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The pricing of multiple exercisable American-style real options

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THE PRICING OF MULTIPLE EXERCISABLE AMERICAN-STYLE REAL OPTIONS

by

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ABSTRACT

Real options embedded in a project provide management with the flexibility to alter initial investment decisions, thus making them a practical tool for project planning and budgeting. Additional values are contributed to the underlying project due to the flexibilities that are provided by real options.

This dissertation presents two models for pricing multiple exercisable American real options, one that employs the binomial tree method and the other one that employs the finite difference method. Different examples of multiple exercisable real options are discussed to demonstrate the two pricing models. Interactions between options and reality constraints are also considered. These two methods are compared with each other at the end.

This dissertation also addresses the problem of tracking early exercise boundaries in pricing American-style real options. It is shown that both models provide effective numerical solutions to the free boundary problem.

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1. INTRODUCTION

One of the most frequently used techniques for the appraisal of capital investment projects is the traditional discounted cash flows approach. For instance, the net present value (NPV) of a potential investment is determined by discounting a time series of future cash flows by a discount rate. The standard NPV method assumes that forecasted cash flows are static and investment decisions cannot be changed once they are made. Such assumptions ultimately ignore the flexibility embedded in investment opportunities. In reality, however, management often has options to alter initial business decisions as more information about market conditions and future cash flows becomes available (Trigeorgis, 1996). Real options, which borrow the mathematical techniques from financial theories, were introduced to overcome the inflexibility of the NPV method and thus, offer a better alternative method for capital investment appraisal.

1.1. FINANCIAL OPTIONS

In finance, an option is a derivative which has become increasingly popular in the last 30 years. A derivative can be defined as a financial instrument whose value depends on (or derives from) the values of other underlying assets, for instance, a stock (Hull, 2009). The assets underlying derivatives are usually tradable. A financial option is a contract that guarantees a future transaction of an asset between two parties for a specified price (i.e., the strike price or exercise price). There are two types of options. A call option gives the holder the right to buy an asset for the strike price on or before a certain date, which is called the expiration date. A put option gives the holder the right to sell an asset for the strike price on or before a specified expiration date. Note that the buyer of the option gains the right to buy or sell the asset; however, on the other hand, the seller of the option has the obligation to fulfill the transaction. Options are categorized as European options if they can only be exercised on the expiration date; or as American options if they can be exercised at any time up to the expiration date. An option that isn't exercised by expiration becomes void. When an option is exercised, the payoff is then realized and its value equals the difference between the stock price and the strike price.

There are six factors that affect the price of an option (Copeland and Antikarov, 2001; Hull, 2009).

- The current price of the underlying asset. The payoff of a call option is the amount by which the stock price exceeds the strike price, and vice versa for a put option. When the stock price increases, the call option becomes more valuable and the put option becomes less valuable.
- The strike price. The value of a call option decreases as the strike price increases. The value of a put option increases as the strike price increases.
- The time to expiration. Both American calls and American puts get more exercise opportunities open when they have a longer time to expiration. An American option with a longer life is usually more valuable than, or at least worth as much as, an option with a shorter life. Although, in the case of a dividend-paying European call, the dividend will cause the decline in the stock price and thus make the option less valuable; generally speaking the values of European calls and puts increase as the time to expiration increases.

- The volatility of the underlying asset's price. The volatility of a stock price is the standard deviation of the continuous compounding return of the stock in one year. It is a measure of the extent of future stock price movements. In other words, it is a measure of the risk associated with the stock price. As volatility increases, the chance that an option will be exercised increases. Therefore, both calls and puts become more valuable as volatility increases.
- The risk-free interest rate. The expected return from the stock tends to increase as the risk-free interest rate increases. The value of a call option increases as the interest rate increases, and the value of a put option decreases as the interest rate increases.
- The dividends. The stock price can be reduced by the dividends on the ex-dividend date. Therefore the value of a call option is negatively impacted by the future dividends to be paid out during the life of the option, and the value of a put option is positively impacted by the anticipated future dividends.

The six factors and their effects on the option values are summarized in Table 1.1.

1.2. REAL OPTIONS

Real options extend the principles of financial options to "access capital investment opportunities in real assets such as land, buildings, plants, and equipment" (Hull, 2009). By analogy, real options give management the right, but not the obligation, to undertake a certain business decision before or on a specified expiration date when

Table 1.1. Summary of the Effect on the Price of a Stock Option by Increasing One Variable while Keeping All Others Fixed

Variables	*European call*	*European put*	*American call*	*American put*
Current stock price	+	−	+	−
Strike price	−	+	−	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	−	+	−
Amount of future dividends	−	+	−	+

+ : indicates that an increase in the variable causes the option price to increase;

-: indicates that an increase in the variable causes the option price to decrease;

?: indicates that the relationship is uncertain.

Source: Hull, J.C., *Options, Futures, and Other Derivatives*. New Jersey: Pearson Prentice Hall, (2009). pp. 202.

favorable information for exercising the right arrives. The underlying assets of financial options are usually tradable securities, such as a stock, while the underlying of real options are tangible assets such as a project or an investment.

The term "real options" was first coined by Steward Myers (1977). After the fundamental Black-Scholes model was developed by Fisher Black, Myron Scholes, and Robert Merton (1973) for valuing financial options, Myers pointed out the similarities between financial options and real options and that the analytical techniques for financial options could be applied to real options (Zeng and Zhang, 2011).

The valuation of real options is also known as real option analysis. Real option analysis has been an active area for academic research for the past two decades. Trigeorgis (1996), Copeland (2001), Dixit (1994), and Mun (2006) are among those who have published influential books and articles in this area.

One of the most important characteristics of real option analysis is that it takes uncertainty into account, thus making it a practical tool for making decisions under uncertainty. The traditional capital budgeting techniques often treat an investment as a now-or-never decision, thus ruling out any possibility of future change in management. In practice, many investment decisions may be put off until more information about market conditions becomes available. This gives decision makers more flexibility when confronted with unexpected market developments. Real options provide a framework to analyze strategic capital investment by viewing management flexibility as valuable opportunities (Dixit and Pindyck, 1994).

Many applications of this discipline in the business world include R&D (Lewis, 2004), patent valuation (Macro, 2005), energy management (Tseng and Lin, 2007), and workforce management (Nembhard et al., 2005).

Trieorgis (1996) classified real options into eight categories and gave definitions and examples of each in his book. The most commonly seen real options include:

- The option to defer investment. A deferral option allows management to put off a decision for up to a certain time period. Management will only commit to the investment when the price of the underlying asset rises sufficiently;

otherwise, it will decline the project. Just before the end of the specified time period, the investment opportunity's value will be max $(V - I_1, 0)$, where V is the gross present value of the completed project's expected operating cash flows and I_1 is the exercise price that equals to the required cost. The option to defer is thus analogous to an American call option. Examples of deferral options can be found in resource extraction industries, farming, paper products, and real estate developments (Macdonald and Siegel, 1986; Paddock et al., 1988; Tourinho, Titman, 1985; Ingersoll and Ross, 1992).

- The option to expand. Expansion options allow management to expand production scale by a certain percentage $(x\%)$ when market conditions are in favor of an investment. An expansion option is similar to a call option to acquire an additional part $(x\%)$ of the base project (V) by paying a follow-on cost (I_E). The value of the investment opportunity with the option to expand is therefore $V + \max(xV - I_E, 0)$. For example, when an oil company purchases vacant undeveloped land or builds a small plant in a new geographic location in anticipation of the rise of oil price, it essentially installs an option for future growth to take advantage of a developing large market. Examples of expansion options are popular in mining, facilities planning, and construction in cyclical industries, consumer goods, commercial real estate, fashion apparel, etc. (Trigeorgis and Mason, 1987; Pindyck, 1988; McDonald and Siegel, 1985; Brennan and Schwartz, 1985).

- The option to contract. Contraction options allow management to contract production scale by a certain percentage when market conditions go against an investment. If it turns out that the market goes weaker than expected, management can operate below capacity, or even reduce the scale of operations (by c%), thereby saving part of the planned investment cost (I_C). An option to contract is similar to a put option on part (c%) of the base project (V), with exercise equal to the potential cost savings (I_C), giving max ($I_C - cV, 0$). Contraction options, like expansion options, may be particularly valuable for new product introduction in uncertain markets.

- The option to abandon for salvage value. When an investment suffers unfavorable market conditions or poor operation, it can be valuable to have an option to abandon the project permanently in exchange for its salvage value, so that management does not have to continue incurring the fixed costs. An abandonment option can be viewed as an American put option on the base project (V), with an exercise price equal to the salvage value (A), giving max ($A - V, 0$). Examples of abandonment options can be found in capital-intensive industries such as airlines and railroads, in financial services, and in new-product introductions in uncertain markets (Myers and Majd, 1990).

Besides the four options described above, there are the time-to-build option, the option to switch use, the option to shut down and restart operations, and the corporate growth option. The descriptions of these options can also be found in Trigeorgis' book.

In practical situations, an investment opportunity may offer more than one option. A generic example might include a collection of real options that would offer management the flexibility to defer the project, abandon the project, contract or expand the operation scale, extend the life of the investment, or switch the use of the investment.

Such a generic project, with its multiple real options, could characterize many real life situations (Trigeorgis, 1996). See Figure 1.1.

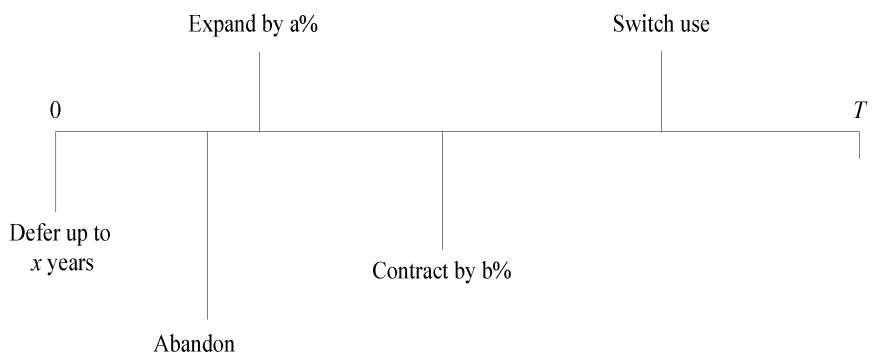

Figure 1.1. A Generic Project with Multiple Real Options

Source: adapted from Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, Cambridge: The MIT Press, (1996), pp. 9.

1.3. OPTION PRICING

The value of a real option lies in the managerial flexibility provided to the investor who holds the option. Unlike a financial option, which doesn't affect the price of the underlying security, real options contribute new values to a real asset when they are exercised.

Similar to financial options, the valuation of real options often involves five primary variables: the value of the underlying risky asset (a project, or an investment opportunity), the exercise price (implementation cost), the time to expiration date, the risk-free rate of interest over the life of the option, and the volatility of future returns (Copeland and Antikarov, 2001; Lewis, 2004).

The approaches developed for valuing financial options may be adapted to price real options where certain modifications are required to make them more plausible for real business practice. The most basic model for option pricing is the Black-Scholes-Merton model. Black and Scholes derived a differential equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. By constructing a risk neutral portfolio to replicate the returns of holding an option, Black and Scholes produced a closed-form solution for a European option's theoretical price. Benaroch (1999) examined the validity and constraints of the Black-Scholes option pricing model in assessing information technology project investments. However, due to its assumptions of non-dividend paying, constant volatility and a constant interest rate, the implementation of the Black-Schole model is of limited use and only suitable for the case of European options. Generally speaking, closed form solutions, including modifications to the Black-Scholes model, are hard to find for American options. Nonetheless, the derivation of the Black-Scholes-Merton model is one of the most important breakthroughs for the existing financial market and provides foundations for other pricing techniques, such as the finite difference methods.

There have been some useful and practical numerical approaches developed for option pricing problems, including partial differential equation (PDE) methods, lattice methods (e.g., binomial trees, trinomial trees, etc.), and simulation process (e.g., Monte Carlo simulation). The binomial tree method is one technique that is most commonly used for pricing real options due to several advantages. First, it can be used to price American Options. Second, it is easy to implement without much advanced mathematical background. Third, it is able to handle more complicated situations where multiple options exist. Brandão et al. (2005) used dynamic programming to solve a binomial decision tree with risk-neutral probabilities that approximated the uncertainty associated with the changes in the value of a project over time.

Finite difference methods for partial differential equations can formulate the value of a real option. Finite difference methods typically require mathematical sophistication and, thus, are not as straightforward as the binomial tree approach. However, they are often chosen for models and securities that are more complicated. The additional computational cost needed by the finite difference methods is mostly compensated for by the additional accuracy in the American option prices (Broadie and Detemple, 2004).

Many research works have attempted to solve the evaluation problem involving a single real option. Berger et al. (1996), for example, used an empirical approach to price an abandonment option based on a firm’s stochastic liquidation value after controlling for the relation between the market value and the present value of expected cash flows. Luehrman (1998) applied a Black-Scholes option pricing table to value an expansion option as a European call by creating new metrics. Kellogg and Charnes (2000) used the decision tree method as well as the binomial lattice method to evaluate a biotechnology company. In the real world, however, an investment opportunity often includes more than one option. In such cases, properly valuing multiple real options as a collection presents quite a challenge, due to the complexity of this problem. Chiara et al. (2007) studied the pricing of simple multiple exercisable real options that could only be exercised at discrete points over a predetermined time period. The value of the options satisfied a Bellman equation by dynamic programming arguments. The least-square Monte Carlo method was employed to approximate the option value. Leung and Sircar (2009) studied American option pricing with multiple exercises in an incomplete market. The value function (option price) was given recursively and the optimal exercise boundary was characterized via indifference prices for holding multiple options.

1.4. CONTRIBUTIONS

This dissertation provides two fast and accurate methods for pricing multiple exercisable American-type real options. The first one employs the binomial tree method and the other one employs the finite-difference method. Both methods are adaptable to different combinations of multiple options and, thus, can apply to various real life situations.

This dissertation also addresses the problem of tracking early exercise boundaries in pricing American-style real options, which is not always covered by other option pricing models. The flexibility of both the binomial tree method and the finite-difference method makes it possible to examine for early exercise at every point in an option's life.

It is shown that both methods provide effective numerical solutions to the free boundary problem. The free boundaries that result from the solution are crucial for decision support.

The algorithms used in this dissertation are novel and provide important insights for project planning and budgeting. Industry is intrigued by option pricing, but is reluctant to incorporate it into its decision making. This study is to help bridge the gap between existing academic knowledge and industry practice.

2. PRICING MODEL WITH BINOMIAL TREES

The objective of this study is to investigate the effectiveness of the binomial tree method in both valuing multiple options embedded in an investment opportunity and accounting for the interactions among them.

The rest of this chapter is organized as follows. Section 2.1 briefly reviews the binomial tree methods with applications in financial options. Section 2.2 explains in detail the scheme of valuing multiple excisable real options with binomial trees. Section 2.3 presents the results from the pricing model. Section 2.4 provides final conclusive remarks.

2.1. BINOMIAL TREES

One popular technique for pricing financial options is the binomial tree method developed by Cox, Ross, and Rubinstein (1979). The derivation of this method involves establishing a risk-free portfolio. This portfolio consists of both a long position (buy) in a number of shares of a stock and a short position (sell) in one call option on that stock. The stock price will either move up by a certain percentage or down by a certain percentage for a specified time period. The underlying assumption is that arbitrage opportunities do not exist for this portfolio, which guarantees that there will be only two possible outcomes and, therefore, rules out any uncertainty about the value of the portfolio at the end of the time period (Hull, 2009). The binomial tree method uses the risk-neutral valuation, which means that the return of the portfolio equals the risk-free interest rate since it has no risk. This makes it possible to work out the cost of setting up the portfolio by discounting the value with the risk-free interest rate at the end of the time period. The option price can then be easily obtained.

Binomial tree pricing models are widely used by practitioners in the option markets. The Black-Scholes model, though mathematically sophisticated, cannot handle options that are more complex than a non-dividend European option. The binomial tree method, on the contrary, is relatively simple, can be easily implemented and handles a variety of conditions, including American options and dividend paying options.

According to Cox et al. (1979), because both the Binomial tree method and the Black-Scholes model use similar assumptions, as well as risk-neutral valuation, the former can work as a discrete time approximation for the continuous process underlying the latter. In a limiting case, for pricing a European option without dividends, the result from the binomial tree method converges to that from the Black-Schole model when the time steps increase.

2.1.1. Geometric Brownian Motion. To understand the pricing of financial options, the behavior of underlying stocks needs to be investigated first. The assumption underlying the binomial tree valuation is that the stock price follows a random walk (Cootner, 1964).

A stock market price is said to follow a stochastic process called geometric Brownian motion, a specific form of the generalized Wiener process.

The following overview of stochastic processes can be found in Hull’s book (2009).

A variable x is said to follow a Wiener process if it has the following properties:

- The change Δx during a small period of time Δt is

$$\Delta x = \epsilon\sqrt{\Delta t} \tag{2.1}$$

where ϵ has a standard normal distribution $\phi(0,1)$.

- The values of Δx are independent for any two different intervals of time Δt.

From the first property, it can be concluded that Δx has a normal distribution with a mean of zero and a standard deviation of $\sqrt{\Delta t}$ or a variance of Δt.

A generalized Wiener process for a variable z is defined as

$$dz = mdt + ndx \tag{2.2}$$

where both m and n are constant, and x follows the basic Wiener process defined above. For a stochastic process, the change in the mean value per unit of time is called the drift rate, and the variance per unit of time is called the variance rate. From the first property of the basic Wiener process, dx has a drift rate of zero and a variance rate of 1.0. The generalized Wiener process has a drift rate of m and a variance rate of n^2.

The most widely used model for the price of a non-dividend-paying stock is known as the geometric Brownian motion, which has the form as

$$dS = \mu Sdt + \sigma Sdx \tag{2.3}$$

or

$$\frac{dS}{S} = \mu dt + \sigma dx \quad (2.4)$$

Because binomial trees are applied as a numerical method for modeling discreet variables, it is more appropriate to use the discrete-time version of the geometric Brownian motion

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta B_t \quad (2.5)$$

where

$$\Delta S_t = S_{t+\Delta t} - S_t$$

$$\Delta B_t \equiv B_{t+\Delta t} - B_t$$

For a small time interval Δt, the variable ΔS_t is the change in the stock price S_t, and the variable ΔB_t follows a normal distribution with a mean of zero and a variance of Δt. The parameter μ is the expected rate of return of the stock and the parameter σ is the volatility of the stock price.

2.1.2. One-Step Binomial Trees. Binomial trees are diagrams constructed to present different paths that the stock price might follow over the life of an option (Hull, 2009).

To construct a binomial tree, the life of an option is divided into a finite number of time steps of equal length. In each time step, there is a certain probability that the stock price will move up by a certain percentage and a certain probability that it will move down by another percentage. The rest of this section demonstrates the computational scheme of binomial trees. More details can be found in Hull (2009).

A generalized one-step binomial tree is illustrated in Figure 2.1. Consider a risk-free portfolio consisting of a long position in Δ shares of a stock and a short position in one call option. At time 0, the present price of the stock is S_0 and the present price of the option is *f*. The option has an expiration of time *T*. During the life of the option, the stock price can either move up from S_0 to a higher level, S_0u, where u > 1, or down from S_0 to a lower level, S_0d, where d < 1. The probability of an upward movement is q, and the probability of downward movement is 1-q. When the stock price reaches S_0u, the payoff from the option is f_u; when the stock price reaches S_0d, the payoff from the option is f_d. It is assumed that there are no arbitrage opportunities present in this situation.

The number of time steps is denoted by *n*, and the length of the time interval for a one-step tree is calculated as

$$\Delta t \equiv \frac{T}{n} \equiv T \tag{2.6}$$

The parameters *u* and *d* can be chosen to match the volatility of the stock price *σ*. The values of *u* and *d*, as proposed by Cox, Ross, and Rubinstein (1979), are

$$u = e^{\sigma\sqrt{\Delta t}} \tag{2.7}$$

and

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{2.8}$$

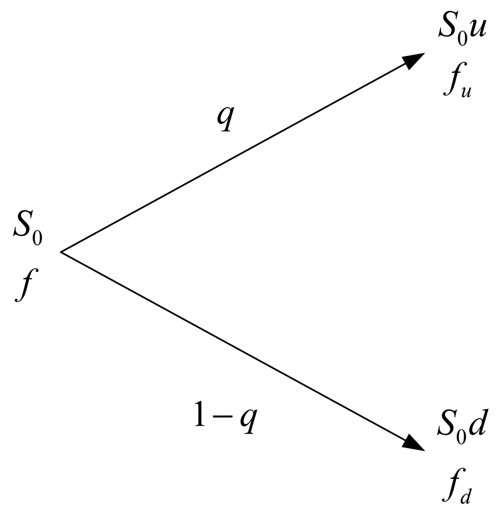

Figure 2.1. A General One-Step Binomial Tree

Source: Adapted from Hull, J.C., *Options, Futures, and Other Derivatives.* New Jersey: Pearson Prentice Hall, (2009). pp. 239.

At the end of time T, the value of the portfolio when S_0 reaches S_0u is

$$\Delta S_0 u - f_u \ ,$$

and the portfolio value when S_0 reaches S_0d is

$$\Delta S_0 d - f_d \ ,$$

respectively.

Since the portfolio is riskless so the final value is the same for both outcomes, which means

$$\Delta S_0 u - f_u = \Delta S_0 d - f_d \tag{2.9}$$

or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{2.10}$$

Delta is called the hedging ratio, which is the ratio between the change in the option value and the change in the stock price between nodes.

The cost of setting up the portfolio is

$$\Delta S_0 - f \ .$$

Since the portfolio is risk-free and there is no arbitrage opportunity, the rate of return must equal the risk-free rate, which is denoted by *r*, and the present value of the portfolio is

$$e^{-rT}(\Delta S_0 u - f_u) .$$

It follows that

$$\Delta S_0 - f = e^{-rT}(\Delta S_0 u - f_u) \tag{2.11}$$

which leads to

$$f = \Delta S_0(1 - ue^{-rT}) + e^{-rT} f_u \tag{2.12}$$

Substituting equation (2.10) for delta, equation (2.12) becomes

$$f = \frac{f_u(1-e^{-rT}d)}{(u-d)} + \frac{f_d(e^{-rT}u-1)}{(u-d)} \tag{2.13}$$

which leads to

$$f = e^{-rT}[\frac{f_u(e^{rT}-d)}{(u-d)} + \frac{f_d(u-e^{rT})}{(u-d)}] \tag{2.14}$$

If p is set as

$$p = \frac{e^{r\Delta t} - d}{u - d} \equiv \frac{e^{rT} - d}{u - d} \tag{2.15}$$

then equation (2.14) can be simplified as

$$f = e^{-rT}[pf_u + (1 - p)f_d] \tag{2.16}$$

Suppose the exercise price of a call option is K. If the stock price goes up to S_0u, then the payoff of the call option, f_u, is

$$f_u = \max(S_0u - K, 0) \tag{2.17}$$

If the stock price moves down to S_0d, then the payoff of a call option, f_d, is

$$f_d = \max(S_0d - K, 0) \tag{2.18}$$

Substituting equation (2.16) with equations (2.17) and (2.18), the option value, f, at time 0 is discounted as

$$f = e^{-rT}[\max(S_0u - K, 0)\, p + \max(S_0d - K, 0)\, (1 - p)] \tag{2.19}$$

According to Cox, Ross and Rubinstein (1979), the equation for the option price has several notable features. First, the probability of an upward movement, q, does not appear in the equation, which implies that, even if different investors have different subjective probabilities about an upward or downward movement in the stock, the stock price will come out the same. It is natural to assume that the price of a call option increases as the probability of an upward movement does. However, these two variables are irrelevant. The reason is that the stock price itself already incorporates the probabilities of up or down movements. As a result, one will only need to take the stock price into consideration when calculating the option value. The second notable feature is that the option price does not depend on investors' risk preferences. It is not surprising that any individual prefers more wealth to less wealth and is willing to take riskless arbitrage opportunities to earn more profits. However, in this case, the underlying assumption is that arbitrage opportunities are eliminated for all investors. Therefore, the option pricing formula is obtained regardless of investors' attitudes towards risk. Finally, the only random variable that the option price depends on is the price of the underlying stock. The value of an option is calculated in terms of S, u, d, and r.

2.1.3. Two-Step Binomial Trees. The derivation of the option pricing formula can be extended to the valuation of an American call using a generalized two-step binomial tree. The evolution of the stock price is shown in Figure 2.2.

The length of the time intervals for a two-step tree is

$$\Delta t \equiv \frac{T}{n} \equiv \frac{T}{2} \tag{2.20}$$

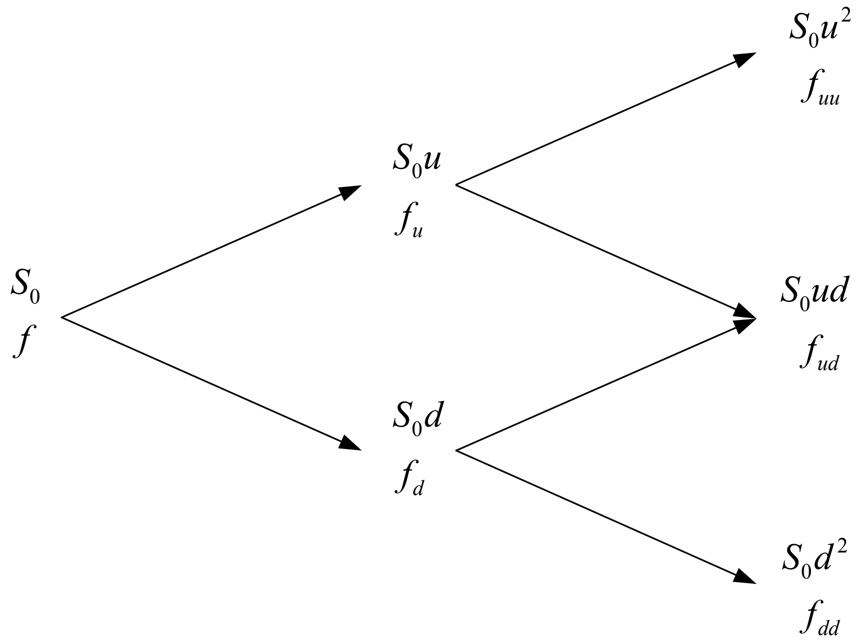

Figure 2.2. A Two-Step Binomial Tree

Source: Hull, J.C., *Options, Futures, and Other Derivatives.* New Jersey: Pearson Prentice Hall, (2009). pp. 245.

For an American call option, the payoffs of the option at the final nodes, f_{uu}, f_{ud}, and f_{dd}, are calculated as follows

$$f_{uu} = \max\,(S_0u^2 - K, 0) \tag{2.21}$$

$$f_{ud} = \max\,(S_0 ud - K, 0) \tag{2.22}$$

$$f_{dd} = \max\,(S_0 d^2 - K, 0) \tag{2.23}$$

The payoffs at the first time step, f_u and f_d, are

$$f_u = \max\,\{e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}], (S_0 u - K)\} \tag{2.24}$$

and

$$f_d = \max\,\{e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}], (S_0 u - K)\} \tag{2.25}$$

At time 0, the value of the American call option, f, is

$$f = \max\,\{e^{-r\Delta t}[pf_u + (1-p)f_d], (S_0 - K)\} \tag{2.26}$$

Substituting from equations (2.24) and (2.25) into equation (2.26) yields the value of the American call option

$$f = \max\,\{e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}], (S_0 - K)\} \tag{2.27}$$

The n-step ($n > 2$) binomial tree will be constructed in a similar way, and will be studied in the following section.

2.2. PRICING MULTIPLE OPTIONS

This section demonstrates the general framework for pricing multiple options using different combinations of real options. Because deferral option, expansion option, contraction option, and abandonment option are the most commonly seen real options, the combinations of two or more of these options can usually characterize many real life situations. An example consisting of an expansion option and a contraction option will be discussed in details, and the same procedure can be easily applied to similar multiple options.

The first example is a multiple option consisting of an expansion option and a contraction option. Suppose management has the option to expand the scale of operation with an investment cost, K_E, and the option to contract the scale of operation for a cost savings, K_C, during the life of the project. Since the two options can be exercised at or before the end of the project, they both fall into the category of American option. The expansion option can be considered an American call, and the contraction option can be considered an American put. The payoff of the expansion is$(aS_t - K_E)^+$, with aS_t being the new production capacity after expansion. The payoff function for the contraction option is $(K_C - bS_t)^+$, with bS_t being the new production capacity after contraction.

A binomial tree consisting of n stages will be constructed for the purpose of calculation. At each node in a single stage, the combined value of the contraction option and the expansion option, $f_{E\&C}$, is calculated backward using the values from a later stage. The process will be repeated until the starting point is reached. If n is large enough, the exercise areas and the hold area can be determined. The computation process is illustrated in Figures 2.3, 2.4, and 2.5.

First, the life of the project is divided into n steps. The gross project value S_{00} evolves over 5 years, as shown in Figure 2.3. The term $S_{i,j}$ denotes the project value at the jth node (j=0,1,…, i) at time $i\Delta t$ (i=0,1,…,n), and it is calculated as $S_0 u^i d^{\,i-j}$ (see note in Figure 2.3).

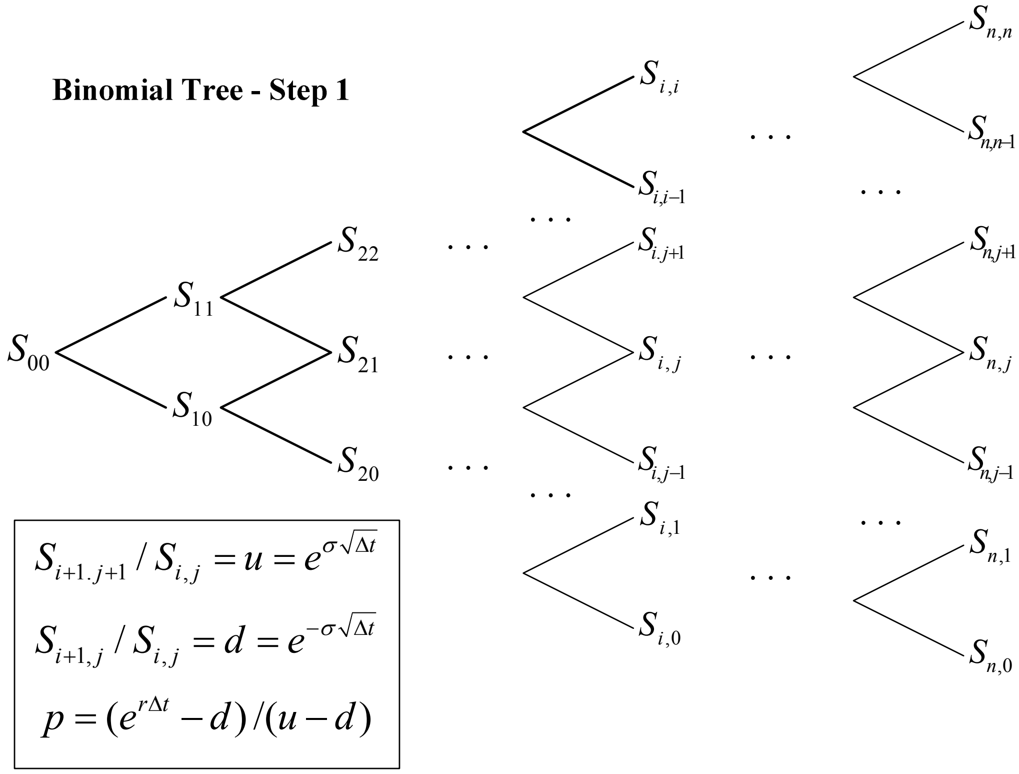

Figure 2.3. Lattice Evolution of the Price of Underlying Asset

Next, the valuation of the first option, the expansion option, is worked back through the binomial tree from the end to the beginning. Since this is an American option, each node needs to be examined to see if early exercise is optimal. The option value at final nodes, $C_{n,j}$, is the same for the American option as it is for the European option and, thus, can be determined by the call payoff function as

$$C_{n,j} = \max\,(aS_{n,j} - K_E, 0) \tag{2.27}$$

where a is the factor by which the production scale of the project is expanded with a cost K_E.

At earlier nodes, i.e., the nodes at time $i\Delta t$, the value of the option $C_{i,j}$ is the greater of

- the expected value at time $(i+1)\Delta t$ discounted for a time period Δt at rate r, which is given by equation (2.16), or
- the payoff from early exercise.

Thus $C_{i,j}$ can be obtained as follows

$$C_{i,j} = \max\,\{aS_{i,j} - K_E, e^{-r\Delta t}\left[pC_{i+1,j+1} + (1-p)C_{i+1,j}\right]\} \tag{2.28}$$

Further, the option value at time 0, C_{00}, is

$$C_{00} = \max\,\{aS_{00} - K_E, e^{-r\Delta t}[pC_{11} + (1-p)C_{10}]\} \tag{2.29}$$

Binomial Tree - Step 2

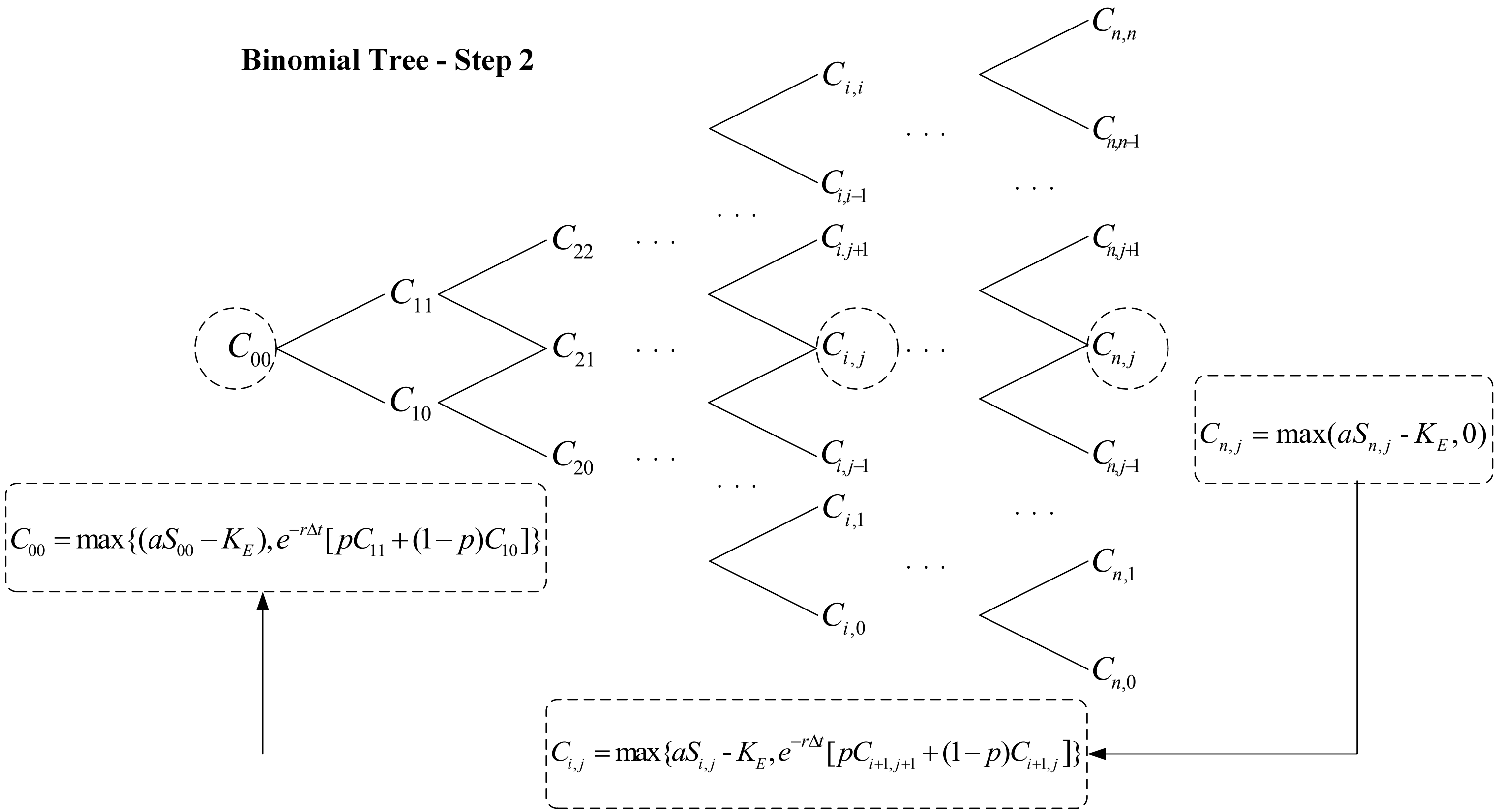

Figure 2.4. Evaluation of an Expansion Option

Last, the multiple options of the expansion option and the contraction option need to be evaluated. The procedure is similar to that previously described for the expansion option. The option value at the final nodes is the greater of the payoffs of both options, calculated as follows

$$f_{n,j} = \max\,(K_C - bS_{n,j}, C_{n,j}) \tag{2.30}$$

where b is the factor by which the production scale of the project is contracted for a cost savings of K_C.

For the nodes at time $i\Delta t$, the value of the option $f_{i,j}$ is the greatest of

- the expected value at time $(i+1)\Delta t$ discounted for a time period Δt at rate r,
- the payoff from exercising the expansion option, or
- the payoff from exercising the contraction option.

Thus, $f_{i,j}$ can be obtained by the following

$$f_{i,j} = \max\,(M_1) \tag{2.31}$$

where

$M_1 = [x_1\ x_2\ x_3]$

$x_1 = K_C - bS_{i,j}$

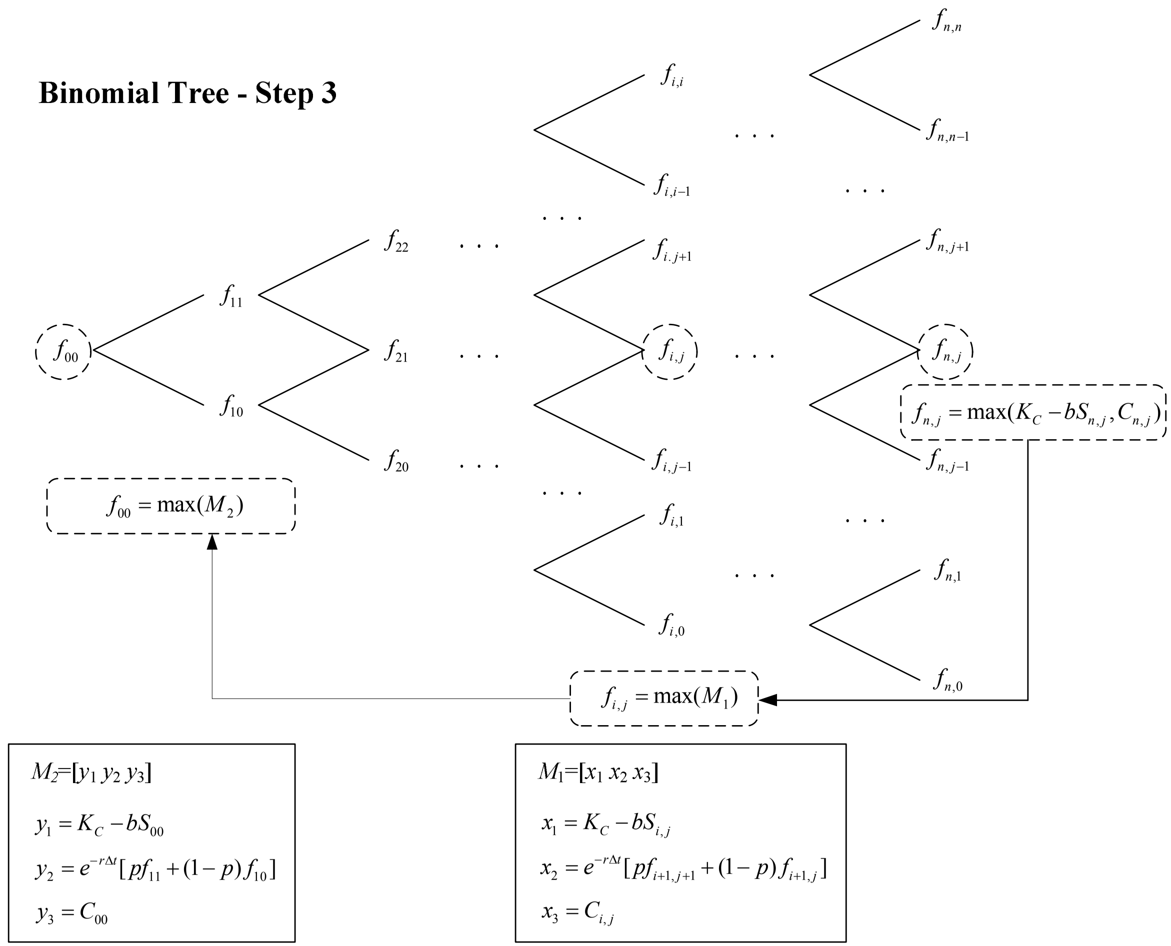

Figure 2.5. Evaluation of a Multiple Option of Expansion and Contraction

$x_2 = e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]$

$x_3 = C_{i,j}$

Finally, the value of the multiple options at time 0, f_{00}, is the greatest of three values as in equation (2.32):

$$f_{00} = \max(M_2) \tag{2.32}$$

where

$M_2 = [y_1\ y_2\ y_3]$

$y_1 = K_C - bS_{00}$

$y_2 = e^{-r\Delta t}[pf_{11} + (1-p)f_{10}]$

$y_3 = C_{00}$

The second example is the combination of an expansion option and an abandonment option. The abandonment option is the option to abandon the project for a salvage value, K_A. The payoff function for the abandonment option is $(K_A - S_t)^+$. The valuation of the option combination is similar to that shown in Figure 2.5. The option value at the final nodes is the greater of the payoffs of both options as

$$f_{n,j} = \max\,(K_A - S_{n,j}, C_{n,j}) \tag{2.33}$$

The value of the combination option can be obtained by working through the binomial tree in the same manner as in the first example.

The last example is the combination of a deferral option and an expansion option. When the market conditions for an investment opportunity are unclear, management would choose to defer the investment until the market conditions become more favorable. If the strike price is K_D, then management will only invest when the underlying project value goes beyond the strike price. The payoff function for the deferral option is therefore $(S_t - K_D)^+$. The option value at the final nodes is the greater of the payoffs of both options as

$$f_{n,j} = \max\,(S_{n,j} - K_D, C_{n,j}) \tag{2.34}$$

2.3. RESULTS

Experiments have been performed to test the effectiveness of binomial trees in evaluating different multiple options. Consider a 5-year long project that contains more than one option. The value of the base project is $400 million. The volatility of the project return is assumed to be 35%. The risk-free rate for the next 5 years is assumed to be 7%. The rate of paid cash inflows is 2% of the project value.

2.3.1. Expansion and Contraction. In the first example, the firm has an option to expand its scale of production by 20% (the parameter *a*), for a cost of $30 million (the strike price K_E), and an option to contract the production scale by 25% (the parameter *b*), for a cost saving of $250 million (the strike price K_C), at any time over the next 5 years. Therefore, this example consists of an American call and an American put. The expansion option can be exercised prior the contraction, and vice versa. The parameters are listed in Table 2.1.

Table 2.1. Parameters for the Multiple Options of Expansion and Contraction

Parameter	**Value**
K_E	30
K_C	120
a	20%
b	25%

The 10,000-steps binomial trees previously illustrated are used to calculate the prices of the options. The price of the expansion is $52.9585 million, the price of the contraction is $31.4210 million, and the price of the multiple options is $73.1619 million.

Clearly the price of the multiple options is greater than the price of either individual option; however, it is less than the sum of the two option prices.

Sensitivity analysis is performed to examine the impact of the price of the underlying project on the prices of the options. The results are illustrated in Figures 2.6, 2.7, 2.8, and 2.9. In Figure 2.6, it can be seen that the price of the expansion increases with the value of the base project. The higher the project value, the more likely the call is to be exercised and yield a higher payoff, and the more valuable the call becomes. The reverse is true for the contraction, as shown in Figure 2.7, as the contraction is treated as a put option. The put option behaves in the opposite direction as the call option does when the value of the underlying project changes.

For a multiple options of one call and one put, such as the expansion and the contraction, the combined value decreases initially and then increases as the project value goes beyond a certain point. As in Figure 2.8, when the project value is smaller, the call is less likely to be exercised and, thus, is less valuable; therefore, the combined value depends primarily on the value of the put, which declines as the project value increases. When the project value becomes large enough, the put is no longer in favor and becomes worthless, the call then takes over and the combined value is heavily determined by its value, which increases as the project value increases.

The sensitivity analysis is performed on the incremental value by the contraction which is considered as a second option in the project. Figure 2.9 shows a negative correlation between the additional value added by a put option and the value of the underlying project.

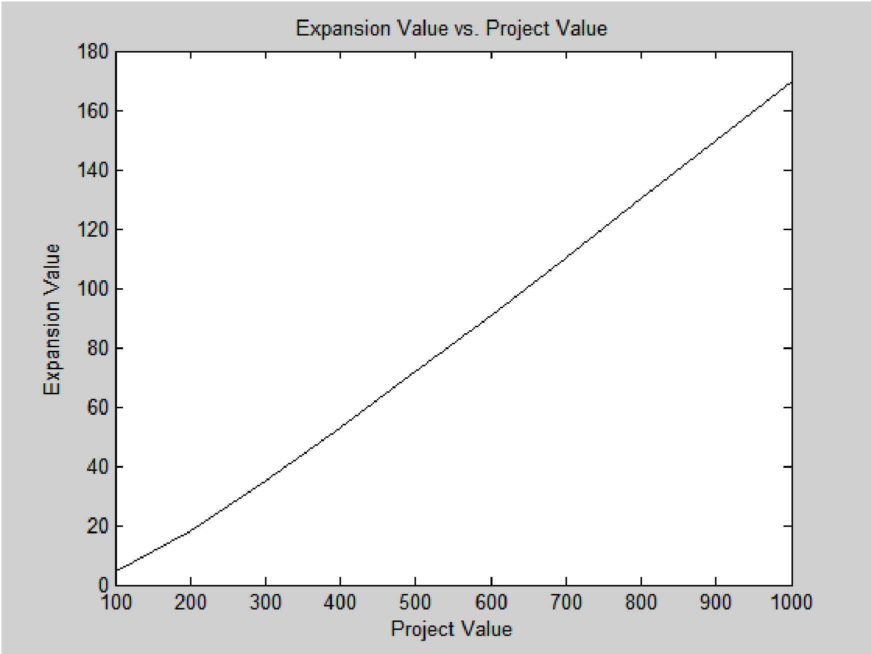

Figure 2.6. Sensitivity Analysis of Impact of Project Value on Expansion

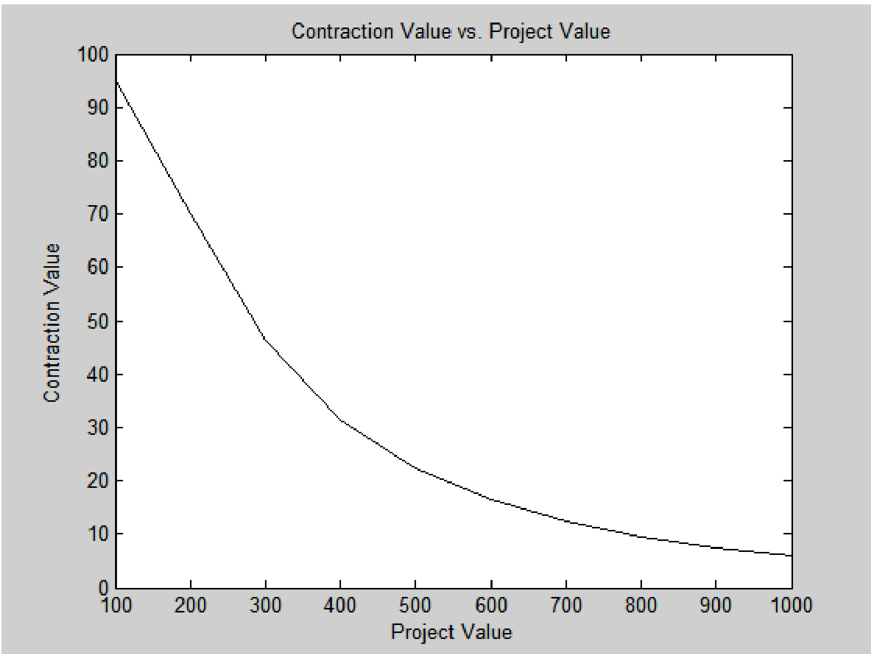

Figure 2.7. Sensitivity Analysis of Impact of Project Value on Contraction

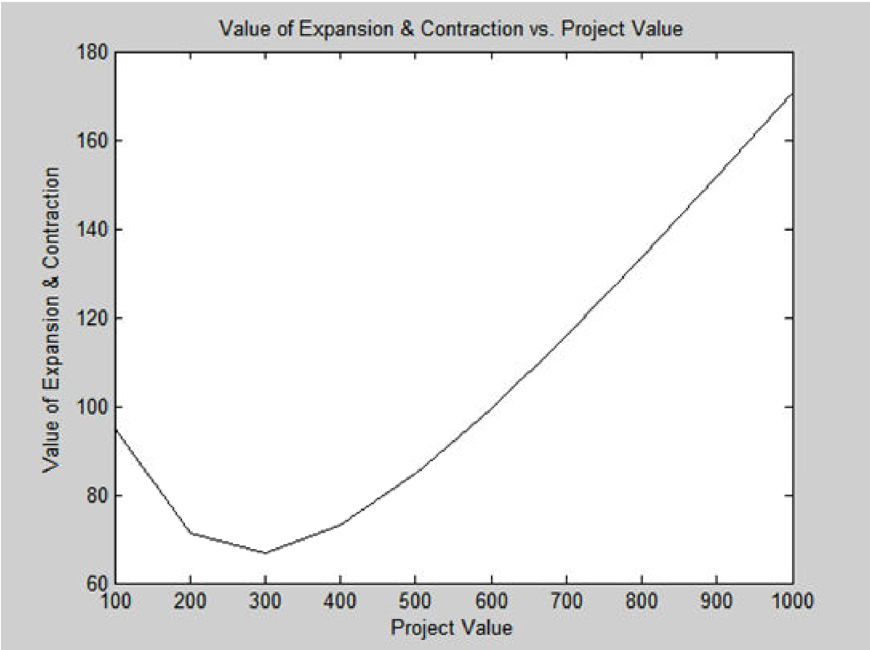

Figure 2.8. Sensitivity Analysis of Impact of Project Value on Multiple Options of Expansion and Contraction

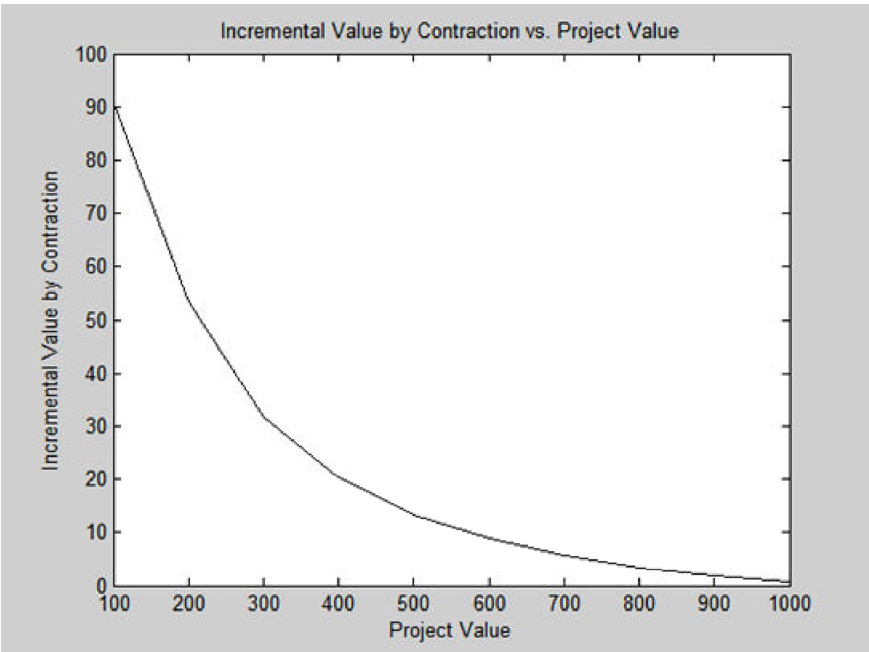

Figure 2.9. Sensitivity Analysis of Impact of Project Value on Incremental Value by Contraction

The free boundaries for the expansion and the contraction options are identified in Figure 2.10. Free boundaries consist of the points where early exercise is optimal. The upper curve is the boundary for the expansion option and the lower curve is the boundary for the contraction option. These curves provide the information regarding the project value as well as the time left to expiration for optimal exercise of either of the two options. When the project value moves beyond the free boundary of the expansion, it is optimal to exercise that option. The area above the free boundary of the expansion is the area in which to exercise the expansion option. When the project value moves below the free boundary of the contraction, it is optimal to exercise the contraction option. The area below the free boundary of the contraction is the exercise area of the contraction. In addition, the region between the boundaries is identified as the "hold" area in which neither of the options is to be exercised.

2.3.2. Expansion and Deferral. The second example is a combination of an option to expand the production scale by 50% (the parameter a) with a cost of \$200 million (K_E) and an option to defer the project for up to 1 year with a strike price of \$450 million (K_D). The parameters are listed in Table 2.2.

This example consists of two American calls. First, consider a situation in which no order of exercise is specified and, therefore, the expansion option and the deferral option are treated as two individual, non-related options. The expansion price is \$71.5903 million, the deferral price is \$43.6778 million, and the combined value of the multiple options is \$80.078 million.

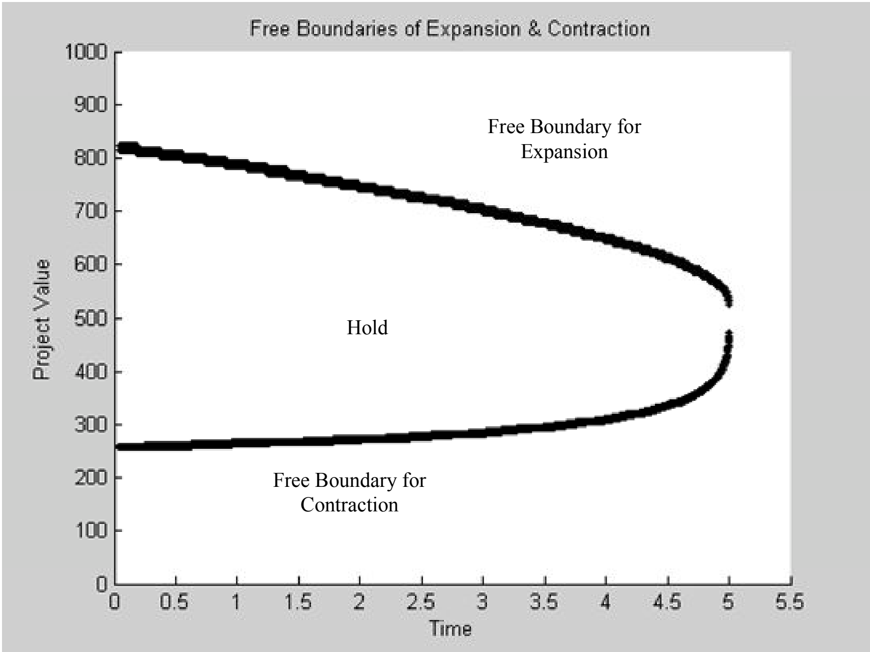

Figure 2.10. Free Boundaries for Expansion and Contraction

Table 2.2. Parameters for the Multiple Options of Expansion and Deferral

Parameter	Value
K_E	200
K_D	450
a	50%

Figures 2.11 and 2.12 illustrate that there is a positive correlation between the prices of the call options and the value of the base project. As a result, in Figure 2.13 the price of the multiple options is positively correlated to the project value as well. The free boundaries are tracked in Figure 2.14. Exercise areas for the expansion and the deferral are specified, as well as the hold area where neither option is to be exercised.

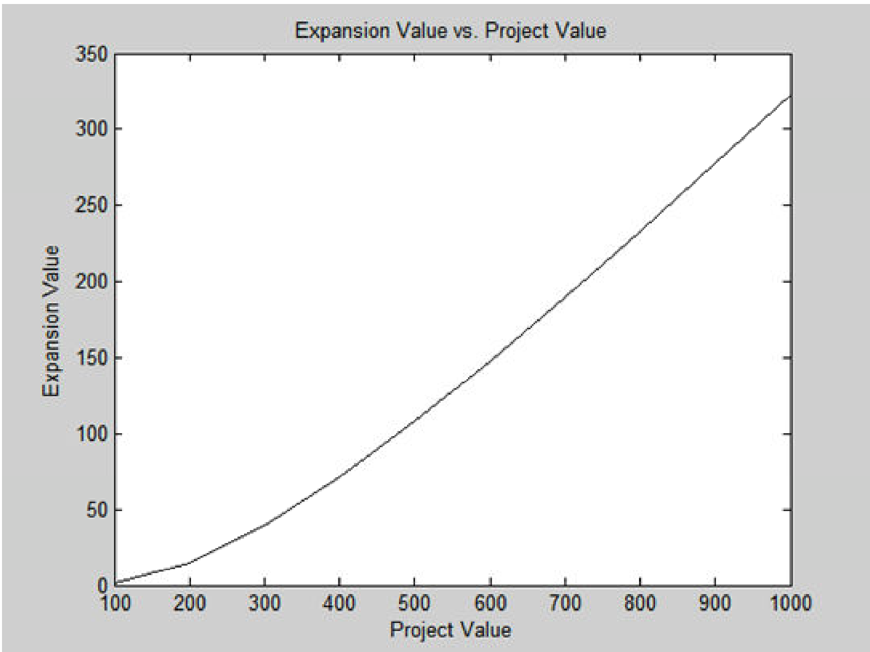

Figure 2.11. Sensitivity Analysis of Impact of Project Value on Expansion

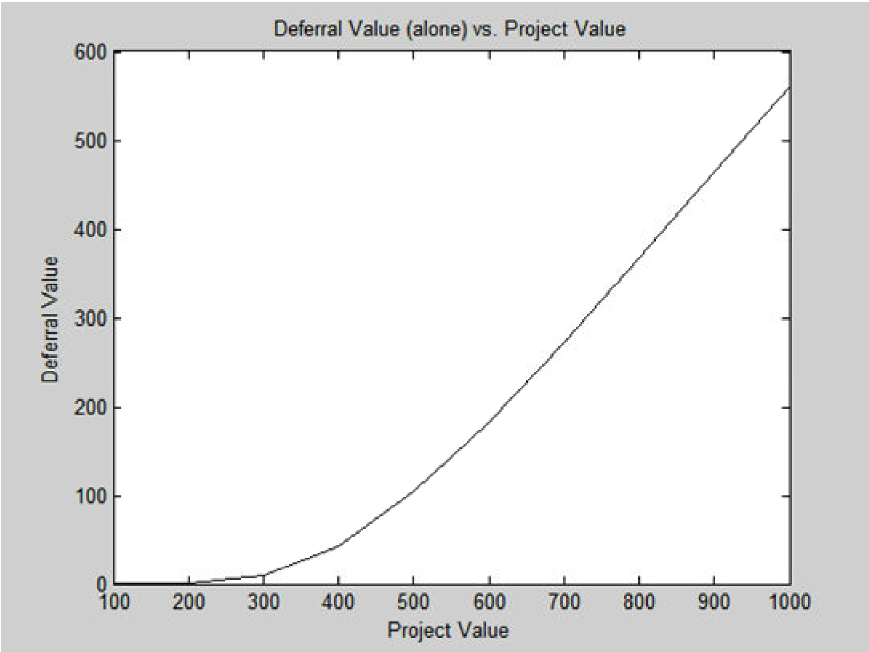

Figure 2.12. Sensitivity Analysis of Impact of Project Value on Individual Deferral

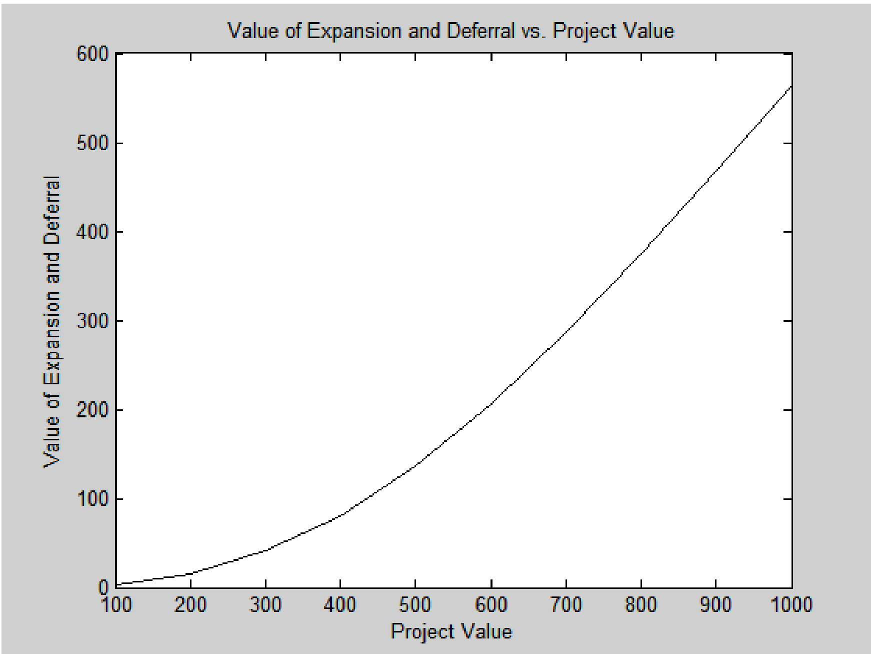

Figure 2.13. Sensitivity Analysis of Impact of Project Value on Multiple Options of Deferral and Expansion

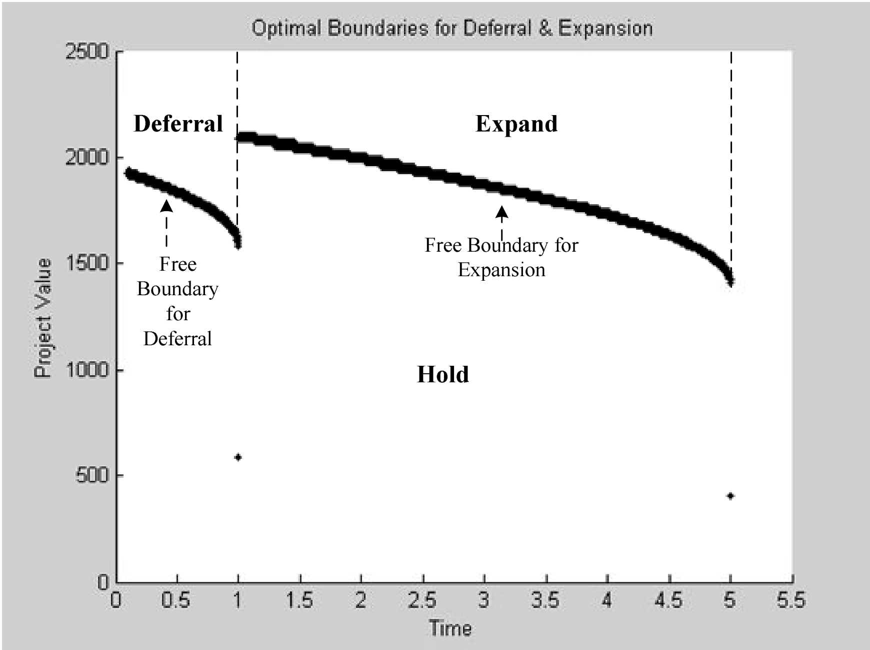

Figure 2.14. Free Boundaries for Expansion and Deferral

It should be noted that the deferral option has a shorter expiration than the expansion option. Therefore, it is inappropriate to ignore the order of exercise due to the fact that the deferral option can only be exercised in the first year and before the exercise of the expansion option. Hence, the deferral option can be treated as a second option that allows management to defer the project that embeds the expansion option. In this case, the price of the deferral is calculated on a new project value which incorporates the price of the expansion instead of the base project value alone. The prices obtained for the expansion, the deferral, and the multiple options are \$71.5903 million, \$84.8873 million, and \$104.99 million, respectively.

Sensitivity analysis is also performed to determine the impact of the project value on the value of the multiple options as well as on the incremental value by the additional option. Positive correlation is shown to exist between the option prices and the project value, as illustrated in Figure 2.15 and Figure 2.16. When the project value becomes larger, both calls are more likely to be exercised and, therefore, be worth more, and it makes a lot of sense that the combined value of two calls increases with the project value. The incremental value added by the deferral option to the project is also positively correlated with the project value, as depicted in Figure 2.17.

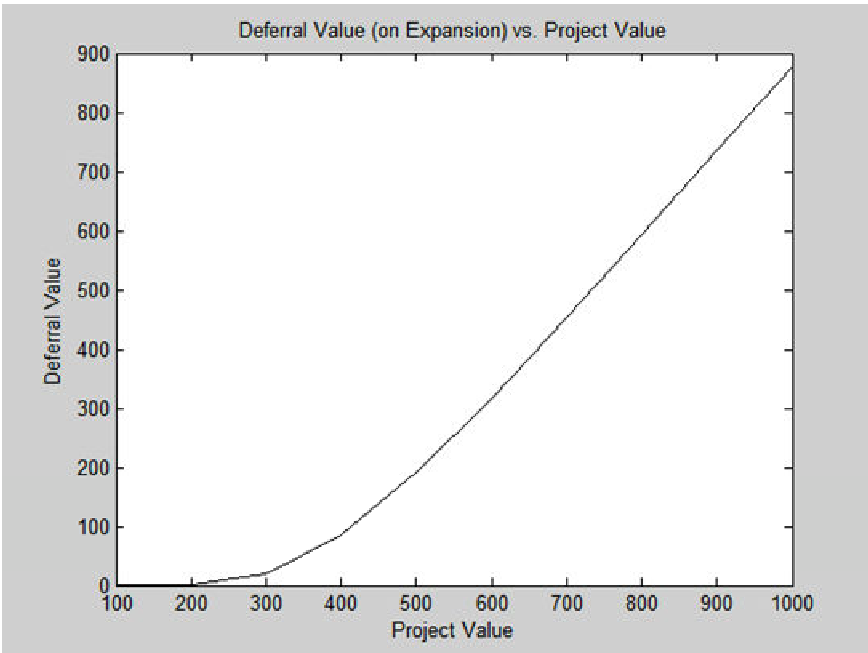

Figure 2.15. Sensitivity Analysis of Impact of Project Value on Deferral (Exercise Order Considered)

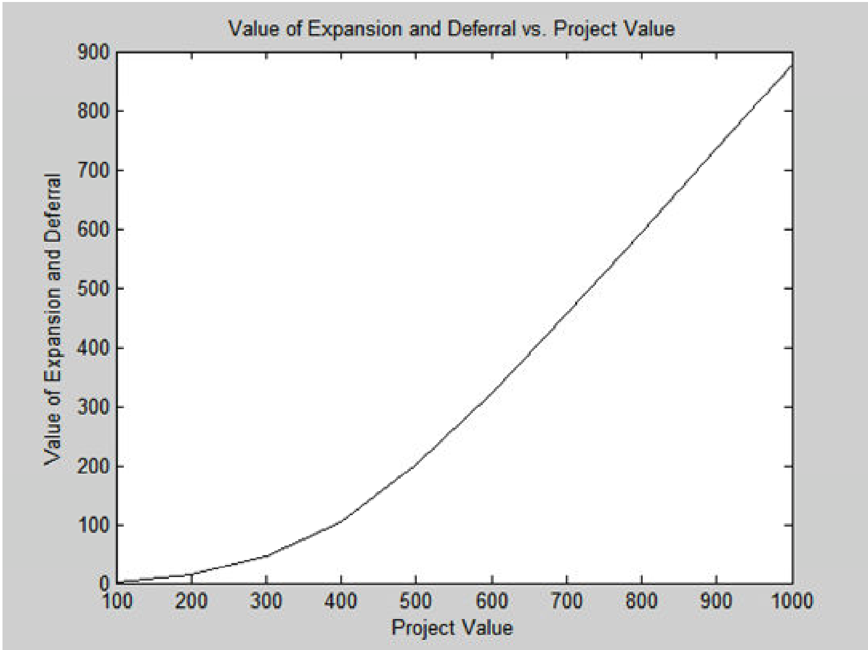

Figure 2.16. Sensitivity Analysis of Impact of Project Value on Multiple Options of Deferral and Expansion (Exercise Order Considered)

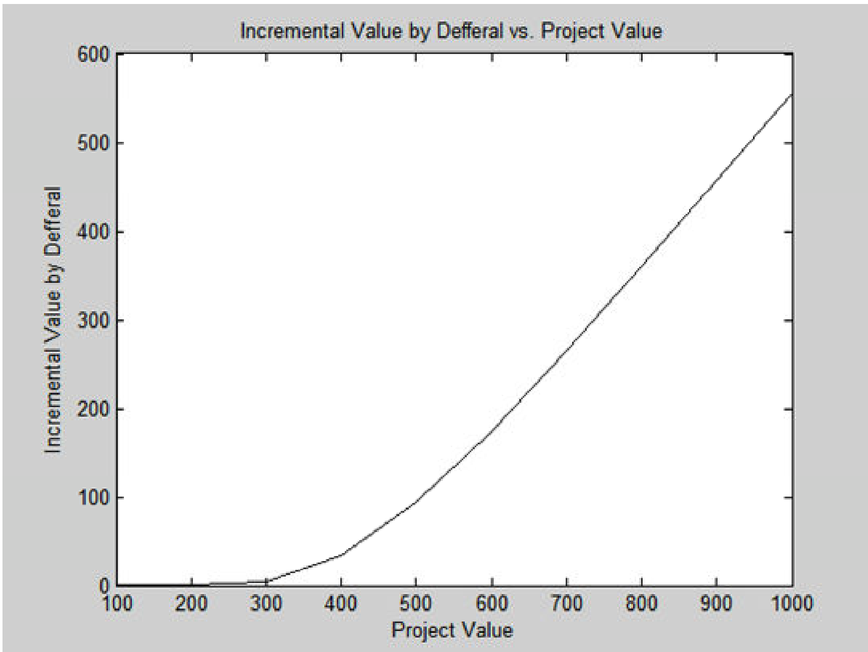

Figure 2.17. Sensitivity Analysis of Impact of Project Value on Incremental Value by Deferral (Exercise Order Considered)

Free boundaries are tracked in Figure 2.18. The free boundary for the deferral option appears to be lower than the one that takes no order of exercise into consideration. The exercise area of the deferral is the area above the deferral option's free boundary, and the exercise area of the expansion is the area above the expansion option's free boundary. Clearly, it is only optimal to exercise the deferral in the first year, if either option is to be exercised at all.

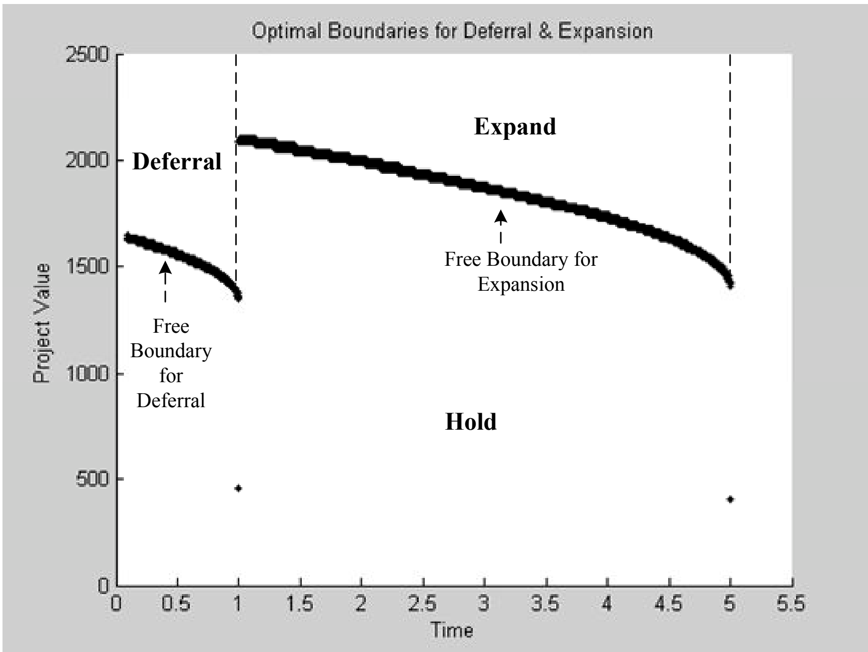

Figure 2.18. Free Boundaries for Deferral and Expansion
(Exercise Order Considered)

2.3.3. Expansion and Abandonment. The third example includes an expansion option to expand the production scale by 50% (the parameter *a*), with a cost of \$200 million (the strike price K_E), and an option to abandon the whole project for a salvage value of \$450 million (the strike price K_A). The parameters are listed in Table 2.3.

Table 2.3. Parameters for the Multiple Options of Expansion and Deferral

Parameter	Value
K_E	200
K_D	450
a	50%

This example consists of an American call and an American put. A constraint on the order of exercise is imposed on calculating the price of the multiple options due to the fact that the expansion option cannot be exercised after the project is abandoned.

Figure 2.19 shows what the free boundaries look like when the order constraint is neglected. It can be seen that the paths of the two free boundaries cross each other and that there is an overlap in the exercise areas of the expansion and the abandonment options, which means that both options are exercisable in the overlapping area. However, this is not plausible in the real world. The expansion must be void once the abandonment is exercised. In Figure 2.20 the free boundary for the expansion is eliminated after the

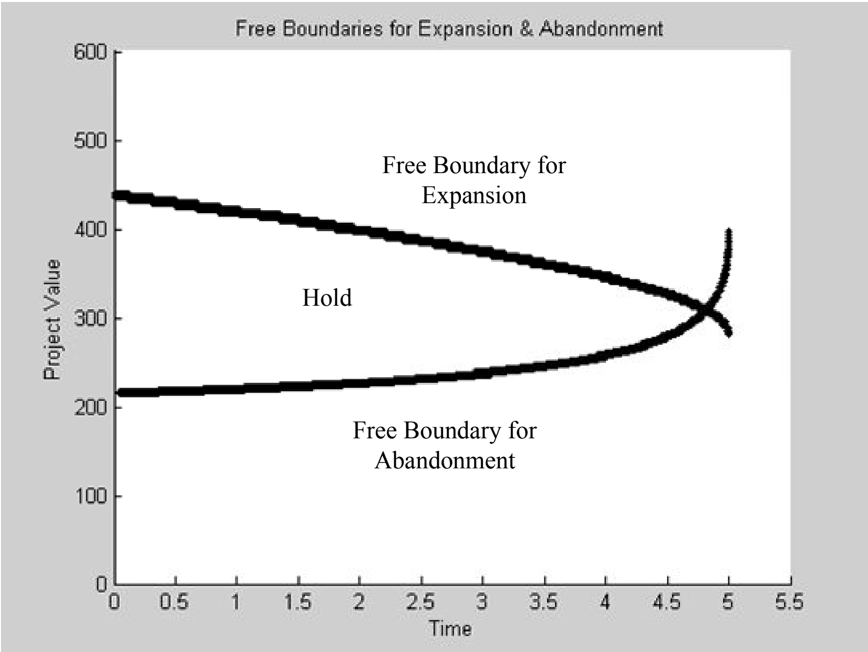

Figure 2.19. Free Boundaries for Expansion and Abandonment

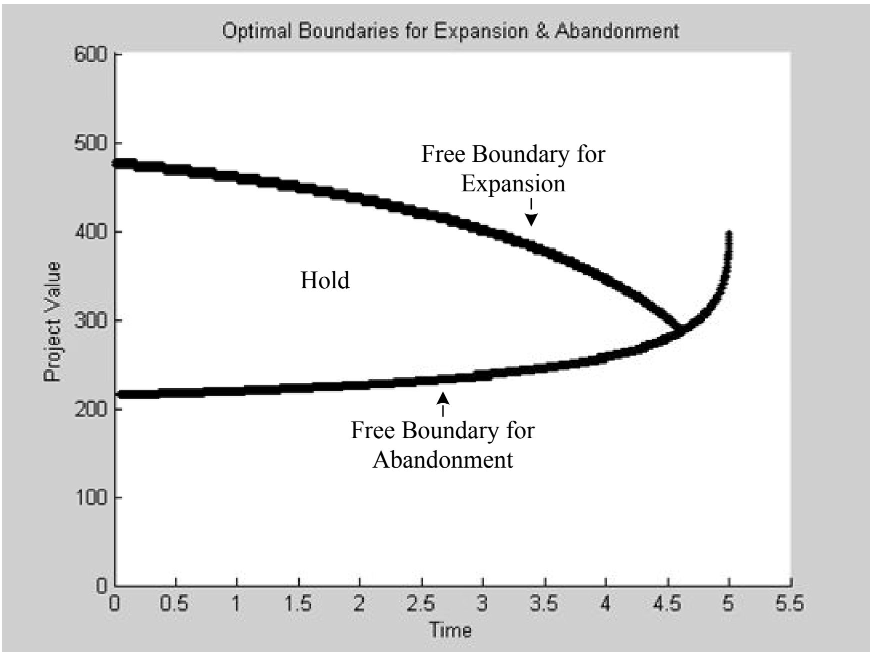

Figure 2.20. Free Boundaries for Expansion and Abandonment
(Exercise Order Considered)

boundaries intersect. When calculating the price of the multiple options, the values at the nodes after the intersection are replaced by the values of the abandonment.

The values for the expansion, the abandonment, and the multiple options are \$160 million, \$79.6224 million, and \$162.6823 million, respectively, without the order constraint. The price of the multiple options becomes \$162.2486, when the order constraint is taken into consideration. When there is no exercise constraint imposed on the multiple options, the value tends to be higher since the options give the management more flexibility.

The results from the sensitivity analysis are displayed in Figures 2.21, 2.22, and 2.23. The value of abandonment is shown to have a negative correlation with the project value. The value of the multiple options at first decreases and then later increases as the project value increases. The incremental value by the abandonment is more prominent when the project value is smaller than \$200 million, but becomes almost insignificant when the project value is greater than \$400 million. This results from the fact that the abandonment option would be more likely to be exercised when the project value decreases.

2.3.4. Numerical Results. Table 2.4 shows the value of each individual option without the presence of the other option. Note that Expansion 1 stands for the value of the expansion in the first example, Expansion 2 for the value in the second example, and Expansion 3 for the value in the third example.

Table 2.5 shows values of different multiple exercisable options. Clearly, the value of a multiple exercisable option is smaller than the sum of two individual option values, confirming the general rule that values of options present in combination

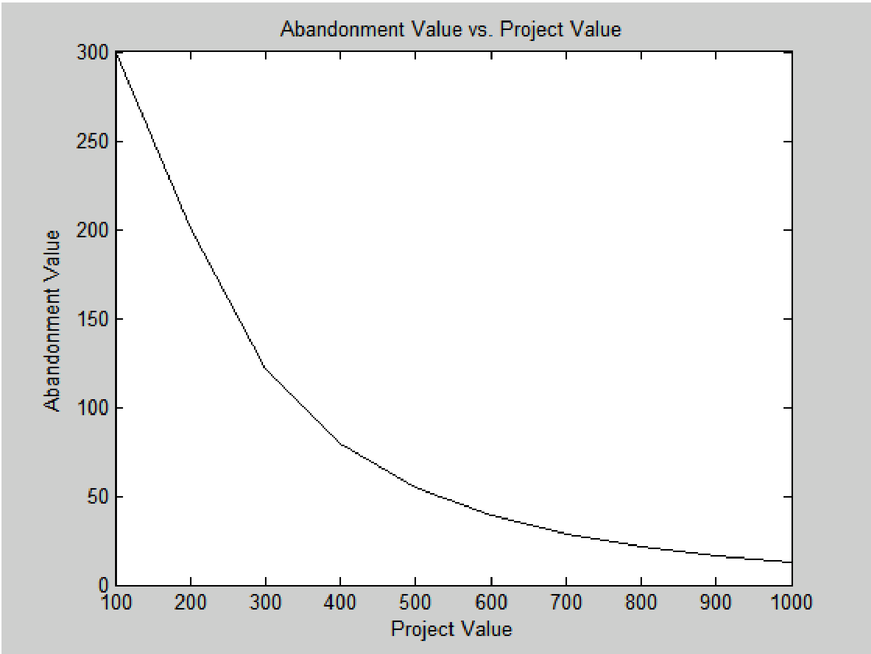

Figure 2.21. Sensitivity Analysis of Impact of Project Value on Abandonment (Exercise Order Considered)

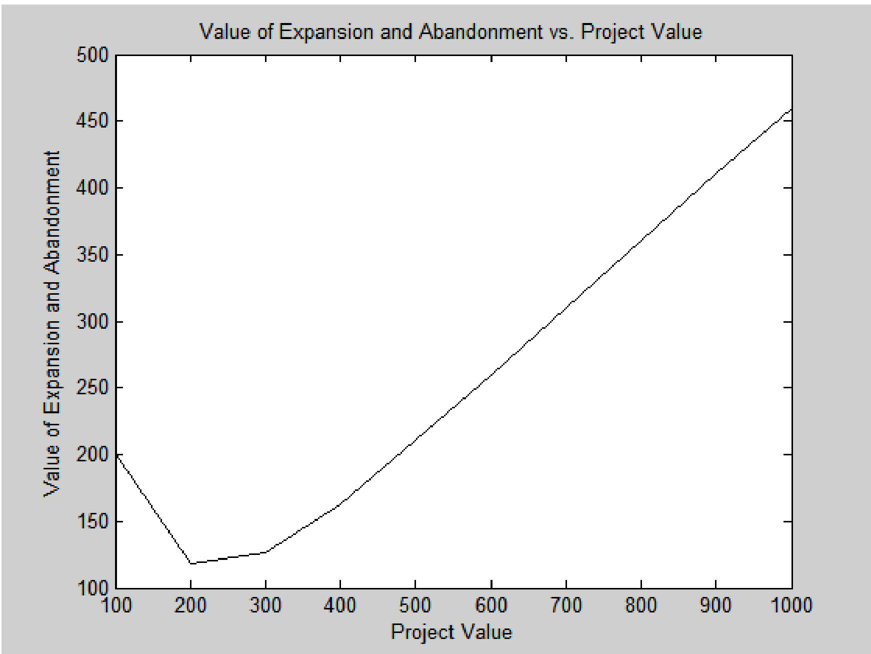

Figure 2.22. Sensitivity Analysis of Impact of Project Value on Multiple Options of Expansion and Abandonment (Exercise Order Considered)

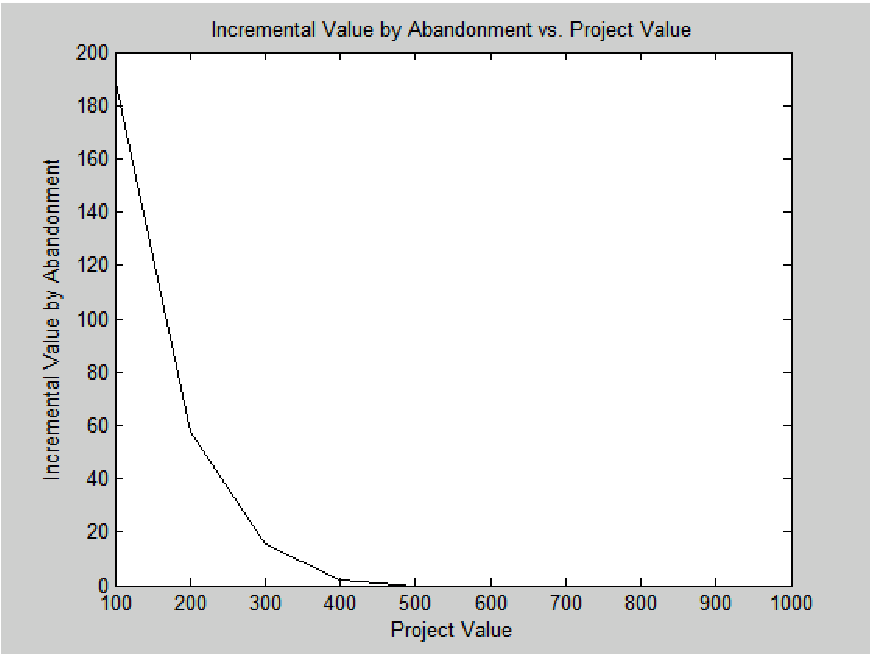

Figure 2.23. Sensitivity Analysis of Impact of Project Value on Incremental Value by Abandonment (Exercise Order Considered)

are not additive due to the interaction among options. For instance, the sum of the expansion option and the contraction option is 84.3795; however, the combination of these two options is only worth of 73.1619.

The incremental value in Table 2.6 is the difference between the values of the multiple exercisable options and the value of the expansion option, and it represents the value contributed to the project by the additional option. The incremental value contributed by the second option is smaller than the option price itself. It can be seen that the contract option is worth 31.421, but the value contributed by this option to the project is only 20.2034.

Table 2.4. Values of Individual Options

Option Value (in million $):	
Expansion1 (E1)	52.9585
Expansion2 (E2)	71.5903
Expansion3 (E3)	160
Contraction (C)	31.421
Deferral (D)	43.6778
Deferral (D)*	84.8873
Abandonment(A)	79.6224

Deferral (D): the value of the deferral option without interaction
Deferral (D)*: the value of the deferral option with interaction

Table 2.5. Values of Multiple Options

Values of Multiple Options (in million $):	
E1&C	73.1619
E2&D	80.078
E2&D*	104.99
E3&A	162.6823
E3&A*	162.2486

E2&D: the value of the multiple options of expansion and deferral without interaction
E2&D*: the value of the multiple options of expansion and deferral with interaction
E3&A: the value of the multiple options of expansion and abandonment without interaction
E3&A*: the value of the multiple options of expansion and abandonment with interaction

Table 2.6. Incremental Values of Additional Options

Incremental Values of Additional Options (in million $):	
Contraction (C)	20.2034
Deferral (D)	8.4877
Deferral (D)*	33.3997
Abandonment (A)	2.6823
Abandonment (A)*	2.2486

Deferral (D): the incremental value by the deferral option without interaction
Deferral (D)*: the incremental value by the deferral option with interaction
Abandonment (A): the incremental value by the abandonment option without interaction
Abandonment (A)*: the incremental value by the abandonment option with interaction

2.4. CONCLUSIONS

This study provides a general framework for the valuation of multiple exercisable real options using the binomial tree method to overcome the inflexibility of the more traditionally used discounted cash flow method, thus providing a better alternative for capital budgeting and project planning. Three different combinations of the expansion option, the contraction option, the abandonment option, and the deferral option are used to investigate and determine the effectiveness of the binomial tree method in pricing multiple options as well as accounting for interactions among them.

The results of these experiments show that the binomial tree model for pricing financial options is also applicable to pricing real options. The advantage of this approach is that it is not only effective for pricing a single option, but it is also promising for pricing multiple exercisable options. Therefore, the binomial tree method is capable of handling different combinations of real options in business practice.

3. PRICING MODEL WITH FINITE DIFFERENCE METHOD

3.1. INTRODUCTION

This chapter provides a general framework for pricing multiple exercisable American-style real options by obtaining numerical solutions to partial differential equations (PDEs) with finite difference methods. Two examples of multiple options, one consisting of an expansion option and a contraction option, the other consisting of an expansion option and an abandonment option, are discussed in this chapter.

The rest of this chapter is organized as follows. Section 3.1 gives the derivation of the Balck-Scholes differential equation for option price and a brief overview about finite difference methods. In Section 3.2, the Black–Scholes equations for a call option and a put option are transformed and then discretized using finite-difference grids. The explicit finite difference method is applied to evaluate two combinations of multiple options in Section 3.3. Numerical results are presented in Section 3.4 and some conclusions are drawn in Section 3.5.

3.1.1. Black-Sholes-Merton Differential Equation. The famous Black-Scholes-Merton model is the building block of the (real) option pricing Theory. The following is to give an overview on the derivation of the Black-Scholes-Merton differential equation, which can be found in Hull (2009).

It is stated in Section 2 that a stock price is assumed to follow a stochastic path called geometric Brownian motion, the continuous form of which is

$$dS = \mu S dt + \sigma S dx \tag{3.1}$$

with a drift rate of μ and a variance rate of σ^2. The term x is a Brownian motion.

Since the price of a stock option is a function of the stock price and time, the behavior of the functions of stochastic variables should be investigated first. An important result in this area is known as Itô's lemma, which was discovered by Kiyosi Itô in 1951.

Suppose a variable y follows the Itô process

$$dy = a(y,t)dt + b(y,t)dz \tag{3.2}$$

where a and b are functions of y and t, and z follows the Wiener process described in Section 2. The variable y then has a drift rate of a and a variance rate of b^2. According to Itô's lemma, a function F of y and t follows the process

$$dF = (\frac{\partial F}{\partial y}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}b^2)dt + \frac{\partial F}{\partial y}bdz \tag{3.3}$$

where z is the same Wiener process as in equation (3.2). Therefore the function F also follows an Itô process which has a drift rate of

$$\frac{\partial F}{\partial y}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}b^2 \ ,$$

and a variance rate of

$$b^2(\frac{\partial F}{\partial y})^2 .$$

Suppose that f is the price of a call option. Hence from equation (3.1) and Itô's lemma,

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dx \quad (3.4)$$

The discrete version of equation (3.1) and (3.4) are

$$\Delta S = \mu S \Delta t + \sigma S \Delta x \quad (3.5)$$

and

$$\Delta f = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta x \quad (3.6)$$

where ΔS and Δf are the changes in S and f during a small time interval Δt. To eliminate the term Δx in equation (3.5) and (3.6), a portfolio which consists of a long position in $\partial f/\partial S$ share of stocks and a short position in one call option is established. Define Π as the value of the portfolio, then

$$\Pi = \frac{\partial f}{\partial S}S - f \quad (3.7)$$

If $\Delta\Pi$ is the change in the portfolio value during the small time interval Δt, it follows

$$\Delta\Pi = \frac{\partial f}{\partial S}\Delta S - \Delta f \tag{3.8}$$

Substituting equation (3.5) and equation (3.6) into equation (3.8) gives

$$\Delta\Pi = -(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\,\Delta t \tag{3.9}$$

Hence the Wiener process Δx is removed and the portfolio must be riskless during Δt. The underlying no-arbitrage assumption implies that the portfolio earns the same rate of return as other short-term risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{3.10}$$

where r is the risk-free interest rate. Substituting equation (3.10) with equation (3.7) and (3.9) yields

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r(\frac{\partial f}{\partial S}S - f)\,\Delta t \tag{3.11}$$

And it follows that

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \tag{3.12}$$

3.1.2. Finite Difference Methods. The price of a derivative can be evaluated with finite difference methods by solving the differential equation that the derivative satisfies. The differential equation is converted into a set of difference equations which are to be solved iteratively (Hull, 2009).

Finite difference method is a classical, straightforward way to numerically solve partial differential equations. The finite difference approach was first proposed for option pricing by Brennan and Schwartz (1977, 1978). It works in a manner similar to the lattice methods, and the trinomial tree method which, in particular, can be viewed as an explicit finite difference method. However, the lattice methods do not require the spatial or side boundary conditions that the finite difference methods do. A major advantage of the finite difference approach for option pricing is that an abundance of existing theory, algorithms, and numerical software is available for solving the problem. Another advantage is that the generality of finite difference methods makes it possible to extend beyond the constant coefficients of the Black-Scholes-Merton model. Finite difference methods are capable of handling processes with time-varying coefficients, Itô processes that are more general, jump and SV models, single and multifactor interest rate models, etc. (Broadie and Detemple, 2004)

The following explains the calculation schemes of the finite difference methods. More detailed description can be found in Wilmott et al. (1995)

First, the PDE to be solved is discretized and reduced to a set of finite difference equations subject to appropriate boundary conditions. Next, the continuous domain of the state variables is transformed by a grid of discrete points, as illustrated in Figure 3.1. Both the x-axis and the τ-axis are divided into equally spaced time steps by a distance of

Δx and $\Delta\tau$, respectively, thus dividing the plane into a grid where the nodes are denoted by $(i\Delta x, j\Delta\tau)$. Boundary conditions need to be specified for the grid in order to solve the finite difference equations. Finally, the value for each node at each time step is calculated using either a forward- or a backward- difference approximation.

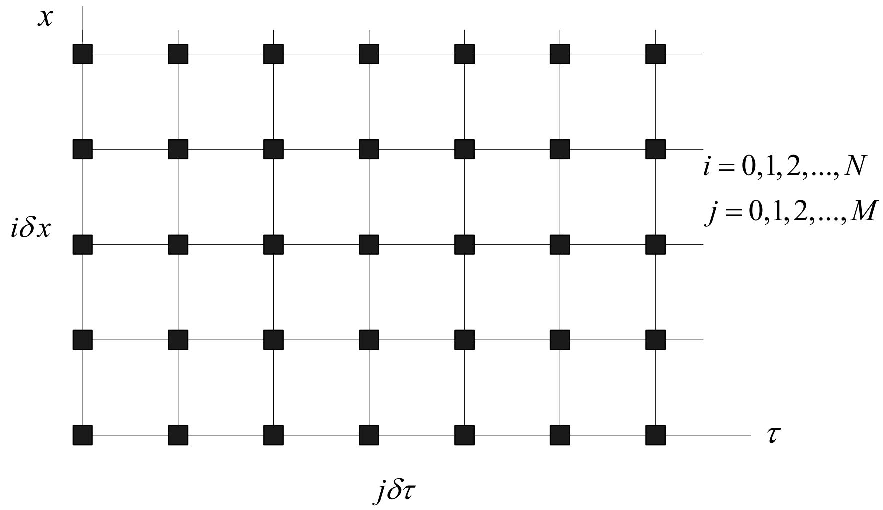

Figure 3.1. Finite-Difference Grid

3.1.2.1 Finite difference approximation. The basic idea of finite difference methods is to replace the partial derivatives in partial differential equations by approximation, based on the Taylor series expansions of functions near the point or points of interest. The partial derivative $\boldsymbol{\partial v/\partial \tau}$ may be defined as the limiting difference

$$\frac{\partial v}{\partial \tau}(x,\tau) = \lim_{\delta\tau \to 0} \frac{v(x,\tau+\delta\tau)-v(x,\tau)}{\delta\tau} \tag{3.13}$$

Considering $\delta\tau$ as being really small but nonzero, equation (3.13) can be approximated to be

$$\frac{\partial v}{\partial \tau} \approx \frac{v(x,\tau+\Delta\tau)-v(x,\tau)}{\Delta\tau} + O(\Delta\tau) \tag{3.14}$$

This is called a finite-difference approximation of $\partial v/\partial \tau$ because it involves small differences of the dependent variable$v$. In particular, it is a forward difference approximation since the differencing is in the forward τ direction.

If the differencing is the backward τ direction, then the limiting difference may be defined as

$$\frac{\partial v}{\partial \tau}(x,\tau) = \lim_{\delta\tau \to 0} \frac{v(x,\tau)-v(x,\tau-\delta\tau)}{\delta\tau} \tag{3.15}$$

which can be approximated to be

$$\frac{\partial v}{\partial x} \approx \frac{v(x,\tau)-v(x-\Delta x,\tau)}{\Delta x} + O(\Delta x) \tag{3.16}$$

This is called a backward difference. Besides the forward and the backward differences, there is the third approximation—central differences. Define the limiting difference of $\partial v/\partial \tau$ to be

$$\frac{\partial v}{\partial \tau}(x,\tau) = \lim_{\delta\tau\to 0} \frac{v(x,\tau+\delta\tau)-v(x,\tau-\delta\tau)}{2\delta\tau} \tag{3.17}$$

which gives rise to the approximation

$$\frac{\partial v}{\partial \tau}(x,\tau) \approx \frac{v(x,\tau+\delta\tau)-v(x,\tau-\delta\tau)}{2\delta\tau} + O((\delta\tau)^2) \tag{3.18}$$

However, the central differences in the form (3.18) are never used in practice because they usually result in bad numerical schemes. Instead, the central differences of the form arise in the Crank-Nicolson (Crank and Nicolson, 1947) finite difference scheme

$$\frac{\partial v}{\partial \tau}(x,\tau) \approx \frac{v(x,\tau+\delta\tau/2)-v(x,\tau-\delta\tau/2)}{\delta\tau} + O((\delta\tau)^2) \tag{3.19}$$

When applied to partial differential equations, forward difference approximation leads to explicit finite difference scheme, and backward difference approximation leads to fully implicit finite difference scheme. The finite difference approximations for the partial derivative of $\partial v/\partial x$ can be defined in exactly the same way as described above.

For second partial derivatives, such as $\partial^2 v/\partial x^2$, a symmetric central difference approximation can be defined as the forward difference of backward difference approximations to the first derivative or as the backward difference of forward difference approximations to the first derivative with the form

$$\frac{\partial^2 v}{\partial x^2}(x,\tau) \approx \frac{v(x+\delta x,\tau)-2v(x,\tau)+v(x-\delta x,\tau)}{(\delta x)^2} + O((\delta x)^2) \tag{3.20}$$

The geometric interpretation of forward-, backward-, and central- differences is shown in Figure (3.2).

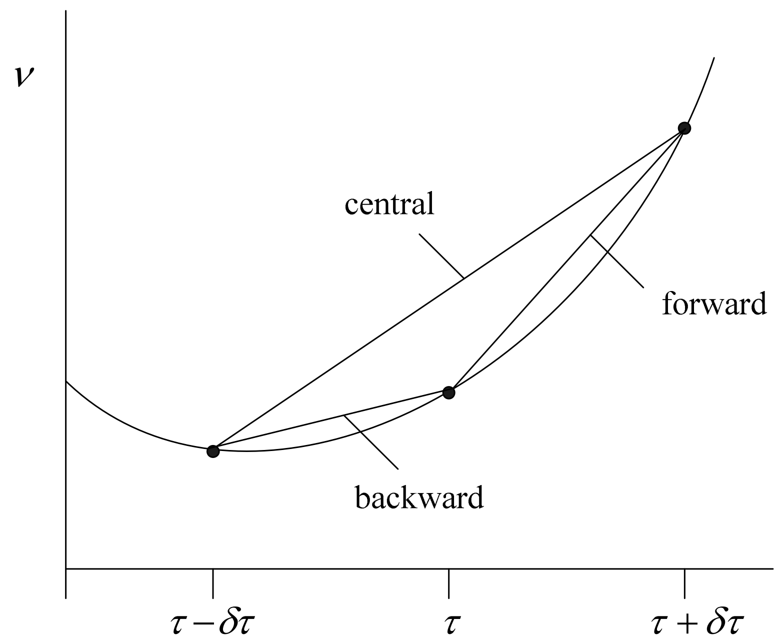

Figure 3.2. Forward-, Backward-, and Central- Differences Approximations

Source: Wilmott, P., Howison, S. and Dewynne, J., *The Mathematics of Financial Derivatives*. Cambridge University Press, (1995), pp. 137.

3.1.2.2 Explicit and implicit finite difference methods. The difference between the explicit and the implicit finite difference methods is as illustrated in Figure 3.3.

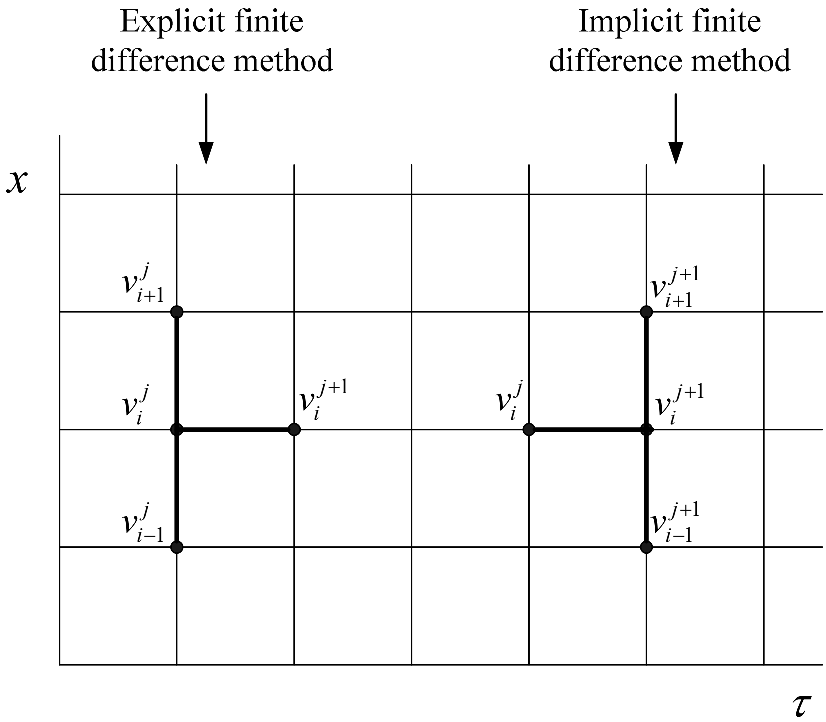

Figure 3.3. Explicit and Implicit Finite Difference Methods

Source: adapted from Wilmott, P., Howison, S. and Dewynne, J., *The Mathematics of Financial Derivatives*. Cambridge University Press, (1995), pp. 141 and 145.

The explicit finite difference method uses a forward difference approximation for the term $\partial v/\partial x$ and a symmetric central difference for the term $\partial^2 v/\partial x^2$. In this scheme, the value of v_i^{j+1} depends only on the values of v_{i+1}^j, v_i^j and v_{i-1}^j, which means that if all values of v_i^j are known at time step j, then v_i^{j+1} can be calculated explicitly.

The fully implicit finite difference method, which is usually known as the implicit finite difference method, uses a forward difference approximation for the term $\partial v/\partial x$ and a symmetric central difference for the term $\partial^2 v/\partial x^2$. In this scheme, the values of v_{i+1}^{j+1}, v_i^{j+1} and v_{i-1}^{j+1} depend on the value of v_i^j in an implicit manner. Therefore, the new values of v_i^j cannot be separated out and calculated explicitly in terms of the old values.

The problem of the evaluation of real options can be formulated using the explicit finite-difference approximation, as shown in Figure 3.4. The Black-Schole equation is first discretized to finite-difference equations, subject to appropriate boundary conditions. Then, a grid of the potential current and future prices of the underlying asset, *S*, is specified. The boundaries for the grid are the payoffs of the option at expiry and where *S* reaches its potential minimum or its potential maximum. At time-step *j*, if the value of the option f is known for each node $(i\Delta S, j\Delta t)$, then f at next time-step *j*+1 can be calculated explicitly. For instance, the value of f_i^{j+1}, depends only on the values of f_{i+1}^j, f_i^j, and f_{i-1}^j.

3.2. OPTION PRICING WITH FINITE DIFFERENCE METHOD

The following notation is used to formulate the models for pricing American-style real options. The derivation of the finite difference equations is based on the book by

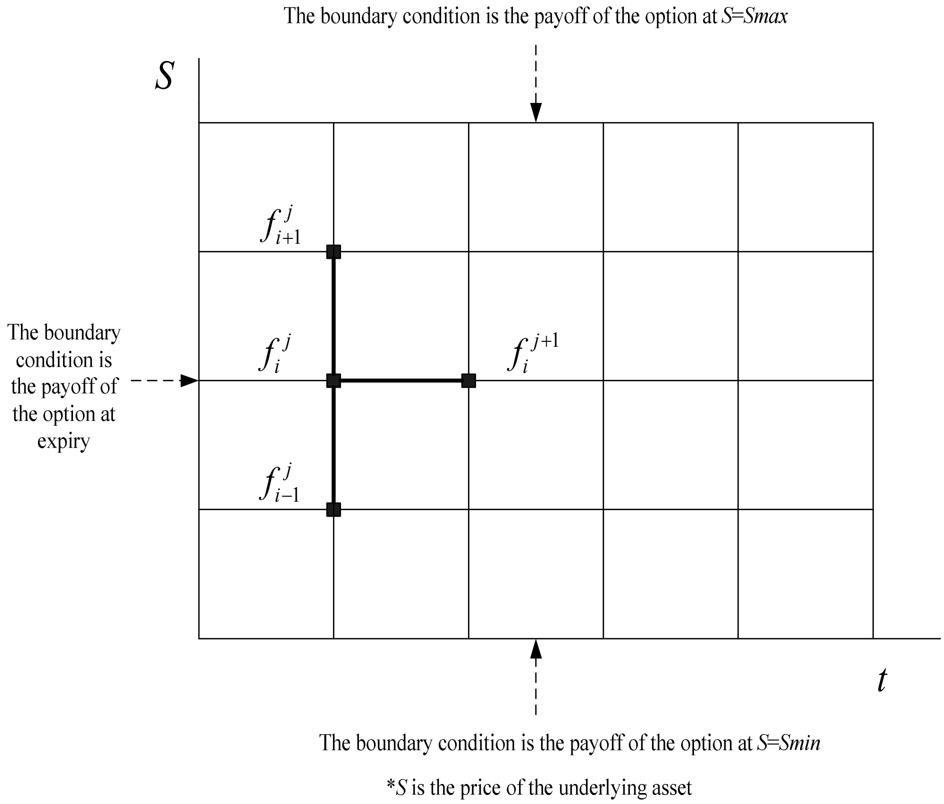

Figure 3.4. Explicit Finite-Difference Approximation for Real Options

Willmott et al. (1995). However, modification in the finite difference approximation scheme is required to make it more proper in the case of American options.

- S: value of underlying asset.
- r: risk-free interest rate.
- σ: volatility of the return of underlying asset.
- D_0: dividend rate.
- E: exercise price.

- t: a time in the life of option. Present=0, T=expiry.
- $C(S,t)$: price of a call option as a function of time and underlying asset.
- $P(S,t)$: price of a put option as a function of time and underlying asset.
- a: percentage by which the product capacity is expanded
- b: percentage by which the product capacity is contracted

For a European call option, the Black-Schole equation is

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r-D_0)S\frac{\partial C}{\partial S}-rC=0 \tag{3.21}$$

and the boundary conditions are

$C(0,t)=0$, $C(S,t)\sim S$ as $S\to\infty$, and $C(S,T)=\max(S-E,0)$

where C(S,t) is the option value at time t throughout the life of a call option.

However, it is difficult to find closed-form solutions for the Black-Scholes equation, especially in the case of real options, as they are usually of American style. It is desirable to reduce terms in the Black-Scholes equation and then transform the B-S equation to a dimensionless diffusion equation, which can later be approximated by finite difference methods.

Assuming $r > D_0 > 0$, define x, τ, and c as

$$x = \ln(S/E) \tag{3.22}$$

$$\tau = \frac{1}{2}\sigma^2(T - t) \tag{3.23}$$

and

$$c(x,\tau) = \frac{E - S + C(S,t)}{E} \tag{3.24}$$

It follows that

$$S = Ee^x\ ,\ t = T - \tau \Big/ \frac{1}{2}\sigma^2\text{, and } C(S,t) = S - E + Ec(x,\tau)$$

Taking the first partial derivative of $C(S,t)$, with respect to S and t, respectively, yields

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 E\frac{\partial c}{\partial \tau} \tag{3.25}$$

and

$$\frac{\partial C}{\partial S} = 1 + (E/S)\frac{\partial c}{\partial x} \tag{3.26}$$

Taking the second partial derivative of $C(S,t)$ with respect to S, gives

$$\frac{\partial^2 C}{\partial S^2} = E(-\frac{1}{S^2}\frac{\partial c}{\partial x} + \frac{1}{S^2}\frac{\partial^2 c}{\partial x^2}) \tag{3.27}$$

Substitute equations (3.25), (3.26), and (3.27) into the Black-Scholes equation (3.21) and rearrange the terms, and the result is

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + \left(\frac{r - D_0}{\frac{1}{2}\sigma^2} - 1\right)\frac{\partial c}{\partial x} - \frac{rc}{\frac{1}{2}\sigma^2} + \frac{(r - D_0)e^x}{\frac{1}{2}\sigma^2} - \frac{re^x}{\frac{1}{2}\sigma^2} + \frac{r}{\frac{1}{2}\sigma^2} \tag{3.28}$$

Let the two dimensionless parameters *k* and *k'* be

$$k = r \Big/ \frac{1}{2}\sigma^2 \tag{3.29}$$

$$k' = (r - D_0) \Big/ \frac{1}{2}\sigma^2 \tag{3.30}$$

and define function $f(x)$ as

$$f(x) = (k' - k)e^x + k \tag{3.31}$$

then the differential equation (3.28) can be simplified as

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1)\frac{\partial c}{\partial x} - kc + f(x) \tag{3.32}$$

for $-\infty < x < \infty$, $\tau > 0$, with

$$c(x,0) = \max(1 - e^x, 0) = \begin{cases} 1 - e^x, & x < 0 \\ 0, & x \geq 0 \end{cases} \tag{3.33}$$

An upwind scheme is applied to the approximation (see appendix), and the finite difference equation for a call option is

$$c_n^{m+1} = [\alpha + (k'-1)\frac{\Delta\tau}{\Delta x}]c_{n+1}^m + [(1-k\Delta\tau) - (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]c_n^m + \alpha c_{n-1}^m + \Delta\tau f(x) \tag{3.34}$$

where

$$\alpha = \frac{\Delta\tau}{(\Delta x)^2} \tag{3.35}$$

The finite difference equation for an American put can also be derived in a similar manner. The Black-Scholes equation for a European put option is

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S\frac{\partial P}{\partial S} - rP = 0 \tag{3.36}$$

with boundary conditions

$P(0,t) = 0$, $P(S,t) \sim E$ as $S \to 0$, and $P(S,T) = \max(E - S, 0)$

Define x and τ as in equation (3.22) and (3.23), and $p(x,\tau)$ as

$$p(x,\tau) = \frac{S-E+P(S,t)}{E} \tag{3.37}$$

then the Black-Scholes equation (3.36) can be simplified as

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2} + (k'-1)\frac{\partial p}{\partial x} - kp - f(x) \tag{3.38}$$

for $-\infty < x < \infty, \tau > 0$, with

$$p(x,0) = \max(e^x - 1, 0) = \begin{cases} e^x - 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{3.39}$$

and the terms k, k', and $f(x)$ are the same as in equations (3.29), (3.30), and (3.31).

After the same transformation (as described above for a call option), the finite difference equation for a put option is

$$p_n^{m+1} = \alpha p_{n+1}^m + [(1-k\Delta\tau) + (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]p_n^m + [\alpha - (k'-1)\frac{\Delta\tau}{\Delta x}]p_{n-1}^m - \Delta\tau f(x) \tag{3.40}$$

Early exercise needs to be taken into consideration in the case of American options. For an American call, early exercise is optimal when the following is satisfied

$$C = S - E \ , \ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S\frac{\partial C}{\partial S} - rC < 0 \, .$$

And similarly, for an American put, early exercise is optimal when the following is satisfied

$$P = E - S \ , \ \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP < 0 \, .$$

After the evolution lattice for option price is developed by finite-difference approximation, the goal is to obtain the free boundary for optimal early exercise. For instance, the call price C_n^m can then be compared with the payoff function $S_n^m - E$. The free boundary is formed by the price of the underlying asset at different times that satisfies the equation $C_n^m = S_n^m - E$. Similarly, the free boundary for a put option is formed by the price of the underlying asset at different times that satisfies the equation $P_n^m = E - S_n^m$.

3.3. PRICING MULTIPLE REAL OPTIONS

Two examples were established to demonstrate the real options pricing technique with the finite difference method. The first example consists of an expansion option and a contraction option, and the second example consists of an expansion option and an abandonment option where the expansion option will be void when the abandonment is exercised.

The values of the parameters in the models are listed in Table 3.1. Note that E_e, E_c, and E_a are the strike prices of the expansion, the contraction, and the abandonment, respectively. All prices listed are in million dollar increments. The life of the underlying project is 5 years. Notations $(1)^*$ and $(2)^*$ represent the expansion option in the first example and in the second example, respectively.

In the first example, the production capacity of the underlying project can be expanded by a certain percentage (e.g., 20%) or contracted by a certain percentage (e.g., 25%) at anytime during the five years of the project life. Therefore, the expansion is considered an American call option and the contraction is considered an American put option.

Table 3.1. Values of the Parameters for the Multiple Options

Parameter	**Value**
S	400
r	0.07
σ	0.35
D_0	0.02
T	5
$E_e\ (1)^*$	30
$E_e\ (2)^*$	40
E_c	120
E_a	400
$a\ (1)^*$	20%
$a\ (2)^*$	50%
b	25%

First, divide the (x, τ) plane into a grid. Then, by using the finite difference equations (3.34) and (3.40) derived above for the American call and put, the value at each grid nodes can be easily calculated. These values can later be converted back to option prices using the preset equations $C(S,t) = aS - E + Ec(x,\tau)$ and $P(S,t) = E - bS + Ep(x,\tau)$. Note that the coefficients a and b are applied to the general forms of the preset equations above to make them more suitable for the case of real options.

Each grid node should be examined if either option is exercised. If no option is exercised at one node, the value at that particular node, CV, is

$$CV_n^{m+1} = \left[\alpha + (k'-1)\frac{\Delta\tau}{\Delta x}\right] CV_{n+1}^m + [1 - k\Delta\tau) - (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]CV_n^m + \alpha CV_{n-1}^m + \Delta\tau f(x) \quad (3.41)$$

or

$$CV_n^{m+1} = \alpha CV_{n+1}^m + [(1 - k\Delta\tau) + (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]CV_n^m + \left[\alpha - (k'-1)\frac{\Delta\tau}{\Delta x}\right] CV_{n-1}^m - \Delta\tau f(x)^* \quad (3.42)$$

where $f(x)$ and $f(x)^*$ are from the first option (call) and the second option (put), respectively. However, the results tend to converge better if the CV_n^{m+1} takes the arithmetic average of the right-hand side of equation (3.41) and (3.42), hence

$$CV_n^{m+1} = \left[\alpha + \frac{1}{2}(k'-1)\frac{\Delta\tau}{\Delta x}\right] CV_{n+1}^m + [(1 - k\Delta\tau) - 2\alpha]CV_n^m + [\alpha - \frac{1}{2}(k'-1)\frac{\Delta\tau}{\Delta x}]CV_{n-1}^m + \frac{1}{2}\Delta\tau[f(x) - f(x)^*] \quad (3.43)$$

Therefore, the combined value of the multiple options at each node is the greatest of the following values:

- the payoff from exercising the first option
- the payoff from exercising the second option
- the value determined by equation (3.43)

The free boundaries are obtained by connecting the nodes where either option is exercised. When the free boundaries are tracked, the (S,t) plane is divided into three distinct regions: the exercise region for the expansion, the exercise region for the contraction, and the hold region where neither option is exercised.

In the second case, the abandonment option, which is considered an American put, imposes a different challenge. The abandonment option ends the underlying project once it is exercised. Therefore, a constraint must be put on this problem in order that the expansion option cannot be exercised after the abandonment option is exercised. However, the abandonment option can be exercised after the expansion option. The free boundaries for the expansion option and the abandonment option will also divide the (S,t) plane into three distinct regions: the exercise regions for the expansion option, the exercise region for the abandonment option, and the hold region where neither option is exercised.

3.4. RESULTS

Results are presented in this section and are compared with those derived by the binomial tree method. Figure 3.5 illustrates the free boundaries of the multiple options of expansion and contraction. The boundary of the expansion descends over the time axis.

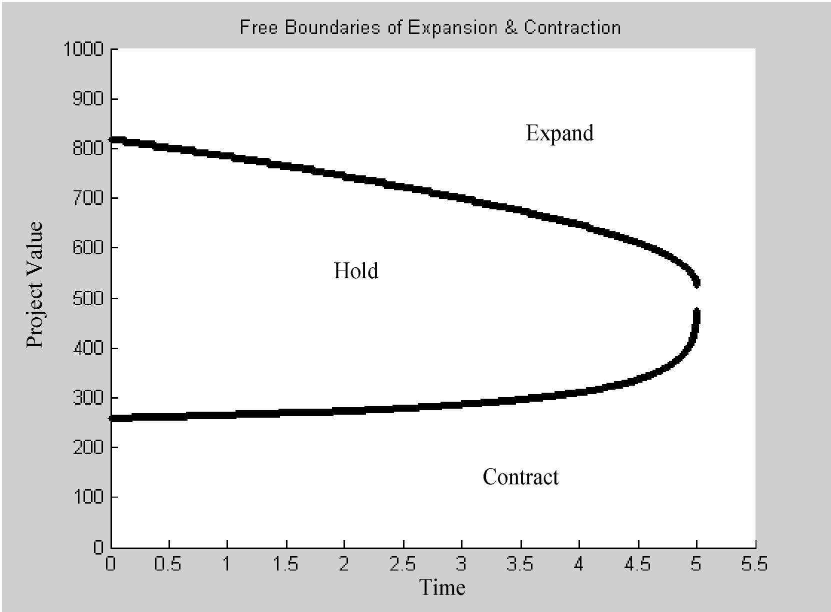

Figure 3.5. The Free Boundaries of the Multiple Options of Expansion and Contraction

On the contrary, the boundary of the contraction ascends over the time axis. The two distinct exercise regions, one for each individual option, as well as the hold region, are also determined. Figure 3.6 illustrates the free boundaries of the multiple options of expansion and abandonment. The value of the multiple options must take into account the impact of the abandonment on the expansion. As a result, there will be no expansion option after the two free boundaries encounter each other. After intersection, the value of each node is replaced by the value of the abandonment option, and the free boundary of the expansion option is tracked based on the new values. For comparison purposes, Figure 3.7 illustrates the two distinct free boundaries of the expansion and the abandonment without considering the interaction between these two options. It can be

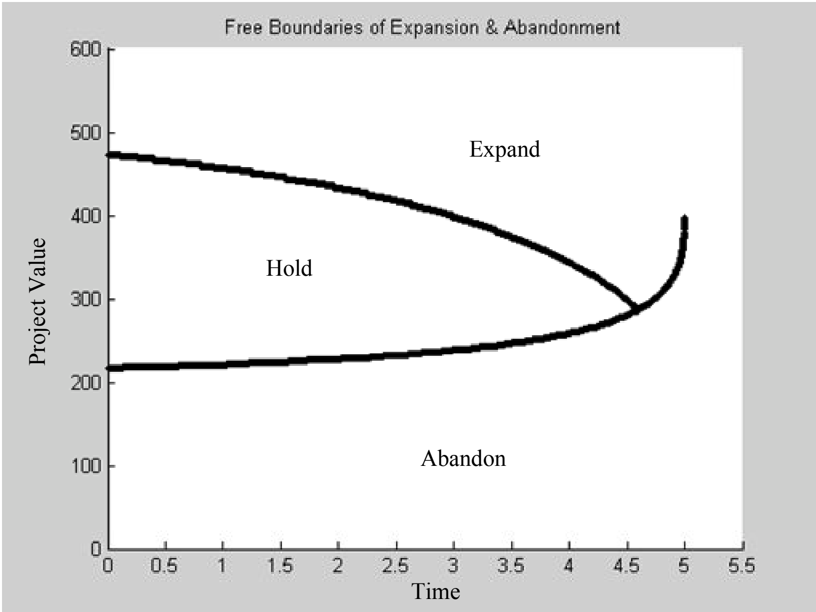

Figure 3.6. The Free Boundaries of the Multiple Options of Expansion and Abandonment

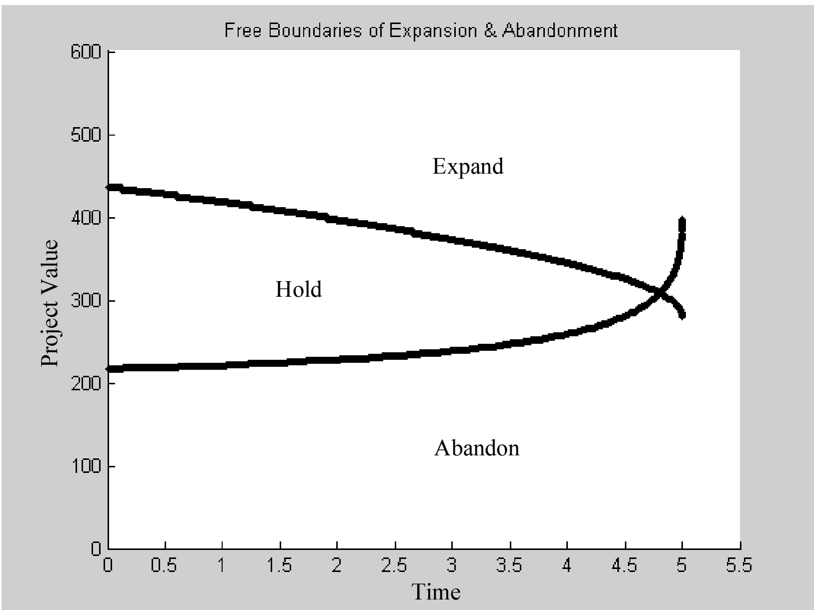

Figure 3.7. The Free Boundaries of the Expansion and the Abandonment without Interaction

seen in Figure 3.6 that the pattern of the free boundary for the expansion is notably altered under the impact of the abandonment. The free boundaries for the multiple options appear to be highly consistent with the ones tracked by the binomial tree method in Section 2.

The values of both the individual options and the combined values of the multiple options are displayed in Table 3.2. For comparison, the values obtained by the binomial tree method are also listed. The value of the multiple options of expansion and abandonment, without considering the interactions between the options, is calculated to be \$162.5748 million. This value is slightly greater than the value shown in the table, \$161.8980 million. That is because, when the constraint that the expansion is void after the abandonment is exercised is neglected, the multiple options give management more flexibility. Thus the “face value” appears to be worth more. This scenario is, however, not practical in reality.

The results show that both the binomial tree method and the finite difference method are effective in determining the value of multiple American options. The binomial tree method is easy to understand and simple to implement. However, the size of a tree can grow exponentially with the increase in time steps, making the computation time excessive. The finite difference method, on the other hand, requires rather sophisticated mathematical understandings. Nonetheless, this method is efficient and accurate, and able to handle more complex situations.

Table 3.2. Prices of the Individual Options and the Multiple Options

	PDE (50000×1000)	**Binomial (10000)**
Expansion (1)	52.9580	52.9585
Expansion (2)	160	160
Contraction	31.4338	31.4210
Abandonment	79.6815	79.6224
Expansion (1) & Contraction	73.1004	73.1619
Expansion (1) & Abandonment	161.8980	162.2486

3.5. CONCLUSIONS

This chapter investigated the pricing of American-style multiple exercisable real options by obtaining numerical solutions to partial differential equations with the finite difference method. Detailed derivations from Black-Scholes equations to finite difference equations are provided in this chapter, and an upwind scheme is applied to evaluate two combinations of the multiple options. The model is demonstrated to be adaptable to different combinations of multiple options and, thus, can handle various real life situations. It also addressed the problems of possible early exercise and the tracking of free boundaries, which are not always covered by some other real options pricing models.

When compared with the binomial tree method, this method is demonstrated to be fast and accurate. The problem tackled in this chapter is important for real life capital budgeting and project planning, and it also provides insights for business decision making.

4. CONCLUSIONS AND FUTURE WORK

4.1. CONCLUSIONS

This dissertation presents two numerical methods for pricing multiple exercisable American-style real options.

The first method is the binomial tree method, which discretizes the stochastic process followed by the price of the underlying asset. Different examples of multiple options are used to demonstrate the calculation scheme of the binomial trees. Both the numerical solutions and the free boundaries for early exercise are obtained and then discussed with consideration of the interactions between the options as well as the reality constraints.

The second method is the finite difference method, which discretizes the Black-Scholes differential equation for option prices using finite difference grids. The numerical solutions and the free boundaries obtained by this model show higher order accuracy than those from the binomial tree model.

It can be concluded that both methods are effective in pricing multiple options of American-style. These methods are therefore practical tools for project budgeting and planning. In addition, the free boundaries also provide much-needed information for business decision-making.

4.2. FUTURE WORK

The complexity in pricing multiple options increases substantially when the number of options embedded in a project increases. Interactions among multiple options can be overwhelming. As a result, it can be very difficult to quantify and account for these interactions. However, future research can extend the models to address pricing problems with more than two options.

Another potential topic for new research is to explore options with multiple underlying assets instead of only one. Variations of the finite difference method may be used for this problem.

APPENDIX

The following scheme is used to approximate the diffusion equation (3.32). Using a forward difference for $\partial v/\partial \tau$, a forward or a backward difference for $\partial v/\partial x$, and a symmetric central difference for $\partial^2 v/\partial x^2$, we have

$$\frac{\partial v}{\partial \tau} \approx \frac{v(x,\tau+\Delta\tau)-v(x,\tau)}{\Delta\tau} + O(\Delta\tau) = \frac{v_n^{m+1}-v_n^m}{\Delta\tau} + O(\Delta\tau) \quad \text{(A.1)}$$

$$\frac{\partial v}{\partial x} \approx \frac{v(x+\Delta x,\tau)-v(x,\tau)}{\Delta x} + O(\Delta x) = \frac{v_{n+1}^{m}-v_n^m}{\Delta x} + O(\Delta x) \quad \text{(A.2)}$$

or

$$\frac{\partial v}{\partial x} \approx \frac{v(x,\tau)-v(x-\Delta x,\tau)}{\Delta x} + O(\Delta x) = \frac{v_{n}^{m}-v_{n-1}^m}{\Delta x} + O(\Delta x) \quad \text{(A.3)}$$

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{v(x+\Delta x,\tau)-2v(x,\tau)+v(x-\Delta x,\tau)}{(\Delta x)^2} + O((\Delta x)^2) = \frac{v_{n+1}^{m}-2v_n^m+v_{n-1}^m}{(\Delta x)^2} + O((\Delta x)^2) \quad \text{(A.4)}$$

An upwind scheme is used to overcome the artificial oscillations (Seydel, 2006). When the term (*k'*-1) in equation (3.32) is less than zero, a forward difference is used to approximate $\partial v/\partial x$ as in equation (A.2); otherwise a backward difference is used as in equation (A.3).

Substitute equation (3.32) with equations (A.1), (A.2), (A.3) and (A.4), and, by omitting the terms of $O(\Delta\tau)$, $O(\Delta x)$ and $O(\Delta x^2)$, we have a new difference equation.

Bvhju If k'-1 > 0,

$$\frac{c_n^{m+1}-c_n^m}{\Delta\tau}=\frac{c_{n+1}^m-2c_n^m+c_{n-1}^m}{(\Delta x)^2}+(k'-1)\frac{c_n^m-c_{n-1}^m}{(\Delta x)}-kc_n^m+f(x) \tag{A.5}$$

which leads to

$$c_n^{m+1}=\alpha c_{n+1}^m+[(1-k\Delta\tau)+(k'-1)\frac{\Delta\tau}{\Delta x}-2\alpha]c_n^m+[\alpha-(k'-1)\frac{\Delta\tau}{\Delta x}]c_{n-1}^m+\Delta\tau f(x) \tag{A.6}$$

If k'-1 < 0,

$$\frac{c_n^{m+1}-c_n^m}{\Delta\tau}=\frac{c_{n+1}^m-2c_n^m+c_{n-1}^m}{(\Delta x)^2}+(k'-1)\frac{c_{n+1}^m-c_n^m}{(\Delta x)}-kc_n^m+f(x) \tag{A.7}$$

which leads to

$$c_n^{m+1}=[\alpha+(k'-1)\frac{\Delta\tau}{\Delta x}]c_{n+1}^m+[(1-k\Delta\tau)-(k'-1)\frac{\Delta\tau}{\Delta x}-2\alpha]c_n^m+\alpha c_{n-1}^m+\Delta\tau f(x) \tag{A.8}$$

For a put option, if k'-1 > 0, the diffusion equation (3.38) is transformed into

$$\frac{p_n^{m+1}-p_n^m}{\Delta\tau}=\frac{p_{n+1}^m-2p_n^m+p_{n-1}^m}{(\Delta x)^2}+(k'-1)\frac{p_{n+1}^m-p_n^m}{(\Delta x)}-kp_n^m-f(x) \tag{A.9}$$

which gives

$$p_n^{m+1} = [\alpha + (k'-1)\frac{\Delta\tau}{\Delta x}]p_{n+1}^m + [(1-k\Delta\tau) - (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]p_n^m + \alpha p_{n-1}^m - \Delta\tau f(x) \qquad \text{(A.10)}$$

Or if *k'*-1 < 0,

$$\frac{p_n^{m+1} - p_n^m}{\Delta\tau} = \frac{p_{n+1}^m - 2p_n^m + p_{n-1}^m}{(\Delta x)^2} + (k'-1)\frac{p_n^m - p_{n-1}^m}{(\Delta x)} - kp_n^m - f(x) \qquad \text{(A.11)}$$

which becomes

$$p_n^{m+1} = \alpha p_{n+1}^m + [(1-k\Delta\tau) + (k'-1)\frac{\Delta\tau}{\Delta x} - 2\alpha]p_n^m + [\alpha - (k'-1)\frac{\Delta\tau}{\Delta x}]p_{n-1}^m - \Delta\tau f(x) \qquad \text{(A.12)}$$

BIBLIOGRAPHY

Benaroch, M. and Kauffman, R.J., A Case for Using Real Options Pricing Analysis to Evaluate Information Technology Project Investments. *Information Systems Research*, Vol. 10, No. 1, (1999), pp. 70-86.

Berger, P.G., Ofek, E. and Swary, I. Investor Valuation of the Abandonment Option. *Journal of Financial Economics*, Vol. 42, No. 2, (1996), pp. 257-287.

Black, F. and Scholes, M., The Pricing of Options on Corporate Liabilities. *The Journal of Political Economy*, Vol. 81, No. 3, (1973), pp. 637-659.

Bodie, Z., Kane, A. and Marcus, A.J. *Investment*. New York: McGraw-Hill, (2004).

Brandão, L.E., Dyer, J.S. and Hahn, W.J. Using Binomial Decision Trees to Solve Real-Option Valuation Problems. *Decision Analysis*, Vol. 2, No. 2, (2005) , pp. 69–88.

Brennan, M.J. and Schwartz, E.S., Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis. *Journal of Financial and Quantitative Analysis*, XIII, 3, (1978), pp. 461-474.

Brennan, M. J. and Schwarz, E. S., Evaluating Natural Resource Investments. *Journal of Business*, Vol. 58(2), (1985), pp. 135-158.

Broadie, M. and Detemple, J.B., Option Pricing: Valuation Models and Applications. *Management Science*, Vol. 50, No. 9, (2004), pp. 1145-1177.

Chiara, N., Garvin, M.J. and Vecer, J., Valuing Simple Multiple-Exercise Real Option in Infrastructure Projects. *Journal of Infrastructure Systems*, Vol.13(2), (2007) pp.97-104.

Cooter, P., *The Random Character of Stock Market Prices*. Cambridge: The MIT Press (1964).

Copeland, T.E. and Antikarov, V. *Real Options: A Practitioner's Guide*. (2001)

Cox, J.C., Ross, S.A. and Rubinstein, M. Option Pricing: A Simplified Approach. *Journal of Financial Economics*, Vol.7, (1979) pp. 229-263.

Crank, J. and Nicolson, P., A Practical Method for Numerical Evaluation of Solutions of Partial Differential Equations of the Heat Conduction Type. *Proc. Camb. Phil. Soc,* 43 (1), (1947), pp. 50–67.

Dixit, A.K. and Pindyck, R.S., *Investment under Uncertainty*. New Jersey: Princeton University Press (1994)

Hull, J.C., *Options, Futures, and Other Derivatives.* New Jersey: Pearson Prentice Hall, (2005).

Ingersoll, J. E. and Ross, S. A., Waiting to Invest: Investment and Uncertainty. *The Journal of Business*, Vol. 65 (1), (1992), pp. 1-29.

Kellogg, D. and Charnes, *J.M.* Real-Options Valuation for a Biotechnology Company. *Financial Analysts Journal*, Vol. 56, No. 3, (2000), pp. 76-84.

Leung, T. and Sircar, R., Exponential Hedging with Optimal Stopping and Application to ESO Valuation. *SIAM Journal on Control and Optimization*, Vol.48(3), (2009), pp. 1422-1451.

Lewis, N. A., Project Valuation for the Strategic Management of Research and Development. *Ph.D. dissertation*, Missouri University of Science & Technology, (2004).

Luehrman, T.A., Investment Opportunities as Real Options: Getting Started on the Numbers. *Harvard Business Review*, Vol. 76, No. 4, (1998), pp. 51-67.

Marco, A.C., The option value of patent litigation: Theory and evidence. *Review of Financial Economics*, vol. 14(3-4), (2005), pp. 323-351.

McDonald, R. and Siegel, D., The Value of Waiting to Invest. *The Quarterly Journal of Economics*, Vol. 101(4), (1986), pp. 707-728

Merton, R. C. Theory of Rational Option Pricing. *Bell Journal of Economics & Management*, Vol. 4, No. 1, (1973), pp. 141-183.

Mun, J., *Real options analysis: tools and techniques for valuing strategic investments and decisions*, John Wiley & Sons (2005).

Myers, S. C. "Determinants of Corporate Borrowing," Journal of Financial Economics, Vol. 5. No. 2. 1977, pp. 147-175.

Nembhard, D.A., Nembhard, H.B. and Qin, R., A Real Options Model for Workforce Cross-Training. *The Engineering Economist*, Vol.50, No.2, (2005), pp. 95-116.

Paddock, L. J., Siegel, D. R. and Smith, J. L., Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases. *The Quarterly Journal of Economics*, Vol. 103(3), (1988), pp. 479-508

Pindyck, R. S., 1988. Irreversible Investment, Capacity Choice, and the Value of the Firm. *American Economic Review*, American Economic Association, Vol. 78(5), (1988), pp. 969-985.

Schwartz, E.S., The Valuation of Warrants: Implementing a New Approach. *Journal of Financial Economics*, 4, (1977), pp. 79-93.

Seydel, R., *Tools for Computational Finance*, Springer (2006).

Smit, H.T.J. and Trigeorgis, L., *Strategic investment: real options and games*, Princeton University Press (2004).

Timan, S., Urban Land Prices Under Uncertainty. *American Economic Review*, Vol. 75, No. 3, (1985), pp. 505-514.

Tourinho, O.A.F., The Valuation of Reserves of Natural Resources: An Option Pricing Approach. *Ph.D. Dissertation*, University of California, Berkeley, Ph.D. Dissertation, (1979).

Trigeorgis, L. and Mason, S. P., Valuing Managerial Flexibility. *Midland Corporate Finance Jounal*, Vol. 5, No.1, (1987), pp. 14-21.

Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, Cambridge: The MIT Press (1996).

Tseng, C. and Lin, K. Y., A Framework Using Two-Factor Price Lattices for Generation Asset Valuation. *Operations Research*, Vol. 55, No. 2, (2007), pp. 234–251.

Wilmott, P., Howison, S. and Dewynne, J., *The Mathematics of Financial Derivatives*. Cambridge University Press (1995).

Zeng, S. and Zhang, S., Real Options Literature Review. *iBussiness*, Vol.3(1), (2011), pp. 43-48.

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