Inventory Management in Supply Chain with Stochastic Inputs

by

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ABSTRACT

Title: Inventory Management in Supply Chain with Stochastic Inputs

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This thesis studies and proposes some new ways to manage inventory in supply chains with stochastic demand and lead time. In particular, it uses queuing principles to determine the parameters of supply chain stations with delayed differentiation (typical assemble-to-order systems) and went on to apply some previously known results of steady state of some queuing systems to the management of flow and work in process inventory in supply chain stations. Consideration was also given to the problem of joint replenishment in partially dependent demand conditions. The first chapter introduces the important concepts of supply chain, the role of inventory in a supply chain, and developing stochastic models for such system. It then went on to review the pertinent literature that has been contributed to the inventory management, especially using stochastic models.

Chapter two presents a perishable inventory model with a multi-server system, where some services, having an exponentially distributed lead time, have to be done on the product before it is delivered to the customer. Customers whose demands are not met immediately are put in an orbit from where they send in random retrial requests for selection. The input stream follows a Markov Arrival Process, $\boldsymbol{MAP}$, and another flow of negative customers (typical of a competitive environment with customer poaching), also following an $\boldsymbol{MAP}$, takes customers away from the orbit. An $(\boldsymbol{s}, \boldsymbol{S})$ replenishment policy was used. The joint probability distribution of the number of busy servers, the inventory level and the number of customers in the orbit is obtained in the steady state. Various measures of

stationary system performance are computed and the total expected cost per unit time is calculated. Numerical illustrations were made.

Chapter three is also a continuous review retrial inventory system with a finite source of customers and identical multiple servers in parallel. The customers are assumed to arrive following a quasi-random distribution. Items demanded are also made available after some service, exponentially distributed, has been done on the demanded item. Customers with unsatisfied orders join an orbit from where they can make retrials only if selected following a special rule. Replenishment follows an $(\boldsymbol{s}, \boldsymbol{S})$ policy and also has an exponentially distributed lead time. The intervals separating two successive repeated attempts are exponentially distributed with rate $\theta + iv$, when the orbit has i customers $i \geq 1$. The joint probability distribution of the number of customers in the orbit, the number of busy servers and the inventory level is obtained in the steady state case. Various measures of stationary system performance are computed and the total expected cost per unit time is calculated.

Chapter four is a two-commodity continuous review inventory system, with three customer input flows, following the $\boldsymbol{MAP}$; one for individual demand for product 1; another for bulk demand for product 2; and the third for a joint individual demand for product 1 and bulk demand for product 2. The ordering policy is to place orders for both commodities when the inventory levels are below prefixed levels for both commodities, using $(\boldsymbol{s_1 s_2}, \boldsymbol{S_1 S_2})$ replenishment. The replenishment lead time is assumed to have phase type distribution and the demands that occur during stock out period are assumed to be lost. The joint probability distribution for both commodities is obtained in the steady state case. Various measures of system performance and the total expected cost rate in the steady state are derived. Numerical illustrations were then done.

Chapter five is a model that shows how the steady state parameters of a typical queuing system can be used in the dynamic management of flow and buffer in a Theory of Constraints ($\boldsymbol{TOC}$) environment. This chapter is in two parts, and the typical $M/M/1/\infty$ production environment with $0 < \rho < 1$ was assumed. The optimal feed rate for maximum profit was obtained. In the first part, the model was considered without consideration for shortage cost. This model was then extended in the second part to a case where a fixed cost is charged for every unit shortage from the desired production level. Part A result was shown to be a special case of part B result; the unit shortage cost has been implicitly taken to be zero in part A.

Chapter six is the concluding chapter, where the various possible applications of the models developed and opportunities for possible future expansions of models and areas of research were highlighted.

The main contributions of this work are in the Supply Chain area of delayed differentiation of products and service lead time. Others include management of joint replenishment and optimisation of flow in a TOC environment. The key contributions to knowledge made in this thesis include:

- A model of a multi-server retrial queue with MAP arrival and negative arrival, and deteriorating inventory system in which inventory items are made available only after some work has been done on the inventory item before it is delivered to the customer. No previous model is known to have considered any queuing system with such multi-server system ahead of this chapter.
- A model of a retrial queuing system with multi-server rule based in which the arrival pattern is quasi-random, the calling population is finite, and an exponentially distributed system service is done on the inventory item before being delivered to the customer. It has not been found in literatures that such models have been developed elsewhere.
- A stochastic model of joint replenishment of stocks in which two products are being ordered together; one of such is ordered in bulk and the other in single units, but both could be ordered together and unfilled order during the replenishment lead-time is lost. No published work is known to have also addressed such systems.
- The management of flow in a theory of constraint environment, which explicitly utilises the holding cost, shortage cost, product margin, the level of utilisation of the resource and the effect of such on the stocks (WIP inventory) build up in the system. Such flows are then explicitly considered in the process of buffering the system. Most works have been known to focus on buffer and not the flow of the products in order to optimise the system profit goal.

Some of the insights derived include

- An understanding of how the system cost rate is affected by the choice of the replenishment policy in systems with $\boldsymbol{MAP}$ arrival pattern so that controlling policies (reorder point and capacity) could be chosen to optimise system profit
- The effect of correlated arrival in $\boldsymbol{MAP}$ input system on the cost rate of the system
- How the nature of input pattern and their level of correlation affect the fraction of the retrials in a retrial queue in a competitive environment that are successful and how many of such customers are likely to be poached away by the randomly arriving competitors. This has direct effect on the future market size.
- The nature of utilisation, blocking and idleness of servers in typical retrial queues, such that there could be yet-to-be-served customers in the orbit while there are still idle serves in such systems
- Management of utilisation of resources in stochastic input and processing environment with respect to the throughput rate of such systems. It was shown that it may not be profitable to strive to always seek to fully utilise the full capacity of a Capacity Constrained Resource, even in the face of unmet demands. Increase in utilisation should always be considered in the light of the effect of such on the throughput time of the products and the consequence on the system's profit goal. This decision is also important in determining the necessity and level of buffers allowable in the production system.

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1.

CHAPTER 1

INTRODUCTION

1.1. SUPPLY CHAIN CONCEPTS

1.1.1. Production Eras and Challenges

The challenges of production of goods and services after the Second World War have gone through three main chronological stages as outlined by Hopp and Wallace (2008). The first era focuses mainly on productivity, and this leads to the traditional focus on cost. Some visible developments in this era included fast paced development in scientific management, especially the reductionist techniques of work study, and more pervasive development and deployment of financial ratios for monitoring the health of firms. This was dated back to manufacturing itself, but received boost immediately after Second World War till the seventies. This trend was supported by the relatively sole strong position of the American economy at the time.

The productivity era was succeeded by the era of quality movement, which was dated back to the seventies and eighties, although the pioneering work appears to have been done as far back as 1931 by Shewhart. Some of the important tools of this holistic management era included the Total Quality Management (TQM) and Just in Time (JIT). These were later revived again in the Six Sigma and Lean movement. This movement was bolstered by the advent of competing nations like Japan and Germany among others that have started emerging from the rubbles of the war and are entering the same market that has been hitherto dominated by America.

The latest era appears to be that of integration, and this is assumed to have commenced in the nineties. This development was driven especially by the rapid development in the Information and Communication Technology (ICT) that makes the whole world to become more integrated than it has ever been. This globalisation trend has been further enhanced by the changing econo-political structure in most Asian, Latin American, Eastern Europe and African countries (from centrally planned to market driven philosophies), and the advancement of the World Wide Web that makes countries to locate their various offices where ever they feel is most appropriate for their businesses.

The Asian Tigers' miracle at the Han River and the emergence of China as strong manufacturing centres, with the later entrance of the Indo-Brazil and South African centres has created massive international competition.

With the possibility of savings by focusing only on the traditional methods of work reductionism thinning out, more focus shifts to the total manufacturing system (the network: from the supplier's supplier to the customer's customer), especially since most international legal, econo-political, fiscal and technical barriers are being constantly lowered. This era birthed the current production trend of Supply Chain Management (SCM). Some other related ideas in this era include Business Process Modelling (BPM) and Enterprise Resource Planning (ERP) amongst others. Supply Chain has been defined severally by a number of authors, but one definition that seems succinct but exhaustive in this thesis' context is presented next.

1.1.2. Supply Chain Definition and Concepts

A supply chain has been defined as a goal oriented network of processes and stock points used to deliver goods and services to customers (Hopp, 2008). This definition highlights the key features of any supply chain to be: the goal, the network, the stock points, the process stations, the products (goods and/or services) and the customers. This definition actually summarises all that is done in a supply chain (especially from the market perspective). This is further explored.

The basic goal of most organisations is profit. Two paths usually lead to increase in profit: cost reduction or growth in market size. But progress along one of these paths may actually degrade the other. So, organisations need to decide how much efforts are put into these two paths to realise the organisational goal of profit, both in the short and the long run. This makes the goal to be closely related to the strategy of the organisation, which is done at the highest planning level, and decides how much of what is traded off to achieve the other, and thereby , hopefully, placing the organisational plan on some sort of "efficient frontier".

Customers are important in the chain because they are the market. The second path to profitability implies ensuring that they are satisfied and delighted. But only if their needs (i.e. demand and timing) are known well in advance would managing the whole chain become easy and all unnecessary costs could be easily eliminated (or reasonably reduced). But, unfortunately, these customers are not so predictable, and hence comes in variability into the system. This is the first level of variability in the supply chain; which is related to the management of the external influences on the chain. This comes in the form of uncertain external demands and lead times.

Supply Chain Process Points, or the work stations, are the resources that actually get out goods and services ready for the customers one wishes to delight. These are the transformation centres that, in the word of Langley et al. (2009) add utilities of form to the input material by transforming its form (or may be servicing the customer). These process centres also contribute the second level of uncertainty, which in this case is internal to the system. This is in the form of uncertain process times of the process centres as products are transformed at these centres. This unavoidable variability in the system forces the strategic deployment of reserves in the supply chain. These reserves have been referred to by Webster (2008) as system slacks. These slacks are in the form of extra capacities or inventories. Therefore, the process points also serve as strategic capacity reserve points while stock points serve as strategic material reserve points. This leads to the discussion of stock points.

Stock points are positions in the supply chain network where inventories of materials or goods are found. These points exist due to two reasons: firstly, they may exist as a result of deliberate plan to keep some materials in some identified locations in the network, e.g. finished goods, some important raw materials, etc. The second reason is because some inventory build up in the system and are controlled by some natural laws like the Little's law. These form part of the work-in-process inventory and cannot be controlled directly but by regulation of flow through the system. Flows are now discussed next.

Flows are the actual products (or even customers) that are processed at the processing centres. They are generated basically by actual orders or demand forecasts. Above the decoupling point, they are driven by a push (production plan/forecast) while below the decoupling point, they are driven by a pull (customer orders). Another very close term is scheduling. Management of flows are very important in any supply chain that would be successful. Flows through the chain or the stations are usually stochastic, and this affects deployment and management of slacks of capacity and inventory. Decisions about full or under-utilisation of capacity affect the inventory cost and profitability of the system. Also, decisions about level of inventory necessary to support a given level of flow are crucial because this affects the level of customer service as well as operating cost of the whole network. These are all inter-related decisions that must be made in the production context. The decisions could sometimes be simplified (howbeit to some level) by choosing a suitable management philosophy (or a mix of such) to adopt. These philosophies are briefly discussed later.

1.1.3. The Goal of a Supply chain

One key issue about which most stakeholders in a supply chain have a common agreement is the provision of superior customer service. Doing this at a low cost is another important thing, and so, the interest in the landed cost of the product and not just the production cost.

Making goods available to customers when needed (referred to as the utility of time) could be achieved through two main means: superior transport service or keeping stock near customers. Two focus areas concerned about this are transport management and inventory management. It is therefore not surprising that transport and inventory costs have been identified as the two major costs of any supply chain. (Langley et al., 2009).

1.1.4. Importance of Inventory

Inventory occupies a strategic position in a production system. Apart from being a major means of fulfilling customer orders, it also has a major effect on the books of the company in that it affects both the balance sheet and income statement; hence its effective management is crucial. The main function, though, is like insurance in the production system, absorbing the variability shocks. Based on the function it performs, it has been classified as cycle stock, safety stock, contingency stock, process stock etc (Jacobs et al., 2009). It is generally true that the level of uncertainty of demand and lead time are the two main parameters that affect the modelling of its behaviour.

1.1.5. Some Production Management Philosophies

Production management philosophies are developed to guide management through effective decision making in the processes of production management that involves intricate and dependent trade-offs. The main difference between all these philosophies is the perception and treatment of slacks in the system. Both slacks cost the system, but one is usually more acceptable than the other depending on the philosophy. Three basic philosophies to be considered are Material Requirement Planning (MRP), Lean Manufacturing and the Theory of Constraints (TOC).

Lean is very critical of inventory, and in ideal Lean environment, the batch size is equal to the actual demand. It works by pure pull and rather tolerates extra capacity than extra inventory. Inventory there is hardly zero, however, but the Kanban controls both the scheduling and the effective quantity of inventory in the system. The MRP accepts more slacks of inventory and tends to utilise capacity more than Lean. Inventory is also used to support capacity utilisation. Theory of constraint, however, is built entirely around flow. Inventory is placed in strategic locations to support the critical resources, while the capacity slacks in the non-critical resources are also used to support flow through the entire system; especially through the critical resources.

1.2. SUPPLY CHAIN SYSTEMS AND MODELLING

Systems have many definitions depending both on the discipline and the issue of interest. In the current context, the system is basically some sort of processes of interest. Systems have some state variables of interest, in this case the level of inventory present in the system. Usually, these state variables can only be manipulated indirectly through the control of some other variables called the control variables. Systems have decision variables, in this case the order policy, order quantity, or the rate of flow through the system, all of which could be manipulated to affect the positions of the state variables, which in turn determine the overall system performance. These state variables together with the parameters, which in many cases are constants or variables with known patterns are what determine the values of the system performance indicators. Such indicators in this context include system cost, level of customer service, utilisation, etc. It is usually necessary to have models that represent these systems so that the behaviour of the systems could be understood through the behaviour of these models.

The contextual and semantic definition of model is quite diverse, but a succinct definition for the current context is that a model is a representation of a system that allows for investigation of the properties of the system and in some cases prediction of the future outcomes.

Models are important in systems analysis and engineering, and the complexity could be viewed along the two dimensional axes of time changes and level of certainty. This makes all systems to be reasonably captured in a four quadrant space of deterministic-static, deterministic-dynamic, stochastic-static and stochastic-dynamic regions. This makes the system whose variables are in the deterministic-static quadrant the most tractable in respect of their mathematical computation, while the stochastic-dynamic models are the least tractable problems. The quadrant to which a problem falls also usually determines the type of models that would be most appropriate for it. Usually, most typical supply chain models fall in the stochastic region and so may need some sophisticated level of mathematical manipulation.

Most models presented in this work are Markovian, so, the problems require the instruments of probability theory, and in some instance matrix mathematics, or some level of differential calculus.

Modelling is both an art and a science. It is an art because the dexterity often improves with usage. It is a science because most techniques have logical sequences and formal methods that are followed. A good modeller knows the level of complexity at which to pitch the modelling of a system. Sometimes, it suffices to use simple models and allow for the inclusion of the simplifying assumptions in the interpretation of the results. This saves a great deal of modelling and solution efforts while still effective at achieving the intention of the model. But in certain instances, there may be the need to develop some more complex models without which some important characteristics of the systems would be sacrificed. These facts have been well noted by Sterman (2000) and Zipkin (2000) and were taken note of in the development of models in this work. It, thus, became necessary to employ the probability tools while solving for the steady state probability distribution of the input parameters of the selected problems, and the use of simple differential calculus in determining the optimal flow parameters given that the system is operating at the steady state.

Supply chain modelling has utilised many analytical tools for the management of stock level and flow of products in the entire chain or at a station in the chain. These techniques include classical optimisation tools, mathematical programming, simulation modelling and probability models. Cases where one or more input into the system (usually the demand or/and lead time) are stochastic have always called for the use of probability techniques, either as simulation models, or in the estimation of the equilibrium properties of the system.

1.3. LITERATURE REVIEW

Various analytical tools have been used in the analysis of production systems to optimise the levels of stock (inventory) it holds. The type of tool depends on the assumptions made about the nature of product flow through the system. This ranges from the deterministic-static type to the dynamic-stochastic type discussed earlier. Such tools include classical optimisation tools, mathematical programming tools, probability models and simulation. Some popular works have been produced in each category.

1.3.1. The Harris Model

The use of deterministic optimisation techniques in the management of the appropriate stock levels to keep in a production environment is pervasive. The seminal model in this category is the Economic Order Quantity (EOQ) model, developed by Harris (1913) and popularised by Wilson (1934). This model is deterministic and static. It also has many other assumptions including zero (or deterministic) lead time, shortages and backlogging not allowed, unit purchase price independent of order quantity, infinite product life, instantaneous product availability (infinite capacity), perfect order quality, fixed set up cost, single item, and probably more. This model has been modified in diverse ways by relaxing one or more of its assumptions. And it is the relaxation of some such assumptions that made the use of classical optimisation techniques inadequate for analysis in certain instances.

There have been some major groups of extensions to this classic work. The Dynamic Economic Lot (DEL) Model by Wagner and Whitten (1958, 2004) removes the static demand assumption, but still assumes the future demand pattern is known with certainty. The Silver-Meal heuristics is another seminal work in this direction. Another interesting extension is in that of single item assumption. The Joint Replenishment Problem (JRP) has been studied by many authors. Goyal and Soni (1969) and Goyal (1974) are notable. Other contributors include Van Eijs (1993), Viswanathan (2002), Fung and Ma (2001), Chan, Cheung and Langevin (2002) and Federgruen and Zheng (1992).

Multi-echelon inventory is another area that has generated much interest, starting from Clark and Scarf (1960). Others include Graves (1985), Erkip, et al (1990) Chen (2000), Rau et al (2003) and Viswanathan and Piplani (2001).

1.3.2. Deteriorating inventory

An area that has enjoyed an extensive research is the deteriorating inventory studies. Starting from the seminal work by Ghare and Shrader (1963) which is a deterministic demand model, much work has followed since. Nahmias (1982) made a detailed survey of the work done on deteriorating inventory up until that time. He summarised the contribution of the various authors reviewed and classified the work into five main areas based on:

- Fixed life perishability,
    - deterministic demand and stochastic demand, single and multi products, exact and approximate solutions, single and multi echelon
- Random lifetime models
    - Periodic review and exponential decay models
- Queuing models with impatience
- Applications.

Raafat (1991) extended the survey to the contributions made after Nahmias. While most of the models reviewed by Nahmias are fixed lifetime models, Raafat extended the survey to cover a lot more random deterioration models. Raafat classified the literatures as single or multiple items, deterministic or probabilistic demand, static or varying demand, single or multiple period, purchase or production model, availability of quantity discount(s), allowance for shortages, constant or varying deterioration rate.

Since the two compendia are quite detailed, effort would be concentrated on reviewing some of the more recent works done after Raafat. Goh et al (1993) presented a model in which inventory deteriorates in two stages. The arrival is a Poisson process with rate λ and the demand rates are μ_1 for stage 1 (fresh) product and μ_2 for the product older than stage 1 but not yet obsolete. Various system parameters were considered in this model. The model was modified in Yadavalli, et al. (2004) with the inclusion of lead time with arbitrary distribution and solved for the various system parameters. Vaughan (1994) presented a customer realised product expiration, in which he treated the expiration date of the product as a decision variable, and the product life time is treated as a random variable.

Kalpakam and Sapna (1995) dealt with a base stock policy, where the lead time is stochastic and correlated with the possibility of lost demand. Products are taken out of the system due to failure or demand. The system parameters were determined. Hariga (1996) developed an EOQ model for deteriorating inventory with time varying demand and with shortages allowed and completely backlogged. The performance of the model with linear and exponential demand inputs was analysed. Yadavalli et al (2006) also presented a model for two component production-inventory assembly system in which products are assembled from two components. A component is produced with the lead time following an arbitrary distribution and the other component is purchased with an exponential lead time. System parameters were estimated.

Chakrabathy et al (1998) presented a model in which the deterioration of inventory follows a three parameter Weibull distribution. The demand is assumed to be time varying and shortages are allowed in the system. Lee and Wu (2002) is a model with Weibull distribution deterioration and power demand with complete backlogging of shortages, and this model was extended by Dye (2004) to a general type time-proportional backlogging rate model. The backlogging rate was defined as a function of the waiting time. Chiao et al (2008) presented a model with two storage facilities, partial backlogging and quantity discount. In this model, the excess product is kept in a rented warehouse due to capacity constraint in own warehouse.

Cases of joint demand have also been investigated by Yadavalli et al. (2004) where there is capacity constraint on stored items and each has different reorder points, but the reorder for one item triggers reorder of all other items. In another paper, Yadavalli et al. (2006) considered a case where two products have individual Poisson demand, and the demand for the first item also generates demand for one of the second. Systems parameters were evaluated. A case of substitutable products with joint demand and joint ordering policy was also considered in Yadavalli et al (2005b). A multi-item inventory with fuzzy deterministic demand has also been considered. (Yadavalli et al. 2005a)

Lee and Hsu (2009) is a model of a two-warehouse inventory management of a free form time dependent demand, where both the replenishment rate and planning horizon are finite. They used an approach which permits variation in production cycle time to determine the number of production cycles and time of replenishment during a finite planning horizon. Ferguson et al (2007) showed that EOQ model with nonlinear holding cost is an approximation of optimal order policy for perishable goods sold in small to medium size grocery stores where there are delivery surcharges due to infrequent ordering, and managers frequently utilize markdowns to stabilize demand as the product's expiration date gets nearer. They showed how the holding cost curve parameters can be estimated via a regression approach from the product's usual holding cost (storage plus capital costs), lifetime, and markdown policy.

Ho et al (2007) considered the effects of deteriorating inventory on lot-sizing in material requirements planning systems. They used simulation studies to evaluate the performance of five existing heuristics using three factors: rate of inventory deterioration, percentage of periods with zero demand, and setup cost. Hwang and Hahn (2000) investigated an optimal procurement policy for items with an inventory level-dependent demand rate and fixed lifetime, being a case for a fish cake retailer. Lin and Gong (2006) considers the impact of random machine breakdowns on the classical Economic Production Quantity (EPQ) model for a product, manufactured in batches, and subject to exponential decay and under a no-resumption ($\boldsymbol{NR}$) inventory control policy. The time-to-breakdown also follows an exponential distribution.

Chung and Wee (2007) developed an integrated deteriorating inventory policy for a single-buyer, single-supplier model with multiple $\boldsymbol{JIT}$ deliveries considering the transportation cost, inspection cost and the cost of less flexibility. Shah and Shukla

(2009a) presented an algorithm and models for a retailer's optimal procurement quantity and the number of transfers from the warehouse to the display area are determined when demand is decreasing due to recession and items in inventory are subject to deterioration at a constant rate. They also presented a deterministic inventory model in Shah and Shukla (2009b) where items are subject to constant deterioration and shortages are allowed. The unsatisfied demand is backlogged as a function of time.

Baten and Kamil (2010) presented a continuous review model for the control of production-inventory system subject to generalised Pareto distributed deterioration. They used the principle of control theory to determine what should be the optimal level of inventory in the system. Benhadid, et al (2008) also used control theory to show how to manage inventory in a production system with deteriorating items and dynamic costs.

Inventory models with Markov Arrival Processes ($\boldsymbol{MAP}$) and/or retrial queues have not been fully studied. The study of systems with $\boldsymbol{MAP}$ input systems have been focused mainly in telephone network systems. This has been highlighted in Gomez-Corral (2006) and Artalejo (1999). The only inventory related $\boldsymbol{MAP}$ input literatures documented is in Gomez-Corral (2006), and it was done by Krishnamoorthy et al. (2003, 2004) and even then, the inventory focus is also related to communication system as well. Some works have started being reported in this area. Yang and Templeton (1987) is another review. Lian, Liu and Zhao (2009) presented a continuous review model for a one item product where the demand has a distribution that is the Markov Arrival Process. The lifetime of the product is exponentially distributed with a constant failure rate λ. All arrival demand requests only one unit of item and all unmet demand is backordered.

Manuel et al. (2007) developed a continuous review perishable $(\boldsymbol{s}, \boldsymbol{S})$ model where there is an $\boldsymbol{MAP}$ arrival and $\boldsymbol{PH}$ service time. There is also a negative flow of unsatisfied customer, following the $\boldsymbol{RCE}$ policy for removal of customers. System parameters were determined. Yadavalli et al (2006) have also presented a model of service facilities where customers do not receive services immediately but have to wait till some services are performed on these products being waited for before the product is brought into stock. Two cases were considered: first where the product is brought in immediately after service; and the second case was where the product is brought in only at the next epoch. System parameters were determined. A model of perishable inventory in a random environment according to an alternating renewal process has also been studied in Yadavalli and Van Schoor (2004). The rate of perishing depends on the state of the system. Generally, it does not appear as if a lot has been done in deteriorating inventory systems with $\boldsymbol{MAP}$ arrival pattern and/or $\boldsymbol{PH}$ service pattern.

1.4. STOCHASTIC PROCESSES

Lindsey (2004) defined a stochastic process as some phenomenon that evolves over time (i.e. a process) and that involves a random component. It involves some response variable x_t that takes values varying randomly and in some way over time $t = 1 \dots T \; or \; 1 \dots.$ and/or space $n = 1 \dots n \; or \; 1 \dots.$ The variable may also be a scalar or vector. The observation of a state (or a change of state) is called an event. Usually, the probabilities of possible events would be conditional on the state of the process. The main properties, among other things, distinguishing a stochastic process are:

- The frequency or periodicity with which observations are made
- The set of all its observable values (state space)
- The sources and forms of randomness present, including the nature of the dependence among the values in a series of realisations
- The number of copies of the process available (only one or several)

1.4.1. Distribution and Transformation of the Random Variable

A random variable can be defined as a real-valued function defined over a sample space. The distribution of a random variable is the sample space of all its possible outcomes and the probability of each one occurring. The distribution function of a random variable plays an important role in the determination of the various parameters of the system in which it occurs.

Solving the state equations of a variable, especially since it is usually a joint distribution, could be quite challenging. It usually necessitates the need to transform the variable from one form to another in which it could be handled in a more straight forward manner. Bocharov et al. (2004) has used the term characteristic transform to describe all the transformations that are used in such manner. This term, he stated, comprises of the characteristic function, Laplace-Steiltjes transform and the moment generating function, depending on which ever is best applied.

1.4.2. Other Properties of the Stochastic Process

Some other issues that would be worth mentioning, apart from the randomness of the variable(s) and its distribution, are state dependence, serial dependence, stationarity, equilibrium, ergodicity, and regeneration point.

A stochastic process is said to be state dependent if the probability of being in a future state is dependent on the present state in which the state is found. This principle is exploited in Markov processes.

A stochastic process is said to have serial dependence if some parameters of the system depend not directly on the previous state of the system, but somehow on the previous state and the prediction at that time. It is a useful mechanism in time series analysis. Such dependencies could be on the location parameter, as in most such models, or on the spread parameter as in heteroscedastic models.

A stochastic process is said to be strictly stationary if sequences of consecutive responses of equal length in time have identical distributions. This means the values of the statistical parameters of the process are assumed constant with respect to time.

A process is said to be in equilibrium if the flow of a parameter of interest (including probability) into and out of a space (or point) balances out. The process may not be in equilibrium when it starts, but may enter a state of equilibrium over time, making it possible to observe its behaviour before entering equilibrium (i.e. while in transit – transient properties) and when it has entered equilibrium. In other words, if equilibrium has been reached, the probability that the process is in a given state, or the proportion of time spent in a given state, has converged to a constant that does not depend on the initial condition, and in essence the system become quite stationary.

Ergodicity is a concept quite related to equilibrium. Ergodic theorems provide identities between probability averages, such as an expected value, and the long run averages over a single realisation of a process. Thus, if the equilibrium probability of being in a given state equals the proportion of a long time period spent in that state, it is called an ergodic property of the process.

A regeneration point is a time instant at which the process returns to a specific state such that the future evolution of the process does not depend on how that state was reached. This means whenever a process arrives at the regeneration point, all of its previous history is forgotten. The renewal process, describing the time between recurrent events, is a well known case of such.

1.4.3. Types of stochastic processes and methods of observation

Basically, there are two main types of stochastic processes: survival processes and recurrent processes. The basic natures of each of these processes also affect the natures of its observations.

Survival processes are those that involve entering into a final state at which the process could be assumed to have terminated. Such processes are very useful in reliability studies in which the process of interest may not have the opportunity to regenerate itself. This limits the type of methods available for its study.

Recurrent processes are characterised by the possibility of the occurrence of more than one event (usually taken as two states in regeneration processes) over the time of study. One state is assumed to dominate while the other occurs occasionally. The latter that sparsely occurs is treated as a point event, and by focusing on its process of occurrence, the process is referred to as a stochastic point process. In contrast to a survival process, the point process only signals a transitory stage such that the event does not really signal a change of state. A binary indicator can, therefore, be used to signify a 1 if the point process occurs and a 0 otherwise. The process can, thus, be called a binary point process.

1.4.4. Method of Observation, Replications and Stopping Time

Two approaches could be used to observe accurate information from a stochastic process.

- One series for a long enough period (if it is reasonably stable)
- Several short replications of the process (if they are reasonably similar)

The nature of survival processes has confined their observation strictly to the second method since the process enters into an absorbing state. But for recurrent processes, one may use either of the two. Using the second method in a recurrent system raises the question of specifying an appropriate time origin. But in a stationary process, the principle of ergodicity makes it fairly simple to use the first method. The regeneration point process then acts as the appropriate time origin from which a datum could be taken for the initialisation of the observation process again.

Cinlar (1975) has defined a stopping time as any random time, T, having the property that for every $n \in N$ the occurrence or non-occurrence of an event $\{T \leq n\}$ can be determined by looking at the values of $x_o \ldots . x_n$.

1.4.5. Observation of Variables of interest

The variable of interest in a stochastic process could be one or more of the following:

- The inter occurrence time i.e. the duration between the occurrence of two consecutive events of interest, e.g. the time between two consecutive regeneration points
- The count of the number of occurrence of an event in a given interval e.g. the number of regenerations or renewals that have occurred between two periods of time
- The cumulative number of events of interest that occurred till date

The subject of renewal theory seeks to answer these questions. A summary of an overview of Renewal process, Markov theory and Queuing theory is included in Appendix 2.

1.5. POPULAR MANAGEMENT PHILOSOPHIES

Production managers have different perceptions about the importance and significance of the different system slacks. While some would not accept the presence of significant idle capacities, others are more critical of excess inventory. The decision about which one appears more critical is also dependent on the production philosophy. But the philosophies address not only issues of system slacks, but also issues of quality and job scheduling among others. This is because these are surrogate issues to the issues of slacks themselves.

Inventory is present in these systems, both as a stock build up, consequent to the job scheduling and flow management techniques as well as a result of deliberate actions of building up strategic reserves as an insurance against demand and lead time uncertainties. While there could be many other ideologies considered as management philosophies, the discussion here is limited to Lean Manufacturing, material Requirement Planning ($\boldsymbol{MRP}$) and the Theory of Constraints ($\boldsymbol{TOC}$). Just an overview of these would be provided also. Volmann et al. (2005), Jacobs et al. (2009), Goldratt and

Cox (2004) and Jonsson and Mattsson (2009) are good further readings for the interested reader for further treatment of the philosophies.

Lean manufacturing is a system that would prefer to pull entirely through the system. It apparently is more critical of excess inventory than spare capacity. In the ideal Lean environment, replacement of outputs or inputs should be lot for lot. This does not give consideration to issues of set up (both of purchase and production). To achieve this, effort goes into eliminating causes of bad quality as well as lead time variation in the system. Efforts are also put into managing demand so that the production rate is quite level. Kanban is used both to control the level of allowable inventory as well as scheduling tasks. Efforts for continuous reduction of set up times are also made consistently in Lean systems.

The Material Requirement Planning ($\boldsymbol{MRP}$), however, has a less critical view of inventory. Inventory is used to support utilisation of resources. Production is back-scheduled. Extra inventory is allowable as safety stock along various points in the network, and capacity utilisation is usually higher than that obtained in Lean.

Theory of Constraints ($\boldsymbol{TOC}$) also has a critical view of inventory in a manner probably similar to the Lean technique. It also would, however, not only allow for spare capacities in the various locations in the production network, but believes they are good. These spare capacities are used to break the production batches of such systems further down to the end that the average work-in-process inventory is further minimised. Strategic reserves are allowed in certain parts of the network where they are used to support the most critical station.

In a $\boldsymbol{TOC}$ environment, the critical station should be fully exploited, but only to the point where it does not also create an unnecessary inventory (finished good or work-in-process). Productivity is different from activation of resources. Productivity is about actual sales and not hours worked. Throughput is only about products that the market is ready to absorb and convert to money, and not just finished product. Finished product not going for sale is just another “undesirable” inventory. Scheduling is about creating an imaginary rope from the strategic buffer locations to the entry point to the flow line, and that suffices to control the flow through all processing stations in the entire line.

An important issue is the treatment of the statistical variations in the processing time and the complex stochastic and dynamic nature of demand that are basically not directly implied in all these models. Determinism is somehow implied to a large extent in the deployment of all these processes. This is the cause of system nervousness in such processes and their treatment has not been fully studied by researchers.

Of particular interest is the determination of the ideal buffer size to place ahead of the critical work station. This station could be a Bottleneck ($\boldsymbol{BN}$) or a Capacity Constrained Resource ($\boldsymbol{CCR}$) depending on if it has demand for production that is more than its capacity or close to its capacity respectively. While $\boldsymbol{TOC}$ seeks to eliminate unnecessary inventory in the system, it deliberately keeps time buffers ahead of the critical station to eliminate unplanned resource idleness and at junctions where other lines meet the critical line to eliminate waiting for parts or components along the critical line. The determination of this buffer size and its relationship to the flow rate in a $\boldsymbol{TOC}$ environment is an issue that still needs investigation, especially in the light of possible variation in resource processing time.

1.6. RESEARCH FOCUS AND CONTRIBUTION

1.6.1. Area of Interest

It has been stated that the aim of the supply chain management is a holistic approach for managing production throughout the entire production network, whereby some of its issues focus on the management of stations and some on the links. Issues of interest in station management relate to those of the traditional productivity and quality issues while issues of link management are those of logistics and information systems.

The focus of this work is on some of the station management principles. The main focus in stations is actually on the management of flows. Of particular interest is in the strategic management of inventories in the system as a result of the variability in the supply chain. Inventory has been mentioned earlier as strategic reserves of materials. They are said to occur both as deliberate strategic stocks and accumulation of flows in the production network.

Queuing principles are the basic tools used throughout this work. In some instances, it was used to determine the steady state parameters of some selected systems of interest. In other instances, the steady state parameters of some queuing processes were used to derive the control parameters (optimal feed rate) of some specific queuing processes considering a particular Operations Management principle.

1.6.2. Contributions to Knowledge

The purpose of this research in station flows in a supply chain is two pronged:

a) The first main contribution in this work is to the body of knowledge in the area of management of production system due to the nature of input system (i.e. pattern or arrival of demand from outside the production network). This involves the understanding of how the system behaves due to the nature of the demand and the characteristics of the processing centre. Zipkin (2000) has noted quite well that the only time in a supply chain when variability in input or processing time becomes important is during lead time, when there is a reasonable possibility of not meeting demand due to non availability of stock, and the attendant cost implication. So, the modelling interest is to understand the joint distribution of demand and lead time so that the steady state distribution of such system is determined, and from there, the system parameters can be calculated.

   This area is actually well researched, and there exists many probability models that have been developed as such. But the area is not yet full researched as there are still cases of some possible input types and demand characteristics not yet solved (e.g. the various $\boldsymbol{MAP}$ and $\boldsymbol{PH}$ distribution considerations being done in this thesis). The theoretical probability distribution of some such Markov processes were developed in this regards in chapters 2 to 4.

b) The second main contribution is in the area of management and accumulation of flows. The Theory of Constraints philosophy was particularly used as the reference philosophy. Contributions are made in the management of flow in a production environment that utilises this theory. This area appears to have an enormous potential for studies by applying the solutions of some of the steady state parameters of the various queuing processes already derived in regulating flows in such production environment. But the area does not appear well researched, and so, considered in this work.

1.7. CHAPTER OVERVIEW

The first chapter of this work contains the background to the study and a review of the relevant literature. The focus of the research is defined and the anticipated contributions to the field of learning were stated.

In chapter two, a multi-server service facility of a perishable inventory system with negative customer is presented. The item demanded is presented to the customer only after some service has been performed on the item. The inventory is depleted at the service rate rather than the demand rate. The arrival of customers follows a Markov Arrival Process ($\boldsymbol{MAP}$) and the service time has an exponential distribution. The ordering policy is $(\boldsymbol{s}, \boldsymbol{S})$, and the lead time has exponential distribution. A customer whose service could not be provided immediately moves into an orbit of infinite size, from where requests are sent back to the system at random intervals characterised by exponential distribution. In addition, a second flow of negative customers following an $\boldsymbol{MAP}$ removes one of the customers from the orbit. The joint probability of the number of busy servers, the inventory level and the number of customers in the orbit is obtained at the steady state. Various stationary system performance measures were calculated, and the result illustrated numerically.

Chapter three is a study of a continuous review retrial inventory system with a finite source of customers and identical multiple servers in parallel. The customers arrive according to a quasi-random distribution. The customers demand unit items which are then delivered after some service has been performed on the items. The re-ordering policy is $(\boldsymbol{s}, \boldsymbol{S})$, and its distribution is assumed to be exponential. A customer with unfulfilled order joins an orbit from which only customers selected based on certain rules can reapply for service. The joint probability distribution of the number of customers in the orbit and the steady state number of busy servers and inventory level are obtained. Measures of system performance were derived.

Chapter four is a study of two-commodity perishable inventory with bulk demand for one commodity. It is a continuous review process in which three flows of customers could demand single item of the first, bulk item of the second or both single item of the first and bulk of the second. The arrival pattern is assumed to be $\boldsymbol{MAP}$. Order policy is to place order for both items when inventory levels are below the fixed levels for both commodities. The lead time is assumed to have a phase type distribution and the demands that occur during the stock out period are lost. The joint probability distribution for both commodities is determined and the various measures of system parameters and the total expected cost rate in the steady state are derived and numerical illustration was done.

Chapter five studies the management of flow in a production environment managed through the Theory of Constraints approach. The system is a continuous or discontinuous flow process with a Poisson input flow and an exponential service time. The system is assumed to have only a Capacity Constrained Resource and no Bottle neck. The option of using a regulated input flow to dynamically control the buffer placed ahead of the critical resource to cover for variations in processing time was shown to provide better management approach than a case where a predetermined buffer size is placed ahead of the resource. This model was further modified to incorporate payment of penalty charges for cases of lost throughput. A formula for determining the optimal flow rate to allow in the system to maximise the system profit was developed. The effect of shortages on the system parameters was illustrated graphically.

Chapter six is basically the concluding overview, the contextualisation of some possible applications of the models developed in the thesis, and the identification of some suggested areas for further future research.

2. *

CHAPTER 2

A MULTI-SERVER PERISHABLE INVENTORY SYSTEM WITH NEGATIVE CUSTOMER

* A modified version of this chapter has been submitted to Computers and Industrial Engineering Journal. The revision has been completed and re-submitted.

2.1. INTRODUCTION

Stochastic inventory models in which the demanded item is not immediately delivered to the customer are being considered by many authors. As the item in the stock may require some time for installation or preparation etc, the time taken to deliver to the customers is positive and usually random. As this causes formation of queues, the inventory manager needs to consider the queue length as well as the waiting time apart from the mean inventory level, holding time, etc to evaluate the system performance and hence to implement various control policies.

Berman et al (1993) considered an inventory management system at a service facility which uses one item of the inventory for each service provided. They assumed that both demand and service rates are deterministic and constant, and queues can form only during the stock outs. They determined optimal order quantity that minimises the total cost rate. Berman and Kim (1999) analysed a problem in a stochastic environment where customers arrive at a service facility according to a Poisson process. The service times are exponentially distributed with mean inter arrival time which is assumed to be larger than the mean service time. Each service requires one item from the inventory. Under both the discounted and average cost cases, the optimal policy of both finite and infinite time horizon problems is a threshold ordering policy.

A logically related model was studied by He et al. (1998), who analysed a Markovian Inventory-Production system, in which the demands are processed by a single machine in a batch size of one. Berman and Sapna (2000) studied an inventory control problem at a service facility which requires one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They analysed the system with a finite waiting room. Under a specified cost structure the optimal ordering quantity that minimises the long run expected cost per unit time has been derived.

Sivakumar and Arivarignan (2006) considered an inventory system with service facility and negative customers. Schwarz et al (2006) have considered an inventory system with

Poisson demand, exponentially distributed service time and deterministic and randomised ordering policies. Manuel et al (2008) analysed an inventory system with service facility and finite waiting hall. They assumed the customers arrive according to a Markovian arrival process, the service times have phase-distribution, the lead time of the reorder and the life time of each item are exponential. When the waiting hall is full, an arriving customer joins the orbit of infinite size and after a random time, the customer tries his/her luck. Yadavalli et al (2008) considered an inventory system with service facility and infinite waiting hall. They assumed that demands occur according to a renewal process with instantaneous supply of reorders.

In all the above models, the authors assume that the service facility had a single server. But in many real life situations, the service facility has more than one server, and this is incorporated in this paper by assuming multiple servers. It was also assumed that any arriving customers who find all the servers are busy or all the items are in service enters into an orbit of infinite size to try their luck again sometime later.

Queues in which customers are allowed to conduct retrials have been widely used to model many problems in production/manufacturing engineering, communication engineering, etc. A complete description of situations where queues with retrial customers arise can be found in Falin and Templeton (1997). A classified biography is given in Artalejo (1999). For more details on multi-server retrial queues, see Anisimov and Artalejo (2001), Artalejo and Gomez-corral (2008), Artalejo et al (2001,2007), and Chakravarthy and Dudin (2002).

The rest of the paper is organised as follows. The next section gives a description of the mathematical model and the notations used. The steady state analysis of the model is presented in section 3. In section 4, various system performance measures in the steady state were derived. In the final section, the total expected cost rate in the steady state was derived and the results are illustrated using numerical examples.

2.2. MODEL DESCRIPTION

Consider the service facility which can stock a maximum of S units and $c(\geq 1)$ identical servers. The customers arrive according to a Markovian Arrival Process ($\boldsymbol{MAP}$) with representation (C_0, C_1) where C's are of order $m_1 \text{x}\, m_1$. The underlying Markov Chain $J_1(t)$ of the $\boldsymbol{MAP}$ has the generator $C(= C_0 + C_1)$ and a stationary distribution vector v_1 of length m_1. The stationary arrival rate is given by $\lambda_1 = v_1 C_1 \boldsymbol{e}$**,** where $\boldsymbol{e}$ is a column vector of appropriate dimension containing all ones. For more details on $\boldsymbol{MAP}$ and their properties, the reader may refer to Neuts (1995). If a new customer finds that anyone of the servers is idle, he/she immediately accedes to the service. The customer who finds either that all servers are busy or there is no service item (excluding those in service) in stock enters into an orbit of infinite size. These orbiting customers send requests at random time points for possible selection of their demands. The interval time between two successive request-time points is assumed to have exponential distribution with parameter θ. It is assumed that the access from the retrial group to the service facility is governed by the constant retrial policy described in Falin and Templeton (1997); i.e. the probability of repeated attempt during the interval $(t, t + \Delta t)$, is given by that $\theta\Delta t + o(\Delta t)$ as $\Delta t \rightarrow 0$. The service times have exponential distribution with rate μ both for primary customers and successful repeat customers. The items are perishable in nature and the life time of each item has a negative exponential distribution with parameter $\gamma(> 0)$. It is also assumed that the servicing item cannot perish. The operating policy is as follows: as soon as the inventory level drops to $s(> c)$, a replenishment order for $Q(= S - s > s)$ items is placed. The lead time is assumed to have exponential distribution with parameter $\beta(> 0)$.

In addition to the regular customers, a second flow of negative arrival following a $\boldsymbol{MAP}$ with representation (D_0, D_1) where D's are of order $m_2 \text{x} m_2$ is also considered. The underlying Markov Chain $J_2(t)$ of the $\boldsymbol{MAP}$ has the generator $D(= D_0 + D_1)$ and a stationary distribution vector v_{-1} of length m_2. The stationary arrival rate is given by $\lambda_{-1} = v_{-1} D_1 \boldsymbol{e}$. A negative customer has the effect of removing a customer from the orbit. The removal policy adopted is $\boldsymbol{RCE}$, (removal of a customer from the end of the queue).

Notations

$[A]_{i,j}$:	The element/sub matrix at $(i,j)th$ position of A
$\mathbf{0}$:	Zero matrix
$e_n(m)$:	A column vector of dimension n with 1 in the m^{th} position
I:	An identity matrix
I_k:	An identity matrix of order k.
$A \otimes B$:	Kronecker product of matrices A and B
$A \oplus B$:	Kronecker sum of matrices A and B
W	$= \{0,1,\dots,\}$
$h(x)$	$= \begin{cases} 1, & if \quad x \geq 0; \\ 0, & if \quad x < 0; \end{cases}$
$\delta_{(i,j)}$	$= \begin{cases} 1, & if \quad i = j; \\ 0, & otherwise; \end{cases}$
$\overline{\delta}_{(i,j)}$	$= 1 - \delta_{(i,j)}$
E_i	$= \{1,2,\dots,i\}$
E_i^0	$= \{0,1,\dots,i\}$

2.3. ANALYSIS

Let $X(t), L(t), Y(t), J_1(t)$ and $J_2(t)$, respectively, denote the number of customers in the orbit, the on-hand inventory level, the number of busy servers, the phase of the arrival of ordinary demand process and the phase of the arrival of the negative demand process at time t. From the assumptions made on the input and output processes, it can be shown that the stochastic process $\{X(t), L(t), Y(t), J_1(t), J_2(t); t \geq 0\}$ is a Markov process with state space given by

$$E = \{(i,k,m,u_1,u_2); i \in W, k \in E_{c-1}^0, m \in E_k^0, u_1 \in E_{m_1}^0, u_2 \in E_{m_2}^0\}$$
$$\cup \ \{(i,k,m,u_1,u_2); i \in W, k \in E_S \backslash E_{c-1}, m \in E_c^0, u_1 \in E_{m_1}^0, u_2 \in E_{m_2}^0\}.$$

Define the following ordered sets:

$$< i,k,m,u_1 > = ((i,k,m,u_1,1),(i,k,u_1,2),\dots(i,k,u_1,m_2)),$$

$$< i,k,m > = (< i,k,m,1 >, < i,k,m,2 >, \dots < i,k,m,m_1 >),$$

$$< i,k > = \begin{cases} (< i,k,0 >, < i,k,1 >, \dots < i,k,k >) & k \in E_{c-1}^0, \\ (< i,k,0 >, < i,k,1 >, \dots < i,k,c >) & k \in E_S \backslash E_c, \end{cases}$$

$$< i > = (< i,0 >, < i,1 >, \dots < i,S >).$$

Then the state space can be ordered as $(< 0 >, < 1 >, \dots)$.

The infinitesimal generator, P, of this process can be written in block partitioned form where the rows and columns correspond to $(< 0 >, < 1 >, \dots)$.

$$P = \begin{pmatrix} B_1 & A_0 & 0 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \tag{2.1}$$

where

$$A_0 = diag(H_0, H_1, \dots, H_{c-1}, H_c, H_c, \dots H_c)$$

$$H_v = e_{v+1}(v+1)e_{v+1}^T(v+1) \otimes (C_1 \otimes I_{m_2}), \quad v \in E_c^0$$

$$A_2 = diag(F_0, F_1, \dots, F_{c-1}, F_c, F_c, \dots F_c)$$

$$F_0 = I_{m1} \otimes D_1 \tag{2.2}$$

For $v \in E_c$

$$[F_v]_{k,l} = \begin{cases} I_{m1} \otimes D_1, & l = k, & k \in E_S^0 \\ \theta I_{m1} \otimes I_{m2}, & l = k+1, & k \in E_{v-1}^0 \\ 0, & & otherwise \end{cases} \tag{2.3}$$

$$[A_1]_{k,l} = \begin{cases} M_k, & l = k, & k \in E_S^0 \\ N_k, & l = k-1, & k \in E_S \\ G_k, & l = k+Q, & k \in E_c^0 \\ G_c, & l = k+Q, & k \in E_S \backslash E_c \\ 0 & & otherwise \end{cases} \tag{2.4}$$

$$G_k = J_k \otimes (\beta I_{m1} \otimes I_{m2}), \quad k \in E_c^0 \tag{2.5}$$

$$J_l = \begin{array}{c|c} & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & l & l+1 & \cdots & c \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ l \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \end{array}, \quad l \in E_c^0 \tag{2.6}$$

$$N_1 = \begin{pmatrix} \gamma I_{m1} \otimes I_{m2} \\ \mu I_{m1} \otimes I_{m2} \end{pmatrix} \tag{2.7}$$

For $v \in E_c \backslash E_1$

$$[N_v]_{k,l} = \begin{cases} (v-k)\gamma I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_{v-1}^0 \\ k\mu I_{m1} \otimes I_{m2}, & l = k-1, \quad k \in E_v \\ \mathbf{0}, & otherwise \end{cases} \tag{2.8}$$

For $v \in E_S \backslash E_c$

$$[N_v]_{k,l} = \begin{cases} (v-k)\gamma I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_c^0 \\ k\mu I_{m1} \otimes I_{m2}, & l = k-1, \quad k \in E_c \\ \mathbf{0}, & otherwise \end{cases} \tag{2.9}$$

$$M_0 = C_0 \oplus D_0 - \beta I_{m1} \otimes I_{m2} \tag{2.10}$$

For $v \in E_{c-1}$,

$$[M_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{v-1}^0 \\ C_0 \oplus D_0 - (v\gamma + \beta + \theta) I_{m1} \otimes I_{m2}, & l = k, \quad k = 0 \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \beta + \theta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_{v-1} \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \beta) I_{m1} \otimes I_{m2}, & l = k, \quad k = v \\ \mathbf{0} & otherwise \end{cases} \tag{2.11}$$

For $v \in E_s \backslash E_{c-1}$,

$$[M_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{c-1}^0 \\ C_0 \oplus D_0 - (v\gamma + \beta + \theta) I_{m1} \otimes I_{m2}, & l = k, \quad k = 0 \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \beta + \theta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_{c-1} \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \beta) I_{m1} \otimes I_{m2}, & l = k, \quad k = c \\ \mathbf{0} & otherwise \end{cases} \tag{2.12}$$

For $v \in E_S \backslash E_s$,

$$[M_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{c-1}^0 \\ C_0 \oplus D_0 - (v\gamma + \theta) I_{m1} \otimes I_{m2}, & l = k, \qquad k = 0 \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \theta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_{c-1} \\ C_0 \oplus D_0 - ((v-k)\gamma + k\mu + \theta) I_{m1} \otimes I_{m2}, & l = k, \qquad k = c \\ \mathbf{0} & \qquad otherwise \end{cases} \quad (2.13)$$

$$[B_1]_{k,l} = \begin{cases} \widetilde{M}_k, & l = k, \quad k \in E_S^0 \\ N_k, & l = k-1, \quad k \in E_S \\ G_k, & l = k+Q, \quad k \in E_c^0 \\ G_c, & l = k+Q, \quad k \in E_s \backslash E_c \\ 0 & \qquad otherwise \end{cases}$$

$$\widetilde{M}_0 = C_0 \oplus D - \beta I_{m1} \otimes I_{m2}$$

For $v \in E_{c-1}$,

$$[\widetilde{M}_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{v-1}^0 \\ C_0 \oplus D - (v\gamma + \beta) I_{m1} \otimes I_{m2}, & l = k, \qquad k = 0 \\ C_0 \oplus D - ((v-k)\gamma + k\mu + \beta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_v \\ \mathbf{0} & \qquad otherwise \end{cases} \quad (2.14)$$

For $v \in E_s \backslash E_{c-1}$

$$[\widetilde{M}_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{c-1}^0 \\ C_0 \oplus D - (v\gamma + \beta) I_{m1} \otimes I_{m2}, & l = k, \qquad k = 0 \\ C_0 \oplus D - ((v-k)\gamma + k\mu + \beta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_c \\ \mathbf{0} & \qquad otherwise \end{cases} \quad (2.15)$$

For $v \in E_S \backslash E_s$

$$[\widetilde{M}_v]_{k,l} = \begin{cases} C_1 \otimes I_{m2}, & l = k+1, \quad E_{c-1}^0 \\ C_0 \oplus D - v\gamma I_{m1} \otimes I_{m2}, & l = k, \qquad k = 0 \\ C_0 \oplus D - ((v-k)\gamma + k\mu + \beta) I_{m1} \otimes I_{m2}, & l = k, \quad k \in E_c \\ \mathbf{0} & \qquad otherwise \end{cases} \quad (2.16)$$

It may be noted that A_0, A_1, A_2, B_1 are square matrices of order $\left(c, \frac{c+1}{2}\right) m_1 m_2 + (S-c)(c+1) m_1 m_2$, $F_i, H_i, i \in E_c^0$ are square matrices of order $(i+1) m_1 m_2$, $\widetilde{M}_i$, $M_i, i \in E_{c-1}^0$ are square matrices of order $(i+1) m_1 m_2$, $\widetilde{M}_i, M_i, i \in E_S \backslash E_{c-1}$ are square matrices of order $(c+1) m_1 m_2$, $N_i, i \in E_c^0$ are of order $(i+1) m_1 m_2 \text{ x } i m_1 m_2$, $N_i, i \in E_S \backslash E_c$ are square matrices of order $(c+1) m_1 m_2$, $G_i, i \in E_{c-1}^0$ are of order $(i+1) m_1 m_2 \text{ x } (c+1) m_1 m_2$, and G_c is a square matrix of order $(c+1) m_1 m_2$.

2.3.1. Stability Analysis

To discuss the stability condition of the process, consider $A = A_0 + A_1 + A_2$ which is given by

$$[A]_{k,l} = \begin{cases} \widehat{M}_k, & l = k, \quad k \in E_S^0 \\ N_k, & l = k-1, \quad k \in E_S \\ G_k, & l = k+Q, \quad k \in E_c^0 \\ G_c, & l = k+Q, \quad k \in E_S \backslash E_c \\ \mathbf{0} & otherwise \end{cases} \tag{2.17}$$

where

$$\widehat{M}_k = \begin{cases} M_k + F_k + H_k, & k \in E_{c-1}^0 \\ M_k + F_c + H_c, & k \in E_S \backslash E_{c-1} \end{cases} \tag{2.18}$$

Let Π denote the steady state probability vector of A, which satisfies

$$\Pi A = \mathbf{0}, \Pi \boldsymbol{e} = 1$$

The vector Π can be represented by

$$\Pi = (\pi^{(0)}, \pi^{(1)}, \cdots, \pi^{(S)})$$

where

$$\pi^{(i)} = \begin{cases} (\pi_{(i,0)}, \pi_{(i,1)}, \dots, \pi_{(i,i)}), & i \in E_{c-1}^0 \\ (\pi_{(i,0)}, \pi_{(i,1)}, \dots, \pi_{(i,c)}), & i \in E_S \backslash E_{c-1} \end{cases} \tag{2.19}$$

with

$$\pi_{(i,k)} = (\pi_{(i,k,1)}, \pi_{(i,k,2)}, \dots, \pi_{(i,k,m1)}), \quad i \in E_S^0, k \in E_c^0$$

and

$$\pi_{(i,k,l)} = (\pi_{(i,k,l,1)}, \pi_{(i,k,l,2)}, \dots, \pi_{(i,k,l,m2)}), \quad i \in E_S^0, k \in E_c^0, l \in E_{m1}$$

It can be easily shown that

$$\pi^{(i)} = \pi^{(Q)} \Omega_i, i \in E_S^0 \tag{2.20}$$

where

$$\Omega_i = \begin{cases} (-1)^{Q-i} N_Q \widehat{M}_{Q-1}^{-1} N_{Q-1} \dots N_{i+1} \widehat{M}_i^{-1} & i = 0,1,2,\dots,Q-1 \\ I, & i = Q \\ (-1)^{S-i+1} \Omega_S \left[\sum_{j=0}^{S-c} \psi(s,j) G_c \eta(S-j,i) + \sum_{j=i-Q}^{c-1} \psi(s,j) G_c \eta(Q+j,i) \right], & i = Q+1, Q+2, \dots, Q+c-1 \\ (-1)^{S-i+1} \Omega_S \sum_{j=0}^{S-i} \psi(s,j) G_c \eta(S-j,i) & i = Q+c, Q+c+1, \dots, S \end{cases} \quad (2.21)$$

with

$$\psi(i,j) = \begin{cases} N_i \widehat{M}_{i-1}^{-1} N_{i-1} \dots \widehat{M}_{i-j}^{-1}, & j \geq 1 \\ I & j = 0 \end{cases}$$

$$\eta(i,j) = \widehat{M}_i N_i \widehat{M}_{i-1}^{-1} \dots \widehat{M}_j^{-1}. \quad (2.22)$$

and $\pi^{(Q)}$ can be obtained by solving

$$\pi^{(Q)} \left(\Omega_{Q+1} N_{Q+1} + \widehat{M}_Q + \Omega_0 G_0 \right) = \mathbf{0}.$$

and

$$\pi^{(Q)} \left(I + \sum_{\substack{k=0 \\ k \neq Q}}^{S} \Omega_k \right) \boldsymbol{e} = 1 \quad (2.23)$$

Now the following result obtains on the stability condition.

Lemma 1 The stability condition of the system under the study is given by

$$\sum_{i=0}^{c-1} \pi_{(i,i)} (C_1 \otimes I_{m2}) \boldsymbol{e} + \sum_{i=c}^{S} \pi_{(i,c)} (C_1 \otimes I_{m2}) \boldsymbol{e} < \begin{pmatrix} \sum_{i=0}^{c-1} \pi_{(i,i)} (I_{m1} \otimes D_1) \boldsymbol{e} + \sum_{i=c}^{S} \pi_{(i,c)} (I_{m1} \otimes D_1) \boldsymbol{e} \\ + \sum_{i=1}^{c-1} \sum_{j=0}^{i-1} \pi_{(i,j)} (I_{m1} \otimes D_1 + \theta I_{m1} \otimes I_{m2}) \boldsymbol{e} \\ + \sum_{i=c}^{S} \sum_{j=0}^{c-1} \pi_{(i,j)} (I_{m1} \otimes D_1 + \theta I_{m1} \otimes I_{m2}) \boldsymbol{e} \end{pmatrix} \quad (2.24)$$

Proof: From the well known result of Neuts (1994) on the positive recurrence of P, there exists

$$\Pi A_0 \boldsymbol{e} < \Pi A_2 \boldsymbol{e}$$

and by exploiting the structure of the matrices A_0 and A_2 and Π, the stated result follows.

2.3.2. Steady State Analysis

It can be seen from the structure of the rate matrix P and from the Lemma 1 that the Markov process $\{(X(t), L(t), Y(t), J_1(t), J_2(t))\ t \geq 0\}$ on E is regular. Hence, the limiting distribution is defined by

$$\emptyset^{(i,k,m,u_1,u_2)} = \lim_{t\to\infty} Pr[X(t) = i, L(t) = k, Y(t) = m, J_1(t) = u_1, J_2(t) = u_2 |\ X(0), L(0), Y(0), J_1(0), J_2(0)], \tag{2.25}$$

where $\emptyset^{(i,k,m,u_1,u_2)}$ is the steady-state probability for the state (i, k, m, u_1, u_2), exists and is independent of the initial state.

The probabilities $\emptyset^{(i,k,m,u_1,u_2)}$ can be grouped as follows:

$$\emptyset^{(i,k,l,u_1)} = (\emptyset^{(i,k,l,u_1,1)}, \emptyset^{(i,k,l,u_1,2)}, \dots , \emptyset^{(i,k,l,u_1,m_2)}),\ i \in W, k \in E_0^S, l \in E_c^0, u_1 \in E_{m1}$$

$$\emptyset^{(i,k,l)} = \left(\emptyset^{(i,k,l,1)}, \emptyset^{(i,k,l,2)}, \dots , \emptyset^{(i,k,l,m_1)}\right),\ i \in W, k \in E_0^S, l \in E_0^c$$

$$\emptyset^{(i,k)} = \begin{cases} \emptyset^{(i,k,0)}, \emptyset^{(i,k,1)}, \dots, \emptyset^{(i,k,k)}, & k \in E_{c-1}^0 \\ \emptyset^{(i,k,0)}, \emptyset^{(i,k,1)}, \dots, \emptyset^{(i,k,c)}, & k \in E_S \backslash E_{c-1} \end{cases}$$

and finally, write

$$\Phi^{(i)} = \left(\emptyset^{(i,0)}, \emptyset^{(i,1)}, \dots , \emptyset^{(i,S)}\right),\ i = 0,1,2, \dots \tag{2.26}$$

The limiting probability distribution $\Phi = (\Phi^{(1)}, \Phi^{(2)}, \dots)$ satisfies

$$\Phi P = 0,\ \Phi \boldsymbol{e} = 1. \tag{2.27}$$

Theorem 1: When the stability condition (2.24) holds good, the steady state probability vector, Φ, is given by

$$\Phi^{(j)} = \Phi^{(0)} R^{(j)},\ j = 0,1, \dots \tag{2.28}$$

where the matrix R satisfies the quadratic equation

$$R^2 A_2 + R A_1 + A_0 = \mathbf{0} \tag{2.29}$$

and the vector $\Phi^{(0)}$ is obtained by solving

$$\Phi^{(0)}(B_1 + R A_2) = \mathbf{0}. \tag{2.30}$$

subject to the normalising condition

$$\Phi^{(0)}(1-R)^{-1}\boldsymbol{e} = 1. \tag{2.31}$$

Proof: The theorem follows from the well known result of the matrix-geometric methods (Neuts, 1994).

2.3.2.1. Computation of the R matrix

In this subsection, an algorithmic procedure for computing the R matrix is presented, which is the main ingredient for discussing the qualitative behaviour of the system under study.

Due to the special structure of the coefficient matrices appearing in (2.29), the square matrix R of dimension $\left(\frac{c(c+1)}{2}\right)m_1m_2 + (S-c)m_1m_2$ can be computed as follows: Note that $A_0\boldsymbol{e}$ is of the form

$$A_0\boldsymbol{e} = \begin{array}{c} 0 \\ 1 \\ \vdots \\ c-1 \\ c \\ c+1 \\ \vdots \\ S \end{array}\begin{pmatrix} H_0\boldsymbol{e} \\ H_1\boldsymbol{e} \\ \vdots \\ H_{c-1}\boldsymbol{e} \\ H_c\boldsymbol{e} \\ H_c\boldsymbol{e} \\ \vdots \\ H_c\boldsymbol{e} \end{pmatrix}, \; H_i\boldsymbol{e} = \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \end{array}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ (C_1 \otimes I_{m2})\boldsymbol{e} \end{pmatrix}, \; i = 0,1,2,\dots,c \tag{2.32}$$

Due to the special structure of A_0 matrix, the matrix R has only$(S+1)m_1m_2$ rows of nonzero entries as shown below

$$R = \begin{pmatrix} R_{(0,0)} & R_{(0,1)} & \cdots & R_{(0,S)} \\ R_{(1,0)} & R_{(1,1)} & \cdots & R_{(1,0)} \\ \vdots & \vdots & \ddots & \vdots \\ R_{(S,0)} & R_{(S,1)} & \cdots & R_{(S,S)} \end{pmatrix} \tag{2.33}$$

where

$$R_{(0,i)} = 0 \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & i \end{array} \\ \begin{pmatrix} R_{(0,i)}^{(0)} & R_{(0,i)}^{(1)} & \cdots & R_{(0,i)}^{(i)} \end{pmatrix} \end{array}, \quad i = 0,1,\dots,c-1$$

$$R_{(0,i)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & c \end{array} \\ 0 & \begin{pmatrix} R^{(0)}_{(0,i)} & R^{(1)}_{(0,i)} & \cdots & R^{(i)}_{(0,i)} \end{pmatrix} \end{array}, \quad i = c, c+1, \dots, S$$

$$R_{(i,i)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & i \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,i)} & R^{(1)}_{(i,i)} & \cdots & R^{(i)}_{(i,i)} \end{pmatrix} \end{array}, \quad i = 1,2, \dots, c-1$$

$$R_{(i,i)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & c \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ c \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,i)} & R^{(1)}_{(i,i)} & \cdots & R^{(c)}_{(i,i)} \end{pmatrix} \end{array}, \quad i = c, c+1, \dots, S$$

$$R_{(i,j)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & j \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,j)} & R^{(1)}_{(i,j)} & \cdots & R^{(j)}_{(i,j)} \end{pmatrix} \end{array}, \quad \begin{array}{c} i = 1,2, \dots, c-1 \\ j = i+1, i+2, \dots, c \end{array}$$

$$R_{(i,j)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & c \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,j)} & R^{(1)}_{(i,j)} & \cdots & R^{(c)}_{(i,j)} \end{pmatrix} \end{array}, \quad \begin{array}{c} i = 1,2, \dots, c-1 \\ j = c+1, c+2, \dots, S \end{array}$$

$$R_{(i,j)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & j \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,j)} & R^{(1)}_{(i,j)} & \cdots & R^{(j)}_{(i,j)} \end{pmatrix} \end{array}, \quad \begin{array}{c} i = 1,2, \dots, c-1 \\ j = 0,1, \dots, i-1 \end{array}$$

$$R_{(i,j)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & j \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ c \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{(0)}_{(i,j)} & R^{(1)}_{(i,j)} & \cdots & R^{(j)}_{(i,j)} \end{pmatrix} \end{array}, \quad \begin{array}{c} i = c, c+1, \dots, S \\ j = 0,1, \dots, c-1 \end{array}$$

$$R_{(i,j)} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & j \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ c \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & \\ R^{0}_{(i,j)} & R^{1}_{(i,j)} & \cdots & R^{c}_{(i,j)} \end{pmatrix} \end{array}, \quad \begin{array}{c} i = c, c+1,, \dots, S \\ j = c, c+1,, \dots, S \\ i \neq j \end{array} \qquad (2.34)$$

The matrix R^2 is also of the form R with only $(S+1)m_1m_2$ nonzero rows. This form is exploited in the computation of R using (2.29). The relevant equations are given in the appendix.

2.4. SYSTEM PERFORMANCE MEASURES

In this section, some stationary performance measures of the system are derived. Using these measures, the total expected cost per unit time can be constructed.

2.4.1. Mean Inventory Level

Let Γ_I denote the mean inventory level in the steady state. Since $\emptyset^{(i,k)}$ denotes the steady state probability vector for kth inventory level with each component specifying a particular combination of the number of customers in the orbit, the number of busy servers, the phase of the ordinary arrival process and the phase of the negative arrival process, the quantity $\emptyset^{(i,k)}\boldsymbol{e}$ gives the probability that the inventory level is k in the steady state. Hence, the mean inventory level is given by

$$\Gamma_I = \sum_{i=0}^{\infty} \sum_{k=1}^{S} k\emptyset^{(i,k)}\boldsymbol{e} \tag{2.35}$$

2.4.2. Expected Reorder Rate

Let Γ_R denote the expected reorder rate in the steady state. Note that a reorder is triggered when the inventory level drops from $s+1$ to s. The steady state probability vector $\emptyset^{(i,s+1,l)}$ gives the rate at which $s+1$ is visited. After the system reaches the inventory level $s+1$, either a service completion of any of the l servers if $l > 0$ or a failure of anyone of $s+1-l$ items trigger the reorder event. This leads to

$$\Gamma_R = \sum_{i=0}^{\infty} \sum_{l=1}^{c} l\mu\emptyset^{(i,s+1,l)}\boldsymbol{e} + \sum_{i=0}^{\infty} \sum_{l=1}^{c} (s+1-l)\gamma\emptyset^{(i,s+1,l)}\boldsymbol{e} \tag{2.36}$$

2.4.3. Mean Perishable Rate

Since $\emptyset^{(i,k,l)}$ is a vector of probabilities with i customers in the orbit, the inventory level is k and l busy servers, the mean perishable rate, Γ_P in the steady state is given by

$$\Gamma_P = \sum_{i=0}^{\infty}\sum_{k=1}^{c}\sum_{l=0}^{k-1}(k-l)\gamma\emptyset^{(i,k,l)}\boldsymbol{e} + \sum_{i=0}^{\infty}\sum_{k=c+1}^{S}\sum_{l=0}^{c}(k-l)\gamma\emptyset^{(i,k,l)}\boldsymbol{e} \qquad (2.37)$$

2.4.4. Mean number of customers in the Orbit

Let Γ_O denote the expected number of customers in the orbit. Since $\Phi^{(i)}$ is the steady state probability vector for i customers in the orbit with each component specifying a particular combination of the inventory level, number of busy servers, the phase of the ordinary customers arrival process and the phase of the negative customers arrival process, the quantity $\Phi^{(i)}$ gives the probability that the number of customers in the orbit is i in the steady state. Hence, the expected number of customers in the orbit is given by

$$\begin{aligned}\Gamma_O &= \sum_{i=1}^{\infty} i\,\Phi^{(i)}\boldsymbol{e}.\\ &= \Phi^{(0)}R(I-R)^{-2}\boldsymbol{e}. \end{aligned} \qquad (2.38)$$

2.4.5. Mean Rate of Arrival of Negative Customers

Let Γ_N denote the mean arrival rate of negative demand in the steady state. This is given by

$$\Gamma_N = \frac{1}{\lambda_{-1}}\sum_{i=1}^{\infty}\left[\emptyset^{(i,0,0)}(I_{m1}\otimes D_1)\boldsymbol{e} + \sum_{k=1}^{c-1}\sum_{l=0}^{k}\emptyset^{(i,k,l)}(I_{m1}\otimes D_1)\boldsymbol{e} + \sum_{k=c}^{S}\sum_{l=0}^{c}\emptyset^{(i,k,l)}(I_{m1}\otimes D_1)\boldsymbol{e}\right] \qquad (2.39)$$

2.4.6. The overall Rate of Retrials

If Γ_{OR} is the overall rate of retrials in the steady state, then overall rate of trials at which the orbiting customers request service is given by

$$\Gamma_{OR} = \theta \sum_{i=1}^{\infty} \Phi^{(i)} \boldsymbol{e}$$
$$= \theta \Phi^{(0)} R(1-R)^{-1} \boldsymbol{e} \qquad (2.40)$$

2.4.7. The Successful Rate of Retrials

Let Γ_{SR} denote the successful rate of retrials in the steady state. Note that the orbiting customer can enter the service if there is at least one free server and there is at least one item which is not in service. Hence, the successful rate of retrial, Γ_{SR}, is given by

$$\Gamma_{SR} = \theta\left[\sum_{i=1}^{\infty} \sum_{k=1}^{c-1} \sum_{l=0}^{k-1} \emptyset^{(i,k,l)} \boldsymbol{e} + \sum_{i=1}^{\infty} \sum_{k=c}^{S} \sum_{l=0}^{c-1} \emptyset^{(i,k,l)} \boldsymbol{e}\right] \qquad (2.41)$$

2.4.8. The Fraction of Successful Rate of Retrial

The fraction of successful rate of retrial is given by

$$\Gamma_{FSR} = \frac{\Gamma_{SR}}{\Gamma_{OR}} \qquad (2.42)$$

2.4.9. The Expected Number of Busy Servers

If Γ_{BS} denotes the mean number of busy servers in the steady state, it is given by

$$\Gamma_{BS} = \sum_{i=1}^{\infty} \sum_{k=1}^{c-1} \sum_{l=0}^{k} l\emptyset^{(i,k,l)} \boldsymbol{e} + \sum_{i=1}^{\infty} \sum_{k=c}^{S} \sum_{l=1}^{c} l\emptyset^{(i,k,l)} \boldsymbol{e} \qquad (2.43)$$

2.4.10. The Expected Number of Idle Servers

If Γ_{IS} denotes the expected number of idle servers in the steady state, then Γ_{IS} is given by

$$\Gamma_{IS} = \mathrm{c} - \Gamma_{BS} \tag{2.44}$$

2.4.11. The Blocking Probability

Let Γ_B denote the blocking probability in the steady state. This is given by

$$\Gamma_B = \sum_{\mathrm{i}=0}^{\infty} \sum_{\mathrm{k}=0}^{\mathrm{c}-1} \emptyset^{(i,k,k)} \boldsymbol{e} + \sum_{\mathrm{i}=0}^{\infty} \sum_{\mathrm{k}=\mathrm{c}}^{\mathrm{S}} \emptyset^{(i,k,c)} \boldsymbol{e} \tag{2.45}$$

2.5. COST ANALYSIS

The total expected cost per unit time (expected cost rate) in the steady state for this model is defined to be

$$TC(S,s,c) = c_h \Gamma_I + c_P \Gamma_P + c_S \Gamma_R + c_w \Gamma_O + c_{ne} \Gamma_N \tag{2.46}$$

where

c_s: Setup cost per order

c_h: Inventory carrying cost per unit item per unit time

c_p: Perishable cost per unit item per unit time

c_w: Backlogging cost per unit time

c_{ne}: Loss per unit time due to arrival of a negative customer

Substituting Γs the cost rate becomes

$TC(S,s,c) = c_h\{\sum_{i=0}^{\infty}\sum_{k=1}^{s} k\emptyset^{(i,k)}\mathbf{e}\} + c_p\{\sum_{i=0}^{\infty}\sum_{k=1}^{c}\sum_{l=0}^{k-1}(k-l)\gamma\emptyset^{(i,k,l)}\mathbf{e} +$

*i=0∞k=c+1Sl=0ck−lγØi,k,l**e**+csi=0∞l=1clμØ(i,s+1,l)**e**+i=0∞l=0c(s+1−l)γØ(i, s+1,l)**e**+cwi=1∞iΦ(i)**e**+cne*1λ−1*i=1∞Øi,0,0Im1⊗D1**e**+k=1c−1l=0k−1Øi,k,lIm 1⊗D1**e**+k=cSl=0cØi,k,lIm1⊗D1**e***

(2.47)

Since the computation of the Ø's involve recursive equations, it is difficult to study the qualitative behaviour of the total expected cost rate analytically. However, the following numerical examples are presented to demonstrate the computability of the results derived in this work.

2.6. NUMERICAL ILLUSTRATIONS[†]

As the total expected cost rate is obtained in a complex form, one cannot study the qualitative behaviour of the total expected cost rate by the analytical methods. Hence, some 'simple' numerical search procedures have been used to find the "local" optimal values by considering a small set of integer values for the decision variables. With a large number of numerical examples, it was found out that the total cost rate per unit time in the long run is either a convex function or an increasing function of any one variable.

Consider the following $\boldsymbol{MAP}$'s for arrivals of regular demands as well as of negative demands. These processes can be normalised so as to have specific demand rate λ_1 (or λ_{-1}) when considered for arrivals of regular (negative) demands. Each of the $\boldsymbol{MAP}$ will be represented by (Z_0, Z_1), where Z_i's will represent C's for regular (positive) demands and D's for negative demands.

[†] Tables (2.2 to 2.19) referenced but not included in the body of this chapter could be found in Appendix 3

1 $\boldsymbol{Exponential\ (Exp)}$

$$Z_0 = (-1) \quad Z_1 = (1)$$

2 $\boldsymbol{Erlang\ (Erl)}$

$$Z_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3 $\boldsymbol{Hyper - exponential\ (HExp)}$

$$Z_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \quad Z_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}$$

4 $\boldsymbol{MAP}$ with negative correlation ($\boldsymbol{MNC}$)

$$Z_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 25.25 & 0 & 55.75 \\ 55.75 & 0 & 25.25 \end{pmatrix}$$

5 $\boldsymbol{MAP}$ with positive correlation ($\boldsymbol{MPC}$)

$$Z_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 55.25 & 0 & 25.75 \\ 25.75 & 0 & 55.25 \end{pmatrix}$$

All the above $\boldsymbol{MAP}$s are qualitatively different in that they have different variance and correlation structures. The first three processes are special cases of renewal processes and the correlation between the arrival times is 0. The demand process labelled $\boldsymbol{MNC}$ has correlated arrivals with correlation coefficient -0.1254 and the demands corresponding to the process $\boldsymbol{MPC}$ has positive correlation coefficient of 0.1213. Since $\boldsymbol{Erl}$ has the least variance among the five arrival processes considered here, the ratios of the variances of the other four processes labelled $\boldsymbol{Exp}, \boldsymbol{HExp}$, $\boldsymbol{MNC}$ and $\boldsymbol{MPC}$ above, with respect to the $\boldsymbol{Erl}$ process are 3.0, 15.1163, 8.1795, 8.1795 respectively. The ratios are given rather than the actual values since the variance depends on the arrival rate which is varied in the discussion. The parameters and values have been chosen in such a way that the system is stable.

In the following discussions, the notations $\boldsymbol{MAP}+, \boldsymbol{Exp}+, \boldsymbol{Erl}+,$ ... were used when the $\boldsymbol{MAP}s, \boldsymbol{EXP}, \boldsymbol{Erl}, \ldots$ were consider respectively for positive demands. When the process for negative demand were considered, the $+$ were replaced by $-$. For example, when a case with $\boldsymbol{HExp}$ were considered for positive demands and $\boldsymbol{MPC}$ for negative demands, this will be denoted by $(\boldsymbol{HEXP}+, \boldsymbol{MPC}-)$.

Example 2.1: In the first example, the optimum values, S^* and s^* that minimise the expected total cost rate were given for each of the five $\boldsymbol{MAP}$s for arrivals of regular demands considered against each of the five $\boldsymbol{MAP}$s for negative demands (see table 2.1). The associated expected total cost values are also given. The lower entry in each cell gives the optimal expected cost rate and the upper entries give the corresponding S^* and s^*. Fixing $\lambda_1 = 10, \lambda_{-1} = 4, c = 3, \beta = 3, \mu = 5, \gamma = 0.6, \theta = 5, c_h = 0.1, c_s = 10, c_p = 1, c_w = 9, c_{ne} = 10$, the following were observed:

1. For the case $(\boldsymbol{Erl}+, \boldsymbol{Erl}-)$, the optimal total cost rate and the optimal inventory level are smaller
2. For the case $(\boldsymbol{MPC}+, \boldsymbol{Hexp}-)$, the optimal cost rate is large
3. For the case $(\boldsymbol{HExp}+, \boldsymbol{HExp}-)$, the optimal inventory level is large
4. For the case $(\boldsymbol{Erl}+, \boldsymbol{Erl}-)$, the optimal inventory level is smaller

Example 2.2: The effect of correlation among positive demands and the correlation among negative demands on the total expected cost rate is studied in this example. Fixing $S = 25, s = 6, \lambda_1 = 6, \lambda_{-1} = 4, \beta = 3, \mu = 5, \gamma = 0.6, \theta = 5, c_h = 0.1, c_s = 10, c_p = 1,$ $c_w = 9, c_{ne} = 10$, the following were observed:

1. When the correlation coefficient of demands of the $\boldsymbol{MAP}+$ increases, the total expected cost rate increases. The same result is observed for $\boldsymbol{MAP}-$.
2. If the correlation among the positive demands increases, the total expected cost rates when computed for each of the $\boldsymbol{MAP}$s of negative demands increase. This

trend is observed for $c = 1,2,3$ and 4. But all the curves become almost equal when $c = 4$.

3. When the correlation among the negative demands increases, the total expected cost rate corresponding to $\boldsymbol{HExp}+$ approaches that of $\boldsymbol{MNC}+$. When the number of servers and the correlation in the $\boldsymbol{MAP}+$ increases, the difference between the total expected cost rate corresponding to $\boldsymbol{MPC}+$ and $\boldsymbol{MPC}-$ increases.
4. The total expected cost rates for $(\boldsymbol{MAP}+, \boldsymbol{Erl}-)$ for all $\boldsymbol{MAP}+$, have smaller value. The same is observed for $(\boldsymbol{Erl}+, \boldsymbol{MAP}-)$.
5. The total expected cost rates for $(\boldsymbol{MAP}+, \boldsymbol{HExp}-)$ and for $(\boldsymbol{MPC}+, \boldsymbol{MAP}-)$ have high values.

Table 2.1: MAP of arrivals

		MAP of negative arrivals									
		Exp-		Erl-		HExp-		MNC-		MPC-	
MAP of positive arrivals	Exp+	32.6872		31.1528		39.3456		35.5992		37.5572	
		34	8	33	7	37	10	35	9	36	9
	Erl+	25.9807		24.9220		30.2158		28.0187		29.0862	
		32	6	31	6	35	9	34	8	34	8
	HExp+	63.6298		60.7149		77.5237		69.0841		74.1758	
		41	12	40	12	43	13	42	13	42	13
	MNC+	52.2187		49.5678		65.0810		57.1639		61.5312	
		37	10	36	10	41	12	38	11	39	11
	MPC+	82.0489		78.8221		98.6941		88.0139		94.0573	
		41	12	40	12	42	13	41	13	42	13

Example 2.3: In this example, the effect of each of the following were illustrated: the positive demand rate λ_1, the negative demand rate λ_{-1}, the lead time β, the service rate μ, the retrial rate θ, the perishable rate γ, the number of servers, $(\boldsymbol{MAP}+, \boldsymbol{HExp}-)$, on the fraction of the successful rate of retrial, Γ_{FSR}. From tables 2.2-2.7, the following were observed:

1. As λ_1 increases, Γ_{FSR} increases, except for the $(\boldsymbol{MPC}+, \boldsymbol{Erl}-)$.
2. Except $c = 1$, the values of Γ_{FSR} decreases as λ_{-1} increases for the model $(\boldsymbol{RP}+, \boldsymbol{RP}-)$, where $\boldsymbol{RP}$ represents the renewal processes, $\boldsymbol{Exp}, \boldsymbol{Erl}$ and $\boldsymbol{HExp}$. (In each of these cases, there is no correlation among the arrivals of demands).

3. In the case of correlated demand processes, i.e. those cases of $(\boldsymbol{NRP}+, \boldsymbol{NRP}-)$, where $\boldsymbol{NRP} = \boldsymbol{MNC\ or\ MPC}$, Γ_{FSR} decreases with $\beta$and increases with θ, when $c \neq 4$.
4. But Γ_{FSR} increases with γ for all c values.
5. It was noted that for all values of c, Γ_{FSR} assumes low value when the input nature is $(\boldsymbol{Erl}+, \boldsymbol{Erl}-)$. It was also noted that this value approaches zero as c increases.

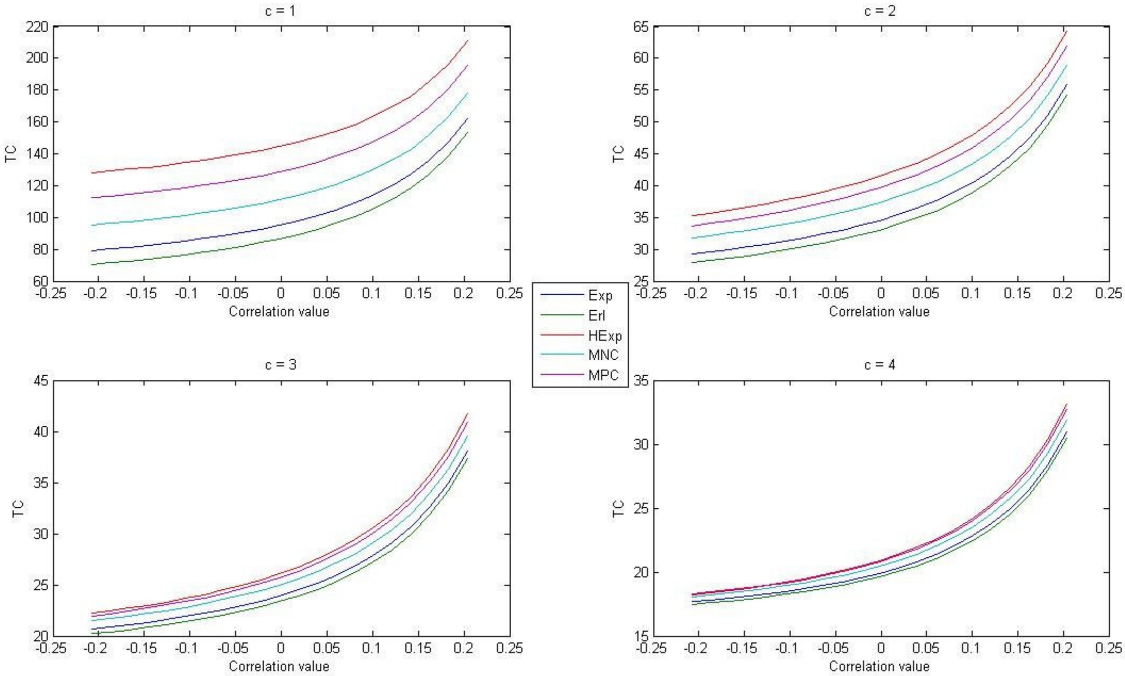

Figure 2.1: The effect of positive demand correlation on TC

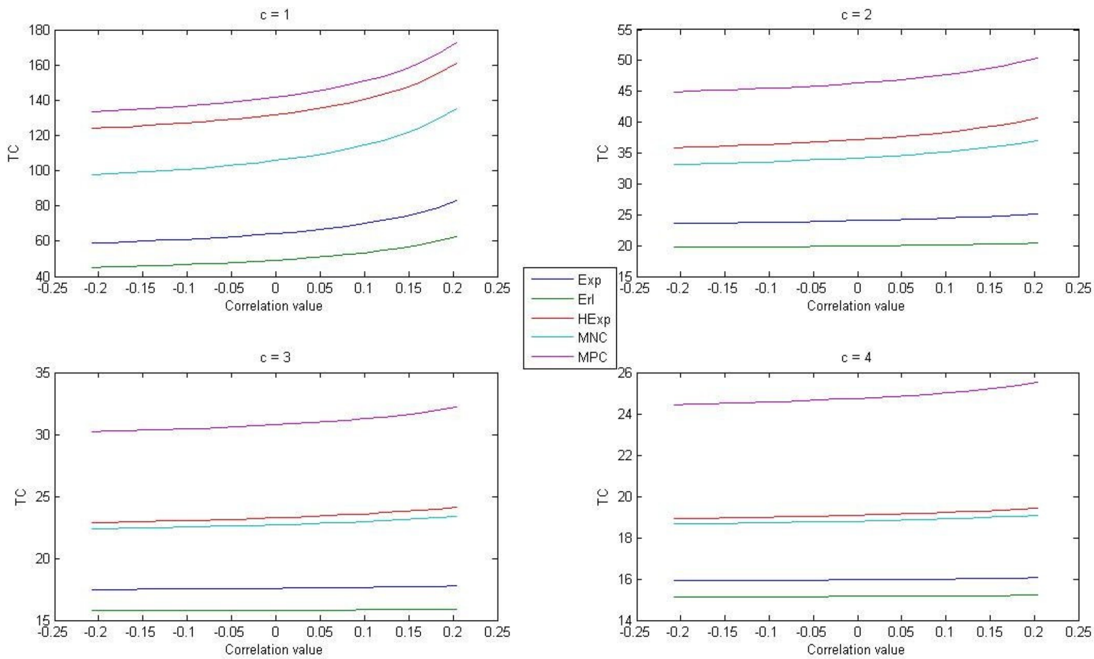

Figure 2.2: The influence of negative demand correlation on T C

Example 2.4: The influences of λ_1, λ_{-1}, β, μ, θ, γ, c and $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ on the blocking probability Γ_B is presented in this example. From tables 2.8 – 2.13, the following were observed:

1. Except for $c = 1$, as $\lambda_1$increases, Γ_B increases for each of the $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ process. For the single server case, as λ_1 increases, Γ_B decreases. The same behaviour is observed when θ increases.
2. Except for $c = 1$, Γ_B decreases when λ_{-1} increases.
3. Γ_B increases when the lead time rate β increases for each of the $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ process.
4. Whenever the number of servers is more than one, Γ_B increases with μ.
5. Γ_B increases with β for each of the $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ process.

Example 5: In this example, the effect of λ_1, λ_{-1}, β, μ, θ, γ, c, $\boldsymbol{MAP}+$ and $\boldsymbol{MAP}-$on the expected number of idle servers, Γ_{IS} were studied. From tables 2.14 - 2.19, the following were noted:

1. As is to be expected, as λ_1 increases, Γ_{IS} decreases except for single server case. This can be explained intuitively as follows. When the rate of positive customers increase, more number of servers would be engaged. This leads to decrease in the number of idle servers. For $c = 1$, Γ_{IS} increases with λ_1. This pattern is also observed for μ, θ.
2. Except for the single server case, Γ_{IS} increases as λ_2 increases. This is because as the negative customers frequently enter the orbit, they remove more customers from the orbit. Therefore, the number of retrying customers in the orbit decreases. Note that the servers will be occupied by both the positive demand and retrial customers. If the retrial customers' level decreases, then naturally, the customers from the orbit will also decrease. This forces the expected number of idle servers to increase.
3. As is to be expected, Γ_{IS} increases as β increases for each of the $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ process.
4. Except for $c = 4$, Γ_{IS} decreases as μ increases.
5. When γ increases, Γ_{IS} decreases for each of the $(\boldsymbol{MAP}+, \boldsymbol{MAP}-)$ process.

CONCLUSION

A continuous review perishable inventory system in a service facility with multi servers is studied in this work. The customers who could not get their demands attended to due to non-availability of items in stock or all the servers are busy join an orbit of infinite size. These customers attempt for service at random times. The customers are removed one by one by negative customers who could be touts of competing organisations. The novel attempt made in this work is to assume independent Markovian Arrival Processes $(\boldsymbol{MAP})$ for the positive demands and negative demands. By assuming $(\boldsymbol{MAP})$, one can also consider non renewal processes with correlated arrivals. Though, algorithmic solution is provided for this model, extended numerical examples were provided to discuss the behaviour of the expected total cost rate and the system performance measures due to changes or variations in the parameters.

3.

CHAPTER 3

A FINITE SOURCE MULTI-SERVER INVENTORY SYSTEM WITH SERVICE FACILITY

3.1. INTRODUCTION

One implicit assumption made by many previous stochastic inventory models is that the item whose inventory is kept is made available to the customer immediately it is demanded. This is not generally true, however, as many items are delivered only after some work has been done on them. This is a particularly growing trend as many organisations are strategically shifting their production approach from a make-to-stock system to an assemble-to-order system. Such systems have longer lead time but maintain smaller inventory levels than the make-to-stock system. The implication of such increase in lead time on the level of service available to customers is an area that is now being actively researched by many authors.

Berman et al (1993) considered an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service rates are deterministic and constant and queues can form only during stock outs. They determined optimal order quantity that minimizes the total cost rate. Berman and Kim (1999) analysed a problem in a stochastic environment where customers arrive at a service facility according to a Poisson process. The service times are exponentially distributed with mean inter-arrival time which is assumed to be larger than the mean service time. Under both the discounted and the average cost cases, the optimal policy of both the finite and infinite time horizon problem is a threshold ordering policy. A logically related model was studied by He et al. (1998), who analyzed a Markovian inventory - production system, in which demands are processed by a single machine in a batch of size one. Berman and Sapna (2000) studied an inventory control problem at a service facility which requires one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They assumed that their the system has finite waiting room. Under a specified cost structure, the optimal ordering quantity that minimizes the long-run expected cost per unit time was derived. Schwarz et al. (2006) considered an inventory system with Poisson demand and exponentially distributed service time with deterministic and randomized ordering policies.

In all the above models the authors assumed that the service facility had a single server. But in many real life situations the service facility may provide more than one server so that more customers are handled at a time. Moreover if a customer's request cannot be processed for want of stock or free server he/she may prefer to leave the system and make an attempt at later time. The concept of having unserviced customers in an orbit and allowing them to retry for the service have been considered in queueing systems. A complete description of situations where queues with retrial customers arise can be found in Falin and Templeton (1997). A classified bibliography is given in Artalejo (1999). For more details on multi-server retrial queues see Anisimov and Artalejo (2001), Artalejo et al. (2001) and Chakravarthy and Dudin (2002).

Multi server inventory system with service facility was considered by Arivarignan et al (2008). They assumed a continuous review (s, S) perishable inventory system in which the customers arrive according to a Markovian arrival process. The service time, the lead time for the reorders and the life time of the items were assumed to be exponential. The customer who arrive during the stock-out period or all the items in the inventory are in service or all the servers are busy entered into the orbit of infinite size and these customers compete for their service after an exponentially distributed time interval. Using matrix geometric method, they derived the steady state probabilities and under a suitable cost structure, they calculated the long run total expected cost rate.

In this chapter, the focus is on the case in which the population of demanding customers under study is finite so that each individual customer generates his own flow of primary demand. The inventory system with finite source was received only a little attention. This concept was introduced by Sivakumar (2009). But the analysis of finite source retrial queue in continuous time have been considered by many authors, the interested reader see Falin and Templeton (1997), Artalejo (1998) and Falin and Artalejo (1998) Almasi et al., (2005) and Artalejo and Lopez-Herero (2007) and references therein. The chapter utilises the quasi-random distribution for the arrival process. A good reading on quasi-random distribution is Sharafali et al (2009).

The rest of the chapter is organized as follows. In the next section, the mathematical model and the notation used were described. The steady state analysis of the model is presented in section 3. In section 4, the various system performance measures in the steady state were derived. In the final section, the total expected cost rate in the steady state were calculated.

Notations :

$[A]_{i,j}$: element/sub-matrix at ith row, jth column of the matrix A.

$\mathbf{0}$: zero vector.

$\mathbf{I}$: identity matrix.

$e^T = (1,1, \dots ,1)$.

$E_i^0 = \{0,1, \dots , i\}$.

$E_i^1 = \{1,2, \dots , i\}$.

$\delta_{ij} = \begin{cases} 1, & if j = i, \\ 0, & otherwise. \end{cases}$

$\bar{\delta}_{ij} = 1 - \delta_{ij}$.

3.2. MODEL DESCRIPTION

Consider a service facility which can stock a maximum of S units and c (≥ 1) identical servers. It is assumed that the arrival process of customers is quasi random with parameter α. The number of sources that generate the customers is assumed to be N. The customers demand a single item and the item is delivered to the customer after performing some service on the item. The service time is assumed to have exponential distribution. If a customer finds any one of the server is idle and at least one item is not in service, then he/she immediately accedes to the service. The customer who finds either all the servers are busy or all the items are in service enters the orbit of unsatisfied customers. These orbiting customers send requests at random time points for possible selection of their demands. The time intervals describing the repeated attempts are assumed to be independent and exponentially distributed with rate $\theta\bar{\delta}_{0j} + i\nu$, when there are i customers in orbit. The service times are independent exponential random variables with rate μ. As and when the on-hand inventory level drops to a prefixed level $s(\geq c)$, an order for $Q(= S - s > s)$ units is placed. The lead time distribution is exponential with parameter $\beta(> 0)$. The streams of arrival of customers, intervals separating successive repeated attempts, service times and lead times are assumed to be mutually independent.

3.3. ANALYSIS

Let $X(t), L(t)$ and $Y(t)$, respectively, denote the number of customers in the orbit, the on-hand inventory level (including those items that are in the service) and the number of busy servers at time t. From the assumptions made on the input and output processes, it may be verified that the stochastic process $\{(X(t), L(t), Y(t)), t \geq 0\}$ is a Markov process with the state space given by

$$\begin{aligned}\Omega = &\{(i,j,k); i \in E_{N-c}^0, j \in E_c^0, k \in E_j^0\} \cup \{(i,j,k); i \in E_{N-c}^0, j \in E_S \backslash E_c, k \in E_c^0\} \\ &\cup \{(i,j,k); i \in E_N \backslash E_{N-c}, j \in E_{N-i}^0, k \in E_j^0\} \\ &\cup \{(i,j,k); i \in E_N \backslash E_{N-c}, j \in E_S \backslash E_{N-i}, k \in E_{N-i}^0\}\end{aligned}$$

The infinitesimal generator of this process, defined by

$$P = (\ p((i,j,k),(l,m,n))\ ), \qquad (i,j,k),(l,m,n) \in E,$$

can be easily calculated and is given by

$$
\begin{cases}
\begin{array}{lll}
(N-i-k)\alpha, & l=i, & i \in E^0_{N-c-1}, \\
 & m=j, & j \in E_S, \\
 & n=k+1, & k \in E^0_{\min(j-1,c-1)}, \\
 & & or \\
 & l=i, & i \in E_{N-1} \backslash E_{N-c-1}, \\
 & m=j, & j \in E_S, \\
 & n=k+1, & k=0,1,\dots,\min(j-1,N-i-1), \\
 & & or \\
 & l=i+1, & i \in E^0_{N-c-1}, \\
 & m=j, & j \in E^0_S, \\
 & n=k, & k=\min(j,c), \\
 & & or \\
 & l=i+1, & i \in E_{N-1} \backslash E_{N-c-1}, \\
 & m=j, & j \in E^0_S, \\
 & n=k, & k=\min(j,N-i), \\
\theta+i\nu, & l=i-1, & i \in E_{N-c-1}, \\
 & m=j, & j \in E_S, \\
 & n=k+1, & k=0,1,\dots,\min(j-1,c-1), \\
 & & or \\
 & l=i-1, & i \in E_N \backslash E_{N-c-1}, \\
 & m=j, & j \in E_S, \\
 & n=k+1, & k=0,1,\dots,\min(j-1,N-i-1),
\end{array}
\end{cases}
\qquad (3.1)
$$

$$\begin{cases} \beta, & l = i, & i \in E_{N-c-1}, \\ & m = j + Q, & j \in E_S^0, \\ & n = k, & k = 0,1,\dots,\min(j,c), \\ & & or \\ & l = i, & i \in E_N \backslash E_{N-c-1}, \\ & m = j, & j \in E_S^0, \\ & n = k, & k = 0,1,\dots,\min(j,N-i), \\ \mu & l = i, & i \in E_{N-c-1}, \\ & m = j - 1, & j \in E_S, \\ & n = k - 1, & k = 1,2,\dots,\min(j,c), \\ & & or \\ & l = i, & i \in E_N \backslash E_{N-c-1}, \\ & m = j - 1, & j \in E_S, \\ & n = k - 1, & k = 1,2,\dots,\min(j,N-i), \\ -\big((N-i-k)\alpha + k\mu & l = i, & i \in E_{N-c-1}^0, \\ +h(s-j)\beta + \bar{\delta}_{i0}\bar{\delta}_{j0}(\theta + iv)\big), & m = j, & j \in E_S^0, \\ & n = k, & k = 0,1,\dots,\min(j,c), \\ & & or \\ -\big((N-i-k)\alpha + k\mu & l = i, & i \in E_N \backslash E_{N-c-1}, \\ +h(s-j)\beta + \bar{\delta}_{j0}(\theta + iv)\big), & m = j, & j \in E_S^0, \\ & n = k, & k = 0,1,\dots,\min(j,N-i), \\ 0, & & otherwise. \end{cases} \tag{3.2}$$

Define the following ordered sets

$$\begin{aligned} &For\ \ i = 0,1,\dots,N-c, \\ &< i,j > \ = \begin{cases} ((i,j,0),(i,j,1),\dots,(i,j,j)), & j = 0,1,\dots,c, \\ ((i,j,0),(i,j,1),\dots,(i,j,c)), & j = c+1,c+2,\dots,S, \end{cases} \\ &For\ i = N-c+1, N-c+2,\dots,N, \\ &< i,j > \ = \begin{cases} ((i,j,0),(i,j,1),\dots,(i,j,j)), & j = 0,1,\dots,N-i, \\ ((i,j,0),(i,j,1),\dots,(i,j,N-i)), & j = N-i+1,N-i+2,\dots,S, \end{cases} \\ &< i > \ = \ (< i,0 >, < i,1 >,\dots, < i,S >), i = 0,1,\dots,N. \end{aligned} \tag{3.3}$$

Then the state space can be ordered as $(< 0 >, < 1 >, \dots, < N >)$.

The infinitesimal generator P of this process may be expressed conveniently as a block partitioned matrix with entries

$$[P]_{il} = \begin{cases} U_i, & l = i, \quad i = 0,1,\dots,N, \\ V_i, & l = i+1, \quad i = 0,1,\dots,N-1, \\ W_i, & l = i-1, \quad i = 1,2,\dots,N, \\ \mathbf{0}, & otherwise. \end{cases} \tag{3.4}$$

More explicitly,

where

$$\begin{aligned}
&For\ \ i = 0,1,\dots,N-c-1, \\
&[V_i]_{jm} = \begin{cases} H_{ij}, & m = j, \quad j = 0,1,\dots,c-1, \\ H_{ic}, & m = j, \quad k = c, c+1,\dots,S, \\ \mathbf{0}, & otherwise. \end{cases} \\
&For\ \ i = N-c, N-c+1,\dots,N-1, \\
&[V_i]_{jm} = \begin{cases} H_{ij}, & m = j, \quad j = 0,1,\dots,N-i-1, \\ \mathbf{0}, & otherwise. \end{cases} \\
&For\ \ i = 0,1,\dots,N-c-1, j = 0,1,\dots,c \\
&[H_{ij}]_{kn} = \begin{cases} (N-i-k)\alpha, & n = k, \quad k = j, \\ 0, & otherwise. \end{cases} \\
&For\ \ i = N-c, N-c+1,\dots,N-1, j = 0,1,\dots,N-i, \\
&[H_{ij}]_{kn} = \begin{cases} (N-i-k)\alpha, & n = k, \quad k = j, \\ 0, & otherwise. \end{cases} \\
&For\ \ i = 1,2,\dots,N-c, \\
&[W_i]_{jm} = \begin{cases} M_{ij}, & m = j, \quad j = 1,2,\dots,c-1, \\ M_{ic}, & m = j, \quad j = c, c+1,\dots,S, \\ \mathbf{0}, & otherwise. \end{cases} \\
&For\ \ i = N-c+1, N-c+2,\dots,N-1, \\
&[W_i]_{jm} = \begin{cases} M_{ij}, & m = j, \quad j = 1,2,\dots,N-i-1, \\ M_{i(N-i)}, & m = j, \quad j = N-i, N-i+1,\dots,S, \\ \mathbf{0}, & otherwise. \end{cases} \\
&[W_N]_{jm} = \begin{cases} M_{i0}, & m = j, \quad j = 1,2,\dots,S, \\ \mathbf{0}, & otherwise. \end{cases} \\
&For\ \ i = 1,2,\dots,N-c, j = 1,2,\dots,c, \\
&[M_{ij}]_{kn} = \begin{cases} \theta + iv, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0, & otherwise. \end{cases} \\
&For\ \ i = N-c+1, N-c+2,\dots,N, j = 1,2,\dots,N-i+1, \\
&[M_{ij}]_{kn} = \begin{cases} \theta + iv, & n = k+1, \quad k = 0,1,\dots,j, \\ 0, & otherwise. \end{cases}
\end{aligned} \tag{3.5}$$

$For\ i = 0,1,\dots,N-c,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m=j, & j=0,1,\dots,c-1,\\ D_{ic}, & m=j, & j=c,c+1,\dots,s,\\ D_{i(s+1)}, & m=j, & j=s+1,s+2,\dots,S,\\ F_{ij}, & m=j, & j=1,2,\dots,c,\\ F_{i(c+1)}, & m=j, & j=c+1,c+2,\dots,S,\\ G_{ij}, & m=j+Q, & j=0,1,\dots,c-1,\\ G_{ic}, & m=j+Q, & j=c,c+1,\dots,S,\\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2,\dots,N-1,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m=j, & j=0,1,\dots,N-i-1,\\ D_{i(N-i)}, & m=j, & j=N-i,N-i+1,\dots,s,\\ D_{i(N-i+1)}, & m=j, & j=s+1,s+2,\dots,S,\\ F_{ij}, & m=j, & j=1,2,\dots,N-i,\\ F_{i(N-i+1)}, & m=j, & j=N-i+1,N-i+2,\dots,S,\\ G_{ij}, & m=j+Q, & j=0,1,\dots,N-i-1,\\ G_{i(N-i)}, & m=j+Q, & j=N-i,N-i+1,\dots,s,\\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = N,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m=j, & j=0,\\ D_{i1}, & m=j, & j=1,2,\dots,s,\\ D_{i2}, & m=j, & j=s+1,s+2,\dots,S,\\ G_{i0}, & m=j+Q, & j=0,1,\dots,s,\\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = 0,1,\dots,N, j = 0,1,\dots,\min(c, N-i),$

$$[G_{ij}]_{kn} = \begin{cases} \beta & n=k, \quad k=0,1,\dots,j,\\ 0, & otherwise. \end{cases}$$

$For\ i = 0,1,\dots,N-c, j = 1,2,\dots,c,$

$$[F_{ij}]_{kn} = \begin{cases} k\mu & n=k-1, \quad k=1,2,\dots,j,\\ 0, & otherwise. \end{cases}$$

$For\ i = 1,2,\dots,N-c,$

$$[F_{i(c+1)}]_{kn} = \begin{cases} k\mu & n=k-1, \quad k=1,2,\dots,c,\\ 0, & otherwise. \end{cases}$$

$For\ i = N-c+1,1,\dots,N, j = 1,2,\dots,N-i,$

$$[F_{ij}]_{kn} = \begin{cases} k\mu & n=k-1, \quad k=1,2,\dots,j,\\ 0, & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2, \dots, N-1,$

$$[F_{i(N-i+1)}]_{kn} = \begin{cases} k\mu, & n = k-1, \quad k = 1,2,\dots,N-i, \\ 0, & otherwise. \end{cases}$$

$$D_{00} = -(N\alpha + \beta),$$

$For\ j = 1,2,\dots,c,$

$$[D_{0j}]_{kn} = \begin{cases} -((N-k)\alpha + k\mu + \beta), & n = k, \quad k = 0,1,\dots,j, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{0(c+1)}]_{kn} = \begin{cases} -((N-k)\alpha + k\mu), & n = k, \quad k = 0,1,\dots,c, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$For\ i = 1,2,\dots,N-c,$

$$D_{i0} = -((N-i)\alpha + \beta),$$

$For\ j = 1,2,\dots,c,$

$$[D_{ij}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kj}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,j, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(c+1)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,c, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2, \dots, N-1,$

$$D_{i0} = -((N-i)\alpha + \beta),$$

$For\ j = 1,2,\dots,N-i-2,$

$$[D_{ij}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kj}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,j, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(N-i-1)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,N-i-1, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(N-i)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,N-i-1, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$$D_{N0} = -\beta,$$
$$D_{N1} = -((\theta + N\nu) + \beta),$$
$$D_{N2} = -(\theta + N\nu).$$

In table 3.1, the size of the sub matrices listed above were given.

Table 3.1: The submatrices and their size

Matrix	Size
$U_i, i = 0,1,\ldots,N-c$, $V_i, i = 0,1,\ldots,N-c-1$, $W_i, i = 1,2,\ldots,N-c$,	$\frac{c(c+1)}{2} + (S-c+1)(c+1) \times \frac{c(c+1)}{2} + (S-c+1)(c+1)$
$U_i, i = N-c+1, N-c+2,\ldots,N$,	$\frac{j(j+1)}{2} + (S-j+1)(j+1) \times \frac{j(j+1)}{2} + (S-j+1)(j+1)$, $j = N-i$
$V_i, i = N-c, N-c+1,\ldots,N-1$,	$\frac{j(j+1)}{2} + (S-j+1)(j+1) \times \frac{(j+1)(j+2)}{2} + (S-j)(j+2)$, $j = N-i$
$W_i, i = N-c+1, N-c+2,\ldots,N$,	$\frac{j(j-1)}{2} + (S-j+2)j \times \frac{j(j+1)}{2} + (S-j+1)(j+1)$, $j = N-i$
$H_{ij}, i = 0,1,\ldots,N-c-1$, $j = 0,1,\ldots,c$	$(j+1) \times (j+1)$
$H_{ij}, i = N-c, N-c-1,\ldots,N-1, j = 0,1,\ldots,N-i$	$(j+1) \times (j+1)$
$M_{ij}, i = 0,1,\ldots,N-c$, $j = 1,2,\ldots,c$	$(j+1) \times (j+1)$
$M_{ij}, i = N-c+1, N-c+2, \ldots, N, j = 1,2,\ldots,N-i+1$	$(j+1) \times (j+2)$
$G_{ij}, i = 0,1,\ldots,N-c$, $j = 0,1,\ldots,c$	$(j+1) \times (c+1)$
$G_{ij}, i = N-c+1, N-c+2,\ldots,N,\ j = 0,1,\ldots,N-i$	$(j+1) \times (N-i+1)$
$F_{ij}, i = 0,1,\ldots,N-c$, $j = 1,2,\ldots,c$	$(j+1) \times j$

$F_{ij}, i = 0,1, \dots, N-c,$ $j = c+1,$	$(c+1) \times (c+1)$
$F_{ij}, i = N-c+1, N-c+2, \dots, N-1, j = 1,2, \dots, N-i$	$(j+1) \times j$
$F_{ij}, i = N-c+1, N-c+2, \dots, N-1, j = N-i+1$	$j \times j$
$D_{ij}, i = 0,1, \dots, N-c,$ $j = 0,1, \dots, c$	$(j+1) \times (j+1)$
$D_{ij}, i = 0,1, \dots, N-c,$ $j = c+1,$	$(c+1) \times (c+1)$
$D_{ij}, i = N-c+1, N-c+2, \dots, N, j = 0,1,2, \dots, N-i$	$(j+1) \times (j+1)$
$D_{ij}, i = N-c+1, N-c+2, \dots, N,\ j = N-i+1$	$j \times j$

3.3.1. Steady State Analysis

It can be seen from the structure of the infinitesimal generator P that the time-homogeneous Markov process $\{(X(t), L(t), Y(t)); t \geq 0\}$ on the finite state space E is irreducible. Hence the limiting distribution

$$\phi_{(i,j,k)} \quad = \quad \lim_{t \to \infty} Pr[X(t) = i, L(t) = j, Y(t) = k | X(0), L(0), Y(0)]$$

exists. Let

$$\phi_{(i,j)} = \begin{cases} (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,j)}), & j = 0,1, \dots, c, \qquad i = 0,1, \dots, N-c, \\ (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,c)}), & j = c+1, c+2, \dots, S, \quad i = 0,1, \dots, N-c, \end{cases}$$

$$\phi_{(i,j)} = \begin{cases} (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,j)}), & j = 0,1, \dots, N-i, \\ & i = N-c+1, N-c+2, \dots, N, \\ (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,N-i)}), & j = N-i+1, N-i+2, \dots, S, \\ & i = N-c+1, N-c+2, \dots, N, \end{cases} \tag{3.6}$$

$$\phi_{(i)} = \quad (\phi_{(i,0)}, \phi_{(i,1)}, \dots, \phi_{(i,S)}),$$
$$and$$
$$\Phi \quad = \quad (\phi_{(0)}, \phi_{(1)}, \dots, \phi_{(N)}).$$

Then the vector of limiting probabilities Φ satisfies

$$\Phi P = 0 \quad and \quad \Phi e = 1. \tag{3.7}$$

From the structure of P, it is seen that the Markov process under study falls into the class of birth and death process in a Markovian environment as discussed by Gaver et al. (1984). Hence using the same argument, the limiting probability vectors can be calculated. For the sake of completeness, the algorithm is provided here.

Algorithm :

Determine recursively the matrices

$$\begin{aligned} Z_0 &= U_0 \\ Z_i &= U_i + W_i(-Z_{i-1}^{-1})V_0, \quad i = 1,2,\dots,N. \end{aligned} \tag{3.8}$$

Compute recursively the vectors $\phi_{(i)}$ using

$$\phi_{(i)} = \phi_{(i+1)} W_{i+1}(-Z_i^{-1}), \quad i = N-1, N-2, \dots, 0, \tag{3.9}$$

Solve the system of equations

$$\phi_{(N)} Z_N = \mathbf{0} \tag{3.10}$$

and

$$\textstyle\sum_{i=0}^{N} \phi_{(i)} e = 1. \tag{3.11}$$

From the system of equations (3.9) – (3.11), vector $\phi_{(N)}$ could be determined uniquely, up to a multiplicative constant.

3.4. SYSTEM PERFORMANCE MEASURES

In this section, some stationary performance measures of the system under study were derived. Using these measures, the total expected cost per unit time can be constructed.

3.4.1. Expected Inventory Level

Let ζ_I denote the expected inventory level in the steady state. Since ϕ_i is the steady state probability vector of i −th customer level with each component specifying a particular combination of the on-hand inventory level and the number of busy servers, the mean inventory level is given by

$$\begin{aligned}\zeta_I &= \sum_{i=0}^{N} \sum_{j=1}^{S} \mathrm{j}\phi_{(i,j)}\mathbf{e} \\ &= \sum_{i=0}^{N-c} \left(\sum_{j=1}^{c} \mathrm{j}\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=0}^{c} \mathrm{j}\,\phi_{(i,j,k)}\right) \\ &\quad + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=0}^{j} \mathrm{j}\,\phi_{(i,j,k)} + \sum_{j=N-i+1}^{S} \sum_{k}^{N-i} \mathrm{j}\,\phi_{(i,j,k)}\right) \\ &\quad + \sum_{j=1}^{S} \mathrm{j}\phi_{(N,j,0)}.\end{aligned} \tag{3.12}$$

3.4.2. Expected Reorder Rate

Let ζ_R denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops to s. The steady state probability $\phi_{(i,s+1,k)}$ gives the rate at which $s+1$ is visited. After the inventory level reaches $s+1$, a service completion of any one of k servers if $k > 0$ takes the inventory level to s. This leads to

$$\zeta_R = \sum_{i=0}^{N-c} \sum_{k=1}^{c} k\mu\phi_{(i,s+1,k)} + \sum_{i=N-c+1}^{N-1} \sum_{k=1}^{N-i} k\mu\phi_{(i,s+1,k)} \tag{3.13}$$

3.4.3. Expected Customer Levels in the Orbit

Let ζ_O denote the expected number of customers in the orbit. Since ϕ_i is the steady state probability vector of i −th customer level with each component specifying a particular combination of the on-hand inventory level and the number of busy servers, the quantity $\phi_i\mathbf{e}$ gives the probability that the inventory level is i in the steady state. Hence, the expected customer level in the orbit is given by

$$\zeta_O = \sum_{i=1}^{N} i\phi_{(i)}\mathbf{e}. \tag{3.14}$$

3.4.4. Overall Rate of Retrials

Let ζ_{OR} denote the expectation of overall rate of retrials. This is given by

$$\zeta_{OR} \quad = \quad \sum_{i=1}^{N} (\theta + i\nu)\phi_{(i)}\mathbf{e}. \tag{3.15}$$

3.4.5. Successful Rate of Retrials

Let ζ_{SR} denote the expectation of successful rate of retrials. Note that a customer from the orbit enters into the service only when any one of the server is idle and at least one item is not in service. This lead to

$$\begin{aligned}\zeta_{SR} \quad = \quad & \sum_{i=1}^{N-c} \left(\sum_{j=1}^{c} \sum_{k=0}^{j-1} (\theta + i\nu)\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=0}^{c-1} (\theta + i\nu)\phi_{(i,j,k)}\right) \\ & + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=0}^{j-1} (\theta + i\nu)\phi_{(i,j,k)} + \sum_{j=N-i}^{S} \sum_{k=0}^{N-i-1} (\theta + i\nu)\phi_{(i,j,k)}\right) \\ & + \sum_{j=1}^{S} (\theta + N\nu)\phi_{(N,j,0)}.\end{aligned} \tag{3.16}$$

3.4.6. Fraction of Successful Rate of Retrials

The fraction of successful rate of retrials ζ_{FSR} is given by

$$\zeta_{FSR} \quad = \quad \frac{\zeta_{SR}}{\zeta_{OR}}. \tag{3.17}$$

3.4.7. Number of Busy Servers

Let ζ_{BS} denote the expected number of busy servers in the steady state. Then ζ_{BS} is given by

$$\begin{aligned}\zeta_{BS} \quad = \quad & \sum_{i=0}^{N-c} \left(\sum_{j=1}^{c} \sum_{k=1}^{j} k\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=1}^{c} k\phi_{(i,j,k)}\right) \\ & + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=1}^{j} k\phi_{(i,j,k)} + \sum_{j=N-i+1}^{S} \sum_{k=1}^{N-i} k\phi_{(i,j,k)}\right).\end{aligned} \tag{3.18}$$

3.4.8. Expected Number of Idle Servers

Let ζ_{IS} denote the expected number of idle servers in the steady state which is given by

$$\zeta_{IS} \quad = \quad c - \zeta_{BS} \tag{3.19}$$

3.5. TOTAL EXPECTED COST

The long-run expected cost rate for this model is defined to be

$$TC(S,s) = c_h\zeta_I + c_s\zeta_R + c_w\eta_O \tag{3.20}$$

where

c_h : The inventory carrying cost/unit/unit time.

c_s : The setup cost/order.

c_w : Waiting cost of a customer/unit time.

Substituting the values of ζ, we get the value of $TC(S,s)$.

Since the computation of the ϕ's are recursive, it is quite difficult to show the convexity of the total expected cost rate analytically.

3.6. CONCLUSION

In this chapter, a continuous review retrial inventory system with a finite source of customers and identical multiple servers in parallel was studied. The customers arrive according a quasi-random distribution. The customers demand unit item and the demanded items are delivered after performing some service which is distributed as exponential. The ordering policy is (s, S) policy, that is, once the inventory level drops to a prefixed level, say s, an order for $Q(= S - s)$ items would be placed. The lead times for the orders are assumed to have an exponential distribution. The arriving customer who finds all the servers are busy or all the items are in service joins an orbit of unsatisfied customers. The orbiting customers form a queue such that only a customer selected according to a certain rule can re-apply for service. The intervals separating two successive repeated attempts are exponentially distributed with rate $\theta + i\nu$, when the orbit has i customers $i \geq 1$. The joint probability distribution of the number of customer in the orbit, the number of busy servers and the inventory level is obtained in the steady state case. Various measures of stationary system performance are computed and the total expected cost per unit time is calculated.

4. [‡]

CHAPTER 4

TWO-COMMODITY PERISHABLE INVENTORY SYSTEM WITH BULK DEMAND FOR ONE COMMODITY

[‡] A modified version of this chapter has been published in the South African Journal of Industrial Engineering, Volume 21 N0 1, 2010

4.1. INTRODUCTION

One of the factors that contribute to the complexity of the present day inventory system is the multitude of items stocked and this necessitated the multi-commodity inventory systems. In dealing with such systems, in the earlier days, many models were proposed with independently established reorder points. But in situations where several products compete for limited storage space or share the same transport facility or are produced on (procured from) the same equipment (supplier) the above strategy overlooks the potential savings associated with joint ordering and, hence, will not be optimal. Thus, the coordinated approach, or what is known as joint replenishment, reduces the ordering and setup costs and allows the user to take advantage of quantity discounts, if any. Various models and references may be found in Miller (1971), Agarwal (1984), Silver (1974), Thomstone and Silver (1975), Kalpakam and Arivarignan (1993) and Srinivasan and Ravichandran (1994) and the references contained therein.

In continuous review inventory systems, Balintfy (1964) and Silver (1974) have considered a coordinated reordering policy which is represented by the triplet (S, c, s), where the three parameters S_i, c_i and s_i are specified for each item i with $s_i \leq c_i \leq S_i$, under the unit sized Poisson demand and constant lead time. In this policy, if the level of i-th commodity at any time is below s_i, an order is placed for $S_i - s_i$ items and at the same time, any other item $j(\neq i)$ with available inventory at or below its can-order level c_j, an order is placed so as to bring its level back to its maximum capacity S_j. Subsequently many articles have appeared with models involving the above policy and another article of interest is due to Federgruen, Groenevelt and Tijms (1984), which deals with the general case of compound Poisson demands and non-zero lead times.

The work on methods to solve the joint replenishment problem throughout the years has been extensive. Some further notable references include the publications of Fung and Ma (2001), Goyal (1973,1974,1988), Goyal and Satir (1989), Kaspi and Rosenblatt (1991), Nilsson et al. (2007), Nilsson and Silver (2008), Olsen (2005), Silver (1976), Van Eijs (1993), Viswanathan (1996,2002,2007) and Wildeman et al. (1997) and references therein.

Kalpakam and Arivarignan (1993) have introduced (s,S) policy with a single reorder level s defined in terms of the total number of items in the stock. This policy avoids separate ordering for each commodity and hence a single processing of orders for both commodities has some advantages in situation wherein procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

In the case of two-commodity inventory systems, Anbazhagan and Arivarignan (2000,2001a,2001b,2003) have proposed various ordering policies. Yadavalli et al. (2005b) have analyzed a model with joint ordering policy and variable order quantities. Sivakumar et al. (2005) have considered a two commodity substitutable inventory system in which the demanded items are delivered after a random time. Sivakumar et al. (2006) have considered a two commodity perishable inventory system with joint ordering policy.

There are some situations in which a single item is demanded for one commodity and multiple items are demanded for another commodity. For instance, a customer may buy a single razor or set of blades or both. Another example is the sales of DVD writer and set of DVDs. It may be noted that the seller would be placing a joint order for both commodities as these will be available from the same source. Moreover, a seller may not be willing to place orders frequently and may prefer to have one order to replenish his/her stock in a given cycle. These situations are modelled in this work by assuming demand processes that require single item for one commodity, multiple items for the other commodities or both commodities and by assuming a joint reorder for both commodities.

This paper is organized as follows: in section 2, the mathematical model and notations followed in the rest of the chapter were described. The steady state solution of the joint probability distribution for both commodities , the phase of the demand process and the phase of the lead time process is given in section 3. In section 4, the various measures of system performance in the steady state were derived and the total expected cost rate is calculated in section 5. Section 6 presents the cost analysis of the model using numerical examples.

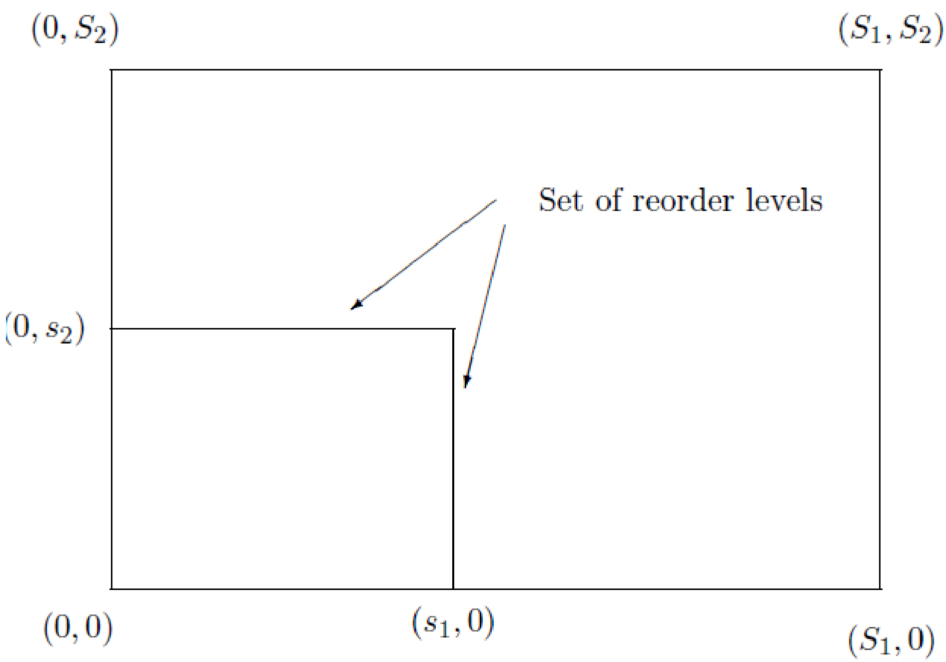

Figure 4.1: Space of Inventory levels

Notations

$\mathbf{0}$: zero matrix

I : an identity matrix

$$H(x)=\begin{cases} x & if \quad x>0 \\ 0 & if \quad x\leq 0 \end{cases}$$

E_i = $\{1,2,\ldots,i\}$

E_i^0 = $\{0,1,\ldots,i\}$

e = a column vector of ones.

4.2. MODEL DESCRIPTION

Consider a two-commodity perishable inventory system with the maximum capacity S_i units for i-th commodity $(i=1,2)$. Assume that the demand for the first commodity is for single item and the demand for the second commodity is for bulk items. An arriving customer may demand only the first commodity or only the second commodity or both. The number of items demanded for the second commodity at any demand point is a random variable Y with probability function $p_k = Pr\{Y=k\}$, $k=1,2,3,\ldots$. The three type of demands for these two commodities occur according to a Markovian arrival process $\boldsymbol{MAP}$. The life time of each commodity is exponential with parameter $\gamma_i(i=1,2)$ The reorder level for the i-th commodity is fixed at s_i $(1\leq s_i \leq S_i)$ and the ordering quantity for the i-th commodity is Q_i $(=S_i - s_i > s_i + 1)$ items when both the inventory levels are less than or equal to their respective reorder levels. It is assumed that demands during stock-out period as well as unsatisfied demands are lost. The requirement $S_i - s_i > s_i + 1$, ensures that after a replenishment the inventory levels of both commodities will always be above the respective reorder levels. Otherwise, it may not be possible to place any reorder (according to this policy) which will lead to perpetual shortage. That is, if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder is made when $L_1(t) \leq s_1$ and $L_2(t) \leq s_2$ (see figure 1). The time to deliver the items are assumed to be of phase (***PH***) type with representation (α, T) of order m_2. It can be noted that the phase type distribution is defined as the time until absorption in a finite state irreducible Markov chain with one absorbing state. The mean of the phase type distribution (α, T) is given by $\alpha(-T)^{-1}\mathbf{e}$. Let β denote the reciprocal of this mean. That is, $\beta = \left[\alpha(-T)^{-1}\mathbf{e}\right]^{-1}$ gives the rate of replenishment once an order is placed. Let T^0 be such that $T\mathbf{e} + T^0 = \mathbf{0}$.

For the description of the demand process, the description of ***MAP*** as given in Lucantoni (1991) was used. Consider a continuous-time Markov chain on the state space $1, 2, \ldots, m_1$. The demand process is constructively defined as follows. When the chain enters a state $i, 1 \leq i \leq m_1$, it stays for an exponential time with parameter θ_i. At the end of the sojourn time in state i, there are four possible transitions: with probabilities $a_{ij}, 1 \leq j \leq m_1$, the chain enters the state j when a demand for the first commodity occurs; with probabilities $b_{ij}, 1 \leq j \leq m_1$, the chain enters the state j when a demand for the second commodity occurs; with probabilities $c_{ij}, 1 \leq j \leq m_1$, the chain enters the state j when a demand for both commodities occurs; with probabilities $d_{ij}, 1 \leq j \leq m_1, i \neq j$, the transitions corresponds to no demand and the state of the chain is j. Note that the Markov chain can go from state i to state i only through a demand. Define the square matrices $D_k, k=0,1,2,12$ of size $m_1 \times m_1$ by $[D_0]_{ii} = -\theta_i$ and $[D_0]_{ij} = \theta_i d_{ij}, i \neq j$, $[D_1]_{ij} = \theta_i a_{ij}, [D_2]_{ij} = \theta_i b_{ij}$ and $[D_{12}]_{ij} = \theta_i c_{ij}$, $1 \leq i, j \leq m_1$. It is easily seen that $D = D_0 + D_1 + D_2 + D_{12}$ is an infinitesimal generator of a continuous-time Markov chain. It is assumed that D is irreducible and $D_0 e \neq 0$.

Let ζ be the stationary probability vector of the continuous-time Markov chain with generator D. That is, ζ is the unique probability vector satisfying

$$\zeta D = 0, \zeta e = 1.$$

Let η be the initial probability vector of the underlying Markov chain governing the $\boldsymbol{MAP}$. Then, by choosing η appropriately the time origin can be modelled to be

1. an arbitrary arrival point;
2. the end of an interval during which there are at least k arrivals;
3. the point at which the system is in specific state such as the busy period ends or busy period begins;

The important case is the one where one gets the stationary version of the $\boldsymbol{MAP}$ by $\eta = \zeta$. The constant $\lambda = \zeta(D_1 + D_2 + D_{12})e$, referred to as the **fundamental rate** gives the expected number of demands per unit of time in the stationary version of the $\boldsymbol{MAP}$. The quantities $\lambda_1 = \zeta D_1 e$, $\lambda_2 = \zeta D_2 e$ and $\lambda_{12} = \zeta D_{12} e$, give the arrival rate of demand for first commodity, second commodity and for both respectively. Note that $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

For further details on $\boldsymbol{MAP}$ and phase-type distributions and their usefulness in Stochastic modelling, the following are good references: Chapter 2 in Neuts (1994), Chapter 5 in Neuts (1989), Ramaswami (1981), Lucantoni (1991, 1993), Lucantoni et al. (1990), Latouche and Ramaswami (1999), Li and Li (1994), Lee and Jeon (2000) and Chakravarthy and Dudin (2003) and references therein for a detailed

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introduction of the $\boldsymbol{MAP}$ and phase-type distribution. Some recent reviews can be found in Neuts (1995) and Chakravarthy (2001).

Let $J_1(t)$ and $J_2(t)$, respectively, denote the phase of the demand process and the phase of the lead time process. Then the stochastic process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ has the state space,

$$\Omega = \{(i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, 0), i_1 \in E^0_{s_1}, \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, i_4), i_1 \in E^0_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}, i_4 \in E_{m_2}\}$$

From the assumptions made on the demand and the replenishment processes, it can be shown that $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ is a Markov process on the state space Ω. By ordering the sets of state space in lexicographic order, the infinitesimal generator of the Markov chain governing the system, in block partitioned form, is given by

$$[P]_{ij} = \begin{cases} A_i, & j = i, \qquad i = 0, 1, \ldots, S_1, \\ B_i, & j = i - 1, \quad i = 1, 2, \ldots, S_1, \\ C, & j = i + Q_1, \quad i = 0, 1, \ldots, s_1, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.1}$$

where

$$[C]_{ij} = \begin{cases} I_{m_1} \otimes T^0, & j = i + Q_2, \quad i = 0, 1, \ldots, s_2, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.2}$$

$For \quad k = s_1 + 2, s_1 + 3, \ldots, S_1,$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = 1,2,\ldots,S_2, \\ D_1 + D_{12} + k\gamma_1 I_{m_1}, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,S_2, \\ p_i^{'} D_{12}, & j = 0, & i = 1,2,\ldots,S_2, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.3}$$

$$p_n^{'} = \sum_{i=n}^{\infty} p_i$$

$For \quad k = s_1 + 1,$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = s_2+1, s_2+2,\ldots,S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, & i = 1,2,\ldots,s_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = s_2+1, s_2+2,\ldots,i-1, & i = s_2+2, s_2+3,\ldots,S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1,2,\ldots,s_2, & i = s_2+2, s_2+3,\ldots,S_2, \\ & or & \\ & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,s_2+1, \\ p_i^{'} D_{12} \otimes \alpha, & j = 0, & i = 1,2,\ldots,S_2, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.4}$$

$For \quad k = 1,2,\ldots,s_1$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = s_2+1, s_2+2,\ldots,S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, & i = 1,2,\ldots,s_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = s_2+1, s_2+2,\ldots,i-1, & i = s_2+2, s_2+3,\ldots,S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1,2,\ldots,s_2, & i = s_2+1, s_2+2,\ldots,S_2, \\ p_i^{'} D_{12} \otimes \alpha, & j = 0, & i = s_2+1, s_2+2,\ldots,S_2, \\ p_{i-j} D_{12} \otimes I_{m_2}, & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,s_2, \\ p_i^{'} D_{12} \otimes I_{m_2}, & j = 0, & i = 1,2,\ldots,s_2, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.5}$$

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$For \quad k = s_1 + 1, s_1 + 2, \ldots, S_1,$

$$[A_k]_{ij} = \begin{cases} p_1 D_2 + k\gamma_2 I_{m_1}, & j = i-1, \quad i = 2,3,\ldots,S_2, \\ p_{i-j} D_2, & j = 1,2,\ldots,i-2, \quad i = 3,4,\ldots,S_2, \\ p_i' D_2, & j = 0, \quad i = 1,2,\ldots,S_2, \\ D_0 - (k\gamma_1 + i\gamma_1) I_{m_1} & j = i, \quad i = 1,2,\ldots,S_2, \\ D_0 + D_2 - k\gamma_1 I_{m_1}, & j = i, \quad i = 0, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.6}$$

$For \quad k = 1,2,\ldots,s_1$

$$[A_k]_{ij} = \begin{cases} p_1 D_2 + i\gamma_2 I_{m_1}, & j = i-1, \quad i = s_2+2, s_2+3, \ldots, S_2, \\ (p_1 D_2 + i\gamma_2 I_{m_1}) \otimes \alpha, & j = i-1, \quad i = s_2+1, \\ (p_1 D_2 + i\gamma_2 I_{m_1}) \otimes I_{m_2}, & j = i-1, \quad i = 2,3,\ldots,s_2, \\ p_{i-j} D_2, & j = s_2+1, s_2+2, \ldots, i-2, \quad i = s_2+3, s_2+4, \ldots, S_2, \\ p_{i-j} D_2 \otimes \alpha, & j = 1,2,\ldots,i-2, \quad i = s_2+1, s_2+2, \\ & \quad or \\ & j = 1,2,\ldots,s_2, \quad i = s_2+3, s_2+4, \ldots, S_2, \\ p_{i-j} D_2 \otimes I_{m_2}, & j = 1,2,\ldots,i-2, \quad i = 3,4,\ldots,s_2, \\ p_i' D_2 \otimes \alpha, & j = 0, \quad i = s_2+1, s_2+2, \ldots, S_2, \\ p_i' D_2 \otimes I_{m_2}, & j = 0, \quad i = 1,2,\ldots,s_2, \\ D_0 - (k\gamma_1 + i\gamma_2) I_{m_1}, & j = i, \quad i = s_2+1, s_2+2, \ldots, S_2, \\ D_0 \oplus T - (k\gamma_1 + i\gamma_2) I_{m_1} \otimes I_{m_2}, & j = i, \quad i = 1,2,\ldots,s_2, \\ (D_0 + D_2) \oplus T - k\gamma_1 I_{m_1} \otimes I_{m_2}, & j = i, \quad i = 0, \\ \mathbf{0}, & otherwise \end{cases} \tag{4.7}$$

For $k=0$

$$[A_k]_{ij} = \begin{cases} p_1(D_2+D_{12})+i\gamma_2 I_{m_1}, & j=i-1, & i=s_2+2,s_2+3,\ldots,S_2, \\ (p_1(D_2+D_{12})+i\gamma_2 I_{m_1})\otimes\alpha, & j=i-1, & i=s_2+1, \\ (p_1(D_2+D_{12})+i\gamma_2 I_{m_1})\otimes I_{m_2}, & j=i-1, & i=2,3,\ldots,s_2, \\ p_{i-j}(D_2+D_{12}), & j=s_2+1,s_2+2,\ldots,i-2, & i=s_2+3,s_2+4,\ldots,S_2, \\ p_{i-j}(D_2+D_{12})\otimes\alpha, & j=1,2,\ldots,i-2, & i=s_2+1,s_2+2, \\ & \quad or & \\ & j=1,2,\ldots,s_2, & i=s_2+3,s_2+4,\ldots,S_2, \\ p_{i-j}(D_2+D_{12})\otimes I_{m_2}, & j=1,2,\ldots,i-2, & i=3,4,\ldots,s_2, \\ p_i'(D_2+D_{12})\otimes\alpha, & j=0, & i=s_2+1,s_2+2,\ldots,S_2, \\ p_i'(D_2+D_{12})\otimes I_{m_2}, & j=0, & i=1,2,\ldots,s_2, \\ D_0+D_1-(i\gamma_2)I_{m_1}, & j=i, & i=s_2+1,s_2+2,\ldots,S_2, \\ (D_0+D_1)\oplus T-(i\gamma_2)I_{m_1}\otimes I_{m_2}, & j=i, & i=1,2,\ldots,s_2, \\ D\oplus T, & j=i, & i=0, \\ \mathbf{0}, & otherwise & \end{cases} \quad (4.8)$$

It may be noted that the matrix C is of order $(Q_1m_1+(s_1+1)m_1m_2)\times(S_2+1)m_1$, the matrices $B_i, i=s_1+2,s_1+3,\ldots,S_1$, are of order $(S_2+1)m_1\times(S_2+1)m_1$, the matrix B_{s_1+1} is of order $(S_2+1)m_1\times(Q_1m_1+(s_1+1)m_1m_2)$, the matrices $B_i, i=1,2,\ldots,s_1$, are of order $(Q_1m_1+(s_1+1)m_1m_2)\times(Q_1m_1+(s_1+1)m_1m_2)$, the matrices $A_i, i=0,1,\ldots,s_1$ are of order $(Q_1m_1+(s_1+1)m_1m_2)\times(Q_1m_1+(s_1+1)m_1m_2)$, and the matrices $A_i, i=s_1+1,s_1+2\ldots,S_1$ are of order $(S_2+1)m_1\times(S_2+1)m_1$.

4.3. STEADY STATE ANALYSIS

It can be seen from the structure of P that the homogeneous Markov process $\{(L_1(t),L_2(t),J_1(t),J_2(t)),t\geq 0\}$ on the finite state space Ω is irreducible.

Hence, the limiting distribution $\phi_{(i,k,j_1,j_2)}$ =

$$\lim_{t\to\infty} Pr\left[L_1(t)=i, L_2(t)=k, J_1(t)=j_1, J_2(t)=j_2 \mid L_1(0), L_2(0), J_1(0), J_2(0)\right]$$

exists. Let

$$\phi_{(i,k,j_1)} = \begin{cases} (\phi_{(i,k,j_1,1)}, \phi_{(i,k,j_1,2)}, \ldots, \phi_{(i,k,j_1,m_2)}), & (i,k,j_1) \in F_1, \\ (\phi_{(i,k,j_1,0)}), & (i,k,j_1) \in F_2, \end{cases}$$

where

$$F_1 = \left\{(i_1, i_2, i_3), i_1 \in E^0_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\right\}$$

$$F_2 = \left\{(i_1, i_2, i_3), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\cup \left\{(i_1, i_2, i_3), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\cup \left\{(i_1, i_2, i_3), i_1 \in E^0_{s_1}, \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\phi_{(i,k)} = \left(\phi_{(i,k,1)}, \phi_{(i,k,2)}, \ldots, \phi_{(i,k,m_1)}\right), k \in E_2, i \in E_1,$$

$$\phi^{(i)} = \begin{cases} (\phi_{(i,0)}, \phi_{(i,1)}, \ldots, \phi_{(i,S_2)}), & i \in E_1 \end{cases}$$

and

$$\Phi = \left(\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(S_1)}\right).$$

Then the vector of limiting probabilities Φ satisfies

$$\Phi P = \mathbf{0} \quad and \quad \Phi \mathbf{e} = 1. \tag{4.9}$$

The first equation of the above yields the following set of equations:

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i = 0, i = 0, 1, \ldots, Q_1 - 1, \tag{4.10}$$

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i + \Phi^{(i-Q_1)} C = 0, i = Q_1, \tag{4.11}$$

$$\Phi^{(i+1)}B_{i+1}+\Phi^{(i)}A_i+\Phi^{(i-Q_1)}C=0, i=Q_1+1, Q_1+2,\ldots,S_1-1, \tag{4.12}$$

$$\Phi^{(i)}A_i+\Phi^{(i-Q_1)}C=0, i=S_1. \tag{4.13}$$

The equations (except (4.11)) can be recursively solved to get

$$\Phi^{(i)}=\Phi^{(Q_1)}\theta_i,\quad i=0,1,\ldots,S_1, \tag{4.14}$$

where

$$\theta_i=\begin{cases}(-1)^{Q_1-i}B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{i+1}A_i^{-1}, & i=0,1,\ldots,Q_1-1,\\ I, & i=Q_1,\\ (-1)^{2Q_1-i+1}\sum_{j=0}^{S-i}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1}\right.\\ \quad\left.\left(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{i+1}A_i^{-1}\right)\right], & i=Q_1+1,\ldots,S_1.\end{cases} \tag{4.15}$$

Substituting the values of θ_i in equation (4.11) and in the normalizing condition thr following is obtained

$$\begin{aligned}\Phi^{(Q_1)}\Big[&(-1)^{Q_1}\sum_{j=0}^{s-1}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1}\right.\\ &\left.\left(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{Q_1+2}A_{Q_1+1}^{-1}\right)\right]B_{Q_1+1}+A_{Q_1}\\ &+(-1)^{Q_1}B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_1A_0^{-1}C\Big]=0,\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\Phi^{(Q_1)}\Big[&\sum_{i=0}^{Q_1-1}\left((-1)^{Q_1-i}B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{i+1}A_i^{-1}\right)+I\\ &+\sum_{i=Q_1+1}^{S_1}\left((-1)^{2Q_1-i+1}\sum_{j=0}^{S-i}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1}\right.\right.\\ &\left.\left.(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{i+1}A_i^{-1})\right)\boldsymbol{e}\Big]=1\end{aligned} \tag{4.17}$$

From the equation (4.16), the value of $\Phi^{(Q)}$ can be obtained up to a constant multiplication. This constant can be determined by substituting the value of $\Phi^{(Q)}$ in the equation (4.17). Substituting the value of $\Phi^{(Q)}$ in the equation (4.14) leads to the values of $\Phi^{(i)}, i = 0,1,\ldots,S.$

4.4. SYSTEM PERFORMANCE MEASURES

In this section, some stationary performance measures of the system were derived. Using these measures, the total expected cost per unit time can be constructed.

4.4.1. Mean Inventory level

Let η_{I_k} denote the mean inventory level of $k-$ th commodity in the steady state $(k=1,2)$. Since $\phi_{(i,j)}$ is the steady state probability vector for inventory level of first commodity i and the second commodity j, then

$$\eta_{I_1} = \sum_{i=1}^{S_1}\sum_{j=0}^{S_2} i\phi_{(i,j)}\mathbf{e}. \tag{4.18}$$

and

$$\eta_{I_2} = \sum_{i=0}^{S_1}\sum_{j=1}^{S_2} j\phi_{(i,j)}\mathbf{e}. \tag{4.19}$$

4.4.2. Mean Reorder Rate

A reorder for both commodities is made when the joint inventory level drops to either (s_1, s_2) or $(s_1, j), j < s_2$ or $(i, s_2), i < s_1$. Let η_R denote the mean reorder rate for both commodities in the steady state and it is given by

$$\eta_R = \sum_{k=0}^{s_1}\sum_{j=1}^{Q_2}\phi_{(k,s_2+j)}\sum_{u=j}^{\infty}p_u(D_2 \otimes \alpha)\mathbf{e} + \sum_{k=0}^{s_2}\phi_{(s_1+1,k)}(D_1 \otimes \alpha)\mathbf{e}$$

$$+\sum_{k=1}^{s_1+1}\sum_{j=1}^{Q_2}\phi_{(k,s_2+j)}\sum_{u=j}^{\infty}p_u(D_{12} \otimes \alpha)\mathbf{e} + \sum_{k=0}^{s_1}(s_2+1)\gamma_2\phi_{(k,s_2+1)}\mathbf{e} \tag{4.20}$$

$$+\sum_{k=0}^{s_2}(s_1+1)\gamma_1\phi_{(s_1+1,k)}\mathbf{e}.$$

4.4.3. Mean Shortage Rate

Let η_{Sh_i} denote the mean shortage rate of $i-$th type demand in the steady state $(i = 1,2,12)$. Then

$$\eta_{Sh_1} = \sum_{k=0}^{S_2}\phi_{(0,k)}D_1\mathbf{e}. \tag{4.21}$$

$$\eta_{Sh_2} = \sum_{i=0}^{S_1}\sum_{j=0}^{S_2}\phi_{(i,j)}\sum_{k=j+1}^{\infty}p_kD_2\mathbf{e}. \tag{4.22}$$

and

$$\eta_{Sh_{12}} = \left(\sum_{k=0}^{S_2}\phi_{(0,k)}D_{12}\mathbf{e} + \sum_{i=0}^{S_1}\sum_{j=0}^{S_2}\phi_{(i,j)}\sum_{k=j+1}^{\infty}p_kD_{12}\mathbf{e}\right). \tag{4.23}$$

4.4.4. Mean Failure Rate

Let the mean failure rate of commodity-i in the steady state be denoted by $\eta_{F_i}, (i=1,2)$. A failure occurs when any one of the stocked items cease to work or perish. Since the rate of failure of a single item is γ_j for the commodity j, the rate at which any one of i items for $j-th$ commodity fails is given by $i\gamma_j, (j=1,2)$. When the process is in state (i,k,j_1,j_2), the rate of failure of any one of item of first commodity is given by $i\gamma_1$ (provided $i>0$) and the failure rate of any one item of second commodity is $k\gamma_2$ (provided $k>0$).

Therefore

$$\eta_{F_1} = \sum_{i=1}^{S_1}\sum_{k=0}^{S_2} i\gamma_1\phi_{(i,k)}\mathbf{e}. \tag{4.24}$$

and

$$\eta_{F_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2} k\gamma_2\phi_{(i,k)}\mathbf{e}. \tag{4.25}$$

4.5. COST ANALYSIS

The total expected cost per unit time (total expected cost rate) in the steady-state for this model is defined to be $TC(S_1,s_1,S_2,s_2)$

$$= c_{h_1}\eta_{I_1} + c_{h_2}\eta_{I_2} + c_s\eta_R + c_{sh_1}\eta_{Sh_1} + c_{sh_2}\eta_{Sh_2} + c_{sh_{12}}\eta_{Sh_{12}} + c_{f_1}\eta_{F_1} + c_{f_2}\zeta_{F_2} \tag{4.26}$$

where

c_{h_i} : The inventory carrying cost of i-th commodity per unit item per unit time

$(i = 1,2)$

c_s : Joint ordering cost per order.

c_{f_i} : The failure cost of i-th commodity per unit item per unit time $(i = 1,2)$.

c_{sh_i} : Shortage cost due to type i demand per unit time $(i = 1,2,12)$.

Since the total expected cost rate is known only implicitly, the analytical properties such as convexity of the total expected cost rate cannot be carried out in the present form. However the following numerical examples were presented to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

4.6. ILLUSTRATIVE NUMERICAL EXAMPLES

As the total expected cost rate is obtained in a complex form, the convexity of the total expected cost rate cannot be studied by the analytical methods. Hence the use `simple' numerical search procedures to find the ``local" optimal vales for any two of the decision variables $\{S_1, s_1, S_2, s_2\}$ by considering a small set of integer values for these variables. With a large number of numerical examples, it was found that the total cost rate per unit time in the long run is either convex function of both variables or an increasing function of any one variable.

The following five $\boldsymbol{MAP}$s for arrival of demands are considered and it may be noted that these processes can be normalized to have a specific (given) demand rate λ when considered for arrival of demands.

1. **Exponential (Exp)**

$$H_0 = (-1) H_1 = (1)$$

2. **Erlang (Erl)**

$$H_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$H_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}$$

4. **MAP with Negative correlation (MNC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 25.25 & 0 & 55.75 \\ 55.75 & 0 & 25.25 \end{pmatrix}$$

5. **MAP with Positive correlation (MPC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 55.25 & 0 & 25.75 \\ 25.75 & 0 & 55.25 \end{pmatrix}$$

All the above $\boldsymbol{MAPs}$ are qualitatively different in that they have different variance and correlation structures. The first three processes are special cases of renewal processes and the correlation between arrival times is 0. The demand process labelled as $\boldsymbol{MNC}$

has correlated arrivals with correlation coefficient -0.1254 and the demands corresponding to the process labelled $\boldsymbol{MPC}$ has positive correlation coefficient 0.1213. Since Erlang has the least variance among the five arrival processes considered here, the ratios of the variances of the other four arrival processes, labelled as $\boldsymbol{Exp}, \boldsymbol{HExp}, \boldsymbol{MNC}$ and $\boldsymbol{MPC}$ above, with respect to the Erlang process are, 3.0, 15.1163, 8.1795, 8.1795, respectively. The ratios were given rather than the actual values since the variance depends on the arrival rate which is varied in the discussion.

For the lead time distribution, the following three $\boldsymbol{PH}$ distributions were considered. Again these processes can be normalized to have a specific (given) rate β when considered for replenishment.

1. **Exponential (Exp)**

$$\alpha = (1) T = (-1)$$

2. **Erlang (Erl)**

$$\alpha = (1,0,0,0) T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$\alpha = (0.9,0.1) T = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1: This example is to illustrate the effect of the demand rate λ, the lead time rate β, the five types of demand processes and the three types of lead time processes on the optimal values (S_1^*, S_2^*) and the optimal cost rate $TC(S_1^*, 2, S_2^*, 4)$. The following fixed values were assumed for the parameters and costs:

$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.8, \gamma_2 = 0.6, p_i = 0.6*0.4^{i-1}, i = 1,2,\ldots,$
$c_{h_1} = 0.05, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

Table 4.1 gives the optimum values, S_1^* and S_2^*, that minimize the total expected cost rate for each of the five $\boldsymbol{MAP}$s for arrivals of demands considered against each of the three $\boldsymbol{PH}$s for lead times. The associated total expected cost rate values are also given in the table. The lower entry in each cell gives the optimal expected cost rate and the upper entries are corresponding to S_1^* and S_2^*. The following observations were noticed from the table 1:

1. As λ increases the optimal total cost rate decreases for all the five demand processes and for all the three lead time processes. Similarly as β increases the optimal total cost rate decreases.
2. The optimal total expected cost rate has higher value for demand process having hyper-exponential distribution and has lower value for Erlang demand process.
3. The lead time distributed as Erlang has low optimal total cost rate except for $\boldsymbol{HExp}$ distributed demand process and $\boldsymbol{HExp}$ distributed lead time has high optimal total cost rate except for **HExp** distributed demand process. For $\boldsymbol{HExp}$ distributed demand process this observation reverse, i.e., $\boldsymbol{HExp}$ distributed lead time has low optimal total cost rate and $\boldsymbol{Erl}$ distributed lead time has high optimal total cost rate.

Example 2: This example serves to illustrate the effect of the arrival rate λ, the lead time rate β and the type of arrival and lead time processes on the optimal values (s_1^*, s_2^*) and optimal cost rate $TC(15, s_1^*, 30, s_2^*)$. The following fixed values were assumed for the parameters and cost:

$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55 * 0.45^{i-1}, i = 1,2,\ldots,$
$ch_1 = 0.01, ch_2 = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

The optimum values, s_1^* and s_2^*, that minimizes the expected total cost for each of the five $\boldsymbol{MAP}$s for arrivals of demands considered against each of the three $\boldsymbol{PH}$s for lead times is given in the table 4.2. The associated total expected cost rate values are also given. The lower entry in each cell gives the optimal expected cost rate and the upper entries correspond to s_1^* and s_2^*. The key observations are summarized below.

1. As λ increases, the optimal total cost rate increases except for $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{Hexp}$ distributed demand process, the optimal total cost rate decreases as the demand rate λ increases.
2. When β increases, the optimal total cost rate increases for all combination of five arrival processes and three demands processes.
3. The optimal cost rate is high in the cases wherein the demand process is $\boldsymbol{Hexp}$ and it is low when the demand process is Erlang.
4. The optimal total cost rate is low when the lead time is $\boldsymbol{Erl}$ except for the $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{Hexp}$ distributed lead time the optimal total cost rate is high except for $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{HExp}$ distributed demand process this observation reverse., i.e., $\boldsymbol{Hexp}$ distributed lead time is associated with low optimal total cost rate and $\boldsymbol{Erl}$ is associated with high optimal total cost rate.

Table 4.1: Total expected cost rate as a function of (S_1, S_2)

			Lead time distribution					
	β		10			15		
	λ		Exp	Erl	HExp	Exp	Erl	HExp
MAP demands distributions	6	Exp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.0236	8.0177	8.0277	8.2027	8.1999	8.2047
		Erl	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			7.9967	7.9905	8.0009	8.1838	8.1809	8.1858
		HExp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.1568	8.1570	8.1567	8.2977	8.2978	8.2976
		MNC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.0736	8.0680	8.0774	8.2379	8.2352	8.2397
		MPC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.1267	8.1214	8.1303	8.2753	8.2727	8.2770
	8	Exp	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.5224	10.5175	10.5258	10.8125	10.8101	10.8141
		Erl	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.4959	10.4908	10.4994	10.7939	10.7915	10.7956
		HExp	(17,58)	(17,58)	(17,58)	(17,59)	(17,59)	(18,60)
			10.6604	10.6608	10.6601	10.9104	10.9106	10.9103
		MNC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.5717	10.5670	10.5749	10.8470	10.8448	10.8486
		MPC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.6260	10.6215	10.6291	10.8852	10.8830	10.8867

Table 4.2: Total expected cost rate as a function of (s_1, s_2)

Lead time distribution									
	β		10				15		
	λ		Exp	Erl	HExp		Exp	Erl	HExp
MAP Demands Distributions	6	Exp	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2328	7.2286	7.2356		7.3624	7.3604	7.3638
		Erl	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2080	7.2037	7.2111		7.3450	7.3429	7.3464
		HExp	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.3579	7.3598	7.3567		7.4518	7.4527	7.4512
		MNC	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2787	7.2748	7.2814		7.3947	7.3929	7.3960
		MPC	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.3282	7.3245	7.3307		7.4296	7.4279	7.4309
	8	Exp	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.4935	9.4902	9.4957		9.7144	9.7129	9.7155
		Erl	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.4694	9.4660	9.4717		9.6977	9.6961	9.6988
		HExp	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.6194	9.6211	9.6183		9.8030	9.8038	9.8025
		MNC	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.5381	9.5351	9.5402		9.7455	9.7441	9.7465
		MPC	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.5876	9.5847	9.5895		9.7799	9.7786	9.7809

Example 3: Next, the impact of c_{f_1} and c_{f_2} on the total expected cost rate was considered. For this, the following values were considered for the parameters and costs: $D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \lambda = 8, \beta = 0.5, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55*0.45^{i-1}$, $i = 1,2,\ldots, c_{h_1} = 0.01, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1.$

The graphs of the total expected cost rate as a function of c_{f_1} and c_{f_2} were plotted for the three lead time processes and the five demand processes in figures 4.2 – 4.6. In all the figures the lead time distributions $\boldsymbol{Exp}, \boldsymbol{Erl}$ and $\boldsymbol{HExp}$ are coloured as blue, black and red respectively. The following were noted:

- In all the five arrival processes, as c_{f_1} and c_{f_2} increase simultaneously, the total expected cost rate increases. But the increasing rate for c_{f_2} is high compared to c_{f_1}.

- The Erlang lead time process is associated with low total expected cost rate and for the hyper exponential lead time process case the total expected cost rate is high.

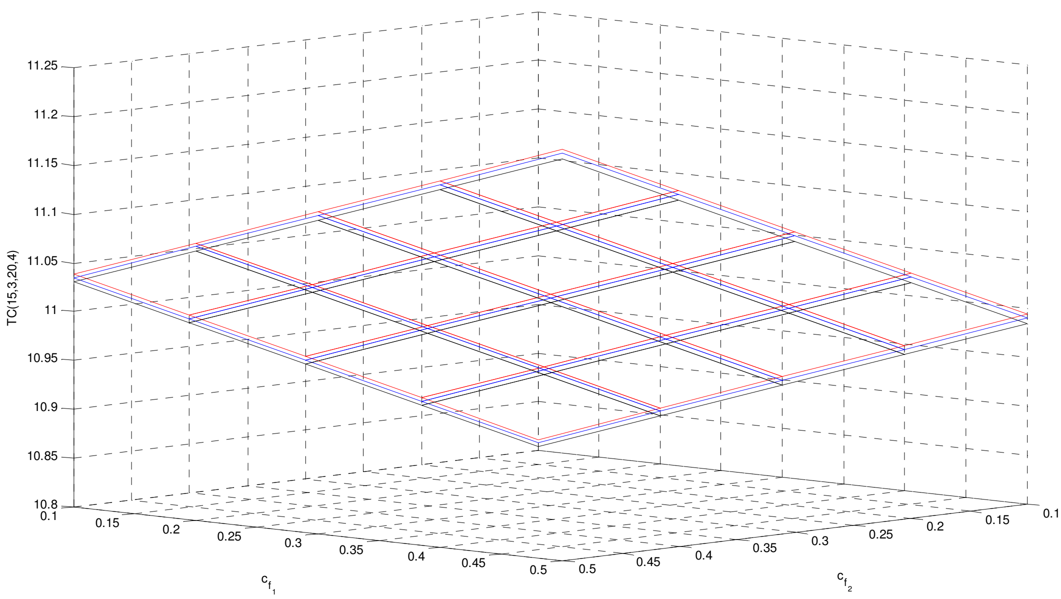

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Figure 4.2: $\boldsymbol{Exp}$ **demand process**

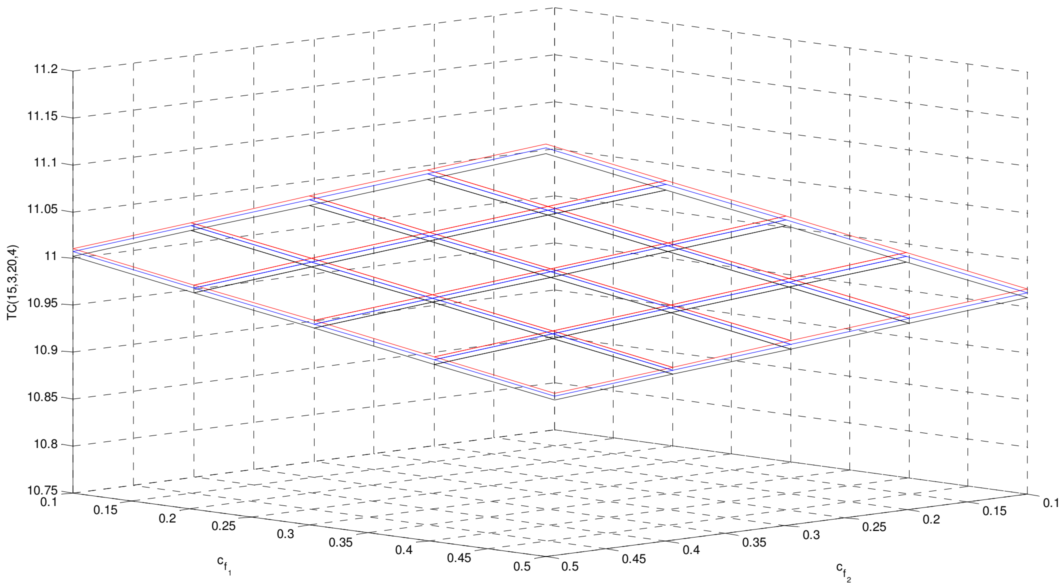


Figure 4.3: $\boldsymbol{Erl}$ **demand process**

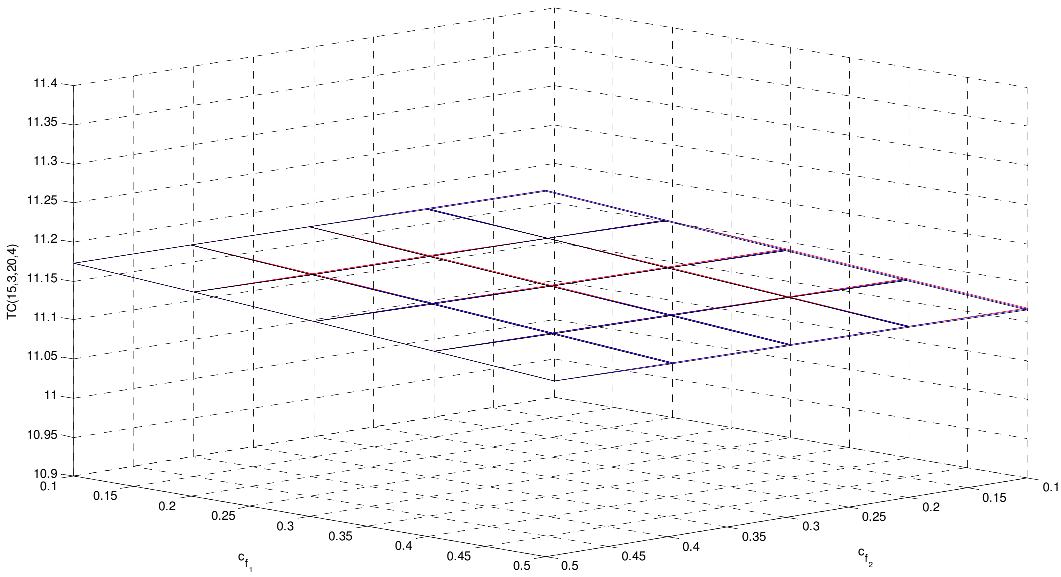


Figure 4.4: $\boldsymbol{HExp}$ **demand process**

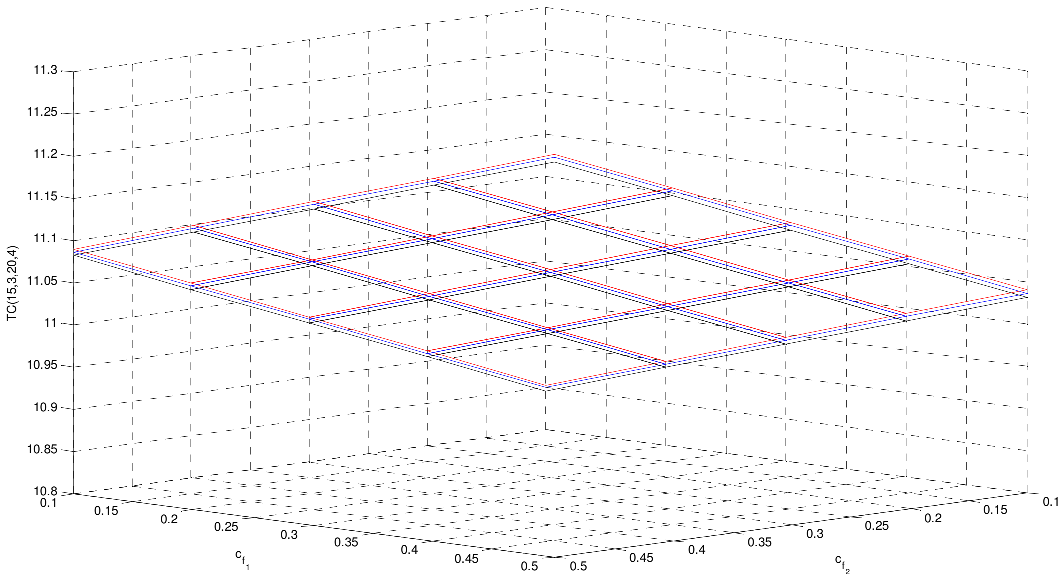


Figure 4.5: MNC demand process

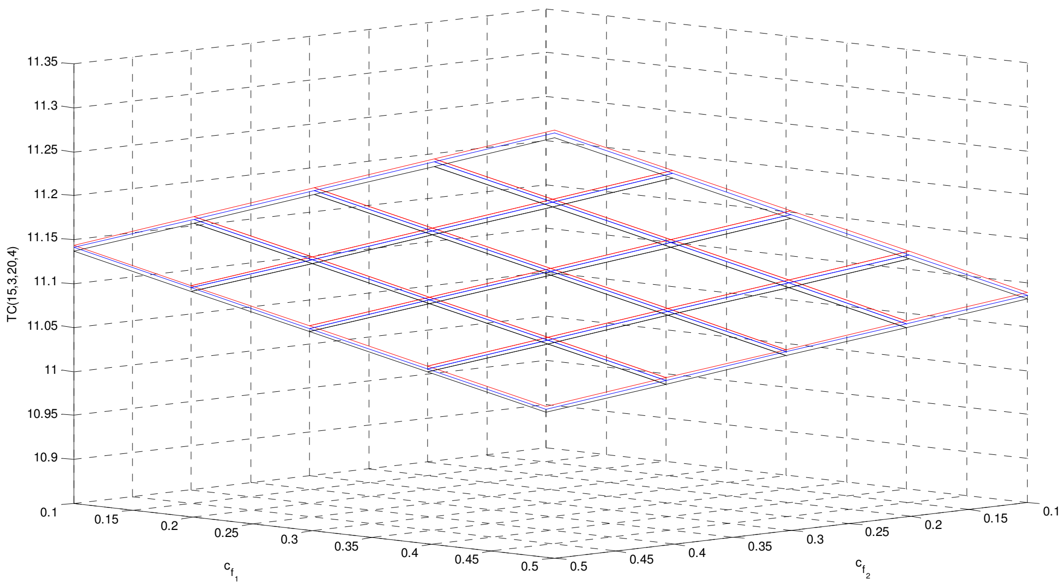


Figure 4.6: MPC demand process

Example 4: In the final example, the impact of c_{h_1} and c_{h_2} on the total expected cost rate was shown. The following values were considered for the parameters and costs: $D_0 = H_0, D_1 = 0.3H1, D_2 = 0.4H1, D_{12} = 0.3H1, \lambda = 15, \beta = 2, \gamma_1 = 0.8, \gamma_2 = 0.4, p_i = 0.6*0.4^{i-1}$, $i = 1,2,\ldots, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

The graphs of the total expected cost rate as a function of c_{f_1} and c_{f_2} were plotted for the three lead time processes and the five demand processes in figures 4.7 – 4.11. In all the figures the plots for the lead time distributions $\boldsymbol{Exp}$, $\boldsymbol{Erl}$ and $\boldsymbol{HExp}$ are coloured as blue, black and red respectively. The following were observed:

- In all the five arrival processes, as c_{h_1} and c_{h_2} increase, the total expected cost rate increases. But the increasing rate for c_{h_2} is high compared to that of c_{h_1}.

- For all the demand process, the Erlang lead time process has low total expected cost rate and hyper exponential lead time process has high total expected cost rate.

- The difference between the total expected cost rate for any two lead time process is high except for $\boldsymbol{HExp}$ demand process. For the $\boldsymbol{HExp}$ demand process, the difference between the total expected cost rate for any two lead time process is low.

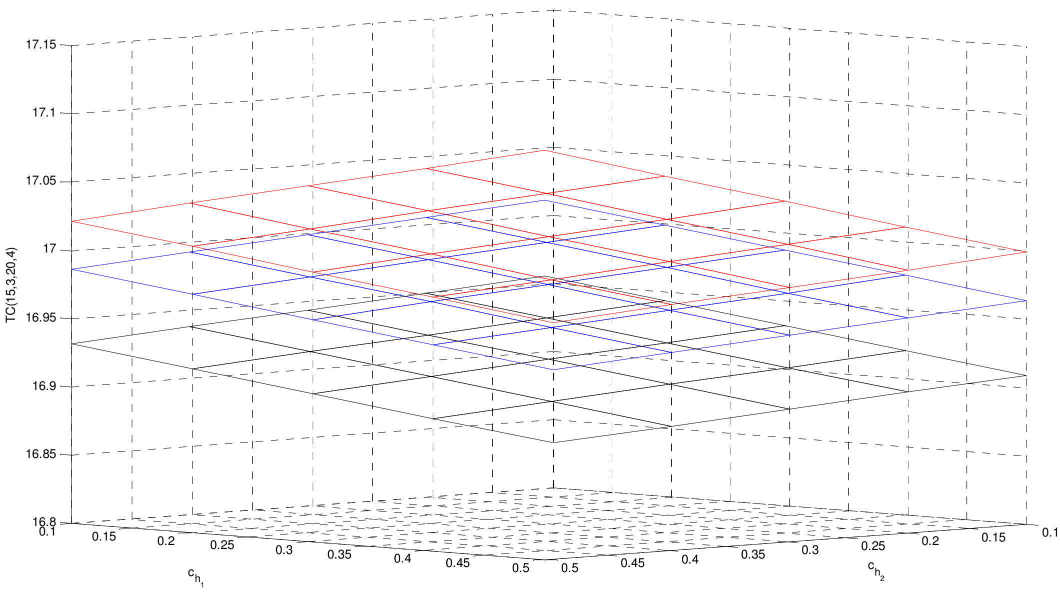


Figure 4.7: $\boldsymbol{Exp}$ demand process

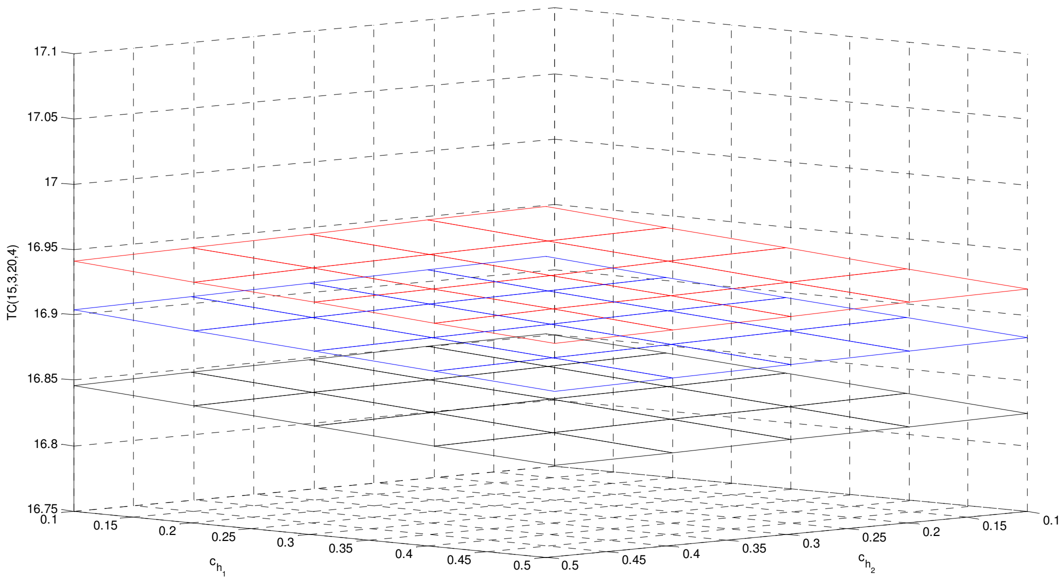


Figure 4.8: Erl **demand process**

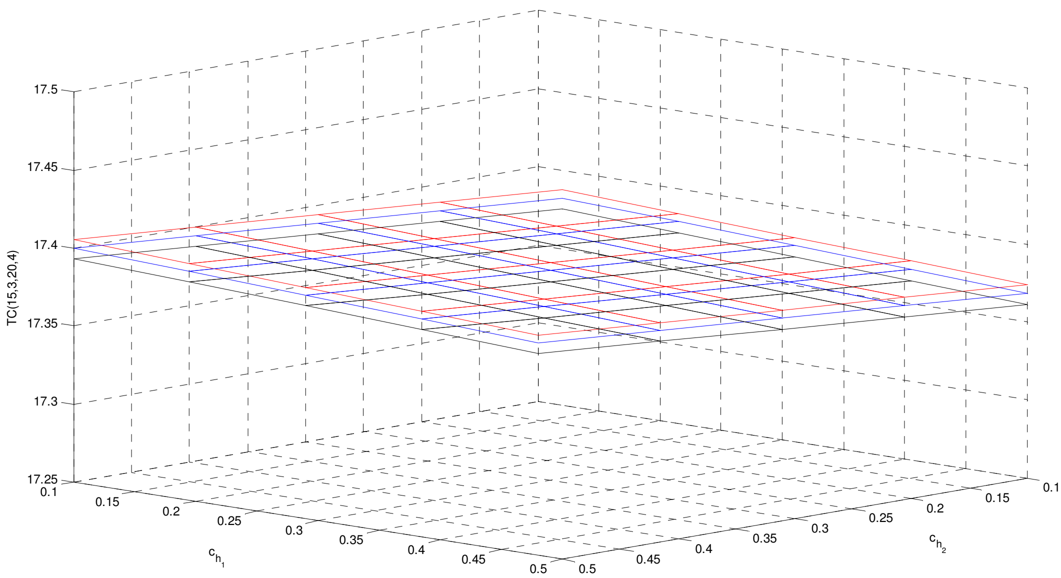


Figure 4.9: $HExp$ **demand process**

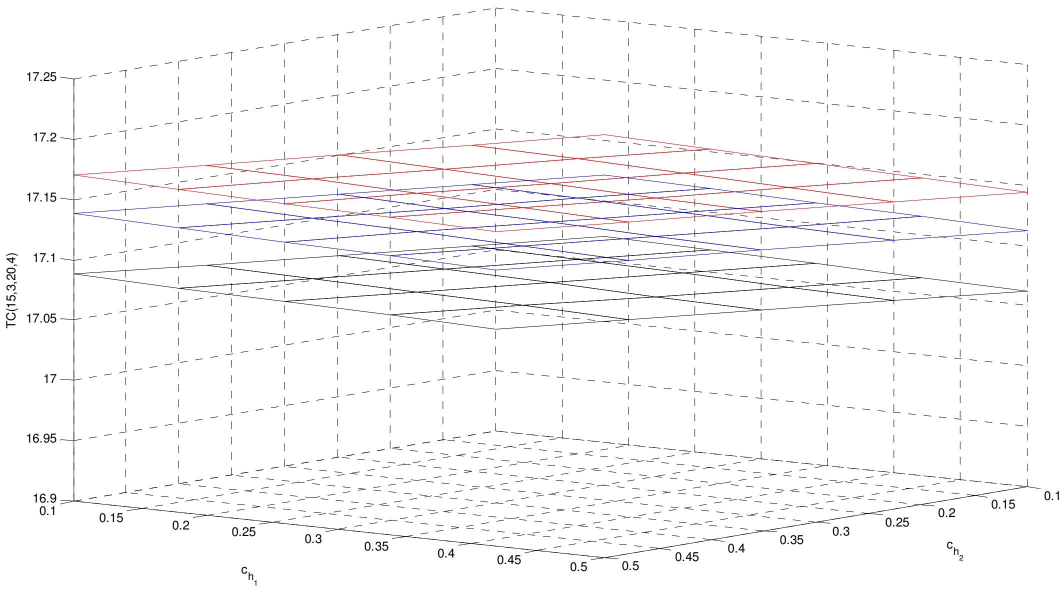


Figure 4:10.: MNC demand process

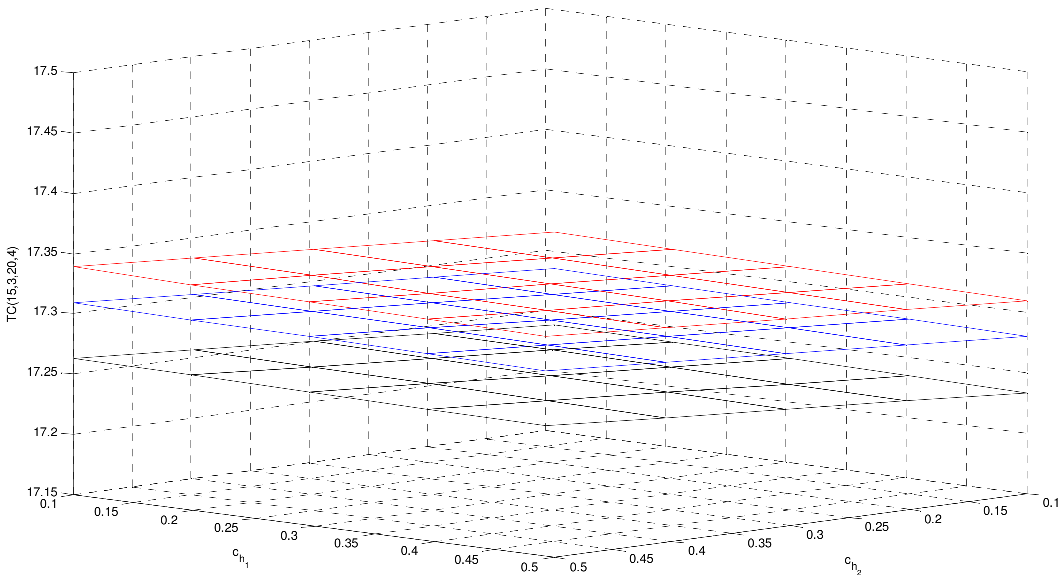


Figure 4.11: MPC demand process

4.7. CONCLUSION

The existing work on two-commodity continuous review inventory system have been extended by introducing the perishability for both commodities, Markov Arrival Process for demand time points and phase type distribution for lead time. It was also assumed that one of the commodities may accept bulk demands. Steady state solutions for the joint distribution of inventory levels have been provided. Under suitable cost structure, the total expected cost rate in steady state have been constructed. To demonstrate the computability of results derived here, ample numerical illustrations have been provided. The effect of the parameters and costs on the total expected cost rate have also been numerically analyzed.

5. [§]

CHAPTER 5

DYNAMIC BUFFERING OF A CAPACITY CONSTRAINED RESOURCE VIA THE THEORY OF CONSTRAINT

[§] A modified version of this first section of this chapter has been submitted to IEOM conference, a peer reviewed international conference holding at Kuala Lumpur, Malaysia in January 2011

5.1. PART A: BUFFERING WITH ZERO SHORTAGE COST

5.1.1. INTRODUCTION

The determination of the size of an inventory buffer placed ahead of the critical resource is one of the main issues deserving of attention in the application of the Theory of Constraints ($\boldsymbol{TOC}$). This seems justified since excess inventory is a perennial problem that the technique is meant to address. Such production systems of interest have some level of (natural) statistical fluctuations in the processing time such that if the resource has an unplanned idle time, planned throughput may be lost. Since it is almost impossible to completely eliminate all forms of uncertainty, there is always a need to accommodate some slack in a system of the nature under consideration. A slack is usually either in the form of reserve capacity or inventory. System slack serves to ameliorate the effects of natural variations that could otherwise lead to the loss of system throughput.

The Theory of Constraints opts to employ the slack of excess capacity to respond to system contingencies that arise due to the natural variations in its processes. It is, however, still impossible to eliminate buffer inventory completely from such systems. It is essential to have a level of inventory necessary to decouple the system in some critical areas of the production network. Such critical stations are allowed time-buffers to maintain throughput, which is the arguably one of the most important measures of the system. The definitions of terms such as throughput, inventory and operating expense are strictly in the context of Goldratt's Theory of Constraints.

The implication of the foregoing is that the level of inventory held in strategic positions is very important in the achievement of the system profit goal. This may explain why a lot of effort in improving the practical potency of the Theory of Constraints has been devoted to managing this type of inventory. The importance is emphasised by the use of the synonym "Drum-Buffer-Rope ($\boldsymbol{DBR}$) system" for this Philosophy of Management, the where the drum is essentially the critical station, and the buffer ahead of it is used to construct a name together with the third word, the rope, which also indicates how the entire system's production is scheduled.

An important question to address at the outset relates to the principal function of the buffer in this system. This question is important since it essentially relates to the buffer size, which has been dealt with extensively by the relevant literature on Inventory Control. The obvious answer is that it serves to protect the critical station which is either the Bottleneck ($\boldsymbol{BN}$) or the most Capacity Constrained Resource ($\boldsymbol{CCR}$) against loss of throughput.

While this answer seems adequate, further elucidation is required on the loss of throughput. The answer that does not seem to always be obvious, is whether the loss is due to the natural process variations that are inherent to the entire system as a result of the variation of the processing time of each work station, or the breakdown of any of the machines that are upstream to the critical station.

Another important issue is the relationship between the Work in Process ($\boldsymbol{WIP}$) Inventory and the flow rate of the system. The amount of inventory that is present ahead of any workstation is not only a function of the strategic buffer placed ahead of such station, but also of the rate of flow of the products through that station. The effect of resource utilisation on the average throughput time and consequently the average number of inventory in the system is well documented in literatures. Some good references are Hopp (2008, pp22-37) and Hopp and Spearman (2009, pp264-349).

A well known equation is the little's law that states that

$$Work - In - Process\ Inventory\ = Throughput\ time\ X\ Throughput\ rate$$

This shows that the quantity of inventory ahead of the critical station cannot be determined as if being independent of the flow rate through the station, especially as the station works close to its full capacity. The effect of utilisation, termed as the curse of utilisation by some authors (Webster, 2008) is presented in figure 5.2.1. This diagram represents the behaviour of an $\boldsymbol{M/M/1}/\infty$ queue before it becomes a bottleneck

$(i.e. 0 < \rho < 1)$. It could be seen that the queue length grows exponentially as the resource transits from a Non-Bottleneck ($\boldsymbol{NBN}$), to a $\boldsymbol{CCR}$ and towards a $\boldsymbol{BN}$. The graph slopes up very quickly as the level of utilisation approaches full utilisation of the resource. This makes it imperative for every manager to place this effect in context as consideration is given to the loading of the system to cover more throughputs and balance the return from such increase in utilisation to have more system throughput against a possible "skyrocketing" cost of holding inventory in the system. That is about the main thrust of this chapter.

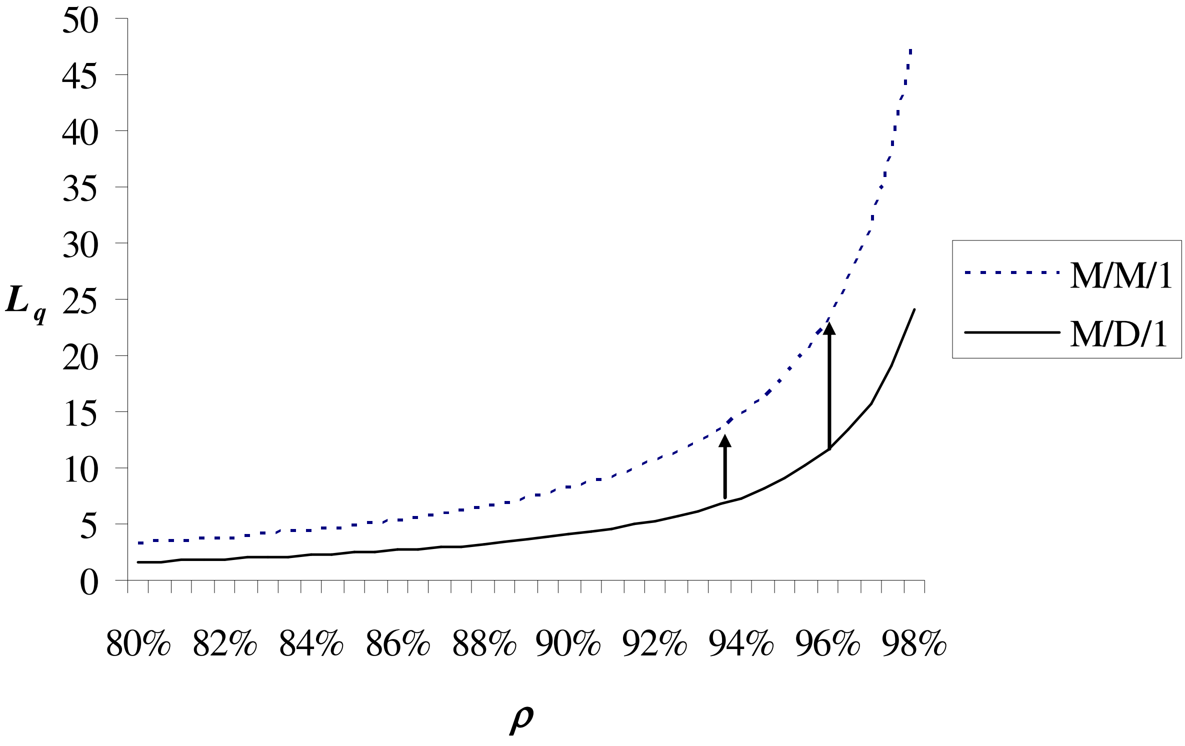


Figure 5.2.1: Curse of utilisation and variance (Webster S. 2008, pg 176)

5.1.2. Some Relevant Salient Features of the TOC

Ronen and Starr (1990) stated some outstanding features of the $\boldsymbol{OPT}$ technique (now commonly referred to as the $\boldsymbol{TOC}$). Two of these are the "unavoidable" statistical fluctuation of the input arrival and service times; and the dependence of processes one on the other, which further worsens the problems of variability. These then dovetail into the effect of such on the $\boldsymbol{WIP}$ discussed earlier.

Another important feature is that this technique can work only in an environment that has a stable schedule, i.e. the product mix (volume and variety) have been stabilised. This is apparent because without such stability, it will be difficult to designate a manufacturing resource as the critical one since its criticality will depend on the current production schedule of the company. This chapter, therefore, assumes a stable production environment and chooses the simplest of such case, perhaps where only one product is produced, and uses that to illustrate how the flow and the buffer in such systems are jointly determined, in tandem with a previous work done assuming a typical $\boldsymbol{M/M/1}$ queuing environment as a reference.

The organisation of the remaining sections of this part of the chapter is as follows. First is a review of some pertinent literature in this area, while trying to identify the purpose of the buffers considered in such literature. Next is the presentation of the model. The next section presents some motivations for considering the process flow rate as an important variable when buffering decisions are being made. This is then followed by a section on numerical example, and then, the suggested areas for further research and conclusions.

5.1.3. Literature

Various authors have written about the applications of the $\boldsymbol{TOC}$ in diverse contexts. But the review here would be limited to those applications that have focused on the determination of the buffer size to be used in the management of the network or the critical station of the system, especially in a quantitative manner.

Many researchers have proposed various heuristics ranging from using the work equivalence of half the manufacturing lead time, a quarter of total lead time or even stating that initial estimation is unnecessary since it is an ongoing improvement process (Spencer, 1991).

Most authors that estimated buffer size quantitatively have been motivated by the failure of the upstream section of the critical resource. Among such papers are Han and Ye (2008) that used the reliability theory to model the machines in the system as having two states of up and down to construct a relationship between the feeder and the fed machines. Page and Louw (2004) used a $\boldsymbol{GI/G/m}$ queues and a queuing network analysis of multiproduct open queuing network modelling method together with the assumption of normality of flow times and a chosen service level to determine the buffer size. So (1989,1997) reports an approximation scheme to determine buffer capacities required to achieve the target performance level in a general flexible manufacturing system with multiple products and another on the optimal buffer allocation problem of minimizing the average work-in-process subject to a minimum required throughput and a constraint on the total buffer space. Simon and Hopp (1991) studied a balanced assembly line system being fed from storage buffers. Processing time is assumed deterministic. Battini et al (2009) developed efficiency simulative study for the allocation of storage capacity in serial production lines and an experimental cross matrix was provided as a tool to determine the optimal buffer size. Li and Tu (1998) presented a constraint time buffer determination model. The model first proposes a machine-view's bill of routing representing a structure that serves as a fundamental structure for formulating and computing the maximum time buffer. By incorporating the Mean-Time-To-Repair ($\boldsymbol{MTTR}$) of each feeder machine, a mathematical relationship was formulated and the time buffer computed. Powel and Pyke (1996) studied the problem of buffering serial lines with moderate variability and a single bottleneck. The focus was essentially on how large variations in mean processing times on machines affect placement of equal buffers between stations.

Not much authors appear to have focussed on buffering exclusively for the purpose of process variation and not resource failure, and to this author's knowledge, none considers, explicitly, managing flow in a $\boldsymbol{TOC}$ environment with considerations for the cost of keeping $\boldsymbol{WIP}$ inventory relative to the gain of achieving such level of utilisation. This directly affects the level of inventory, which is also supposed to be managed by the buffer size, in any system with stochastic input and processing time as typified in an $\boldsymbol{M/M/1}$ queue. The work that appears to have focused exclusively on the critical work station only and in a stochastic processing time environment seems to be that of Radovilsky (1998). This section seeks to build on Radovilsky's work, considering Radovilsky to be good for a $\boldsymbol{BN}$ system but not ideal for a $\boldsymbol{CCR}$ system.

5.1.4. Model Presentation

In the models presented in the literature survey, the goal, generally, seems to be to determine the optimal size of the buffers (constraint or others). These models presuppose that covering the throughputs to meet the market demand to the best of the capacity of the constraint resource would always generate profit for the company. But this may not always be true. While profit may always be realised from the sale of every extra unit of product, the cost that would have resulted from the $\boldsymbol{WIP}$ inventory held in the system as a result of the curse of utilisation might have contributed more expense that the profit realised. This is an often ignored reality in most models. The goal here is to rather seek to determine the optimal flow rate and study how the system profit goal behaves as a result of this flow.

This chapter, therefore, seeks to contribute to how decisions about flow should be made in an $\boldsymbol{M/M/1}$ arrival and processing system in a $\boldsymbol{TOC}$ environment. This is then placed in the context of strategic buffer placement in such environment, bearing in mind the contributions the unit profit per product, unit holding cost per unit product per unit time, and the resource utilisation, ρ, on the profit goal of the organisation. The implication of the Markovian environment is that the holding cost may indirectly be an exponential function, since it is affected by the rate of growth of the queue size ahead of the critical station.

The variables and notations adopted in this paper are consistent with the ones used in Radovilsky (1998). This is to allow for ease of comparison. So, an optimal flow rate is being sought to maximise the profit function of the system. From this, the average queue size is to be retrieved. Other decisions about what size of buffer to allow would then be made based on these functions. It is also assumed that only one product is being produced in this system, and a processing centre is involved. This is to simplify the analysis without loss of generalisation. The objective is the maximisation of the Net Profit function which is defined as

$$NP = TH - OE \tag{5.1.1}$$

$$TH = \mu(1 - P_0)C_{TH} \tag{5.1.2}$$

$$OE = L_S C_{OE} \tag{5.1.3}$$

where NP is the Net Profit,

TH is the throughput rate,

OE is the Operating Expense (incurred during the same time window as the throughput, and is assumed here to be made up of only the holding cost)

μ is the rate of service at the resource over a stated time interval

$P_0$is the probability that constraint buffer of the resource is empty

C_{TH} is the profit earned from selling a unit of output

L_S is the average queue length on the resource

C_{OE} is the inventory cost per unit (product-time)

K is the buffer size

D is the demand rate from the market

ρ_D is the level of utilisation based on D defined as the ratio $^D/_\mu$.

The process is assumed to follow the $\boldsymbol{M/M/1/\infty}$ queue and so, P_0 and L_S are substituted with the following in the $\boldsymbol{NP}$ equation:

$$P_0 = 1 - \rho \tag{5.1.4}$$

$$L_S = \frac{\rho}{1-\rho} \tag{5.1.5}$$

So, the net profit equation becomes

$$NP = \mu\rho C_{TH} - \frac{\rho C_{OE}}{1-\rho} \tag{5.1.6}$$

This makes the optimal ρ to be

$$\rho^* = 1 - \sqrt{\frac{C_{OE}}{\mu C_{TH}}} \tag{5.1.7}$$

Recovering the optimal buffer size simply becomes associated with the steady state queue length, L_S, corresponding to ρ^*, and this is

$$L_S = \sqrt{\frac{\mu C_{TH}}{C_{OE}}} - 1 \qquad 5.1.8$$

And the optimal net profit, $\boldsymbol{NP}^*$, function becomes,

$$NP^* = \left(\sqrt{\mu C_{TH}} - \sqrt{C_{OE}}\right)^2 \qquad 5.1.9$$

Radovilsky (1998) had derived a similar equation for the optimal buffer size for considering the process to be an $\boldsymbol{M/M/1/K}$ for case $\rho = 1$. The results are that

$$K^* = \sqrt{\frac{2\mu C_{TH}}{C_{OE}}} - 1 \quad (\rho = 1) \qquad 5.1.10$$

and
$$NP^* = \frac{1}{2}\left(\sqrt{2\mu C_{TH}} - \sqrt{C_{OE}}\right)^2 \quad (\rho = 1) \qquad 5.1.11$$

Radovilsky's assumptions connote the $\boldsymbol{BN}$ condition, hence, solving the case $\rho = 1$. He also did some numerical analysis for the case $\rho > 1$.

5.1.5. Benefits of optimising with respect to the ρ

Before analysing and making deductions from the model proposed in this paper, some benefits of optimising the profit with respect to the flow rather than the buffer size would be pointed out.

Firstly, the effect of possible exponentially increasing queuing time on the system profit as the flow rate gets closer to the full utilisation of the resource capacity is more easily observed. It may be more profitable to allow lost throughput than to buffer for process variability. This will be further discussed. Secondly, it is easier to extend the model to other queuing cases. This is because ρ is a more pervasive variable than K. While K is found in capacitated queues only, ρ is the main variable of interest of all queuing types. This will make it possible to utilise other types of queues, e.g. queues with balking, perishable input, etc. Thirdly, controlling the buffer may be simply reduced to controlling the flow rate rather than monitoring the position of the buffer. The former would be easier.

5.1.6. ANALYSIS AND DEDUCTIONS

From equation 5.1.7, one could notice that as C_{OE} decreases, other things being equal, ρ edges closer to unity indicating higher utilisation of resource. The corresponding effect is seen in L_S in equation 5.1.8 because the average queue length increases, meaning more inventory is allowed. The effect of C_{TH} is the reverse; increase in C_{TH} leads to increase in in both the flow rate and average queue length. Also, optimal buffer size increases with increase in service rate (or capacity) of the system. The effects of increase or decrease in C_{TH}, C_{OE} and μ are also apparent in equation 5.1.9; as either of μ and C_{TH} increases, net profit also increases, and as C_{OE} increases, net profit decreases as expected.

5.1.7. Numerical Analysis

The effect of using the dynamic buffering approach proposed is compared to the result from Radovilsky's model. This is done using a numerical example. But before the numerical analysis is done, an observation is raised.

In any $M/M/1$ queuing model, working at 100 percent utilisation is not theoretically unachievable because of the corrupting influence of variability on the build up of $\boldsymbol{WIP}$ ahead of the critical station. This has been explained with the curse of utilisation, and the implication is that inventory could theoretically build up ahead of the critical station infinitely. With $\rho = 1 \equiv \lambda = \mu$, a Markov chain in which all the states are recurrent null results, and the expected time of return to any of the states it has ever visited is infinite. This implies that the queue would grow on perpetually. (An interested reader may refer to Hopp (2008, section 1.3 pg 15) and Cinlar (1975, Chapter 6, Lemma 5.33 pg 176).

There will be periods of blocking for as long as $\rho \geq 1$ in a series system that includes the critical resource somewhere along its line except there is an infinite space in between the critical resource and the feeding resource. For there not to be blocking in the queue type considered at a specified probability level, the buffer size in equation 5.1.10 to be greater than kL_s, for most μ in equation 8. $k = 2$ for about 95 percent level. This means

$$\sqrt{\frac{\mu C_{TH}}{C_{OE}}} - 1 < \frac{1}{2}\sqrt{\frac{2\mu C_{TH}}{C_{OE}}} - 1 \qquad 5.1.12$$

The condition for this to happen is that

$$\mu < \frac{1}{2(3-2\sqrt{2})}\frac{C_{OE}}{C_{TH}} \qquad 5.1.13$$

This implies that the processing rate has to be quite small compared to the cost of inventory relative to the unit profit. It should be noted that the unit of μ is $1/time$, the unit of C_{TH} is $money$ while that of C_{OE} is $1/(money.time)$. This means that the flow rate per time must be less than the ratio of the inventory cost per unit product per time to the profit made from a unit product, divided by $1/[2(3-2\sqrt{2})]$. Very few products will probably fulfil this. This makes it imperative to seek to optimise ρ in the $\boldsymbol{CCR}$.

Figure 5.1.2 shows the behaviour of the system net profit before and after the optimal flow rate. This picture shows that the net profit increases somehow linearly until the maximum at the optimal flow rate, but declines very rapidly after the optimal flow rate. This shows that the curse of utilisation kicks in very strongly once the optimal flow rate is exceeded, and every marginal gain in profit is quickly eroded by the ballooning inventory cost. This indicates that it might be better not to meet all the customer demands that are between ρ^* and ρ_D. This gives a guide as to making trade off decisions in a $\boldsymbol{CCR}$ environment.

Next is presented the results of some numerical analysis in graphical form. Since Radovilsky's model uses $\rho = 1$, there is the need to scale the model so that an effective comparison can be made. It was noted earlier that full utilisation would perpetually build up finished goods inventory which, theoretically, could increase the buffer size to infinity. This would mean the cost also grows to infinity, thereby decreasing productivity accordingly (in line with $\boldsymbol{TOC}$'s technical definitions). This implies that the throughput in Radovilsky could have been overstated because it was assumed there that all output at $\rho = 1$ is throughput.

A benign alternative is to imagine that the full capacity of the station mentioned in Radovilsky is actually μ', a down-scaled portion of the actual μ, which is determined by $\mu' = \rho.\mu = \lambda$. It would also be assumed that this μ' is the production output that is guaranteed to be purchased by the market, and is the actual throughput in the context of $\boldsymbol{TOC}$. This means the constraint moves from the market to the production facility and the $\boldsymbol{CCR}$ "behaves" like the $\boldsymbol{BN}$ which now runs at 100 percent utilisation. The capacity then changes to $\mu\rho$, where ρ is what the new model determines as the actual feed rate to control the entire system to build the dynamic buffer ahead of the $\boldsymbol{CCR}$. This second scenario is, therefore, taken here as the upper bound for the Net Profit using Radovilsky's model. Based on this modification, the comparison was done.

For the purpose of this numerical illustration, arbitrary values were chosen as follows: Service rate = 50 items per time; Profit from unit sale = 50 units of money; Unit inventory holding cost = 20 units of money. For some dynamic analysis to track the behaviour of the model as a given parameter changes while others are kept fixed, an upper limit as set for the three variables that determine ρ, K and $\boldsymbol{NP}$ are as follows: Service rate = 100; Profit from unit sale = 150 units of money; Unit inventory holding cost = 100 units of money.

With all other variables held constant, figure 5.1.3 shows that optimal feed rate increases with increasing service rate; figure 5.1.4 shows that optimal buffer size increases with increasing service rate; figure 5.1.5 shows that optimal buffer size increases with increasing profit per unit sale; figure 5.1.6 shows that optimal buffer size decreases with increasing unit holding cost. It is worth mentioning that the effect of decreasing holding cost seems more drastic than those of other parameters on the optimal buffer size. This would be further buttressed when the graph of the Net Profit function is also interpreted. This is noticeable from the slopes of each of the curves. The same pattern is observed for the effect of each of the parameters on the average inventory and as such, the diagrams were not repeated.

The impact of the three key variables on Net Profit is examined in figures 5.1.7 to 5.1.10. Holding all other parameters constant, it can be seen from figure 5.1.7 that the net profit increases with increase in service rate; figure 5.1.9 shows that net profit increases with increasing profit per unit sale; figure 5.1.10 shows that net profit decreases with increase in unit holding cost. It can also be seen that the rate of decrease in net profit per unit increase in holding cost is more drastic, buttressing the initial observation with the buffer size. This is actually why the optimal buffer size drops sharply with every increase in unit holding cost.

One can also observe from the net profit function graphs that if adjustment is made for the fact that not all products made for full utilisation could be sold if the demand is less than the capacity, then, the profit margin for the proposed model seems higher than that of Radovilsky in the range $0 < \rho < 1$.

5.1.8. CONCLUSION

In conclusion, a model has been presented that has the potential for more profit in a $\boldsymbol{CCR}$ system than that which was done earlier. The focus of the model is on buffering a $\boldsymbol{DBR}$ system for statistical process fluctuations, without breakdown of upstream stations. More so, it is easier to control such system with the dynamic buffering approach through ρ than it would likely be in Radovilsky's model because it is not necessary to build up any inventory ahead of the $\boldsymbol{CCR}$ before regulating the feed rate of the $\boldsymbol{CCR}$ line. With the optimal ρ already determined, the system dynamically adjusts the optimal time buffer accordingly. Also, the optimal buffer size was retrieved indirectly from the optimum ρ. The elimination of the need to have the optimal buffer length involved in the derivation of the optimal Net Profit function makes it easy to extend the model to other more interesting areas like deteriorating inventory and network buffer balancing, which are some of the interesting areas of research to be explored after this work.

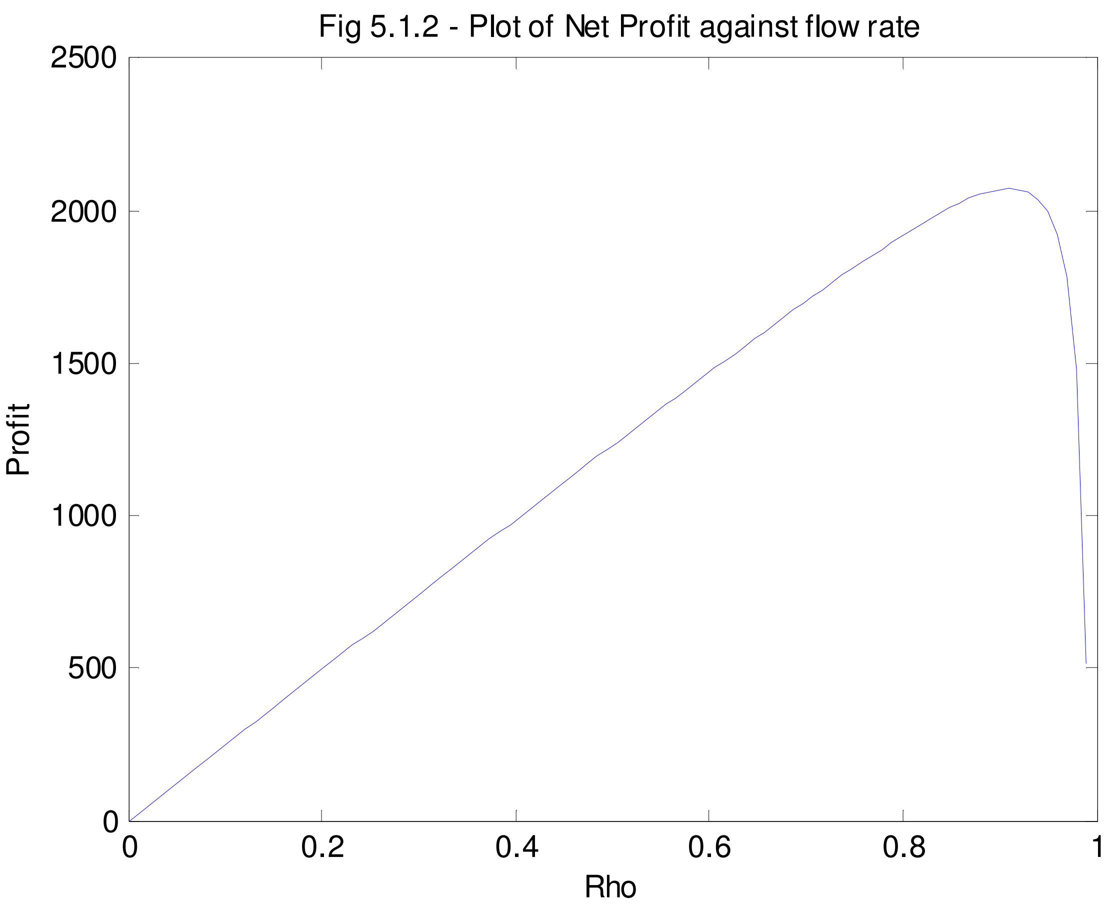


Figure 5.1.2: Net profit change with rho for $0 < \rho < 1$

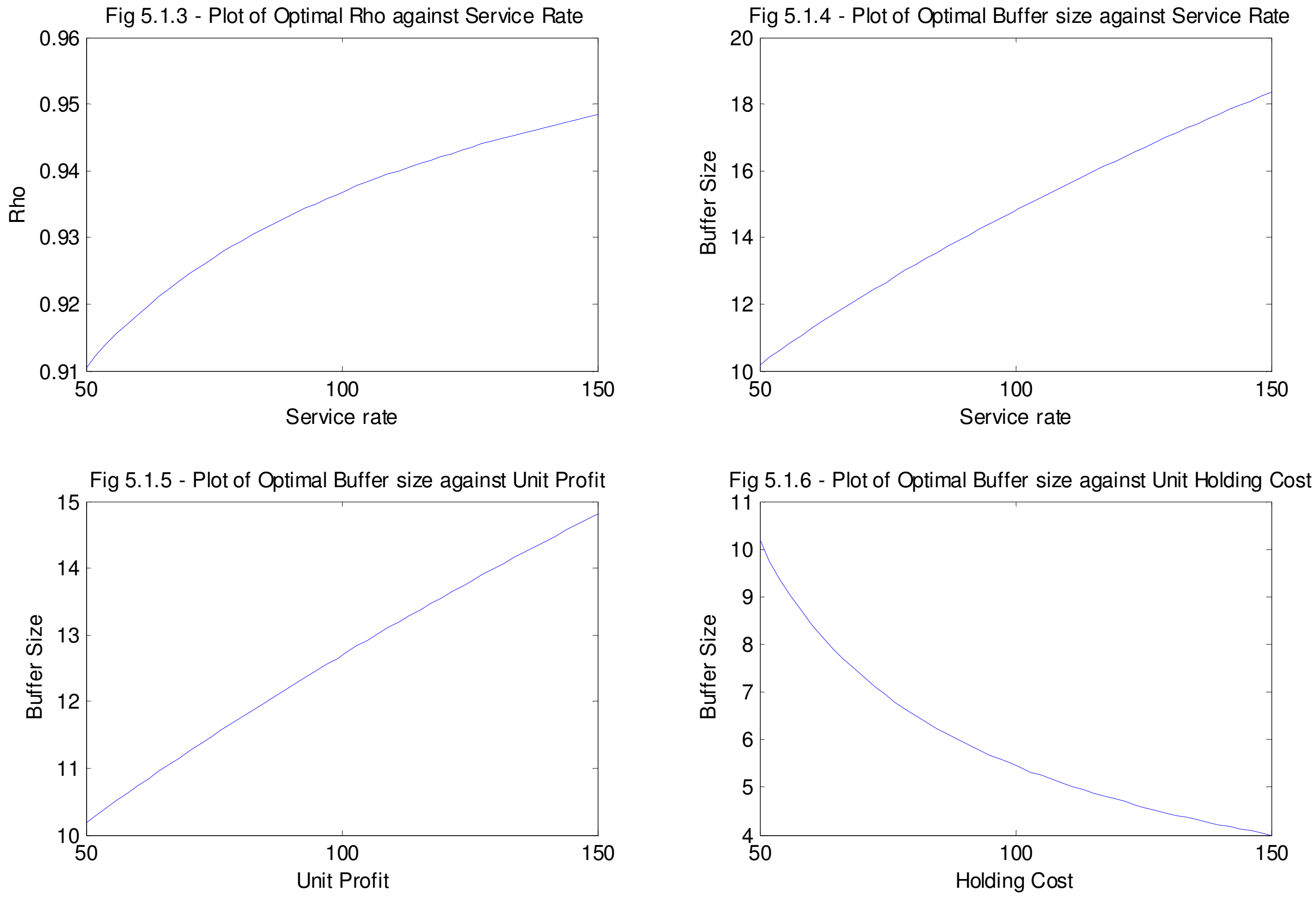
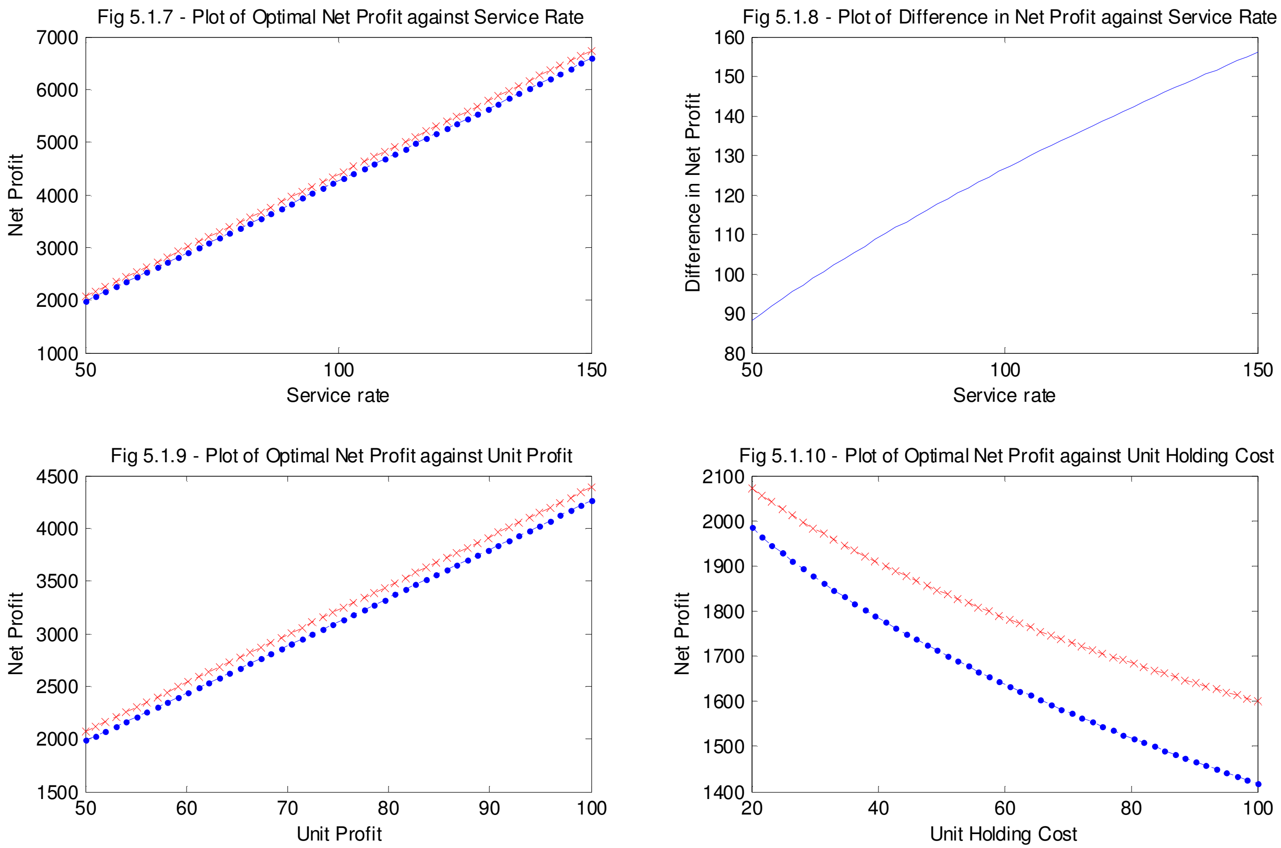


Figure 5.1.3 to 5.1.6: Changes in rho and buffer size with input parameters



Figures 5.1.7 – 5.1.10: Changes in net profit with input parameters

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OPTIMISING FLOW IN AN M/M/1 SYSTEM WITH SHORTAGE COST: A THEORY OF CONSTRAINTS APPROACH

** A modified version of this part of this chapter has been accepted for presentation at the IASTED conference, a peer reviewed international conference, holding at Gaborone, Botswana in September, 2010.

5.2. PART B: BUFFERING WITH POSITIVE SHORTAGE COST

5.2.1. INTRODUCTION

Excessive build up of Inventory in a production system is one of the critical wastes that the Theory of Constraints seeks to attack. Based on this principle, the focus of a production system should be on maintaining flow rather than keeping inventory in the system. Inventory should only be kept ahead of the most critical work station and at some strategic points where the most critical line meet other lines in such a manner that other resources are scheduled to support this critical resource. The determination of the appropriate buffer size to place ahead of this critical resource and at the strategic points in the network is an area that has generated diverse interests, but most authors have not discussed issues of optimising flow through these lines.

In this section, the problem of the determination of the optimal rate of flow in a production system is being further considered. Such flow would automatically build up inventory ahead of the critical station, which in this case is a Capacity Constrained Resource ($\boldsymbol{CCR}$), in a production management environment utilising the Theory of Constraints ($\boldsymbol{TOC}$), and where every unit of lost production throughput has a stipulated cost. This seems plausible because, based on queuing theory, they are jointly determined, and the optimal value of one implies that of the other. Decision for any extra inventory may be made, however, based on marginal return of such extra inventory. In deriving this model, it was assumed the cost paid is once off, and not time dependent, for every throughput that is lost. This model is an extension of that derived in the previous section (5.1), and which was compared to that developed by Radovilsky (1998).

This second section, therefore, presents a more generalised model. The model in section 5.1 is a particular case of this extended model where it is implicitly taken that the unit shortage cost is zero.

5.2.2. LITERATURE REVIEW

Much work has been done on buffering in a manufacturing flow process. The majority appears to have focussed on integrated (automated and semi-automated) systems. This makes the focus of most such articles to be the solution to the design problem of the space to be allowed in-between processing centres in such an integrated environment which needs to be determined before construction, which is different from the problem of the management of the actual production process flow.

Some of the early contributions to this area include the paper by Hunt (1956), which was an analysis of a system where service is to be done in stages. This work was different from phase type process earlier done by Jackson (1954) in that simultaneity and blocking are allowed in the processes. Poisson input and exponential service time was assumed and the model is basically Markovian. Others include machine reliability approach by Enginarlar et al (2002) and Bartini et al (2009), and Production system with three unbalanced stations by Powell (1994) amongst others.

Something common to almost all these papers is that all the machines in the production network were being buffered. The approach, therefore, seems rather different from that being advocated by the Theory of Constraint ($\boldsymbol{TOC}$), where buffers are included only in strategic locations and not ahead of all machines/processing centres as in almost all the cases reported earlier. $\boldsymbol{TOC}$ advocates the presence of spare capacities in many areas of the production system but disapproves of holding inventories except where necessary.

Also, most of the works done seem to be buffering for the failure of feeder machines upstream to the critical station. Buffering for the purpose of the statistical fluctuations in the input and processing times seems not to be the main concern. Only Radovilsky (1998) appears to be quite applicable to buffering for the flow of the process, and it explicitly includes unit profit and unit holding cost in the model.

In summary, a review of literatures on the determination of an appropriate buffer size to place ahead of the critical resource in a production environment utilising the Theory of Constraint has been done by in a previous paper in section 5.1. The summary of the contributions of several authors like Faria et al (2006), Han and Ye (2008), Li and Tu (1998), Powell and Pyke (1996) and Radovilsky (1998) were discussed amongst others.

The effect of utilisation on the Work-in-Process ($\boldsymbol{WIP}$) inventory and its implication on the system cost appears not yet fully researched. Most authors that have written on buffering the relevant stations of the theory of constraints appear to have assumed that all the demands from the market should be met. But in order to meet these demands sometimes, the utilisation of the resources may need to be quite high. This has been discussed in section 5.1 and illustrated with figure 5.1.1.

Radovilsky (1998) has shown how the buffer size to support the bottle neck ($\boldsymbol{BN}$) station could be estimated using the capacitated queue M/M/1/K approach, where he found the derivative ofthe profit function relative to the queue capacity, K, and derived the optimal queue size.

While this is a good attempt, it has two key drawbacks. Firstly, it is difficult to extend this model to a case where other types of inputs (e.g. deteriorating inputs or balking inputs) could be considered. Secondly, it is difficult to include the range $0 < \rho < 1$ in the analysis. This has also been discussed in section 5.1, where it was shown that a solution to both drawbacks could be to optimise the flow rather than the buffer. The optimal buffer size can then be obtained from the steady state size of the queue once the maximum allowable shortage is specified (this can be a policy matter). Controlling the production system should also become easier since the feed rate controls the whole production line rather than just one machine. This makes the management of the system easier. This, actually, is in full sync with the philosophy of the $\boldsymbol{TOC}$, where the focus should be on the flow rather than the capacity of the system, and hence the Drum-Buffer-Rope approach.

The previous works did not consider the possibility of paying some cost for every throughput lost. A more realistic model will seem to be the one that accounts for the possibility of paying for every lost throughput. The least that could be paid is the opportunity cost of the revenue that should have been earned. In addition to this, there could be other penalties imposed on the company by its clients, especially in a case where it has one or more major client(s) that are responsible for the purchase of the bulk of its output. This scenario is not farfetched today where supply chain management ($\boldsymbol{SCM}$) is rife and many major global companies are implementing lean techniques and having their inputs delivered Just-in-Time and probably Just-in-Sequence.

The need to account for this cost of failure to deliver output as needed necessitates this extension. The shortage cost here is, however, assumed to be a fixed cost paid per unit product of output not supplied to the customer as and when needed and not increasing with the length of time for which the output was not available.

5.2.3. MODEL PRESENTATION

In this section, the net profit function is defined to include some cost of shortages. The net profit function then becomes

$$NP = TH - OE - SC \tag{5.2.1}$$

$$TH = \mu(1 - P_0)C_{TH} \tag{5.2.2}$$

$$OE = L_s C_{OE} \tag{5.2.3}$$

$$SC = \mu P_0 C_{SH} \tag{5.2.4}$$

where NP is the Net Profit,

TH is the throughput,

OE is the Operating Expense

SC is the Shortage Cost

μ is the rate of service at the station

$P_0$is the probability that waiting buffer of the resource is empty

C_{TH} is the profit earned from selling a unit of output

L_S is the average queue length on the resource

C_{OE} is the inventory cost per unit (product-time)

C_{SH} is the shortage cost for every unit throughput lost

An implicit assumption in the models in section 5.1 and Radovilsky (1998) is that this cost of shortages is actually zero. This can be seen by looking at equation 1. The new term introduced, $\boldsymbol{SC}$, as seen in equation 4, must be zero if we must have equation 1 appearing in the initial form. For this term to be zero, at least one of μ or P_0 or C_{SH} equals 0. Since it is not reasonable for either μ or P_0 to be zero, else the first term, $\boldsymbol{TH}$, would have also been zero or the resource becomes a bottleneck, so then, only C_{SH} could have been zero.

From the solution to $\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1}/\infty$, queue P_0 and L_S are:

$$P_0 = 1 - \rho \tag{5.2.5}$$

$$L_S = \frac{\rho}{1-\rho} \tag{5.2.6}$$

Having done this, the net profit equation becomes

$$NP = \mu\rho C_{TH} - \frac{\rho C_{TH}}{1-\rho} - \mu(1-\rho)C_{SH} \tag{5.2.7}$$

Differentiating equation 7 with respect to ρ and setting the derivative to zero to obtain the optimal ρ gives

$$\rho^* = 1 - \sqrt{\frac{C_{OE}}{\mu(C_{TH}+C_{SH})}} \tag{5.2.8}$$

The optimal buffer size can then be recovered from the optimal steady state queue length, L_S, corresponding to ρ*, and this is obtained by substituting equation 8 into equation 6 to obtain

$$L_S = \sqrt{\frac{\mu(C_{TH}+C_{SH})}{C_{OE}}} - 1 \tag{5.2.9}$$

Putting 5.2.8 and 5.2.9 into 5.2.7 and solving for NP*, the maximum profit function,

$$NP^* = \left(\sqrt{\mu(C_{TH}+C_{SH})} - \sqrt{C_{OE}}\right)^2 - \mu C_{SH} \tag{5.2.10}$$

One can see that this model is similar to the one obtained for the case where shortage cost was not considered in section 1 and reproduced here as equation 5.2.11.

$$NP^* = \left(\sqrt{\mu C_{TH}} - \sqrt{C_{OE}}\right)^2 \qquad 5.2.11$$

5.2.4. DEDUCTIONS FROM THE OPTIMAL NP EQUATION

One could easily see from equation 5.2.10 that if C_{TH} is zero, the solution is the same as that obtained in the previous section. But since the cost of shortages is hardly ever zero, then the model presented in this paper should give a more realistic profit estimate than equation 5.2.11.

The effects of C_{TH}, C_{OE} and μ are easily observed from the optimal ρ, optimal L_S and optimal $\boldsymbol{NP}$ equations. One could see that as C_{TH} increases, the optimal ρ, the optimal L_S as well as the Net Profit increase. One can also notice that as C_{OE} increases, the optimal ρ decreases, the optimal L_S decreases and the expected net profit decreases as well.

The effect of the unit shortage cost is easily seen for both the optimal ρ and optimal L_S. One can see that as C_{SH} increases, both the optimal ρ and the optimal L_S increase. But the effect of an increase in C_{SH} on the optimal $\boldsymbol{NP}$ is not so obvious from equation 5.2.10 since C_{SH} is in the two terms of the NP function, where its increase will tend to have an increasing effect due to the first one and a decreasing effect due to the other.

The effect of the unit shortage cost on the new profit function would be done in the section where numerical analysis is carried out, but it is worth exploring how the new variable affects the overall profit function. The effect of C_{SH} on the optimal profit function could be analytically studied by assuming one function is greater than the other and finding the condition under which that could be true. Intuitively, one can assume that including the shortage cost in the equation should reduce the profit function as shown in equation 5.2.12.

$$\left(\sqrt{\mu(C_{TH} + C_{SH})} - \sqrt{C_{OE}}\right)^2 - \mu C_{SH} < \left(\sqrt{\mu C_{TH}} - \sqrt{C_{OE}}\right)^2 \qquad 5.2.12$$

Solving the inequality and find the condition under which that could be true. This gives

$$\mu C_{SH} C_{OE} > 0 \qquad 5.2.13$$

Since it has been established that both μ and C_{OE} are not zero (actually positive), the condition in inequality 5.2.12 can only be true for C_{SH} greater than zero. The same conclusion could have been easily reached by simply looking at equation 5.2.7 and noting that $C_{SH} < 0$ increases the profit function, $C_{SH} > 0$ decreases the profit function, while $C_{SH} = 0$ makes the profit function to be equal to the model in equation 5.2.11.

This means that the expression in equation 5.2.10 is equal to the expression in equation 11 only when C_{SH} is zero. If C_{SH} is negative, then the expression in equation 5.2.10 is always greater than that equation 5.2.11 and if C_{SH} is positive, the expression in equation 5.2.10 is always less than that in equation 5.2.11. Since having negative C_{SH} is unreasonable, the value of C_{SH} can only range from zero to positive. This means the Net Profit function is of 5.2.10 always less than that in equation 5.2.11 for as long as there is cost of shortages, which makes intuitive sense.

The models derived in equations 5.2.8 and 5.2.9 therefore give guidance for how to select the optimal feed rate to optimise the net profit in a system that has a Capacity Constrained Resource but no Bottleneck when applying the Theory of Constraints in a production system, and/or where buffering is being made for statistical fluctuation in processing time and not for breakdown of the upstream stations to the critical resource.

5.2.5. NUMERICAL ANALYSIS

The effect of the inclusion of shortage cost in the model on the net profit is shown here. The net profit realised with shortage cost included is compared to that the dynamic buffering approach in section 1.

Figure 5.2.1 shows that as the flow rate moves towards the optimal rate, the difference between the model with and that without the shortage cost narrows. This shows that the effect of shortage cost becomes more pronounced as the system operates below the optimal level. But as the utilisation moves towards unity, the effect of shortage cost fizzles away. An explanation for this is that the possibility of shortage becomes almost zero as the queue length increases tremendously. This is because it is almost impossible to have shortages as a result of an idle resource as the probability of being idle goes towards zero. Also, the holding cost term dominates the profit function.

Next, the effect of changes in the various input parameters on the optimal utilisation (intensity), ρ, and the optimal average queue length, L_S, were graphically evaluated. For the purpose of our analysis, starting values were randomly chosen for the input variables. All of them were initialised to 50. With every other variable kept constant, the effect of each of the input variable on the optimal output values were observed by varying only the variable of interest.

Figures 5.2.2 to 5.2.5 show the effects of the changes in the values of the input variables on the optimal value of ρ. From these, optimal ρ increases with every of the input except the holding cost, and this is easily seen from equation 5.2.8. Also, both the shape and the slope of the curves of change in unit profit and change in unit shortage cost are the same. This can also be easily deduced from equation 5.2.8. It can also be seen that the effect of the service rate and the holding cost are more dramatic than those of unit profit and unit shortage costs. As each of the input variable quadruples from 50 to 200, one would notice that the rate of change in value of ρ for both the holding cost and the service rate are double those of unit profit and shortage cost. This is also apparent from equation 5.2.8. The effects of each of the input variables on the optimal average buffer build up is exactly the same as that noticed in ρ, and this is seen from figures 5.2.6 to 5.2.9.

Figures 5.2.10 to 5.3.13 show the effects of changes in the values of the input variables on the optimal net profit. It could be seen that while net profit increases with increasing service rate and unit profit, it decreases with increasing unit holding cost and unit shortage cost.

The net profit functions of the models with and without shortages have been plotted on the same axes. The diagram suggests that if the effect of shortage cost is neglected as done in the previous models, the changes in unit profit appears to have less effect on difference in profit predicted by the model with shortage cost and the one without it. But changes in holding cost appear to have the most dramatic effect. This can be explained by looking at equation 5.2.13.

In figure 5.2.7, the net profit function changes relative to changes in unit shortage cost is seen as a straight line for the model without shortages since C_{SH} has been taken as zero here. But the effect of increasing the holding cost on the net profit appears more drastic than that of the shortage cost.

Following the analyses of the effects of the various input variables on the computed output parameters, the holding cost appears to be the most important variable whose changes should be monitored to make the necessary flow adjustments to keep the system optimal.

5.2.6. CONCLUSION

The model of dynamic buffering of a $\boldsymbol{TOC}$ with shortage cost has been presented. It was assumed that the cost of shortage is a once off unit cost charged per unit product short. The previous model without shortages was shown to be a particular case of this model where the cost of shortages could be taken as zero. This model should be more realistic than a model without shortage cost included.

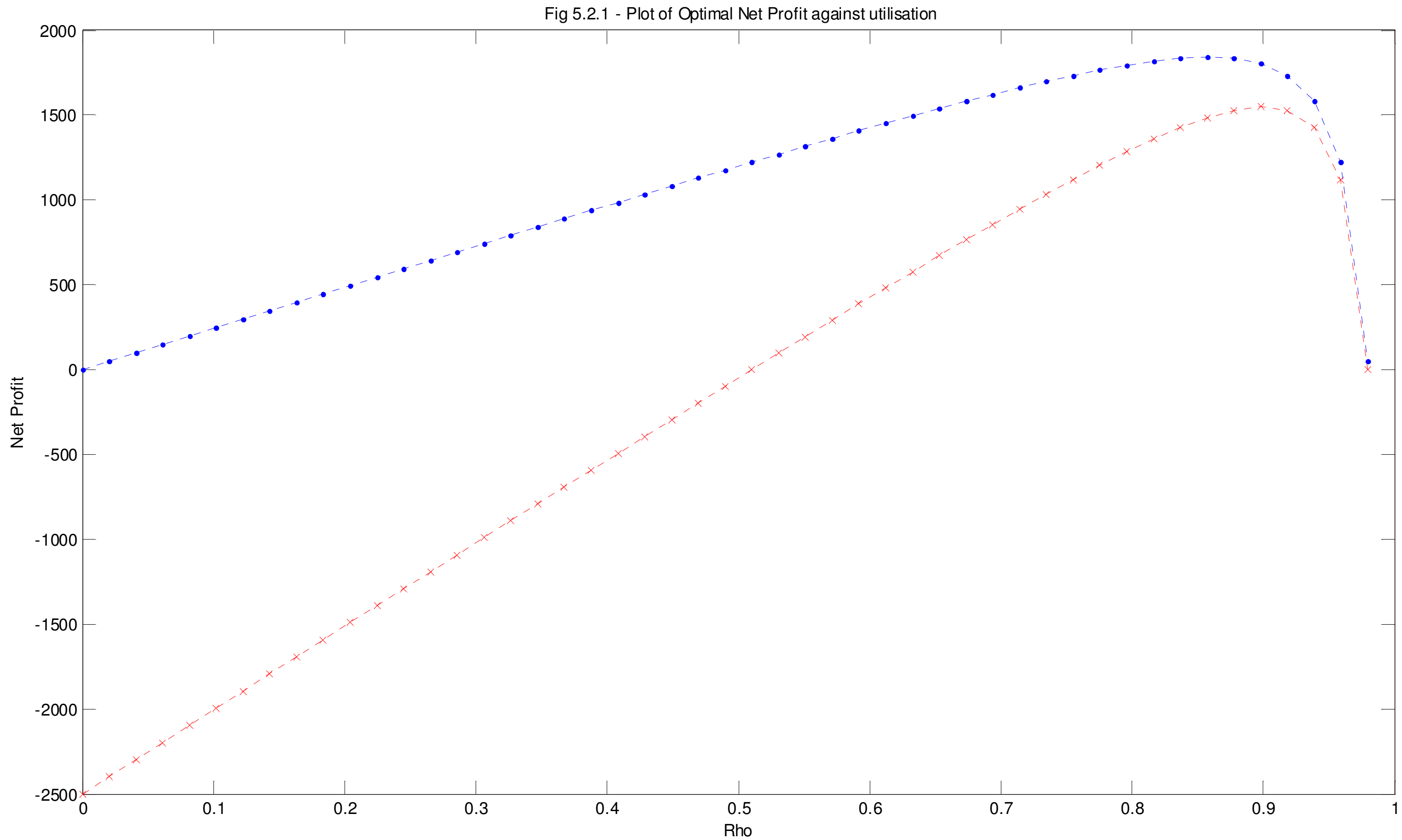

Figure 5.2.1: Changes in profit with rho $(0 \leq \rho \leq 1)$

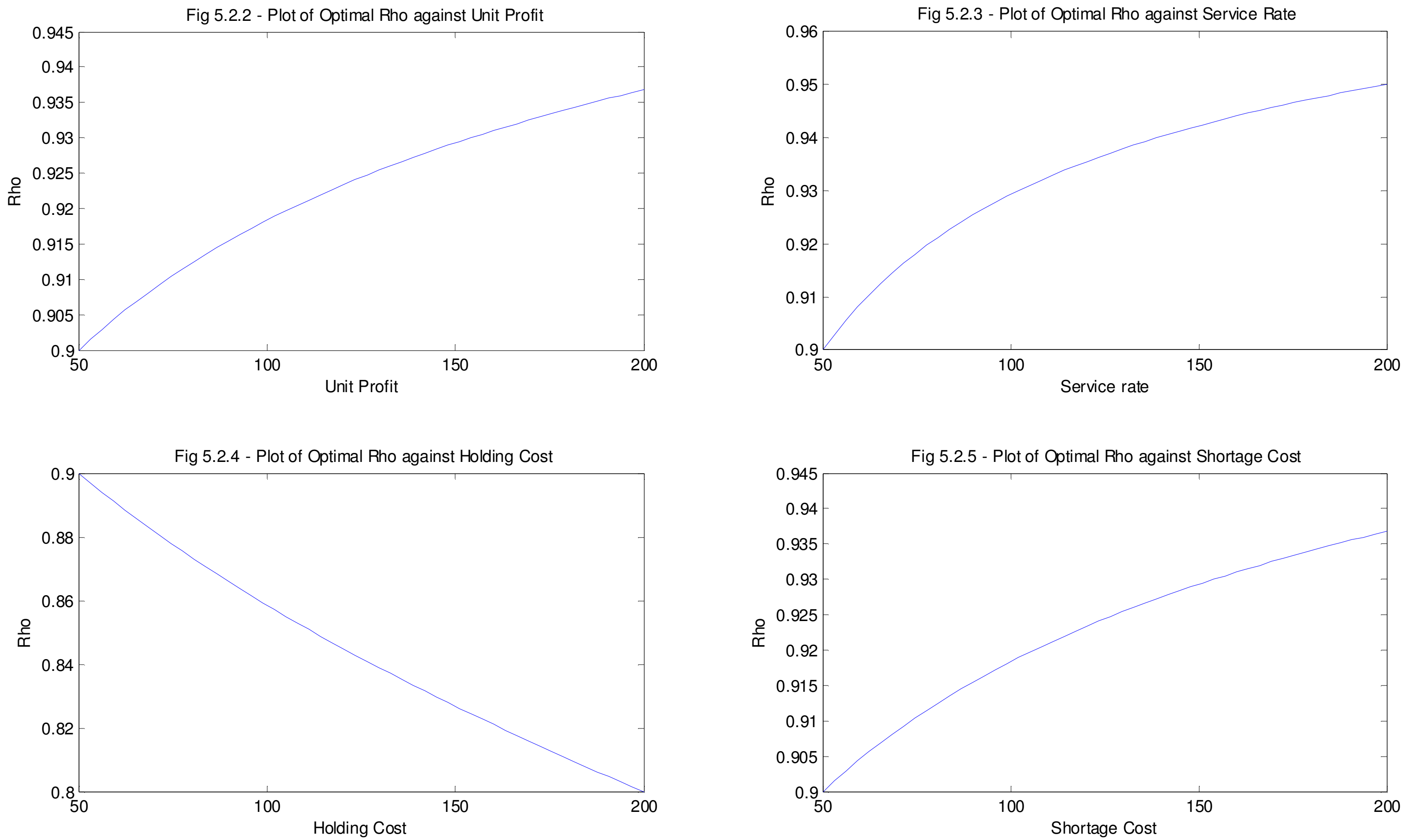

Figure 5.2.2 – 5.2.5: Changes in rho with input parameters

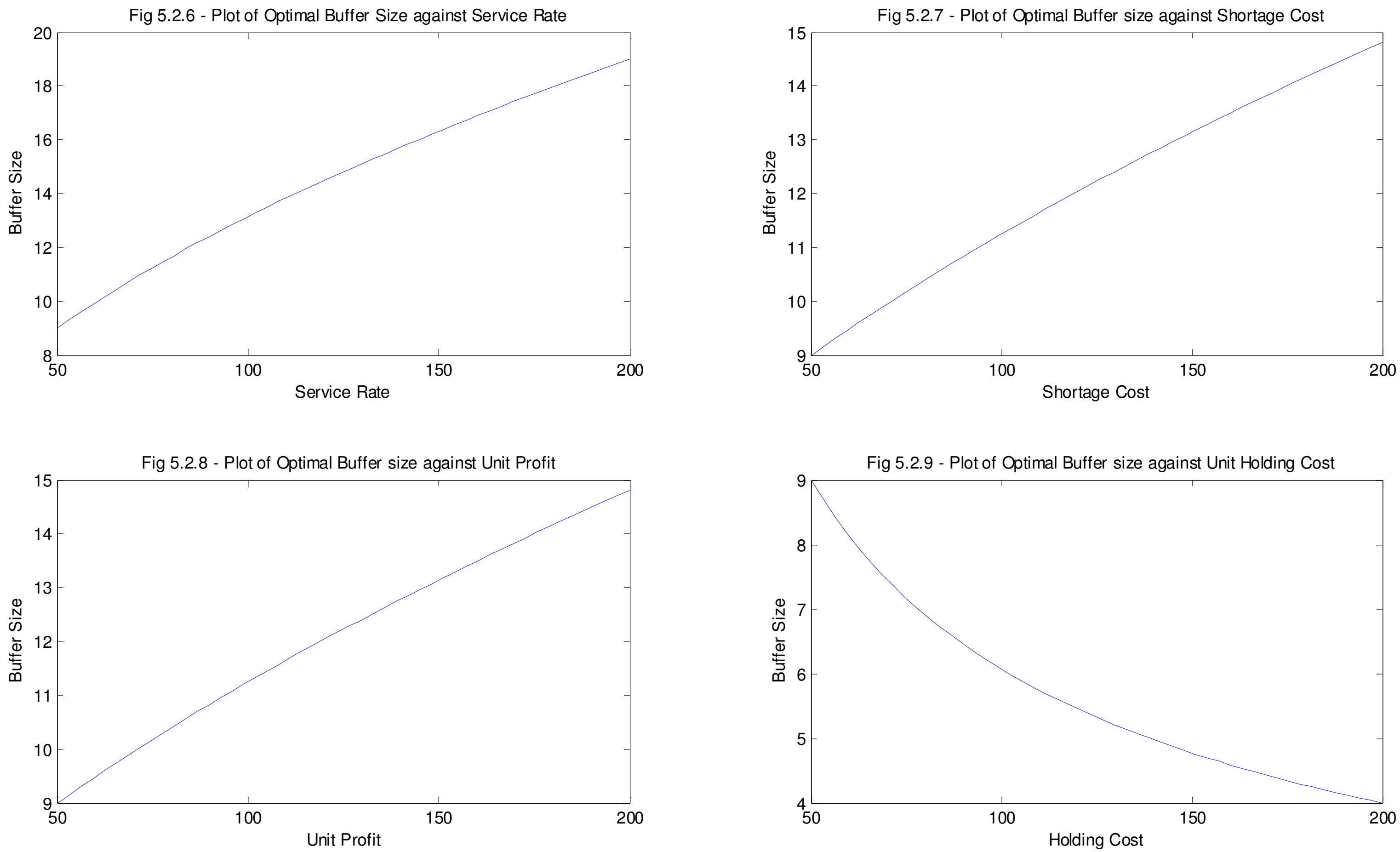

Figure 5.2.6 – 5.2.9: Changes in buffer size with input parameters

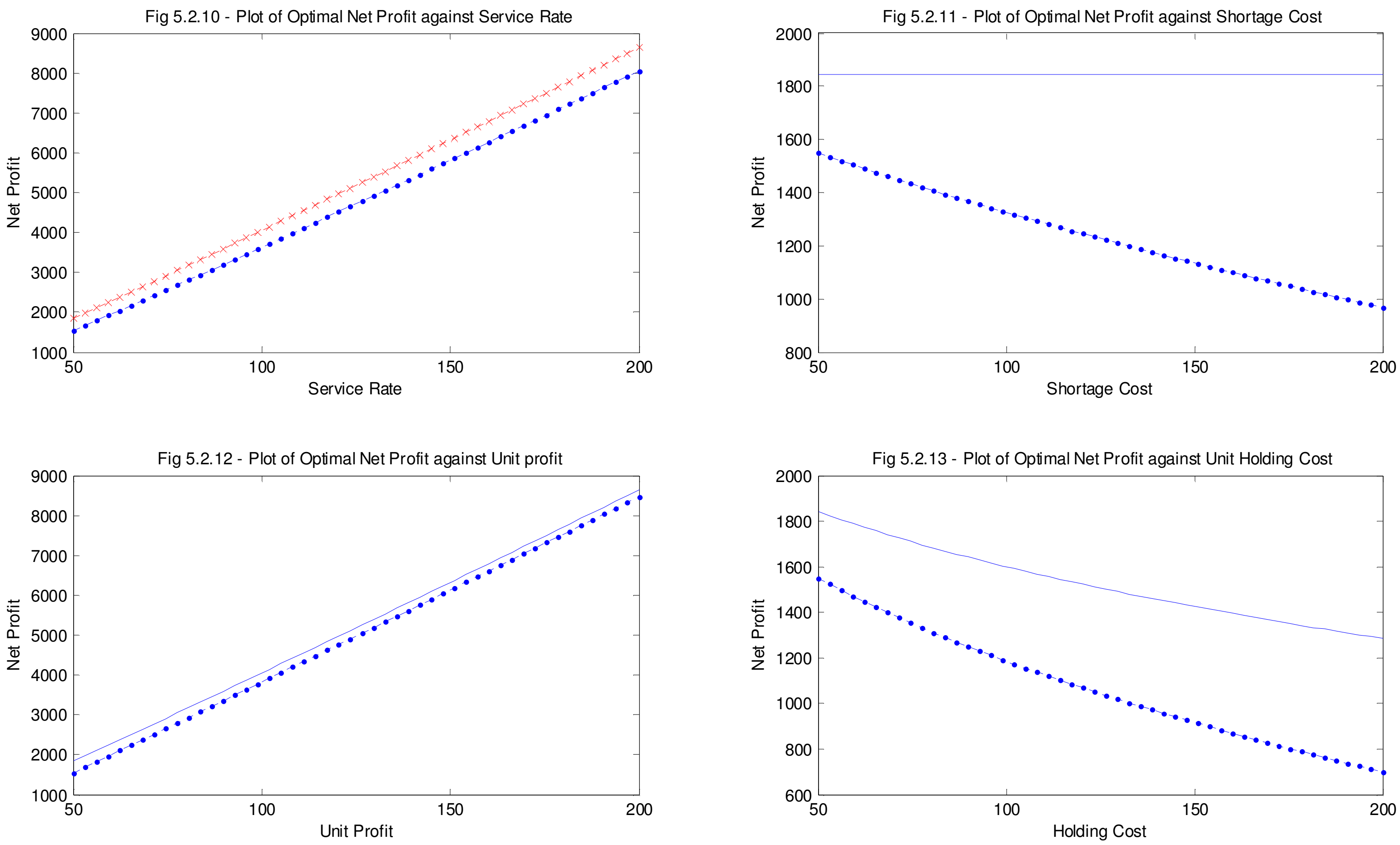

Figure 5.2.10 – 5.2.13: Changes in buffer size with input parameters

6.

CHAPTER 6

CONCLUSION

6.1. CONCLUDING OVERVIEW

Two common threads can be found in the compendium of works presented in this document. The first is that queuing principles with stochastic parameters have been used to analyse or applied to the various types of systems considered. The second is that the performance of the inventory management system has been studied directly or indirectly throughout. The focus and applications and/or contributions of each chapter can be summarised as follows. The work in the first three chapters have made particular use of the Markov Arrival Process (MAP) that makes it possible to expand the basic Poisson input stream to various practical environments that have more complex input systems, but that could still take advantage of the memorylessness properties of the attendant exponential distribution to simplify the calculations.

6.2. SOME POSSIBLE APPLICATIONS OF DERIVED MODELS

Chapters 2 and 3 contain the analyses of systems where products are not delivered immediately in response to demands, but where some services are further done on the items to be delivered before actual delivery. Exponential distributions were assumed for the lead time between the order placement and actual delivery. These types of systems are currently pervasive. A common knowledge today is the need to decide if the production system is to be managed as a make-to-stock, make-to-order, or assemble-to-order (or even engineer-to-order) system. This decision is usually dependent on the level of trade off desirable between long supply lead time and explosive inventory level.

While making to stock generally guarantees high responsiveness, it usually implies carrying a large volume of inventory. On the other hand, making to order reduces the inventory level drastically but leads to high response (lead) time. A recent best practice is that of delayed differentiation of products, which is some form of assembling to order. This type of environment usually leads to some final services being done on the inventory stock before being delivered. This implies that inventory is depleted at the rate of the services performed on the stock rather than directly on the demand for such products. Such systems seek to find some form of compromise between managing explosive inventory levels and having a long supply lead time.

With the general shift in the production environment towards lean manufacturing and - assembling-to-order, models developed for such systems (as in this work) would start having more applications, as compared to the traditional queuing systems that implicitly assumes that items are produced to stock and orders are immediately fulfilled from stocks. Herein lays the importance of the first two models presented in this work. The distributions and steady state parameters of some such systems have been studied in chapters 2 and 3. These steady state parameter estimates could be used in further applied probability contexts in many systems. This will be further discussed briefly in section 3 of this chapter.

Chapter 4 is a contribution to the field of Joint Replenishment Planning ($\boldsymbol{JRP}$**)**. Such systems are more practical in many real life instances than the typical assumptions around which some $\boldsymbol{ERP}$ systems are built. There are usually advantages in seeking how two or more products could be ordered together (usually from the same source) or produced together on the same machine. This may lead to savings in order (or set up) cost and thus overall reduction in the total production cost. Chapter 4 furthers the work done in this area.

While chapters 2, 3 and 4 are focused on the derivation of system parameters using queuing principles, chapter 5 is an application of the parameters derived in an $\boldsymbol{M/M/1}$ environment in the management of flows in a production system utilising the theory of constraint. The first part shows that determining the optimal buffer size indirectly by first determining the optimal flow rate, leads to further simplification of the application of optimisation techniques, and probably a more optimal profit function as compared to the previously documented approach of optimising the profit function directly with respect to the buffer size. This approach has been referred to in this book as dynamic buffering.

A more interesting observation made from this indirect approach is that it makes it easier to notice if it is actually necessary to seek to meet all customer demands in the first instance. It then makes it possible to obtain the optimal buffer size for more general systems other than the $M/M/1$ because such can also be indirectly retrieved since the flow intensity is a more pervasive parameter in all queuing models, while models explicitly containing a buffer size parameter are limited. This makes it possible to generalise the model to other types of systems. This was illustrated with a simple modification of the $M/M/1$ model initially presented to a case where there is shortage cost included.

6.3. POSSIBLE AREAS FOR FUTURE RESEARCH

The field of queuing theory is very popular and has enjoyed (and still enjoys) tremendous research focus, partially because if the ubiquity of queues, and therefore, the applicability of its theory. But it is possible to extend its applicability in many other ways, for instance, with the MAP input stream replacing the traditional Poisson input flow, and the PH service time model extending the traditional exponential model, as is currently being done by many authors, and in this work as well. The stochastic JRP system that has an MAP input like in chapters 2 and 3 are possible areas for further research. Models with input recovery system are another area that seems, for instance, yet to be explored. Such models would have another input stream recovered from the imperfections in removal of deteriorated items from wholesome stocks. This has generally not been considered in any work hitherto.

Also, the application of the steady state distributions and parameter estimates of the first three models considered in work, like many other such results by diverse authors, are fertile areas for improvement of the relevant areas in many production management philosophies. For instance, the application of some Phase distribution models like the Erlang, Hyper-exponential and Hyper-Erlang seems like possible candidates for resolving the issues of determining the transfer batch sizes in the Theory of Constraints environment. No application of stochastic processes appears to have been made in these areas. Others include management of system nervousness due to non-deterministic demand and lead times in the $\boldsymbol{MRP}$.

Steady state queue solutions, including those developed here, appear to have possible applications in such systems. While it is pertinent to state ahead that many such models may not have closed form solutions due to the nature of the solutions derived for the parameter estimates from many complex systems, it is anticipated that numerical iterative solutions would be useful tools in solving such problems. Such problems are being considered as part of the possible areas to explore by this author going forward from here.

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APPENDIX 1

To compute the R matrix, we use the following set of non linear equations. This can be solved by using Gauss-Siedel iterative process. The equations are derived by exploiting the coefficient matrices appearing in chapter 2 (2).

For $i = 0$,

$$Z_{(i,i)}^{(0)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(0)}[C_0 \oplus D_0 - \beta(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(0)}\gamma(I_{m1} \otimes I_{m2})$$

$$+R_{(i,i+1)}^{(1)}\mu(I_{m1} \otimes I_{m2}) + C_1 \otimes I_{m2} = 0,$$

For $i = 1,2, \dots ,c-1$

$$Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - (i\gamma + \beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i + 1 - k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,i+1)}^{(k+1)}(k + 1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 0$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - ((i-k)\gamma$$
$$+ k\mu + \beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i + 1 - k)\gamma(I_{m1} \otimes I_{m2})$$
$$+ R_{(i,i+1)}^{(k+1)}(k + 1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 1,2, \dots , i - 1$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - ((i-k)\gamma$$
$$+ k\mu + \beta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i + 1 - k)\gamma(I_{m1} \otimes I_{m2})$$
$$+ R_{(i,i+1)}^{(k+1)}(k + 1)\mu(I_{m1} \otimes I_{m2}) + C_1 \otimes I_{m2} = 0$$

$$k = i$$

For $i = c, c+1, \ldots, Q-1$

$$Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - (i\gamma + h(s-i)\beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 0$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - ((i-k)\gamma + k\mu + h(s-i)\beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 1,2, \ldots, c-1$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - ((i-k)\gamma + k\mu + h(s-i)\beta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1} \otimes I_{m2}) + C_1 \otimes I_{m2} = 0$$

$$k = c$$

For $i = Q, Q+1, \ldots, Q+c-1$

$$Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - (i\gamma + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) + R_{(i,i-Q)}^{(k)}\beta(I_{m1} \otimes I_{m2}) = 0$$

$$k = 0$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) Z_{(i,i)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,i)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,i)}^{(k)}[C_0 \oplus D_0 - ((i-k)\gamma + k\mu + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) + h(i-Q-k)R_{(i,i-Q)}^{(k)}\beta(I_{m1} \otimes I_{m2}) = 0$$

$$k = 1,2, \ldots, c-1$$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1}\otimes I_{m2})+Z_{(i,i)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,i)}^{(k-1)}(C_1\otimes I_{m2})+R_{(i,i)}^{(k)}[C_0\oplus D_0-((i-k)\gamma+k\mu)(I_{m1}\otimes I_{m2})]+R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1}\otimes I_{m2})+C_1\otimes I_{m2}=0$$

$k=c$

For $i=Q+c,Q+C+1,\dots,S$

$$Z_{(i,i)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,i)}^{(k)}[C_0\oplus D_0-(i\gamma+\theta)(I_{m1}\otimes I_{m2})]+\bar{\delta}_{(i,S)}R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1}\otimes I_{m2})+\bar{\delta}_{(i,S)}R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1}\otimes I_{m2})+R_{(i,i-Q)}^{(k)}\beta(I_{m1}\otimes I_{m2})=0$$

$k=0$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1}\otimes I_{m2})+Z_{(i,i)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,i)}^{(k-1)}(C_1\otimes I_{m2})+R_{(i,i)}^{(k)}[C_0\oplus D_0-\big((i-k)\gamma+k\mu+\theta\big)(I_{m1}\otimes I_{m2})]+\bar{\delta}_{(i,S)}R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1}\otimes I_{m2})+\bar{\delta}_{(i,S)}R_{(i,i+1)}^{(k+1)}(k+1)\mu(I_{m1}\otimes I_{m2})+R_{(i,i-Q)}^{(k)}\beta(I_{m1}\otimes I_{m2})=0$$

$k=1,2,\dots,c-1$

$$Z_{(i,i)}^{(k-1)}\theta(I_{m1}\otimes I_{m2})+Z_{(i,i)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,i)}^{(k-1)}(C_1\otimes I_{m2})+R_{(i,i)}^{(k)}[C_0\oplus D_0-((i-k)\gamma+k\mu)(I_{m1}\otimes I_{m2})]+\bar{\delta}_{(i,S)}R_{(i,i+1)}^{(k)}(i+1-k)\gamma(I_{m1}\otimes I_{m2})+R_{(i,i-Q)}^{(k)}\beta(I_{m1}\otimes I_{m2})+C_1\otimes I_{m2}=0$$

$k=c$

For $i=0,1,\dots,c-1,\ j=i+1,i+2,\dots,c\ \ or\ i=1,2,\dots,c,j=0,1,\dots,i-1$

$$Z_{(i,j)}^{k}(I_{m1}\otimes D_1)+R_{(i,j)}^{k}[C_0\oplus D_0-(j\gamma+\beta+\theta)(I_{m1}\otimes I_{m2})]+R_{(i,j+1)}^{k}(j+1-k)\gamma(I_{m1}\otimes I_{m2})+R_{(i,j+1)}^{k+1}\mu(I_{m1}\otimes I_{m2})=0$$

$k=0$

For $i = 0,1, \ldots\ c-1, j = c+1, c+2, \ldots, Q-1,\ \ or\ i = c, c+1, \ldots\ , Q-2, j = i+1, i+2, \ldots, Q-1\ \ or\ i = c+1, c+2, \ldots, Q, j = c, c+1, \ldots, i-1\ \ or\ i = Q+1, Q+2, \ldots, S\ \ j = c, c+1, \ldots, Q-1,$

$$Z_{(i,j)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,j)}^{(k)}[C_0 \oplus D_0 - (j\gamma + h(s-j)\beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 0$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,j)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,j)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,j)}^{(k)}[C_0 \oplus D_0 - ((j-k)\gamma + k\mu + h(s-j)\beta + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) = 0$$

$$k = 1,2, \ldots, c-1$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1} \otimes I_{m2}) + Z_{(i,j)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,j)}^{(k-1)}(C_1 \otimes I_{m2}) + R_{(i,j)}^{(k)}[C_0 \oplus D_0 - ((j-k)\gamma + k\mu + h(s-j)\beta)(I_{m1} \otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1} \otimes I_{m2}) = 0$$

$$k = c$$

For $i = 0,1, \ldots\ c-1, j = Q, Q+1, \ldots, Q+c-1,\ \ or\ i = c, c+1, \ldots\ , Q-1, j = Q, Q+1, \ldots, Q+c\ \ \ or\ i = Q, Q+1, \ldots, Q+c-1, j = i+1, i+2, \ldots, Q+c\ \ or\ i = Q+1, Q+2, \ldots, Q+c,\ \ j = Q, Q+1, \ldots, i-1,\ \ or\ i = Q+c+1, Q+c+2, \ldots, S, j = Q, Q+1, \ldots, Q+c$

$$Z_{(i,j)}^{(k)}(I_{m1} \otimes D_1) + R_{(i,j)}^{(k)}[C_0 \oplus D_0 - (j\gamma + \theta)(I_{m1} \otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1} \otimes I_{m2}) + R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1} \otimes I_{m2}) + R_{(i,j-Q)}^{(k)}\beta I_{m1} \otimes I_{m2} = 0$$

$$k = 0$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1}\otimes I_{m2}) + Z_{(i,j)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,j)}^{(k-1)}(C_1\otimes I_{m2}) + R_{(i,j)}^{(k)}[C_0\oplus D_0-((j-k)\gamma+k\mu + \theta)(I_{m1}\otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1}\otimes I_{m2}) + R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1}\otimes I_{m2}) + h(j-Q-k)R_{(i,j-Q)}^{(k)}\beta I_{m1}\otimes I_{m2} = 0$$

$$k=1,2,\dots,c-1$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1}\otimes I_{m2})+Z_{(i,j)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,j)}^{(k-1)}(C_1\otimes I_{m2}) + R_{(i,j)}^{(k)}[C_0\oplus D_0-((j-k)\gamma + k\mu)(I_{m1}\otimes I_{m2})] + R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1}\otimes I_{m2}) = 0$$

$$k=c$$

For $i=0,1,\dots\ c-1, j=Q+c,Q+c+1,\dots,S\ \ or\ i=c,c+1,\dots,Q-1, j=Q+c+1,Q+c+2,\dots,S\ \ or\ i=Q,Q+1,\dots,Q+c-1, j=Q+c+1,Q+c+2,\dots,S,\ \ or\ i=Q+c,Q+c+1,\dots,S-1, j=i+1,i+2,\dots,S$

$$Z_{(i,j)}^{(k)}(I_{m1}\otimes D_1) + R_{(i,j)}^{(k)}[C_0\oplus D_0-(j\gamma+\theta)(I_{m1}\otimes I_{m2})] + \bar{\delta}_{(j,S)}R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1}\otimes I_{m2}) + \bar{\delta}_{(j,S)}R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1}\otimes I_{m2}) + R_{(i,j-Q)}^{(k)}\beta I_{m1}\otimes I_{m2} = 0$$

$$k=0$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1}\otimes I_{m2}) + Z_{(i,j)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,j)}^{(k-1)}(C_1\otimes I_{m2}) + R_{(i,j)}^{(k)}[C_0\oplus D_0-((j-k)\gamma+k\mu + \theta)(I_{m1}\otimes I_{m2})] + \bar{\delta}_{(j,S)}R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1}\otimes I_{m2}) + \bar{\delta}_{(j,S)}R_{(i,j+1)}^{(k+1)}(k+1)\mu(I_{m1}\otimes I_{m2}) + R_{(i,j-Q)}^{(k)}\beta I_{m1}\otimes I_{m2} = 0$$

$$k=1,2,\dots,c-1$$

$$Z_{(i,j)}^{(k-1)}\theta(I_{m1}\otimes I_{m2})+Z_{(i,j)}^{(k)}(I_{m1}\otimes D_1)+R_{(i,j)}^{(k-1)}(C_1\otimes I_{m2})+R_{(i,j)}^{(k)}[C_0\oplus D_0-((j-k)\gamma + k\mu)(I_{m1}\otimes I_{m2})]+\bar{\delta}_{(j,S)}R_{(i,j+1)}^{(k)}(j+1-k)\gamma(I_{m1}\otimes I_{m2})+R_{(i,j-Q)}^{(k)}\beta I_{m1}\otimes I_{m2}=0$$

$$k=c$$

APPENDIX 2

1. Renewal Processes

A renewal process is a sequence of independent non-negative random variables having identical distributions. Formally, if $\{N(i), i > 0\}$ is a counting process with $N(0) = 0$, and $N(i) = \sum_{j=1}^{i} x_j$ and $x_n = 1,2 \ldots$ the time between the $(n-1)$th and nth event of this process, $n \geq 1$. Let $\left\{g_i^{(n)} = P\{x_n = i\},\ i \geq 0\right\}$ be the distribution series of $x_n,\ n \geq 1$. If the sequence of $\{x_1, x_2 \ldots\}$ is independently and identically distributed from the second one, then the random sequence $v_n = max_{i \geq 0}\{i: N(i) \leq n\},\ n \geq 0$ is called the general discrete renewal process. This means $v_n$is the number of renewals until the instant n, inclusive.

The renewal process, v_n is said to be simple if $g_i^{(1)} = g_i,\ i \geq 0$. Also, v_n said to be stationary if the distribution series $\left\{g_i^{(1)},\ i \geq 0\right\}$ of the first instant of renewal $N(1) = x_1$ obeys the formula

$$g_0^1 = 0, \quad g_i^1 = \frac{1}{g} \sum_1^{\infty} g_j\,,\ \ i \geq 1\ \ and\ g = \boldsymbol{E} x_n = \sum_{i=1}^{\infty} i g_i\,,\ \ g < \infty\ .$$

The random variable v_n has moments of any order, and for any renewal process has moments of any order, and for any renewal process has moments of any order, and for any renewal process $\{v_n,\ n \geq 0\}$, and each $n \geq 0$, there exists a number $C = C(n)$, such that $E v_n^k \leq C^k k!\ \ \forall\, k \geq 0$.

1.1. The renewal function

The renewal function, H_n, is the number of renewals up until the instant n inclusive and is given by $H_n = \boldsymbol{E} v_n$. The renewal series is the number of renewal at b, and is given by $h_n = H_n - H_{n-1},\ n \geq 1$. h_n can be considered to be the probability that a renewal occurs at the instant n.

The renewal series satisfies the renewal equation

$$h_n = g_n^{(1)} + \sum_{i=1}^{n} h_i g_{n-i}\,,\ n \geq 0;$$

Solving this equation with the generating function h_z defined over $z < 1$,

$$H_z = G_z^{(1)} + H_z G_z$$

From which

$$H_z = \frac{G_z^{(1)}}{1-G_z}$$

In the stationary case, this equation becomes

$H_z = \lambda \frac{z}{1-z}$, where $\lambda = {}^{1}\!/_{g}$.

From Blackwell and Smith theorems, as $n \to \infty$, if the skip is defined to mean the instant of the first after the nth renewal and the nth renewal, the distribution of the skip coincides with distribution of the instance of the first renewal and becomes

$\lambda \sum_{j=0}^{\infty} g_{j+i} = \frac{1}{g} \sum_{i}^{\infty} g_j$. This is the key renewal theorem for discrete case.

The above formulae easily generalise to the continuous case and becomes

$$h_t = g_t^{(1)} + \int_0^t g_{n-i} dh_i$$

And solving using the Laplace-Stieltjes transform

$\alpha(s) = \frac{\gamma^{(1)}(s)}{1-\gamma(s)}$, s being the Laplace variable

And with Blackwell and Smith theorems the stationary distribution of the skip becomes

$\int_0^t g(t-x) dH(x) \underset{t\to\infty}{\longrightarrow} \lambda \int_0^{\infty} g(x)dx$. This is the key renewal theorem for continuous case.

2. Markov Processes

2.1. Markov Chain

A Markov chain is sequence of discrete random variables such that for any n, x_{n+1} is conditionally independent of $x_0, \dots x_{n-1}$ given x_n. This means the future is independent of the past given the current state irrespective of how the current of how the current state has been reached.

Formally, this can be written as follows. Suppose a probability space (Ω, x, P) is defined such that $x_n : \Omega \to S,\ where\ S = 1, \dots N\ or\ S = 1, \dots$ i.e. S is finite or countably infinite.

$$\mathrm{P}\{\mathrm{x}_{\mathrm{n}+1} = \mathrm{j} \mid \mathrm{x}_0, \dots, \mathrm{x}_{\mathrm{n}}\} = \mathrm{P}\{\mathrm{x}_{\mathrm{n}+1} = \mathrm{j} \mid \mathrm{x}_{\mathrm{n}}\} \quad \forall\, \mathrm{j} \in \mathrm{S} \text{ and } \mathrm{n} \in \mathrm{N}.$$

The Markov chain has a transition matrix, P, made up of classes of states that could be transient, recurrent null or recurrent non-null. This classification is important for solving problems using P.

The Markov property simplifies the manipulation of the Transition matrix such that For any $m, n \in N$,

$$P\{x_{n+m} = j \mid x_n = i\} = P^m(i, j).$$

The Chapman-Kolmogorov equation is important in manipulating the Markov chain. This provides that

$$P^{m+n} = P^m P^n$$

P can be used to find the potential matrix, R, of the variable x, and F, the time of first visit to a state, which are also useful in solving for the equilibrium distribution of its probabilities.

The matrices $R(i, j)$ = the potential matrix or expected number of visits to a state j from another state i and $F_k(i, j)$ = the time of first visit of state j from state i are important in classifying and also solving for the equilibrium distribution of the probabilities. They are defined as

$$F_k = \begin{Bmatrix} P(i, j) & k = 1 \\ P(i, b) F_{k-1}(b, j) & k \geq 2 \end{Bmatrix}$$

$$R(j, j) = E_j N_j = (I - P)^{-1}$$

where P is as defined earlier (the) one step transition matrix. Cinlar (1975) gives a detailed treatment of the foregoing.

2.2. Markov Process

A Markov chain is silent about the length of time spent in a given state, j. To address this, the time variable, t, is defined such that this variable, together with the Markov chain is used to define another random variable called the Markov process. The variable, t, would be taken such that

$$t_n: \Omega \rightarrow R_+, \ i.e. R = [0, \infty].$$

The process

$$\mathrm{P}\{\mathrm{x}_{\mathrm{t+m}} = \mathrm{j} \mid \mathrm{x}_{\mathrm{u}}; \mathrm{u} \leq \mathrm{t}\} = \mathrm{P}\{\mathrm{x}_{\mathrm{t+m}} = \mathrm{j} \mid \mathrm{x}_{\mathrm{t}}\} \quad \forall \, \mathrm{j} \in \mathrm{S} \text{ and } \mathrm{t}, \mathrm{s} \ \in \mathrm{R}_+.$$

The Markov process such that

$$P\{x_{n+m} = j \mid x_n = i\} = P_n(i, j)$$

or in the matrix form

$$P(m + n) = P(m)P(n)$$

holds is said to be time homogeneous, where P_n is the probability of being in state n. The Kolmogorov-Chapman equation still holds. The function $P_n(i, j)$ is called the transition function. The set of successive states visited by the process forms a Markov chain with the corresponding transition matrix, P, and the time of sojourn in each state has a probability distribution, which usually could be taken as exponential.

2.3. The Infinitesimal matrix

If it is assumed that the following holds everywhere

$$P_\Delta(i, j) \underset{\Delta \downarrow 0}{\longrightarrow} \delta(i, j) \quad i, j \in S, \ \delta(i, j) \ is\ the\ Kroneker\ symbol$$

Then there exists the limits

$$a(i, j) = \lim_{\Delta \downarrow 0} \frac{P_\Delta(i, j)}{\Delta}, \quad i, j \in S, \ i \neq j$$

$$-a(i, i) = \lim_{\Delta \downarrow 0} \frac{P_\Delta(i, i) - 1}{\Delta}, \quad i \in S,$$

And

$0 \leq a(i,i) \leq \infty, \;\; o \leq a(i,j) \leq \infty, \;\; i,j \in S, \;\; i \neq j$

$\sum_{j \in S} a(i,j) \leq 0, \;\;\; i \in S$

For a conservative (i.e. locally regular) matrix, the equation becomes

$\sum_{j \in S} a(i,j) = 0, \;\;\; i \in S.$

The parameter $a(i,j)$ is the intensities of transition from state i to state j. Also, $a(i,i)$ is the intensity of exit from state i. The matrix $A = a(i,j)$ is the matrix of transition intensities or the infinitesimal matrix.

The transition matrix can be constructed from

$$q(i,j) = \begin{cases} \frac{a(i,j)}{a(i,i)}, & i \neq j \\ 0, & i = j \end{cases}$$

This is called the embedded Markov chain of the Markov process. The process is assumed to be conservative.

It is important that $a(i,i) > 0$, and also, to guarantee regularity of Markov chain, either

- $a(i,i)$ should be uniformly bounded, i.e. $a(i,i) < c < \infty, \;\; \forall\, i \in S$ or
- all the states of the Markov process should be recurrent.

3. Queuing Theory

Queuing is one of the areas in which stochastic processes in general and Markov processes in particular have had extensive applications. The essence of studying queuing is to understand how the properties of the system of interest will behave in the steady state and/or the transient state. Optimisation is not the actual goal of such analysis, but the results of such systems parameters as the expected queue length, expected waiting time, expected throughput time, facility utilisation etc (all usually expressed as a function of the traffic intensity) could find application in optimisation processes. Queuing techniques are particularly suitable in systems where there are flows, and where stocks are built up as a result of flows through such systems. This is actually characteristic of most production systems.

3.1. Properties and Classification of Queues

The idea of properties and classifications of queues are closely tied because queues are classified based on the values of these characteristics. The properties are: input pattern, service pattern, number of servers, location and sizes of buffer, the service discipline and the size of the calling source.

There have been many classifications based on all these properties, but the classification effort usually regarded as the first documented attempt was that of Kendall (1953). This makes use of the first three properties. Lee G was credited to have added the fourth property of service discipline. There are still other classifications depending on the problem being addressed.

3.2. Constructive Description of Model

Queuing models could be constructed by considering all the means through which entities enter (i.e. the birth process) and exit (i.e. the death process) the system of interest. This is summarised in the birth and death process of such queues and this immediately leads to the generation of the infinitesimal matrix of the process.

A more generalised and powerful approach for a conservative process is through the use of the global, local and partial (where necessary) balance of flows of probabilities between two states of the system. This approach is premised on the fact that at dynamic equilibrium, the ergodic properties of the system ensures that the flow of probabilities out of and into a stage cancels out.

If the states of a process are represented as nodes and all nodes that could be reached in one step of transition are connected by arcs, then the total flow into and out of a state of such system constitutes the global balance of flow. This is represented as

$$a_i p_i = \sum_{j \in S \setminus \{i\}} a_{ji} p_j$$

The local balance concerns flow between any two states. At equilibrium, the flow from a state m to another state n should be equal to the flow from state n back to state m. This is the same as looking at the ark between these two states and equating the flow across it. Formally,

$$\sum_{i \in S_1} \sum_{i \in S_2} a_{ij} p_i = \sum_{i \in S_1} \sum_{i \in S_2} a_{ji} p_j$$

The partial balance can be formally written as

$$a_{ij} p_i = a_{ji} p_j$$

The partial balance is not usually satisfied, but whenever it is satisfied, it gives some important consequences. In particular, it implies

$$p_i = \frac{a_{ji} p_j}{a_{ij}}$$

3.3. Solving the Flow Problems

The importance of the characteristic transformations in solving the problems encountered using the various distributions has been highlighted in the body of this thesis. But the two that are mostly applied appear to be the moment generating function when the random variable is defined on the integer space due to its simplicity, and the Laplace transform since it is defined on the Real field, and is simpler to handle than the characteristic function. The characteristic function is the only one applicable on the complex field. Other theorems and functions that are useful during the transformation process include the derivative function, the shifting theorems and the convolution theorem.

The transform of the derivatives, stated in general terms as

$$L[f^n](s) = SF(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

This is usually considered up until the first derivative only.

i.e. $L[f'](s) = S^n F(s) - f(0)$

The first shifting theorems addresses a shift in the s variable of Laplace function and is written as

$$L(e^{at} f)(s) = F(s - a)$$

This means the Laplace transform of a function multiplied by an exponential function simply shifts the Laplace transform back by the coefficient of the exponent's variable. The inverse is also true.

The Heaviside, or unit step, function, in general, is

$$H(t - a) = 0 \quad if \;\; t < a, \qquad and \qquad H(t - a) = 1 \;\; if \;\; t \geq a.$$

So, multiplying a function $f(x)$ by the Heaviside $H(t - a)$ turns off $f(x)$ if $x < a$ and on if $x \geq a$. Also, Combining two Heaviside functions $H(t - a) - H(t - b)$ produces the pulse function that turns off x before a, turns it on between a and b, and then off again after b.

The Heaviside function can be combined with the first shifting theorem to produce the Heaviside shifted function to produce another shifting theorem:

$$L[f(t - a)H(t - a)](s) = e^{-as} F(s).$$

These theorems are useful in manipulating the joint distribution of many Markovian random variables, seeing that the exponential distribution has the general form $1 - e^{\lambda t} \equiv 1 - e^{at}$.

3.4. Inputs Flows, Service Pattern and Nature of Queue

The Kendall classification, making use of the pattern of the input flow, service pattern and Queue size and location has been about the most important system of classification. A queue is said to be Markovian if the distribution of the input and output parameters conform to models that could be said to have Markov properties. This usually means the arrival pattern is Poisson (or compound Poisson) while the service pattern is exponential. But there are other input patterns said to follow the Markovian Arrival Pattern (MAP) that have become important. Also, the repertoire of Markov queuing models has been extended by the service pattern said to be Phase (PH) distribution. Another class extension of the queue type is the class of virtual queues called the retrial queues. These three extensions have further enriched the study of queuing systems and expanded the scope of applications of queuing principles to problems encountered daily.

3.5. System and Queue Structure

The system could be such that once a customer or job has been served in a facility, the customer or job exits the system. Such systems are referred to as single stage systems. Some other systems are such that when a customer has been served at one stage, the customer might move to another stage for another service. Such systems are referred to as multistage queuing systems, or in some instances, network systems.

Buffers refer to places where jobs or customers still (may be in process) are kept. There could be no buffers, real buffers or virtual buffers in a system. Systems without buffers are special cases of balking queues. If the buffers are real, it could have finite or infinite capacity. This is characteristic of most queues. In a virtual buffering system, the system does not have an actual place where customers waiting to be served could stay. Such customers would join a virtual buffer (sometimes called an orbit) where they could make subsequent attempts or leave the system altogether. Such type of buffering is characteristic of retrial queues.

3.6. Pattern of Input Flow

There are two main ways of describing the nature of the random input flow into a queuing system. The first is through the joint distribution of the times between the subsequent arrivals. If $\tau_1, \tau_{2\ldots,}\tau_{1\geq}0$, is a sequence of non-decreasing time of occurrence of certain event, and $\xi_i = \tau_i - \tau_{i-1}$ is the time between the $i - 1th$ and the ith arrival, then this is represented as

$$F_{\xi 1, \xi 2, \dots, \xi k}(x_1, x_2, \dots, x_n) = P(\xi_1 < x_1, \xi_2 < x_2, \dots, \xi_k < x_k,$$

where ξ_k is the time of arrival of the kth customer and x_k is a stopping time.

The second approach is based on the consideration of the likelihood of an event of interest occurring in some set of families of intervals $[0, t_i), [t_1, t_2), \dots [t_{k-1}, t_k),\ k \geq 1$ and defining the joint distribution function as

$$G(m_1, m_2, \dots, m_k; t_1, t_2, \dots, t_k) = P(\varsigma_1 = m_1, \varsigma_2 = m_2, \dots, \varsigma_k < m_k,$$

where m_k is the interval $[t_{k-1}, t_k)$ and ς_k is the arrival of the kth customer.

3.7. Poisson Input Flow

The Poisson input flow is the assumption of most Markov models, and the pattern of input flow is said to be Poisson of the probability, $p_i(t)$ of the ith customer arriving at time t is

$$p_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

The distribution parameter is λ and the time between arrivals is exponentially distributed exponentially with the same parameter λ. Since the Poisson flow is stationary and memoryless, with another assumption of ordinariness, the transition intensity matrix becomes

$$a_{ij} = \begin{cases} -\lambda, & j = i \\ \lambda, & j = i + 1 \\ 0 & otherwise \end{cases}$$

The compound (or superposed) Poisson process has the arrival rate

$\lambda = \sum_{i=1}^{n} \lambda_i, \ \sum \lambda_i = 1$

where λ_i is the weight of the component i of the superposed flows.

The convolution theorem comes in handy to solve the problem of the product of two functions. Unlike the addition function, the Laplace transform of the product of two functions is not equal to the product of the Laplace transform of the functions i.e. $L[f * g] = L[f] * L[g]$. But the convolution of two functions defined as

$$(f©g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

has the property that $L[f©g] = L[f]L[g]$. This makes it to handle the problems of the renewal equation which has that general form. O'neil (1995) treats this to further details.

3.8. Markov Input Flow

Some systems demand input flow that is more complex than the ordinary or compound Poisson, but still Markovian. An example of such flow is the Markov Arrival Process (MAP). They are a generalisation of the Poisson and compound Poisson flows.

If $v(t)$ is the number of customers that arrive in the time interval $[0, t)$, and $\tau_1, \tau_{2\ldots,} \tau_{1\geq} 0$ the instants of their arrival, and there exists a Markov process $\xi(t)$ defined on the finite state $S = \{1,2, \ldots, l\}$. Also, define $\eta(t) = \{\xi(t), v(t)\}$. Then the process state set $\{\eta(t), t \geq 0\}$ is representable as

$\bigcup_{k=0}^{\infty} S_k$, where $S_k = \{(i,k), \ i = 1,2, \ldots l, \ \ k \geq 0\}$.

$\eta(t)$ is said to be in state $(i, k), \ i = 1,2, \ldots l, \ \ k \geq 0$ if k customers arrive at the instant t, and the process $\xi(t), t \geq 0$ is in the state i.

The flow $\{\tau_j, j \geq 1\}$ is said to be a Markov flow with respect to the process $\{\xi(t), t \geq 0\}$ if the random process $\{\eta(t), t \geq 0\}$ is a homogenous Markov process and its matrix, A, of transition intensities is of the block form

$$A = \begin{bmatrix} \Upsilon & N & 0 & 0 & 0 & 0 & \ldots \ldots \\ 0 & \Upsilon & N & 0 & 0 & 0 & \ldots \ldots \\ 0 & 0 & \Upsilon & N & 0 & 0 & \ldots \quad . \\ . & . & . & . & . & . & . \end{bmatrix}$$

where Υ and N are square matrices of order l,. $\Upsilon^* = \Upsilon + N$ is the the matrix of transition intensities of the Markov process $\{\xi(t), t \geq 0\}$.

Other Markov models can be seen as special cases of this matrix. For instance, if l is 1, then flow is the ordinary Poisson process. If N is a diagonal matrix, then the flow is a Markov Modulated Poisson process. With l = 2 for matrix N and only one non-zero and strictly positive diagonal matrix, the flow is the Interrupted Poisson process. If N is representable as $N = v\alpha^T$, where v and α are column vectors of dimension l and α is a probability vector, then the flow is called the phase type (PH) flow.

It should be noted that if $\{\xi(t), t \geq 0\}$ is a stationary Markov process, then the resulting flow from $\{\eta(t), t \geq 0\}$ is also stationary.

3.9. Distribution of service time

Basically, the service time in Markov models is assumed to be exponentially distributed. Formally, the distribution and the density function respectively are

$$F(x) = 1 - e^{\mu x} \text{ and } f(x) = \mu e^{\mu x}$$

where μ is the service rate.

But some other possible distributions include: Erlang, which is useful for cases where service time is made up of a series of some exponentially distributed stages; hyper-exponential, hyper-erlang and phase type distributions.

Formally, the Erlang density function is of the form

$$f(x) = \frac{\mu^m x^{m-1}}{(m-1)!} e^{-\mu x}, \;\; x > 0, \;\; m = 1,2, \ldots, \;\; 0 < \mu < \infty$$

The hyper-exponential distribution is

$B(x) = \sum_{j=1}^{m} \beta_j (1 - e^{-\mu_j x})$ where $x > 0,\ \beta_j > 0,\ 0 < \mu_j < \infty,\ j = 1,2, \dots, m,$ $\Sigma\beta_j = 1$

And the hyper-Erlang distribution is

$B(x) = \sum_{j=1}^{m} \beta_j E_{mj}(x),$ where $\beta_j > 0,\ j = 1,2, \dots, m,\ \Sigma\beta_j = 1$ and E_{mj} is the Erlang distribution with the parameter m_j and μ_j.

In fact, Erlang, hyper-exponential and hyper-Erlang distributions are special cases of a more general class of distributions said to have fictitious phases, as coined by Erlang, or commonly called the phase type distributions.

3.10. PH distribution of service time

Some Markov models have flows that are more generalised than those discussed earlier. These can be got from the PH distribution. Generally, PH distributions admit the form

$$F(x) = 1 - \boldsymbol{f}^T e^{Gx} \mathbf{1},\ x > 0$$

where $\boldsymbol{f}$ is a probability vector, G is a probability matrix, $\sum_{j=1}^{m} f_j \le 1,\ f_j \ge 0,\ j = 1,2, \dots m,\ \sum_{j=1}^{m} G_{ij} \le 0,\ i \ne j,\ G_{ii} < 0,\ i,j = 1,2, \dots m,$ and $\sum_{j=1}^{m} G_{ij} < 0$ for at least one i. The pair $(\boldsymbol{f}, G)$ is called the PH-representation of order m of the distribution function $F(x)$.

The distribution function of the PH type of a non-negative random variable admits probabilistic interpretation based on the concept of phase. Let $v_i, i = 1 \dots m,\ v_i \ge -G_{ii}$ be some real numbers, the numbers $\theta_{ij}, i,j = 1,2, \dots m$ obey the formula

$$\theta_{ij} = \begin{cases} 1 + \frac{G_{ii}}{v_i}, & i = j \\ \frac{G_{ij}}{v_i} & i \neq j \end{cases}$$

This is synonymous to the embedded Markov chain of the Markov process. And the matrix of transitional intensities becomes

$$G_{ij} = \begin{cases} v_i(\theta_{ii} - 1), & i = j \\ v_i\theta_{ij}, & i \neq j \end{cases}$$

The matrix of transitional intensities satisfies the set of Kolmogorov differential equations

$$\frac{d}{dt}P(t) = P(t)G$$

With the initial condition $P(0) = I$ and the solution obeying the formula $P(t) = e^{Gt}$. And so,

$$P\{\tau < x\} = 1 - \boldsymbol{f}^T e^{Gx} 1.$$

$F(x)$ indicates the distribution of the customer sojourn in the queue network.

3.11. Solution Methods

Solving the problems of models with PH distribution requires special mathematical machinery which is found in the matrix theoretic functions. The Kronecker product of two matrices, A and B, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \cdot & & \cdot \\ \cdot & \cdot & \cdot \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$$

The Kronecker sum of two matrices, A and B, is defined as

$A \oplus B = A \otimes I_n + I_m \otimes$

where I is the identity matrix, where m and n are the orders of the matrices A and B respectively.

The Kronecker product has many properties like scalar multiplication of the entries of the matrices, distributivity, associativity, identity matrices, zero matrices, transposition, inverse matrices, mixed product of matrices, vectorisation, eigen factors and vectors, determinants etc.

Some properties of the Kronecker sum and products that make them very useful, however, are that the products and sums are defined irrespective of the orders of the matrices A and B involved, and probably more importantly that while the expression

$$e^{A+B} = e^{A} * e^{B}$$

is true if and only if A and B commute, the expression

$$e^{A\oplus B} = e^{A\otimes I} * e^{I\otimes B}$$

is true irrespective of commutativity. This property makes the Kronecker product and sum very useful in the manipulation of PH distributed variables. Detailed treatment of Matrix theoretic functions are contained in Graham (1975) and Latouche and Ramaswami (1999).

APPENDIX 3

Table 2.2: Fraction of successful rate of retrials

$S = 25$, $s = 8$, $\lambda_{-1} = 2.5$, $\beta = 3$, $\mu = 10$, $\gamma = 0.1$, $\theta = 5$.

λ_1	c		Exp-	Erl-	HExp-	MNC-	MPC-
4	1	Exp+	0.420649	0.420426	0.421149	0.420994	0.421106
		Erl+	0.413698	0.411711	0.417077	0.415944	0.416344
		HExp+	0.428533	0.429046	0.427033	0.427875	0.427426
		MNC+	0.419197	0.419555	0.418566	0.418736	0.418499
		MPC+	0.419571	0.41978	0.419045	0.419251	0.419029
	2	Exp+	0.485599	0.482846	0.488151	0.489082	0.489177
		Erl+	0.40133	0.390861	0.416056	0.414357	0.413997
		HExp+	0.516639	0.515427	0.516175	0.518226	0.518234
		MNC+	0.508838	0.510211	0.505402	0.507208	0.506529
		MPC+	0.510315	0.5116	0.506864	0.508413	0.507263
	3	Exp+	0.401298	0.387547	0.421536	0.419646	0.419622
		Erl+	0.07422	0.059093	0.123982	0.099355	0.104827
		HExp+	0.477132	0.464921	0.492656	0.493288	0.494844
		MNC+	0.57606	0.576866	0.571744	0.575422	0.574373
		MPC+	0.596288	0.59917	0.588457	0.592109	0.589718
	4	Exp+	0.152172	0.129549	0.211352	0.185908	0.194186
		Erl+	0.027208	0.011397	0.092098	0.055607	0.065265
		HExp+	0.29284	0.26799	0.342831	0.32796	0.337393
		MNC+	0.562215	0.553123	0.575204	0.575434	0.576236
		MPC+	0.687067	0.688647	0.682241	0.685003	0.683503
4.5	1	Exp+	0.422597	0.422295	0.423307	0.423047	0.423229
		Erl+	0.416912	0.414786	0.420653	0.419324	0.419877
		HExp+	0.429211	0.429685	0.427853	0.428586	0.428131
		MNC+	0.419622	0.419857	0.419252	0.419323	0.419161
		MPC=	0.419908	0.420045	0.419582	0.419695	0.419544
	2	Exp+	0.489948	0.486681	0.493598	0.494093	0.494419
		Erl+	0.408121	0.395695	0.42552	0.423192	0.423394
		HExp+	0.522583	0.521683	0.521688	0.523836	0.523829
		MNC+	0.511748	0.512884	0.508933	0.510426	0.50983
		MPC+	0.51167	0.512708	0.508889	0.510125	0.509155
	3	Exp+	0.407207	0.391574	0.430702	0.427854	0.428886
		Erl+	0.097437	0.077082	0.15461	0.128575	0.13588
		HExp+	0.493367	0.481972	0.507929	0.508694	0.510701
		MNC+	0.577522	0.577711	0.574611	0.5777	0.576937
		MPC+	0.596487	0.598866	0.589923	0.59301	0.590933
	4	Exp+	0.177112	0.151344	0.240088	0.214447	0.224531
		Erl+	0.044207	0.022633	0.11876	0.079291	0.091906
		HExp+	0.331235	0.30736	0.377949	0.365196	0.374875
		MNC+	0.565337	0.555503	0.580159	0.579569	0.581233
		MPC+	0.687487	0.68858	0.683919	0.686072	0.684942
5	1	Exp+	0.424699	0.424358	0.425494	0.425195	0.42543
		Erl+	0.420374	0.41822	0.42421	0.42283	0.423529
		HExp+	0.429535	0.429936	0.428404	0.428989	0.428566
		MNC+	0.420284	0.420424	0.42009	0.420103	0.420003
		MPC+	0.42048	0.420564	0.420298	0.420349	0.420254
	2	Exp+	0.494902	0.491266	0.499425	0.499545	0.500113
		Erl+	0.416833	0.402967	0.436063	0.433359	0.434136
		HExp+	0.527899	0.527373	0.52646	0.528721	0.528638
		MNC+	0.514858	0.515783	0.512594	0.513806	0.513298
		MPC+	0.513523	0.514357	0.511304	0.512277	0.511466
	3	Exp+	0.416118	0.399252	0.441691	0.438292	0.440294
		Erl+	0.124192	0.099401	0.185896	0.160052	0.168928
		HExp+	0.509105	0.498813	0.522305	0.523227	0.525513
		MNC+	0.579764	0.579456	0.578037	0.580608	0.580129
		MPC+	0.59742	0.599373	0.591959	0.59455	0.592763
	4	Exp+	0.204386	0.176595	0.268793	0.243957	0.255451
		Erl+	0.065624	0.039159	0.146214	0.105902	0.121116
		HExp+	0.367325	0.344936	0.410367	0.3995	0.409157
		MNC+	0.570096	0.559872	0.586099	0.584909	0.587309
		MPC+	0.722301	0.720412	0.728029	0.72524	0.727214

Table 2.3: Fraction of successful rate of retrials

$S = 25$, $s = 8$, $\lambda = 5$, $\beta = 3$, $\mu = 10$, $\gamma = 0.1$, $\theta = 5$.

λ_{-1}	c		Exp-	Erl-	HExp-	MNC-	MPC-
2.5	1	Exp+	0.424699	0.424358	0.425494	0.425195	0.42543
		Erl+	0.420374	0.41822	0.42421	0.42283	0.423529
		HExp+	0.429535	0.429936	0.428404	0.428989	0.428566
		MNC+	0.420284	0.420424	0.42009	0.420103	0.420003
		MPC+	0.42048	0.420564	0.420298	0.420349	0.420254
	2	Exp+	0.494902	0.491266	0.499425	0.499545	0.500113
		Erl+	0.416833	0.402967	0.436063	0.433359	0.434136
		HExp+	0.527899	0.527373	0.52646	0.528721	0.528638
		MNC+	0.514858	0.515783	0.512594	0.513806	0.513298
		MPC+	0.513523	0.514357	0.511304	0.512277	0.511466
	3	Exp+	0.416118	0.399252	0.441691	0.438292	0.440294
		Erl+	0.124192	0.099401	0.185896	0.160052	0.168928
		HExp+	0.509105	0.498813	0.522305	0.523227	0.525513
		MNC+	0.579764	0.579456	0.578037	0.580608	0.580129
		MPC+	0.59742	0.599373	0.591959	0.59455	0.592763
	4	Exp+	0.204386	0.176595	0.268793	0.243957	0.255451
		Erl+	0.065624	0.039159	0.146214	0.105902	0.121116
		HExp+	0.367325	0.344936	0.410367	0.3995	0.409157
		MNC+	0.570096	0.559872	0.586099	0.584909	0.587309
		MPC+	0.688112	0.688782	0.685658	0.687277	0.686493
3	1	Exp+	0.424236	0.423805	0.425171	0.424833	0.425082
		Erl+	0.418926	0.416387	0.423444	0.421743	0.422452
		HExp+	0.430661	0.431176	0.429145	0.429986	0.429486
		MNC+	0.420678	0.420912	0.420312	0.420393	0.420244
		MPC+	0.420939	0.421077	0.420619	0.42073	0.420587
	2	Exp+	0.492639	0.488047	0.498434	0.498222	0.498808
		Erl+	0.411212	0.396107	0.433891	0.429285	0.429796
		HExp+	0.52706	0.525758	0.526211	0.528674	0.528728
		MNC+	0.51557	0.51646	0.512907	0.514551	0.514029
		MPC+	0.515374	0.516383	0.512507	0.513882	0.512935
	3	Exp+	0.407721	0.388812	0.438088	0.432295	0.43398
		Erl+	0.108785	0.085701	0.177118	0.144507	0.15286
		HExp+	0.501356	0.48825	0.518882	0.518511	0.521037
		MNC+	0.579717	0.578749	0.578074	0.581243	0.580724
		MPC+	0.601037	0.603265	0.594285	0.597771	0.59575
	4	Exp+	0.182832	0.154249	0.257382	0.224562	0.236167
		Erl+	0.044882	0.021858	0.133629	0.083737	0.098757
		HExp+	0.344754	0.318419	0.398409	0.381895	0.392501
		MNC+	0.562977	0.550732	0.582746	0.580216	0.582648
		MPC+	0.689763	0.690446	0.686752	0.688904	0.688021
3.5	1	Exp+	0.423811	0.42327	0.424889	0.424514	0.424776
		Erl+	0.417655	0.414815	0.422788	0.420778	0.421482
		HExp+	0.431583	0.432163	0.429747	0.430829	0.430284
		MNC+	0.421068	0.421391	0.420514	0.420684	0.420494
		MPC+	0.421407	0.421601	0.420931	0.421118	0.420928
	2	Exp+	0.490536	0.485064	0.49754	0.496979	0.497559
		Erl+	0.406691	0.391045	0.43197	0.425741	0.425913
		HExp+	0.525948	0.523757	0.525933	0.528437	0.528634
		MNC+	0.516073	0.516825	0.513138	0.515176	0.514664
		MPC+	0.517085	0.518242	0.5136	0.515383	0.514324
	3	Exp+	0.400912	0.380702	0.434977	0.427164	0.428388
		Erl+	0.097879	0.077658	0.169225	0.131923	0.139338
		HExp+	0.494083	0.478355	0.515908	0.51404	0.516711
		MNC+	0.579501	0.57778	0.578082	0.581772	0.581219
		MPC+	0.60435	0.606787	0.596402	0.60077	0.598559
	4	Exp+	0.165834	0.137955	0.247285	0.208087	0.219285
		Erl+	0.030232	0.011885	0.122218	0.065774	0.079954
		HExp+	0.324474	0.294937	0.387858	0.365761	0.376986
		MNC+	0.55675	0.542772	0.579843	0.576077	0.578415
		MPC+	0.69132	0.691983	0.687776	0.690469	0.689503

Table 2.4: Fraction of successful rate of retrials

S = 25, s = 8, λ = 5, λ−1 = 2, β= 3, μ = 10, γ = 0.3, θ = 3.

μ	c		Exp-	Erl-	HExp-	MNC-	MPC-
10	1	Exp+	0.487758	0.487521	0.488273	0.488107	0.488283
		Erl+	0.483721	0.482082	0.486377	0.485646	0.48627
		HExp+	0.489472	0.489586	0.489164	0.489315	0.489182
		MNC+	0.485473	0.485519	0.485408	0.485407	0.485356
		MPC+	0.485418	0.485443	0.485356	0.485375	0.485338
	2	Exp+	0.567485	0.562344	0.574365	0.574354	0.575461
		Erl+	0.4517	0.431961	0.477431	0.47573	0.478222
		HExp+	0.599593	0.598368	0.600738	0.601675	0.602219
		MNC+	0.605164	0.60551	0.604363	0.604893	0.604635
		MPC+	0.606362	0.606763	0.605424	0.605744	0.605313
	3	Exp+	0.430634	0.409823	0.462658	0.458819	0.464102
		Erl+	0.174204	0.142482	0.236018	0.216877	0.230031
		HExp+	0.555182	0.544349	0.571402	0.571214	0.575242
		MNC+	0.648973	0.646531	0.652211	0.653029	0.653674
		MPC+	0.684333	0.684978	0.682823	0.683404	0.682766
	4	Exp+	0.271087	0.240744	0.331204	0.313398	0.328415
		Erl+	0.162945	0.126146	0.238481	0.212774	0.230613
		HExp+	0.438024	0.418016	0.476583	0.467996	0.479591
		MNC+	0.60289	0.590112	0.624668	0.621917	0.627027
		MPC+	0.754603	0.753086	0.75832	0.757064	0.758286
11	1	Exp+	0.488842	0.488574	0.489429	0.489239	0.489427
		Erl+	0.483631	0.481732	0.486645	0.485853	0.486487
		HExp+	0.491195	0.491337	0.490818	0.491004	0.49085
		MNC+	0.486676	0.486735	0.486593	0.48659	0.486526
		MPC+	0.486625	0.486658	0.486561	0.48657	0.486526
	2	Exp+	0.562341	0.556418	0.57013	0.570204	0.571372
		Erl+	0.421074	0.398857	0.450726	0.448365	0.451455
		HExp+	0.598419	0.596728	0.600219	0.601174	0.60187
		MNC+	0.608125	0.60853	0.60721	0.607795	0.607501
		MPC+	0.609787	0.610256	0.608703	0.609063	0.608569
	3	Exp+	0.401582	0.379403	0.436687	0.43173	0.437846
		Erl+	0.161662	0.129979	0.224599	0.20439	0.218329
		HExp+	0.533881	0.521672	0.552462	0.551917	0.556498
		MNC+	0.644886	0.642009	0.648763	0.649588	0.650331
		MPC+	0.686251	0.68694	0.684675	0.685263	0.684594
	4	Exp+	0.260157	0.229583	0.321615	0.302832	0.318618
		Erl+	0.165573	0.128363	0.241526	0.215887	0.233903
		HExp+	0.41988	0.399606	0.460019	0.450326	0.462772
		MNC+	0.585109	0.571241	0.608933	0.605719	0.611345
		MPC+	0.751069	0.74929	0.755366	0.753945	0.755349
12	1	Exp+	0.489772	0.489474	0.490422	0.490214	0.490411
		Erl+	0.483292	0.481127	0.48663	0.485814	0.486451
		HExp+	0.492764	0.492931	0.492323	0.492542	0.492371
		MNC+	0.487774	0.487849	0.487674	0.487668	0.487592
		MPC+	0.487733	0.487775	0.487656	0.487664	0.487611
	2	Exp+	0.556093	0.54935	0.564863	0.565003	0.566245
		Erl+	0.389838	0.36549	0.423312	0.42003	0.423874
		HExp+	0.596045	0.59384	0.598558	0.599533	0.600382
		MNC+	0.610457	0.610906	0.609456	0.610086	0.609765
		MPC+	0.61268	0.613214	0.611463	0.611857	0.611305
	3	Exp+	0.375521	0.352295	0.413315	0.407203	0.414154
		Erl+	0.154371	0.122685	0.217984	0.197154	0.2116
		HExp+	0.513104	0.499735	0.533841	0.532852	0.537991
		MNC+	0.639922	0.636582	0.644489	0.645305	0.646159
		MPC+	0.687357	0.688072	0.685758	0.686342	0.685658
	4	Exp+	0.25293	0.222184	0.315356	0.295874	0.312239
		Erl+	0.168034	0.13047	0.244328	0.218765	0.23692
		HExp+	0.406244	0.38585	0.44755	0.436931	0.45008
		MNC+	0.568124	0.553288	0.593846	0.590142	0.596276
		MPC+	0.746607	0.744543	0.751526	0.749934	0.751535

Table 2.5: Fraction of successful rate of retrials

S = 25, s = 8, λ =5, $\lambda-1$ = 2, μ= 10, γ = 0.3, θ = 5.

β	c		Exp-	Erl-	HExp-	MNC-	MPC-
3	1	Exp+	0.487758	0.487521	0.488273	0.488107	0.488283
		Erl+	0.483721	0.482082	0.486377	0.485646	0.48627
		HExp+	0.489472	0.489586	0.489164	0.489315	0.489182
		MNC+	0.485473	0.485519	0.485408	0.485407	0.485356
		MPC+	0.485418	0.485443	0.485356	0.485375	0.485338
	2	Exp+	0.567485	0.562344	0.574365	0.574354	0.575461
		Erl+	0.4517	0.431961	0.477431	0.47573	0.478222
		HExp+	0.599593	0.598368	0.600738	0.601675	0.602219
		MNC+	0.605164	0.60551	0.604363	0.604893	0.604635
		MPC+	0.606362	0.606763	0.605424	0.605744	0.605313
	3	Exp+	0.430634	0.409823	0.462658	0.458819	0.464102
		Erl+	0.174204	0.142482	0.236018	0.216877	0.230031
		HExp+	0.555182	0.544349	0.571402	0.571214	0.575242
		MNC+	0.648973	0.646531	0.652211	0.653029	0.653674
		MPC+	0.684333	0.684978	0.682823	0.683404	0.682766
	4	Exp+	0.271087	0.240744	0.331204	0.313398	0.328415
		Erl+	0.162945	0.126146	0.238481	0.212774	0.230613
		HExp+	0.438024	0.418016	0.476583	0.467996	0.479591
		MNC+	0.60289	0.590112	0.624668	0.621917	0.627027
		MPC+	0.754603	0.753086	0.75832	0.757064	0.758286
3.5	1	Exp+	0.472677	0.472592	0.472913	0.472824	0.472901
		Erl+	0.469953	0.46887	0.471739	0.471245	0.471668
		HExp+	0.473971	0.47408	0.473689	0.473814	0.473681
		MNC+	0.470198	0.470261	0.470087	0.470103	0.470026
		MPC+	0.470077	0.470109	0.469992	0.470023	0.469974
	2	Exp+	0.556055	0.553307	0.559422	0.559843	0.560369
		Erl+	0.475621	0.461846	0.492227	0.492089	0.493259
		HExp+	0.578784	0.578526	0.578347	0.579461	0.57952
		MNC+	0.577624	0.578413	0.576053	0.576705	0.576208
		MPC+	0.576118	0.576624	0.574915	0.57534	0.574797
	3	Exp+	0.467484	0.451333	0.490012	0.489048	0.491945
		Erl+	0.180173	0.149466	0.236103	0.220782	0.230845
		HExp+	0.563506	0.555436	0.574336	0.575379	0.577863
		MNC+	0.636986	0.636751	0.636564	0.637907	0.63769
		MPC+	0.654967	0.656134	0.65224	0.653247	0.652154
	4	Exp+	0.278936	0.249145	0.333956	0.319793	0.331985
		Erl+	0.143099	0.106822	0.215467	0.191681	0.207221
		HExp+	0.446092	0.426312	0.480926	0.475126	0.484652
		MNC+	0.626588	0.617291	0.641306	0.64049	0.643596
		MPC+	0.742802	0.742502	0.743643	0.743422	0.743619
4	1	Exp+	0.460268	0.460282	0.460314	0.460282	0.460291
		Erl+	0.458307	0.457591	0.459493	0.459179	0.45947
		HExp+	0.461266	0.461363	0.461029	0.461125	0.461004
		MNC+	0.457975	0.458049	0.457832	0.457861	0.457767
		MPC+	0.457796	0.457832	0.457697	0.457736	0.45768
	2	Exp+	0.54035	0.539088	0.541647	0.542262	0.542474
		Erl+	0.485744	0.476583	0.496101	0.496655	0.497232
		HExp+	0.557496	0.557845	0.556148	0.557302	0.557081
		MNC+	0.552466	0.553492	0.550506	0.551197	0.550575
		MPC+	0.549518	0.55007	0.548209	0.54867	0.54808
	3	Exp+	0.494754	0.483391	0.509306	0.509909	0.51134
		Erl+	0.199658	0.170729	0.248655	0.237324	0.244472
		HExp+	0.565116	0.559674	0.571544	0.57321	0.574612
		MNC+	0.618968	0.620218	0.616247	0.617807	0.617078
		MPC+	0.625527	0.627002	0.622125	0.623342	0.62199
	4	Exp+	0.297645	0.269239	0.346632	0.336057	0.34553
		Erl+	0.128148	0.092751	0.197091	0.175174	0.188695
		HExp+	0.459031	0.440262	0.48952	0.48611	0.493663
		MNC+	0.640549	0.63447	0.649547	0.649849	0.651535
		MPC+	0.725455	0.726025	0.724348	0.724763	0.724269

Table 2.6: Fraction of successful rate of retrials

$S = 25$, $s = 8$, $\lambda = 5$, $\lambda_{-1} = 2$, $\beta = 4$, $\mu = 10$, $\gamma = 0.3$.

θ	c		Exp-	Erl-	HExp-	MNC-	MPC-
3	1	Exp+	0.460268	0.460282	0.460314	0.460282	0.460291
		Erl+	0.458307	0.457591	0.459493	0.459179	0.45947
		HExp+	0.461266	0.461363	0.461029	0.461125	0.461004
		MNC+	0.457975	0.458049	0.457832	0.457861	0.457767
		MPC+	0.457796	0.457832	0.457697	0.457736	0.45768
	2	Exp+	0.54035	0.539088	0.541647	0.542262	0.542474
		Erl+	0.485744	0.476583	0.496101	0.496655	0.497232
		HExp+	0.557496	0.557845	0.556148	0.557302	0.557081
		MNC+	0.552466	0.553492	0.550506	0.551197	0.550575
		MPC+	0.549518	0.55007	0.548209	0.54867	0.54808
	3	Exp+	0.494754	0.483391	0.509306	0.509909	0.51134
		Erl+	0.199658	0.170729	0.248655	0.237324	0.244472
		HExp+	0.565116	0.559674	0.571544	0.57321	0.574612
		MNC+	0.618968	0.620218	0.616247	0.617807	0.617078
		MPC+	0.625527	0.627002	0.622125	0.623342	0.62199
	4	Exp+	0.297645	0.269239	0.346632	0.336057	0.34553
		Erl+	0.128148	0.092751	0.197091	0.175174	0.188695
		HExp+	0.459031	0.440262	0.48952	0.48611	0.493663
		MNC+	0.640549	0.63447	0.649547	0.649849	0.651535
		MPC+	0.725455	0.726025	0.724348	0.724763	0.724269
4	1	Exp+	0.461157	0.461096	0.461336	0.46127	0.461324
		Erl+	0.458284	0.457448	0.459673	0.45932	0.459644
		HExp+	0.462867	0.462965	0.462611	0.462728	0.462609
		MNC+	0.459821	0.459901	0.459666	0.459695	0.459598
		MPC+	0.459679	0.459721	0.459576	0.459609	0.459547
	2	Exp+	0.539029	0.537359	0.540858	0.541549	0.541837
		Erl+	0.477489	0.469291	0.486663	0.48779	0.488231
		HExp+	0.559354	0.559143	0.5591	0.559961	0.560026
		MNC+	0.557211	0.557861	0.555959	0.556458	0.556066
		MPC+	0.557246	0.557696	0.556212	0.556551	0.556076
	3	Exp+	0.480343	0.470249	0.492946	0.494478	0.495609
		Erl+	0.171678	0.149662	0.209045	0.201371	0.206869
		HExp+	0.557082	0.551373	0.563984	0.565882	0.567393
		MNC+	0.621728	0.622113	0.620532	0.621782	0.62138
		MPC+	0.636838	0.637962	0.634362	0.63517	0.634131
	4	Exp+	0.260879	0.238188	0.299249	0.292482	0.300122
		Erl+	0.092667	0.066625	0.143854	0.128228	0.138604
		HExp+	0.427444	0.410849	0.453384	0.452146	0.458931
		MNC+	0.627813	0.621971	0.636051	0.637086	0.638625
		MPC+	0.729295	0.729529	0.728875	0.729104	0.728841
5	1	Exp+	0.461845	0.461714	0.462146	0.462053	0.462147
		Erl+	0.458168	0.457238	0.459699	0.459338	0.459683
		HExp+	0.46427	0.464355	0.464032	0.464154	0.464052
		MNC+	0.461408	0.46148	0.461276	0.461295	0.461211
		MPC+	0.461381	0.461422	0.46128	0.461309	0.461249
	2	Exp+	0.537068	0.535154	0.539187	0.539985	0.540314
		Erl+	0.46992	0.462579	0.478067	0.479551	0.47987
		HExp+	0.559939	0.559327	0.560427	0.561142	0.561404
		MNC+	0.560427	0.560765	0.559726	0.56012	0.559909
		MPC+	0.563307	0.563653	0.562533	0.562773	0.56241
	3	Exp+	0.467087	0.458165	0.478002	0.480105	0.480964
		Erl+	0.151006	0.133994	0.180012	0.174624	0.17885
		HExp+	0.548487	0.542772	0.555405	0.557556	0.559081
		MNC+	0.622225	0.622012	0.622003	0.623156	0.622977
		MPC+	0.645099	0.645926	0.643338	0.643881	0.643102
	4	Exp+	0.232153	0.213881	0.262668	0.258248	0.264398
		Erl+	0.068331	0.048922	0.106915	0.09541	0.1034
		HExp+	0.399992	0.385327	0.422291	0.422415	0.428475
		MNC+	0.614959	0.609477	0.622411	0.623955	0.625316
		MPC+	0.730643	0.730622	0.730696	0.730842	0.730751

Table 2.7: Fraction of successful rate of retrials

$S = 25$, $s = 8$, $\lambda = 5$, $\lambda_{-1} = 2$, $\beta = 4$, $\mu = 10$, $\theta = 5$.

γ	c		Exp-	Erl-	HExp-	MNC-	MPC-
0.2	1	Exp+	0.436775	0.436613	0.437144	0.437026	0.43714
		Erl+	0.432755	0.431715	0.434517	0.434054	0.43444
		HExp+	0.440201	0.440329	0.439838	0.440022	0.439865
		MNC+	0.436056	0.436134	0.435914	0.435929	0.435841
		MPC+	0.436155	0.436204	0.43603	0.43607	0.435998
	2	Exp+	0.50383	0.502276	0.505483	0.506271	0.506548
		Erl+	0.449861	0.444245	0.455989	0.457368	0.457559
		HExp+	0.528796	0.528434	0.528807	0.529628	0.529763
		MNC+	0.522528	0.522907	0.521755	0.522164	0.521958
		MPC+	0.525345	0.525725	0.524472	0.524758	0.524366
	3	Exp+	0.465914	0.459122	0.473806	0.476083	0.476425
		Erl+	0.163422	0.149332	0.187381	0.183416	0.186025
		HExp+	0.53225	0.52765	0.537453	0.539676	0.540736
		MNC+	0.58998	0.590247	0.588994	0.590204	0.589885
		MPC+	0.607385	0.6084	0.605165	0.605882	0.604947
	4	Exp+	0.238128	0.22233	0.264437	0.261036	0.265509
		Erl+	0.050832	0.035261	0.084827	0.073229	0.079946
		HExp+	0.394155	0.380758	0.413989	0.414712	0.41961
		MNC+	0.606515	0.602511	0.611597	0.613374	0.614088
		MPC+	0.702292	0.702796	0.701175	0.701698	0.701173
0.3	1	Exp+	0.461845	0.461714	0.462146	0.462053	0.462147
		Erl+	0.458168	0.457238	0.459699	0.459338	0.459683
		HExp+	0.46427	0.464355	0.464032	0.464154	0.464052
		MNC+	0.461408	0.46148	0.461276	0.461295	0.461211
		MPC+	0.461381	0.461422	0.46128	0.461309	0.461249
	2	Exp+	0.537068	0.535154	0.539187	0.539985	0.540314
		Erl+	0.46992	0.462579	0.478067	0.479551	0.47987
		HExp+	0.559939	0.559327	0.560427	0.561142	0.561404
		MNC+	0.560427	0.560765	0.559726	0.56012	0.559909
		MPC+	0.563307	0.563653	0.562533	0.562773	0.56241
	3	Exp+	0.467087	0.458165	0.478002	0.480105	0.480964
		Erl+	0.151006	0.133994	0.180012	0.174624	0.17885
		HExp+	0.548487	0.542772	0.555405	0.557556	0.559081
		MNC+	0.622225	0.622012	0.622003	0.623156	0.622977
		MPC+	0.645099	0.645926	0.643338	0.643881	0.643102
	4	Exp+	0.232153	0.213881	0.262668	0.258248	0.264398
		Erl+	0.068331	0.048922	0.106915	0.09541	0.1034
		HExp+	0.399992	0.385327	0.422291	0.422415	0.428475
		MNC+	0.614959	0.609477	0.622411	0.623955	0.625316
		MPC+	0.730643	0.730622	0.730696	0.730842	0.730751
0.4	1	Exp+	0.482632	0.482491	0.482953	0.482855	0.482957
		Erl+	0.479071	0.478156	0.480554	0.480229	0.480566
		HExp+	0.484591	0.484645	0.484435	0.484522	0.484461
		MNC+	0.482488	0.482543	0.482395	0.482403	0.482337
		MPC+	0.482433	0.482464	0.482367	0.482379	0.482335
	2	Exp+	0.563523	0.561085	0.56634	0.567141	0.567569
		Erl+	0.481098	0.471785	0.491574	0.493174	0.493709
		HExp+	0.586571	0.58564	0.587632	0.588248	0.588676
		MNC+	0.592583	0.592806	0.5921	0.592447	0.592283
		MPC+	0.596068	0.596351	0.595458	0.595628	0.595328
	3	Exp+	0.462938	0.451876	0.47695	0.478819	0.480354
		Erl+	0.147514	0.127804	0.180546	0.174434	0.18006
		HExp+	0.560939	0.554105	0.569605	0.571663	0.573707
		MNC+	0.647334	0.646497	0.648104	0.64918	0.649224
		MPC+	0.676958	0.677544	0.675758	0.676102	0.675534
	4	Exp+	0.233412	0.212984	0.26725	0.262298	0.269899
		Erl+	0.087464	0.064646	0.129614	0.118668	0.127738
		HExp+	0.407888	0.392172	0.432232	0.431885	0.438997
		MNC+	0.619003	0.612034	0.628852	0.630148	0.632241
		MPC+	0.753679	0.753098	0.754944	0.754722	0.755106

Table 2.8: Blocking Probability

$S = 25, \quad s = 8, \quad \lambda_{-1} = 2, \quad \beta = 4, \quad \mu = 10, \quad \gamma = 0.3, \theta = 5.$

λ_1	c		Exp-	Erl-	HExp-	MNC-	MPC-
4.5	1	Exp+	0.494634	0.494672	0.494592	0.494559	0.494531
		Erl+	0.493193	0.493182	0.493252	0.493206	0.49321
		HExp+	0.501079	0.501252	0.500602	0.500774	0.500556
		MNC+	0.496617	0.496756	0.496376	0.496359	0.496213
		MPC+	0.496934	0.497025	0.496717	0.496764	0.496641
	2	Exp+	0.30636	0.306303	0.306431	0.306469	0.306486
		Erl+	0.303884	0.303871	0.303896	0.303912	0.303913
		HExp+	0.314138	0.314047	0.314265	0.314317	0.314371
		MNC+	0.309648	0.309568	0.309786	0.309801	0.309863
		MPC+	0.309572	0.309515	0.3097	0.309682	0.309754
	3	Exp+	0.232376	0.232366	0.232386	0.232401	0.232404
		Erl+	0.229905	0.229904	0.229904	0.22991	0.229911
		HExp+	0.239222	0.239184	0.239267	0.239302	0.239318
		MNC+	0.236747	0.236695	0.236816	0.23685	0.236879
		MPC+	0.236997	0.236937	0.237107	0.237114	0.237179
	4	Exp+	0.176361	0.176354	0.176368	0.176377	0.176381
		Erl+	0.177323	0.177319	0.177325	0.177331	0.177333
		HExp+	0.177598	0.177575	0.17763	0.177647	0.177662
		MNC+	0.175403	0.175356	0.175464	0.175495	0.17552
		MPC+	0.176122	0.176027	0.176291	0.176307	0.176399
5	1	Exp+	0.494296	0.494304	0.494308	0.494274	0.494267
		Erl+	0.493185	0.493136	0.493318	0.493264	0.49329
		HExp+	0.499859	0.500013	0.499431	0.499585	0.499371
		MNC+	0.495688	0.495788	0.495518	0.495503	0.495387
		MPC+	0.495919	0.495987	0.495759	0.495791	0.495695
	2	Exp+	0.307628	0.307546	0.30774	0.307787	0.307815
		Erl+	0.305195	0.305171	0.305222	0.305246	0.305249
		HExp+	0.315348	0.315239	0.315512	0.315565	0.315641
		MNC+	0.310565	0.310461	0.310754	0.310764	0.310851
		MPC+	0.310222	0.31015	0.310392	0.310361	0.310455
	3	Exp+	0.233717	0.2337	0.233734	0.233754	0.233759
		Erl+	0.231004	0.231003	0.231003	0.231012	0.231013
		HExp+	0.240939	0.240884	0.241009	0.241052	0.241078
		MNC+	0.238167	0.238098	0.238263	0.238302	0.238343
		MPC+	0.238133	0.23806	0.238275	0.238276	0.238358
	4	Exp+	0.179362	0.179352	0.179373	0.179384	0.17939
		Erl+	0.180171	0.180168	0.180174	0.180181	0.180184
		HExp+	0.180859	0.180825	0.180909	0.180929	0.180951
		MNC+	0.178608	0.178547	0.178689	0.178726	0.178759
		MPC+	0.179216	0.179105	0.17942	0.179431	0.179543
5.5	1	Exp+	0.494197	0.49418	0.494255	0.494219	0.494235
		Erl+	0.493416	0.493331	0.493614	0.493553	0.493608
		HExp+	0.498712	0.498836	0.498366	0.498488	0.4983
		MNC+	0.495003	0.495066	0.494898	0.494884	0.494805
		MPC+	0.495157	0.495202	0.495053	0.495072	0.495005
	2	Exp+	0.309193	0.309081	0.309355	0.309409	0.309454
		Erl+	0.306852	0.306811	0.306904	0.306936	0.306943
		HExp+	0.316728	0.316604	0.316926	0.316976	0.317076
		MNC+	0.311722	0.311595	0.311965	0.311966	0.312081
		MPC+	0.311128	0.311042	0.311339	0.311294	0.311411
	3	Exp+	0.235368	0.235342	0.235397	0.235422	0.235431
		Erl+	0.232429	0.232427	0.23243	0.232439	0.232441
		HExp+	0.242926	0.242851	0.243027	0.243077	0.243117
		MNC+	0.239837	0.23975	0.239966	0.240008	0.240064
		MPC+	0.23951	0.239423	0.239687	0.239679	0.239781
	4	Exp+	0.182529	0.182516	0.182547	0.182559	0.182568
		Erl+	0.183149	0.183145	0.183154	0.183162	0.183166
		HExp+	0.184335	0.184288	0.184408	0.184432	0.184463
		MNC+	0.181982	0.181907	0.182089	0.182131	0.182175
		MPC+	0.18245	0.182324	0.182693	0.182695	0.182829

Table 2.9: Blocking Probability

$S = 25$, $s = 8$, $\lambda_1 = 5$, $\beta = 4$, $\mu = 10$, $\gamma = 0.3$, $\theta = 5$.

λ_{-1}	c		Exp-	Erl-	HExp-	MNC-	MPC-
2	1	Exp+	0.494296	0.494304	0.494308	0.494274	0.494267
		Erl+	0.493185	0.493136	0.493318	0.493264	0.49329
		HExp+	0.499859	0.500013	0.499431	0.499585	0.499371
		MNC+	0.495688	0.495788	0.495518	0.495503	0.495387
		MPC+	0.495919	0.495987	0.495759	0.495791	0.495695
	2	Exp+	0.307628	0.307546	0.30774	0.307787	0.307815
		Erl+	0.305195	0.305171	0.305222	0.305246	0.305249
		HExp+	0.315348	0.315239	0.315512	0.315565	0.315641
		MNC+	0.310565	0.310461	0.310754	0.310764	0.310851
		MPC+	0.310222	0.31015	0.310392	0.310361	0.310455
	3	Exp+	0.233717	0.2337	0.233734	0.233754	0.233759
		Erl+	0.231004	0.231003	0.231003	0.231012	0.231013
		HExp+	0.240939	0.240884	0.241009	0.241052	0.241078
		MNC+	0.238167	0.238098	0.238263	0.238302	0.238343
		MPC+	0.238133	0.23806	0.238275	0.238276	0.238358
	4	Exp+	0.179362	0.179352	0.179373	0.179384	0.17939
		Erl+	0.180171	0.180168	0.180174	0.180181	0.180184
		HExp+	0.180859	0.180825	0.180909	0.180929	0.180951
		MNC+	0.178608	0.178547	0.178689	0.178726	0.178759
		MPC+	0.179216	0.179105	0.17942	0.179431	0.179543
2.5	1	Exp+	0.494332	0.494349	0.494335	0.494297	0.494286
		Erl+	0.493151	0.493112	0.493284	0.493219	0.49324
		HExp+	0.500498	0.50068	0.499976	0.500184	0.499952
		MNC+	0.496095	0.496231	0.495859	0.495853	0.495713
		MPC+	0.496349	0.496438	0.496136	0.496186	0.496069
	2	Exp+	0.307528	0.307436	0.307663	0.307703	0.307734
		Erl+	0.305168	0.30514	0.3052	0.305225	0.305228
		HExp+	0.315147	0.315018	0.315354	0.315396	0.315481
		MNC+	0.310359	0.310242	0.310587	0.310581	0.310676
		MPC+	0.309964	0.309882	0.310168	0.31012	0.310226
	3	Exp+	0.23369	0.233669	0.233712	0.233734	0.23374
		Erl+	0.230997	0.230994	0.230996	0.231007	0.231009
		HExp+	0.24085	0.240785	0.240938	0.240979	0.241009
		MNC+	0.23805	0.237971	0.238171	0.238203	0.23825
		MPC+	0.237914	0.23783	0.238091	0.238077	0.238171
	4	Exp+	0.179343	0.17933	0.179357	0.179369	0.179376
		Erl+	0.180162	0.180156	0.180166	0.180175	0.180178
		HExp+	0.180794	0.180754	0.180856	0.180874	0.180899
		MNC+	0.178508	0.178438	0.178611	0.178642	0.17868
		MPC+	0.178925	0.178798	0.179179	0.179167	0.179295
3	1	Exp+	0.494368	0.494394	0.494362	0.494322	0.494307
		Erl+	0.493133	0.493105	0.49326	0.493189	0.493205
		HExp+	0.500969	0.501163	0.500386	0.500636	0.500402
		MNC+	0.496438	0.496605	0.496143	0.496149	0.495992
		MPC+	0.496705	0.496812	0.496447	0.496514	0.496383
	2	Exp+	0.307443	0.307343	0.307598	0.307629	0.307662
		Erl+	0.305143	0.305111	0.305181	0.305205	0.305209
		HExp+	0.314974	0.314829	0.315218	0.315248	0.315339
		MNC+	0.310184	0.310058	0.310445	0.310423	0.310524
		MPC+	0.309742	0.309652	0.309975	0.309911	0.310025
	3	Exp+	0.233666	0.233641	0.233693	0.233715	0.233723
		Erl+	0.230989	0.230983	0.23099	0.231002	0.231005
		HExp+	0.240772	0.240699	0.240876	0.240914	0.240947
		MNC+	0.237948	0.23786	0.23809	0.238114	0.238166
		MPC+	0.237722	0.237628	0.237929	0.2379	0.238003
	4	Exp+	0.179325	0.17931	0.179343	0.179355	0.179362
		Erl+	0.180153	0.180144	0.18016	0.180168	0.180172
		HExp+	0.180736	0.180692	0.180809	0.180824	0.180852
		MNC+	0.178421	0.178343	0.178543	0.178566	0.178609
		MPC+	0.178671	0.178531	0.178969	0.178934	0.179074

Table 2.10: Blocking Probability

$S = 25, \quad s = 8, \quad \lambda_1 = 5, \quad \lambda_{-1} = 3, \quad \mu = 10, \quad \gamma = 0.3, \quad \theta = 5.$

β	c		Exp-	Erl-	HExp-	MNC-	MPC-
4	1	Exp+	0.494368	0.494394	0.494362	0.494322	0.494307
		Erl+	0.493133	0.493105	0.49326	0.493189	0.493205
		HExp+	0.500969	0.501163	0.500386	0.500636	0.500402
		MNC+	0.496438	0.496605	0.496143	0.496149	0.495992
		MPC+	0.496705	0.496812	0.496447	0.496514	0.496383
	2	Exp+	0.307443	0.307343	0.307598	0.307629	0.307662
		Erl+	0.305143	0.305111	0.305181	0.305205	0.305209
		HExp+	0.314974	0.314829	0.315218	0.315248	0.315339
		MNC+	0.310184	0.310058	0.310445	0.310423	0.310524
		MPC+	0.309742	0.309652	0.309975	0.309911	0.310025
	3	Exp+	0.233666	0.233641	0.233693	0.233715	0.233723
		Erl+	0.230989	0.230983	0.23099	0.231002	0.231005
		HExp+	0.240772	0.240699	0.240876	0.240914	0.240947
		MNC+	0.237948	0.23786	0.23809	0.238114	0.238166
		MPC+	0.237722	0.237628	0.237929	0.2379	0.238003
	4	Exp+	0.179325	0.17931	0.179343	0.179355	0.179362
		Erl+	0.180153	0.180144	0.18016	0.180168	0.180172
		HExp+	0.180736	0.180692	0.180809	0.180824	0.180852
		MNC+	0.178421	0.178343	0.178543	0.178566	0.178609
		MPC+	0.178671	0.178531	0.178969	0.178934	0.179074
5	1	Exp+	0.476097	0.476283	0.475753	0.475768	0.475634
		Erl+	0.47393	0.474039	0.473774	0.473739	0.473691
		HExp+	0.483688	0.484003	0.482801	0.483139	0.482759
		MNC+	0.479561	0.479882	0.478931	0.479003	0.47868
		MPC+	0.479802	0.480003	0.479284	0.47944	0.479182
	2	Exp+	0.286944	0.286894	0.287021	0.287036	0.287053
		Erl+	0.2843	0.284283	0.28432	0.284331	0.284333
		HExp+	0.295113	0.295046	0.295202	0.29524	0.295279
		MNC+	0.290553	0.290521	0.290622	0.290615	0.290644
		MPC+	0.290527	0.290511	0.29057	0.29056	0.290584
	3	Exp+	0.215053	0.215038	0.215068	0.215081	0.215084
		Erl+	0.212316	0.212313	0.212316	0.212322	0.212322
		HExp+	0.22224	0.222195	0.222299	0.222328	0.222346
		MNC+	0.219619	0.219571	0.219694	0.219711	0.219739
		MPC+	0.219651	0.219603	0.21975	0.219742	0.219796
	4	Exp+	0.164032	0.164024	0.164041	0.164048	0.164051
		Erl+	0.165064	0.165061	0.165067	0.16507	0.165072
		HExp+	0.164801	0.164772	0.164842	0.164857	0.164871
		MNC+	0.162822	0.162765	0.162907	0.162928	0.162956
		MPC+	0.162859	0.162751	0.163085	0.163061	0.163167
6	1	Exp+	0.46356	0.463856	0.462986	0.463037	0.462822
		Erl+	0.460758	0.460961	0.460408	0.460397	0.460305
		HExp+	0.471829	0.472226	0.470733	0.471131	0.47065
		MNC+	0.467981	0.468407	0.46712	0.467237	0.4668
		MPC+	0.468205	0.46847	0.467508	0.467726	0.467379
	2	Exp+	0.272511	0.272496	0.272536	0.27254	0.272545
		Erl+	0.269584	0.269578	0.269592	0.269595	0.269596
		HExp+	0.28119	0.281178	0.281173	0.281216	0.281221
		MNC+	0.276765	0.276801	0.276701	0.276703	0.276683
		MPC+	0.277042	0.277078	0.276951	0.276979	0.27694
	3	Exp+	0.201652	0.201644	0.201661	0.201669	0.201671
		Erl+	0.198761	0.19876	0.198761	0.198764	0.198764
		HExp+	0.209108	0.209079	0.209141	0.209163	0.209173
		MNC+	0.206528	0.206506	0.206559	0.206571	0.206584
		MPC+	0.206772	0.206755	0.206801	0.206807	0.206827
	4	Exp+	0.152823	0.152818	0.152829	0.152833	0.152834
		Erl+	0.15382	0.153818	0.153821	0.153822	0.153823
		HExp+	0.153492	0.153472	0.15352	0.153532	0.15354
		MNC+	0.151595	0.15155	0.151662	0.151679	0.1517
		MPC+	0.151514	0.151426	0.1517	0.15168	0.151767

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Table 2.11: Blocking Probability

S = 25, s = 8, λ_1 = 5, λ_{-1} = 3, β = 4, γ = 0.3, θ = 5.

µ	c		Exp-	Erl-	HExp-	MNC-	MPC-
10	1	Exp+	0.494368	0.494394	0.494362	0.494322	0.494307
		Erl+	0.493133	0.493105	0.49326	0.493189	0.493205
		HExp+	0.500969	0.501163	0.500386	0.500636	0.500402
		MNC+	0.496438	0.496605	0.496143	0.496149	0.495992
		MPC+	0.496705	0.496812	0.496447	0.496514	0.496383
	2	Exp+	0.307443	0.307343	0.307598	0.307629	0.307662
		Erl+	0.305143	0.305111	0.305181	0.305205	0.305209
		HExp+	0.314974	0.314829	0.315218	0.315248	0.315339
		MNC+	0.310184	0.310058	0.310445	0.310423	0.310524
		MPC+	0.309742	0.309652	0.309975	0.309911	0.310025
	3	Exp+	0.233666	0.233641	0.233693	0.233715	0.233723
		Erl+	0.230989	0.230983	0.23099	0.231002	0.231005
		HExp+	0.240772	0.240699	0.240876	0.240914	0.240947
		MNC+	0.237948	0.23786	0.23809	0.238114	0.238166
		MPC+	0.237722	0.237628	0.237929	0.2379	0.238003
	4	Exp+	0.179325	0.17931	0.179343	0.179355	0.179362
		Erl+	0.180153	0.180144	0.18016	0.180168	0.180172
		HExp+	0.180736	0.180692	0.180809	0.180824	0.180852
		MNC+	0.178421	0.178343	0.178543	0.178566	0.178609
		MPC+	0.178671	0.178531	0.178969	0.178934	0.179074
11	1	Exp+	0.496514	0.496496	0.496598	0.496547	0.496564
		Erl+	0.495456	0.495399	0.495634	0.495566	0.495592
		HExp+	0.503266	0.503418	0.502785	0.503009	0.502831
		MNC+	0.498327	0.498455	0.498123	0.498107	0.497998
		MPC+	0.498646	0.498727	0.498458	0.498501	0.498406
	2	Exp+	0.309519	0.309427	0.309656	0.309692	0.30972
		Erl+	0.307358	0.307333	0.307384	0.307408	0.30741
		HExp+	0.31683	0.316681	0.317077	0.317111	0.317197
		MNC+	0.3121	0.31196	0.312376	0.312361	0.312466
		MPC+	0.311647	0.311546	0.311906	0.311839	0.311965
	3	Exp+	0.235584	0.235562	0.235604	0.235626	0.235632
		Erl+	0.233222	0.233217	0.233222	0.233235	0.233237
		HExp+	0.242184	0.24212	0.242272	0.242309	0.242336
		MNC+	0.239558	0.239472	0.239693	0.239719	0.239767
		MPC+	0.239372	0.239275	0.239582	0.239555	0.23966
	4	Exp+	0.179234	0.17922	0.17925	0.179261	0.179268
		Erl+	0.180191	0.180182	0.180197	0.180206	0.18021
		HExp+	0.18048	0.180442	0.180543	0.180556	0.180581
		MNC+	0.178173	0.178103	0.178281	0.178304	0.178342
		MPC+	0.178521	0.178387	0.178802	0.178774	0.178906
12	1	Exp+	0.498395	0.498339	0.498552	0.498495	0.498535
		Erl+	0.497488	0.497411	0.497696	0.497638	0.497667
		HExp+	0.505282	0.505392	0.5049	0.505097	0.504971
		MNC+	0.499973	0.500064	0.499853	0.499818	0.499751
		MPC+	0.500342	0.5004	0.500218	0.500239	0.500176
	2	Exp+	0.311299	0.311213	0.31142	0.311459	0.311483
		Erl+	0.309272	0.309252	0.30929	0.309312	0.309314
		HExp+	0.318412	0.318263	0.318654	0.318692	0.318772
		MNC+	0.313729	0.31358	0.314015	0.314007	0.314114
		MPC+	0.313274	0.313163	0.313552	0.313484	0.313619
	3	Exp+	0.237245	0.237226	0.237262	0.237282	0.237288
		Erl+	0.235169	0.235163	0.235169	0.235181	0.235183
		HExp+	0.243411	0.243355	0.243486	0.243521	0.243545
		MNC+	0.240925	0.240842	0.241053	0.241081	0.241126
		MPC+	0.240778	0.240678	0.240991	0.240967	0.241073
	4	Exp+	0.179158	0.179146	0.179173	0.179184	0.179191
		Erl+	0.180225	0.180217	0.180231	0.18024	0.180244
		HExp+	0.180265	0.180231	0.180321	0.180333	0.180356
		MNC+	0.177954	0.17789	0.178051	0.178073	0.178107
		MPC+	0.17839	0.17826	0.178656	0.178633	0.178758

Table 2.12: Blocking Probability

$S = 25, \quad s = 8, \quad \lambda_1 = 5, \quad \lambda_{-1} = 2, \quad \beta = 6, \quad \mu = 10, \quad \gamma = 0.3.$

θ	c		Exp-	Erl-	HExp-	MNC-	MPC-
5	1	Exp+	0.46356	0.463856	0.462986	0.463037	0.462822
		Erl+	0.460758	0.460961	0.460408	0.460397	0.460305
		HExp+	0.471829	0.472226	0.470733	0.471131	0.47065
		MNC+	0.467981	0.468407	0.46712	0.467237	0.4668
		MPC+	0.468205	0.46847	0.467508	0.467726	0.467379
	2	Exp+	0.272511	0.272496	0.272536	0.27254	0.272545
		Erl+	0.269584	0.269578	0.269592	0.269595	0.269596
		HExp+	0.28119	0.281178	0.281173	0.281216	0.281221
		MNC+	0.276765	0.276801	0.276701	0.276703	0.276683
		MPC+	0.277042	0.277078	0.276951	0.276979	0.27694
	3	Exp+	0.201652	0.201644	0.201661	0.201669	0.201671
		Erl+	0.198761	0.19876	0.198761	0.198764	0.198764
		HExp+	0.209108	0.209079	0.209141	0.209163	0.209173
		MNC+	0.206528	0.206506	0.206559	0.206571	0.206584
		MPC+	0.206772	0.206755	0.206801	0.206807	0.206827
	4	Exp+	0.152823	0.152818	0.152829	0.152833	0.152834
		Erl+	0.15382	0.153818	0.153821	0.153822	0.153823
		HExp+	0.153492	0.153472	0.15352	0.153532	0.15354
		MNC+	0.151595	0.15155	0.151662	0.151679	0.1517
		MPC+	0.151514	0.151426	0.1517	0.15168	0.151767
6	1	Exp+	0.463344	0.463638	0.462792	0.462817	0.462607
		Erl+	0.460607	0.460814	0.460268	0.460235	0.460146
		HExp+	0.471739	0.472124	0.470689	0.471052	0.47059
		MNC+	0.467632	0.468055	0.466793	0.466878	0.466449
		MPC+	0.467945	0.468211	0.467259	0.467459	0.467115
	2	Exp+	0.272539	0.272522	0.272567	0.272573	0.272579
		Erl+	0.26959	0.269584	0.269598	0.269602	0.269603
		HExp+	0.281297	0.281273	0.281311	0.281345	0.281358
		MNC+	0.276854	0.276876	0.276824	0.276816	0.276807
		MPC+	0.277245	0.277267	0.277194	0.277206	0.277184
	3	Exp+	0.201664	0.201655	0.201672	0.201681	0.201683
		Erl+	0.198763	0.198762	0.198764	0.198766	0.198766
		HExp+	0.209159	0.209129	0.209193	0.209218	0.209229
		MNC+	0.206596	0.206571	0.206632	0.206646	0.206661
		MPC+	0.206969	0.206944	0.207014	0.207018	0.207046
	4	Exp+	0.152828	0.152823	0.152833	0.152838	0.15284
		Erl+	0.153821	0.15382	0.153822	0.153824	0.153824
		HExp+	0.153516	0.153496	0.153541	0.153555	0.153563
		MNC+	0.151638	0.151594	0.151698	0.151719	0.151739
		MPC+	0.151684	0.151597	0.151852	0.151847	0.15193
7	1	Exp+	0.463172	0.46346	0.462647	0.462646	0.462443
		Erl+	0.460481	0.460688	0.460156	0.460104	0.460016
		HExp+	0.471689	0.472061	0.470691	0.47102	0.470578
		MNC+	0.467373	0.467788	0.466569	0.466622	0.466206
		MPC+	0.467773	0.468035	0.467107	0.467287	0.46695
	2	Exp+	0.272565	0.272546	0.272595	0.272603	0.272609
		Erl+	0.269596	0.26959	0.269603	0.269608	0.269609
		HExp+	0.281393	0.281361	0.281429	0.281459	0.281478
		MNC+	0.276942	0.276952	0.276936	0.276924	0.276923
		MPC+	0.277439	0.277449	0.27742	0.27742	0.277413
	3	Exp+	0.201674	0.201665	0.201682	0.201691	0.201693
		Erl+	0.198765	0.198763	0.198766	0.198768	0.198768
		HExp+	0.209203	0.209172	0.209238	0.209265	0.209277
		MNC+	0.206658	0.20663	0.206696	0.206713	0.206729
		MPC+	0.207149	0.207118	0.207205	0.207209	0.207243
	4	Exp+	0.152832	0.152827	0.152837	0.152842	0.152844
		Erl+	0.153822	0.153821	0.153823	0.153825	0.153825
		HExp+	0.153536	0.153516	0.153559	0.153574	0.153582
		MNC+	0.151673	0.151631	0.151727	0.151751	0.15177
		MPC+	0.151826	0.151742	0.151979	0.151985	0.152066

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Table 2.13: Blocking Probability

$S = 25$, $s = 8$, $\lambda_1 = 5$, $\lambda_{-1} = 3$, $\beta = 6$, $\mu = 10$, $\theta = 5$.

γ	c		Exp-	Erl-	HExp-	MNC-	MPC-
0.2	1	Exp+	0.441381	0.441649	0.440882	0.440909	0.440717
		Erl+	0.4388	0.438979	0.438513	0.438488	0.438409
		HExp+	0.450281	0.450677	0.44918	0.449581	0.449105
		MNC+	0.445723	0.446143	0.444908	0.444992	0.444569
		MPC+	0.446044	0.446308	0.445371	0.445568	0.445229
	2	Exp+	0.253259	0.253227	0.253312	0.253319	0.25333
		Erl+	0.250641	0.250629	0.250656	0.250662	0.250664
		HExp+	0.261439	0.261401	0.261482	0.261512	0.261535
		MNC+	0.256996	0.256998	0.257006	0.256995	0.257001
		MPC+	0.257183	0.257192	0.257166	0.257166	0.257161
	3	Exp+	0.185234	0.185223	0.185247	0.185256	0.185258
		Erl+	0.1826	0.182598	0.182601	0.182603	0.182603
		HExp+	0.191957	0.19192	0.192005	0.192027	0.192041
		MNC+	0.189593	0.189556	0.189651	0.189662	0.189683
		MPC+	0.189556	0.189519	0.189635	0.189626	0.189668
	4	Exp+	0.139303	0.139297	0.139309	0.139312	0.139314
		Erl+	0.140447	0.140446	0.140448	0.140449	0.14045
		HExp+	0.139381	0.139359	0.13941	0.139423	0.139431
		MNC+	0.137767	0.137718	0.137839	0.137856	0.137878
		MPC+	0.137271	0.137173	0.137478	0.137453	0.137549
0.25	1	Exp+	0.453116	0.453404	0.452567	0.452609	0.4524
		Erl+	0.450377	0.450572	0.45005	0.450032	0.449945
		HExp+	0.461814	0.462217	0.460697	0.461105	0.460618
		MNC+	0.457564	0.457993	0.456712	0.456816	0.456379
		MPC+	0.457835	0.458104	0.457139	0.457351	0.457003
	2	Exp+	0.263277	0.263255	0.263314	0.26332	0.263327
		Erl+	0.260481	0.260473	0.260492	0.260497	0.260498
		HExp+	0.271848	0.271824	0.271857	0.271894	0.271907
		MNC+	0.267322	0.267344	0.26729	0.267286	0.267277
		MPC+	0.267578	0.267603	0.267518	0.267535	0.26751
	3	Exp+	0.193741	0.193731	0.193751	0.193759	0.193761
		Erl+	0.190968	0.190966	0.190969	0.19097	0.190971
		HExp+	0.200929	0.200897	0.200969	0.200991	0.201004
		MNC+	0.198385	0.198356	0.198428	0.19844	0.198457
		MPC+	0.198511	0.198485	0.198563	0.198562	0.198592
	4	Exp+	0.146304	0.146299	0.14631	0.146314	0.146316
		Erl+	0.147379	0.147378	0.14738	0.147381	0.147381
		HExp+	0.146712	0.14669	0.14674	0.146753	0.146761
		MNC+	0.144921	0.144874	0.144991	0.145008	0.145029
		MPC+	0.144647	0.144554	0.144842	0.14482	0.144911
0.3	1	Exp+	0.46356	0.463856	0.462986	0.463037	0.462822
		Erl+	0.460758	0.460961	0.460408	0.460397	0.460305
		HExp+	0.471829	0.472226	0.470733	0.471131	0.47065
		MNC+	0.467981	0.468407	0.46712	0.467237	0.4668
		MPC+	0.468205	0.46847	0.467508	0.467726	0.467379
	2	Exp+	0.272511	0.272496	0.272536	0.27254	0.272545
		Erl+	0.269584	0.269578	0.269592	0.269595	0.269596
		HExp+	0.28119	0.281178	0.281173	0.281216	0.281221
		MNC+	0.276765	0.276801	0.276701	0.276703	0.276683
		MPC+	0.277042	0.277078	0.276951	0.276979	0.27694
	3	Exp+	0.201652	0.201644	0.201661	0.201669	0.201671
		Erl+	0.198761	0.19876	0.198761	0.198764	0.198764
		HExp+	0.209108	0.209079	0.209141	0.209163	0.209173
		MNC+	0.206528	0.206506	0.206559	0.206571	0.206584
		MPC+	0.206772	0.206755	0.206801	0.206807	0.206827
	4	Exp+	0.152823	0.152818	0.152829	0.152833	0.152834
		Erl+	0.15382	0.153818	0.153821	0.153822	0.153823
		HExp+	0.153492	0.153472	0.15352	0.153532	0.15354
		MNC+	0.151595	0.15155	0.151662	0.151679	0.1517
		MPC+	0.151514	0.151426	0.1517	0.15168	0.151767

Table 2.14: Mean number of Idle Servers

$S = 25$, $s = 8$, $\lambda_{-1} = 2$, $\beta = 4$, $\mu = 10$, $\gamma = 0.3$, $\theta = 5$.

λ_1	c		Exp-	Erl-	HExp-	MNC-	MPC-
4.5	1	Exp+	0.506841	0.506744	0.507007	0.507023	0.507097
		Erl+	0.508277	0.508228	0.508337	0.508372	0.508395
		HExp+	0.50086	0.500659	0.5014	0.501222	0.501478
		MNC+	0.504654	0.504479	0.50498	0.50498	0.505171
		MPC+	0.50439	0.504277	0.50467	0.504602	0.504758
	2	Exp+	1.130471	1.130619	1.130283	1.130185	1.130139
		Erl+	1.132947	1.132985	1.132913	1.132865	1.132861
		HExp+	1.118357	1.118562	1.118055	1.117958	1.117836
		MNC+	1.127076	1.127328	1.126644	1.126599	1.126411
		MPC+	1.12813	1.128324	1.127695	1.127761	1.127522
	3	Exp+	1.75473	1.754793	1.754671	1.754593	1.75457
		Erl+	1.753625	1.753641	1.753622	1.753581	1.753572
		HExp+	1.749937	1.750093	1.749741	1.749618	1.749545
		MNC+	1.756176	1.756476	1.755758	1.755611	1.755446
		MPC+	1.762416	1.762811	1.761649	1.761669	1.761252
	4	Exp+	2.267027	2.267072	2.266996	2.266915	2.266885
		Erl+	2.243383	2.243411	2.243373	2.243313	2.243297
		HExp+	2.292411	2.292504	2.292294	2.292185	2.292112
		MNC+	2.306713	2.306956	2.306405	2.306246	2.30611
		MPC+	2.319806	2.320332	2.318866	2.318812	2.318306
5	1	Exp+	0.507566	0.507491	0.507695	0.507707	0.507774
		Erl+	0.508727	0.508698	0.508753	0.508785	0.508804
		HExp+	0.502437	0.502259	0.502922	0.502761	0.503013
		MNC+	0.505872	0.505737	0.506124	0.506122	0.506284
		MPC+	0.505685	0.505595	0.505906	0.505853	0.505982
	2	Exp+	1.126544	1.126741	1.126276	1.126163	1.126093
		Erl+	1.128428	1.128488	1.128363	1.128299	1.12829
		HExp+	1.115979	1.116219	1.115599	1.115511	1.115344
		MNC+	1.12417	1.124468	1.123631	1.123602	1.123359
		MPC+	1.125717	1.12594	1.125198	1.125293	1.125006
	3	Exp+	1.744545	1.744629	1.74446	1.744364	1.744333
		Erl+	1.742773	1.742788	1.74277	1.742718	1.742707
		HExp+	1.742887	1.743093	1.742604	1.742469	1.742361
		MNC+	1.747152	1.747511	1.74663	1.746474	1.746262
		MPC+	1.754146	1.754587	1.753257	1.753311	1.752824
	4	Exp+	2.248576	2.248626	2.248539	2.248442	2.248405
		Erl+	2.223203	2.223228	2.2232	2.223122	2.223103
		HExp+	2.280753	2.280878	2.280579	2.280461	2.280362
		MNC+	2.290425	2.290715	2.290038	2.289863	2.289691
		MPC+	2.304108	2.304692	2.303024	2.303001	2.302412
5.5	1	Exp+	0.50809	0.508037	0.508179	0.508191	0.508244
		Erl+	0.508987	0.508979	0.508981	0.509011	0.50902
		HExp+	0.503946	0.503803	0.50434	0.504211	0.504433
		MNC+	0.506873	0.506779	0.507047	0.507046	0.507169
		MPC+	0.506752	0.506688	0.506909	0.506871	0.506968
	2	Exp+	1.122283	1.122534	1.121922	1.1218	1.121697
		Erl+	1.123541	1.123633	1.123432	1.123349	1.123334
		HExp+	1.113437	1.113707	1.112978	1.112908	1.112692
		MNC+	1.120918	1.121258	1.120274	1.120269	1.119967
		MPC+	1.122888	1.123135	1.12229	1.122416	1.122084
	3	Exp+	1.734204	1.734313	1.734083	1.73397	1.733927
		Erl+	1.73178	1.731796	1.731775	1.731714	1.7317
		HExp+	1.735776	1.73604	1.735388	1.735248	1.735096
		MNC+	1.737918	1.738336	1.737283	1.737127	1.73686
		MPC+	1.745552	1.746035	1.744543	1.744635	1.74408
	4	Exp+	2.230683	2.230737	2.230636	2.230527	2.230482
		Erl+	2.203849	2.203868	2.203853	2.203761	2.203738
		HExp+	2.269533	2.269696	2.269284	2.269164	2.269032
		MNC+	2.274301	2.274641	2.273828	2.273642	2.27343
		MPC+	2.288393	2.28903	2.287169	2.287182	2.286512

Table 2.15: Mean number of Idle Servers

S = 25, s = 8, λ_1 = 5, β = 4, μ = 10, γ = 0.3, θ = 5.

λ_{-1}	c		Exp-	Erl-	HExp-	MNC-	MPC-
2	1	Exp+	0.507566	0.507491	0.507695	0.507707	0.507774
		Erl+	0.508727	0.508698	0.508753	0.508785	0.508804
		HExp+	0.502437	0.502259	0.502922	0.502761	0.503013
		MNC+	0.505872	0.505737	0.506124	0.506122	0.506284
		MPC+	0.505685	0.505595	0.505906	0.505853	0.505982
	2	Exp+	1.126544	1.126741	1.126276	1.126163	1.126093
		Erl+	1.128428	1.128488	1.128363	1.128299	1.12829
		HExp+	1.115979	1.116219	1.115599	1.115511	1.115344
		MNC+	1.12417	1.124468	1.123631	1.123602	1.123359
		MPC+	1.125717	1.12594	1.125198	1.125293	1.125006
	3	Exp+	1.744545	1.744629	1.74446	1.744364	1.744333
		Erl+	1.742773	1.742788	1.74277	1.742718	1.742707
		HExp+	1.742887	1.743093	1.742604	1.742469	1.742361
		MNC+	1.747152	1.747511	1.74663	1.746474	1.746262
		MPC+	1.754146	1.754587	1.753257	1.753311	1.752824
	4	Exp+	2.248576	2.248626	2.248539	2.248442	2.248405
		Erl+	2.223203	2.223228	2.2232	2.223122	2.223103
		HExp+	2.280753	2.280878	2.280579	2.280461	2.280362
		MNC+	2.290425	2.290715	2.290038	2.289863	2.289691
		MPC+	2.304108	2.304692	2.303024	2.303001	2.302412
2.5	1	Exp+	0.507404	0.507314	0.507561	0.507569	0.507641
		Erl+	0.508672	0.508632	0.508711	0.508745	0.508766
		HExp+	0.501666	0.501452	0.502262	0.502044	0.50232
		MNC+	0.505317	0.505136	0.505655	0.505637	0.505831
		MPC+	0.505111	0.504995	0.505401	0.505323	0.505479
	2	Exp+	1.126787	1.127009	1.126462	1.126367	1.126289
		Erl+	1.128502	1.128573	1.128422	1.128354	1.128344
		HExp+	1.116404	1.116683	1.115934	1.115876	1.115691
		MNC+	1.124744	1.125081	1.124092	1.124114	1.123847
		MPC+	1.12649	1.126744	1.125861	1.126013	1.125692
	3	Exp+	1.744681	1.744786	1.74457	1.744469	1.74443
		Erl+	1.742825	1.742855	1.742816	1.742753	1.742738
		HExp+	1.743229	1.743467	1.742881	1.742757	1.742633
		MNC+	1.747725	1.748133	1.747076	1.746972	1.746731
		MPC+	1.755378	1.75588	1.754281	1.754443	1.753895
	4	Exp+	2.248711	2.248786	2.248655	2.248543	2.248497
		Erl+	2.223285	2.223334	2.223269	2.223176	2.223151
		HExp+	2.281067	2.281219	2.280851	2.280724	2.280605
		MNC+	2.290924	2.291262	2.290435	2.290289	2.290089
		MPC+	2.305595	2.306263	2.304247	2.304356	2.303688
3	1	Exp+	0.507271	0.507168	0.507452	0.507454	0.507528
		Erl+	0.508623	0.508573	0.508674	0.508709	0.508731
		HExp+	0.501084	0.50085	0.501757	0.501493	0.501777
		MNC+	0.504854	0.504637	0.505269	0.505232	0.505447
		MPC+	0.50464	0.504503	0.504987	0.504886	0.50506
	2	Exp+	1.126994	1.127234	1.12662	1.126546	1.126463
		Erl+	1.128567	1.128651	1.128474	1.128405	1.128393
		HExp+	1.116767	1.117074	1.11622	1.116194	1.115997
		MNC+	1.125229	1.125593	1.124482	1.124556	1.124273
		MPC+	1.127152	1.12743	1.126431	1.126635	1.12629
	3	Exp+	1.744805	1.744929	1.744668	1.744566	1.744521
		Erl+	1.742877	1.742925	1.742857	1.742787	1.742769
		HExp+	1.74353	1.743794	1.743124	1.743016	1.742878
		MNC+	1.74822	1.748661	1.74746	1.747413	1.74715
		MPC+	1.756445	1.756992	1.75517	1.755438	1.754843
	4	Exp+	2.248841	2.248942	2.248761	2.24864	2.248584
		Erl+	2.223366	2.22344	2.22333	2.22323	2.223199
		HExp+	2.281353	2.281533	2.281097	2.280964	2.280828
		MNC+	2.291364	2.29174	2.290782	2.290675	2.29045
		MPC+	2.306892	2.30762	2.305312	2.305555	2.304826

Table 2.16: Mean number of Idle Servers

$S = 25$, $s = 8$, $\lambda_1 = 5$, $\lambda_{-1} = 3$, $\mu = 10$, $\gamma = 0.3$, $\theta = 5$.

β	c		Exp-	Erl-	HExp-	MNC-	MPC-
4	1	Exp+	0.507271	0.507168	0.507452	0.507454	0.507528
		Erl+	0.508623	0.508573	0.508674	0.508709	0.508731
		HExp+	0.501084	0.50085	0.501757	0.501493	0.501777
		MNC+	0.504854	0.504637	0.505269	0.505232	0.505447
		MPC+	0.50464	0.504503	0.504987	0.504886	0.50506
	2	Exp+	1.126994	1.127234	1.12662	1.126546	1.126463
		Erl+	1.128567	1.128651	1.128474	1.128405	1.128393
		HExp+	1.116767	1.117074	1.11622	1.116194	1.115997
		MNC+	1.125229	1.125593	1.124482	1.124556	1.124273
		MPC+	1.127152	1.12743	1.126431	1.126635	1.12629
	3	Exp+	1.744805	1.744929	1.744668	1.744566	1.744521
		Erl+	1.742877	1.742925	1.742857	1.742787	1.742769
		HExp+	1.74353	1.743794	1.743124	1.743016	1.742878
		MNC+	1.74822	1.748661	1.74746	1.747413	1.74715
		MPC+	1.756445	1.756992	1.75517	1.755438	1.754843
	4	Exp+	2.248841	2.248942	2.248761	2.24864	2.248584
		Erl+	2.223366	2.22344	2.22333	2.22323	2.223199
		HExp+	2.281353	2.281533	2.281097	2.280964	2.280828
		MNC+	2.291364	2.29174	2.290782	2.290675	2.29045
		MPC+	2.306892	2.30762	2.305312	2.305555	2.304826
5	1	Exp+	0.524477	0.524261	0.52489	0.524859	0.525016
		Erl+	0.52666	0.52652	0.526888	0.526907	0.52697
		HExp+	0.517097	0.516766	0.518018	0.517676	0.518076
		MNC+	0.520894	0.520555	0.52157	0.521486	0.521831
		MPC+	0.520677	0.520465	0.521229	0.52106	0.521334
	2	Exp+	1.183111	1.183298	1.182817	1.182766	1.182704
		Erl+	1.186663	1.186727	1.186583	1.186542	1.186535
		HExp+	1.167882	1.168112	1.167509	1.167453	1.167312
		MNC+	1.178451	1.178687	1.177967	1.178014	1.177828
		MPC+	1.179282	1.17945	1.178849	1.17897	1.178757
	3	Exp+	1.844566	1.844669	1.844443	1.844375	1.844348
		Erl+	1.846282	1.846305	1.846267	1.84624	1.846233
		HExp+	1.833504	1.833753	1.833138	1.833034	1.832927
		MNC+	1.842686	1.84309	1.842006	1.841948	1.841722
		MPC+	1.848828	1.849313	1.847714	1.847933	1.84741
	4	Exp+	2.378727	2.37879	2.378668	2.378607	2.37858
		Erl+	2.357812	2.357844	2.357791	2.357755	2.357744
		HExp+	2.398879	2.399032	2.398673	2.398572	2.398488
		MNC+	2.414239	2.41459	2.413701	2.413602	2.413421
		MPC+	2.427574	2.428284	2.426048	2.426271	2.425575
6	1	Exp+	0.536663	0.536354	0.537266	0.537207	0.537433
		Erl+	0.53946	0.539244	0.539841	0.539845	0.539943
		HExp+	0.5285	0.528096	0.52961	0.529211	0.529699
		MNC+	0.532198	0.531764	0.533077	0.532955	0.533401
		MPC+	0.531986	0.531715	0.532695	0.532473	0.532826
	2	Exp+	1.224637	1.224762	1.224437	1.224405	1.224364
		Erl+	1.229577	1.229621	1.229519	1.229496	1.229492
		HExp+	1.206238	1.20638	1.206049	1.205973	1.20589
		MNC+	1.217753	1.217855	1.217536	1.217561	1.217473
		MPC+	1.217686	1.217745	1.217535	1.217574	1.217493
	3	Exp+	1.921137	1.921223	1.921033	1.920982	1.920963
		Erl+	1.925554	1.925566	1.925544	1.925531	1.925529
		HExp+	1.903387	1.903607	1.903072	1.902977	1.902891
		MNC+	1.915155	1.915498	1.914587	1.914528	1.914341
		MPC+	1.919385	1.919785	1.918482	1.918645	1.918215
	4	Exp+	2.482824	2.482866	2.482781	2.482746	2.482733
		Erl+	2.465847	2.46586	2.465837	2.465823	2.465819
		HExp+	2.493508	2.493637	2.493339	2.493258	2.493202
		MNC+	2.512287	2.512609	2.511801	2.511706	2.511556
		MPC+	2.523356	2.52402	2.521941	2.522135	2.521492

Table 2.17: Mean number of Idle Servers

$S = 25$, $s = 8$, $\lambda_1 = 5$, $\lambda_{-1} = 3$, $\beta = 4$, $\gamma = 0.3$, $\theta = 5$.

μ	c		Exp-	Erl-	HExp-	MNC-	MPC-
10	1	Exp+	0.507271	0.507168	0.507452	0.507454	0.507528
		Erl+	0.508623	0.508573	0.508674	0.508709	0.508731
		HExp+	0.501084	0.50085	0.501757	0.501493	0.501777
		MNC+	0.504854	0.504637	0.505269	0.505232	0.505447
		MPC+	0.50464	0.504503	0.504987	0.504886	0.50506
	2	Exp+	1.126994	1.127234	1.12662	1.126546	1.126463
		Erl+	1.128567	1.128651	1.128474	1.128405	1.128393
		HExp	1.116767	1.117074	1.11622	1.116194	1.115997
		MNC+	1.125229	1.125593	1.124482	1.124556	1.124273
		MPC+	1.127152	1.12743	1.126431	1.126635	1.12629
	3	Exp+	1.744805	1.744929	1.744668	1.744566	1.744521
		Erl+	1.742877	1.742925	1.742857	1.742787	1.742769
		HExp+	1.74353	1.743794	1.743124	1.743016	1.742878
		MNC+	1.74822	1.748661	1.74746	1.747413	1.74715
		MPC+	1.756445	1.756992	1.75517	1.755438	1.754843
	4	Exp+	2.248841	2.248942	2.248761	2.24864	2.248584
		Erl+	2.223366	2.22344	2.22333	2.22323	2.223199
		HExp+	2.281353	2.281533	2.281097	2.280964	2.280828
		MNC+	2.291364	2.29174	2.290782	2.290675	2.29045
		MPC+	2.306892	2.30762	2.305312	2.305555	2.304826
11	1	Exp+	0.50528	0.505214	0.505382	0.505397	0.50544
		Erl+	0.50644	0.506418	0.506439	0.506474	0.506483
		HExp+	0.499047	0.498846	0.499634	0.499395	0.49963
		MNC+	0.503097	0.502912	0.503435	0.503418	0.503591
		MPC+	0.502843	0.502728	0.50313	0.50305	0.503193
	2	Exp+	1.123003	1.123218	1.122684	1.1226	1.122532
		Erl+	1.124186	1.124252	1.124123	1.124058	1.124049
		HExp+	1.113578	1.113875	1.113062	1.113023	1.112847
		MNC+	1.121778	1.122152	1.121036	1.121089	1.120813
		MPC+	1.123841	1.124134	1.123098	1.123298	1.122943
	3	Exp+	1.739418	1.739525	1.739308	1.739211	1.739171
		Erl+	1.736326	1.736371	1.736308	1.73624	1.736222
		HExp+	1.740169	1.740392	1.739839	1.73973	1.739614
		MNC+	1.744405	1.744815	1.743717	1.743658	1.743423
		MPC+	1.753161	1.7537	1.751927	1.752171	1.751599
	4	Exp+	2.249043	2.249139	2.248971	2.248852	2.248798
		Erl+	2.222677	2.222751	2.222641	2.222543	2.222512
		HExp+	2.282566	2.282725	2.282347	2.282216	2.282089
		MNC+	2.293367	2.293706	2.292853	2.292742	2.292538
		MPC+	2.310065	2.310763	2.30858	2.308786	2.308103
12	1	Exp+	0.503542	0.503509	0.50358	0.503602	0.503622
		Erl+	0.504531	0.504531	0.504493	0.504526	0.504526
		HExp+	0.49728	0.497113	0.497784	0.49757	0.497761
		MNC+	0.501575	0.501421	0.501843	0.501843	0.501979
		MPC+	0.501284	0.501189	0.501518	0.501456	0.501572
	2	Exp+	1.119604	1.119798	1.11933	1.119242	1.119185
		Erl+	1.120416	1.120468	1.120372	1.120312	1.120304
		HExp+	1.110901	1.111183	1.110423	1.110372	1.110217
		MNC+	1.118892	1.119271	1.118161	1.118196	1.117929
		MPC+	1.121089	1.121393	1.120331	1.120526	1.120164
	3	Exp+	1.734785	1.734879	1.734695	1.734601	1.734564
		Erl+	1.730633	1.730677	1.730617	1.730549	1.730531
		HExp+	1.737288	1.737479	1.737015	1.736905	1.736804
		MNC+	1.74126	1.741642	1.74063	1.740564	1.74035
		MPC+	1.750551	1.751082	1.749335	1.749578	1.749026
	4	Exp+	2.249332	2.249424	2.249265	2.249148	2.249095
		Erl+	2.222195	2.222269	2.22216	2.222062	2.222032
		HExp+	2.283725	2.283868	2.283533	2.283403	2.283282
		MNC+	2.295266	2.295576	2.294805	2.294693	2.294506
		MPC+	2.313106	2.313776	2.311705	2.311879	2.311236

Table 2.18: Mean number of Idle Servers

S = 25, s = 8, λ_1 = 5, λ_{-1} = 2, β = 6, μ = 10, γ = 0.3.

θ	c		Exp-	Erl-	HExp-	MNC-	MPC-
5	1	Exp+	0.536663	0.536354	0.537266	0.537207	0.537433
		Erl+	0.53946	0.539244	0.539841	0.539845	0.539943
		HExp+	0.5285	0.528096	0.52961	0.529211	0.529699
		MNC+	0.532198	0.531764	0.533077	0.532955	0.533401
		MPC+	0.531986	0.531715	0.532695	0.532473	0.532826
	2	Exp+	1.224637	1.224762	1.224437	1.224405	1.224364
		Erl+	1.229577	1.229621	1.229519	1.229496	1.229492
		HExp+	1.206238	1.20638	1.206049	1.205973	1.20589
		MNC+	1.217753	1.217855	1.217536	1.217561	1.217473
		MPC+	1.217686	1.217745	1.217535	1.217574	1.217493
	3	Exp+	1.921137	1.921223	1.921033	1.920982	1.920963
		Erl+	1.925554	1.925566	1.925544	1.925531	1.925529
		HExp+	1.903387	1.903607	1.903072	1.902977	1.902891
		MNC+	1.915155	1.915498	1.914587	1.914528	1.914341
		MPC+	1.919385	1.919785	1.918482	1.918645	1.918215
	4	Exp+	2.482824	2.482866	2.482781	2.482746	2.482733
		Erl+	2.465847	2.46586	2.465837	2.465823	2.465819
		HExp+	2.493508	2.493637	2.493339	2.493258	2.493202
		MNC+	2.512287	2.512609	2.511801	2.511706	2.511556
		MPC+	2.523356	2.52402	2.521941	2.522135	2.521492
6	1	Exp+	0.536892	0.536584	0.537477	0.537445	0.537666
		Erl+	0.539619	0.539398	0.53999	0.540016	0.540112
		HExp+	0.528617	0.528223	0.529684	0.529319	0.529791
		MNC+	0.53257	0.532138	0.533431	0.53334	0.533779
		MPC+	0.532271	0.532	0.532974	0.532767	0.533119
	2	Exp+	1.224534	1.22466	1.224343	1.224298	1.224258
		Erl+	1.229549	1.229591	1.229497	1.229469	1.229465
		HExp+	1.205957	1.206118	1.205723	1.205651	1.205555
		MNC+	1.217452	1.217577	1.217194	1.217216	1.217113
		MPC+	1.217105	1.217191	1.216888	1.216944	1.216831
	3	Exp+	1.921083	1.921164	1.92099	1.920933	1.920914
		Erl+	1.925543	1.925555	1.925533	1.925519	1.925517
		HExp+	1.903191	1.903405	1.902903	1.902786	1.902703
		MNC+	1.914868	1.915202	1.914352	1.914254	1.914075
		MPC+	1.918645	1.919046	1.917789	1.917896	1.91747
	4	Exp+	2.482785	2.482827	2.482745	2.482706	2.482692
		Erl+	2.465834	2.465849	2.465823	2.465808	2.465804
		HExp+	2.493361	2.493484	2.493213	2.493118	2.493063
		MNC+	2.51212	2.512422	2.511698	2.511571	2.51143
		MPC+	2.522815	2.523444	2.521571	2.521654	2.521053
7	1	Exp+	0.537077	0.536773	0.537638	0.537631	0.537845
		Erl+	0.539752	0.539529	0.540111	0.540156	0.540251
		HExp+	0.528691	0.52831	0.52971	0.529378	0.529832
		MNC+	0.53285	0.532424	0.53368	0.53362	0.534047
		MPC+	0.532469	0.5322	0.533154	0.532966	0.533312
	2	Exp+	1.224445	1.224571	1.224265	1.22421	1.22417
		Erl+	1.229525	1.229566	1.229478	1.229447	1.229443
		HExp+	1.205715	1.205889	1.205454	1.205379	1.205275
		MNC+	1.217187	1.21733	1.216905	1.216919	1.216806
		MPC+	1.2166	1.216706	1.216338	1.2164	1.216264
	3	Exp+	1.921039	1.921117	1.920955	1.920894	1.920876
		Erl+	1.925534	1.925547	1.925524	1.92551	1.925507
		HExp+	1.90303	1.903237	1.902767	1.902634	1.902553
		MNC+	1.91464	1.914961	1.914169	1.914042	1.913872
		MPC+	1.918059	1.918456	1.917255	1.917314	1.916896
	4	Exp+	2.482754	2.482795	2.482715	2.482675	2.48266
		Erl+	2.465824	2.465839	2.465812	2.465797	2.465793
		HExp+	2.49324	2.493358	2.493109	2.493004	2.49295
		MNC+	2.511999	2.512281	2.511629	2.511482	2.511349
		MPC+	2.522457	2.523047	2.521357	2.521358	2.520797

Table 2.19: Mean number of Idle Servers

S = 25, s = 8, λ_1 = 5, λ_{-1} = 3, β = 6, μ = 10, θ = 5.

γ	c		Exp-	Erl-	HExp-	MNC-	MPC-
0.2	1	Exp+	0.558737	0.558461	0.559255	0.559223	0.559421
		Erl+	0.56131	0.561123	0.561617	0.561636	0.561719
		HExp+	0.549917	0.549516	0.551027	0.550625	0.551107
		MNC+	0.554372	0.553947	0.555198	0.555111	0.555539
		MPC+	0.554059	0.553793	0.55474	0.55454	0.554882
	2	Exp+	1.280978	1.281161	1.280685	1.280642	1.280582
		Erl+	1.285306	1.285369	1.285223	1.28519	1.285184
		HExp+	1.263517	1.263745	1.263138	1.263095	1.262952
		MNC+	1.275332	1.275541	1.274883	1.274943	1.274772
		MPC+	1.275562	1.275703	1.275178	1.2753	1.275114
	3	Exp+	2.014689	2.014793	2.014559	2.014502	2.01448
		Erl+	2.018676	2.018688	2.018664	2.018654	2.018652
		HExp+	1.998994	1.999268	1.99859	1.998487	1.998379
		MNC+	2.010126	2.01056	2.009397	2.009332	2.009097
		MPC+	2.015747	2.016262	2.014554	2.014795	2.014238
	4	Exp+	2.616494	2.616538	2.616446	2.616414	2.616401
		Erl+	2.599141	2.599151	2.599132	2.599122	2.599119
		HExp+	2.630792	2.630945	2.630587	2.630502	2.630437
		MNC+	2.647079	2.647449	2.646514	2.646412	2.646241
		MPC+	2.66141	2.662193	2.659721	2.659975	2.659216
0.25	1	Exp+	0.547049	0.546751	0.547622	0.547574	0.54779
		Erl+	0.549781	0.549575	0.550133	0.550145	0.550237
		HExp+	0.538444	0.538035	0.539573	0.539164	0.539657
		MNC+	0.542568	0.542133	0.543435	0.543327	0.543771
		MPC+	0.542307	0.542035	0.543013	0.542798	0.54315
	2	Exp+	1.251672	1.251823	1.251431	1.251394	1.251345
		Erl+	1.256368	1.25642	1.256299	1.256272	1.256266
		HExp+	1.233369	1.233549	1.233097	1.233034	1.232924
		MNC+	1.245277	1.245426	1.244958	1.245	1.244876
		MPC+	1.245294	1.245388	1.245042	1.245118	1.244991
	3	Exp+	1.966311	1.966405	1.966194	1.96614	1.96612
		Erl+	1.970544	1.970556	1.970533	1.970521	1.970519
		HExp+	1.94918	1.949426	1.948824	1.948724	1.948628
		MNC+	1.96094	1.961326	1.960297	1.960236	1.960026
		MPC+	1.965771	1.966224	1.964733	1.964932	1.964443
	4	Exp+	2.547198	2.547241	2.547153	2.547119	2.547106
		Erl+	2.529999	2.530011	2.529989	2.529978	2.529974
		HExp+	2.559356	2.559496	2.559169	2.559087	2.559026
		MNC+	2.577246	2.577592	2.576723	2.576625	2.576464
		MPC+	2.589827	2.590548	2.588282	2.588504	2.587806
0.3	1	Exp+	0.536663	0.536354	0.537266	0.537207	0.537433
		Erl+	0.53946	0.539244	0.539841	0.539845	0.539943
		HExp+	0.5285	0.528096	0.52961	0.529211	0.529699
		MNC+	0.532198	0.531764	0.533077	0.532955	0.533401
		MPC+	0.531986	0.531715	0.532695	0.532473	0.532826
	2	Exp+	1.224637	1.224762	1.224437	1.224405	1.224364
		Erl+	1.229577	1.229621	1.229519	1.229496	1.229492
		HExp+	1.206238	1.20638	1.206049	1.205973	1.20589
		MNC+	1.217753	1.217855	1.217536	1.217561	1.217473
		MPC+	1.217686	1.217745	1.217535	1.217574	1.217493
	3	Exp+	1.921137	1.921223	1.921033	1.920982	1.920963
		Erl+	1.925554	1.925566	1.925544	1.925531	1.925529
		HExp+	1.903387	1.903607	1.903072	1.902977	1.902891
		MNC+	1.915155	1.915498	1.914587	1.914528	1.914341
		MPC+	1.919385	1.919785	1.918482	1.918645	1.918215
	4	Exp+	2.482824	2.482866	2.482781	2.482746	2.482733
		Erl+	2.465847	2.46586	2.465837	2.465823	2.465819
		HExp+	2.493508	2.493637	2.493339	2.493258	2.493202
		MNC+	2.512287	2.512609	2.511801	2.511706	2.511556
		MPC+	2.523356	2.52402	2.521941	2.522135	2.521492