

DOCTORAL THESIS

**Stability, Robustness and Switching
Performance of Vibrational Control
Systems**

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Abstract

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Stability, Robustness and Switching Performance of Vibrational Control Systems

by Xiaoxiao CHENG

The thesis focuses on the stability and robustness analysis of a class of vibrational control systems, which take advantages of high-frequency dithers to provide an extra design freedom. By carefully revisiting the vibrational stabilizability in literature, a new definition of vibrational stabilizability is introduced, which plays as a basis of this thesis. With this help of this definition, it is possible to obtain more general vibrational stability properties such as semi-globally practically vibrational stability for a large class of nonlinear engineering systems. Stability criteria for the new definition of vibrational stabilizability have been obtained. This new definition can provide a link between the vibrational stabilizability and the standard stability or robustness definition with respect to a given equilibrium point. Hence it makes possible to analyze the robustness properties of the vibrational control system. By using averaging technique and perturbation theory, it has been shown that both linear and nonlinear vibrational control systems are robust with respect to bounded additive disturbances. In particular, when disturbances are much faster compared to the frequency of dithers, it is shown that the system can handle disturbances with a large amplitude while the ultimate bound of the state trajectories can be reduced. Moreover, the transient response of vibrational control systems in the presence of disturbances is fully investigated. Finally, in order to reduce the energy consumption, novel switching laws are proposed. Guidelines for the design of switching laws and parameters tuning are provided for the linear and nonlinear switched vibrational control systems separately. Dynamical behaviours of the switched vibrational control systems have been discovered. Numeric simulations of several application examples have been executed to verify the effectiveness of theoretic analysis.

Declaration of Authorship

I, Xiaoxiao CHENG, declare that this thesis titled, 'Stability, Robustness and Switching Performance of Vibrational Control Systems' and the work presented in it are my own.

I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- The thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

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Date: *3/1/2019*

Preface

The outcomes of the thesis are published or in progress of publication in the following journals and conferences. For all these publications, the student contributed greater than 50% of the content of each publication and is the primary author i.e. the student was responsible primarily for the planning, execution and preparation of the work for publication. However, the student benefited from his supervisors and colleagues from group meeting sessions where they provided technical comments and guidance. The publications and the contribution of each author are listed in the following:

- X. Cheng, Y. Tan, and I. Mareels. On robustness analysis of linear vibrational control systems. *Automatica*, 87 (2018), 202-209.
First author: problem formulation; finding an suitable approach based on literature review; tackling the problem and completing the mathematics proofs; simulations verification; writing the paper; revision.
Second author: supervision; provide technical comments; proofreading; paper polishing.
Third author: consultation.
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First author: problem formulation; finding an suitable approach based on literature review; tackling the problem and completing the mathematics proofs; simulations verification; writing the paper; revision.
Second author: supervision; provide technical comments; proofreading; paper polishing.
Third author: consultation.
- X. Cheng, Y. Tan, and I. Mareels. On non-local vibrational stabilization of non-linear systems. In *Control Conference (ASCC), 2017 11th Asian* (pp. 286-291). IEEE, 2017.
First author: problem formulation; finding an suitable approach based on literature review; tackling the problem and completing the mathematics proofs; simulations verification; writing the paper; revision.
Second author: supervision; provide technical comments; proofreading; paper polishing.
Third author: consultation.
- X. Cheng, Y. Tan, and I. Mareels. On robustness analysis of a vibrational control system: Input-to-state practical stability. In *Control and Automation (ICCA)*,

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Second author: supervision; provide technical comments; proofreading; paper polishing.

Third author: consultation.

- X. Cheng, Y. Tan, and I. Mareels. Robustness of nonlinear vibrational control systems based on sampling Lyapunov method. In American Control Conference (ACC), 2018 IEEE International Conference on. IEEE, 2018.

First author: problem formulation; finding an suitable approach based on literature review; tackling the problem and completing the mathematics proofs; simulations verification; writing the paper; revision.

Second author: supervision; provide technical comments; proofreading; paper polishing.

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Abbreviations

LTI	Linear Time Invariant
SISO	Single Inut Single Output
AP	Almost Periodic
ISS	Iput-to-State Stable
SPVS	Semi-globally Practically Vibrationally Stable
SPAS	Semi-globally Practically Asymptotically Stable
LAS	Locally Asymptotically Stable
GAS	Globally Asymptotically Stable
US	Uniformly Stable
GES	Globally Exponentially Stable
US	Uniformly Stable
UAS	Uniformly Asymptotically Stable
GUAS	Globally Uniformly Asymptotically Stable
GUES	Globally Uniformly Exponentially Stable
VS	Vibrationally Stabilizable
TVS	Totally Vibrationally Stabilizable
SA	Strong Average
WA	Weak Average
LVCS	Linear Vibrational Control System
NVCS	Nonlinear Vibrational Control System
DOA	Domain Of Attraction
CSTR	Continuous Stirred Tank Reactor

Chapter 1

Introduction

1.1 A Motivational Example

A motivational example of vibrational control algorithm is stabilizing the inverted pendulum without a feedback. The mechanism is shown in Figure 1.1. Assume a massless slider is connected to the suspension pin of an inverted pendulum by a revolution joint. The slider acts as an actuator to guide the vertical movement of suspension pin. To stabilize the pendulum, an open-loop high-frequency sinusoidal dither is inserted to the suspension point in the vertical direction as shown in Figure 1.1.

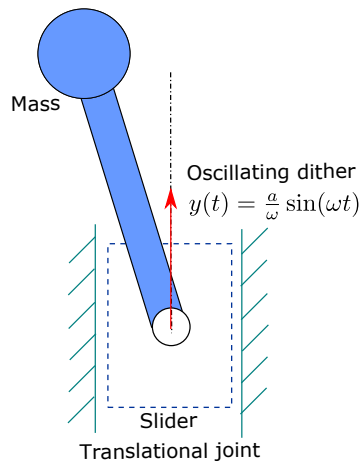


FIGURE 1.1: Inverted pendulum stabilized by injecting high-frequency oscillations

The dynamics equation of the inverted pendulum after injecting dithers is

$$ml\ddot{\theta} + (mg - maw \sin \omega t) \sin \theta + kl\dot{\theta} + ka \cos \omega t \sin \theta = 0,$$

where the variable θ is the angle of pendulum to the lower equilibrium point. The upper equilibrium point of the inverted pendulum then becomes $\theta_e = \pi$, which is open-loop

unstable. The system parameter m is the mass, l is the length of the pendulum and k is the viscous friction coefficient. The vibrational control parameter a can change the amplitude of sinusoidal dither and ω is the frequency.

Representing the system in state-space by letting $x_1 = \theta$, $x_2 = \dot{\theta}$ and introducing the small positive parameter $\varepsilon = \frac{1}{\omega}$, the system becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \left(\frac{a}{\varepsilon l} \sin\left(\frac{t}{\varepsilon}\right) - \frac{ka}{ml} \cos\left(\frac{t}{\varepsilon}\right)\right) \sin x_1 \end{bmatrix}}_{g\left(\frac{t}{\varepsilon}, x, \varepsilon\right)}. \quad (1.1)$$

The simulated trajectories of the dynamics model from different initial positions are depicted in Figure 1.2. It can be seen that all trajectories converge to three equilibrium points $[0, 0]$, $[\pi, 0]$, $[2\pi, 0]$. Some trajectories around $[\pi, 0]$ converge to it which means the originally unstable upper equilibrium point has been stabilized after using the dithers. The dithers injected are open-loop signals so this method can be free from the on-line measurements of the state used in the traditional feedback control.

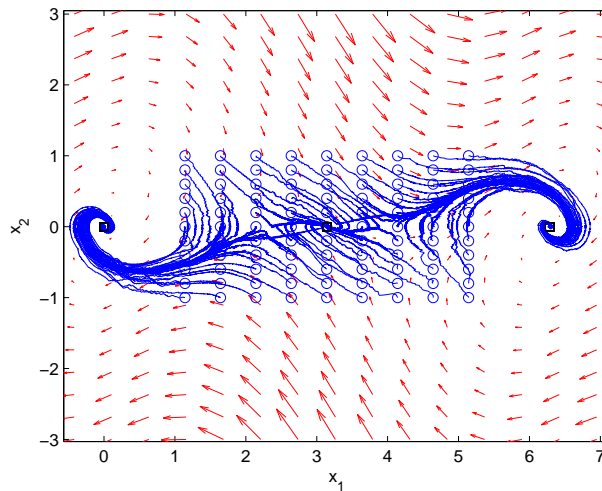


FIGURE 1.2: Phase portrait of the inverted pendulum stabilized by vibrational control

1.2 Literature Review

1.2.1 Background

Vibrational control algorithm was motivated from the aforementioned example of the stabilization of inverted pendulum by injecting vertically high-frequency oscillations. The first publication dates back to 1908 when A. Stephenson [4] firstly demonstrated the stability of inverted pendulum by using dithers. In 1950s, P.L. Kapitsa [5] developed

the rigorous stability conditions for the stabilization of inverted pendulum when using harmonic oscillations. Hence such a stabilized inverted pendulum is also called Kapitza pendulum. Later, N.N. Bogoliubov [6] approximated solutions of Kapitza pendulum and provided the stability conditions by using averaging theory.

1.2.2 Linear Vibrational Control Systems

S.M. Meerkov [7] firstly introduced vibrational control as an open-loop control algorithm that can stabilize an open-loop unstable system by using oscillating dithers. His published papers [7–9], built foundations of vibrational control systems.

In general, for an unstable linear-time-invariant system (LTI) $\dot{x} = Ax$, after injecting dithers, linear vibrational control systems are assumed to have the following form:

$$\dot{x} = (A + B(t))x, x(t_0) = x_0 \in \mathbb{R}^n, \quad (1.2)$$

where $A \in \mathbb{R}^{n \times n}$ is a square matrix and $B : [t_0, \infty) \rightarrow \mathbb{R}^{n \times n}$ is a periodic matrix with a zero mean value, which comes from injecting oscillating dithers to the system.

The vibrationally stabilizability is introduced to indicate the stabilizability of the system $\dot{x} = Ax$ by adding oscillations:

Definition 1.1. [9] The system $\dot{x} = Ax$ is called vibrationally stabilizable if there exists a periodic matrix $B(t)$ with a zero mean value such that the trivial solution $x = 0$ is asymptotically stable.

Remark 1.1. For linear systems, the definition shows that after applying vibrational control, the origin would be transited from an unstable equilibrium point to an asymptotically stable one. This definition would be generalized later for nonlinear vibrational control systems in Chapter 2, where the systems could be stabilized to a limit cycle around the desired equilibrium point.

A necessary and sufficient condition for vibrationally stabilizability was then given:

Theorem 1.1. [9] *Let the system $\dot{x} = Ax$ be observable in principle. Then for this system to be vibrationally stabilizable it is necessary and sufficient that the trace of matrix A be negative.*

Remark 1.2. The system $\dot{x} = Ax$ is observable in principle means that the matrix A is a similar matrix to a controllable canonical matrix, so there is a non-singular matrix P such that $\Xi = P^{-1}AP$ where Ξ is a controllable canonical matrix.

The trace of the matrix is necessary to be negative for the vibrational stabilization. From linear system theory [10, 11] it is known that the system $\dot{x} = Ax$ is stable if all the real parts of eigenvalues are negative and unstable if there exists an eigenvalue with a positive real part. Besides, the trace is defined as the summation of eigenvalues, so it means that although there exist eigenvalues with positive real parts but overall the summation is negative. This provides the possibility for the periodic matrix $B(t)$ to relocate all the real parts of eigenvalues to negative plane such that the system becomes stable. One feasible structures for $B(t)$ is the quasi-triangular structure (see more details in [9]).

The main idea of the proof of the theorem is transforming the system (1.2) into a time-invariant system $\dot{\bar{x}} = (A + \bar{B})\bar{x}$ where \bar{B} is a time-invariant matrix. It is achieved by showing the stability of the time-invariant systems and the closeness of solutions. Although not explicitly demonstrated in [9], this time-invariant system is actually the averaged system.

Later, the work [12, 13] extended the open-loop linear vibrational control systems (1.2) to a closed-loop form by considering the state feedback:

$$\dot{x} = Ax + B_1\phi(t)u, \quad (1.3)$$

and the feedback control $u = Kx$, where $\phi(\cdot)$ is a periodic function s.t. $\phi(t) = \phi(t + T)$. Then the closed-loop system is

$$\dot{x} = (A + B_1\phi(t)K)x. \quad (1.4)$$

In the open-loop form, the periodic matrix $B(t)$ comes from adding oscillations to the component of the matrix A , however the existence of such a $B(t)$ matrix is not always satisfied in applications. In the closed-loop form, the feedback control provides an extra freedom to design the feedback gain K such that $B(t) = B_1\phi(t)K$ is a qualified matrix to stabilize the system $\dot{x} = Ax$. Essentially once the feedback gain K is designed, the system would have the form of (1.4) so the stability analysis is the same. A parameterization approach [14] was used to capture the stabilizability with designed controller to achieve the desired time-domain specifications.

The closed-loop vibrational control systems discussed in [12–14], use time-invariant feedback gain while the periodic function $\phi(t)$ comes from oscillation of structural components, as illustrated in (1.4). In 2004, Luc Moreau discussed the output feedback stabilization with periodic feedback gain for a class of single-input-single-output (SISO)

linear systems [15]:

$$\begin{aligned}\dot{x} &= Ax + bu(t), \\ y &= cx, \\ u(t) &= k(t)y,\end{aligned}\tag{1.5}$$

where $k(t) = m + n\omega\cos(\omega t)$. The closed-loop form of the system (1.5) is in the form of vibrational control systems (1.2) and the conditions for stabilization with periodic feedback were provided. It shows that the introduction of periodic feedback provides an extra design freedom for the pole assignment.

The work [16, 17] addresses the capability of poles relocation of the vibrational control algorithm. For a desired region, necessary and sufficient conditions for poles assignment of the linear vibrational control system (1.2) were given when the system $\dot{x} = Ax$ has both real and complex eigenvalues. The vibrational control algorithm was also proven to be powerful in zero placement, as shown in [12, 18]. It can be applied to the problems of finite gain margin and decentralized fixed modes as illustrated in [19–21]. Similar periodic controllers in discrete systems were shown to have capabilities for gain margins improvement [18, 22, 23] in linear system and be useful to achieve absolute stability of nonlinear system with memoryless uncertainties [24].

1.2.3 Nonlinear Vibrational Control Systems

The framework of nonlinear vibrational control systems was established by R. Bellman, J. Bentsman and S.M. Meerkov. In their seminar work, the definition of vibrational stabilization was extended to capture the stabilization in nonlinear systems. The corresponding criteria of stabilization, controllability and transient behaviour for different types of nonlinear vibrational control systems were addressed in [25–28]. A general form of nonlinear vibrational control system is

$$\dot{x} = f(x) + \frac{1}{\epsilon}g\left(\frac{t}{\epsilon}, x\right),\tag{1.6}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, $g : \mathbb{R}_+, \mathbb{R}^n \rightarrow \mathbb{R}^n$ is T -periodic in its first argument and continuously derivable. ϵ is a sufficiently small positive coefficient which serves as a tuning parameter.

Three types of vibrational control $g(\frac{t}{\epsilon}, x)$ and conditions for vibrational stabilizability were discussed respectively in [26] and [29]:

- when $g(\frac{t}{\epsilon}, x) = B(\frac{t}{\epsilon})x$, it is called linear multiplicative as the vibrational control mapping is linear;
- when $g(\frac{t}{\epsilon}, x) = L(\frac{t}{\epsilon})$, it is called almost periodic (AP) forcing;
- when $g(\frac{t}{\epsilon}, x) = B(\frac{t}{\epsilon})g(x)$, it is called nonlinear multiplicative.

Remark 1.3. In nonlinear stabilization, the oscillating dithers steer the trajectories to a limit cycle around the desired equilibrium point, which is a relaxed stability property compared to the asymptotic stability in linear systems. A non-singular coordinate transformation was introduced to capture the transient behaviour of system (1.6), where averaging technique is applicable. Based on their work, our work considers disturbances exist in the system and analyses the robustness performance. More details about vibrational stabilizability in R. Bellman's work will be revisited in Chapter 2.

Theorem 1.1 shows that in linear vibrational control systems, vibrational stabilization is only possible if the trace of the matrix A is negative, however, B. Shapiro [30] proved that the nonlinear vibrational control system (1.6) could be stabilized even if the Jacobian matrix of the original dynamics $\dot{x} = f(x)$ has a positive trace. This makes it possible to apply the vibrational control algorithm to a large class of engineering systems. K.R. Schneider [31] discussed the vibrational stabilizability of the system with fast and slow variables using the persistence theory of normally hyperbolic invariant manifolds. A. Balestrino [32] provided an alternative averaging method based on Taylor series expansion to resolve the computational difficulties in nonlinear system. The link between the amplitude and frequency of vibrational dither with the amplitude of the steady-state oscillation of system state was established in [33] when vibrational control is AP forcing, making it more practical for the controller design. Also from the design perspective, J.M. Berg [34] introduced a design framework based on stability maps for second-order periodic systems. Stability conditions were extended to vibrational control systems with time lags in [35] indicating that the stabilization is able to handle some sufficiently small state-delay. Vibrational control systems with arbitrarily large but bounded delay [36, 37] were shown to be stable when dithers are sufficiently fast, by applying averaging theory of time delayed differential equations [38]. As an extension of the application of the vibrational control to finite systems with ordinary differential equations, J. Bentsman and K.S. Hong [39] applied it to stabilize a class of distributed parameter systems governed by parabolic partial differential equations with Neumann boundary conditions. The stability criterion and transient response of the specific systems were discussed in [40] and [41] separately.

1.2.4 Robustness Analysis of Vibrational Control Systems

Although various studies have been done to show the stability of vibrational control systems, very limited work has addressed the robustness with respect to disturbances or model uncertainties, especially for nonlinear systems. S.M. Meerkov [9] considered the existence of disturbances in the linear vibrational control systems

$$\begin{aligned}\dot{x} &= Ax + B(t)x + rw, \\ y &= cx,\end{aligned}\tag{1.7}$$

where w belongs to a class of bounded disturbances. The output invariance with respect to disturbances is introduced to capture the robustness which means the output is invariant under different disturbances w_i, w_j i.e.

$$y_{w_i}(t) = y_{w_j}(t), \forall t \geq t_0, w_i \neq w_j.\tag{1.8}$$

The output invariance is obtained in the assumptions that the system is vibrationally stabilizable and (r, c) are orthogonal vectors, which prevents the disturbances from affecting the output.

The modelling uncertainty of linear vibrational control systems was considered in [42]:

$$\dot{x} = (A + \Delta A)x + B(t)x.$$

An upper bound of allowable unstructured uncertainty was derived to preserve the vibrational stabilizability. The work [43] considered disturbances in the closed-loop linear vibrational control systems and discussed disturbance decoupling problem with respect to output, which means by designing the feedback gain, the closed-loop transfer function from disturbances to output is set as zero.

However, even if the disturbances rejection can be shown for special output of vibrational control systems, the influence of disturbances to the state of the vibrational control systems needs further exploration as the state can become unstable even if the output converges. Therefore it is worthwhile to exploit the state-trajectories behaviour in the presence of disturbances. A central robustness concept in the nonlinear systems is input-to-state stable (ISS), which was formulated in [44]. It can estimate the bound of the trajectories of a dynamic system in the presence of disturbances. To the best of our knowledge, there has not been results to reveal the ISS properties of vibrational control systems. Recently, D. Nesić and A.R. Teel [45] developed strong and weak averaging techniques which provide useful tools for robustness analysis of nonlinear time-varying

systems. It naturally becomes a feasible tool for the robustness analysis of disturbed vibrational control systems.

1.2.5 Applications

1.2.5.1 Hamiltonian mechanical systems

Because vibrational control algorithms could provide extra capability for stabilizing the system, it has been useful and successfully applied to stabilize a class of under-actuated robotic, i.e. a manipulator that has more degrees of freedom than control inputs, see [1, 46–49] for examples. In general, it contains a class of mechanical Hamiltonian systems with conservative forces that are integrable in the existence of both holonomic and nonholonomic constraints [50–53].

In Section 1.2.3, the stability analysis of the vibrational control system is mainly dependent on the averaging theory. In classic averaging theory, the linearization matrix of averaged system is usually assumed to be Hurwitz in order to obtain the stability of original time-varying systems and guarantee the closeness of solutions in infinite time domain [54, 55]. However, geometry mechanics provide another possibility for system analysis. As it links to the features of mechanic systems, it becomes natural for Hamiltonian systems. The work [50, 56] analyzed the stability of Hamiltonian systems by both classic averaging theory and geometric analysis. The connection between them is the averaged potential which is an energy-like geometric quantity defined from averaged Hamiltonian, which is an unchanged quantity through averaging. The control design criteria and stability conditions are derived based on averaged potential technique [57, 58], which indicates that the equilibrium is vibrational stabilizable if it is an minimum point in the averaged potential. The geometric method is appealing because it is suitable for a class of systems where vibrational stabilizability could not be directly obtained by classic averaging. J.M. Coron [59] discussed the stability for a specific vibrational control system where there exists some eigenvalue of averaged system lying on the imaginary axis. Although in this case the averaging method is unable to show the stability directly, the geometric approach indicates the stability and provides a guidance for the controller design through the averaged potential. Similar results could be found in [48, 60].

S. Tahmasian [61] considered the existence of different frequencies of the vibrational control inputs in mechanical systems. His results showed that using multi-frequency inputs would result in lower control authority compared to single-frequency inputs. Later in [62], the optimal input waveform shape of vibrational systems in a control-affine form was discussed by transforming the problem into a constrained optimization problem.

It turned out the square waves require the smallest amplitudes while sinusoidal input waveforms consumes the least energy. Combined with a state feedback control, vibrational control inputs have been used to control the flight of a biomimetic air vehicle [63] and track a prescribed trajectory. Simulation results showed that the proposed control algorithms have robustness properties to overcome modelling error and parameter uncertainties.

The control of non-actuated bodies in the under-actuated system relies on the kinematic or dynamic couplings with the actuated parts. For instance, this coupling behaviour exists in the example of stabilizing inverted pendulum where the high-frequency oscillations in the suspension point are connected with the motion of the pendulum, which forms a state-involved vibrational control function in the format of (1.6). Based on this idea, vibrational control algorithm has been applied to steer and stabilize pendulum-based robotic manipulators. In [1, 46], it is used to steer and stabilize the planar 2R manipulator as shown in Figure 1.3.

The dynamics model of the 2R planar manipulator can be obtained from Lagrange equation [1]:

$$\begin{aligned} M_{11}(\theta_2)\ddot{\theta}_1 + M_{12}(\theta_2)\ddot{\theta}_2 + C_1(\theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \tau(t), \\ M_{12}(\theta_2)\ddot{\theta}_1 + M_{22}(\theta_2)\ddot{\theta}_2 + C_2(\theta_2, \dot{\theta}_1, \dot{\theta}_2) &= 0, \end{aligned} \quad (1.9)$$

where M_{ij} is the mass matrices, C_i is the Coriolis vector functions, τ is driven torque from actuator in the first joint. The partially linearised system is obtained after following the same procedure in [1]:

$$\begin{aligned} \ddot{\theta}_1 &= u, \\ \ddot{\theta}_2 &= -(1 + p \cos \theta_2)\ddot{\theta}_1 - p\dot{\theta}_1^2 \sin \theta_2 - f_v \dot{\theta}_2, \end{aligned} \quad (1.10)$$

where $p = m_2 l_1 l_{2c} / (m_2 l_{2c}^2 + I_2)$ is a constant and u is the control input. The first part of the control algorithm is driving the active joint to the desired position by designing proper u . Then vibration control is introduced by inserting oscillations in the active joint: $\theta_1(t) = -\frac{\alpha}{\omega} \cos \omega t + \theta_{1d}$. The dynamics of unactuated joint is derived by the second equation of (1.10), where the dithers are coupled with the pendulum dynamics. Experiments have been executed to verify the feasibility of the algorithm. The reachable and stabilizable area for a vertical two-link underactuated manipulator was found in [47] and the influence of direction of gravity was clarified through nonlinear characteristics of bifurcations. The stabilization of n-pendulum manipulator was discussed in [48].

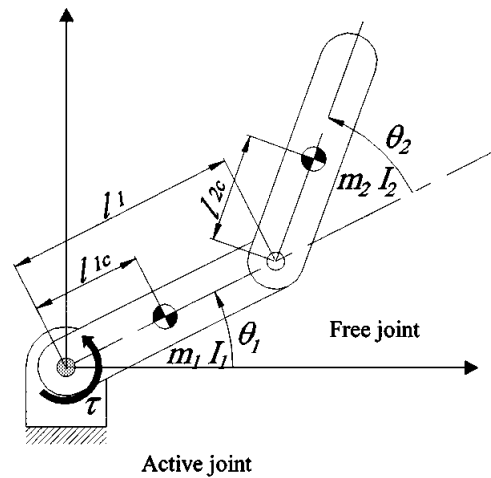


FIGURE 1.3: A Planar 2R manipulator with under-actuated joint [1].

1.2.5.2 Industry applications

Vibrational control has been successfully implemented into some industry applications and shown to be useful in the situations where on-line measurement is difficult and expensive for the feedback control. As introduced in [64], Arrhenius system stands for a large class of chemical reactors, the behaviours of which are characterized by two positive parameters. There exists an instability area due to its original dynamics as well as technique limits such as the maximum admissible temperature. However, the state in the unstable domain offers optimal economical output. Hence it is needed to design a controller to stabilize the state. Traditional feedback is difficult to be applied to such a system because on the one hand, on-line measurement of state is expensive and inaccurate. On the other hand, the system dynamics are too fast to be controlled by the feedback considering the existence of a large delay in the control input [64].

The vibrational control algorithm provides an alternative way to stabilize the Arrhenius system. The work [64] showed that, by adding fast and zero mean oscillations in the input flow rate, the originally unstable equilibriums can be stabilized and sufficient conditions for the stability were given. Experiment results [2] in an exothermic continuous stirred-tank reactor (CSTR, see Figure 1.4) verify the theoretic findings, showing that the introduction of vibrational control algorithm reduces the negative slope part where the system is unstable and improves the system characteristics.

Similar applications, to which the feedback cannot be directly applicable, exist. For example, in the stabilization of the beam of particles along the axis of particle accelerators, it is hard to measure the state [65]. It was also applied to the stabilization of CO_2 laser system [66] where on-line measurement is expensive. In 2003, J.A. Holyst applied vibrational control into a typical financial market and found that if periodic

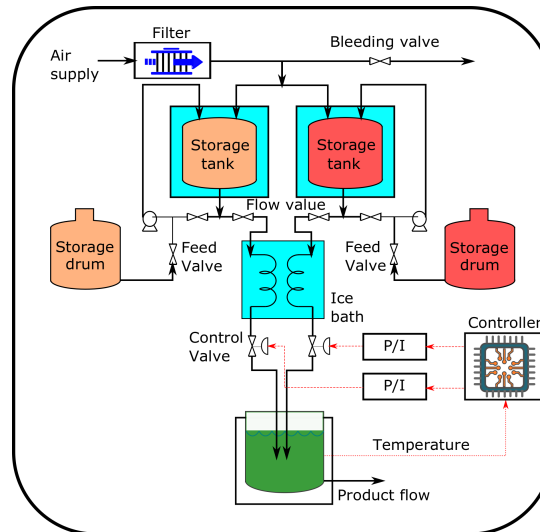


FIGURE 1.4: Diagram of experimental reactor system [2].

perturbations are introduced in demand or supply, the equilibrium price of a product can be increased [67].

It is interesting to know that the vibrational control algorithm has been applied to suppress the oscillations or resonance happened in some systems. It is shown as an effective method by experimental investigations to stabilize the boundary layer flow and panel vibration in [68, 69]. In the transient control of boundary layer, the periodic heating and cooling of the wall results in a parametric oscillation of the fluid viscosity which stabilizes the flow successfully [68]. In the panel stabilization, an additive oscillating force applied in the longitudinal direction with high-frequency could suppress the subharmonic vibrations [69]. In 2003, P.L. Chow generalized the method to stabilize the nonlinear elastic panel excited by the periodic wall-pressure fluctuation in a boundary-layer flow [70]. The control inputs consist of both high-frequency parametric vibrations and the force amplitude modulation to stabilize the unstable periodic motion. This demonstrates the stabilization capacities of vibrational control in infinite-dimensional systems. Similar studies on harmonic vibration suppression by vibrational control have been done for a rotor-bearing system [71] and a helicopter [43].

1.2.6 Summary of the Literature Review

From the literature review, the framework for stability of vibrational control has been built by R. Bellman, J. Bentsman and S.M. Meerkov for both linear and non-linear systems. For linear vibrational systems, a necessary and sufficient condition for stability have been discovered. In nonlinear systems, the vibrational stabilizability concept were proposed to capture the convergence of trajectories to a limit cycle. Stability criteria

based on class averaging theory has been introduced. However, as the linearisation technique is used for the averaged system to be exponentially stable, the obtained results are only valid in a local region around the desired equilibrium point. With the development of recent non-local averaging technique [72], it is possible to extend the local vibrational stabilization to a non-local version, which has a large domain of attraction, making the stabilization available for a larger set of initial conditions.

Although some work has addressed the robustness of the vibrational stabilization with respect to the disturbances or uncertainties, most of them are limited to the output rejection to linear systems. When disturbances exist, the influence to the state-trajectories is worthwhile to explore because they are closely related to the system stability. Besides, there is few work to discuss the robustness analysis of nonlinear vibrational control systems. As one of the most important system performance requirements, a framework for robustness of vibrational control systems is needed to ensure the disturbances handling.

Another potential improvement of the vibrational control method which has not been discussed in literature, is how to reduce energy consumption coming from injecting high-frequency dithers. One way is introducing a switching signal which turns off the control input while it is not necessary in the stabilization process. Then the stability and the dynamic performance of switched vibrational control systems would be worthwhile for more attention.

Overall these tasks which could improve different aspects of the vibrational control, make it more applicable and attractive would be our main objectives in the thesis.

1.3 Contributions of the Thesis

This thesis has the following contributions:

- We extend the current local definition of vibrational stabilizability for nonlinear vibrational control systems to a non-local version by proposing a new definition called semi-globally practically vibrational (SPV) stabilizability, in which the domain of attraction can be arbitrarily large. Sufficient criteria for SPV stabilizability are derived which require the averaged system is globally asymptotically stable. This extends the feasibility of vibrational control to a larger class of engineering systems. The new proposed definition is also useful to be generalized to characterize the robustness of the vibrational control systems. These results have been published in [73].

- A robustness framework of vibrational control systems is built which addresses different types of disturbances for both linear and nonlinear systems. When the system stabilized by vibrational control is linear, input-to-state stability (ISS) is obtained for any bounded disturbances. In particular, when disturbances are also periodic, ultimate bound can be attenuated by high frequencies. In other words, a higher frequency leads to a smaller ultimate bound. By using strong and weak averaging techniques, a sufficient condition is given to show that the vibrational stabilization can handle a more general class of disturbances that are coupled with the state. These results have been published in [74, 75].

For nonlinear vibrational control systems, by applying the sampling-data Lyapunov method the local robustness properties are obtained with respect to a class of additive constrained disturbances. Then the found local robustness properties are extended to more general robustness results. By considering a relatively weak stability condition, similar robustness conclusions can be made for a large class of systems with either local or non-local stability domain. When the disturbances are periodic, the vibrational control system can handle large disturbances if they are fast-varying and the average of the disturbances is small. These results have been included in [76] and a journal paper recently submitted to *Automatica*.

- The energy consumption of vibrational control algorithm is relatively high as the high-frequency dithers are used, which have potential damages to the actuators. To reduce the energy consumption, novel switching laws are introduced to both linear and nonlinear vibrational control systems. Performance of switched vibrational control systems in consideration of disturbances is explored. The guidelines for the switching law design and parameters tuning are provided to reduce the ultimate bound and increase the convergence speed. Parts of these results are included in the aforementioned submitted journal paper.

1.4 Organization of the Thesis

The remaining part of this thesis is organized as follows. In Chapter 2, the Lyapunov stability for both equilibrium points and periodic solutions is reviewed. Then the general format of nonlinear vibrational control systems and the definition of vibrational stabilizability are introduced. An important coordinate transformation for stability and robustness analysis is introduced before discussing the local and non-local stabilization results.

Chapter 3 and 4 address the robustness of vibrational control systems. In Chapter 3, both the original dynamics and vibrational control function are assumed to be linear. An

important robustness property called input-to-state stability (ISS) will be introduced followed by the strong and weak averaging techniques, which are useful robustness analysis tools for a general nonlinear time-varying system. As a natural choice for the robustness analysis, the first robustness result is based on strong and weak averaging techniques, which is applicable to slowly varying disturbances. Subsequently, a more general robustness is obtained by applying averaging and perturbation techniques. Last part of Chapter 3 discusses the system performance in the presence of a more general class of disturbances that are coupled with state.

The robustness analysis for nonlinear vibrational control systems is given in Chapter 4. Firstly a local robustness conclusion is obtained from extending the local vibrational stabilization by considering the existence of additive disturbances. With weak stability conditions, more general robustness results with respect to additive disturbances are found for a large class of vibrational control systems with either local or non-local stability domain. When disturbances are periodic, the robustness properties of vibrational control method become stronger. By adapting the weak averaging technique, the vibrational control systems can handle arbitrarily large bounded disturbances under some conditions.

The system performance of switched vibrational control systems is introduced in Chapter 5. The basic knowledge on the switched systems in literature is reviewed, including different switching schemes and the stability and robustness analysis methods of switched systems. A switching signal with average dwell time is introduced to the linear vibrational control systems and the stability of these switched systems is discussed. To avoid the trajectories escaping the domain of attraction for nonlinear vibrational control systems, a periodic switching scheme is used. Dynamic behaviour of the switched vibrational control systems is analysed in consideration of additive disturbances. The guidelines for the switching laws design and parameters tuning are carefully explained.

Finally, the conclusions and an outline of possible future research directions are included in Chapter 6. The references are listed in the Bibliography and the Appendices contain detailed derivations of some theorems in this thesis.

Chapter 2

Vibrational Stabilization of Nonlinear Systems

2.1 Overview

This chapter will address the stabilization of nonlinear systems with vibrational control method. First of all, a stability definition called vibrational stabilizability is introduced to characterize a class of nonlinear systems that can be stabilized by using dithers. Vibrational stabilizability shows that an open-loop unstable system can be stabilized to a limit cycle around the desired equilibrium point. To show the system is vibrationally stabilizable, a coordinate change is needed to convert the problem from the stability analysis of a limit cycle to an equilibrium point, thus the stability analysis tools for an equilibrium point can be applied.

Averaging is a key technique to show the stability of a nonlinear time-varying system by transforming it to an autonomous system. Based on the classic averaging tool, a local vibrational stabilizability result in literature will be reviewed, which assumes that the linearization matrix of the averaged system is Hurwitz. The system is shown to be vibrationally stabilizable within a local domain of attraction.

However, when the Hurwitz condition is unsatisfied, for example, some eigenvalues lie on the imaginary axis of complex plane, it is interesting to discuss whether the systems still can be stabilized by vibrational control. A motivational example is found to support the hypothesis. The averaged system in the example is globally asymptotically stable although its linearisation matrix is not Hurwitz. Simulations results indicate the satisfaction of vibrational stabilizability so we seek an alternative averaging technique to extend the results to make it applicable to a large class of systems.

Thus we propose a non-local vibrational stabilization concept called semi-globally practically vibrational (SPV) stabilizability which means the system can be stabilized by vibrational control algorithm for any compact initial domain. We shows that the SPV stabilizability can be obtained if the averaged system is globally asymptotically stable, uniformly in the tuning parameter.

The chapter is organized as follows. In Section 2.2, the Lyapunov stability for an equilibrium point as well as periodic solutions is introduced. As the key analysis tool for stability analysis of vibrational control systems, both local and non-local averaging technique will be introduced and explained. A general nonlinear vibrational control systems is formulated in Section 2.3 before the definition of vibrational stabilizability is introduced. Next, the procedure to make a coordinate change is explained after introducing several auxiliary systems. Based on that, the local vibrational stabilizability result is put forward. In Section 2.6, the conditions of semi-globally practically asymptotically vibrational stabilizability is proposed, which are verified by numeric simulations. Section 2.7 summarizes the chapter.

2.2 Preliminaries

2.2.1 Lyapunov Stability

2.2.1.1 Autonomous systems

In this section, we will introduce the Lyapunov stability. Firstly let's consider an autonomous systems

$$\dot{x} = F(x), x(t_0) \in R^n \quad (2.1)$$

where $F : D \rightarrow R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ to R^n . A function is Lipschitz if there exists a constant L such that for $x, y \in D$ the following inequality is satisfied

$$|f(x) - f(y)| \leq L|x - y|^1. \quad (2.2)$$

Without losing generality, suppose the origin $x = 0$ is the equilibrium point of the system such that $F(0) = 0$. Then the stability of the origin $x = 0$ is defined as follows

Definition 2.1. [77] The equilibrium point $x = 0$ is

¹In this thesis, we use the notation $|\cdot|$ to present the Euclidean norm and $\|\cdot\|$ for the norm in the functional space.

- stable if for each $\delta > 0$ there exists $\Delta > 0$ such that

$$|x(t)| < \delta, \forall t \geq 0 \quad (2.3)$$

whenever $|x(0)| < \Delta$.

- unstable if it is not stable.
- asymptotically stable if it is stable and Δ can be chosen such that

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad (2.4)$$

whenever $|x(0)| < \Delta$.

- globally asymptotically stable if it is asymptotically stable and Δ can be arbitrarily large.

A positive Lyapunov function can be used for determine the stability of the equilibrium point:

Theorem 2.1. [77] *Let $x = 0$ be an equilibrium point for (2.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \rightarrow R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$. If $\dot{V}(x) \leq 0$ for all $x \in D$ then $x = 0$ is stable. Moreover, if $\dot{V} < 0$ for all $x \in D - \{0\}$ then $x = 0$ is asymptotically stable.*

If the Lyapunov function is radially unbounded and the derivative is negative, the equilibrium is globally asymptotically stable:

Theorem 2.2. [77] *Let $x = 0$ be an equilibrium point for (2.1). Let $V : R^n \rightarrow R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$. If $V(x)$ is radially unbounded i.e. $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $\dot{V} < 0$ for all $x \neq 0$, then $x = 0$ is globally asymptotically stable.*

Normally the derivative of Lyapunov function along the system (2.1) is required to be negative to guarantee the asymptotic stability of the equilibrium point, however in some system with the additional knowledge about the behaviour of the solutions, the negativity can be relaxed by the LaSalle's invariance principle. Next, the definition of invariant set and the LaSalle's theorem are stated.

Definition 2.2. A set M is said to be an invariant set with respect to (2.1) if

$$x(t) \in M, \forall t \geq 0 \quad (2.5)$$

for all $x(0) \in M$.

Theorem 2.3. [78] Let $x = 0$ be an equilibrium point for (2.1). Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable radially unbounded function such that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. Suppose that the derivative along the solutions of the system (2.1) satisfies $\dot{V} \leq 0$. Let M be the largest invariant set contained in the set $\{x : \dot{V}(x) = 0\}$. Then the system (2.1) is stable and every solution that remains bounded for $t \geq 0$ approaches M as $t \rightarrow \infty$. In particular, if all solutions remain bounded and $M = \{0\}$, then the system (2.1) is globally asymptotically stable.

Remark 2.1. The boundedness of solutions can be satisfied by the condition $\dot{V} \leq 0$. To show the equilibrium point $x = 0$ is globally asymptotically stable it is required to demonstrate that no solutions can stay in the set $\{x : \dot{V}(x) = 0\}$ except for $x = 0$. We will use the LaSalle Theorem later in the example 2.2 to show the averaged system is globally asymptotically stable.

The stability of an equilibrium point of a nonlinear autonomous system (2.1) can be indicated by investigating the stability of linearised system associated with the equilibrium point. The follow theorem is known as Lyapunov's indirect method:

Theorem 2.4. [77] Let $x = 0$ be an equilibrium point for the nonlinear system (2.1). Let the linearised matrix

$$A = \left. \frac{\partial f}{\partial x}(x) \right|_{x=0}.$$

Then

- The origin is asymptotically stable if $\text{Re}(\lambda_i) < 0$ for all eigenvalues of A , where $\text{Re}(\lambda_i)$ means the real part of some eigenvalue.
- The origin is unstable if $\text{Re}(\lambda_i) > 0$ for one or more of the eigenvalues of A .

2.2.1.2 Time-varying systems

Next, we will introduce the stability of the equilibrium point for a nonlinear time-varying system. The following time-varying system is considered:

$$\dot{x} = F_1(t, x), x(t_0) \in \mathbb{R}^n \quad (2.6)$$

where $F_1 : [0, \infty) \times D \rightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, then there exists a constant L such that for all $x, y \in D$ and $t \geq 0$ the following inequality satisfies:

$$|F_1(t, x) - F_1(t, y)| \leq L|x - y|, \quad (2.7)$$

where $D \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for (2.6) if $F_1(t, 0) = 0, \forall t \geq 0$.

Comparison functions are introduced to characterize the stability of time-varying systems.

Definition 2.3. [77] A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class \mathcal{K}_∞ if $\alpha : [0, \infty) \rightarrow [0, \infty)$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Definition 2.4. [77] A continuous function $\beta : [0, a) \times [0, \infty)$ is said to belong to class \mathcal{KL} if for each fixed s the mapping $\beta(r, s)$ belongs to class \mathcal{K} with respect to r and for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

The definitions of uniform stability, uniform asymptotic stability are given by using class \mathcal{K} and \mathcal{KL} functions.

Definition 2.5. [77] The equilibrium point $x = 0$ of (2.6) is

- uniformly stable (US) if there exists a class \mathcal{K} function α , and a positive constant c , independent of t_0 , such that

$$|x(t)| \leq \alpha(|x(t_0)|), \forall t \geq t_0 \geq 0, \forall |x(t_0)| \leq c; \quad (2.8)$$

- uniformly asymptotically stable (UAS) if there exists a class \mathcal{K} function α , a class \mathcal{KL} function β and a positive constant c , independent of t_0 , such that

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \forall t \geq t_0 \geq 0, \forall |x(t_0)| \leq c; \quad (2.9)$$

- uniformly exponentially stable (UES) if there exist positive constants c, k and λ such that

$$|x(t)| \leq k|x(t_0)|e^{-\lambda(t-t_0)}, \forall |x(t_0)| \leq c; \quad (2.10)$$

- globally uniformly asymptotically (GUAS) stable if the inequality (2.9) is satisfied for any initial state $x(t_0) \in \mathbb{R}^n$.

When the system is parametrized, the stability could depend on the tuning of the parameter. Next we consider a class of parametrized nonlinear time-varying systems

$$\dot{x} = F_2(t, x, \varepsilon), x(t_0) \in \mathbb{R}^n, \quad (2.11)$$

where x_e is the equilibrium point uniformly in t such that $F_2(t, x_e, 0) = 0$. The semi-globally practically asymptotic stability for system (2.11) is defined as follows:

Definition 2.6. [45] Let $\beta \in \mathcal{KL}$. The equilibrium point x_e of a time-varying system $\dot{x} = F_2(t, x, \varepsilon)$ is said to be semi-globally practically asymptotically (SPA) stable uniformly in ε if for each pair of strictly positive real numbers (Δ, δ) , there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of $\dot{x} = F_2(t, x, \varepsilon)$ satisfy

$$|x(t) - x_e| \leq \beta(|x_0 - x_e|, t - t_0) + \delta, \forall t \geq t_0 \geq 0, \quad (2.12)$$

whenever $|x_0 - x_e| \leq \Delta$.

Remark 2.2. Compared to uniform asymptotic stability defined in Definition 2.5, SPA stability is dependent on a tuning parameter such that it holds when the parameter ε is sufficiently small. According to the definition, the domain of attraction is captured by the parameter Δ which can be arbitrarily large. All trajectories starting in the domain of attraction converge to a neighbourhood that can be arbitrarily small around the equilibrium point, as characterized by the parameter δ . This neighbourhood circling the trajectories of steady-states is called ultimate bound.

2.2.1.3 Stability of periodic solutions

When periodic solutions exist in the time-varying systems

$$\dot{x} = F_1(t, x), x(t_0) \in \mathbb{R}^n, \quad (2.13)$$

the defined stability concepts for an equilibrium point can be adapted to characterize the stability of periodic solutions. Let $u(t)$ be a periodic solution of $\dot{x} = F_1(t, x)$ thus it satisfies $u(t + T) = u(t)$ and $\dot{u}(t) = F_1(t, u(t))$. Introducing $v(t) = x(t) - u(t)$, in the new coordinate, the system (2.13) becomes

$$\dot{v} = F_1(t, v + u(t)) - F_1(t, u(t)) = \tilde{F}_1(t, v). \quad (2.14)$$

Hence, in the new coordinate, $v = 0$ becomes an equilibrium point of the system (2.14). Thus the definition of uniformly asymptotic stability for an equilibrium point can be used to describe the asymptotic stability of periodic solutions.

Definition 2.7. Let D be a domain in \mathbb{R}^n which contains origin. The periodic solution $u(t)$ of nonlinear time-varying systems $\dot{x} = F_1(t, x)$ is called uniformly asymptotically stable, if there exists a class- \mathcal{KL} function $\beta(\cdot, \cdot)$ such that in the new coordinate $v = x - u$,

the solutions of $v(t)$ satisfy

$$|v(t)| \leq \beta(|v(t_0)|, t - t_0), \forall v(t_0) \in D.$$

When a family of parametrized systems $\dot{x} = F_2(t, x, \varepsilon)$ is considered, the periodic solution $u(t)$ is also parametrized by ε , i.e., $u(t) = u_\varepsilon(t)$. The asymptotic stability uniformly in ε is defined as follows:

Definition 2.8. Let D be a domain in \mathbb{R}^n which contains origin. The periodic solutions $u_\varepsilon(t)$ of nonlinear time-varying systems $\dot{x} = F_2(t, x, \varepsilon)$ is called uniformly asymptotically stable, uniformly in ε , if there exists a class- \mathcal{KL} function $\beta(\cdot, \cdot)$ such that there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, in the new coordinate $v_\varepsilon = x_\varepsilon - u_\varepsilon$, the solutions $w_\varepsilon(t)$ satisfy

$$|v_\varepsilon(t)| \leq \beta(|v_\varepsilon(t_0)|, t - t_0), \forall v_\varepsilon(t_0) \in D.$$

Remark 2.3. The stability definition of periodic solutions is generated from stability definition of the equilibrium point by analyzing the stability of origin in system (2.14). The trajectories of periodic solutions $u_\varepsilon(t)$ form a closed and time-invariant orbit Ω_u in state plane such that $u_\varepsilon(t)$ stays in the orbit Ω_u for all $t \geq t_0$ if the initial condition $u_\varepsilon(t_0)$ lies in the orbit Ω_u . If periodic solutions $u_\varepsilon(t)$ are locally asymptotically stable, the closed orbit will attract all solutions in its neighbourhood. The Δ -domain of attraction is then defined as a neighbourhood of Ω_u : $ROA = \left\{ x \in \mathbb{R}^n \mid \inf_{y \in \Omega_u} |x - y| < \Delta \right\}$.

Correspondingly, next definition defines uniformly global asymptotic stability uniformly in parameter ε .

Definition 2.9. The periodic solution $u_\varepsilon(t)$ of nonlinear time-varying systems $\dot{x} = F_2(t, x, \varepsilon)$ is called uniformly globally asymptotically stable, uniformly in ε , if there exists a class- \mathcal{KL} function $\beta(\cdot, \cdot)$ such that there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, in the new coordinate $w_\varepsilon = x_\varepsilon - u_\varepsilon$, the solutions of $w_\varepsilon(t)$ satisfy

$$|w_\varepsilon(t)| \leq \beta(|w_\varepsilon(t_0)|, t - t_0), \forall w_\varepsilon(t_0) \in \mathbb{R}^n.$$

2.2.2 Averaging

Averaging is an approximation method that estimates the solutions $x(t)$ of a time-varying system by calculating the solutions $x_{av}(t)$ of a time-invariant averaged system [6], [55]. The approximation error between $x(t)$ and $x_{av}(t)$ can be reduced by tuning the system parameters, thus the analysis of the time-varying system behaviour or stability is dependent on the time-invariant averaged system that is normally simpler than the original time-varying system.

The order of magnitude notation will be used to describe the approximation error, which is defined in the following:

Definition 2.10. $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ if there exist positive constants k and c such that

$$|\delta_1(\varepsilon)| \leq k|\delta_2(\varepsilon)|, \forall |\varepsilon| < c. \quad (2.15)$$

Remark 2.4. In the averaging theory, the error between the solutions $x(t)$ and $x_{av}(t)$ can be in the order of ε where ε is a positive tuning parameter in the system. Expressed as $x(t) - x_{av}(t) = O(\varepsilon)$, the error satisfies the inequality $|x(t) - x_{av}| \leq k\varepsilon$, $\forall \varepsilon \leq c$ where k, c are positive constants.

Next we will consider the parametrized time-varying systems and introduce the Averaging results. The following parametrized time-periodic system is considered:

$$\dot{x} = \varepsilon F_2(t, x, \varepsilon), x(t_0) \in \mathbb{R}^n, \quad (2.16)$$

where ε is a small positive parameter and there exists $T > 0$ such that

$$F_2(t + T, x, \varepsilon) = F_2(t, x, \varepsilon), \forall (t, x, \varepsilon) \in [0, \infty) \times D \times [0, \varepsilon^*], \quad (2.17)$$

for some domain $D \in \mathbb{R}^n$. The autonomous averaged system associated with the system (2.16) is defined as:

$$\dot{x} = \varepsilon F_{av}(x), x(t_0) \in \mathbb{R}^n, \quad (2.18)$$

where $F_{av}(x) = \frac{1}{T} \int_0^T F_2(\tau, x, 0) d\tau$.

The closeness of solutions between time-varying systems (2.16) and its approximated time-invariant system (2.18) is characterized in the next theorem. Furthermore, if the averaged system (2.18) is locally exponentially stable then stability of time-varying system (2.16) could be concluded.

Theorem 2.5. [77, Theorem 10.4] *Let $F_2(t, x, \varepsilon)$ and its partial derivatives with respect to (x, ε) up to the second order be continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [0, \varepsilon^*]$ for every compact set $D_0 \in D$, where D is a domain in \mathbb{R}^n . Suppose F is T -periodic in t for some $T > 0$ and ε is a positive parameter. Let $x_\varepsilon(t)$ and $x_{av,\varepsilon}(t)$ denote the solutions of (2.16) and (2.18) respectively.*

- *If $x_{av,\varepsilon}(t) \in D \forall t \in [0, b/\varepsilon]$ and $x_\varepsilon(0) - x_{av,\varepsilon}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x_\varepsilon(t)$ is defined and*

$$x_\varepsilon(t) - x_{av,\varepsilon}(t) = O(\varepsilon) \quad \text{on } [0, b/\varepsilon].$$

- If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the averaged system (2.18), $\Omega \in D$ is a compact subset of its domain of attraction, $x_{av,\varepsilon}(0) \in \Omega$, and $x_\varepsilon(0) - x_{av,\varepsilon}(0) = O(\varepsilon)$, then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $x_\varepsilon(t)$ is defined and

$$x_\varepsilon(t) - x_{av,\varepsilon}(t) = O(\varepsilon) \quad \text{on } [0, \infty].$$

- If the origin $x = 0 \in D$ is an exponentially stable equilibrium point of the averaged system (2.18), then there exist positive constants ε^* and k such that for all $0 < \varepsilon < \varepsilon^*$, (2.16) has a unique, exponentially stable, T -periodic solution $x_\varepsilon(t)$ with the property $|x_\varepsilon(t)| \leq k\varepsilon$.

Remark 2.5. The first sub-result shows that if the initial conditions of the time-varying system (2.16) and the averaged system (2.18) are close, by tuning the parameter ε sufficiently small, the closeness of solutions can be arbitrarily small in a finite time interval. Besides, if the origin is an exponentially stable equilibrium point for the averaged system (2.18), the closeness of solutions are sufficiently small for infinite time range. Moreover, in this case the time-varying system (2.16) has exponentially stable periodic solutions, which converge to a neighbourhood of the origin. The parameter ε plays an important role for the system behaviour of (2.16) and (2.18) as when ε tends to zero both trajectories change slowly, behaving more like a constant. Actually in this special form of systems, the solutions of the time-varying system (2.16) can be regarded as the solutions of the averaged system (2.18) perturbed by a small perturbation which is in the order of ε . More generally, the periodic requirement for the time-varying system (2.18) can be relaxed to almost periodic one, where similar closeness and stability can be reached by assuming stability of the corresponding averaged system (see more details in [77, Theorem 10.5]). However, considering that the averaged system is assumed to be exponentially stable in a local region, the obtained closeness of solutions and stability results are both restricted in the predefined local domain.

A non-local stability can be derived for the time-varying system (2.16) if the averaged system (2.18) is globally asymptotically stable, which is introduced in [45, 72].

Theorem 2.6. [45] Suppose $F_2(t, x, \varepsilon)$ is Lipschitz for $x \in D$ uniformly in t and ε , where D is a domain in R^n . If the averaged system (2.18) is globally asymptotically stable, the actual system $\dot{x} = \varepsilon F_2(t, x, \varepsilon)$ is semi-globally practically asymptotically stable, uniformly in ε .

Remark 2.6. Compared to the local averaging results in Theorem 2.5, Theorem 2.6 supposes that the averaged system is globally asymptotically stable, consequently the stability of periodic system can be guaranteed in the semi-global region. In the next

sections, we will use both local and non-local averaging result to indicate the stability of nonlinear vibrational control systems.

2.3 Nonlinear Vibrational Control Systems and Stabilization

The objective of vibrational control method is to stabilize an unstable equilibrium point x_e for the time-invariant system:

$$\dot{x} = f(x), x(t_0) = x_0 \in \mathbb{R}^n, \quad (2.19)$$

where nonlinear mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and the notion x_e denotes a point in \mathbb{R}^n such that $f(x_e) = 0$.

After introducing the dither signals, the general format of nonlinear parametrized vibrational control systems have been formulated in [29, 79]. It satisfies the following additive form:

$$\dot{x} = f(x) + \frac{1}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right), x(t_0) = x_0 \in \mathbb{R}^n, \quad (2.20)$$

for all $t \geq t_0 \geq 0$. Here $g : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and T -periodic in t , locally Lipschitz in x , uniformly in t . The parameter ε is a small positive constant.

Remark 2.7. As discussed in [29], vibrational control systems have a specifically additive structure which consists of two parts: the first term represents open-loop system dynamics that could be unstable while the second term is related to high-frequency dither signals. This special structure originates from the stabilization of inverted pendulum without using feedback. In that example, a sinusoidal dither is added to the suspension pin vertically as an open loop control input, which results in a closed-loop-like form (2.20) as dither signals are naturally coupled with states.

Remark 2.8. The time-periodic and dither related mapping $g(t, x)$ is the key to stabilize the unstable equilibrium point by injecting oscillations, however there still lacks of a standard procedure to design a suitable $g(\cdot, \cdot)$ for a given unstable system $f(x)$. Normally it relies on the practical experience to find a component to add the dither signal which couples suitably with the system dynamics. Another possible way is considering parameter perturbations, as shown in the stabilization of Rayleigh and Duffing equation [27]. Moreover, it can also result from time-varying feedback control gain as illustrated in the example of periodic output feedback control systems [15].

The idea of vibrational stabilizability is to show that by inserting dither signals, the solutions of the system (2.20) converge to asymptotically stable periodic solutions around the desired equilibrium point. The definition is given below:

Definition 2.11. [26] Let $D \in \mathbb{R}^n$ be a domain containing the equilibrium point x_e as an interior point. The time-invariant system (2.19) is said to be vibrationally stabilizable (v-stabilizable) if there exists an almost periodic and zero-mean function $g(t, x)$ in t such that the system (2.20) has an almost periodic, asymptotically stable solution $x(t)$ characterized by

$$|\bar{x} - x_e| \leq \delta, \forall x_0 \in D, \quad (2.21)$$

where $\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} x(\tau) d\tau$.

Remark 2.9. The definition is modified from [26], compared to that, the domain of attraction is introduced to specify the initial domain. The definition requires that the vibrational control system (2.20) has asymptotically stable periodic solutions, which would form a limit cycle in the state-space. Besides, it requires the average of the solutions converges arbitrarily close to the equilibrium point. From a geometric view, the equilibrium point is thus the averaged center of the limit cycle formed by the periodic solutions.

Remark 2.10. The definition of vibrational stabilizability captures the properties of the original dynamics (2.19) to be stabilized by using dithers. We sometimes also call it vibrational stability, which addresses the stability of periodic solutions of the vibrational control systems (2.20) after injecting dithers. Thus it is obvious that if the system (2.20) is vibrationally stable, then the original dynamics (2.19) is vibrationally stabilizable. To obtain the vibrational stabilizability, it is normally more convenient to show that the system (2.20) is vibrationally stable for some dither function $g(\cdot, \cdot)$. In other words, the problem in stability analysis is converted to show that limit cycle of the vibrational system (2.20) is asymptotically stable.

In some situations, not only the averaged trajectories of the vibrational control systems (2.20) converge to the equilibrium point x_e , but also does the real trajectories when the limit cycle shrinks to a point along a trajectory to the equilibrium point. This kind of vibrational stability property is called totally vibrationally stabilizability:

Definition 2.12. [26] Let $D \in \mathbb{R}^n$ be a domain containing the equilibrium point x_e as an interior point. The equilibrium point x_e of $f(x)$ is said to be totally vibrationally stabilizable (tv-stabilizable) if for any $\delta > 0$, there exists periodic $g(t, x)$ in time with zero-mean-value such that there exists t^* s.t. the solutions $x(t)$ satisfy

$$|x(t) - x_e| < \delta, \forall t > t^*,$$

whenever $x_0 \in D$.

Remark 2.11. Totally vibrationally stabilizable system has a better performance as the trajectories converge to the desired equilibrium point which satisfies the control objective. In this special case, the vibrational stabilizability for limit cycle has the same meaning with asymptotically stability for an equilibrium point. Although it has special requirements for the system, it's not a rare phenomena, for example, the stabilization of inverted pendulum by injecting dithers satisfies this definition. We will give the conditions for both vibrational stabilizability and totally vibrational stabilizability in the next section.

2.4 Coordinate Transformation

In order to analyze the stability of the limit cycle of system (2.20), a coordinate change is introduced to transform the system (2.20) into a standard average form (2.16). Following the procedures in [26], after introducing a new time ' $\tau = \frac{t}{\varepsilon}$ ', the system (2.20) becomes

$$\frac{dx}{d\tau} = \varepsilon f(x) + g(\tau, x), x(\tau_0) = x_0 \in \mathbb{R}^n. \quad (2.22)$$

It is noticeable that the dynamics of the system (2.22) is dominated by the periodic function $g(\tau, x)$ when ε is sufficiently small, so we will analyze the behaviour of the auxiliary system first:

$$\frac{d\xi}{d\tau} = g(\tau, \xi), \xi(\tau_0) = c. \quad (2.23)$$

Here we make a mild assumption that the solution $\xi(\tau) = h(\tau, c)$ of the time-periodic system (2.23) is also periodic:

Assumption 2.1. Let Ω be a compact set in \mathbb{R}^n containing the origin. For any given constant $c \in \Omega$, the nonlinear function $h(\tau, c)$ is continuous and T -periodic with respect to τ , locally Lipschitz continuous with respect to c .

Under this assumption, the behaviour of the system (2.23) is a limit cycle, the size or position of which is decided by the initial constant c . The overall behaviour of the system (2.22) can be regarded as the behaviour of the dominant periodic vibrational function $g(\tau, x)$ perturbed by the original dynamics $\varepsilon f(x)$ such that it keeps updating the initial constant c , causing a transient movement of the limit cycle $h(\tau, c)$. To capture this transient behaviour, a coordinate change is introduced which substitutes the initial constant c with new coordinate y :

$$x(\tau) = h(\tau, y). \quad (2.24)$$

As shown in [26], $\left\{\frac{\partial h}{\partial y}\right\}^{-1}$ exists in Ω , by taking derivative in both side, the dynamics of transient y -system satisfy the following equation

$$\frac{dy}{d\tau} = \varepsilon \left\{\frac{\partial h}{\partial y}\right\}^{-1} f(h(\tau, y)) = \varepsilon f_1(\tau, y). \quad (2.25)$$

Remark 2.12. The coordinate change provides a way to analyze the dynamics behaviour of the system (2.22). We analyze y -system (2.25) first to see transient behaviour $y(t)$. As $x(t)$ and $y(t)$ are linked by the coordinate change, the behaviour of $x(t)$ can be obtained consequently once $y(t)$ is known. To guarantee the convergence of the $x(t)$, $y(t)$ is supposed to converge to some point y_e such that the $x(t)$ converge to the limit cycle $h(t, y_e)$. The problem is then converted to analyze the stability of transient system (2.25), which is also periodic in time. Averaging is a useful method to show the stability of such a time-periodic system by approximating the behaviour with an autonomous averaged system.

The averaged system of the transient system (2.25) is

$$\frac{dz}{d\tau} = \varepsilon f_{1,av}(z), \quad (2.26)$$

where $f_{1,av}(z) := \frac{1}{T} \int_0^T \left\{\frac{\partial h}{\partial z}\right\}^{-1} f(h(\tau, z)) d\tau$ and z_e is the equilibrium point of $f_{1,av}(z)$ such that $f_{1,av}(z_e) = 0$.

Remark 2.13. According to the averaging theory (Theorem 2.5 and 2.6), when the averaged system (2.26) holds some stability conditions, the corresponding stability of transient system can be concluded. In the next section, the local averaging technique is firstly used to produce vibrational stabilizability in a local region, then we extend the results to non-local vibrational stabilizability by applying non-local averaging method.

Remark 2.14. Coordinate change is an essential bedding for the use of averaging technique, otherwise the averaging technique is unable to prove the stability if directly applied to the overall system (2.20). Due to the prerequisite $g(t, x)$ is zero mean function, which would become zero after averaging, the averaged system is thus composed of the original dynamics i.e. $\dot{x} = f(x)$ that has an unstable equilibrium point. When the averaged system is unstable, we could neither conclude the time-varying system is stable or not. The reason why the transient system (2.25) in y -coordinate can be stable is that the dithers are coupled with original dynamics so they have a chance to stabilize the averaged system (2.26) by tuning the parameter. Proved in the [29], the coordinate change is non-singular so the stability properties of the overall system (2.20) and transient system (2.25) are equivalent.

2.5 Local Vibrational Stabilization

The vibrational stability of the system (2.20) relies on the stability analysis of transient y -system (2.25), the behaviour of which is approximated by its averaged system (2.26). As the coordinate change is non-singular, the limit cycle in the system (2.20) can be shown to be stable if the averaged system (2.26) holds certain stability. In [26], the linearization matrix of the averaged system (2.26) is assumed to be Hurwitz, leading to the fact that the averaged system is locally exponentially stable. Besides, the following assumptions are needed.

Assumption 2.2. Let Ω be a compact set in R^n . The nonlinear mapping $f_1(\cdot, y)$ in the dynamics of $y(\tau)$ in (2.25) is locally Lipschitz continuous for all $y \in \Omega$.

Assumption 2.3. The nonlinear mapping $f_{1,av}(\cdot)$ in the averaged system (2.26) is continuously differentiable for all $z \in \Omega$.

Denote z_e as the equilibrium point of the averaged system (2.26). To guarantee that the averaged trajectories converge to desired equilibrium point x_e , the equilibrium points are supposed to satisfy the following assumption:

Assumption 2.4. There exists an equilibrium point z_e of (2.26) such that

$$\frac{1}{T} \int_0^T h(\tau, z_e) d\tau = x_e. \quad (2.27)$$

Therefore, the local vibrational stabilization of system (2.20) is summarized in the following Theorem.

Theorem 2.7. [26] *Let Ω be a compact set in R^n . Suppose that Assumptions 2.1 - 2.4 hold. If the linearization matrix of the averaged system (2.26) $\bar{A} = \left[\frac{\partial f_{1,av}}{\partial z} \right]_{z=z_e}$ is Hurwitz, then there exist $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, the system (2.19) is vibrationally stabilizable for all $x_0 \in \Omega$.*

Remark 2.15. Theorem 2.7 shows that if the condition (2.27) holds and the averaged system (2.26) is locally exponentially stable, then the system (2.19) is vibrationally stabilizable. The proof employed the averaging theory (Theorem 2.5) to show the closeness of solutions, i.e. $|y(\tau) - z(\tau)| \leq k\varepsilon, \forall t \geq t_0$ for some positive k . As the averaged system (2.26) is locally exponentially stable, $y(\tau)$ exponentially converges to a neighbourhood of equilibrium point z_e . The coordinate change $x(\tau) = h(\tau, y)$ then guarantees that trajectories of the system (2.20) converge to a limit cycle $h(\tau, z_e)$, which is asymptotically stable. Condition (2.27) ensures that the average of the limit cycle is x_e thus the averaged centre requirement in the Definition 2.11 is satisfied. The stability of the

system (2.20) can be guaranteed when the initial condition starts in a neighbourhood of the limit cycle $h(t, z_e)$.

Remark 2.16. If $h(\tau, \cdot)$ is almost periodic, condition (2.27) can be slightly modified by taking the limit of T and with the general averaging results (see [77, Theorem 10.4] for more details), Theorem 2.7 can still be derived.

Remark 2.17. In [26], it also indicates that if \bar{A} is an unstable matrix, the system (2.19) cannot be vibrational stabilizable. However, there is a special case when \bar{A} is neither Hurwitz nor unstable, there lacks of results to show the stability properties in this case. We will see an example with all the eigenvalues on the imaginary axis and show that it is also vibrational stabilizable in the next section.

When the equilibrium points have closer relationship by the coordinate change, totally vibrational stabilizability could be achieved.

Corollary 2.1. Let Ω be a compact set in R^n . Suppose that Assumptions 2.1 - 2.3 hold. If the points x_e and z_e satisfy the coordinate change such that $x_e = h(t, z_e)$, and the linearization matrix of the averaged system $\bar{A} = \left[\frac{\partial f_{1,av}}{\partial z} \right]_{z=z_e}$ is Hurwitz, then there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, the system (2.19) is totally vibrationally stable for all $x_0 \in \Omega$.

Example 2.1. In the stabilization of the inverted pendulum by vibrational control, the coordinate change used is

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 - \frac{a}{l} \cos \tau \sin y_1 \end{cases} \quad (2.28)$$

In this example, $x_e = z_e = [\pi, 0]^T$. They satisfy the coordinate change $x_e = h(t, z_e)$ such that totally vibrational stabilization is achieved. It happens because as y_1 gets close to π , the amplitude of oscillation $\frac{a}{l} \sin y_1$ becomes smaller such that the oscillation diminishes so the $x(t)$ converges to the equilibrium point instead of a limit cycle. For other z_e , the oscillating part $\frac{a}{l} \cos \tau \sin y_1$ would cause a limit cycle of $x(t)$ in the steady states. ◦

2.6 Non-local Vibrational Stabilization

In Theorem 2.7, the linearization technique is used to show the stability of averaged system such that it is only guaranteed in a local domain. In this section, we will seek whether the vibrational stabilizability can be extended to a larger non-local initial domain. We propose a non-local definition of vibrational stabilizability called semi-globally practically vibrational (SPV) stabilizability in which the domain of attraction can be

arbitrarily large. SPV stabilizability can be achieved if the averaged system is globally asymptotically stable, which is formally stated in Theorem 2.8. First of all, a motivational example will be given, whose linearisation matrix is not Hurwitz. In this case, Theorem 2.7 cannot be applied to show the vibrational stabilizability so new analysis tools needs to be established to indicate the stability of such a class of system.

Example 2.2. Consider the following vibrational control system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2^3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{2}{\varepsilon} \sin\left(\frac{t}{\varepsilon}\right) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (2.29)$$

This system belongs to the category of a linear multiplicative vibrational control scheme [26] because the vibrational function $g\left(\frac{t}{\varepsilon}, x\right)$ has a linear form.

The original time-invariant system $f(x_1, x_2) = [x_2, x_1 - x_2^3]^T$ is unstable as the eigenvalues of linearization matrix on origin are ± 1 where positive eigenvalue exists.

Following steps presented in Section 2.4, in the new time τ , the coordinate transformation is obtained as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 \cos(\tau) & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (2.30)$$

By applying the averaging technique as introduced in Section 2.2.2, the averaged system (2.26) after transformation is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -8z_1^2 z_2 - z_2^3 \end{bmatrix}. \quad (2.31)$$

In this example, the linearization matrix \bar{A} of the averaged system (2.31) have eigenvalues $\pm i$ on the imaginary axis. It indicates that \bar{A} is not a Hurwitz. Theorem 2.7 is thus not applicable.

Construct the quadratic Lyapunov function $V(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2)$. The derivative of Lyapunov function along the trajectories of the averaged system is $\dot{V} = -z_2^4 - 8z_1^2 z_2^2 \leq 0$. Let $S = \{z \in \mathbb{R}^2 | \dot{V}(z) = 0\}$, as $\dot{V} = 0 \Rightarrow z_1 = 0, z_2 = 0$, no solution can stay identically in S other than the trivial solution $z_1(t) = 0, z_2(t) = 0$.

As the Lyapunov function $V(z_1, z_2)$ is radially unbounded in \mathbb{R}^2 , applying the LaSalle's Theorem 2.3, the origin of the averaged system (2.31) is globally asymptotically stable. The numerical solutions of the example shown in Figure 2.1 indicate vibrationally stabilizable behaviors. \circ

This example shows that when the averaged system (2.2) is globally asymptotically stable instead of locally exponentially stable, the vibrational control systems (2.20) still have some stability properties due to closeness of solutions between trajectories of the averaged system and the trajectories of the actual system. New stability results thus are needed.

2.6.1 Semi-globally Practically Vibrational Stabilizability

Based on the idea of semi-globally practical stability in Definition 2.6, the semi-globally practical vibrational stabilizability is introduced for the vibrational control systems (2.20).

Definition 2.18. Assume that there exists $x_e \in R^n$ such that $f(x_e) = 0$ in the system (2.19). The equilibrium x_e is said to be semi-globally practically vibrationally (SPV) stabilizable if there exist $\beta \in \mathcal{KL}$ and a time-periodic and zero-mean $g(t, x)$ such that for any positive real pair (δ, Δ) there exists a positive ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$ the system (2.20) has an asymptotically stable periodic solution $x(t)$ characterized by

$$|x_{av}(t) - x_e| \leq \beta(|x_{av}(t_0) - x_e|, t - t_0) + \delta, \quad (2.32)$$

for all $|x_0 - x_e| \leq \Delta$, where $x_{av}(t) = \frac{1}{T} \int_t^{t+T} x(\tau) d\tau, \forall t \geq t_0 \geq 0$.

Remark 2.19. Compared to the vibrational stabilizability in Definition 2.11, SPV stabilizability allows that the system (2.20) has arbitrarily large domain of attraction Δ and arbitrarily small ultimate bound δ by tuning the parameter ε appropriately. This definition has shown its usefulness in investigating the stability of vibrational control systems when the averaged system (2.26) is uniformly globally asymptotically stable as shown in Theorem 2.8.

Remark 2.20. SPV stabilizability introduced in Definition 2.18 is a non-local version of v-stabilizability in Definition 2.11. The subtle difference is that Definition 2.11 only characterizes the steady-state behaviours of the system (2.20) while Definition 2.18 characterizes both the transient response and the steady-state response. When taking the average over $[t_0, \infty)$ as in Definition 2.11, the transient response can not be presented. With the introduction of the moving average of the trajectories of the system (2.20), it is possible to use the \mathcal{KL} function $\beta(\cdot, \cdot)$ to characterize the convergence of averaged trajectories. These two definitions are consistent in terms of steady-state behaviours.

Remark 2.21. The definition lays the foundation for robustness analysis in this thesis as it can easily link to the well-known robustness results. In the next chapter, we will introduce a key concept to describe the robustness concept called input-to-state stable

(ISS) for the vibrational control systems in the presence of disturbances. The refined definition with dynamics bound can be directly extended to the ISS by adding an extra disturbances-related bound.

Next, the conditions for the system (2.19) to be SPV stabilizable are presented:

Theorem 2.8. *Suppose Assumptions 2.1 - 2.3 hold and there exists an equilibrium point z_e of the averaged system (2.26) such that the condition (2.27) holds. If z_e of the system (2.26) is a globally asymptotically stable equilibrium, then the system (2.19) is semi-globally practically vibrationally stabilizable.*

Proof. For a given x_e and Δ , the initial condition set is defined as $\Omega_{x_0} = \{x \mid |x - x_e| \leq \Delta\}$. In [29], it indicates that for any given constant $c \in \Omega_{x_0}$, $h^{-1}(t, c)$ exists and it is periodic in t so it is bounded for all $c \in \Omega_{x_0}$.

As z_e be an equilibrium point of (2.26), we denote that

$$\Delta_1 = \sup_{\tau \in [\tau_0, \infty), x_0 \in \Omega_{x_0}} |h^{-1}(\tau, x_0) - z_e|.$$

Accordingly, the initial condition set for y coordinate is

$$\Omega_{y_0} = \{y \mid |y - z_e| \leq \Delta_1\}.$$

As the equilibrium z_e of averaged system (2.26) is globally asymptotically stable and Assumption 2.3 hold, from Theorem 2.6, there exists $\beta \in \mathcal{KL}$ such that for each pair of strictly positive numbers (δ_1, Δ_1) , there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of system (2.25) satisfy the following inequality:

$$|y(\tau) - z_e| \leq \beta(|y_0 - z_e|, \tau - \tau_0) + \delta_1, \forall \tau \geq \tau_0 \geq 0, \quad (2.33)$$

whenever $|y_0 - z_e| \leq \Delta_1$.

From Assumption 2.2, it is clear that $f_1(\cdot, \cdot)$ is continuous and locally Lipschitz in Ω_{y_0} with a Lipschitz constant L . By using condition (2.27), it has

$$\begin{aligned} |x_{av}(\tau) - x_e| &= \left| \frac{1}{T} \int_{\tau}^{\tau+T} x(s) ds - x_e \right| \\ &\leq \left| \frac{1}{T} \int_{\tau}^{\tau+T} (h(s, y) - h(s, z_e)) ds \right| \\ &\leq \frac{1}{T} \int_{\tau}^{\tau+T} L |y(s) - z_e| ds. \end{aligned} \quad (2.34)$$

Combining inequalities (2.33) and (2.34) yields

$$\begin{aligned} \left| \frac{1}{T} \int_{\tau}^{\tau+T} x(s) ds - x_e \right| &\leq \frac{1}{T} \int_{\tau}^{\tau+T} L(\beta(|y_0 - z_e|, s - \tau_0) + \delta_1) ds \\ &\leq \frac{1}{T} \int_{\tau}^{\tau+T} L(\beta(|y_0 - z_e|, s - \tau_0)) ds + L\delta_1, \end{aligned} \quad (2.35)$$

$$\forall \tau \geq \tau_0 \geq 0, |y_0 - z_e| \leq \Delta.$$

Let $\tilde{\beta}(r, \tau) = \frac{1}{T} \int_{\tau}^{\tau+T} L(\beta(r, s)) ds$. It is easily to see that $\tilde{\beta}(r, t)$ belongs to class \mathcal{K} for fixed τ . It will be shown that for fixed r , $\tilde{\beta}(r, \tau)$ decrease to zero as τ goes to infinity.

Let $t_1 > t_2$,

$$\begin{aligned} &\tilde{\beta}(r, t_1) - \tilde{\beta}(r, t_2) \\ &= \frac{L}{T} \left[\int_{t_1}^{t_1+T} (\beta(r, s - t_0)) ds - \int_{t_2}^{t_2+T} (\beta(r, s - t_0)) ds \right] \\ &= \frac{L}{T} \left[\lim_{\delta t \rightarrow 0} \sum_{t=t_1}^{t_1+T} \beta(r, t) \delta t - \lim_{\delta t \rightarrow 0} \sum_{t=t_2}^{t_2+T} \beta(r, t) \delta t \right] \\ &= \frac{L}{T} \left[\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{T}{n} \left(\beta \left(r, t_1 + k \frac{T}{n} \right) - \beta \left(r, t_2 + k \frac{T}{n} \right) \right) \right]. \end{aligned} \quad (2.36)$$

As $\beta \in \mathcal{KL}$ is decreasing with respect to the second argument, $\beta \left(r, t_1 + k \frac{T}{n} \right) < \beta \left(r, t_2 + k \frac{T}{n} \right)$ for any $k > 0$, hence $\tilde{\beta}(r, t_1) < \tilde{\beta}(r, t_2)$, that is to say $\tilde{\beta}(r, \tau)$ is decreasing with respect to τ for fixed r . Moreover, it can be verified $\tilde{\beta}(r, \tau) \rightarrow 0$ as $\tau \rightarrow \infty$. Hence it is shown that $\tilde{\beta}(r, \tau)$ belongs to \mathcal{KL} .

According to (2.34) and (2.35), it follows that

$$|x_{av}(\tau) - x_e| \leq \tilde{\beta}(|y_0 - z_e|, \tau - \tau_0) + L\delta_1. \quad (2.37)$$

After selecting $\delta_1 = \frac{\delta}{L}$, there exists $\tilde{\beta} \in \mathcal{KL}$ such that

$$|x_{av}(\tau) - x_e| \leq \tilde{\beta}(|x_{av}(\tau_0) - x_e|, \tau - \tau_0) + \delta \quad (2.38)$$

Therefore the equilibrium x_e of the system (2.19) is semi-globally practically vibrationally stabilizable. \square

Remark 2.22. Theorem 2.8 provides sufficient conditions to achieve SPV stabilizability. Instead of requiring the local exponential stability of the averaged system, it requires the averaged system (2.26) is GAS, uniformly in ε . With a globally stable averaged system, the original dynamics (2.19) is then semi-globally practically vibrationally stabilizable, which indicates that for an arbitrarily large initial domain Δ , the average of trajectories

converges to an arbitrarily small δ -neighbourhood near the equilibrium x_e when the parameter ε is sufficiently small.

2.6.2 Simulations

Next, we will verify the semi-globally practical vibrational stabilizability and the proposed criteria by numerically solving Example 2.2. In order to show SPVS properties, the system behaviour of Example 2.2 will be simulated from enlarging sets of initial conditions.

Let the tuning parameter $\varepsilon = 0.01$ in the system (2.29) initially. Figure 2.1 shows the state trajectories starting at 8 different initial positions from $[1, 1]$, $[2, 2]$, ..., to $[8, 8]$ where the topmost branch is the one from $[8, 8]$. It can be seen that all the branches converge to a neighborhood of the origin with a small-amplitude oscillation while the average of each branch is converging to a neighbourhood of the origin, satisfying the definition of vibrational stabilizability.

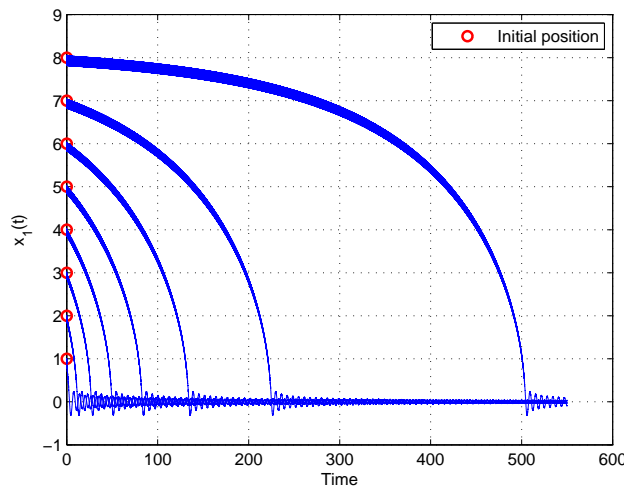


FIGURE 2.1: State trajectory $x_1(t)$ of vibrational control system (2.29) with different initial positions from $[1, 1]$, $[2, 2]$, ..., $[8, 8]$ when $\varepsilon = 0.01$.

Next we will show that it is also uniformly semi-globally practically vibrationally stable. By enlarging the initial position to $[9, 9]$, the system behaviour in Figure 2.2(a) shows that the vibrational system is not stable anymore with the current tuning parameter value. After reducing the parameter ε to 0.001, the system could be again stabilized starting at the same initial position as shown in Figure 2.2(b). This indicates that when initial domain Δ is enlarged, if the tuning parameter ε is sufficiently small, the system could be vibrationally stabilized, which is consistent with the definition of SPVS.

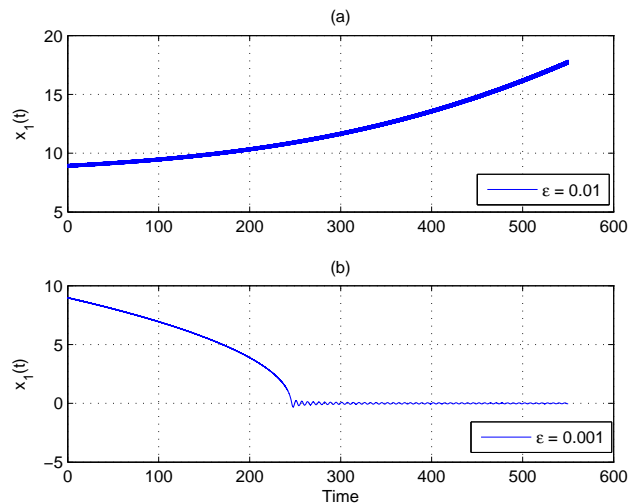


FIGURE 2.2: State trajectory $x_1(t)$ of vibrational control system (2.29) starting at initial positions $[9, 9]$ with $\epsilon = 0.01$ and $\epsilon = 0.001$ respectively.

2.7 Summary

Vibrational stabilizability properties were discussed in this chapter. In the beginning, necessary stability analysing tools were introduced. The Lyapunov stability for an equilibrium point of both autonomous and time-varying systems was addressed and then extended to capture the stability of periodic solutions. Averaging technique is a useful method for the stability analysis of a time-varying system, which approximates the system behaviour with the solutions of a simpler time-invariant averaged system. The stability of time-varying systems can be obtained if the averaged system is assumed to be either locally exponentially stable or globally asymptotically stable.

The definition of vibrational stabilizability proposed by R. Bellman was introduced, which characterizes a class of nonlinear systems that can be stabilized by injecting dithers. The idea is steering the trajectories to a limit cycle around the desired equilibrium point by injecting high-frequency dither signals into the system. The average center of the limit cycle is close to the equilibrium point such that the overall behaviour of trajectories in the steady state is similar to the ones converging to it.

The local vibrational stabilization results in literature were reviewed which applies standard averaging technique, however all stability results obtained are only limited to a local domain as linearisation technique are used for the averaged system. A motivational example was found to show that even if the averaged system does not satisfy the local vibrational stabilizability conditions, the system can still have vibrational stabilizability property with a large domain of attraction. Hence new tools to address a class of systems with non-local vibrational stabilizability are needed. Motivated by this

example, we extended the definition of local vibrational stabilizability to semi-globally practically vibrational (SPV) stabilizability. Our derived results show that when the averaged system is globally asymptotically stable uniformly in the parameter, the non-linear vibrational control systems are SPV stable, in which the domain of attraction can be an arbitrarily large compact set. The obtained results can be generalized to show the robustness when disturbances exist by appropriately applying averaging technique and perturbation theory. In the next chapter, we will consider a class of additive disturbances existing in linear vibrational control systems and discuss the system robustness properties.

Chapter 3

Robustness of Linear Vibrational Control Systems

3.1 Overview

In this chapter, the robustness of linear vibrational control systems with respect to a class of additive disturbances will be investigated. By assuming that the linear system is vibrational stabilizable such that the original linear dynamics can be stabilized by injecting dithers, then we explore the perturbed performance after introducing the additive disturbances. The first question needed to be answered is whether the system stabilized by vibrational control can handle some types of disturbances or not. If they can, what is the tolerance of disturbances while preserving the stability. At last, we will explore the system performance with different types of disturbances such as estimating the transient behaviour and the ultimate bound of state-trajectories.

One of key techniques used in the stability analysis of vibrational control systems is averaging, so the robustness of averaging techniques need to be explored. Strong and weak average techniques [45] are useful tools to study the robustness of a class of general nonlinear time-varying systems, extending the classic averaging techniques to be applicable to the systems with disturbances. When considering the linear vibrational control systems with disturbances, the strong and weak averaging techniques are natural choices of tools for the robustness analysis. If the strong averaged system exists, it is possible to get stronger robustness of the original time-varying system, however usually it is relatively easier to find the weak averaged system for a time-varying dynamic system in the presence of disturbances. The results in Theorem 3.3 show that strong averaged system does not exist while weak averaged system exists. By applying the

weak average technique, the system is shown to be robust with respect to a class of slowly time-varying disturbances.

It is worthwhile to note that strong and weak averaging techniques are applicable to a class of nonlinear time-varying systems with lumped uncertainties or (the effect of disturbances to the system is nonlinear). Hence when additive disturbances are considered, the results obtained using strong or weak averaging might be too conservative. From simulation results observed, it is clear that the linear vibrational control systems are robust to fast disturbances. More interestingly, it was observed that the linear vibrational systems can attenuate fast disturbances. Besides, strong and weak averaging require that the averaged system is Lyapunov-ISS which is a strong assumption in the real engineering systems.

Therefore, we relax the assumption by only assuming the system is vibrationally stabilizable and treating disturbances as some perturbation to the system. By combining the averaging technique and perturbation theory, updated results in Theorem 3.4 show that the linear vibrational control system is robust to both slow and fast disturbances provided that the disturbances are bounded. The linear vibrational system has input-to-state stability properties, meaning that the trajectories converge to an ultimate bound decided by the disturbances. When the disturbances are periodic, we further analyze frequency-spectrum composition of the ultimate bound by using Fourier series of disturbances. The results indicate that the frequency of disturbances has attenuation influence to the ultimate bound such that higher frequency leads to smaller ultimate bound. In the last part of this chapter, the existence of a more general type of disturbances called states-dependent disturbances is considered and sufficient conditions to guarantee the system robustness are provided.

The chapter is organized as follows. In Section 3.2, a well-known robustness concept for the disturbed systems called input-to-state stability will be revisited. The robustness analysis tool strong and weak average will also be covered there. Initial robustness results of linear vibrational control systems derived from strong and weak average are presented in Section 3.3. Subsequently the robustness results are strengthened in Section 3.4 by using averaging and perturbation methods. When the disturbances are also periodic, the influence of the frequencies of disturbances is discussed, followed by the robustness conclusion with respect to state-dependent disturbances. Section 3.5 summarizes the chapter.

3.2 Preliminaries

3.2.1 Input-to-State Stable

Define L-infinity norm $\|w\|_\infty := \text{ess sup}_{t \geq 0} |w(t)|$. If $\|w\|_\infty < \infty$, it can be called that $w \in \mathcal{L}_\infty$. Consider a time-varying system in the presence of disturbances:

$$\dot{x} = F(t, x, w), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad (3.1)$$

where $F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ is piecewise continuous in t and locally Lipschitz in x and w . Here $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ are the state and disturbance respectively. The undisturbed system is:

$$\dot{x} = F(t, x, 0), \quad x(t_0) = x_0 \in \mathbb{R}^n. \quad (3.2)$$

Without losing generality, here we suppose that origin is the equilibrium point for the undisturbed system (3.2) such that $F(t, 0, 0) = 0$. The system (3.1) can be treated as a perturbation of the system (3.2), the input-to-state stability (ISS) is introduced to describe the stability properties of the perturbed system (3.1):

Definition 3.1. [77] The system (3.1) is said to be input-to-state stable (ISS) with gain $\gamma \in \mathcal{K}$ if there exists $\beta \in \mathcal{KL}$ such that for each $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^n$, the solutions starting at (x_0, t_0) exist and satisfy:

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|w\|_\infty), \quad \forall t \geq t_0 \geq 0. \quad (3.3)$$

Remark 3.1. Inequality (3.3) indicates that the trajectories of the system (3.1) are bounded if the disturbances are bounded. The first composition of bound is time-decreasing, which is dependent on the initial condition while the second part is time-invariant that is related to the L-infinity norm of disturbances. As time goes to the infinity, the trajectories converge to the ultimate bound, which is dependent on the disturbances. Specifically, if the disturbances $w = 0$, as we know the property of class- \mathcal{K} functions $\gamma(0) = 0$, the trajectories converge to origin, which means the origin is a globally asymptotically stable equilibrium point according to Definition 2.5. This is an important corollary of ISS which infers that the the equilibrium point of undisturbed system (3.2) is GUAS.

A sufficient condition to ensure that the system (3.1) is ISS if it is ISS in Lyapunov sense.

Definition 3.2. [45] The system (3.1) is called Lyapunov-ISS with the gain γ if there exists a continuous differentiable function $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that the following

inequalities hold

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, w) \leq -\alpha_3(|x|), \forall |x| > \gamma_0(\|w\|_\infty),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\gamma_0 \in \mathcal{K}$.

Theorem 3.1. [77, Theorem 4.19] *The system (3.1) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \gamma_0$ if it is Lyapunov ISS in Definition 3.2.*

Remark 3.2. From the definition of Lyapunov ISS, the derivative of the Lyapunov function for the system (3.1) decreases when the trajectories is outside the ball $\gamma_0(\|w\|_\infty)$. As $|x| \leq \alpha_1^{-1}(V(t, x))$, the trajectories of the system (3.1) decrease until hit the ultimate bound and then keep staying inside.

3.2.2 Strong and Weak Averaging Techniques

As shown in Section 2.2.2, averaging is a powerful tool to study the stability of a class of time-varying dynamic systems (2.16). To address system robustness with respect to disturbances, D. Nesic and A.R. Teel [45, 80] proposed the concept of strong and weak average. The following parametrized time-varying system with disturbances is considered:

$$\dot{x} = \epsilon F(t, x, w), \quad x(t_0) = x_0 \in \mathbb{R}^n. \quad (3.4)$$

The definitions of strong and weak averaging are given below separately:

Definition 3.3. A locally Lipschitz function $F_{sa} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be the strong average of $F(t, x, w)$ if there exist $\beta_{sa} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq 0$, for all $w \in \mathcal{L}_\infty$, for all $T > T^*$, the following holds:

$$\left| \frac{1}{T} \int_t^{t+T} [F_{sa}(x, w(s)) - F(s, x, w(s))] ds \right| \leq \beta_{sa}(\max\{|x|, \|w\|_\infty, 1\}, T). \quad (3.5)$$

Definition 3.4. A locally Lipschitz function $F_{wa} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be the weak average of $F(t, x, w)$ if there exist $\beta_{wa} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \geq 0$, for all $T > T^*$ the following holds:

$$\left| F_{wa}(x, w) - \frac{1}{T} \int_t^{t+T} F(s, x, w) ds \right| \leq \beta_{wa}(\max\{|x|, |w|, 1\}, T). \quad (3.6)$$

Remark 3.3. One of the major difference between strong and weak averaging techniques is the allowed changing rate of disturbances. The disturbances in the definition of strong average are treated as a time-varying variable in the integral while they are treated as constants in the weak averaged system. It means that the changing rate of disturbances

in the weak averaged systems should be slower than the system dynamics while the one for strong averaged system can be arbitrarily fast.

As shown in the following lemma, the strong average only exists for a limited class of nonlinear time-varying systems where time and disturbances can be decoupled.

Lemma 3.1. [45] Suppose that $F(t, x, w)$ is continuous and periodic in t of period $T > 0$. Then there exists a strong average $F_{sa}(x, w)$ if and only if $F(t, x, w) = F_1(t, x) + F_2(x, w)$ for some continuous functions F_1 and F_2 where the average of $F_1(t, x)$ exists.

Since strong average exists for systems with a specific structure, weak average is suitable to analyse the robustness for a more general time-varying systems (3.4). Later we will show that the vibrational control systems after coordinate transformation only have weak average. The next theorem shows that if the weak average of system (3.4) exist and it is Lyapunov-ISS, we can conclude the robustness properties of the system (3.4).

Theorem 3.2. [45] Assume that the system (3.4) is locally Lipschitz in x and w , uniformly in t and there exists $c \geq 0$ such that $|F(t, 0, 0)| \leq c, \forall t \geq 0$. If the weak average of (3.4) exists and is Lyapunov-ISS with gain $\gamma \in \mathcal{K}_\infty$, then there exists $\beta \in \mathcal{KL}$ and given any strictly positive real numbers $\Delta, k_w, k_{\dot{w}}, \delta$, there exists $\epsilon^* > 0$ such that $\forall \epsilon \in (0, \epsilon^*)$, the solutions of (3.4) satisfy: $\forall t \geq t_0 \geq 0$

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|w\|_\infty)\} + \delta$$

whenever $|x(t_0)| \leq \Delta, \|w\|_\infty \leq k_w, w(t)$ is absolutely continuous and $\|\dot{w}\|_\infty \leq k_{\dot{w}}$.

Remark 3.4. An important assumption in the Theorem 3.2 is that the weak averaged system is Lyapunov-ISS, which means the autonomous weak averaged system is already robust to the disturbances. Based on that, the theorem indicates that the original time-varying system (3.4) can be practically ISS. Compared to the conclusion when strong average exists, the changing rate of disturbances needs to be bounded in Theorem 3.2, which means the system can only handle slow disturbances.

3.3 Robustness based on Strong and Weak Averaging Techniques

3.3.1 Robustness analysis by Strong and Weak Average

This section presents the robustness analysis for LTI systems with a periodic state feedback. Such a system is called linear multiplicative vibrational control type in [29].

We consider external disturbances exist in the system:

$$\dot{x} = Ax + \frac{1}{\epsilon} B_1 \begin{pmatrix} t \\ \epsilon \end{pmatrix} x + B_2 w(t), \quad x(t_0) = x_0 \in \mathbb{R}^n \quad (3.7)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B_1 : [t_0, \infty) \rightarrow \mathbb{R}^{n \times n}$ and is periodic in t . $B_2 \in \mathbb{R}^{n \times m}$, $w : [t_0, \infty) \rightarrow \mathbb{R}^m$ and ϵ is a positive tuning parameter. It is assumed that the matrices A and $B_1(t)$ satisfy the following assumptions:

Assumption 3.1. The matrix A has the observable canonical form satisfying $\text{trace}(A) < 0$.

Assumption 3.2. The matrix $B_1(t)$ is continuous and periodic with zero mean value, i.e. there exists a positive number T such that $B_1(t+T) = B_1(t)$ and $\frac{1}{T} \int_t^{t+T} B_1(\tau) d\tau = 0$.

Remark 3.5. Assumption 3.1 and 3.2 guarantees that the system is vibrationally stabilizable from Theorem 1.1 such that there exists a suitable periodic $B_1(t)$ that stabilizes the system. We will next explore to what extent the stabilization of the original dynamics $\dot{x} = Ax$ by vibrational control can handle the disturbances and what's the performance of the system (3.7) under different types of disturbances.

Following the steps in the coordinate transformation, the system (3.7) can be re-written in a new time $\tau = \frac{t}{\epsilon}$:

$$\frac{dx}{d\tau} = \epsilon Ax + B_1(\tau)x + \epsilon B_2 w(\epsilon\tau). \quad (3.8)$$

Next, we introduce the following auxiliary variable ξ coming from the following linear-time-varying dynamic system:

$$\frac{d\xi}{d\tau} = B_1(\tau)\xi, \quad \xi(\tau_0) \in \mathbb{R}^n, \forall \tau \geq \tau_0 \geq 0$$

with its state transition matrix $\Phi(\tau, \tau_0)$ satisfying the following homogeneous relations:

$$\begin{cases} \frac{d\Phi(\tau, \tau_0)}{d\tau} = B_1(\tau)\Phi(\tau, \tau_0) \\ \Phi(\tau_0, \tau_0) = I_n \end{cases}. \quad (3.9)$$

By introducing the following linear coordinate transformation

$$x(\tau) = \Phi(\tau, \tau_0)y(\tau), \quad (3.10)$$

where $\Phi(\tau, \tau_0)$ is defined in (3.9), it yields:

$$\frac{dx}{d\tau} = \frac{d\Phi(\tau, \tau_0)}{d\tau} y(\tau) + \Phi(\tau, \tau_0) \frac{dy}{d\tau}.$$

Substituting $x(\tau)$ and its derivative with respect to $y(\tau)$ and making use of the homogeneous property of $\Phi(\tau, \tau_0)$, the transformed system becomes:

$$\frac{dy}{d\tau} = \epsilon(\Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)y(\tau) + \Phi^{-1}(\tau, \tau_0)B_2w(\epsilon\tau)), y(\tau_0) = x_0. \quad (3.11)$$

Remark 3.6. Discussed in [29], the inverse matrix of the state transition matrix $\Phi^{-1}(\tau, \tau_0)$ exists such that this transformation is nonsingular and preserves the stability of system (3.7). Thus the robustness analysis focuses on the transformed system (3.11), in which the new disturbances $\tilde{w}(\tau) = \Phi^{-1}(\tau, \tau_0)B_2w(\epsilon\tau)$ exist. When disturbances exist, Strong and weak averaging techniques are the natural consideration for robustness analysis.

The averaged system without disturbances can be expressed as follows:

$$\frac{dz}{d\tau} = \epsilon\bar{A}z, z(\tau_0) = y(\tau_0), \quad (3.12)$$

where $\bar{A} = \frac{1}{T} \int_t^{t+T} \Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)d\tau$.

From Theorem 1.1, for any matrix A satisfying Assumption 3.1, there exists an periodic matrix $B_1(\tau)$ with zero average such that the matrix \bar{A} in the averaged system (3.12) is Hurwitz. Moreover, the corresponding state transition matrix $\Phi(\tau, \tau_0)$ is periodic in τ , such that the $\Phi^{-1}(\tau, \tau_0)$ is periodic and bounded so the new disturbance $\tilde{w}(\tau)$ in system (3.11) is bounded. Firstly we check the existence of strong and weak averaged system:

Proposition 3.1. The strong average of the system (3.11) doesn't exist while the weak average exists. The weak average is

$$\dot{y}_{wa} = f_{wa}(y_{wa}, w) = \bar{A}y_{wa} + \bar{B}_2w, y_{wa}(\tau_0) = y(\tau_0) \quad (3.13)$$

where $\bar{B}_2 = \frac{1}{T} \int_t^{t+T} \Phi^{-1}(\tau, \tau_0)B_2d\tau$.

Proof: It can be concluded directly from Lemma 3.1 that strong average doesn't exist because the time and disturbances are coupled in the transformed system (3.11). Proposed system (3.13) can be verified to satisfy the Definition 3.4 of the weak average. **Q.E.D.**

Remark 3.7. The coupling between disturbances and time comes from the time-varying coordinate transformation. This leads to the non-existence of strong average of the transformed system (3.11) because from Lemma 3.1 we know that the strong average only exists when the system configuration satisfies $F(t, x, w) = F_1(t, x) + F_2(x, w)$, where the disturbance and time are decoupled. In fact strong average exists directly applying the averaging technique to the system (3.7) in original coordinate. As $B_1(t)$ is zero mean it will disappear after averaging, the strong averaged system would be $\dot{x} = Ax + B_2w$.

However, the original dynamics matrix A is unstable so the averaged system is impossible to satisfy the Lyapunov-ISS condition in the strong and weak averaging technique thus no robustness conclusion could be reached in this way. From these reasons we can see that the transformation is an important bridge in the robustness analysis, which converts the system (3.7) to a suitable format that is easier to analyse.

After showing the existence of weak averaged system, next we will demonstrate that it is Lyapunov-ISS to obtain the practical ISS properties, which is stated in the next Proposition.

Proposition 3.2. The weak averaged system (3.13) is Lyapunov-ISS with gain γ .

Proof: According to Theorem 1.1, there exists a periodic matrix $B_1(\tau)$ such that \bar{A} is Hurwitz. Hence for any positive-definite symmetric matrix Q , there exists a unique positive-definite symmetric matrix P satisfying $P\bar{A} + \bar{A}^T P = -Q$. Select $V(y_{wa}) = y_{wa}^T P y_{wa}$ as a Lyapunov candidate, it has:

$$\lambda_{\min}(P)|y_{wa}|^2 \leq V(y_{wa}) \leq \lambda_{\max}(P)|y_{wa}|^2.$$

The derivative of $V(y_{wa})$ along the trajectory of system (3.13) can be derived:

$$\begin{aligned} \dot{V} &= -y_{wa}^T (\bar{A}^T P + P \bar{A}) y_{wa} + 2\bar{B}_2 w^T P y_{wa} \\ &= -y_{wa}^T Q y_{wa} + 2\bar{B}_2 w^T P y_{wa} \\ &\leq -\frac{1}{2} y_{wa}^T Q y_{wa} - \frac{1}{2} \lambda_{\min}(Q) |y_{wa}|^2 + 2|\bar{B}_2 w| \|P\| |y_{wa}| \\ &\leq -\frac{1}{2} y_{wa}^T Q y_{wa}, \forall |y_{wa}| \geq \frac{4\|\bar{B}_2\| \|w\| \|P\|}{\lambda_{\min}(Q)} = \rho(\|w\|) \end{aligned}$$

Therefore the averaged system (3.13) is Lyapunov-ISS with gain $\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \gamma_0(r) = \frac{4\|\bar{B}_2\| \|P\| \sqrt{\lambda_{\max}}}{\lambda_{\min}(Q) \sqrt{\lambda_{\min}}} r$. **Q.E.D.**

Based on Proposition 3.1 and 3.2, Theorem 3.3 states the robustness from Weak averaging indication:

Theorem 3.3. Suppose Assumption 3.1 and 3.2 hold. There exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, positive numbers M_1 and M_2 such that for any given strictly positive real numbers Δ , k_w , $k_{\dot{w}}$, δ , there exists ϵ^* s.t. $\forall \epsilon \in (0, \epsilon^*)$ the solutions of the system (3.7) satisfy:

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|w\|_\infty) + \delta, \quad t \geq t_0 \geq 0 \quad (3.14)$$

whenever $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq k_w$ and $\|\dot{w}\|_\infty \leq k_{\dot{w}}$.

Proof: Since Assumption 3.1 holds, $\Phi(\tau, \tau_0)$ and $\Phi^{-1}(\tau, \tau_0)$ are periodic and bounded, let their bounds be M_1 and M_2 respectively, i.e. $\sup_{\tau \geq \tau_0} \|\Phi\| = M_1, \sup_{\tau \geq \tau_0} \|\Phi^{-1}\| = M_2$.

Let $\tilde{w}(\tau) = \Phi^{-1}(\tau, \tau_0)w(\tau)$, then $\tilde{w}(\tau)$ is also absolutely continuous with bounded rates because

$$\|\tilde{w}\| \leq \sup_{\tau \geq \tau_0} \|\Phi^{-1}\| \|w\|_\infty \leq M_2 k_w,$$

$$\left\| \frac{d\tilde{w}}{d\tau} \right\| \leq \sup_{\tau \geq \tau_0} \left\| \frac{d\Phi^{-1}}{d\tau} \right\| k_w + \sup_{\tau \geq \tau_0} \|\Phi^{-1}\| k_{\dot{w}} =: k_{\dot{w}_1}.$$

Therefore, from Theorem 3.2, there exist $\beta_1 \in \mathcal{KL}$, $\gamma_1 \in \mathcal{K}$ such that for any given $\delta_1 > 0$, there exist ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of the system (3.11) satisfy

$$|y(\tau)| \leq \max\{\beta_1(|y(\tau_0)|, \tau - \tau_0), M_2 \gamma_1(\|w\|_\infty)\} + \delta_1,$$

for all $\tau \geq \tau_0 \geq 0$.

Since $x(\tau) = \Phi(\tau, \tau_0)y(\tau)$ and $x(\tau_0) = y(\tau_0)$, $\forall \tau \geq \tau_0 \geq 0$ the solutions of system (3.7) satisfy:

$$\begin{aligned} |x(\tau)| &\leq M_1 \max\{\beta_1(|x(\tau_0)|, \tau - \tau_0), M_2 \gamma_1(\|w\|_\infty)\} + M_1 \delta_1, \\ &\leq \max\{\beta(|x(\tau_0)|, \tau - \tau_0), \gamma(\|w\|_\infty)\} + \delta. \end{aligned} \quad (3.15)$$

where $\beta(r, s) = M_1 \beta_1(r, s)$, $\gamma(r) = M_1 M_2 \gamma_1(r)$ and δ_1 is selected as δ/M_1 .

Representing the above inequality to original time scale $t = \varepsilon\tau$, the inequality (3.14) thus is obtained. **Q.E.D.**

Remark 3.8. Theorem 3.3 shows that if the disturbance is changing slowly with some bounded rate, trajectories of the system (3.7) converge to a neighbourhood of the origin. The radius of the ultimate bound is $\gamma(\|w\|_\infty) + \delta$. However, the obtained results are only valid for disturbances that are sufficiently slow compared to the dither signal. The system behaviour of the system (3.7) with fast disturbances needs further exploration. Another limitation is the existence of practical term in the ultimate bound. For linear systems, the asymptotically stability and exponentially stability are equivalent, so without considering disturbances, the exponential stability of averaged system indicates that the original system is also exponentially stable from averaging theory (Theorem 2.5). Theoretically, this practical term could be handled in the linear vibrational control systems.

Remark 3.9. Without considering the disturbances ($w = 0$), the upper bound in inequality (3.14) indicates the trajectories of the system (3.7) is totally vibrationally stabilizable as defined in the Definition 2.12. This result is linked to and consistent with the Corollary 2.1 which characterizes the systems that are totally vibrationally stabilizable. As shown in the Corollary 2.1, totally vibrational stability happens when the equilibrium points x_e and z_e satisfy the coordinate change i.e. $x_e = h(t, z_e)$. This condition is automatically satisfied when vibrational control systems (2.20) are linear because the

coordinate change is a time-periodic linear mapping $x(t) = \Phi(t, t_0)y(t)$, where the origin is the equilibrium point for both systems. When totally vibrational stability is achieved, the the vibrational system (3.7) has better performance as its trajectories converge to the ultimate bound instead of only the averaged trajectories. Therefore, in the rest of this chapter we can address the trajectories behaviour directly. It is obvious that the averaged trajectories have the similar behaviour accordingly.

Remark 3.10. A straightforward way to extend Theorem 3.3 to nonlinear vibrational systems is to explore local robustness by using standard linearization techniques. Moreover, strong and weak averaging techniques are useful tools to characterize the robustness for a large class of nonlinear systems, thus they can be used to show the vibrational stability for the nonlinear time-varying systems, for which weak average exists and is Lyapunov-ISS.

3.3.2 Simulation results

To illustrate obtained results, the linearised Duffing system stabilized by vibrational control with disturbances is analyzed. The state-space model of such a system is:

$$\begin{aligned}\dot{x}_1 &= x_2 + w_1(t) \\ \dot{x}_2 &= -ax_2 + bx_1 + \frac{k}{\epsilon} \sin\left(\frac{t}{\epsilon}\right) x_1 + w_2(t),\end{aligned}$$

where a, b are strictly positive values, ϵ is the reciprocal of dither frequency and k is the amplitude, $w_1(t)$ and $w_2(t)$ are disturbances.

When writing this system into the standard form as in (3.7), we have

$$A = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}, B_1\left(\frac{t}{\epsilon}\right) = \begin{bmatrix} 0 & 0 \\ k \sin\left(\frac{t}{\epsilon}\right) & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It could be seen that matrix A is not stable and $B_1(t)$ has quasi-lower triangular form.

In the following simulations, parameters of linearised Duffing system are selected as $a = 1, b = 1$ and vibrational control parameters are chosen as $k = 20, \epsilon = 0.01$. The simulation results in Figure 3.1 show that without disturbance, the original dynamics $\dot{x} = Ax$ is stabilized by using the dithers $B_1\left(\frac{t}{\epsilon}\right)x$.

Next, three different types of disturbances will be considered: $d_1 = l \sin(\omega t)$, $d_2 = l$, $d_3 = le^{-t}$. By applying Theorem 3.3, the linear vibrational control system will have ISS-like stability properties if the disturbances are slowly varying.

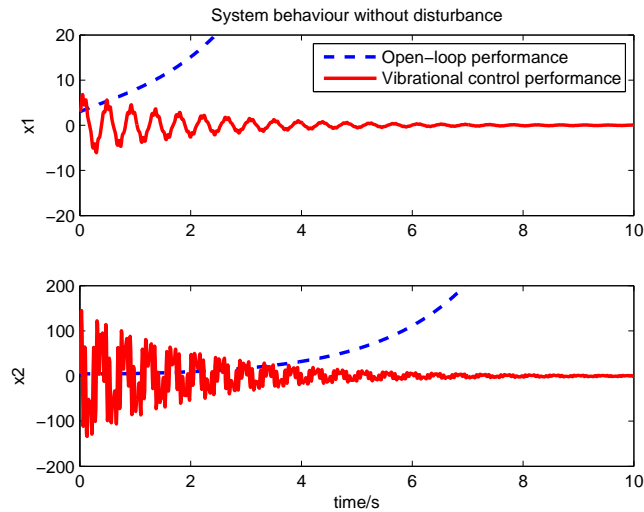
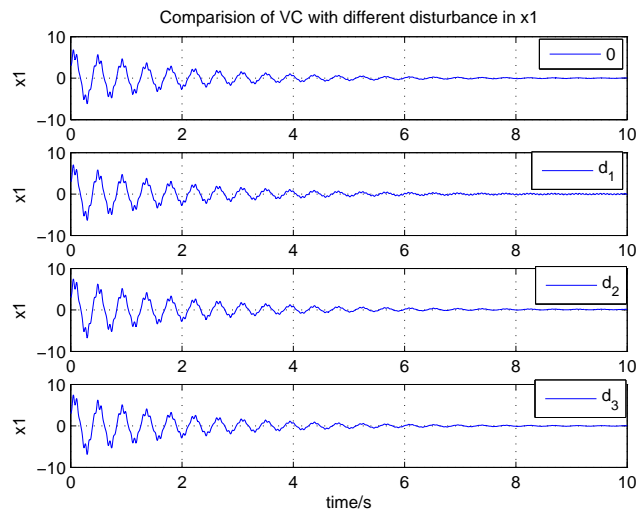


FIGURE 3.1: Effect of vibrational control to linearised Duffing equation.

FIGURE 3.2: Comparison of vibrational control system with different kinds of disturbance in x_1 .

From the simulation results in Figure 3.2 and Figure 3.3, it can be seen that both states eventually enter a bounded region in all cases when disturbances are relatively slow. Therefore, simulation results support theoretic results of Theorem 3.3.

According to Theorem 3.3, the disturbance should be relatively slow compared with the dither signal to ensure the robustness of the vibrational systems. However, Theorem 3.3 only provides sufficient conditions. We found in simulations that the vibrational control system is not sensitive to high frequency disturbances either. Figure 3.4 shows the performance of the vibrational control systems in the presence of the sinusoid disturbances applied to the system $d_1(\omega t)$ with different ω . When ω is quite large, the vibrational control system still keeps ISS like performance.

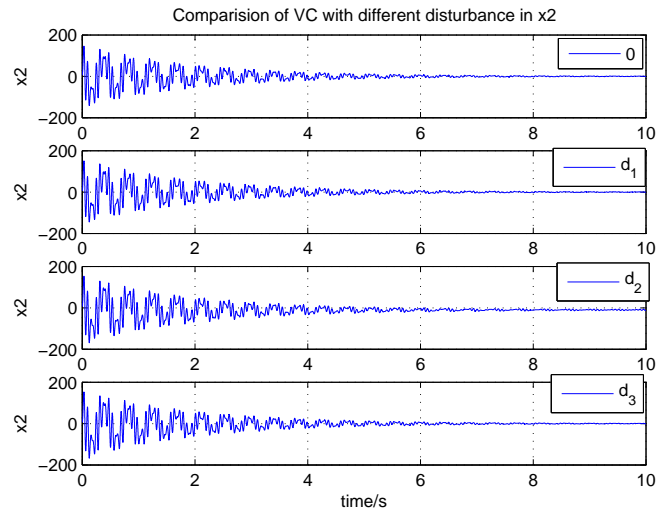


FIGURE 3.3: Comparison of vibrational control system with different kinds of disturbance in x_2 .

It is also interesting to observe the ultimate bound is getting smaller while the frequency of the disturbances increases, see Figure 3.4 and Table 3.1. This indicates that the current two averaging tools (strong and weak average) might not be sufficient enough to fully capture the robustness properties of averaging technique. New averaging tools are needed to cover the situation when the disturbance is much faster than the frequency of the system.

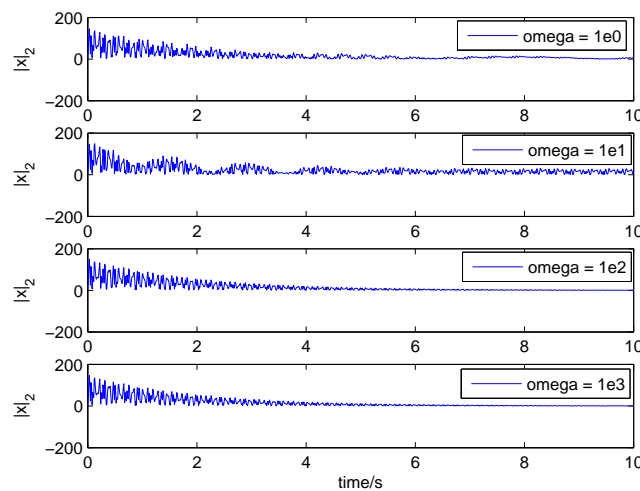


FIGURE 3.4: Comparison of bound with different frequency of sinusoidal disturbance.

TABLE 3.1: Ultimate bound with different frequencies

Frequency	10^0	10^1	10^2	10^3
Ultimate bound	6.5101	29.5637	2.1253	1.5836

3.3.3 Summary of this section

In this section, the robustness of linear vibrational control systems is addressed considering the existence of additive disturbances. Because strong and weak averaging techniques characterize the robustness of a large class of nonlinear time-varying systems with disturbances, it is straightforward to apply it to the robustness analysis of the linear vibrational control system, which is a special case of a general nonlinear time-varying system. For a linear vibrational system in the form of (3.7), the strong average doesn't exist while the weak average exists and be Lyapunov ISS. By using the weak average result, the linear vibrational control systems are shown to have ISS-like properties which means that the trajectories converge to a neighbourhood of the equilibrium point. The ultimate bound is related to the L -infinity norm of disturbances and a practical term δ . The domain of initial points can be designed arbitrarily large and the practical term δ can be arbitrarily small by tuning the parameter.

However, the robustness result derived from weak averaging is limited to slowly varying disturbances. Numeric simulations indicate the linear vibrational control systems are also robust when fast disturbances exist. Hence when the disturbances are fast time-varying, further robustness analysis are needed by using alternative tools. Moreover, simulations also show that high frequencies of disturbances will attenuate the ultimate bound of state-trajectories such that it becomes smaller in the presene of higher-frequency disturbances. In this case, using L -infinity norm of disturbances to characterize the ultimate bound, as used in most ISS definitions, might not be necessary. A less conservative estimation of the ultimate bound related to the frequencies of disturbances is needed. In next sections, we will continue exploring the robustness of linear vibrational control systems by addressing these points.

3.4 Strengthened Robustness based on Averaging and Perturbation

3.4.1 Input-to-State Stability

The previous robustness properties via weak averaging are limited to slow disturbances thus new tools are needed for a more general class of disturbances. Next, we will exploit the strengthened robustness by applying averaging and perturbation techniques.

Consider linear vibrational control systems with additive disturbances (3.7), with matrices A and $B_1(t)$ satisfying Assumptions 3.1 and 3.2 respectively.

Transformed system (3.11) can be rewritten as

$$\frac{dy}{d\tau} = \varepsilon(\tilde{A}(\tau)y(\tau) + \tilde{w}(\tau)), y(\tau_0) = x(\tau_0) \in \mathbb{R}^n, \quad (3.16)$$

where $\tilde{A}(\tau) = \Phi^{-1}(\tau, \tau_0)A\Phi(\tau, \tau_0)$ and $\tilde{w}(\tau) = \Phi^{-1}(\tau, \tau_0)B_2w(\varepsilon\tau)$. Since $\Phi(\tau, \tau_0)$ and $\Phi^{-1}(\tau, \tau_0)$ are T -periodic functions, $\tilde{A}(\tau)$ is also T -periodic. Moreover, as $\Phi^{-1}(\tau, \tau_0)$ is periodic and continuous, then $\|\Phi^{-1}(\tau, \tau_0)\|$ is bounded for any $\tau \geq \tau_0 \geq 0$. As $w \in \mathcal{L}_\infty$, the boundedness of $\Phi^{-1}(\cdot, \cdot)$ and the constant B_2 matrix lead to $\tilde{w} \in \mathcal{L}_\infty$.

The subsequent robustness analysis is based on the transformed system (3.16). Perturbation technique is used to show that the solutions of system satisfy the following inequality for bounded disturbances.

Theorem 3.4. *Suppose Assumption 3.1 holds and $w \in \mathcal{L}_\infty$. There exists $B_1(t)$ satisfying Assumption 3.2 such that there exists $\varepsilon^* > 0$ s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of system (3.7) satisfy:*

$$|x(t)| \leq MN|x_0|e^{-\lambda(t-t_0)} + \frac{MN\|w\|_\infty}{\lambda}, \quad (3.17)$$

where N, M, λ are strictly positive constants.

Proof: Introduce the following auxiliary system:

$$\frac{dy_1}{d\tau} = \varepsilon\tilde{A}(\tau)y_1(\tau), y_1(\tau_0) = y(\tau_0). \quad (3.18)$$

The averaging result in Theorem 2.5 indicates there exists a positive pair (ε^*, k) such that for all $\varepsilon \in (0, \varepsilon^*)$, thus the following inequality holds:

$$|y_1(\tau) - z(\tau)| \leq k\varepsilon. \quad (3.19)$$

where $z(\tau)$ is the solution of averaged system (3.12) without disturbances. According to Theorem 1.1, if the matrix A satisfies Assumption 3.1, there exists $B_1(t)$ satisfying Assumption 3.2 such that the matrix \bar{A} is Hurwitz. Then we have that $|z(\tau)| \leq e^{\lambda_{max}(\varepsilon\bar{A})\tau}|z(\tau_0)|$, where $\lambda_{max}(\varepsilon\bar{A})$ is the largest eigenvalue of $\varepsilon\bar{A}$. Thus for any given positive number δ , there exists positive ε^* such that when $\varepsilon \in (0, \varepsilon^*)$ the solutions of system (3.18) satisfy:

$$|y_1(\tau)| \leq e^{\lambda_{max}(\varepsilon\bar{A})\tau}|z(\tau_0)| + \delta.$$

It means we can find positive numbers N, λ such that $|y_1(\tau)| \leq Ne^{-\varepsilon\lambda\tau}|y_1(\tau_0)|$. Consequently, the upper bound of the trajectories of (3.16) is obtained as

$$\begin{aligned} |y(\tau)| &\leq Ne^{-\varepsilon\lambda\tau}|y(\tau_0)| + \varepsilon \int_0^\tau Ne^{-\varepsilon\lambda(\tau-s)}|\tilde{w}(s)|ds \\ &\leq Ne^{-\varepsilon\lambda\tau}|y(\tau_0)| + \frac{1}{\lambda}N\|\tilde{w}\|_\infty(1 - e^{-\varepsilon\lambda\tau}) \\ &\leq N|y(\tau_0)|e^{-\varepsilon\lambda\tau} + \frac{1}{\lambda}N\|\tilde{w}\|_\infty. \end{aligned}$$

From (3.10), $x(\tau) = \Phi(\tau, \tau_0)y(\tau)$, therefore, the solutions of system (3.7) satisfy the following inequality:

$$|x(t)| \leq MN|x(\tau_0)|e^{-\lambda t} + \frac{MN}{\lambda}\|w\|_\infty, \forall t \geq t_0,$$

where $M = M_1M_2$, $M_1 = \sup_{\tau \geq \tau_0} \|\Phi(\tau, \tau_0)\|$ and $M_2 = \sup_{\tau \geq \tau_0} \|\Phi^{-1}(\tau, \tau_0)\|$. **Q.E.D.**

Remark 3.11. In the proof, the transformed system (3.16) can be treated as the system (3.18) perturbed by disturbances. The closeness of solutions between system (3.18) and averaged system (3.12) are guaranteed in infinite time interval by averaging theory because the averaged system is assumed to be locally exponentially stable. The perturbation of disturbances leads to an ultimate bound in the size of disturbances.

Remark 3.12. Compared to Theorem 3.3 derived from weak average, Theorem 3.4 removes the constraint of the derivative of disturbances so the vibrational control systems can handle not only slow additive disturbances but also fast additive disturbances. In addition, the trajectories converge to the ultimate bound that is only composed of the L -infinity norm of disturbances so the estimation is more accurate after removing the practical term. This is reasonable because strong and weak averaging techniques capture the robustness for a more general nonlinear time-varying system so they need the practical term to bound some nonlinear terms. By applying perturbation technique to the linear vibrational systems (3.7) with additive disturbances, less conservative results can be obtained.

Without disturbances, the averaged system of (3.16) becomes the system (3.12). The closeness of solutions between (3.16) and (3.12) is stated in the following proposition:

Proposition 3.3. (Closeness of solutions) Suppose Assumptions 3.1 and 3.2 are satisfied. There exists ε^* and T^* such that whenever $\varepsilon \in (0, \varepsilon^*)$, solutions of (3.16) $y(\tau, \varepsilon)$ and solutions of averaged system (3.12) $z(\tau, \varepsilon)$ satisfy:

$$|y(\tau, \varepsilon) - z(\tau, \varepsilon)| < \delta + \tilde{\gamma}(\|w\|_\infty), \forall \tau \geq T^*,$$

where $\tilde{\gamma}(\|w\|_\infty) = MN\|w\|_\infty/\lambda$, and M and N are defined in (3.17).

Proof: From closeness of solutions by averaging technique, it can be shown that there exists positive real number ε_1^* and k such that for all $\varepsilon \in (0, \varepsilon_1^*)$, it has (3.18),

$$|y_1(\tau) - z(\tau)| \leq k\varepsilon.$$

where $y_1(\tau)$ comes from (3.11) and $z(\tau)$ comes from (3.18). Let $e(\tau) = y(\tau) - y_1(\tau)$, then take derivative in both sides:

$$\frac{de}{d\tau} = \varepsilon(\tilde{A}(\tau)e(\tau) + \tilde{w}(\tau)).$$

Using the similar procedure as the proof of Theorem 3.4, there exist positive numbers N and λ such that the upper bound of the solution $e(\tau)$ satisfies:

$$|e(\tau)| \leq N|e(\tau_0)|e^{-\varepsilon\lambda\tau} + \frac{1}{\lambda}N\|\tilde{w}\|_\infty.$$

Noting that $|y(\tau) - z(\tau)| \leq |y(\tau) - y_1(\tau)| + |y_1(\tau) - z(\tau)|$, the closeness of solutions between (3.11) and (3.12) is thus established:

$$\begin{aligned} |y(\tau) - z(\tau)| &\leq N|e(\tau_0)|e^{-\varepsilon\lambda\tau} + \frac{N}{\lambda}\|\tilde{w}\|_\infty + k\varepsilon \\ &\leq N|e(\tau_0)|e^{-\varepsilon\lambda\tau} + \frac{N}{\lambda}\|\tilde{w}\|_\infty + k\varepsilon. \end{aligned}$$

Consequently, there exists T^* such that $N|e(\tau_0)|e^{-\varepsilon\lambda T^*} < \frac{\delta}{2}$ and let $\varepsilon_2^* = \frac{\delta}{2k}$. If we choose $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$, for all $\varepsilon \in (0, \varepsilon^*)$, it follows that $|y(\tau, \varepsilon) - z(\tau, \varepsilon)| < \delta + \gamma(\|w\|_\infty)$, $\forall \tau \geq T^*$, which completes the proof. **Q.E.D.**

Remark 3.13. Proposition 3.3 indicates by assuming that the averaged system (3.12) is exponentially stable, although perturbed by disturbances, the solutions of transformed system remains close to the solutions of averaged system, which shows good robustness performance. The distance between the solutions of the system (3.16) $y(\tau, \varepsilon)$ and solutions of averaged system (3.12) $z(\tau, \varepsilon)$ is described by the size of disturbances $\tilde{\gamma}(\|w\|_\infty)$.

3.4.2 The Influence of Disturbances Frequency on the Robustness Performance

When the bounded additive disturbances are periodic, the robustness of the linear vibrational system (3.7) can be linked to the frequency of disturbances. In particular, we will show that high frequency of disturbances will attenuate the perturbation to trajectories of the disturbed vibrational system (3.7) such that the ultimate bound of trajectories becomes smaller.

Fourier series expansion is used to get the frequency spectrum of disturbances. It is well-known that the Fourier series expansion of a periodic signal exists if it satisfies Dirichlet conditions (see [81] for more details). Most periodic signals in engineering applications satisfy it. Basically we analyze the system response to each component in the frequency spectrum and obtain a less conservative estimation of the upper bound compared to taking the L -infinity norm of disturbances directly. The main results are summarized in the following theorem:

Theorem 3.5. *Let Assumption 3.1 hold and $w \in \mathcal{L}_\infty$. Suppose disturbance $w(t)$ is a T_w -periodic function and its norm can be expressed in Fourier series: $|w(t)| = a_0/2 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$ where $\omega_0 = 2\pi/T_w$. Then there exist $B_1(t)$ satisfying Assumption 3.2 and there exist strictly positive real numbers M , N , λ and ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of system (3.7) satisfy:*

$$\begin{aligned} |x(t)| &\leq MN e^{-\lambda t} |x_0| + \frac{a_0 MN}{2\lambda} (1 - e^{-\lambda t}) \\ &+ \sum_{k=1}^{\infty} \frac{MN a_k}{\lambda^2 + k^2 \omega_0^2} [\lambda \cos(k\omega_0 t) - \lambda e^{-\lambda t} + k\omega_0 \sin(k\omega_0 t)] \\ &+ \sum_{k=1}^{\infty} \frac{MN b_k}{\lambda^2 + k^2 \omega_0^2} [\lambda \sin(k\omega_0 t) - k\omega_0 \cos(k\omega_0 t) + k\omega_0 e^{-\lambda t}]. \end{aligned} \quad (3.20)$$

Proof: From Proof of Theorem 3.4, there exist strictly positive real numbers N and λ s.t.

$$|y(\tau)| \leq N e^{-\varepsilon \lambda \tau} |y(\tau_0)| + \varepsilon \int_0^\tau N M_2 e^{-\varepsilon \lambda (\tau-s)} |w(\varepsilon s)| ds.$$

where N , M_2 and λ are defined the same as in Theorem 3.4. Since $|w(t)|$ could be expressed in Fourier series: $|w(t)| = |w(\varepsilon \tau)| = a_0/2 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 \varepsilon \tau) + b_k \sin(k\omega_0 \varepsilon \tau)]$.

It leads to

$$\begin{aligned}
|y(\tau)| &\leq N e^{-\lambda\tau} |y(\tau_0)| + \frac{a_0 N M_2 \varepsilon}{2} \int_0^\tau e^{-\varepsilon\lambda(\tau-s)} ds \\
&\quad + N M_2 a_k \sum_{k=1}^{+\infty} \varepsilon \int_0^\tau \cos(k\omega_0 \varepsilon s) e^{-\varepsilon\lambda(\tau-s)} ds \\
&\quad + N M_2 b_k \sum_{k=1}^{+\infty} \varepsilon \int_0^\tau \sin(k\omega_0 \varepsilon s) e^{-\varepsilon\lambda(\tau-s)} ds.
\end{aligned}$$

By calculations, we have:

$$\begin{aligned}
\varepsilon \int_0^\tau e^{-\varepsilon\lambda(\tau-s)} ds &= \frac{1}{\lambda} (1 - e^{-\lambda\varepsilon\tau}), \\
\varepsilon \int_0^\tau \cos(k\omega_0 \varepsilon s) e^{-\varepsilon\lambda(\tau-s)} ds &= \varepsilon \int_0^\tau \frac{e^{jk\omega_0 \varepsilon s} + e^{-jk\omega_0 \varepsilon s}}{2} e^{-\varepsilon\lambda(\tau-s)} ds \\
&= \frac{1}{\lambda^2 + k^2\omega_0^2} \left(\lambda \cos(k\omega_0 \varepsilon \tau) + k\omega_0 \sin(k\omega_0 \varepsilon \tau) - \lambda e^{-\lambda\varepsilon\tau} \right), \\
\varepsilon \int_0^\tau \sin(k\omega_0 \varepsilon s) e^{-\varepsilon\lambda(\tau-s)} ds &= \varepsilon \int_0^\tau \frac{e^{jk\omega_0 \varepsilon s} - e^{-jk\omega_0 \varepsilon s}}{2} e^{-\varepsilon\lambda(\tau-s)} ds \\
&= \frac{1}{\lambda^2 + k^2\omega_0^2} \left(\lambda \sin(k\omega_0 \varepsilon \tau) - k\omega_0 \cos(k\omega_0 \varepsilon \tau) + k\omega_0 e^{-\lambda\varepsilon\tau} \right).
\end{aligned}$$

Noting that $|x(\tau)| \leq \sup_{\tau \geq \tau_0} \|\Phi(\tau, \tau_0)\| |y(\tau)|$, solutions of $x(t)$ satisfy

$$\begin{aligned}
|x(t)| &\leq M N e^{-\lambda t} |x(t_0)| + \frac{a_0 N}{2\lambda} (1 - e^{-\lambda t}) \\
&\quad + N \sum_{k=1}^{\infty} \frac{a_k}{\lambda^2 + k^2\omega_0^2} [\lambda \cos(k\omega_0 t) - \lambda e^{-\lambda t} + k\omega_0 \sin(k\omega_0 t)] \\
&\quad + N \sum_{k=1}^{\infty} \frac{b_k}{\lambda^2 + k^2\omega_0^2} [\lambda \sin(k\omega_0 t) - k\omega_0 \cos(k\omega_0 t) + k\omega_0 e^{-\lambda t}]
\end{aligned}$$

where $M = M_1 M_2$. This completes the proof. **Q.E.D.**

Remark 3.14. When disturbances are not only bounded but also continuously periodic, the ultimate bound of trajectories obtained in Theorem 3.4 can be expanded by Fourier series of disturbances. Instead of considering the L -infinity norm of disturbances, the frequency analysis improves the resolution of boundary calculation such that it becomes a more accurate estimation. As shown in (3.20), there are two major components in the estimate of the ultimate bound: one is related to DC term a_0 and the other is related to

the frequency of the periodic disturbances. The upper bound related to the frequency indicates that disturbances with a higher frequency would have a smaller ultimate bound.

3.4.3 State-dependent Disturbances

In some situations, the disturbances are coupled with system states, for example, in the stabilization of the inverted pendulum, the viscous friction coefficient varies as humidity changes resulting in state-dependent disturbances exist in the system. This section aims at analyzing the robustness of the vibrational control systems in the presence of a more general class of disturbances:

$$\dot{x} = Ax + \frac{1}{\varepsilon} B_1 \begin{pmatrix} t \\ \varepsilon \end{pmatrix} x + \eta(x, w), \quad x(t_0) = x_0 \in \mathbb{R}^n. \quad (3.21)$$

In (3.21), matrices A and B_1 are defined the same as in (3.7) and the nonlinear function $\eta(x, w)$ represents the state-dependent disturbances. It is assumed that $\eta : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous and locally Lipschitz in x and w satisfying $\eta(0, 0) = 0$.

Due to existence of the nonlinear term $\eta(\cdot, \cdot)$, the standard averaging technique cannot be applied directly. The perturbation techniques cannot be applied either. Thus strong average and weak average techniques are used to show the robustness of (3.21).

Similar to previous sections, by using the coordinate transformation (3.10), the system (3.21) is transformed into the following form:

$$\frac{dy}{d\tau} = \varepsilon (\Phi^{-1}(\tau, \tau_0) A \Phi(\tau, \tau_0) y + \Phi^{-1}(\tau, \tau_0) \eta(\Phi(\tau, \tau_0) y, w)). \quad (3.22)$$

Proposition 3.4. The strong average of the system (3.22) doesn't exist but the weak average exists. The weak averaged system is:

$$\dot{y}_{wa} = \varepsilon (\bar{A} y_{wa} + \tilde{w}_{wa}), \quad y_{wa}(\tau_0) = y(\tau_0), \quad (3.23)$$

where $\bar{A} = \frac{1}{T} \int_{\tau}^{\tau+T} \Phi^{-1}(s, \tau_0) A \Phi(s, \tau_0) ds$, $\tilde{w}_{wa} = \frac{1}{T} \int_{\tau}^{\tau+T} \Phi^{-1}(s, \tau_0) \eta(\Phi(s, \tau_0) y, w) ds$.

The proof of Proposition 3.4 is similar to the proof of Proposition 3.1, thus it is omitted.

As the weak average of the system (3.22) exists, if it is Lyapunov-ISS, by applying weak average Theorem 3.2, the following result is obtained.

Theorem 3.6. Suppose Assumption 3.1 holds and $w \in \mathcal{L}_{\infty}$. Assume the weak averaged system (3.23) is Lyapunov-ISS with gain $\hat{\gamma}$. There exist $B_1(t)$ satisfying Assumption 3.2 and $\beta \in \mathcal{KL}$, for any given strictly positive real numbers Δ , k_w , $k_{\dot{w}}$, δ , there exist

positive constants ε^* , M_1 , M_2 s.t. $\forall \varepsilon \in (0, \varepsilon^*)$ the solutions of (3.21) satisfy:

$$|x(t)| \leq M_1 \max\{\beta(|x_0|, t - t_0), M_2 \tilde{\gamma}(\|w\|_\infty)\} + M_1 \delta, \quad (3.24)$$

whenever $t \geq t_0 \geq 0$, $|x_0| \leq \Delta$, $\|w\|_\infty \leq k_w$ and $\|\dot{w}\|_\infty \leq k_{\dot{w}}$.

Proof: Since Assumption 3.1 holds, $\Phi(\tau, \tau_0)$ and $\Phi^{-1}(\tau, \tau_0)$ are periodic and bounded, let their supremum norm be M_1 and M_2 respectively i.e. $M_1 = \sup_{\tau \geq \tau_0} \|\Phi(\tau, \tau_0)\|$ and $M_2 = \sup_{\tau \geq \tau_0} \|\Phi^{-1}(\tau, \tau_0)\|$

The dynamics of transformed system (3.22) $f(\tau, y, w) = \varepsilon (\Phi^{-1}(\tau)A\Phi(\tau)y + \Phi^{-1}(\tau)\eta(\Phi(\tau)y, w))$, then

$$\begin{aligned} |f(t, y_1, w) - f(t, y_2, w)| &\leq \|\Phi^{-1}A\Phi\| |y_1 - y_2| + \|\Phi^{-1}\| |\eta(\Phi y_1, w) - \eta(\Phi y_2, w)| \\ &\leq M_1 M_2 \left(\max_{i \in \{1, 2, \dots, n\}} \{|\lambda_i(A)|\} + L_y \right) |y_1 - y_2|, \end{aligned}$$

$$|f(t, y, w_1) - f(t, y, w_2)| \leq M_2 L_w |w_1 - w_2|.$$

So the system (3.22) is Lipschitz in y and w uniformly in τ . Therefore, applying Theorem 3.3, there exists $\beta \in \mathcal{KL}$ such that the solutions of system (3.22) satisfy: $\forall \tau \geq \tau_0 \geq 0$

$$|y(\tau)| \leq \max\{\beta(|y(\tau_0)|, \tau - \tau_0), M_2 \tilde{\gamma}(\|w\|_\infty)\} + \delta.$$

Since $x(\tau) = \Phi(\tau, \tau_0)y(\tau)$ and $x(\tau_0) = y(\tau_0)$, $\forall \tau \geq \tau_0 \geq 0$, it leads to

$$|x(\tau)| \leq M_1 \max\{\beta(|x(\tau_0)|, \tau - \tau_0), M_2 \tilde{\gamma}(\|w\|_\infty)\} + M_1 \delta.$$

Q.E.D.

Remark 3.15. The assumption that weak averaged system (3.23) is Lyapunov-ISS in Theorem 3.6 is not very restrictive due to the fact that \bar{A} is Hurwitz from Assumption 3.1. Next, Corollary 3.1 provides a sufficient condition for the nonlinear mapping $\eta(\cdot, \cdot)$ to guarantee that the weak averaged system (3.23) is Lyapunov-ISS. \circ

Corollary 3.1. Suppose Assumption 3.1 holds. If the nonlinear function $\eta(\cdot, \cdot)$ in (3.22) satisfies the following inequality

$$|\eta(x, w)| \leq |x|^c |w|, \quad (3.25)$$

for some $c \in [0, 1)$, then that weak averaged system (3.23) of is Lyapunov ISS.

Proof: As Assumption 3.1 holds, Theorem 1.1 indicates that there exists $B(\frac{t}{\varepsilon})$ such that \bar{A} is Hurwitz. According to [77, Theorem 4.6], for any positive definite symmetric matrix Q , there exists positive definite symmetric P s.t. $PA_{av} + A_{av}^T P = -Q$. Choosing

$Q = I$, a Lyapunov candidate is selected as $V(y) = y^T P y$. This Lyapunov candidate is used to show that the weak averaged system (3.23) is Lyapunov-ISS.

$$\begin{aligned}
\dot{V}(t) &= -y^T y + 2y^T P \frac{1}{T} \int_t^{t+T} \Phi^{-1}(\tau) \eta(\Phi(\tau)y, w) d\tau \\
&\leq -y^T y + 2|y| \|P\| \|\Phi\|_\infty \sup_{t \geq t_0} |\eta(\Phi(t)y, w)| \\
&\leq -|y|^2 + 2|y| \|P\| \|\Phi\|_\infty \|\Phi^{-1}\|_\infty^c |y|^c |w| \\
&\leq -(1 - \theta) |y|^2,
\end{aligned} \tag{3.26}$$

whenever $|y| \geq \left(\frac{2\|P\| \|\Phi\|_\infty \|\Phi^{-1}\|_\infty^c |w|}{\theta} \right)^{1/(1-c)}$ for $0 \leq c < 1$ and $0 < \theta < 1$. Applying [77, Theorem 4.19] directly, it is concluded that the system (3.23) is Lyapunov-ISS.

Q.E.D.

Remark 3.16. By applying weak average, vibrational control systems (3.7) are shown to be robust to a more general states-dependent disturbances and it has ISS-like stability with bounded and slow disturbances when the weak averaged system is Lyapunov ISS. The Lyapunov ISS condition is not strict for the linear systems, which can be satisfied if the coupling disturbances function $\eta(x, w)$ is sub-linear.

3.4.4 Simulation Example: Linearised Inverted Pendulum

A strengthened robustness analysis is given in the Section 3.4, 3.4.2 and 3.4.3, which shows that the vibrational control system has ISS like properties. Next, the model of linearised inverted pendulum serves as an illustrate example to verify these robustness performances with numeric simulations.

The state-space model of the inverted pendulum after linearization has the form of (3.7) with

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}, B_1 \left(\frac{t}{\varepsilon} \right) = \begin{bmatrix} 0 & 0 \\ \frac{a}{l\varepsilon} \cos \frac{t}{\varepsilon} + \frac{ka}{ml} \sin \frac{t}{\varepsilon} & 0 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned} \tag{3.27}$$

It can be easily verified that the matrix A satisfies Assumption 3.1 and $B_1(t)$ satisfies Assumption 3.2. Without disturbances, Theorem 1.1 indicates that vibrational control system is stable if ε is sufficiently small.

In the following simulations, parameters of linearised inverted pendulum are selected as $l = 0.185, m = 0.2, k = 1, g = 9.8$ and vibrational control parameters are chosen as

$a = 20, \varepsilon = 0.0032$. Figure 3.5 shows that the vibrational controller can stabilize the inverted pendulum while the original system is unstable.

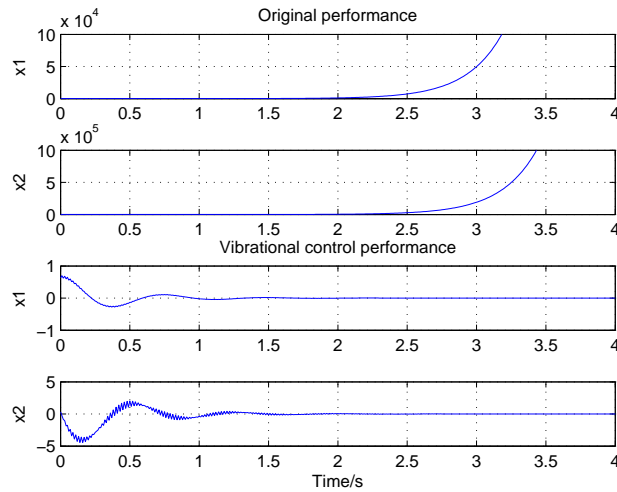


FIGURE 3.5: Linearised inverted pendulum is stabilized by vibrational control while original system is unstable

Next it will be verified that this vibrational control algorithm is robust to different types of disturbances. Assumed that there are exogenous forces disturbing the stabilization of inverted pendulum, the system has additive, but state-independent disturbances in this situation. As Theorem 3.4 indicates, the vibrational control system is ISS if disturbances are bounded. The first disturbance is selected as

$$w_0(t) = \frac{t^2}{1+t^2}, \quad (3.28)$$

which satisfies $\|w_0\|_\infty = 1$. The simulated trajectories in l_2 -norm in Figure 3.6(a) show that the states converge to a neighborhood of the origin.

Theorem 3.5 provides a less conservative estimation of trajectories bound when the state-independent disturbances are periodic. Under such a situation, both Theorem 3.4 and 3.5 are applicable. In order to compare the ultimate bounds for two results, $w_2(t) = \sin 50t$ is applied. By applying Theorem 3.4, the bound of the trajectories estimated from (3.17) is around 300. By applying Theorem 3.5, the bound from (3.20) is shown in Figure 3.7, which is much less conservative (compared with 300 obtained from Theorem 3.4).

Other than providing a less conservative bound for trajectories, Theorem 3.5 also indicates that when disturbances are periodic, the ultimate bound is almost inversely proportional to the disturbance frequency when it is sufficiently large. Figure 3.8 shows

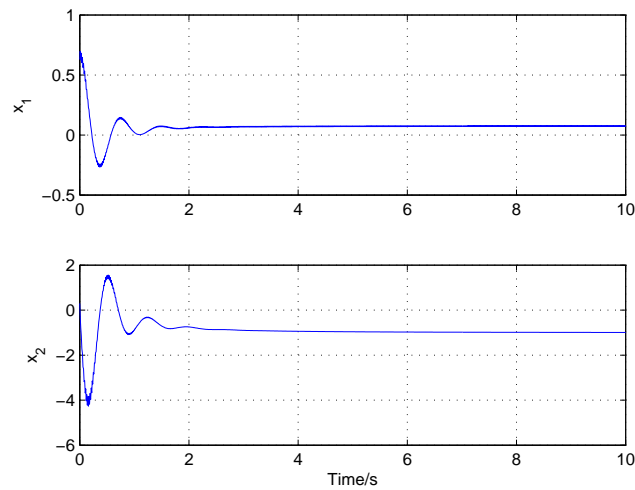


FIGURE 3.6: Trajectories of the inverted pendulum in the existence of disturbances $w_0(t)$

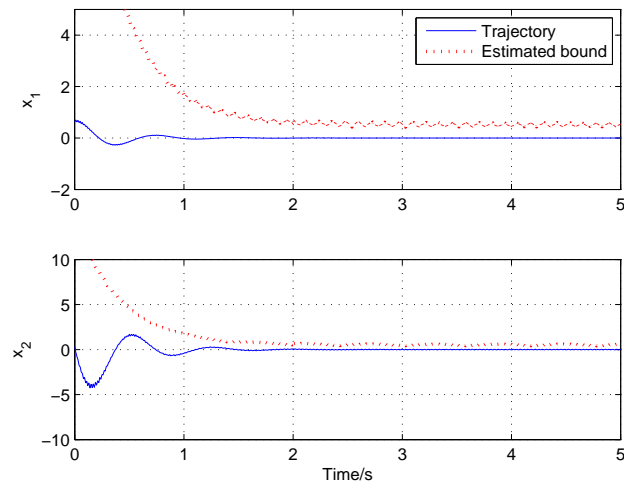


FIGURE 3.7: Trajectories of x_1, x_2 and the bound of the trajectories from Theorem 3.5

simulated trajectories with different-frequency sinusoidal disturbances (the angular frequency is 10, 50, 10^2 , 10^3 , 10^4 respectively). It indicates that the ultimate bound will reduce as the frequency of disturbances increases.

It is possible to have the scenario when disturbances are coupled with states. For example, the friction coefficient may vary as humidity changes, leading to state-dependent disturbances.

Suppose state-dependent disturbance $\eta(x, w)$ takes the form of $\eta(x, w) = \left[\frac{|x|^{0.5} w_0(t)}{\sqrt{2}}, \frac{|x|^{0.5} w_0(t)}{\sqrt{2}} \right]^T$, where w_0 comes from (3.28). Obviously, the condition in Corollary 3.1 is satisfied. Thus the weak average (3.23) is Lyapunov-ISS. As the disturbance $w(t)$ is bounded and slowly

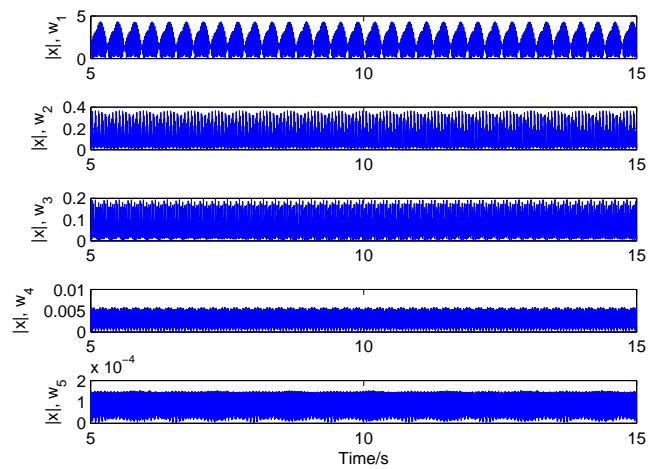


FIGURE 3.8: The relationship between the ultimate bound and the frequency of disturbances

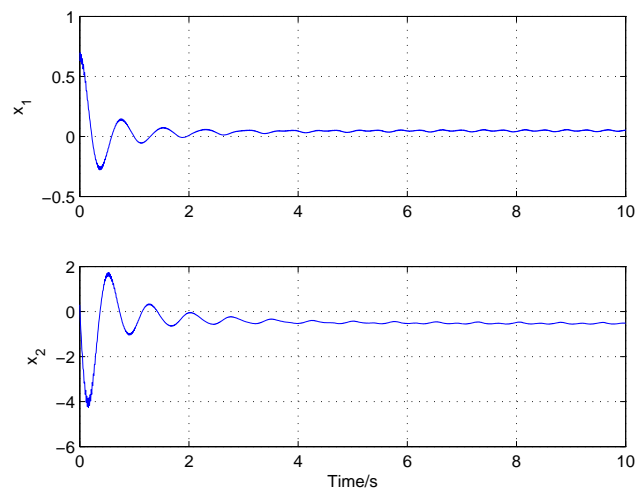


FIGURE 3.9: System is practically input-to-state stable in the existence of state-dependent disturbance.

time-varying, Theorem 3.6 shows that the trajectories of the system will be practically ISS. The trajectories of the vibrational control system with such a disturbance are shown in Figure 3.9. These simulation results are consistent with theoretic results.

3.5 Summary

In this chapter, the robustness properties of linear vibrational control systems (LVCS) with respect to different types of bounded disturbances were discussed. Input-to-state

stability is an important concept to characterize the robustness of systems when disturbances exist. It is straightforward to explore the robustness by applying the strong and weak average techniques which show that LVCS has ISS-like stability properties when the disturbances are bounded and sufficiently slow. Next, the robustness result was generalized to demonstrate LVCS could also handle fast disturbances. By applying the averaging and perturbation technique, the robustness was also strengthened to show that LVCS is input-to-state stable such that the ultimate bound is only decided by the L -infinity norm of disturbances.

To get a less conservative estimation of the bound of trajectories when additive disturbances are periodic, the frequency spectrum of ultimate bound was analysed by taking Fourier series of disturbances. It indicated that DC component and low-frequency components of disturbances constitute the ultimate bound more compared to high-frequency components. And also it shows that higher frequency of disturbances leads to smaller ultimate bound. Finally, a class of state-dependent additive disturbances are considered. The obtained results show that LVCS is robust with respect to a class of state-coupling disturbance functions if the disturbances are slowly time-varying. Simulation results agree with theoretic analysis.

Chapter 4

Robustness of Nonlinear Vibrational Control Systems

4.1 Overview

In this chapter, we will continue discussing the robustness of vibrational control systems with respect to additive disturbances, extending the obtained results to the general non-linear vibrational systems.

First of all, we will extend the local vibrational stability in Theorem 2.7 to local vibrational robustness by considering the existence of additive disturbances. It is shown in Theorem 2.7 that if the linearized matrix of the averaged system is Hurwitz, the original dynamics is vibrationally stabilizable in the absence of disturbances. Based on that, we will explore the perturbed performance when disturbances exist such as the transient behaviour, the steady state and the ultimate bound.

The robustness analysis is basically representing the disturbed nonlinear vibrational control systems as the stable averaged system and several perturbation terms. We adapt the sample-data approach used in [45, 82] in the proof to show that a quadratic Lyapunov function decays exponentially at sampling time instances. After showing the closeness between sampled trajectories and actual trajectories, the robustness of nonlinear vibrational control systems is concluded in Theorem 4.1. The main result shows that the averaged trajectories of nonlinear vibrational control systems have locally practically ISS properties.

Next, we will explore the non-local vibrational robustness properties of nonlinear vibrational control systems with respect to additive disturbances. We consider that the averaged system is asymptotically stable in any domain $D \subset R^n$, which has a weaker

stability requirement compared to exponentially stable assumption used in the local robustness. Main result in Theorem 4.2 shows that the averaged trajectories of time-varying system have ISS-like stability properties for a non-local domain of attraction, which can be an arbitrarily large if D is the global domain.

When the bounded disturbances are periodic and fast-varying, the nonlinear vibrational control system has stronger robustness properties as shown in Theorem 4.3. It shows that the nonlinear vibrational control system can handle large amplitude of disturbances in these cases. Besides, the averaged trajectories converge to a smaller ultimate bound. In the definition of input-to-state stability, the trajectories converge to a ultimate bound decided by the L -infinity norm of the disturbances, however we have shown that the ultimate bound here is only related to the average of the disturbances such that the estimation of the steady-states is less conservative.

The standing point of the above results is assuming that the nonlinear vibrational control systems in the absence of disturbances is vibrationally stable. The disturbances are regarded as perturbations to the stabilized system such that they (or their average) should be bounded by some value to avoid driving the trajectories out of the domain of attraction.

In the last section, we considers the weak averaged system (the averaged system with disturbances) exists and is Lyapunov-ISS, then nonlinear vibrational control systems can handle arbitrarily large disturbances. Besides, the averaged trajectories satisfy semi-globally practically ISS properties. We adapt the weak average results to show that for disturbed nonlinear vibrational control systems, the obtained robustness results are valid for both fast and slow disturbances.

The chapter is organized as follows. In Section 4.2, local robustness results are obtained. Robustness based on a weak stability condition is presented in Section 4.3, applicable to a large class of systems. A stronger robustness conclusion when periodic disturbances exist is also stated. weak average technique is adapted in Section 4.4 to show that the system can handle arbitrarily large disturbances when the weak averaged system is Lyapunov-ISS. Section 4.5 summarizes the chapter.

4.2 Local Vibrational Robustness with respect to Constrained Additive Disturbances

4.2.1 Local Robustness Analysis in the presence of Additive Disturbances

In section 2.6.2, the local vibrational stabilization is discussed when the linearised matrix of averaged system is Hurwitz. As an extension, next we consider the exogenous disturbances w exist in the vibrational control system (2.20) with an additive form:

$$\dot{x} = f(x) + \frac{1}{\varepsilon} g\left(\frac{t}{\varepsilon}, x, \varepsilon\right) + d(w), x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \quad (4.1)$$

where $w \in \mathbb{R}^m$ and the disturbance function $d: \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies the Assumption 4.1.

Assumption 4.1. Let Ω_d be a compact set in \mathbb{R}^m , in which the origin is an interior point. There exists $\gamma \in \mathcal{K}$ such that the disturbance mapping satisfies

$$|d(w)| \leq \gamma(|w|) \quad (4.2)$$

for all $w \in \Omega_d$.

Remark 4.1. The assumption is to guarantee that the influence of disturbances to the systems is caused by its largest value of disturbances. When disturbances tends to zero, the influence to the system also diminishes.

By transforming the disturbed vibrational control systems (4.1) with the coordinate change (2.24), we have

$$\begin{aligned} \frac{dy}{d\tau} &= \varepsilon \left\{ \frac{\partial h}{\partial y} \right\}^{-1} (f(h(\tau, y)) + d(w(\varepsilon\tau))) \\ &= \varepsilon(f_1(\tau, y) + d_1(\tau, y, w)). \end{aligned} \quad (4.3)$$

where $d_1(\tau, y, w) = \left\{ \frac{\partial h}{\partial y} \right\}^{-1}(\tau, y)d(w(\varepsilon\tau))$.

Remark 4.2. The disturbance in new coordinate $d_1(\cdot, \cdot, \cdot)$ also satisfies the Assumption 4.1 because $\left\{ \frac{\partial h}{\partial y} \right\}^{-1}(\tau, y)$ is periodic in τ and continuous in y , thus it is bounded for all y in a compact set Ω_{y0} in \mathbb{R}^n . Based on assumptions and conditions used in Theorem 2.7, we will analyse the robustness of vibrational control systems by treating $d_1(\tau, y, w)$ as a perturbation to the nominal system $f_1(\tau, y)$.

The averaged system of the system (4.3) without considering disturbances is

$$\frac{dz}{d\tau} = \varepsilon f_{1,av}(z), \quad (4.4)$$

where $f_{1,av}(z) := \frac{1}{T} \int_0^T \left\{ \frac{\partial h}{\partial z} \right\}^{-1} f(h(\tau, z)) d\tau$ and z_e is the equilibrium point of $f_{1,av}(z)$ such that $f_{1,av}(z_e) = 0$.

Next, we apply the standard sampling technique as introduced in [45, 82] to estimate the original trajectory bound by computing the values of a Lyapunov function at sampled time instances. Next Lemma shows a quadratic Lyapunov function exists and decreases at sampled time instances for all initial points inside the domain of attraction.

Lemma 4.1. Suppose Assumptions 2.1 - 2.3 and 4.1 hold. Assume that there exists an equilibrium point z_e of (4.4) such that $\bar{A} = \left[\frac{\partial f_{y,av}}{\partial z} \right]_{z=z_e}$ is Hurwitz, there exists positive definite matrix P , positive constants ε^* , μ , ρ_0 , $\nu(\rho_0)$ and $\tilde{\gamma} \in \mathcal{K}$ such that $\forall \varepsilon \in (0, \varepsilon^*)$, $\forall |y_0| < \rho_0$ and $\forall \|w\|_\infty \leq \nu$, Lyapunov function $V(y) = y^T P y$ satisfies the following condition:

$$V_{k+1} - V_k \leq -\mu V_k \varepsilon T, \quad \forall V_k \geq \tilde{\gamma}(\|w\|_\infty) \quad (4.5)$$

where T is the period of dither signal in time τ and $V_k = V(y(t_k))$, $t_k = t_0 + k\varepsilon T$, $k = 0, 1, \dots, n$.

Proof. Without losing generality, suppose the equilibrium point of the transform system (2.25) is origin. If the equilibrium point is not the origin, by coordinate change it can be shifted to the origin. According to Mean Value Theorem, there is $\xi \in (0, y)$ such that the i -th component of the transformed system (4.3) $f_1(\tau, y)$ can be expressed as

$$\begin{aligned} f_{y,i}(\tau, y) &= f_{y,i}(\tau, y) - f_{y,i}(\tau, 0) \\ &= \frac{\partial f_{y,i}}{\partial y}(\tau, \xi) y \\ &= \frac{\partial f_{y,i}}{\partial y}(\tau, 0) y + \left(\frac{\partial f_{y,i}}{\partial y}(\tau, \xi) - \frac{\partial f_{y,i}}{\partial y}(\tau, 0) \right) y. \end{aligned} \quad (4.6)$$

For $\frac{\partial f_{y,i}}{\partial y}(\tau, y)$ is assumed to be locally Lipschitz continuous, for a given compact set, there exists positive numbers L_i such that

$$\left| \frac{\partial f_{y,i}}{\partial y}(\tau, y_1) - \frac{\partial f_{y,i}}{\partial y}(\tau, y_2) \right| \leq L_i |y_1 - y_2|.$$

Separating the linear term in this way, we can rewrite the transformed system (4.3) as

$$\frac{dy}{d\tau} = \varepsilon(A(\tau)y + \eta(\tau, y) + d_1(\tau, y, w)), \quad (4.7)$$

where $A(\tau) = \frac{\partial f_1}{\partial y}(\tau, 0)$, $\eta(\tau, y) = \left(\frac{\partial f_1}{\partial y}(\tau, \xi) - \frac{\partial f_1}{\partial y}(\tau, 0) \right) y$. From the Lipschitz condition we can see that $|\eta(\tau, y)| \leq L|y|^2$ where $L = \sqrt{\sum_{i=1}^n L_i^2}$. As the averaged matrix \bar{A} is Hurwitz, there exist positive-definite matrices P and Q s.t. the following Lyapunov

equation is satisfied:

$$P\bar{A} + \bar{A}^T P = -Q. \quad (4.8)$$

The transformed system (4.7) can be further written as:

$$\frac{dy}{d\tau} = \epsilon(\bar{A}y + (A(\tau) - \bar{A})y + \eta(\tau, y) + d_1(\tau, y, w)). \quad (4.9)$$

Select a Lyapunov function $V(y) = y^T P y$. Since the matrix P is positive definite, there exist positive constants c_1, c_2 such that $c_1|y|^2 \leq V(y) \leq c_2|y|^2$.

The derivative of $V(y)$ along the system (4.9) is

$$\begin{aligned} \frac{dV}{d\tau}(\tau) &= \epsilon y^T P [\bar{A}y + (A(\tau) - \bar{A})y + \eta(\tau, y) + d_1(\tau, y, w)] \\ &\quad + \epsilon [\bar{A}y + (A(\tau) - \bar{A})y + \eta(\tau, y) + d_1(\tau, y, w)]^T P y \\ &= \epsilon y^T (P\bar{A} + \bar{A}^T P)y + \epsilon y^T P(A(\tau) - \bar{A})y \\ &\quad + \epsilon y^T (A(\tau) - \bar{A})^T P y + 2\epsilon \eta^T(\tau, y) P y + 2\epsilon d_1^T(\tau, y, w) P y \\ &= -\epsilon y^T Q y + \epsilon y^T P(A(\tau) - \bar{A})y + \epsilon y^T (A(\tau) - \bar{A})^T P y \\ &\quad + 2\epsilon \eta^T(\tau, y) P y + 2\epsilon d_1^T(\tau, y, w) P y. \end{aligned} \quad (4.10)$$

Let $\varphi(\tau) = A(\tau) - \bar{A}$. Since $A(\tau)$ is periodic and \bar{A} is a constant matrix then $\varphi(\tau)$ is also periodic. According to the definition of averaged system $\bar{A} = \frac{1}{T} \int_{\tau}^{\tau+T} A(s) ds$, the average of $\varphi(\tau)$ is

$$\frac{1}{T} \int_{\tau}^{\tau+T} \varphi(s) ds = 0, \quad (4.11)$$

which indicates $\varphi(\tau)$ is zero mean.

The equation (4.10) is then rewritten in the original time scale:

$$\dot{V}(t) = -y^T Q y + y^T P \varphi\left(\frac{t}{\epsilon}\right) y + y^T \varphi\left(\frac{t}{\epsilon}\right)^T P y + 2\eta^T\left(\frac{t}{\epsilon}, y\right) P y + 2d_1^T\left(\frac{t}{\epsilon}, y, w\right) P y. \quad (4.12)$$

The upper bound of $V(t)$ can be estimated by integrating its derivative:

$$\begin{aligned}
V(t) &= V(t_0) + \int_{t_0}^t \dot{V}(s) ds \\
&= V(t_0) + \int_{t_0}^t [-y^T Q y + 2\eta^T(s, y) P y + 2d_1^T(s, y, w) P y] ds \\
&\quad + \int_{t_0}^t \left[y^T P \varphi\left(\frac{s}{\epsilon}\right) y + y^T \varphi\left(\frac{s}{\epsilon}\right)^T P y \right] ds \\
&\leq V(t_0) + \int_{t_0}^t [-c_3 |y|^2 + 2Lc_2 |y|^3 + 2b|d(w)|c_2 |y|] ds + \int_{t_0}^t \left[y^T P \varphi\left(\frac{s}{\epsilon}\right) y + y^T \varphi\left(\frac{s}{\epsilon}\right)^T P y \right] ds \\
&\leq V(t_0) - \int_{t_0}^t (1 - \theta_1 - \theta_2)c_3 |y|^2 ds - \int_{t_0}^t (\theta_1 c_3 |y|^2 - 2Lc_2 |y|^3) ds \\
&\quad - \int_{t_0}^t (\theta_2 c_3 |y|^2 - 2b\gamma(\|w\|_\infty)c_2 |y|) ds + \int_{t_0}^t \left[y^T P \varphi\left(\frac{s}{\epsilon}\right) y + y^T \varphi\left(\frac{s}{\epsilon}\right)^T P y \right] ds \\
&\leq V(t_0) + \int_{t_0}^t \left[-(1 - \theta_1 - \theta_2)c_3 |y|^2 + 2y^T P \varphi\left(\frac{s}{\epsilon}\right) y \right] ds, \\
\forall \tilde{\gamma}(\|w\|_\infty) &\leq |y| \leq \rho_0,
\end{aligned} \tag{4.13}$$

where $\rho_0 = \frac{\theta_1 c_3}{2Lc_2}$, $\tilde{\gamma}(\|w\|_\infty) = \frac{2b\gamma(\|w\|_\infty)c_2}{\theta_2 c_3}$, $b = \sup_{|y| \leq \rho_0} \left| [\partial h / \partial y]^{-1}(\tau, y) \right|$ and θ_1, θ_2 are positive real numbers satisfying $\theta_1 + \theta_2 \in (0, 1)$.

According to the continuity of solutions, for all initial values $\{y_0 \in \mathbb{R}^n \mid |y_0| < \rho_0\}$ and disturbances $\{w \in \mathbb{R}^m \mid \|w\|_\infty \leq \tilde{\gamma}^{-1}(\rho_0)\}$, there are positive constants M and ε_1^* such that for all $\varepsilon \in (0, \varepsilon_1^*)$, the solutions of transformed system (4.3) satisfy:

$$|y(t) - y(t_0)| \leq M(t - t_0), \forall t \in [t_0, t_0 + \varepsilon T]. \tag{4.14}$$

An upper bound of the term in the equation (4.13) can be found in the following way:

$$\begin{aligned}
&\int_{t_0}^{t_0 + \varepsilon T} y^T(s) P \varphi\left(\frac{s}{\epsilon}\right) y(s) ds \\
&= \int_{t_0}^{t_0 + \varepsilon T} \left[y^T(s) P \varphi\left(\frac{s}{\epsilon}\right) y(s) - y_0^T P \varphi\left(\frac{s}{\epsilon}\right) y_0 \right] ds \\
&\quad + \int_{t_0}^{t_0 + \varepsilon T} \left[y^T(s) P \varphi\left(\frac{s}{\epsilon}\right) y_0 - y^T(s) P \varphi\left(\frac{s}{\epsilon}\right) y_0 \right] ds \\
&= \int_{t_0}^{t_0 + \varepsilon T} \left[y^T(s) P \varphi\left(\frac{s}{\epsilon}\right) (y(s) - y_0) + (y(s) - y_0)^T P \varphi\left(\frac{s}{\epsilon}\right) y_0 \right] ds \\
&\leq \int_{t_0}^{t_0 + \varepsilon T} (|y(s)| + |y_0|) \|P\| |y(s) - y_0| \left\| \varphi\left(\frac{s}{\epsilon}\right) \right\| ds \\
&< 2\rho_0 c_2 \xi_0 M(\varepsilon T)^2, \forall t_0 \leq t \leq t_0 + \varepsilon T,
\end{aligned} \tag{4.15}$$

where $\sup_{\tau \geq 0} \|A(\frac{\tau}{\epsilon}) - \bar{A}\| = \xi_0$. Thus there exists $\epsilon_2^* = \frac{|y_0|}{2MT}$ such that $|y(t)|^2 \geq \frac{|y_0|^2}{4}$, for all $t \in [t_0, t_0 + \epsilon T]$. Combined with inequality (4.13) we can see that

$$V(t_0 + \epsilon T) \leq V(t_0) - \frac{1}{4}(1 - \theta_1 - \theta_2 - \theta_3)c_3|y_0|^2\epsilon T + 4\rho_0c_2\xi_0M(\epsilon T)^2 - \frac{1}{4}\theta_3c_3|y_0|^2\epsilon T. \quad (4.16)$$

Then there exists $\epsilon_3^* = \frac{\theta_3c_3|y_0|^2}{16\rho_0c_2\xi_0M}$ such that $4\rho_0c_2\xi_0M(\epsilon T)^2 - \frac{1}{4}\theta_3c_3|y_0|^2\epsilon T \leq 0$. Select $\epsilon^* = \min(\epsilon_1^*, \epsilon_2^*, \epsilon_3^*)$ therefore for every $\epsilon \in (0, \epsilon^*)$, for all $y_0 \in \Omega$, $\forall t \in [t_0, t_0 + \epsilon T]$

$$V(t_0 + \epsilon T) \leq V(t_0) - \frac{1}{4}(1 - \theta_1 - \theta_2 - \theta_3)c_3|y(t_0)|^2\epsilon T < V(t_0).$$

We can see that the Lyapunov function decreases over a period ϵT . In the same way we could see that it decreases at each sampled time instance $t_k = t_0 + k\epsilon T$ where $k \in N$. If $V_k \geq c_2(\gamma(\|w\|_\infty))^2 = \gamma_1(\|w\|_\infty)$, there exists ϵ^* s.t. for every $\epsilon \in (0, \epsilon^*)$, the following inequality holds

$$V_{k+1} \leq V_k - \frac{1}{4}(1 - \theta_1 - \theta_2 - \theta_3)c_3|y_k|^2\epsilon T < V_k, t \in [t_k, t_{k+1}]. \quad (4.17)$$

In the condition of $V_k \geq c_1|y_k|^2$, the difference of Lyapunov function at two sequential sampled time instances is

$$V_{k+1} - V_k \leq -\mu V_k \epsilon T, t \in [t_k, t_{k+1}], \quad (4.18)$$

where $\mu = \frac{(1-\theta_1-\theta_2-\theta_3)c_3}{4c_1}$. □

Remark 4.3. As \bar{A} is Hurwitz, for any positive definite matrix Q , there exists a positive definite matrix P such that $P\bar{A} + \bar{A}^T P = -Q$. Although the derivative of proposed Lyapunov function $V(y) = y^T P y$ cannot be shown as negative, Lemma 4.1 shows that it decreases at sampled time instances along the trajectories of system (4.3), based on which the contraction could be obtained. The estimated domain of attraction is $DOA = \{y \in R^n \mid |y - z_e| \leq \rho_0\}$. The L_∞ norm of disturbances that the system can handle will depend on the domain of attraction. For disturbances outside of the estimated bound, it might drive the trajectories of the system outside the DOA.

Based on the fact that Lyapunov function decreases at sampled time instances, Lemma 4.2 shows the sampled trajectory also decreases.

Lemma 4.2. Suppose the inequality (4.5) holds from Lemma 4.1, there exist $\beta \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}$ such that the sampled trajectories of (4.3) satisfy

$$|y(t_k) - z_e| \leq \max\{\beta(|y_0 - z_e|, k\epsilon T), \hat{\gamma}(\|w\|_\infty)\}. \quad (4.19)$$

Proof. Without losing the generality, let $z_e = 0$. Then we introduce a new variable $u(s) = V_k + (\frac{s}{\varepsilon T} - k)(V_{k+1} - V_k)$, where $s \in [t_k, t_{k+1}]$, $t_k = t_0 + k\varepsilon T$ and V_k satisfies inequality (4.5). The function $u(s)$ is an absolutely continuous, piecewise linear function, then it is differentiable for almost all $s \geq 0$. As $u(s) \leq V_k$, if $u(s) \geq \tilde{\gamma}(\|w\|_\infty)$ we have $V_k \geq \tilde{\gamma}(\|w\|_\infty)$. According to (4.5),

$$\frac{du}{ds} = (V_{k+1} - V_k) \frac{1}{\varepsilon T} \leq -\mu V_k \leq -\mu u(s), \forall u(s) \geq \tilde{\gamma}(\|w\|_\infty). \quad (4.20)$$

According to Comparison Lemma in [77, Lemma 3.4],

$$|u(s)| \leq \max\{e^{-\mu s}|u_0|, \tilde{\gamma}(\|w\|_\infty)\}.$$

As $u(t_k) = V_k \geq c_1|y(t_k)|^2$, $|u_0| = V_0 \leq c_2|y_0|^2$, where c_1 and c_2 are related to the positive definite matrix P . Then there exists $\beta \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}_\infty$ such that the following inequality holds

$$|y(t_k)| \leq \max\{\beta(|y_0|, k\varepsilon T), \hat{\gamma}(\|w\|_\infty)\}, \quad (4.21)$$

where $\beta(|y_0|, k\varepsilon T) = \sqrt{\frac{c_2}{c_1}} e^{-0.5\mu k\varepsilon T} |y_0|$ and $\hat{\gamma}(\|w\|_\infty) = \sqrt{\frac{c_2}{c_1}} \tilde{\gamma}(\|w\|_\infty)$. \square

Remark 4.4. The sampled trajectories converge exponentially because $\beta(\cdot, s)$ is an exponentially decaying function with respect to s . This is a natural result from the assumption that \bar{A} is Hurwitz which indicates the averaged system without disturbances is locally exponentially stable. In the existence of disturbances, the sampled trajectories will converge to a neighbourhood related to the L_∞ norm of disturbances.

After showing that the inter-sampling behaviour between sampled points could be made sufficiently small by tuning the parameter ε , the main result of this section is given in Theorem 4.1.

Theorem 4.1. *Suppose Assumptions 2.1 - 2.4 and 4.1 hold. If there exists an equilibrium point z_e of (4.4) such that $\bar{A} = \left[\frac{\partial f_{y,av}}{\partial z} \right]_{z=z_e}$ is Hurwitz, there exist positive numbers $a_1, a_2, \lambda, \mathcal{K}_\infty$ functions $\hat{\gamma}_1, \hat{\gamma}_2$ and positive constants $\rho, v(\rho), \delta^*(\rho)$ such that for any $\delta \in (0, \delta^*)$ there exists ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of the system (4.1) exist for $t \geq t_0$ and satisfy*

$$|x_{av}(t) - x_e| \leq \max\{a_1|x_{av}(t_0) - x_e|e^{-\lambda(t-t_0)}, \hat{\gamma}_1(\|w\|_\infty)\} + \delta \quad (4.22)$$

for all $t \geq t_0$, whenever $|x_0 - x_e| \leq \rho$ and $\|w\|_\infty \leq \nu$, where $x_{av}(t) = \int_t^{t+T} x(\tau) d\tau$.

Moreover, if $x_e = h(t, z_e)$ for all $t \geq t_0$, the solutions of the system (4.1) satisfy

$$|x(t) - x_e| \leq \max\{a_2|x(t_0) - x_e|e^{-\lambda(t-t_0)}, \hat{\gamma}_2(\|w\|_\infty)\} + \delta, \quad (4.23)$$

for all $t \geq t_0$, whenever $|x_0 - x_e| \leq \rho$ and $\|w\|_\infty \leq \nu$.

Proof. Since Assumptions 2.1 - 2.3, 4.1 hold, sampled trajectory of transformed system (4.3) satisfies (4.19) in Lemma 4.2. Without losing the generality, let $z_e = 0$. As indicated in (4.14), for any $\delta > 0$ there exists $\varepsilon^* > 0$ s.t. for all $\varepsilon \in (0, \varepsilon^*)$,

$$|y(t)| \leq |y(t_k)| + \delta, \forall t \in [t_k, t_{k+1}].$$

Combining the above inequality with (4.21), it shows that

$$|y(t)| \leq \max\{\beta(|y_0|, k\varepsilon T), \hat{\gamma}(\|w\|_\infty)\} + \delta, \forall t \geq t_0 \geq 0. \quad (4.24)$$

where $\beta(|y_0|, k\varepsilon T) = \sqrt{\frac{c_2}{c_1}} e^{-0.5\mu k\varepsilon T} |y_0|$. Then there exists positive a_0 such that for all $\varepsilon \in (0, \varepsilon^*)$, the following inequality satisfies

$$a_0 \geq \sqrt{\frac{c_2}{c_1}} e^{-0.5\mu\varepsilon T}. \quad (4.25)$$

Therefore, the trajectories between two sampling instances can be bounded by

$$\begin{aligned} |y(t)| &\leq \max\left\{\sqrt{\frac{c_2}{c_1}} e^{-0.5\mu k\varepsilon T} |y_0|, \hat{\gamma}(\|w\|_\infty)\right\} + \delta, \\ &\leq \max\{a_0 e^{-0.5\mu(k+1)\varepsilon T} |y_0|, \hat{\gamma}(\|w\|_\infty)\} + \delta. \end{aligned} \quad (4.26)$$

As $(k+1)\varepsilon T \geq t - t_0$ for any $t \in [t_k, t_{k+1}]$,

$$|y(t)| \leq \max\{a_0 e^{-0.5\mu(t-t_0)} |y_0|, \hat{\gamma}(\|w\|_\infty)\} + \delta. \quad (4.27)$$

The averaged trajectory is defined as $x_{av}(t) = \frac{1}{T} \int_t^{t+T} x(\tau) d\tau$, then the closeness to the equilibrium point could be found using the following inequality:

$$\begin{aligned} |x_{av}(t) - x_e| &= \frac{1}{T} \int_t^{t+T} (h(\tau, y) - h(\tau, 0)) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} (L|y(\tau)|) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} L \left(\max\{a_0 e^{-0.5\mu(\tau-t_0)} |y_0|, \hat{\gamma}_2(\|w\|_\infty)\} + \delta \right) d\tau \\ &\leq \max\{a_1 |x_{av}(t_0) - x_e| e^{-\lambda(t-t_0)}, \hat{\gamma}_1(\|w\|_\infty)\} + \delta. \end{aligned} \quad (4.28)$$

where $\lambda = -0.5\mu$, $a_1 = \frac{L}{\lambda T} (1 - e^{-\lambda T}) \sup_t \{h^{-1}(t, y_0 - z_e)\}$ and $\hat{\gamma}_1 \in \mathcal{K}$.

Moreover, suppose $x_e = h(t, 0)$ holds. Since $h(t, c)$ is locally Lipschitz then for any $\delta > 0$, we have

$$\begin{aligned} |x(t) - x_e| &= |h(t, y) - h(t, 0)| \\ &\leq L|y(t)| \\ &\leq \max\{a_2|x(t_0) - x_e|e^{-\lambda(t-t_0)}, \hat{\gamma}_2(\|w\|_\infty)\} + \delta, \forall t \geq t_0 \geq 0, \end{aligned} \quad (4.29)$$

where $a_2 = La_0 \sup_t \{h^{-1}(t, y_0 - z_e)\}$. This completes the proof. \square

Remark 4.5. The moving average of the system trajectories is used in Theorem 4.1. Compared to averaging the trajectory over an infinity time in Definition 2.11, it characterizes the transient behaviour of the averaged trajectory such as the decaying rate. It shows that the averaged trajectory converges arbitrarily close to the ultimate bound $\hat{\gamma}_1(\|w\|_\infty)$. In the special case of $w = 0$, the averaged trajectory will converge arbitrarily close to the equilibrium point which means the system is vibrationally stable. When the equilibriums x_e and z_e satisfy condition (2.27), the original trajectory instead of averaged trajectory of the solutions will converge to the ultimate bound.

4.2.2 An Illustrative Example: Vertically Moving Inverted Pendulum to Track a Moving Target

Vibrational control has been shown to be a useful method to stabilize the inverted pendulum without using a feedback as demonstrated in the motivational example in the Section 1.1. Next, instead of stabilizing the pendulum to a still upper point, we assume the targeted upper point of pendulum is moving vertically at a constant speed. Thus besides the sinusoidal dither, an extra displacement signal $\sigma(t) = \sigma t$ which casts the moving target will be added to the motion of the slider, where σ is the speed of the target.

After deriving the equations of motions from Lagrange dynamics modeling and representing the system in state-space by letting $x_1 = \theta$, $x_2 = \dot{\theta}$, the system becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ (\frac{a}{\varepsilon l} \sin(\frac{t}{\varepsilon}) - \frac{ka}{ml} \cos(\frac{t}{\varepsilon})) \sin x_1 \end{bmatrix}}_{g(\frac{t}{\varepsilon}, x, \varepsilon)} + \underbrace{\begin{bmatrix} 0 \\ -\frac{k\sigma}{ml} \sin x_1 \end{bmatrix}}_{d(\sigma, x)}. \quad (4.30)$$

Remark 4.6. It can be seen from equation (4.30) that stabilizing the moving inverted pendulum with vibrational control brings in an additive term $d(\sigma, x)$. The moving speed is then regarded as an additive disturbance to the vibrational control system. As $d(\sigma, x)$ has the property $|d(\sigma, x)| \leq \frac{k\sigma}{ml}$ such that Assumption 4.1 is satisfied, Theorem 4.1 is

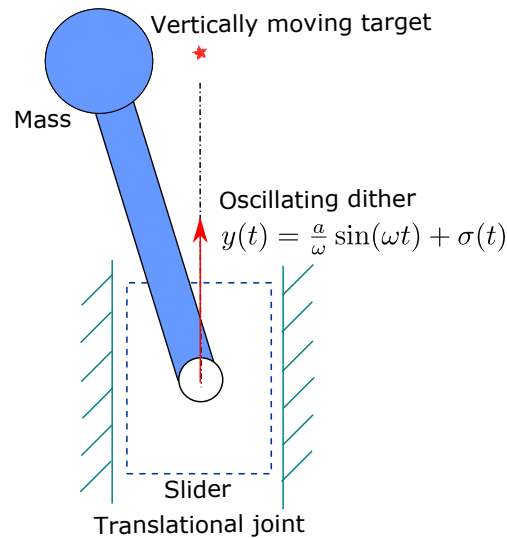


FIGURE 4.1: Moving inverted pendulum stabilized by vibrational control method.

applicable to show the robustness, though the disturbance is coupled with states. There are other cases where state-independent additive disturbances exist. For example, a disturbance force F_w is acting on the mass where $d(F_w) = [0, F_w l]^T$.

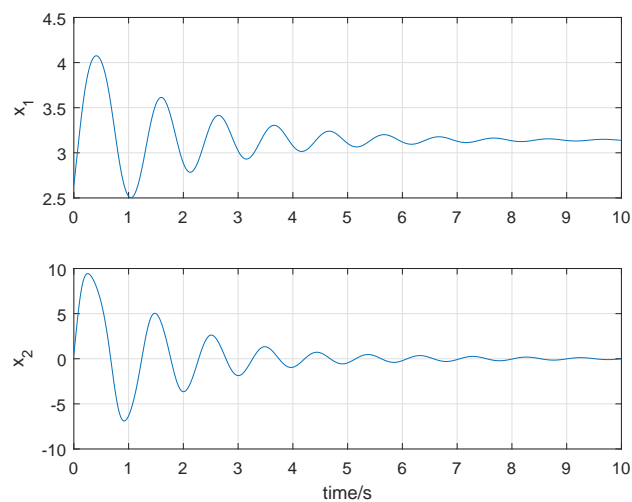


FIGURE 4.2: The trajectories of inverted pendulum stabilized by vibrational control algorithm.

Next, we will simulate the inverted pendulum with a moving target to verify the robustness of vibrational control systems which is indicated in Theorem 4.1. All the parameters used in the simulations are given in the Table 4.1.

The simulated angular positions and velocities of the pendulum with respect to time are shown in Figure 4.2. The inverted pendulum starts from a neighbourhood from its equilibrium position where $x_e = [\pi, 0]^T$. Although the moving speed σ exists as

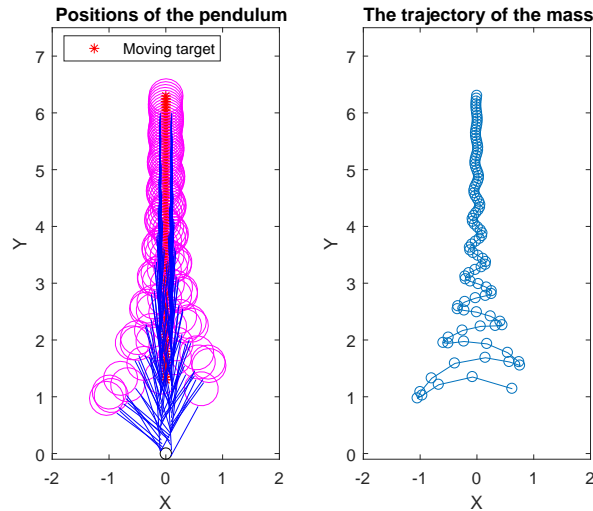


FIGURE 4.3: (a) Positions of inverted pendulum at different time instances; (b) trajectory of the mass.

TABLE 4.1: System parameters of inverted pendulum used in the simulation

Classification	Parameter	Value	Unit
Pendulum	mass m	1	Kg
	viscous coefficient k	0.5	N/m·s ⁻¹
	length l	1	m
	gravitational acceleration g	9.8	m/s ²
Vibrational controller	amplitude a	10	m
	tuning parameter ε	0.01	s
Initial conditions	angular displacement θ_0	2.64	rad
	angular velocity $\dot{\theta}_0$	0.5	rad/s
Disturbance	moving speed σ	0.5	m/s

disturbances, the states converge close to the equilibrium positions which shows the robustness of vibrational control algorithm. By computations, the equilibrium points in the transformed systems is $y_e = z_e = [\pi, 0]^T$. As the coordinate transformation is

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 - \frac{a}{l} \cos \tau \sin y_1. \end{cases} \quad (4.31)$$

It means that the equilibrium points x_e, z_e satisfy the condition of $x_e = h(\tau, z_e)$ in Theorem 4.1. Hence the trajectory instead of its average converges arbitrarily close to the equilibrium point. The trajectory behaviours in Figure 4.2 are consistent with the theoretic analysis.

The positions of inverted pendulum at different time instances are shown in Figure

4.3(a). It shows that the sensorless inverted pendulum with vibrational controller could track the moving target in an open-loop fashion. The trajectory of the mass is shown in Figure 4.3(b).

Overall, the simulation results support the theoretic analysis which verify the local robustness of vibrational control systems.

4.2.3 Summary of this Section

In this section, we presented the robustness analysis of nonlinear vibrational control systems with respect to additive disturbances when the linearized matrix of the averaged system is Hurwitz. The Lyapunov sample-data method was used to show the convergence of solutions. It indicates that for all initial points from the domain of attraction, the periodic solutions will converge to an ultimate bound near the equilibrium point, provided that disturbances are constrained in a compact set. The ultimate bound of solutions is related to the size of disturbances and can be estimated by applying perturbation technique. Numerical simulations supported theoretic findings. In the next section, a weaker stability condition will be used for robustness analysis, which assumes the averaged system is asymptotically stable in some domain of attraction that can be either local or global region.

4.3 Non-local Vibrational Robustness with respect to Constrained Additive Disturbances

In Section 4.2, a direct extension of local vibrational stabilization to local vibrational robustness was made considering the existence of additive disturbances when the averaged system is assumed to be locally exponentially stable (LES). However, the LES assumption is too strong for some practical system, for example, in Section 2.6 we have seen that vibrational stabilization could be achieved for systems without holding the LES condition. In the following section, we will provide a more general robustness analysis where the averaged system is asymptotically stable in any domain of attraction in R^n . The result is not limited to the local region but can also work when the averaged system has globally asymptotic stability. When disturbances are bounded and periodic, taking the L -infinity norm of disturbances to estimate the ultimate bound might be too conservative. For those fast time-varying periodic disturbances, the ultimate bound will be shown related to the average of disturbances instead of the L -infinity norm.

4.3.1 Non-local Robustness Analysis in the presence of Bounded Additive Disturbances

Consider disturbances w exist in the vibrational control system (2.20) with an additive form:

$$\dot{x} = f(x) + \frac{1}{\varepsilon}g\left(\frac{t}{\varepsilon}, x\right) + d(w), x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \quad (4.32)$$

where $w \in \mathbb{R}^m$ and the disturbance mapping $d: \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies the Assumption 4.1.

By transforming the disturbed vibrational control systems (4.32) with the coordinate change (2.24), we have

$$\begin{aligned} \frac{dy}{d\tau} &= \varepsilon \left\{ \frac{\partial h}{\partial y} \right\}^{-1} (f(h(\tau, y)) + d(w(\varepsilon\tau))) \\ &= \varepsilon(f_1(\tau, y) + d_1(\tau, y, w(\varepsilon\tau))), \end{aligned} \quad (4.33)$$

where $d_1(\tau, y, w) = \left\{ \frac{\partial h}{\partial y} \right\}^{-1} (\tau, y)d(w(\varepsilon\tau))$.

Remark 4.7. The additive form is preserved in transformed systems (4.33), in which the dominant term $f_1(\tau, y)$ is the same as (2.25). Then system (4.33) can be treated as the averaged system (4.4) with additional perturbations. The first perturbation comes from the difference between time periodic functions $f_1(\tau, y) - f_{y,av}(y)$ while the other one comes from the disturbances. Next we will show that under some conditions, the system keeps stable in the existence of these perturbations.

Let D_z be the domain of attraction that is mapped to D_x by the transformation (2.24) and D_{x_0} be a compact subset of D_x .

Theorem 4.2. *Suppose Assumptions 2.1 - 2.4 and 4.1 hold. If there exists an equilibrium point z_e of (4.4) such that the averaged system (4.4) is asymptotically stable for all $z(t_0) \in D_z$, there exist positive constants ν, δ^* such that for any $\delta \in (0, \delta^*)$ there exists ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of the system (4.32) exist for $t \geq t_0$ and satisfy*

$$|x_{av}(t) - x_e| \leq \max\{\hat{\beta}_1(|x_{av}(t_0) - x_e|, t - t_0), \hat{\gamma}_1(\|w\|_\infty)\} + \delta \quad (4.34)$$

for all $t \geq t_0$, whenever $x_0 \in D_{x_0}$ and $\|w\|_\infty \leq \nu$, where $\hat{\beta}_1 \in \mathcal{KL}$, $\hat{\gamma}_1 \in \mathcal{K}_\infty$.

Proof: see Proof of Theorem 4.2 in Appendix A.1. □

Remark 4.8. As the stability properties of the averaged system (4.4) are defined in the domain of attraction D_z , the disturbances in (4.32) cannot be arbitrarily large as the existence of disturbance might drive the trajectories of the system (4.32) outside the domain of the attraction. When the disturbances are sufficiently small, the trajectories

of the vibrational control system (4.32) still preserve some stability properties. The averaged trajectories of system converge to a neighbourhood of the equilibrium x_e . The ultimate bound of the trajectories have two parts: one is related to disturbances or $\|w\|_\infty$, while the other term δ comes from the closeness of trajectories between the original system and the averaged system. The value of δ can be tuned arbitrarily small by reducing the value of the parameter ε .

Remark 4.9. Compared with Theorem 4.1, Theorem 4.2 extends the local robustness from linearisation to the non-local robustness where the domain of attraction can be arbitrarily large. Besides, the averaged system (4.4) is relaxed to a weak assumption that it is asymptotically stable instead of locally exponentially stable.

If asymptotic stability of the averaged system is satisfied globally, the initial domain could be arbitrarily large while trajectories satisfying (4.34) for any bounded disturbances. It is stated in the following corollary.

Corollary 4.1. If the system (4.4) is globally asymptotically stable for all $z(t_0) \in R^n$ and all the other assumptions of Theorem 4.2 hold, for any given Δ and ν , the solutions of the system (4.32) satisfy (4.34) whenever $|x_0 - x_e| \leq \Delta$ and $\|w\|_\infty \leq \nu$.

Remark 4.10. Corollary 4.1 can be indicated from Theorem 4.2 when the domain of attraction of the averaged system (4.4) is global, then the time-varying system (4.32) keeps stable for any bounded initial set. While disturbances are zero, the inequality (4.34) indicates the system is uniformly semi-globally practically vibrational stable, which can be linked to SPV stability results in Theorem 2.8.

With an extra condition $x_e = h(t, z_e)$, which means that equilibrium point x_e of the original dynamics $\dot{x} = f(x)$ coincides with the one after the transformation, the following corollary shows a stronger result.

Corollary 4.2. Suppose all the assumptions and conditions in Theorem 4.2 are satisfied. In addition, if $x_e = h(t, z_e)$ for all $t \geq t_0$, the solutions of the system (4.32) satisfy

$$|x(t) - x_e| \leq \max\{\hat{\beta}_2(|x_0 - x_e|, t - t_0), \hat{\gamma}_2(\|w\|_\infty)\} + \delta, \quad (4.35)$$

for all $t \geq t_0$, where $\hat{\beta}_2 \in \mathcal{KL}$, $\hat{\gamma}_2 \in \mathcal{K}_\infty$.

The proof of Corollary 4.2 is straightforward, thus it is omitted.

4.3.2 Frequency-related Robustness in the presence of Bounded and Periodic Additive Disturbances

As high frequency dither signals are injected to the system (2.19), it is possible that the dither signals might excite some other periodic disturbances that satisfy the following assumption.

Assumption 4.2. Disturbances $w(t)$ are bounded and periodic i.e. $w \in L_\infty$ and there exists $T_w > 0$ such that $w(t + T_w) = w(t)$.

The disturbances are also required to be bounded as in Assumption 4.2 such that Theorem 4.2 is applicable. In Theorem 4.2, the estimation of ultimate bound comes from the worst case of disturbances $\|w\|_\infty$. However, when more information of disturbance is known, a less conservative estimation of the ultimate bound could be obtained. When disturbances are periodic with the frequency ω , they can be represented as

$$w\left(\omega\left(t + \frac{T_w}{\omega}\right)\right) = w(\omega t).$$

By letting $\eta = \frac{1}{\omega}$, the systems (4.32) can be rewritten into the following form with two time scales:

$$\dot{x} = f(x) + \frac{1}{\varepsilon}g\left(\frac{t}{\varepsilon}, x, \varepsilon\right) + d_w\left(\frac{t}{\eta}\right), x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0 \geq 0, \quad (4.36)$$

where $d_w(t) = d \circ w(t)$ is written in the composition form.

By transforming the disturbed vibrational control systems (4.32) with the coordinate change (2.24), we have

$$\begin{aligned} \frac{dy}{d\tau} &= \varepsilon \left\{ \frac{\partial h}{\partial y} \right\}^{-1} (f(h(\tau, y)) + d(w(\varepsilon\tau))) \\ &= \varepsilon \left(f_1(\tau, y) + d_1\left(\tau, y, w\left(\frac{\varepsilon\tau}{\eta}\right)\right) \right), \end{aligned} \quad (4.37)$$

where $d_1(\tau, y, w) = \left\{ \frac{\partial h}{\partial y} \right\}^{-1}(\tau, y) d_w\left(\frac{\varepsilon\tau}{\eta}\right)$.

Next Theorem shows that if the disturbances are faster than dither signal, a less conservative estimation of trajectories bound could be obtained.

Theorem 4.3. Suppose Assumptions 2.1 - 2.4, 4.1 - 4.2 hold. If there exists an equilibrium point z_e of (4.4) such that the system (4.4) is asymptotically stable for all $z(t_0) \in D_z$, there exist positive constants ν, δ^* such that for any $\delta \in (0, \delta^*)$, there exists ε^* s.t. for any $\varepsilon \in (0, \varepsilon^*)$, there exists $\eta^* < \varepsilon$ for any $\eta \in (0, \eta^*)$, the solutions of the

system (4.32) exist and satisfy

$$|x_{av}(t) - x_e| \leq \hat{\beta}_3(|x_{av}(t_0) - x_e|, t - t_0) + \hat{\gamma}_3(|\bar{d}_w|) + \delta, \quad (4.38)$$

for all $t \geq t_0$, whenever $x_0 \in D_{x_0}$ and $|\bar{d}_w| \leq \nu$, where $\bar{d}_w = \frac{1}{T_w} \int_t^{t+T_w} d_w(\tau) d\tau$. Moreover, if $x_e = h(t, z_e)$ for all $t \geq t_0$, the solutions of the system (4.32) satisfy

$$|x(t) - x_e| \leq \hat{\beta}_4(|x_0 - x_e|, t - t_0) + \hat{\gamma}_4(|\bar{d}_w|) + \delta, \quad (4.39)$$

for all $t \geq t_0$, where $\hat{\beta}_3, \hat{\beta}_4 \in \mathcal{KL}$, $\hat{\gamma}_3, \hat{\gamma}_4 \in \mathcal{K}_\infty$.

Proof: see Proof of Theorem 4.3 in Appendix A.2. □

Remark 4.11. According to equation (A.37) in the proof, the practical term δ in ultimate bound can be represented as $O(\epsilon) + O\left(\frac{\eta}{\epsilon}\right) + O(\eta)$. To constrain the trajectories within δ , firstly dither frequency ϵ is chosen to be sufficiently small such that it takes part of δ . It is satisfied if the frequency of disturbances is high enough such that $O\left(\frac{\eta}{\epsilon}\right) + O(\eta)$ takes the rest of threshold. Normally δ is smaller than 1, so η should accordingly be smaller than ϵ which means the frequency of disturbances is supposed to be higher than that of dither signal.

Remark 4.12. Compared to Theorem 4.2, the ultimate bound is dependent on the average of disturbances other than its L -infinity norm. This estimation could be much less conservative because the average is possible to be much smaller than its L -infinity norm, for example when the disturbances are periodic with zero mean, the trajectories of system converge to the δ -neighbourhood that is independent of the L -infinity norm of disturbances. In other words, the equilibrium point of the system keeps vibrational stabilizable in the existence of disturbances if they are periodic with zero average and fast varying.

Remark 4.13. Strong and weak average provides robustness analysis tools for a general nonlinear time-varying systems in [45]. By applying them to vibrational control systems in [75], strong average doesn't exist while weak average exists. From weak average results, trajectories boundary similar as (4.35) could be estimated if the disturbances are slowly varying. Theorem 4.3 can be regarded as the complement which works for fast disturbances having a less conservative estimation.

Remark 4.14. The idea of Theorem 4.3 is similar to that presented in Theorem 3.5 as both of them consider the frequency components of disturbances. Subtle differences exist. As Fourier series expansion of $|w(t)|$ was used in the proof of Theorem 3.5 thus the result obtained is more conservative compared using the periodicity property of the disturbances in Theorem 4.3. On the other hand, when applying Theorem 4.3, the periodic disturbances don't necessarily need to be fast.

When the average of the periodic disturbances are zero, it is possible to obtain stronger stability results under some conditions.

Proposition 4.1. Suppose Assumptions 2.1 - 2.3 and 4.1 - 4.2 hold. Assume that there exists positive integers m and n such that the common period $T_c = m\varepsilon T = n\eta T_w$ and (4.4) is asymptotically stable for all $z(t_0) \in D_z$. Moreover, if $\frac{1}{T_c} \int_0^{T_c} d_1(\tau, y) d\tau = 0$, for any given δ there exists ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$ solutions of the system (4.32) exist and satisfy

$$|x_{av}(t) - x_e| \leq \hat{\beta}_5 (|x_{av}(t_0) - x_e|, t - t_0) + \delta \quad (4.40)$$

for all $t \geq t_0$, whenever $x_0 \in D_{x_0}$, where $\hat{\beta}_5 \in \mathcal{KL}$

Proof. By using the Lyapunov candidate in the proof of Theorem 4.3, the Lyapunov function taking the values at sampling instance $t_k = t_0 + km\varepsilon T$ satisfies (A.25). For $\frac{1}{T_c} \int_0^{T_c} d_1(\tau, y) d\tau = 0$, the second integral in (A.25) is bounded by

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) c\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) ds \\ &= \int_{t_k}^{t_{k+1}} \left(\frac{\partial V}{\partial y}(y) - \frac{\partial V}{\partial y}(y_k) \right) c\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) ds \\ &\leq \int_{t_k}^{t_{k+1}} L|y - y_k| \left| c\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) \right| ds \\ &\leq LMm^2(\varepsilon T)^2 \phi_0. \end{aligned} \quad (4.41)$$

Take (A.46), (A.47) and (4.41) into (A.25), the value of the Lyapunov function at t_{k+1} is bounded by:

$$V(t_{k+1}) \leq V(t_k) - \frac{1}{3} \alpha_3 (0.5|y_k|) \varepsilon T, \quad (4.42)$$

whenever $|y_k| \geq O(\varepsilon)$. Then for any given δ there exists ε^* such that the solutions of the system (4.32) satisfy (4.40), following the procedures in the proof of Theorem 4.2 after (A.16). \square

Remark 4.15. The estimated bound of the trajectories of the system (4.4) (see (4.40)) is smaller than that in (4.38). In order to achieve the better performance with smaller trajectory bound, the knowledge of the frequency information of disturbances is needed.

The Proposition 4.1 can be extended to almost periodic disturbances, though it is not the major scope of this paper. A straightforward way is applying general averaging to the transformed system (4.33) and the system is shown to be vibrationally stabilizable:

Proposition 4.2. Suppose that Assumptions 2.1 - 2.3, 4.1 - 4.2 hold. Assume that

$$\bar{A} = \left[\frac{\partial f_{y,av}}{\partial z} \right]_{z=z_e}$$

is Hurwitz. In addition, if $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d_1(\tau, y) d\tau = 0$, the system (4.33) is vibrationally stable.

Proof. The average of system (4.33) is system (4.4) because $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d_1(\tau, y) d\tau = 0$. For \bar{A} is Hurwitz, there exists a compact set D which z_e is an interior point such that the average system (4.4) is locally exponentially stable. According to [77, Theorem 10.5], there exists $\gamma, k > 0$ such that the solutions of original systems (4.33) satisfy

$$|y(t) - z_e| \leq |y_0 - z_e| e^{-\gamma(t-t_0)} + k\varepsilon$$

So the closeness between averaged solutions of the system (4.32) and the equilibrium point is bounded by:

$$\begin{aligned} |x_{av}(t) - x_e| &= \frac{1}{T} \int_t^{t+T} (h(\tau, y) - h(\tau, z_e)) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} (L|y(\tau) - z_e|) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} L (|y_0 - z_e| e^{-\gamma(t-t_0)} + k\varepsilon) d\tau \\ &\leq N|x_0 - x_e| e^{-\gamma(t-t_0)} + \delta, \end{aligned} \tag{4.43}$$

where $N = \frac{L}{\gamma T}(1 - e^{-\gamma T})$. □

Remark 4.16. Another possible way to explore the stabilization when disturbances are almost periodic is using the partial averaging technique introduced in [83]. It discusses the stability of the time-varying system $\dot{x}(t) = f(x(t), t, \alpha t)$ where α is a sufficiently large parameter. The original system can be concluded to be locally exponentially stable if the partially averaged system $\dot{x}(t) = f_{pav}(x(t), t)$ is locally exponentially stable. Applying the partial averaging technique into transformed system (4.37), the partially averaged system is simply relied on the $f_1(\tau, y)$ if $\lim_{T_w \rightarrow \infty} \frac{1}{T_w} \int_0^{T_w} d_w(\tau) d\tau = 0$ because the other parts are regarded as constant while doing the partial averaging. In this way, the vibrational stabilization could also be obtained in the assumption of the zero-mean condition on disturbances although it demands the disturbances are sufficiently fast time-varying.

4.3.3 Simulation Verification: a 2R Planar Manipulator Steered by Vibrational Control while Disturbances Exist.

Vibrational control has been shown to provide extra degrees of freedom by inserting the high-frequency dither signals. It is applied to stabilize and steer the underactuated

planar manipulators which is introduced in Section 1.2.5.1. A planar manipulator with two joints is shown in Figure 4.4 where the second joint is unactuated. The linearised system is not controllable at the operating point such that the manipulator cannot be steered and stabilized with conventional feedback control. The method used in [1] is firstly drive the active joint to the desired position by applying partially linearising control technique before exerting vibrational control approach to the actuated joint to guide the unactuated joint to its target position.

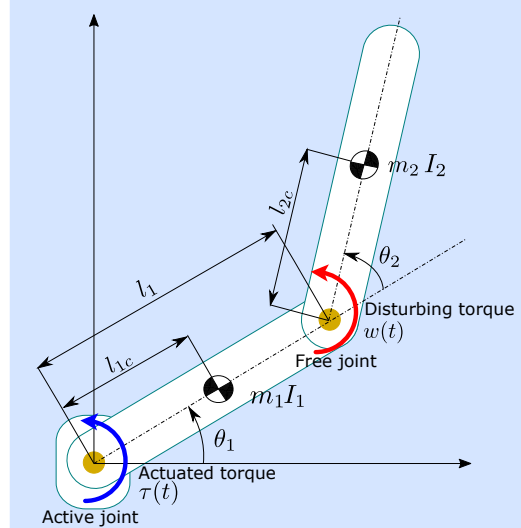


FIGURE 4.4: A 2R planar manipulator steered by vibrational control with disturbances.

Representing the system in state-space by letting $x_1 = \theta_2$, $x_2 = \dot{\theta}_2$ and introducing the small positive parameter $\varepsilon = \frac{1}{\omega}$, the system becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -p\alpha^2 \sin x_1 \sin^2 \frac{t}{\varepsilon} - \frac{\alpha}{\varepsilon} \cos \frac{t}{\varepsilon} (1 + p \cos x_1) - f_v x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{p}{m_2 l_{2c} l_1} w \end{bmatrix}, \quad (4.44)$$

which satisfies the additive form of disturbed vibrational control systems (4.33). The parameters used in the simulations are given in Table 4.2.

Next, it will be verified that vibrational control systems are robust to different types of disturbances. The first category considered is a class of bounded disturbances. As indicated by Theorem 4.2, the system keeps stable if the L -infinity norm is sufficiently small. Let $w_1(t) = \frac{0.2t^2}{1+t^2}$ and $w_2(t) = 0.2$. The system behaviours are shown in Figure 4.5 which indicates that both trajectories of the system converge to a neighbourhood of equilibrium point. According to (4.34), the ultimate bound is dependent on the L -infinity norm of disturbances, where $\|w_1\|_\infty = \|w_2\|_\infty = 0.2$ for the given ones, numeric solutions verify that they have the same ultimate bound.

When disturbances are bounded and periodic, Theorem 4.3 signifies the ultimate bound

TABLE 4.2: System parameters of the planar 2R manipulator used in the simulations

Classification	Parameter	Value	Unit
Manipulator	mass of link 1 m_1	0.200	Kg
	mass of link 2 m_2	0.255	Kg
	inertia of link 1 I_1	3.097	Kg·m ²
	inertia of link 2 I_2	3.499	Kg·m ²
	length of link 1 l_1	0.185	m
	length of link 2 l_2	0.135	m
	Distance to mass center l_{1c}	0.103	m
	Distance to mass center l_{2c}	0.060	m
Vibrational controller	viscous friction coefficient f_v	0.100	Nm·s
	amplitude α	2	m
Initial conditions	tuning parameter ε	0.01	s
	angular displacement θ_{20}	1.5	rad
	angular velocity $\dot{\theta}_{20}$	0.1	rad/s

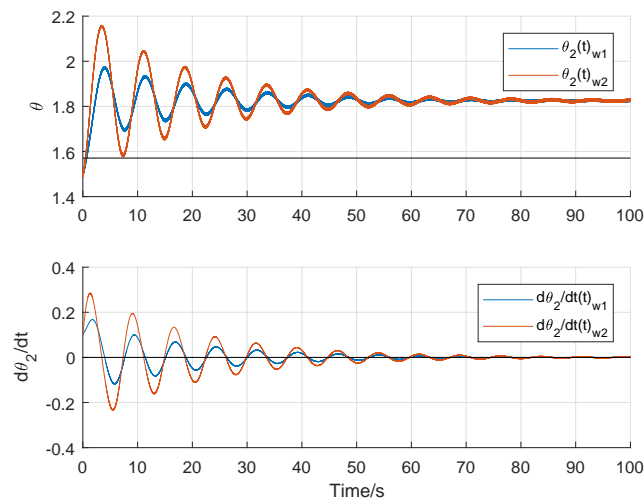


FIGURE 4.5: System behaviour in the existence of bounded disturbances.

is determined by the average of disturbances while practical term δ can be reduced for faster disturbances. Take the sinusoidal disturbance $w_3(t) = 0.2 + 5 \sin(\omega_d t)$ as an example where $\|w_3\|_\infty = 5.2$ which is much larger than the average $|\bar{w}_3| = 0.2$. Theorem 4.2 may be unavailable to show the stability because the L -infinity norm of disturbance can exceed the tolerance. Even if it is within the allowed bound of disturbances, the estimation of ultimate bound (4.34) is very conservative as it takes the L -infinity norm. Theorem 4.3 provides a better option which only has constraint of the average of disturbance that can be satisfied more easily and provides a more accurate approximation of ultimate bound related to the average.

System behaviours for three frequencies ($\omega_d = 1$, $\omega_d = 20$, and $\omega_d = 1000$) are displayed

in Figure 4.6 respectively. It can be seen that when disturbances are slow ($\omega_d = 1$), the equilibrium point of inverted pendulum is no longer stabilized but keeps stable for relatively fast disturbances ($\omega_d = 20$). To compare the ultimate bound, the system behaviour in the existence of $w_1(t)$ is also included. Even though the L -infinity norm of $w_3(t)$ is larger than that of $w_1(t)$, the ultimate bound remains the same because it relies on the average of disturbances instead of L -infinity norm in the fast and periodic case. When the frequency increases, the ultimate bound is further reduced. Overall, these simulation results are consistent with theoretic analysis.

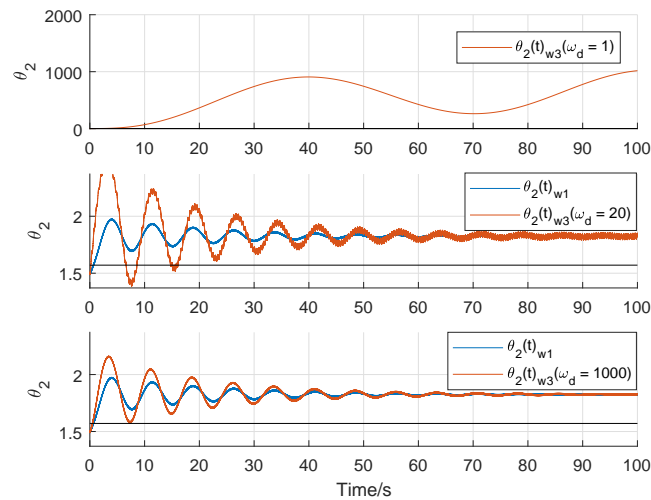


FIGURE 4.6: System behaviour in the existence of bounded and periodic disturbances with three different frequencies.

4.3.4 Summary of this Section

This section focuses on the robustness analysis of nonlinear vibrational control systems (NVBS) in the presence of additive disturbances when the average system is assumed to be asymptotically stable which is a weaker condition than the exponential stability assumption in the previous section. The robustness analysis is based on the perturbation method by viewing the NVBS as the averaged system perturbed by time-periodic difference system and disturbances. By using the Lyapunov function of averaged system, established from converse Lyapunov theory, the domain of attraction and ultimate bound of the NVBS is estimated. The domain of attraction will set up an upper bound on the disturbances which the system can handle. To show the convergence of solutions to the equilibrium point, the sample-data approach displays the decrease of Lyapunov

function at sampled instances. After showing the closeness between sampled trajectories and actual trajectories, the robustness of NVBS is concluded. It indicates that the averaged trajectories of the systems have practically ISS properties. That means for all initial conditions from the domain of attraction, disturbances constrained within the estimated bound, the solutions of system converge arbitrarily close to the ultimate bound that determined by the L -infinity norm of disturbances.

When the bounded disturbances are periodically fast time-varying, Theorem 4.3 denotes that the ultimate bound estimation can be less conservative. By dividing the integral of disturbances on the dither period into finite summations of the integral on the disturbance period, previous ultimate bound estimation determined by the L -infinity norm of disturbances can be replaced by the average of disturbances and terms inversely proportional to the frequency of disturbances. It indicates that when the frequency of the disturbance is increased, the ultimate bound can be reduced. Specifically when the average of disturbances is zero and the frequency is sufficiently large, the influence of disturbances to the system behaviour is negligible.

4.4 Semi-global Vibrational Robustness with respect to Arbitrarily Large Additive Disturbances

The robustness analysis in Section 4.2 and 4.3 is based on the stability of averaged system while disturbances are treated as a perturbation to the stabilized system. The bound of disturbances is thus constrained by the domain of attraction. For large disturbances, the trajectories will be dragged out of the domain of attraction then the system can become unstable. In this section, we assume the weak average exists for nonlinear vibrational control system and it is Lyapunov ISS. In this setup, the system (4.33) can handle arbitrarily large bounded disturbances if the parameters are tuned sufficiently small.

The weak average of transformed system (4.3) is

$$\frac{dz}{d\tau} = \varepsilon(f_{y,av}(z) + d_{y,wa}(z, w)), \quad (4.45)$$

where the weak average of disturbances $d_{y,wa}(z, w) = \frac{1}{T} \int_{t_0}^{t_0+T} d_1(\tau, z, w) d\tau$. $f_{y,av}(z)$ is previously defined as in (4.4).

Next theorem reveals that if the weak average of vibrational control systems (4.33) exists and is Lyapunov ISS, the original system can be semi-globally practically ISS where initial domain and disturbances can be arbitrarily large:

Theorem 4.4. *Suppose Assumptions 2.1 - 2.4 and 4.1 hold. If there exists an equilibrium point z_e of (4.4) such that the weak averaged system (4.45) is Lyapunov ISS with gain γ , there exists a positive constant δ^* , class \mathcal{KL} function $\hat{\beta}_6$, class \mathcal{K} function α_1, α_2 such that for any $\delta \in (0, \delta^*)$, $\Delta > 0$, $v > 0$, there exists ε^* s.t. for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of the system (4.32) exist for $t \geq t_0$ and satisfy*

$$|x_{av}(t) - x_e| \leq \max\{\hat{\beta}_6(|x_{av}(t_0) - x_e|, t - t_0), \hat{\gamma}_6(\|w\|_\infty)\} + \delta \quad (4.46)$$

for all $t \geq t_0$, whenever $|x_0 - x_e| \leq \Delta$ and $\|w\|_\infty \leq v$, where $\hat{\beta}_6 \in \mathcal{KL}$ and $\hat{\gamma}_6 = \alpha_1 \circ \alpha_2^{-1} \circ \gamma$.

Proof: See Proof of Theorem 4.4 in Appendix A.3 □

Remark 4.17. Theorem 4.4 indicates the trajectories converge to the a neighbourhood of the equilibrium point, which is composed of the L -infinity norm of disturbances for any large initial compact set and arbitrarily large bounded disturbances. Compared to the weak average results in Theorem 3.2, it relaxes the constraint for the derivative of disturbances, i.e. the system is robust to both fast and slow disturbances provided that they are bounded. This relaxation is achieved by taking advantage of the special additive structure of the system.

Remark 4.18. The Lyapunov ISS assumption of weak averaged system is stronger than the regionally asymptotic stability of averaged system in Theorem 4.2 because while disturbances are zero Lyapunov ISS signifies that the averaged system has global asymptotic stability. Also, the disturbances can be arbitrarily large leading to a relative large ultimate bound when the weak averaged system is Lyapunov ISS.

Remark 4.19. Although the ultimate of Theorem 4.2 and 4.4 is both related to the L -infinity norm of disturbances, they origins from different way. On the one hand, the ultimate bound of Theorem 4.2 is derived by treating the disturbances as perturbation in the integral of sampling instances (see the derivation of (A.17), on the other hand, the ultimate bound of Theorem 4.4 comes from the Lyapunov ISS (A.38). Although another ultimate bound (A.49) is produced in the sampling process, it is related to the tuning parameter ε , which can be made smaller than the one generated from Lyapunov ISS.

4.5 Summary

In this chapter, we presented the robustness of nonlinear vibrational control systems. First of all, the local vibrational stabilization was extended to local robustness by considering the existence of additive disturbances. By applying the Lyapunov sample-data

approach, the averaged trajectories of nonlinear vibrational control systems are shown to have locally practically ISS properties for constrained disturbances. When capturing non-local robustness, we considered a weak stability condition that the averaged system without disturbances is asymptotically stable in a domain of attraction that can be either local or non-local. Disturbances are treated as the perturbations to the stabilized system so the L -infinity norm is supposed to be bounded by a small value to keep the system stable. The allowed disturbances bound is decided by the domain of attraction of the averaged system, the tuning parameter as well as the initial set. The ultimate bound of the trajectories is determined by the L -infinity norm of disturbances accordingly. The state-trajectories are shown to converge to the ultimate bound when the tuning parameter is sufficiently small. When the bounded disturbances are also periodic, the ultimate bound estimation can be less conservative. Using the periodicity of disturbances, the ultimate bound is relied on the average of disturbances while other parts can be largely reduced with high frequencies of disturbances. Lastly, the weak averaging technique is used to show that if the weak averaged system exists and is Lyapunov-ISS, the nonlinear vibrational control system can handle arbitrarily large disturbances. Serving as an illustrative example, a 2R planar manipulator steered by vibrational control is analysed in the consideration of different types of disturbances. Numeric simulations are consistent with theoretic analysis.

Chapter 5

Performance of Switched Vibrational Control System

5.1 Overview

The power of vibrational control systems is the extra design freedom for stabilization coming from high-frequency dither signals, but it would lead to high energy consumption of actuators and cause potential damages to actuators. One possible solution is switching off high-frequency dither signals for some time when it is not necessary. As indicated in previous chapters, the original dynamics of vibrational control systems is usually unstable such that the high-frequency dither cannot be switched-off all the time. Hence an appropriate switching law is needed in order to ensure the stability of the switched vibrational control systems.

Different switching schemes have been proposed in literature such as state-dependent switching v.s. time-dependent switching or arbitrarily switching v.s. slow switching [3]. For the vibrational control, which is regarded as an open-loop control method, there lack of sensors for state measurement so time-dependent switching is the preferred way to introduce the switching signals. A switching signal is thus needed to be designed off-line to activate the subsystems according to predefined time sequences.

The autonomous system (2.19) to be stabilized is unstable so the trajectories will be divergent from the desired equilibrium point x_e when the oscillating dither is switched off. The existence of the unstable subsystem decides that the arbitrarily switching scheme is not possible because when the switching signal stays in the index of the unstable subsystem there is no chance for the overall switched system to become stable, therefore we will use the slow switching methods. For linear vibrational control systems,

a switching signal with average dwell time is introduced, which restricts the number of switching for a given time interval. For the contraction of trajectories only happens in the domain of attraction in nonlinear vibrational control systems, a periodic switching law is then designed to regulate the sequence of stable subsystem and unstable subsystem specifically to avoid the trajectories jumping out of the domain of attraction.

In the stability or robustness analysis of switched systems, Lyapunov method serves as a key technique to show the convergence of solutions to an equilibrium point. Common Lyapunov Function method assumes that all the subsystems have a common Lyapunov function [84], whose derivative along all the subsystems is negative, then the switched system can be shown to be global uniformly asymptotically stable. Relaxed from the Common Lyapunov Function method, Multiple Lyapunov Function method allows the subsystems to have independent Lyapunov functions but requires that each Lyapunov function after switching back to a given subsystem decays [85]. For vibrational control systems however, the behaviours of the subsystems vary largely. The original dynamics have an unstable equilibrium point while after adding the oscillation signals the vibrational control system has an asymptotically stable limit cycle. Besides, the stability analysis methods for different subsystems are also very different. For example, in order to show the stability of vibrational control systems we need to use a coordinate change and apply the averaging technique, which is very complicated. Under this scenario, showing the stability of the overall switched vibrational control systems by constructing common/multiple Lyapunov functions directly is more than difficult. Thus the Lyapunov-based method is not suitable for the stability or robustness analysis for switched vibrational control systems.

Alternatively, we seek the trajectories-based method to show the stability after introducing the time-dependent switching signal. For the switched linear vibrational control systems, Theorem 5.6 shows that the trajectories are decaying exponentially to the equilibrium point if the average dwell time is sufficiently long and the ratio of stable and unstable duration is sufficiently large. For the periodically switched nonlinear system, Theorem 5.7 shows that the trajectories are decreasing at the end of each switching period. Geometric series are constructed to capture the decaying of trajectories at the sampling instances. In the last part of this chapter, the influence of switching signal to the robustness of vibrational control systems will be discussed. Although the trajectories can converge to the neighbourhood of the equilibrium point, it shows that the ultimate bound is amplified after using switching signal, which is correlated to the duration of the unstable subsystem. To have a smaller ultimate bound, the duration of the unstable subsystem needs to be reduced. Simulation results from the inverted pendulum and planar robotic manipulator with switching signals verify the theoretic analysis.

The chapter is organized as follows. In Section 5.2, the preliminaries on switched systems are stated. The stability of switched vibrational control systems are discussed in Section 5.3 for both linear and nonlinear systems, followed by the robustness analysis of the switched system in Section 5.4. Section 5.5 summarizes the chapter.

5.2 Preliminaries

5.2.1 Switched Systems

Suppose there are a family of functions $f_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $p \in \mathcal{P}$ and \mathcal{P} is an index set, for example $\mathcal{P} = \{1, 2, 3, 4, \dots, n\}$. A system can be composed of the family of functions f_p [3]:

$$\dot{x} = f_p(x), x(t_0) = x_0, p \in \mathcal{P}. \quad (5.1)$$

To define the switching system, a switching signal needs to be introduced to describe the function sequences, which is denoted as a piecewise constant function $\sigma : [t_0, \infty) \rightarrow \mathcal{P}$. The number of discontinuities of switching function σ indicates the switching times on a compact time interval. The switching signal σ regulates the index $\sigma(t) \in \mathcal{P}$ of the active subsystems at each time instance t which means the active function in system (5.1) that determines the dynamics. The switching signal is assumed to be continuous from the right: $\sigma(t_i) = \lim_{\tau \rightarrow t_i^+} \sigma(\tau)$ where t_i is a switching time. For example, a switching signal between three subsystems is shown in Figure 5.1, in this case $\mathcal{P} = \{1, 2, 3\}$.

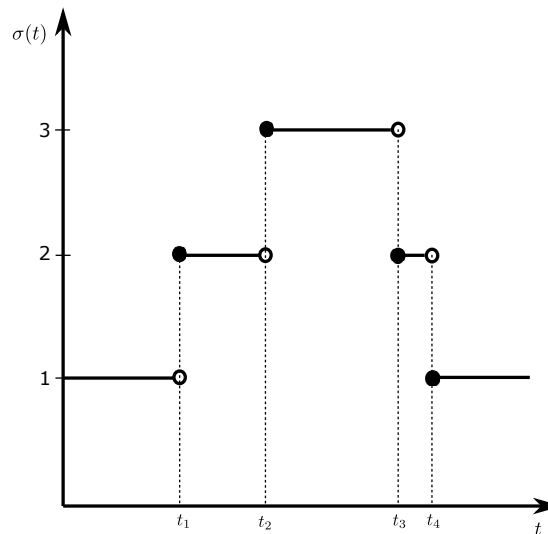


FIGURE 5.1: An example of time-dependent switching signal [3].

Based on the defined switching signal, a switched system can be characterized by the equation:

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), x(t_0) = x_0. \quad (5.2)$$

Specifically, when all subsystems are linear, the switched linear system is written as

$$\dot{x}(t) = A_{\sigma(t)}x(t), x(t_0) = x_0. \quad (5.3)$$

The solution of switched linear systems under the switching signal in Figure 5.1 can be expressed as

$$x(t) = e^{A_1(t-t_4)}e^{A_2(t_4-t_3)}e^{A_3(t_3-t_2)}e^{A_2(t_2-t_1)}e^{A_1(t_1-t_0)}x_0, \forall t > t_4. \quad (5.4)$$

5.2.2 Stability under Arbitrarily Switching with Common Lyapunov Function

Given the family of functions (5.1), to guarantee the asymptotic stability for every switching signal, all subsystems is supposed to be asymptotically stable otherwise the switched system would be unstable if the switching signal stays on the unstable subsystem. So for stability under arbitrarily switching, the asymptotic stability for each subsystem is a necessary condition presumed [3].

Suppose origin is a common equilibrium point for all subsystems i.e. $f_p(0) = 0$ for all $p \in \mathcal{P}$. The stability definitions for switched systems (5.2) are adapted from stability definition of non-autonomous systems (Definition 2.5) and given below:

Definition 5.1. [3] The equilibrium point of switched system (5.2) is called

- uniformly asymptotically stable if there exists a positive constant δ and a class \mathcal{KL} function β such that for all switching signals σ the solutions of (5.2) with $|x(0)| \leq \delta$ satisfy the inequality

$$|x(t)| \leq \beta(|x_0|, t - t_0), \forall t \geq t_0. \quad (5.5)$$

- uniformly exponentially stable if the function β takes the form $\beta(r, s) = cre^{-\lambda s}$ for some $c, \lambda > 0$ such that the solutions satisfy

$$|x(t)| \leq c|x_0|e^{-\lambda(t-t_0)}, \forall t \geq t_0. \quad (5.6)$$

- Globally uniformly asymptotically stable (GUAS) and globally uniformly exponentially stable (GUES) if (5.5) and (5.6) are satisfied for all switching signals and all initial conditions $x_0 \in R^n$.

Uniform stability of the switched system (5.2) can be obtained by assuming the existence of a Lyapunov function whose derivative along solutions of all subsystems satisfy certain conditions. The definition of common Lyapunov function is defined first:

Definition 5.2. [3] Given a positive definite continuously differentiable function $V : R^n \rightarrow R$, it is called a common Lyapunov function for the family of systems (5.1) if there exists a positive definite continuous function $W : R^n \rightarrow R_{\geq 0}$ such that the following inequality satisfies:

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x), \forall x, \forall p \in \mathcal{P}. \quad (5.7)$$

The next Theorem concludes the GUAS through the existence of the common Lyapunov function:

Theorem 5.1. [3] *If all systems in the family (5.1) share a radially unbounded common Lyapunov function satisfying (5.7), then the switched system (5.2) is GUAS.*

Remark 5.1. The upper bound by the $-W(x)$ is necessary to guarantee the switched systems (5.2) have sufficient decreasing rate which is independent of switching signals. A counter example is given in [3] to show that $\frac{\partial V}{\partial x} f_p(x) < 0$ is not enough to conclude the asymptotic stability uniformly in the switching signals. A converse theorem exists showing that when the switched system (5.2) is GUAS, the family of systems (5.1) shares a common Lyapunov function. Although a common Lyapunov function is very beneficial to show the stability of the switched system, it is a strong assumption that is not always satisfied in reality. In next section, the stability analysis method with the existence of a multiple Lyapunov function will be introduced.

5.2.3 Stability under Arbitrarily Switching with Multiple Lyapunov Functions

Multiple Lyapunov Functions method can be an alternative tool to analyse the stability of the switched system in cases a common Lyapunov function does not exist. Suppose that all the subsystems in the family of systems (5.1) are globally asymptotically stable, from the Converse Lyapunov Theorem there exists positive definite Lyapunov function $V_p, p \in \mathcal{P}$ for each system satisfying $\frac{\partial V_p}{\partial x} f_p(x) \leq -W_p(x)$ where $W_p(x)$ is a positive definite continuous function. It is clear that the Lyapunov function will decrease along

the solutions of the corresponding subsystem but the values of Lyapunov functions may have experience an increase when switching to some subsystem with a different Lyapunov function. Next theorem indicates that the switched system is asymptotically stable if a switching law makes values of a Lyapunov function V_p decrease after switching back to the same subsystem.

Theorem 5.2. [78] *Let (5.1) be a family of globally asymptotically stable systems and $V_p, p \in \mathcal{P}$ be a family of corresponding radially unbounded Lyapunov functions. Suppose that there exists a family of positive definite continuous functions $W_p, p \in \mathcal{P}$ with the property that for every pair of two consecutive switching times (t_i, t_j) , $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{P}$ and $\sigma(t_k) \neq p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)). \quad (5.8)$$

Then the switched system (5.2) is globally asymptotically stable.

Remark 5.2. To apply the Theorem 5.2, values of the Lyapunov function at switching instances need to be estimated such that the condition (5.8) is satisfied. This means the solutions information are required. When there is some constraint on the switching signals such as switching frequency, it provides a useful tool for stability analysis as the upper bound of solutions can be estimated.

5.2.4 Stability under Slow Switching with Dwell Time

When all subsystems are asymptotically stable, the switched systems are stable if the switching is relatively slow. A straightforward way to introduce a slow switching signal is introducing a number τ_d that restricts the least duration between two consecutive switching instances. Assuming the switching instances are t_1, t_2, \dots, t_n , it is required that $t_{i+1} - t_i \geq \tau_d$ for all i . The least duration between switchings τ_d is called dwell time. Dwell time provides a way to show the stability of switched systems and an explicit lower bound can be derived if the subsystems are exponentially stable [86]. However, it exerts quite a strict condition for the switching system that no switching is allowed during the dwell period after a switching.

The concept of dwell time is extended to averaged dwell time later. It allows the fast switching during a certain period but reduces the switching frequency later to constrain the averaged dwell time. The definition of average dwell time is given below:

Definition 5.3. [87] Suppose the number of discontinuities of a switching signal σ on an interval (t, T) is denoted by $N_\sigma(T, t)$. The switching signal σ is called to have average

dwell time τ_a if there exist two positive number N_0 and τ_a such that

$$N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_a}, \quad \forall T \geq t \geq 0. \quad (5.9)$$

Next theorem provides an lower bound of average dwell time to guarantee the global asymptotic stability of the switched system (5.2).

Theorem 5.3. [87] Consider the family of systems (5.1). Suppose that there exist continuous functions $V_p : R^n \rightarrow [0, \infty)$, $p \in \mathcal{P}$, class \mathcal{K}_∞ functions α_1 and α_2 , and a positive number λ_0 such that we have

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|), \quad \forall x, \quad \forall p \in \mathcal{P} \quad (5.10)$$

and

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -2\lambda_0 V_p(x), \quad \forall x, \quad \forall p \in \mathcal{P}. \quad (5.11)$$

Suppose also that there exists positive number μ such that

$$V_p(x) \leq \mu V_q(x), \quad \forall x, \quad \forall p, q \in \mathcal{P}. \quad (5.12)$$

Then the switched system (5.2) is globally asymptotically stable for every switching signal σ with average dwell time

$$\tau_a > \frac{\log \mu}{2\lambda_0}. \quad (5.13)$$

Remark 5.3. Theorem 5.3 can be specified to the switching signal with dwell time. When N_0 equals 1, the definition of averaged dwell time requires that there's no switching allowed on any interval of length smaller than τ_a which satisfies the definition of dwell time. It is noted that the choice of N_0 will not affect the stability so the switched system with a switching signal that has dwell time is also stable when all the conditions in the theorem are satisfied.

5.2.5 Input-to-State Stability of Switched Systems with Averaged Dwell Time

While noises/disturbances exist in the switched systems, the input-to-state stability has been addressed in [88, 89], where all the subsystems are ISS but there does not exist a common Lyapunov function. Conditions are derived for input-to-state stability of the switched system under slow switching signal with average dwell time [88] and averaged average dwell time [89] respectively. Consider the family of nonlinear systems with disturbances:

$$\dot{x} = f_p(x, w), p \in \mathcal{P}, \quad (5.14)$$

where the state $x \in \mathbb{R}^n$ and the disturbances $w \in \mathbb{R}^l$, and \mathcal{P} is an index set. Suppose that for each $p \in \mathcal{P}$, f_p is locally Lipschitz and $f_p(0,0) = 0$. A switched system with disturbances is composed of the family of systems (5.14):

$$\dot{x} = f_\sigma(x, w), \quad (5.15)$$

where σ is the switching signal introduced in Section 5.2.1.

Definition 5.4. [90] The switched system (5.15) is input-to-state stable (ISS) if there exists a function $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for each $t_0 \geq 0$, each $x_0 \in \mathbb{R}^n$ and each input $w \in L_\infty$, the solutions of switched system (5.15) satisfy the following inequality

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|w\|_\infty), \forall t \geq t_0. \quad (5.16)$$

As an extension of Theorem 5.3, next Theorem shows that the switched system (5.15) is ISS if each subsystem is Lyapunov-ISS and the dwell time of the switching signal is sufficiently long .

Theorem 5.4. [89] Suppose that there exist continuous functions $V_p : \mathbb{R}^n \rightarrow [0, \infty)$, $p \in \mathcal{P}$ satisfying conditions (5.10) and (5.12), and there exists class \mathcal{K}_∞ functions γ and positive number λ_0, μ such that the following inequality is satisfied

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -\lambda_0 V_p(x) + \gamma(\|w\|_\infty), \quad \forall x, \quad \forall p \in \mathcal{P}. \quad (5.17)$$

Then the switched system (5.2) is ISS for every switching signal σ with average dwell time

$$\tau_a > \frac{\log \mu}{\lambda_0}. \quad (5.18)$$

The above ISS results can be extended to the cases where not all subsystems of switched system (5.2) are ISS, for example there may exist some unstable subsystems. Let \mathcal{P}_s denote the set of subsystems which are ISS and \mathcal{P}_u be the set of subsystems that are not ISS, so we have $\mathcal{P}_s \cup \mathcal{P}_u = \mathcal{P}$ and $\mathcal{P}_s \cap \mathcal{P}_u = \emptyset$. Denote $T_u(t_0, t)$ the total activation time of the systems in \mathcal{P}_u and $T_s(t_0, t)$ the total activation time of the systems in \mathcal{P}_s during the time interval $[t_0, t)$. Obviously we can have $T_u(t_0, t) + T_s(t_0, t) = t - t_0$. The conditions for the switched system (5.2) to be ISS are summarized in the following theorem:

Theorem 5.5. [90] Consider the family of systems (5.2). Suppose there exist functions $\gamma \in \mathcal{K}_\infty$ continuously differentiable functions $V_p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and constants $\lambda_s, \lambda_u > 0$, $\mu \geq 1$ such that (5.10) and (5.12) hold for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ and further more,

the following holds:

$$\begin{aligned}\frac{\partial V_p}{\partial x} f_p(x) &\leq -\lambda_s V_p(x), \quad \forall x, \quad \forall p \in \mathcal{P}_s, \\ \frac{\partial V_p}{\partial x} f_p(x) &\leq \lambda_u V_p(x), \quad \forall x, \quad \forall p \in \mathcal{P}_u,\end{aligned}\tag{5.19}$$

for all $|x| \geq \gamma(\|w\|_\infty)$. If there exist constants $\rho, T_0 \geq 0$ satisfying

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u},\tag{5.20}$$

such that for all $t \geq t_0 \geq 0$, it satisfies

$$T_u(t, t_0) \leq T_0 + \rho(t - t_0)\tag{5.21}$$

and if σ is a switching signal with average dwell time satisfying:

$$\tau_a > \frac{\ln \mu}{\lambda_s(1 - \rho) - \lambda_u \rho},\tag{5.22}$$

then the switched system (5.2) is ISS.

Remark 5.4. Theorem 5.5 shows that even if some subsystems are not stable, the switched systems (5.2) can be ISS if the activation duration of non-ISS subsystems is short and the divergent speed of the unstable subsystems is bounded. This gives some intuitive ideas for the stabilization of the switched vibrational control systems with a unstable subsystem.

5.3 Stability Analysis of Switched Vibrational Control Systems

5.3.1 Switched Vibrational Control Systems

Next, the switched vibrational control systems will be introduced and the stability of the system will be discussed. A switching signal σ has been added to the vibrational control system (2.20):

$$\dot{x} = f(x) + \frac{\sigma(t)}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right), \quad x(t_0) = x_0 \in \mathbb{R}^n,\tag{5.23}$$

where the nonlinear mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous while $g : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and T -periodic in t , locally Lipschitz in x , uniformly in t . The parameter ε is a small positive constant. Here $\sigma : [t_0, \infty) \rightarrow \{0, 1\}$ is the switching signal which

switches on and off the vibrational control input. When $\sigma = 0$, the active subsystem becomes

$$\Sigma_u : \dot{x} = f(x), x(t_0) = x_0, \quad (5.24)$$

which represents the original unstable dynamics. While $\sigma = 1$ the active subsystem becomes the vibrational control systems (2.20):

$$\Sigma_s : \dot{x} = f(x) + \frac{1}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right), x(t_0) = x_0. \quad (5.25)$$

To achieve the stability of switched vibrational control systems (5.23), the subsystem with vibrational control Σ_s is assumed to be stable.

Remark 5.5. The stability conditions have been discussed in Chapter 2 so here we assume that this subsystem has been stabilized by finding a suitable vibrational mapping $g(\cdot, \cdot)$ and parameter ε . The domain of attraction will be specified in the following switched vibrational stability analysis.

The first task to stabilize switched vibrational control systems is designing the switching signal $\sigma(t)$. As discussed in Section 5.2, the asymptotic stability is necessary for each subsystem to allow arbitrarily switching. For the existence of an unstable subsystem, arbitrarily switching is not available for the vibrational control systems. Thus the slow switching is chosen as our switching scheme. Next, the switched linear vibrational control systems are discussed first and a switching signal with average dwell time is introduced.

5.3.2 Stability of Switched Linear Vibrational Control Systems under a Switching Signal with Average Dwell Time

Switched linear vibrational control systems (5.23) have the following form:

$$\dot{x} = Ax + \frac{\sigma(t)}{\varepsilon} B_1 \left(\frac{t}{\varepsilon}\right) x, x(t_0) = x_0 \in \mathbb{R}^n, \quad (5.26)$$

where A and $B(t)$ are defined in the equation (3.7). Suppose Assumption 3.1 and 3.2 are satisfied such that when $\sigma = 1$ Theorem 3.4 holds. It indicates that there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$ the solutions of the subsystem when $\sigma = 1$ satisfy:

$$|x(t)| \leq a_1 |x_0| e^{-\lambda_1(t-t_0)}, x(t_0) = x_0 \in \mathbb{R}^n. \quad (5.27)$$

where constants $a_1 > 1$ and $\lambda_1 > 0$. When $\sigma = 0$, the activated subsystem becomes time invariant $\dot{x} = Ax$ that is unstable such that there exist constants $a_2 > 1$, $\lambda_2 > 0$

such that the solutions satisfy:

$$|x(t)| \leq a_2|x_0|e^{\lambda_2(t-t_0)}. \quad (5.28)$$

Suppose $\sigma(t)$ is a piecewise constant switching signal that has an average dwell time. Let $T_{\sigma 1}$ denotes the total activation duration for the stable subsystem Σ_s and $T_{\sigma 0}$ be the duration for unstable subsystem Σ_u . Similar to [91], the following assumption is needed for the switching signal.

Assumption 5.1. Let λ_1, λ_2 are two positive constants. For a given $\lambda^* \in (0, \lambda_1)$, suppose the switching signal σ satisfies

$$\frac{T_{\sigma 1}}{T_{\sigma 0}} \geq \frac{\lambda_2 + \lambda^*}{\lambda_1 - \lambda^*}, \quad (5.29)$$

on any interval $[t_0, T]$.

According to the definition of switching signal with an average dwell time (Definition 5.3), the discontinuities $N_\sigma(T, t)$ of switching on an interval (t, T) satisfies

$$N_\sigma(T, t) \leq N_0 + \frac{T-t}{\tau_a}, \quad \forall T \geq t \geq 0.$$

Let $N_{\sigma 1}$ and $N_{\sigma 0}$ denote the numbers of modes $\sigma = 1$ and $\sigma = 0$ respectively. The solutions of switched linear vibrational control systems (5.26) can be bounded by

$$|x(t)| \leq b_{i+1}e^{\iota_{i+1}(t-t_i)}b_i e^{\iota_i(t_i-t_{i-1})} \dots b_1 e^{\iota_1(t_1-t_0)}|x_0|. \quad (5.30)$$

where t_i is the switching instances, $b_i \in \{a_1, a_2\}$, $\iota_i \in \{-\lambda_1, \lambda_2\}$. It can be further written as

$$|x(t)| \leq a_1^{N_{\sigma 1}} a_2^{N_{\sigma 0}} e^{-\lambda_1 T_{\sigma 1} + \lambda_2 T_{\sigma 0}} |x_0| \leq a^{N_{\sigma 1} + N_{\sigma 0}} e^{-\lambda_1 T_{\sigma 1} + \lambda_2 T_{\sigma 0}} |x_0| \quad (5.31)$$

where $a = \max\{a_1, a_2\}$ and it is easily verified that $N_{\sigma 1} + N_{\sigma 2} \leq N_\sigma(t, t_0)$. By using the Assumption 5.1, we obtain that $-\lambda_1 T_{\sigma 1} + \lambda_2 T_{\sigma 0} \leq -\lambda^*(t - t_0)$,

$$\begin{aligned} |x(t)| &\leq a^{N_\sigma(t, t_0)} e^{-\lambda^*(t-t_0)} |x_0| \leq a^{N_0 + (t-t_0)/\tau_a} e^{-\lambda^*(t-t_0)} |x_0| \\ &\leq a^{N_0} e^{(\ln a / \tau_a - \lambda^*)(t-t_0)}. \end{aligned} \quad (5.32)$$

Choosing a $\bar{\lambda} \in (0, \lambda^*)$, whenever $\tau_a \geq \ln a / (\lambda^* - \bar{\lambda})$ we have

$$|x(t)| \leq c e^{-\bar{\lambda}(t-t_0)} |x_0|, x(t_0) = x_0 \in \mathbb{R}^n, \forall t \geq t_0, \quad (5.33)$$

where $c = a^{N_0}$. Therefore, the switched system is globally exponentially stable. It can be summarised in the following theorem:

Theorem 5.6. *Suppose when activated, linear vibrational control systems satisfy Assumption 3.1 and 3.2 and the switching signal $\sigma(t)$ satisfies Assumption 5.1. There exists τ_a^* such that the system is globally exponentially stable for any average dwell time $\tau_a \geq \tau_a^*$ and any chatter bound N_0 .*

Remark 5.6. The exponentially decaying rate is $\bar{\lambda}$, which is upper bounded by the λ^* . Thus one way to increase the convergence speed is to enlarge λ^* to have more space for increasing $\bar{\lambda}$. From Assumption 5.1, the increase of λ^* means the ratio between stable duration and unstable duration becomes larger. It means the switched system stays longer in the stable subsystem for a given time interval so the trajectories converge faster.

5.3.3 Stability of Switched Nonlinear Vibrational Control Systems under a Periodic Switching Signal

Designing the switching law becomes more challenging for nonlinear vibrational systems compared to the linear systems. The stability of the subsystem Σ_s is valid only within the domain of attraction so the slow switching scheme with average dwell time used in the switched linear vibrational control systems is not going to work any more. For example, if the subsystem is active initially from unstable system, the trajectories could escape from the domain of attraction, failing to hold the stability. Hence avoiding the trajectories escaping is a necessary requirement in the design of switching law. In this work, we will use a periodic switching signal to regulate the activating sequences of the stable and unstable subsystems.

Verifying the stability of the nonlinear vibrational control systems with the designed switching law is also difficult. In nonlinear systems due to the special concept of vibrational stability, the trajectories of the stable subsystem converge to a limit cycle while their average converges to the equilibrium point. So the stability analysis with such a special subsystem is more complicated. To the best of our knowledge, the stability analysis of the switched system when limit cycles exist in some subsystems has not been addressed. In our work, the stability analysis will be based on the behaviour estimation of the averaged trajectories to show that vibrational stability would be preserved after switching.

The local stability of vibrational systems is assumed such that when $\sigma = 1$ the averaged trajectories converge exponentially if they are in the domain of attraction $D \in R^n$.

There exists $a_1 \geq 1, \lambda_1 > 0$ such that the averaged solutions satisfy

$$|x_{av}(t)| \leq a_1 e^{-\lambda_1(t-t_0)} |x_0| + \delta, \quad (5.34)$$

where $\delta > 0$ is the pre-defined ultimate bound. Here the equilibrium point is assumed to be origin without losing generality. On the other hand, when $\sigma = 0$ the subsystem becomes

$$\dot{x} = f(x), x(t_0) = x_0 \in R^n. \quad (5.35)$$

Actually there's no stability prerequisite for the above original system, however we can assume that for a given compact set $D_{x_0} \in R^n$ the function $f(x)$ is locally Lipschitz. The solutions of the original subsystem are

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t f(x(s)) ds \\ &= x(t_0) + \int_{t_0}^t (f(x(s)) - f(0)) ds. \end{aligned} \quad (5.36)$$

By using the Lipschitz condition,

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^t \lambda_2 |x(s)| ds. \quad (5.37)$$

Applying the Gronwall-Bellman inequality [77, Lemma A.1] to $|x(t)|$, the upper bound of the solutions can be obtained

$$|x(t)| \leq a_2 |x_0| e^{\lambda_2(t-t_0)}, \quad (5.38)$$

where $a_2 \geq 1$ and λ_2 is the Lipschitz constant in D_{x_0} . It is worthwhile to note that the upper bound of unstable subsystem exists exclusively for $x \in D_{x_0}$. Outside of this local region we cannot find a constant λ_2 to bound the solutions exponentially. We can guarantee the trajectories stay inside such a compact set by properly designing the switching signal such that the inequality (5.39) holds. Consequently, the averaged trajectories satisfy:

$$|x_{av}(t)| \leq a_2 |x_0| e^{\lambda_2(t-t_0)}. \quad (5.39)$$

Next we will use a periodic switching law which pre-defined the switching sequences and then achieve the stability of the switched system by designing the suitable parameters to attain the desired trajectories behaviour. A periodic switching signal is introduced as follows

$$\sigma(t) = \begin{cases} 1, & t \in [k(T_s + T_u), k(T_s + T_u) + T_s); k \in \mathbb{N} \\ 0, & t \in [k(T_s + T_u) + T_s, (k+1)(T_s + T_u)); k \in \mathbb{N} \end{cases} \quad (5.40)$$

where T_s , T_u are the durations when $\sigma = 1$ and $\sigma = 0$ respectively. The period for the switching signal is $T_s + T_u$ as a result.

To achieve the stability of switched system (5.23) under the periodic switching law, intuitively the duration of the stable system should be sufficiently long compared to the duration for the unstable system so the trajectories converge toward the origin. The requirement for stable and unstable duration is discussed in details next.

Selecting $\lambda^* \in (0, \lambda_1)$, then there exists T_s^* s.t. for all $T_s > T_s^*$, the following inequality holds:

$$(\lambda_1 - \lambda^*)T_s - \ln a_1 - \ln a_2 > 0, \quad (5.41)$$

and $T_s^* = \frac{\ln a_1 + \ln a_2}{\lambda_1 - \lambda^*}$. Hence there exists T_u^* such that for all $T_u \in (0, T_u^*)$, we have:

$$\ln a_1 + \ln a_2 + (\lambda^* - \lambda_1)T_s + (\lambda_2 + \lambda^*)T_u \leq 0. \quad (5.42)$$

This leads to an upper bound of T_u :

$$T_u^* = \frac{(\lambda_1 - \lambda^*)T_s - \ln a_1 - \ln a_2}{\lambda_2 + \lambda^*}.$$

Based on the above context, the stability of switched nonlinear vibrational control systems is summarised in the following theorem:

Theorem 5.7. *Suppose all conditions in Theorem 4.1 are satisfied such that solutions of the stable subsystem (5.25) satisfy (5.34), for all $x_0 \in D_{x_0} \in R^n$. Consider $f(x)$ is locally continuously Lipschitz for all $x \in D_{x_0}$. Assume the switching signal $\sigma(t)$ is periodic, satisfying (5.42). Selecting $\lambda^* \in (0, \lambda_1)$, there exists T_s^* and for any $T_s \geq T_s^*$ there exists $T_u^*(T_s)$ such that for all $T_u \in (0, T_u^*]$ the solutions of the switched systems (5.23) satisfy:*

$$|x_{av}(t) - x_e| \leq a_1 e^{-\lambda^* t} |x_0 - x_e| + a_1 a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - e^{-\lambda^*(T_s + T_u)}} + \delta, \forall t \geq 0. \quad (5.43)$$

where $x_{av}(t) = \frac{1}{T} \int_t^{t+T} x(s) ds$.

Proof. See Proof of Theorem 5.7 in Appendix A.4. □

Remark 5.7. The estimated bound of averaged trajectory is composed of the exponential decaying transient response and the ultimate bound. The transient response is related to the initial condition whose decaying rate is λ^* , which is bounded by λ_1 . To have a larger decaying rate λ^* , (5.41) indicates the unstable duration T_s should be tuned larger and the unstable duration T_u needs to be reduced as shown in (5.42). The ‘new’ ultimate bound is relevant with the unstable duration T_u , stable duration T_s and the previous

ultimate bound of the stable subsystem δ . Thus in order to have a small ultimate bound, the unstable duration T_u needs to be small enough while the duration of stable mode is sufficiently large.

Remark 5.8. Several aspects are different in switched nonlinear vibrational systems. Firstly, the slow switching law becomes periodic other than switching with average dwell time because the sequence of stable and unstable mode has to be fixed to avoid that the trajectories escaping the domain of attraction. Besides, instead of the original trajectories, only the averaged trajectories can be shown to converge to the equilibrium point. Moreover, in the linear case, it is only required that the ratio of stable and unstable duration is larger than some value, however in nonlinear switched system the stable duration has a lower bound while the unstable duration has an upper bound for one switching period. The last difference is that the averaged trajectories of nonlinear switched system only converge to an neighbourhood of the equilibrium point due to the existence of δ in the ultimate bound while in linear system the trajectories converge to the equilibrium point.

Remark 5.9. The vibrational stability is preserved after switching because δ and the switching duration T_u, T_s can be tuned to have an arbitrarily small ultimate bound, satisfying the definition of vibrational stability.

After showing the convergence of the averaged trajectories, next we will discuss the convergence of the real trajectories of the switched nonlinear vibrational systems (5.23).

The closeness between the trajectories $x_s(t)$ of the stable subsystem Σ_s and the desired equilibrium point can be expressed with

$$x_s(t) - x_e = h(t, y) - x_e = h(t, y) - h(t, z_e) + h(t, z_e) - x_e. \quad (5.44)$$

According to the stability analysis in Chapter 2, it can be bounded by

$$\begin{aligned} |x_s(t) - x_e| &\leq |h(t, y) - h(t, z_e)| + |h(t, z_e) - x_e| \\ &\leq a_1 e^{-\lambda_1 t} |x_0 - x_e| + \delta + |h(t, z_e) - x_e|. \end{aligned} \quad (5.45)$$

Let $H(t) = h(t, z_e) - x_e$, then it is periodic and zero mean as

$$\frac{1}{T} \int_0^T H(t) dt = \frac{1}{T} \int_0^T h(t, z_e) dt - x_e = 0 \quad (5.46)$$

based on the fact that $x_e = \frac{1}{T} \int_0^T h(t, z_e) dt$.

Let the maximum value of $H(t)$ be $h_{max} = \max_{t \geq t_0} H(t)$, so the upper bound of the trajectories is

$$|x_s(t)| \leq a_1 e^{-\lambda_1 t} |x_0| + \delta + h_{max}, \quad (5.47)$$

where the equilibrium point x_e is set as origin without losing generality. It has the same form of the inequality (5.34). The upper bound of the unstable subsystem is unchanged with inequality (5.39). Therefore, the upper bound of the trajectories $x(t)$ of the switched vibrational systems (5.23) can be obtained by modifying Theorem 5.7.

Corollary 5.1. Suppose the conditions in Theorem 4.1 are satisfied such that the solutions of the stable subsystem Σ_s (5.25) satisfy (5.45), for all $x_0 \in D_{x_0}$. Then the solutions of switched systems (5.23) satisfy:

$$|x(t)| \leq a_1 e^{-\lambda^* t} |x_0| + a_1 a_2 (\delta + h_{max}) e^{\lambda_2 T_u} \frac{1}{1 - e^{-\lambda^* (T_s + T_u)}} + \delta + h_{max}, \forall t \geq 0, \quad (5.48)$$

where T_u , T_s and λ^* are defined in (5.43). Specially when $x_e = h(t, z_e)$, the solutions $x(t)$ satisfy:

$$|x(t)| \leq a_1 e^{-\lambda^* t} |x_0| + a_1 a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - e^{-\lambda^* (T_s + T_u)}} + \delta, \forall t \geq 0. \quad (5.49)$$

Corollary 5.1 shows that the trajectories have a larger ultimate bound other than the ultimate bound of the averaged trajectories in (5.43) because they converge to the limit cycle that is bounded by h_{max} . Considering that the trajectories of the stable subsystem can converge arbitrarily close to the limit cycle, next we will introduce a special periodic switching signal to reduce the ultimate bound.

Since $H(t)$ is a continuously T -periodic function with zero mean, there exists t_0^* such that $|H(t_0^*)| = h_{min} = \inf_{t \geq t_0} |H(t)|$. Design the following switching periodic signal, where for some $m \in \mathbb{N}$, the stable duration is mT and unstable one is T :

$$\sigma(t) = \begin{cases} 1, & t \in [t_0^* + k(m+1)T, t_0^* + k(m+1)T + mT); k \in \mathbb{N} \\ 0, & t \in [t_0^* + k(m+1)T + mT, t_0^* + (k+1)(m+1)T); k \in \mathbb{N} \end{cases} \quad (5.50)$$

Similarly, there exists m^* such that for all $m > m^*$, the following inequality holds:

$$\ln a_1 + \ln a_2 + (\lambda^* - \lambda_1)mT + (\lambda_2 + \lambda^*)T \leq 0. \quad (5.51)$$

As (5.45) shows, the trajectories of stable subsystem can be bounded by

$$|x_s(t)| \leq a_1 e^{-\lambda_1(t-t_0)} |x_0| + \delta + |H(t)|. \quad (5.52)$$

Then the upper bound of sampled trajectories at following sampling point can be bounded by

$$|x_s(t_0^* + nT)| \leq a_1 e^{-\lambda_1(nT)} |x_0| + \delta + h_{min}. \quad (5.53)$$

Following the proof in Theorem 5.7, next Corollary can be obtained:

Corollary 5.2. Suppose the conditions in Theorem 4.1 are satisfied such that the solutions of the stable subsystem Σ_s (5.25) are bounded by (5.52), for all $x_0 \in D_{x_0}$. There exists t_0^*, m^* such that for all $m > m^*$ when the switching signal (5.50) is applied to the system, the solutions of switched systems (5.23) satisfy:

$$|x(t)| \leq a_1 e^{-\lambda^*(t-t_0^*)} |x_0| + a_1 a_2 (\delta + h_{min}) e^{\lambda_2 T} \frac{1}{1 - e^{-\lambda^*(m+1)T}} + \delta + h_{max}, \forall t \geq t_0^*, \quad (5.54)$$

where T is the inherent frequency of the dither signals.

Remark 5.10. The upper bound (5.52) of trajectories of stable subsystem is periodic as $|H(t)|$ is a periodic function so it is varying within the switching period. The switching law (5.50) activates the unstable subsystem when the bound of trajectories reaches the smallest value. Consequently, the first term in ultimate bound can be reduced from h_{max} to h_{min} . This is extremely useful for a large difference between h_{max} and h_{min} . Specially, when h_{min} is close to zero, the trajectories of switched systems can converge very close to the limit cycle.

5.4 Robustness of Switched Nonlinear Vibrational Control Systems

5.4.1 Switched Vibrational Control Systems with Additive Disturbances

Next, the existence of additive disturbances is considered in the switched system. The switched vibrational control system with additive disturbances are expressed as

$$\dot{x} = f(x) + \frac{\sigma(t)}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right) + d(w), x(t_0) = x_0 \in \mathbb{R}^n, \quad (5.55)$$

Two subsystems are denoted as $\Sigma_{s,w}$ and $\Sigma_{u,w}$ separately:

$$\Sigma_{s,w} : \dot{x} = f(x) + \frac{1}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right) + d(w), x(t_0) = x_0 \in \mathbb{R}^n \quad (5.56)$$

$$\Sigma_{u,w} : \dot{x} = f(x) + d(w), x(t_0) = x_0 \in \mathbb{R}^n \quad (5.57)$$

From Theorem 4.2, if the averaged system (2.26) of the stable subsystem (5.56) is asymptotically stable in a domain of attraction $D \in \mathbb{R}^n$, the averaged trajectories satisfy

$$|x_{av}(t) - x_e| \leq \max\{\hat{\beta}_1(|x_{av}(t_0) - x_e|, t - t_0), \hat{\gamma}_1(\|w\|_\infty)\} + \delta, \quad (5.58)$$

where $\hat{\beta}_1 \in \mathcal{KL}$ and $\hat{\gamma}_1 \in \mathcal{K}$.

After estimating the upper bound for the unstable subsystem (5.56), by properly designing the stable and unstable duration, the switched vibrational systems are robust with respect to disturbances, which is concluded in the next theorem:

Theorem 5.8. *Suppose all the assumptions and conditions of Theorem 4.2 hold such that solutions of $\Sigma_{s,w}$ -subsystem (5.56) satisfy inequality (5.58) for sufficiently small ε . For any $\delta > 0$ there exists $T_s^* > 0$ such that for a designed $T_s \geq T_s^*$ there exists T_u^* s.t. under the periodic switching law (5.40), for all $T_u < T_u^*$ solutions of the system (5.23) satisfy*

$$|x_{av}(t) - x_e| \leq \bar{\beta}(|x_0 - x_e|, t - t_0) + \bar{\gamma}(\|w\|_\infty) + \delta. \quad (5.59)$$

whenever $x_0 \in D_x$ and $\|w\|_\infty \leq \nu$, where $\bar{\beta}(\cdot, \cdot) \in \mathcal{KL}$ and $\bar{\gamma}(\cdot) \in \mathcal{K}$.

Proof: see Proof of Theorem 5.8 in Appendix A.5. □

Remark 5.11. Stability of switched systems with both stable and unstable subsystems, has been investigated in literature for time-invariant systems e.g. [90, 91] have addressed the switching stability for linear and nonlinear systems respectively. However there has little work dealing with switching between subsystems with limit cycles and subsystems with unstable equilibrium points. This is also the first time as far as we know that the stability and robustness of switched vibrational control systems are addressed.

Remark 5.12. Even though only two parameters are used in the switching law, the tuning is quite complicated. The stability or the robustness properties hold with a correct tuning sequence as stated in Theorem 5.8. First of all, $\Sigma_{s,w}$ -subsystem (5.25) should be vibrationally stable with a given domain of attraction such that the trajectories converge towards the desired equilibrium point for the first T_s seconds. When it switches to the unstable $\Sigma_{u,w}$ -subsystem (5.24), the trajectories keep going away for T_u seconds so we need to make sure that at time $T_s + T_u$, the trajectories of the system have a suitable contraction compared to the initial condition. The contraction condition we used in the tuning guidance of switched vibrational system (5.55) is $|x_{av}(T_s + T_u)| \leq p|x_{av}(0)|$ for some $p \in (0, 1)$, which guarantees the convergence of the overall trajectories.

Remark 5.13. Due to the existence of the unstable subsystem (5.24) and switching, the ultimate bound of the switched system is larger than the previous ultimate bound of stable subsystem (5.25). As shown in the proof of Theorem 5.8, the ‘new’ ultimate bound can be estimated as $\tilde{\delta} = e^{LT_u}\delta + \frac{\gamma(\|w\|_\infty)}{L}(e^{LT_u} - 1) > \delta$ as shown in (A.73).

5.4.2 Simulation Verification

The example of inverted pendulum stabilized by vibrational control (4.30) is used as an illustrative example to verify the stability and robustness of switched system. The

switching signal is added to the amplitude $a(t) = a_0\sigma(t)$ where $\sigma(t)$ is defined as (5.40).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \left(\frac{a_0\sigma(t)}{\varepsilon l} \sin\left(\frac{t}{\varepsilon}\right) - \frac{ka_0\sigma(t)}{ml} \cos\left(\frac{t}{\varepsilon}\right) \right) \sin x_1 \end{bmatrix}}_{g\left(\frac{t}{\varepsilon}, x, \varepsilon\right)} + \underbrace{\begin{bmatrix} 0 \\ -\frac{w(t)}{ml} \end{bmatrix}}_{d(w(t))}. \quad (5.60)$$

The switching period $T_s + T_u$ is set as 1 second. According to the Theorem 5.8, the system is stable if the duration of unstable mode (5.24) is sufficient small compared with the stable one (5.25). Figure 5.2 shows that when $T_u = 0.2s$, the switched system is stable with trajectories converging to the neighbourhood of the equilibrium point. While the duration of unstable mode increases to $0.3s$, the system becomes unstable as illustrates in Figure 5.3.

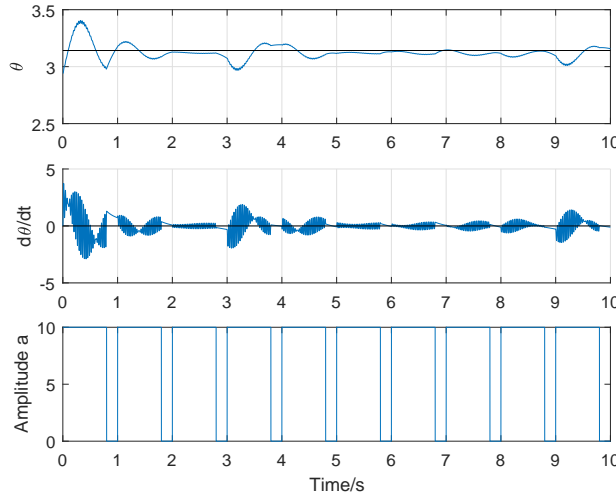


FIGURE 5.2: Switched vibrational control system is stable when $T_s + T_u = 1s$ and $T_u = 0.2s$.

The trajectories under different unstable duration are shown in Figure 5.4, where undisturbed solutions are included for comparison. It can be seen that the trajectories converge to the equilibrium point without neither switching nor disturbances. In the presence of disturbances, the trajectories converge to a neighbourhood of equilibrium point, where the amplitude of oscillations in steady states is small. While switching signal with small unstable duration ($T_u = 0.1$) is introduced to the system, the trajectories start to hover around the previous ultimate bound, with a larger oscillation amplitude. When the unstable duration is further increased ($T_u = 0.2$), the amplitude of the oscillations becomes larger. This is consistent of the comments in Remark 5.13 because it shows that the ultimate bound will be amplified by the e^{LT_u} .

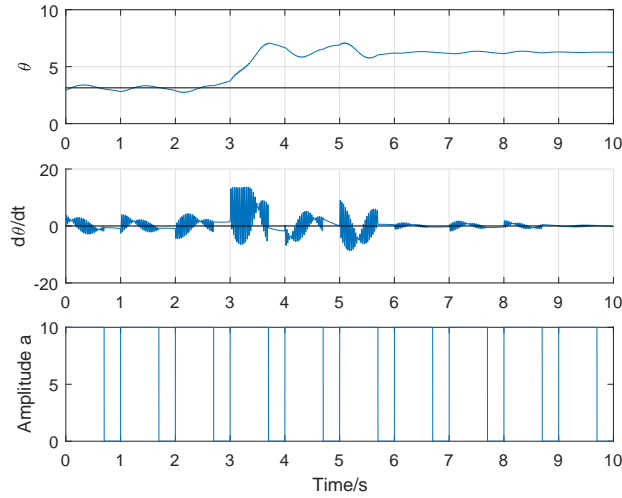
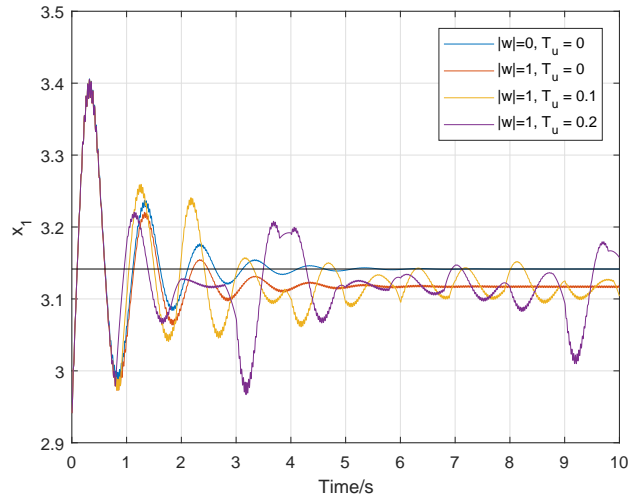
FIGURE 5.3: Switched vibrational is unstable when $T_s + T_u = 1s$ and $T_u = 0.3s$.

FIGURE 5.4: Trajectories of inverted pendulum stabilized by vibrational control with switching signal under different unstable durations.

In the example of inverted pendulum (5.60), the energy consumed by the actuator includes two parts: the first part is changing the mechanical energy of the system i.e. $\Delta E = E(t) - E(t_0)$, where $E(t) = \Lambda(t) + \Xi(t)$ and kinetic energy $\Lambda(t) = \frac{1}{2}mv^2(t)$, gravitational potential energy $\Xi(t) = mgh(t)$ and $E(t_0)$ is the initial mechanical energy. The velocity of the mass $v(t)$ and the height of the mass $h(t)$ can be related to the θ and $\dot{\theta}$ as well as the oscillation $\frac{a}{\varepsilon} \sin\left(\frac{t}{\varepsilon}\right)$. The second part of the energy is consumed by the friction during the stabilization process, which can be calculated as $W_f(t) = \int_{t_0}^t kv^2(t)dt$. Then the total energy consumed can be expressed as $W_{actuator}(t) = \Delta E(t) + W_f(t)$. Figure 5.5 shows the simulation results of energy comparison between non-switched system and switched system with $T_u = 0.2$, which is largest allowed unstable duration we

found. It can be seen that after introducing the switching signal, the energy consumed by vibrational control is reduced by almost 20% in both transient process and steady states.

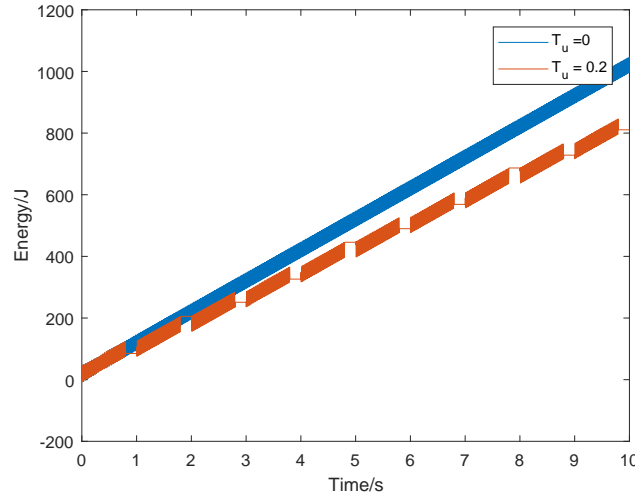


FIGURE 5.5: Comparison of energy consumption between non-switched and switched vibrational control systems.

It is shown in Theorem 5.8 that the allowed maximum unstable duration T_u^* depends on the choice of stable duration T_s , the divergence speed of the unstable system as well as the decaying rate of the stable system. Next the dynamics behaviour of 2R planar manipulator (4.44) with switching signal (5.40) which has a smaller divergence speed serves as an comparison. The switching period $T_s + T_u$ is also set as 1 second. The trajectories with different unstable duration are shown in Figure 5.6. It shows that with a smaller divergence speed, the allowed unstable duration can be largely increased from 0.2 to 0.5 second. Although the stable duration decreases from 0.8 to 0.5 second, the switched system stays stable. These simulation results are consistent with theoretic analysis.

5.5 Summary

This chapter presented the stability and robustness analysis of switched vibrational control systems. The key feature of vibrational control systems is the high-frequency oscillations. This requires high energy consumption with possible damages to actuators. To reduce the energy consumption, novel switching laws are introduced to turn-off the dither signals for a certain time when it's not necessary in the process of stabilization.

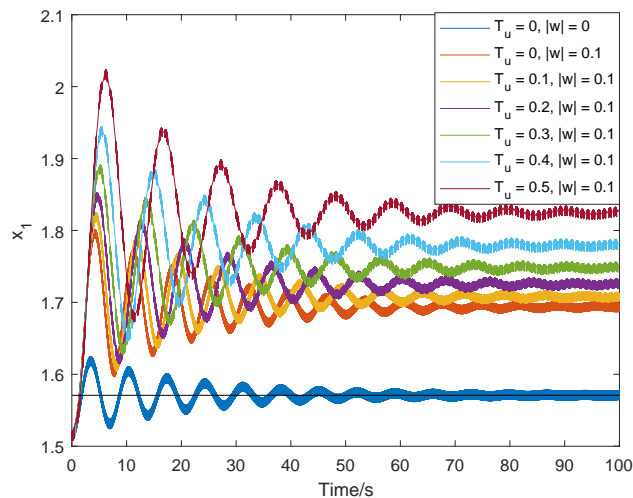


FIGURE 5.6: Trajectories of the switched vibrational planar robotic manipulator under different unstable durations.

For linear vibrational control systems, a switching law with average dwell time that limits the switching times during a certain time interval was introduced. It was shown that the switched linear vibrational control systems under this slowly varying switching signal have a globally uniformly exponentially stable equilibrium point if the ratio of stable duration and unstable duration is sufficiently large.

When vibrational control systems are nonlinear, a periodic switching signal was used to regulate activating sequences of stable and unstable subsystems. This periodic switching scheme avoids the trajectories escaping from the domain of attraction. To guarantee that the trajectories converge toward the equilibrium point, the duration of stable subsystem and unstable subsystem was carefully tuned. It shows that the averaged trajectories converge arbitrarily close to the equilibrium point which means the vibrational stabilizability is preserved after using switching. Besides, the real trajectories converges to a neighbourhood of a limit cycle. By properly choosing the switching instance, the ultimate bound to the limit cycle can be reduced. Finally, the switched vibrational control systems are shown to be robust to a class of bounded disturbances. The guidelines for reducing the ultimate bound were proposed by either reducing the ultimate bound of stable subsystem or tuning the switching duration. Simulation results from an inverted pendulum and a planar robotic manipulator illustrate the effectiveness of the estimated trajectories bounds. Besides, the energy consumed by vibrational control can be reduced after switching.

Chapter 6

Conclusion

In the first part of the thesis, we presented the stability analysis of vibrational control systems with more general definitions. The local vibrational stabilization results for non-linear systems have been extended to establish non-local vibrational stability criteria with a new definition. Next, the robustness under different types of disturbances was discussed in both linear and non-linear vibrational control systems. The perturbed system performance such as the trajectories convergence, the transient behaviour and the ultimate bound was addressed. To reduce the energy consumption by inserting high-frequency dithers, novel switching laws were introduced to turn-off the control input for a period in the stabilization process. The stability and robustness of switched vibrational control systems were discussed. Numeric simulations support the theoretic findings. In the rest of this chapter, three main contributions will be summarized below, followed by suggestions for future work.

6.1 Summary of Contributions

6.1.1 Non-local Vibrational Stabilization

In order to obtain non-local vibrational stabilization for nonlinear vibrational control systems, the concept of semi-global practical vibrational stabilizability was presented as an extension of well-known results in literature. Our derived result showed that when the averaged system is globally asymptotically stable uniformly in the parameter, the nonlinear vibrational control systems are semi-globally practically vibrationally stable, where the domain of attraction can be an arbitrarily large compact set. The obtained results can be generalized to show the robustness when disturbances exist by applying the average technique and perturbation theory.

6.1.2 Robustness Framework of Vibrational Control Systems with Bounded Disturbances

For linear vibrational control systems, when the disturbances are state-independent, perturbation techniques can be used to show that the linear vibrational control system is input-to-state stable (ISS) with respect to additive disturbances. In particular, when disturbances are periodic, a higher frequency leads to a smaller ultimate bound. When state-dependent disturbances are considered, weak averaging technique is used to show the robustness of vibrational control systems when disturbances are slowly time-varying.

For nonlinear vibrational control systems, the local vibrational stabilization was extended to local robustness by considering the existence of additive disturbances. When capturing non-local robustness, we considered a weak stability condition that suits a large class of systems. The averaged trajectories of nonlinear vibrational control systems were shown to have practically ISS properties for constrained disturbances. When the bounded disturbances are also periodic, the ultimate bound estimation can be less conservative. Lastly, the weak averaging technique is used to show that the nonlinear vibrational control system can handle arbitrarily large disturbances if the weak averaged system is Lyapunov-ISS.

6.1.3 System Performance of Switched Vibrational Control Systems

The key feature of vibrational control systems is using high-frequency dithers to provide an extra design freedom in stabilization. This requires high energy consumption with possible damages to actuators. To reduce the energy consumption, novel switching laws were introduced to turn-off the dither signals during a certain period when it's not necessary in the process of stabilization.

For linear vibrational control systems, a switching law with average dwell time that limits the switching times during a period was introduced. It was shown that the switched linear vibrational systems under this slowly varying switching signal have a globally uniformly exponentially stable equilibrium point if the ratio of stable duration and unstable duration is sufficiently large.

When vibrational control systems are nonlinear, a periodic switching signal was used to regulate activating sequences of stable and unstable subsystems. This periodic switching scheme avoids the trajectories escaping from the domain of attraction. To guarantee that the trajectories converge toward the equilibrium point, the duration of the stable subsystem and the unstable subsystem was carefully tuned. It shows that the averaged

trajectories converge arbitrarily close to the equilibrium point thus the vibrational stabilizability is preserved after using switching. Besides, the real trajectories converges to a neighbourhood of a limit cycle. By properly choosing the switching instance, the ultimate bound to the limit cycle can be reduced. Finally, the switched vibrational control systems were shown to be robust to a class of bounded disturbances. The guidelines for reducing the ultimate bound were proposed by either reducing the ultimate bound of stable subsystem or tuning the switching duration.

6.2 Suggested Future Work

6.2.1 A Systematic Procedure to Design the Vibrational Control Function

As shown in Chapter 2, the general vibrational control systems have the following additive form

$$\dot{x} = f(x) + \frac{1}{\varepsilon} g\left(\frac{t}{\varepsilon}, x\right), x(t_0) = x_0 \in \mathbb{R}^n, \quad (6.1)$$

where $g : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and T -periodic in t , locally Lipschitz in x , uniformly in t .

Vibrational control function $g(\cdot, \cdot)$ currently comes from a naturally coupling between the oscillating dithers and the system dynamics. But how to insert the dithers to have a suitable vibrational control function still relies on the practice experiences. For example, in the inverted pendulum, if the dithers are inserted in the horizon direction, it fails to stabilize the system. In the stability analysis, we have shown that the stabilization can be successful if the dithers are inserted in the vertical direction and the obtained stability criteria explain why it happens. However, the stability criteria need to check the stability of the averaged system after transformation which is not practical to give a guidance on the dither injecting in implementation. Actually inserting dithers in two directions results in two different vibrational control functions, so we expect that the successful stabilizing function would satisfy some properties. In other words, there should exist a link between the obtained stability criteria and these ‘stabilizing properties’ of the vibrational function. After transferring the stability criteria to the vibrational function, it gives more information about the successful coupling thus becomes more applicable.

The constructed Lyapunov function in the sample-data method in Chapter 4 is a possible solution to build the link. It has been shown that the Lyapunov function contains the information of the stability of the averaged system (the derivative along the averaged

system is upper bounded by a negative function), so by tracking backward through the transformation it could pass the stability criteria to the vibrational function.

6.2.2 Algorithm Implementation

We find some new features of the vibrational control algorithm in this work, for example we show that the stabilization could be semi-global if the averaged system is globally asymptotically stable. Besides, the stabilization could handle a class of disturbances and interesting frequency attenuation to the ultimate bound is found if the disturbances are periodic. Moreover, different switching schemes are introduced to the control input systems to reduce the energy consumption while keeps the systems stable. Although the numeric simulation results verify the effectiveness of these theoretic findings, more work is needed to implement the algorithm with these new features to demonstrate the effectiveness in applications.

However, designing an engineering device in application to verify the new findings obtained from this thesis can be very challenging. For example, to assure the semi-globally practically vibrational stabilizability, we need to find a mechanism such that after inserting the dithers the averaged system is globally asymptotically stable, but it would be challenging to find such a nonlinear mechanism.

Exploring the switching performance in applications is also an interesting task because different systems would have different convergence speeds so their abilities to handle switching are different. It is expected that the system with higher convergence speed could save more energy by using switching. It is also worthwhile to verify how conservative is the estimated trajectories bound obtained from theoretic derivation.

Appendix A

Proofs

A.1 Proof of Theorem 4.2

Step 1: decrease of Lyapunov functions at sampled instances

In this proof, the equilibrium position z_e is assumed to be origin without losing generality. Since the averaged system (2.26) is asymptotically stable in the region of attraction D_z , then according to converse Lyapunov theorem [77, Theorem 4.16], there is a continuously differentiable function $V : [t_0, \infty) \times D_z \rightarrow R$ that satisfies the inequalities:

$$\begin{aligned}\alpha_1(|z|) &\leq V(z) \leq \alpha_2(|z|) \\ \frac{\partial V}{\partial z} f_{y,av}(z) &\leq -\alpha_3(|z|) \\ \left| \frac{\partial V}{\partial z} \right| &\leq \alpha_4(|z|).\end{aligned}\tag{A.1}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are class \mathcal{K} functions.

The disturbed vibrational control systems in transformed coordinate (4.33) can be rewritten as

$$\frac{dy}{d\tau} = \varepsilon(f_{y,av}(y) + (f_y(\tau, y) - f_{y,av}(y)) + d_y(\tau, y, w)).\tag{A.2}$$

The derivative of Lyapunov function $V(y)$ along trajectories of system (A.2) is:

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial y} \left(f_{y,av}(y) + f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) + d_y \left(\frac{s}{\varepsilon}, y, w \right) \right) \\ &\leq -\alpha_3(|y|) + \frac{\partial V}{\partial y} \left(f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) + d_y \left(\frac{s}{\varepsilon}, y, w \right) \right)\end{aligned}\tag{A.3}$$

The upper bound of $V(t)$ can be estimated by integrating its derivative:

$$\begin{aligned} V(t) &\leq V(t_0) - \int_{t_0}^t \alpha_3(|y(s)|) ds + \int_{t_0}^t \frac{\partial V}{\partial y}(y) d_y \left(\frac{s}{\varepsilon}, y, w \right) ds \\ &\quad + \int_{t_0}^t \frac{\partial V}{\partial y}(y) \left(f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) \right) ds. \end{aligned} \quad (\text{A.4})$$

Next we will use sampling-instances method introduced in [45] to show that the Lyapunov function decreases at each sampling step $t_k = t_0 + k\varepsilon T$, where $k \in N$ and T is period of the dither signal. According to the continuity of solutions, for all initial values $y_0 \in D_z$ and disturbances $w \in L_\infty$, there is positive constants M and ε_1^* such that for all $\varepsilon \in (0, \varepsilon_1^*)$, the solutions of transformed system (A.2) satisfy:

$$|y(t) - y(t_0)| \leq M(t - t_0), \forall t \in [t_0, t_0 + \varepsilon T]. \quad (\text{A.5})$$

As D_z is a compact set in R^n , there exists ρ such that the domain $D_{z0} = \{y \in R^n \mid |y| \leq \rho\}$ is a subset of D_z . Then for all $y_0 \in D_{z0}$, there exists ε_2^* such that $|y_0| \geq 2M\varepsilon T$ for all $\varepsilon \in (0, \varepsilon_2^*)$, then for all $t \in [t_0, t_0 + \varepsilon T]$:

$$-\alpha_3(|y(t)|) \leq -\alpha_3(\max\{|y_0| - M\varepsilon T, 0\}) \leq -\alpha_3\left(\frac{1}{2}|y_0|\right) \quad (\text{A.6})$$

Then the upper bound of the integral of $-\alpha_3(|y(t)|)$ in a period is:

$$-\int_{t_0}^{t_0+\varepsilon T} \alpha_3(|y(s)|) ds \leq -\alpha_3\left(\frac{1}{2}|y_0|\right) \varepsilon T. \quad (\text{A.7})$$

Secondly, $\frac{\partial V}{\partial y}(y) d_y \left(\frac{s}{\varepsilon}, y, w \right)$ could be bounded as

$$\begin{aligned} \frac{\partial V}{\partial y}(y) d_y \left(\frac{s}{\varepsilon}, y, w \right) &\leq \left| \frac{\partial V}{\partial y}(y) d_y \left(\frac{s}{\varepsilon}, y, w \right) \right| \\ &\leq \alpha_4(|y|) \left| \left\{ \frac{\partial h}{\partial y} \right\}^{-1} \left(\frac{t}{\varepsilon}, y \right) \right| |d(w)|. \end{aligned} \quad (\text{A.8})$$

For all $t \in [t_0, t_0 + \varepsilon T]$, the trajectory $y(t)$ could be preserved in D_{z0} , then there exists ρ_0 such that $\alpha_4(|y|) \leq \rho_0$. As $\left\{ \frac{\partial h}{\partial y} \right\}^{-1} \left(\frac{t}{\varepsilon}, y \right)$ is continuously periodic in t and continuous in y , it will be bounded for all $t \in [t_0, t_0 + \varepsilon T]$ and all $y \in D_{z0}$. By letting

$$\sup_{t \geq t_0, y \in D_{z0}} \left| \left\{ \frac{\partial h}{\partial y} \right\}^{-1} \left(\frac{t}{\varepsilon}, y \right) \right| = \xi$$

and supposing the holding of Assumption 4.1, we can estimate the upper bound:

$$\int_{t_0}^{t_0+\varepsilon T} \left| \frac{\partial V}{\partial y}(y) d_y \left(\frac{s}{\varepsilon}, y, w \right) \right| ds \leq \rho_0 \xi \gamma(\|w\|_\infty) \varepsilon T. \quad (\text{A.9})$$

The residual term in (A.3) could be divided into the following parts:

$$\begin{aligned}
& \frac{\partial V}{\partial y}(y) \left(f_y \left(\frac{t}{\varepsilon}, y \right) - f_{y,av}(y) \right) \\
&= \frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y \right) - \frac{\partial V}{\partial y}(y) f_{y,av}(y) \\
&+ \frac{\partial V}{\partial y}(y_0) f_y \left(\frac{t}{\varepsilon}, y_0 \right) - \frac{\partial V}{\partial y}(y_0) f_y \left(\frac{t}{\varepsilon}, y_0 \right) + \frac{\partial V}{\partial y}(y_0) f_{y,av}(y_0) - \frac{\partial V}{\partial y}(y_0) f_{y,av}(y_0) \\
&= \left[\frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y \right) - \frac{\partial V}{\partial y}(y_0) f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right] \\
&+ \left[\frac{\partial V}{\partial y}(y_0) f_{y,av}(y_0) - \frac{\partial V}{\partial y}(y) f_{y,av}(y) \right] + \frac{\partial V}{\partial y}(y_0) \left(f_y \left(\frac{t}{\varepsilon}, y_0 \right) - f_{y,av}(y_0) \right).
\end{aligned} \tag{A.10}$$

It can be shown that the following boundary exists:

$$\begin{aligned}
& \left| \frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y \right) - \frac{\partial V}{\partial y}(y_0) f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right| \\
&\leq \left| \frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y \right) - \frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right| + \left| \frac{\partial V}{\partial y}(y) f_y \left(\frac{t}{\varepsilon}, y_0 \right) - \frac{\partial V}{\partial y}(y_0) f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right| \\
&\leq |\alpha_4(y)| \left| \left(f_y \left(\frac{t}{\varepsilon}, y \right) - f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right) \right| + \left| f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right| \left| \left(\frac{\partial V}{\partial y}(y) - \frac{\partial V}{\partial y}(y_0) \right) \right| \\
&\leq 2KL|y - y_0|
\end{aligned} \tag{A.11}$$

for all $y \in D_{z_0}$ and

$$K = \max \left\{ \sup_{y \in D_{z_0}} \{ \alpha_4(y) \}, \sup_{t \geq t_0, y_0 \in D_{z_0}} \left\{ f_y \left(\frac{t}{\varepsilon}, y_0 \right) \right\} \right\}.$$

Similarly, we have

$$\left| \frac{\partial V}{\partial y}(y) f_{y,av}(y) - \frac{\partial V}{\partial y}(y_0) f_{y,av}(y_0) \right| \leq 2KL|y - y_0|. \tag{A.12}$$

From the definition of averaging system,

$$\frac{1}{\varepsilon T} \int_{t_0}^{t_0 + \varepsilon T} \left(f_y \left(\frac{s}{\varepsilon}, y_0 \right) - f_{y,av}(y_0) \right) ds = 0. \tag{A.13}$$

Combining inequalities (A.5), (A.11) - (A.13), we can get the following upper bound:

$$\int_{t_0}^{t_0 + \varepsilon T} \frac{\partial V}{\partial y}(y) \left(f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) \right) ds \leq 4KLM(\varepsilon T)^2. \tag{A.14}$$

From (A.7), (A.9) and (A.14), the change of $V(t)$ in a period (A.4) is

$$\begin{aligned} V(t_0 + \varepsilon T) &\leq V(t_0) - \alpha_3 \left(\frac{1}{2} |y_0| \right) \varepsilon T + \rho_0 \xi \gamma(\|w\|_\infty) \varepsilon T + 4KLM(\varepsilon T)^2, \\ &\leq V(t_0) - \frac{1}{3} \alpha_3 \left(\frac{1}{2} |y_0| \right) \varepsilon T, \end{aligned} \quad (\text{A.15})$$

whenever $|y_0| \geq 2\alpha_3^{-1}(3\rho\xi\gamma(\|w\|_\infty)) + 2\alpha_3^{-1}(12KLM\varepsilon T) := \gamma_1(\|w\|_\infty) + O(\varepsilon)$.

We can see that the Lyapunov function decreases over a period εT at the initial point. In the same way we could see that it decreases in each sampled instance $t_k = t_0 + k\varepsilon T$ where $k \in N$ if $|y_k| \geq \gamma_1(\|w\|_\infty) + O(\varepsilon)$ which means the difference of Lyapunov function at two sequential sampled time instances satisfies:

$$V_{k+1} - V_k \leq -\frac{1}{3} \alpha_3(0.5|y_k|)\varepsilon T. \quad (\text{A.16})$$

Since $\alpha_1(|y_k|) \leq V_k \leq \alpha_2(|y_k|)$, then

$$V_{k+1} - V_k \leq -\frac{1}{3} \alpha_3(0.5\alpha_2^{-1}(V_k))\varepsilon T, \quad (\text{A.17})$$

whenever $V_k \geq \alpha_2 \circ \gamma_1(\|w\|_\infty) + O(\varepsilon)$.

Step 2: boundary of sampled values

Introduce a new variable $u(s) = V_k + \left(\frac{s}{\varepsilon T} - k\right)(V_{k+1} - V_k)$ where $s \in [t_k, t_{k+1}]$ and V_k satisfies (A.17). According to the definition, the variable $u(s)$ is a continuous, piecewise linear function, then it is differentiable for almost all $s \geq 0$. It is noted that $0 \leq V_{k+1} \leq u(s) \leq V_k$ for all $s \in [t_k, t_{k+1}]$, hence if $u(s) \geq \alpha_2 \circ \gamma_1(\|w\|_\infty) + O(\varepsilon)$, we have $V_k \geq \alpha_2 \circ \gamma_1(\|w\|_\infty) + O(\varepsilon)$. Combining with the inequality (A.17), the derivative of $u(s)$ satisfies:

$$\frac{du}{ds} = (V_{k+1} - V_k) \frac{1}{\varepsilon T} \leq -\frac{1}{3} \alpha_3(0.5\alpha_2^{-1}(u(s))), \quad (\text{A.18})$$

whenever $u(s) \geq \alpha_2 \circ \gamma_1(\|w\|_\infty) + O(\varepsilon)$. According to the standard comparison Lemma [77, Lemma 3.4], there exists $\beta \in \mathcal{KL}$ such that

$$|u(s)| \leq \max\{\beta(|u_0|, s), \alpha_2 \circ \gamma_1(\|w\|_\infty + O(\varepsilon))\}. \quad (\text{A.19})$$

Since $u(t_k) = V_k \geq \alpha_1(|y_k|)$, then there exists $\beta_1(u, v) = \alpha^{-1}(\beta(\alpha_2(u), v))$ such that the following inequality holds:

$$|y(t_k)| \leq \max\{\beta_1(|y_0|, k\varepsilon T), \hat{\gamma}(\|w\|_\infty) + O(\varepsilon)\} \quad (\text{A.20})$$

where $\hat{\gamma}(\|w\|_\infty) = \alpha_1^{-1} \circ \alpha_2 \circ \gamma_1(\|w\|_\infty)$.

Step 3: boundary of trajectories

As indicated in (A.5), the closeness of sampled values with trajectories could be bounded

by

$$|y(t)| \leq |y(t_k)| + O(\varepsilon), \forall t \in [t_k, t_{k+1}].$$

Combining the above inequality with (A.20), it shows that

$$|y(t)| \leq \max\{\beta_1(|y_0|, k\varepsilon T), \hat{\gamma}(\|w\|_\infty) + O(\varepsilon)\}, \forall t \geq t_0 \geq 0. \quad (\text{A.21})$$

For any given $\delta > 0$, there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$

$$|y(t)| \leq \max\{\beta_1(|y_0|, k\varepsilon T), \hat{\gamma}(\|w\|_\infty) + \delta\}, \forall t \geq t_0 \geq 0. \quad (\text{A.22})$$

As shown in [45, Lemma 5], for any $\beta \in \mathcal{KL}$, there exists $\tilde{\beta} \in \mathcal{KL}$ s.t. $\beta_1(s, k\varepsilon T) \leq \tilde{\beta}(s, (k+1)\varepsilon T)$, $\forall s, \forall k$. As $t - t_0 < (k+1)\varepsilon T$ therefore

$$|y(t)| \leq \max\{\tilde{\beta}(|y_0|, t - t_0), \hat{\gamma}(\|w\|_\infty)\} + \delta, \forall t \geq t_0 \geq 0. \quad (\text{A.23})$$

Step 4: boundary of original trajectories

The averaged trajectory is defined as $x_{av}(t) = \frac{1}{T} \int_t^{t+T} x(\tau) d\tau$, then the closeness to the equilibrium point could be found

$$\begin{aligned} |x_{av}(t) - x_e| &= \frac{1}{T} \int_t^{t+T} (h(\tau, y) - h(\tau, 0)) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} (L|y(\tau)|) d\tau \\ &\leq \frac{1}{T} \int_t^{t+T} L \left(\max\{\tilde{\beta}(|y_0|, \tau - t_0), \hat{\gamma}_2(\|w\|_\infty)\} + \delta \right) d\tau \\ &\leq \max\{\hat{\beta}_1(|x_{av}(t_0) - x_e|, t - t_0), \hat{\gamma}_1(\|w\|_\infty)\} + \delta. \end{aligned} \quad (\text{A.24})$$

where $\hat{\beta}_1 \in \mathcal{KL}$ and $\hat{\gamma}_1 \in \mathcal{K}$. This completes the proof.

Q.E.D.

A.2 Proof of Theorem 4.3

As assumed that the averaged system is asymptotically stable in a compact set D_z within region of attraction, then there exists continuously differentiable function $V : [t_0, \infty) \times D_z \rightarrow R$ that satisfies (A.1).

Next we will apply instance-sampling method as illustrated in the proof of Theorem 4.2.

By taking derivative of the Lyapunov function along the trajectories of systems (4.36)

and doing integral from an arbitrary point $y_k \in D_z$ at sampling instance $t_k = t_0 + k\varepsilon T$:

$$\begin{aligned} V(t_{k+1}) &\leq V(t_k) + \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) c\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) ds \\ &\quad - \int_{t_k}^{t_{k+1}} \alpha_3(|y(s)|) ds + \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(f_y\left(\frac{s}{\varepsilon}, y\right) - f_{y,av}(y)\right) ds. \end{aligned} \quad (\text{A.25})$$

where $c(t, y) = \left\{\frac{\partial h}{\partial y}\right\}^{-1}(t, y)$ which is continuous and periodic in t . As proved in (A.7) and (A.14), the following inequalities are satisfied:

$$- \int_{t_k}^{t_{k+1}} \alpha_3(|y(s)|) ds \leq -\alpha_3\left(\frac{1}{2}|y_k|\right) \varepsilon T. \quad (\text{A.26})$$

$$\int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(f_y\left(\frac{s}{\varepsilon}, y\right) - f_{y,av}(y)\right) ds \leq 4KLM(\varepsilon T)^2. \quad (\text{A.27})$$

The term related to disturbances in (A.25) can be rewritten by adding a new term related to its average:

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) ds = \\ &\int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds + \int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \bar{d}_w ds \end{aligned} \quad (\text{A.28})$$

where $\phi(t, y) = \frac{\partial V}{\partial y}(y) c(t, y)$ that is continuous and periodic in t and $\bar{d}_w = \frac{1}{T_w} \int_t^{t+T_w} d_w(\tau) d\tau$. The second integral in the right hand side of (A.28) could be bounded as

$$\int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \bar{d}_w ds \leq \phi_0 \bar{d}_w \varepsilon T \quad (\text{A.29})$$

where $\phi_0 = \sup_{t \in [t_0, \infty), y \in D_z} |\phi(t, y)|$.

Considering the first term, it could be bounded by taking advantage of the definition of the average \bar{d}_w .

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \\ &= \sum_{i=0}^{i=N_0} \int_{t_k+i\eta T_w}^{t_k+(i+1)\eta T_w} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \\ &\quad + \int_{t_k+(N_0+1)\eta T_w}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \end{aligned} \quad (\text{A.30})$$

where N_0 is the largest nonnegative active integer such that $(N_0 + 1)\eta T_w \leq \varepsilon T$. Define $t_{ki} = t_k + i\eta T_w$

$$\begin{aligned} & \int_{t_{ki}}^{t_{k(i+1)}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \\ &= \int_{t_{ki}}^{t_{k(i+1)}} \left[\phi\left(\frac{s}{\varepsilon}, y\right) - \phi\left(\frac{t_{ki}}{\varepsilon}, y\right)\right] \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \\ &+ \int_{t_{ki}}^{t_{k(i+1)}} \left[\phi\left(\frac{t_{ki}}{\varepsilon}, y\right) - \phi\left(\frac{t_{ki}}{\varepsilon}, y_{ki}\right)\right] \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds, \end{aligned} \quad (\text{A.31})$$

where the property that

$$\int_{t_{ki}}^{t_{k(i+1)}} \phi\left(\frac{t_{ki}}{\varepsilon}, y_{ki}\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds = 0$$

is used which comes from the definition of $\bar{d}_w = \frac{1}{T_w} \int_t^{t+T_w} d_w(\tau) d\tau$. By mean value theorem, there exists $\xi_{ki} \in [t_{ki}, t_{k(i+1)})$ such that

$$\phi\left(\frac{s}{\varepsilon}, y\right) - \phi\left(\frac{t_{ki}}{\varepsilon}, y\right) = \phi'(\xi_{ki}, y) \left(\frac{s - t_{ki}}{\varepsilon}\right). \quad (\text{A.32})$$

In the assumption the $c(t, y)$ is locally Lipschitz, $\phi(t, y)$ is locally Lipschitz, then the following property holds:

$$\left| \phi\left(\frac{t_{ki}}{\varepsilon}, y\right) - \phi\left(\frac{t_{ki}}{\varepsilon}, y_{ki}\right) \right| \leq L|y - y_{ki}|. \quad (\text{A.33})$$

Taking advantage of property (A.32) and (A.33), (A.31) is bounded as

$$\begin{aligned} & \left| \int_{t_{ki}}^{t_{k(i+1)}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) ds \right| \\ & \leq \int_{t_{ki}}^{t_{k(i+1)}} |\phi'(\xi_{ki}, y)| \left| \left(\frac{s - t_{ki}}{\varepsilon}\right) \right| \left| \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) \right| ds \\ & + \int_{t_{ki}}^{t_{k(i+1)}} L|y - y_{ki}| \left| \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) \right| ds \\ & \leq \phi'_0 \frac{\eta T_w}{\varepsilon} d_{we} \eta T_w + ML\eta T_w d_{we} \eta T_w, \end{aligned} \quad (\text{A.34})$$

where $\phi'_0 = \sup_{t \in [t_0, \infty), y \in D_z} |\phi'(t, y)|$, $d_{we} = \max_{t \in [t_0, \infty)} \left| \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w\right) \right|$.

By using the inequality of (A.34) and the fact that $|t_{k+1} - (t_k + (N_0 + 1)\eta T_w)| < \eta T_w$,

the left hand side of (A.30) could be bounded by

$$\begin{aligned} & \left| \int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) \left(d_w\left(\frac{s}{\eta}\right) - \bar{d}_w \right) ds \right| \\ & \leq \phi'_0 \frac{\eta T_w}{\varepsilon} d_{we}(N_0 + 1) \eta T_w + ML \eta T_w d_{we}(N_0 + 1) \eta T_w + \phi_0 d_{we} \eta T_w \\ & \leq \kappa_1 \frac{\eta}{\varepsilon} \varepsilon T + \kappa_2 \eta \varepsilon T + \kappa_3 \eta. \end{aligned} \quad (\text{A.35})$$

By taking the property (A.35) into (A.28) to get the boundary:

$$\left| \int_{t_k}^{t_{k+1}} \phi\left(\frac{s}{\varepsilon}, y\right) d_w\left(\frac{s}{\eta}\right) ds \right| \leq \left(\frac{\kappa_1 \eta}{\varepsilon} + \kappa_2 \eta + \phi_0 |\bar{d}_w| \right) \varepsilon T + \kappa_3 \eta. \quad (\text{A.36})$$

Take (A.36), (A.46) and (A.47) into (A.25), the change of Lyapunov function could be bounded by

$$\begin{aligned} V(t_{k+1}) & \leq V(t_k) - \alpha_3 (0.5|y_k|) \varepsilon T + 4KLM(\varepsilon T)^2 \\ & \quad + \kappa_1 \frac{\eta}{\varepsilon} \varepsilon T + \kappa_2 \eta \varepsilon T + \kappa_3 \eta + \phi_0 |\bar{d}_w| \varepsilon T \\ & \leq V(t_k) - \frac{1}{5} \alpha_3 (0.5|y_k|) \varepsilon T, \end{aligned} \quad (\text{A.37})$$

whenever $|y_k| \geq \bar{\kappa}_1 \varepsilon + \bar{\kappa}_2 \frac{\eta}{\varepsilon} + \bar{\kappa}_3 \eta + \gamma(|\bar{d}_w|)$ where $\gamma(x) = 2\alpha_3^{-1}(5\phi_0 x) \in \mathcal{K}_\infty$. Follow the same procedures in the proof of Theorem 4.2 after (A.16), there exists $\hat{\beta}_3, \hat{\beta}_4 \in \mathcal{KL}$, $\hat{\gamma}_3, \hat{\gamma}_4 \in \mathcal{K}_\infty$ such that for given $\delta > 0$ there exists ε^* and $\eta^* \ll \varepsilon^*$ such that for all $\varepsilon \in (0, \varepsilon^*)$ and $\eta \in (0, \eta^*)$, (4.38) and (4.39) hold respectively. **Q.E.D.**

A.3 Proof of Theorem 4.4

The equilibrium position z_e is assumed to be origin without losing generality. Since the weak averaged system (4.45) is Lyapunov ISS, then according to definition of Lyapunov ISS 3.2, there is a continuously differentiable function $V : [t_0, \infty) \times R^n \rightarrow R$ that satisfies the inequalities:

$$\alpha_1(|z|) \leq V(z) \leq \alpha_2(|z|), \quad (\text{A.38})$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z} (f_{y,av}(z) + d_{y,wa}(z, w)) \leq -\alpha_3(|z|), \forall |z| > \rho(\|w\|_\infty), \quad (\text{A.39})$$

where $\alpha_1, \alpha_2, \alpha_3$ and ρ are class \mathcal{K} functions.

The disturbed vibrational control systems in transformed coordinate (4.33) can be rewritten as

$$\frac{dy}{d\tau} = \varepsilon (f_{y,av}(y) + d_{y,wa}(y, w) + f_y(\tau, y) - f_{y,av}(y) + d_y(\tau, y, w) - d_{y,wa}(y, w)). \quad (\text{A.40})$$

The derivative of Lyapunov function $V(y)$ along trajectories of system (A.40) is:

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial y} \left(f_{y,av}(y) + d_{y,wa}(y, w) + f_y \left(\frac{t}{\varepsilon}, y \right) - f_{y,av}(y) + d_y \left(\frac{t}{\varepsilon}, y, w \right) - d_{y,wa}(y, w) \right) \\ &\leq -\alpha_3(|y|) + \frac{\partial V}{\partial y} \left(f_y \left(\frac{t}{\varepsilon}, y \right) - f_{y,av}(y) + d_y \left(\frac{t}{\varepsilon}, y, w \right) - d_{y,wa}(y, w) \right)\end{aligned}\quad (\text{A.41})$$

Let $k_y = \sup_{|x-x_e| \leq k_x, t \geq t_0} h^{-1}(t, x)$ be the boundary of domain of attraction mapped by transformation. For any point $y_k \in \{y \in R^n \mid |y - y_e| \leq k_y\}$ at sampling time instance $t = t_k$, the Lyapunov value at the next sampling instance can be bounded by

$$\begin{aligned}V(t_{k+1}) &\leq V(t_k) - \int_{t_k}^{t_{k+1}} \alpha_3(|y(s)|) ds + \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(d_y \left(\frac{s}{\varepsilon}, y, w \right) - d_{y,wa}(y, w) \right) ds \\ &\quad + \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) \right) ds.\end{aligned}\quad (\text{A.42})$$

The second integral containing disturbances can be further bounded by

$$\begin{aligned}&\int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(d_y \left(\frac{s}{\varepsilon}, y, w \right) - d_{y,wa}(y, w) \right) ds \\ &\leq \int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(c \left(\frac{s}{\varepsilon}, y \right) - c_{av}(y) \right) d(w(s)) ds \\ &\leq \int_{t_k}^{t_{k+1}} \left| \frac{\partial V}{\partial y}(y) \left(c \left(\frac{s}{\varepsilon}, y \right) - c_{av}(y) \right) \right| |d(w(s))| ds \\ &\leq \int_{t_k}^{t_{k+1}} \left| \frac{\partial V}{\partial y}(y) \left(c \left(\frac{s}{\varepsilon}, y \right) - c_{av}(y) \right) \right| ds \gamma(\|w\|_\infty)\end{aligned}\quad (\text{A.43})$$

where $c \left(\frac{t}{\varepsilon}, y \right) = \left\{ \frac{\partial h}{\partial y} \right\}^{-1}(\tau, y)$ that is periodic in t and continuous in y and $c_{av}(y) = \frac{1}{T} \int_{t_0}^{t_0+T} c \left(\frac{s}{\varepsilon}, y \right) ds$.

Similar to the proof of (A.14), the following inequality satisfies:

$$\int_{t_k}^{t_{k+1}} \left| \frac{\partial V}{\partial y}(y) \left(c \left(\frac{s}{\varepsilon}, y \right) - c_{av}(y) \right) \right| ds \leq 4\tilde{K}\tilde{L}\tilde{M}(\varepsilon T)^2. \quad (\text{A.44})$$

Substitute (A.44) into (A.43):

$$\int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(d_y \left(\frac{s}{\varepsilon}, y, w \right) - d_{y,wa}(y, w) \right) ds \leq 4\tilde{K}\tilde{L}\tilde{M}(\varepsilon T)^2 \gamma(k_w). \quad (\text{A.45})$$

As proved in (A.7) and (A.14), the following inequalities are satisfied:

$$- \int_{t_k}^{t_{k+1}} \alpha_3(|y(s)|) ds \leq -\alpha_3 \left(\frac{1}{2} |y_k| \right) \varepsilon T. \quad (\text{A.46})$$

$$\int_{t_k}^{t_{k+1}} \frac{\partial V}{\partial y}(y) \left(f_y \left(\frac{s}{\varepsilon}, y \right) - f_{y,av}(y) \right) ds \leq 4KLM(\varepsilon T)^2. \quad (\text{A.47})$$

Then the Lyapunov value at t_{k+1} from (A.42) can be bounded by

$$\begin{aligned} V(t_{k+1}) &\leq V(t_k) - \alpha_3 \left(\frac{1}{2}|y_k| \right) \varepsilon T + 4\tilde{K}\tilde{L}\tilde{M}(\varepsilon T)^2\gamma(k_w) + 4KLM(\varepsilon T)^2, \\ &\leq -\frac{1}{3}\alpha_3(0.5|y_k|)\varepsilon T \end{aligned} \quad (\text{A.48})$$

whenever

$$|y_k| \geq 2\alpha_3^{-1}(12\tilde{K}\tilde{L}\tilde{M}\gamma(k_w)\varepsilon T) + 2\alpha_3^{-1}(12KLM\varepsilon T) \quad (\text{A.49})$$

The ultimate bound is determined by the maximum value of (A.49) and ultimate bound of Lyapunov ISS function. Since (A.49) is related to parameter ε , when ε is tuned sufficiently small, (A.49) can be made smaller than the other candidate, then the final ultimate bound is decided by (A.38). Follow the step 2 - 4 in the Appendix A.1, Theorem 4.4 can be proved. **Q.E.D.**

A.4 Proof of Theorem 5.7

The initial values of the averaged trajectories at the end of each periods $t = k(T_s + T_u)$ and in the middle of periods $t = k(T_s + T_u) + T_s$ can be calculated:

$$\begin{aligned} |x_{av}(T_s + T_u)| &\leq a_1 a_2 e^{-\lambda_1 T_s + \lambda_2 T_u} |x_0| + \delta a_2 e^{\lambda_2 T_u}; \\ |x_{av}((T_s + T_u) + T_s)| &\leq a_1 a_2 e^{-\lambda_1 T_s + \lambda_2 T_u} |x_0| a_1 e^{-\lambda_1 T_s} + a_1 a_2 e^{-\lambda_1 T_s + \lambda_2 T_u} \delta + \delta \\ |x_{av}(2(T_s + T_u))| &\leq a_1^2 a_2^2 e^{-2(\lambda_1 T_s + \lambda_2 T_u)} |x_0| + a_1 a_2 e^{-\lambda_1 T_s + \lambda_2 T_u} \delta a_2 e^{\lambda_2 T_u} + \delta a_2 e^{\lambda_2 T_u} \\ |x_{av}(2(T_s + T_u) + T_s)| &\leq a_1^2 a_2^2 e^{-2(\lambda_1 T_s + \lambda_2 T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + \sum_{i=0}^2 a_1^i a_2^i e^{i(-\lambda_1 T_s + \lambda_2 T_u)} \delta \end{aligned} \quad (\text{A.50})$$

The upper bound of trajectory value in the middle of k -th period can be calculated by iteration:

$$\begin{aligned} |x_{av}(k(T_s + T_u) + T_s)| &\leq a_1^k a_2^k e^{k(-\lambda_1 T_s + \lambda_2 T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + \sum_{i=0}^k a_1^i a_2^i e^{i(-\lambda_1 T_s + \lambda_2 T_u)} \delta \\ &\leq e^{k(\ln a_1 + \ln a_2 - \lambda_1 T_s + \lambda_2 T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + \sum_{i=0}^k e^{i(\ln a_1 + \ln a_2 - \lambda_1 T_s + \lambda_2 T_u)} \delta. \end{aligned} \quad (\text{A.51})$$

According to (5.42), $\ln a_1 + \ln a_2 - \lambda_1 T_s + \lambda_2 T_u \leq -\lambda^*(T_s + T_u)$, then $|x_{av}(k(T_s + T_u))|$ is bounded by

$$\begin{aligned} |x_{av}(k(T_s + T_u) + T_s)| &\leq e^{-k\lambda^*(T_s+T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + \sum_{i=0}^k e^{-i\lambda^*(T_s+T_u)} \delta \\ &\leq e^{-k\lambda^*(T_s+T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + \delta \frac{1 - q^{k+1}}{1 - q}, k = 0, 1, \dots \end{aligned} \quad (\text{A.52})$$

where $q = e^{-\lambda^*(T_s+T_u)}$.

As a result, the solutions at $t = k(T_s + T_u)$ can be bounded by:

$$|x_{av}(k(T_s + T_u))| \leq e^{-k\lambda^*(T_s+T_u)} |x_0| + \delta a_2 e^{\lambda_2 T_u} \frac{1 - q^k}{1 - q}, k = 0, 1, \dots \quad (\text{A.53})$$

For $t \in [k(T_s + T_u), k(T_s + T_u) + T_s)$, the upper bound of the trajectories are

$$\begin{aligned} |x_{av}(t)| &\leq |x_{av}(k(T_s + T_u))| a_1 e^{-\lambda_1(t-k(T_s+T_u))} + \delta \\ &\leq \left(e^{-k\lambda^*(T_s+T_u)} |x_0| + \delta a_2 e^{\lambda_2 T_u} \frac{1 - q^k}{1 - q} \right) a_1 e^{-\lambda_1(t-k(T_s+T_u))} + \delta \\ &\leq e^{-k\lambda^*(T_s+T_u)} |x_0| a_1 e^{-\lambda_1(t-k(T_s+T_u))} + \delta a_1 a_2 e^{\lambda_2 T_u} \frac{1 - q^k}{1 - q} + \delta \\ &\leq a_1 e^{-\lambda^* t} |x_0| + a_1 a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q} + \delta. \end{aligned} \quad (\text{A.54})$$

For $t \in [k(T_s + T_u) + T_s, (k + 1)(T_s + T_u))$, the upper bound can be found in the similar way

$$\begin{aligned} |x_{av}(t)| &\leq a_2 e^{\lambda_2(t-k(T_s+T_u)-T_s)} e^{-k\lambda^*(T_s+T_u)} |x_0| a_1 e^{-\lambda_1 T_s} + a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q} \\ &\leq e^{-k\lambda^*(T_s+T_u)} |x_0| e^{\ln a_1 + \ln a_2 + \lambda_2(t-k(T_s+T_u)-T_s) - \lambda_1 T_s} + a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q}. \end{aligned} \quad (\text{A.55})$$

According to the the constraint of stable duration and unstable duration (5.42),

$$\ln a_1 + \ln a_2 - \lambda_1 T_s \leq -\lambda^*(T_s + T_u) - \lambda_2 T_u.$$

Also it is noted that $t - k(T_s + T_u) - T_s \leq T_u$, then

$$\begin{aligned} |x_{av}(t)| &\leq e^{-(k+1)\lambda^*(T_s+T_u)} |x_0| e^{-\lambda_2(T_u - (t-k(T_s+T_u)-T_s))} + a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q} \\ &\leq e^{-\lambda^* t} |x_0| + a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q}. \end{aligned} \quad (\text{A.56})$$

Therefore, the trajectories can be bounded by

$$|x_{av}(t)| \leq a_1 e^{-\lambda^* t} |x_0| + a_1 a_2 \delta e^{\lambda_2 T_u} \frac{1}{1 - q} + \delta, \forall t \geq 0, \quad (\text{A.57})$$

which completes the proof. **Q.E.D.**

A.5 Proof of Theorem 5.8

Since solutions of systems (5.23) satisfy inequality (4.35) while $\sigma = 1$, thus for $t \in [k(T_s + T_u), k(T_s + T_u) + T_s]$, $k \in \mathbb{N}$, when $|x_{av}(t) - x_e| \geq \hat{\gamma}_1(\|w\|_\infty)$

$$|x_{av}(t) - x_e| \leq \hat{\beta}_1(|x_0 - x_e|, t - t_0) + \delta,$$

Let $\tilde{x}_{av}(t) = x_{av}(t) - x_e$. When $t = k(T_s + T_u) + T_s$

$$|\tilde{x}_{av}(k(T_s + T_u) + T_s)| \leq \hat{\beta}_1(|x_k|, T_s) + \delta. \quad (\text{A.58})$$

When $\sigma = 0$, the active system becomes time invariant:

$$\dot{\tilde{x}} = f(\tilde{x} + x_e) + d(w), \tilde{x}(t_0) = x_0 - x_e. \quad (\text{A.59})$$

From Assumption 4.1, $|d(w)| \leq \gamma(|w|)$. Let $\gamma_w =: \sup_{\|w\|_\infty \leq v} \gamma(|w|) = \gamma(\|w\|_\infty)$. The solutions of system (A.59) can be written as

$$\begin{aligned} \tilde{x}(t) &= \tilde{x}(t_0) + \int_{t_0}^t f(\tilde{x}(s) + x_e) ds + \int_{t_0}^t d(w(s)) ds, \\ &= \tilde{x}(t_0) + \int_{t_0}^t (f(\tilde{x}(s) + x_e) - f(x_e)) ds + \int_{t_0}^t d(w(s)) ds \end{aligned} \quad (\text{A.60})$$

Let B_ρ be the largest ball inside the initial set: $B_\rho = \{x \in D_{x_0} | |x - x_e| \leq \rho + H\}$ where $H = \max_t |h(t, z_e)|$. As assumed that $f(x)$ is Lipschitz in D_{x_0} , then there exists constant L such that the following hold:

$$|f(\tilde{x}(s) + x_e) - f(x_e)| \leq L|\tilde{x}(s)|. \quad (\text{A.61})$$

The upper bound of the solutions could be estimated as follows:

$$|\tilde{x}(t)| \leq |\tilde{x}(t_0)| + \gamma_w(t - t_0) + \int_{t_0}^t L|\tilde{x}(s)| ds \quad (\text{A.62})$$

After applying Gronwall-Bellman inequality ([77, Lemma A.1]) to $|\tilde{x}(t)|$,

$$|\tilde{x}(t)| \leq |\tilde{x}_0| + \gamma_w(t - t_0) + \int_{t_0}^t L(|\tilde{x}_0| + \gamma_w)(s - t_0) e^{L(t-s)} ds.$$

Integrating the right hand side, we obtain when $\sigma = 0$

$$|\tilde{x}(t)| \leq |\tilde{x}_0|e^{L(t-t_0)} + \frac{\gamma w}{L}(e^{L(t-t_0)} - 1). \quad (\text{A.63})$$

Therefore, the averaged trajectories of unstable subsystem satisfy:

$$|\tilde{x}_{av}(t)| \leq |\tilde{x}_0|e^{L(t-t_0)} + \frac{\gamma w}{L}(e^{L(t-t_0)} - 1). \quad (\text{A.64})$$

For a point in $\tilde{B}_\rho = \{\tilde{x} \in \mathbb{R}^n \mid |\tilde{x}| \leq \rho + H\}$ where $t_k = k(T_s + T_u)$, the upper bound of the value at t_{k+1} can be estimated if $|\tilde{x}_{av}(t)| \geq \hat{\gamma}_1(\|w\|_\infty) + \delta$ for all $t \in [t_k, t_{k+1}]$:

$$|\tilde{x}_{av,k+1}| \leq \hat{\beta}_1(|\tilde{x}_{av,k}|, T_s)e^{LT_u} + \left(\frac{\gamma w}{L} + \delta\right)e^{LT_u} - \frac{\gamma w}{L}. \quad (\text{A.65})$$

Solutions of the switched system converge to the boundary $\{x_{av} \in \mathbb{R}^n \mid |x_{av} - x_e| = \hat{\gamma}_1(\|w\|_\infty) + \delta\}$ if there exists $p \in (0, 1)$ and T_u such that the following conditions are satisfied:

$$\hat{\beta}_1(|\tilde{x}_{av,k}|, T_s)e^{LT_u} \leq p|\tilde{x}_{av,k}| \quad (\text{A.66})$$

$$p|\tilde{x}_{av,k}| + \left(\frac{\gamma}{L} + \delta\right)e^{LT_u} - \frac{\gamma}{L} \leq |\tilde{x}_{av,k}| \quad (\text{A.67})$$

To guarantee the existence of T_u for (A.66) and (A.67), there exists T_s^* such that the following inequality is true for all $T_s > T_s^*$

$$\inf_{|x| \in [\hat{\gamma}_1(\|w\|_\infty) + \delta, \rho]} \frac{\hat{\beta}_1(|\tilde{x}|, T_s)}{|\tilde{x}|} < \frac{\hat{\gamma}_1(\|w\|_\infty)}{\hat{\gamma}_1(\|w\|_\infty) + \delta} \quad (\text{A.68})$$

Selecting $p \in \left[\inf_{|\tilde{x}| \in [\hat{\gamma}_1(\|w\|_\infty) + \delta, \rho]} \frac{\hat{\beta}_1(|\tilde{x}|, T_s)}{|\tilde{x}|}, \frac{\hat{\gamma}_1(\|w\|_\infty)}{\hat{\gamma}_1(\|w\|_\infty) + \delta} \right]$, the conditions (A.66) and (A.67) could be satisfied for sufficient small T_u . Therefore, from the conditions (A.66) and (A.67), the critical value of T_u could be found: $T_u^* = \min\{T_{u1}^*, T_{u2}^*\}$ which

$$T_{u1}^* = \frac{1}{L} \ln \inf_{|\tilde{x}| \in [\hat{\gamma}_1(\|w\|_\infty) + \delta, \rho]} \frac{p|\tilde{x}|}{\hat{\beta}_2(|\tilde{x}|, T_s)}; \quad (\text{A.69})$$

$$T_{u2}^* = \frac{1}{L} \ln \frac{1}{\gamma/L + \delta} \left((1-p)\hat{\gamma}_1(\|w\|_\infty) + \frac{\gamma}{L} \right). \quad (\text{A.70})$$

Therefore, for all $T_s > T_s^*$ and $T_u < T_u^*$, conditions (A.66) and (A.67) are satisfied such that $|\tilde{x}_{av,k+1}| < |\tilde{x}_{av,k}|$ whenever $|\tilde{x}_{av}(t)| \in [\hat{\gamma}_1(\|w\|_\infty) + \delta, \rho]$, combing the fact that $|\tilde{x}_{av}(t)| < |\tilde{x}_{av,k}|$ for $t \in [k(T_s + T_u), (k+1)(T_s + T_u))$, there exists $\bar{\beta} \in \mathcal{KL}$ such that $|\tilde{x}_{av}(t)| \leq \bar{\beta}(|\tilde{x}_0|, t - t_0)$, whenever $|\tilde{x}_{av}| \geq \hat{\gamma}_1(\|w\|_\infty) + \delta$. Let \check{t} and \hat{t} denote the time that averaged trajectories enter and leave the boundary $\{x \in \mathbb{R}^n \mid |x| = \hat{\gamma}_1(\|w\|_\infty) + \delta\}$

respectively. If $\hat{t} = \infty$ then the averaged trajectories can be bounded by

$$|\tilde{x}_{av}(t)| \leq \bar{\beta}(|\tilde{x}_0|, t - t_0) + \hat{\gamma}_1(\|w\|_\infty) + \delta, \forall t \geq t_0. \quad (\text{A.71})$$

When $\hat{t} < \infty$, the averaged trajectories can be bounded for $t \in [\hat{t}, \hat{t} + T_u]$:

$$\begin{aligned} |\tilde{x}_{av}(t)| &\leq (\hat{\gamma}_1(\|w\|_\infty) + \delta) e^{L(t-t_0)} + \frac{\gamma_w}{L}(e^{LT_u} - 1), \\ &\leq (\hat{\gamma}_1(\|w\|_\infty) + \delta) e^{LT_u} + \frac{\gamma_w}{L}(e^{LT_u} - 1) \end{aligned} \quad (\text{A.72})$$

For $t > \hat{t} + T_u$, the averaged trajectories $|\tilde{x}_{av}(t)| \leq \bar{\beta}(|\tilde{x}_0|, t - t_0)$ until it enters the boundary $\{x \in R^n \mid |x| = \hat{\gamma}_1(\|w\|_\infty) + \delta\}$ again, and the above procedure occurs repeatedly so the trajectories can be bounded for all $t \geq t_0$ as follows:

$$\begin{aligned} |\tilde{x}_{av}(t)| &\leq \bar{\beta}(|\tilde{x}_0|, t - t_0) + (\hat{\gamma}_1(\|w\|_\infty) + \delta) e^{LT_u} + \frac{\gamma_w}{L}(e^{LT_u} - 1), \\ &\leq \bar{\beta}(|\tilde{x}_0|, t - t_0) + \bar{\gamma}(\|w\|_\infty) + e^{LT_u}\delta + \frac{\gamma_w}{L}(e^{LT_u} - 1). \end{aligned} \quad (\text{A.73})$$

For any given $\tilde{\delta} > 0$, the practical term of ultimate bound can be achieved by letting $\delta = \tilde{\delta}/2$ and choosing T_u^{**} such that $e^{LT_u}\tilde{\delta}/2 + \frac{\gamma_w}{L}(e^{LT_u} - 1) \leq \tilde{\delta}$ for all $T_u \leq T_u^{**}$. Therefore, when $T_u \leq \min\{T_u^*, T_u^{**}\}$, the trajectories satisfy

$$|\tilde{x}_{av}(t)| \leq \bar{\beta}(|\tilde{x}_0|, t - t_0) + \hat{\gamma}_3(\|w\|_\infty) + \tilde{\delta}, \quad (\text{A.74})$$

By using the fact $x_{av}(t) = \tilde{x}_{av} + x_e$, (5.59) could be obtained which completes the proof.

Q.E.D.

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