MODELLING AND ESTIMATION STUDY OF COMPLEX RELIABILITY SYSTEMS

By

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To my husband
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SUMMARY

To improve the reliability of a system, the following two well-known methods are used:

1. Provision of redundant units, and

2. Repair maintenance

In a redundant system more units are made available for performing the system function when fewer are required actually. The provision of redundant units could be performed mainly in three ways, namely, series, parallel and standby. This thesis deals with these three types.

Following are some classical assumptions that are made in the analysis of redundant systems.

1. The life time and the repair time distributions are assumed to be exponential.

2. The repair rate is assumed to be constant.

3. There is a single repair facility.

4. The repair facility will continuously available.

5. The system under consideration is needed all the time.

6. The lifetime or repair time of the units are assumed to be independent.

7. Usage of only conventional methods for the analysis of the estimated reliability systems.

8. Switch is perfect in the sense that the switching device does not fail.

9. The switchover time required to transfer a unit from the standby state to the online state is negligible.

10. There is no human error when we handle the machines and no common cause of failures.

11. The repair rate is independent of the number of failed units.

We frequently come across systems where one or more of these assumptions have been dropped.

This is the motivation of the detailed study of the models presented in this thesis. We present several models of redundant repairable systems relaxing one or more assumptions (1-11) simultaneously. More specifically, it is a study of stochastic models of redundant repairable systems with a single repair facility.

The estimation study of the system measures is focused in some chapters. Imperfect switch, non-instantaneous switchover, variating repair rate and common cause of failure with human errors, etc. are some of the aspects focused in the thesis.

Chapter 1 is essentially an introductory in nature and contains a brief description of the mathematical techniques used in the analysis of redundant systems.

In **Chapter 2** assumptions (1), (2) and (4) are relaxed. Here we deal with an n - unit warm standby system with varying repair rate. We first consider a model in which the repair rate of a failed unit is constant depending on the number of failed units at the epoch of commencement of each repair and the vacation period is introduced after each repair completion. Introducing a profit function, the optimal number of standby units is also determined. A special case is obtained by suspending the vacation period.

In **Chapter 3**, we have relaxed an assumption (6). A three unit warm standby system with dependent structure, wherein the lifetimes of online unit, standby units and the repair time of failed units are governed by quadrivariate exponential law is studied. Measures of system performance such as, reliability, MTSF, availability and steady state availability are also obtained. A $100(1-\alpha)\%$ confidence interval for the steady state availability of the system and an estimator of system reliability based on moments are obtained. Numerical work is carried out to illustrate the behaviour of the system reliability based on moments by simulating samples from quadrivariate exponential distribution. Generalization of the above results to a $n(\geq 4)$ unit warm standby system with $r(\geq 2)$ repair facilities is investigated.

In **Chapter 4**, a slight modification of an assumption (4) is studied. This chapter deals with the study of three unit system where unit 1 is connected in series and the other two units are connected in parallel. The significant feature of this chapter is modification of an assumption (4) by assuming the repair facility gives priority to the repair of the unit 1 in the sense that whenever the unit 1 fails in the operable state, and at that instant if there is already unit 2 or unit 3 under repair, the repair of unit 1 starts immediately keeping the unit under repair in queue, and the repair of which is taken afresh immediately after the repair of unit 1 is completed.

In **Chapter 5**, a two unit cold standby system with constant failure rate and two stage Erlangian repair is studied. Measures of system performance such as reliability, MTSF, availability and steady state availability are obtained. Furthermore confidence limits for the steady availability of the system, ML estimator of system reliability and Bayes estimator of MTSF are derived. Numerical illustration is carried out to study the performance of the Bayes estimator of MTSF.

A three unit series-parallel system with preparation time is studied in **Chapter 6**. Unit 1 is given a priority over unit 2 and 3 for repair as it is connected to a series system. The expressions for system measures like availability and reliability are obtained.

In **Chapter 7,** two unit warm standby system with imperfect switch and preparation time is studied. The switching device will have a head-of-line priority over the units for repair. Assuming various arbitrary distributions for some of the random variables involved, MTSF and A_∞ are obtained.

CHAPTER 1

Introduction

1.1 INTRODUCTION

Some of the most important characteristics of a reliable person are consistency, loyalty, dependability, honesty, truthfulness and trustworthiness. Assuming a world without these significant human attributes is itself a peculiar supposition. This is the reason that industry, commerce and generally society wish to associate most with the reliable persons. Humanity desires for things which are consistent and predictable. Much as human beings treasure reliability in human behaviour, it is hard to define its characteristics or to be able to measure it with precision. In practice, there is no hard and fast rule to judge a person who is reliable and the one who happens not to be. However, a judgment can be made whether an individual is reliable or not on the basis of definite human functions. For example the reliability of an individual working with an organization may be assessed on the basis of punctuality of arriving at work or participation in the activities of the organization.

Reliability in a general sense may be regarded as a measure of performance. Persons who are able to finish their work on time are described as reliable. Those people who keep time i.e., they are at the right place at the right time may be considered reliable as well as they fulfil their commitments. Reliability of human beings therefore depends on time at which or in which they perform any particular task and hence may be taken as an important measure.

Reliability does not only apply to man's activities but also to tools and machinery he uses. We have seen that reliability has been applied to man's actions but when it comes to the objects he has made or invented, expectations of reliability are even higher. This is because it does not only frustrate his/her feelings but wastes time, money and endangers life. According to Green and Bourne (1978), the consequences of unreliability have led to man's greater interests in reliability and more desire to acquire or use more reliable products.

Advancements in information, communications technology and military systems have made systems even more complicated. These complications and intricacies have attracted the attention of number of researchers and scientists from various disciplines especially the system engineers, software engineers and applied probabilists. These developments have resulted in the emergence of reliability theory another scientific discipline dealing with methods and techniques to ensure maximum effectiveness of systems (from known qualities of their components). This theory has now become one of the important areas in operational research and system engineering.

Gnedenko et al (1969) pointed out that reliability theory assigns quantitative indices to qualities of production which are computed from the design stage through manufacturing process to use and storage of manufactured goods and operating systems. Increased reliability of manufactured goods and operating systems is a challenge to governments, engineers and scientists. According to Lloyd and Lopow (1962) unreliability costs money, time wasted and inconveniences the users, in some cases may jeopardize personal and national security. The year 1963 saw the birth of the journal on reliability known as IEEE-Transactions on Reliability.

Mathematical models help the system designers who are faced with the problems of evaluation of several measures of system performance, methods of improving them and determination of optimum preventive maintenance schedule. These models explain the various operational and theoretical features of the system under consideration taking into account its essential features. Since unavailability and breakdowns of a system are becoming more and more unacceptable, the demand for systems that perform better but cost less is on the increase. It is common knowledge that repairing failed units and providing redundancy are two important methods of improving the performance of a system.

Reliability theory is multidisciplinary in nature since problem handling requires methods of probability theory and mathematical statistics such as in information theory, queuing theory, linear and nonlinear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, etc. Reliability theory has been applied in contemporary medical science, software systems, geo astronomy, irregular interactions of physiological systems, spontaneous single neon discharge, phase dependence of population growth, fluctuation in business investments, etc. In addition, mathematical models relying on probability theory and stochastic processes are used in making realistic modelling for mobility of individuals and industrial labour, advancement in education and diffusion of information. According to Watson and Galton (1874) biological sciences stochastic models were first introduced in the study of extinction of families. This was followed by its application in population genetics, branching process, birth and death processes, recovery, relapse, cell survival irradiation, the flow of particles through organs, etc. These analytical models have been used in the purchasing behaviour of the individual consumer, credit risk and term structures, income determination etc. The traffic flow studies have also used the theory of stochastic models for traffic of pedestrians, freeways, parking lots, intersections, etc.

Various problems have emerged in the design of highly reliable technical systems which include: the creation classes of probability-statistical models which may be used in description of the reliability behaviour of the system, and the development mathematical methods for the assessment of the reliability characteristics of system.

These problems encouraged the development of high-accuracy methods of reliability analysis. Gnedenko et al (1969), Barlow (1984), Gertsbakh (1989), Kovalenko et al (1997) and El- Sherbeny (2010) considered redundant systems and the classical examples are the methods of Markov processes with finite sets of states such as birth and death processes. Cox (1962) studied renewal process method and Cinler (1975) studied semi-Markov process method and its generalizations, Rubinstein (1981) generalized semi markov process (GSMP) method while Aven (1996) looked at special models for coherent systems. Ozekici (1996), Finkelstein (1999 a,b,c) and Chandrasekhar et al (2005) studied systems with random environment.

In many areas of research, a suitable form of reliability may be introduced. Stochastic analysis is based on good probability models with ultimate aim of giving numerical estimates of reliability characteristics. Reliability theory also offers solutions to a number of problems not handled by the usual standard probability theoretical approach. According to Gertsbakh (1989) reliability of a system depends on the reliability of its components, provides a mathematical expression of aging process, offers well-developed method of renewal theory, introduces redundant systems to optimize the performance of standby components. Gnedenko et al (1969), provides the theory of optimal preventive maintenance and is also a study of inferential statistics often of censored data.

Reliability theory of technical objects and survival analysis of biological entities are similar with the exceptions of notations. Therefore the term "lifetime" is applicable to engineering systems, components, units etc., and to the disciplines like biological, financial etc., with minor modifications.

1.2 FAILURE

Failure is one of the basic and useful concepts of reliability theory. The ability of a unit not to have failures throughout specified period of time is called failure free operation. According to Gertsbakh (1989), failure is a result of joint action of many unpredictable, random processes going on inside the operating system as well as in the environment in which the system itself is operating. Failure is stochastic in nature and its operation gets seriously impeded or completely stopped at a certain point in time. Determination of failure may be easily detected in some cases just through observation known as well-defined failure but in others it is very difficult since these units deteriorate continuously and the actual moment of failure is not as easy to determine (known as partial or relative failure). A typical example of a component having well defined failure is an electric bulb which has two states: either working or not working at all. However in some cases, the concept of failure is extremely relative. Failure of electronic component such as resistor is an example of a partial failure. We assume that failure is exactly observable in this thesis and failure is known as a disappointment or a death. When a system fails it enters a down state which may also be called a system breakdown (Finkelstein (1999a)). According to Zacks (1992): data is of two types: from continuously monitored units for failure and from observation of failure made at discrete points in time.

Villemeur (1992) cited a number of possible failures and their causes, which fall into two categories: random individual independent failures and interdependent failures. Failures are either catastrophic or drift depending on whether their parameters fall sharply or gradually as a result of wear and fatigue.

1.3 REPAIRABLE SYSTEMS

In case of failure of a unit, it is renewed. The renewal can assume various forms: it can be replaced with a new unit that is identical to it or it can be subjected to maintenance or repair that completely restores all its original properties. Although the replacement of a failed units of a system with new ones is a good option, repair is always more feasible because of the cost involved in buying new ones. Some systems are repairable while others are not.

Repairable system is the system which may be made operable by a repair facility once it is in a down state as a result of a failure. A renewed system has its service time increased as a result of its reliability increased. In case the repair facility is not free then the failed units queue up for repair.

In this thesis the lifetime of a unit while on line, standby or under repair are considered both independent and dependent variables. We assume that the distributions of these random variables are known with probability density functions. Investigations of repairable systems have been there for ages.

The random variables considered in these investigations are as below:

1. Availability and reliability
2. Time necessary for repair
3. Repair (numbers) that can be handled
4. Switchover time for the repair facility
5. Possibility of a vacation time for repair facility

The repair facility problems have much in common with queuing problems (Barlow (1962)). The problems of locating an optimum value of m out of n: G system of maximum reliability was conducted by Rau (1964). Ascher (1968) mentioned some inconsistencies in modelling of repairable system of renewal theory.

Buzacott (1970), Shooman (1968), Barlow and Porschan (1965, 1975), Sandler (1963), and Doyan and Berssenbrugge (1968) and many others used continuous time discrete state Markov process model for modelling the behaviour of repairable system.

Despite the simplicity of these systems conceptually, their practicability in large number of states is not feasible. A semi-Markov process was used for computation of reliability of a system with exponential failures by Gaver (1963), Gnedenko et al (1969), Srinivasan (1966), and Osaki (1970a). Osaki (1969) used signal flow graphs to analyse a two unit system while Kumagi (1971) applied a semi Markov process to determine the impact of different failures distribution on availability through numerical computation.

A semi- Markov process was used by Branson and Shah (1971) to study a repairable system with arbitrary distributions. Srinivasan and Subramanian (1980), Venkatakrishnan (1975), Ravichandran (1979), Natarajan (1980) and Sarma (1982) applied regeneration point techniques to study repairable systems using arbitrary distributions.

A number of papers have been written in this field and related topics as seen in Subba Rao and Natarajan(1970), Osaki and Nakagawa (1976), Pierskalla and Voelker (1976), Lie et al (1977), Kumar and Agarwal (1980), Birolini (1985), Yearout et al (1986) and Finkelstein (1993a, 1993b). In order to improve the efficiency of the system, Jain and Jain (1994) introduced the regulation of up and down times of repairable systems.

1.4 REDUNDANCY AND DIFFERENT TYPES OF REDUNDANT SYSTEMS

Redundancy is one of the basic methods to increase system reliability. This is introduced in a system by building into it more units than is actually required for the system to perform properly. There are two types of redundancy namely parallel and standby redundancy. Parallel redundancy is when the units form part of the system from the start while in a series redundancy a standby system does not form part of the system until when it is required.

1.4.1 PARALLEL SYSTEMS

A parallel redundant system is defined as one with n-units which are all functioning concurrently, despite the fact that system operation needs at least one unit to be in operation. That is, the system works as long as even a single component is still alive. In this case system failure occurs only when all the components have failed.

Let k be a non-negative integer such that $k \leq n$, counting the number of units in n-unit system. This system is normally referred to as a *k*-out-of-*n* system.

***K*-out-of-*n*: F-System**

A *k*-out-of-*n*: F system is a redundant system composed of n-units and the system fails if *k* units fail in a *k*-out-of-*n* system. Sfakianakis and Papastavridis (1993) pointed out that the functioning of a minimum number of units ensures that the system is operating and Chao et al (1995) surveyed such systems.

***K*-out-of-*n*: G-System**

A *k*-out-of-*n*: G system is a redundant system composed of *n*-units and the system functions if and only if at least *k* units out of the n-units of the system are operational. Zhang and Lam (1998) and Liu (1998) have recently studied such systems, for example a radar network system has n radar control stations. Covering a certain area in which the system can be operable if and only if at least k of these stations are operable in this case a minimum number of units, k is essential for the functioning of the system.

Attention has shifted to load - sharing of *k*-out-of-*n*: G systems of late, where serving units share the load and the failure rate of components is affected by the magnitude of the load it shares.

***n*-out-of-*n*: G-System**

An *n*-out-of-*n* G-system is basically a series system that consists of n-units and failure of any one unit causes the system to fail. This type of system is not really redundant since all the units are in series and have to be operational for the system to operate however; it is still called a special case of a *k*-out-of-*n* system.

Scheuer (1988) looked at reliability of shared-load in *k*-out-of-*n*: G systems and pointed out that there is an increasing failure rate in survivors, assumed identically distributed components with constant failure rates. Shao and Lamberson (1991) introduced imperfect switching to the same case. A paper by Liu (1998) considered the influence of work-load sharing in non-identical, non-repairable components, each having an arbitrary failure time distribution. His assumptions were that failure time distribution of the components may be represented by an accelerated failure time model, which happens to be a proportional hazards model when Weibull base-line reliability is used.

1.4.2 STANDBY REDUNDANCY

Standby redundancy is one of the basic methods of increasing reliability. Standby redundancy consists of an attachment to an operating unit one or more redundant (standby) units, which, when failure occur, take the place of basic operating unit and fulfil its function. These units may be classified as cold, warm or hot (Gnedenko et al (1969)).

1. A cold standby is not hooked up hence completely inactive, it cannot in (theory) fail until it is put to use by replacing a primary unit. Assume that since it is not in operation its reliability will not change when it is put into operation.

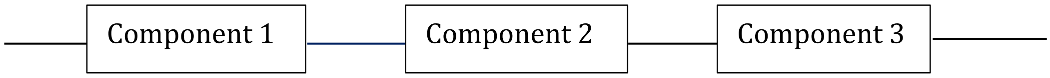
2. A warm standby is when a unit is partially energized hence has a diminished load. The on-line unit and standby unit are not subject to the same loading conditions. The failure of standby unit is attributable to some extraneous random influence. The probability of failure of the warm standby unit is smaller than the probability of failure of the on-line unit. This is the most general type of standby due to the high failure rate of the hot standby unit and possible lapse before it is operable in the case of the cold standby unit.

3. A hot standby is fully energized and active in the system although redundant and the possibility of failure of a hot standby is the same as that of an operating unit in the standby state. A hot standby's reliability is independent of the instant at which it takes place in the operable unit.

1.4.3 SERIES SYSTEM

A series system is an arrangement of components such that the failure of any of the system component results the entire system failure. That is, a system in which all the components have to work for the system to work. A simple computer consists of a processer, a bus and a memory, chains made out of links; highways that maybe closed to traffic due to accidents at different locations, the food chains of certain animal species and layered company organizations in which information is passed from one hierarchical level to the next are some examples of a system in series.

Following is a graphical description of a series system:



In a series system of n component the success of the system is equivalent to the success of every individual component and hence series system's reliability decreases as the number of component increases. The reliability of series system is easily calculated from the reliability of its components. If there are n components in series where the reliability of i-th component is R_i the system reliability is $R_s = R_1 \times R_2 \times \ldots \times R_n$.

The implications of the above equation are that the combined availability of n components in series is always lower than the availability of its individual components. As a matter of fact a series system is one that is as weak as its weakest link.

1.5 MEASURES OF SYSTEM PERFORMANCE

The aim of this section is to discuss some important measures of system performance as applicable in different situations (Barlow and Proschan (1965), Gnedenko et al (1969)).

1.5.1 RELIABILITY

The study of reliability has attracted the attention of engineers, mathematicians, economists and industrial managers over the past decades mainly because of the development of the high risk and complex system (Tijms (1988), Beichelt and Fatti (2002)). Reliability is a kind of quantitative measure of operational efficiency. The reliability of a product is therefore a measure of its ability to perform its functions expected, when it is required, for a specific time, in a particular environment. It is measured in terms of probability and comprises of four parts, namely**:**

1. System expected function
2. System operating environment (climate, packaging, transportation, storage, installation, pollution)
3. Time which is often negatively correlated with reliability
4. Probability, which is time dependent

There are two types of reliability namely**:**

1. Mission reliability is when a device is made for the performance of one mission only.
2. Operational reliability is when a system is turned on and of intermittently for the purpose of performing a certain specified function.

The latter case is known as an intermittently used system (Kapur and Kapoor (1978)), Kapoor and Kapur (1980). Ordinarily the period of time intended for use in (0, t] is called a need period.

Let $\{\psi(t) : t \geq 0\}$ be the performance process of the system.

For fixed t, $\psi(t)$ is a binary random variable which takes on the value 0 if the system operates satisfactorily at time t and takes value 1 otherwise.

Reliability $R(t)$ is then given as

$$R(t) = P\,\{system\ is\ up\ in\ (0,t]\}$$

$$= P\,\{\psi(u) = 0\ for\ all\ u \in (0,t]\}.$$

The performance measure for the interval of reliability in case the number of system failures in the interval (t, t +x] is considered as

$$R\ (t,x)\ =\ P\ \{\psi(u)\ =\ 0\ for\ all\ u\ \in\ (t,t\ +x]\}.$$

When $t = 0$, the interval reliability becomes the reliability $R(x)$. The limiting interval reliability is the limit of $R\ (t,x)$ as $t \to \infty$ and it is indicated as $R_\infty(x)$.

The mean time to system failure (MTSF) is the expectation of random variable $\psi\ (t)$. It represents the duration of the time measured from the point the systems commences operation until the instant when it fails for the first time and it can be computed from $R(t)$ as given below

$$MTSF\ = \int_0^t R(u)du.$$

1.5.2 AVAILABILITY

Availability is also a measure of system performance, which is the probability that the system will be operational (in operable condition or available for use) at the given time t. It implies that the system is either in active operation or is able to operate if required and consists of aspects of reliability, maintainability and maintenance support.

Availability is applicable only to systems which undergo repair and are restored after failure. In theory availability $A(t)$ should be 100 % but in practice, even equipment coming directly out of storage may be defective. Availability is very important and high availability may be obtained either by increasing the average operational time until the next failure, or by improving maintainability of the system.

Pointwise availability is a point function which describes the probability that a system will be able to operate at a given instant of time (Gnedenko and Ushakov (1995)). Klaassen and Van Peppen (1989), Beasley (1991).

In symbols

$$A\,(t) \;=\; P\{\psi(t) \;=\; 0\,\}.$$

According to Barlow and Proschan (1965) steady state or asymptotic availability is a limiting availability A_∞ and it is defined as expected fraction of time that the system operates satisfactorily in the long run.

$$A_\infty = \lim_{t\to\infty} A(t).$$

The joint availability is the probability that the system is operating at t and t +τ, that is, we have

$$A\,(t,\tau) \;=\; P\,\{\psi(t) \;=\; 0, \psi\,(t+\tau) \;=\; 0\}.$$

Just as reliability and interval reliability are related availability and joint availability satisfies the following relation

$$A\,(t) \;=\; A\,(0,t).$$

The expected number of visits by the repair facility is a widely used concept in queuing theory of the server taking vacations and a lot of research has been done on server vacation models (see, for example Doshi (1986), Kella (1989)). The server takes vacations according to some specified assumptions, whenever the busy period of service station terminates. We assume that the cost structure whenever the server starts its busy cycle. We consider the idea of server vacation in reliability modeling and compute the expected number of visits by the repair facility in the arbitrary interval of time by supposing that repair facility takes vacation whenever the repair facility becomes free and that it returns back only at the epoch of the next failure.

In addition to estimating some of the above measures, a few other interesting, important and useful performance measures characteristics to each model are also derived in this thesis.

1.6 COST FUNCTION

There are a number of constraints facing the designer of a system. Some consideration has to be made system's reliability and availability, its usefulness and effectiveness. Due to the complexity of the present-day systems, measures such as reliability, availability etc. alone are not sufficient. In addition, cost and profit have become the guiding principles in every industrial and social management endeavor. Hence cost optimization has become one important criterion for system designers.

We shall focus, in this thesis to the construction of comprehensive cost function for each of the models considered. Since they are highly nonlinear, analytical optimization of these functions become impracticable, if not impossible. Hence we resort to numerical optimization; assuming that the control parameters are within certain specific intervals, we obtain numerically their optimal values.

1.6.1 MEAN NUMBER OF EVENTS IN $(0, t]$

Let $N(a,t)$ denotes the number of a particular type of an event such as a disappointment, system recovery, system down, etc., in $(0, t]$. The mean number of events in (0, t] is shown below

$$E[N(a,t)] = \int_0^t h_1(u)du,$$

where h_1 (u) is the first order product density of the events (defined in section 1.7.4). The mean stationary rate of occurrence of these events is

$$E[N(a)] = \lim_{t\to\infty} \frac{E[N(a,t)]}{t}.$$

1.6.2 CONFIDENCE LIMITS FOR THE STEADY STATE AVAILABILITY

A 100 $(1-\alpha)$ % confidence interval for A_∞ is stated as

$$P\ [a\ <\ A_\infty\ <\ b]\ =\ 1-\ \alpha\ .$$

Appropriate statistical tables are used to determine the numbers a and b $(a\ <\ b)$.

A_∞ is a function of parameters of operating time distribution, repair time, need and no need period distributions (Yadavalli et al (2002, a, b), Yadavalli et al (2001), Yadavalli et al (2005)).

1.7 STOCHASTIC PROCESSES USED IN THE ANALYSIS OF REDUNDANT SYSTEMS

In previous section, different types of redundant system and the various measures of system performance were studied. The purpose of this section is to discuss some techniques used in the analysis of redundant repairable systems.

1.7.1 RENEWAL THEORY

In renewal theory we are interested in the lifetime of the unit, there exists times commonly random from which onward the future of the process is probabilistic replica of the original process. At the beginning $t = 0$ a repairable unit is put into operation and functioning. The unit is replaced by a new one of the same type and subjected to maintenance that completely restores it to an as good as new condition upon failure. This process is repeated upon failure and a replacement time is considered negligible. These results in a sequence of life times and this study is restricted to these renewal points. The number of renewals N_t upto some time t is the probability object in these sums of non-negative i.i.d. random variables.

A number of researchers have studied specific reliability problems using renewal processes. The homogeneous Poisson process has received considerable attention and happens to be the simplest renewal process. The time parameter may be taken as either discrete or continuous. A proper lead for the discrete case was conducted by Feller (1950) followed by a very lucid account of Cox (1962) for the continuous case (he provided an introduction to renewal theory in the case of a repair facility not being available and failed units queuing up for repair).

Barlow (1962) applied in his research on repairable systems queuing theory. Some operating characteristics of a one unit system were studied by Srinivasan (1971) while Gnedenko et al (1969) worked out the mean time to system failure of a two unit standby system. Some priority redundant systems were studied by Buzacott (1971).

In renewal systems the system starts a new cycle after each renewal which is independent of the previous ones despite its possibility of taking on different forms. In case of repair time which is both random variables with individual distributions repair time may be considered as a fixed time) this process is known as**:**

1. An ordinary renewal process if the time origin is the initial installation of the system and the repair time is taken as negligibly small in comparison with the life time of the unit renewal is taken as instantaneous or
2. A general renewal process if the time origin is some point after the initial installation of the system (Cox (1962)). Hoyland and Rausand (1994) named this a modified renewal process, while Feller (1957) calls this process considering the residual life time of a system at an arbitrary chosen time origin as a delayed renewal process.

(a) Ordinary renewal process: Instantaneous renewal

This is when a basic model of continuous operation is considered whose unit begins operating at instant $t = 0$ and stays operational for a random time T_1 and then fails. At this instant the unit is replaced by a new and statistically identical unit which operates for a length of time T_2 then fails and is again replaced etc. These random components life lengths $T_1, T_2, T_3 \ldots T_r$ ... of the identical units are independent, nonnegative and identically distributed that constitute ordinary renewal process.

Let

$$P[T_i \leq t] = F(t);\ t > 0, \qquad i = 1,2, \ldots$$

be considered as an underlying distribution of the renewal process. The time taken until the r^{th}- renewal is given by

$$t_r = T_1 + T_2 + \ldots + T_r = \sum_{i=1}^{r} T_i.$$

Let $N(t)$ be a random variable where $N(t) = max\{r: R_r \leq t\}$ which denotes the number of time the renewal takes place in the interval $(0, t]$, then the number of renewal in an arbitrary time interval $(t_1, t_2]$ is equal to

$$N\,(t_2) - N\,(t_1).$$

A renewal function $H\,(t)$ which is the expected value of $N\,(t)$ in the time interval $(0, t]$ can now be defined as

$$\begin{aligned} H\,(t) &= E[N(t)] \\ &= \sum_{r=1}^{\infty} F^{(r)}\,(t) \\ &= F\,(t) + \int_0^t H\,(t-x)\, d(F(x), \end{aligned}$$

where $F^{(r)}(.)$ is the r- fold convolution of F

The renewal density function is

$$h\,(t) = \sum_{n=1}^{\infty} f^{r}(t)$$

and these renewal density function h(t) satisfies the equation

$$h\,(t) = f(t) + \int_0^t h\,(t-x) f(x) dx.$$

It indicates that the renewal density $h(t)$ basically differs from the hazard rate $h^0(t)$ as

$$h^0(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)}.$$

(b) Random renewal time

In case the time for renewal is not instantaneous but it is taken as a random variable that is included in the subsequent time period, or cycles, of the system performance, each cycle will then comprise of a time for the failure or the time for repair.

The failure or repair time will both be stochastic in nature. The instant of failure and cycles of renewal can be determined.

Let $F(t)$ be the life time distribution and $G(x)$ be the repair time function with respective probability density function $f(t)$ and $g(x)$. Therefore the density function of a cycles C of the life time, say $k(t)$ is estimated using the convolution formula

$$k(t) = \int_0^t f(x)g(t-x)dx.$$

Let $N_F(t)$ count the number of failures and $N_R(t)$ the number of repairs in the interval $(0, t]$, define

$$W(t) = E\left(N_F(t)\right),\ V(t) = E(N_R(t)) \text{ and let}$$

$$Q(t) \ = W(t) - V(t), \qquad \text{for all} \quad t,$$

assuming that $w(t) = W'(t)$ and $v(t) = V'(t)$.

The failure and repair intensities can then be respectively be defined as $\lambda(t) = \frac{w(t)}{A(t)}$), where $A(t)$ is availability function, and

$$\mu(t) = \frac{v(t)}{Q(t)}, \qquad \text{where} \quad Q(t) \neq 0.$$

(c) Alternating renewal processes

Takacs (1957) was the first to study in details alternating renewal processes and then many text books have discussed it further (Ross (1970)). A generalization of the ordinary renewal process discussed previously follows where the state of the unit is given by the binary variable

$$X(t) = \begin{cases} 0, & if\ the\ unit\ is\ functioning\ at\ time\ t, \\ 1, & otherwise. \end{cases}$$

The two alternating states may be taken as 'system up' and 'system down'. If these alternating independent renewal processes are distributed according to F(x) and G(x), there are two renewal processes embedded in them for the different transitions from 'system up' to 'system down'. Usually one-item repairable structure are considered as alternating renewal processes under the assumption that after each repair the item is as good as new.

(d) Age and Remaining life time of a unit

Let t_r indicates the random component life time, that is,

$$t_r = T_1 + T_2 + ... + T_r = \sum_{i=1}^{r} T_i.$$

Let $R_r, r \in N$, represent the length of the r^{th}- repair time, then the sequence T_1, R_1,T_2, R_2, … form an alternating renewal process. Define

$$t_n = T_1 + \sum_{r=1}^{n-1}(R_r + T_{r+1});\ n \epsilon N \quad \text{and}$$

$$t_n^0 = \sum_{r=1}^{n}(R_r + T_r) \text{ and set } t_0 = t_0^0 = 0.$$

This sequence t_n generates a delayed renewal process.

If $B_1(t)$ denotes the forward recurrence time at time t, then

$$B_1(t) = t_{N_{t+1}} - t, \quad \text{or}$$

$$B_1(t) = t_{N_t^0+1} - t.$$

Hence

- $B_1(t)$ equals the time to the next failure time if the system is up at time t, or
- $B_1(t)$ equals the time to complete the repair if the system is down at time t.

Also,

- $B_2(t)$ equals the age of the unit if the system is up at time t, or
- $B_2(t)$ equals the duration of the repair if the system is down at time t.

Feller (1941) defined the elementary renewal theorem as an ordinary renewal process with underlying exponential distribution (parameter λ and $H\ (t)\ = \lambda\ t$).

$$\lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{\mu}$$

with $\mu = E(T_i) = \frac{1}{\lambda}$, the mean life time.

In case the renewal match the component failure, the mean number of failure in (0, t] is approximately (for t large).

$$\begin{aligned} H(t) &= E\big(N(t)\big) \\ &\approx \frac{1}{\mu} = \frac{1}{MTSF}. \end{aligned}$$

1.7.2 SEMI MARKOV AND MARKOV RENEWAL PROCESSES

We shall study the general description of a process where a system

- Moves from one state to another with random sojourn times in between
- The successive states visited form a Markov chain

- The sojourn times have a distribution which depends both on the present state and the next state.

It is considered a Markov chain if all the sojourn times are equal to one and a Markov process if the distribution of the sojourn times are all exponential and independent of the next state. It is renewal process if there is only one state allowing an arbitrary distribution of the sojourn times.

The state space may be denoted by the set of non- negative integers $\{1,2,\ldots\}$ and transition probabilities by $p_{ij}\,; i,j = 0,1,2,\ldots$. If $F_{ij}(t), t > 0$ is the conditional distribution of the sojourn time in state i, given that the next transition will be into the state j, let

$$Q_{ij}(t) = p_{ij}F_{ij}(t), i,j = 0,1,2,\ldots.$$

Denote the probability that the process makes a transition into state j in an amount of time less than or equal to t, given that it just entered state i at t=0. The functions $Q_{ij}(t)$ satisfy the conditions which follow:

$$\begin{cases} Q_{ij}(0) = 0, Q_{ij}(\infty) = p_{ij} \\ Q_{ij}(t) \geq 0;\; i,j = 0,1,2,\ldots \\ \displaystyle\sum_{j=0}^{\infty} Q_{ij}(t) = 1. \end{cases}$$

Denote initial state and the state after the n^{th} transition occurs by J_0 and J_n respectively. The embedded Markov chain $\{J_n, n = 0,1,2,\ldots\}$ then becomes the Markov chain with transition probabilities p_{ij}.

If $N_i(t)$ represents the number of transitions into state i in (0, t] and

$$\mathrm{N(t)} = \sum_{\mathrm{i=0}}^{\infty} \mathrm{N_i(t)}.$$

A semi Markov process (SMP) is a stochastic process $\{X(t), t \geq 0\}$ with $X(t) = i$ representing the process in state i at time t and it indicates that $X(t) = J_{N(t)}$. A SMP is a pure jump process and all the states are regenerative states. The subsequent states form a time homogenous Markov chain process without memory at the transition point from one state to the next. A Markov renewal process (MRP) is a vector stochastic process $\{N_1(t), N_2(t), \dots\}\, for\ t \geq 0$. A SMP records the state process at each time point while the MRP is a counting process keeping track of the number of visits to each state.

Suppose the time interval in which the random variable $X(t)$ continues to remain in the n-point state are independently distributed such that

$$\lim_{\Delta \to 0} P[X(t+x) = j,\ \ X(t+u) = i \colon \forall\, u \leq x\ |X(t) = i, X(t-\Delta) \neq i]$$

$$= f_{ij}, i, j = 1,2, \dots, n.$$

A Markov chain with a randomly transformed time scale is called a MRP, if the transition $X(t)$ is characterised by a change of state and the qualities $f_{ii}(.)$ are zero functions.

In order to remove $f_{ii}(.) = 0$, another definition of a MRP can be given, namely considering it as regenerative stochastic process $\{X(t)\}$ in which the epochs at which

$X(t)$ visits any member of a certain countable set of states are regenerative points, the visits become regenerative events.

In order to obtain more powerful tool than either a Markov chain or a renewal process the two are combined to form a SMP. Levy (1954) and Smith (1955) introduced SMP independently. Pyke (1961 a, b), Cinler (1975) and Ross (1970) have used both SMP and MRP extensively while Barlow and Proschan (1965) applied these processes to determine the MTSF of a two unit system. In their discussion of certain reliability problems, Cinler (1975), Osaki (1970 a, b), Arora (1976 a, b), Nakagawa and Osaki (1976) and Nakagawa (1974) have used the theory of SMP.

1.7.3 REGENERATIVE PROCESSES

A sequence $t_0, t_1, \dots$ of stopping times such that $t = \{t_n : n \in N\}$ is a renewal process in a regenerative stochastic process $X(t)$. In case a point of regeneration occurs at $t = t_1$, then the knowledge of the history of the process prior to t_1 loses its predictive value; the future of the process is totally independent of its past. It therefore implies that $X(t)$ regenerates itself repeatedly at these stopping times and the times between consecutive renewals are known as regeneration times. Renewal theory is an important tool in elementary probability theory because of its application to regenerative processes.

Delayed renewal process is stated as follows: if $\hat{t} = \{t_n - t_0; n \in N\}$ is a renewal process such that $t_0 \geq 0$ is independent of $\hat{t}$ which implies that the time t_0 of the first renewal is not necessarily the time origin. A delayed renewal process is formed by a delayed regenerative process which is a process with a sequence $t = \{t_n\,; n \in N\}$ of stopping times. For instance for any initial state i, the times of subsequent entrances to a fixed state j in a Markov process become a delayed process.

In general non-exponentially distributed repair times and/ or failure free operating times lead to processes with only a few regenerations states (or even to non-regenerative processes) with the exception of few cases when it may lead to semi Markov processes. The focus of recent research is on Brownian motion with interest in the random set of all regeneration times and on the excursions of the process between generations.

1.7.4 STOCHASITC POINT PROCESS

Point processes are widely used in reliability theory to model the appearance of events in time among discrete stochastic processes. A renewal process is used as a mathematical model to describe the flow of failures in time. It is a point process known to be with restricted memory and each event is a regeneration point. In practical applications to reliability problems, the interest is focused on the behavior of a renewal process in a stationary regime, that is, when $t \rightarrow \infty$, as repairable systems enter an almost stationary regime very quickly. Alternating renewal process is a generalization of a renewal process, which comprises of two types of i.i.d. random variables alternating with each other in turn.

Point processes have been defined by different individuals in the different areas of application since recurrent events has had applications in a number of fields including physics, biology, management sciences, cyber metrics and many other areas. Wold (1948) and Bartlett (1954) first studied the properties of stationary point processes to which we attribute the current terminology. Moyal (1962) provided a formal and wellknit theory of the subject and even extended it to cover non Euclidean spaces. Srinivasan (1974), Srinivasan and Subramanian (1980) and Finkelstein (1998, 1999c) applied extensively point processes in reliability theory.

Our concern in point processes majors on those applications which, in general, lead to the development of multivariate point processes. In this particular case, a point process can be defined as a stochastic process whose realization are related to the series of point events occurring in a continuous one- dimensional parameter space (such as, time, etc.). The time series $\{t_n\}$ are the renewal epochs which generate the point process. The two random variables of concern are the number of points that fall in the interval $(t, t + x]$ and the time that has lapsed since the n^{th} point after (or before) t.

Characterization property of stationarity applies certain point processes, such as the density function of the number of observed events in a time interval which does not depend on its position on the time axis, but only on the length of the interval (Srinivasan and Subramanian (1980)). Point process models for software reliability are studied by Finkelstein (1999c)

(a) Multivariate point processes

Multivariate stationary processes have been applied in many fields and the properties of these processes have been investigated widely by Cox and Lewis (1970). A stationary point process is obtained by relaxing the constraint of independent it results in a multivariate stationary point process.

The product density technique as a sophisticated tool for the study of point processes was developed, analyzed and perfected by Ramakrishnan (1954). A point process is denoted by the triplet $(\emptyset, B, P)$, where P is a probability distribution on some $\sigma -$ field

B of subsets of the space ∅ of all states. A point x of a fixed set of points X describes the state of a set of objects.

Suppose for X is the real number line for this discussion and define A_k as intervals and $N(.)$ as a counting measure which is uniquely associated with a series of point $\{t\}$ such that

$$N(A) \;=\; \text{the number of points in the sequence } \{t_i : t_i \in A\},$$

$$N(t,x) \;=\; \text{the number of points (events) in the interval } (t : t \;+\; x]$$

$$N(t,x) \;=\; \text{the number of points (events) in } (t+x, t+x+\Delta].$$

The central quality of interest in the product density technique is this $N'(t,x)$, representing the number of entities with parametric values between x and $x+\Delta$ at time t.

Resulting the factorial moment distribution the product density of order n, which denotes the probability of an event in each of the intervals $(x_1, x_1+\Delta_1)$, $(x_2, x_2+\Delta_2)$,,… , $(x_n, x_n+\Delta_n)$, can be defined. It is symbolized by the product of the density of expectation measures at different points as shown below,

$$h_n(x_1, x_2, \dots, x_n) = \lim_{\Delta_1, \Delta_2, \dots \Delta_n} \frac{E[\prod_{i=1}^{n} N(x_i, \Delta_i)]}{\Delta_1 \Delta_2 \dots \Delta_n}; x_1 \neq x_2 \neq \cdots \neq x_n.$$

Or, equivalently

$$h_n(x_1, x_2, \dots, x_n) = \lim_{\Delta_1, \Delta_2, \dots \Delta_n} \frac{P[N(x_i, \Delta_i) > 1, i = 1,2, \dots, n]}{\Delta_1 \Delta_2 \dots \Delta_n}; x_1 \neq x_2 \neq \cdots \neq x_n.$$

The density $h_n(\dots)$ is known as a product density because it is essentially a product of the density of expectation measures at different points. The renewal function $H(t)$ is the expected number of random points in the interval $(0, t]$. Revise the process by allocation of all integral values to $\{t_i\}$ and suppose a matching sequence of points on the real line. The resultant point process generated by the random variables $\{t_i\}$, the counting process $N(t, x)$ denotes the number of points in the interval $(t, t + x]$ and the product density is

$$h_m(x_1, x_2, \dots, x_n) = E[N'(t, t_1), N'(t, t_2) \dots N'(t, t_m)].$$

A product density of degree m is defined as follows:

$$h_m(t, t_1, t_2, \dots, t_m) = E[h_1(t, t_1), h(\ t_2 - t_3) \dots h(\ t_m - t_{m-1})(t_1 < t_2 < \dots < t_m).$$

CHAPTER 2

Maintenance Analysis of an n-Unit Warm Standby System with Varying Repair Rate and Vacation Period for the Repair Facility

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J. Amadi-Echendu et al, 9th WCEAM Research papers.

2.1 INTRODUCTION

In the last four decades there has been an increasing need for the development of complex systems containing large number of units and hence it has become necessary to study multiple unit systems. Though various types of two - unit systems have been studied extensively in the past, multiple unit redundant systems have not received sufficient attention. This is due to the complexity of the analysis of these systems in contrast with the analysis of two – unit systems.

Initially MTSF or equivalently reliability was considered as the measure of system performance. However it was found to be an incomplete measure of effectiveness because it could not take into consideration of maintainability – another important aspect of system performance. Also the total system failure is a catastrophe in such cases and hence an infinite cost is associated with a system failure. With increasing complexity and the resulting high operation and maintenance costs, greater emphasis is placed on reducing cost of system maintenance while improving reliability. Operating characteristics like the expected number of repairs completed and system recoveries over an arbitrary interval are important measures. These quantities also play an important role in the study of cost structure / profit analysis of the system. While point wise availability has received considerable attention, (Srinivasan and Gopalan (1973a, b)) Birolini(1974,1975), [Osaki and Nakagawa (1976), Kumar and Agarwal (1980), Subramanian and Natarajan (1981), Subramanian et al (1984), Lie et al (1977), Huang et al (2006), El-Sherbeny (2012), Trivedi (2002)], Yuan and Xu (2011a, b), Yadavalli et al (2002 a, b) studied the reliability analysis or the maintenance optimization with a consideration of vacation period for the repair facility for a two unit cold standby system. Subramanian et al, (1984) have performed a profit analysis for an n-unit system with a constant repair rate without vacation period for repair facility.

It is generally assumed that repair facility will remain available every time without rest. In practical situations the repair facility also needs to prepare itself for the next repair. This vacation period occurs after each repair completion. Also, the repair rate of a unit depends on the number of failed units at the epoch of the commencement of the repair.

Hence an attempt is made in this chapter to consider a model in which, the repair facility is not available for a random time after each repair completion.

The aim of this chapter is to study maintenance analysis of an n-unit warm standby system with varying repair rate and the vacation period for the repair facility. The life time of a unit while on line is arbitrarily distributed random variable, while in standby has a constant failure rate.

Identifying suitable regeneration points, expressions for the availability, reliability and the profit function are derived.

This chapter contributes to the study of maintenance systems in two ways: (1) the introduction of vacation period (i.e., the vacation for the repair facility will be given just after the completion of each repair). (2) Varying repair rate (the repair rate depends on the number of failed units). A numerical example provided to illustrate the results obtained.

2.2 SYSTEM DESCRIPTION AND ASSUMPTION

In this section, we set the following assumptions and notations needed in the sequel.

1. The system consists of n identical units; one operates online and the others are warm standbys.

2. There is a single repair facility and the repairs are taken up in FIFO order.

3. The repair facility may be on vacation for some time.

4. Each unit is new after repair.

5. Switch is perfect and switchover is instantaneous.

6. At $t = 0$ there is a system recovery; i.e. system entering the upstate from the down state. This event is denoted by the symbol E.

7. The failure rate of a unit while in standby is a constant denoted by b.

8. The life time of a unit while online is an arbitrary distributed random variable with pdf $f(.)$.

9. The repair rate of a failed unit is a constant which depends upon the number of failures at the epoch of commencement of a repair. If there are j failed units at the epoch of commencement of repair, the repair rate is μ_j. We also define $\mu_0 = 0$.

10. The vacation time for the repair facility (RF) is an exponentially distributed random variable with parameter ν.

Following notations are also needed in the sequel**:**

$Z(t) = j$ is state of the repair facility at a time t, if the repair rate of the unit under repair is μ_j, $Z(t) = 0$ implies that the repair facility (RF) is free and can see any unit for repair as and when it fails.

$Z(t) = n$, if RF is under vacation at time t.

$N(\eta, t)$ = Number of η events in $(0, t]$.

$R(t) = P$ [system is up in $(0, t]$ | $E\ at\ t = 0$].

$A(t) = P$ [system is available at t | $E\ at\ t = 0$].

$\lambda_i = (n - 1 - i)\, b$.

© = Convolution symbol.

$C^{(n)}(t) = n$ – fold convolution of the function $c(t)$.

$C(t) = 1 - \int_0^t c(u)\, du$.

$\emptyset^*(s)$ = Laplace Transform of an arbitrary function $\emptyset(t)$.

2.3 THE SUBSYSTEM

When one unit is continuously operating online, the behavior of the other units can be studied independently. We call the system consisting of all the units other than the one operating online and the repair facility as the subsystem. If $Y(t)$ denotes the number of failed units at time t in the subsystem then its state at any time t can be described by the ordered pair (i, j), where $Y(t) = i,\ Z(t) = j$. If there are i failed units and repair facility is under vacation; i.e., $Y(t) = i,\ Z(t) = n$. The state space of the stochastic process describing the behavior of the subsystem is as follows:

Let

$$A = A_1 \cup A_2 \cup A_3,$$

Where

$$A_1 = \{(0,0)\},$$

$$A_2 = \{(i,j);\quad 1 \le j \le i \le n - 1\},$$

$$A_3 = \{(i,n);\quad 0 \le i \le n - 1\}. \tag{2.3.1}$$

We note that A is the state space of the stochastic process describing the subsystem.

A contains $K = \frac{n\,(n+1)}{2} + 1$ elements.

Now let B be the set of positive integers less than or equal to K, we define a bijective mapping $\pi: A \rightarrow B$ as follows

$$\pi\,(i,j) = \frac{i\,(i-1)}{2} + j + 1. \tag{2.3.2}$$

The function $\pi^{-1}\,(.)$ is determined from the following rule;

$$\pi^{-1}\,(k) = (0,0)\ \ if\ \ k = 1\,. \tag{2.3.3}$$

If $k > 1$, find the smallest positive integer i such that $k \leq S_i + 1$ where

$$S_i = \frac{i(i+1)}{2}. \quad (2.3.4)$$

This fixes i; then j is determined from the relation

$$j = i + k - 1 - S_i \quad (2.3.5)$$

Let, for fixed *t*,

$$W(t) = \pi(i,j) \text{ if } Y(t) = i,\ Z(t) = j. \quad (2.3.6)$$

Then

$$W(t) = k \Rightarrow Y(t) = i,\ Z(t) = j,$$

Where $\pi^{-1}(k) = (i,j)$. It is clear that the behavior of the subsystem is also described by the stochastic process $\{W(t), t \geq 0\}$.

For studying the behavior of the system, the following auxiliary functions are required, which will be considered only during the period of operation of an online unit or part thereof

$$p_{kk'}(t) = P[W(t) = k' \mid W(t) = k], k, k' \in B \quad (2.3.7)$$

$$p_k^T(t) = [p_{k1}(t), p_{k2}(t), \ldots, p_{kK}(t)].$$

THEOREM:

For $\alpha, \beta \in B$, let $\pi^{-1}(\alpha) = (i_1, j_1)$ and $\pi^{-1}(\beta) = (i_2, j_2)$. Then the functions $p_k(t)$ are given by

$p_k(t) = exp(Dt)p_k(0)$ where $p_k(0) = e_k$ is the column vector whose kth element is one and all the others are zero, and $D = [d_{\alpha\beta}]$ is a $K \times K$ matrix.

The elements of D are given by

$$d_{\alpha,\alpha} = -\left(\lambda_{i_1} + \mu_{i_1}\right);\ \alpha = 1,2,3,\dots,K \tag{2.3.8}$$

For $\alpha \neq \beta$,

$$d_{\alpha,\beta} = \{\lambda_{i_1} \quad if\ i_2 = i_1 + 1\, and\ j_2 = j_1\}$$

$$= \{\mu_{j_1} \quad if\ i_2 = i_1 - 1\ and\ j_2 = n\, or\ j_1 = n\, and\, i_1 = i_2 = j_2\}.$$

(2.3.9)

PROOF:

We observe that

$$P\,[Y\,(t + \Delta) = i_2\,, Z\,(t + \Delta) = j_2 \mid Y\,(t) = i_1\ Z\,(t) = j_1]$$

$$= \lambda_{i_1}\,\Delta + o\,(\Delta)\ if\ i_2 = i_1 + 1\ and\, j_2 = j_1,$$

$$= \mu_{j_1}\Delta + o\,(\Delta)\ if\ i_2 = i_1 - 1\ and\, j_2 = n\, or\, j_1 = n\ and\ i_1 = i_2 = j_2$$

$$= 1 - (\lambda_{i_1} + \mu_{j_1})\Delta + o(\Delta)\ if\ i_2 = i_1\, and\ j_2 = j_1$$

$$= o\,(\Delta)\ \ for\, other\, values\, of\, i_2\, and\, j_2. \tag{2.3.10}$$

$$For\ i_1 = 0 = j_1$$

$$P\,[Y\,(t,t + \Delta) = i_2\,, Z\,(t,t+\Delta) = j_2 \mid Y(t) = 0\,, Z(t) = 0]$$

$$= (n-1)\,b + o(\Delta)\, if\ i_2 = 1\, and\ j_2 = 1$$

$$= 1 - (n - 1)\, b + o(\Delta)\; if \;\; i_2 = 0 = j_2$$

$$= o(\Delta)\; for\; other\; values\; of\; i_2\; and\; j_2. \tag{2.3.11}$$

Using these observations and considering the behavior of the subsystem in the interval $(t, t + \Delta)$, we arrive at the following matrix differential equation:

$$\frac{d}{dt} p_k(t) = D\, p_k(t). \tag{2.3.12}$$

The solution of this matrix differential equation is

$$p_k(t) = exp\,(Dt) p_k\,(0).$$

Having identified the subsystem, we are now in a position to analyze the main system.

2.4 THE MAIN SYSTEM

The subsystem taken together with the unit operating online will be called the main system. To study its behavior, we define the following events.

E_{ij} at t : event that one unit is just online at t and $Y(t+) = i$,

$$Z(t+) = j;\; and\;\; (Y\,(t+), Z\,(t+)) \in A.$$

We also require the following auxiliary functions h $(t, i', j' \mid i, j)$ to describe the behavior of the main system when it is in upstate.

$$h\,(t\,, i', j' \mid i\,, j\,) = \lim_{\Delta \to 0} Pr\,[\,E_{i', j'}\;\; in\,(\,t\,, t + \Delta),$$

$$system\; is\;\; up\; in\,(0\,, t\,] \mid E_{ij}\;\,] \,/\, \Delta.$$

Next we derive the equation satisfied by $h\,(t, i', j' \mid i, j)$.

For convenience we let

$$P\,(t\,, i'\,, j' \mid i\,, j) \;=\; p_{kk'}\,(t),$$

where $(i\,, j) \;=\; k \;\; and \;(i', j') \;=\; k'$.

Observe that $h\,(t, i', j' \mid i, j)\,\Delta$ is the probability of occurrence of an $E_{i',j'}$ event in $(t, t\; + \Delta\,)$ given an E_{ij} at $t \;=\; 0$.

Hence for the occurrence of $E_{i',j'}$ in $(t\,, t\; +\; \Delta)$, the unit which is operating online must fail in $(t, t\; +\; \Delta)$. This failure may be the one which was put on line at $t \;=\; 0$, or a subsequent one.

For all $(\,i\,, j\,) \,\epsilon\, A \;\; and \;\; (\,i'\,, j'\,) \,\epsilon\, C\,, we \;\; have$

$$h\,(\,t\,, i'\,, j' \mid i\,, j\,) \;=\; f(t)\, p\,(t\,, i' - 1\,, j' \mid i\,, j\,)$$

$$+ \sum_{(i_2, j_2) \in C} f(t)\;\; p(\,t\,, i_2\,, j_2 \mid i\,, j\,) © h\,(t, i', j' \mid i_2 + 1\,, j_2), \quad (2.4.1)$$

where

$C_1 \;=\; \{\,(\,0\,, 0\,)\,\}$,

$C_2 \;=\; \{\,(\,i\,, j\,)\,;\, 1 \;\leq\; j \;\leq\; i < \;\; n - 1\,\}\, and$

$C_3 \;=\; \{\,(\,i\,, n\,)\,;\;\; 0 \;\;\leq\;\; i \;\;<\;\; n - 1\,\}$

and

$C \;=\; C_1\, U\, C_2\, U\, C_3.$ (2.4.2)

For fixed (i', j') above equation can be solved for $h^*(s, i', j' \mid i, j)$ by Laplace transform technique.

2.5 RELIABILITY OF THE SYSTEM

We now drive an expression for the reliability R (t) of the system. Considering the mutually exclusive and exhaustive cases that the unit which was put online at t = 0,

(1) Does not fail up to time

(2) Fails before the time t, we obtain the expression

$$R\,(t) \;=\; \bar{F}(t) \;+\; \sum_{(i,j)\in C} h(t\,, i\,, j\,|n-1, n)© \,\bar{F}(t), \qquad (2.5.1)$$

The mean time to system failure (MTSF) can be obtained using the relation R*(0).

2.6 AVAILABILITY OF THE SYSTEM

While computing the expression for the availability of the system, we have to permit system downs also. Hence we have to introduce some auxiliary functions. Suppose that

$$\emptyset(t) = \lim_{\Delta\to 0} Pr\,[E\;in\;(t\,, t+\Delta\,)\,, N(\,E\,, t\,) = \;0|\;E\;at\;t\;=\;0\,]$$

and

$$\bar{\Phi}\,(t) \;=\; 1 - \int_0^t \emptyset(u)du.$$

The expression for $\emptyset(t)$ is derived by considering the fact that system may or may not enter the down state in $(0, t]$;

$$\overline{\Phi}(t) = R(t) + \sum_{(i,j)\in C} h(t,i,j\,|n-1,n) © [\sum_{k=1}^{n-1} f(t)p(t,n-1,k\,|i,j) © e^{-\mu_k t}$$

$$+ f\,(t)\,p(t,n-1,n\,|\,i,j) ©\{\, e^{-\mu_k t} + \, e^{-\gamma t} © e^{-\mu_k t}\}]$$

(2.6.1)

Noting that the interval $(0, t]$ may be intercepted by an event E or not, the availability of the system is obtained as

$$A\,(t) \;=\; R\,(t) \;+\; \textstyle\sum_{n=1}^{\infty} \emptyset^{(n)}\,(t) © R\,(t). \qquad (2.6.2)$$

The steady state availability A_∞ can be obtained from the relation

$$A_\infty = \lim_{t\to\infty} A(t) = \lim_{s\to 0} sA^*\,(s).$$

From equation (2.6.2) we get

$$A_\infty = \frac{R^*(0)}{\overline{\Phi}^*(0)}\ .$$

2.7 PROFIT ANALYSIS

In this section we calculate the profit from the system per unit time, when the system has reached the steady state.

Let 'r' be the return rate from the system when it is operable. Then the gross return from the system per unit time is $A_\infty \times r$.

Next we consider the expenditure incurred per unit time. Let 'd' be the fixed cost per unit time associated with each of the unit in the system. Then the fixed expenditure per unit time is $n \times d$.

Since the repair rate of a unit depends on the number of failed units at the epoch of commencement of its repair, the cost incurred for a repair should also depend on the repair rate. This repair rate is likely to be different for different repairs. In order to give due weightage to this fact, we define the event E_j as follows.

E_j : Event that the repair for a unit commences and the number of failed unit is 'j',

where, $j = 1, 2, 3, \ldots, n$.

Let $N\ (E_j)$ be the stationary rate of the $E_j$events and C_j be the cost associated with its occurrence. Then the expenditure per unit time corresponding to the repairs is $C_j \times N\ (E_j)$.

The net profit function

$$\psi(n) = r.\ A_\infty - \left[\sum_{j=1}^{\infty} \mathrm{N}(E_j).\, d_j + n.\, d\right]. \qquad (2.7.1)$$

To determine $\psi\ (n)$, it remain for us to find an expression for $N\ (E_j)$.

For this purpose define,

$$\emptyset_j\ (t) = \lim_{\Delta \to 0} \Pr[\ E_j \text{ in } (\mathrm{t}, \mathrm{t} + \Delta),\ N(E, t) = 0 |\ at\ t = 0]\ /\Delta.$$

Note that $\emptyset_j(t)$ is the first order product density. To obtain an expression for $\emptyset_j\ (t)$ we make use of the following auxiliary function;

$$D^{\emptyset_j}(\mathrm{t}) \;=\; [1 - \delta_{j,n-1} \sum_{k=1}^{j+1} P(t, j+1, k \mid n-1, n)\, \mu_k$$

$$+ p(t, j, n \mid n-1, n)\gamma\,]\; \bar{F}(t) \;+\; \sum_{(i',\; j') \in C} h(t, i', j' \mid n-1, n)$$

$$© \, [(1 - \delta_{j,n-1}) \textstyle\sum_{k=1}^{j+1} p(t, j+1, k \mid i',\, j')\gamma]\bar{F}(t) \;+ \delta_{j\,1}[\{\, p(t, 0, 0 \mid n-1, n)$$

$$(n-1)\, b \;+\; p(t, 1, n \mid n-1, n)\gamma\}\bar{F}(\mathrm{t})$$

$$+ \sum_{(i',\; j') \in C} h(t, i', j' \mid n-1, n)\, ©\{\, p(t, 0, 0 \mid i',\; j')(n-1)b$$

$$+ p(t, 1, n \mid i', j')\,\gamma\}\bar{F}(t)] \;\; + \delta_{jn}\; [p(t, n-1, n) f(t) © \gamma\, e^{-\gamma t}$$

$$+ \sum_{(i', j') \in C} h(t, i', j' \mid \; n-1, n) © p(t, n-1, n \mid i', j') f(t) © \, \gamma\, e^{-\gamma t}\,. \qquad (2.7.2)$$

The above expression $D^{\emptyset_j}(\mathrm{t})$ is obtained by considering the following mutually exclusive and exhaustive cases:

(i) The online unit does not fail up to t.

(ii) The online unit fails before t.

Next we drive the following expression for $\emptyset_j(t)$ by considering the fact that a system recovery has occurred or not in $(0, t]$:

$$\emptyset_j(t) = D^{\emptyset_j(t)} + \delta_{j,n-1}\emptyset(t) + \sum_{n=1}^{\infty} \emptyset^{(n)}(t) © [D^{\emptyset_j(t)} + \delta_{j,n-1}\emptyset_j(t)]. \quad (2.7.3)$$

The expected number of E_j events in $(0, t]$ is given by

$$E[N_j(t)] = \int_0^t \emptyset_j(u)\, du.$$

The stationary rate of occurrence of E_j events is given by

$$\begin{aligned} E[N_j] &= \lim_{t\to\infty} \frac{E[N_j(t)]}{t} \\ &= lim_{s\to 0}\, s\emptyset^*(s) \\ &= \frac{D^{\emptyset_j\,*}(0)}{\overline{\Phi}^{\,*}(0)}. \end{aligned} \qquad (2.7.4)$$

2.8 PARTICULAR CASE

When $\mu_j = \mu$ and $\nu \to \infty$, we get the results corresponding to the system in which the repair rate of a unit is constant. In this case the model is in agreement with the model developed in [Subramanian et al (1984)].

CHAPTER 3

Applications of Quadrivariate Exponential Distribution to a Three Unit Warm Standby System with Dependent Structure

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3.1 INTRODUCTION

Two unit warm standby systems have been elaborately dealt with in the literature. However, the study of standby systems with more than two units, though very relevant in state-of-art practical situations, has received little attention because of mathematical intricacies involved in analysing them. Also, such systems have been studied assuming (i) the lifetime or repair time of the units to be exponential or (ii) the lifetime and repair time to be independent.

As pointed out by Srinivasan and Subramanian (2006), the study of three unit warm standby systems is challenging because of the built-in intricacies involved in their analysis. Several authors have extensively studied two unit standby redundant systems in the past. Osaki and Nakagawa (1976) gave a bibliography of the work on two unit systems. Most of the studies on two unit warm standby systems are confined to obtaining expressions for various measures of system performance and do not consider the associated inference problems. Chandrasekhar and Natarajan (1994), Yadavalli et al (2002a) have considered a two unit cold standby system and obtained the exact confidence limits for the steady state availability of the system under the assumption that the lifetime of online unit and the repair time of a failed unit are independent. Subsequently asymptotic confidence limits for a two-unit parallel system was studied by Yadavalli et al (2002b)

In general, the failure time and repair time need not be independent always. A system or a component that fails frequently within a short time interval has to be analysed thoroughly and the time taken to repair such a system will be more. The dependency between life time / failure time and repair time can be modelled by assuming a suitable form of a bivariate density function. In the past, a number of bivariate exponential distributions have been proposed and studied well in the literature. But the bivariate exponential distribution of Marshall and Olkin (1967) is widely accepted among many bivariate exponential distributions proposed in the statistical literature because of its nice properties.

The present work is an attempt to analyse a three unit warm standby system under the assumption that the joint distribution of the lifetimes of online unit, standby units and the repair time of a failed unit in the system is quadrivariate exponential. Further, it is assumed that the lifetimes of the units kept in standby are identical. The model and the assumptions, expressions for reliability, availability and associated statistical inference together with numerical illustration are discussed in detail in this chapter.

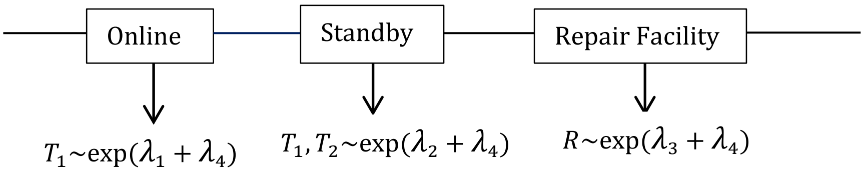
The present contribution is an improvement in the state-of-art in the sense that three unit warm standby systems with dependent structure is shown to be capable of comprehensive analysis.

In this chapter, the system reliability and availability are studied. An associated statistical inference for a three unit warm standby system with dependent structure is also discussed. The work presented in this chapter is an extension of the results obtained by Chandrasekhar et al (2013) for a two unit warm standby system with dependent structure.

3.2. THE MODEL AND ASSUMPTIONS

The system under consideration is a three unit warm standby system with a single repair facility.

Fig 3.2.1



System Configuration: Three unit warm standby system with repair.

Precisely the following are the assumptions.

(i) The units are similar and statistically not independent. One unit is operating online and other two units are kept as warm standby. The three units have constant failure rates say $(\lambda_1 + \lambda_4)$ while online and $(\lambda_2 + \lambda_4)$ while in standby. Further, each failed unit has a constant repair rate say$(\lambda_3 + \lambda_4)$.

(ii) There is only one repair facility.

(iii) Let T_1,T_2 and T_3denote the lifetimes of the three units and R the repair time of a failed unit in the system. As the system consists of three units with single repair, it is appropriate to consider the following Marshall-Olkin (1967) quadrivariate exponential (QVE) distribution for T_1, T_2 ,T_3 and R with the survival function given by

$$\bar{F}(t_1, t_2, t_3, t_4) = e^{-[\lambda_1 t_1 + \sum_{i=2}^{3} \lambda_2 t_i + \lambda_3 t_4 + \lambda_4 . max(t_1, t_2, t_3, t_4)]},$$

$$t_i > 0, \quad i = 1,2,3,4; \lambda_i > 0,$$

$$i = 1,2,3; \ \lambda_4 \geq 0 \qquad (3.2.1)$$

and is denoted by

$$(T_1, T_2, T_3, R) \sim \text{QVE}(\lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_4).$$

See Marshall and Olkin (1967).

(iv) Each unit is new after repair and

(v) Switch is perfect and the switchover is instantaneous.

Note:

1. The lifetimes of units T_1,T_2 and T_3 are exponential random variables each with the parameters $(\lambda_1 + \lambda_4), (\lambda_2 + \lambda_4)$ and $(\lambda_2 + \lambda_4)$ respectively.

2. The repair time R is exponential with the parameter $(\lambda_3 + \lambda_4)$.

3. $\text{E}(T_1) = \dfrac{1}{(\lambda_1 + \lambda_4)}$

$$\text{E}(T_i) = \frac{1}{(\lambda_2 + \lambda_4)}, i = 2,3 \quad \text{and}$$

$$\text{E(R)} = \frac{1}{(\lambda_3 + \lambda_4)}$$

$$\text{Var}(T_1) = \frac{1}{(\lambda_1 + \lambda_4)^2}$$

$$\text{Var}(T_i) = \frac{1}{(\lambda_2 + \lambda_4)^2}, i = 2,3 \quad \text{and}$$

$$\text{Var(R)} = \frac{1}{(\lambda_3 + \lambda_4)^2} .$$

4. The joint distribution of $(T_1, T_j),\ j = 2,3$ is bivariate exponential (BVE) with the parameters $(\lambda_2, \lambda_2, \lambda_4)$ and joint distribution of (T_2, T_3) is bivariate exponential (BVE) with the parameters $(\lambda_2, \lambda_2, \lambda_4)$. Similarly, the joint distribution of (T_1, R) is BVE with the parameters $(\lambda_1, \lambda_3, \lambda_4)$ and that of $(T_i, R), i = 2,3$ is also BVE with the parameters $(\lambda_2, \lambda_3, \lambda_4)$.

5. The covariance between T_1 and T_j is given by

$$\mathrm{Cov}(T_1, T_j) = \frac{\lambda_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4)}\ , j = 2,3.$$

Similarly, the following results can be established.

$$\mathrm{Cov}(\mathrm{T}_2, \mathrm{T}_3) = \frac{\lambda_4}{(\lambda_2 + \lambda_4)^2(2\lambda_2 + \lambda_4)}$$

$$\mathrm{Cov}(\mathrm{T}_1, \mathrm{R}) = \frac{\lambda_4}{(\lambda_1 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4)}$$

$$\mathrm{Cov}(\mathrm{T_i}, \mathrm{R}) = \frac{\lambda_4}{(\lambda_2 + \lambda_4)\ (\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)}, \mathrm{i} = 2,3.$$

6. The lifetimes T_1, T_2 and T_3 and the repair time R are independent if and only if $\lambda_4 = 0$.

7. If $(X_i, Y_i), i = 1,2, \dots, n$ is a random sample of size n from a BVE population with parameters $(\lambda_1, \lambda_2, \lambda_3)$, then covariance between the sample means $\bar{X}$ and $\bar{Y}$ is given by

$$\mathrm{Cov}(\bar{\mathrm{X}}, \bar{\mathrm{Y}}) = \frac{\lambda_3}{n(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}\ .$$

8. The joint distribution of (T_1, T_2, T_3) is trivariate exponential (TVE) with the parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Further, the joint distributions of $(T_1, T_j, R), j = 2,3$

and (T_2, T_3, R) are (TVE) with the parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\lambda_2, \lambda_2, \lambda_3, \lambda_4)$ respectively.

3.3 ANALYSIS OF THE SYSTEM

To analyse the behaviour of the system, define $X(t)$ as the number of failed units at time t. The stochastic process $\{X(t), t \geq 0\}$ with the state space given by $E = \{0,1,2,3\}$ denotes the state of the system at time t. Since quadrivariate exponential distribution has exponential marginals and satisfies lack of memory property, it follows that the stochastic process describing the behaviour of the system is a Markov process with infinitesimal generator Q given by

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & -(\lambda_1 + 2\lambda_2 + 3\lambda_4) & (\lambda_1 + 2\lambda_2 + 3\lambda_4) & 0 & 0 \\
1 & (\lambda_3 + \lambda_4) & -(\lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4) & (\lambda_1 + \lambda_2 + 2\lambda_4) & 0 \\
2 & 0 & (\lambda_3 + \lambda_4) & -(\lambda_1 + \lambda_3 + 2\lambda_4) & (\lambda_1 + \lambda_4) \\
3 & 0 & 0 & (\lambda_3 + \lambda_4) & -(\lambda_3 + \lambda_4)
\end{array}
\qquad (3.3.1)
$$

It may be noted that the system upstates are 0, 1, 2, while state 3 is the system downstate. Let $p_i(t) = \Pr[X(t) = i] \; \forall \; i \in E$ represent the probability that the system is in state i at time t with the initial condition $p_0(0) = 1$. We assume that initially all the three units are operable and obtain the measures of system performance as follows:

3.4 SYSTEM RELIABILITY

The system reliability $R(t)$ is the probability of failure free operation of the system in $(0, t]$.To derive an expression for the reliability of the system, we restrict the transitions of the Markov process to the upstates namely 0, 1 and 2. Using the infinitesimal generator of the process given in (3.3.1), pertaining to these upstates, we derive the following differential–difference equations:

$$\frac{dp_0(t)}{dt} = -(a + 2b)p_0(t) + cp_1(t) \tag{3.4.1}$$

$$\frac{dp_1(t)}{dt} = (a + 2b)p_0(t) \; - (a + b + c)p_1(t) + cp_2(t) \tag{3.4.2}$$

$$\frac{dp_2(t)}{dt} = (a + b)p_1(t) \; - (a + c)p_2(t), \tag{3.4.3}$$

Where

$a = (\lambda_1 + \lambda_4), b = (\lambda_2 + \lambda_4)$ and $c = (\lambda_3 + \lambda_4)$.

Let $L_i(s)$ be the Laplace transform of $p_i(t), i = 0,1,2$. Taking Laplace transforms on both sides of the differential-difference equations given in (3.4.1), (3.4.2) and (3.4.3), solving for $L_i(s), i = 0,1,2$ and inverting, we get $p_0(t)$, $p_1(t)$ and $p_2(t)$.Thus the system reliability is given by

$$R(t) = \sum_{i=1}^{3} \frac{e^{\alpha_i t}\left\{\begin{matrix}[\alpha_i^2 + (2a+b+2c)\alpha_i + (a^2+ab+ac+c^2)] + [(a+2b)(\alpha_i + a + c)] \\ +[(a+b)(a+2b)]\end{matrix}\right\}}{\prod_{j=1, j\neq i}^{3}(\alpha_i - \alpha_j)}, \tag{3.4.4}$$

where α_1, α_2 and α_3 are the roots of the cubic equation

$$s^3 + (3a + 3b + 2c)s^2 + (3a^2 + 2b^2 + c^2 + 6ab + 2bc + 2ca)s + a(a^2 + 3ab + 2b^2) = 0$$

3.5 MEAN TIME TO SYSTEM FAILURE (MTSF)

The system MTSF is the expected or average time to failure and is given by

$$MTSF = R^*(0) = L_0(0) + L_1(0) + L_2(0),$$

where $R^*(s)$ is the Laplace transform of $R(t)$ at s.

Hence,

$$MTSF = \frac{(3a^2 + 2b^2 + c^2 + 6ab + 2bc + 2ca)}{a(a^2 + 3ab + 2b^2)}. \tag{3.5.1}$$

3.6 SYSTEM AVAILABILITY

The system availability $A(t)$ is the probability that the system operates within the tolerances at a given instant of time t and is obtained by solving for $p_i(t), i \in E$. The following system of differential–difference equations are obtained by using the infinitesimal generator given in (3.3.1).

$$\frac{dp_0(t)}{dt} = -(a+2b)p_0(t) + c\,p_1(t) \tag{3.6.1}$$

$$\frac{dp_1(t)}{dt} = (a+2b)p_0(t) \; -(a+b+c)p_1(t) + \; c\,p_2(t) \tag{3.6.2}$$

$$\frac{dp_2(t)}{dt} = (a+b)p_1(t) \; -(c+a)p_2(t) + c\,p_3(t) \tag{3.6.3}$$

$$\frac{dp_3(t)}{dt} = ap_2(t) \; -c\,p_3(t) \tag{3.6.4}$$

Using Laplace transform on both sides of the above differential-difference equations and solving, the expressions for $p_i(t),\ i = 0,1,2,3$ are obtained respectively as

$$\frac{-c^3}{\alpha_1\alpha_2\alpha_3} + c(a+2b)\sum_{i=1}^{3}\frac{[\alpha_i^2 + (a+2c)\alpha_i + c^2]}{\alpha_i(\alpha_i + a + 2b)\prod_{j=1,j\neq i}^{3}(\alpha_i - \alpha_j)}e^{\alpha_i t} \tag{3.6.5}$$

$$\frac{-(a+2b)c^2}{\alpha_1\alpha_2\alpha_3} + (a+2b)\sum_{i=1}^{3}\frac{[\alpha_i^2 + (a+2c)\alpha_i + c^2]}{\alpha_i\prod_{j=1,j\neq i}^{3}(\alpha_i - \alpha_j)}e^{\alpha_i t} \tag{3.6.6}$$

$$\frac{-(a+b)(a+2b)c}{\alpha_1\alpha_2\alpha_3} + (a+b)(a+2b)\sum_{i=1}^{3}\frac{(\alpha_i + c)}{\alpha_i\prod_{j=1,j\neq i}^{3}(\alpha_i - \alpha_j)}e^{\alpha_i t} \tag{3.6.7}$$

$$\frac{-a(a+b)(a+2b)}{\alpha_1\alpha_2\alpha_3} + a(a+b)(a+2b)\sum_{i=1}^{3}\frac{1}{\alpha_i\prod_{j=1,j\neq i}^{3}(\alpha_i-\alpha_j)}e^{\alpha_i t} \qquad (3.6.8)$$

where α_1, α_2 and α_3 are the roots of the equation

$$s^3 + 3(a+b+c)s^2 + [(a+2c)(a+2b+c) + (a+b)(2a+2b+c) + c^2]\,s$$

$$+[c^3 + (a+2b)(a^2+c^2+ab+bc+ca)] = 0.$$

Hence, the system availability is given by

$$A(t) = p_0(t) + p_1(t) + p_2(t). \qquad (3.6.9)$$

3.7 STEADY STATE AVAILABILITY

The system steady state availability is the expected fractional amount of time in a continuum of operating time that the system is in upstate and is given by

$$A_\infty = \lim_{t\to\infty} A(t)$$

$$= \frac{c[c^2 + (a+2b)(a+b+c)]}{[c^3 + (a+2b)(a^2+c^2+ab+bc+ca)]}. \qquad (3.7.1)$$

3.7.1 Particular Case

The equations for system reliability, MTSF, system availability and steady state availability when the lifetimes of online and standby units and the repair time of a failed unit are independent can be obtained by taking $\lambda_4 = 0$ in (3.4.4), (3.5.1), (3.6.9) and (3.7.1), respectively.

3.8 CONFIDENCE INTERVAL FOR STEADY STATE AVAILBILITY OF THE SYSTEM

Let $(Y_{1i}, Y_{2i}, Y_{3i}, Y_{4i}), i = 1,2, \dots, n$ be a random sample of size n drawn from a quadrivariate exponential lifetimes and repair time population with the survival function given by (3.2.2). It is clear that $\bar{Y}_1, \bar{Y}_2, \bar{Y}_3 (= \bar{Y}_2)$ and $\bar{Y}_4$ are the moment estimators of $\frac{1}{(\lambda_1+\lambda_4)}, \frac{1}{(\lambda_2+\lambda_4)}, \frac{1}{(\lambda_2+\lambda_4)}$ and $\frac{1}{(\lambda_3+\lambda_4)}$ respectively, where $\bar{Y}_1, \bar{Y}_2$, $\bar{Y}_3$ $(= \bar{Y}_2)$ and $\bar{Y}_4$ are the sample means of lifetimes of online and standby units and repair time of a failed unit respectively.

Let $\theta_i = \frac{1}{(\lambda_i+\lambda_4)}$, $i = 1,2,3$. Using $\theta_i, i = 1,2,3$ in (3.7.1) and substituting its corresponding moment estimators, the estimator of the steady state availability of the system A_∞ based on moments is obtained as

$$\hat{A}_\infty = \frac{\bar{Y}_1[\bar{Y}_1{}^2\bar{Y}_2{}^2 + \bar{Y}_4(2\bar{Y}_1 + \bar{Y}_2)(\bar{Y}_1\bar{Y}_2 + \bar{Y}_2\bar{Y}_4 + \bar{Y}_4\bar{Y}_1)]}{[\bar{Y}_1{}^3\bar{Y}_2{}^2 + \bar{Y}_2{}^2\bar{Y}_4{}^3 + \bar{Y}_4\bar{Y}_1(\bar{Y}_4+\bar{Y}_1)\left(2\bar{Y}_4\bar{Y}_1 + \bar{Y}_2{}^2 + 3\bar{Y}_2\bar{Y}_4\right) + 2\,\bar{Y}_1{}^3\bar{Y}_2\bar{Y}_4]} \qquad (3.8.1)$$

It may be noted that $\hat{A}_\infty$ given in (3.8.1) is a real valued function in $\bar{Y}_1, \bar{Y}_2$ and $\bar{Y}_4$, which is also differentiable. Consider the following multivariate central limit theorem. See Radhakrishna Rao (1974).

3.8.1 Multivariate Central Limit Theorem

Suppose $T_1', T_2', T_3', \ldots$ are independent and identically distributed k-dimensional random variables such that

$$T_n' = (T_{1n}, T_{2n}, \ldots, T_{kn}), \quad n = 1, 2, 3, \ldots$$

having the first and second order moments $E(T_n) = \mu$ and $\operatorname{var}(T_n) = \Sigma$. Define the sequence of random variables

$$\bar{T}_n' = (\bar{T}_{1n}, \bar{T}_{2n}, \ldots, \bar{T}_{kn}), n = 1, 2, 3, \ldots,$$

Where

$$\bar{T}_{in} = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, i = 1, 2, \ldots, k.$$

Then

$$\sqrt{n}(\bar{T}_n - \mu) \xrightarrow{d} N_k(0, \Sigma) \text{ as } n \to \infty.$$

3.8.2 CONSISTANTLY ASYPMTOTIC NORMAL (CAN) ESTIMATOR

By applying the multivariate central limit theorem given in section 3.8.1, it is seen that

$$\sqrt{n}[(\bar{Y}_1, \bar{Y}_2, \bar{Y}_2, \bar{Y}_4) - (\theta_1, \theta_2, \theta_2, \theta_3)] \xrightarrow{d} N_4(0, \Sigma), \text{ as } n \to \infty,$$

where the dispersion matrix $\Sigma = ((\sigma_{ij}))$ is given by

$$\begin{array}{c|cccc} & \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3(=\bar{Y}_2) & \bar{Y}_4 \\ \hline \bar{Y}_1 & {\theta_1}^2 & \frac{\lambda_4\theta_1^2\theta_2^2}{(\theta_1+\theta_2-\lambda_4\theta_1\theta_2)} & \frac{\lambda_4\theta_1^2\theta_2^2}{(\theta_1+\theta_2-\lambda_4\theta_1\theta_2)} & \frac{\lambda_4\theta_1^2\theta_3^2}{(\theta_1+\theta_3-\lambda_4\theta_1\theta_3)} \\ \bar{Y}_2 & \frac{\lambda_4\theta_1^2\theta_2^2}{(\theta_1+\theta_2-\lambda_4\theta_1\theta_2)} & {\theta_2}^2 & \frac{\lambda_4\theta_2^3}{(2-\lambda_4\theta_2)} & \frac{\lambda_4\theta_2^2\theta_3^2}{(\theta_2+\theta_3-\lambda_4\theta_2\theta_3)} \\ \bar{Y}_3(=\bar{Y}_2) & \frac{\lambda_4\theta_1^2\theta_2^2}{(\theta_1+\theta_2-\lambda_4\theta_1\theta_2)} & \frac{\lambda_4\theta_2^3}{(2-\lambda_4\theta_2)} & \theta_2^2 & \frac{\lambda_4\theta_2^2\theta_3^2}{(\theta_2+\theta_3-\lambda_4\theta_2\theta_3)} \\ \bar{Y}_4 & \frac{\lambda_4\theta_1^2\theta_3^2}{(\theta_1+\theta_3-\lambda_4\theta_1\theta_3)} & \frac{\lambda_4\theta_2^2\theta_3^2}{(\theta_2+\theta_3-\lambda_4\theta_2\theta_3)} & \frac{\lambda_4\theta_2^2\theta_3^2}{(\theta_2+\theta_3-\lambda_4\theta_2\theta_3)} & \theta_3^2 \end{array} \quad (3.8.2)$$

Thus,

$$\sqrt{n}\left(\hat{A}_\infty - A_\infty\right) \xrightarrow{d} N_4\left(0, \sigma^2(\theta)\right),$$

where $\theta = (\theta_1, \theta_2, \theta_2, \theta_3)$.

And

$$\sigma^2(\theta) = \left[\theta_1^2\left(\frac{\partial A_\infty}{\partial\theta_1}\right)^2 + \frac{4\theta_2^2}{(2-\lambda_4\theta_2)}\left(\frac{\partial A_\infty}{\partial\theta_2}\right)^2 + \theta_3^2\left(\frac{\partial A_\infty}{\partial\theta_3}\right)^2\right]$$

$$+4\lambda_4\theta_2^2\left[\frac{\theta_1^2}{(\theta_1+\theta_2-\lambda_4\theta_1\theta_2)}\left(\frac{\partial A_\infty}{\partial\theta_1}\right)\left(\frac{\partial A_\infty}{\partial\theta_2}\right) + \frac{\theta_3^2}{(\theta_2+\theta_3-\lambda_4\theta_2\theta_3)}\left(\frac{\partial A_\infty}{\partial\theta_2}\right)\left(\frac{\partial A_\infty}{\partial\theta_3}\right)\right]$$

$$+2\frac{\lambda_4\theta_1^2\theta_3^2}{(\theta_1+\theta_3-\lambda_4\theta_1\theta_3)}\left(\frac{\partial A_\infty}{\partial\theta_1}\right)\left(\frac{\partial A_\infty}{\partial\theta_3}\right). \quad (3.8.3)$$

By substituting for the partial derivatives $\left(\frac{\partial A_\infty}{\partial \theta_i}\right)$, $i = 1,2,3$ in (3.8.3), we get an expression for $\sigma^2(\theta)$. Thus $\hat{A}_\infty$ is a CAN estimator of A_∞.

3.8.3 CONFIDENCE INTERVAL FOR THE STEADY STATE AVAIABILITY OF THE SYSTEM

Let $\sigma^2(\hat{\theta})$ be an estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely $\hat{\theta} = (\bar{Y}_1, \bar{Y}_2, \bar{Y}_2, \bar{Y}_4)$. Let $\hat{\sigma}^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2(\theta)$. i.e. $\hat{\sigma}^2 \xrightarrow{P} \sigma^2(\theta)\ as\ n \to \infty$. By Slutsky theorem, we have

$$\frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$$

that is,

$$\Pr\left(-k_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} < -k_{\frac{\alpha}{2}}\right) = (1 - \alpha),$$

where $k_{\frac{\alpha}{2}}$ is obtained from normal tables. Hence a $100(1 - \alpha)\%$ confidence interval for A_∞ is given by $\hat{A}_\infty \pm k_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}$, where $\hat{\sigma}$ is obtained from (3.8.3).

3.8.4 AN ESTIMATOR OF SYSTEM RELIABILITY BASED ON METHOD OF MOMENTS

We have already seen that $\bar{Y}_1, \bar{Y}_2, \bar{Y}_2$ and $\bar{Y}_4$ are the moment estimators of $\frac{1}{(\lambda_1+\lambda_4)}$, $\frac{1}{(\lambda_2+\lambda_4)}$, $\frac{1}{(\lambda_2+\lambda_4)}$ and $\frac{1}{(\lambda_3+\lambda_4)}$ respectively, where $\bar{Y}_1, \bar{Y}_2, \bar{Y}_2$ and $\bar{Y}_4$are the sample means of life time of online unit, lifetimes of standby units and repair time of a failed unit respectively. Hence an estimator of system reliability can be obtained by using the moment estimators in (3.4.4) and is given by

$$\hat{R}(t) = \hat{p}_0(t) + \hat{p}_1(t) + \hat{p}_2(t), \tag{3.8.4}$$

where

$$\hat{p}_0(t) = \frac{1}{\bar{Y}_1^{\,2}\bar{Y}_2\bar{Y}_4^{\,2}} \sum_{i=1}^{3} \frac{\begin{bmatrix} \hat{\alpha}_i^2\left(\bar{Y}_1^{\,2}\bar{Y}_2\bar{Y}_4^{\,2}\right) + \bar{Y}_1\bar{Y}_4(2\bar{Y}_4\bar{Y}_2 + \bar{Y}_4\bar{Y}_1 + 2\bar{Y}_1\bar{Y}_2) \\ + \left(\bar{Y}_1\bar{Y}_4^{\,2} + \bar{Y}_1^{\,2}\bar{Y}_2 + \bar{Y}_2\bar{Y}_4^{\,2} + \bar{Y}_1\bar{Y}_2\bar{Y}_4\right) \end{bmatrix}}{\prod_{j=1, j\neq i}^{3}\left(\hat{\alpha}_i - \hat{\alpha}_j\right)} e^{\hat{\alpha}_i t} \tag{3.8.5}$$

$$\hat{p}_1(t) = \frac{(2\bar{Y}_1 + \bar{Y}_2)}{(\bar{Y}_1^{\,2}\bar{Y}_2\bar{Y}_4)} \sum_{i=1}^{3} \frac{[\hat{\alpha}_i\bar{Y}_1\bar{Y}_4 + \bar{Y}_4 + \bar{Y}_1]}{\prod_{j=1, j\neq i}^{3}\left(\hat{\alpha}_i - \hat{\alpha}_j\right)} e^{\hat{\alpha}_i t} \tag{3.8.6}$$

$$\hat{p}_2(t) = \frac{(\bar{Y}_1 + \bar{Y}_2)(2\bar{Y}_1 + \bar{Y}_2)}{(\bar{Y}_1\bar{Y}_2)^2} \sum_{i=1}^{3} \frac{1}{\prod_{j=1, j\neq i}^{3}\left(\hat{\alpha}_i - \hat{\alpha}_j\right)} e^{\hat{\alpha}_i t} \tag{3.8.7}$$

where $\hat{\alpha}_1$,$\hat{\alpha}_2$ and $\hat{\alpha}_3$are the roots of

$$\bar{Y}_1{}^3\bar{Y}_2{}^2\bar{Y}_4{}^2 s^3 + \bar{Y}_1{}^2\bar{Y}_2\bar{Y}_4(2\bar{Y}_1\bar{Y}_2 + 3\bar{Y}_1\bar{Y}_4 + 3\bar{Y}_2\bar{Y}_4)s^2$$

$$+\bar{Y}_1\left(\bar{Y}_1{}^2\bar{Y}_2{}^2 + 3\bar{Y}_2{}^2\bar{Y}_4{}^2 + 2\bar{Y}_1{}^2\bar{Y}_4{}^2 + 6\bar{Y}_1\bar{Y}_2\bar{Y}_4{}^2 + 2\bar{Y}_1{}^2\bar{Y}_2\bar{Y}_4 + 2\bar{Y}_1\bar{Y}_2{}^2\bar{Y}_4\right)s$$

$$+\bar{Y}_4{}^2\left(2\bar{Y}_1{}^2 + 3\bar{Y}_1\bar{Y}_2 + \bar{Y}_2{}^2\right) = 0.$$

3.9 NUMERICAL ILLUSTRATION

In this section, numerical illustration of the behaviour of the reliability of the system is provided by generating random samples of size $n = 10000$ each from the quadrivariate exponential distribution using R language version 3.0.2 by fixing the values of the various parameters as $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 3$ and $\lambda_4 = 1$ respectively. The estimated values of the system reliability ($\hat{R}(t)$) based on moments given in (3.8.4) is evaluated for various choices of time periods $t = 1.6, 1.4, \ldots, 3.2$. The following are the values of $\hat{R}(t)$ obtained for various choices of t.

t	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2
$\hat{R}(t)$	0.00398	0.00332	0.00267	0.00211	0.00164	0.00126	0.00097	0.00074	0.00056

The line plot of $(t, \hat{R}(t))$ is shown below.

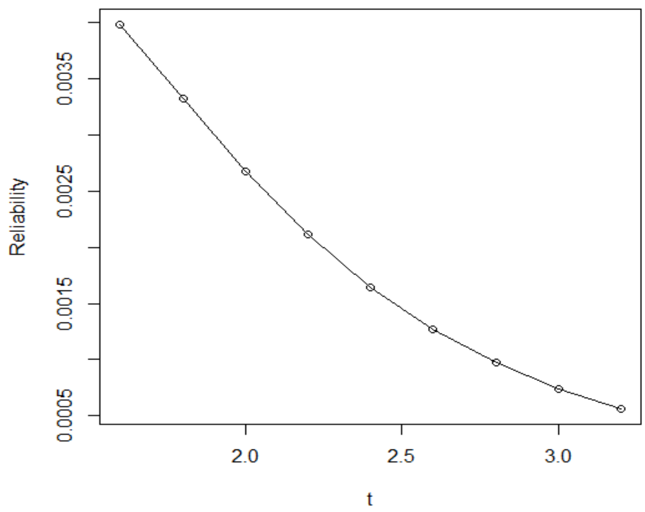


Fig 3.9.1: Line plot of the estimated values of the system reliability based on moments.

It is evident from the plot that as t increases, the value of $\hat{R}(t)$ decreases agreeing with the theoretical results.

CHAPTER 4

A Three Unit Series-Parallel System With Pre-emptive Priority Repair

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4.1. INTRODUCTION

Two unit standby redundant systems have been studied extensively in the past (see Nakagawa and Osaki (1976), Kumar and Agarwal (1980), Srinivasan and Subramanian (1980), Birolini (1985), Yearout et al (1986) and Dhillon (1993)). However the study of n ($\geq$ 3) unit standby redundant systems, though very important, has received much less attention, possibly because of the built-in difficulties in analysing such systems. Subba Rao and Natarajan (1970) have investigated the reliability characteristics of a single unit system with spares and repair facilities. Kistner and Subramanian (1974) considered an n - unit warm standby redundant system with a single repair facility and their results were extended to cover the case of several repair facilities by Subramanian et al (1976). Its dual problem, viz., the n- unit system in which the p.d.f. of the repair time is arbitrary, was studied by Gopalan (1975).

Kalpakam et al (1987) have considered a multi-component system in which n identical units connected in series are needed for the system function, there being m spares and a single repair facility. Gupta et al (1986) have studied the cost-benefit analysis of a single server three unit redundant system with inspection, delayed replacement and two types of repairs. Gupta and Bansal (1991) have analysed a cost function in the case of a three unit standby system subject to random shocks and linearly increasing failure rates. Subramanian and Anantharaman (1995) obtained several system measures for a three unit cold standby system.

Further in a three unit system, the units can be connected in series or they can be connected in parallel or one on line unit with two standbys or two online units with one standby or one unit connected in series with the two units which are connected in parallel and so on. There are many applications of these systems; for example, in a music system an amplifier may be connected in series with two speakers which are connected in parallel. Such systems have not been studied in detail, probably because of the complex nature of the underlying stochastic processes. Only a few authors have studied 3-unit systems (see Nakagawa (2008)). Abuelma'atti and Qamber (1997) have considered a broadcasting system formed of two transmitters which are connected from a common power supply unit (with no provision for repair of the failed units). SPICE circuit simulation program was developed and the state probabilities are obtained.

The three unit system considered by Birolini (1985) consists of a single unit connected in series to a two unit parallel system with a single repair facility and the failed units are taken up for repair in the order in which they arrive. He considered two models. In one model all the units have constant failure rates and constant repair rates. In the other, the repair time of a failed unit has an arbitrary distribution. Expressions are obtained for the reliability and availability of the system. In both models, the system will be in the down state whenever the single unit connected in series is in the failed state. Hence it is not desirable to keep this unit to wait for repair whenever it is in the failed state. It can be noticed that by adopting pre-emptive priority policy for the repair of this unit (see Jaiswal (1968)), the system down time can be reduced and the system availability can be increased. Study of the 3-unit series parallel system with this assumption thus becomes very essential to describe more realistic models.

The aim of this chapter is to consider a 3-unit system in which unit 1 is connected in series and the other two units, unit 2 and unit 3 are connected in parallel. The life-times of all the units are assumed to be exponentially distributed. There is a single repair facility. Whenever a unit fails, the repair for it commences immediately if there is no unit under repair already in the repair facility. However the repair facility gives priority to the repair of the unit 1. That is, whenever the unit 1 fails in the operable state, and at that instant if there is already a unit (either unit 2 or unit 3) under repair, the repair of unit 1 commences immediately keeping the unit under repair in queue, and the repair of which is taken afresh immediately after the repair of unit 1 is completed.

The repair times are exponentially distributed. All the units are assumed to be cold. The reliability and availability of the system are examined. A numerical illustration is also provided.

4.2 SYSTEM DESCRIPTION

We consider a 3-unit system in which unit 1 is connected in series and the other two units, unit 2 and unit 3 are connected in parallel as described in the block diagram given below.

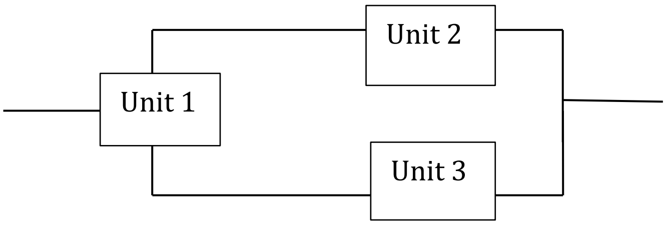


Fig.4.2.1: Block diagram of the series-parallel system

We assume here that the life-times of the unit 1, unit 2 and unit 3 are exponentially distributed with parameters λ_1, λ_2 and λ_3 respectively. There is a single repair facility. We assume that the repair times of the units, unit 1, unit 2 and unit 3 are exponentially distributed with parameters μ_1, μ_2 and μ_3 respectively. In order to have the system operable, the system requires at least one of the units unit 2 or unit 3 in the up-state and the unit 1 in the up-state. We assume that no unit can fail in the system down state. Whenever a unit fails, the repair for it commences immediately if there is no unit under repair already in the repair facility. But the repair facility gives priority to the repair of the unit 1 in the sense that whenever the unit 1 fails in the operable state, and at that instant if there is already a unit (either unit 2 or unit 3) under repair, the repair of unit 1 commences immediately keeping the unit under repair in queue, and the repair of which is taken afresh immediately after the repair of unit 1 is completed. This type of repair-priority is known in the literature as pre-emptive priority repair (see Jaiswal (1968)). At any instant of time, the priority unit is either in the operable state or in the down state undergoing repair while a non-priority unit is either in the operable state or in the failed state waiting for repair or in the failed state undergoing repair.

4.3 RELIABILITY OF THE SYSTEM

We observe that the failure of unit 1 causes a system down. Hence, we observe that for the continuous operation of the system, unit 1 should not fail. Accordingly, to find the reliability of the system, we first consider the reliability of the subsystem consisting of unit 2 and unit 3. This subsystem is a two unit parallel system with constant failure rates and constant repair rates. We define

$R_0(t)$: The reliability of the subsystem given that both the units are operable at time 0.

$R_2(t)$: The reliability of the subsystem given that the unit 3 is in the operable state and the unit 2 is in the failed state at time 0.

$R_3(t)$: The reliability of the subsystem given that the unit 2 is in the operable state and the unit 3 is in the failed state at time 0.

Then, using probabilistic arguments, we obtain

$$R_0(t) = e^{-(\lambda_2+\lambda_3)t} + \lambda_2 e^{-(\lambda_2+\lambda_3)t} © R_2(t) + \lambda_3 e^{-(\lambda_2+\lambda_3)t} © R_3(t) \tag{4.3.1}$$

$$R_2(t) = e^{-(\lambda_3+\mu_2)t} + \mu_2 e^{-(\lambda_3+\mu_2)t} © R_0(t) \tag{4.3.2}$$

$$R_3(t) = e^{-(\lambda_2+\mu_3)t} + \mu_3 e^{-(\lambda_2+\mu_3)t} © R_0(t) \tag{4.3.3}$$

Taking Laplace transformations of the equations (4.3.1), (4.3.2) and (4.3.3), we get

$$R_0^*(s) = \frac{1}{s+\lambda_2+\lambda_3} + \frac{1}{s+\lambda_2+\lambda_3}\{\lambda_2 R_2^*(s) + \lambda_3 R_3^*(s)\}, \tag{4.3.4}$$

$$R_2^*(s) = \frac{1}{s+\lambda_3+\mu_2} + \frac{\mu_2 R_0^*(s)}{s+\lambda_3+\mu_2}, \tag{4.3.5}$$

$$R_3^*(s) = \frac{1}{s+\lambda_2+\mu_3} + \frac{\mu_3 R_0^*(s)}{s+\lambda_2+\mu_3}. \tag{4.3.6}$$

Solving the equations (4.3.4), (4.3.5) and (4.3.6) for $R_0^*(s)$, we obtain

$$R_0^*(s) = \frac{\begin{vmatrix} 1 & -\lambda_2 & -\lambda_3 \\ 1 & s+\lambda_3+\mu_2 & 0 \\ 1 & 0 & s+\lambda_2+\mu_3 \end{vmatrix}}{\begin{vmatrix} s+\lambda_2+\lambda_3 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ 1 & s+\lambda_3+\mu_2 & 0 \\ 1 & 0 & s+\lambda_2+\mu_3 \end{vmatrix}}. \quad (4.3.7)$$

Taking inverse Laplace transform of (4.3.7), we get $R_0(t)$. Let all the units be operable at time t=0 and let the reliability of the main system be $R(t)$. Then using the probabilistic arguments, we get

$$R(t) = e^{-\lambda_1 t} R_0(t). \quad (4.3.8)$$

Taking Laplace transformation of (4.3.8), we get

$$R^*(s) = R_0^*(s+\lambda_1)$$

$$= \frac{\begin{vmatrix} 1 & -\lambda_2 & -\lambda_3 \\ 1 & s+\lambda_1+\lambda_3+\mu_2 & 0 \\ 1 & 0 & s+\lambda_1+\lambda_2+\mu_3 \end{vmatrix}}{\begin{vmatrix} s+\lambda_1+\lambda_2+\lambda_3 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ 1 & s+\lambda_1+\lambda_3+\mu_2 & 0 \\ 1 & 0 & s+\lambda_1+\lambda_2+\mu_3 \end{vmatrix}} \quad (4.3.9)$$

Inversion of equation (4.3.9) yields $R(t)$. We can easily obtain the Mean time to system failure (MTSF) as follows. Since the (MTSF) is given by

$$MTSF = \int_0^\infty R(t)dt = \lim_{s\to 0}\int_0^\infty e^{-st}R(t)dt = R^*(0).$$

We have

$$MTSF = \frac{(\lambda_1 + \lambda_2 + \lambda_3 + \mu_2)(\lambda_1 + \lambda_2 + \lambda_3 + \mu_3) - \lambda_2\lambda_3}{(\lambda_1 + \lambda_2 + \lambda_3 + \mu_2 + \mu_3)(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + \lambda_1\mu_2\mu_3}. \quad (4.3.10)$$

When $\lambda_3 = \lambda_2$ and $\mu_3 = \mu_2 = \mu$, we have

$$MTSF = \frac{(\lambda_1 + 3\lambda_2 + \mu)(\lambda_1 + \lambda_2 + \mu)}{(\lambda_1 + 2\lambda_2 + 2\mu)(\lambda_1 + \lambda_2)^2 + \lambda_1\mu^2}$$

$$= \frac{\lambda_1 + 3\lambda_2 + \mu}{{\lambda_1}^2 + \lambda_1(3\lambda_2 + \mu) + 2{\lambda_2}^2}. \quad (4.3.11)$$

Further, when $\lambda_1 = \lambda_2 = \lambda$, we have

$$MTSF = \frac{4\lambda + \mu}{\lambda(6\lambda + \mu)} \quad (4.3.12)$$

Now we proceed to provide the availability analysis.

4.4. AVAILABILITY ANALYSIS

We denote by 0 the up-state of a unit and by 1 the down-state of a unit undergoing repair. The failed state of a non-priority unit waiting for repair is denoted by 2. We observe that, when a non-priority unit is in state 2, either the unit 1 or the other non-priority unit is in state 1. The state space of unit 1 is $\{0,1\}$ and that of a non-priority unit is $\{0,1,2\}$. Let $X_i(t)$ represent the state of unit i at time t, i = 1, 2, 3. Then, for the priority unit 1, we have

$$X_1(t) = \begin{cases} 0 \; if \; the \; unit \; 1 \; is \; in \; the \; operable \; state; \\ 1 \; if \; the \; unit \; 1 \; is \; in \; the \; failed \; state, \end{cases}$$

and for the non- priority units

$$X_i(t) = \begin{cases} 0 \; ifthe \; unit \; i \; is \; in \; the \; operable \; state; \\ 1 \; if \; the \; unit \; i \; is \; in \; the \; failed \; state \; undergoing \; repair; \\ 2 \; if \; the \; unit \; i \; is \; in \; the \; failed \; state \; waiting \; for \; repair, \end{cases}$$

where i = 2, 3. Now the state of the system is described by the vector stochastic process

$$\xi(t) = (X_1(t), X_2\,(t), X_3(t)).$$

We observe that the state space S of the process $\xi(t)$ is given by

$$\mathrm{S} = \{(0,0,0), (0,0,1), (0,1,0)\,, (1,0,0)\,, (0,1,2), (0,2,1)\,, (1,2,0)\,, (1,0,2)\}.$$

The system up-states are given by

$$S_U = \{(0,0,0), (0,0,1), (0,1,0)\}$$

and the system down-states by

$$S_D = \{(1,0,0), (1,2,0), (1,0,2), (0,1,2), (0,2,1)\}.$$

The state transition diagram is shown below:

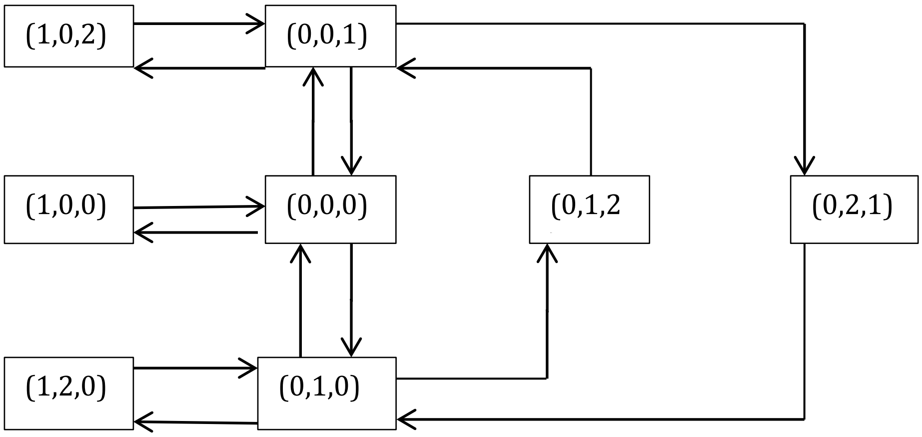


Fig. 2: State transition diagram of the series-parallel system.

The probability mass function of the vector process $\xi(t)$ is defined by

$$p_{i,j,k}(t) = \Pr\{\xi(t) = (i,j,k)\} = \Pr\{X_1(t) = i, X_2(t) = j, X_3(t) = k\}, (i,j,k) \in S.$$

Let $Z(t)$ be the stochastic process defined by

$$Z(t) = \begin{cases} 0 & if the\ system\ is\ up\ at\ time\ t; \\ 1 & if\ the\ system\ is\ down\ at\ time\ t. \end{cases}$$

The probability mass function Z(t) is defined by

$$q_i(t) = \Pr\{Z(t) = i\}, i = 0, 1.$$

Since we have

$$\{Z(t) = 0\} = \Pr\{\xi(t) \in S_U\},$$

$$\{Z(t) = 1\} = \Pr\{\xi(t) \in S_D\}.$$

Thus, we obtain that

$$q_0(t) = p_{0,0,0}(t) + p_{0,1,0}(t) + p_{0,0,1}(t),$$

$$q_1(t) = p_{1,0,0}(t) + p_{1,2,0}(t) + p_{1,0,2}(t) + p_{0,1,2}(t) + p_{0,2,1}(t).$$

It is clear that $q_0(t)$ represents the probability that the system is available; that is, the system is in the operable state at time *t*. Accordingly the availability function A(*t*) of the system is given by

$$A(t) = q_0(t), t > 0. \qquad (4.4.1)$$

To obtain $A(t)$, we assume that all the units are in the operable state at time $t = 0$ so that

$$p_{i,j,k}(0) = \begin{cases} 1 & if\ i = j = k = 0; \\ 0 & otherwise\ , \end{cases} \qquad (4.4.2)$$

and A(0)=1. We now proceed to derive a system of differential equations satisfied by the functions $p_{i,j,k}(t),\ (i,j,k) \in S$.

Using the laws of probability theory and the state transition diagram in Fig. 2, we obtain for the system's up states

$$p_{0,0,0}(t + \Delta t) = p_{0,0,0}(t)[1 - (\lambda_1 + \lambda_2 + \lambda_3)\Delta t] + p_{1,0,0}(t)\mu_1\Delta t$$
$$+p_{0,1,0}(t)\mu_2\Delta t + p_{0,0,1}(t)\mu_3\Delta t + o(\Delta) \qquad (4.4.3)$$

$$p_{0,1,0}(t + \Delta t) = p_{0,1,0}(t)[1 - (\mu_2 + \lambda_1 + \lambda_3)\Delta t] + p_{0,0,0}(t)\lambda_2\Delta t$$
$$+p_{1,2,0}(t)\mu_1\Delta t + p_{0,2,1}(t)\mu_3\Delta t + o(\Delta) \qquad (4.4.4)$$

$$p_{0,0,1}(t+\Delta t) = p_{0,0,1}(t)[1-(\lambda_1+\lambda_2+\mu_3)\Delta t] + p_{0,0,0}(t)\lambda_3\Delta t + p_{1,0,2}(t)\mu_1\Delta t + p_{0,1,2}(t)\mu_2\Delta t + o(\Delta) \quad (4.4.5)$$

Similarly, for the systems down states, we have

$$p_{1,0,0}(t+\Delta t) = p_{1,0,0}(t)[1-\mu_1\Delta t] + p_{0,0,0}(t)\lambda_1\Delta t + o(\Delta) \quad (4.4.6)$$

$$p_{1,2,0}(t+\Delta t) = p_{1,2,0}(t)[1-\mu_1\Delta t] + p_{0,1,0}(t)\lambda_1\Delta t + o(\Delta) \quad (4.4.7)$$

$$p_{1,0,2}(t+\Delta t) = p_{1,0,2}(t)[1-\mu_1\Delta t] + p_{0,0,1}(t)\lambda_1\Delta t + o(\Delta) \quad (4.4.8)$$

$$p_{0,1,2}(t+\Delta t) = p_{0,1,2}(t)[1-\mu_2\Delta t] + p_{0,1,0}(t)\lambda_3\Delta t + o(\Delta) \quad (4.4.9)$$

$$p_{0,2,1}(t+\Delta t) = p_{0,2,1}(t)[1-\mu_3\Delta t] + p_{0,0,1}(t)\lambda_2\Delta t + o(\Delta). \quad (4.4.10)$$

Simplifying equations (4.4.3) to (4.4.10), we have

$$p'_{0,0,0}(t) = -(\lambda_1+\lambda_2+\lambda_3)p_{0,0,0}(t) + p_{1,0,0}(t)\mu_1 + p_{0,1,0}(t)\mu_2 + p_{0,0,1}(t)\mu_3 \quad (4.4.11)$$

$$p'_{0,1,0}(t) = -(\mu_2+\lambda_1+\lambda_3)p_{0,1,0}(t) + p_{0,0,0}(t)\lambda_2 + p_{1,2,0}(t)\mu_1 + p_{0,2,1}(t)\mu_3 \quad (4.4.12)$$

$$p'_{0,0,1}(t) = -(\lambda_1+\lambda_2+\mu_3)p_{0,0,1}(t) + p_{0,0,0}(t)\lambda_3 + p_{1,0,2}(t)\mu_1 + p_{0,1,2}(t)\mu_2 \quad (4.4.13)$$

$$p'_{1,0,0}(t) = -\mu_1 p_{1,0,0}(t) + p_{0,0,0}(t)\lambda_1 \tag{4.4.14}$$

$$p'_{1,2,0}(t) = -\mu_1 p_{1,2,0}(t) + p_{0,1,0}(t)\lambda_1 \tag{4.4.15}$$

$$p'_{1,0,2}(t) = -\mu_1 p_{1,0,2}(t) + p_{0,0,1}(t)\lambda_1 \tag{4.4.16}$$

$$p'_{0,1,2}(t) = -\mu_2 p_{0,1,2}(t) + p_{0,1,0}(t)\lambda_3 \tag{4.4.17}$$

$$p'_{0,2,1}(t) = -\mu_3 p_{0,2,1}(t) + p_{0,0,0}(t)\lambda_2 \tag{4.4.18}$$

Taking Laplace transform on both sides of (4.4.14) to (4.4.18) and using the initial condition (4.4.2), we obtain

$$p^*_{1,0,0}(s) = \frac{\lambda_1}{s + \mu_1}\, p^*_{0,0,0}\,(s) \tag{4.4.19}$$

$$p^*_{1,2,0}(s) = \frac{\lambda_1}{s + \mu_1}\, p^*_{0,1,0}\,(s), \tag{4.4.20}$$

$$p^*_{1,0,2}(s) = \frac{\lambda_1}{s + \mu_1}\, p^*_{0,0,1}(s), \tag{4.4.21}$$

$$p^*_{0,1,2}(s) = \frac{\lambda_3}{s + \mu_2}\, p^*_{0,1,0}(s), \tag{4.4.22}$$

$$p^{*}{}_{0,2,1}(s) = \frac{\lambda_2}{s + \mu_3}\, p^{*}_{0,0,1}(s). \tag{4.4.23}$$

Taking Laplace transform on both sides of (4.4.11) to (4.4.13) and using the initial condition (4.4.2) and the equations (4.4.19) to (4.4.23), we obtain

$$\left\{(s + \lambda_1 + \lambda_2 + \lambda_3) - \frac{\mu_1\lambda_1}{s + \mu_1}\right\} p^{*}{}_{0,0,0}(s) - \mu_2 p^{*}{}_{0,1,0}(s) - \mu_3 p^{*}{}_{0,0,1}(s) = 1 \tag{4.4.24}$$

$$-\lambda_2\, p^{*}{}_{0,0,0}(s) + \left\{(s + \lambda_1 + \lambda_3 + \mu_2) - \frac{\mu_1\lambda_1}{s + \mu_1}\right\} p^{*}{}_{0,1,0}(s) - \frac{\mu_3\lambda_2}{s + \mu_3} p^{*}{}_{0,0,1}(s) = 0 \tag{4.4.25}$$

$$-\lambda_3\, p^{*}{}_{0,0,0}(s) - \frac{\mu_2\lambda_3}{s + \mu_2} p^{*}{}_{0,1,0}(s) + \left\{(s + \lambda_1 + \lambda_2 + \mu_3) - \frac{\mu_1\lambda_1}{s + \mu_1}\right\} p^{*}{}_{0,0,1}(s) = 0. \tag{4.4.26}$$

Solving the simultaneous equations (4.4.24) to (4.4.26), we obtain explicit expressions for $p^{*}{}_{0,0,0}(s)$, $p^{*}{}_{0,1,0}(s)$ and $p^{*}{}_{0,0,1}(s)$ and inverting them, we obtain time-dependent expressions for the up-state probabilities $p_{0,0,0}(t)$, $p_{0,1,0}(t)$ and $p_{0,0,1}(t)$. Then the availability is given by

$$A(t) = p_{0,0,0}(t) + p_{0,1,0}(t) + p_{0,0,1}(t) \tag{4.4.27}$$

The first order product density of unit 1-failure is given by

$$\{ p_{0,0,0}(t) + p_{0,1,0}(t) + p_{0,0,1}(t)\}\, \lambda_1. \tag{4.4.28}$$

Also, the first order product density of system failure is given by

$$p_{0,0,0}(t)\lambda_1 + \ p_{0,1,0}(t)(\lambda_1 + \lambda_3) + p_{0,0,1}(t)(\lambda_1 + \lambda_2). \qquad (4.4.29)$$

As the calculations are quite unwieldy, we provide the steady state analysis in the next section.

4.5 STEADY STATE ANALYSIS

First, we proceed to obtain the steady state expressions for the state probabilities. To this end, we define

$$P((i,j,k)) = \lim_{t\to\infty} p_{i,j,k}(t),\ (i,j,k) \in S. \qquad (4.5.1)$$

By Tauberian theorem, we have

$$\lim_{t\to\infty} p_{i,j,k}(t) = \lim_{s\to 0} sp^{*}{}_{i,j,k}(s)$$

Consequently, we obtain from the equations (4.4.24), (4.4.25) and (4.4.26),

$$\begin{pmatrix} \lambda_2+\lambda_3 & -\mu_2 & -\mu_3 \\ -\lambda_2 & \lambda_3+\mu_2 & -\lambda_2 \\ -\lambda_3 & -\lambda_3 & \lambda_2+\mu_3 \end{pmatrix} \begin{pmatrix} P((0,0,0)) \\ P((0,1,0)) \\ P((0,0,1)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (4.5.2)$$

The equation (4.5.2) reduces to

$$\begin{pmatrix} \lambda_2+\lambda_3 & -\mu_2 & -\mu_3 \\ -\lambda_2 & \lambda_3+\mu_2 & -\lambda_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P((0,0,0)) \\ P((0,1,0)) \\ P((0,0,1)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (4.5.3)$$

Hence we obtain

$$\frac{P((0,0,0))}{\lambda_2\mu_2 + \mu_3(\lambda_3 + \mu_2)} = \frac{P((0,1,0))}{\lambda_2\mu_3 + \lambda_2(\lambda_2 + \lambda_3)} = \frac{P((0,0,1))}{\lambda_3(\lambda_2 + \lambda_3 + \mu_2)} = \alpha. \qquad (4.5.4)$$

From the equations (4.4.19) to (4.4.23), we obtain

$$P\big((1,0,0)\big) = \frac{\lambda_1}{\mu_1} P\big((0,0,0)\big) \qquad (4.5.5)$$

$$P\big((1,2,0)\big) = \frac{\lambda_1}{\mu_1} \big((0,1,0)\big) \qquad (4.5.6)$$

$$P\big((1,0,2)\big) = \frac{\lambda_1}{\mu_1} P\big((0,0,1)\big) \qquad (4.5.7)$$

$$P\big((0,1,2)\big) = \frac{\lambda_3}{\mu_2} P\big((0,1,0)\big) \qquad (4.5.8)$$

$$P\big((0,2,1)\big) = \frac{\lambda_2}{\mu_3} P\big((0,0,1)\big) \qquad (4.5.9)$$

From (4.5.4) to (4.5.9), we obtain

$$P\big((0,0,0)\big) = \alpha\{\lambda_2\mu_2 + \lambda_3\mu_3 + \mu_2\mu_3\} \qquad (4.5.10)$$

$$P\big((0,1,0)\big) = \alpha\big\{\lambda_2\mu_3 + {\lambda_2}^2 + \lambda_3\lambda_2\big\} \qquad (4.5.11)$$

$$P\big((0,0,1)\big) = \alpha\{\lambda_3\lambda_2 + \lambda_3{}^2 + \lambda_3\mu_2\} \tag{4.5.12}$$

$$P\big((1,0,0)\big) = \frac{\lambda_1}{\mu_1}\alpha\{\lambda_2\mu_2 + \lambda_3\mu_3 + \mu_2\mu_3\} \tag{4.5.13}$$

$$P\big((1,2,0)\big) = \frac{\lambda_1}{\mu_1}\alpha\{\lambda_2\mu_3 + \lambda_2{}^2 + \lambda_2\lambda_3\} \tag{4.5.14}$$

$$P\big((0,1,2)\big) = \frac{\lambda_3}{\mu_2}\alpha\{\lambda_2\mu_3 + \lambda_2{}^2 + \lambda_2\lambda_3\} \tag{4.5.15}$$

$$P\big((1,0,2)\big) = \frac{\lambda_1}{\mu_1}\alpha\{\lambda_2\lambda_3 + \lambda_3{}^2 + \lambda_3\mu_2\} \tag{4.5.16}$$

$$P\big((0,2,1)\big) = \frac{\lambda_2}{\mu_3}\alpha\{\lambda_2\lambda_3 + \lambda_3{}^2 + \lambda_3\mu_2\} \tag{4.5.17}$$

Using the condition $\sum_{(i,j,k)\in S} P\big((i,j,k)\big) = 1$, we obtain

$$\alpha = \Big\{(\lambda_2\mu_2 + \lambda_3\mu_3 + \mu_2\mu_3)\left(1 + \frac{\lambda_1}{\mu_1}\right) + (\lambda_2\mu_3 + \lambda_2{}^2 + \lambda_3\lambda_2)\left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_3}{\mu_2}\right) +$$

$$\big(\lambda_3\lambda_2 + \lambda_3{}^2 + \lambda_3\mu_2\big)(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_3}))\Big\}^{-1}.$$

Now we consider three particular cases.

Case 1: When the non-priority units have the same life-time and the same repair rate, we have

$$\lambda_3 = \lambda_2 \; and \; \mu_2 = \mu_3.$$

Hence we obtain

$$P((0,0,0)) = \frac{\mu_1 {\mu_2}^2}{\mu_1\mu_2(\lambda_2+\mu_2)+(\lambda_1\mu_2+\lambda_2\mu_1)(2\lambda_2+\mu_2)}$$

$$P\big((0,1,0)\big) = P(0,0,1) = \frac{\lambda_2\mu_1\mu_2}{\mu_1\mu_2(\lambda_2+\mu_2)+(\lambda_1\mu_2+\lambda_2\mu_1)(2\lambda_2+\mu_2)}$$

$$P((1,0,0)) = \frac{\lambda_1}{\mu_1}P((0,0,0))$$

$$P\big((1,2,0)\big) = P\big((1,0,2)\big) = \frac{\lambda_1}{\mu_1}P((0,1,0))$$

$$P\big((0,1,2)\big) = P\big((0,2,1)\big) = \frac{\lambda_2}{\mu_2}P((0,1,0))\}.$$

The mean stationary rate of occurrence of failure of unit 1 is given by

$$\Theta_1 = \{P\big((0,0,0)\big) + P\big((0,1,0)\big) + P\big((0,0,1)\big)\}\lambda_1$$

$$= \frac{\lambda_1\mu_1\mu_2(2\lambda_2+\mu_2)}{\mu_1\mu_2(\lambda_2+\mu_2)+(\lambda_1\mu_2+\lambda_2\mu_1)(2\lambda_2+\mu_2)}. \qquad (4.5.18)$$

We note that Θ_1 is also the mean stationary rate of occurrence of system failure due to the failure of unit 1. On the other hand, the mean stationary rate of occurrence of system failure due to the failure of unit 2 or unit 3 is given by

$$\Theta_2 = \{P((0,1,0)) + P((0,0,1))\}\lambda_2$$

$$= \frac{2{\lambda_2}^2\mu_1\mu_2}{\mu_1\mu_2(\lambda_2+\mu_2)+(\lambda_1\mu_2+\lambda_2\mu_1)(2\lambda_2+\mu_2)}. \tag{4.5.19}$$

The mean stationary rate of occurrence of system revival is

$$\Theta_3 = \{P((1,0,0)) + P((1,2,0)) + P((1,0,2))\}\mu_1 + \{P((0,1,2)) + P((0,2,1))\}\mu_2$$

$$= \{P((0,0,0)) + 2P((0,1,0))\}\lambda_1 + 2P((0,1,0))\lambda_2$$

$$= \frac{\mu_1\mu_2\,\{\lambda_1(2\lambda_2+\mu_2)+2{\lambda_2}^2\}}{\mu_1\mu_2(\lambda_2+\mu_2)+(\lambda_1\mu_2+\lambda_2\mu_1)(2\lambda_2+\mu_2)}\,. \tag{4.5.20}$$

Case 2: When the non-priority units have the same life-time and the repair times of all the units both priority and non-priority units are same, we have

$$\lambda_3 = \lambda_2 \; and \; \mu_1 = \mu_2 = \mu_3 = \mu$$

Consequently, the equations (4.4.13) to (4.4.17) yield

$$P((0,0,0)) = \frac{\mu^2}{\mu(\lambda_2+\mu)+(\lambda_1+\lambda_2)(2\lambda_2+\mu)} \tag{4.5.21}$$

$$P((0,1,0)) = P(0,0,1) = \frac{\lambda_2\mu}{\mu(\lambda_2+\mu)+(\lambda_1+\lambda_2)(2\lambda_2+\mu)} \tag{4.5.22}$$

$$P\big((1,0,0)\big) = \frac{\lambda_1 \mu}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)}, \tag{4.5.23}$$

$$P\big((1,2,0)\big) = P\big((1,0,2)\big) = \frac{\lambda_1 \lambda_2}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)} \tag{4.5.24}$$

$$P\big((0,1,2)\big) = P\big((0,2,1)\big) = \frac{{\lambda_2}^2}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)}. \tag{4.5.25}$$

The rates Θ_1 , Θ_2 and Θ_3 are given by

$$\Theta_1 = \frac{\lambda_1 \mu(2\lambda_2 + \mu)}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)} \tag{4.5.26}$$

$$\Theta_2 = \frac{2{\lambda_2}^2 \mu}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)} \tag{4.5.27}$$

$$\Theta_3 = \frac{\mu\{\lambda_1(2\lambda_2 + \mu) + 2{\lambda_2}^2\}}{\mu(\lambda_2 + \mu) + (\lambda_1 + \lambda_2)(2\lambda_2 + \mu)} \tag{4.5.28}$$

Case 3: When all the units (both priority and non-priority units) have the same life-time and the repair times of all the units are same, we have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \; and \; \mu_1 = \mu_2 = \mu_3 = \mu$$

In this case we have

$$P\big((0,0,0)\big) = \frac{\mu^2}{4\lambda^2 + 3\mu\lambda + \mu^2} \tag{4.5.29}$$

$$P\big((0,1,0)\big) = P(0,0,1) = P(1,0,0) = \frac{\lambda\mu}{4\lambda^2 + 3\mu\lambda + \mu^2} \tag{4.5.30}$$

$$P\big((1,2,0)\big) = P\big((1,0,2)\big) = P\big((0,1,2)\big) = P\big((0,2,1)\big) = \frac{\lambda^2}{4\lambda^2 + 3\mu\lambda + \mu^2}\,, \tag{4.5.31}$$

The rates θ_1 , θ_2 and θ_3 are given by

$$\theta_1 = \frac{\lambda\mu(2\lambda + \mu)}{\mu(\lambda + \mu) + 2\lambda(2\lambda + \mu)} \tag{4.5.32}$$

$$\theta_2 = \frac{2\lambda^2\mu}{\mu(\lambda + \mu) + 2\lambda(2\lambda + \mu)} \tag{4.5.33}$$

$$\theta_3 = \frac{\mu\{\lambda(2\lambda + \mu) + 2\lambda^2\}}{\mu(\lambda + \mu) + 2\lambda(2\lambda + \mu)} \tag{4.5.34}$$

4.6 COST ANALYSIS

In this section, we follow the notations given below.

e: Event of occurrence of system failure when the failure is due to the failure of unit 1;

f: Event of occurrence of system failure when the failure is due to the failure of unit 2 or unit 3;

$N_1(t)$: The number of e- events that have occurred in the interval $(0, t]$;

$N_2(t)$: The number of f- events that have occurred in the interval $(0, t]$;

C_1: The cost of occurrence of an e-event;

C_2: The cost of occurrence of an f-event.

The total cost $C(t)$ due to system failure at time t is given by

$$C(t) = \int_0^t C_1 dN_1(u) + \int_0^t C_2 dN_2(u). \quad (4.6.1)$$

Then the expected value of $C(t)$ is given by

$$E\,[C(t)] = C_1 \int_0^t h_{1e}(u)du + C_2 \int_0^t h_{1f}(u)du, \quad (4.6.2)$$

where $h_{1e}(t)$ is the first order product density of the point process of e-events and $h_{1f}(t)$ is the first order product density of the point process of f-events. We observe that first order product densities are given by

$$h_{1e}(t) = \{\, p_{0,0,0}(t) + \, p_{0,1,0}(t) + p_{0,0,1}(t)\}\, \lambda_1, \quad (4.6.3)$$

$$h_{1f}(t) = p_{0,1,0}(t)\lambda_3 + p_{0,0,1}(t)\,\lambda_2. \qquad (4.6.4)$$

We now proceed to obtain the mean-stationary rate of the cost which is given by

$$C_\infty = \lim_{t\to\infty} E\left[\frac{C(t)}{t}\right].$$

Using Tauberian theorem, we have

$$C_\infty = C_1 \lim_{t\to\infty} \frac{1}{t}\int_0^t h_{1e}(u)du + C_2 \lim_{t\to\infty} \frac{1}{t}\int_0^t h_{1f}(u)du.$$

$$= C_1 \lim_{t\to\infty} h_{1e}(t) + C_2 \lim_{t\to\infty} h_{1f}(t) \qquad (4.6.5)$$

$$= C_1\{P\big((0,0,0)\big) + P\big((0,1,0)\big) + P\big((0,0,1)\big)\}\lambda_1 + C_2\{P\big((0,1,0)\big)\lambda_3 + P\big((0,0,1)\big)\lambda_2\}$$

$$= C_1\theta_1 + C_2\theta_2.$$

4.7 NUMERICAL ILLUSTRATION

In the particular case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\mu_1 = \mu_2 = \mu_3 = \mu$, we have

$$C_\infty = \frac{\lambda\mu\{(2\lambda + \mu)C_1 + 2\lambda C_2\}}{\mu^2 + 3\lambda\mu + 4\lambda^2}.$$

From the condition

$$\frac{dC_\infty}{d\mu} = 0,$$

we observe that the optimum value of μ is given by the cubic equation

$$(2C_2 - C_1)\mu^2 - 8\lambda C_1\mu - 8\lambda^2(C_1 + C_2) = 0.$$

As an illustration, choosing $\lambda = 0.5$, $C_1 = 50$, $C_2 = 150$, the optimum valued of μ is 1.727 and the corresponding cost is 37.6179.

CHAPTER 5

Classical And Bayesian Estimation Study of Standby System with An Erlangian Repair

Modified version of this chapter is accepted in Pakistan Journal of Statistics.

5.1 INTRODUCTION

Reliability theory is concerned with statistical description of various measures of system performance such as reliability, MTSF, point availability, steady state availability and so on and have been studied in detail using their respective failure time and repair time density functions. The failures and repairs in any system are influenced by several factors such as system configuration, the environmental conditions under which the system operates and the varying failures (minor and major) and so on, which cannot be controlled or assessed well in advance. For a detailed study of systems operating in random environments, see Chandrasekhar and Natarajan (2001) and Chandrasekhar et al (2005). There is an urgent need to avail the sample information in order to draw valid inference about the measures of system performance. System configurations in Reliability Theory are often used in the design and analysis of telecommunication systems, traffic systems and so on. In real life situations, in the problems involving system configurations, it is essential to carry out a practical analysis of measures of system performance. These problems often require the applications of statistical tools such as point estimation, interval estimation, hypotheses testing and Bayesian inference. The subjective Bayesian inference is very recent; An important aspect of Reliability Theory is to estimate lifetime and repair time parameters for which both classical and Bayesian approaches are useful. It is often the case that some information is available on the parameters of lifetime and repair time distributions from prior experiments or prior analysis of the lifetime or repair time data. Bayesian approach provides the methodology for the formal incorporation of the prior information with the current data. Analysis of several systems in these directions has not received much attention in the past.

In recent times, there has been great interest in analysing the system from a Bayesian perspective (see Yadavalli et al (2001), Yadavalli et al (2005)). However, all the Bayesian research work till date has been on the usual constant failure and service rates. Our interest in this chapter is on statistical inference procedures of a standby system with two stage Erlangian repair. The choice of the Erlangian distribution is motivated by the fact that an Erlangian variate with shape parameter k is the sum of k independent and identically distributed (iid) exponential variates. Hence, an Erlangian repair model can be thought of as a model with repair in k exponential phases, where repair at each phase is exponential with rate μ.

In our model, we perform a simple experiment by observing m lifetimes and n repair times. Given this experiment, the likelihood is of the form

$$\begin{aligned} L(\text{parameters}\,|\text{data}) &= \lambda^{m} e^{-\lambda u}\left(\frac{\mu^{2n}}{\Gamma(2)^{n}} e^{-\mu v} \prod_{j=1}^{n} y_j\right) \\ &= \mu^{2n} e^{-(\lambda u+\mu v)} \lambda^{m} \prod_{j=1}^{n} y_j \quad , \qquad (5.1.1) \end{aligned}$$

where u and v are the sums of m observed lifetimes and n repair times respectively. For the system under consideration, we have described maximum likelihood and Bayesian procedures. Flexible priors for lifetime and repair time parameters are introduced under the assumption that priors for life time and repair time parameters are independent. By using these conjugate prior distributions, we evaluate the posterior distributions along with Bayes estimators.

Several authors have studied extensively two unit standby redundant systems in the past. Osaki and Nakagawa (1976) gave a bibliography of the work on two unit systems.

In this chapter, we discuss in detail a two unit cold standby system with constant failure rate λ and constant repair rate μ (both unknown) and two repair stages. Measures of system performance such as reliability, mean time to system failure (MTSF), point availability and steady state availability of a two unit cold standby system with constant failure rate and two stage Erlangian repair time distribution are obtained. Further maximum likelihood estimator (MLE) of the system reliability, asymptotic confidence limits for steady state availability of the system and Bayes estimator of MTSF are derived.

The model and the assumptions, expressions for system reliability, MTSF, availability and associated statistical inference together with numerical illustration are discussed in detail in the following sections.

5.2 THE MODEL AND ASSUMPTIONS

The system under consideration is a two unit cold standby system with a single repair facility. Precisely, we have the following assumptions:

(i) The units are similar and statistically independent. Each unit has a constant failure rate, say λ.

(ii) A unit in standby will not fail. In other words, the failure rate of the standby unit is zero.

(iii) There is only one repair facility and the repair time distribution is a two stage Erlangian with probability density function (pdf) given by

$$g(t) = \mu^2 \, t \, e^{-\mu t}, \; 0 < t < \infty, \; \mu > 0. \qquad (5.2.1)$$

It may be noted that the density given in (5.2.1) corresponds to the sum of two iid exponential varieties each with the parameter μ.

(iv) Each unit is new after repair.

(v) Switch is perfect and the switchover is instantaneous.

5.3 ANALYSIS OF THE SYSTEM

To analyse the behaviour of the system, we note that any time t, the system will be found in any one of the following mutually exclusive and exhaustive states S_i , i=0,1,…,4.

State **0 (S_0):** One unit is operating online and the other is kept in the standby.

State **1 (S_1):** One unit is operating online and the other unit is in the first stage of repair.

State **2 (S_2):** One unit is operating online and the other unit is in the second stage of repair.

State **3 (S_3):** One unit is in the first stage of repair and the other unit is waiting for repair.

State **4 (S_4):** One unit is in the second stage of repair and the other unit is waiting for repair.

Since the two stage Erlangian distribution corresponds to the sum of two iid exponential variates and exponential distribution satisfies lack of memory property (LMP), it follows that the stochastic process describing the behaviour of the system is a Markov process with the infinitesimal generator given by

$$
Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 \\
0 & -(\lambda+\mu) & \mu & \lambda & 0 \\
\mu & 0 & -(\lambda+\mu) & 0 & \lambda \\
0 & 0 & 0 & -\mu & \mu \\
0 & \mu & 0 & 0 & -\mu
\end{pmatrix} \end{array}
\tag{5.3.1}
$$

It may be noted that the system upstates are 0,1 and 2, while the state's 3 and 4 are the system down states. Let $p_i(t)$ be the probability that the system is in state S_i , i=0,1,…,4 at time t with the initial condition $p_0(0)=1$. We assume that initially all the two units are operable and obtain the measures of system performance as follows.

5.4 SYSTEM RELIABILITY

The system reliability R(t) is the probability of failure free operation of the system in (0, t]. To derive an expression for R(t), we restrict the transitions of the Markov process to the upstates namely 0,1 and 2. Using the infinitesimal generator of the process given in (5.3.1), pertaining to these upstates, we derive the following differential – difference equations.

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_2(t) \tag{5.4.1}$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t)-(\lambda+\mu)p_1(t) \tag{5.4.2}$$

$$\frac{dp_2(t)}{dt} = \mu p_1(t)-(\lambda+\mu)p_2(t) \tag{5.4.3}$$

Let $L_i(s)$ denote the Laplace transform of $p_i(t)$, i=0,1,2. Taking Laplace transform on both the sides of differential – difference equations given in (5.4.1) – (5.4.3), solving for $L_i(s)$, i=0,1,2 and inverting, we get $p_i(t)$, i=0,1,2 as follows.

$$p_0(t) = \sum_{i=1}^{3} \frac{(\alpha_i+\lambda+\mu)^2}{\prod_{j=1,j\neq i}^{3}(\alpha_i-\alpha_j)} e^{\alpha_i t} \tag{5.4.4}$$

$$p_1(t) = \lambda \sum_{i=1}^{3} \frac{(\alpha_i+\lambda+\mu)}{\prod_{j=1,j\neq i}^{3}(\alpha_i-\alpha_j)} e^{\alpha_i t} \tag{5.4.5}$$

$$p_2(t) = \lambda\mu \sum_{i=1}^{3} \frac{1}{\prod_{j=1,j\neq i}^{3}(\alpha_i-\alpha_j)} e^{\alpha_i t} \tag{5.4.6}$$

Thus, the system reliability R(t) is given by

$$R(t) = p_0(t) + p_1(t) + p_2(t) \tag{5.4.7}$$

where α_i, i = 1, 2, 3 are the roots of the cubic equation

$$s^3+(3\lambda+2\mu)s^2+(3\lambda^2+4\lambda\mu+\mu^2)s+\lambda^2(\lambda+2\mu)=0.$$

5.5 MEAN TIME TO SYSTEM FAILURE

The system MTSF is the expected or average time to failure and is given by

$$MTSF = L_0(0) + L_1(0) + L_2(0)$$

$$= \frac{(2\lambda^2+4\lambda\mu+\mu^2)}{\lambda^2(\lambda+2\mu)}. \tag{5.5.1}$$

5.6 SYSTEM AVAILABILITY

The system availability A(t) is the probability that the system operates within the tolerances at a given instant of time and is obtained as follows. From the infinitesimal generator given in (5.3.1), we have the following system of differential – difference equations.

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)+\mu p_2(t) \tag{5.6.1}$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t)-(\lambda+\mu)p_1(t)+\mu p_4(t) \tag{5.6.2}$$

$$\frac{dp_2(t)}{dt} = \mu p_1(t)-(\lambda+\mu)p_2(t) \tag{5.6.3}$$

$$\frac{dp_3(t)}{dt} = \lambda p_1(t)-\mu p_3(t) \tag{5.6.4}$$

$$\frac{dp_4(t)}{dt} = \lambda p_2(t) + \mu p_3(t)-\mu p_4(t). \tag{5.6.5}$$

Solving the system of differential – difference equation (5.6.1) – (5.6.5) using the fact that $\sum_{i=0}^{4} p_i\,(t)=1$, we obtain the solution as follows.

$$p_0(t) = \frac{\mu^4}{\prod_{i=1}^{4}\alpha_i} + \lambda\mu^2 \sum_{i=1}^{4} \frac{(\alpha_i+\mu)^2}{\alpha_i(\alpha_i+\lambda)\prod_{j=1,j\neq i}^{4}(\alpha_i-\alpha_j)} e^{\alpha_i t} \tag{5.6.6}$$

$$p_1(t)=\frac{\lambda\mu^2(\lambda+\mu)}{\prod_{i=1}^{4}\alpha_i}+\lambda\sum_{i=1}^{4}\frac{(\alpha_i+\mu)^2(\alpha_i+\lambda+\mu)}{\alpha_i\prod_{j=1,j\neq i}^{4}(\alpha_i-\alpha_j)}e^{\alpha_i t} \tag{5.6.7}$$

$$p_2(t)=\frac{\lambda\mu^3}{\prod_{i=1}^{4}\alpha_i}+\lambda\mu\sum_{i=1}^{4}\frac{(\alpha_i+\mu)^2}{\alpha_i\prod_{j=1,j\neq i}^{4}(\alpha_i-\alpha_j)}e^{\alpha_i t} \tag{5.6.8}$$

$$p_3(t)=\frac{\lambda^2\mu(\lambda+\mu)}{\prod_{i=1}^{4}\alpha_i}+\lambda^2\sum_{i=1}^{4}\frac{(\alpha_i+\mu)(\alpha_i+\lambda+\mu)}{\alpha_i\prod_{j=1,j\neq i}^{4}(\alpha_i-\alpha_j)}e^{\alpha_i t} \tag{5.6.9}$$

$$p_4(t)=\frac{\lambda^2\mu(\lambda+2\mu)}{\prod_{i=1}^{4}\alpha_i}+\lambda^2\mu\sum_{i=1}^{4}\frac{[\lambda+2(\alpha_i+\mu)]}{\prod_{j=1,j\neq i}^{4}(\alpha_i-\alpha_j)}e^{\alpha_i t} \tag{5.6.10}$$

where α_i, i=1,2,3,4 are the roots of the equation

$$s^4+(3\lambda+4\mu)s^3+\left(3\lambda^2+10\lambda\mu+6\mu^2\right)s^2+(\lambda^3+8\lambda^2\mu+9\lambda\mu^2+4\mu^3)s$$
$$+\mu(2\lambda^3+4\lambda^2\mu+2\lambda\mu^2+\mu^3) = 0.$$

Hence, the system availability is given by

$$A(t) = p_0(t)+p_1(t)+p_2(t)$$
$$= \frac{\mu^2(\lambda+\mu)^2}{\prod_{i=1}^4 \alpha_i} + \lambda\mu^2 \sum_{i=1}^{4} \frac{(\alpha_i+\mu)^2}{\alpha_i(\alpha_i+\lambda)\prod_{j=1,j\neq i}^4(\alpha_i-\alpha_j)} e^{\alpha_i t}$$

$$+\lambda \sum_{i=1}^{4} \frac{(\alpha_i+\mu)^2(\alpha_i+\lambda+\mu)}{\alpha_i \prod_{j=1,j\neq i}^4(\alpha_i-\alpha_j)} e^{\alpha_i t} + \lambda\mu \sum_{i=1}^{4} \frac{(\alpha_i+\mu)^2}{\alpha_i \prod_{j=1,j\neq i}^4(\alpha_i-\alpha_j)} e^{\alpha_i t}. \qquad (5.6.11)$$

5.7 STEADY STATE AVAILABILITY

The system steady state availability is the expected fractional amount of time in a continuum of operating time that the system is in an upstate and is given by

$$A_\infty = \lim_{t\to\infty} A(t) = \frac{\mu(\lambda+\mu)^2}{(2\lambda^3+4\lambda^2\mu+2\lambda\mu^2+\mu^3)}, \qquad (5.7.1)$$

which is in agreement with Chandrasekhar and Natarajan (1994).

5.8 ML ESTIMATOR OF SYSTEM RELIABILITY

Let $X_1,X_2,\ldots,X_n$ and $Y_1,Y_2,\ldots,Y_n$ be two random samples each of size n drawn from exponential failure and two stage Erlangian repair time populations with the pdf given by (5.1.1). It is clear that $\overline{X}$ and $\frac{\overline{Y}}{2}$ are the ML estimators of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively, where $\overline{X}$ and $\overline{Y}$ are the sample means of failure times and repair times of a failed unit respectively. Hence, the ML estimator of system reliability is given by

$$\hat{R}(t)=\sum_{i=1}^{3}\frac{[(2\overline{X}+\overline{Y}+\hat{\alpha}_i\overline{X}\,\overline{Y})^2+\overline{Y}(4\overline{X}+\overline{Y}+\hat{\alpha}_i\overline{X}\,\overline{Y})]}{(\overline{X}\,\overline{Y})^2\prod_{j=1,j\neq i}^{3}(\alpha_i-\alpha_j)}e^{\hat{\alpha}_i t} \tag{5.8.1}$$

where $\hat{\alpha}_i$, i=1,2,3 are the roots of the cubic equation

$$\overline{X}^3\overline{Y}^2s^3+\overline{X}^2\overline{Y}(4\overline{X}+3\overline{Y})s^2+\overline{X}(2\overline{X}+\overline{Y})(2\overline{X}+3\overline{Y})s+\overline{Y}(4\overline{X}+\overline{Y})=0. \tag{5.8.2}$$

5.9 CONFIDNECE INTERVAL FOR STEADY STATE AVAILABILITY OF THE SYSTEM

In section 5.8, we have seen that $\overline{X}$ and $\frac{\overline{Y}}{2}$ are the ML estimators of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively. Let $\theta_1=\frac{1}{\lambda}$ and $\theta_2=\frac{1}{\mu}$. Clearly, the steady state availability of the system given in (5.7.1) is simplified to

$$A_\infty=\frac{\theta_1(\theta_1+\theta_2)^2}{(\theta_1^3+2\theta_1^2\theta_2+4\theta_1\theta_2^2+2\theta_2^3)} \tag{5.9.1}$$

and hence ML estimator of A_∞ is given by

$$\widehat{A}_\infty = \frac{\overline{X}(2\overline{X}+\overline{Y})^2}{(4\overline{X}^3+4\overline{X}^2\overline{Y}+4\overline{X}\,\overline{Y}^2+\overline{Y}^3)}. \tag{5.9.2}$$

It may be noted that $\widehat{A}_\infty$ given in (5.9.2) is real valued function in $\overline{X}$ and $\overline{Y}$, which is also differentiable. By applying the multivariate central limit theorem (See Radhakrishna Rao (1974)), it is seen that

$$\sqrt{n}\left[\left(\overline{X}, \frac{\overline{Y}}{2}\right) - (\theta_1,\theta_2)\right] \xrightarrow{d} N_2(\mathbf{0},\Sigma) \text{ as } n\to\infty,$$

where the dispersion matrix $\Sigma=\left((\sigma_{ij})\right)$ is given by $\Sigma=\text{diag}(\theta_1, \frac{\theta_2^2}{2})$.

Again from Radhakrishna Rao (1974), we have

$$\sqrt{n}\left[\widehat{A}_\infty - A_\infty\right] \xrightarrow{d} N(0,\sigma^2(\theta)) \text{ as } n\to\infty,$$

where $\theta=(\theta_1,\theta_2)$ and

$$\sigma^2(\theta)=\theta_1^2\left(\frac{\partial A_\infty}{\partial\theta_1}\right)^2+\frac{\theta_2^2}{2}\left(\frac{\partial A_\infty}{\partial\theta_2}\right)^2 \tag{5.9.3}$$

It may be noted that the partial derivatives $\left(\frac{\partial A_\infty}{\partial\theta_i}\right)$, i=1,2 are given by

$$\left(\frac{\partial A_\infty}{\partial\theta_1}\right) = \frac{2\theta_2^2(3\theta_1^3+6\theta_1^2\theta_2+4\theta_1\theta_2^2+\theta_2^3)}{(\theta_1^3+2\theta_1^2\theta_2+4\theta_1\theta_2^2+2\theta_2^3)^2} \tag{5.9.4}$$

$$\left(\frac{\partial A_\infty}{\partial \theta_2}\right) = \frac{-2\theta_1\theta_2(3\theta_1^3+6\theta_1^2\theta_2+4\theta_1\theta_2^2+\theta_2^3)}{(\theta_1^3+2\theta_1^2\theta_2+4\theta_1\theta_2^2+2\theta_2^3)^2} \tag{5.9.5}$$

Substituting (5.9.4) and (5.9.5) in (5.9.3) and simplifying, we get

$$\sigma^2(\theta) = \frac{6\theta_1^2\theta_2^4(3\theta_1^3+6\theta_1^2\theta_2+4\theta_1\theta_2^2+\theta_2^3)^2}{(\theta_1^3+2\theta_1^2\theta_2+4\theta_1\theta_2^2+2\theta_2^3)^4}. \tag{5.9.6}$$

Thus, $\widehat{A}_\infty$ is a CAN estimator of A_∞. There are several methods for generating CAN estimators and the method of moments and the method of maximum likelihood are commonly used to generate such estimators, see Sinha (1986).

Let $\sigma^2(\hat{\theta})$ be an estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator namely, $\hat{\theta} = \left(\overline{X}\,,\, \frac{\overline{Y}}{2}\right)$. Further, let $\widehat{\sigma^2} = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\widehat{\sigma^2}$ is a consistent estimator of $\sigma^2(\theta)$. i.e., $\widehat{\sigma^2} \xrightarrow{P} \sigma^2(\theta)$ as $n \to \infty$. By Slustky theorem, we have $\frac{\sqrt{n}[\hat{A}_\infty - A_\infty]}{\hat{\sigma}} \xrightarrow{d} N(0,1)$. Hence a $100(1\text{-}\alpha)\%$ confidence interval for A_∞ is given by $\hat{A}_\infty \pm k_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}$,where $k_{\frac{\alpha}{2}}$ is the upper $100\left(1\text{-}\frac{\alpha}{2}\right)\%$ quantile of standard normal distribution and $\hat{\sigma}$ is obtained from (5.9.6) and is given by

$$\hat{\sigma} = \sqrt{\frac{3\overline{X}^2\,\overline{Y}^4(24\overline{X}^3+24\overline{X}^2\,\overline{Y}+8\overline{X}\,\overline{Y}^2+\overline{Y}^3)^2}{2(4\overline{X}^3+4\overline{X}^2\,\overline{Y}+4\overline{X}\,\overline{Y}^2+\overline{Y}^3)^4}} \tag{5.9.7}$$

In the next section, Bayes estimator of MTSF under squared error loss function is obtained using the same sample observations as in section 5.3

5.10 BAYES ESTIMATION OF MTSF IN A TWO UNIT STANDYBY SYSTEM

In this section, we derive the Bayes estimator of MTSF by considering Gamma distributions with parameters (α,β) and (δ,ω) as natural conjugate priors for the lifetimes and repair times respectively. In other words, λ and μ have the following prior distributions with the probability density functions as follows.

$$\tau_1(\lambda|\alpha,\beta)=\frac{\alpha^{\beta}}{\Gamma(\beta)}e^{-\alpha\lambda}\lambda^{\beta-1},\ 0<\lambda<\infty;\ \alpha,\ \beta>0, \tag{5.10.1}$$

$$\tau_2(\mu|\delta,\omega)=\frac{\delta^{\omega}}{\Gamma(\omega)}e^{-\delta\mu}\mu^{\omega-1},\ 0<\mu<\infty;\ \delta,\ \omega>0. \tag{5.10.2}$$

Bayes estimator is usually obtained by minimizing the expected loss function, where the expectation is taken with respect to the joint distribution of sample observations and the parameters. Point estimators, which are known as Bayes estimators in the Bayesian framework are the expected values of posterior distribution under squared error loss function (SELF). Bayes estimators are usually obtained under (i) SELF (ii) entropy loss function (ELF) and (iii) precautionary loss function (PLF).

It can be shown that the posterior distributions of λ and μ given the sample observations $X_1, X_2,\ldots,X_m$ and $Y_1, Y_2,\ldots,Y_n$ are respectively given by

$$q_1(\lambda|x_1,x_2,\ldots,x_m)=\frac{(\alpha+u)^{m+\beta}}{\Gamma(m+\beta)}e^{-\lambda(\alpha+u)}\lambda^{(m+\beta)-1},\ 0<\lambda<\infty;\ \alpha,\ u,\ m,\ \beta>0, \tag{5.10.3}$$

$$q_2\left(\mu|y_1, y_2,\ldots,y_n\right) = \frac{(\delta+v)^{2n+\omega}}{\Gamma(2n+\omega)} e^{-\mu(\delta+v)}\mu^{(2n+\omega)-1},\ 0 <\mu < \infty;\ \delta,\ v,\ n,\ \omega > 0\ . \qquad (5.10.4)$$

In other words, λ and μ are distributed as Gamma with parameters $(\alpha+u,\ m+\beta)$ and $(\delta+v,\ 2n+\omega)$ respectively.

Bayes estimator of MTSF say $MTSF^*$, given the sample observations is defined as

$MTSF^*$ = E [MTSF | sample observations]

$$= \int_0^\infty \int_0^\infty \frac{(2\lambda^2+4\lambda\mu+\mu^2)}{\lambda^2(\lambda+2\,\mu)}\ q_1(\lambda|x_1,x_2\ldots,x_m)q_2\left(\mu|y_1,\ldots,y_n\right) d\lambda\, d\mu \qquad (5.10.5)$$

$$= \int_0^\infty \int_0^\infty \left(\frac{1}{\mu}+\frac{2}{\lambda}+\frac{\mu}{2\lambda^2}\right) \sum_{j=0}^{\infty}\frac{(-1)^j}{2^j}\left(\frac{\lambda}{\mu}\right)^j q_1(\lambda|x_1,x_2,\ldots,x_m)\ q_2\left(\mu|y_1,y_2,\ldots,y_n\right) d\lambda\ d\mu,$$

$$\lambda<\mu$$

$$= \frac{1}{\Gamma(m+\beta)\Gamma(2n+\omega)}\left[\begin{array}{l} \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j}{2^j}\frac{(\delta+v)^{j+1}}{(\alpha+u)^j}\ \Gamma(m+\beta+j)\Gamma(2n+\omega-j-1) \\ \displaystyle +2\sum_{j=0}^{\infty}\frac{(-1)^j}{2^j}\frac{(\delta+v)^j}{(\alpha+u)^{j-1}}\ \Gamma(m+\beta+j-1)\Gamma(2n+\omega-j) \\ \displaystyle +\frac{1}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{2^j}\frac{(\delta+v)^{j-1}}{(\alpha+u)^{j-2}}\ \Gamma(m+\beta+j-2)\Gamma(2n+\omega-j+1)\end{array}\right] \qquad (5.10.6)$$

5.11 NUMERICAL ILLUSTRATION

In this section, the performance of the Bayes estimate of MTSF i.e., $MTSF^*$ is illustrated through simulated data. The estimates are obtained using (5.10.4). Monte Carlo integration method is used to evaluate the integrals in (5.10.4) in two steps.

First, the inner integral is evaluated by generating random samples using the posterior density of λ treating μ as unknown. The outer integral is then evaluated using random samples generated from the posterior density of μ. The values of hyper parameters in the posterior density functions are fixed as $m = n = 50$; $\alpha = 2.5$; $\beta = 3.0$; $\delta = 0.75$; $\omega = 1.5$. u and v are determined by taking the sums of iid samples of sizes m and n generated from exponential and Erlangian distributions respectively as given in pdf (5.2.1). For generating samples, the following choices of λ and μ are used namely, $\lambda = 3, 6, 9, 12$ and $\mu = 2, 4, 6, 8$. The results of the simulation based on 10,000 Monte Carlo runs are presented below.

Table 5.11.1: Bayes estimate of MTSF

μ \ λ	3.0	6.0	9.0	12.0
2.0	0.01662	0.03390	0.07491	0.11261
4.0	0.02634	0.05689	0.14142	0.15921
6.0	0.02723	0.11658	0.16159	0.26393
8.0	0.03311	0.14515	0.26254	0.39131

From the above table, it can be observed that for fixed repair rate (μ), the Bayes estimate of MTSF increases as the failure rate (λ) increases. Similarly, for fixed λ, the Bayes estimate of MTSF increases as μ increases. In other words, whenever the two unit cold standby system with single repair facility under consideration exhibits high failure and repair rates, then the associated mean time before failure is also high.

CHAPTER 6

Three Unit Series-Parallel System with Preparation Time for the Repair Facility

6.1 INTRODUCTION

Two unit standby redundant systems remained under the focus of several researchers working in various fields such as industry and system engineering. A bibliography of the work on two unit system is given by Osaki and Nakagawa (1976) (see also, Kumar and Agarwal (1980), Srinivasan and Subramanian (1980), Birolini (1985), Yearout et al (1986) and Dhillon (1993)). However the study of n ($\geq$3) unit standby redundant systems, though very important, has received less attention, possibly because of the built-in difficulties in analysing such systems. Further in a three unit system, the units can be connected in series or they can be connected in parallel or one on line unit with two standbys or two online units with one standby or one unit connected in series with the two units which are connected in parallel and so on. There are many applications of these systems; for example, in a music system an amplifier may be connected in series with two speakers which are connected in parallel. Such systems have not been studied in detail, probably because of the complex nature of the underlying stochastic processes. Only a few authors have studied 3-unit systems (see Nakagawa (2008)).

Abuelma'atti and Qamber (1997) have considered a broadcasting system formed of two transmitters which are connected from a common power supply unit (with no provision for repair of the failed units). SPICE circuit simulation program was developed and the state probabilities are obtained. A 3-unit series parallel system was considered by Birolini (1985). The system considered by him consists of a single unit connected in series to a two unit parallel system with a single repair facility and the failed units are taken up for repair in the order in which they arrived. He considered two models. In one model all the units have constant failure rates and constant repair rates. In the other, the repair time of a failed unit has an arbitrary distribution.

Sarma and Pervez (1984) studied a three unit system in which all the distributions are assumed to be discrete. Muller (2005) studied a three unit standby system when the lifetime and repair time distributions are assumed to be arbitrary, and obtained expressions for reliability and availability. In all the above models, it is clear that they have assumed that the repair facility is continuously available to attend the repair of the failed unit (Bon and Paltanea (2001), Krishnamoorthy et al (2002), Frostig and Levikson (2002), Ke and Pearn (2004), Wang et al (2009), Wang et al (2011).

But it is reasonable to expect that a preparation might be needed to get the repair facility ready before the next repair could be taken up. If this preparation is started only when a unit arrives for repair, it is easy to solve the problem, since the preparation time plus the actual repair time may be taken as total repair time. But this preparation time starts immediately after each repair completion, so that the facility becomes available at the earliest. Two unit parallel systems with two-dissimilar units and preparation time were studied by Sarma (1982). He assumed that the repair times and preparation times are to be non-markovian. The confidence limit for a two-unit parallel system was subsequently studied by Yadavalli et al (2002a, b).

A 3-unit system is considered in which unit 1 is connected in series and the other two units, unit 2 and unit 3 are connected in parallel. The life-times of all the units are assumed to be exponentially distributed. There is a single repair facility. That is, whenever a unit fails, the repair for it commences immediately if there is no unit under repair already in the repair facility. However, it is reasonable to expect that a preparation might be needed to get the repair facility ready before the next repair could be taken up. In this chapter, a three units series parallel system is studied in which the repair facility is not available for a random time after each repair completion.

The reliability and availability of the system are examined.

6.2 SYSTEM DESCRIPTION AND NOTATION

A three unit series parallel system in which one unit is connected in series with a 2-unit parallel system is considered. The configuration of the system is given in Figure 6.2.1.

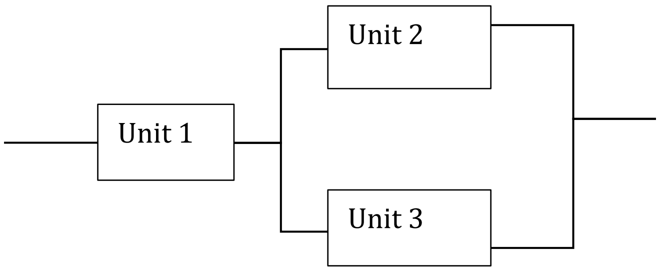


Fig. 6.2.1: Block diagram of the series parallel system

We assume here that the life-times of the unit 1, unit 2 and unit 3 are exponentially distributed with parameters λ_1, λ_2 and λ_3 respectively. There is a single repair facility. We assume that the repair times of the units, unit 1, unit 2 and unit 3 are exponentially distributed with parameters μ_1, μ_2 and μ_3 respectively. In order to have the system operable, the system requires at least one of the units unit 2 or unit 3 in the up-state and the unit 1 in the up-state. We assume that no unit can fail in the system down state. There is a single repair facility (RF) and failed units are repaired. After completion of each repair of a failed unit preparation time is required for the commencement of repair of the next failed unit. On completion of the preparation time RF becomes free to take up any failed unit for repair. If a unit fails during the preparation time of the RF the failed unit will wait in a queue till RF becomes ready to take up repair. This preparation time is assumed to be an exponentially distributed random variable with parameter ν. whenever a unit fails, the repair for it commences immediately if there is no unit under repair already in the repair facility and the RF is ready to take up a repair. But the repair facility gives priority to the repair of the unit 1 in the sense that whenever the unit 1 fails, and at that instant if there is already a unit (either unit 2 or unit 3) is under repair, the repair of unit 1 commences immediately keeping the unit under repair in queue, and the repair of which is taken afresh immediately after the repair of unit 1 is completed. However preparation time is required at the commencement of repair of that unit. This type of repair-priority is known in the literature as pre-emptive priority repair (see Jaiswal (1968)). If the repair of unit 2 or unit 3 is stopped in the middle unit 1 will be taken up for repair immediately and no preparation time is required. However if the RF is under preparation time period at the epoch of failure of unit 1 then unit 1 will also wait and will be taken up for repair immediately as soon as the RF becomes ready to take up a repair. We proceed to find the reliability and availability of the system.

6.3 THE RELIABILITY OF THE SYSTEM

We observe that the failure of unit 1 causes a system down. Hence, we observe that for the continuous operation of the system, unit 1 should not fail. Accordingly, to find the reliability of the system, we first consider the reliability of the subsystem consisting of unit 2 and unit 3. This subsystem is a two unit parallel system with constant failure rates, constant repair rates and preparation time for the repair facility.

We define the following:

$\boldsymbol{R_0(t)}$ **:** The reliability of the subsystem given that both the units are operable at time 0 and the preparation time just commences for the repair facility.

$\mathbf{R_2(t)}$: The reliability of the subsystem given that the unit 3 is in the operable state and unit 2 is in the failed state at time 0 and repair just commences for it.

$\mathbf{R_3(t)}$: The reliability of the subsystem given that the unit 2 is in the operable state and unit 3 is in the failed state at time 0 and repair just commences for it.

Then, using probabilistic arguments, we obtain

$$R_0(\mathrm{t}) = e^{-(\lambda_2 + \lambda_3)\mathrm{t}} + e^{-\lambda_3 t}[\lambda_2 e^{-\lambda_2 t}(1 - e^{-\nu t}) + \nu e^{-\nu t}(1 - e^{-\lambda_2 t})©R_2(\mathrm{t})]$$

$$+e^{-\lambda_2 t}[\lambda_3 e^{-\lambda_3 t}(1 - e^{-\nu t}) + \left[\nu e^{-\nu t}\left(1 - e^{-\lambda_3 t}\right)©R_3(\mathrm{t})\right] \quad (6.3.1)$$

$$R_2(\mathrm{t}) = e^{-(\lambda_3 + \mu_2)t} + \mu_2 e^{-(\lambda_3 + \mu_2)t}©R_0(\mathrm{t}) \quad (6.3.2)$$

$$R_3(\mathrm{t}) = e^{-(\lambda_2 + \mu_3)t} + \mu_3 e^{-(\lambda_2 + \mu_3)t}©R_0(\mathrm{t}). \quad (6.3.3)$$

Taking Laplace transformations of the equations (6.3.1), (6.3.2) and (6.3.3), we get

$$R_0^*(s) = \frac{1}{s + \lambda 2 \ + \lambda 3} + \left[\lambda_2 \left\{\frac{1}{s + \lambda_2 + \lambda_3} - \frac{1}{s + \lambda_2 + \lambda_3 + v}\right\} + v\left\{\frac{1}{s + \lambda_3 + v} - \frac{1}{s + \lambda_2 + \lambda_3 + v}\right\}\right] R_2^*(s) + \left[\lambda_3 \left\{\frac{1}{s + \lambda_2 + \lambda_3} - \frac{1}{s + \lambda_2 + \lambda_3 + v}\right\} + v\left\{\frac{1}{s + \lambda_2 + v} - \frac{1}{s + \lambda_2 + \lambda_3 + v}\right\}\right] R_3^*(s) \tag{6.3.4}$$

$$R_2^*(s) = \frac{1}{s + \lambda_3 + \mu_2} + \frac{\mu_2 R_0^*(s)}{s + \lambda_3 + \mu_2} \tag{6.3.5}$$

$$R_3^*(s) = \frac{1}{s + \lambda_3 + \mu_3} + \frac{\mu_3 R_0^*(s)}{s + \lambda_3 + \mu_3} \tag{6.3.6}$$

Solving the equations (6.3.4), (6.3.5) and (6.3.6) for $R^*(0)$, we obtain

$$R_0^*(s) = \frac{\begin{vmatrix} 1 & \frac{\lambda_2 v(\lambda_2+v+2(s+\lambda_3))}{(s+\lambda_2+\lambda_3+v)(s+\lambda_3+v)} & \frac{\lambda_3 v(\lambda_3+v+2(s+\lambda_2))}{(s+\lambda_2+\lambda_3+v)(s+\lambda_2+v)} \\ s+\lambda_3+\mu_2 & s+\lambda_3+\mu_2 & 0 \\ s+\lambda_2+\mu_3 & 0 & s+\lambda_2+\mu_3 \end{vmatrix}}{\begin{vmatrix} s+\lambda_2+\lambda_3 & \frac{\lambda_2 v(\lambda_2+v+2(s+\lambda_3))}{(s+\lambda_2+\lambda_3+v)(s+\lambda_3+v)} & \frac{\lambda_3 v(\lambda_3+v+2(s+\lambda_2))}{(s+\lambda_2+\lambda_3+v)(s+\lambda_2+v)} \\ -\mu_2 & s+\lambda_3+\mu_2 & 0 \\ -\mu_3 & 0 & s+\lambda_2+\mu_3 \end{vmatrix}} \qquad (6.3.7)$$

Taking inverse Laplace transform of (6.3.7), we get $R_0(t)$.

Now we consider the main system. Let all the units be operable at time $t = 0$ and preparation time commences for the repair facility. Let the reliability of the main system be $R(t)$. Then, using the probabilistic arguments, we get

$$R(t) = e^{-\lambda_1 t} R_0(t). \qquad (6.3.8)$$

Taking Laplace transformation of (6.3.8), we get

$$R^*(s) = R_0^*(s + \lambda_1)$$

$$= \frac{\begin{vmatrix} 1 & \frac{\lambda_2 v(\lambda_2+v+2(s+\lambda_1+\lambda_3))}{(s+\lambda_1+\lambda_2+\lambda_3+v)(s+\lambda_1+\lambda_3+v)} & \frac{\lambda_3 v(\lambda_3+v+2(s+\lambda_1+\lambda_2))}{(s+\lambda_1+\lambda_2+\lambda_3+v)(s+\lambda_1+\lambda_2+v)} \\ s+\lambda_1+\lambda_3+\mu_2 & s+\lambda_1+\lambda_3+\mu_2 & 0 \\ s+\lambda_1+\lambda_2+\mu_3 & 0 & s+\lambda_1+\lambda_2+\mu_3 \end{vmatrix}}{\begin{vmatrix} s+\lambda_1+\lambda_2+\lambda_3 & \frac{\lambda_2 v(\lambda_2+v+2(s+\lambda_1+\lambda_3))}{(s+\lambda_1+\lambda_2+\lambda_3+v)(s+\lambda_1+\lambda_3+v)} & \frac{\lambda_3 v(\lambda_3+v+2(s+\lambda_1+\lambda_2))}{(s+\lambda_1+\lambda_2+\lambda_3+v)(s+\lambda_1+\lambda_2+v)} \\ -\mu_2 & s+\lambda_1+\lambda_3+\mu_2 & 0 \\ -\mu_3 & 0 & s+\lambda_1+\lambda_2+\mu_3 \end{vmatrix}} \qquad (6.3.9)$$

Inversion of the equation (6.3.9) yields R(t). We can easily obtain the Mean time to system failure (MTSF) as follows. Since the (MTSF) is given by

$$MTSF = \int_0^{\infty} R(t)dt = \lim_{s\to 0} \int_0^{\infty} e^{-st} R(t)dt = R^*(0).$$

6.4 THE AVAILABILITY OF THE SYSTEM

For convenience we assume $\lambda_3 = \lambda_2$ and $\mu_3 = \mu_2$ and define

X (t) = State of the units at time 't' represented as 0,1,2 ,3,4

= 0 ⇒ all the three units are operable

= 1 ⇒ unit 1 is operable and one of the units in the subsystem is in the failed state

= 2 ⇒ unit 1 is operable and both the units in the subsystem are in failed state

= 3 ⇒ unit 1 is in the failed state and the other two units are operable

= 4 ⇒ unit 1 is in the failed state and one of the units in the subsystem is in the failed state.

Y (t) = State of the repair facility at time 't' represented as 0,1

= 0 RF is available

= 1 RF is under preparation period.

$Z(t)$ the vector process $\{X(t), Y(t)\}$ representing the State of the system at time t.

We note that the stochastic process $Z(t)$ is a Markov process with state space.

$$E = \{(i, j): i = 0, 1, 2, 3, 4 : j = 0, 1\}.$$

Also we write $E = E_1 \cup E_2$ where

$$E_1 = \{(i, j): i = 0,1: j = 0, 1\}.$$

And

$$E_2 = \{(i, j): i = 2,3,4 : j = 0, 1\}.$$

E_1 represents the system up states and E_2 the set of down system states. Now we consider the different states and the possible transitions from them.

State (0,0): all the three units are operable and the RF is available. The possibilities for a transition from this state are (i) one of the units unit 2 or unit 3 fails or (ii) unit 1 fails. That is a transition to state (1,0) occurs with rate $2\lambda_2$ or a transition to state (3,0) occurs with rate λ_1.

Similarly we consider all other states and the possible transitions from them and give the results in the following table. For convenience the state (i, j) is denoted as (ij) ;

Transition table 6.4.1.

From State	To State	Rate
(0,0)	(1,0)	$2\lambda_2$
(0,0)	(3,0)	λ_1
(1,0)	(0,1)	μ_2
(1,0)	(2,0)	λ_2
(1,0)	(4,0)	λ_1
(2,0)	(1,1)	μ_2
(3,0)	(0,1)	μ_1
(4,0)	(1,1)	μ_1
(0,1)	(1,1)	$2\lambda_2$
(0,1)	(3,1)	λ_1
(0,1)	(0,0)	ν
(1,1)	(1,0)	ν
(1,1)	(2,1)	λ_2
(1,1)	(4,1)	λ_1
(2,1)	(2,0)	ν
(3,1)	(3,0)	v
(4,1)	(4,0)	v

We also get the reliability of the system when $\lambda_2 = \lambda_3$ and $\mu_3 = \mu_2$ directly. Assume that the system has entered state (01) initially at time $t = 0$. Define

$$R_{ij}(t) = \Pr\{\text{the system is up in } (0,t] \mid \text{the system has entered state } (i,j) \text{ at } t = 0\}, (i,j) \in E_1.$$

We now derive equation for the reliability of the system

$$R_{00}(t) = e^{-(\lambda_1+2\lambda_2)t} + 2\lambda_2 e^{-(\lambda_1+2\lambda_2)t} © R_{10}(t) \tag{6.4.2}$$

$$R_{10}(t) = e^{-(\lambda_1+\lambda_2+\mu_2)t} + \mu_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © R_{01}(t) \tag{6.4.3}$$

$$R_{01}(t) = e^{-(\lambda_1+2\lambda_2)t+v} + 2\lambda_2 e^{-(\lambda_1+2\lambda_2+v)t} © R_{11}(t)$$
$$+ ve^{-(\lambda_1+2\lambda_2+v)t} © R_{00}(t) \tag{6.4.4}$$

$$R_{11}(t) = e^{-(\lambda_1+\lambda_2+v)t} + ve^{-(\lambda_1+\lambda_2+v)t} © R_{10}(t). \tag{6.4.5}$$

Solving the set of equations (6.4.2) $-$ (6.4.5) after taking Laplace transforms we get the Laplace transform $R^*_{01}(s)$ of the reliability of the system. This result is in agreement with equation (6.3.9) if λ_3 and μ_3 are respectively replaced with λ_2 and μ_2 and simplified.

6.5 AVAILABILITY

Define

$$A_{ij}(t) = Pr\{\text{the system is up at time } t \mid \text{the system}$$
$$\text{has entered state } (i,j) \text{ at } t = 0\}, (i,j) \in E.$$

We now derive equation for the availability of the system

$$A_{00}(t) = e^{-(\lambda_1+2\lambda_2)t} + 2\lambda_2 e^{-(\lambda_1+2\lambda_2)t} © A_{10}(t) + \lambda_1 e^{-(\lambda_1+2\lambda_2)t} © A_{30}(t) \qquad (6.5.1)$$

$$A_{10}(t) = e^{-(\lambda_1+\lambda_2+\mu_2)t} + \mu_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © A_{01}(t)$$

$$\lambda_2 e^{-(\lambda_1+\lambda_2+\mu_2)t} © A_{20}(t) + \lambda_1 e^{-(\lambda_1+\lambda_2+\mu_2)t} © A_{40}(t) \qquad (6.5.2)$$

$$A_{01}(t) = e^{-(\lambda_1+2\lambda_2)t+v} + 2\lambda_2 e^{-(\lambda_1+2\lambda_2+v)t} © A_{11}(t) + v e^{-(\lambda_1+2\lambda_2+v)t} © A_{00}(t)$$

$$+ \lambda_1 e^{-(\lambda_1+\lambda_2+v)t} © A_{31}(t) \qquad (6.5.3)$$

$$A_{11}(t) = e^{-(\lambda_1+\lambda_2+v)t} + v e^{-(\lambda_1+\lambda_2+v)t} © A_{10}(t)$$

$$+ \lambda_1 e^{-(\lambda_1+2\lambda_2+v)t} © A_{41}(t) + \lambda_2 e^{-(\lambda_1+\lambda_2+v)t} © A_{21}(t) \qquad (6.5.4)$$

$$A_{20}(t) = \mu_2 e^{-\mu_2 t} © A_{11}(t) \qquad (6.5.5)$$

$$A_{30}(t) = \mu_1 e^{-\mu_1 t} © A_{01}(t) \qquad (6.5.6)$$

$$A_{40}(t) = \mu_1 e^{-\mu_1 t} © A_{11}(t) \qquad (6.5.7)$$

$$A_{21}(t) = v e^{-vt} © A_{20}(t) \qquad (6.5.8)$$

$$A_{31}(t) = v e^{-vt} © A_{30}(t) \qquad (6.5.9)$$

$$A_{41}(t) = v e^{-vt} © A_{40}(t). \qquad (6.5.10)$$

After taking Laplace transforms the set of equations (6.5.1)-(6.5.10) can be solved for $A^*_{01}(s)$ which gives the Laplace transform of the availability of the system.

6.6 STEADY STATE AVAILABILITY

Let

$$Pij\ (t) = \Pr\{Z\ (t) = (i,j)|Z\ (0) = (0,1)\}$$

and let

$$\lim_{t\to\infty} P_{ij}(t) = P_{ij}.$$

Using the principle of flow we have

$$(\lambda_1 + 2\lambda_2)P_{00} = vP_{01} \tag{6.6.1}$$

$$(\lambda_1 + 2\lambda_2)P_{01} = \mu_1 P_{30} + \mu_2 P_{10} \tag{6.6.2}$$

$$(\lambda_1 + \lambda_2 + \mu_2)P_{10} = vP_{11} \tag{6.6.3}$$

$$(\lambda_1 + \lambda_2 + v)P_{11} = \mu_1 P_{40} + \mu_2 P_{20} + 2\lambda_2 P_{01} \tag{6.6.4}$$

$$\mu_2 P_{20} = \lambda_2 P_{10} + vP_{21} \tag{6.6.5}$$

$$vP_{21} = \lambda_2 P_{11} \tag{6.6.6}$$

$$\mu_1 P_{30} = \lambda_1 P_{10} + vP_{31} \tag{6.6.7}$$

$$vP_{31} = \lambda_1 P_{01} \tag{6.6.8}$$

$$\mu_1 P_{40} = \lambda_1 P_{00} \tag{6.6.9}$$

$$vP_{41} = \lambda_1 P_{11} \tag{6.6.10}$$

The set of equations can be solved for P_{ij} and state availability is given by

$$A_\infty = \sum_{(ij)\in E_1} P_{ij}.$$

CHAPTER 7

Two Unit Standby Systems with Imperfect Switch and Preparation Time

7.1 INTRODUCTION

Two unit standby redundant systems have been studied extensively, as can be seen from the following published bibliographies, Osaki and Nakagawa (1976), Lie et al (1977), Kumar and Agarwal (1980), Sarma (1982).

If the system is automatically controlled then the system must include a switching device that will detect when the online unit has failed and will switch the standby unit, if operable, online. The repair facility will then, when available repair the failed unit. Gnedenko (1969) in an expository text discussed several reasons why the switching device might also fail. Mathematical modelling of such two-unit standby systems with an imperfect switch has been done by several researchers, in some models the switching device and the units are repaired by the same repair facility and in others each has its own repair facility (Sarma (1982), Pan (1998), Jain (2016), Jain and Rani (2013), Wang et al (2012), Osaki (1972)).

Even if the switch does not fail the switch over time might not be negligible such models with non-instantaneous switch over time have been investigated by Khalil (1977), Sarma (1982), Botha (2000).

Sarma (1982), Botha (2000), Yadavalli et al, (2005) considered the situation whereby the repair facility might not always be available. In their models, after each repair, either of a switching device or of a unit, the repair facility will be unavailable for a period of time which is called preparation time.

Following kendall's notations in queuing theory **A|B|C** where

A: Online life time distribution

B: Life time distribution of the standby unit

C: Repair time distribution we apply them to reliability models

M: Exponential distribution

G: General / arbitrary distribution

7.2 MODEL (G|M|M SYSTEM WITH IMPERFECT SWITCH AND PREPARATION TIME)

7.2.1 SYSTEM DISCRIPTIONS AND NOTATION;

1. The system consists of 2 identical units; either unit perform the system function satisfactorily. When one unit is operating online and the other unit is kept standby.
2. At $t = 0$ the new unit is just put online and another new unit is kept in standby. The repair facility is just available and switching device is operable.
3. The unit and the switching device share the repair facility and they are new after each repair.
4. Switchover is instantaneous.
5. After each time point, the switch is repaired and the repair facility is not available for random time which is denoted as preparation time.
6. If the switch device and a unit are both in the failed state then the switching device will be given head-of-line priority (Jaiswal (1968)) for repair over the unit.

7. The life of a unit while operating on line is an arbitrarily distributed r.v. with p.d.f. f(.).
8. The life time of a unit while in standby the repair time of a unit, the life time and the repair time of a switching device for a repair facility is exponentially distributed r.vs with parameters λ_b, μ, λ_s, μ_s $and\ d$ respectively.

7.2.3 SUBSYSTEM

In order to obtain subsystem measure for the main system, which consists of two units, the switching device and the repair facility the following subsystem will be considered. When one unit is operating online continuously, the subsystem will consist of off line unit, the switching device and the repair facility. The state of the subsystem is denoted by the stochastic process $\{Z\ (t);\ t > 0\}$. There are different exhaustive states that $Z(t)$ can take, depending on the state of the offline unit, the switching device and the repair facility as shown in the table 7.2.1. The following auxiliary function is defined to facilitate the study of the behaviour of the process $\{Z\ (t);\ t > 0\}$.

Table 7.2.1

States of Subsystem

State	Off line Unit	State of Switching Devices	Repair facility
0	o	o	A
1	r	o	A
2	q.r	o	Na
3	q.r	q.r	Na
4	q.r	r	A
5	r	q.r	A
6	o	o	Na
7	o	r	A
8	o	q.r	Na

Where o **:** operable; r **:** under repair; q.r **:** queering for repair; a **:** available; na : not available.

Table 7.2.2

Event	One Unit	State of Other Unit	Switching Device	Regenerative Results
E_0	J.o	o	o	A
E_1	J.o	r	o	A
E_2	J.o	q.r	o	Na

J.o **:** Just online.

$$P_{ij}(t) = P[Z(t) = j \mid Z(0) = i] \tag{7.2.1}$$

$$P_{ij}(0) = \delta_{ij} = \begin{Bmatrix} 1, & if\ i = j \\ 0, & if\ i \neq j \end{Bmatrix} \tag{7.2.2}$$

The difference - differential equations are obtained by using the state transition diagram Figure 7.2.1 and the Chapman-Kolmogorov equations

$$P_{ij}(t+\Delta) = \sum_{k \epsilon Z(t)} P_{ik}(t)P_{kj}(\Delta) \tag{7.2.3}$$

$$P'_{i0}(t) = -(\lambda_s + \lambda_b)P_{i0}(t) + dP_{i0}(t) + \mu P_{i1}(t) \tag{7.2.4}$$

$$P'_{i1}(t) = -(\mu + \lambda_s)P_{i1}(t) + \lambda_s P_{i0}(t) + dP_{i2}(t) \tag{7.2.5}$$

$$P'_{i2}(t) = -(d + \lambda_s)P_{i2}(t) + \mu_s P_{i4}(t) + \lambda_b P_{i6}(t) \tag{7.2.6}$$

$$P'_{i3}(t) = -dP_{i3}(t) + \lambda_s P_{i2}(t) + \lambda_b P_{i8}(t) \tag{7.2.7}$$

$$P'_{i4}(t) = -\mu_s P_{i4}(t) + \lambda_b P_{i7}(t) + dP_{i3}(t) \tag{7.2.8}$$

$$P'_{i5}(t) = -\mu P_{i5}(t) + \lambda_s P_{i1}(t) \tag{7.2.9}$$

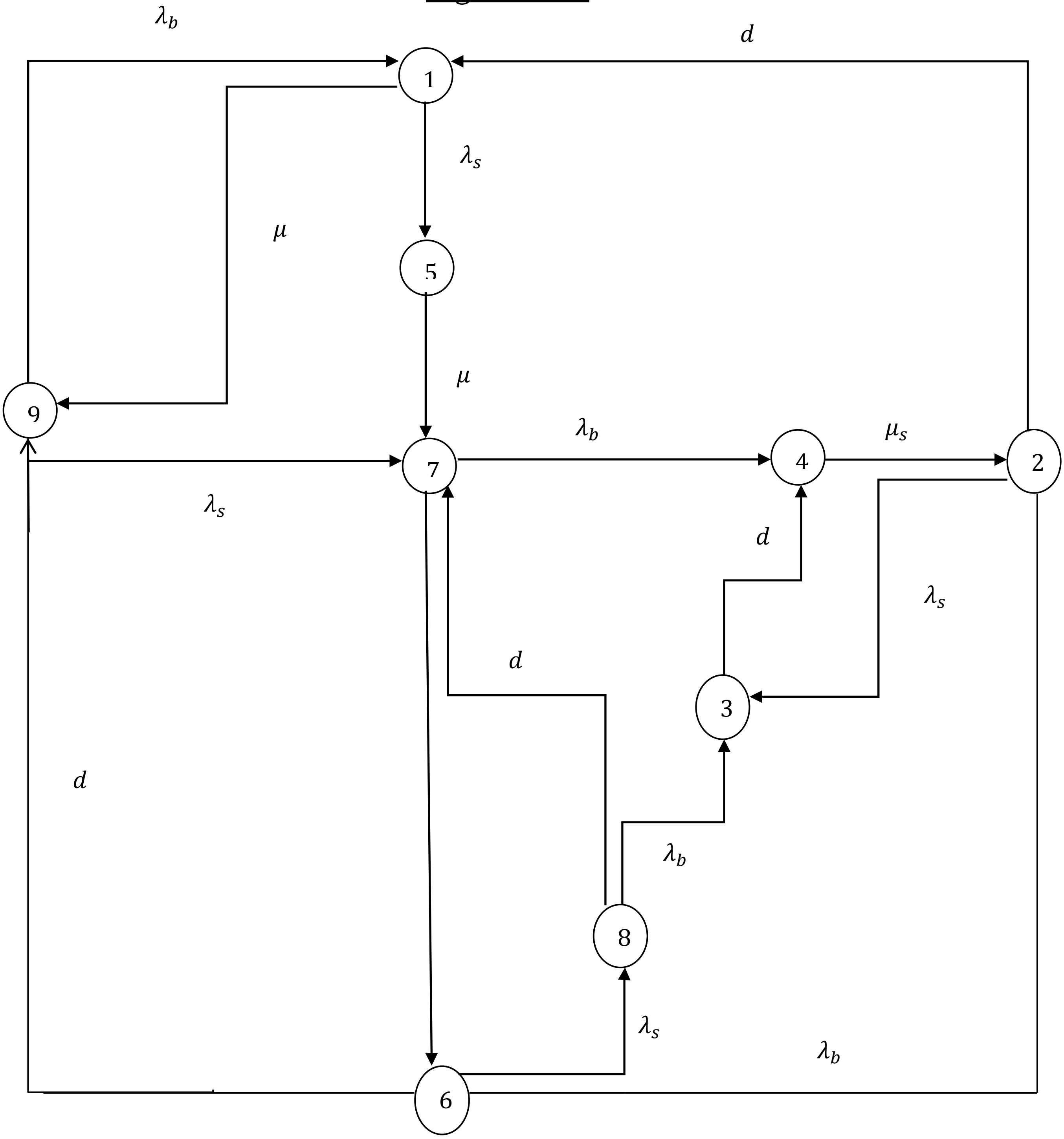
$$P'_{i6}(t) = -(\lambda_s + d + \lambda_b)P_{i6}(t) + \mu_s P_{i7}(t) \tag{7.2.10}$$

$$P'_{i7}(t) = -(\lambda_b + \mu_s)P_{i7}(t) + \lambda_s P_{i0}(t) + \mu P_{i5}(t) + dP_{i8}(t) \tag{7.2.11}$$

$$P'_{i8}(t) = -(\lambda_b + d)P_{i8}(t) + \lambda_s P_{i6}(t) \tag{7.2.12}$$

$$i = 0,1,2,\ldots$$

Figure 7.2.1


λ_b
d
1
λ_s
μ
5
μ
9
λ_b
μ_s
7
4
2
λ_s
d
λ_s
d
3
d
λ_b
8
λ_s
λ_b
6

7.2.4 AVAILABILITY ANALYSIS

The regenerative events given in table 7.2.2 are required for the availability analysis, and Figure 7.2.2 gives various mutually exclusive and exhaustive possibilities used to derive the availability equations $A_i(t), i = 0,1,2, \ldots$

$$A_i(t) = \bar{F}(t) + [f(t)P_{i0}(t)]©A_1(t) + [f(t)P_{i6}(t)]©A_2(t)$$
$$+ \sum_{\substack{j=1 \\ j\neq 6}}^{8} [f(t)P_{ij}(t)]©\Pi_{j0}(t)©A_1(t)$$
$$+ \sum_{\substack{j=1 \\ j\neq 6}}^{8} [f(t)P_{ij}(t)]©\Pi_{j6}(t)©A_2(t) \qquad (7.2.13)$$

Where

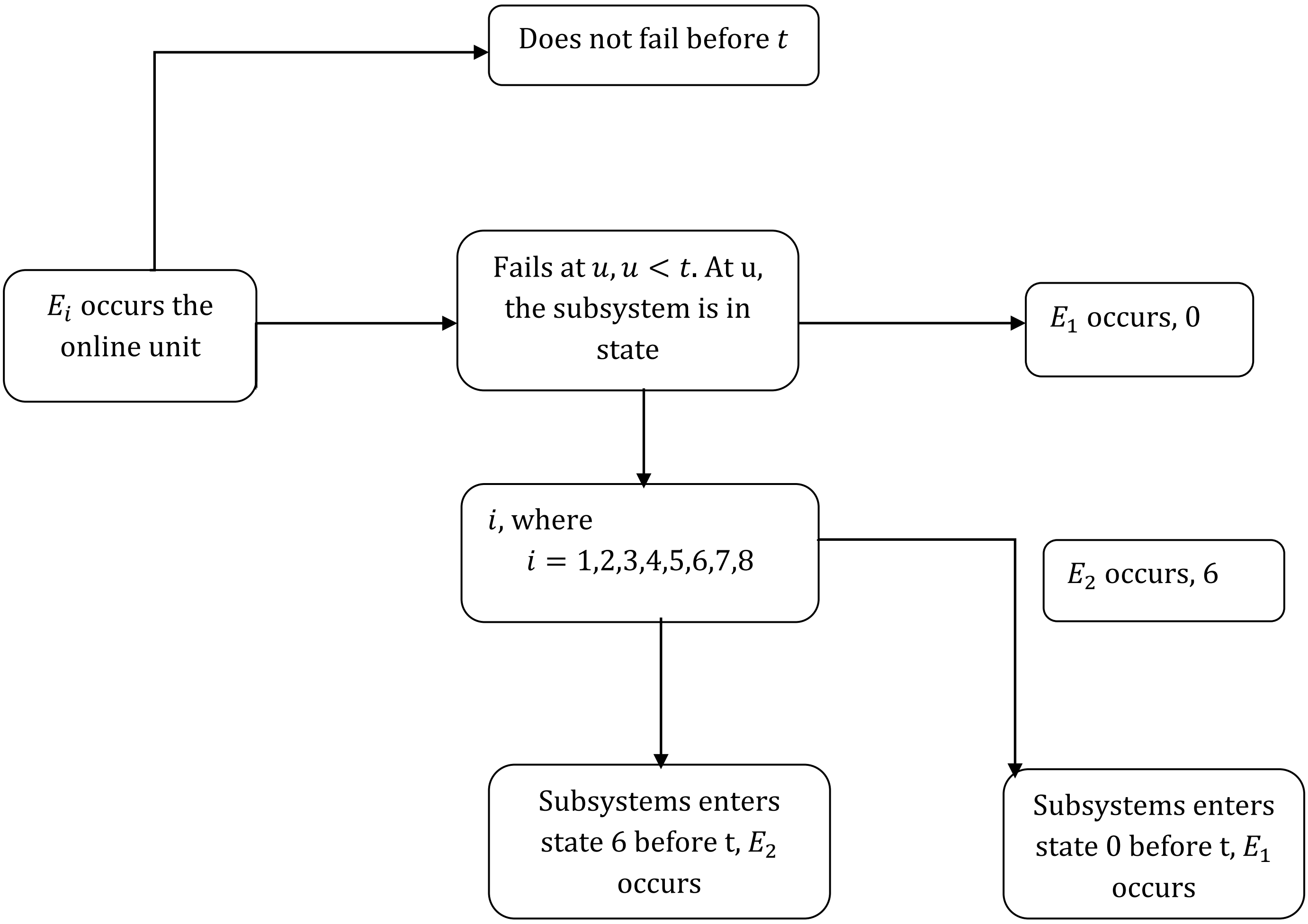
$$\Pi_{10}(t) = \mu e^{-(\mu+\lambda_s)t} + \left[\mu e^{-\mu t}\left\{1 - e^{-\lambda_s t}\right\}\right]©\Pi_{70}(\mathrm{t}) \qquad (7.2.14)$$

$$\Pi_{16}(t) = \left[\mu e^{-\mu t}\left\{1 - e^{-\lambda_s t}\right\}\right]©\Pi_{46}(\mathrm{t}) \qquad (7.2.15)$$

$$\Pi_{20}(t) = de^{-(d+\lambda_s)t}©\left[\mu e^{-\mu t}\left\{1 - e^{-\lambda_s t}\right\}\right]©\Pi_{70}(\mathrm{t}) + de^{-(d+\lambda_s)t}©\mu e^{-(\mu+\lambda_s)t}$$
$$+\left[de^{-dt}\left\{1 - e^{-\lambda_s t}\right\}\right]©\Pi_{40}(\mathrm{t}) \qquad (7.2.16)$$

$$\Pi_{26}(t) = de^{-(d+\lambda_s)t}©\mu e^{-(\mu+\lambda_s)t}©\Pi_{76}(\mathrm{t})$$
$$+\left[de^{-dt}\left\{1 - e^{-\lambda_s t}\right\}\right]©\Pi_{46}(\mathrm{t}) \qquad (7,2,17)$$

Figure 7.2.2



$$\Pi_{30}(t) = de^{-dt} © \Pi_{40}(t) \tag{7.2.18}$$

$$\Pi_{36}(t) = de^{-dt} © \Pi_{46}(t) \tag{7.2.19}$$

$$\Pi_{40}(t) = \mu_s e^{-\mu_s t} © \Pi_{20}(t) \tag{7.2.20}$$

$$\Pi_{46}(t) = \mu_s e^{-\mu_s t} © \Pi_{26}(t) \tag{7.2.21}$$

$$\Pi_{50}(t) = \mu e^{-\mu t} © \Pi_{70}(t) \tag{7.2.22}$$

$$\Pi_{56}(t) = \mu e^{-\mu t} © \Pi_{76}(t) \tag{7.2.23}$$

$$\Pi_{70}(t) = \mu_s e^{-\mu_s t}\left[1 - e^{-\lambda_b t}\right] © \Pi_{20}(\mathrm{t}) \quad (7.2.24)$$

$$\Pi_{76}(t) = \mu_s e^{-\mu_s t}\left[1 - e^{-\lambda_b t}\right] © \Pi_{26}(\mathrm{t}) + \mu_s e^{-(\mu_s + \lambda_b)t} \quad (7.2.25)$$

$$\Pi_{80}(t) = d e^{-(d+\lambda_b)t} © \Pi_{70}(\mathrm{t}) + d e^{-dt}\left[1 - e^{-\lambda_b t}\right] © \Pi_{40}(\mathrm{t}) \quad (7.2.26)$$

$$\Pi_{86}(t) = d e^{-(d+\lambda_b)t} © \Pi_{76}(\mathrm{t}) + d e^{-dt}\left[1 - e^{-\lambda_b t}\right] © \Pi_{46}(\mathrm{t}) \quad (7.2.27)$$

By the Laplace transform technique the equation (7.2.13) can be solved. The steady state availability A_∞ can be obtained from the relation

$$A_\infty = \lim_{t\to\infty} A_i(t) = \lim_{s\to 0} sA_i^*(s) = \frac{N}{D}$$

Where

$$\mathrm{N} = \{[1 - U_{10}^*(0) + U_{00}^*(0)][1 - U_{26}^*(0) + U_{16}^*(0)] - [U_{00}^*(0) - U_{20}^*(0)]\,[U_{06}^*(0) - U_{i6}^*(0)]\}\int_0^\infty tf(t)dt$$

$$D = -U_{26}^{*'}(0)[1 - U_{10}^*(0)] - U_{10}^{*'}(0)[1 - U_{26}^*(0)] - U_{16}^*(0)U_{20}^{*'}(0) - U_{20}^*(0)U_{16}^{*'}(0)$$

$$U_{ij}^{*'}(0) = \frac{d}{ds}U_{ij}^*(s)\Big|_{\mathrm{s}=0}$$

$$U_{ij}^*(s) = V_{ij}^*(s) + \sum_{\substack{k=1\\k\neq 6}}^{8} V_{ik}^*(s)\Pi_{kj}^*(s) \qquad \begin{array}{l} i = 0,1,2 \\ j = 0,6. \end{array}$$

$$V_{ij}(t) = f(t)P_{ij}(t), \qquad i,j = 0,1,2,\ldots,8.$$

7.2.5 RELIABILITY ANALYSIS

The reliability equation can easily be found from the availability equations by noting that there must not be a system failure in (0,t], and omitting those terms where the system is in down state

$$R_i(t) = \bar{F}(t) + [f(t)P_{i0}(t)]©R_1(t) + [f(t)P_{i6}(t)]©R_2(t) \qquad (7.2.28)$$

By taking Laplace transform for the equations (7.2.28) we can solve for $R_0^*(s)$.

The mean time to system failure can be obtained by the relation on

$$MTSF = R_0^*(0) = \frac{N_R}{D_R}.$$

$$N_R = [\{1 - V_{i0}^*(0) + V_{00}^*(0)\}\{1 - V_{26}^*(0) + V_{16}^*(0)\} - \{V_{00}^*(0) - V_{20}^*(0)\}\{V_{06}^*(0) - V_{16}^*(0)\}] \int_0^\infty tf(t)dt$$

$$D_R = [1 - V_{10}^*(0)][1 - V_{26}^*(0)] - V_{16}^*(0)V_{20}^*(0).$$

7.2.6 PARTICULAR CASE

When we take $\lambda_s = 0$, i.e., the switching device is perfect. This means that $\mu_s = d = 0$ (a perfect switching device does not need repair and the preparation time only occur after the switch is repaired).

$$A_\infty = \frac{\int_0^\infty tf(t)dt}{\int_0^\infty tf(t)dt + \frac{\lambda_b + \mu f^*(\mu + \lambda_b)}{\mu(\mu + \lambda_b)}}.$$

$$MTSF = \frac{(\mu+\lambda_b)[1-f^*(\mu+\lambda_b)]}{\lambda_b+\mu f^*(\mu+\lambda_b)} \int_0^\infty tf(t)dt,$$

which is in agreement with Subramanian (1975).

7.3 MODEL 2 (DUAL MODEL OF 1)

M|M|G system with imperfect switch and preparation time.

7.3.1 SYSTEM DESCRIPTION AND NOTATION

Assumptions 1-6 of section 7.2.1 will also hold for model 2. Assumptions 7 and 8 will be as follows**:**

7: The lifetime of the online unit, offline unit and the switching device are exponentially distributed r.vs.

8: The repair time of a unit of the switching device and preparation time of the repair facility after repairing the switch are arbitrarily distributed r.vs with probability density function g (.), g_s(.) and k(.) respectively**.**

7.3.2 AVAILABILITY ANALYSIS

In order to find the availability analysis of the system, the regenerative events in Table 7.3.1 are required

Table 7.3.1

Event	Outline Unit	State of Offline Unit	Switching Device	Repair facility
E_0	o	o	O	a
E_1	o	o	O	J.na
E_2	o	o	J.R	a
E_3	o	J.R	O	a
E_4	o	q.r	O	J.na
E_5	o	q.r	J.R	a
E_6	q.r	J.R	O	a
E_7	q.r	q.r	O	J.na
E_8	q.r	o	J.R	a
E_9	q.r	q.r	J.R	a

O **:** operable; q.r **:** queuing for repair; a **:** available; J.R **:** repair just started; J.na **:** just not available.

To derive the availability equations $A_0(t)$, the following exhaustive and mutually exclusive events in (0, t] are considered.

1. Neither unit nor switching device fails before t.
2. The online unit fails first.
3. The offline unit fails first.
4. The switching device fails first.

The set of exhaustive and mutually exclusive events in (0,t] used to find $A_i(t)$ are given below (on next page). Figures 7.3.1 and 7.3.2 give various possibilities to be considered while deriving the equations for $A_2(t)$ *and* $A_3(t)$.

Figure 7.3.1

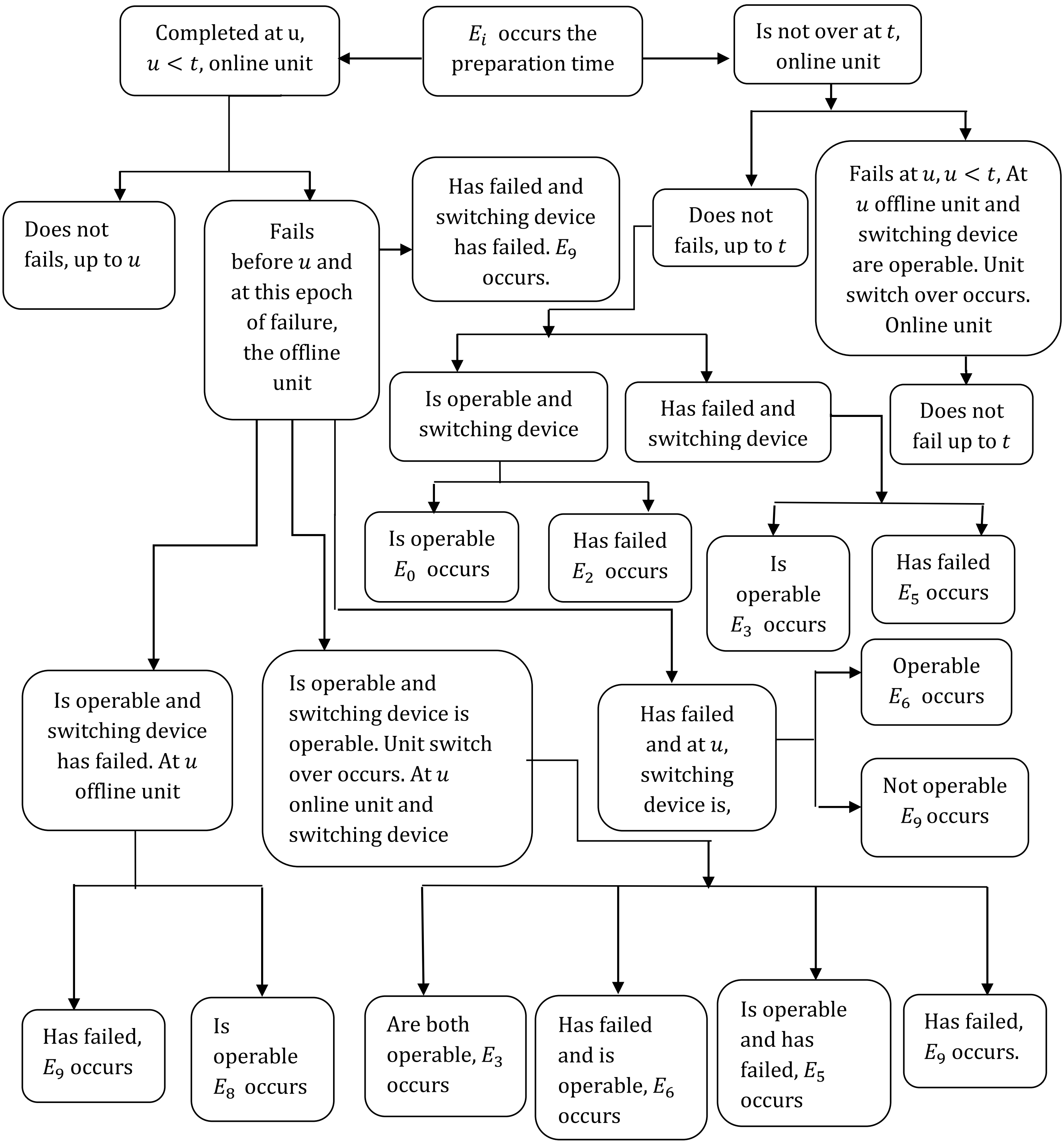
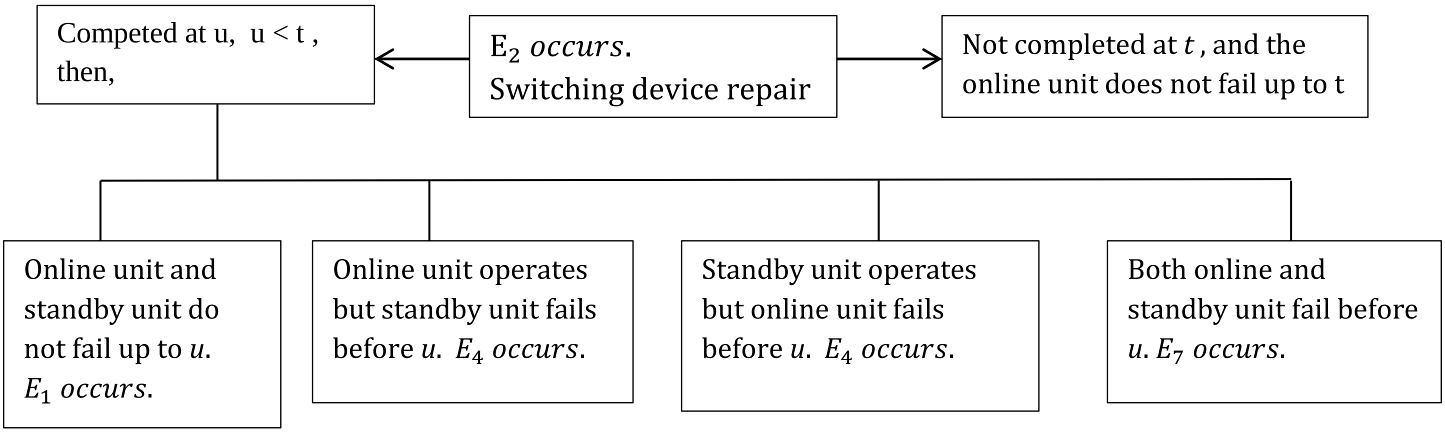

Completed at u, $u < t$, online unit
E_i occurs the preparation time
Is not over at t, online unit
Does not fails, up to u
Fails before u and at this epoch of failure, the offline unit
Has failed and switching device has failed. E_9 occurs.
Does not fails, up to t
Fails at $u, u < t$, At u offline unit and switching device are operable. Unit switch over occurs. Online unit
Is operable and switching device
Has failed and switching device
Does not fail up to t
Is operable E_0 occurs
Has failed E_2 occurs
Is operable E_3 occurs
Has failed E_5 occurs
Operable E_6 occurs
Not operable E_9 occurs
Is operable and switching device has failed. At u offline unit
Is operable and switching device is operable. Unit switch over occurs. At u online unit and switching device
Has failed and at u, switching device is,
Has failed, E_9 occurs
Is operable E_8 occurs
Are both operable, E_3 occurs
Has failed and is operable, E_6 occurs
Is operable and has failed, E_5 occurs
Has failed, E_9 occurs.

Figure 7.3.2



$$A_0(t) = e^{-(\lambda+\lambda_b+\lambda_s)t} + \lambda e^{-(\lambda+\lambda_b+\lambda_s)t} \copyright A_2(t) \qquad (7.3.1)$$

$$\begin{aligned}
A_1(t) = {} & \bar{k}(t)[e^{-\lambda t} + \lambda e^{-(\lambda+\lambda_b+\lambda_s)t} \copyright e^{-\lambda t}] + \left[k(t)e^{-(\lambda+\lambda_b+\lambda_s)t}\right] \copyright A_0(t) \\
& + \left[k(t)e^{-(\lambda+\lambda_b)t}\{1-e^{-\lambda_s t}\}\right] \copyright A_2(t) \\
& + \left[k(t)e^{-(\lambda+\lambda_s)t}\{1-e^{-\lambda_b t}\}\right] \copyright A_3(t) \\
& + \left[k(t)e^{-\lambda t}\{1-e^{-\lambda_s t}\}\{1-e^{-\lambda_b t}\}\right] \copyright A_5(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_s+\lambda_b)t} \copyright e^{-(\lambda+\lambda_s)t}\}\right] \copyright A_9(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_s+\lambda_b)t} \copyright e^{-\lambda_s t}\{1-e^{-\lambda t}\}\}\right] \copyright A_6(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_s+\lambda_b)t} \copyright e^{-\lambda t}\{1-e^{-\lambda_s t}\}\}\right] \copyright A_5(t) \\
& + \left[k(t)\left\{\lambda e^{-(\lambda+\lambda_s+\lambda_b)t} \copyright \{1-e^{-\lambda_s t}\}\{1-e^{-\lambda t}\}\right\}\right] \copyright A_9(t) \\
& + \left[k(t)\left\{\lambda e^{-(\lambda+\lambda_b)t}\{1-e^{-\lambda_s t}\} \copyright \{1-e^{-\lambda_b t}\}\right\}\right] \copyright A_9(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_b)t}\{1-e^{-\lambda_s t}\} \copyright e^{-\lambda_b t}\}\right] \copyright A_8(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_s)t}\{1-e^{-\lambda_b t}\} \copyright \{1-e^{-\lambda_s t}\}\}\right] \copyright A_9(t) \\
& + \left[k(t)\{\lambda e^{-(\lambda+\lambda_s)t}\{1-e^{-\lambda_b t}\} \copyright e^{-\lambda_s t}\}\right] \copyright A_6(t) \\
& + \left[k(t)\{\lambda e^{-\lambda t} \copyright \{1-e^{-\lambda_s t}\}\{1-e^{-\lambda_b t}\}\} \copyright 1\right] \copyright A_9(t)
\end{aligned}$$

(7.3.2)

$$A_2(t) = \overline{G_s}(t)e^{-\lambda t} + \left[g_s(t)e^{-(\lambda+\lambda_b)t}\right]©A_1(t) + \left[g_s(t)e^{-\lambda t}\left\{1 - e^{-\lambda_b t}\right\}\right]©A_4(t)$$
$$+ \left[g_s(t)e^{-\lambda_b t}\left\{1 - e^{-\lambda t}\right\}\right]©A_4(t)$$
$$+ \left[g_s(t)\left\{1 - e^{-\lambda_b t}\right\}\left\{1 - e^{-\lambda t}\right\}\right]©A_7(t). \qquad (7.3.3)$$

$$A_3(t) = \overline{G_s}(t)e^{-\lambda t} + \left[g(t)\left\{1 - e^{-\lambda t}\right\}\left\{1 - e^{-\lambda_s t}\right\}\right]©A_8(t)$$
$$+ \left[g(t)e^{-(\lambda+\lambda_s)t}\right]©A_0(t) + \left[g(t)e^{-\lambda t}\left\{1 - e^{-\lambda_s t}\right\}\right]©A_2(t)$$
$$+ \left[g(t)e^{-\lambda_s t}\left\{1 - e^{-\lambda t}\right\}\right]©A_3(t). \qquad (7.3.4)$$

$$A_4(t) = \bar{k}(t)e^{-\lambda t} + \left[k(t)e^{-\lambda t}\left\{1 - e^{-\lambda_s t}\right\}\right]©A_5(t)$$
$$+ \left[k(t)e^{-\lambda_s t}\left\{1 - e^{-\lambda t}\right\}\right]©A_6(t) + \left[k(t)e^{-(\lambda+\lambda_s)t}\right]©A_3(t)$$
$$+ \left[k(t)e^{-\lambda_s t}\left\{1 - e^{-\lambda t}\right\}\left\{1 - e^{-\lambda_s t}\right\}\right]©A_9(t). \qquad (7.3.5)$$

$$A_5(t) = \overline{G_s}(t)e^{-\lambda t} + \left[g_s(t)e^{-\lambda t}\right]©A_4(t)$$
$$+ \left[g_s(t)\left\{1 - e^{-\lambda t}\right\}\right]©A_7(t). \qquad (7.3.6)$$

$$A_6(t) = \left[g(t)e^{-\lambda_s t}\right]©A_9(t) + \left[g(t)\left\{1 - e^{-\lambda_s t}\right\}\right]©A_8(t). \qquad (7.3.7)$$

$$A_7(t) = \left[k(t)e^{-\lambda_s t}\right]©A_6(t) + \left[k(t)\left\{1 - e^{-\lambda_s t}\right\}\right]©A_9(t). \qquad (7.3.8)$$

$$A_8(t) = \left[g_s(t)e^{-\lambda_b t}\right]©A_4(t) + \left[g_s(t)\{1 - e^{-\lambda_b t}\}\right]©A_7(t). \tag{7.3.9}$$

$$A_9(t) = g_s(t)©A_7(t). \tag{7.3.10}$$

7.3.3 RELIABILTY ANALYSIS

The reliability equations can easily be found from the availability equations by noting that there must not be a system down in $(0, t]$, and omitting those terms which imply that a system failure has occurred.

$$R_0(t) = e^{-(\lambda+\lambda_s+\lambda_b)t} + \lambda e^{-(\lambda+\lambda_s+\lambda_b)t}©R_2(t). \tag{7.3.11}$$

$$\begin{aligned} R_1(t) = {} & \bar{k}(t)\left[e^{-\lambda t} + \lambda e^{-(\lambda+\lambda_s+\lambda_b)t}©e^{-\lambda t}\right] + \left[k(t)\lambda e^{-(\lambda+\lambda_s+\lambda_b)t}©R_0(t)\right] \\ & + \left[k(t)\lambda e^{-(\lambda+\lambda_b)t}\{1 - e^{-\lambda_s t}\}\right]©R_2(t) \\ & + \left[k(t)\lambda e^{-(\lambda+\lambda_b)t}\{1 - e^{-\lambda_b t}\}\right]©R_3(t) \\ & + \left[k(t)e^{-\lambda t}\{1 - e^{-\lambda_b t}\}\{1 - e^{-\lambda_s t}\}\right]©R_5(t) \\ & + \left[k(t)\{e^{-(\lambda+\lambda_s+\lambda_b)t}©e^{-(\lambda+\lambda_s)t}\}\right]©R_9(t) \\ & + \left[k(t)\left\{\lambda e^{-(\lambda+\lambda_s+\lambda_b)t}©\left\{e^{-\lambda t}\left(1 - e^{-\lambda_s t}\right)\right\}\right\}\right]©R_5(t). \end{aligned} \tag{7.3.12}$$

$$\begin{aligned} R_2(t) = {} & \overline{G_s}(t)e^{-\lambda t} + \left[g_s(t)e^{-(\lambda+\lambda_b)t}\right]©R_1(t) \\ & + \left[g_s(t)e^{-\lambda t}\{1 - e^{-\lambda_b t}\}\right]©R_4(t). \end{aligned} \tag{7.3.13}$$

$$\begin{aligned} R_3(t) = {} & \overline{G_s}(t)e^{-\lambda t} + \left[g(t)e^{-(\lambda+\lambda_s)t}\right]©R_0(t) + \left[g(t)e^{-\lambda t}\{1 - e^{-\lambda_s t}\}\right]©R_2(t) \\ & + \left[g(t)e^{-\lambda_s t}\{1 - e^{-\lambda t}\}\right]©R_3(t). \end{aligned} \tag{7.3.14}$$

$$R_4(t) = \bar{k}(t)e^{-\lambda t}\left[k(t)e^{-\lambda t}\left\{1 - e^{-\lambda_s t}\right\}\right]©R_5(t) + \left[k(t)e^{-(\lambda+\lambda_s)t}\right]©R_3(t). \quad (7.3.15)$$

$$R_5(t) = \overline{G_s}(t)e^{-\lambda t} + \left[g_s(t)e^{-\lambda t}\right]©R_4(t). \quad (7.3.16)$$

These equations can be solved by using Laplace transforms and the MTSF can be obtained by the relation,

$$MTSF = R_0^*(0).$$

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