right]^{\mathrm{T}}$. Then, the state tracking error dynamics in (122) becomes:

$$\dot{e}(t) \quad = \quad A_{\mathrm{m}}e(t) + B\Lambda\tilde{W}_e^{\mathrm{T}}(t)\beta_e(\beta(x(t)), x(t), r(t)) \tag{130}$$

Defining

$$\Omega_{2e}(t) \quad \equiv \quad W_e(t) - \Omega_{1e}W_e(t-\tau) \tag{131}$$

where $\Omega_{1e} \equiv \mathrm{diag}[\Omega_1, \Xi_{11}, \Xi_{21}]$ and $||\Omega_{2e}(t)|| \leq \delta_e^*$, $\tilde{W}_e(t)$ can be written as

$$\tilde{W}_e(t) \quad = \quad \Omega_{1e}\tilde{W}_e(t-\tau) + \Omega_{2e}(t) - \hat{\Omega}_{21}(t) \tag{132}$$

where $\Omega_{21}(t) \equiv [\hat{\Omega}_2^{\mathrm{T}}(t), \Xi_{12}^{\mathrm{T}}(t), \Xi_{22}^{\mathrm{T}}(t)]^{\mathrm{T}}$. The state tracking error dynamics in (130) can be rewritten using (132) as:

$$\dot{e}(t) \quad = \quad A_{\mathrm{m}}e(t) + B\Lambda\left[\Omega_{1e}\tilde{W}_e(t-\tau) + \Omega_{2e}(t) - \hat{\Omega}_{21}(t)\right]^{\mathrm{T}}(t)\beta_e(\cdot) \tag{133}$$

To show that the closed-loop system given by (132) and (133) is UUB, consider the Lyapunov-Krasovskii functional

$$\mathcal{V}(e(t), \tilde{W}_{e_t}) = e^{\mathrm{T}}(t)Pe(t) + \rho\ \mathrm{tr}\Big[\int_{t-\tau}^{t} \tilde{W}_e^{\mathrm{T}}(s)\tilde{W}_e(s)\big(\Lambda + \varepsilon I_m\big)\mathrm{d}s\Big] \tag{134}$$

where $\rho > 0$, $\tilde{W}_{e_t}$ represents $\tilde{W}(t)$ over the time interval $t-\tau$ to t, and $\varepsilon > 0$ is sufficiently small. By following identical steps presented in the proof of Theorem 4.1, the directional derivative of (134) along the closed-loop system trajectories of (132) and (133) is given by

$$\dot{\mathcal{V}}(e(t), \tilde{W}_{e_t}) \quad \leq \quad -\bar{c}_1\|e(t)\|^2 - \bar{c}_2\|\tilde{W}_e(t)\|^2 - \bar{c}_3\|\tilde{W}_e(t-\tau)\|^2 + d_e \tag{135}$$

where the constants $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, and $\bar{d}$ are:

$$\bar{c}_1 = \lambda_{\min}(Q_{\mathrm{m}}) > 0 \tag{136}$$

$$\bar{c}_2 = \rho\underline{\vartheta}\xi > 0 \tag{137}$$

$$\bar{c}_3 = \rho\underline{\vartheta}\lambda_{\min}(I - \Omega_{1e}^{\mathrm{T}}\Omega_{1e}) > 0 \tag{138}$$

$$d_e = \rho\overline{\vartheta}(\eta + \eta^2/\gamma)\delta_e^{*^2} > 0 \tag{139}$$

where $\underline{\vartheta} = \lambda_{\min}(\Lambda + \varepsilon I_m)$ and $\overline{\vartheta} = \lambda_{\max}(\Lambda + \varepsilon I_m)$. Either $|e(t)| > \bar{\Psi}_1$ or $\|\tilde{W}_e(t)\| > \bar{\Psi}_2$ or $\|\tilde{W}_e(t-\tau)\| > \bar{\Psi}_3$ renders $\dot{\mathcal{V}}(e(t), \tilde{W}_{e_t}) < 0$, where $\bar{\Psi}_1 = \sqrt{d_e/\bar{c}_1}$, $\bar{\Psi}_2 = \sqrt{d_e/\bar{c}_2}$, and $\bar{\Psi}_3 = \sqrt{d_e/\bar{c}_3}$. Hence, it follows that $e(t)$ and $\tilde{W}_e(t)$ are UUB. □

Remark 4.13. For the simplicity of presentation, Theorem 4.4 assumes that the derivative-free weight update laws given by (76), (125), and (126) use the same $\tau > 0$ and $\kappa_2 > 0$ design parameters. To add greater flexibility in the design, (125), (126), (128), and (129) can be replaced by:

$$\hat{K}_{1e}(t) = \Xi_{11}\hat{K}_{1e}(t - \tau_{1e}) + \hat{\Xi}_{12}(t), \quad \tau_{1e} > 0 \tag{140}$$

$$\hat{K}_{2e}(t) = \Xi_{21}\hat{K}_{2e}(t - \tau_{2e}) + \hat{\Xi}_{22}(t), \quad \tau_{2e} > 0 \tag{141}$$

$$\hat{\Xi}_{12}(t) = \kappa_{12}x(t)e^{\mathrm{T}}(t)PB, \quad \kappa_{12} > 0 \tag{142}$$

$$\hat{\Xi}_{22}(t) = \kappa_{22}r(t)e^{\mathrm{T}}(t)PB, \quad \kappa_{22} > 0 \tag{143}$$

respectively. In this case, the proof of Theorem 4.4 follows a similar line by considering

$$\begin{aligned} \mathcal{V}(e(t), \tilde{W}_{e_t}) = {} & e^{\mathrm{T}}(t)Pe(t) + \rho \ \mathrm{tr}\Big[\int_{t-\tau}^{t} \tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\big(\Lambda + \varepsilon I_m\big)\mathrm{d}s\Big] \\ & + \rho_{1e} \ \mathrm{tr}\Big[\int_{t-\tau_{1e}}^{t} \tilde{K}_{1e}^{\mathrm{T}}(s)\tilde{K}_{1e}(s)\big(\Lambda + \varepsilon I_m\big)\mathrm{d}s\Big] \\ & + \rho_{2e} \ \mathrm{tr}\Big[\int_{t-\tau_{2e}}^{t} \tilde{K}_{2e}^{\mathrm{T}}(s)\tilde{K}_{2e}(s)\big(\Lambda + \varepsilon I_m\big)\mathrm{d}s\Big] \end{aligned} \tag{144}$$

where $\rho_{1e} > 0$ and $\rho_{2e} > 0$.

Remark 4.14. Under the conditions of Theorem 4.4, an estimate for the ultimate bound and convergence rate to this ultimate bound can be expressed in a form similar to Corollaries 4.1 and 4.2, respectively. For the case of constant ideal weights in Assumption 4.1, Theorem 4.4 with $\Omega_1 = I$, $\Xi_{11} = I$, and $\Xi_{21} = I$ guarantees that $e(t)$, $\tilde{W}(t)$, and $\tilde{D}(t)$ approach a subspace in these error variables in which $e(t) = 0$ and $B\Lambda\tilde{W}_e^{\mathrm{T}}(t)\beta_e(\cdot) = 0$, similar to the result given by Theorem 4.2. Modifications to the derivative-free weight update laws in (76), (125), and (126) can be employed, analogous to given in Table 1.

4.6 Examples on a First Order System

In this section we compare the standard MRAC law given by (7) with the proposed DF-MRAC law given by (76) on a simple model for the rolling dynamics of an aircraft [53].

4.6.1 Uncertainty with Time-Varying Ideal Weights

Consider the scalar dynamics

$$\dot{x}(t) = L_p x(t) + L_\delta[u(t) + \Delta(t, x(t))] \tag{145}$$

where $x(t)$ represents roll rate, $u(t)$ represents aileron deflection, $L_p = -1$, $L_\delta = 1$, and $\Delta(x(t)) = w(t)x(t)$ with $w(t)$ being a square wave having an amplitude of 1.0, and a period of 5.0 seconds. While aircraft dynamics do not behave this manner, their stability derivatives can undergo a sudden change in the event of damage to the airframe. So this example should be regarded in this context. In both the MRAC and DF-MRAC architectures, we choose as a reference model $A_{\mathrm{m}} = -2$ and $B_{\mathrm{m}} = 2$. Note that for this example $\varepsilon(x) = 0$, so we may use the MRAC law given by (7) without a modification term, and choose $\beta(x) = x$. Furthermore, we consider two adaptation gains, $\gamma = 10^2$ and $\gamma = 10^4$. For the DF-MRAC law given by (76), we set $\tau = 0.01$ seconds, $\Omega_1 = 0.5$ which satisfies (77) , and $\kappa_2 = 200$ in (78). $Q_{\mathrm{m}} = 2$ was used in (8) to solve for P in both architectures.

Remark 4.15. It is important to mention that we implemented standard MRAC in *analog* form. Therefore, all discrete time implementations used to realize the integration (see, for example, Remark 4.3) will produce nearly the same results, since they all represent an *approximation* to an integration performed in continuous time, and then sampled at the control update rate.

Figure 34 shows the performance of the nominal control design without adaptation. In Figure 35, the standard MRAC architecture is used with $\gamma = 10^2$ and $\gamma = 10^4$, respectively. Tracking performance is not improved by increasing the adaptation gain beyond 10^2, and increasing gain causes high frequency oscillations in the control response that would be unacceptable in a real system. Figure 36 shows the case in which the DF-MRAC adaptive law in (76) was used with $\kappa_2 = 200$. It can be seen that tracking performance is excellent, and the control time history is reasonable, and in this case the estimated weight is close to the ideal value.

As a variation of the previous example, the ideal weight history was changed to a sinusoidal function in which the frequency was varied from 0.5 rad/s up to 50 rad/s. The results are shown in Figures 37–39. In Figure 37, note that the adaptive controller does not significantly improve the response at the low setting for adaptation gain, and gives an even worse response when using the high setting for adaptation gain. Figures 38 and 39 show that the DF-MRAC case gives an excellent response, and the estimated weight is very close to the true weight, even at the high frequency end (see Figure 39). Inspired by this result we decided to try a case in which $w(t)$ is a band-limited white noise signal. The associated results are shown in Figures 40 and 41. In particular, Figure 41 shows that the estimated weight is remarkably close to to $w(t)$.

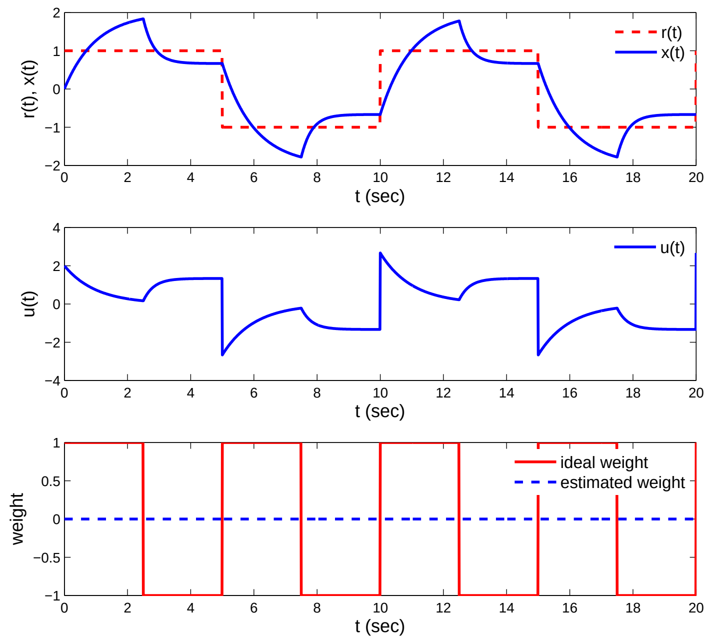

Figure 34: Responses with nominal controller for the square wave ideal weight.

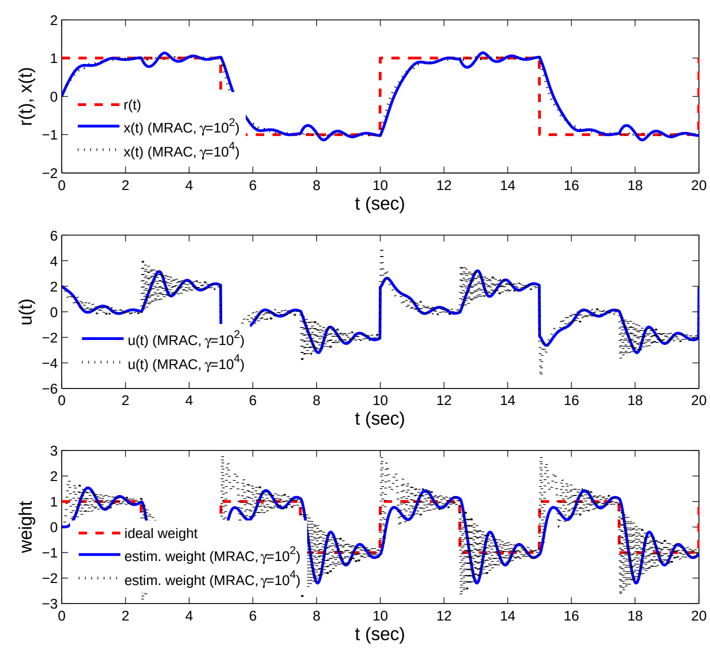

Figure 35: Responses with standard MRAC using $\gamma = 10^2$ and $\gamma = 10^4$.

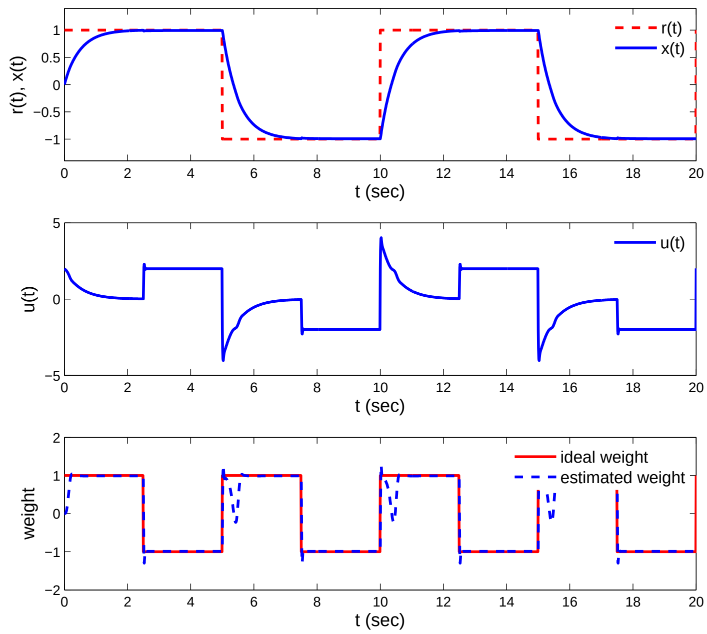

Figure 36: Responses with DF-MRAC for the square wave ideal weight.

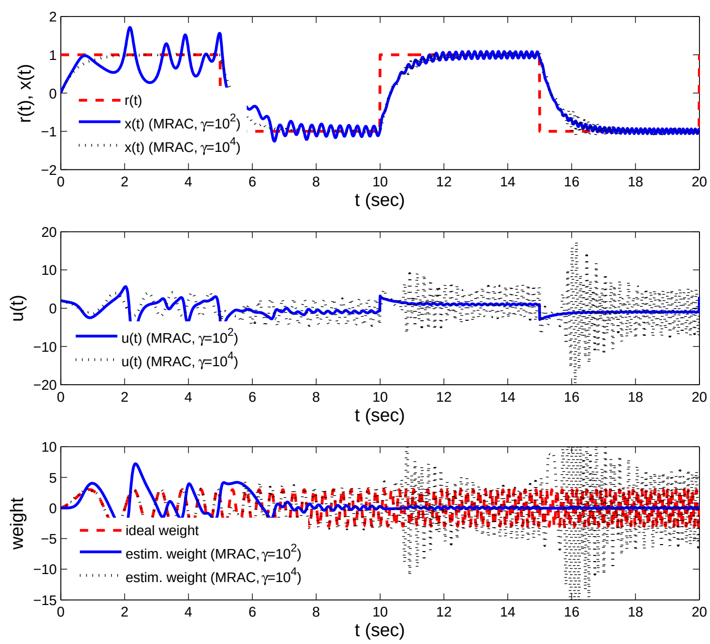

Figure 37: Responses with standard MRAC using $\gamma = 10^2$ and $\gamma = 10^4$.

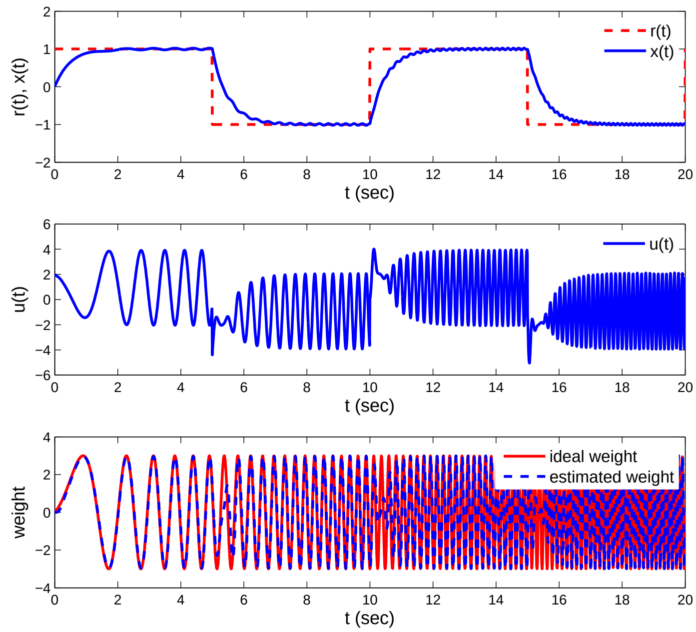

Figure 38: Responses with DF-MRAC for a sinusoidal ideal weight.

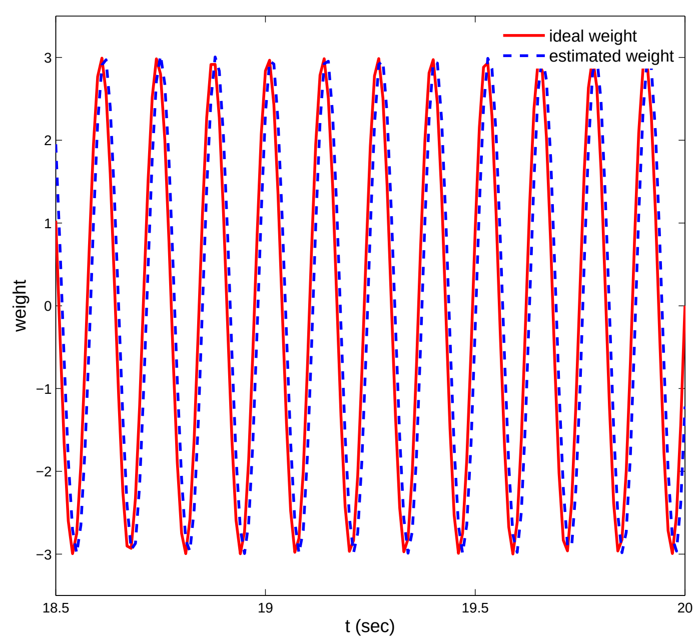

Figure 39: Expanded view of the estimate of the ideal weight.

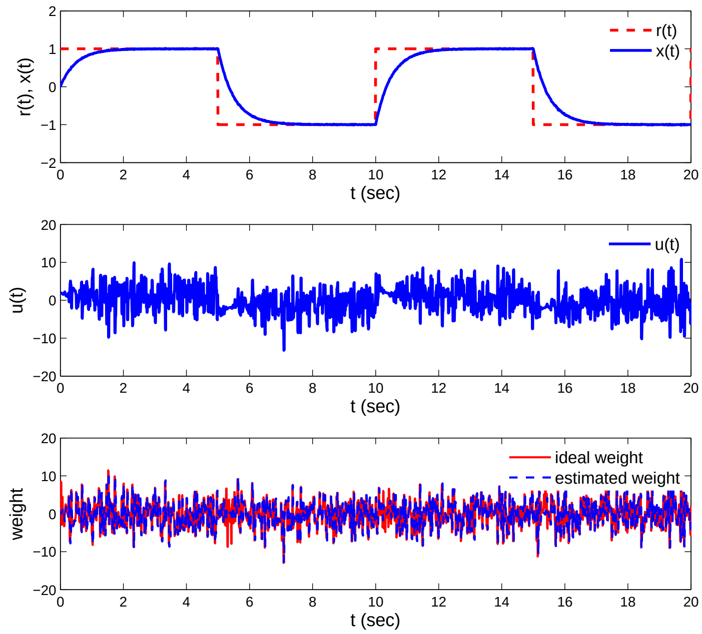

Figure 40: Responses with DF-MRAC for a band limited white noise ideal weight.

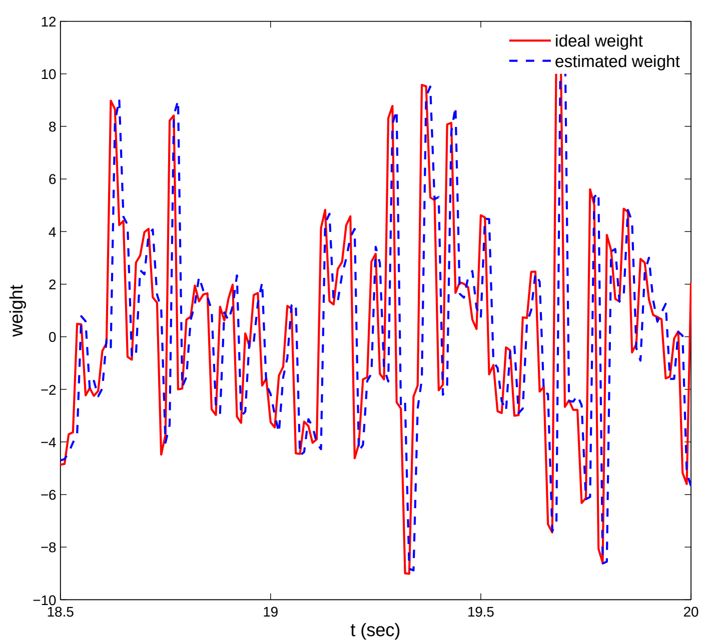

Figure 41: Expanded view of the estimate of the ideal weight.

4.6.2 Uncertainty with Constant Ideal Weights

Consider the scalar dynamics given by (145) with $\Delta(t, x) = 1 + 5x$. To illustrate the point that conventional MRAC is problematic when augmenting a nominal controller that has integral action, we introduce an integrator state

$$\dot{x}_{\mathrm{i}}(t) = -x(t) + r(t) \tag{146}$$

and consider the augmented dynamics

$$\dot{x}_{\mathrm{a}}(t) = \begin{bmatrix} L_{\mathrm{p}} & 0 \\ -1 & 0 \end{bmatrix} x_{\mathrm{a}}(t) + \begin{bmatrix} L_{\delta} \\ 0 \end{bmatrix} \Big(u(t) + \big[1 + 5x(t)\big]\Big) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \tag{147}$$

where $x_{\mathrm{a}}^{\mathrm{T}}(t) = [x(t)\ x_{\mathrm{i}}(t)]$, and choose a PI form for the nominal controller:

$$u_{\mathrm{n}}(t) = -[k_x\ k_{\mathrm{i}}]x_{\mathrm{a}}(t) \tag{148}$$

Then, the reference model in (4) becomes:

$$\dot{x}_{\mathrm{m}}(t) = \begin{bmatrix} L_{\mathrm{p}} - L_{\delta}k_x & -L_{\delta}k_{\mathrm{i}} \\ -1 & 0 \end{bmatrix} x_{\mathrm{m}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \tag{149}$$

The adaptive control is chosen in the form of (6), with $\beta^{\mathrm{T}}(x(t)) = [1\ x_{\mathrm{a}}^{\mathrm{T}}(t)]$, and uses the same weight update law as in the previous example. We again let $L_{\mathrm{p}} = -1$ and $L_{\delta} = 1$, and the PI gains are set to $k_x = 1.5$ and $k_{\mathrm{i}} = 3$. For the MRAC law given by (7), we consider two adaptation gains, $\gamma = 20$ and $\gamma = 200$. For the DF-MRAC law, the parameter settings are the same as in the previous example. Figures 42 and 43 present results that support the statement made in Remark 4.8. Figure 42 illustrates the conflict that can arise when a nominal control law with integral action is augmented with an adaptive controller that employs an integrator in its weight update law. Note the slowly varying tracking error during the first transient phase. In more complex problems this slow variation can be persistent. Figure 43 shows that the DF-MRAC law works well, even during the first transient phase.

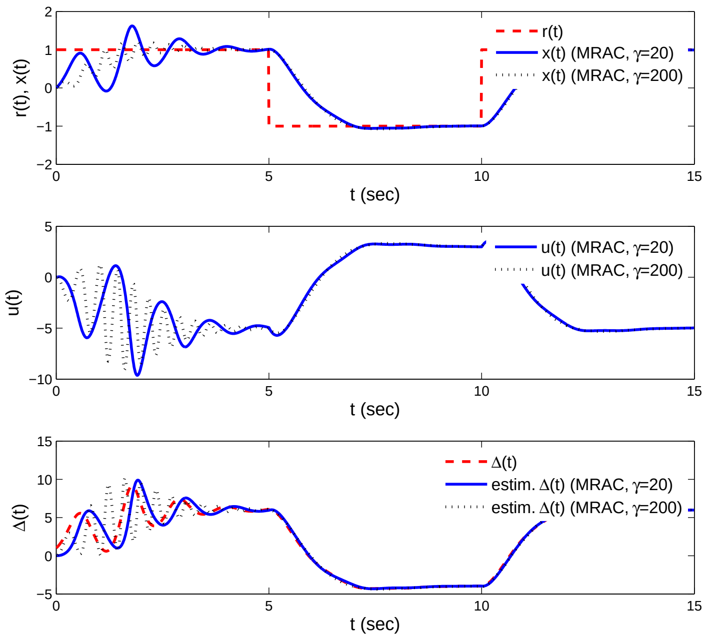

Figure 42: Responses with standard MRAC for an uncertainty with constant ideal weights.

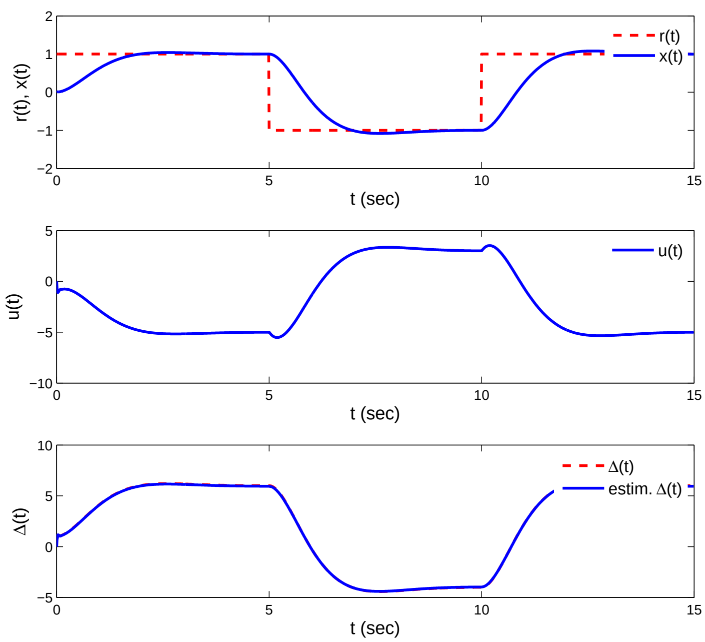

Figure 43: Responses with DF-MRAC for an uncertainty with constant ideal weights.

4.6.3 Input Uncertainty Case

Consider the scalar dynamics

$$\dot{x}(t) = L_p x(t) + L_\delta[\Lambda u(t) + \Delta(t, x(t))] \tag{150}$$

where $\Lambda = 0.1$ represents an input uncertainty. Let $w(t)$ be a sinusoidal time-varying ideal weight with an amplitude of 1 and a frequency of 2.5 seconds. Furthermore, consider the same adaptive control design in Section VI.A for (76) and choose $\Xi_{11} = \Xi_{21} = 0.5$ for (125) and (126). Figure 44 shows the result when (76) is employed and Figure 45 shows the result when (76), (125), and (126) are employed. There is a dramatic improvement when (76), (125), and (126) are employed.

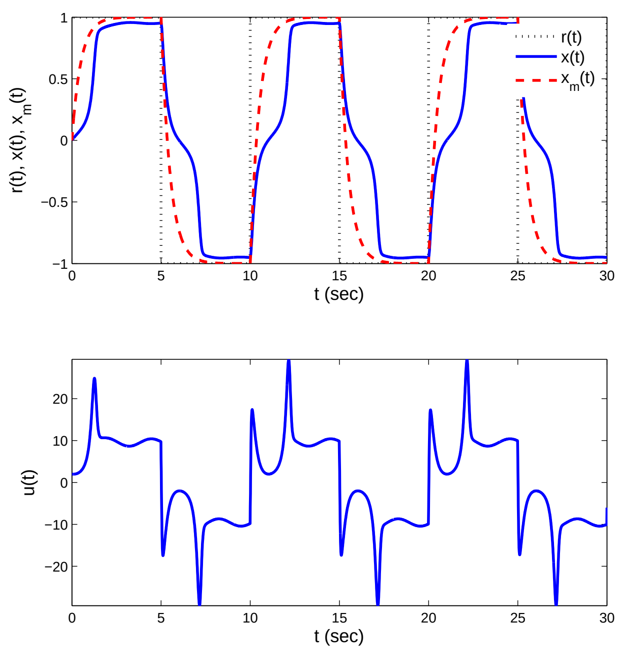

Figure 44: Responses with DF-MRAC for input uncertainty case when (76) is employed.

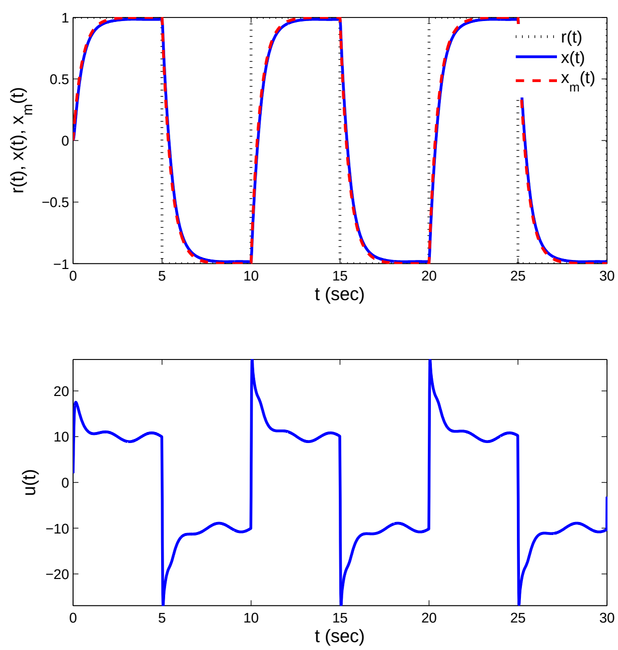

Figure 45: Responses with DF-MRAC for input uncertainty case when (76), (125), and (126) are employed.

4.7 Examples on a Generic Transport Model

This section presents a DF-MRAC design for the NASA GTM, and evaluates this design for several damage scenarios. GTM is a high-fidelity scaled transport aircraft model developed by NASA Langley Research Center [1]. A linearized model for GTM at an angle of attack of 2 degrees and 10^4 ft altitude is obtained in the form of (1). The primary sources of uncertainty are any one of a set of possible damage conditions that are included as a part of the modeling in GTM.

A nominal controller is first designed for the linearized model using a robust servomechanism LQR approach that incorporates integral control [79], with the objective of tracking roll rate, pitch rate, and yaw rate commands. Including the integral states, the linearized GTM model is 9^{th}-order with the state vector $x(t) = \big[u(t)\ v(t)\ w(t)\ p(t)\ q(t)\ r(t)\ \phi(t)\ \theta(t)\ q_{\text{int}}(t)\ p_{\text{int}}(t)\ w_{\text{int}}(t)\big]^{\text{T}}$, where $u(t)$, $v(t)$, $w(t)$ are velocity components; $p(t)$, $q(t)$, $r(t)$ are body angular rates about the roll, pitch and yaw body axes; $\phi(t)$ and $\theta(t)$ are roll and pitch attitude; and $q_{\text{int}}(t)$, $p_{\text{int}}(t)$, $w_{\text{int}}(t)$ are the integrator states. In this simulation study, tracking of roll and pitch rate commands are considered, and yaw rate command is set to zero. Roll and pitch attitude are not used in the design. Figure 46 shows the performance of the nominal controller under normal operating conditions.

Since this design has redundant actuation and damage conditions that may include loss of one or more actuator channels, it is necessary to generalize the form in (117) to

$$\dot{x}(t) = Ax(t) + B\Lambda\big[Gu_{\text{n}}(t) - u_{\text{ad}}(t)\big] + B\Delta(t, x(t)) \tag{151}$$

$$u_{\text{ad}}(t) = \hat{W}^{\text{T}}(t)\beta(x(t)) + G\hat{K}_{1e}^{\text{T}}(t)x(t) + G\hat{K}_{2e}^{\text{T}}(t)r(t) \tag{152}$$

where $G \in \mathrm{I\!R}^{M\times 3}$ is a control allocation matrix in which $M > 3$ denotes the number of independent control channels, $u_{\text{n}} \in \mathrm{I\!R}^3$ is the *effective* nominal control for the roll, pitch, and yaw axes, respectively, and $u_{\mathrm{a}} \in \mathrm{I\!R}^M$ is the adaptive control. Note that while the nominal control law must operate through the control allocation matrix, a portion of the adaptive controller has direct access to each independent channel of actuation. The quantity $\Lambda \in \mathrm{I\!R}^{M\times M}$ is nominally an identity matrix, and loss of actuation is represented by setting one or more of its diagonal elements to zero, and $\Delta : [0, \infty) \times \mathrm{I\!R}^n \to \mathrm{I\!R}^n$ is the uncertainty, which primarily enters the p, q, and r state equations. It is used to represent uncertainty in the stability derivatives. In general the portion of the uncertainty that remains matched under actuator failure corresponds to the projection of Δ onto the column space of $B\Lambda$. For aircraft flight control applications the assumption that Δ remains matched under actuator failure amounts to assuming that $B\Lambda$ and Δ primarily influence the moments acting on the aircraft, and that control of moments in all three axes is maintained under actuator failure. For this study only the spoilers are independently controlled, so there is a total of $M = 6$ independent control channels: elevator, aileron, rudder, left and right spoilers, and stabilizer. Since there are 6 independent control surfaces and only 3 moment axes, it is conceivable that up to 3 control surfaces could fail with minimal change in the column space of B. In general, this is approximately true in aircraft applications where the means of control is primarily through moment generation. Therefore, Assumption 4.2 is not restrictive for these cases when we have actuator redundancy. On the other hand, for the cases when there is no actuator redundancy, this is a restrictive assumption. Servo dynamics and position and rate limits are included in the GTM model. The stabilizer servo has a relatively low bandwidth and low value for its rate limit, and is useful primarily for maintaining trim in the pitch axis. The nominal control design is performed using BG in place of B in (1) when doing the design.

For the adaptive design, neural network sigmoidal type functions are used in the form $\beta(x) = [1,\ \beta_1(x),\ \beta_2(x), \ldots,\ \beta_6(x)]^{\mathrm{T}}$, where $\beta_i(x) = (1 - e^{-x_i})/(1 + e^{-x_i})$, $i =$

$1, \ldots, 6$, and P in (8) is computed using $Q_{\rm m} = \text{diag}\big[10^{-4}\ 10^{-2}\ \ 10^{-3}\ 60\ 30\ 15\ 10\ 10\ 10\big]$. Taking into consideration Theorem 4.4, weight update laws with ALR modification are:

$$\begin{aligned}\hat{W}(t) &= \Omega_1 \hat{W}(t-\tau) + \Big[\beta(x(t))e^{\rm T}PB - \kappa_{\rm m}\big[\beta_x(x(t))\beta_x^{\rm T}(x(t))\hat{W}(t) \\ &\qquad + \beta_x(x(t))\hat{K}_{1e}(t)G^{\rm T}\big]\Big]\Gamma_{\kappa_2}, \quad \Gamma_{\kappa_2} > 0 \end{aligned} \tag{153}$$

$$\hat{K}_{1e}(t) = \Xi_{11}\hat{K}_{1e}(t-\tau_{1e}) + \kappa_{21}\big[x(t)e^{\rm T}(t)PB\big] \tag{154}$$

$$\hat{K}_{2e}(t) = \Xi_{21}\hat{K}_{2e}(t-\tau_{2e}) + \kappa_{22}\big[r(t)e^{\rm T}(t)PB\big] \tag{155}$$

We chose $\Omega_1 = 0.1$, $\Gamma_{\kappa_2} = \text{diag}\big[10\ 1\ 10\ 10\ 50\ 10\big]$, and $\tau = 0.5$ for (153), and $\Xi_{11} = 0.1I_3$, $\Xi_{21} = 0.1I_2$, $\kappa_{12} = \kappa_{22} = 0.05$, and $\tau_{1e} = \tau_{2e} = 0.5$, for (154) and (155). The ALR modification gain in (153) was chosen as $\kappa_{\rm m} = 5.0$.

Remark 4.16. The form of the ALR modification term in (153) is different from the form given in Table 1. Following the procedure in Ref. 10, the form in Table 1 is obtained by taking the gradient of $||\partial u_{\rm ad}(\hat{W}, x)/\partial x||^2/2$ with respect to $\hat{W}$, with $u_{\rm ad}(\hat{W}, x)$ defined by (6), whereas the form in (153) is obtained with $u_{\rm ad}(\hat{W}, x)$ defined by (152).

Remark 4.17. The form of the adaptation gain, $\Gamma_{\kappa_2} > 0$, in (153) is more general that the scalar form used in (78). This was done to add greater flexibility in the design. The proof of Theorem 4.1 can be generalized to this case using:

$$\mathcal{V}(e(t), \tilde{W}_t) = e^{\rm T}(t)Pe(t) + \text{tr}\Big[\int_{t-\tau}^{t} \tilde{W}^{\rm T}(s)\tilde{W}(s)\Gamma_\rho \text{d}s\Big], \quad \Gamma_\rho > 0 \tag{156}$$

4.7.1 Missing Left Wing Tip

In the missing left wing tip damage scenario there is 25% loss of outboard left wing tip and the left aileron is missing, therefore available roll control effectiveness is reduced [1]. Figure 47 shows the degree of degraded performance when the nominal controller is evaluated for this damage case. Figure 48 shows the improvement in response obtained when DF-MRAC is employed. In Figure 49, 0.01 seconds of time delay is added to the rudder channel. In this case, the performance using DF-MRAC alone is not satisfactory. Figure 50 shows that the addition of ALR modification significantly improves the robustness of the adaptive controller to time delay.

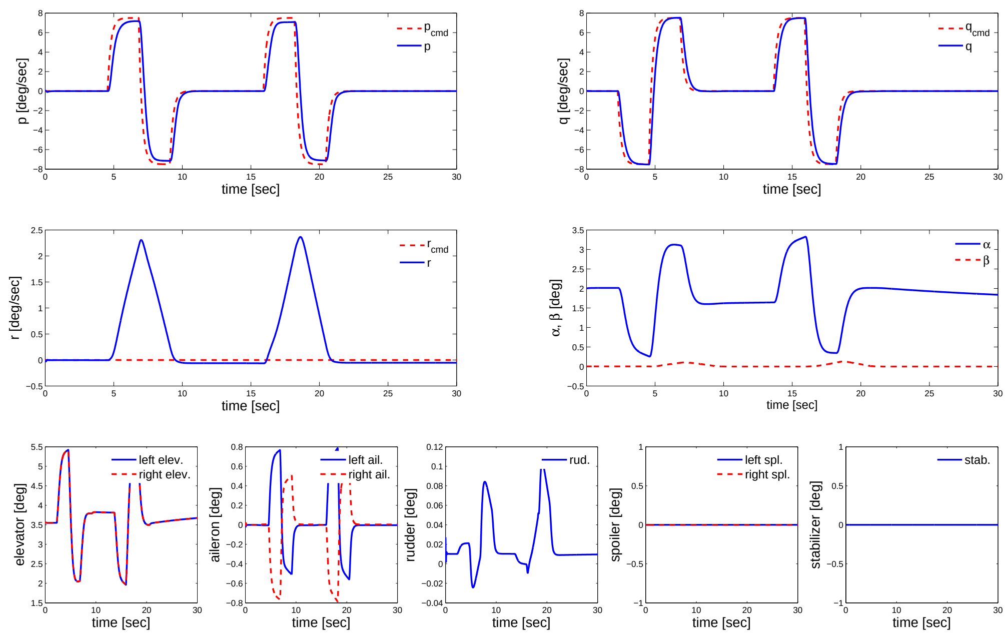

Figure 46: GTM nominal control response for nominal operating conditions.

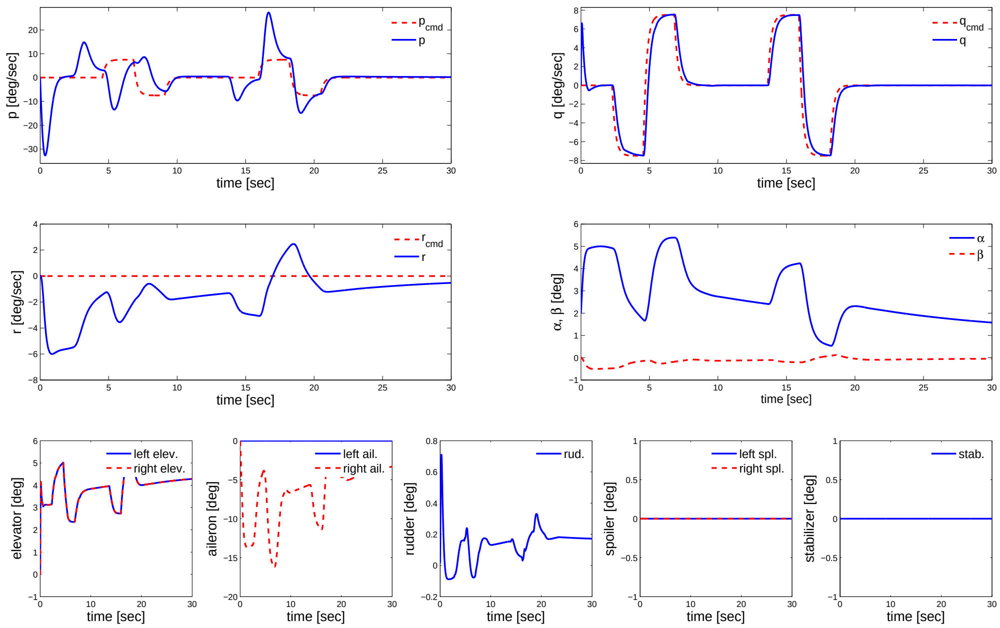

Figure 47: GTM nominal control response for the missing left wing tip case.

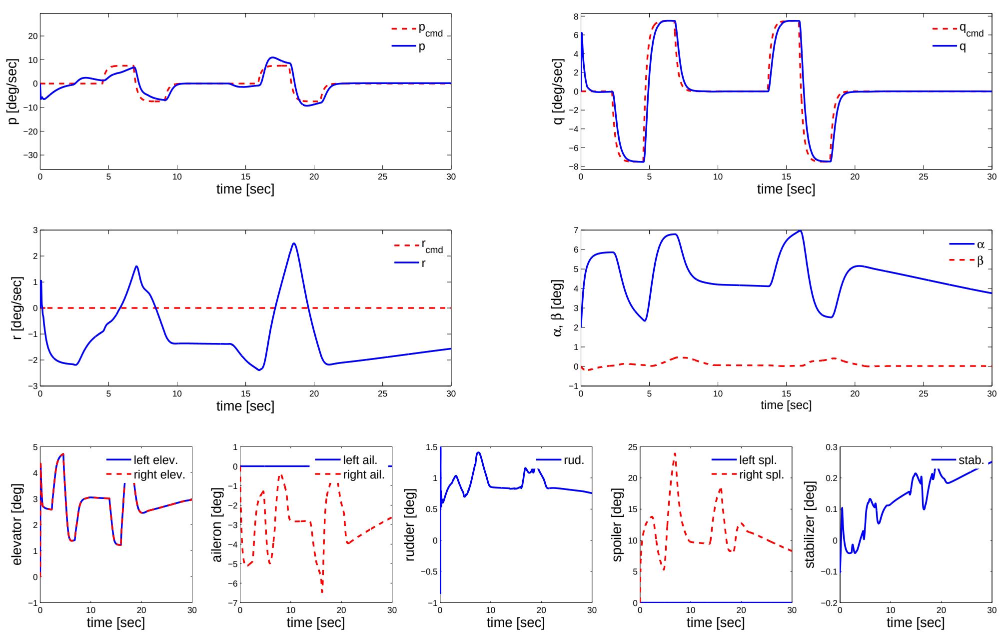

Figure 48: GTM DF-MRAC response for the missing left wing tip case.

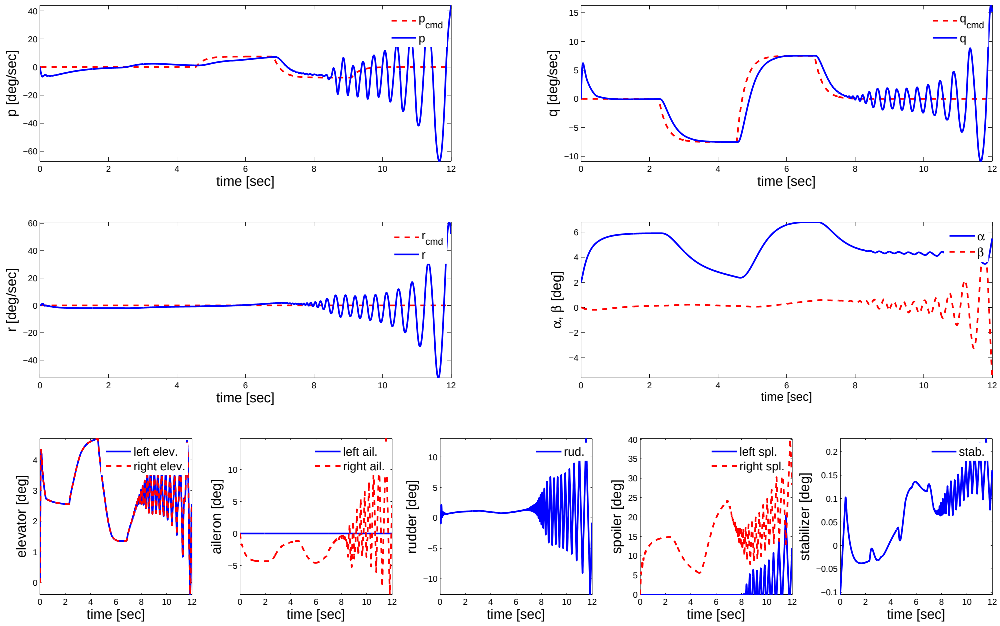

Figure 49: GTM DF-MRAC response for the missing left wing tip case with 0.01 seconds of time delay in the rudder channel.

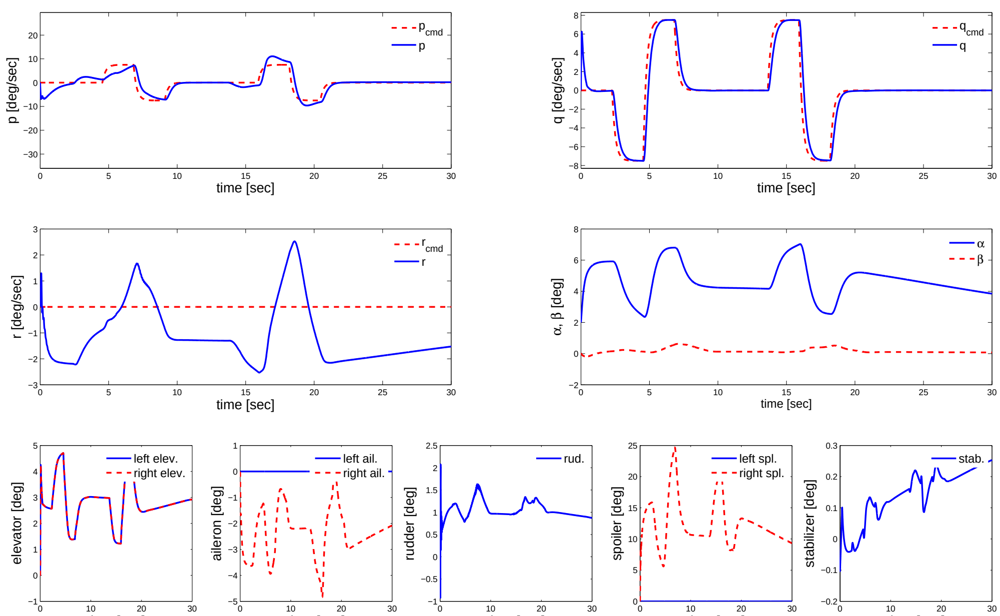

Figure 50: GTM DF-MRAC response with ALR modification term for the missing left wing tip case with 0.01 seconds of time delay in the rudder channel.

4.7.2 Missing Vertical Tail

In the missing vertical tail damage scenario the entire vertical tail is missing, therefore there is a loss in directional stability and a complete loss in rudder control effectiveness [1]. Figure 51 shows that the nominal controller response for this damage case is unstable. Figure 52 shows that the DF-MRAC controller provides upset recovery and satisfactory tracking performance for this damage scenario. Figure 53 shows a result for the same damage case in Figure 52 with 0.07 seconds of time delay in the right aileron channel. The performance of the DF-MRAC controller is not satisfactory with this amount of time delay. Figure 54 shows that ALR modification improves the time delay margin of the DF-MRAC controller design for this failure case.

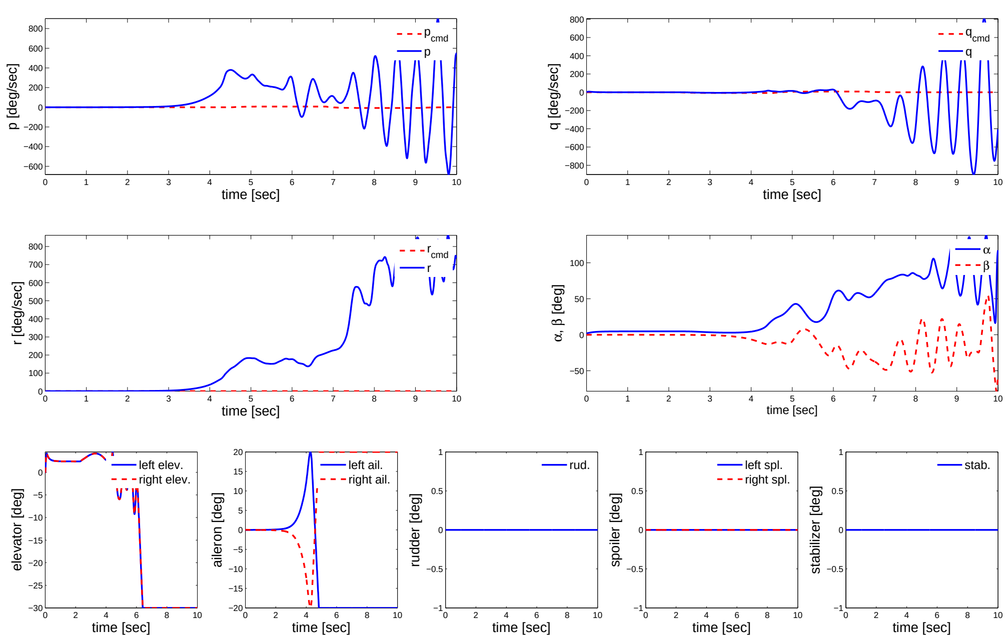

Figure 51: GTM nominal control response for the missing vertical tail case.

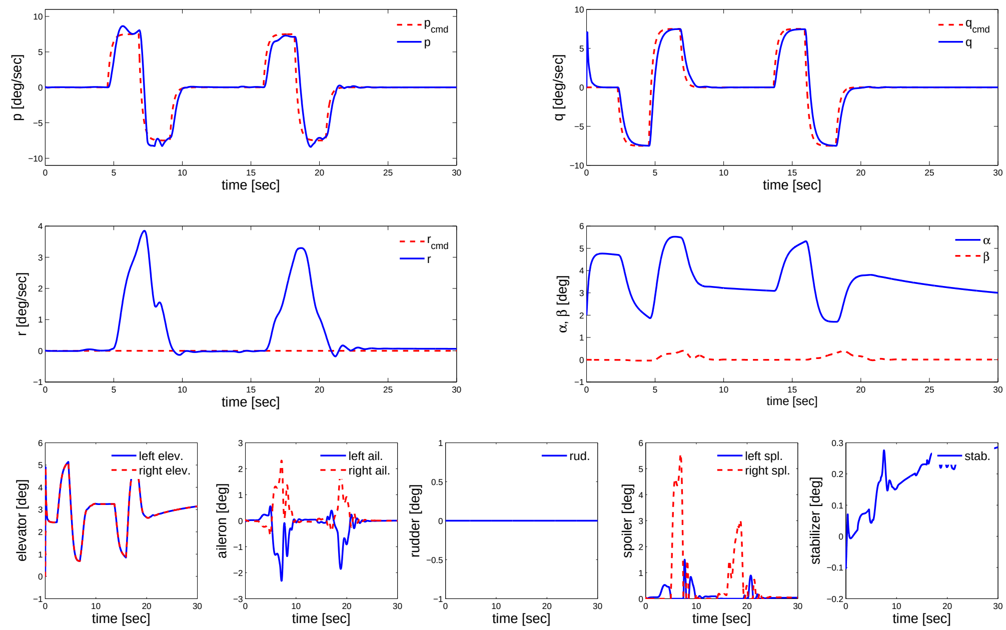

Figure 52: GTM DF-MRAC response for the missing vertical tail case.

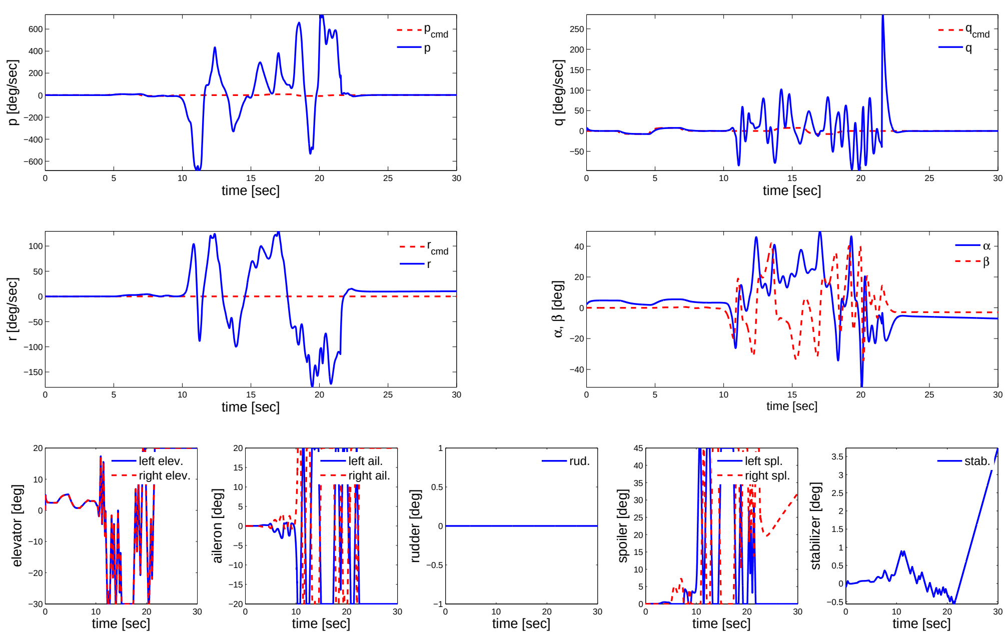

Figure 53: GTM DF-MRAC response for the missing vertical tail case with 0.07 seconds of time delay in the right aileron channel.

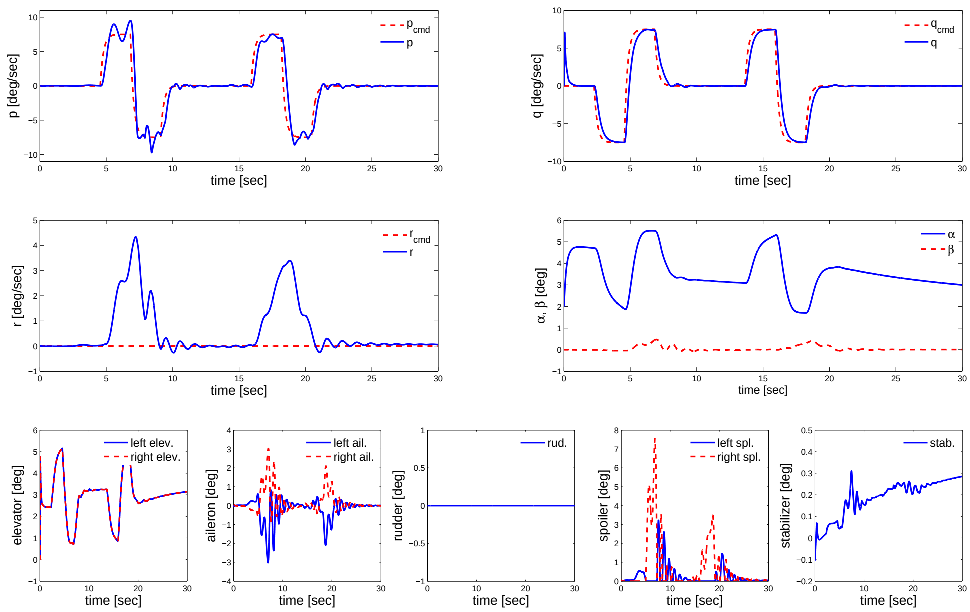

Figure 54: GTM DF-MRAC response with ALR modification term for the missing vertical tail case with 0.07 seconds of time delay in the right aileron channel.

4.8 Flight Test Results on NASA AirSTAR

In this section, we analyze flight test results obtained employing a derivative-free adaptive controller with ALR modification on the NASA Airborne Subscale Transport Aircraft Research (AirSTAR) flight test vehicle [41, 42] shown in Figure 55, which is designed to provide a flexible research environment for testing control algorithms under severe uncertainties and failures. The previous section presented results on GTM, which is a high fidelity model of AirSTAR.

In-flight damage was represented by introducing additional feedback loops to change stability derivatives that represent unknown destabilizing changes in stiffness and damping terms in the longitudinal and lateral dynamics. Here, we use the wording *adaptive controller* to denote DF-MRAC with ALR modification. Figures 56–61 present samples of the results. Each figure has four subfigures. The upper two subfigures show roll rate and pitch rate responses for a baseline, non-adaptive control design. The lower subfigures show the roll rate and pitch rate responses for the adaptive design. In all the figures, the dashed red line shows the pilot command and the solid blue line shows the actual AirSTAR responses.

4.8.1 Case 1: Employing Baseline and Adaptive Controllers

Figure 56 compares the results when baseline and adaptive controllers are first engaged. Here, the responses of baseline and adaptive controllers in roll and pitch rates look nearly the same since there was no induced uncertainty, and pilot commands occurred primarily in the pitch axis.

4.8.2 Case 2: Roll and Pitch Rate Commands

Figure 57 compares the results when a roll rate doublet and a sinusoidal pitch rate are simultaneously commanded, without an induced uncertainty. This case shows improvement in pitch rate response with adaptive control most likely due to modeling error associated with cross axis coupling.

4.8.3 Case 3: Latency Emulation

Figure 58 compares the roll and pitch rate responses of baseline and adaptively controlled AirSTAR, when a time delay up to 0.1 seconds is injected to all input channels. It is crucial to note here that DF-MRAC with ALR modification preserves the time delay margin of the baseline controller. In addition, if we look at the magnitude of the commands, it can be seen that harsher commands were applied to the adaptively controlled AirSTAR especially in roll rate. It should also be noted that the flight test results shown in this figure were taken on different days, and that the wind conditions for the results with DF-MRAC were considerably more variable and greater in magnitude. Unfortunately we do not have a flight test result that compares DF-MRAC with and without ALR when subjected to latency in order to validate that ALR modification does significantly improve time delay margin of an adaptive control design, as illustrated previously in the simulated results of Figures 6 and 7.

4.8.4 Case 4: C_{m_α} and $C_{\mathrm{l}_\mathrm{p}}$ Reduction by 50% and +0.2 under Pitch Rate Command

Figure 59 compares the roll and pitch rate responses of baseline and adaptively controlled AirSTAR under parametric uncertainty (induced by introducing additional feedback of α and p), when a pitch rate doublet is commanded. Also near the end of the interval a pulse in the roll rate command is applied. The response with adaptive controller is better than the baseline controller's response. The baseline controller's roll rate response shown in upper left side of Figure 59 is much more oscillatory and varies approximately from -15 to +20. The adaptive controller's roll rate response shown in the bottom left side of Figure 59 is better by more than a factor of 2. The responses in pitch rate look nearly the same.

Figure 55: Flight test vehicle.

4.8.5 Case 5: C_{m_α} and C_{l_p} Reduction by 50% and +0.2 under Roll Rate Command

Figure 60 compares the roll and pitch rate responses of baseline and adaptively controlled AirSTAR under parametric uncertainty, when a roll rate doublet is commanded. Roll rate tracking with the adaptive controller is again improved by roughly a factor of 2, and the pitch rate tracking is also improved, similar to what was observed in Figure 57.

4.8.6 Case 6: C_{m_α} and C_{l_p} Reduction by 75% and +0.3 under Pitch Rate Command

Figure 61 compares the roll and pitch rate responses of baseline and adaptively controlled AirSTAR under a harsher parametric uncertainty, when a pitch rate doublet is commanded. In this case the performance in terms of tracking error in both pitch and roll rate is reduced by at least a factor of 2 with the adaptive controller.

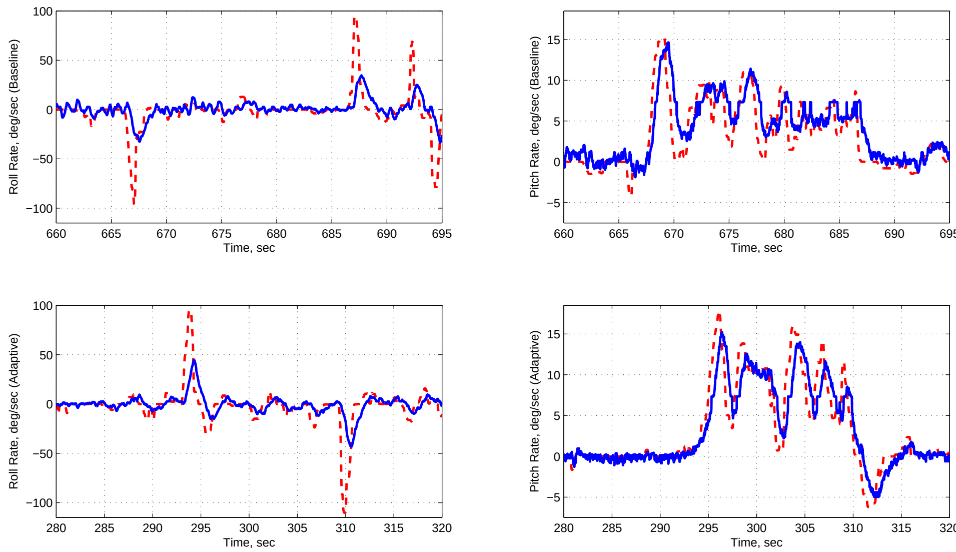

Figure 56: Comparison of baseline and adaptive controller responses for Case 1.

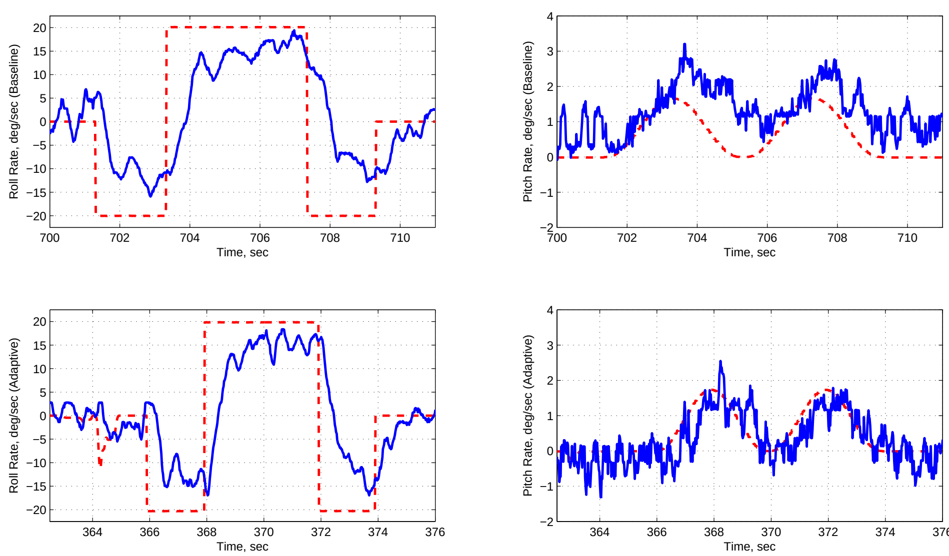

Figure 57: Comparison of baseline and adaptive controller responses for Case 2.

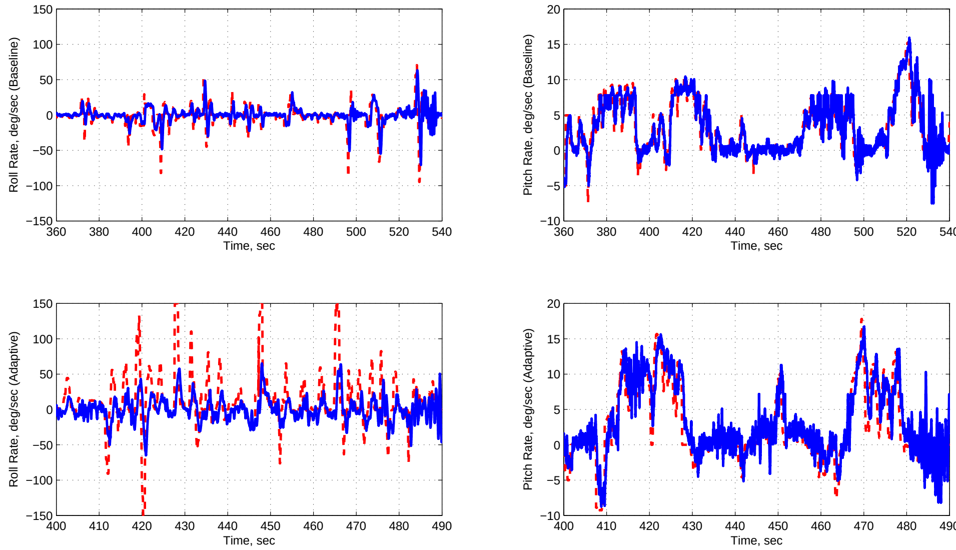

Figure 58: Comparison of baseline and adaptive controller responses for Case 3.

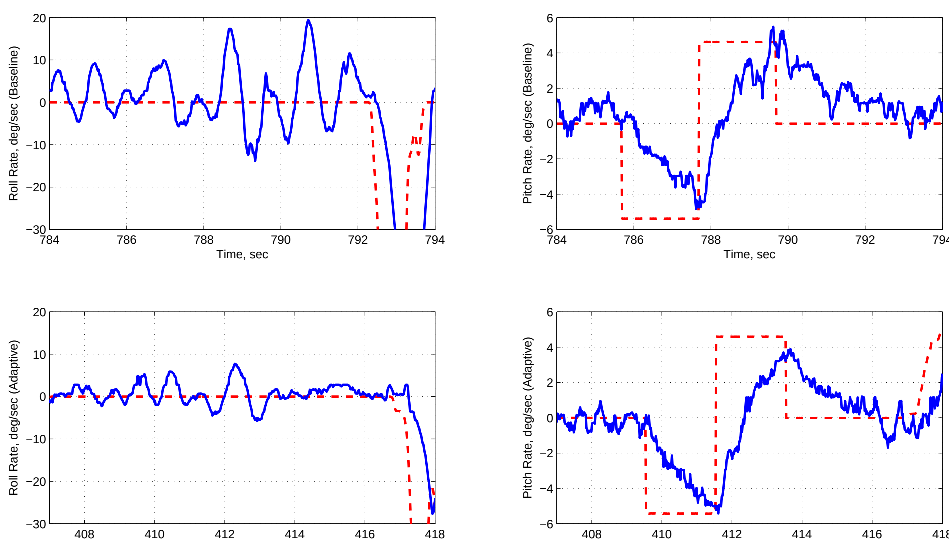

Figure 59: Comparison of baseline and adaptive controller responses for Case 4.

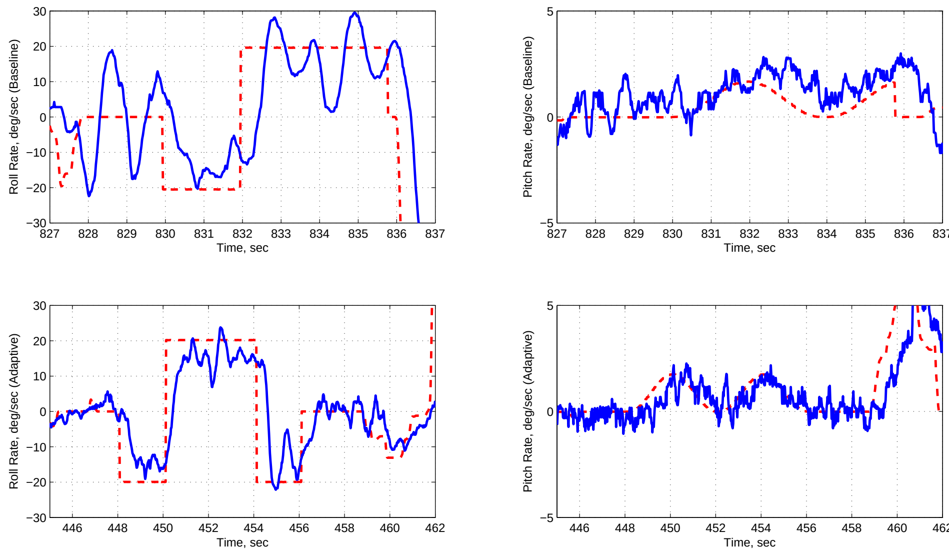

Figure 60: Comparison of baseline and adaptive controller responses for Case 5.

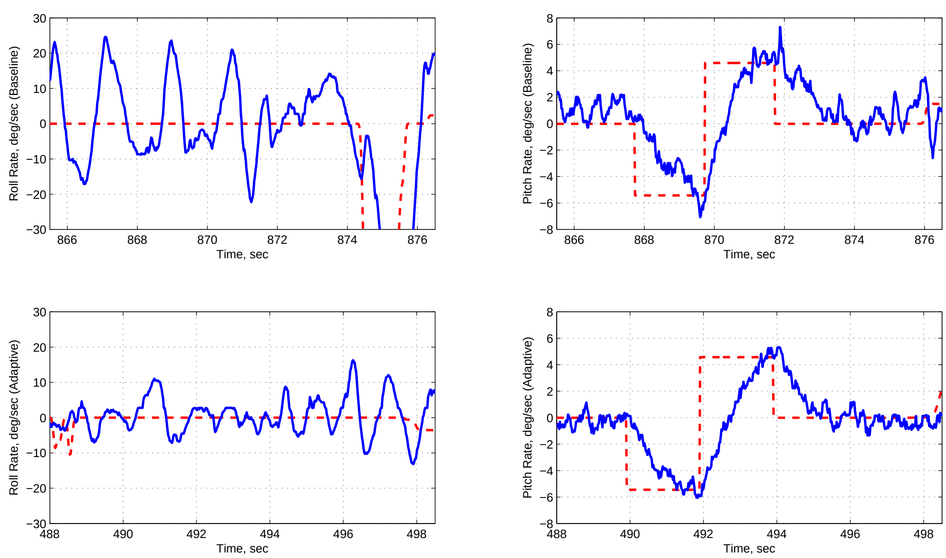

Figure 61: Comparison of baseline and adaptive controller responses for Case 6.

4.9 Conclusion

This chapter presents a derivative-free model reference adaptive control law for uncertain systems. The key feature is that the stability analysis is performed under the assumption that the ideal weights are bounded, but otherwise arbitrarily time varying. The approach is particularly useful for situations in which the nature of the system uncertainty cannot be adequately represented by a set of basis functions with unknown constant weights. The system error signals, including the state tracking error and the weight update error, are proven to be uniformly ultimately bounded using a Lyapunov-Krasovskii functional without the introduction of modification terms. We have shown that the errors approach the ultimate bound exponentially in time. We have also provided an analysis that shows how the design parameters in the adaptive law can influence the ultimate bound, and the exponential rate of convergence to the bound. The examples illustrate the superior performance of derivative free adaptation in comparison to a conventional adaptive law, when it is evaluated for a variety of cases in which there is a sudden or rapidly varying set of dynamics, and in disturbance rejection. We have also shown how the derivative-free adaptive law can be modified using many of the existing modification terms. The simulation results show that dramatic improvement in robustness to time delays can be achieved by modifying the derivative-free adaptation law using a method of adaptive loop recovery. Furthermore, the performance of the proposed approach is successfully evaluated in flight on AirSTAR. Finally, the proposed approach is also successfully evaluated in flight on TwinSTAR and the results are reported in Ref. 16.

CHAPTER V

Derivative-Free Adaptive Control: The Output Feedback Case

5.1 Introduction

This chapter presents an output feedback adaptive control architecture for continuous-time uncertain systems based on state observer and derivative-free delayed weight update law. The observer is employed in the adaptive part of the design in place of a reference model. A derivative-free weight update law in output feedback form ensures that the estimated states follow both the reference model states and the true states so that both the state estimation error and the state tracking error are bounded. The stability analysis uses a Lyapunov-Krasovskii functional that entails the solution of a parameter dependent Riccati equation [46], rather than a Lyapunov equation, to show that all the error signals are UUB. The complexity of the proposed approach is less than many other output feedback adaptive control architectures available in the literature and it is intended to be used to augment an existing state observer based linear controller. The proposed approach is particularly advantageous for applications to systems that can undergo a sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, docking, or structural damage.

5.2 *Derivative-Free Output Feedback Adaptive Control Architecture*

Consider the uncertain system given by

$$\dot{x}(t) = Ax(t) + B\big[u(t) + \Delta(t, x(t))\big] \quad (157)$$

$$y(t) = Cx(t) \quad (158)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ are known system matrices, $x(t) \in \mathbb{R}^n$ is the unknown state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\Delta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is the matched uncertainty, and $y(t) \in \mathbb{R}^m$ is the *regulated* output vector. Furthermore, the triple (A, B, C) is minimal and the control input vector is restricted to the class of admissible controls consisting of measurable functions.

Remark 5.1. The system given by (157) and (158) assumes that the control input vector and the regulated output vector have the same dimension. For the case when the dimension of the control input vector is larger than the dimension of the regulated output vector and there redundancy in actuation, one can use either matrix inverse and pseudoinverse approaches, constrained control allocation, pseudocontrols, or daisy chaining [7,24,78] to reduce the dimension of the control input vector to the dimension of the regulated output vector. Furthermore, the system can have a *sensed* output vector denoted by

$$y_{\mathrm{s}}(t) = C_{\mathrm{s}}x(t) \quad (159)$$

where $y_{\mathrm{s}}(t) \in \mathbb{R}^l$, $C_{\mathrm{s}} \in \mathbb{R}^{l\times n}$, $l \geq m$, such the elements of $y(t)$ are a subset of the elements of $y_{\mathrm{s}}(t)$.

Consider the state observer based nominal control law given by

$$u_{\mathrm{n}}(t) = -K_1\hat{x}(t) + K_2 r(t) \quad (160)$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ are known feedback and feedforward matrices, respectively, $r(t) \in \mathbb{R}^r$ is the regulated output command vector, and $\hat{x}(t) \in \mathbb{R}^n$ is an observer estimate of $x(t)$ given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{\rm n}(t) + L\big[y_{\rm s}(t) - C_{\rm s}\hat{x}(t)\big] \tag{161}$$

with $L \in {\rm I\!R}^{n\times l}$ being the observer gain matrix designed such that $A_e \equiv A - LC_{\rm s} \in {\rm I\!R}^{n\times n}$ is Hurwitz. Define the reference model as in (4) subject to Assumption 2.2.

Remark 5.2. The above implies that the gains K_1 and K_2 have been designed using the certainty equivalence principle ($\hat{x}(t) = x(t)$) with $\Delta(t, x(t)) = 0$, so that $C\hat{x}(t)$ tracks $r(t)$ to within some set of specifications on both the transient and steady state performance.

Assumption 5.1. The matched uncertainty in (157) can be linearly parameterized as

$$\Delta(t, x) = W^{\rm T}(t)\beta(x), \quad \forall x \in \mathcal{D}_x \subset {\rm I\!R}^n \tag{162}$$

where $W(t) \in {\rm I\!R}^{s\times m}$ is the unknown time-varying weight matrix that satisfies $\|W(t)\| \leq \bar{\omega}$, $\beta : {\rm I\!R}^n \to {\rm I\!R}^s$ is a known Lipschitz continuous basis vector of the form $\beta(x) = [\beta_1(x), \beta_2(x), ..., \beta_s(x)]^{\rm T}$ with $|\beta(x)| \leq \bar{\beta}$, and $\mathcal{D}_x$ is sufficiently large.

Remark 5.3. Assumption 5.1 expands the class of uncertainties that can be represented by a given set of basis functions. That is, an adaptive law design subject to Assumption 5.1 can be more effective than an adaptive law designed subject to

$$\Delta(x) = W^{\rm T}\beta(x) + \varepsilon(x), \quad \forall x \in \mathcal{D}_x \tag{163}$$

where W is a unknown constant ideal weight and $\varepsilon(x)$ is the residual error. It also permits an explicit dependence of the uncertainty on time.

Remark 5.4. Assumption 5.1 does not place any restriction on the time derivative of the weight matrix. However the degree of time dependence will depend on how $\beta(x)$ is chosen.

The adaptive control objective is to design a control law $u(\cdot)$ for the system given by (157) and (158) so that $x(t)$ tracks $x_{\rm m}(t)$ with bounded error. For this purpose, the nominal control law $u_{\mathrm{n}}(t)$ given by (160) is augmented with the adaptive control law $u_{\mathrm{ad}}(t)$ as:

$$u(t) = u_{\mathrm{n}}(t) - u_{\mathrm{ad}}(t) \tag{164}$$

$$u_{\mathrm{ad}}(t) = \hat{W}^{\mathrm{T}}(t)\beta(\hat{x}(t)) \tag{165}$$

Note that the state observer given by (161) is regarded as a part of the nominal control design. However, our viewpoint below is that L may be altered for purposes of adaptively augmenting the nominal controller.

Denote $\tilde{x}(t) \equiv x(t) - \hat{x}(t)$ for the state estimation error, $\hat{e}(t) \equiv \hat{x}(t) - x_{\mathrm{m}}(t)$ for the estimated state tracking error, and $\tilde{W}(t) \equiv W(t) - \hat{W}(t)$ for the weight estimate error. From (157) and (161), the dynamics for the state estimation error can be written in the form

$$\dot{\tilde{x}}(t) = A_e\tilde{x}(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(\hat{x}(t)) + Bg(x(t), \hat{x}(t)) \tag{166}$$

where $g(x(t), \hat{x}(t)) \equiv W^{\mathrm{T}}(t)\big[\beta(\hat{x}(t)) - \beta(x(t))\big]$ with $|g(x(t), \hat{x}(t))| \leq \bar{w}L_\beta|\tilde{x}(t)|$, where $L_\beta > 0$ is the Lipschitz constant for the known basis vector. Likewise, from (161) and (4) the dynamics for the estimated state tracking error can be written in the form:

$$\dot{\hat{e}}(t) = A_{\mathrm{m}}\hat{e}(t) + LC\tilde{x}(t) \tag{167}$$

Next, consider the derivative-free weight update law given by

$$\hat{W}(t) = \Omega_1\hat{W}(t-\tau) + \hat{\Omega}_2(t) \tag{168}$$

where $\tau > 0$, and $\Omega_1 \in \mathbb{R}^{s\times s}$ and $\hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s\times m}$ satisfy:

$$0 \leq \Omega_1^{\mathrm{T}}\Omega_1 < \kappa_1 I, \quad 0 \leq \kappa_1 < 1/(1+\mu), \quad \mu > 0 \tag{169}$$

$$\hat{\Omega}_2(t) \equiv \kappa_2\beta(\hat{x}(t))\tilde{y}^{\mathrm{T}}(t), \quad \kappa_2 > 0 \tag{170}$$

where $\tilde{y}(t) \equiv y(t) - \hat{y}(t)$, where $\hat{y}(t) = C\hat{x}(t)$.

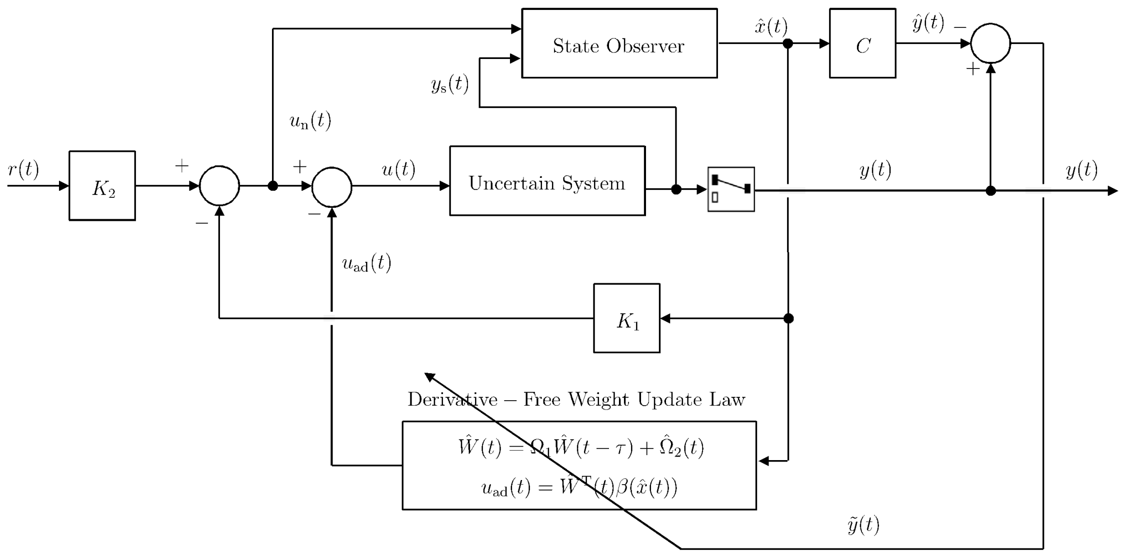

Figure 62: Derivative-free output feedback adaptive control architecture

A visualization of the derivative-free output feedback adaptive control architecture is given in Figure 62. Note that the state observer serves as the reference model. Its dynamics are the same as the reference model if $u_{\rm ad}(t)$ cancels $\Delta(t, x(t))$, and in this case the observer error transient $\tilde{y}(t)$ goes to zero. The components that are added to an observer based nominal controller architecture, in order to realize the adaptive control, reduce to computing the basis functions and implementing the adaptation law in (168).

5.3 *A Parameter Dependent Riccati Equation*

In this section we summarize the properties of a parameter dependent Riccati equation [46] that is needed for the stability analysis of the following section. Consider the following quadratic equation in P

$$0 = A_e^{\rm T} P + P A_e + Q \tag{171}$$

$$Q = Q_0 + (\kappa_2 + 1/\mu)\bar{\beta}^2 N N^{\rm T} = Q_0 + \nu N N^{\rm T} \tag{172}$$

$$N = C^{\rm T} - PB \tag{173}$$

in which $Q_0 \in \mathbb{R}^{n\times n} > 0$, and μ and $\bar{\beta}$ have previously been defined by (169) and Assumption 5.1, respectively.

Remark 5.5. Let $0 < \nu < \bar{\nu}$ define the largest set within which there exists a positive definite solution for P. Since $P > 0$ for $\nu = 0$ and P depends continuously on ν, the existence of $P(\nu) > 0$ for $0 < \nu < \bar{\nu}$ is assured. Furthermore, this implies that $\kappa_2 < \bar{\nu}/\bar{\beta}^2 - 1/\mu \equiv \bar{\kappa}_2$.

Remark 5.6. If $N = 0$ in (173), then it follows from (171)–(173) that we have

$$0 = A_e^{\mathrm{T}} P + P A_e + Q_0 \tag{174}$$

$$0 = C^{\mathrm{T}} - PB \tag{175}$$

which implies that the transfer function associated with the system $G(s) = C(sI - A_e)^{-1}B$ is positive-real [30]. In this case (171) reduces to a Lyapunov equation associated with the error dynamics in (166), which is commonly employed in the stability analysis of adaptive systems.

Lemma 5.1. If A_{m} has no repeated eigenvalues and the state observer gain matrix L of (161) is designed using pole placement such that

$$\lambda(A_e) = k\lambda(A_{\mathrm{m}}) \tag{176}$$

then for $\nu < \bar{\nu}$, we can denote $P(k)$ as the corresponding positive definite solution for P of the parameter dependent Riccati equation given by (171)–(173). In this case:

$$\lim_{k\to\infty} P(k) = 0 \tag{177}$$

Proof. Let S_{m} and S_e be diagonalizing transformations for A_{m} and A_e, respectively. Then:

$$A_{\mathrm{m}} = S_{\mathrm{m}} \mathrm{diag}[\lambda(A_{\mathrm{m}})] S_{\mathrm{m}}^{-1} \tag{178}$$

$$A_e = k S_e \mathrm{diag}[\lambda(A_{\mathrm{m}})] S_e^{-1} \tag{179}$$

With $S_{\mathrm{s}} \equiv S_e S_{\mathrm{m}}^{-1}$, A_e can be written as:

$$A_e = k S_{\mathrm{s}} A_{\mathrm{m}} S_{\mathrm{s}}^{-1} = k A_{\mathrm{s}} \tag{180}$$

Using (180) and denoting $P_k \equiv kP(k)$ and $N_k \equiv C^{\mathrm{T}} - k^{-1} P_k B$, the parameter dependent Riccati equation in (171)–(173) can be written as:

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P_k + P_k A_{\mathrm{s}} + Q_0 + \nu N_k N_k^{\mathrm{T}} \tag{181}$$

Taking the limit of (181) as $k \to \infty$, it follows that:

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P_\infty + P_\infty A_{\mathrm{s}} + Q_0 + \nu C C^{\mathrm{T}} \tag{182}$$

Since the solution of (182) is finite, it follows that for all finite k, $P(k) > 0$, and:

$$\lim_{k\to\infty} P(k) = \lim_{k\to\infty} \frac{P_k}{k} = 0 \tag{183}$$

□

Remark 5.7. Lemma 5.1 implies that $||PB||$ can be made arbitrarily small by making the observer dynamics sufficiently fast, while preserving the property that $P > 0$. This property will become important in the stability analysis of Section IV.

The next lemma shows that for $\nu < \bar{\nu}$, (171)–(173) can reliably be solved for $P > 0$ using the Potter approach given in Ref. 67. This also implies that $\bar{\nu}$ can be determined by searching for the boundary value that results in a failure of the algorithm to converge. We employ the notation ric$(\cdot)$ and dom(ric) as defined in Ref. 22.

Lemma 5.2. Let P satisfy the parameter dependent Riccati equation given by (171)–(173) and let the modified Hamiltonian be given by

$$H \equiv \begin{bmatrix} A_e - \nu BC & \nu R \\ -Q & -\left(A_e - \nu BC\right)^{\mathrm{T}} \end{bmatrix} \tag{184}$$

where $Q \equiv Q_0 + \nu CC^{\mathrm{T}}$ and $R \equiv BB^{\mathrm{T}}$. Then, for all $0 < \nu < \bar{\nu}$, $H \in \mathrm{dom(ric)}$ and $P = \mathrm{ric}(H)$.

Proof: The proof follows from Lemmas 1 and 2 of Ref. 22. $\square$

5.4 *Stability Analysis*

This section presents a stability analysis for the derivative-free output feedback adaptive control architecture.

Theorem 5.1. Consider the controlled uncertain dynamical system given by (157) and (158) subject to Assumption 5.1. Consider, in addition, the feedback control law given by (164), with the nominal feedback control component given by (160) and (161), and the adaptive feedback control component given by (165) and (168) subject to the conditions in (169) and (170), and with $\kappa_2 < \bar{\kappa}_2$. Then, $\tilde{x}(t)$ and $\tilde{W}(t)$ are UUB.

Proof. Using (168) and defining

$$\Omega_2(t) \equiv W(t) - \Omega_1 W(t-\tau) \tag{185}$$

where $||\Omega_2(t)|| \leq \bar{\delta}$, $\bar{\delta} \equiv \bar{w}(1 + ||\Omega_1||)$, the weight update error $\tilde{W}(t)$ can be rewritten as:

$$\tilde{W}(t) = \Omega_1 \tilde{W}(t-\tau) + \Omega_2(t) - \hat{\Omega}_2(t) \tag{186}$$

Using (186), the state estimation error in (166) becomes:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_e \tilde{x}(t) + B\big[\Omega_1 \tilde{W}(t-\tau) + \Omega_2(t) - \hat{\Omega}_2(t)\big]^{\mathrm{T}} \beta(\hat{x}(t)) \\ &\quad + Bg(x(t), \hat{x}(t)) \end{aligned} \tag{187}$$

Note that $\hat{\Omega}_2(t)$, given by (170), can be equivalently written using (173) as:

$$\begin{aligned} \hat{\Omega}_2(t) &= \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) C^{\mathrm{T}} \\ &= \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) \big[PB + N\big] \\ &= \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) PB + \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) N \end{aligned} \tag{188}$$

To show that the closed-loop system given by (186) and (187) is UUB, consider the Lyapunov-Krasovskii functional [31]

$$\mathcal{V}(\tilde{x}(t),\tilde{W}_t) = \tilde{x}^{\mathrm{T}}(t)P\tilde{x}(t) + \rho\ \mathrm{tr}\Big[\int_{t-\tau}^{t} \tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\Big] \tag{189}$$

where $\rho > 0$, $\tilde{W}_t$ represents $\tilde{W}(t)$ over the time interval $t-\tau$ to t, and P satisfies (171) with $\nu < \bar{\nu}$, which implies $\kappa_2 < \bar{\kappa}_2$ as noted in Remark 5.5. The directional derivative of (189) along the closed-loop system trajectories of (186) and (187) is given by

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;=\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \\ & +2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t)) \\ & -2\tilde{x}^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t)) \\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) + \eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) - \tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau)\big]\end{aligned} \tag{190}$$

where $\eta \equiv 1+\xi$, $\xi > 0$. Using (186) to expand the term $\mathrm{tr}[\eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)]$ in (190) produces:

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;=\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \\ & +2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t)) \\ & \boxed{-2\tilde{x}^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(\hat{x}(t))}^{\,\mathrm{a}} + 2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t)) \\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) - \tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau) \\ & +\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau) \\ & \boxed{+\eta\hat{\Omega}_2^{\mathrm{T}}(t)\hat{\Omega}_2(t)}^{\,\mathrm{a}} + \eta\Omega_2^{\mathrm{T}}\Omega_2\ \boxed{-2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_1\tilde{W}(t-\tau)}^{\,\mathrm{a}} \\ & \boxed{+2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2}^{\,\mathrm{b}}\ \boxed{-2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_2}^{\,\mathrm{a}}\big]\end{aligned} \tag{191}$$

Consider $|a^{\mathrm{T}}b| \leq \gamma a^{\mathrm{T}}a + b^{\mathrm{T}}b/4\gamma$, $\gamma > 0$, that follows from Young's inequality [5] extended to the vector case for any vectors a and b. This can be further generalized to $|\mathrm{tr}\big[A^{\mathrm{T}}B\big]| = |\mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(B)| \leq \gamma\mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(A) + \mathrm{vec}(B)^{\mathrm{T}}\mathrm{vec}(B)/4\gamma =$

$\gamma \text{tr}\big[A^{\rm T}A\big]+\text{tr}\big[B^{\rm T}B\big]/4\gamma$, $\gamma > 0$, for any matrices A and B with appropriate dimensions. Using this, we can write:

$$\begin{aligned} \text{tr}\big[2\eta\tilde{W}^{\rm T}(t-\tau)\Omega_1^{\rm T}\Omega_2(t)\big] &\leq \text{tr}\big[\gamma\tilde{W}^{\rm T}(t-\tau)\Omega_1^{\rm T}\Omega_1\tilde{W}(t-\tau)\big] \\ &\quad +\text{tr}\big[\eta^2\Omega_2^{\rm T}(t)\Omega_2(t)/\gamma\big], \quad \gamma > 0 \end{aligned} \tag{192}$$

Using (188) for the terms in $\boxed{\cdot}^{\,\rm a}$ and (192) for the term in $\boxed{\cdot}^{\,\rm b}$, (191) becomes:

$$\begin{aligned} \dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \leq\; & -\tilde{x}^{\rm T}(t)Q\tilde{x}(t) + 2\tilde{x}^{\rm T}(t)PBg(x(t),\hat{x}(t)) \\ & \boxed{+2\tilde{x}^{\rm T}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\rm T}\beta(\hat{x}(t))}^{\,\rm c} \\ & \boxed{-2\tilde{x}^{\rm T}(t)PB[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)PB]^{\rm T}\beta(\hat{x}(t))}^{\,\rm f} \\ & \boxed{-2\tilde{x}^{\rm T}(t)PB[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)N]^{\rm T}\beta(\hat{x}(t))}^{\,\rm d} \\ & \boxed{+2\tilde{x}^{\rm T}(t)PB\Omega_2^{\rm T}\beta(\hat{x}(t))}^{\,\rm e} \\ & +\rho\ \text{tr}\ \big[-\xi\tilde{W}^{\rm T}(t)\tilde{W}(t)\big] \\ & \boxed{+\rho\ \text{tr}\ \big[-\tilde{W}^{\rm T}(t-\tau)\tilde{W}(t-\tau)\big]}^{\,\rm g} \\ & \boxed{+\rho\ \text{tr}\ \big[\eta\tilde{W}^{\rm T}(t-\tau)\Omega_1^{\rm T}\Omega_1\tilde{W}(t-\tau)\big]}^{\,\rm g} \\ & \boxed{+\rho\ \text{tr}\ \big[\eta\kappa_2^2B^{\rm T}P\tilde{x}(t)\beta^{\rm T}(\hat{x}(t))\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)PB\big]}^{\,\rm f} \\ & \boxed{+2\rho\ \text{tr}\ \big[\eta\kappa_2^2B^{\rm T}P\tilde{x}(t)\beta^{\rm T}(\hat{x}(t))\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)N\big]}^{\,\bar{\rm d}} \\ & +\rho\ \text{tr}\ \big[\eta\kappa_2^2N^{\rm T}\tilde{x}(t)\beta^{\rm T}(\hat{x})\beta(\hat{x})\tilde{x}^{\rm T}(t)N\big] \\ & \boxed{+\rho\ \text{tr}\ \big[\eta\Omega_2^{\rm T}\Omega_2\big]}^{\,\rm h} \\ & \boxed{+\rho\ \text{tr}\ \Big[-2\eta\big[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)PB\big]^{\rm T}\Omega_1\tilde{W}(t-\tau)\Big]}^{\,\bar{\rm c}} \\ & +\rho\ \text{tr}\ \Big[-2\eta\big[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)N\big]^{\rm T}\Omega_1\tilde{W}(t-\tau)\Big] \\ & \boxed{+\rho\ \text{tr}\ \big[\gamma\tilde{W}^{\rm T}(t-\tau)\Omega_1^{\rm T}\Omega_1\tilde{W}(t-\tau)\big]}^{\,\rm g} \\ & \boxed{+\rho\ \text{tr}\ \big[\eta^2\Omega_2^{\rm T}\Omega_2/\gamma\big]}^{\,\rm h} \\ & \boxed{+\rho\ \text{tr}\ \Big[-2\eta\big[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)PB\big]^{\rm T}\Omega_2\Big]}^{\,\bar{\rm e}} \\ & +\rho\ \text{tr}\ \Big[-2\eta\big[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\rm T}(t)N\big]^{\rm T}\Omega_2\Big] \end{aligned} \tag{193}$$

Since $\kappa_2 = 1/\rho\eta$, $\boxed{\cdot}^{\,\mathrm{c}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{c}}}$, $\boxed{\cdot}^{\,\mathrm{d}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{d}}}$, and $\boxed{\cdot}^{\,\mathrm{e}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{e}}}$ in the above expression. Furthermore, grouping the terms in $\boxed{\cdot}^{\,\mathrm{f}}$ as $-\kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}(t)\,\beta^{\mathrm{T}}(\hat{x}(t))\,\beta(\hat{x}(t))$, the terms in $\boxed{\cdot}^{\,\mathrm{g}}$ as $-\rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big]$, and the terms in $\boxed{\cdot}^{\,\mathrm{h}}$ as $\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2]$ yields

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \leq\ & -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \\ & -\kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t)) \\ & -\rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] - \rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] \\ & +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2] \\ & +\kappa_2\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t)\bar{\beta}^2 - 2\ \mathrm{tr}[N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_1\tilde{W}(t-\tau)] \end{aligned} \tag{194}$$

where using Young's inequality again for the last term in (194) produces:

$$\begin{aligned} 2\ \mathrm{tr}[N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_1\tilde{W}(t-\tau)] \leq\ & \mathrm{tr}[\mu\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)] \\ & +\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t)\bar{\beta}^2/\mu, \quad \mu > 0 \end{aligned} \tag{195}$$

Using (195) in (194) results in

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \leq\ & -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \\ & -\kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t)) - \rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] \\ & -\rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma+\mu)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] \\ & +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2] \\ & \big(\kappa_2+1/\mu\big)\bar{\beta}^2\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t) \end{aligned} \tag{196}$$

Applying the definitions of Q and ν in (172), and enforcing $\kappa_2 < \bar{\kappa}_2$, the first and last terms in (196) can be combined to produce the following inequality

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \leq\ & -\tilde{x}^{\mathrm{T}}(t)Q_0\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \\ & -\kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t)) - \rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] \\ & -\rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma+\mu)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] \\ & +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2] \end{aligned} \tag{197}$$

Define $\kappa_1 \equiv 1/(\eta+\gamma+\mu)$. Note that $\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] = \mathrm{vec}\big(\tilde{W}(t-\tau)\big)^{\mathrm{T}}\mathrm{vec}\big([I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big) = \mathrm{vec}\big(\tilde{W}(t-\tau)\big)^{\mathrm{T}}\big(I\otimes[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\big)\mathrm{vec}\big(\tilde{W}(t-\tau)\big) \geq \lambda_{\min}\big(I\otimes[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\big)||\tilde{W}(t-\tau)||^2 = \lambda_{\min}(I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1)||\tilde{W}(t-\tau)||^2$. Using this and (172) in (197) with $|g(x(t),\hat{x}(t))| \leq \bar{g}$, $\bar{g} \equiv 2\bar{w}\bar{\beta}$, yields

$$\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;\leq\; -c_1|\tilde{x}(t)|^2 - c_2||\tilde{W}(t)||^2 - c_3||\tilde{W}(t-\tau)||^2 + d_1 + d_2|\tilde{x}(t)| \quad (198)$$

where $c_1 \equiv \lambda_{\min}(Q_0) > 0$, $c_2 \equiv \rho\xi > 0$, $c_3 \equiv \rho\lambda_{\min}(I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1) > 0$, $d_1 \equiv \rho(\eta+\eta^2/\gamma)\bar{\delta}^2 \geq 0$, and $d_2 \equiv 2(||PB||\bar{g} + ||N||||\Omega_2||\bar{\beta}) > 0$. We can further arrange (198) as

$$\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;\leq\; -\big[\sqrt{c_1}|\tilde{x}(t)| - d_2/2\sqrt{c_1}\big]^2 - c_2||\tilde{W}(t)||^2 - c_3||\tilde{W}(t-\tau)||^2 + d \quad (199)$$

where $d \equiv d_1 + d_2^2/4c_1$. Either $|\tilde{x}(t)| > \Psi_1$ or $\|\tilde{W}(t)\| > \Psi_2$ or $\|\tilde{W}(t-\tau)\| > \Psi_3$ renders $\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) < 0$, where $\Psi_1 \equiv \sqrt{d/c_1} + d_2/2c_1$, $\Psi_2 \equiv \sqrt{d/c_2}$, and $\Psi_3 \equiv \sqrt{d/c_3}$. Hence, it follows that $\tilde{x}(t)$ and $\tilde{W}(t)$ are UUB. □

Remark 5.8. The derivative-free weight update law given by (168) does not require a modification term to prove the error dynamics, including the weight errors, are UUB.

Remark 5.9. Derivative-free adaptive control does not employ an integrator in its weight update law. This is advantageous from the perspective of augmenting a nominal controller that employs integral action to ensure that the regulated output variables track $r(t)$ for constant disturbances, regardless of how these disturbances may enter the system. An example that illustrates this advantage is provided for full-state feedback case in Section V of Ref. 85.

Define $q(t) \equiv [\tilde{x}^{\mathrm{T}}(t), \tilde{v}(t,\tau)]^{\mathrm{T}}$, where $\tilde{v}^2(t,\tau) \equiv \mathrm{tr}\big[\int_{-\tau}^{0}\tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\big]$, and let $\mathcal{B}_r = \{q(t) : |q(t)| < r_\alpha\}$, such that $\mathcal{B}_r \subset \mathcal{D}_q$ for a sufficiently large compact set $\mathcal{D}_q$. Then, we have the following corollary.

Corollary 5.1. Under the conditions of Theorem 5.1, an estimate for the ultimate bound for $q(t)$ is given by

$$\underline{r} = \sqrt{\frac{\lambda_{\max}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\lambda_{\min}(\tilde{P})}} \tag{200}$$

where $\tilde{P} = \mathrm{diag}[P, \rho]$.

Proof. Denote $\Omega_\alpha = \{q(t) \in \mathcal{B}_r : q^{\mathrm{T}}(t)\tilde{P}q(t) \leq \hat{\alpha}\}$, $\hat{\alpha} = \min_{||q(t)||=r} q^{\mathrm{T}}(t)\tilde{P}q(t) = r_\alpha^2\lambda_{\min}(\tilde{P})$. Since

$$\begin{aligned} \mathcal{V}(\tilde{x}(t), \tilde{W}_t) &= q^{\mathrm{T}}(t)\tilde{P}q(t) \\ &= \tilde{x}^{\mathrm{T}}(t)P\tilde{x}(t) + \rho \ \mathrm{tr}\big[\int_{t-\tau}^{t} \tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\big] \end{aligned} \tag{201}$$

it follows that Ω_α is an invariant set if and only if:

$$\hat{\alpha} \geq \lambda_{\max}(P)\Psi_1^2 + \rho\tau\Psi_2^2 \tag{202}$$

Thus, the minimum size of $\mathcal{B}_r$ that ensures this condition has radius given by (200).

□

Remark 5.10. The proofs of Theorem 5.1 and Corollary 5.1 assume that the sets $\mathcal{D}_x$ and $\mathcal{D}_q$ are sufficiently large. If we define $\mathcal{B}_{r^*}$ as the largest ball contained in $\mathcal{D}_q$, and assume that the initial conditions are such that $q(0) \subset \mathcal{B}_{r^*}$, then from Figure 2 we have added the condition that $\underline{r} < r^*$, which implies an lower bound on ρ. It can be shown that in this case the lower bound must be such that $\lambda_{\min}(\tilde{P}) = \rho$. With $\underline{r}$ defined by (200) and $\lambda_{\min}(\tilde{P}) = \rho$, the condition $\underline{r} < r^*$ implies:

$$\rho \ > \ \frac{\lambda_{\max}(P)\Psi_1^2}{r^{*2} - \tau\Psi_2^2} \tag{203}$$

Since $\kappa_2 = 1/\rho\eta$, $\eta > 1$, it follows from (203) that r^* should ensure that:

$$\kappa_2 \ < \ \frac{r^{*2} - \tau\Psi_2^2}{\lambda_{\max}(P)\Psi_1^2} \tag{204}$$

Therefore, the meaning of $\mathcal{D}_q$ *sufficiently large* in Corollary 5.1 is that

$$r^* > \sqrt{\kappa_2\lambda_{\max}(P)\Psi_1^2 + \tau\Psi_2^2} \tag{205}$$

and $q(0) \subset \mathcal{D}_{r^*}$. The meaning of $\mathcal{D}_x$ *sufficiently large* is difficult to characterize precisely since $x(t)$ depends on both $r(t)$ and $x(0)$. Nevertheless it can be seen that increasing κ_2 implies increasing the require size of the set $\mathcal{D}_x$.

Lemma 5.3. If $\tilde{x}(t)$ is bounded, then the state tracking error defined as $e(t) \equiv x(t) - x_{\mathrm{m}}(t)$ is bounded.

Proof.

$$\begin{aligned} |e(t)| &= |x(t) - x_{\mathrm{m}}(t)| \\ &= |x(t) - \hat{x}(t) + \hat{x}(t) - x_{\mathrm{m}}(t)| \\ &\leq |x(t) - \hat{x}(t)| + |\hat{x}(t) - x_{\mathrm{m}}(t)| \\ &= |\tilde{x}(t)| + |\hat{e}(t)| \end{aligned} \tag{206}$$

where if $\tilde{x}(t)$ is bounded then from (167) $\hat{e}(t)$ is bounded. This implies that $e(t)$ is bounded. □

Corollary 5.2. Consider the system of equations given by (166) and (167). If $\tilde{x}(t)$ is UUB by r, then $e(t)$ is UUB by $r(1+v)$ where

$$v \equiv \frac{2||P_{\mathrm{m}}LC||}{\lambda_{\min}(Q_{\mathrm{m}})} \tag{207}$$

and $P_{\mathrm{m}} \in \mathrm{I\!R}^{n\times n}$ being a positive-definite solution to the Lyapunov equation given by

$$0 = A_{\mathrm{m}}^{\mathrm{T}} P_{\mathrm{m}} + P_{\mathrm{m}} A_{\mathrm{m}} + Q_{\mathrm{m}} \tag{208}$$

for a given positive-definite matrix $Q_{\mathrm{m}} \in \mathrm{I\!R}^{n\times n}$.

Proof. To show that $\hat{e}(t)$ is UUB, consider the Lyapunov function given by

$$\mathcal{V}(\hat{e}(t)) = \hat{e}^{\mathrm{T}}(t) P_{\mathrm{m}} \hat{e}(t) \tag{209}$$

where P_{m} satisfies (208). We can write the directional derivative of (209) along the trajectories of $\hat{e}(t)$ given by (167) as

$$\begin{aligned} \dot{\mathcal{V}}(\hat{e}(t)) &= -\hat{e}^{\mathrm{T}}(t) Q_{\mathrm{m}} \hat{e}(t) + 2\hat{e}^{\mathrm{T}}(t) P_{\mathrm{m}} LC\tilde{x}(t) \\ &\leq -|\hat{e}(t)|\Big[\lambda_{\min}(Q_{\mathrm{m}})|\hat{e}(t)| - 2||P_{\mathrm{m}}LC||\ |\tilde{x}(t)|\Big] \end{aligned}$$

so that $\dot{\mathcal{V}}(\hat{e}(t)) < 0$ for all $|\hat{e}(t)| > rv$, since $\tilde{x}(t)$ is UUB by r and v is given by (207). Hence, it follows from Lemma 5.3 that $|e(t)|$ is UUB by $r(1+v)$. □

Remark 5.11. It follows from Corollaries 4.1 and 4.2 that $|e(t)|$ is UUB by $r_{e_1} \equiv r(1+v)$, where:

$$r_{e_1} = \sqrt{\frac{\lambda_{\max}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\lambda_{\min}(\tilde{P})}}\left[1 + \frac{2||P_{\mathrm{m}}LC||}{\lambda_{\min}(Q_{\mathrm{m}})}\right] \tag{210}$$

5.5 *Examples on a Model of Wing Rock Dynamics*

This section gives several examples of the architecture described in Figure 62 using a model of wing rock dynamics. A two state model for wing rock dynamics can be written in the form given by (32) and

$$y_{\mathrm{s}}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + w(t) \tag{211}$$

where $w(t)$ is measurement noise disturbance and

$$\begin{aligned} \Delta(t, x(t)) = {} & \big[\alpha_1 + f_1(t)\big]x_1(t) + \big[\alpha_2 + f_2(t)\big]x_2(t) + \alpha_3|x_1(t)|x_2(t) \\ & + \alpha_4|x_2(t)|x_2(t) + \alpha_5 x_1^3(t) + d(t) \end{aligned} \tag{212}$$

with *constant* coefficients $\alpha_1 = 0.2314$, $\alpha_2 = 0.6918$, $\alpha_3 = -0.6245$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0214$, and *time varying* coefficients $f_1(t)$ and $f_2(t)$, and an external disturbance $d(t)$. We chose $K_1 = [2.56, 2.56]$ and $K_2 = 2.56$ for the nominal controller design, and $L^{\mathrm{T}} = [12.8\ 64.0]$ for the observer gain which corresponds to $k = 5$ in (176). For the adaptive control design, we used sigmoidal basis functions of the form $\beta(x(t)) = \big[0.5,\ \beta_1(x_1(t)),\ \beta_2(x_2(t))\big]$, where $\beta_i(x_i(t)) = \frac{1-e^{-x_i(t)}}{1+e^{-x_i(t)}}$, $i = 1, 2$. Since $|\beta_i(x_i)| \le 1$, it follows that $\bar{\beta} = 1.5$. For $\mu = 0.05$ and $Q_0 = 0.25I_2$, it was determined by Lemma 5.2 that $\bar{\nu} = 124.6$ which implies that the adaptation gain in (170) must satisfy $\kappa_2 < 35.4$. Furthermore, we set $\Omega_1 = 0.95I_3$, $\kappa_2 = 35$, and $\tau = 0.01$ seconds.

We consider the command tracking problem in the following subsections with the initial conditions for the dynamics in (32) set zero. Figure 63 shows the nominal and adaptive control responses for the case when the ideal weights are constant $\big(f_1(t) = f_2(t) = 0\big)$, and in the absence of disturbances $\big(d(t) = w(t) = 0\big)$. Overall performance is significantly improved with adaptation. Improvement is measured by comparing how well the responses follow $y_{\mathrm{m}}(t)$. Figure 64 compares the responses for the case when the ideal weights are time varying, with $f_1(t)$ being a square wave having an amplitude of 0.5 and a period of 15 seconds and $f_2(t) = 0.5\sin(1.5t)$. The improvement with adaptation is greater in comparison to the improvement in Figure 63. The effect of $f_1(t)$ is clearly evident in the nominal case.

Next we include the effect of a process disturbance and measurement noise. In this case $d(t)$ is a square wave having an amplitude of 0.1 and a period of 6 seconds. To model sensor noise we let $w(t)$ be a band-limited white noise process with a correlation time constant of 0.01 seconds, and a noise power level of 0.0001. These processes are depicted in Figure 65. Figure 66 shows that tracking performance is significantly improved with adaptation, and that the control time history is well behaved (sensor noise is not amplified). Figure 67 includes a comparison between $\Delta(t, x(t))$ and $u_{\mathrm{ad}}(t)$.

5.6 *Examples on an Aeroelastic Generic Transport Model*

This section illustrates an application of the proposed approach on an aeroelastic model of longitudinal dynamics for a generic transport model. The form of the derivative-free, output feedback adaptive control differs from the one given in Section 5.2 in that it is designed to augment a nominal controller with integral action. For this purpose, we consider the uncertain system given by

$$\dot{x}_{\mathrm{p}}(t) = A_{\mathrm{p}} x_{\mathrm{p}}(t) + B_{\mathrm{p}}\big[u(t) + \Delta(t, x_{\mathrm{p}}(t))\big] \quad (213)$$

$$y(t) = C_{\mathrm{p}} x_{\mathrm{p}}(t) \quad (214)$$

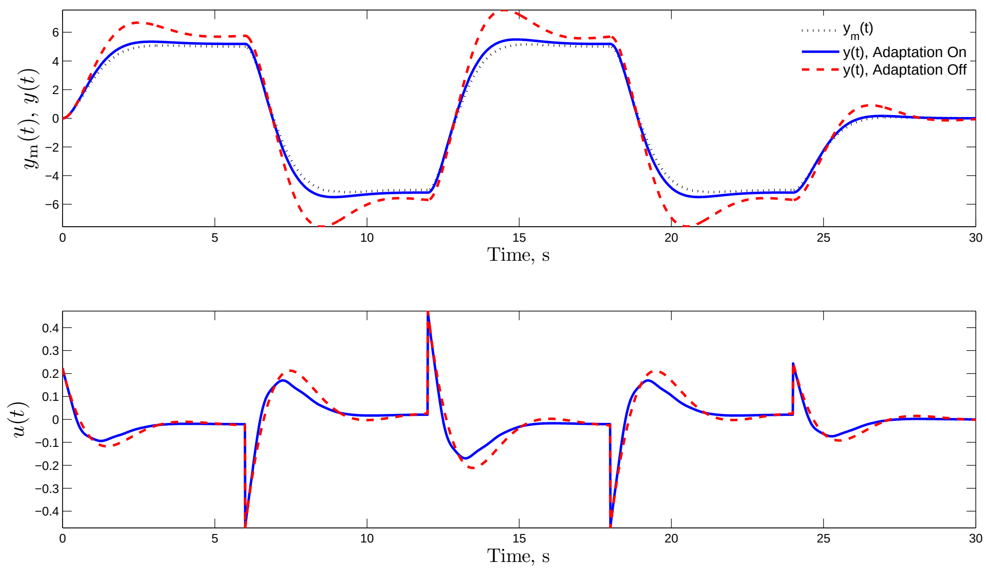

Figure 63: Nominal and adaptive control responses for the case of constant ideal weights

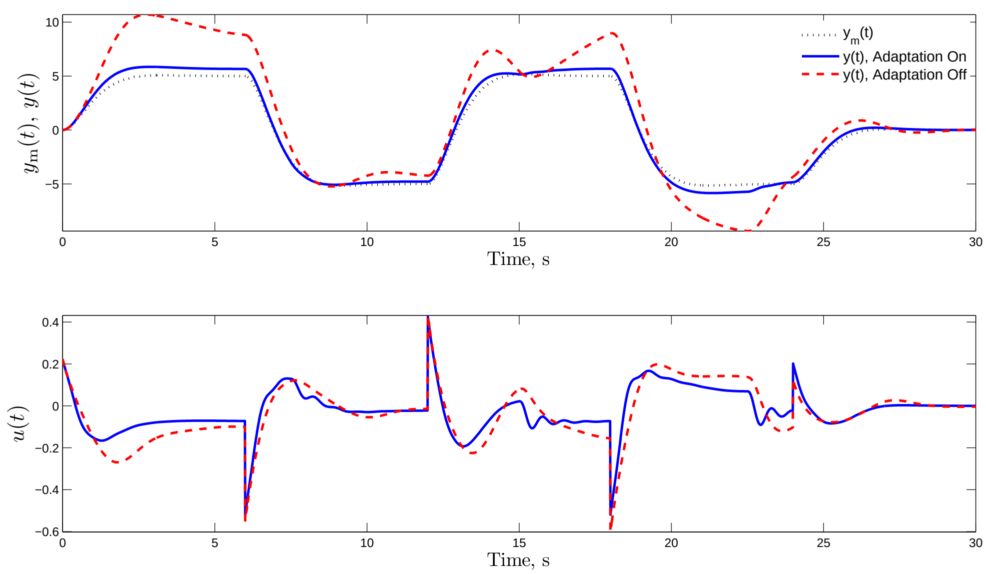

Figure 64: Nominal and adaptive control responses for the case of time varying ideal weights

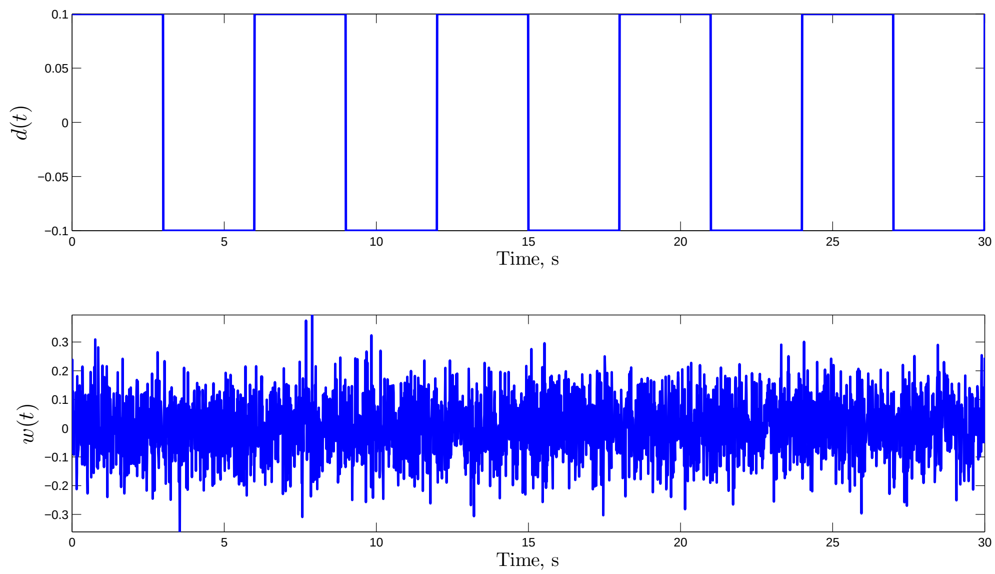

Figure 65: Depiction of $d(t)$ and $w(t)$

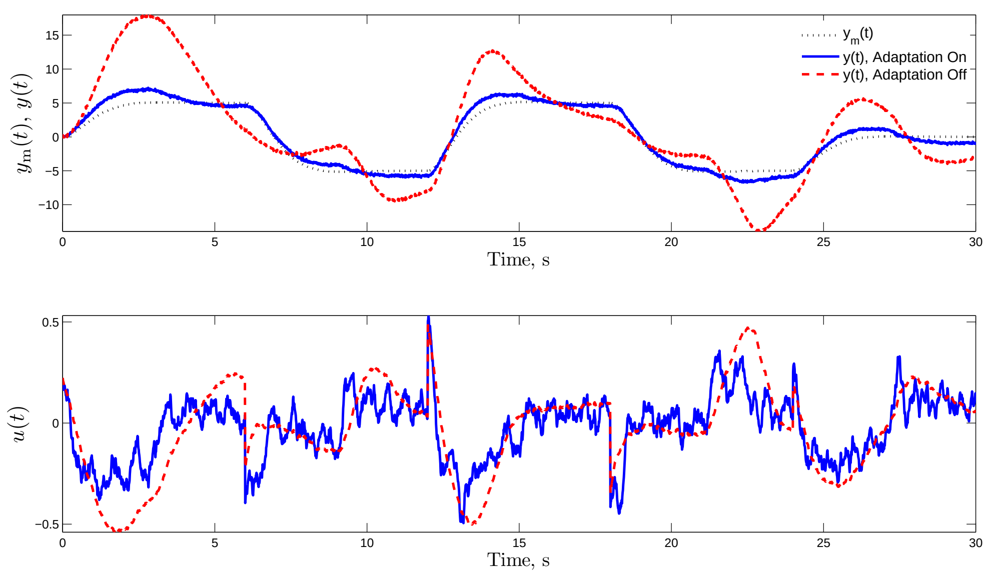

Figure 66: Nominal and adaptive control responses with disturbances for the case of time varying ideal weights

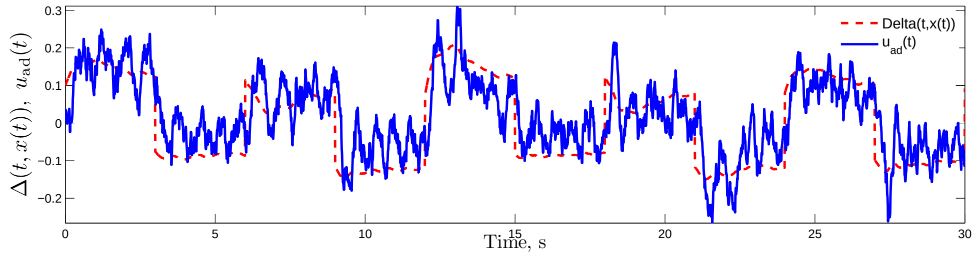

Figure 67: $\Delta(t,x(t))$ and $u_{\rm ad}(t)$

where $A_{\rm p} \in \mathbb{R}^{n_{\rm p}\times n_{\rm p}}$, $B_{\rm p} \in \mathbb{R}^{n_{\rm p}\times m}$, and $C_{\rm p} \in \mathbb{R}^{m\times n_{\rm p}}$ are known system matrices with $(A_{\rm p}, B_{\rm p}, C_{\rm p})$ being the minimal triple, $x_{\rm p}(t) \in \mathbb{R}^{n_{\rm p}}$ is the unknown state vector, $u(t) \in \mathbb{R}^m$ is the control input vector restricted to the class of admissible controls consisting of measurable functions, $\Delta : \mathbb{R} \times \mathbb{R}^{n_{\rm p}} \to \mathbb{R}^m$ is the matched uncertainty satisfying Assumption 5.1, and $y(t) \in \mathbb{R}^m$ is the *regulated* output vector. Furthermore, the system can have a *sensed* output vector denoted by

$$y_{\rm s}(t) \quad = \quad C_{\rm s} x_{\rm p}(t) \tag{215}$$

where $y_{\rm s}(t) \in \mathbb{R}^{l_{\rm p}}$, $C_{\rm s} \in \mathbb{R}^{l_{\rm p}\times n_{\rm p}}$, $l_{\rm p} \geq m$, such the elements of $y(t)$ are a subset of the elements of $y_{\rm s}(t)$.

We will consider the situation in which there is a state observer based nominal controller in which the control of the regulated outputs that are commanded include integral action, whereas the regulated outputs that are not commanded are subject to proportional control. Let

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{\rm p1} \\ C_{\rm p2} \end{bmatrix} x_{\rm p}(t) \tag{216}$$

where $y_1(t) \in \mathbb{R}^r$, $r \leq m$, is regulated with proportional and integral control to track a given command vector $r(t) \in \mathbb{R}^r$, $y_2(t) \in \mathbb{R}^{m-r}$ is regulated with proportional control, $C_{\rm p1} \in \mathbb{R}^{r\times n_{\rm p}}$, and $C_{\rm p2} \in \mathbb{R}^{(m-r)\times n_{\rm p}}$. The integrator dynamics are defined by

$$\dot{x}_{\rm int}(t) \quad = \quad -y_1(t) + r(t) \quad = \quad -C_{\rm p1} x_{\rm p}(t) + I_r r(t) \tag{217}$$

where $x_{\rm int}(t) \in {\rm I\!R}^r$. Considering (213), (214), and (217), the augmented dynamics become:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} A_{\rm p} & 0 \\ -C_{\rm p1} & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} B_{\rm p} \\ 0 \end{bmatrix}}_{B} \Big[u(t) + \Delta(t, x_{\rm p}(t))\Big] + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{B_{\rm m}} r(t) \quad (218)$$

$$y(t) = \underbrace{\begin{bmatrix} C_{\rm p} & 0 \end{bmatrix}}_{C} x(t) \quad (219)$$

where $x(t) = [x(t),\ x_{\rm int}(t)]^{\rm T} \in {\rm I\!R}^n$, $A \in {\rm I\!R}^{n\times n}$, $B \in {\rm I\!R}^{n\times m}$, $B_{\rm m} \in {\rm I\!R}^{n\times r}$, $C \in {\rm I\!R}^{m\times n}$, and $n = n_{\rm p} + r$. In addition, the augmented *sensed* output vector becomes

$$\bar{y}_{\rm s}(t) = \begin{bmatrix} y_{\rm s}(t) \\ x_{\rm int}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C_{\rm s} & 0 \\ 0 & I_r \end{bmatrix}}_{\bar{C}_{\rm s}} x(t) \quad (220)$$

where $\bar{y}_{\rm s}(t) \in {\rm I\!R}^l$, $\bar{C}_{\rm s} \in {\rm I\!R}^{l\times n}$, $l = l_{\rm p} + r$. Consider the state observer based nominal control law given by

$$u_{\rm n}(t) = -K\hat{x}(t) \quad (221)$$

where $K \in {\rm I\!R}^{m\times n}$ is known feedback matrix and $\hat{x}(t) \in {\rm I\!R}^n$ is an observer estimate of $x(t)$ given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{\rm n}(t) + B_{\rm m}r(t) + L\big[\bar{y}_{\rm s}(t) - \bar{C}_{\rm s}\hat{x}(t)\big] \quad (222)$$

with $L \in {\rm I\!R}^{n\times l}$ being the observer gain matrix designed such that $A_e \equiv A - L\bar{C}_{\rm s} \in {\rm I\!R}^{n\times n}$ is Hurwitz.

Define the reference model as in (4) subject to $A_{\rm m} = A - BK$ being Hurwitz by design. This implies that the gain K has been designed using the certainty equivalence principle ($\hat{x}(t) = x(t)$) with $\Delta(t, x_{\rm p}(t)) = 0$, so that $y(t)$ is regulated and $y_1(t)$ tracks $r(t)$ to within some set of specifications on both the transient and steady state performance.

The adaptive control objective is to design a control law $u(\cdot)$ for the system given by (218) and (219) so that $x(t)$ tracks $x_{\mathrm{m}}(t)$. For this purpose, the nominal control law $u_{\mathrm{n}}(t)$ given by (221) is augmented with the adaptive control law $u_{\mathrm{ad}}(t)$ as in (164) with

$$u_{\mathrm{ad}}(t) \quad = \quad \hat{W}^{\mathrm{T}}(t)\beta(\hat{x}_{\mathrm{p}}(t)) \tag{223}$$

where $\hat{x}_{\mathrm{p}}(t) = \left[I_{n_{\mathrm{p}}},\ 0_{n_{\mathrm{p}}\times m}\right]\hat{x}(t)$.

Consider the derivative-free weight update law given by (168), where $\tau > 0$, and $\Omega_1 \in \mathbb{R}^{s\times s}$ and $\hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s\times m}$ satisfy (169) and

$$\hat{\Omega}_2(t) \equiv \kappa_2\beta(\hat{x}_{\mathrm{p}}(t))\tilde{y}^{\mathrm{T}}(t), \quad \kappa_2 > 0 \tag{224}$$

where $\tilde{y}(t) \equiv y(t) - \hat{y}(t)$, where $\hat{y}(t) = C\hat{x}(t)$.

A visualization of the derivative-free output feedback adaptive control architecture is given in Figure 68. Its dynamics are the same as the reference model if $u_{\mathrm{ad}}(t)$ cancels $\Delta(t, x_{\mathrm{p}}(t))$, and in the case the observer error transient $\tilde{y}(t)$ goes to zero.

Denote $\tilde{x}(t) \equiv x(t) - \hat{x}(t)$ for the state estimation error, $\hat{e}(t) \equiv \hat{x}(t) - x_{\mathrm{m}}(t)$ for the estimated state tracking error, and $\tilde{W}(t) \equiv W(t) - \hat{W}(t)$ for the weight estimate error. From (218) and (222), the dynamics for the state estimation error can be written in the form

$$\dot{\tilde{x}}(t) \quad = \quad A_{\mathrm{e}}\tilde{x}(t) + B\tilde{W}^{\mathrm{T}}(t)\beta(\hat{x}(t)) + Bg(x_{\mathrm{p}}(t), \hat{x}_{\mathrm{p}}(t)) \tag{225}$$

where $g(x_{\mathrm{p}}(t), \hat{x}_{\mathrm{p}}(t)) \equiv W^{\mathrm{T}}(t)\left[\beta(\hat{x}_{\mathrm{p}}(t)) - \beta(x_{\mathrm{p}}(t))\right]$ with $|g(x_{\mathrm{p}}(t), \hat{x}_{\mathrm{p}}(t))| \le \bar{w}L_\beta|\hat{x}_{\mathrm{p}}(t) - x_{\mathrm{p}}(t)| \le \bar{w}L_\beta|\tilde{x}(t)|$, where $L_\beta > 0$ is the Lipschitz constant for the known basis vector. Likewise, from (222) and (4) the dynamics for the estimated state tracking error can be written in the form:

$$\dot{\hat{e}}(t) \quad = \quad A_{\mathrm{m}}\hat{e}(t) + LC\tilde{x}(t) \tag{226}$$

Hence, it follows identical to the proof of Theorem 5.1 that $\tilde{x}(t)$ and $\tilde{W}(t)$ are UUB, and then to the proof of Corollary 5.2 that $e(t)$ is UUB.

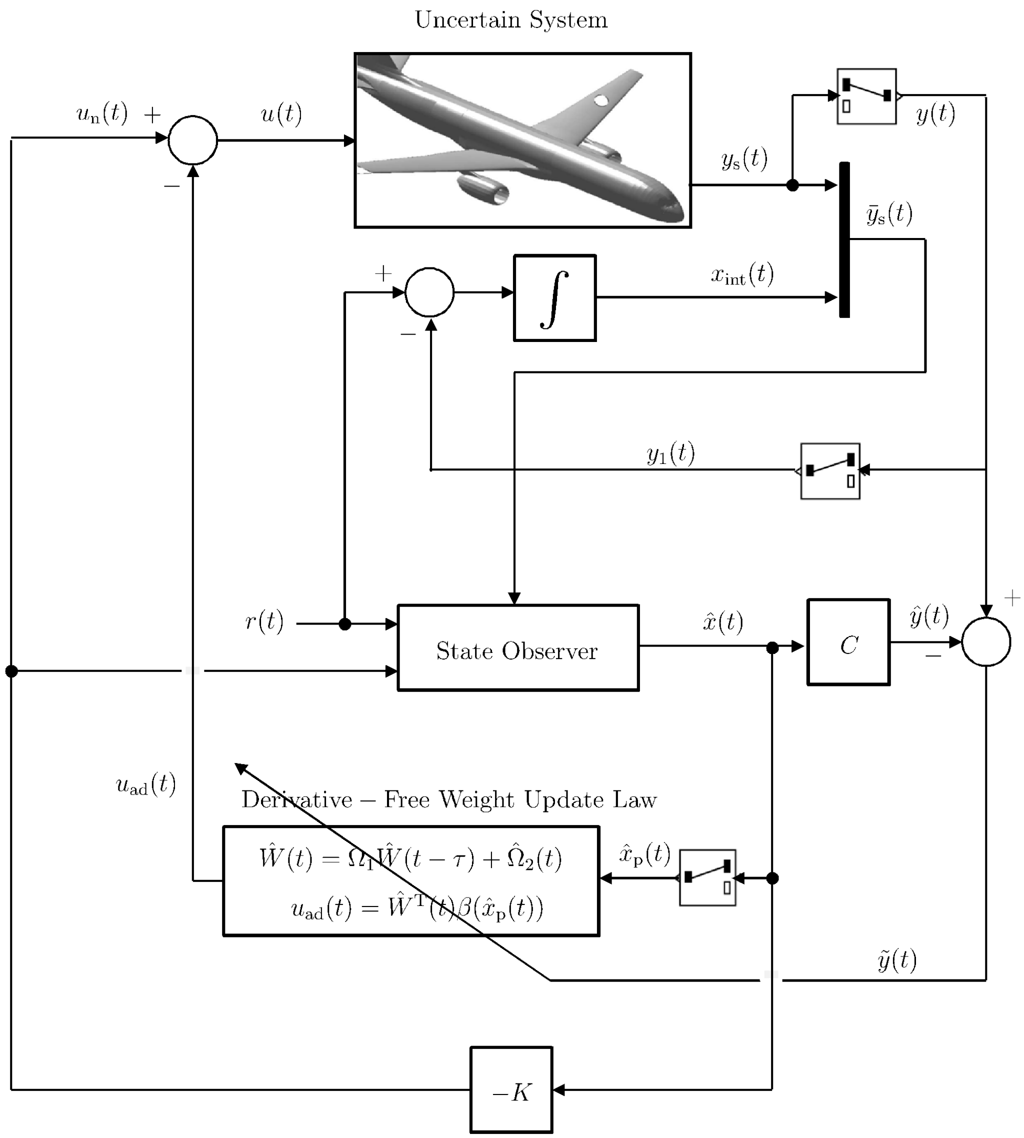

Figure 68: Derivative-free output feedback adaptive control architecture

We can now formulate a state space model of the rigid body pitch dynamics coupled with the flexible body dynamics. The details for this modeling are given in Ref. 59. For the configuration with 50% fuel, and with an altitude of 30000 feet and a Mach number of 0.8, a linearized model under nominal conditions $\big(\Delta(t, x_{\rm p}(t)) = 0\big)$

is obtained in the form of (213) and (214) with

$$A_{\rm p} = \begin{bmatrix} -8.01\times10^{-1} & 9.65\times10^{-1} & 1.26\times10^{-2} & 5.09\times10^{-1} & 5.46\times10^{-4} & -2.42\times10^{-3} \\ -2.45 & -9.14\times10^{-1} & 1.75\times10^{-1} & 7.39 & 9.11\times10^{-3} & -3.11\times10^{-2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1.42\times10^{3} & 3.96\times10^{1} & -3.13\times10^{1} & -1.40\times10^{3} & -3.25 & -8.26 \\ -2.62\times10^{2} & -5.63 & 3.78 & -1.89\times10^{2} & 2.52\times10^{-1} & -3.75 \end{bmatrix} \tag{227}$$

$$B_{\rm p} = \begin{bmatrix} 6.50\times10^{-2} & 3.52 & 0 & 0 & 0 & 0 \end{bmatrix}^{\rm T} \tag{228}$$

$$C_{\rm p} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1.98\times10^{1} & -8.47\times10^{-1} & 3.11\times10^{-1} & 1.26\times10^{1} & 1.35\times10^{-2} & -6.00\times10^{-2} \end{bmatrix} \tag{229}$$

with the state vector being defined as $x_{\rm p}(t) = [\alpha(t),\ q(t),\ w(t),\ \theta(t),\ \dot{w}(t),\ \dot{\theta}(t)]^{\rm T}$, where $\alpha(t)$ denotes the angle of attack, $q(t)$ denotes the pitch angular rate, $w(t)$ denotes the bending modal amplitude, and $\theta(t)$ denotes the torsional modal amplitude. The control input and the sensed output are defined as $u(t) = \delta_{\rm e}(t)$ and $y_{\rm s}(t) = [q(t),\ A_z(t)]^{\rm T}$, where $\delta_{\rm e}(t)$ denotes the elevator deflection and $A_z(t)$ denotes the normalized acceleration at the aircraft center of gravity. The measurement equation for $A_z(t)$ was obtained using the relationship

$$A_z(t) = U_0\big(\dot{\alpha}(t) - q(t)\big)/g \tag{230}$$

where U_0 is the equilibrium speed and g is the acceleration due to gravity. Only $q(t)$ is regulated, so the dimension of $y(t) = y_1(t)$ in (216) and of $r(t)$ in (4) is one ($r = 1$). Therefore, the dimension of the augmented sensed output vector, $\bar{y}_{\rm s}(t)$ in (220), is three ($l = 3$). Note that since the second entry of $B_{\rm p}$ in (228) is approximately 50 times greater than its first entry, the uncertainty $\Delta(t, x_{\rm p}(t))$ introduced in the following subsections is approximately matched.

We included the effect of actuator dynamics, measurement noise, and analog pre-filtering of the $y_s(t)$ in our simulation. The actuator model for the elevator has a bandwidth of 12Hz, an amplitude saturation of ± 30 degrees, and a rate saturation of ± 100 degrees per second. To model sensor noise, we assumed independent band-limited white noise processes with correlation time constants of 0.001 seconds and noise power levels of 1.5×10^{-8} and 1×10^{-6}, respectively. The pre-filters for $q(t)$ and $A_z(t)$ each have a bandwidth of 8Hz, which is well beyond the bandwidth of the nominal control design.

The eigenvalues of the rigid aircraft's short period mode, obtained from the 2×2 upper left matrix partition of (227), are $-0.85 \pm 1.53\jmath$. The eigenvalues of the aircraft's two flexible modes, obtained from the 4×4 lower right matrix partition of (227), are $-3.32 \pm 7.56\jmath$ and $-0.18 \pm 12.83\jmath$. These correspond to a bending and a torsion mode, respectively. The spectrum of A_p is $\rho(A_p) = \{-4.53 \pm 8.07\jmath,\ 0.71 \pm 13.08\jmath, -2.63$, and $1.56\}$. This shows that the linearized model under nominal conditions is unstable due to coupling effects and the low damping of the torsion mode. The frequency responses from the control input to the two sensed outputs for this model are given in Figure 69. The spike in the bode plot of the transfer function from $\delta_e(t)$ to $q(t)$ is due to a near pole/zero cancellation associated with the torsion mode. The resonance due to the torsion mode is evident in the transfer function from $\delta_e(t)$ to $A_z(t)$.

LQG/LTR theory was used to design the nominal controller [23]. The controller gain matrix (K) was obtained using $Q_K = \text{diag}[1, 25, 0.0001, 5, 0.0001, 5, 125]$ to penalize $x(t)$, and $R_K = 10$ to penalize $u(t)$. The observer gain matrix (L) was obtained using $Q_L = \varsigma^2 BB^{\text{T}}$ as the process noise matrix where $\varsigma = 6$ is the LTR gain, and $R_L = \text{diag}[I_2,\ 0.01]$ as the measurement noise matrix. The resulting gain matrices

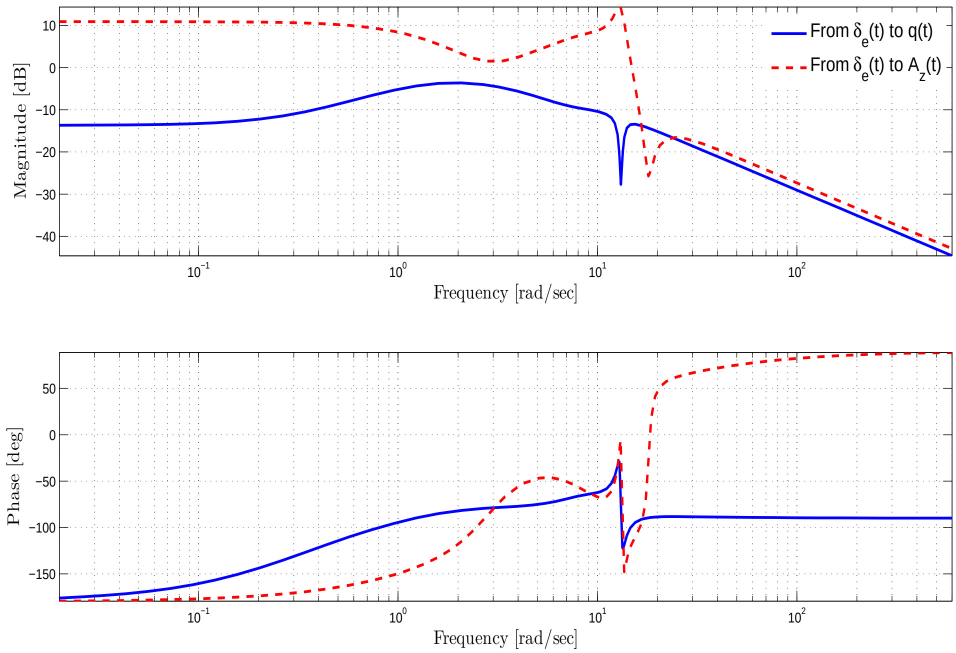

Figure 69: Frequency response of the linearized model

for this design are:

$$K = \begin{bmatrix} 18.55 & 3.09 & -0.24 & 12.45 & -0.02 & 0.02 & -3.53 \end{bmatrix} \tag{231}$$

$$L = \begin{bmatrix} 0.45 & 12.48 & 1.45 & 0.21 & 5.42 & -2.25 & -0.23 \\ -0.81 & -16.29 & 1.73 & 0.29 & -42.10 & 10.77 & 0.53 \\ -3.58 & -23.13 & -105.45 & 0.01 & -168.11 & 31.89 & 3.53 \end{bmatrix}^{\mathrm{T}} \tag{232}$$

Figure 70 shows the frequency response of the loop transfer functions with the loop broken at the plant input for both full-state feedback and LQG/LTR loops, where it can be seen that the margins of the full-state feedback loop are nearly recovered.

We consider the command tracking problem with the initial conditions set zero. Figures 71 and 72 show the responses with nominal control in the absence of uncertainty, and without disturbances. The upper part of Figure 71 compares the pitch rate command ($q_{\mathrm{cmd}}(t)$), the reference model response ($q_{\mathrm{m}}(t)$), and the actual plant

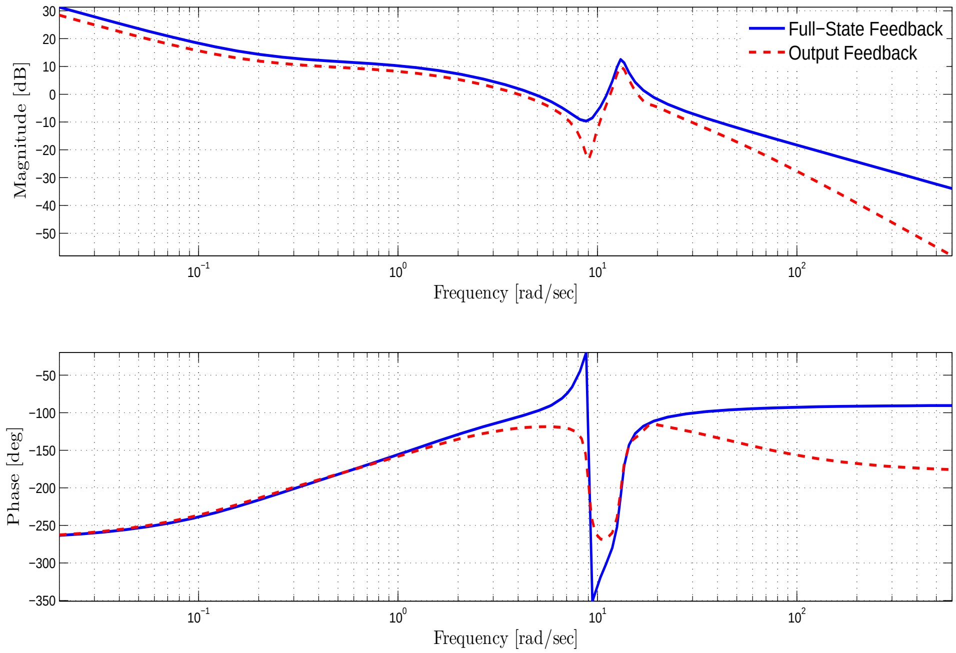

Figure 70: Frequency response of the loop transfer functions with the loop broken at the plant input for both full-state feedback and LQG/LTR loops

response ($q(t)$). The lower portion compares the command to the actuator ($\delta_{\rm e}(t)$ cmd) with the actuator response. The difference cannot be distinguished at this scale. The upper left portion of Figure 72 shows that there is a significant amount of sensor noise. The remainder of this figure shows the performance of the observer. The effect of sensor noise is most evident in the rate of change of the torsional amplitude.

For the adaptive control design we used a combination of bias and sigmoidal basis functions of the form $\beta(\hat{x}(t)) = \left[1,\ \beta_1(\hat{x}_1(t)),\ \ldots,\ \beta_6(\hat{x}_6(t))\right]$, where $\beta_i(\hat{x}_i(t)) = \frac{1-e^{-\hat{x}_i(t)}}{1+e^{-\hat{x}_i(t)}}$, $i = 1,\ldots,6$. For this choice of basis functions, $\bar{\beta} = \sqrt{7}$. For a chosen $\mu = 0.95$, the upper bound for κ_1 in (169) was computed as $\bar{\kappa}_1 = 0.51$. Furthermore, for a chosen $Q_0 = 1.2I_7$, it was determined by Lemma 2.1 that $\bar{\nu} = 228.7$ which results in the upper bound $\bar{\kappa}_2 = 31.6$ for κ_2 in (224). Given these bounds, we set $\Omega_1 = 0.5I_7$ and $\kappa_2 = 5$. Finally, we chose $\tau = 0.05$ seconds.

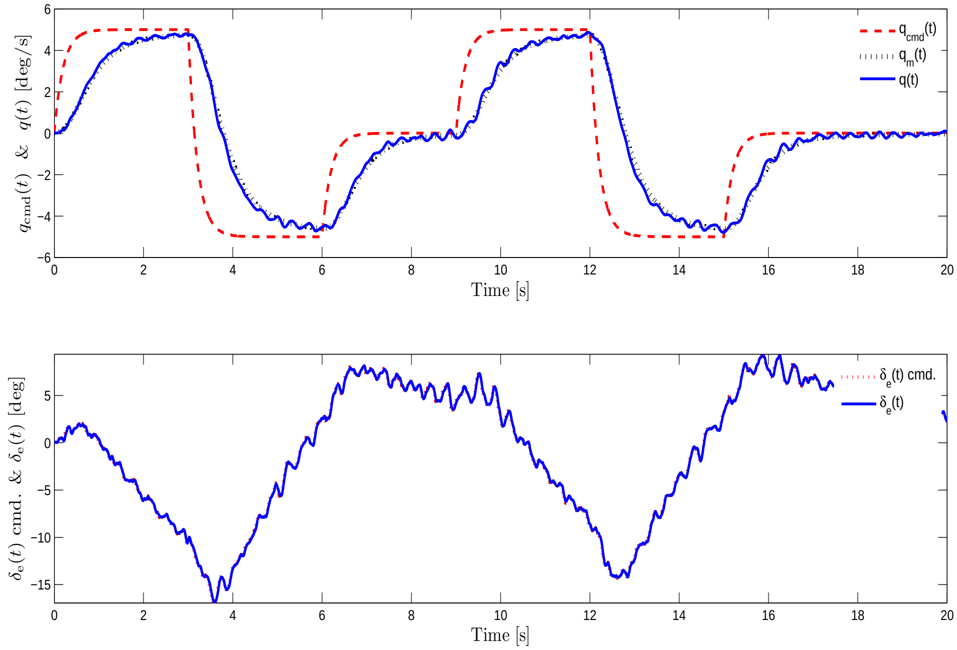

Figure 71: Pitch rate and elevator responses with nominal control in the absence of uncertainty

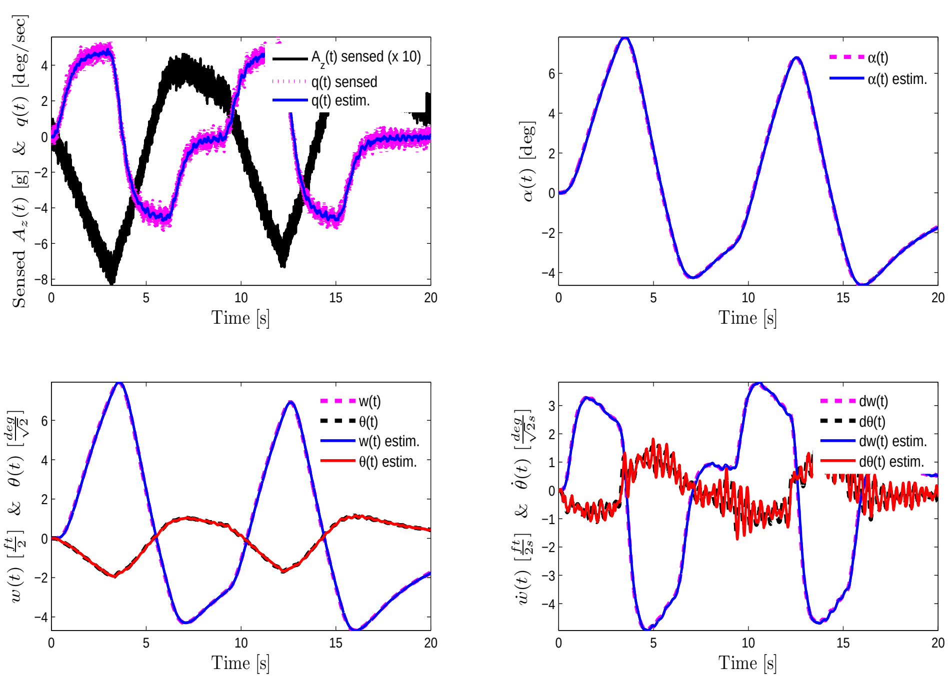

Figure 72: Measurement and state responses and their estimates with nominal control in the absence of uncertainty

5.6.1 Nonlinear Uncertainty

We first consider the case when there exists a nonlinear uncertainty of the form

$$\begin{aligned}\Delta(x(t)) &= 2.5\alpha(t)q(t) + 2.5\alpha^2(t) + 1.5q^2(t) + 1.25w(t)\theta(t) + 0.25\dot{w}(t)\dot{\theta}(t) \\ &\quad + 2.5\theta(t)\dot{\theta}(t) - 2.5\dot{\theta}^2(t) + 0.00025\dot{w}^2(t) \qquad (233)\end{aligned}$$

Figure 73 shows the responses with nominal control and Figure 74 shows the responses with adaptive control. The response with nominal control eventually goes unstable. The response with adaptation is stable, and tracking performance is nearly as good as that observed in Figure 71 without uncertainty. Also, the level of noise present in the elevator command for the adaptive result is comparable to that observed for the nominal control without uncertainty in Figure 71.

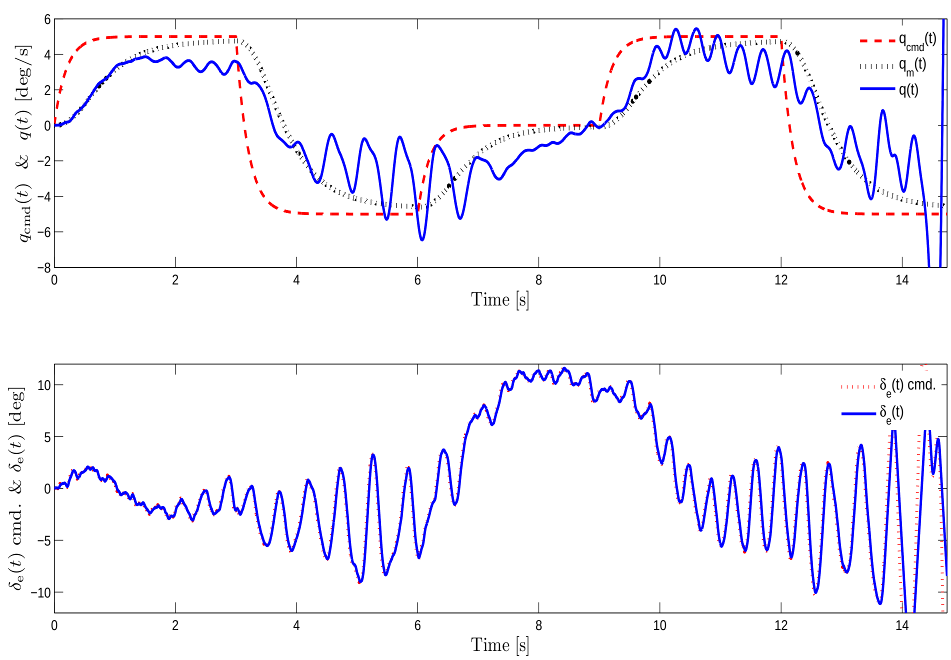

Figure 73: Pitch rate and elevator responses with nominal control for the case of nonlinear uncertainty

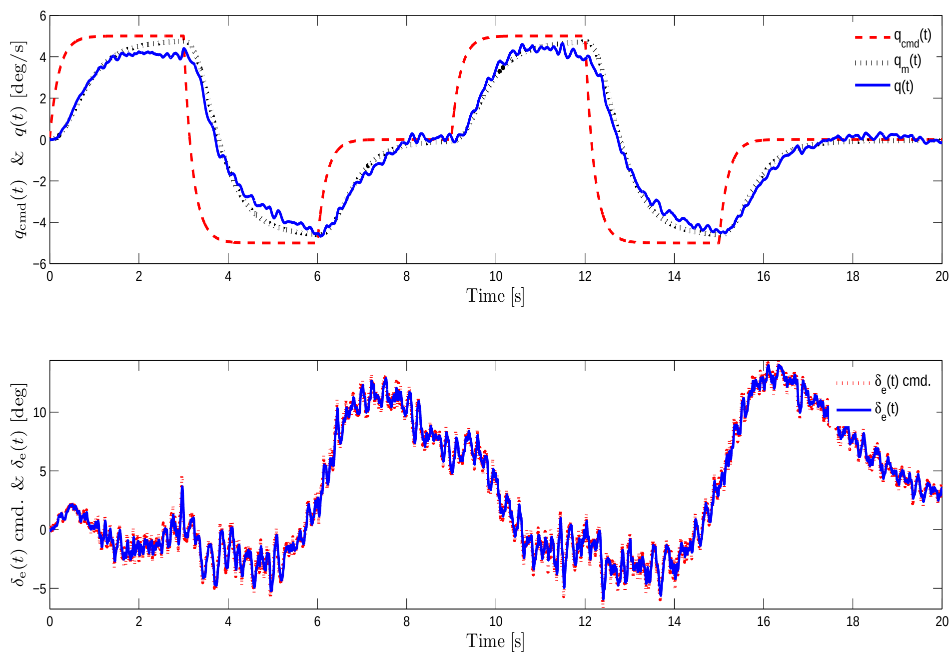

Figure 74: Pitch rate and elevator responses with adaptive control for the case of nonlinear uncertainty

5.6.2 External Disturbance

Next we consider the case when a disturbance $d(t) = 4\sin(3t)$ is applied to the system through the control input. Figure 75 shows the responses with nominal control and Figure 76 shows the responses with adaptive control. The tracking performance is significantly improved with adaptation.

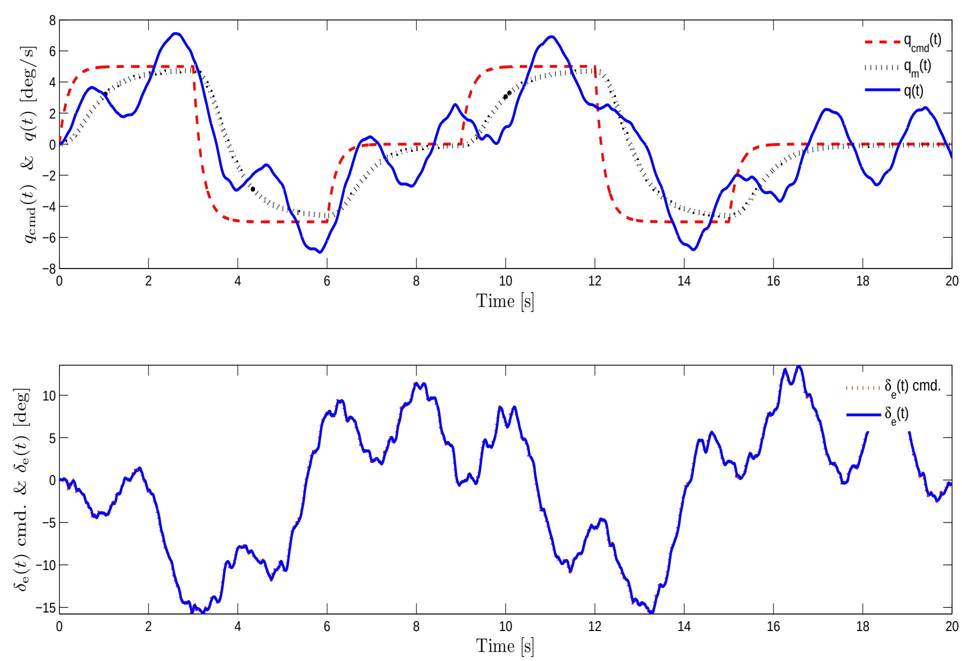

Figure 75: Pitch rate and elevator responses with nominal control for the case of external disturbance

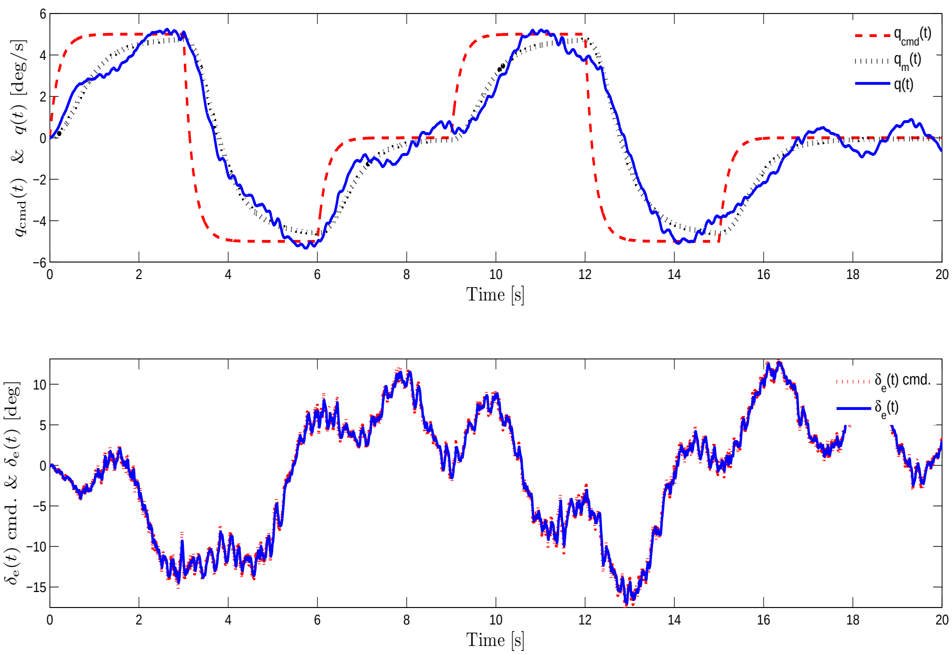

Figure 76: Pitch rate and elevator responses with adaptive control for the case of external disturbance

5.6.3 Nonlinear Uncertainty and External Disturbance

The uncertainties in the previous two cases are combined in this example. Figure 77 shows that the response with adaptation remains stable, and adequate tracking performance is still maintained.

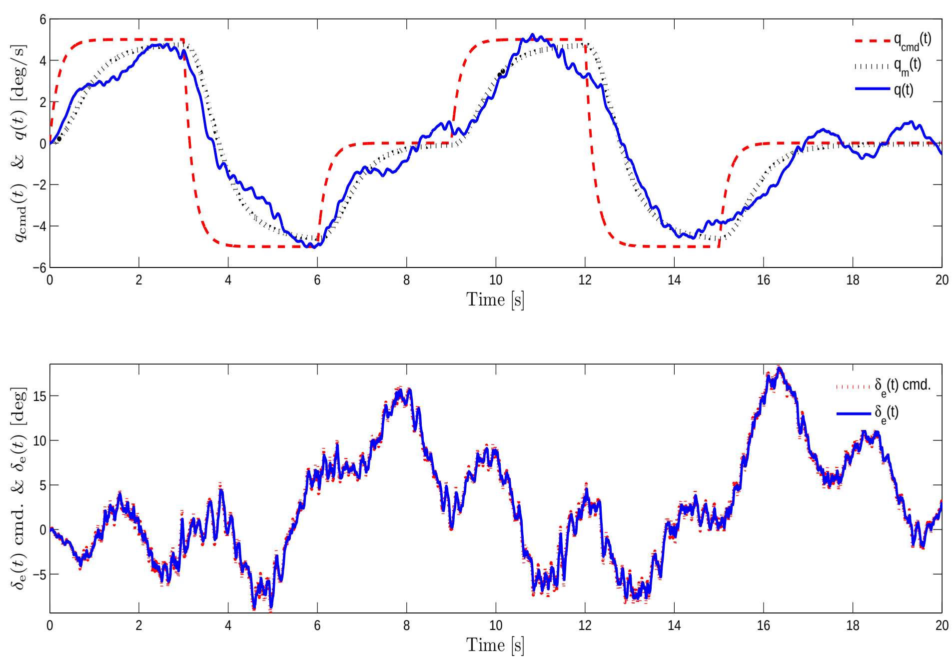

Figure 77: Pitch rate and elevator responses with adaptive control for the case of nonlinear uncertainty and external disturbance

5.6.4 Sudden Change in A_{p}

In this subsection, we illustrate an extreme case of parameter change, as might be caused by structural damage, in which the second row of A_{p} changes sign at $t = 6$ seconds. Figure 78 shows that the response with nominal control becomes unstable, whereas Figure 79 shows that the effect of this sudden change is hardly noticeable in the response with adaptive control. We include Figure 80 to show that the estimation performance of the observer is somewhat degraded by the uncertainty. In all the other cases the effect of uncertainty on estimation performance was barely noticeable.

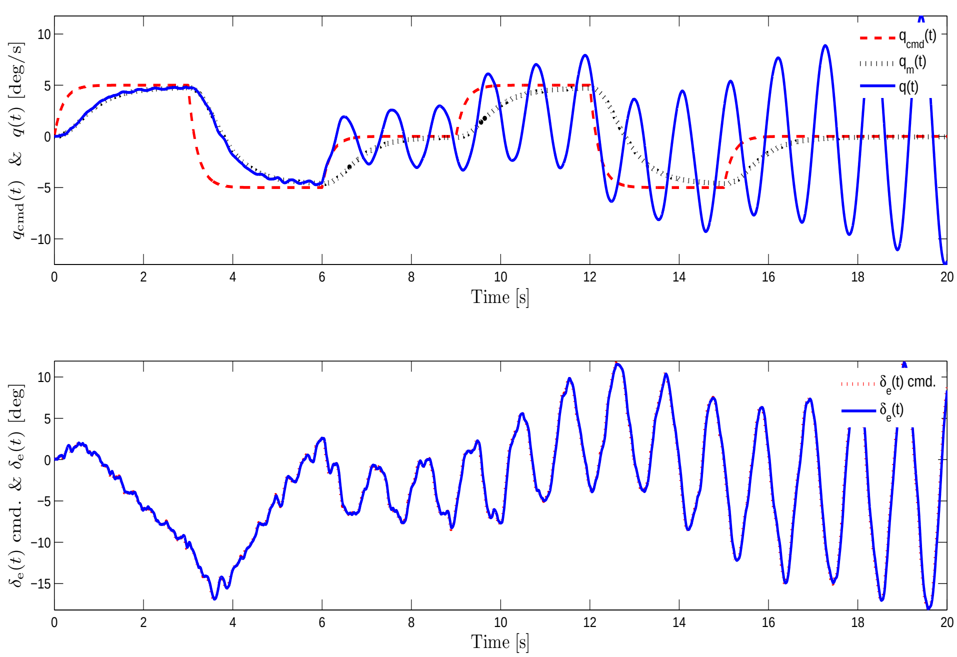

Figure 78: Pitch rate and elevator responses with nominal control for the case of sudden change in the sign of second row of A_{p}

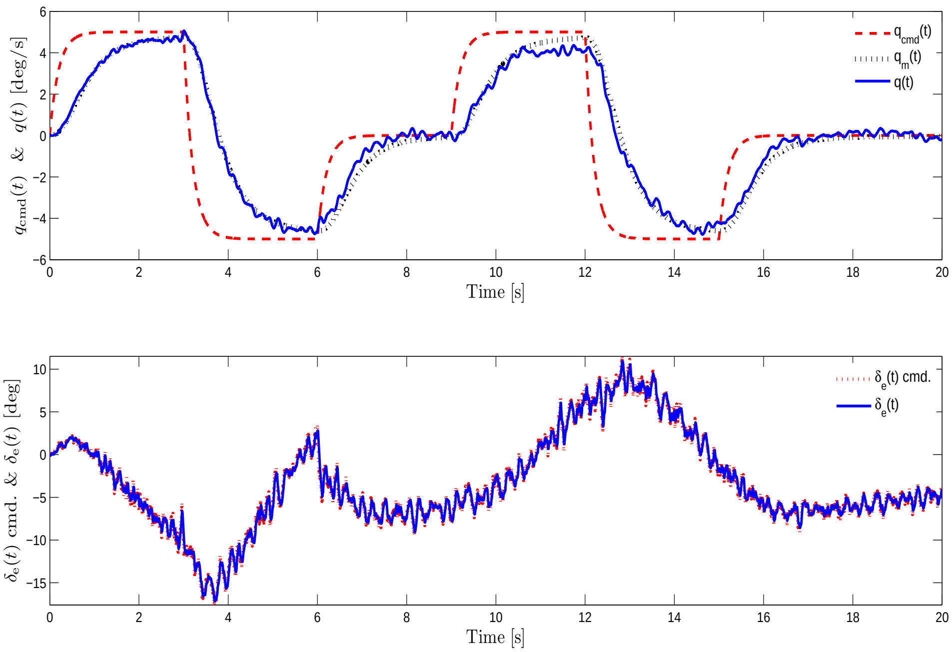

Figure 79: Pitch rate and elevator responses with adaptive control for the case of sudden change in the sign of second row of A_{p}

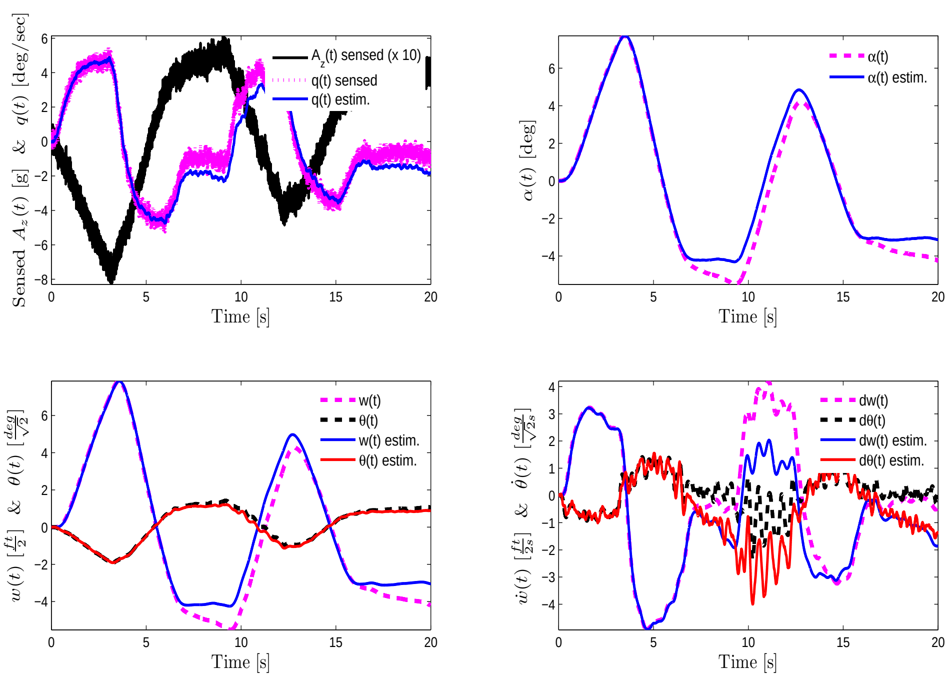

Figure 80: Measurement and state responses and their estimates with adaptive control for the case of sudden change in the sign of second row of A_{p}

5.6.5 Sudden Change in C_{m_α}

Finally we illustrate additional two cases. For the first case the pitch stiffness, C_{m_α}, becomes zero at $t = 6$ seconds. Figure 81 shows that the response with nominal control, whereas Figure 82 shows that the effect of this sudden change is hardly noticeable in the response with adaptive control. Figure 83 shows that the estimation performance of the observer is also as good as that observed in Figure 72 without uncertainty. For the second case C_{m_α} changes sign at $t = 6$ seconds. Figure 84 shows that the response with nominal control becomes unstable, whereas Figure 85 shows that the tracking performance is nearly as good as that observed in Figure 71 without uncertainty. Once again, Figure 86 shows that the estimation performance of the observer is as good as that observed in Figure 72.

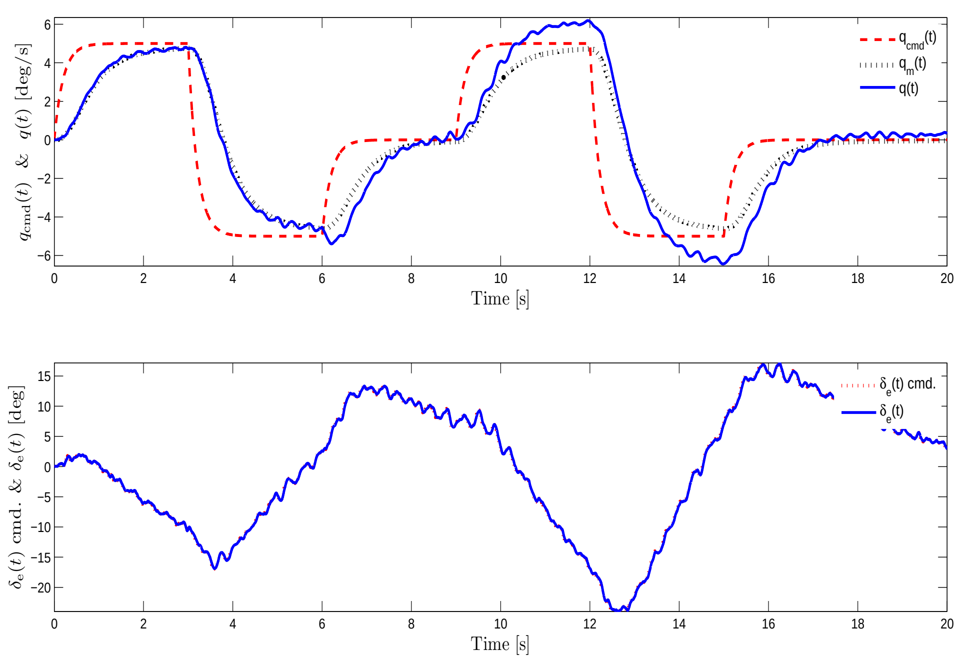

Figure 81: Pitch rate and elevator responses with nominal control for the case of sudden change in C_{m_α}

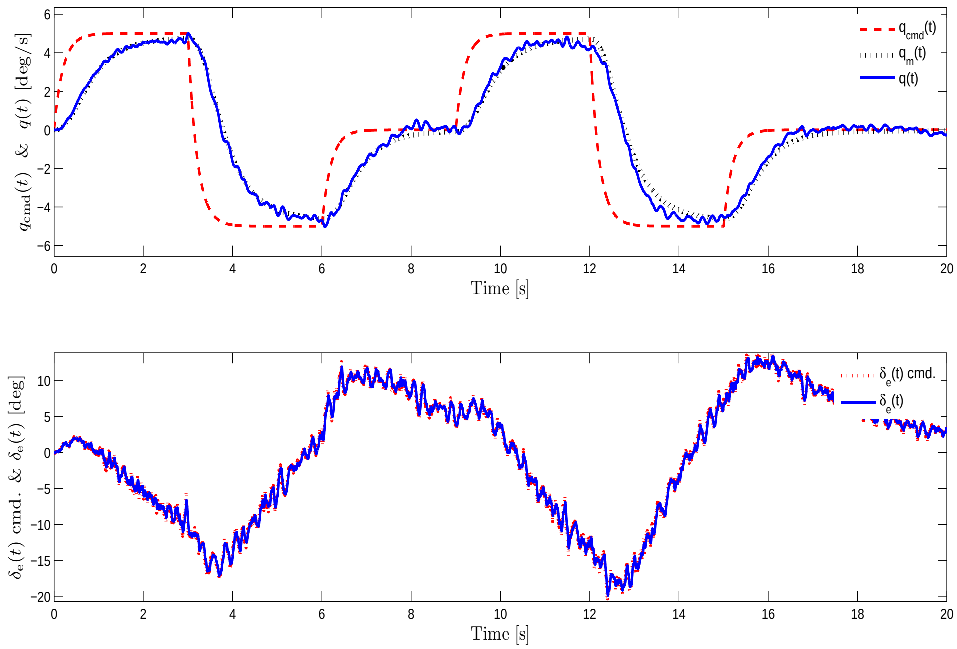

Figure 82: Pitch rate and elevator responses with adaptive control for the case of sudden change in C_{m_α}

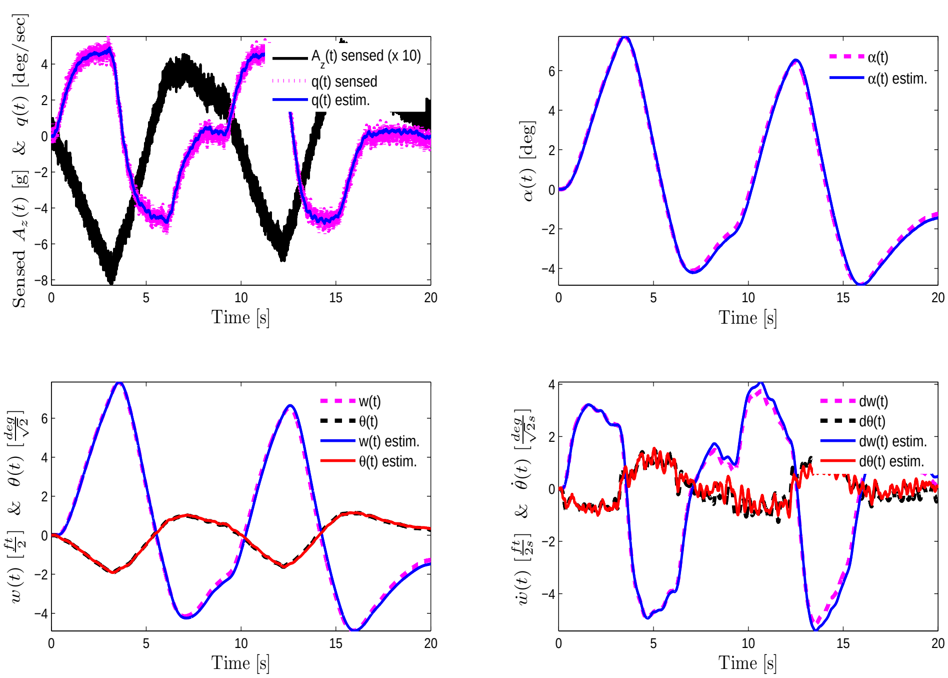

Figure 83: Measurement and state responses and their estimates with adaptive control for the case of sudden change in C_{m_α}

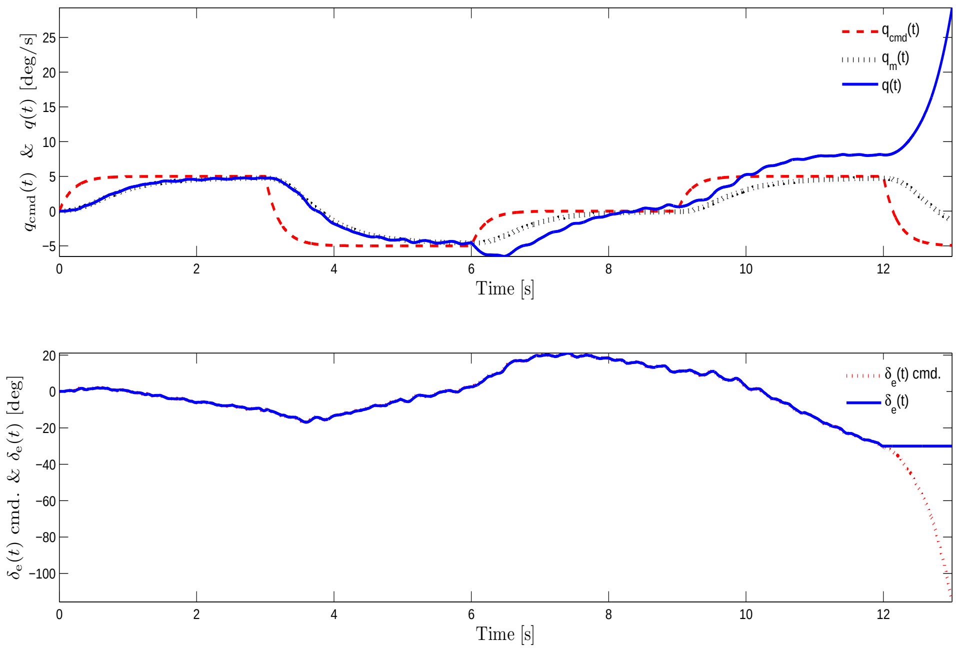

Figure 84: Pitch rate and elevator responses with nominal control for the case of sudden change in the sign of C_{m_α}

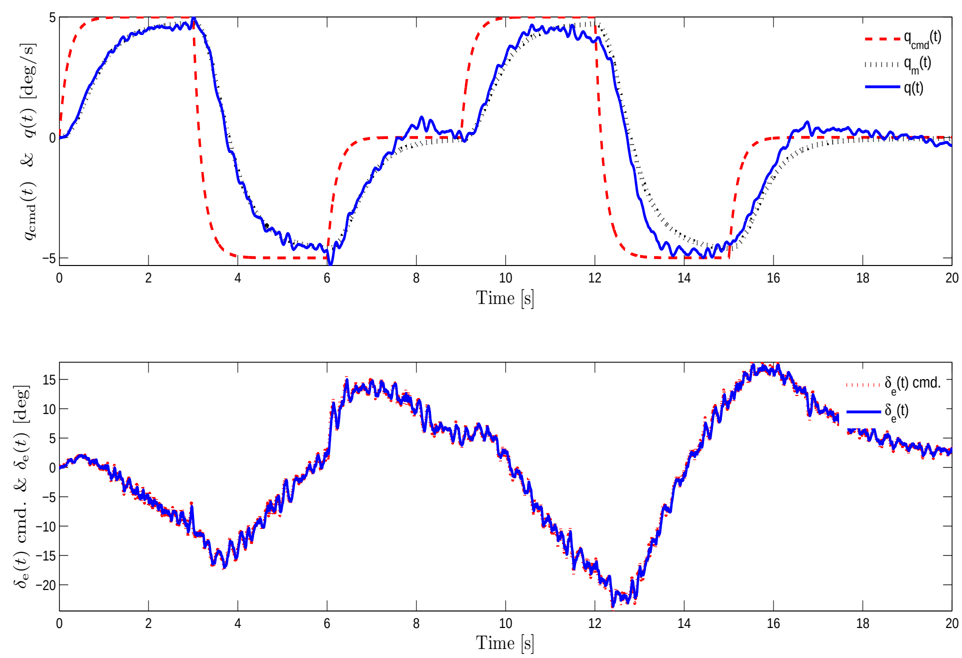

Figure 85: Pitch rate and elevator responses with adaptive control for the case of sudden change in the sign of C_{m_α}

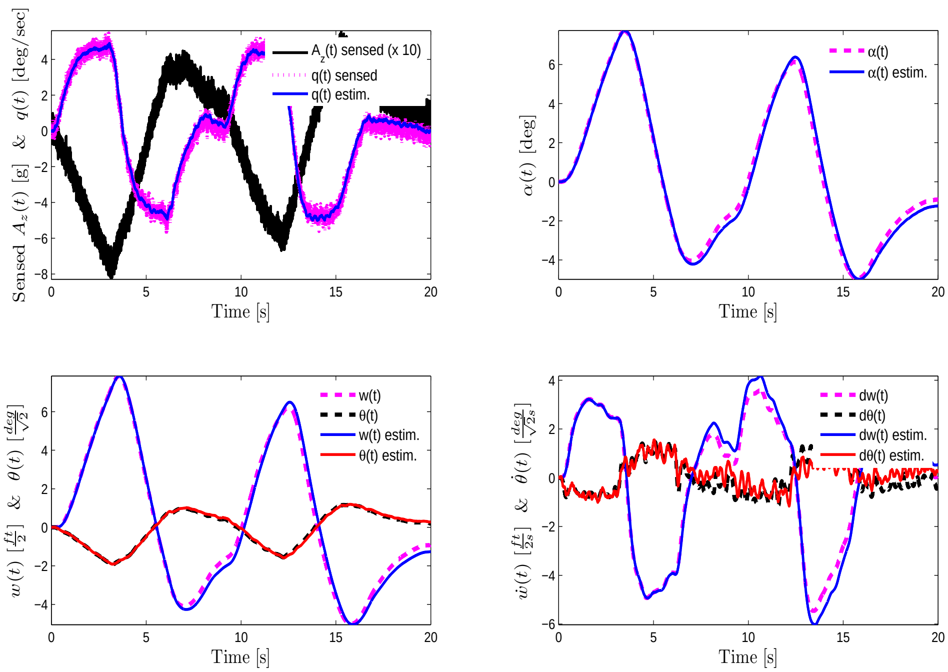

Figure 86: Measurement and state responses and their estimates with adaptive control for the case of sudden change in the sign of C_{m_α}

5.7 Conclusion

In this chapter we extended derivative-free adaptive control to the output feedback case. The level of complexity of the output feedback form is far less than many other methods, and it can be implemented in a form that augments an observer based linear controller architecture. Illustrative examples on a model of wing rock dynamics and an aeroelastic model of longitudinal dynamics of a generic transport model confirm that this is a promising approach, particularly for applications in which there is a combination of modeling uncertainty and external disturbances affecting the plant, or there is uncertainty in the aeroelastic model of the aircraft.

CHAPTER VI

Concluding Remarks and Future Research

6.1 Concluding Remarks

The intent of this dissertation has been to present new approaches to improve standard designs in adaptive control theory and novel adaptive control architectures. For this purpose, we first present a procedure that approximately enforces linear constraints to modify an existing adaptive control design by using a Kalman filter optimization approach in Chapter II. One key difference in the Kalman filter based approach to modification is that the resulting gain on the modification term is optimal and time varying. The proposed approach to deriving a modification term for an adaptive controller can be used in place of all modification terms that can be equivalently viewed as the gradient of a norm measure of a linear constraint on the adaptation gains. We then present a Kalman filter based adaptive controller approach in Chapter III. This approach is an extension of the results in Chapter II where instead of optimizing the modification gain, the adaptation gain is optimized. This approach results in a time-varying adaptation gain. The simulation results on a model of wing rock dynamics illustrate the presented theory and show significant improvement over other gradient based modification approaches.

The second major contribution of this dissertation is the development of a novel derivative-free, delayed weight update law for adaptive control in Chapter IV. The key feature is that the stability analysis is performed under the assumption that the ideal weights are bounded, but otherwise arbitrarily time varying. The approach is particularly useful for situations in which the nature of the system uncertainty cannot be adequately represented by a set of basis functions with unknown constant weights.

The system error signals, including the state tracking error and the weight update error, are proven to be uniformly ultimately bounded using a Lyapunov-Krasovskii functional without the introduction of modification terms. We have shown that the errors approach the ultimate bound exponentially in time. We have also provided an analysis that shows how the design parameters in the adaptive law can influence the ultimate bound, and the exponential rate of convergence to the bound. We have also shown how the derivative-free adaptive law can be modified using many of the existing modification terms. These results have been extended to the case of uncertainty in control effectiveness. We show various examples that illustrate the superior performance of derivative free adaptation in comparison to a conventional adaptive law, when it is evaluated for a variety of cases in which there is a sudden or rapidly varying set of dynamics. Furthermore, the performance of the proposed approach is successfully evaluated in flight tests performed by NASA on AirSTAR and the results are analyzed in Section IV.8.

Finally, we develop a novel approach for extending all the methods developed in this dissertation to the case of output feedback in Chapter V. The approach is developed only for the case of derivative-free adaptive control, and the extension of the other approaches developed previously for the state feedback case to output feedback is left as a future research topic. The level of complexity of the output feedback form is far less than many other methods, and it can be implemented in a form that augments an observer based linear controller architecture. Illustrative examples on a model of wing rock dynamics and an aeroelastic model of longitudinal dynamics of a generic transport model confirm that this is a promising approach.

6.2 Recommended Future Research

We recommend the following future research topics: *(i)* The results of Chapters II and III can be extended to output feedback adaptive control. This can be done by using the results presented in Chapter V. *(ii)* The results of Chapters II and III can be used to develop a Kalman filter based derivative-free adaptive control law. This will result in an optimal and time-varying adaptation gain and allow to achieve a given performance criteria without the need for excessive adaptation gain tuning. *(iii)* In Chapter IV, we showed that the closed loop system errors are uniformly ultimately bounded and approach to the ultimate bound exponentially in time. This characterizes the transient and steady-state performance bounds for the controlled uncertain system by derivative-free adaptive control law. Next step towards this direction is to characterize the performance bounds for the output feedback case in Chapter V. *(iv)* We suggest an extension of the derivative-free output feedback adaptive control law to the case when there is an input uncertainty in the system. This extension can be done similar to the extension given in Chapter IV.5. *(v)* An extension for the results of Chapters IV and V is to develop a derivative-free adaptive control law for decentralized control. *(vi)* The results in this dissertation assume that the full dimension of the plant is known. Therefore, they do not apply to the problem of unmodeled dynamics. This problem is an important extension for all the results in this dissertation.

Quantification of transient dynamics and gain/time-delay sensitivity at the system inputs and outputs are two open problems in adaptive control theory. To address these problems, there is a need to develop a systematic procedure to assign rates of adaptation and the design matrices used in the Lyapunov or Riccati equations. As discussed, derivative-free adaptive control is shown to have guaranteed transient and steady-state performance bounds. However, there is still a need to quantify the transient dynamics. Towards this end, we suggest the following future research topic: *(vii)* An analysis based on Singular Perturbation Theory is used to quantify the transient dynamics of an adaptively controlled system in Ref. 52. A similar analysis can be applied to derivative-free adaptive control which should provide a similar result.

In order to address gain/time-delay sensitivity, we combined derivative-free adaptive control with adaptive loop recovery modification as proposed in Ref. 10, where this modification term results in approximate retention of reference model loop properties such as gain and time-delay margins. However, we did this combination only for the full-state feedback control problem. Therefore, we suggest the following future research topic: *(viii)* Adaptive loop recovery modification term can be employed with the proposed derivative-free output feedback adaptive controller in order to improve gain and time-delay margins of the adaptively controlled system.

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