ABSTRACT

Title of thesis: BARKER SEQUENCES
THEORY AND APPLICATIONS

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A Barker sequence is a finite length binary sequence with the minimum possible aperiodic autocorrelation. Currently, only eight known Barker sequences exist and it has been conjectured that these are the only Barker sequences that exist. This thesis proves that long sequences (having length longer than thirteen) must have an even length and be a perfect square. Barker sequences are then used to explore flatness problems related to Littlewood polynomials. These theorems could be used to determine the existence or non-existence of longer sequences. Lastly, an application of Barker sequences is given. Barker sequences were initially investigated for the purposes of pulse compression in radar systems. This technique results in better range and Doppler resolution without the need to shorten a radar pulse, nor increase the power.

Barker Sequences
Theory and Applications

by

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Dedication

Dedicated to my father Charles MacDonald, who has always been supportive and proud of my accomplishments. And to my sister Cynthia MacDonald, who shares my love of mathematics.

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I'd like to thank Professor John Benedetto for giving me the opportunity, guidance, and encouragement to complete my degree. In addition to this, Professor Benedetto got me excited about Barker sequences and harmonic analysis for which I am grateful.

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Chapter 1

Introduction and Motivation

1.1 Introduction

Given a real sequence $\{a_i\}_{i=1}^{n}$ the aperiodic autocorrelation function is defined as

$$c_k = \sum_{i=1}^{n-k} a_i a_{i+k}$$

and the periodic autocorrelation is defined as

$$\gamma_k = \sum_{i=1}^{n-k} a_i a_{(i+k) \bmod n}$$

The modular arithmetic, in the subscript, is computed over $\{1, 2, ..., n\}$, instead of $\{0, 1, ..., n-1\}$. For completeness, define $c_{-k} = c_k$. Also, note that $c_k + c_{n-k} = \gamma_k$.

A Barker sequence, $\{a_i\}_{i=1}^{n}$, is such that $a_i = \pm 1$ and $|c_k| \leq 1$ where $k \geq 1$ and $c_0 = n$. Unfortunately, not too many exist. The restrictions mentioned allows for only eight different sequences known, up to transformations. Below are the known sequences, assuming $a_1 = a_2 = 1$.

$$\begin{array}{lll} n = & 2 & ++ \\ & 3 & ++- \\ & 4 & +++- \\ & & ++-+ \\ & 5 & +++-+ \\ & 7 & +++--+- \\ & 11 & +++---+--+- \\ & 13 & +++++--++-+-+ \end{array}$$

For all the above sequences, since $a_i = \pm 1$, we have $a_i^2 = 1$, which further implies $c_0 = n$. For the above sequences I will show that for non-zero k, $|c_k| \leq 1$.

For $n = 2$, $a = ++$ and $c_1 = 1$.

For $n = 3$, $a = ++-$ and

$c_1 = 1 - 1 = 0$

$c_2 = -1$

For $n = 4$, $a = +++-$ and

$c_1 = 1 + 1 - 1 = 1$

$c_2 = 1 - 1 = 0$

$c_3 = -1$

For $n = 4$, $a = ++-+$ and

$c_1 = 1 - 1 - 1 = -1$

$c_2 = -1 + 1 = 0$

$c_3 = 1$

For $n = 5$, $a = +++-+$ and

$c_1 = 1 + 1 - 1 - 1 = 2 - 2 = 0$

$c_2 = 1 - 1 + 1 = 2 - 1 = 1$

$c_3 = -1 + 1 = 0$

$c_4 = 1$

For $n = 7$, $a = +++--+-$ and

$c_1 = 1 + 1 - 1 + 1 - 1 - 1 = 3 - 3 = 0$

$c_2 = 1 - 1 - 1 - 1 + 1 = 2 - 3 = -1$

$c_3 = -1 - 1 + 1 + 1 = 2 - 2 = 0$

$c_4 = -1 + 1 - 1 = 1 - 2 = -1$

$c_5 = 1 - 1 = 0$

$c_6 = -1$

For $n = 11$, $a = +++---+--+-$ and

$c_1 = 1 + 1 - 1 + 1 + 1 - 1 - 1 + 1 - 1 - 1 = 5 - 5 = 0$

$c_2 = 1 - 1 - 1 + 1 - 1 + 1 - 1 - 1 + 1 = 4 - 5 = -1$

$c_3 = -1 - 1 - 1 - 1 + 1 + 1 + 1 + 1 = 4 - 4 = 0$

$c_4 = -1 - 1 + 1 + 1 + 1 - 1 - 1 = 3 - 4 = -1$

$c_5 = -1 + 1 - 1 + 1 - 1 + 1 = 3 - 3 = 0$

$c_6 = 1 - 1 - 1 - 1 + 1 = 2 - 3 = -1$

$c_7 = -1 - 1 + 1 + 1 = 2 - 2 = 0$

$c_8 = -1 + 1 - 1 = -2 + 1 = -1$

$c_9 = 1 - 1 = 0$

$c_{10} = -1$

For $n = 13$, $a = + + + + + - - + + - + - +$ and

$c_1 = 1 + 1 + 1 + 1 - 1 + 1 - 1 + 1 - 1 - 1 - 1 - 1 = 6 - 6 = 0$

$c_2 = 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1 + 1 + 1 + 1 = 6 - 5 = 1$

$c_3 = 1 + 1 - 1 - 1 + 1 - 1 + 1 + 1 - 1 - 1 = 5 - 5 = 0$

$c_4 = 1 - 1 - 1 + 1 + 1 + 1 - 1 - 1 + 1 = 5 - 4 = 1$

$c_5 = -1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 = 4 - 4 = 0$

$c_6 = -1 + 1 + 1 - 1 + 1 + 1 - 1 = 4 - 3 = 1$

$c_7 = 1 + 1 - 1 + 1 - 1 - 1 = 3 - 3 = 0$

$c_8 = 1 - 1 + 1 - 1 + 1 = 3 - 2 = 1$

$c_9 = -1 + 1 - 1 + 1 = 2 - 2 = 0$

$c_{10} = 1 - 1 + 1 = 2 - 1 = 1$

$c_{11} = -1 + 1 = 0$

$c_{12} = 1$

The transformations $s_1(a_i) = (-1)^i a_i$, $s_2(a_i) = (-1)^{i+1} a_i$, and $s_3(a_i) = -a_i$ transform one Barker sequence into another. These transformation, along with the identity function, forms an abelian group under addition. To not confuse the two separate c_k that will be computere here, denote $A_a(k)$ as c_k for the sequece a.

I. $s_1(a_i) = (-1)^i a_i$

To show that this preserves Barker sequences, let $y_i = s_1(a_i)$ and note that $A_y(k) = \sum_{i=1}^{n-k} y_i y_{n+k}$. Two cases need to be analyzed- one where k even, the other where k odd.

a. k even. $A_y(k) = \sum_{i=2m} a_i a_{i+k} + \sum_{i=2m+1} (-a_i)(-a_{i+k}) = A_a(k)$.

b. k odd. $A_y(k) = \sum_{i=2m} a_i(-a_{i+k}) + \sum_{i=2m+1} (-a_i)a_{i+k} = -A_a(k)$.

II. $s_2(a_i) = (-1)^{i+1} a_i$

As for s_1, there are two cases to consider.

a. k even. $A_y(k) = \sum_{i=2m} (-a_i)(-a_{i+k}) + \sum_{i=2m+1} a_i a_{i+k} = A_a(k)$.

b. k odd. $A_y(k) = \sum_{i=2m} (-a_i)a_{i+k} + \sum_{i=2m+1} a_i(-a_{i+k}) = -A_a(k)$.

III. $s_3(a_i) = -a_i$

$A_y(k) = \sum_{i=1}^{n-k} (-a_i)(-a_{i+k}) = \sum_{i=1}^{n-k} a_i a_{i+k} = A_a(k)$.

This paper will provide motivation for finding Barker sequences, then state and prove the main existence problem known for Barker sequences. Specifically, it is known that few Barker sequences exist and it is widely conjectured that the above sequences are the only Barker sequences that exist.

1.2 Motivation

Signals with low non-zero autocorrelation have many uses in signal processing and mathematics. A signal is defined as a function $f : \mathbb{R} \to \mathbb{R}$, or more generally $f : \mathbb{C} \to \mathbb{C}$, such that $f \in L^2$. This allows for the definition of autocorrelation: $R_f(\tau) = \int f(t)f^*(t+\tau)dt$, where f^* denotes the complex conjugate of f. Also, this can be defined for a periodic discrete signals $a : \mathbb{N} \to \mathbb{C}$, for $a \in l^2$, as $R_x(k) = \sum_i a_i a^*_{i+k}$.

Much of this paper will be the study of finite discrete binary sequences and a variety of applications to engineering and mathematics. A binary sequence is defined as $\{a_i\}_{i=1}^n$, such that $a_i = \pm 1$. These types of signals can be implemented easily as a binary phase shift key signal, with modulation of a constant frequency shifting between 0 and π phase. A 0 phase represents a 1, while a π phase represents a -1. Binary sequences with the lowest autocorrelation are Barker sequences.

Pulse Compression Pulse compression radar techniques take advantage of the received signal strength of long pulse signals, combined with the range resolution of short pulse signals. Clearly, one cannot send a short pulse and long pulse simultaneously. To circumvent this, instead of sending out a simple pulsed signal, the signal would be modulated as well. This is called pulse compression. The trade off between long and short pulses is power consumption by the transmitter. For example, if you want a resolution of 15cm, using a bandwidth of 1GHz, with pulse energy

1mJ, without pulse compression, you need a pulse of 1ns with the transmitter power being 1MW. As a contrast, using pulse compression, requires a pulse of 0.1ms with the transmitter power being 100W. Using sequences with low autocorrelation and modulation 180 degrees out of phase allows longer pulses to be sent, while maintaining the range resolution of the radar system. These restrictions make Barker sequences an ideal choice.

Flat Polynomial Flat polynomials are polynomials such that there exist $0 < a < b$ where for all $|z| = 1$, $a < \frac{|f(z)|}{\sqrt{n}} < b$ and $n-1$ is the degree of the polynomial. For unimodular polynomials, which are polynomials haveing coeffiecients $|a_i| = 1$, for every $\epsilon > 0$ such that $a = 1 - \epsilon$ and $b = 1 + \epsilon$, there is a sequence of flat polynomials. A Littlewood polynomial is a polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ where $a_i = \pm 1$. For Littlewood polynomials, it is known that there exist polynomials satisfying the upper bound where $b = \sqrt{2}$, but it is not known whether there exists a sequence with a lower bound as well. Having the coefficients of the polynomial form a Barker sequence suffices for the polynomial to be a flat Littlewood polynomial. This means that the non-existence of flat Littlewood polynomials implies the non-existence of long Barker sequences.

Chapter 2

Even and Odd Length Sequences

2.1 Even Length

The first thing to do is to see what consequences follow by merely considering whether or not the length of a Barker sequence is even or odd. This section states and proves some interesting and surprising results that follow from the length of the sequence. First, if a sequence has an even length, n, then n is a perfect square. Second, if n is odd, then $n \leq 13$. Surprisingly, the two conditions defining Barker codes- 1) they are binary (± 1) and 2) the magnitude of all non-zero autocorrelation is less than one- are enough to admit seemingly only a finite number of sequences. It is widely conjectured that the Barker sequences listed above are the only ones. Many mathematicians and engineers over the last 60 years have labored to prove just that.

Now, let's start by examining these basic existence theorems. The following results are primarily due to [12] and [2]. Recall that $c_k = \sum_{i=1}^{n-k}(a_i a_{i+k})$. Now we state and prove our first theorem.

Theorem 1: *If $\{a_i\}_{i=1}^{n}$ is a Barker sequence and $n \geq 4$ is even, then $n = 4N^2$.*

Proof:

Fix a Barker sequence $a = \{a_i\}$ of length n.

Also, define $x = \sum \chi_{(a_i a_{i+k}=1)}$ and $y = \sum \chi_{(a_i a_{i+k}=-1)}$. This gives us the following linear equations:

$$x + y = n - k \tag{2.1}$$

$$x - y = c_k \tag{2.2}$$

Subtracting (2.2) from (2.1) yields $2y = n-k-c_k$, which further implies $y = (n-k-c_k)/2$.

Define $d_k = \prod_{i=1}^{n-k} a_i a_{i+k}$. By the above calculation

$$\begin{aligned} d_k &= \prod_{i=1}^{n-k} a_i a_{i+k} \\ &= 1^x(-1)^y \\ &= (-1)^y \\ &= (-1)^{(n-k-c_k)/2} \end{aligned}$$

Consider $c_k + c_{n-k} = \sum_{i=1}^{n-k} a_i a_{i+k} + \sum_{i=1}^{k} a_i a_{i+n-k} = \sum_{i=1}^{n} a_i a_{i+k}$, where $i+k$ in the last term is taken $\pmod n$, over the set $\{1, 2, ..., n\}$, not $\{0, 1, ..., n-1\}$. Combining this calculation with that of d_k results in

$$\begin{aligned} d_k d_{n-k} &= \prod_{i=1}^{n-k} a_i a_{i+k} \prod_{i=1}^{k} a_i a_{i+n-k} \\ &= \prod_{i=1}^{n} a_i a_{i+k \bmod n} \\ &= \prod_{i=1}^{n} a_i^2 \\ &= 1 \end{aligned}$$

Also,

$$\begin{aligned} d_k d_{n-k} &= (-1)^{(n-k-c_k)/2}(-1)^{(n-(n-k)-c_{n-k})/2} \\ &= (-1)^{(n-k-c_k+n-n+k-c_{n-k})/2} \\ &= (-1)^{(n-c_k-c_{n-k})/2} \end{aligned}$$

The above two calculations implies that $1 = (-1)^{(n-c_k-c_{n-k})/2}$, thus $(n - c_k - c_{n-k})/2$ is even. This implies that $c_k + c_{n-k} \equiv n (\text{mod } 4)$, an important identity that will be used throughout this section, not just this theorem.

Assuming $|c_k| \leq 1$ and that $n = 2m$ for some $m \in \mathbb{N}$, consider $c_k = a_1 a_{1+k} + ... + a_{n-k} a_n$. This sum will have an even number of terms. If

$k = 2s$, then $c_k = 2t$. Since $|c_{2s}| \leq 1 \Rightarrow c_{2s} = 0$. Similarly, $|c_{2s+1}| = 1$.

Since n is even, if k is even, then $c_k = 0$ and $c_{n-k} = 0$. Also, since $n \equiv c_k + c_{n-k} (\text{mod } 4)$, then $n \equiv 0 (\text{mod } 4)$. If k is odd, then $c_k + c_{n-k} \in \{-2, 0, 2\}$. Since $-2 \not\equiv 0 (\text{mod } 4)$ and $2 \not\equiv 0 (\text{mod } 4)$, we have $c_k + c_{n-k} = 0$ when k is odd, also. This leads us to the final concluding statement:

$(\sum_{i=1}^{n} a_i)^2 = c_0 + \sum_{i=1}^{n-1} (c_k + c_{n-k}) = c_0 = n$. Thus, when n is even it is a perfect square, so $n = 4N^2$.

QED.

2.2 Odd Length

To prove that $n \leq 13$ if n is odd, let's first take a look at the immediate consequences that n being odd has on c_k. If n is odd, then this implies that for any k, $|c_k + c_{n-k}| = 1$ with $c_{2j+1} = 0$ and $c_{2j} = \pm 1$. By the above theorem, we know that $n \equiv c_k + c_{n-k} (\text{mod } 4)$. This implies that $n \equiv \pm 1 (\text{mod } 4)$.

This gives us two cases in which to investigate c_{2j}. If $n \equiv 1 (\text{mod } 4)$, then $c_{2j} = 1$. If $n \equiv -1 (\text{mod } 4)$, then $c_{2j} = -1$. These two cases taken together implies that $c_{2j} = (-1)^{(n-1)/2}$, which depends only on n and has nothing at all to do with j.

Now make use of d_k defined in the above theorem.

$$\begin{aligned} d_k d_{k+1} &= (-1)^{(n-k-c_k)/2}(-1)^{(n-(k+1)-c_{k+1})/2} \\ &= (-1)^{(2n-2k-1-c_k-c_{k+1})/2} \\ &= (-1)^{n-k-(1+c_k-c_{k+1})/2} \end{aligned}$$

And

$$\begin{aligned} d_k d_{k+1} &= (\prod_{i=1}^{n-k} a_i a_{i+k})(\prod_{i=1}^{n-k-1} a_i a_{i+k+1}) \\ &= [(a_1 a_{1+k})...(a_{n-k} a_n)][(a_1 a_{k+2})...(a_{n-k-1} a_n)] \\ &= (\prod_{i=1}^{n-k-1} a_i^2)(\prod_{i=2+k}^{n-1} a_i^2)(a_{k+1} a_{n-k}) \\ &= a_{k+1} a_{n-k} \end{aligned}$$

These two calculations yield:

$$a_{n-k} a_{k+1} = (-1)^{n-k-(1+c_k+c_{k+1})/2} \tag{2.3}$$

Since $c_k + c_{n-k} \equiv n(\text{mod } 4)$, then $n = x + 4m \Rightarrow c_k + c_{n-k} = x + 4t$, where $x = \pm 1$, and we get the following:

$$a_{n-k} a_{k+1} = (-1)^{n-k-(1+c_k+c_{k+1})/2}$$

$$
\begin{aligned}
&= (-1)^{x+4m-k-(1+x+4t)/2} \\
&= (-1)^{x-k-(1+x)/2} \\
&= (-1)^{-k-(1+x-2x)/2} \\
&= (-1)^{-k+(x-1)/2} \\
&= (-1)^{-k+(x+4m-1)/2} \\
&= (-1)^{k+(n-1)/2}
\end{aligned}
$$

The last equality in the above is true since $(-1)^k = (-1)^{-k}$.

If $a_{n-k}a_{k+1} = (-1)^{k+(n-1)/2}$, then

$a_{n-(i+2j-1)}a_{i+2j} = (-1)^{i+2j-1+(n-1)/2}$, so

$a_{i+2j} = a_{n-(i+2j-1)}(-1)^{i+2j-1+(n-1)/2}$.

A sequence satisfying (2.3) has the property $c_{2j+1} = 0$, $0 < 2j+1 < n$, yields the following:

$$
\begin{aligned}
(-1)^{(n-1)/2} &= c_{2j} \\
&= \sum_{i=1}^{n-2j} a_i a_{i+2j} \\
&= \sum_{i=1}^{n-2j} a_i a_{n-(i+2j-1)}(-1)^{i+2j-1+(n-1)/2}
\end{aligned}
$$

$$= \quad (-1)^{(n-1)/2} \sum_{i=1}^{n-2j} a_i a_{n-(i+2j-1)} (-1)^{i+1}$$

Dividing each side by $(-1)^{(n-1)/2}$ and letting $n - 2j = 2k + 1$, $k \geq 1$, we get

$$\begin{aligned}
1 &= \sum_{i=1}^{2k+1} (a_i a_{2k+2-i})(-1)^{i+1} \\
&= \sum_{i=1}^{k} + \sum_{i=k+1}^{2k+1} \\
&= [a_1 a_{2k+1} - ... + a_k a_{k+2}(-1)^{k+1}] + a_{k+1}^2 (-1)^{k+2} \\
&\quad [a_{k+2} a_k (-1)^{k+1} + ... + a_{2k+1} a_1] \\
&= 2 \sum_{i=1}^{k} [a_i a_{2k+2-i})(-1)^{i+1}] + a_{k+1}^2 (-1)^{k+2}
\end{aligned}$$

whenever $n > 2k + 1 \geq 3$. Since $a_{k+2}^2 = 1$, the above gets us

$1 = 2 \sum_{i=1}^{k} [a_i a_{2k+2-i})(-1)^{i+1}] - (-1)^{k+1}$. Then a little algebra gets us

$$\frac{1 + (-1)^{k+1}}{2} = \sum_{i=1}^{k} [a_i a_{2k+2-i})(-1)^{i+1}] \tag{2.4}$$

Define $P(k) = \frac{1+(-1)^{k+1}}{2}$.

Whenever $1 \leq k < \frac{n-1}{2}$. Now we are ready to state our next lemma, which says that a Barker sequence of odd length will exhibit some periodic behavior.

Lemma: *Let* $\{a_i\}_{i=1}^{n}$, *n odd, be a sequence that satisfies (2.4) for* $1 \leq k \leq t$, $a_i = \pm 1$. *Let* $a_i = 1$ *for* $1 \leq i \leq p$, $a_{p+1} = -1$. *If* $p > 1$ *then*

i) $a_i a_{i+1} = a_{2i} a_{2i+1}$, $1 \leq i \leq t$,

ii) $p \leq 2t + 1$ *implies* p *is odd,*

iii) $pj + r \leq 2t + 1$, $1 \leq r \leq p$ *implies* $a_{p(j-1)+r} = a_{p(j-1)+1}$, *and*

iv) $z_j = a_{p(j-1)+1}$ *satisfies (2.4) for* $k \leq t/p$.

Proof:

Proof of i):

$\sum_{i=1}^{k}[a_i a_{2k+2-i})(-1)^{i+1}] = a_1 a_{2k+1} - a_2 a_{2k} + ... + a_k a_{k+2}(-1)^{k+1}$ is a constant dependent only on k by (2.4) above. Note that if each term $a_i a_{2k+2-i} = 1$, then (2.4) is satisfied. Thus, if any one of the terms, $a_i a_{2k+2-i}$, is -1, there must be another term, $a_j a_{2k+2-j}$, that is -1. Therefore, there are always an even number of terms, $a_i a_{2k+2-i}$, that equal -1.

This leads us to conclude that $\prod_{i=1}^{k} a_i a_{2k+2-i} = 1^a(-1)^b = 1$, where a is the number of terms, $a_i a_{2k+2-i}$, that equals 1 and b is the number of terms, $a_i a_{2k+2-i}$, equal to -1.

$$\begin{aligned} 1 &= \prod_{i=1}^{k}(a_i a_{2k+2-i}) \\ &= (a_1 a_{2k+1})(a_2 a_{2k+2})...(a_k a_{k+2}) \end{aligned}$$

$$= \quad a_1 a_2 ... a_k a_{k+2} ... a_{2k+1}$$

which implies $\prod_{i=1}^{2k+1} a_i = a_{k+1}$. So we get:

$$\begin{aligned}
a_{2k+2}a_{2k+3} &= (\sum_{i=1}^{2k+1} a_i)^2 a_{2k+2}a_{2k+3} \\
&= \prod_{i=1}^{2k+1} a_i \prod_{j=1}^{2k+3} a_j \\
&= a_{k+1}a_{k+2} \\
&= a_{k+1}a_{(k+1)+1}
\end{aligned}$$

Since $a_{2k+2}a_{2k+3} = a_{2(k+1)}a_{2(k+1)+1} = a_{k+1}a_{(k+1)+1}$, i) is proven.

To prove ii), note that if $p = 2s$, then $a_s a_{s+1} = a_{2s}a_{2s+1}$, which means $1 = -1$, a contradiction. Thus p is odd when $p \leq 2t + 1$.

To prove iii) and iv) I use an induction approach. For $t < p$ there is nothing to prove because the statements are true by assumption. Choosing $p = t$, from (2.4) we get

$$\begin{aligned}
1 &= P(p) \\
&= a_1 a_{2p+1} + a_2 a_{2p} + ... + a_p a_{p+2} \qquad (2.5)
\end{aligned}$$

By assumption $a_1 = a_2 = ... = a_p = 1$, so by i) $a_{2k} = a_{2k+1}$ for $p < 2k < 2p$,(2.5) reduces to

$$
\begin{aligned}
1 &= P(p) \\
&= a_{2p+1} - a_{2p} + a_{2p-1} - ... - a_{p+3} + a_{p+2} \\
&= a_{2p+1} - a_{2p} + a_{p+2}
\end{aligned}
$$

And since, by assumption, $a_p = 1$ and $a_{p+1} = -1$, then by i) $-1 = a_p a_{p+1} = a_{2p} a_{2p+1}$, which implies that $a_{2p+1} = -a_p \Rightarrow a_{2p+1} - a_p = 2a_{2p+1}$. Therefore, we get

$1 = 2a_{2p+1} + a_{p+2}$.

Thus

$a_{p+2} = -1$

$a_{2p+1} = 1$

For $2t + 1 \leq 3p$, let $t = p + s$. Using the fact that $a_{2i} = a_{2i+1}$ for $i \not\equiv 0$ $(mod\ p)$,by the induction assumption we have

$$
\begin{aligned}
P(p+s) &= \sum_{1}^{2s+1} + \sum_{2s+2}^{p} + \sum_{p+1}^{2p+s} \\
&= a_{2p+1} - a_{p+1} a_{p+2s+1} + P(s-1)
\end{aligned}
$$

Since p is odd, $P(p+s) = P(s-1)$ and $a_{2p+1} + a_{p+2s+1}$, which implies $a_{p+2s+1} + -1$.

For $2t+1 > 3p$, let $2t+1 = hp+m$ with $1 \leq m \leq p$. We need to consider the case where h even and h odd separately.

Case 1:

$h = 2H+1$ implies $\sum_1^t = \sum_0^{H-1}(\sum_{jp+1}^{jp+m} + \sum_{jp+m+1}^{(j+1)p}) + \sum_{Hp+1}^{Hp+m} + \sum_{Hp+m+1}^{t}$

As above, this equation can be reduced by using the fact that $a_{2i} = a_{xi+1}$ for $i \not\equiv 0 (mod\ p)$ in the first three sums. The last sums are of the form $(-1)^i$ by the induction assumption. So we get

$P(t) = a_p a_{(h-1)p+m+1} - a_{p+1} a_{(h-1)p+m} + a_{p+m} a_{(h-1)p+1} +$

$\sum_2^H z_i z_{h+1-i}(-1)^{i+1} + (-1)^H P(\frac{p+m-1}{2})$.

Since $t = Hp + (p+m-1)/2$, we get

$P(H) - \sum_1^H z_i z_{h+1-i}(-1)^{i+1} = a_{(h-1_p+m+1} + a_{(h-1)p+m} - 2a_{(h-1)p+1}$

By the induction assumption $P(H) - \sum_1^H z_i z_{h+1-i}(-1)^{i+1} = 0$, so

$$a_{(h-1)p+m} = x(h-1)p + 1.$$

Case 2:

$h = 2H$, proceeding as before

$$P(H) - \sum_1^H z_i z_{h+1-i}(-1)^{i+1} = -z_h - a_{hp} + 2a_{(h-1)p+m}$$

for $m < p$. Subtracting successive equations of this type ($m = 1, 3, 5, ...$) gets $a_{(h-1)p+1}$. For m even, the equation follows from ii). For $m = p$, implies $2t + 1 = (2H + 1)p$, which yields

$$P(H) - \sum_1^H z_i z_{h+1-i}(-1)^{i+1} = z_h + a_{hp}.$$

Compare the above equation with the equation for $m = 1$, we get

$$P(H) - \sum_1^H z_i z_{h+1-i}(-1)^{i+1} = 0$$

which gets us $a_{hp} = a_{(h-1)p+1}$.

QED

The above theorem shows that odd Barker sequences exhibit some periodic behavior. The next theorem says that because of this behavior, an odd length Barker sequence must be short.

Theorem 2: *A Barker sequence of odd length n implies $n \leq 13$.*

Proof:

By the transformations mentioned in the introduction, we may assume $a_1 = a_2 = 1$. $P(k)$ satsifies (2.4) for $1 \leq k < (n-1)/2$ and $p < (n-1)/2$ for $n > 3$, by the above lemma.

If $n > 4p$ then by periodic behavior discussed in the above lemma

$a_i = +1$, $1 \leq i \leq p$, $2p+1 \leq i \leq 3p$

$a_i = -1$, $p+1 \leq i \leq 2p$.

By (2.3), $a_{n-k}a_{k+1} = (-1)^{k+(n-1)/2}$, which implies, if $a_i = a_{i+1}$, then $a_{n+1-i} = -a_{n-i}$. So if $n > 4p$,

$a_i = +1$, $1 \leq i \leq p$

$a_i = -1,\ p+1 \leq i \leq 2p$

$a_i = +1,\ 2p+1 \leq i \leq 3p$

and the last three blocks of length p are blocks of alternating $+1$'s and -1's, which implies $n \geq 6p-1$. Similarly, if $n > kp$, then $n \geq 2(k-1)p-3$, which can continues ad infinitem, so $n < 4p$.

Also, $n \neq 3p$ because if $n = 3p$, then

$a_p a_{p+1} = a_{2p} a_{2p+1} = -a_{p+1} a_p$.

Letting $n = 2m-3$ and $b = (-1)^{(n-1)/2}$, then equation (2.4) is

$\frac{1+b}{2} = \sum_1^m a_i a_{n-1-i}(-1)^{i+1} =$

$a_1 a_{n-2} - a_2 a_{n-3} + ... - a_{p-3} a_{n-p+2} + a_{p-2} a_{n-p+1} + ...$

Since the first block of length p is $+1$ and the last block of length p has alternating $+1$'s and -1's, the first $p-2$ elements in this sum are the same; in particular, they equal a. Let N the be number of terms $a_i a_{n-1-i}(-1)^{i+1} = -b$ with $p-2 \leq i \leq m$, then

$\frac{1+b}{2} = (p-2)b + (m-N-(p-2))b - Nb$

$\frac{1+b}{2} + N - (p-2) = m - N - (p-2) \geq 0$

$p - 2 - \frac{b+1}{2} \leq \frac{n-3}{2} - (p-2),$

which implies $4p \leq n+6+b$. Since $n < 3p$ we get $p < 6+b$, which implies $p \leq 5$. By the above lemma, we can construct sequences of length 5, 7, 13. Finally, for $3p < n < 4p$, let $n = 2m+1$ and we get $a_m = -a_{m+2}$ getting us $a_i = -1$ for $p+1 \leq i \leq 2p$. By the lemma, we must have $n - 2 - p \leq m + 1 = (n+1)/2$, which implies $n \leq 2p+5$. Now $n > 3p$ implies $p < 5$, hence $p = 3$ and we get can construct the sequence of length 11.

QED

Mossinghoff [8], and Jedwab [7] have done additional investigation into the length of n. Mossinghoff, using numerical techniques has determined the largest known lower bound on the length of n for a Barker sequences. No Barker sequences exists for lengths n such that $13 < n < 189\ 269\ 268\ 001\ 034\ 441\ 552\ 766\ 781\ 604$. In the next section related topics explore the existence of longer Barker sequences.

Chapter 3

Flat Polynomials

3.1 Definitions and Rudin-Shapiro Polynomials

A given sequence $\{a_i\}_{i=1}^{n}$ can be associated with a polynomial, f, such that $f(z) = \sum_{i=0}^{n-1} a_{i+1} z^i$. For a given $p > 0$, the p norm for a polynomial is defined as

$$\|f\|_p = \left(\int_0^1 |f(e(t))|^p dt \right)^p$$

where $e(t) = e^{2\pi it}$. Below are three properties [6] relating to this norm that will be useful in further analysis.

1) If $\mu(E) = 1$ and $0 < p < q < \infty$, then $\|f\|_p \leq \|f\|_q$.

Let $r = q/p > 1$ with conjugate r', then

$$\begin{aligned} \int_E |f|^p &\leq \left(\int_E |f|^{pr} \right)^{1/r} \left(\int_E 1^{r'} \right)^{1/r'} \\ &= \left(\int_E |f|^q \right)^{p/q} \end{aligned}$$

Since the last equation is finite, the first is as well. Taking the p^{th} root of both sides gets us the desired result.

2) $\lim_{p\to\infty} \|f\|_p = \sup_{0\leq t\leq 1} |f(e(t))|$.

Note that

$$
\begin{aligned}
\left(\int_0^1 |f|^p\right)^{1/p} &\leq \left(\int_0^1 \sup_{0\leq t\leq 1} |f|^p\right)^{1/p} \\
&= \sup_{0\leq t\leq 1} |f| \left(\int_0^1 |1|^p\right)^{1/p} \\
&= \sup_{0\leq t\leq 1} |f| \mu(E)^{1/p} \\
&= \sup_{0\leq t\leq 1} |f|
\end{aligned}
$$

Let $\alpha = \sup_{0\leq t\leq 1} |f|$, then α is an upper bound for the set of $\{\|f\|_p : 0 < p < \infty\}$. Now we need to show that α is the least upper bound. Fix $\epsilon > 0$, and let $A = \{x \in E : |f(x)| \geq \alpha - \epsilon\}$, then

$$
\begin{aligned}
\mu(A)^{1/p}(\alpha - \epsilon) &= \left(\int_A (\alpha - \epsilon)^p\right)^{1/p} \\
&\leq \left(\int_A |f|^p\right)^{1/p} \\
&\leq \left(\int_0^1 |f|^p\right)^{1/p} \\
&= \|f\|_p
\end{aligned}
$$

Letting $p \to \infty$, then $\mu(A)^{1/p} \to 1$, which implies that

$$(\alpha - \epsilon) \quad \leq \quad \lim_{p\to\infty} \|f\|_p$$

which gets us the desired result.

3) $\lim_{p\to 0} \|f\|_p = \exp\left(\int_0^1 \log|f(e^{2\pi it})|dt\right)$. This is the Mahler measure.

Set $g = |f|$, then note that $\log t \leq t - 1$ for all $t > 0$. Let $E = [0, 1]$, so $\mu(E) = 1$. Replacing t with $\frac{g}{\|g\|_1}$ and integrating implies

$$\int_E \log \frac{g}{\|g\|_1} \leq 0$$

implying

$$\int_E \log g \leq \log \int_E g$$

The next thing to do is to see that, by L'Hospital,

$$\lim_{r\to 0^+} \frac{g^r - 1}{r} = \log g$$

Since $\mu(E) = 1$ and $\|g\|_p \leq \|g\|_q$ whenever $p < q$, then

$$\begin{aligned}\lim_{r\to 0^+} r^{-1}\int_E (g^r-1) &= \lim_{n\to\infty} n\int_E (g^{1/n}-1)\\ &= \int_E \lim_{n\to\infty} n(g^{1/n}-1)\\ &= \int_E \log g\end{aligned}$$

by the Bound Convergence Theorem ($\lim_{n\to\infty} n(g(t)^{1/n}-1)$ is a converging sequence for all $t\in[0,1]$, thus bounded). Now by tying everything together we see that

$$\begin{aligned}(1/r)\left[\int_E g^r - 1\right] &\geq (1/r)\log\int_E g^r\\ &\geq (1/r)\int_E \log g^r\\ &= \int_E \log g\end{aligned}$$

The first equation converges to the last equation as $r\to 0^+$, so by the Sandwich Theorem, the second equation $(1/r)\log\int_E g^r = \log\|g\|_r \to \int_E \log g$. Apply the exponential to both sides and the proposition follows.

Polynomials such that $|a_i| = 1$ are called unimodular polynomials and denoted as U. Polynomials such that $a_i = \pm 1$ are called Littlewood polynomials and are denoted as L. Whenever $f(z) = \sum_{i=0}^{n-1} a_{i+1}z^i$, by Parseval, $\|f\|_2^2 = n$. A question that has arisen in relation to unimodular and Littlewood polynomials is the existence of so called flat polynomials. A sequence of polynomials of degree $n-1$ is called flat if there exist two postive numbers α_1 and α_2 such that for all n

$$\alpha_1\sqrt{n} \leq |f(z)| \leq \alpha_2\sqrt{n}$$

whenever $|z| = 1$. For unimodular polynomials it is known that for any $\epsilon > 0$ there is a sequence of flat polynomials such that $\alpha_1 = 1-\epsilon$ and $\alpha_2 = 1+\epsilon$. A sequence of polynomials that satisfies this strict condition is called ultraflat. Less is known for the Littlewood polynomials. For the Rudin-Shapiro polynomials (known also as simply Shapiro polynomials), which are a subset of the Littlewood polynomials, it is known that they satisfy the upper bound of $\alpha_2 = \sqrt{2}$ [11], but no known sequence satisfies the lower bound. Rudin-Shapiro polynomials are defined as

$$P_0(z) = 1$$
$$Q_0(z) = 1$$

Then the rest are defined inductively

$$P_{n+1} = P_n(z) + z^{2^n} Q_n(z)$$
$$Q_{n+1} = P_n(z) - z^{2^n} Q_n(z)$$

From the parallelogram law $|\alpha+\beta|^2+|\alpha-\beta|^2 = 2(|\alpha|^2+|\beta|^2)$ we see that

$$\begin{aligned}
|P_{n+1}(z)|^2+|Q_{n+1}(z)|^2 &= |P_n(z)+z^{2^n}Q_n(z)|^2+|P_n(z)-z^{2^n}Q_n(z)|^2 \\
&= 2(|P_n(z)|^2+|Q_n(z)|^2)
\end{aligned}$$

Since $|P_0(z)|^2+|Q_0(z)|^2=2$, the above recursion means that $|P_n(e^{2\pi ix})| \le \sqrt{2}N^{1/2}$, where $N=2^n$. This means that the upper bound flatness condition of the Rudin-Shapiro polynomials is satisfied by $\alpha_2=\sqrt{2}$. However, $P_{2k+1}(-1)=0$, so there is no $\alpha_1>0$ that satisfies the lower bound flatness condition.

3.2 Littlewood's Problem

So we start by investigating some of the behavior of Littlewood polynomials following the logic used in [2], [3]. If $|z|=1$, then $\overline{z}=1/z$ and since $c_{-k}=c_k^*$ we get

$$\begin{aligned}
\|f\|_4^4 &= \left\| f(z)\overline{f(z)} \right\|_2^2 \\
&= \left\| \sum_{k=1-n}^{n-1} c_k z^k \right\|_2^2 \\
&= |c_0|^2 + 2\sum_{k=1}^{n-1}|c_k|^2 \\
&= n^2 + 2\sum_{k=1}^{n-1}|c_k|^2
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{\|f\|_4}{\sqrt{n}} &= \frac{1}{\sqrt{n}}\left(n^2 + 2\sum_{k=1}^{n-1}|c_k|^2\right)^{1/4} \\
&= \left(1 + \frac{1}{n^2}2\sum_{k=1}^{n-1}|c_k|^2\right)^{1/4} \\
&\leq \left(1 + \frac{1}{n^2}2(\frac{n}{2})\right)^{1/4} \\
&= \left(1 + \frac{1}{n}\right)^{1/4} \\
&< 1 + \frac{1}{4n}
\end{aligned}$$

Therefore, to show that long Barker sequences don't exist, it suffices to show that for all large n, $\|f\|_4 \geq \sqrt{n} + \frac{1}{4\sqrt{n}}$, whenever $f \in L_n$. The following theorem tells us that if there exist longer Barker sequences, then the existence of flat Littlewood polynomials follows. Conversely, if sequences of flat Littlewood polyomials do not exist, then no long Barker sequences exist.

Theorem : *If f is a Littewood polynomial of degree $n-1$ such that the coefficients form a Barker sequence, then*

$$\alpha_1 + O\left(\frac{1}{n}\right) \leq \frac{|f(z)|}{\sqrt{n}} \leq \alpha_2 + O\left(\frac{1}{n}\right)$$

where $|z| = 1$, $\alpha_1 = \sqrt{1-\theta} = 0.524774875...$, $\alpha_2 = \sqrt{1+\theta} = 1.31324459...$, *and*

$$\theta = \sup_{t>0} \frac{\sin^2 t}{t} = 0.7246113537...$$

Proof:

Assume f is Littlewood polynomial of degree $n-1$ and $n > 13$ and write $n = 4N^2 = 4m$. Since the coefficients of f form a Barker sequence of even length, we know that $c_k = -c_{n-k}$, which leads to the following

$$\begin{aligned}
|f(e^{it})|^2 &= \left(\sum_{k=0}^{n-1} a_{k+1}e^{ikt}\right)\left(\sum_{k=0}^{n-1} a_{k+1}e^{-ikt}\right) \\
&= n + \sum_{k=1}^{n-1} c_k(e^{ikt} + e^{-ikt}) \\
&= n + 2\sum_{k=1}^{n-1} c_k \cos(kt) \\
&= n + 2\sum_{k=1}^{4m-1} c_k \cos(kt) \\
&= n + 2\sum_{k=1}^{2m-1} c_k[\cos(kt) - \cos((n-k)t)]
\end{aligned}$$

Using the fact that $\cos(v) - \cos(u) = -2\sin((u+v)/2)\sin((u-v)/2)$ we get

$$\begin{aligned}
|f(e^{it})|^2 - n &= 2\sum_{k=1}^{2m-1} c_k[-2\sin\left(\frac{kt + (4m-k)t}{2}\right)\sin\left(\frac{kt - (tm-k)t}{2}\right) \\
&= 2\sum_{k=1}^{2m-1} c_k[-2\sin(2mt)\sin((k-2m)t)] \\
&= 4\sin(2mt)\sum_{k=1}^{2m-1} c_k \sin((2m-k)t)
\end{aligned}$$

Since $c_{2j} = 0$

$$|f(e^{it})|^2 - n \quad = \quad 4\sin(2mt)\sum_{k=1}^{m} c_{2m-2k+1}\sin((2k-1)t)$$

Dividing both sides by $n = 4m$ we get

$$\begin{aligned} \left|\frac{|f(e^{it})|^2}{n} - 1\right| &= \left|\frac{4\sin(2mt)\sum_{k=1}^{m} c_{2m-2k+1}\sin((2k-1)t)}{4m}\right| \\ &\leq \frac{|\sin(2mt)|}{m}\sum_{k=1}^{m}|\sin((2k-1)t)| \\ &\leq \theta_m \end{aligned}$$

where

$$\theta_m = \max_{0\leq t\leq 2\pi}\frac{|\sin(2mt)|}{m}\sum_{k=1}^{m}|\sin((2k-1)t)|$$

To compute θ_m define ϕ_m and ψ_m as follows

$$\psi_m = \max_{0\leq t\leq \pi/4}\frac{|\sin(2mt)|}{m}\sum_{k=1}^{m}|\sin((2k-1)t)|$$

and

$$\psi_m = \max_{0\le t\le \pi/4} \frac{|\sin(2mt)|}{m} \sum_{k=1}^{m} |\cos((2k-1)t)|$$

so $\theta_m = \max\{\phi_m, \psi_m\}$. Note that the sum in both equations is the midpoint approximation of the integral, i.e.,

$$\frac{1}{m}\sum_{k=1}^{m} |f((2k-1)t)| \approx \int_0^1 |f(2mtx)|dx$$

The error of each approximation per interval, assuming no cusps occur in the inverval, is $O\left(\frac{1}{m^3}\right)$, making the total error $O\left(\frac{1}{m^2}\right)$. If a cusp occurs in the interval, the worst case is when the cusp occurs at the midpoint. If a cusp occurs in the interval, the error is $O(m)$. Thus the error for the approximation is $O\left(\frac{1}{m^2}\cdot m\right) = O\left(\frac{1}{m}\right)$. Therefore,

$$\begin{aligned}
\phi_m &= \max_{0\le t\le \pi/4} |\sin(2mt)| \int_0^1 |\sin(2mtx)|dx + O\left(\frac{1}{m}\right) \\
&\le \sup_{a\ge 0} |\sin(a)| \int_0^1 |\sin(ax)|dx + O\left(\frac{1}{m}\right)
\end{aligned}$$

Letting $a = k\pi + y$, then

$$
\begin{aligned}
\phi_m &= \sup_{k\geq 0} \max_{0\leq y<\pi} |\sin(k\pi+y)| \int_0^1 |\sin((k\pi+y)x)|dx + O\left(\frac{1}{m}\right) \\
&= \sup_{k\geq 0} \max_{0\leq y<\pi} \frac{|\sin(k\pi+y)|}{k\pi+y} \int_0^{k\pi+y} |\sin(x)|dx + O\left(\frac{1}{m}\right) \\
&= \sup_{k\geq 0} \max_{0\leq y<\pi} \frac{\sin(y)}{k\pi+y} \left(k\int_0^{\pi} \sin(x)dx + \int_0^{y} \sin(x)dx \right) + O\left(\frac{1}{m}\right) \\
&= \sup_{k\geq 0} \max_{0\leq x\leq\pi} \frac{(2k+1-\cos x)\sin x}{k\pi+x} + O\left(\frac{1}{m}\right) \\
&= \max_{0\leq x\leq\pi} \frac{(1-\cos x)\sin x}{x} + O\left(\frac{1}{m}\right) \\
&= 0.6639534894... + O\left(\frac{1}{m}\right)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\psi_m &= \max_{0\leq t\leq \pi/4} |\sin(2mt)| \int_0^1 |\cos(2mtx)|dx + O\left(\frac{1}{m}\right) \\
&\leq \sup a \geq 0 |\sin(a)| \int_0^1 |\cos(ax)|dx + O\left(\frac{1}{m}\right) \\
&= \sup_{n\geq 0} \max_{-\pi/2\leq x\leq \pi/2} \frac{(2n-\sin x)|\sin x|}{n\pi+x} + O\left(\frac{1}{m}\right) \\
&= \max_{0\leq x\leq \pi/2} \frac{\sin^2 x}{x} + O\left(\frac{1}{m}\right) \\
&= 0.7246113537... + O\left(\frac{1}{m}\right)
\end{aligned}
$$

Since

$$
\left| \frac{|f(e^{it})|^2}{n} - 1 \right| \leq \theta_m
$$

The theorem follows.

QED.

3.3 Mahler's Problem

Another problem associated with Barker sequences is the problem of maximizing $\frac{\|f\|_0}{\|f\|_2}$. By the three properties of norms developed in the previous section, this ratio will always be less than or equal to 1. This is the so called Mahler problem. Again, for unimodular polynomials, this problem has been solved; meaning that for any $\epsilon > 0$, there exists a unimodular polynomial such that $\frac{\|f\|_0}{\|f\|_2} > 1 - \epsilon$. For Littlewood polynomials, the largest ratio known is given by the polynomial with coefficients from the 13 long Barker sequence above, with the ratio of 0.98636598.... As with the previous theorem, Barker sequences solve this problem. If long Barker sequences exist, this ration gets closer to unity. And, again, if it can be proved that for some N, all Littlewood polynomials of degree $k > N$, then $\frac{\|f\|_0}{\|f\|_2} < 1 - \epsilon$ for some ϵ, then there are only a finite number of Barker sequences.

Theorem: *For a Littlewood polynomial of degree $n-1$, with coefficients forming an n long Barker sequence, then $\frac{\|f\|_0}{\|f\|_2} > 1 - \frac{1}{\sqrt{n}}$*

Let $f(z) = \sum_{k=0}^{n-1} a_{k+1} z^k$, such that$\{a_k\}$ is a Barker sequence. By Parseval, $\|f\|_2^2 = n$, thus $\|f\|_2 = \sqrt{n}$. Also, since we are interested in large n, assume $n > 13$, thus $n = 4N^2$. Since we define $c_{-k} = c_k^*$, we get

$$\begin{aligned}
\|f\|_4^4 &= \left\| f(z)\overline{f(z)} \right\|_2^2 \\
&= \left\| \sum_{k=1-n}^{n-1} c^k z^k \right\|_2^2 \\
&= n^2 + 2\sum_{k=1}^{n-1} |c_k|^2
\end{aligned}$$

In the above sum, there are $n-2$ summands with $n/2$ odd k. Since $c_{2j} = 0$, because n even, and $|c_{2j+1}| = 1$, so the sum equals $n/2$. This implies

$$\|f\|_4^4 = n^2 + n$$

This gets us

$$\begin{aligned}
\int_0^1 \left(\frac{|f(e(t))|^2}{n} - 1 \right) dt &= \int_0^1 \left(\frac{|f(e(t))|^4}{n^2} - 2\frac{|f(e(t))|^2}{n} + 1 \right) dt \\
&= \frac{\|f\|_4^4}{n^2} - 2\frac{\|f\|_2^2}{n} + 1 \\
&= \frac{n^2+n}{n^2} - 2\frac{n}{n} + 1 \\
&= \frac{1}{n}
\end{aligned}$$

Next, since $\log t \leq t-1$, letting $t = a/b$, with $a > b > 0$, we get $\frac{a-b}{b} \geq \log a - \log b$. Letting $a(t) = \max\left\{\frac{|f((t))}{n}, 1\right\}$ and $b(t) = \min\left\{\frac{|f((t))}{n}, 1\right\}$, hence

$$\int_0^1 \left(\frac{|f(e(t))|^2}{n} - 1\right)^2 dt \ \geq \ \int_0^1 b(t)^2 (2\log|f(e(t))| - \log n)^2 dt$$

This integral, combined with the above calculations and the theorem in the previous section yields

$$\frac{1}{n\alpha_1^2} + \frac{1}{n^2} \geq \int_0^1 (2\log|f(e(t))| - \log n)^2 dt$$

where $\alpha_1 = 0.52477...$, from the previous section's theorem. Using Schwarz's inequality yields the following

$$\begin{aligned} \int_0^1 |2\log|f(e(t))| - \log n| dt &\leq \frac{1}{\alpha_1\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \\ \int_0^1 2\log|f(e(t))| - \log n dt &\geq -\frac{1}{\alpha_1\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \\ \int_0^1 \log|f(e(t))| dt &\geq \log\sqrt{n} - \frac{1}{2\alpha_1\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \end{aligned}$$

Since $1/2\alpha_1 = 0.9527895...$ and applying the exponential to each side of the above eqn implies

$$\|f\|_0 \geq \sqrt{n} - \frac{1}{2\alpha_1} + O\left(\frac{1}{n^{3/2}}\right)$$

Further implying

$$\frac{\|f\|_0}{\sqrt{n}} \geq 1 - \frac{1}{2\alpha_1\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) > 1 - \frac{1}{\sqrt{n}}$$

QED

Chapter 4

Physical Application to Radars

4.1 Introduction to Radar Systems

A radar system uses the electromagnetic spectrum to determine the speed and distance of a target. A broader development and discussions of the following topics can be found in [9]. The distance from the radar system and the target called the range is given by the equation, $R = \frac{1}{2}C_p\tau$. Here C_p denotes the radar signal velocity of propogation and τ is the time it takes for the echo of a pulse to reach the radar system; τ is, also, called the delay. Since the radar signal must travel to the target and back to the radar, the equation needs to divide by 2.

To determine the speed at which the target is traveling at the time the signal reaches the target takes a little more thought. Imagine the signal as a sine wave, with leading peak A. Let t_0 be the time A leaves the radar; let R_0 be the range of the target at time t_0; and let Δt be the time it takes for the signal to reach the target. This yields $C_p\Delta t = R_0 + v\Delta t$, where v is the velocity at which the target is traveling. Thus $\Delta t = \frac{R_0}{C_p - v}$. Now let t_1 be the time that A comes back to the radar. This yields $t_1 = t_0 + 2\Delta t = t_0 + \frac{2R_0}{C_p - v}$.

If B it the peak just behind A, then the time that B returns to the radar is given by $t_2 = t_0 + T + \frac{2R_1}{C_p - v}$, where R_1 is the target location at the time B leaves the radar. If T is the period of the sine wave, then $R_1 = R_0 + vT$. Thus the period of the received sine wave is given by

$$T_R = t_2 - t_1 = t_0 + T + \tfrac{2(R_0+vT)}{C_p - v} - (t_0 + \tfrac{2R_0}{C_p - v}) = T\tfrac{C_p+v}{C_p-v}$$

Since the frequency of a signal is given by $1/T$, then the frequency of the received signal is

$$f_R = f_0 \tfrac{1-v/C_p}{1+v/C_p}$$

Since electromagnetic propogation is always much faster than target velocities, we get the following approximation.

$$f_R = f_0 \tfrac{1-v/C_p}{1+v/C_p} = f_0(1 - v/C_p)(1 - v/C_p + (v/C_p)^2 - ...) \approx f_0(1 - 2v/C_p)$$

The last approximation is because $v/C_p >> (v/C_p)^2$. This approximation is rewritten as

$$f_R \approx f_0(1 - 2v/C_p) = f_0 - \tfrac{2v}{C_p/f_0} = f_0 - 2v/\lambda$$

Here λ is the trasmitted wavelength. Hence the Doppler shift is defined as

$$f_D = f_R - f_0 = -2v/\lambda.$$

4.2 Matched Filter

A filter that maximizes the signal to noise ratio (SNR) of the received radar signal is called a matched filter. To study this filter, I will examine a basic narrow bandpass signal and it's envelope. A basic narrow bandpass signal, with a bandlimit of $2W$ and carrier frequence $\pm\omega_c$, is given by

$$s(t) = g(t)\cos(\omega_c t + \phi(t)) \tag{4.1}$$

Here $g(t)$ is called the envelope of the signal and $\phi(t)$ is the instantaneous phase of the signal. It is convenient to write $s(t)$ in various ways depending on the analysis being done. Writing $g_c(t) = g(t)\cos(\phi(t))$ and $g_s(t) = g(t)\cos(\phi(t))$, we can write

$$s(t) = g_c(t)\cos(\omega_c t) - g_s(t)\sin(\omega_c t) \tag{4.2}$$

The equation (4.2) is called the canonical form of the signal. Writing the complex envelope of the signal as $u(t) = g_c(t) + ig_s(t)$, where $i = \sqrt{-1}$, we get a third form of the signal

$$s(t) = \Re(u(t)e^{i\omega_c t}) \tag{4.3}$$

In (4.3), ω_c is chosen such that $u(t)e^{i\omega_c t} = s(t) + i\hat{s}(t)$, where

$$\hat{s}(t) = s(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{s(\tau)}{t-\tau} d\tau$$

the Hilbert transform of $s(t)$.

Now, onto the development of a matched filter. The model used is the signal $s(t)$ with additive white Gaussian noise, with the two sided power spectral density $N_0/2$ added to it. We need to develop the filter, $h(t)$ (or $H(\omega)$, such that

$$\left(\frac{S}{N}\right)_{out} = \frac{|s_0(t_0)|^2}{n_0^2(t)} \tag{4.4}$$

is maximized at some delay t_0 and where $s_0(t)$ is the filtered signal. Thus,

$$s_0(t_0) = \frac{1}{2\pi}\int_{\mathbb{R}} H(\omega)S(\omega)e^{i\omega t_0} d\omega \tag{4.5}$$

Also, the mean squared error, $n_0^2(t)$, which is independent of t and

$$n_0^2(t) = \int_{\mathbb{R}} |H(\omega)|^2 d\omega \tag{4.6}$$

Putting (4.5) and (4.6) together we get

$$\begin{aligned}
\left(\frac{S}{N}\right)_{out} &= \frac{1}{\pi N_0}\frac{|\int_{\mathbb{R}} H(\omega)S(\omega)e^{i\omega t_0}d\omega|^2}{\int_{\mathbb{R}}|H(\omega)|^2 d\omega} && (4.7)\\
&\leq \frac{1}{\pi N_0}\frac{\int_{\mathbb{R}}|H(\omega)|^2 d\omega \int_{\mathbb{R}}|S(\omega)|^2 d\omega}{\int_{\mathbb{R}}|H(\omega)|^2 d\omega} && (4.8)
\end{aligned}$$

$$= \frac{1}{\pi N_0} \int_{\mathbb{R}} |S(\omega)|^2 d\omega \tag{4.9}$$

$$= \frac{2E}{N_0} \tag{4.10}$$

By Parseval

$$E = \int_{\mathbb{R}} |s(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |S(\omega)|^2 d\omega \tag{4.11}$$

The equation (4.8) is due to Cauchy-Schwarz, which states

$$\left| \int_{\mathbb{R}} H(\omega) S(\omega) e^{i\omega t_0} d\omega \right|^2 \leq \int_{\mathbb{R}} |H(\omega)|^2 d\omega \int_{\mathbb{R}} |S(\omega)|^2 d\omega \tag{4.12}$$

with equality if and only if $H(\omega) = KS^*(\omega)e^{-i\omega t_0}$ for some constant K. Thus, $(\frac{S}{N})_{out}$ is maximized when $h(t) = Ks^*(t_0 - t)$. For $h(t)$ to be causal, $t \geq d$, where d is the duration of the signal $s(t)$. With matched filtering, $(\frac{S}{N})_{out} = \frac{2E}{N_0}$, which depends only on the energy, E, of the signal.

Lastly, it should be noted that if $h(t)$ is the filter matched to the signal $s(t)$, then

$$s_0(t) = s(t) * h(t) = \int_{\mathbb{R}} s(\tau) s^*(t_0 - (t - \tau)) d\tau \tag{4.13}$$

This is the autocorrelation function for the signal $s(t)$ whenever $K = 1$ and $t_0 = 0$.

4.3 Development of the Ambiguity Function

From (4.3) we see that $s(t) = \Re(u(t)e^{i\omega_c t}) = 1/2(u(t)e^{i\omega_c t} + u^*(t)e^{-i\omega_c t})$. Using this fourth representation of the signal $s(t)$, then

$$
\begin{aligned}
s_0(t) &= \int_{\mathbb{R}} s(\tau) K s^*[t_0 - (t-\tau)] d\tau \\
&= \frac{K}{4} \int_{\mathbb{R}} [u(\tau) e^{i\omega_c \tau} + u^*(\tau) e^{-i\omega_c \tau}] \\
& \quad [u^*(t_0 - (t-\tau)) e^{-i\omega_c (t_0 - (t-\tau))} + u(t_0 - (t-\tau)) e^{i\omega_c (t_0 - (t-\tau))}] d\tau \\
&= \frac{K}{4} \int_{\mathbb{R}} [u(\tau) u^*(\tau - t + t_0) e^{i\omega_c (t - t_0)} + \\
& \quad u^*(\tau) u^*(\tau - t + t_0)) e^{-i2\omega_c \tau} e^{i\omega_c (t-t_0)} + \\
& \quad u(\tau) u(\tau - t + t_0)) e^{i2\omega_c \tau} e^{-i\omega_c (t-t_0)} + \\
& \quad u(\tau)^* u(\tau - t + t_0) e^{-i\omega_c (t-t_0)}] d\tau \\
&= \frac{K}{2} \Re[e^{i\omega_c (t-t_0)} \int_{\mathbb{R}} u(\tau) u^*(\tau - t + t_0) d\tau] + \\
& \quad \frac{K}{2} \Re[e^{i\omega_c (t-t_0)} \int_{\mathbb{R}} u^*(\tau) u^*(\tau - t + t_0) e^{-i2\omega_c \tau} d\tau]
\end{aligned}
$$

The last equality is due to the fact that in the third equality, the first and last summands are complex cojugates of each other, as are the two middle summands. Also, the second summand in the last equality is the Fourier transform of $u^*(\tau)u^*(\tau - t + t_0)$ evaluated at $2\omega_c$. Since the spectral components of $u(t)$ are cutoff below ω_c this second term can be ignored. This implies

$$
\begin{aligned}
s_0(t) &\approx \frac{K}{2} \Re[e^{i\omega_c (t-t_0)} \int_{\mathbb{R}} u(\tau) u^*(\tau - t + t_0) d\tau] \\
&= \Re\{[\frac{1}{2} K e^{-i\omega_c t_0} \int_{\mathbb{R}} u(\tau) u^*(\tau - t + t_0) d\tau] e^{i\omega_c t_0}\}
\end{aligned}
$$

To get a handle on the notation, define $u_0(t) = K_u \int_{\mathbb{R}} u(\tau) u^*(\tau - t + t_0) d\tau$, where $K_u = 1/2 K e^{-i\omega_c t_0}$. This gets us

$$s_0(t) \approx \Re\{u_0(t)e^{i\omega_c t}\}$$

Now we'll see what happens when Doppler effects are accounted for. Consider $u(t)$ to have the doppler effect ν and define $u_D(t) = u(t)e^{i2\pi\nu t}$. Letting $K_u = 1$ and $t_0 = 0$, we get

$$u_0(t,\nu) = \int_{\mathbb{R}} u(\tau)e^{i2\pi\nu\tau}u^*(\tau - t)d\tau$$

In abuse of the notation reverse the rolls of t and τ in the above equation and changing the sign of τ and define

$$\chi(\tau,\nu) = \int_{\mathbb{R}} u(t)u^*(t+\tau)e^{i2\pi\nu t}dt$$

The ambiguitiy function (AF) is defined as $|\chi(\tau,\nu)|$. A positive value of τ denotes a target at a distance from the radar. A positive value of ν implies that the target is moving towards the radar. The AF is used to analyze radar signals. Signals are searched for to have certain AF properties. Unfortunately, there is no inverse function for the AF. In other words, it is not as straight forward as finding an AF with the desired properties, then inverting to find $u(t)$. Thus, various signals $u(t)$ are studied in the hopes of finding on with an AF with the desired properties.

4.4 Properties of the Ambiguity Function

Four main properties exist that play an important roll in describing the ambiguity function. They describe certain limitation and trade offs when looking for an appropriate radar signal. In investigating these properties, I will assume that $E = \int_{\mathbb{R}} |u(t)|^2 dt = 1$.

Property 1 $|\chi(\tau, \nu)| \leq |\chi(0,0)| = 1$

$$\begin{aligned}
|\chi(\tau, \nu)| &= \left| \int_{\mathbb{R}} u(t) u^*(t+\tau) e^{i2\pi\nu t} dt \right|^2 \\
&\leq \int_{\mathbb{R}} |u(t)|^2 dt \int_{\mathbb{R}} |u^*(t+\tau) e^{i2\pi\nu t}|^2 dt \\
&= \int_{\mathbb{R}} |u(t)|^2 dt \int_{\mathbb{R}} |u^*(x)|^2 dx \\
&= EE \\
&= 1
\end{aligned}$$

where $x = t + \tau$. And since $|\chi(0,0)| = |\int_{\mathbb{R}} u(t) u^*(t) dt| = \int_{\mathbb{R}} |u(t)|^2 dt = 1$, property 1 follows.

Property 2 $\int_{\mathbb{R}} \int_{\mathbb{R}} |\chi(\tau, \nu)| d\tau d\nu = 1$

Let $\nu = -f$, this implies $\chi(\tau, -f) = \int_{\mathbb{R}} u(t) u^*(t+\tau) e^{-i2\pi f t} dt$. Letting $\beta(\tau, t) = u(t) u^*(t+\tau)$, then $\chi(\tau, -f)$ is the Fourier transform of $\beta(\tau, t)$. So by Parseval,

$\int_{\mathbb{R}} |\beta(\tau, t)|^2 dt = \int_{\mathbb{R}} |\chi(\tau, -f)|^2 df = \int_{\mathbb{R}} |\chi(\tau, \nu)|^2 d\nu$. This implies

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\beta(\tau, t)|^2 dt d\tau = \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi(\tau, \nu)|^2 d\nu d\tau = V$$

Let $t = t_1$ and $t + \tau = t_2$ in the left hand side of the above equation, then

$$V = \int_{\mathbb{R}} \int_{\mathbb{R}} |\beta(\tau, t)|^2 dt d\tau = \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t_1)u^*(t_2)|^2 J(t_1, t_2) dt_1 dt_2$$

where

$$J(t_1, t_2) = \begin{vmatrix} \frac{\partial t_1}{\partial t} & \frac{\partial t_1}{\partial \tau} \\ \frac{\partial t_2}{\partial t} & \frac{\partial t_2}{\partial \tau} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus

$$\begin{aligned} V &= \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t_1)u^*(t_2)|^2 dt_1 dt_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t_1)|^2 |u^*(t_2)|^2 dt_1 dt_2 \\ &= \int_{\mathbb{R}} |u(t_1)|^2 dt_1 \int_{\mathbb{R}} |u(t_2)|^2 dt_2 \\ &= EE \\ &= 1 \end{aligned}$$

Proving property 2. This means that modulating a signal to lower the AF in one region of the (τ, ν) plane, the AF must rise in another.

Property 3 $|\chi(-\tau, -\nu)| = |\chi(\tau, \nu)|$

Let $t_1 = t - \tau$, then

$$\begin{aligned}
\chi(-\tau, -\nu) &= \int_{\mathbb{R}} u(t)u^*(t-\tau)e^{-i2\pi\nu t}dt \\
&= \int_{\mathbb{R}} u(t_1+\tau)u^*(t_1)e^{-i2\pi\nu(t_1+\tau)}dt_1 \\
&= e^{-i2\pi\nu\tau}\int_{\mathbb{R}} u(t_1+\tau)u^*(t_1)e^{-i2\pi\nu t_1}dt_1 \\
&= e^{-i2\pi\nu\tau}[\int_{\mathbb{R}} u^*(t_1+\tau)u(t_1)e^{i2\pi\nu t_1}dt_1]^* \\
&= e^{-i2\pi\nu\tau}\chi^*(\tau, \nu)
\end{aligned}$$

Taking the absolute value of both sides proves property 3. Thus, $|\chi(\tau, \nu)|$ is symmetric about the origin.

Property 4 If $u(t) \leftrightarrow |\chi(\tau, \nu)|$, then $u(t)e^{i\pi kt^2} \leftrightarrow |\chi(\tau, \nu - k\tau)|$.

Let $u_1(t) = u(t)e^{i\pi kt^2}$, then $u_1(t) \overset{AF}{\leftrightarrow} |\chi_1(\tau, \nu)|$

$$\begin{aligned}
A_1(\tau, \nu) &= \int_{\mathbb{R}} u_1(t)u_1^*(t+\tau)e^{i2\pi\nu t}dt \\
&= \int_{\mathbb{R}} u(t)e^{i\pi kt^2}u^*(t+\tau)e^{-i\pi k(t+\tau)^2}e^{i2\pi\nu t}dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} u(t)e^{i\pi kt^2}u^*(t+\tau)e^{-i\pi k(t^2+2t\tau+\tau)^2}e^{i2\pi\nu t}dt \\
&= \int_{\mathbb{R}} u(t)u^*(t+\tau)e^{-i\pi k(2t\tau+\tau)^2}e^{i2\pi\nu t}dt \\
&= \int_{\mathbb{R}} u(t)u^*(t+\tau)e^{-i\pi k(2t\tau+\tau)^2}e^{i2\pi\nu t}dt \\
&= e^{-i\pi k\tau^2}\int_{\mathbb{R}} u(t)u^*(t+\tau)e^{i2\pi(\nu-k\tau)t}dt \\
&= e^{-i\pi k\tau^2}\chi(\tau,\nu-k\tau)
\end{aligned}
$$

Taking the absolute value of both sides proves property 4. Modulating $u(t)$ by multiplying by $e^{i\pi kt^2}$ is called linear phase modulation (LFM). In this signal the frequency increases linearly with time. This results in better delay resolution, meaning that uses LFM can tell how far away a target is with better accuracy.

4.5 Basic Radar Signals and Ambiguity of the Signals

A basic radar signal has an envelope, $u(t)$ that is a square pulse of duration T, i.e.

$$
u(t) = \frac{1}{\sqrt{T}}1_{[-T/2,T/2)}(t)
$$

The function $1_E(t) = 1$ whenever $t \in E$ and is 0 whenever $t \notin E$. I analyze the AF for this signal in two separate cases, when τ is positve and when it is negative. In the case that $0 \le \tau \le T$,

$$
|\chi(\tau,\nu)| = \left|\frac{1}{T}\int_{\mathbb{R}} u(t)u^*(t+\tau)e^{i2\pi\nu t}dt\right|
$$

$$
\begin{aligned}
&= \left| \frac{1}{T} \int_{-T/2+\tau}^{T/2} e^{i2\pi\nu t} dt \right| \\
&= \left| \frac{1}{T2\pi i\nu} [e^{2\pi i\nu T/2} - e^{-2\pi i\nu(T/2-\tau)} \right| \\
&= \left| \frac{1}{T2\pi i\nu} e^{-\pi i\nu\tau} [e^{\pi i\nu T(1-\tau/T)} - e^{-\pi i\nu T(1-\tau/T)} \right| \\
&= \left| \frac{\sin\left[\pi i\nu T(1-|\tau|/T)\right]}{\pi T\nu} \right|
\end{aligned}
$$

Similarly, when $-T \leq \tau < 0$, then

$$
\begin{aligned}
|\chi(\tau,\nu)| &= \left| \frac{\sin\left[\pi i\nu T(1+\tau/T)\right]}{\pi T\nu} \right| \\
&= \left| \frac{\sin\left[\pi i\nu T(1-|\tau|/T)\right]}{\pi T\nu} \right|
\end{aligned}
$$

Hence, for all $-T \leq \tau \leq T$ and muliplying by $1 = \frac{1-|\tau|/T}{1-|\tau|/T}$

$$
|\chi(\tau,\nu)| = \left| (1-|\tau|/T) \frac{\sin\left[\pi T\nu(1-|\tau|/T)\right]}{\pi T\nu(1-|\tau|/T)} \right|
$$

The purpose of multiplying by $\frac{1-|\tau|/T}{1-|\tau|/T}$ is to study what happens to the AF as ν approaches 0. Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we see that

$$
|\chi(\tau,0)| = 1 - |\tau|/T
$$

which is the autocorrelation of $u(t)$ and is the triangular function which is zero at $\tau \leq -T$, increasing to 1 at $\tau = 0$, decreasing to 0 at $\tau \geq T$.

Along the Doppler axis, when $\tau = 0$ the AF is

$$|\chi(0, \nu)| = \left| \frac{\sin(\pi \nu T)}{\pi \nu T} \right|$$

a decaying sine wave with nulls at $\nu = n/T$ for all $n \geq 1$.

The delay of this signal can be improved by LFM. Let

$$s(t) = u(t) e^{i\pi k t^2}$$

with $k = \pm \frac{B}{T}$ then by property 4 of the AF

$$|\chi(\tau, \nu)| = \left| (1 - |\tau|/T) \frac{\sin[\pi T(\nu \pm B(\tau/T))(1 - |\tau|/T)]}{\pi T(\nu \pm B(\tau/T))(1 - |\tau|/T)} \right|$$

where $|\tau| \leq T$ and zero elsewere. Thus the autocorrelation function of $s(t)$ is given by

$$|\chi(\tau, 0)| = \left| (1 - |\tau|/T) \frac{\sin[\pi B \tau(1 - |\tau|/T)]}{\pi B \tau(1 - |\tau|/T)} \right|$$

This gives a better range resolution. For example, for the simple pulse $u(t)$ with a pulse width of $T = 5\mu s$, then $\Delta R = C_p T/2 = (1/2)(3 \times 10^8 m/s)(5 \times 10^{-6} s) = 750m$. This resolution is not very good. Using this signal, to tell the difference between two different targets, they would have to be more than half a mile apart. Any closer and the range resolution blurs the two together. Modulating $u(t)$ into $s(t)$ with LFM yields better range resolution based on the product TB, the pulse compression. If $TB = 100$, range resolution reduces to $7.5m$, an acceptable level for many purposes.

4.6 Barker Sequences in Radar Signals

In addition to LFM, another moduation scheme is phase coding. This is done by dividing T into n equally spaced time units and changing the phase, instead of the frequency, of the signal in each time unit. This signal is defined by

$$u(t) = \frac{1}{\sqrt{T}} \sum_{j=1}^{n} u_j 1_{E_j}(t)$$

where E_j is the j^{th} interval in the $[-T/2, T/2)$, which is divided into n equally spaced intervals, and $u_j = e^{i\theta_j}$. You can see where this is going. Letting $\theta_j \in \{0, \pi\}$, then $u_j = \pm 1$. Also, analyzing the AF can be very difficult, which leads to the study of the autocorrelation function. The goal is to find a signal with a spike in the autocorrelation function at $\tau = 0$ and zero everywhere else. This, however, is impossible by property 2 of the AF (the autocorrelation function is simply $\chi(\tau, 0)$). Because of this, the search is made for a signal that has a narrow main lobe with side lobes as small as possible. Making the obvious choice of $\{u_j\}$ a Barker sequence.

The autocorrelation of a signal $u(t)$ is defined by

$$R_u(\tau) = \int_{\mathbb{R}} u(t)u^*(t+\tau)dt$$

Note that for any real signal $R_u(\tau) = R_u(-\tau)$. I'm going to make some simplifying adjustments to the above signal to make the notation a little easier to cope with. I'm going to ignore the normalizing factor, let $T = n$, and divide the interval $[0, n)$ into n itervals with unit length, making $E_j = [j-1, j)$. These simplifications yield

$$\begin{aligned} R_u(\tau) &= \int_{\mathbb{R}} u(t)u^*(t+\tau)dt \\ &= \int_{\mathbb{R}} \left(\sum_{k=1}^{n} u_k 1_{E_k}(t) \sum_{j=1}^{n} u_j 1_{E_j}(t+\tau) \right) dt \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} u_k u_j \int_{\mathbb{R}} 1_{E_k}(t) 1_{E_j}(t+\tau) dt \end{aligned}$$

The integral $\int_{\mathbb{R}} 1_{E_k}(t)1_{E_j}(t+\tau)dt = 1-|j-\tau-k|$ whenever $|j-\tau-k| \leq 1$ and is zero otherwise. Note that this integral takes a maximum when $j - \tau = k \Rightarrow \tau = j - k$. Thus, the integral creates a function of τ that is a triangle with a center at $j - k$ and base width of 2, so the autocorrelation function only needs to be computed for

τ an integer. For non-integer values, $R(\tau)$ is the linear interpolation of the integer values between the $a = \lfloor \tau \rfloor$, the integer floor of τ, and $a + 1$. Setting $\tau = s + \eta$, where s an integer and $0 \leq \eta < 0$. Therefore,

$$\begin{aligned} R_u(\tau) &= R_u(s+\eta) \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} u_k u_j \int_{\mathbb{R}} 1_{E_k}(t) 1_{E_j}(t+s+\eta)dt \\ &= (1-\eta)c_s + \eta c_{s+1} \end{aligned}$$

where for any integer s, $c_s = \sum_{i=1}^{n-s} u_i u_{i+s}$. Therefore, to compute the continuous autocorrelation, $R_u(\tau)$, it is enough to only compute the discrete aperiodic autocorrelation function on the sequence $\{u_i\}_{=1}^{n}$. Since we are looking for a sequence that minimizes non-zero delays in the autocorrelation function, Barker sequences are sought after. The more general formula for the autocorrelation function is

$$R_u(\tau) = \frac{1}{bT}[(b-\eta)c_s + \eta c_{s+1}]$$

where $b = T/n$ and $\tau = bs + \eta$, where s is an integer and $0 \leq \eta < b$. The shape of this autocorrelation function, in contrast to the autocorrelation of a simple pulse, obviously has better range resolution. The autocorrelation of a basic pulse is one wide triangle, with the base being the width of the pulse. Whereas for a phase coded pulse using a Barker sequence, the autocorrelation has a main lobe, where the function attains its maximum value, with a width of T/n and minimal height side lobes. In fact the ratio of side lobes and the main lobe will be less than $1/n$.

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