ABSTRACT

Title of dissertation: TOPOLOGICAL CHARGE OF REAL FINITE-GAP SINE-GORDON SOLUTIONS

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The most basic characteristic of x-quasiperiodic solutions $u(x,t)$ of the sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$ is the *topological charge density.* The real finite-gap solutions $u(x,t)$ are expressed in terms of the Riemann θ-functions of a non-singular hyperelliptic curve Γ and a positive generic divisor D of degree g on Γ, where the spectral data (Γ, D) must satisfy some reality conditions. The problem addressed in this dissertation is: to calculate the topological charge density from the θ-functional expressions for the solution $u(x,t)$. This problem has remained unsolved since it was first raised by S.P. Novikov in 1982. The problem is solved here by introducing a new limit of real finite-gap sine-Gordon solutions, which we call the *multiscale* or *elliptic limit*. We deform the spectral curve to a singular nodal curve, having elliptic curves as components, for which the calculation of topological charges reduces to two special easier cases.

TOPOLOGICAL CHARGE OF REAL FINITE GAP SINE-GORDON SOLUTIONS

by

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Chapter 1

Introduction

1.1 Sine-Gordon equation and Topological Charge

The sine-Gordon equation (SG) is one of the fundamental integrable systems of mathematical physics.

$$u_{tt} - u_{xx} + \sin u = 0, \qquad u = u(x,t), \tag{1.1.1}$$

where a subscript denotes the partial derivative: $u_x := \partial_x u$. In the light cone variables (ξ, η) where

$$x = 2\,(\xi + \eta), \quad t = 2\,(\xi - \eta)$$

$$\partial_\xi = 2\,(\partial_x + \partial_t), \quad \partial_\eta = 2\,(\partial_x - \partial_t),$$

it is written as:

$$u_{\xi\eta} = 4 \sin u, \qquad u = u(\xi, \eta). \tag{1.1.2}$$

In the form (1.1.2) it was derived in geometry (in the 19-th century), where it describes immersions of constant negative curvature surfaces into 3-dimensional Euclidean space ([36], Chapter 3). The SG-equation has several applications in physics [2], for example in superconductivity theory (Josephson junctions), classical relativistic field theory, as a model quantum field theory, nonlinear optics, dislocation in crystals. The quantity u in (1.1.1) and (1.1.2) is understood as an angular quantity. It is the quantum phase in superconductivity, and in geometry it is the angle between asymptotic lines on the constant negative curvature surface.

In this thesis we are concerned with solutions $u(x,t)$ in the class of functions which are quasi-periodic in x. But before that, we briefly describe the most basic *soliton* solutions, that is solutions in the class of functions rapidly decreasing in x. Since u is an angular quantity, rapidly decreasing here means $u_x, u_{xx}, u_t, u_{tt}, u_{xt}, \cdots$ etc. decay to 0 as $|x| \to \infty$ and u itself decays to 0 (mod 2π) as $|x| \to \infty$. Historically, the soliton solutions were found from the trivial solution $u \equiv 0$ by using substitutions known as Bäcklund transformations [25] (which were discovered by L. Bianchi and S. Lie). The basic 1-soliton solutions are the *kink* and the *anti-kink*. The kink satisfies $\lim_{x\to-\infty} u = 0$ and $\lim_{x\to+\infty} u = 2\pi$. The anti-kink satisfies $\lim_{x\to-\infty} u = 2\pi$ and $\lim_{x\to+\infty} u = 0$. Explicit solutions are given by

$$u(x,t) = 4 \arctan \exp(\pm \frac{x - vt - x_0}{\sqrt{1-v^2}}),$$

where the plus sign is taken for the kink and minus for anti-kink. The basic 2-soliton solution is the *breather* and satisfies $\lim_{x\to-\infty} u = 0 = \lim_{x\to+\infty} u$ and shows coupled kink and anti-kink behavior. The phase gained by u as x goes from $-\infty$ to $+\infty$ for these three basic solutions is $+1, -1$ and 0 respectively and is called the *topological charge*. The topological charge, $\frac{1}{2\pi}\int_{-\infty}^{\infty} u_x\, dx$ is the most basic conservation law for the SG-equation in the rapidly decreasing class.

The notion of topological charge is central to this thesis. The precise definitions for periodic and quasi-periodic solutions are as follows. We say that $u(x,t)$ is x-

periodic with period T if

$$u(x+T,t) = u(x,t) + 2n\,\pi, \quad n \in \mathbb{Z}, \quad \text{or} \quad e^{iu(x+T,t)} = e^{iu(x,t)}. \tag{1.1.3}$$

The number n is called **the topological charge**, the ratio $\bar{n} = n/T$ is called **the topological charge density**.

Similarly, by quasi-periodic solutions we mean that $e^{iu(x,t)}$ is quasi-periodic in x. The notion of **topological charge density**, $\bar{n}$ can naturally be extended to quasiperiodic solutions. It is defined as the limit

$$\bar{n} = \lim_{T\to\infty} \frac{u(x+T,t) - u(x,t)}{2\pi\,T}. \tag{1.1.4}$$

We briefly explain, why (1.1.4) is well defined. For fixed t, a g-phase real quasiperiodic solution $u(x,t)$ is of the form $-i\,\log\phi(xU)$ where ϕ is a circle valued function on a g-dimensional real torus, $\phi : \mathbb{R}^g/\mathbb{Z}^g \to S^1 \subset \mathbb{C}$, and $U \in \mathbb{R}^g$. If $U \in \mathbb{Q}^g$ (upto proportionality) then clearly $u(x,t)$ is x-periodic in the sense of (1.1.3) and we have defined $\bar{n} = n/T$ for $u(x,t)$ of the form (1.1.3). In the case when U is not proportional to a rational vector, we take any sequence of rational vectors $\{U_m,\, m \in \mathbb{N}\}$ converging to U. It can be shown that $\bar{n}$ as defined in (1.1.4) will be the limit of corresponding sequence $\{\bar{n}_m,\, m \in \mathbb{N}\}$.

The topological charge density $\bar{n}$ is the most basic and useful characteristic of real solutions. In particular, it is a conservation law for the SG-hierarchy "surviving" generic perturbations of the type

$$u_{tt} - u_{xx} + \sin u = \varepsilon F(\sin u, \cos u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots)$$

1.2 Real Finite-Gap Sine-Gordon Solutions

The SG-equation was one of the first systems to be integrated in the rapidly decreasing class by the *inverse scattering transform* first discovered by Gardner-Greene-Kruskal-Miura [15] in 1967 for the KdV equation. The modern approach to the integration of the SG-equation was developed by Ablowitz, Kaup, Newell and Segur in 1973 [1], in which the *zero-curvature representation* was found for the SG equation: it is represented as the consistency condition for the following pair of overdetermined linear differential equations, depending additionally on a *spectral parameter* λ.

$$\Psi_\xi = \mathcal{U}\,\Psi, \qquad \Psi_\eta = \mathcal{V}\,\Psi, \qquad \text{where} \quad \Psi = \Psi(\lambda,\xi,\eta) = \begin{pmatrix} \psi_1(\lambda,\xi,\eta) \\ \psi_2(\lambda,\xi,\eta) \end{pmatrix}$$

$$\mathcal{U} = \mathcal{U}(\lambda,\xi,\eta) = \begin{bmatrix} \frac{iu_\xi}{2} & 1 \\ -\lambda & -\frac{iu_\xi}{2} \end{bmatrix}, \quad \mathcal{V} = \mathcal{V}(\lambda,\xi,\eta) = \begin{bmatrix} 0 & -\frac{1}{\lambda}e^{iu} \\ e^{-iu} & 0 \end{bmatrix} \tag{1.2.1}$$

The compatibility condition $\Psi_{\xi\eta} = \Psi_{\eta\xi}$ turns out to be independent of the spectral parameter λ and equivalent to the SG-equation:

$$0 = \mathcal{U}_\eta - \mathcal{V}_\xi + [\mathcal{U},\mathcal{V}] = (u_{\xi\eta} - 4\sin u)\begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix},$$

where $[\mathcal{U},\mathcal{V}]$ denotes the commutator $\mathcal{U}\mathcal{V} - \mathcal{V}\mathcal{U}$.

The principal method for constructing periodic or quasiperiodic solutions of soliton equations is the *finite-gap* or *algebro-geometrical* integration technique, first developed by Novikov in 1974 for the KdV equation [30]. Finite-gap SG solutions $u(x,t)$ are constructed from the *spectral data* (Γ, D) where Γ is a hyperelliptic Riemann surface and D (referred to as the *divisor*) is a generic set of g points on Γ.

Finite gap solutions $u(\xi, \eta)$ are written in terms of the Riemann θ-function of Γ and were first constructed by Kozel-Kotlyarov in 1976 [22]. These solutions are in general complex-valued. Applications in physics as well as differential geometry require the solution $u(\xi, \eta)$ to be *real-valued*. In order to obtain real solutions $u(\xi, \eta)$, reality conditions have to be imposed on the spectral data (Γ, D). The reality condition on Γ was also obtained in [22]. However, *in contrast with the KdV equation*, the reality condition on the divisor D, turned out to be a difficult problem. We remark that for problems like KdV, KP-1, defocusing nonlinear Schrödinger equation (NLS), sinh-Gordon equation, the reality condition on the divisor is straight-forward owing to the self-adjointness (in appropriate function space) of the auxiliary linear operators appearing in the zero-curvature representation. This does not hold for SG, KP-2, focusing NLS equations. Due to this difficulty, periodic finite-gap Sine-Gordon theory had relatively few applications for a long time. The reality condition on the divisor D was obtained by Cherednik in 1980 [3]. For brevity we use the term *admissible divisor* for divisors D which yield real solutions. It was shown in [3] that set of admissible divisors is disconnected. Each connectivity component represents a different **topological type** of solutions. In the same work [3], the number of these components was found, and it was proved that *all real finite-gap solutions are automatically non-singular*.

1.3 Statement of Problem

As emphasized in Section 1, the topological charge density is the most basic and useful characteristic of quasiperiodic SG-solutions. Therefore, the first main problem is:

Problem: Calculate the topological charge density of real finite gap SG-solutions in terms of the spectral data (Γ, D).

This problem was first raised and solution attempted in 1982 by Dubrovin and Novikov [8]. In [33] Novikov pointed out, that the approach of [8] fails to work in general. (The proof there only works for spectral curves sufficiently close to degenerate curves.) A solution was obtained, about 20 years later by Grinevich and Novikov in 2001 [16], [17]. However in this solution, no use was made of the θ-functional formulas for $u(x,t)$! Therefore the situation was somewhat paradoxical: We have explicit θ-functional formulae for the solution, but they do not help to answer the most basic questions about the solution. This raises a question on the usefulness or effectiveness of writing θ-functional expressions of solutions of soliton equations in general. As pointed out by Novikov, if the θ-functional form of SG-solutions is to be considered as an effective one, then one should be able to extract from them, a formula for the topological charge density. *Until now, this problem has resisted efforts at solution. The main result of this dissertation is the solution to this problem.*

1.4 Sketch of Solution and Outline

A sketch of the solution is as follows: The calculation of topological charge density requires calculation of certain integers called basic charges. Each component (topological type) of admissible divisors will be seen to be a real torus isomorphic to $\mathbb{R}^g/\mathbb{Z}^g$, in the complex torus which is the Jacobian variety of the spectral curve. There is a function $\phi : \mathbb{R}^g/\mathbb{Z}^g \to S^1 \subset \mathbb{C}$, such that the solution $e^{iu(x,t)}$ is given by $\phi(xU + tV)$, for some $\tilde{U}, \tilde{V} \in \mathbb{R}^g$. Let $\phi_* : H_1(\mathbb{R}^g/\mathbb{Z}^g, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$ be given by the matrix $[n_1, n_2, \cdots, n_g]$ with respect to the standard basic 1-cycles of $\mathbb{R}^g/\mathbb{Z}^g$ and S^1. By the *basic charges*, we mean the integers $n_1, \cdots, n_g$.

The basic charges are easily calculated for spectral curves of genus-1, using elementary properties of elliptic functions. The basic charges are seen to be trivial (zero) for spectral curves with no real branch points (only complex conjugate branch points). We deform the spectral curve to a singular nodal curve, having elliptic curves as components (and possibly an additional hyperelliptic curve), for which the calculation of basic charges reduces to the above two special cases. Since the basic charges are integers, they do not change as the spectral curve is deformed continuously. Therefore the basic charges obtained from the singular limit are valid for the original spectral curve. We call the singular limit of the spectral curve we construct as the *multiscale* or elliptic limit. This limit has not appeared before in the literature of soliton theory.

The outline of the thesis is as follows. In Chapter 2, we describe the construction of complex finite-gap SG solutions. The real solutions and their topological types are also detailed. In Chapter 3, the setup for calculation the topological charge density and the basic charges is described. The multiscale limit is constructed, and the topological charge density is calculated. Directions for future work are discussed.

Chapter 2

Complex and Real Finite-Gap Sine-Gordon Solutions

2.1 Complex solutions

In this chapter we derive θ-functional formulae for the real and complex finite gap SG solutions $u(x,t)$ associated with the spectral data (Γ, D) where Γ is a non-singular hyperelliptic curve of genus g:

$$\Gamma : \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - E_i) \quad \text{with} \quad E_1, \cdots, E_{2g} \quad \text{distinct nonzero complex numbers}$$

$$D = \gamma_1 + \gamma_2 + \cdots + \gamma_g, \quad \text{a } \textit{nonspecial} \text{ divisor with } \gamma_1, \cdots, \gamma_g \in \Gamma \backslash \{0, \infty\} \tag{2.1.1}$$

These formulae can also be found in the articles [22] and [7]. (The term 'non-special' is defined later in the paragraph preceding Fact 2.1.1). We will derive three equivalent expressions for $u(x,t)$. The first expression is called the '*Its-Matveev*' type and it is expressed in terms of the second logarithmic derivatives of the θ function. The second expression is called the '*θ-quotient*' type and is expressed as a ratio of four θ-functions. The third expression is the so-called '*trace formula*'. The trace formula is not expressed in terms of θ-functions, but rather involves coordinates of g moving points on Γ. We will not need the trace formula in this thesis, because it does not meet our requirements of calculating the topological charge density of real solutions $u(x,t)$ from the θ-functional formulae. The trace formula was an essential ingredient in the calculation of topological charge density in the work [17]. We will first derive, in section 2.1.1, the formula of Its-Matveev type for $u(x,t)$. Using this formula, we will derive, in section 2.1.2, the θ-quotient formula. From the θ-quotient formula, the trace formula will be obvious. Our approach in obtaining the Its-Matveev type formula will be using the 'Baker-Akhiezer'-function method first introduced by Krichever [23]. We remark that there is a quick and elegant method to derive both the Its-Matveev type as well as the θ-quotient type formula (simultaneously) for $u(x,t)$ solving the equation $u_{\xi\eta} = 4c \sin u$, (where c is some constant) using a corollary of the Fay trisecant identity. This method (due to Mumford) is given in [29], section 4 of Chapter IIIb. However, in this approach it is hard to pin down the conditions that achieve $c = 1$.

2.1.1 Formula for $u(x,t)$ of 'Its-Matveev' type

In order to derive the formula for $u(x,t)$ we need to introduce some notation. Definitions and details of the notions involved in the discussion below can be found in the survey article [5]. We choose a symplectic basis of cycles $\{a_1, b_1, \cdots, a_g, b_g\}$ in $H_1(\Gamma, \mathbb{Z})$, i.e satisfying $a_i \circ a_j = b_i \circ b_j = 0$ and $a_i \circ b_j = \delta_{ij}$, where $a \circ b$ denotes the intersection index of two 1-cycles a, b. We also choose a basis for the g-dimensional space of holomorphic differentials on Γ: $\vec{\omega} = (\omega_1, \cdots, \omega_g)$, normalized such that $\int_{a_j} \omega_i = \delta_{ij}$. The Riemann matrix of Γ is the matrix defined by $B_{ij} = \int_{b_j} \omega_i$. The matrix B is symmetric and its imaginary part is positive definite. The Riemann theta function associated with B is the entire function of g complex variables $z =$

$(z_1, \cdots, z_g)$ defined by

$$\theta(z|B) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i\, n^t B n)\, \exp(2\pi i\, n^t z) \tag{2.1.2}$$

It satisfies the transformation rule

$$\theta(z + N + BM) = \theta(z)\, \exp\left(-\pi i\, \left[2M^t z + M^t BM\right]\right) \qquad \text{where} \quad N, M \in \mathbb{Z}^g \tag{2.1.3}$$

Let ω_∞ and ω_0 be abelian differentials of the second kind having double poles at $P = \infty$ and $P = 0$ respectively with the principal parts being $-\lambda\, d(1/\sqrt{\lambda})$ and $-1/\lambda\, d\sqrt{\lambda}$ respectively and normalized to have zero a-periods (where by $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$ we mean local coordinates k_0 and k_∞ at $P = 0$ and $P = \infty$ respectively with $k_0^2 = \lambda$ and $k_\infty^2 = 1/\lambda$). We define two vectors $U, V \in \mathbb{C}^g$ by

$$-U_j = \frac{1}{2\pi} \int_{b_j} \omega_\infty, \qquad -V_j = \frac{1}{2\pi} \int_{b_j} \omega_0 \tag{2.1.4}$$

Explicit formulae for U, V can be derived. From the bilinear relations of Riemann for differentials of first and second kind ([13], formula 3.8.2) we immediately obtain

$$U = \left(\frac{i\,\vec{\omega}}{d\sqrt{\lambda^{-1}}}\right)(\infty), \qquad V = \left(\frac{i\,\vec{\omega}}{d\sqrt{\lambda}}\right)(0) \tag{2.1.5}$$

The complex torus $J(\Gamma) = \mathbb{C}^g / \{\mathbb{Z}^g + B\mathbb{Z}^g\}$ is the Jacobian variety of Γ. We define the Abel-map with basepoint $P = \infty$ as

$$A : \Gamma \to J(\Gamma), \quad A(P) = \int_\infty^P \vec{\omega}$$

Let $K \in \mathbb{C}^g$ be the vector of Riemann constants (see [5]), associated with the chosen base-point and basic cycles. In Section 3, we will give an explicit formula for K.

The construction of finite gap SG-solution proceeds via a vector function $\vec{\psi}(\xi,\eta,P) = (\psi_1,\psi_2)$, called 'Baker-Akhiezer' function, on Γ having the following properties: $\vec{\psi}$ is meromorphic on $\Gamma - \{0,\infty\}$ with simple poles at the points of the divisor $D = (\gamma_1 + \cdots + \gamma_g)$, and essential singularities at $P = 0$ and $P = \infty$, having the form

$$\vec{\psi}(\xi,\eta,P) = e^{i\xi\sqrt{\lambda}} \begin{pmatrix} 1 + c_2(\xi,\eta)/\sqrt{\lambda} + O(1/\lambda) \\ i\sqrt{\lambda} + c_3 + O(1/\sqrt{\lambda}) \end{pmatrix} \qquad \text{as } \lambda \to \infty \tag{2.1.6}$$

$$\vec{\psi}(\xi,\eta,P) = e^{-i\eta/\sqrt{\lambda}} \begin{pmatrix} \phi_1(\xi,\eta) + c_1(\xi,\eta)\sqrt{\lambda} + O(\lambda) \\ i\phi_2(\xi,\eta)\sqrt{\lambda} + O(\lambda) \end{pmatrix} \qquad \text{as } \lambda \to 0 \tag{2.1.7}$$

The 'Baker-Akhiezer' function $\vec{\psi}$ is constructed as follows. Let $\hat{\Gamma}$ be the maximal abelian cover of Γ. The Abel-Jacobi map lifts to a map $\hat{A} : \hat{\Gamma} \to \mathbb{C}^g$. Let $\hat{D} = \hat{\gamma}_1 + \cdots + \hat{\gamma}_g$ where $\hat{\gamma}_i \in \hat{\Gamma}$ are any lifts of $\gamma_i \in \Gamma$ to $\hat{\Gamma}$. We define an expression $z(x,t) \in \mathbb{C}^g$ which moves rectilinearly with respect to x and t. The same notation $z(x,t)$ will also be used for the image of $z(x,t)$ in $J(\Gamma)$

$$\begin{aligned} z_0 &= -\hat{A}(\hat{D}) - K \\ z(\xi,\eta) &= z_0 + \eta V - \xi U \end{aligned} \tag{2.1.8}$$

We recall the relation between (x,t) and the variables (ξ,η) (from section 1.1)

$$\begin{aligned} x &= 2\,(\xi+\eta), \qquad t = 2\,(\xi-\eta) \\ \partial_\xi &= 2\,(\partial_x+\partial_t), \qquad \partial_\eta = 2\,(\partial_x-\partial_t) \end{aligned} \tag{2.1.9}$$

Since ω_∞, ω_0 have no polar periods, there is a meromorphic function g on $\hat{\Gamma}$, unique upto an additive constant, satisfying $dg = i\xi\omega_\infty - i\eta\omega_0$. Let $\hat{\infty} \in \hat{\Gamma}$ be the unique point lying over $\infty \in \Gamma$ satisfying $\hat{A}(\hat{\infty}) = 0$. The local coordinate $1/\sqrt{\lambda}$ at ∞ is taken as a local coordinate at $\hat{\infty}$. We fix the ambiguity in g by requiring that:

$$(g - i\,\xi\sqrt{\lambda})(\hat{\infty}) = 0 \tag{2.1.10}$$

Consider the function ψ_1 on $\hat{\Gamma}$:

$$\psi_1(\xi, \eta, \hat{P}) = C(\xi, \eta)\, \exp(g(\hat{P}))\, \frac{\theta(\hat{A}(\hat{P}) + z(\xi, \eta))}{\theta(\hat{A}(\hat{P}) + z_0)} \qquad \text{for } \hat{P} \in \hat{\Gamma} \tag{2.1.11}$$

where $C(\xi, \eta)$ is a normalization factor (to be specified below). We next show that ψ_1 is invariant under the action of $H_1(\Gamma, \mathbb{Z})$ on $\hat{\Gamma}$ and hence defines a function on Γ. Indeed, when $\hat{P} \in \hat{\Gamma}$ is shifted by the action of $\sum_i n_i\, a_i + \sum_j m_j\, b_j \in H_1(\Gamma, \mathbb{Z})$ the ratio of θ-functional terms in (2.1.11) acquires a multiplicative factor of $\exp[-2\pi\, i\, m^t(\eta V - \xi U)]$ (using the transformation rule (2.1.3)). The term $\exp(g(\hat{P}))$, on the other hand gains a multiplicative factor of $\exp[2\pi\, i\, m^t(\eta V - \xi U)]$ using (2.1.4). Thus (2.1.11) is invariant under the action of $H_1(\Gamma, \mathbb{Z})$ on $\hat{\Gamma}$. Next, we fix the normalization factor $C(\xi, \eta)$ by requiring that ψ_1 has the asymptotic behavior at $\hat{P} = \infty$ given by the first row of (2.1.6). Taking (2.1.10) into account we thus obtain:

$$C(\xi, \eta) = \frac{\theta(z_0)}{\theta(z(\xi, \eta))} \tag{2.1.12}$$

We note that ψ_1 also satisfies the first row of (2.1.7): the desired asymptotic at $0 \in \Gamma$. We observe (using 2.1.3) that the ratio:

$$\frac{\theta(z_0)}{\theta(z(\xi, \eta))}\, \frac{\theta(\hat{A}(P) + z(\xi, \eta))}{\theta(\hat{A}(P) + z_0)}$$

is independent of the choice of lift $\hat{D} \in \hat{\Gamma}$ of $D \in \Gamma$. In summary: The function $\psi_1(\xi, \eta, \hat{P})$, although defined in terms of $\hat{P}$ and $\hat{D}$, depends only on P and D.

We next prove that for small enough values of (ξ, η) (to be made precise later) the Baker-Akhiezer function $\psi_1(\xi, \eta, P)$ is unique, and is given by the construction (2.1.11). We have already shown that (2.1.11) has the desired asymptotics at $P = 0, P = \infty$. Before showing that (2.1.11) has poles at most at D, we need a lemma. First, we recall the definitions of θ-divisor and special divisors, and two facts about them. The θ-divisor is defined as:

$$\Theta = \{z \in J(\Gamma) \,|\, \theta(z) = 0\} \tag{2.1.13}$$

Although, θ is a function on $\mathbb{C}^g$, the transformation rule (2.1.3) implies that it is meaningful to talk about the zeros in Γ of $\theta(z)$. A positive divisor $\Delta = P_1 + \cdots + P_g$ of degree g is called special if there is a non-constant meromorphic functions on Γ with poles at most at D. Let Θ_{sp} denote the image $A(\Delta) + K$ in $J(\Gamma)$ of these special divisors Δ (positive, degree g). We will need two facts.

Fact 2.1.1.

1. *([13] pp 310): Θ consists of elements of the form $A(P_1 + \cdots + P_{g-1}) + K$.*
2. *([13] pp 319): $\Theta_{sp} \subset \Theta$. The function $\theta(A(P) - e)$ vanishes identically on Γ if and only if $e \in \Theta_{sp}$. If $e + K \notin \Theta_{sp}$ then the function $\theta(A(P) - e - K)$ has exactly g zeros $Q_1, \cdots, Q_g \in \Gamma$, and they satisfy $A(Q_i) = e$ (Jacobi inversion).*

Lemma 2.1.1.

1. *The conditions on the divisor $D = \gamma_1 + \gamma_2 + \cdots + \gamma_g$ that it is nonspecial and that $\gamma_1, \cdots, \gamma_g \in \Gamma \backslash \{0, \infty\}$ are equivalent to the condition:*

$$A(D) + K \notin \Theta \cup (\Theta + A(0))$$

2. *Let $W \subset \mathbb{R}^2$ be the largest disk around $(0,0)$ in the (ξ,η)-plane, such that $-z(\xi,\eta) = A(D) + K - \eta V + \xi U$ also satisfies the condition*

$$-z(\xi,\eta) \notin \Theta \cup (\Theta + A(0))$$

Then for $(\xi,\eta) \in W$, there exists a divisor $D(\xi,\eta) = \gamma_1(\xi,\eta) + \cdots + \gamma_g(\xi,\eta)$ with $A(D(\xi,\eta)) + K = -z(\xi,\eta)$.

3. *For $(\xi,\eta) \in W$, $\psi_1(\xi,\eta,P)$ (2.1.11) has poles at D and zeros at $D(\xi,\eta)$.*

Proof.

1. Suppose D is non-special and $0,\infty \notin D$, but $A(D) + K \in \Theta \cup (\Theta + A(0))$. From the first fact mentioned above about Θ, we deduce (from Abel-theorem) that $D \sim P_1 + \cdots + P_g$ (where $\sim$ denotes the linear equivalence of divisors) for some P_i with $P_g \in \{0,\infty\}$. By the definition of being non-special, we obtain $D = P_1 + \cdots + P_g$ contradicting the hypothesis that $0,\infty \notin D$. Conversely, suppose $A(D) + K \notin \Theta \cup (\Theta + A(0))$. Using the second fact mentioned above, we obtain that D is non-special. If $0,\infty \in D$, then the first fact above implies that we have $A(D) + K \in \Theta \cup (\Theta + A(0))$, which is a contradiction.

2. If $-z(\xi,\eta) \notin \Theta \cup (\Theta + A(0))$, then the by the second fact (that $\Theta_{sp} \subset \Theta$) we know that $-z(\xi,\eta) \notin \Theta_{sp}$ and hence there is a divisor $D(\xi,\eta) = \gamma_1(\xi,\eta) + \cdots + \gamma_g(\xi,\eta)$ with $A(D(\xi,\eta)) + K = -z(\xi,\eta)$. We note for future use that part 1) applied to the divisor $D(\xi,\eta)$ (instead of D) implies that $0,\infty \notin D(\xi,\eta)$ as well.

3. The above two parts imply that ψ_1 has poles at D and zeros at $D(\xi,\eta)$ provided $(\xi,\eta)\in W$.

□

We need a fact from [5]:

Fact 2.1.2. *(Theorem 3.1.1 in [5])*

The Baker-Akhiezer function ψ_1 with the asymptotics (2.1.6-2.1.7) and poles at D is unique, provided $D(\xi,\eta)$ is non-special.

For $(\xi,\eta)\in W$, $D(\xi,\eta)$ is non-special since $\Theta_{sp}\subset\Theta$ and $A(D(\xi,\eta))+K\notin\Theta$ by Lemma 2.1.1, part 2). Thus, the Baker-Akhiezer function is unique and hence given by the construction (2.1.11). For our needs in section 2.2 (real solutions), W will be shown to be all of $\mathbb{R}^2$. We remark that, although a choice was made for the Riemann matrix (or equivalently choice of basic cycles) and the base point for the Abel map in the construction of ψ_1, it is actually independent of these choices due to the uniqueness of ψ_1.

The second component ψ_2 of the Baker-Akhiezer function is defined by

$$\psi_2=\psi_{1\xi}-\frac{\phi_{1\xi}}{\phi_1}\psi_1 \tag{2.1.14}$$

where ϕ_1 is as in (2.1.7). Although we do not use it, we provide here for completeness a θ-functional formula for ψ_2. Let $\omega_{0\infty}$ be the abelian differential of third kind on Γ, normalized to have a-period zero, and having simple poles at $0,\infty$ with residues $1,-1$ respectively. There is a unique meromorphic function h on $\hat{\Gamma}$ satisfying $\frac{dh}{h}=\omega_{0\infty}$

and such that the principal part of h near $\hat{\infty}$ is given by $\sqrt{\lambda}$. We have:

$$\psi_2(\xi,\eta,\hat{P}) = i\,\exp(g(\hat{P}))\,h(\hat{P})\,\frac{\theta(z_0)}{\theta(z(\xi,\eta)+A(0))}\,\frac{\theta(\hat{A}(P)+z(\xi,\eta)+A(0))}{\theta(\hat{A}(P)+z_0)} \tag{2.1.15}$$

We list some of the useful relations between ψ_1, ψ_2 and the various coefficients appearing in equations (2.1.6-2.1.7). These relations are forced by the uniqueness of ψ_1 :

$$\begin{aligned} -\lambda\psi_1 &= \psi_{2\xi} - \frac{\phi_{2\xi}}{\phi_2}\psi_2 \\ \psi_1 &= \frac{\psi_{2\eta}}{c_{3\eta}} \\ \psi_2 &= \frac{i\lambda\psi_{1\eta}}{c_{2\eta}} \\ \frac{\phi_2}{\phi_1} = c_{3\eta} &= \frac{1}{ic_{2\eta}} = -i(c_1/\phi_1)_\xi \\ \frac{\phi_{1\xi}}{\phi_1} + \frac{\phi_{2\xi}}{\phi_2} &= 0 \end{aligned} \tag{2.1.16}$$

Again using the uniqueness of the Baker-Akhiezer function ψ_1 we can show that:

$$\begin{aligned} (\partial_\xi - \mathcal{U})\,\vec{\psi} &= (\phi_{1\xi}/\phi_1 - iu_\xi/2)\begin{pmatrix}\psi_1\\ -\psi_2\end{pmatrix} \\ (\partial_\eta - \mathcal{V})\,\vec{\psi} &= (-i/c_{2\eta} - e^{-iu})\begin{pmatrix}i\psi_2\,e^{iu}/\lambda\\ -\psi_1\end{pmatrix} \end{aligned} \tag{2.1.17}$$

where $\mathcal{U},\mathcal{V}$ are from the zero-curvature representation (1.2.1) of the SG-equation If we define

$$u(\xi,\eta) = i\log(\phi_2/\phi_1) \tag{2.1.18}$$

then it follows from the fourth and fifth relation in equation (2.1.14) that $\vec{\psi}_\xi = \mathcal{U}\,\vec{\psi}$ and $\vec{\psi}_\eta = \mathcal{V}\,\vec{\psi}$, and therefore $u(\xi,\eta)$ is a solution of the SG-equation. Some comments regarding the interpretation of (2.1.18) are necessary. Let $v(\xi,\eta)$ denote the quantity

$\exp(iu(\xi,\eta)$. We observe that only $v(\xi,\eta)$ and $\frac{dv}{v} = u_\xi$ appear in (2.1.17) and in the matrices $\mathcal{U}, \mathcal{V}$ (1.2.1). The quantity $u(\xi,\eta)$ is itself obtained as $\log v(\xi,\eta)$. For this logarithm to be well defined, we use the fact that the disk W is simply connected and we need to prove that $v(\xi,\eta) = \frac{\phi_1}{\phi_2} \in \mathbb{C}\backslash\{0\}$ (in other words ϕ_1, ϕ_2 are always nonzero). To see that $\phi_1(\xi,\eta) \neq 0$, we note that if it is zero then $\psi_1(\xi,\eta,P)$ has a zero at $P = 0$ (from 2.1.6) , but we have shown in the proof of Lemma 2.1.1, part 3, that $0, \infty \notin D(\xi,\eta)$, the zeros of ψ_1. The relation $i\phi_2 c_{2\eta} = \phi_1$ (the fourth equation in (2.1.16)) shows that $\phi_1 \neq 0$ implies $\phi_2 \neq 0$. Thus $u(\xi,\eta)$ is well defined. We next proceed to develop the Its-Matveev type formula for $u(\xi,\eta)$.

We have

$$\exp(-iu(\xi,\eta)) = \phi_2/\phi_1 \qquad \text{from (2.1.18)},$$

$$\phi_2/\phi_1 = -i\,(c_1/\phi_1)_\xi \qquad \text{from (2.1.16)},$$

$$-i\,(c_1/\phi_1)_\xi = -i\,\frac{d\,\partial_\xi \log\psi_1}{d\sqrt{\lambda}}|_{P=0} \qquad \text{from (2.1.7)},$$

$$-i\,\frac{d\,\partial_\xi \log\psi_1}{d\sqrt{\lambda}}|_{P=0} = \omega_\infty - \partial^2_{\xi\eta}\log\theta(A(0) + z(\xi,\eta)) \quad \text{from (2.1.11)}$$

Similarly we have

$$\exp(i\,u(\xi,\eta)) = \phi_1/\phi_2 \qquad \text{from (2.1.18)},$$

$$\phi_1/\phi_2 = i\,c_{2\eta} \qquad \text{from (2.1.16)},$$

$$i\,c_{2\eta} = i\,\frac{d\,\partial_\eta \log\psi_1}{d\,1/\sqrt{\lambda}}|_{P=\infty} \qquad \text{from (2.1.6)},$$

$$i\,\frac{d\,\partial_\eta \log\psi_1}{d\,1/\sqrt{\lambda}}|_{P=\infty} = \omega_0 - \partial^2_{\xi\eta}\log\theta(z(\xi,\eta)) \quad \text{from (2.1.11)}$$

From the above two equation sets, we obtain the expressions for $u(\xi,\eta)$ of the 'Its-

Matveev' type:

$$\exp(-iu(\xi,\eta)) = C_0 - \partial^2_{\xi\eta}\log\theta(A(0)+z(\xi,\eta)) \tag{2.1.19}$$

$$\exp(+iu(\xi,\eta)) = C_\infty - \partial^2_{\xi\eta}\log\theta(z(\xi,\eta)) \tag{2.1.20}$$

where

$$\begin{aligned} z(\xi,\eta) &= -A(D) - K + \eta V - \xi U \\ C_\infty &= (\omega_0/d\sqrt{\lambda^{-1}})_{|P=\infty} \\ C_0 &= (\omega_\infty/d\sqrt{\lambda})_{|P=0} \quad \text{and} \\ C_0 &= C_\infty \end{aligned} \tag{2.1.21}$$

The equality of C_0 and C_∞ follows from the Riemann bilinear relation for the pair of second kind differentials ω_0 and ω_∞ having zero a-periods. The formula (2.1.19) for $u(\xi,\eta)$ and the first equation in (2.1.21) show that $u(\xi,\eta)$ is meromorphic in (ξ,η). We have already shown that $\exp(i\,u(\xi,\eta)) \in \mathbb{C} - \{0\}$, which implies $u(\xi,\eta)$ has no poles. We summarize the results:

Theorem 2.1.1. *We have a well defined holomorphic map $u : W \to \mathbb{C}$, $(\xi,\eta) \mapsto -i\,\log(e^{iu(\xi,\eta)})$ where $e^{iu(\xi,\eta)}$ is given by (2.1.19).*

2.1.2 θ-quotient formula for $u(x,t)$

In this section we derive a formula for $u(x,t)$ expressing it as a ratio of θ-functions. We will construct a meromorphic function of three variables $(Q,\xi,\eta) \in \Gamma \times W$ denoted $\exp(i\,u_Q(\xi,\eta))$. The SG-solutions $\exp(i\,u(\xi,\eta))$ of (2.1.19) and $\exp(-i\,u(\xi,\eta))$ of (2.1.20) will equal $\exp(i\,u_Q(\xi,\eta))$ for $Q=\infty$ and $Q=0$ respectively. We start with an elementary result that we need. It does not seem to have appeared before in the literature of hyperelliptic Riemann surfaces.

Theorem 2.1.2. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Let $A : X \to J(X)$ be the Abel map with respect to some choice of base-point $P_0 \in X$ and some choice of canonical basic cycles on X. Let K be the associated vector of Riemann constants. Let L denote the divisor class given by $P + \sigma P$ for any $P \in X$, where σ is the hyperelliptic involution of X. Let Θ_{sp} be as defined before Fact 2.1.1, and let W_{g-2} denote the subset of $J(X)$ which is the Abel image of effective divisors of degree $g-2$. Then we have*

$$\Theta_{sp} = W_{g-2} + A(L) + K \tag{2.1.22}$$

If P_0 is chosen to be a Weierstrass point then the term $A(L)$ in (2.1.22) can be dropped.

Proof. Let X be given by $\mu^2 = R_{2g+1}(\lambda)$ where the polynomial $R_{2g+1}(\lambda)$ has distinct roots. First, let $e \in W_{g-2} + A(L) + K$, then $e - K$ can be written as $A(D)$ for $D = P_1 + \cdots + P_g$ with $P_2 = \sigma P_1$. Let $P_i = (\lambda_i, \mu_i)$. Consider the holomorphic differential

$$\alpha = \frac{(\lambda - \lambda_1) \prod_{i=3}^{g} (\lambda - \lambda_i)\, d\,\lambda}{\mu}$$

Clearly α has zeros at D, whence D is special and thus $e \in \Theta_{sp}$.

Conversely, let $D = P_1 + \cdots + P_g$ be a special divisor, we will show that P_2 can be chosen to be σP_1 without loss of generality. Indeed let α be a holomorphic differential having zeros at D. The zeros of α can be written as

$$(Q_1 + \cdots + Q_m) + \sigma(Q_1 + \cdots + Q_m) + 2(g-1-m) \cdot \infty, \qquad 0 \leq m \leq g-1$$

where ∞ denotes the branch point at $\lambda = \infty$. If $D \geq 2 \cdot \infty$ then we select $P_1 = P_2 = \infty$ and we are done. If $\infty \notin D$ then $m \leq g-1$ implies that D must contain Q_i and σQ_i for some i, we set $P_1 = Q_i$ and $P_2 = \sigma Q_i$. If $D \geq \infty$ but $D \not\geq 2 \cdot \infty$, then it follows that $m \leq g-2$ and writing $D = P_1 + \cdots + P_{g-1} + \infty$ with $P_i \neq \infty$, we see that $P_1 + \cdots + P_{g-1}$ must contain Q_i and σQ_i for some i, and we set $P_1 = Q_i$ and $P_2 = \sigma Q_i$. Finally, if the basepoint P_0 is a Weierstrass point then $0 = A(2P_0) = A(L)$, and hence $A(L)$ can be dropped from (2.1.22).

$\square$

We will develop the construction of $u_Q(\xi, \eta)$ in the next Lemma.

Lemma 2.1.2.

1. *Let D be a non-special positive divisor of degree g on Γ. Then for every $Q \in \Gamma - D$, there exists a unique positive divisor D_Q of degree g satisfying $A(D_Q) = A(D) - A(Q)$. The divisor D_Q is not only non-special but also satisfies $A(D_Q) + K \notin \Theta$. This fact is true not just for Γ (2.1.1) but for any genus g compact Riemann surface.*

2. *Further assume $A(D) + K \notin \Theta$. By Lemma 2.1.1 part 2), we know that for all $(\xi, \eta) \in W \subset \mathbb{R}^2$ there exists a divisor $D(\xi, \eta)$ satisfying $A(D(\xi, \eta)) = A(D) - \eta V + \xi U$ and $A(D(\xi, \eta) + K \notin \Theta$. We claim that all $(\xi, \eta) \in W$ and all $Q \in \Gamma - D$, we have $A(D(\xi, \eta)) - A(Q) + K \notin \Theta_{sp}$. In particular there is a non-special positive divisor of degree g, which we denote $D_Q(\xi, \eta)$ satisfying $A(D_Q(\xi, \eta)) = A(D(\xi, \eta)) - A(Q) = A(D_Q) - \eta V + \xi U$.*

3. *The Baker-Akhiezer function corresponding to the divisor* D_Q *instead of* D *(for all* $Q \in \Gamma - D$*) is unique and given by (2.1.11) (with* D *replaced by* D_Q*) for all* $(\xi, \eta) \in W$*. From the uniqueness of the Baker-Akhiezer function, the equations (2.1.16-2.1.20) hold and we thus obtain the following expressions:*

$$\exp(-iu_Q(\xi, \eta)) = C_0 - \partial^2_{\xi\eta} \log \theta(A(Q) + A(0) + z(\xi, \eta)) \tag{2.1.23}$$

$$\exp(+iu_Q(\xi, \eta)) = C_0 - \partial^2_{\xi\eta} \log \theta(A(Q) + z(\xi, \eta)) \tag{2.1.24}$$

where $z(\xi, \eta) = -A(D) - K + \eta V - \xi U$ *and* C_0 *is as in (2.1.21)*

Proof.

1. Let Γ' be the embedding of Γ into $J(\Gamma)$ given by $Q \mapsto A(Q) - A(D) - K$. The non-specialty of D implies via Jacobi inversion (Fact 2.1.2, part 2) that Γ' intersects Θ in exactly the g points of D. Together with $\Theta = -\Theta$ we deduce that for all $Q \notin D$, we have $A(D) - A(Q) + K \notin \Theta$. In particular $A(D) - A(Q) + K \notin \Theta_{sp}$ and thus Jacobi inversion again implies that there is a positive divisor D_Q of degree g with $A(D_Q) = A(D) - A(Q)$. The D_Q is non-special because it even satisfies $A(D_Q) + K \notin \Theta$. We note that the proof works for any genus g compact Riemann surface, not just the spectral curve Γ.

2. Suppose $A(D(\xi, \eta)) - A(Q) + K \in \Theta_{sp}$ for some $(\xi, \eta) \in W$ and some $Q \in \Gamma - D$. We now make (essential) use of Theorem 2.1.2, to write

$$A(D(\xi, \eta)) - A(Q) = A(Q_1 + Q_2 + \cdots + Q_g) \quad \text{with } Q_2 = \sigma Q_1.$$

Now, we use that the fact that $A(\sigma P) = -A(P)$ which follows from the fact that $A(\sigma P + P)$ is independent of the point $P \in \Gamma$, and using $P = \infty$ (the base-point of the Abel map) we get $A(\sigma P + P) = 2A(\infty) = 0$. The above equation can thus be rewritten as

$$A(D(\xi, \eta)) = A(\infty + Q + Q_3 + \cdots + Q_g).$$

The above equation immediately implies that $A(D(\xi, \eta)) + K \in \Theta$ by Fact 2.1.1 part 1, and this contradicts the fact that $A(D(\xi, \eta) + K \notin \Theta$. Thus $A(D(\xi, \eta)) - A(Q) + K \notin \Theta_{sp}$ and by Jacobi inversion, there is a non-special divisor $D_Q(\xi, \eta)$ satisfying $A(D_Q(\xi, \eta)) = A(D(\xi, \eta)) - A(Q)$

3. We recall (from [5]) that the construction of Baker-Akhiezer functions requires the divisor D to be only non-special (even though we required in (2.1.1) further the condition that $0, \infty \notin D$). We denote the Baker-Akhiezer function ψ corresponding to the non-special divisor D_Q instead of D (for $Q \in \Gamma - D$) by $\psi_{1Q}(P, \xi, \eta)$. The non-specialty of $D_Q(\xi, \eta)$ (part 2 above) implies that $\psi_{1Q}(P, \xi, \eta)$ is uniquely determined (Fact 2.1.2) by formula (2.1.11) (with D replaced by D_Q). The formulas (2.1.16-2.1.20) hold because they are derived only using the uniqueness of ψ_1. We thus obtain the expressions (2.1.23-2.1.24)

□

The expressions $\exp(-i\, u_Q(\xi, \eta))$ and $\exp(i\, u_Q(\xi, \eta))$ (2.1.23-2.1.24) are meromorphic functions of $(Q, \xi, \eta) \in (\Gamma - D) \times W$ and are reciprocals of each other there. However the right hand sides of these expressions are meromorphic on all of $\Gamma \times W$.

In fact, it is clear from (2.1.24) that $\exp(i\, u_Q(\xi,\eta))$ has poles at $2D(\xi,\eta)$. Similarly from (2.1.23) we see that $\exp(-iu_Q(\xi,\eta))$ has poles at $2D_0(\xi,\eta)$. The divisor $D_0(\xi,\eta)$ is defined because from part 2) above we know that $D_Q(\xi,\eta)$ exists for all $Q \in \Gamma - D$, and we have assumed in (2.1.1) that $0 \notin D$. Now the fact that $\exp(-iu_Q(\xi,\eta))$ and $\exp(i\, u_Q(\xi,\eta))$ are reciprocal to each other, it follows that the meromorphic (in Q) function $\exp(i\, u_Q(\xi,\eta))$ has divisor :

$$(\, \exp(i\, u_Q(\xi,\eta))\,) = 2\, D_0(\xi,\eta) - 2\, D(\xi,\eta) \tag{2.1.25}$$

However, upto a multiplicative constant, such a function is uniquely determined. The meromorphic function of Q defined by the θ-quotient formula below has the same divisor as (2.1.25). Therefore we conclude:

$$\exp(i\, u_Q(\xi,\eta)) = C_1\, \frac{\theta(A(0)+A(Q)+z(\xi,\eta))\, \theta(-A(0)+A(Q)+z(\xi,\eta))}{\theta^2(A(Q)+z(\xi,\eta))} \tag{2.1.26}$$

In claiming that (2.1.26) has zeros at $2\, D_0(\xi,\eta)$ we have used the fact that $A(2\cdot 0) = A(L) = A(2\cdot\infty) = 0$ (see Theorem 2.1.2 for definition of L), and therefore $A(0)$ is a half-period. We set

$$A(0) = \frac{1}{2}(\epsilon' + B\epsilon) \tag{2.1.27}$$

for some integer vectors ϵ, ϵ'. Next, we determine the constant C_1. By Replacing the divisor D with D_0, the function $\exp(i\, u_Q(\xi,\eta))$ turns into $\exp(-i\, u_Q(\xi,\eta))$. Applying this to (2.1.26) we obtain:

$$\exp(-i\,u_Q(\xi,\eta)) = C_1\,\frac{\theta(2A(0)+A(Q)+z(\xi,\eta))\,\theta(A(Q)+z(\xi,\eta))}{\theta^2(A(0)+A(Q)+z(\xi,\eta))}$$

$$= C_1^2\,\exp(-\pi\,i\epsilon^t(B\epsilon+\epsilon'))\,\frac{\theta^2(A(Q)+z(\xi,\eta))}{C_1\,\theta(A(0)+A(Q)+z(\xi,\eta))\,\theta(-A(0)+A(Q)+z(\xi,\eta))}$$

$$= C_1^2\,\exp(-\pi i\,\epsilon^t(B\epsilon+\epsilon'))\,[\exp(i\,u_Q(\xi,\eta))]^{-1} \tag{2.1.28}$$

Thus (2.1.27) implies that:

$${C_1}^2 = (-1)^{\epsilon^t\epsilon'}\,\exp(\pi i\,\epsilon^t B\epsilon) \quad \text{or} \quad C_1 = (-1)^{\epsilon^t\epsilon'/2}\,\exp(\frac{\pi i}{2}\,\epsilon^t B\epsilon)$$

Further, the function $\exp(iu_Q(\xi,\eta))$ depends only on Q, D. Therefore the right hand side of (2.1.26) must be invariant under $\epsilon' \mapsto \epsilon' + 2m$ for an integer vector m. This forces:

$$C_1 = \left\{\begin{matrix} \pm 1 & \text{if} & \epsilon^t\epsilon' \equiv 0\,(\text{mod}\,2) \\ \pm i & \text{if} & \epsilon^t\epsilon' \equiv 1\,(\text{mod}\,2) \end{matrix}\right\} \tag{2.1.29}$$

Thus C_1 is determined upto sign. We now specialize (2.1.26) to the case $Q = \infty$ and also switch to the (x,t) variables.

Theorem 2.1.3. *The finite-gap SG-solution $u(x,t)$ associated with the divisor D are given by the θ-quotient formula*

$$\exp(i\,u(\xi,\eta)) = C_1\frac{\theta(A(0)+z(\xi,\eta))\,\theta(-A(0)+z(\xi,\eta))}{\theta^2(z(\xi,\eta))} \quad \text{or}$$

$$\exp(i\,u(x,t)) = C_1\frac{\theta(A(0)+z(x,t))\,\theta(-A(0)+z(x,t))}{\theta^2(z(x,t))} \quad \text{where}$$

$$z(\xi,\eta) = -A(D) + \eta V - \xi U - K \qquad \text{or}$$

$$z(x,t) = -A(D) + x(V-U)/4 - t(U+V)/4 - K \quad \text{and} \tag{2.1.30}$$

$$C_1 = \left\{\begin{matrix} \pm 1 & \text{if} & \epsilon^t\epsilon' \equiv 0\,(\text{mod}\,2) \\ \pm i & \text{if} & \epsilon^t\epsilon' \equiv 1\,(\text{mod}\,2) \end{matrix}\right\}, \quad A(0) = \frac{1}{2}(\epsilon' + B\epsilon)$$

2.1.3 Trace formula for $u(x,t)$

In this section we derive the 'trace-formula' for the finite-gap SG solution $u(x,t)$ corresponding to the divisor D (2.1.1). We recall from Lemma 2.1.2 (see proof of part 2)), that for $(\xi,\eta)\in W$, we have $0,\infty\notin D(\xi,\eta)$ and $D(\xi,\eta)$ is non-special. We write $D(\xi,\eta)=(\lambda_1(\xi,\eta),\mu_1(\xi,\eta))+\cdots+(\lambda_g(\xi,\eta),\mu_g(\xi,\eta))$.

Lemma 2.1.3.

1. *There is a unique polynomial $P_{g-1}(\lambda)=P_{g-1}(\lambda,\xi,\eta)$ of degree $g-1$ interpolating the points $(\lambda_i(\xi,\eta),\mu_i(\xi,\eta)/\lambda_i(\xi,\eta))$ for $1\le i\le g$, i.e.,*

$$P_{g-1}(\lambda_i(\xi,\eta))=\frac{\mu_i(\xi,\eta)}{\lambda_i(\xi,\eta)}.$$

2. *There is a meromorphic function on Γ, $f(Q)=f(Q,\xi,\eta)$, unique upto a fourth root of unity , satisfying $f^2(Q)=\lambda\exp(i\,u_Q(\xi,\eta))$, where $Q=(\lambda,\mu)$. It is given by:*

$$f(\lambda,\mu)=\left(\frac{\prod_{i=1}^{g}(-\lambda_i(\xi,\eta))}{\sqrt{E_1E_2\cdots E_{2g}}}\right)^{1/2}\frac{\lambda P_{g-1}(\lambda)+\mu}{\prod_{i=1}^{g}(\lambda-\lambda_i(\xi,\eta))} \tag{2.1.31}$$

3. *We obtain another formula for the function $\exp(i\,u_Q(\xi,\eta))$, uniquely determined upto sign*

$$\exp(i\,u_Q(\xi,\eta))=\frac{\prod_{i=1}^{g}(-\lambda_i(\xi,\eta))}{\sqrt{E_1E_2\cdots E_{2g}}}\,\frac{(\lambda P_{g-1}(\lambda)+\mu)^2}{\lambda\prod_{i=1}^{g}(\lambda-\lambda_i(\xi,\eta))^2} \tag{2.1.32}$$

Proof.

1. We note that $0,\infty\notin D$, therefore μ_i/λ_i are meaningful (finite). Because D is non-special, Theorem 2.1.2 implies that if $(\lambda_i,\mu_i)\in D$ then $(\lambda_i,-\mu_i)\notin D$.

Therefore we get $\lambda_i \neq \lambda_j$ if $i \neq j$. The desired polynomial $P_{g-1}(\lambda)$ is simply the Lagrange interpolation polynomial for the points g points $(\lambda_i, \mu_i/\lambda_i)$, which is unique because the data points are distinct.

2. The divisor of $\lambda \exp(i\, u_Q(\xi,\eta)$ is $2\,D(\xi,\eta) - 2\,D_0(\xi,\eta) + 2\cdot 0 - 2\cdot\infty$ which can be written as $2\cdot\Delta$ for the divisor $\Delta = (D(\xi,\eta)+0) - (D_0(\xi,\eta)+\infty)$. Clearly $A(\Delta) = 0$, whence by Abel's theorem, there is a meromorphic function f, unique upto a multiplicative constant, with divisor Δ. We simply write an expression of f (using elementary facts about the function field of the hyperelliptic curve Γ):

$$f(\lambda,\mu) = C_2 \frac{\lambda P_{g-1}(\lambda) + \mu}{\prod_{i=1}^{g}(\lambda - \lambda_i(\xi,\eta))}$$

Observing that

$$(f^2/\lambda)|_{Q=0} = (\lambda/f^2)|_{Q=\infty} \qquad (= \exp(-i\, u(\xi,\eta)))$$

we determine C_2 to satisfy

$$C_2^2 = \frac{\prod_{i=1}^{g}(-\lambda_i(\xi,\eta))}{\sqrt{E_1 E_2 \cdots E_{2g}}}$$

3. Part 3), follows from part 2) and the definition $\exp(i\, u_Q(\xi,\eta)) = f^2/\lambda$

□

Observing that the solution $\exp(i\, u(\xi,\eta)$ is obtained as the special case of $\exp(i\, u_Q(\xi,\eta))$ for $Q = \infty$, we obtain the trace-formula.

Theorem 2.1.4. *Trace-formula for $u(\xi,\eta)$*

The finite-gap solution $u(\xi,\eta)$ corresponding to the divisor D, is given (upto sign) *in terms of* $D(\xi,\eta) = (\lambda_1(\xi,\eta),\mu_1(\xi,\eta)) + \cdots + (\lambda_g(\xi,\eta),\mu_g(\xi,\eta))$ *by*

$$u(\xi,\eta) = \frac{\prod_{i=1}^{g}(-\lambda_i(\xi,\eta))}{\sqrt{E_1E_2\cdots E_{2g}}} \tag{2.1.33}$$

2.2 Real solutions

In order to obtain real finite-gap solutions $u(x,t)$, we must impose some conditions on the spectral data. The reality condition on Γ (found in [20], [22]) is:

$$\{\overline{E_1},\cdots,\overline{E_{2g}}\} = \{E_1,\cdots,E_{2g}\} \quad \text{and} \quad E_i \in \mathbb{R} \Rightarrow E_i < 0 \tag{2.2.1}$$

Let $2m$ denote the number of real E_i. We order the real branch points as $0 > E_1 > E_2 > \cdots > E_{2m}$ and also assume $E_{2i} = \overline{E_{2i-1}}$ for $m+1 \leq i \leq g$. The reality condition on the divisor found by Cherednik [3] is

$$D + \tau D - 0 - \infty \sim \mathcal{K} \tag{2.2.2}$$

where τ is the anti-holomorphic involution of Γ given by

$$\tau : (\lambda,\mu) \mapsto (\bar{\lambda},\bar{\mu})$$

and $\mathcal{K}$ is the canonical class (i.e. the divisor class of meromorphic differentials), and $\sim$ denotes the relation of linear equivalence of divisors.

Definition 2.2.1. *A positive divisor D of degree g will be called* **admissible** *if it satisfies the condition (2.2.2)*

It was shown in the works [3], [7], [12] that the image in $J(\Gamma)$ of the admissible divisors under the Abel map, consists of 2^m components each of which is a real

g-dimensional torus. We use here a concrete description of these tori, similar to the one in [7]. We choose a special basis of cycles $\{a_1, b_1, \cdots, a_g, b_g\}$ suggested in [8], and depicted in Fig. 2.1 (where the parameter k must be taken to be 1). The picture shows the λ-plane with the thick-dashed lines representing the system of cuts, and the transition from solid to dashed line in the cycles $\{b_{m+1}, \cdots, b_g\}$ indicating change of sheet across a cut.

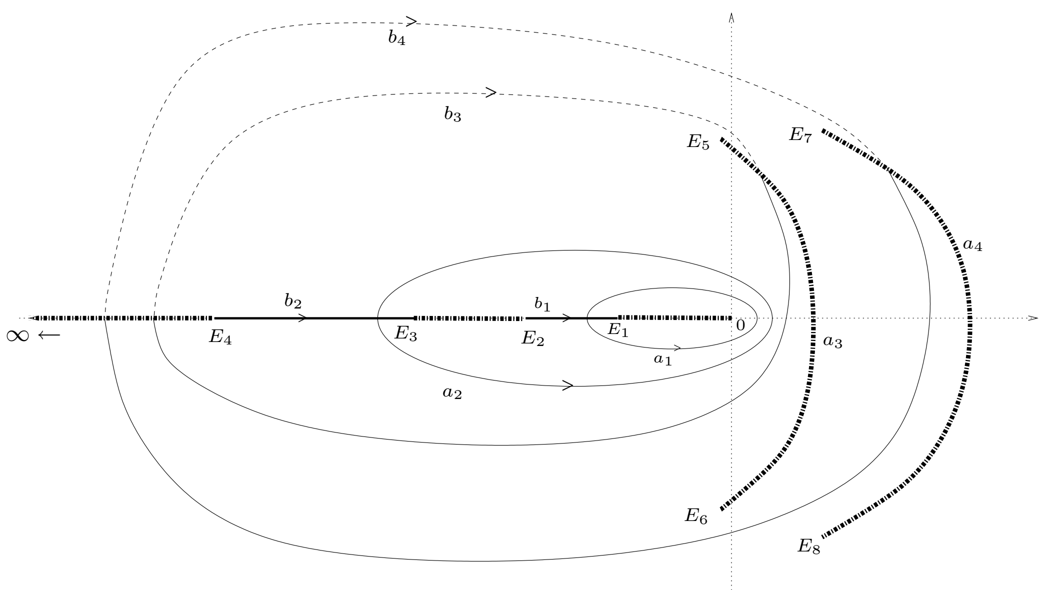

Figure 2.1: Basic cycles and cuts on Γ (shown for $g = 4, m = 2$)

The action of τ on $H_1(\Gamma, \mathbb{Z})$ is given by

$$\begin{aligned} \tau a_i &= -a_i \qquad 1 \le i \le g \\ \tau b_i &= b_i \qquad 1 \le i \le m \\ \tau b_i &= b_i + a_i \quad m+1 \le i \le g \end{aligned} \tag{2.2.3}$$

This immediately implies that the effect of τ on the holomorphic differentials is given by $-\omega_j = \overline{\tau^*\omega_j}$. Therefore

$$A(\tau P) = -\overline{A(P)} \tag{2.2.4}$$

$$-\bar{B} = B + \begin{pmatrix} 0 & 0 \\ 0 & I_{g-m} \end{pmatrix} \quad \text{or} \quad \mathrm{Re}(B) = -1/2 \begin{pmatrix} 0 & 0 \\ 0 & I_{g-m} \end{pmatrix} \tag{2.2.5}$$

where I_{g-m} is the identity matrix of size $g-m$. (From here on, all vectors v written as $(v_1, v_2)^t$ are understood to be split into blocks of length m and $g-m$).

In order to describe the real tori, we need to calculate $A(\mathcal{K})$ and $A(0)$. The divisor $(d\lambda/\mu) = (2g-2)\cdot\infty$ is canonical hence we obtain

$$A(\mathcal{K}) = 0.$$

Let $2\pi i\,\alpha$ and $2\pi i\,\beta$ denote the vectors of a and b-periods of the differential

$$\omega = \frac{1}{2}\,d\log\lambda$$

From the bilinear relations of Riemann applied to the pair of differentials ω, ω_i for $1 \le i \le g$, we immediately obtain

$$(\beta - B\alpha)/2 = A(0)$$

The numbers α_j and β_j are the winding numbers of the projection $\lambda(a_j)$ and $\lambda(b_j)$ around $\lambda = 0$, and are read off from Fig. 2 to be

$$\alpha = (1,0)^t \quad \text{and} \quad \beta = (0,1)^t$$

Thus we obtain:

$$A(\mathcal{K}) = 0 \tag{2.2.6}$$

$$A(0) = \frac{1}{2}\,\epsilon' + \frac{1}{2}\,B\epsilon, \quad \epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \epsilon' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2.7}$$

Every $z \in J(\Gamma)$ can be written as

$$z = x + B\,y \qquad \text{for unique vectors } x, y \in \mathbb{R}^g/\mathbb{Z}^g$$

We will denote by T^g the subset of $J(\Gamma)$ given as

$$T^g = \mathbb{R}^g + B \cdot 0 \subset J(\Gamma)$$

Clearly $T^g \subset J(\Gamma)$ is isomorphic to the real g-dimensional torus $\mathbb{R}^g/\mathbb{Z}^g$. Applying the Abel map to the admissibilty condition (2.2.2) and using (2.2.4), (2.2.6) and (2.2.7) we obtain:

$$A(D) = x + B\begin{pmatrix} s/4 \\ 1/2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ s_2 \\ \cdot \\ s_m \end{pmatrix} \tag{2.2.8}$$
$$\text{for some} \quad x \in T^g \quad \text{and} \quad (s_1, \cdots, s_m) \in \{-1, 1\}^m$$

For each of the 2^m collection of symbols $(s_1, \cdots, s_m) \in \{-1, 1\}^m$, let T_s denote the subset of $J(\Gamma)$ consisting of elements of the form $-A(D) - K$ where $A(D)$ is as in (2.2.8) and K is the vector of Riemann constants.

$$T_s = -T^g - B\begin{pmatrix} s/4 \\ 1/2 \end{pmatrix} - K \tag{2.2.9}$$

The symbol s will be called the **topological type** of the admissible divisor D. In the sequel, the terminology real tori will refer to the T_s. We collect the essential facts (from [3]) about the real tori in the following lemma.

Lemma 2.2.1. *Real Tori*

1. *The real tori T_s do not intersect the divisor $\Theta \cup (\Theta + A(0))$. Therefore by Lemma 2.1.1 part 1), it follows that the admissible divisors are non-special and do not contain $0, \infty$.*

2. *Let D be an admissible divisor of topological type s, and let*

$$z(x,t) = -A(D) + x(V-U)/4 - t(U+V)/4 - K$$

Then $z(x,t) \in T_s$ for all x,t.

3. *The real solutions $u(x,t)$ are non-singular for all x,t. In other words the set W defined in Lemma 2.1.1 part 2), is all of $\mathbb{R}^2$ for admissible divisors.*

Proof.

1. Using the Fact 2.1.1 part 1), together with the fact $\Theta = -\Theta$, it follows that for a divisor D, the condition $-A(D) - K \in \Theta \cup (\Theta + A(0))$ is equivalent to $D \sim \gamma_1 + \cdots + \gamma_g$ for some γ_i with $\gamma_g \in \{0, \infty\}$. Suppose such a divisor D is also admissible, i.e. $D + \tau D - 0 - \infty \sim \mathcal{K}$. Since the Cherednik condition depends only on the divisor class of D, we may assume $D = \gamma_1 + \cdots + \gamma_{g-1} + \gamma_g$ with $\gamma_g \in \{0, \infty\}$. Let $D_1 = \gamma_1 + \gamma_2 + \cdots + \gamma_{g-1}$. The admissibility condition can be rewritten as

$$D_1 + \tau D_1 \pm (0 - \infty) = (\alpha) \quad \text{for some abelian differential } \alpha \tag{2.2.10}$$

If the differential α is holomorphic then (2.2.10) implies that it has an odd number of zeros at ∞. However every holomorphic differential on Γ has an even number of zeroes at ∞. Therefore α is not holomorphic and (2.2.10) now implies that α has a single pole, contradicting the residue theorem. Thus T_s does not intersect $\Theta \cup (\Theta + A(0))$.

2. The second assertion follows from the first as soon as we show that the vectors U, V defined by (2.1.4) are purely real. It is easy to see from the definition

of ω_0, ω_∞ that they are fixed by the involution $\omega \mapsto \overline{\tau^* \omega}$. Equation (2.1.4) together with the fact that $\tau b_i = b_i \bmod \mathrm{a_i}$, now implies that U and V are purely real.

3. The formula (2.1.30) for $u(x,t)$ shows that it is nonsingular as long as $z(x,t) \notin \Theta \cup (\Theta + A(0))$. Thus the first two assertions imply the third one.

□

To conclude this section, we calculate the vector of Riemann constants K for the chosen base point of Abel map ($P = \infty$), and the choice of basic cycles. A formula for K is well-known ([13], pp 325) when the base point of Abel map is a branch point of a 2-sheeted cover $\lambda : \Gamma \to \mathbb{P}^1$. It equals $\sum_i A(P_i)$ where the sum is over those branch points P_i for which $A(P_i)$ is an odd half-period (i.e., of the form $(n+Bm)/2$ with the scalar product $n^t m$ being an odd integer). In the present case, we obtain:

$$K = A(E_1) + A(E_3) + \cdots + A(E_{2g-1}) \tag{2.2.11}$$

Let $2\pi i\, \alpha'$ and $2\pi i\, \beta'$ be the vectors of a and b-periods of the meromorphic differential

$$\tilde{\omega} = \frac{1}{2}\, d \,\log \prod_{i=1}^{g} (\lambda - E_{2i-1})$$

which is a differential of the third kind having residues $+1$ at $E_1, E_3, \cdots, E_{2g-1}$ and residue $-g$ at ∞. From the bilinear relations of Riemann applied to the pair of differentials $\omega_i, \tilde{\omega}$ for $1 \leq i \leq g$, we obtain

$$(\beta' - B\alpha')/2 = A(E_1) + A(E_3) + \cdots + A(E_{2g-1}) = K$$

The numbers α_i' and β_i' being $\frac{1}{2\pi i}\int_{a_i}\tilde{\omega}$ and $\frac{1}{2\pi i}\int_{b_i}\tilde{\omega}$ respectively are equal to the sum of the winding numbers about $E_1, E_3, \cdots, E_{2g-1}$ of the projected curves $\lambda(a_i)$ and $\lambda(b_i)$ respectively. The winding numbers can be read off from Fig. 2.1. We obtain:

$$K = \frac{1}{2}\begin{pmatrix} 1 \\ \nu_2 \end{pmatrix} + \frac{1}{2}\, B \begin{pmatrix} \nu_1 \\ 1 \end{pmatrix} \quad \text{where} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \cdot \\ g \end{pmatrix} \tag{2.2.12}$$

Chapter 3

Calculation of Topological Charge

3.1 Setup

We recall some definitions and formulae from Chapter 1 and Chapter 2. The definition of topological charge density $\bar{n}$ from (1.1.4) is:

$$\bar{n} = \lim_{T\to\infty} \frac{u(x+T,t) - u(x,t)}{2\pi T} \tag{3.1.1}$$

The θ-functional formula for the finite-gap solution $e^{iu(x,t)}$ associated with the spectral data (Γ, D) is, from (2.1.30)

$$\begin{aligned} e^{iu(x,t)} &= C_1 \frac{\theta(A(0)+z(x,t))\,\theta(-A(0)+z(x,t))}{\theta^2(z(x,t))} \quad \text{where} \\ z(x,t) &= -A(D) - x(U-V)/4 - t(U+V)/4 - K \end{aligned} \tag{3.1.2}$$

We recall from Chapter 2, the definition of the real tori T_s, for each of the 2^m collection of symbols $(s_1, \cdots, s_m) \in \{-1, 1\}^m$:

$$T_s = -T^g - B\begin{pmatrix} s/4 \\ 1/2 \end{pmatrix} - K \quad \text{where} \quad T^g = \mathbb{R}^g + B\cdot 0 \subset J(\Gamma) \tag{3.1.3}$$

Definition 3.1.1. *Topological type*

An admissible divisor D has **topological type** *s, if $-A(D) - K \in T_s$ (3.1.3)*

Notational Remark : As in Chapter 2, all vectors written in the form $(v_1, v_2)^t$ are understood to be split into blocks of length m and $g-m$. The vectors written simply as 1 (resp. 0) have all components 1 (resp. 0).

Let $\phi_s : \mathbb{R}^g/\mathbb{Z}^g \to S^1 \subset \mathbb{C}$, be defined by:

$$\phi_s(v) = C_1 \frac{\theta(A(0) - v - A(D) - K)\,\theta(-A(0) - v - A(D) - K)}{\theta^2(-v - A(D) - K)}, \quad v \in \mathbb{R}^g/\mathbb{Z}^g \tag{3.1.4}$$

The solution $e^{iu(x,t)} : T_s \to S^1$ is given by $\phi_s(x(U - V)/4 + t(U + V)/4)$. Let $\phi_{s*} : H_1(\mathbb{R}^g/\mathbb{Z}^g, \mathbb{R}) \to H_1(S^1, \mathbb{R})$ be given by the matrix $[n_1, n_2, \cdots, n_g]$ with respect to the standard basic 1-cycles $\{e_1, \cdots, e_g\}$ of $H_1(\mathbb{R}^g/\mathbb{Z}^g, \mathbb{Z})$ and standard basic cycle $[S^1]$ of $H_1(S^1, \mathbb{Z})$. The integers n_j can be computed as:

$$n_j = \frac{1}{2\pi i} \int_{T=0}^{T=1} d \log \phi_s(Te_j) \tag{3.1.5}$$

Definition 3.1.2. *Basic charges*

The **basic charge** n_j *is given by (3.1.5). (It is the j-th entry of the matrix $[n_1, \cdots, n_g]$ defined above).*

The topological charge density $\bar{n}$ (3.1.1) has the following simple relation to the integer basic charges n_j ([17], Lemma 2.4).

$$\bar{n} = \sum_{j=1}^{g} (U_j - V_j)\, n_j/4 \tag{3.1.6}$$

Intuitively, $\bar{n}$ equals $\phi_{s*}((U - V)/4)$ where $(U - V)/4 \in H_1(T_s, \mathbb{R})$ is the 'direction' of the straight line in T_s along which the x-dynamics evolves.

If D and D' are two admissible divisors in the same real torus T_s, then the cycles $-A(D) - Te_j - K$ and $-A(D') - Te_j - K$ for $0 \le T \le 1$ are homologous in T_s. (since one cycle is obtained from the other by translation in the torus T_s). Thus the charges n_j depend only on the topological type s of the admissible divisor. Thus we may compute n_j using (3.1.4-3.1.5) with $A(D) = B \binom{s/4}{1/2}$

It will also be convenient to replace $\epsilon = (1,0)^t$ in the expression $A(0) = \epsilon'/2 + B\epsilon/2$ (2.2.7) with $\tilde{\epsilon}$ defined by

$$\tilde{\epsilon}_j = \begin{Bmatrix} (-1)^j s_j & \text{if} \quad 1 \le j \le m \\ 0 & \text{if} \quad j > m \end{Bmatrix} \tag{3.1.7}$$

From the transformation rule (2.1.3)

$$\theta(z + BM) = \theta(z)\, \exp(-2\pi i\, M^t z - \pi i\, M^t BM)$$

of the θ-function, we see that this only changes the constant C_1 appearing in (3.1.4). However the constant C_1 does not affect n_j, as the formula (3.1.5) shows. Therefore we may calculate n_j using $\tilde{\epsilon}_j$ in the formula for $A(0)$. We take the new expression for $A(0)$ in (3.1.4) and apply the θ-transformation rule above, and use the resulting expression for $\phi_s(v)$ in the formula (3.1.5), to obtain:

$$n_j = -\tilde{\epsilon}_j + \frac{2}{2\pi i} \int_{T=0}^{T=1} d\log\left(\frac{\theta(B\tilde{\epsilon}/2 - Te_j - K - B\binom{s/4}{1/2})}{\theta(-Te_j - K - B\binom{s/4}{1/2})} \right) \tag{3.1.8}$$

3.2 Topological charge in two special cases

In this section, we calculate n_j using formula (3.1.8) in two special cases. We recall from Section 2.2 that $2m$ is the number of negative real branch points of Γ. The number of ovals of the real hyperelliptic curve Γ is $m+1$.

Lemma 3.2.1. *Basic charges when Γ has only one real oval.*

If the real hyperelliptic curve $\Gamma : \mu^2 = \lambda \prod_{i=1}^{2g}(\lambda - E_i)$ has no real branch points, i.e. $m = 0$, then the basic charges are all zero, $n_j = 0$ for all j. The topological charge density $\bar{n}$ is also zero.

Proof. Since $m = 0$, there is only one real torus. The quantity $\tilde{\epsilon}$, which was defined as $\tilde{\epsilon}_j = (-1)^j s_j$ for $1 \le j \le m$ and $\tilde{\epsilon}_j = 0$ for $j > m$, is clearly the zero vector. Looking at formula (3.1.8) for n_j, we see that both terms in the right side of this equation are zero because $\tilde{\epsilon} = 0$. The formula (3.1.6) for $\bar{n}$ shows that $\bar{n}$ also equal zero in this case. $\square$

Next, we consider genus one curves

$$\Gamma : \mu^2 = \lambda(\lambda - E_1)(\lambda - E_2),$$

with $E_1, E_2 < 0$. In other words, $m = 1$. There are two real tori characterized by the symbol $s_1 \in \{+1, -1\}$. We denote the 1×1 Riemann matrix B by τ. The vector of Riemann constants $K = (1 + \tau)/2$ using formula (2.2.12). The formula (3.1.8) can be rewritten as:

$$n_1 = s_1 + \frac{2}{2\pi i} \int_{T=0}^{T=1} d\log\left(\frac{\theta(-T - (1+\tau)/2 - s_1\tau/4 - s_1\tau/2)}{\theta(-T - (1+\tau)/2 - s_1\tau/4)}\right) \tag{3.2.1}$$

Lemma 3.2.2. *Basic charges when Γ is an elliptic curves with 2 real ovals.*

1. *For the case $g = m = 1$, the basic charge $n_1 = s_1$.*

2. *For later use, we also calculate a related quantity $\tilde{n}_1$ given by:*

$$\tilde{n}_1 = -s_1 + \frac{2}{2\pi i} \int_{T=0}^{T=1} d\log\left(\frac{\theta(-T - 1/2 - s_1\tau/4 + s_1\tau/2)}{\theta(-T - 1/2 - s_1\tau/4)}\right) \tag{3.2.2}$$

   We have $\tilde{n}_1 = -s_1$.

Proof. Let $R \subset \mathbb{C}$ be the region defined by

$$R = \{z \in \mathbb{C} \,|\, -\tau/4 \le \mathrm{Im}(z) \le \tau/4, -1 \le \mathrm{Re}(z) \le 0\}$$

Let N_R denote the number of zeros of $\theta(z) = \theta(z|\tau)$ in the region R. Since the elliptic theta function $\theta(z|\tau)$ vanishes if and only if $z \equiv (1+\tau)/2$ and the region R does not contain any such z, it follows that $N_R = 0$. From elementary complex analysis, it follows that the the integral term in (3.2.1) is equal to $2s_1 N_R$ and the integral term in (3.2.2) is equal to $-2s_1 N_R$. From $N_R = 0$, it follows that $n_1 = s_1$ and $\tilde{n}_1 = -s_1$. $\square$

3.3 Multiscale Limit

We consider a family of hyperelliptic curves $\Gamma(k)$ depending on the real parameter $k \in [1, \infty)$:

$$\Gamma(k) : \mu^2 = \lambda \prod_{i=1}^{m} (\lambda - k^{i-1} E_{2i-1})(\lambda - k^{i-1} E_{2i}) \prod_{i=2m+1}^{2g} (\lambda - k^m E_i)$$

We note that for $k = 1$ the curve $\Gamma(k)$ is just $\Gamma : \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - E_i)$. We note that the $2g + 2$ points on the Riemann sphere given by

$$\{0, \infty\} \cup \{k^{i-1} E_{2i-1}, k^{i-1} E_{2i} \,|\, 1 \le i \le m\} \cup \{k^m E_i \,|\, 2m+1 \le i \le 2g\}$$

are distinct and are mapped to themselves under complex conjugation. Therefore $\Gamma(k)$ is a nonsingular real hyperelliptic curve of the form (2.2.1) for all $k \in [1, \infty)$. The basic cycles $\{a_1(k), b_1(k), \cdots, a_g(k), b_g(k)\}$ and the system of cuts on $\Gamma(k)$ are depicted in Fig. 3.1. They are chosen as follows. If $\Pi_j(k)$ for $m+1 \le j \le g$ denotes the cut on the λ-plane joining the complex conjugate branch points $k^m E_{2j-1}$ and $k^m E_{2j}$, then we require $\Pi_j(k) = k^m \Pi_j(1)$. (The basic cycle and cuts on $\Gamma(1)$ are shown in Fig. 2.1) The remaining cuts lie on the negative real line of the λ-plane and are obvious from Fig. 3.1. As indicated in Fig. 3.1, the cycles $a_j(k), b_j(k)$ are completely determined by their projections $\lambda(a_j(k)), \lambda(b_j(k))$ to the λ-plane. We specify the latter by:

$$\lambda_j(a_j(k)) = \lambda(a_j(1)), \quad \lambda_j(b_j(k)) = \lambda(b_j(1))$$

$$\text{where } \lambda_j = \left\{ \begin{matrix} \lambda/k^{j-1} & \text{if} \quad j \leq m \\ \lambda/k^{m} & \text{if} \quad j > m \end{matrix} \right\} \text{ and } \lambda : \Gamma(k) \to \mathbb{P}^1 \text{ are projections} \tag{3.3.1}$$

where we have introduced rescaled versions λ_j of the coordinate function λ.

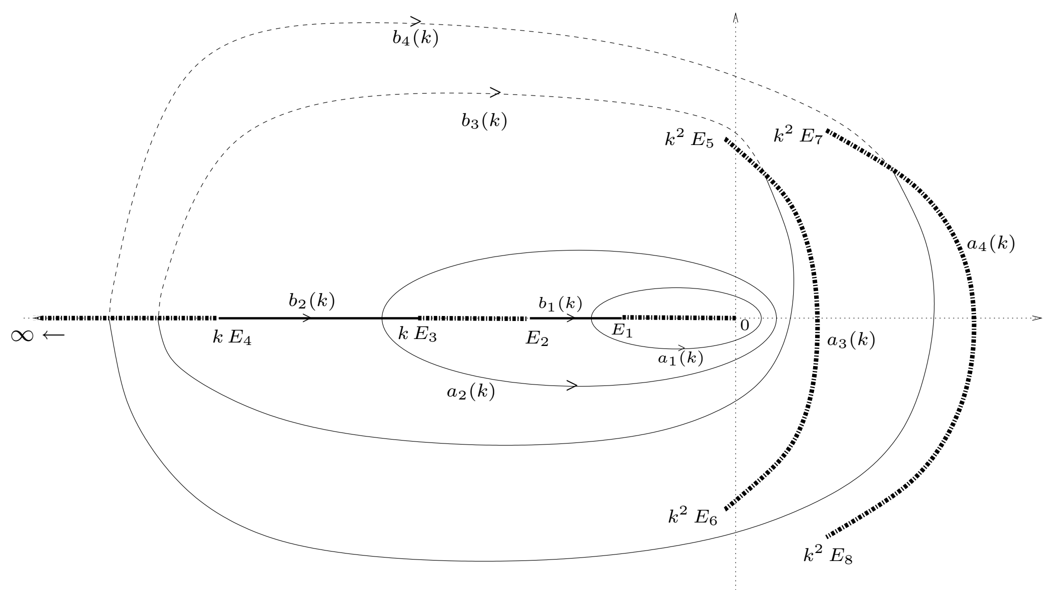

Figure 3.1: Basic cycles and cuts on $\Gamma(k)$ (shown for $g = 4, m = 2$)

We will use the notation $\Gamma(k)^+$ to denote the sheet for which $\mu > 0$ when $\lambda > 0$.

We also associate with the deformation $\Gamma(k)$, m elliptic curves $\mathcal{C}_1, \cdots, \mathcal{C}_m$ and a hyperelliptic curve $\mathcal{C}_{m+1}$.

$$\begin{aligned} \mathcal{C}_j : y_j^2 &= P_j(x_j) = x_j(x_j - E_{2j-1})(x_j - E_{2j}), \quad 1 \leq j \leq m \\ \mathcal{C}_{m+1} : y_{m+1}^2 &= P_{m+1}(x_{m+1}) = x_{m+1} \prod_{i=2m+1}^{2g} (x_{m+1} - E_i). \end{aligned} \tag{3.3.2}$$

The system of cuts on these curves is shown in Fig. 3.2. The cuts decompose each $\mathcal{C}_j$ into two sheets. By the sheet $\mathcal{C}_j^+$ we will mean the sheet for which $y_j > 0$ when $x_j > 0$.

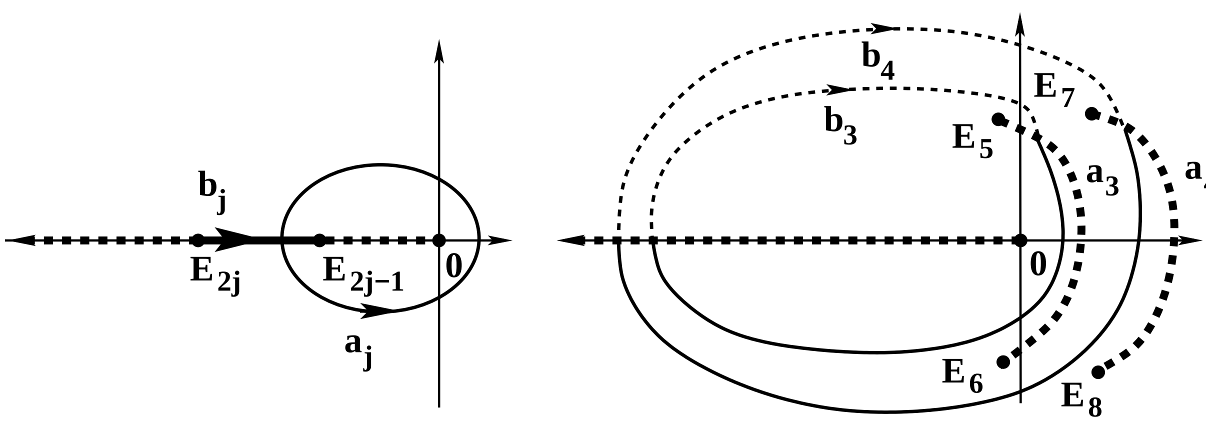

Figure 3.2: Basic cycles and cuts on $\mathcal{C}_j$, $1 \leq j \leq m$, and $\mathcal{C}_{m+1}$ (for $g = 4$, $m = 2$)

Let $B(k)$ denote the Riemann matrix of $\Gamma(k)$ with respect to this choice of basic cycles. We will be concerned with the limit of the pair $(\Gamma(k), B(k))$ as $k \to \infty$. To this end, we decompose $\Gamma(k)$ into $m+1$ open sets R_j, and we also introduce certain rescaled versions of the coordinate function μ on $\Gamma(k)$:

$$\begin{aligned} R_1 &= \{(\lambda, \mu) \in \Gamma(k) \mid 0 \leq |\lambda_1| < k^{2/3}\} \\ R_j &= \{(\lambda, \mu) \in \Gamma(k) \mid k^{-1/2} < |\lambda_j| < k^{2/3}\}, \quad \text{for } 2 \leq j \leq m \\ R_{m+1} &= \{(\lambda, \mu) \in \Gamma(k) \mid k^{-1/2} < |\lambda_{m+1}| \leq \infty\} \end{aligned} \tag{3.3.3}$$

$$\mu_j = \left\{ \begin{array}{cc} \mu / \Big(\sqrt{k}^{3(j-1)} \, k^{(j+\cdots+m-1)} \, k^{m(g-m)} \, \lambda^{j-1} \, (\prod_{l=2j+1}^{2g} E_l)^{0.5} \Big) & \text{if} \quad 1 \le j \le m \\ \mu / \Big(\sqrt{k} \, k^{m(g-m)} \, \lambda^{m} \Big) & \text{if} \quad j = m+1 \end{array} \right\} \tag{3.3.4}$$

where all square-roots occurring in (3.3.4) are positive real numbers. We define maps $\phi(k) = (\phi_1(k), \phi_2(k), \cdots, \phi_{m+1}(k))$ from $\Gamma(k)$ to $(\mathbb{P}^2)^{m+1}$ given by:

$$\phi_j(k)(\lambda, \mu) = (\lambda_j : \mu_j : 1) \tag{3.3.5}$$

We consider the restriction of the maps $\phi_j(k)$ to the regions R_i. Using (3.3.3) and (3.3.4) in (3.3.5) we obtain:

$$\phi_j(k)_{|R_i}(\lambda, \mu) = \left\{ \begin{array}{ll} (o(1/k) : o(1/k) : 1) & \text{if} \quad i < j \\ (o(1/k) : 1 : o(1/k)) & \text{if} \quad i > j \\ (\lambda_j : \sqrt{P_j(\lambda_j)} \, (1 + o(1/k)) : 1) & \text{if} \quad i = j \end{array} \right\} \quad \text{as} \quad k \to \infty \tag{3.3.6}$$

where, for $(\lambda, \mu) \in \Gamma(k)^+$ the expression $(\lambda_j, \sqrt{P_j(\lambda_j)}) \in \mathcal{C}_j^+$. This can be seen by noting that, for $j \le m$, $\operatorname{sgn}(-\mu) = \operatorname{sgn}(\lambda^{j-1}) = (-1)^{j-1}$ on $\Gamma(k)^+$ over (E_{2j}, E_{2j-1}), and therefore (3.3.4) implies that $\operatorname{sgn}(\mu_j) = -1$ on $\Gamma(k)^+$ over (E_{2j}, E_{2j-1}). Also $\operatorname{sgn}(\sqrt{P_j(x_j)}) = -1$ on $\mathcal{C}_j^+$ over $x_j \in (E_{2j}, E_{2j-1})$. Similarly if $j = m+1$, then both quantities μ, λ^m and therefore μ_{m+1} are positive on $\Gamma(k)^+$ over the positive real axis, and $\sqrt{P_{m+1}(x_{m+1})}$ is also positive on $\mathcal{C}_{m+1}^+$ over the positive real axis.

We next consider embeddings $\phi_i : \mathcal{C}_i \to (\mathbb{P}^2)^{m+1}$ given by:

$$\phi_i : (x_i, y_i) \mapsto \{(0:1:0)\}^{i-1} \times (x_j : y_j : 1) \times \{(0:0:1)\}^{m+1-i}$$

(ϕ_{m+1} is singular at $\infty \in \mathcal{C}_{m+1}$ if $g - m > 1$, but this is inessential for us). Let $\mathcal{C} \subset (\mathbb{P}^2)^{m+1}$ be defined as $\cup_{i=1}^{m+1} \phi_i(\mathcal{C}_i)$. Clearly $\mathcal{C}$ is a nodal hyperelliptic curve of genus g having m double points $\{(0:1:0)\}^i \times \{(0:0:1)\}^{m+1-i}$ for $1 \leq i \leq m$. It is also clear from (3.3.6) that:

$$\lim_{k\to\infty} \phi(k)(\Gamma(k)) \; = \; \mathcal{C} \subset (\mathbb{P}^2)^{m+1}$$

$$\text{with} \quad \lim_{k\to\infty} \phi(k)(R_i) = \phi_i(\mathcal{C}_i) \subset \mathcal{C}$$

$$\text{and} \quad \lim_{k\to\infty} \phi(k)(R_i \cap R_{i+1}) = \{(0:1:0)\}^i \times \{(0:0:1)\}^{m+1-i} \quad \text{(the } m \text{ double points)}$$

Let a_j, b_j denote the basic cycles on $\phi_j(\mathcal{C}_j) \subset \mathcal{C}$ (or $\phi_{m+1}(\mathcal{C}_{m+1}) \subset \mathcal{C}$ if $j > m$). Let ω_j be the holomorphic differential on the elliptic curve $\mathcal{C}_j$ (for $j \leq m$) satisfying $\int_{a_j} \omega_j = 1$, and define $\tau_j = \int_{b_j} \omega_j$. We define B_1 to be the diagonal matrix $\mathrm{diag}(\tau_1, \cdots, \tau_m)$. Similarly, let ω_{m+j} for $1 \leq j \leq g-m$ be holomorphic differentials on the hyperelliptic curve $\mathcal{C}_{m+1}$ satisfying $\int_{a_{m+j}} \omega_{m+i} = \delta_{ij}$ and let $(B_2)_{ij} = \int_{b_{m+j}} \omega_{m+i}$ for $1 \leq i,j \leq g-m$. We note from (2.2.2-2.2.5) that $\mathrm{Re}(B_2) = -\frac{1}{2} I_{g-m}$, where I_{g-m} is the $g-m \times g-m$ identity matrix. Define B_∞ to be the block-diagonal matrix $B_\infty = \mathrm{diag}(B_1, B_2)$. It is the Riemann matrix of $\mathcal{C}$ with respect to the basic cycles $\{a_1, b_1, \cdots, a_g, b_g\}$. Using (3.3.6) again, it follows that the component of $\lim_{k\to\infty} \phi(k)(a_j(k))$ in $\phi_i(\mathcal{C}_i)$ is homologous to a_i if $i = j$ and homologous to zero if $i \neq j$. Similarly the component of $\lim_{k\to\infty} \phi(k)(b_j(k))$ in $\phi_i(\mathcal{C}_i)$ is homologous to b_i if $i = j$ and homologous to zero if $i \neq j$.

$$\lim_{k\to\infty} \phi(k)_*(a_j(k)) = a_j, \quad \text{and} \quad \lim_{k\to\infty} \phi(k)_*(b_j(k)) = b_j$$

$$\text{therefore} \quad \lim_{k\to\infty} B(k) = B_\infty$$

We note that in the case when $m = g$ (all $E_i < 0$), there is no need in the above construction for $\mathcal{C}_{m+1}, \lambda_{m+1}, R_{m+1}, \mu_{m+1}$ and B_2. Thus we have proved the theorem:

Theorem 3.3.1. *The limiting curve* $(\Gamma(\infty), B(\infty))$ *is a nodal curve with* m *nodes (double points) when* $m < g$, *and* $m-1$ *nodes when* $m = g$. *The irreducible components of the normalization of* $\Gamma(\infty)$ *are* $\{(\mathcal{C}_i, \tau_i) \,|\, 1 \le i \le m\} \cup (\mathcal{C}_{m+1}, B_2)$ *when* $m < g$, *and* $\{(\mathcal{C}_i, \tau_i) \,|\, 1 \le i \le g\}$ *when* $m = g$. *The limiting Riemann matrix* $B(\infty)$ *is block diagonal* $B(\infty) = diag(B_1, B_2)$ *where* $B_1 = diag(\tau_1, \cdots, \tau_m)$ *and* $\mathrm{Re}(B_2) = -\frac{1}{2} I_{g-m}$.

3.4 Formula for Topological charge

We reduce the calculation of the basic charges n_j in the general case to the two special cases of Lemma 3.2.1 and Lemma 3.2.2, using the multiscale limit of the previous section. As proved in Lemma 2.2.1 part 1), the real tori T_s do not intersect the divisor $\Theta \cup (\Theta + A(0))$. Therefore the integral term in the formula (3.1.8) for n_j stays nonsingular through the deformation $B(k)$. It is also constant because it is continuous in the deformation parameter k and it is integer valued. In other words the basic charges n_j may be calculated from (3.1.8) using the Riemann matrix B_∞ in place of B.

We have the following decomposition property of theta functions:

$$\theta\left((z_1, z_2)^t \,|\, \mathrm{diag}(B_1, B_2)\right) = \theta(z_1 \,|\, B_1)\, \theta(z_2 \,|\, B_2) \tag{3.4.1}$$

Using this decomposition, we see from the integral in formula (3.1.8), that n_j for $1 \le j \le m$ is the basic charge n for the elliptic curve $\mathcal{C}_i$ with $s = (-1)^{j-1} s_j$ (i.e s_j for j oddd and $-s_j$ for j even). Also formula (2.2.12) for the Riemann constants shows that $K_j = (1 + \tau_j)/2$ or $K_j = 1/2$ according as j is odd or even. Therefore, if

j is odd, then n_j equals n_1 of formula (3.2.1) with s_1 there taken to be s_j. On the other hand, if j is even , then n_j equals $\tilde{n}_1$ of formula (3.2.2) with s_1 there taken to be s_j. These are exactly the two situations considered in Lemma 3.2.2. Therefore we obtain $n_j = (-1)^{j-1}s_j$ for $1 \le j \le m$.

In the case $j > m$ again the decomposition formula (3.4.1) shows that n_j is the basic charge n_j for the real hyperelliptic curve $\mathcal{C}_{m+1}$ with no negative real branch points. We have shown in Lemma 3.2.1 that these n_j are zero. Therefore we summarize:

Theorem 3.4.1. *The topological charge density $\bar{n}$ for the real finite-gap solution $u(x,t)$ for the spectral data (Γ, D) with D of topological type s is given by*

$$\bar{n} = \sum_{j=1}^{g} (U_j - V_j)\, n_j/4$$

where the basic charges n_j are:

$$n_j = \begin{Bmatrix} (-1)^{j-1}s_j & \text{if} & 1 \le j \le m \\ 0 & \text{if} & j > m \end{Bmatrix} \qquad (3.4.2)$$

3.5 Comparison with literature, future work

In [17], the admissible divisors and real tori were characterized by certain symbols $\{s'_1, \cdots, s'_m\} \in \{\pm 1\}^m$ defined as follows. Given an admissible divisor $D = \{(\lambda_i, \mu_i) \,|\, 1 \le j \le g\}$ let $P(\lambda)$ be the unique polynomial of degree $g-1$ interpolating the g points $(\lambda_i, \mu_i/\lambda_i)$. Then $P(\lambda)$ is real and s'_j is defined to be the sign of $P(\lambda)$ over $[E_{2j}, E_{2j-1}]$. It was shown in [17] that the charges n_j are equal to $(-1)^{j-1}s'_j$ for $j \le m$ and $n_j =$ for $j > m$. Comparing with the formula (3.4.2), it follows that the symbols s_j' and s_j coincide for all j.

Future work: The multiscale limit of the spectral curve constructed above was used only for a topological argument. The sine-Gordon solutions $u(x,t,k)$ associated with the spectral curve $\Gamma(k)$ (and admissible divisors $D(k)$) depend on the vectors $U(k)$ and $V(k)$ (2.1.4). As $k \to \infty$, some component of $U(k)$ will diverge to ∞. Thus there is no limiting solution. However asymptotic expansion in the parameter k of $u(x,t,k)$ involving elliptic (genus 1) solutions can be written. This will be investigated in a future work.

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